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POINTWISE DISCONTINUOUS FUNCTIONS

by

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INTRODUCTION

The concept of pointwise discontinuity is a fairly recent one in mathematics. Originally introduced as a convenient term in the study of integration, it has quite outgrown its former sphere of usefulness and has had an ever-widening field of application in modern analysis. The appearance in 1899 of the doctor's thesis of M.Baire, in which he investigated the properties of a function approached by continuous functions and found that the necessary and sufficient condition for such approach involves the idea of pointwise discontinuity, firmly grounded the conception in the fundamentals of mathematical theory.

The considerations of the present paper involve a number of investigations into certain phases of the subject of pointwise discontinuity; such as, the construction and classification of pointwise discontinuous functions; their properties, singly and in combination, etc. We have just remarked that this subject is closely related to the question of approach of continuous functions, and this phase of the subject is treated in Chapter III. The final chapter is devoted to a short study of the oscillation function in the general case.

The first chapter is devoted to an exposition of certain concepts and facts which are of fundamental importance in the developments of succeeding chapters. The idea throughout has been to make the treatment such that the thesis will be intelligible to one with an elementary knowledge of the theory of functions. In particular, an understanding of the elements of point set theory is presupposed.

As a matter of convenience for the reader, those portions of the paper which are original are marked with the sign #. In a few cases, proofs of well-known facts are so marked, if the proofs differ enough from those ordinarily given to warrant it. In general, the sign is used to indicate the parts of the work which involve something more than the mere adaptation of material found elsewhere, and it does not mean that the portions so designated are all radically new.

CHAPTER I. GENERAL NOTIONS

1. Functions. A very general definition of function is used in analysis. y is a function of x if for any given value of x there correspond one or more values of y. We shall restrict ourselves in this thesis to the case where a single value of y corresponds to each value of x; y is then said to be a *single valued function* of x.

2. Maximum of a function at a point. About any point A let an interval $(A-h, A+h)$ be constructed. Let $M(x,h)$ be the upper limit of the values of the function in the interval. If $f(x)$ is not bounded above in the interval we say that $M(x,h)=+\infty$ Let h approach 0; $M(x,h)$ never increases and always remains greater than, or at least equal to, $f(A)$; hence, it approaches a finite limit except in the case where it remains infinite. This is called the maximum of the function at the point A, and is denoted by $M(f,A)$, or by $M(A)$. In case $M(x,h)$ remains infinite as h approaches 0, $M(A)=+\infty$.

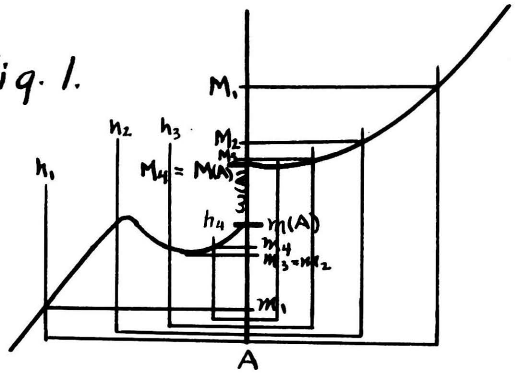
Otherwise stated, the maximum of a function at a point is the lower limit of the maxima of all intervals enclosing the point.

3. Minimum of a function at a point. Let $m(x,h)$ be the lower limit of $f(x)$ in an interval constructed as above. If $m(x,h)$ is not bounded below in the interval we say that $m(x,h)=-\infty$ As h approaches 0, $m(x,h)$ never decreases and always remains less than or at most equal to $f(A)$; hence it approaches a finite limit, except in the case where $m(x,h)=-\infty$ for all values of h. This limit is the minimum of the function at the point A, and is written $m(f,A)$ or $M(A)$. If $m(x,h)=-\infty$ for all values of h, $m(A)=-\infty$

The minimum of the function is the upper limit of the minima of all intervals enclosing the point.

The accompanying figure will make clear the ideas of sections 2 and 3. Several intervals are drawn and the values of M(x,h) and m(x,h) are shown for each.

Fig. 1.



4. About any point A an interval can be found such that, for any given $\varepsilon > 0$, $f(x) < M(A) + \varepsilon$ throughout the interval. For, since M(x,h) never increases and has the limit M(A) as h approaches 0, an interval can be found such that $M(x,h) < M(A) + \varepsilon$. But $f(x) \leqq M(x,h)$ throughout the interval. Therefore $f(x) < M(A) + \varepsilon$.

In a similar way it can be shown that there exists an interval about A in which $f(x) > m(A) - \varepsilon$.

5. Oscillation at a point. The oscillation at the point A, written $\omega(f,A)$ or $\omega(A)$, is defined by the following equation:(1)

$$\omega(A) = M(A) - m(A).$$

If either M(A) or m(A) is infinite, $\omega(A) = +\infty$. Since $M(A) \geqq f(A)$ and $m(A) \leqq f(A)$, it follows that $\omega(A)$ cannot be negative.

6. Semicontinuity. A function is said to be semicontinu-

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(1) Hobson uses the term oscillation somewhat differently, the value of the function at the point in question being left out of consideration. He uses the term saltus for oscillation as here defined. The definition above follows Baire and Borel.

ous above at a point A if $M(A) = f(A)$. It is semicontinuous below at the point if $m(A) = f(A)$. A function is said to be semi-continuous above if it is semi-continuous above at every point; it is semicontinuous below if it is semicontinuous below at every point.

7. It was just shown that there exists an interval about A in which $f(x) < M(A)+\varepsilon$. If the function is semicontinuous above at the point A, $M(A) = f(A)$ and $f(x) < f(A)+\varepsilon$ throughout this interval. Conversely, if, for any preassigned $\varepsilon > 0$, an interval can be found in which $f(x) < f(A)+\varepsilon$ the function is semicontinuous above at A. For, the upper limit of the values of the function in that interval, $M(x,h) < f(A)+\varepsilon$ and, since ε can be made arbitrarily small, the lower limit of the maxima in the intervals resulting is less than or equal to $f(A)$; that is, $M(A) \leqq f(A)$. But $M(A) \geqq f(A)$; therefore, $M(A) = f(A)$, and the function is semicontinuous above at A.

Similarly, if the function is semicontinuous below at A, there exists an interval about A in which $f(x) > f(A)-\varepsilon$, and conversely.

8. Theorem. If $\omega(A) = 0$, A is a point of continuity; and conversely, if A is a point of continuity, $\omega(A) = 0$.

Given any preassigned $\varepsilon > 0$, an interval enclosing A can be found in which $f(x) < M(A)+\varepsilon/2$, and another interval in which $f(x) > m(A)-\varepsilon/2$. In every interval entirely within these two intervals,

$$|f(x) - f(A)| < M(A) - m(A)+\varepsilon = \omega(A)+\varepsilon = \varepsilon,$$

which is the well known condition for continuity.

Conversely, let A be a point of continuity. For a given ε an interval can be found in which,

$$f(A)+\varepsilon/2 > f(x) > f(A)-\varepsilon/2$$

In this interval,

$$M(A) \leqq M(x,h) \leqq f(A)+\varepsilon/2 ,$$

and
$$m(A) \geqq M(x,h) \geqq f(A)-\varepsilon/2 .$$

Then,
$$\omega(A) = M(A) - m(A) \leqq \varepsilon .$$

Since ε may be made arbitrarily small, $\omega(A) = 0$.

Corollary. It follows from the theorem just proved that if $\omega(A) > 0$, A is a point of discontinuity, and conversely.

9. At points of continuity $\omega(A) = M(A) - m(A) = 0$. Then $M(A) = m(A) = f(A)$. In other words, all the points of continuity of $f(x)$ lie on the curves $M(x)$ and $m(x)$.

10. Semicontinuity of $M(x)$ and $m(x)$. It will now be shown that $M(x)$ is semicontinuous above. About any point A we can find an interval for which, with a given ε ,

$$M(A,h) < M(A)+\varepsilon .$$

For any point A' within this interval,

$$M(A') \leqq M(A,h) < M(A)+\varepsilon ,$$

and this is the condition for semicontinuity above of $M(x)$.

Similarly, it can be shown that $m(x)$ is semicontinuous below.

11. The sum of a finite number of functions which are semicontinuous above is semicontinuous above.

Let $F(x) = f_1(x) + f_2(x) + \cdots\cdots\cdots + f_n(x)$, where the f's are semicontinuous above. About any point A we can find, for a given ε , an interval a_1b_1 for which $f_1(x) < f_1(A) + \varepsilon/n$; within this an interval a_2b_2 for which $f_2(x) < f(A) + \varepsilon/n$, and so on. Let ab be an interval enclosed in all the preceding intervals. Then, throughout ab,

$$F(x) = f_1(x) + f_2(x) + \cdots\cdots + f_n(x)$$
$$< f_1(A) + f_2(A) + \cdots\cdots + f_n(A) + n\frac{\varepsilon}{n}$$

That is, $F(x) < F(A) + \varepsilon$,

which is the condition for semicontinuity above.

In a similar way, it can be shown that the sum of a finite number of functions semicontinuous below is semicontinuous below.

12. <u>If f(x) is semicontinuous below at A, −f(x) is semicontinuous above at A.*</u>

For a given ε there exists an interval in which $f(x) > f(A) - \varepsilon$. Multiplying by −1, $-f(x)$ $f(A)$; then f(x) is semicontinuous above at A.

Since $\omega(x)$ is the sum of M(x) and −m(x), two functions semicontinuous above, <u>$\omega(x)$ is semicontinuous above.</u>

13. <u>If a function, f(x), is semicontinuous above the set of points where $f(x) \geqq k$ is closed.</u>

Let A be a limit point of points where $f(x) \geqq k$. Then in any interval, ab, about A, $M(A,ab) \geqq k$. Since M(A) is the lower limit of the maxima of all intervals like ab, it follows that $M(A) \geqq k$. Since f(x) is semicontinuous above, $M(A) = f(A)$, and $f(A) \geqq k$. Thus A belongs to the set; in other words, the set is closed.

In a similar way, it can be shown that if f(x) is semicontinuous below the set of points where $f(x) \leqq k$ is closed.

From what we found of M(x), m(x), and $\omega(x)$ in sections 10 and 12, it follows that:

The set where $M(x) \geqq k$ is closed.

" " " $m(x) \leqq k$ " "

" " " $\omega(x) \geqq k$ " "

CHAPTER II. POINTWISE DISCONTINUOUS FUNCTIONS

Part 1. Definition and Elementary Properties.

14. Definition. A pointwise discontinuous function is a function having a point of continuity in every interval.

Otherwise stated, a pointwise discontinuous function is one in which the points where $\omega=0$ are everywhere dense.

A function which is not pointwise discontinuous is called totally discontinuous.

15. Theorem. In a pointwise discontinuous function the points where $\omega \geqq k$, an arbitrary positive number, form a non dense set; and conversely if the set where $\omega \geqq k$, is non dense for any value of k greater than 0, the function is pointwise discontinuous.

Let K be the set of points where $\omega \geqq k$. Since the set is closed (13), it cannot be dense in any interval without including all the points in the interval. There is then no point of continuity in that interval, for at points of continuity $\omega = 0$. K, then, is non dense in any interval.

To prove the converse, take a sequence of k's approaching 0, $k_1 > k_2 > k_3 > \ldots\ldots > k_n > \ldots\ldots$ and let K_1, K_2, ...K_n, ... be the non dense sets corresponding. In any interval ab there is an interval a_1b_1 containing no points of K_1, since K_1 is non dense. Similarly in a_1b_1 there is an interval a_2b_2 containing no points of K_2, and so on. The intervals ab, a_1b_1, a_2b_2, ... a_nb_n, ... have at least one point A in common. Given any k_n, A lies in a_nb_n, and $\omega(A) < k_n$. Since k_n may be made arbitrarily small, $\omega(A) = 0$; and A is a point of continuity. There is thus a point of continuity in every interval, and the function is pointwise discontinuous.

This property, viz., that the points where $\omega \geqq k$ form a non dense set, is used by Hobson as the definition of a pointwise discontinuous function.(i)

16.# Theorem: In a pointwise discontinuous function $m(\omega, x) = 0$ at every point; conversely, if $m(\omega, x) = 0$ at every point the function is pointwise discontinuous.

The first part of the proposition is evident. ω is never negative, and every interval contains points at which $\omega = 0$. Then $m(\omega, x) = 0$ at every point.

To prove the second part, let K be the set of points for which $\omega \geqq k$. It will be shown that K is non dense. Suppose K to be dense in some interval; then, since K is closed (13), it includes all the points in that interval. Then $\omega \geqq k$ and $m(\omega, x) \geqq k$ throughout the interval. But this is contrary to the hypothesis that $m(\omega, x) = 0$ everywhere. Then K is non dense, and we have just found that this is a sufficient condition for pointwise discontinuity (15).(ii)

17. Harnack(iii) has defined a pointwise discontinuous function with the added restriction that the points of discontinuity shall be of content 0. While this definition is important in the theory of integration, it narrows very much the application of these functions in other fields; and the work of Baire, done since Harnack's time, on the approach to discontinuous functions by

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(i) Hobson, Theory of Functions of a Real Variable, p.243.

(ii) This proposition is not new; but I have not seen a proof along these lines. Baire, Lecons sur les fonctions dixcontinues, pp.74, 75, has a proof that establishes practically the same thing, but is quite extended.

(iii) Math. Annalen, Vol.XIX., 1882, p.242, and Vol.XXIV., 1884, p.218.

continuous ones would require the coining of a new word for pointwise discontinuity as we have defined it above.

Part 2. Sets of the First and Second Categories.

18. Sets which are formed by the union of a countable number of non dense sets, like the K sets of the preceding paragraphs, are of such fundamental importance in the theory of pointwise discontinuous functions that we shall give here some of their properties. Such a set is called by Baire a set of the first category. A set G is of the first category then if it consists of all the points contained in any of the sets G_1, G_2, G_3, ... G_n, ..., each of which is non dense in any interval. A set which cannot be so constituted is of the second category.(1)

19. If G is of the first category, then in any interval there is a point not belonging to G. For, in any interval, ab, there is an interval a_1b_1 containing no points of G_1, since G_1 is non dense. In a_1b_1 there is an interval a_2b_2 containing no points of G_2, and so on. The intervals ab, a_1b_1, a_2b_2, ... a_nb_n, ..., each of which is contained in the preceding, have at least one point A in common. A does not belong to G_n for any value of n since it lies in a_nb_n; hence it does not belong to G.

Corollary. Since a set of the first category does not include all the points of the continuum, the continuum is of the

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(1) This is the definition of sets of the first and second categories as given by Baire who originated the ideas. Borel and W.H.Young follow him. Hobson includes the condition that the component sets shall be closed. His definition has certain advantages of a minor nature, but the definition originally given by Baire is in more general use. See:

Baire, Lecons sur les fonctions discontinues, p.78.
Borel, Lecons sur les fonctions de variables reelles, p.21.
Young, The Theory of Sets of Points, p.70.
Hobson, The Theory of Functions of a Real Variable, p.114.

second category.

20. A countable set is of the first category, for it may be formed by the combination of a countable number of sets each of which consists of an individual point.

A set of the first category may be everywhere dense. A dense countable set, like the rationals, is an example.

21. Theorem: The sum of a finite number or of a countable infinity of sets of the first category is a set of the first category.

The resulting set, G, is formed by the union of all the non dense sets constituting the different sets of the first category. But these non dense sets are countable, for it is a well known theorem that a countable infinity of countable sets is countable. Then G is formed by the union of a countable number of non dense sets, and is therefore of the first category.

22. The set that remains on a line after the removal of a set of the first category——the complement of a set of the first category——is a set of the second category. If this were not true, the sum of the two sets, by the theorem just proved, would be of the first category. But the sum of the two is the whole continuum, and this we found to be of the second category.

23. Theorem: The points of discontinuity of a pointwise discontinuous function constitute a set of the first category.

We have found (15) that the points where $\omega \geqq k$, for $k > 0$, form a non dense set. Consider a sequence of k's approaching 0, $k_1 > k_2 > k_3 > \cdots\cdots > k_n > \cdots$. The points where $\omega \geqq k_1$ form a non dense set. The same is true of the points where $k_1 > \omega \geqq k_2$, $k_2 > \omega \geqq k_3$, and so on. The necessary and sufficient condition for a point of discontinuity is that $\omega > 0$ (8). Hence,

any point contained in any of the above sets is a point of disconuity, and conversely any point of continuity will appear in one of the above sets. The set of all the points of discontinuity is the union of all these non dense sets, and is thus of the first category.

24. The set of the points of continuity, which is the complement of the above set, is a set of the second category. Since a countable set is of the first category, it follows that the points of continuity of a pointwise discontinuous function are more than countable. It can be proved that the points of continuity have the power of the continuum, but to include the proof here would unnecessarily lengthen the treatment. (i)

25. We shall now investigate the question whether any set of the first category is the set of the points of discontinuity of some function, while the complementary set of the second category is the set of its points of continuity. This appears not to be true in general. However, the following sufficient, but not necessary, condition can be stated:

Theorem: If a set G can be broken up into non dense component sets, G_1, G_2, ... G_n, ... , which are closed, a function f(x) can be set up having its points of discontinuity in G and its points of continuity in the complement of G. (ii)

Let α_1, α_2, α_3, ... α_n, ... be a sequence of decreasing positive numbers approaching 0. Define f(x) as follows:

(i) See Young, The Theory of Sets of Points, p.71, where the proposition is established by means of theorems on inner limiting sets.

(ii) Since Hobson's definition of sets of the first category requires that the component sets be closed, the proposition is true for all sets of the first category as he uses the term.

$f(x) = \alpha_1$, for points on G_1;

$f(x) = \alpha_2$, for points on G_2, and not on G_1;

. . . .

$f(x) = \alpha_n$, for points on G_n, and not on any preceding set;

. . . .

$f(x) = 0$, for points not belonging to G.

It will now be shown that every point of G is a point of discontinuity, while every point not belonging to G is a point of continuity. Consider first a point P, belonging to G. P belongs to one or more of the component sets; suppose G_m is the first set in which it appears. Then $f(P) = \alpha_m$, and $M(P) = \alpha_m$. Since the second category points on which $f(x) = 0$ are everywhere dense (19), $m(P) = 0$. The $\omega(P) = \alpha_m > 0$, and P is a point of discontinuity.

Secondly, let A be a point not belonging to G. Since G_1 is closed A cannot be a limit point of G_1 without belonging to the set. Then an interval $a_1 b_1$, including A in its interior and containing no points of G_1, can be constructed. Then $M(A) \gtreqless M(A, a_1 b_1) < \alpha_1$. Within $a_1 b_1$ we can construct another interval about A containing no points of G_2, and so on. In general, $M(A) \gtreqless M(A, a_n b_n) < \alpha_n$, where α_n may be made as small as desired. Then $M(A) = 0$, and $\omega(A) = M(A) - m(A) = 0$. A is thus a point of continuity.

<u>26</u>. # From this proposition it follows that any countable set can be made the set of the points of discontinuity of a pointwise discontinuous function. For, each of the component sets can be taken as composed of a single point or of a finite number of points. Since a finite number of points have no limit point the conditions of the above proof are satisfied.

The proposition of (25) gives us a ready means of constructing pointwise discontinuous functions of considerable complexity. This method will be used in the following section to build up various types of pointwise discontinuous fuhctions.

Part 3. Examples of Pointwise Discontinuous Functions

27. In this section will be given a few typical examples which will be useful for fixing the ideas and for reference in succeeding pages. Pointwise discontinuous functions may be classified into seven groups, the basis of the classification being the number and the distribution along the continuum of the points of discontinuity.

28. Class I. No points of discontinuity. The function is then everywhere continuous. This type is so well known that an example is unnecessary.

29. Class II. Points of discontinuity finite in number. The discontinuous functions of elementary mathematics belong to this class. A familiar example is the function, $f(x) = \sin 1/x$ for $x \neq 0$, $f(0) = 0$. It has a single point of discontinuity; viz., at the origin. The graph of this function is shown in Figure 2. Its oscillation function is given in Figure 3; its value is 0 at every point except at the origin, where it is equal to 2.

Another example is shown in Figure 4, where $f(a) = f(b) = f(c) = f(d) = 1$, and at other points $f(x) = x$. Figure 5 is its oscillation function.

30. Class III. Points of discontinuity infinite in number, countable, and not dense in any interval. Figure 6 is such a function; $f(x) = x^2/2$ for $x = (3/4)^n$, $f(x) = x$ otherwise. Its

Fig. 2.

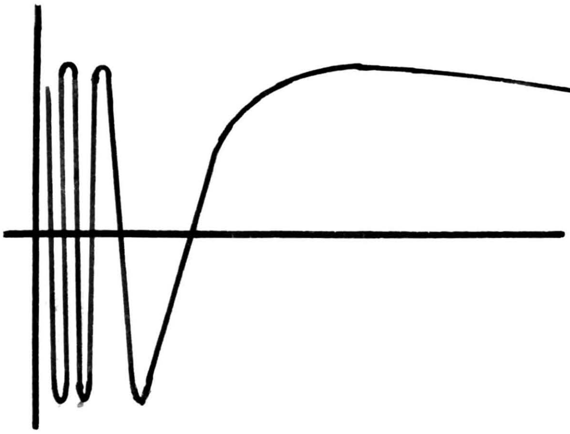


Fig. 4.

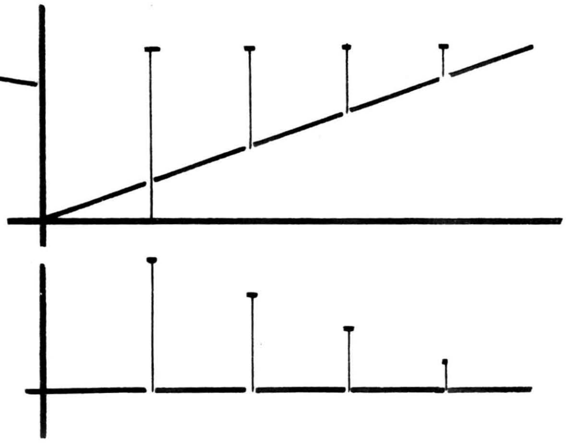


Fig. 5.

Fig. 3.

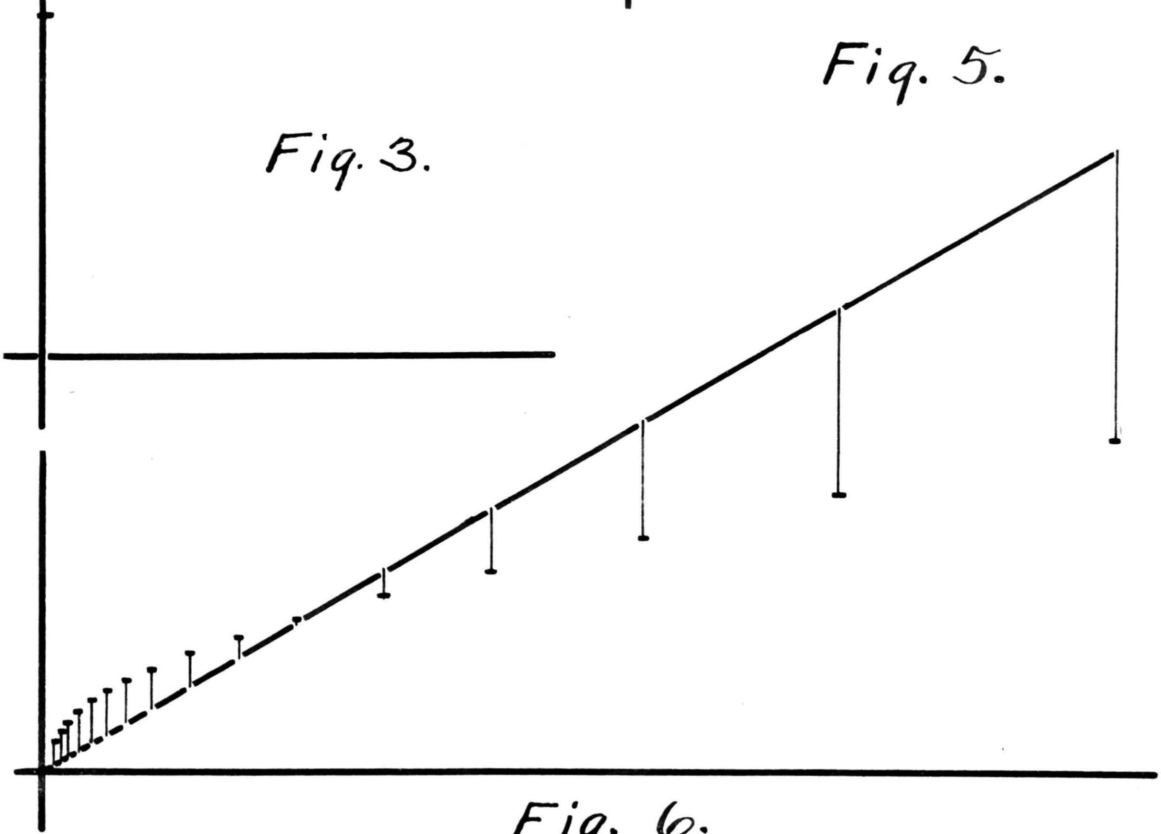


Fig. 6.

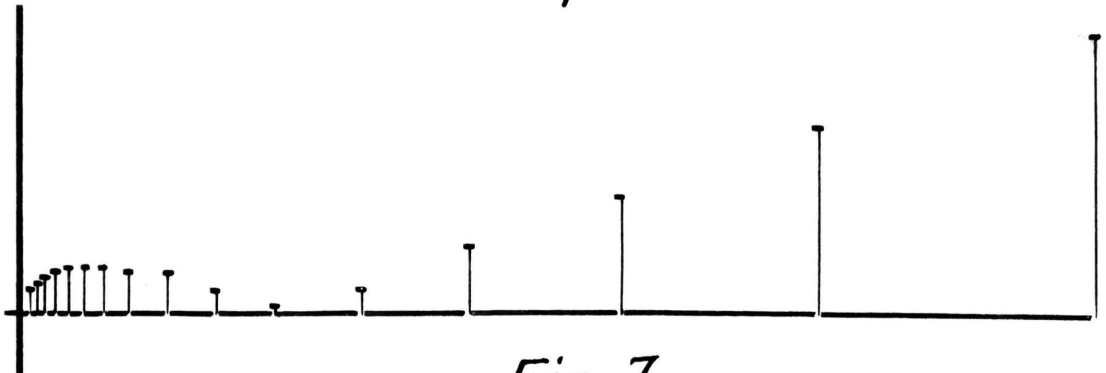


Fig. 7.

oscillation function is given in Figure 7.

$f(x) = 1$, for $x \doteq p^n (p < 1)$; $f(x) = 0$ everywhere else is another. Any set with a finite number of limit points can easily be made the set of points of discontinuity for a function of this class. Most of the discontinuous functions of elementary analysis belong to classes II and III.

31. Class IV. Points of discontinuity countable and dense in some interval. This includes the functions whose points of discontinuity are countable and dense in every interval. The method of (25) can be used to construct a function with points of discontinuity on any dense countable set. Figure 8 shows a function where the points of discontinuity are the rationals. $f(m/n) = 1/n$, m/n being in its lowest terms; $f(0) = 1$; $f(x) = 0$ on the irrationals. Referring to (25) the following component sets were used:

$G_1 = 0, 1; \quad \alpha_1 = 1;$

$G_2 = 1/2, \quad \alpha_2 = 1/2;$

$G_3 = 1/3,\ 2/3, \qquad 1/3.$

. . . . . .

$G_n = 1/n,\ 2/n, \ldots\ldots\ldots (n-1)/n; \quad \alpha_n = 1/n.$

The figure shows the values of the functions on the sets G_1 to G_{12}. $\omega(f, x)$ coincides with $f(x)$.

32. Class V. Points of discontinuity more than countable and not dense in any interval. The classic example of such a function is the one shown in Figure 9. $f(x) = 0$ on the part of the line lying between $x = 1/3$ and $x = 2/3$; also between 1/9 and 2/9, and between 7/9 and 8/9; between 1/27 and 2/27, 7/27 and 8/27, 19/27 and 20/27, 25/27 and 26/27; and so on. $f(x) = a$ at the remaining points. Otherwise stated, the points where

Fig. 8.

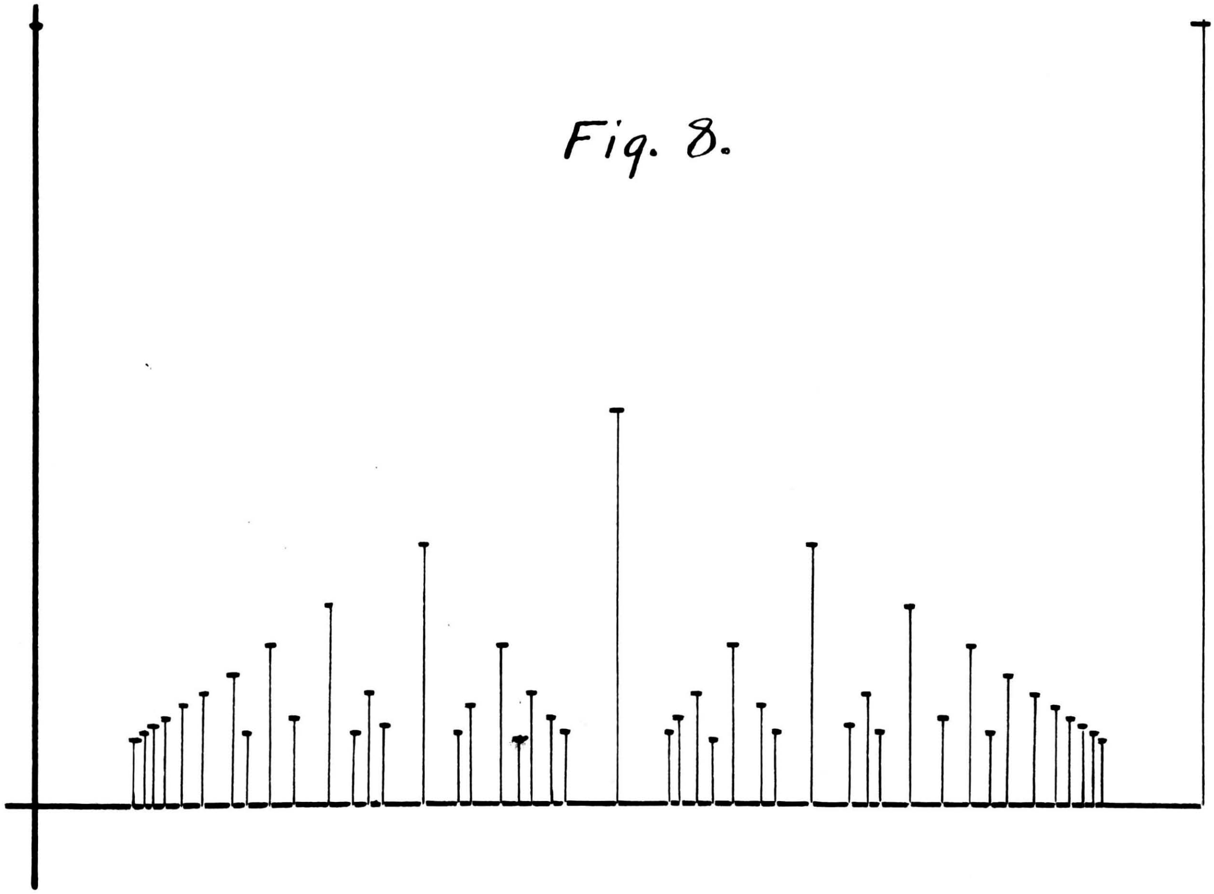
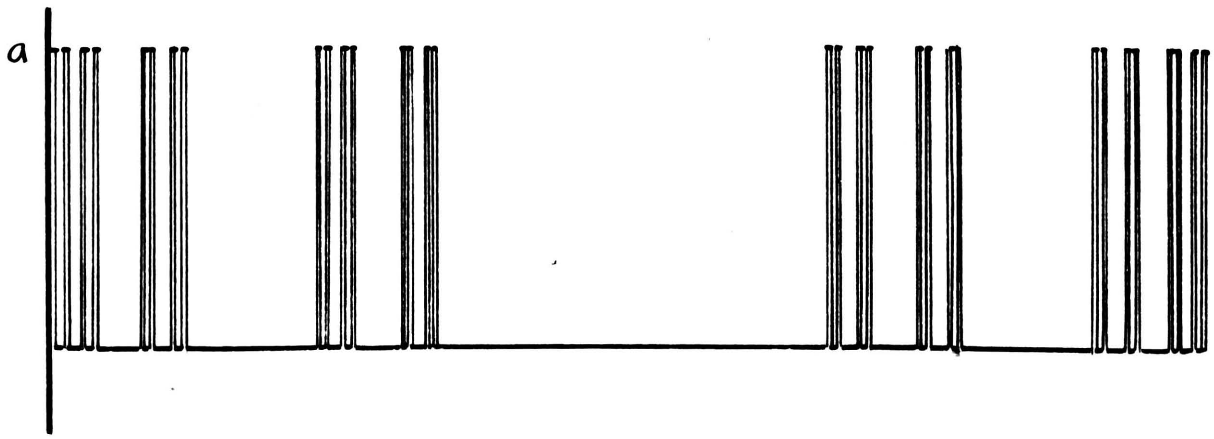


Fig. 9.



$f(x) = a$ are the points that are left after removing from the line the middle third, exclusive of the end points; then removing the middle third of each portion that remains; and so on. The points which are left form a perfect set, which, as is well known, has the power of the continuum. The points of the perfect set will be points of discontinuity, for any interval about any point of the set will have $m(f,x) = 0$, and $\omega = a$. Every other point is a point of continuity, for a finite interval can be drawn about it excluding all points of the perfect set. In this interval, $M(x) = m(x) = 0$, and $\omega(x) = 0$.

This function can be simply defined in another way. If the x coördinates be expressed in triadic fractions, the points where $f(x) = a$ are the following:

The point 0.00;

All terminating in the figure 1 preceded by only the figures 0 and 2; as, .01, .0221, etc.

All composed of 0's and 2's only; as, .202, .0202022....... (A number containing an infinite number of 2's and no 1's or 0's after a given point, like .012222......, is to be replaced by its simpler equivalent, .02). $\omega(x)$ coincides with the original function.

33. Class VI. Points of discontinuity not countable and everywhere dense, but not condensed in any whole interval.(i) We can easily build such a function from the examples of Classes IV and V by the method of (25).

$G_1 =$ the non dense perfect set of V; $f(x) = 1$ on G_1;

$G_2 = 1/2$, $f(x) = 1/2$ on G_2p

$G_3 = 1/3, 2/3$, $f(x) = 1/3$ on G_3;

. . . .

$G_n = 1/n, 2/n, \ldots (n-1)/n$ (fractions not in their lowest terms omitted) $f(x) = 1/n$ on G_n. It is understood that if m/n is also a point of G_1, f(x) is to have the value 1 on that point.

$f(x) = 0$ everywhere else.

34. Class VII. Points of discontinuity condensed on every point is some interval. Figure 10 is an example of such a function. G_1 is the non dense perfect set of (32); $f(x) = 1$ on G_1. G_2 is the set of points gotten by constructing in each of the open intervals left by G_1 a set bearing the same relation to the interval that G_1 does to the interval 0-1; $f(x) = 1/2$ on G_2. G_3 is constructed in a similar way on the intervals left by G_2; $f(x) = (1/2)^2$ on G_3; and so on. Every point is a point of condensation of points of discontinuity; yet, by (25) there is a point of continuity in every interval.

Put in terms of the triadic system of notation, G_1 is:

The point 0.00;

All terminating in 1, preceded by 0's and 2's.

All composed only of 0's and 2's.

G is:

All ending in 1, the numbers preceding containing a single 1 and the remaining figures being 0's and 2's; as, .10201, .0210201.

All having a single 1, the last figure, if there is one, being a 2; as, .102, .021022, .21202202.....

. . . .

. . . .

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(1) A set of points is condensed at a point if every interval enclosing the point contains a more than countable number of points of the set; the set is condensed in a given interval if

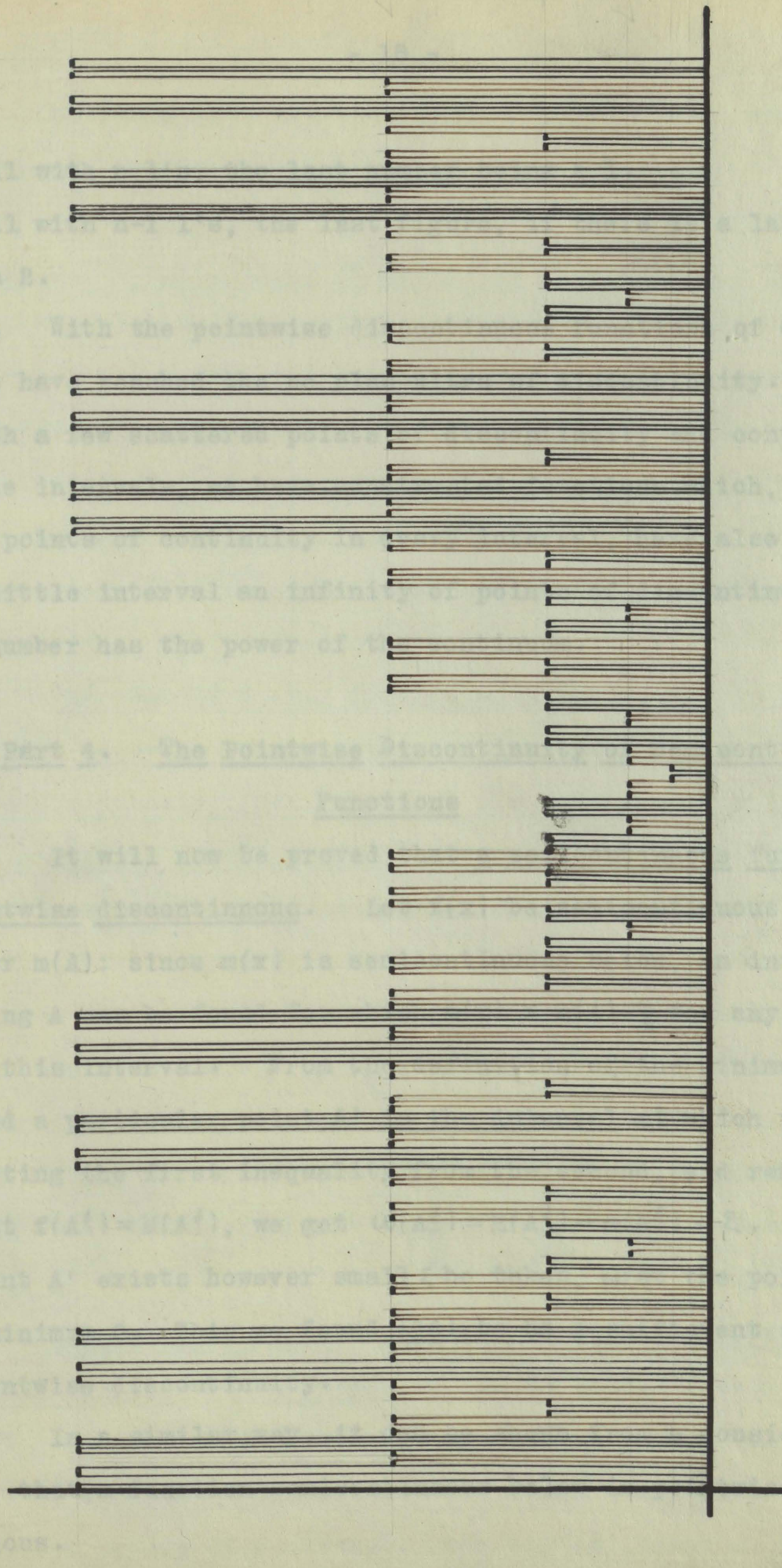

Fig. 10.

G_{η} is:

All with n 1's, the last number being a 1.

All with n-1 1's, the last figure, if there is a last one, being a 2.

35. With the pointwise discontinuous functions of Class VII, we have reached the ne plus ultra of discontinuity. Starting with a few scattered points of discontinuity and continuity by whole intervals, we have constructed functions which, while having points of continuity in every interval, have also in every little interval an infinity of points of discontinuity whose number has the power of the continuum.

Part 4. The Pointwise Discontinuity of Semicontinuous Functions

36. It will now be proved that a semicontinuous function is pointwise discontinuous. Let f(x) be semicontinuous above. Consider m(A): since m(x) is semicontinuous below, an interval enclosing A can be found for which $m(x) > m(A) - \frac{\varepsilon}{2}$ for any point within this interval. From the definition of the minimum we can find a particular point A' in the interval at which $f(A') < m(A) + \frac{\varepsilon}{2}$. Subtracting the first inequality from the second, and remembering that $f(A') = M(A')$, we get $\omega(A') = M(A') - m(A') < \varepsilon$. Since the point A' exists however small ε be taken, ω at the point A has a minimum 0. This we found (16) to be a sufficient condition for pointwise discontinuity.

In a similar way, it can be shown from a consideration of M(A) that a function semicontinuous below is pointwise discontinuous.

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it is condensed at every point of the interval.

We found (10, 12) that $M(x)$, $m(x)$, and $\omega(x)$ are semi-continuous. They are then pointwise discontinuous.

Part 5. Combinations of Pointwise Discontinuous Functions

37. In this section will be developed some of the fundamental facts concerning the combination of pointwise discontinuous functions by addition, subtraction, multiplication, and division. After treating a finite number of functions we shall take up the case of infinite sequences of functions. Attention will first be called to a fact, present mention of which will avoid repetition in the proofs.

38. Theorem: If $f_1(x)$, $f_2(x)$, ...$f_n(x)$, ... is a finite or infinite sequence of pointwise discontinuous functions, there exist points in every interval which are points of continuity of each and every function of the sequence. For, we found in (23) that the points of discontinuity of each function form a set of the first category. We also learned (21) that a finite number or a countable infinity of sets of the first category is a set of the first category; then the set of all the points of discontinuity of the functions is of the first category. There are points in every interval not belonging to this first category set (19); such points are points of continuity of each function.

39. Theorem: The sum (difference) of two pointwise discontinuous functions is a pointwise discontinuous function. For, if two functions are continuous at a point their sum, or difference, is continuous at the point.(1) Since every interval contains common points of continuity, the sum (difference) has points of continuity in every interval; hence it is pointwise

discontinuous.

It follows from this fact that the combination by addition and subtraction of a finite number of pointwise discontinuous functions gives a pointwise function.

40. # The following theorem can be proved in a manner analogous to the preceding: The product of two pointwise discontinuous functions is a pointwise discontinuous function.

It follows from this and the preceding article that any integral rational function of pointwise discontinuous functions is pointwise discontinuous.

41. It can be shown similarly that the quotient of two pointwise discontinuous functions is pointwise discontinuous, provided the function in the denominator is nowhere equal to 0. This provision is not necessary, but it is sufficient.

42. Turning now to an infinite series of functions we shall prove the following important theorem: If $F(x)=f_1(x)+\ldots+f_n(x)+\ldots$ is a uniformly convergent series, any common point of continuity of all the f's is a point of continuity of $F(x)$.

Let A be a point of continuity of all the f's. But $F(x)=F_n(x)+R_n(x)$, where $F_n(x)$ is the sum of the first n terms, and consequently has A as a point of continuity, and $R_n(x)$ is the remainder term. Since the series is uniformly convergent, for a given $\varepsilon>0$, we can find an m such that, for all values of x, $|R_n(x)|<\varepsilon/3$, for $n>m$. Fixing n, and as a consequence $F_n(x)$, we can find an $h>0$ such that $|F_n(x')-F_n(A)|<\varepsilon/3$, for $|x'-A|<h$. Now, $|R_n(x')|<\varepsilon/3$ and $|R_n(A)|<\varepsilon/3$; whence,

$$|F_n(x')+R_n(x')-F_n(A)-R_n(A)|<\varepsilon, \text{ for } |x'-A|<h.$$

That is, $|F(x')-F(A)|<\varepsilon$, for $|x'-A|<h$;

and A is a point of continuity.

43. Corollary 1. A uniformly convergent series of continuous terms is everywhere continuous. This is well known.

44. # Corollary 2. A uniformly convergent series of pointwise discontinuous terms is pointwise discontinuous.

45. A pointwise discontinuous function can be built from any sequence of pointwise discontinuous functions. If the sum of the functions is not a uniformly convergent series, the terms can be multiplied by convergence factors. If all the functions have a least upper bound, any absolutely convergent series, $\alpha_1, \alpha_2, \ldots\ldots\ldots\ldots \alpha_n, \ldots\ldots$ will accomplish the result; and the series will assume the form,

$$F(x) = \alpha_1 f_1(x) + \alpha_2 f_2(x) + \ldots\ldots\ldots\ldots + \alpha_n f_n(x) + \ldots\ldots$$

It is not necessary that the convergence factors be constants. The series may be constructed as follows:

$$F(x) = \phi_1(x) f_1(x) + \phi_2(x) f_2(x) + \ldots\ldots\ldots + \phi_n(x) f_n(x) + \ldots\ldots$$

where $\phi_1(x)$, $\phi_2(x), \ldots\ldots\ldots$ $\phi_n(x), \ldots\ldots$ are continuous or pointwise discontinuous functions so chosen that the series converges uniformly.

46. So far we have been attending to the preservation of the points of continuity; and we found that for integral rational functions and uniformly convergent series every point which is a point of continuity of each and every function is a point of continuity of the function resulting from their combination. We shall now investigate the question of the preservation of the points of discontinuity. In the general case we can say nothing. It is easy, for instance, to set up functions whose points of continuity are very complicated, but whose sum is

(1) The proof of this and similar statements made later are well known, and can be found in many places. See Bocher, Introduction to Higher Algebra, pp. 14-16.

everywhere continuous. Thus, let $f_1(x)$ be any function whatever, pointwise discontinuous or totally discontinuous, and let $f_2(x) = -f_1(x)$. Then the sum of the two equals 0, and is continuous everywhere.

<u>47</u>. # As an aid to the further investigation of the problem we shall prove the following proposition: <u>If $F(x)=f_1(x)+f_2(x)$, then at any point A,</u>

$$\omega(f_1,A)+\omega(f_2,A) \geqq \omega(F,A) \geqq |\omega(f_1,A)-\omega(f_2,A)|$$

From the definitions of maximum and minimum at a point, we can write the following very evident inequalities:

1. $M(F,A) \leqq M(f_1,A)+M(f_2,A);$ $\quad m(F,A) \geqq m(f_1,A)+m(f_2,A).$
2. $M(F,A) \geqq M(f_1,A)+M(f_2,A);$ $\quad m(F,A) \leqq m(f_1,A)+M(f_2,A).$
3. $M(F,A) \geqq m(f_1,A)+M(f_2,A);$ $\quad m(F,A) \leqq M(f_1,A)+m(f_2,A).$

From 1, $\omega(F,A)=M(F,A)-m(F,A) \leqq \omega(f_1,A)+\omega(f_2,A).$

From 2, $\omega(F,A) \geqq \omega(f_1,A)-\omega(f_2,A).$

From 3, $\omega(F,A) \geqq \omega(f_2,A)-\omega(f_1,A).$

The proposition follows at once from these inequalities.

<u>48</u>. # We can extend this, under certain conditions, to the case of a converging infinite series of functions. Let the series be, $F(x)=f_1(x)+f_2(x)+ \cdots\cdots +f_n(x)+ \cdots\cdots$ Suppose that at the points of discontinuity of $f_n(x)$, $\omega(f_n,x)=\alpha_n$ a constant > 0, and suppose further that the series, $\alpha_1+\cdots+\alpha_n+\cdots$ is one in which $R_n < \alpha_n$. Then all the points of discontinuity of the different functions persist as points of $F(x)$. To show this, let A be a point of discontinuity of one or more of the functions, and let $f_r(x)$ be the first function in which A appears as a point of discontinuity. Then the least possible value $\omega(F,x)$ could have would be,

$$\alpha_r - \alpha_{r+1} - \cdots - \alpha_n - \cdots\cdots = \alpha_r - R_r > 0$$

A would thus be a point of discontinuity.

There are many series of the type mentioned; viz., where $R_n < \alpha_n$. The geometrical series, $a + ar + ar^2 + \cdots + ar^n + \cdots$ where $0 < r < 1/2$, is an example. Here $R_n = ar^{n+1}(1 + r + r^2 + \cdots) = ar^{n+1}/(1-r) = ar^n \cdot r/(1-r) < ar^n$. Any other series in which $\alpha_{n+1}/\alpha_n < 1/2$ is an example; for instance, the series,

$$1 + 1/2 + 1/3! + \cdots\cdots\cdots + 1/n! + \cdots\cdots,$$

or the series,

$$1 + 1/2^2 + 1/3^3 + \cdots\cdots\cdots + 1/n^n + \cdots\cdots$$

49. # Let $f_1(x)$, $f_2(x)$, ... $f_n(x)$, ... be a sequence of functions, each being bounded and in each of which $\omega_n = a_n$, a constant, at the points of discontinuity; then a function,

$$F(x) = d_1 f_1(x) + d_2 f_2(x) + \cdots\cdots + d_n f_n(x) + \cdots\cdots$$

can be set up, having as its points of continuity the points of continuity common to the different functions, and as its points of discontinuity every point of discontinuity of each of the functions.

Let B_n be an upper bound of $|f_n(x)|$. We can set up a uniformly convergent series,

$$\phi(x) = b_1 f_1(x) + b_2 f_2(x) + \cdots\cdots + b_n f_n(x) + \cdots\cdots$$

by taking $b_n = c_n/B_n$, where c_1, c_2, ...c_n, ... is any absolutely converging series of positive terms. Then by (42) all the common points of continuity of the terms will be points of continuity of $\phi(x)$. These will still be points of continuity if each term is multiplied by a constant and the sequence of multipliers has an upper bound, for the series will still be uniformly convergent.

It is now necessary to reconstruct the series, on the basis of the last remark, so that the points of discontinuity of the terms shall persist. The oscillation at each point of discontinuity of the term $b_n f_n(x)$ is $b_n a_n$. Beginning with the second term multiply each term by such a constant that $b_{n+1} a_{n+1} / b_n a_n < 1/2$, and let all the multipliers be bounded above (less than 1, for instance). I say that all the points of discontinuity of the separate functions are points of discontinuity of the resulting function,

$$F(x) = d_1 f_1(x) + d_2 f_2(x) + \cdots\cdots + d_n f_n(x) + \cdots\cdots;$$

for we have thus a series of functions whose oscillations form a series of the type mentioned in (48), and in this case we found that the points of discontinuity are preserved.

50. # **We can go further** and state that if the functions are bounded and if $\omega_n(x) \geqq a_n$ at all the points of discontinuity of $f_n(x)$, we can set up a function in which the points of discontinuity of the separate functions persist.

After setting up the uniformly convergent series,

$$\phi(x) = b_1 f_1(x) + b_2 f_2(x) + \cdots\cdots + b_n f_n(x) + \cdots\cdots,$$

as before, which provides for the preservation of the common points of continuity, we choose our constants (less than 1, for example) so that $b_{n+1} B_{n+1} / b_n a_n < 1/2$. Then if A is a point of discontinuity which appears first as a point of discontinuity in $f_r(x)$, the least possible value $\omega(A)$ could have would be,

$$b_r a_r - b_{r+1} B_{r+1} - b_{r+2} B_{r+2} - \cdots\cdots - b_n B_n - \cdots\cdots$$

Since $b_{n+1} B_{n+1} / b_n a_n < 1/2$, $b_{n+1} B_{n+1} / b_n B_n < 1/2$, for $a_n < B_n$; and the series is of the type mentioned in (48). Then the points of discontinuity of the component functions persist in F(x).

51. It should be noted that the work of 47 - 50 applies to all kinds of functions, and not merely to pointwise discontinuous functions. By making the component functions pointwise discontinuous we have a new method of constructing a very complicated pointwise discontinuous function, which has for its points of discontinuity all the points of discontinuity of each of the sequence of functions. Attention should be called to the fact that if the component functions are pointwise discontinuous their points of discontinuity are non dense in any interval; for the requirement that $\omega_n(x) \geqq a_n$ at the points of discontinuity would force $\omega_n(x) \geqq a_n$ at every point, in the whole interval if the points of discontinuity were dense, since the set where $\omega_n(x) \geqq a_n$ is closed (13); and there would be no point of continuity in the interval.

Part 6. Pointwise Discontinuous Functions in n Variables

52. We shall now extend to space of n dimensions the ideas developed for one dimensional space.

Limit Point. A point A is a limit point of a set of points in n dimensions if every sphere with A as center contains points of the set (A not considered). A point inside the sphere satisfied the inequality $(x_1-x_1^{(A)})^2 + (x_2-x_2^{(A)})^2 + \dots\dots$ $+ (x_n-x_n^{(A)})^2 < R^2$, where R is the radius of the sphere. Since we can enclose a parallelopiped by a sphere or a sphere by a parallelopiped, enclosing the point in each case, we can, if it more convenient, use a parallelopiped instead of a sphere. Then if there exists a point such that $x_1^{(A)} - h_1 < x_1 < x_1^{(A)} + h_1$, $x_2^{(A)} - h_2 < x_2 < x_2^{(A)} + h_2$ .... $x_n^{(A)} - h_n < x_n < x_n^{(A)} + h_n$, however small the

positive numbers, $h_1, h_2, \ldots\ldots h_n$ be taken, A is a limit point of the set.

53. Maximum of a function at a point. Let $f(x_1, x_2, \ldots x_n)$ be the function, and let A be the point under consideration. With A as center, describe a sphere with radius R.(1) Let M(A,R) be the upper limit of the values of the function within the sphere. Let R approach 0; M(A,R) never increases and remains always greater than f(A); hence it approaches a limit, and this limit is called the maximum of the function at the point. It is denoted by M(f,A) or M(A) as before.

54. Minimum of a function at a point. By taking the lower limit of the values of the function in the sphere we can define the minimum of the function at the point, in a manner analogous to its definition in 3; likewise the oscillation at the point, $\omega(A) = M(A) - m(A)$. Provision must be made for infinite values of M(A), m(A), and $\omega(A)$, as in 2-4.

55. Without going through the details of the proofs, which are similar to those in the case of a function of a single variable, I will merely state the results that may be arrived at:

1. If $\omega(A) = 0$, A is a point of continuity; and, conversely, if A is a point of continuity, $\omega(A) = 0$.

2. $M(x_1, \ldots x_n)$ and $\omega(x_1 \ldots x_n)$ are semicontinuous above; $m(x_1, \ldots x_n)$ is semicontinuous below.

3. The sets where $M(x_1, \ldots\ldots x_n) \gtreqless k$, where $m(x_1, \ldots\ldots x_n) \lesseqgtr k$, and where $\omega(x_1, \ldots\ldots x_n) \gtreqless k$ are each closed.

56. Pointwise discontinuity. A pointwise discontinuous function in space of n dimensions is one having points of con-

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(1) In two dimensions the sphere becomes a circle in the coordinate plane.

tinuity in every sphere; that is, the points of continuity are everywhere dense.

It can be shown, in the manner of 15, that in a pointwise discontinuous function the set of points where $\omega \geqq k > 0$ is non dense; and, conversely, if the set where $\omega \geqq k$ is non dense whatever value, greater than 0, k be given the function is pointwise discontinuous.

57. Semicontinuous functions. Semicontinuous functions can be shown to be pointwise discontinuous. If in 36 we read the word "sphere" for interval" the proof holds throughout.

58. Sets of the first and second categories. A set G is of the first category if it consists of all the points contained in the sets G_1, G_2, ... G_n, ... each of which is non dense. A set not so constituted is of the second category.

All of the facts concerning sets of the first and second categories can be easily worked out for the case of n-dimensional space, certain obvious modifications being necessary. As an example the following proposition, similar to that of 25 for a single variable, will be established.

59. # If G is a set of points in n dimensional space which can be broken up into component sets, G_1, G_2, ... G_n, ..., which are closed, then a function $f(x_1, x_2, \ldots x_n)$ can be set up, having its points of discontinuity on G and its points of continuity on the complement of G.

Let $\alpha_1, \alpha_2, \ldots\ldots \alpha_n, \ldots$ be a sentence of decreasing positive numbers approaching 0. Define f as follows:

$f(x_1, \ldots\ldots x_n) = \alpha_1$, for points of G_1.

$f(x_1, \ldots\ldots x_n) = \alpha_2$, for points in G_2 and not in G_1.

. . . .

$f(x_1, \ldots\ldots x_n) = 0$, everywhere else.

Consider a point P belonging to G. Suppose G_m is the first set to which P belongs. Then $f(P) = \alpha_m$, and $M(P) \geqq f(P)$. Now $m(P) = 0$, since the points where $f(x \ , \ldots\ldots x \ ) \quad 0$ are everywhere dense. Then $\omega(P) \geqq \alpha_m$, and P is a point of discontinuity.

Let A be a point not belonging to G. Since G_n is closed, A is not a limit point of G_n; and a sphere S_n, having A as center and containing no points of G_n, can be constructed. Then $M(A) \leqq M(S_n) < \alpha_n$, $M(S_n)$ being the upper limit of the values of the function in the sphere. Since α_n can be made as small as we please, $M(A) = 0$. Then $\omega(A) = M(A) - m(A) = 0$, and A is a point of continuity. The proposition is thus proved.

60. <u>Classification of pointwise discontinuous functions in n dimensions.</u> The same division into seven classes that we made for pointwise discontinuous functions of a single variable (27-34) can be made here, and the proposition just established furnishes us a ready means of constructing examples of the various classes. Two examples are shown in the figures.

A function of two variables having points of discontinuity countable and everywhere dense (class IV) is shown in Figure II. G_n= all points with coordinates of the form $(m_1/n, m_2/n)$, with the fractions in their lowest terms; $\alpha_n = (1/2)^n$. $f(x,y) = 0$ on the points not belonging to G, as before. The figure shows the points of discontinuity on the sets up to and including G_7.

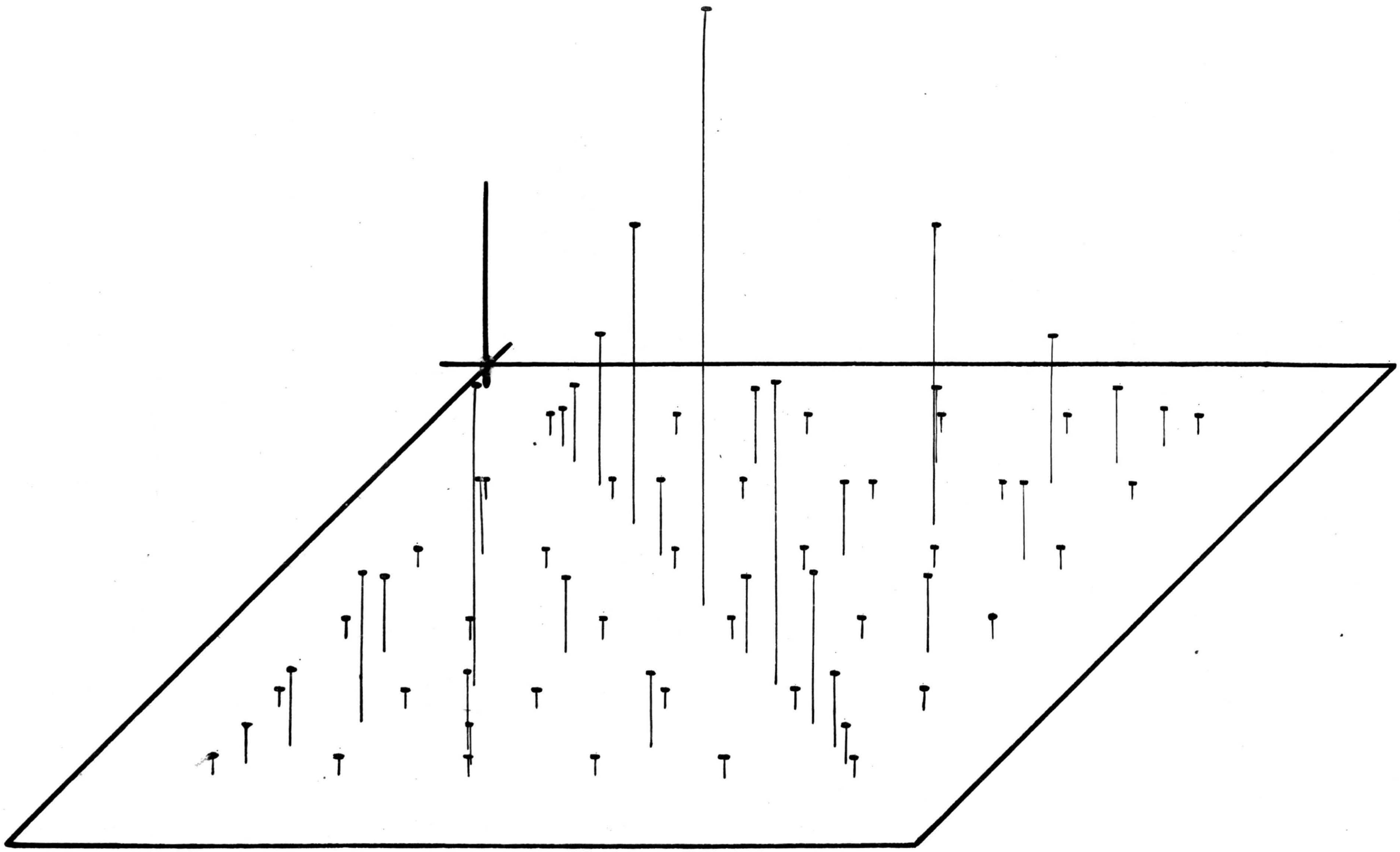

Fig. 11.

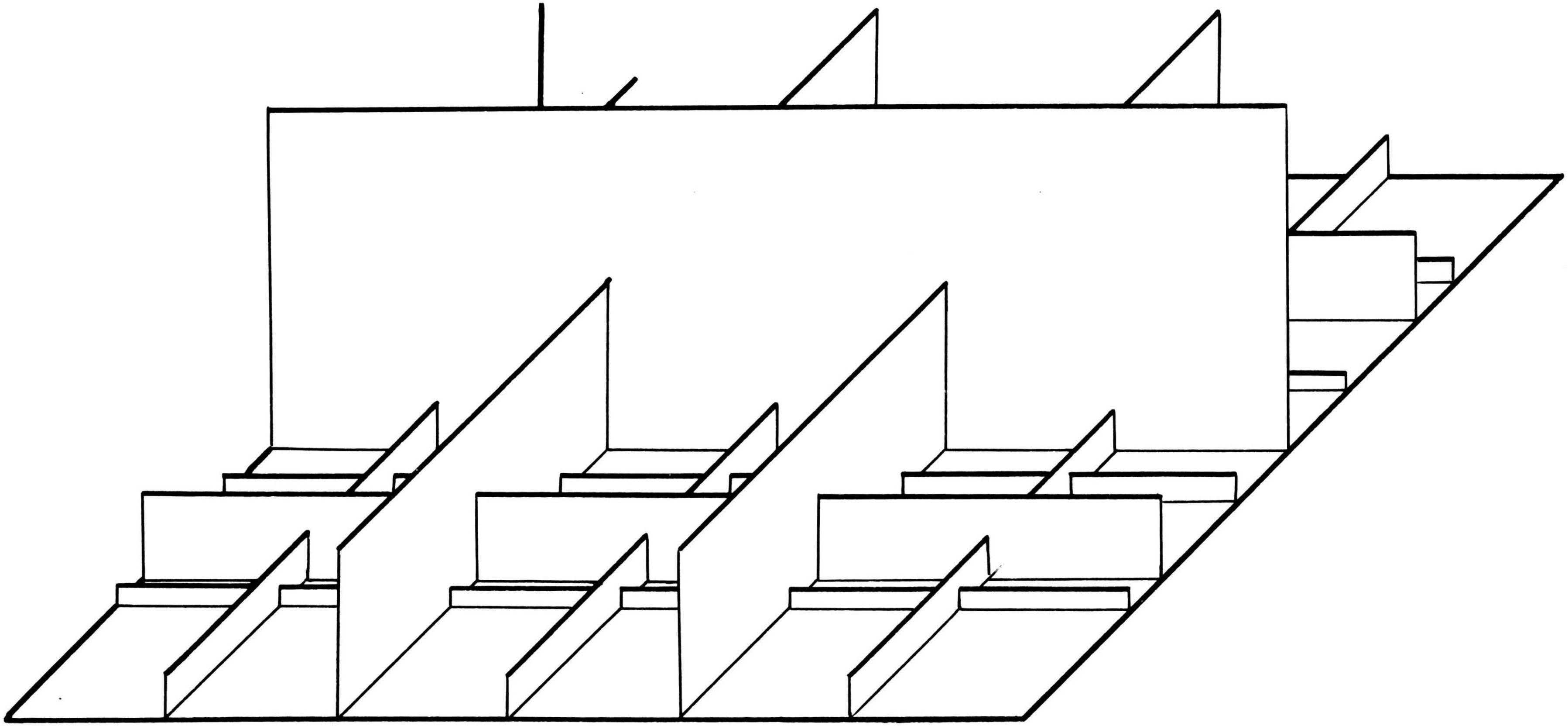

Fig. 12.

Figure 12 shows a function with points of discontinuity everywhere condensed (Class VII). G_1 is the set of points for which $y=1/2$; $\alpha_1=1/2$. G_2 is the set for which $x=1/3$, $x=2/3$; $\alpha_2=(1/2)^2$. G_3 is the set $y=1/4$, $y=3/4$; $\alpha_3=(1/2)^3$; etc. Each set contains all the points on a straight line; hence each is more than countable.

Section 7. Pointwise Discontinuity on Perfect Sets

61. The notion of pointwise discontinuity on a perfect set is of very great importance in a connection that will be discussed later; namely, the necessary and sufficient condition for the approach to discontinuous functions by continuous functions. In this section will be given, as briefly as possible, a few facts necessary for an understanding of this phase of the subject. We shall treat only perfect sets in one dimension, but the ideas can, in general, be extended without difficulty to n-dimensional space.

62. Perfect sets and their construction. A perfect set is a closed set every point of which is a limit point. Any closed interval (an interval including its end points) is a perfect set. If a countable number of open intervals (intervals exclusive of their end points) be removed from a closed interval, the set remaining is a perfect set. In fact, any perfect set can be so constructed. A perfect set was used in the example of 32. Mention was there made of the fact that a perfect set has the power of the continuum.

63. The general notions in this case. If we construct an interval about any point A of the set it will contain an

infinite number of points of the set. We can then define $M(A)$, $m(A)$, and $\omega(A)$ as before, with the understanding that only the values of the function on the points of the perfect set are taken into account. The various properties of $M(x)$, $m(x)$, and $\omega(x)$ follow as before.

64. Pointwise discontinuity. A function is pointwise discontinuous on a perfect set if every interval containing points of the set contains points of the set where $\omega(x) = 0$. As stated in 63, $\omega(x)$ must be defined leaving entirely out of consideration the values of the function at points not belonging to the perfect set. Thus, in the example of 32 $\omega(x) = a$ at points of the perfect set, if the values at other points of the line be considered; however, $f(x) = a$ on the perfect set, a perfectly continuous function, and $\omega(x) = 0$.

65. Sets of the first and second categories. A set G is of the first category if it consists of all the points of a countable number of sets, G_1, G_2, ...G_n, ..., each of which is non dense on the perfect set. In saying that G_n is non dense on the perfect set, we mean that in any interval containing points of the perfect set, we can find another interval containing points of the perfect set but containing no points of G_n. A set not constituted as above is of the second category.

All the facts concerning points of the first and second category work out without difficulty for the case of perfect sets. It is easily shown, for example, that any set of the first category whose component sets are closed can be made the set of points of discontinuity of a function pointwise

discontinuous on the perfect set, the points of the second category remaining on the perfect set being the points of continuity. Using this fact, we can build up functions corresponding to the seven classes found in the case of a function defined on the whole continuum.

66. If a function is semicontinuous on a perfect set, it can be shown to be pointwise discontinuous on the set, by the method used in proving the same fact for functions semicontinuous on the whole continuum (36).

CHAPTER III. THE APPROACH TO DISCONTINUOUS FUNCTIONS BY CONTINUOUS FUNCTIONS

Part 1. Fundamental Ideas of Approach

67. A sequence of functions, $f_1(x)$, $f_2(x)$, ... $f_n(x)$,.... is said to approach a function f(x) if, given any point x′ and any positive number ε, however small, a number m can be found such that $|f(x') - f_n(x')| < \varepsilon$, for $n > m$.

If a value of m exists such that the inequality holds with the same m, for all values of x in the interval under consideration, the approach is said to be uniform.

68. A little more than a decade ago Baire worked out the conditions under which a function may be approached by a sequence of continuous functions. His results may be stated in a single sentence:

Theorem: The necessary and sufficient condition that any function whatever, finite or infinite, be the limit of continuous functions is that it be pointwise discontinuous on every perfect set.(1) This is true for a function of any number of variables.

69. It should be noted that it is not sufficient that the function be merely pointwise discontinuous on the continuum; it must also be pointwise discontinuous on every perfect set. All of the examples of pointwise discontinuous functions that have been given so far have been pointwise discontinuous on every perfect set. An example of a function without this property is the one of 32 if $f(x) = 0$ on the end points of the intervals. While the function would still be pointwise

(1) Baire, Lecons sur les fonctions discontinues, p.124.

discontinuous on the continuum, it would be totally discontinuous on the perfect set which remains after the removal of the given open intervals. For, any interval including a point of the perfect set would contain points where $f(x)=a$ and also end points of the intervals where $f(x)=0$. $\omega(x)=a$ then at every point of the perfect set.

70. Semicontinuous functions. It will now be shown that any semicontinuous function is the limit of continuous functions. Let $f(x)$ be semincontnuous above. It will be shown that $f(x)$ is semicontinuous above on every perfect set. $f(x)=M_c(x)$, where $M_c(x)$ is the maximum of the function defined on the continuum. At any point A on a perfect set, $M(A) \gtreqless M_c(A)=f(A)$, for in the case of the perfect set certain values of the function may be left out of consideration. But in general $M(A) \geqq f(A)$. Then on the perfect set $M(A)=f(A)$, and the function is semicontinuous above. Consequently it is pointwise discontinuous on the perfect set (66), and it can be approached by continuous functions.

In a similar way, it can be shown that if a function is semicontinuous below, it can be approached by continuous functions. It follows that $M(x)$, $m(x)$, and $\omega(x)$ can be approached by a sequence of continuous functions.

71. Sequences and series. Approach by a sequence of functions $f_1(x)$, $f_2(x)$, ... $f_n(x)$, ...can be put in the form of a series, thus, $f(x)=f_1(x)+(f_2(x)-f_1(x))+\ldots\ldots + (f_n(x)-f_{n-1}(x))+\ldots$ The sum of the first n terms is $f_n(x)$ and the remainder term is $f(x)-f_n(x)$. If the sequences approaches $f(x)$ the series converges to the value $f(x)$,

and conversely. Also if there is uniform approach of the sequence there is uniform convergence of the series, and conversely.

72. This enables us to apply the propositions of 42-44 to the case of approach. If the approach is uniformly continuous and the approaching functions are continuous, the function approached is continuous. If the approach is uniformly continuous and the approaching functions are pointwise discontinuous, the function approached is pointwise discontinueus.

It follows, then, that if a pointwise discontinuous function, which is not everywhere continuous, is approached by continuous functions, the approach is not uniform.

73. In 1885 Weierstrass established the proposition that any function continuous in a closed interval can be approached uniformly by a series of polynomials.(1) It will

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(1) Note on Taylor's series. If $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$, where the polynomial is uniformly convergent in an interval, it can be shown by differentiation, since $f'(x)$ is also a uniformly convergent series about the origin, that $a_0 = f(0)$, $a_1 = f'(0)$, etc. It might appear, from the theorem of Weierstrass, that any continuous function is developable in a Taylor's series about the origin. e^{-1/x^2} is a familiar example of a function which cannot be so developed. The apparent contradiction lies in the fact that the Weierstrass polynomials are not in the form implied above; for in the Taylor's series $P_{n+1}(x)$ differs from $P_n(x)$ only in having an added term $a_{n+1}x^{n+1}$, while this is not true in general of the Weierstrass polynomials.

This matter is intimately connected with the use of approximation formulae in the sciences. It is customary to represent a function based upon experimental data in the form $f(x) = a_0 + a_1 x + \ldots\ldots$, where a very few terms generally suffice. According to Weierstrass' theorem, such an approximation can be made to fit the true curve with any desired degree of accuracy, provided the curve is continuous. But this important fact must be kept in mind: If the accuracy of the approximation is increased by the addition of new terms, it

now be shown that <u>any function which is approached by a sequence of continuous functions, $f_1(x), f_2(x), \ldots\ldots f_n(x), \ldots\ldots$, can be approached by a sequence of polynomials</u>. Take a sequence of decreasing positive numbers approaching 0, $\varepsilon_1, \varepsilon_2, \ldots\ldots \varepsilon_n, \ldots\ldots$ Choose a polynomial $P_n(x)$ such that $|f_n(x) - P_n(x)| < \varepsilon_n$. Then the sequence of polynomials $P_1(x), P_2(x), \ldots\ldots P_n(x), \ldots\ldots$ approaches the function $f(x)$. For, given an $\varepsilon > 0$, we can choose an $\varepsilon_r < \frac{\varepsilon}{2}$; then $|f_n(x') - P_n(x')| < \frac{\varepsilon}{2}$ for $n > r$, for any value of x; and with a given x' we can find an s such that $|f(x') - f_n(x')| < \frac{\varepsilon}{2}$ for $n > s$. Then if m is the greater of the two numbers r and s, by adding the two inequalities we get $|f(x') - P_n(x')| < \varepsilon$, for $n > m$, which is the condition for approach to f(x) by the sequence, $P_1(x), \ldots\ldots P_n(x), \ldots\ldots$

Part 2. Methods of Construction Continuous Functions Approaching Discontinuous Functions

<u>74</u>. This section will be devoted to the description, with the aid of figures, of methods which may be conveniently used to construct a sequence of continuous functions approaching a given discontinuous function. The means of accomplishing this result has been suggested by Baire. He says:

"The investigation of a suite of continuous functions having for a limit a function f(x) defined on a segment AB is equivalent to that of a function F(x,y) which reduces to f(x) for $y = 0$ and is continuous with respect to the two variables x and y in a rectangle ABA'B' except on AB and continuous

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is necessary to redetermine all the constants of the polynomial and not merely to find the constants of the added terms, for these additional terms will, in general, change the polynomial thru-ou

with respect to y on every point of AB."[1]

The problem is reduced to that of the construction of a surface whose sections by planes parallel to the yz-plane are continuous curves and whose sections by planes parallel to the xz-plane are continuous curves, except in the case of the plane $y=0$, where the curve shall be $f(x)$. The sections of this surface by the suite of planes $y=k_1$, $y=k_2$, ...$y=k_n$, ..., where k_1, k_2, ...k_n, ... is a series of positive terms approaching 0, will be a sequence of continuous functions approaching $f(x)$. The examples which are given, with the accompanying figures, will make this clearer.

75. Figure 13 shows the approach by continuous functions to the function $f(1/2)=a$, $f(x)=0$ elsewhere. Several sections of the surface by planes parallel to the xz-plane are shown, and each of them is seen to be a continuous curve. It can readily be shown that if a sequence of curves, $f_1(x)$, $f_2(x)$, ...$f_n(x)$..... be cut from the surface by the planes, $y=k_1$, $y=k_2$, ... $y=k_n$, ..., where the k's approach 0 and are, for this figure all positive, these approach the curve $f(x)$. It must be shown that for a given x' and a number $\varepsilon>0$, we can find an m such that $|f(x')-f_n(x')|<\varepsilon$, for $n>m$. This is true for the point of discontinuity, $x=1/2$, for $f_n(1/2)=a$ for all values of n, Then $|f(1/2)-f_n(1/2)|=0<\varepsilon$, for all n. If $x'\neq 1/2$, we can determine a strip of width y' such that $f_n(x')=f(x')=0$ if $k_n<y'$. Since the k's approach 0, we can find an m such that $k_n<y'$ for $n>m$; then $|f(x')-f_n(x')|=0<\varepsilon$ for $n>m$. Figure 14 shows the function $f(x)$ which is approached. It

Fig. 13.

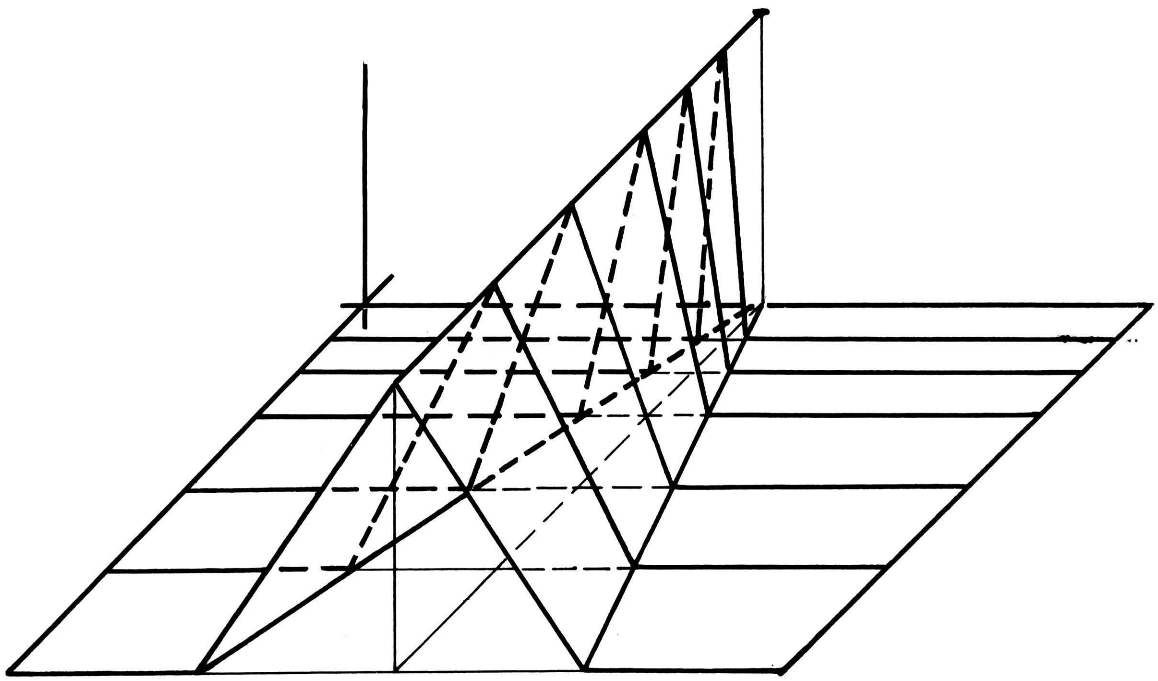


Fig. 14.

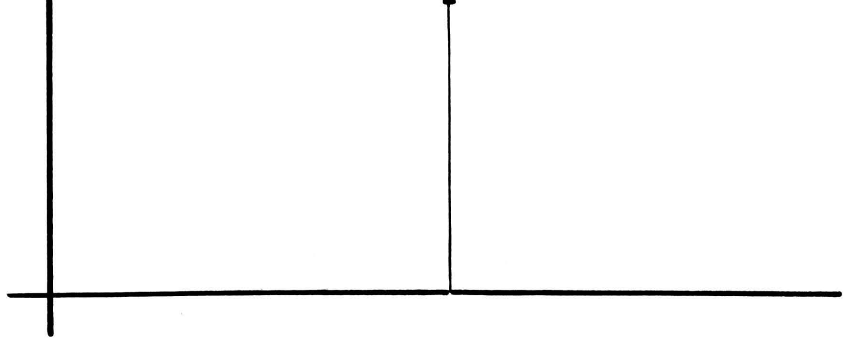


Fig. 15.

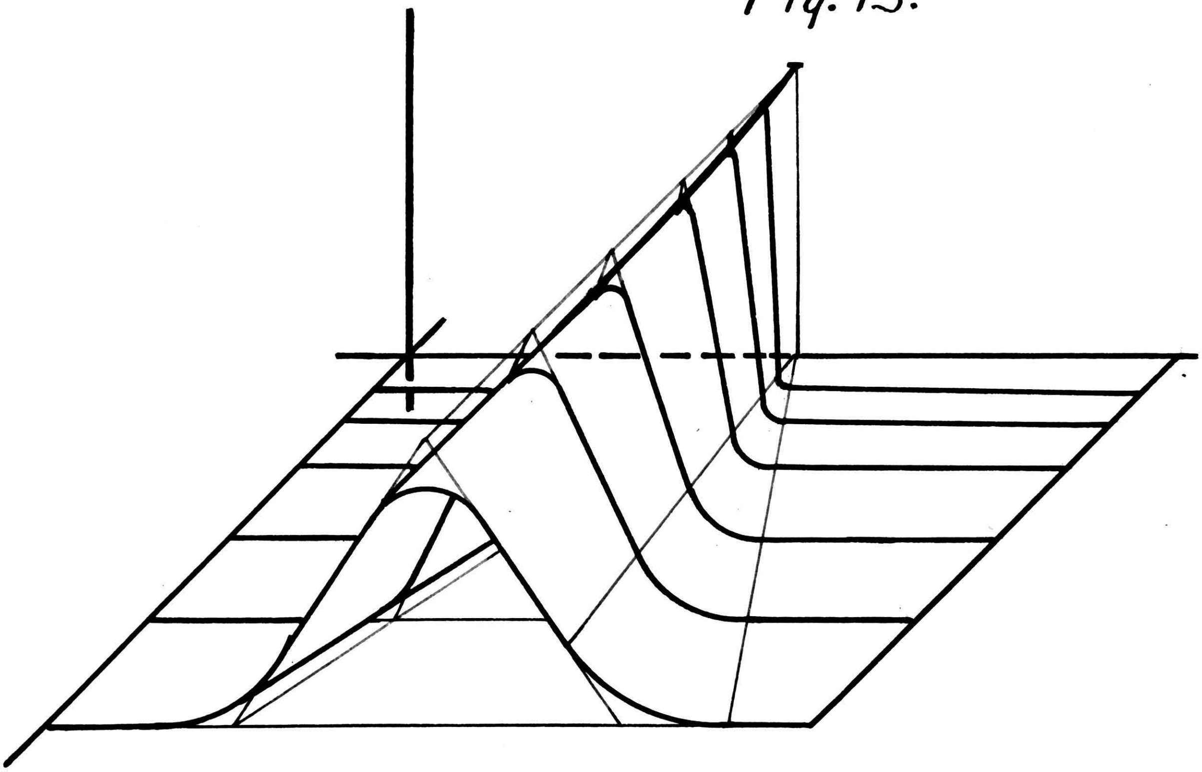
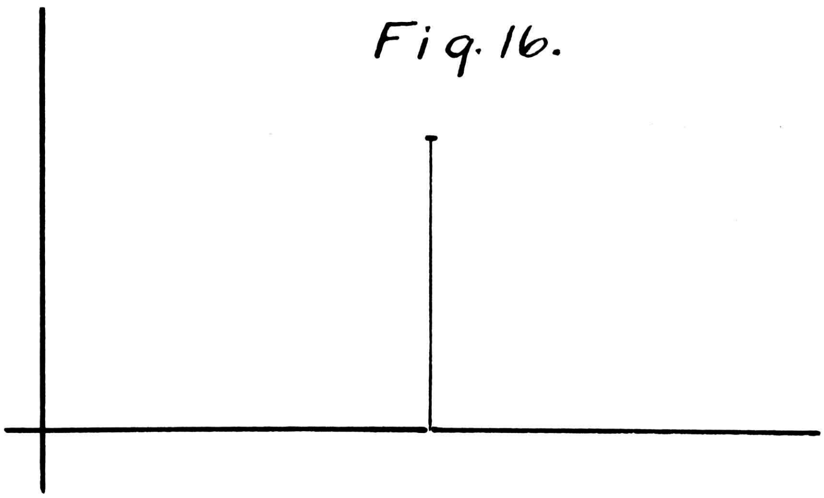


Fig. 16.



is also $\omega(x)$, since the two are identical.

76. Figure 15 shows the approach to the same function by functions which are not only continuous but also have continuous first derivatives. The straight lines of the preceding example are joined by arcs of circles which are tangent to the lines they connect. It is necessary that the radii of these circles approach 0 as the xz-plane is approached. This is accomplished by keeping the points of tangency of the circles between two lines on the surface which converge at the point $x=1/2$, $y=0$. These converging lines are in brown in the figure.

77. Figure 17 shows the approach to the function $f(x)=1/2$, for $x \lesseqgtr 1/2$; $f(x)=0$ for $x>1/2$. It is discontinuous at the point 1/2, where it is semicontinuous above. It is seen from the figure that $f_n(1/2)=f(1/2)=1/2$ for all values of n; that is, for every section of the surface by a plane of the form $y=k_n$.

In Figure 19 is the same function, except that $f(1/2)=0$. It is semicontinuous below at the point 1/2. $f_n(1/2)=f(1/2)=0$ for all values of n.

Figure 21 shows the case where f(x) is neither semicontinuous above nor below at the point 1/2, but takes on an intermediate value a. Here, as before, $f_n(1/2)=f(1/2)=a$ for all values of n. This figure and the two just mentioned illustrate the methods employed to make the continuous functions approach any desired value at the point of dis-

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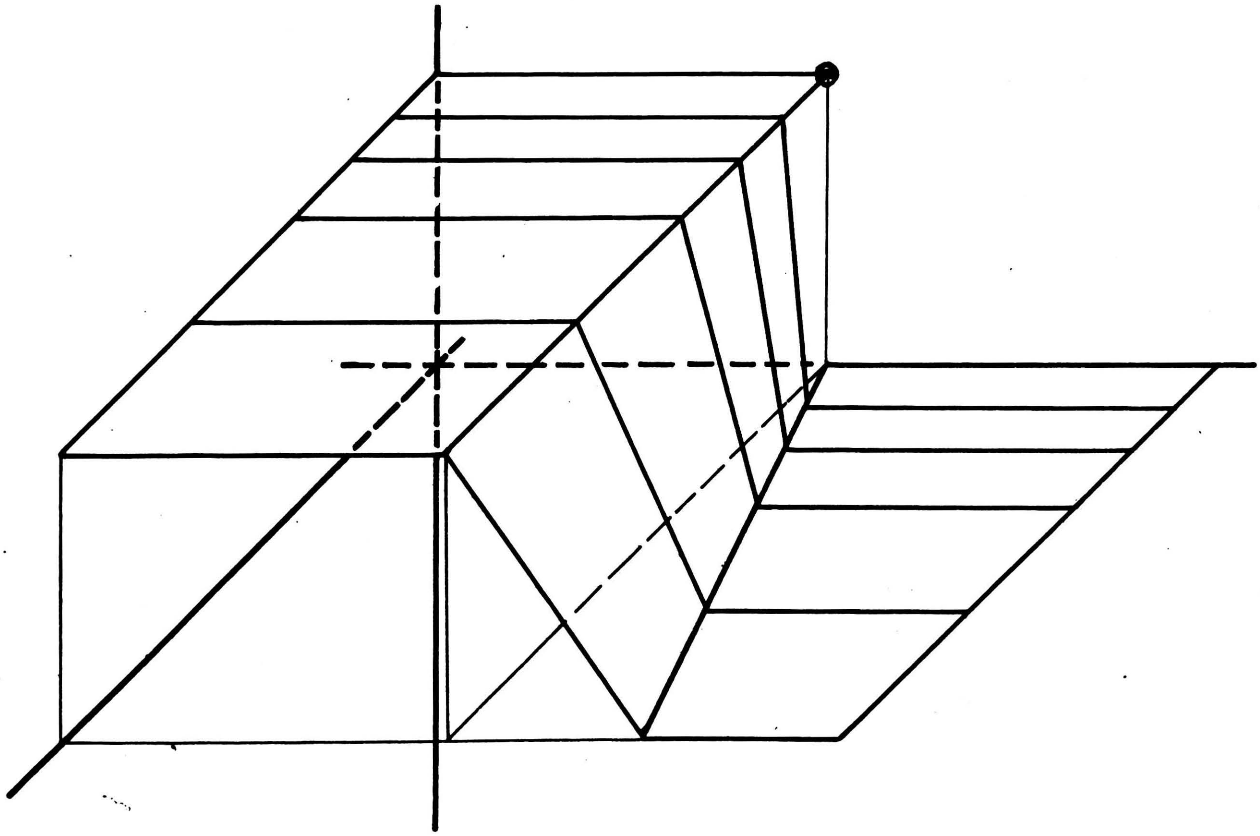
(1) Baire, Lecons sur les fonctions discontinues, p.9.
p La recherche d'une suite de fonctions continues ayant pour limite une fonction f(x) définie sur le segment AB, est absolument équivalente à celle d'une fonction F(x,y) se réduisant à f(x) pour y=0, continue par rapport à l'ensemble

Fig. 17.



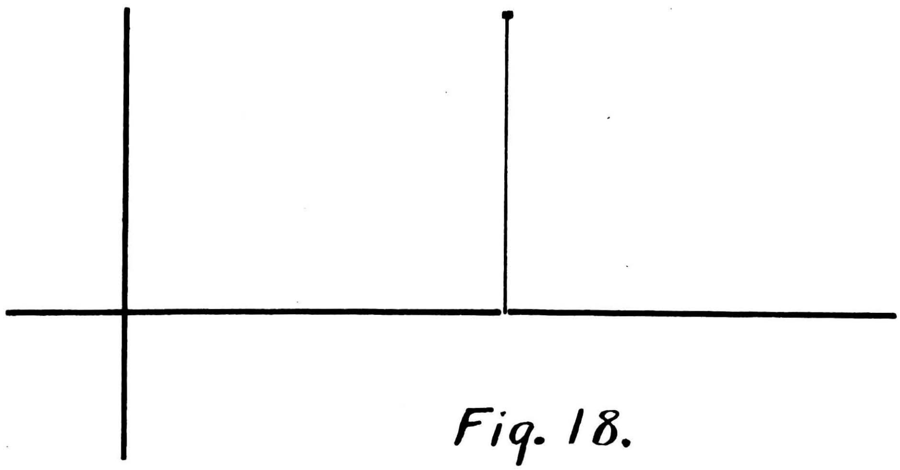
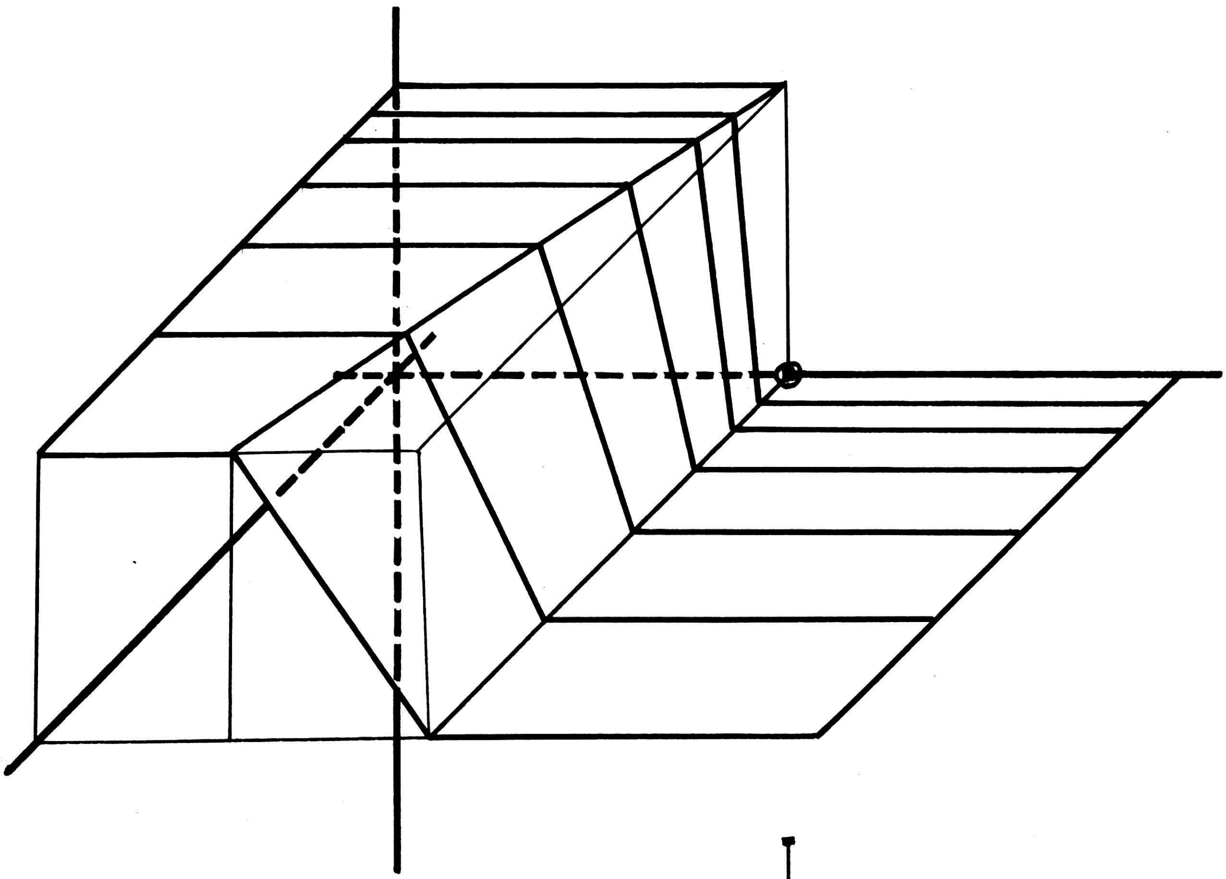


Fig. 18.

Fig. 19.



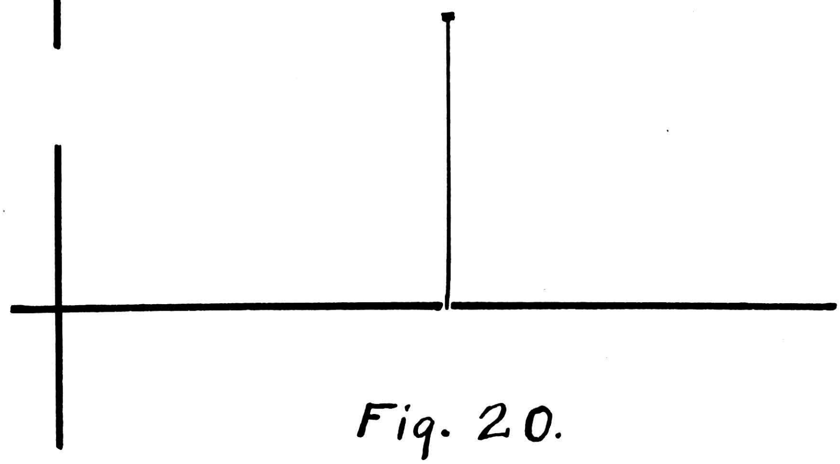


Fig. 20.

Fig. 21.

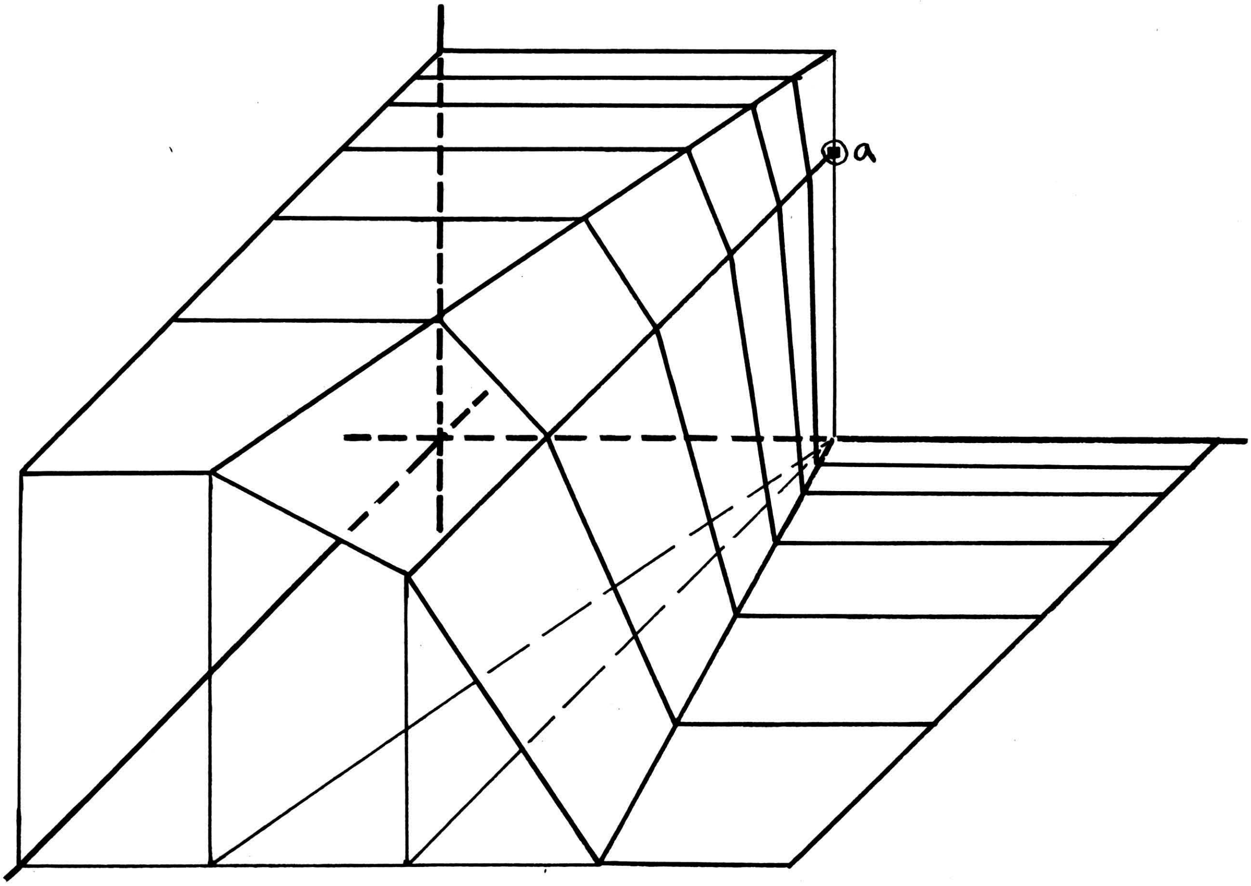
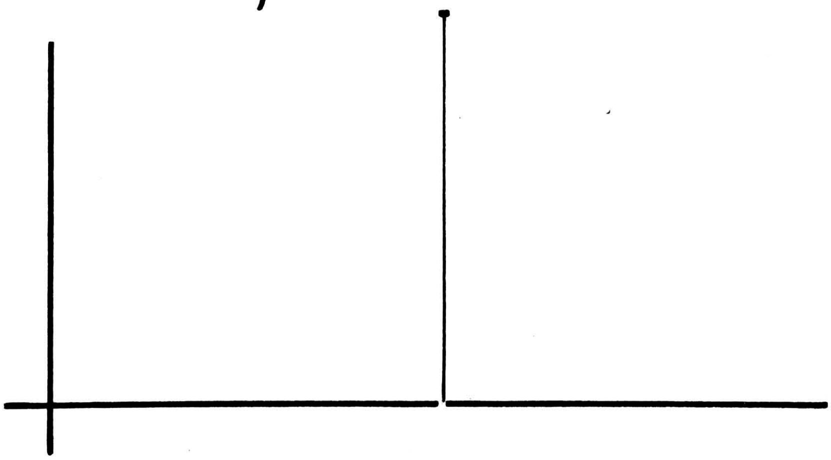


Fig. 22.



continuity. Figures 18, 20, and 22 show the oscillation functions in each of the three cases. The oscillation function is the same in each case.

78. Figure 23 illustrates the case of a function having an infinite number of points of discontinuity which have a single limit point. The function used is $f(0) = 1$, $f(1) = -1$, $f(1 - 1/2^n) = 1/2^n$, $f(x) = 0$ elsewhere. It is seen from an inspection of the figure that for any point x', a point either of continuity or of discontinuity, and a given $\varepsilon > 0$ we can find a strip of width y' such that $|f(x') - f_n(x')| < \varepsilon$ for values of n which throw the section $y = k_n$ within that strip. The figure below (Figure 24) is $\omega(x)$. It is equal to f(x) everywhere except at the point 1; $\omega(1) = -f(1) = 1$.

In a manner entirely similar to the one just shown, we can put in, in the intervals between the points of discontinuity of the last example, an infinite number of other points of discontinuity having these points as limit points. Between these points of discontinuity we can place still others having these as limit points, and so on. Let P be the set of all the points of discontinuity of a function f(x). Let P' be the first derived set of P; that is, the set of the limit points of P. Let P'' be the second derived set of P — the set of the limit points of P', and so on. If $P^{(n)} = 0$ we can use the method just shown to construct the continuous functions approaching f(x). For, since

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des deux variables (x,y) dans le rectangle ABA'B' et enfin continue par rapport à y en tout point de AB.

Fig. 23.

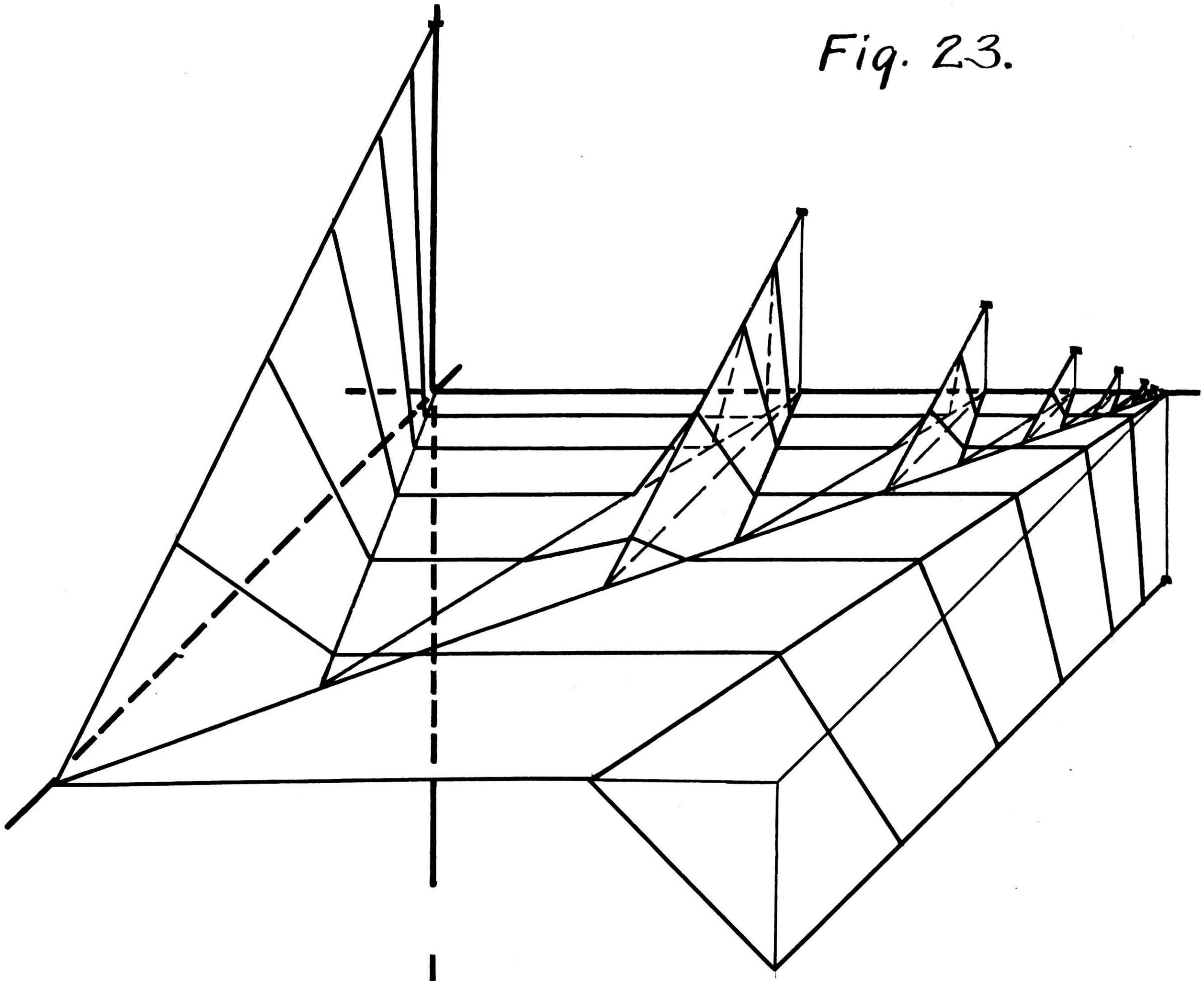
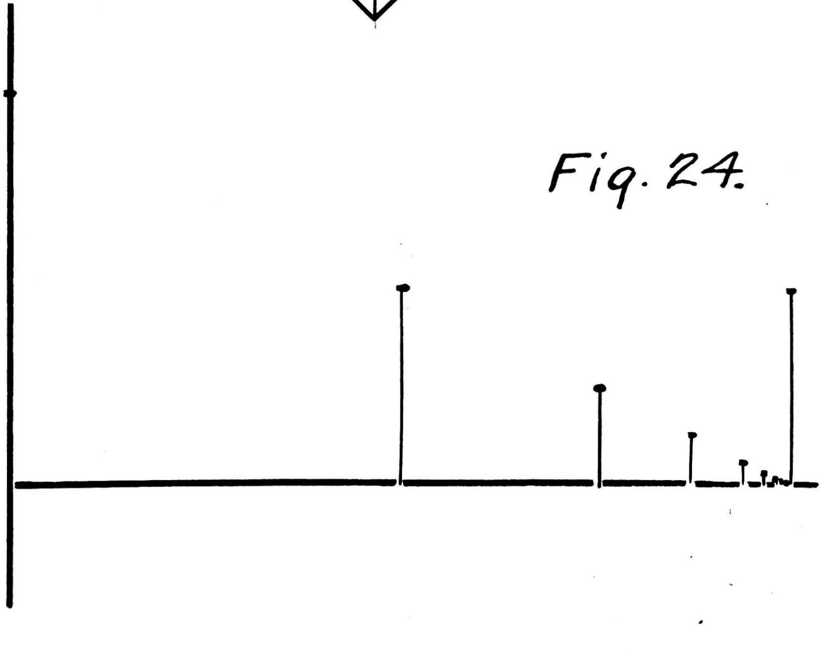


Fig. 24.



$P^{(n)} = 0$, $P^{(n-1)}$ consists of a finite number of points; and, beginning with these, we can proceed in the way indicated back to the points of P.

79. Figure 25 is a function equal to a except on the points of a perfect set, where it is equal to 0. Mention has been made of the fact that any perfect set can be constructed by the removal of a countable number of open intervals (62). In the example shown, the intervals are finite in number. If the perfect set is non dense, the number of intervals will be infinite. The process can evidently be continued, the only requirement being that the lengths in the y direction of the rectangles on which the surface leavesthe xy-plane shall approach 0; so that any section of the form $y = k_n > 0$ shall cut only a finite number of these rectangles. Figure 26 is the oscillation for the function of Figure 25.

80. In Figure 27, f(x) is continuous everywhere except at the point 1. As x approaches 1, f(x) increases without limit; $f(1) = 0$. The construction is evident from the figure. In any section of the surface by $y = k_n$, $f_n(x) = f(x)$ for points not contained in the triangle ABC. For points inside the triangle $f_n(x)$ is the straight line which passes through the point on the surface directly above the section of the plane with AB and is equal to 0 for $x = 1$. The oscillation function is shown in Figure 28; $\omega(x) = 0$ for $x \neq 1$, $\omega(1) = +\infty$.

81. Figures 29-32 indicate the manner of approaching a discontinuous function of two variables. This cannot be shown in a single figure, as was the case of a function of

Fig. 25.

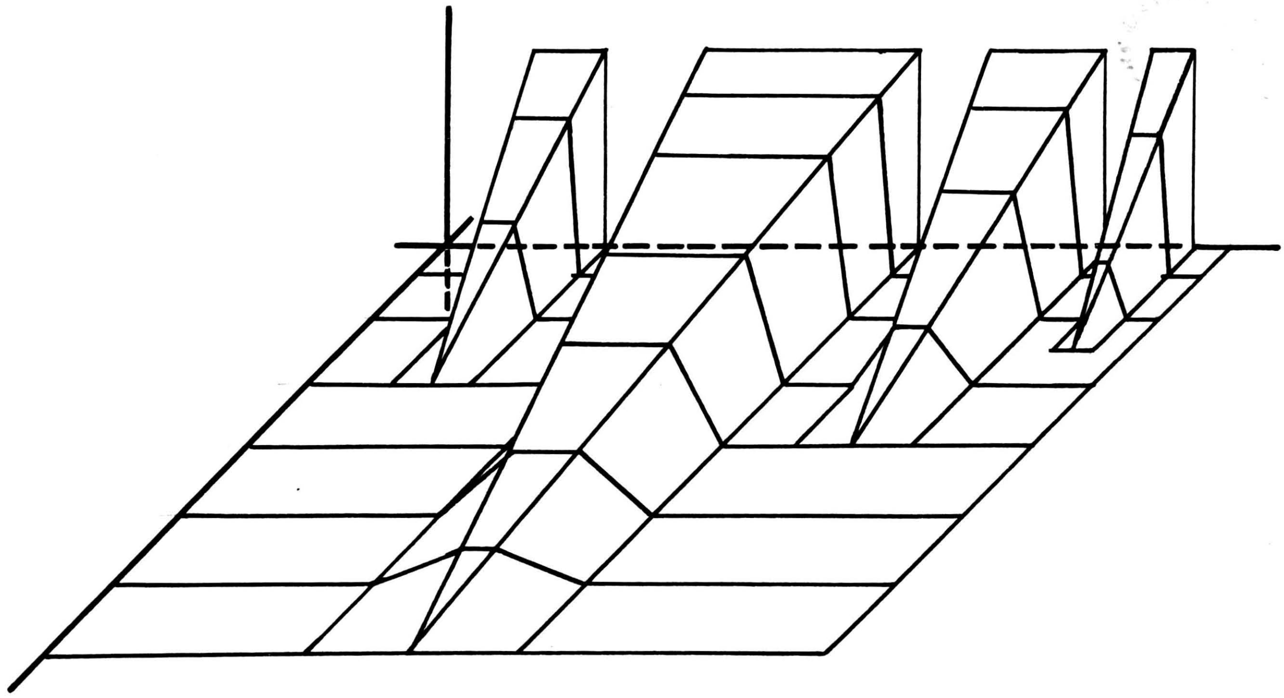
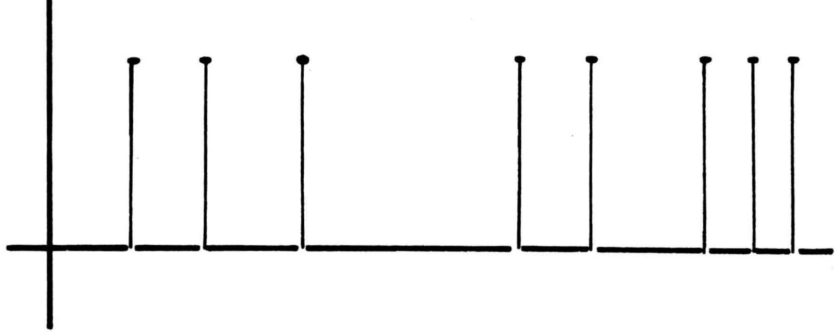


Fig. 26.



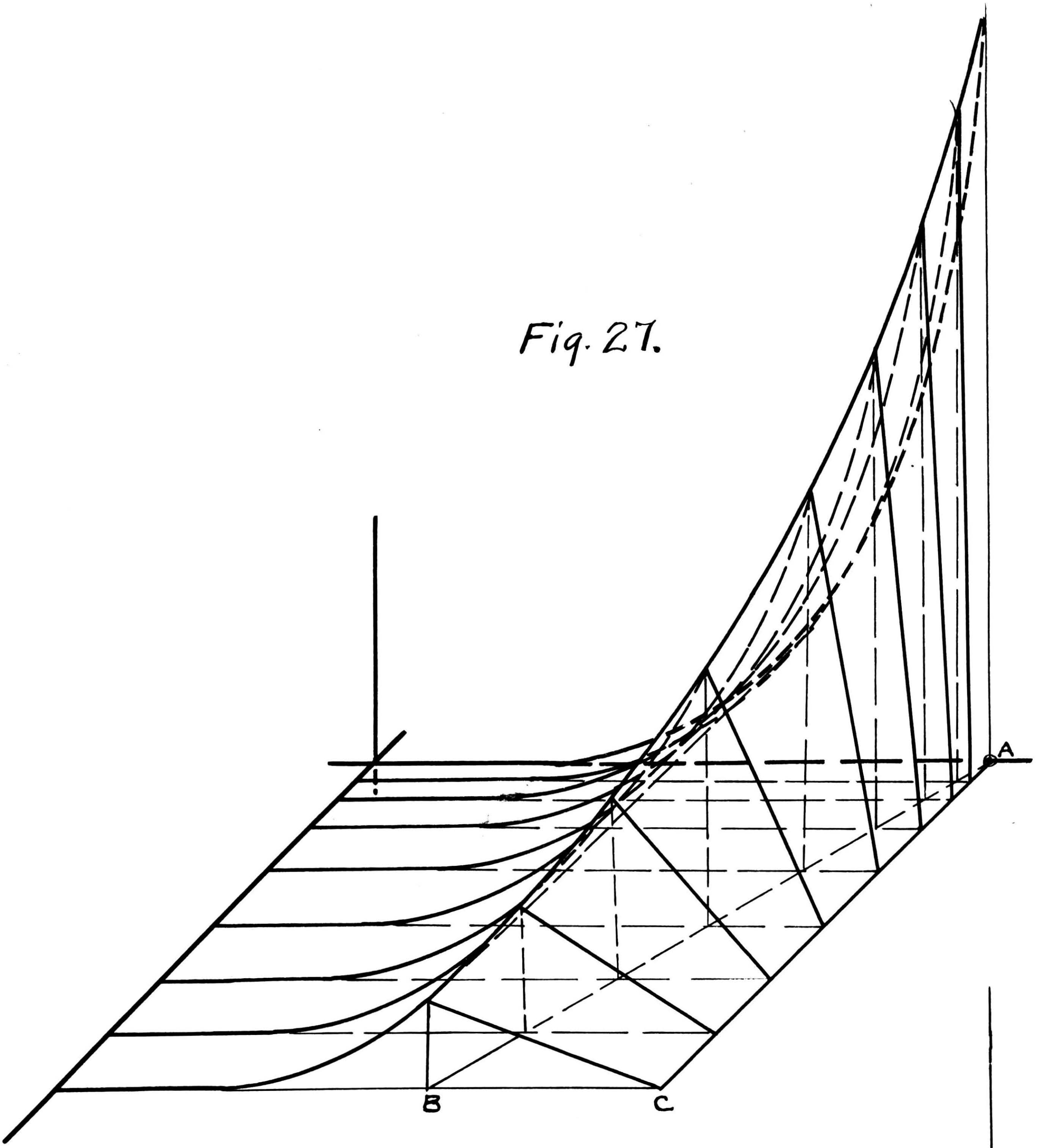


Fig. 27.

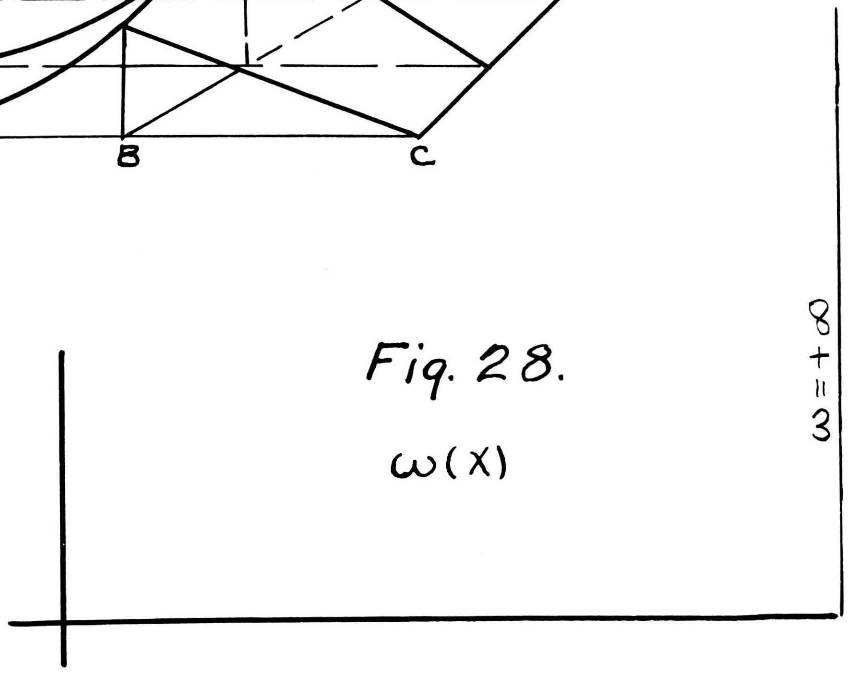


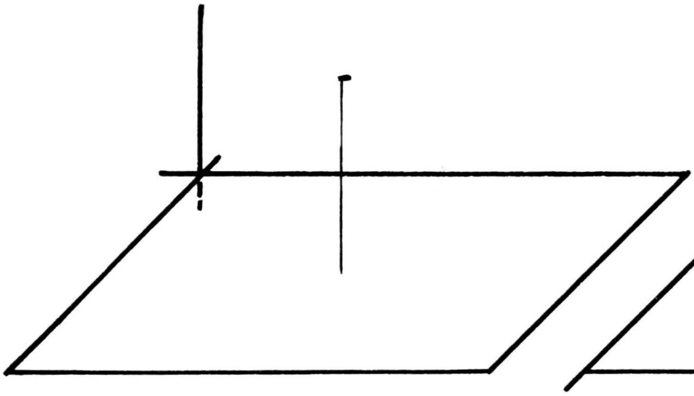
Fig. 28.

one variable, for it would involve a surface in space of four dimensions. The function to be approached is $f(1/2,1/2)=1/2$, $f(x,y)=0$ elsewhere. It is shown in Figure 29. The succeeding figures show $f_1(x,y)$, $f_2(x,y)$, and $f_n(x,y)$. $\omega(x,y)$ is equal to $f(x,y)$, and is thus shown in Figure 29.

82. Following the methods used in the case of functions of a single variable, we can without difficulty devise continuous functions approaching functions of two variables which have an infinite number of points of discontinuity, which are discontinuous on non dense perfect sets, or which are unlimited in certain regions.

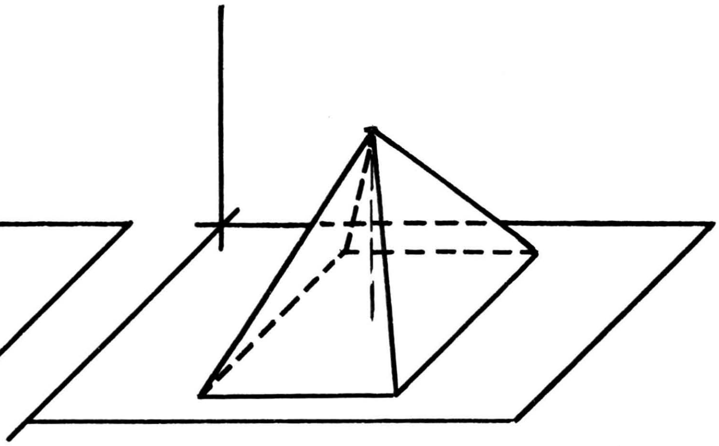
In the case of functions of more than two variables we have no geometrical means of finding a sequence of approaching continuous functions.

Fig. 29.



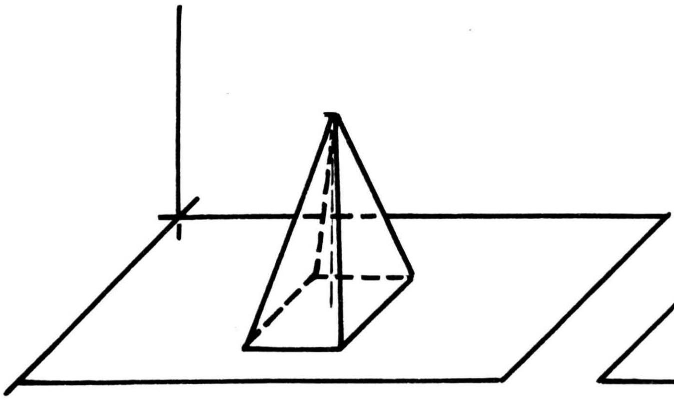
f(x, y)

Fig. 30.



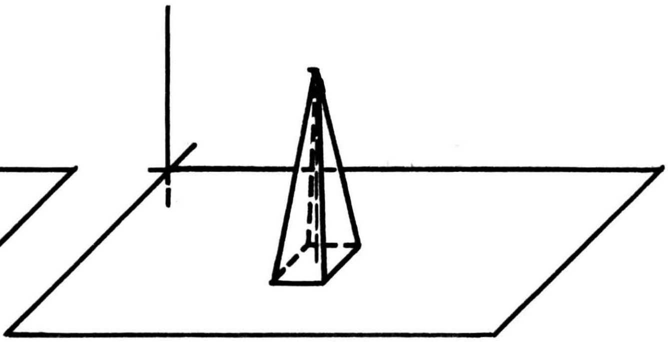
$f_1(x, y)$

Fig. 31.



$f_2(x, y)$

Fig. 32.



$f_3(x, y)$

CHAPTER IV. THE ω FUNCTION

83. In the preceding sections only those functions have been studied in which $m(\omega,x)=0$ at every point; that is, pointwise discontinuous functions. We shall now investigate the properties of ω for any function whatever.

It has already been shown (5) that ω is always positive or 0. It has also been pointed out that ω may have no least upper bound; that is, that $\omega(x)$ may equal $+\infty$ at some points. As a matter of fact, perfectly well defined functions exist in which $\omega(x)=+\infty$ at every point. # The following is an example of such a function:-

$f(x)=0$, for x irrational;

$f(x)=q$, where x is the rational fraction p/q (in its lowest terms).

At any point A, $m(A)=0$, $M(A)=+\infty$, $\omega(A)=+\infty$.

84. # It has been proved that $\omega(x)$ is semicontinuous above, and hence pointwise discontinuous. The question now arises,—Is any function, which is semicontinuous above and everywhere positive or 0, the ω of some function? The answer is in the affirmative, and it will now be shown how a function can be constructed having the given function as its oscillation. Let $\phi(x)$ be the given function. The points of continuity of ϕ, which is pointwise discontinuous, are everywhere dense and have the power of the continuum (24). It is possible to choose from among these points of continuity an everywhere dense countable set C, leaving the remaining points of continuity still everywhere dense, in fact everywhere condensed. Define a function $F(x)$

as follows:

$F(x) = 0$ on the set C;

$F(x) = \phi(x)$, everywhere else.

It will now be shown that $F(x)$ has $\phi(x)$ as its oscillation function. Since $\phi(x)$ is semicontinuous above $M(\phi,x)=\phi(x)$ at all points with the possible exception of the points of the set C. The points of C being points of continuity, $M(\phi,x) = m(\phi,x) = \phi(x)$. Then the removal of the countable number of points of C and putting $F(x) = 0$ on those points still leaves a more than countable set in the region of any point of C, which make $M(F,x) = \phi(x)$ still. C being everywhere dense, $m(F,x) = 0$ everywhere. Then $\omega(F,x) = M(F,x) - m(F,x) = \phi(x)$, which was to be shown.

This countable dense set can be chosen in an infinite number of ways; in fact, by shifting a single point of C we see that the number of sets like C that can be chosen has the power of the continuum. Hence, any function which is semicontinuous above and which has no negative values is the oscillation of an infinity of functions whose number has the power of the continuum.

85. # <u>Theorem: In order that $\omega(x)$ be continuous, it is necessary and sufficient that $M(x)$ and $m(x)$ each be continuous.</u>

It is evidently sufficient; for $\omega(x) = M(x) - m(x)$, and the difference of two continuous functions is continuous.

It is necessary. $\omega(x)$ being continuous, for a given $\varepsilon > 0$ and a given x' there exists an η_1 such that,

$$|\omega(x') - \omega(x' + h)| < \varepsilon/2 \text{ , for } |h| < \eta_1 .$$

That is, $|M(x') - m(x') - (M(x' + h) - m(x+h))| < \varepsilon/2$,

or $\quad |(M(x') - M(x' + h)) + (m(x' + h) - m(x'))| < \varepsilon/2, \quad |h| < \eta_1 .$

This can be written,

$$|M(x') - M(x' + h)| + |m(x' + h) - m(x')| < \varepsilon/2, \qquad \text{(a)}$$

if the quantities $M(x') - M(x' + h)$ and $m(x' + h) - m(x')$ have the same algebraic sign (of if one or both are 0); or it can be written,

$$\big||M(x') - M(x' + h)| - |m(x' + h) - m(x')|\big| < \varepsilon/2, \qquad \text{(b)}$$

if the quantities have different signs. In the former case we have directly,

$$|M(x') - M(x' + h)| < \varepsilon/2 , \quad |h| < \eta_1 ;$$
$$|m(x' + h) - m(x')| < \varepsilon/2 , \quad \text{" } \quad \text{" } .$$

In the latter case we have since $M(x)$ is semicontinuous above (7),

$$M(x' + h) < M(x') + \varepsilon/2 \quad , |h| < \eta_2 \quad .$$

Then, $\quad M(x') - M(x' + h) > -\varepsilon/2 \quad , \text{ " } \quad \text{ " } \quad . \qquad \text{(c)}$

Similarly, $m(x)$ being semicontinuous, we have,

$$m(x' + h) > m(x') - \varepsilon/2 \quad , |h| < \eta_3 \quad ,$$

and, $\quad m(x' + h) - m(x') > -\varepsilon/2 \quad , \text{ " } \quad \text{ " } \quad . \qquad \text{(d)}$

Let $|h| < \eta$, where η is the smallest of the three numbers, η_1 , η_2 , and η_3. Suppose in (b) above that $m(x' + h) - m(x')$ is negative. From (d),

$$|m(x' + h) - m(x')| < \varepsilon/2 \quad , |h| < \eta \quad .$$

Then, $\quad |M(x') - M(x' + h)| < \varepsilon \quad , \text{ " } \quad \text{ " } \quad$, from (b) and(d)

If $M(x') - M(x' + h)$ is negative in (b), we have, from (c),

$$|M(x') - M(x' + h)| < \varepsilon/2 \quad , |h| < \eta \quad ,$$

and, $\quad |m(x' + h) - m(x')| < \varepsilon \quad , \text{ " } \quad \text{ " } \quad$, from (b) and (c).

In all cases, then,—when the two quantities mentioned have

the same sign and when they differ in sign—we are able to find, for a given $\varepsilon > 0$, an η such that,

$$|M(x') - M(x'+h)| < \varepsilon \quad , |h| < \eta ,$$

and, $$|m(x'+h) - m(x')| < \varepsilon \quad , |h| < \eta .$$

Since x' is any point, M(x) and m(x) are continuous everywhere. Their continuity is thus necessary for the continuity of $\omega(x)$.

<u>86</u>. # <u>Successive oscillation functions</u>. The oscillation of f(x), or $\omega(f,x)$ will be called the oscillation of f(x) of the first order. The oscillation of $\omega(f,x)$ will be called the oscillation of f(x) of the second order and will be written $\omega^{(2)}(f,x)$; and so on. We shall now establish a few theorems concerning these oscillations.

<u>87</u>. # <u>Given any function f(x); $\omega^{(n)}(f,x) = \omega^{(2)}(f,x)$, for n equal to or greater than 2.</u>

We shall first show that if $\phi(x)$ is any function semicontinuous above and having a minimum 0 at every point, then $\omega(\phi,x) = \phi(x)$. For, $M(\phi,x) = \phi(x)$, and $m(\phi,x) = 0$; then $\omega(\phi,x) = M(\phi,x) - m(\phi,x) = \phi(x)$.

Now, $\omega(f,x)$ is semicontinuous above, and hence pointwise discontinuous. Then $\omega(\omega,x)$, or $\omega^{(2)}(f,x)$ has a minimum 0 at every point (16). Being an oscillation function it is also semicontinuous above. Then, $\omega^{(2)}(f,x)$ has the properties of $\phi(x)$ above. Hence $\omega^{(3)}(f,x) = \omega^{(2)}(f,x)$; and similarly for higher orders.(1)

---

(1) Attention should be called to the fact that the proof does not go through for the case where $\omega(f,x)$ is infinite by whole intervals, for $\omega^{(2)}(f,x)$ cannot be defined.

88. # <u>A necessary and sufficient condition for pointwise discontinuity of a function f(x) is that the oscillation of f(x) of the second order be equal to its oscillation of the first order.</u>

It is necessary; For, f(x) being pointwise discontinuous, $\omega(f,x)$ has a minimum 0 at every point. It thus has the properties of $\phi(x)$ in (87); then $\omega^{(2)}(f,x) = \omega(f,x)$.

It is also sufficient. Since $\omega(f,x)$ is pointwise discontinuous, $\omega^{(2)}(f,x)$ has a minimum 0 at every point. Then, since $\omega^{(2)}(f,x) = \omega(f,x)$, $\omega(f,x)$ has a minimum 0 at every point, and this is sufficient for pointwise discontinuity of f(x) (16).