GRADE 4-6 STUDENT CONCEPTIONS AND UTILIZATION OF INFORMAL AND FORMAL VARIABLE REPRESENTATIONS ACROSS MATHEMATICALLY EQUIVALENT TASKS

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by
MATT SWITZER

Dr. John Lannin, Dissertation Advisor

MAY 13, 2011

The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

GRADE 4-6 STUDENT CONCEPTIONS AND UTILIZATION OF INFORMAL AND FORMAL VARIABLE REPRESENTATIONS ACROSS MATHEMATICALLY EQUIVALENT TASKS

Presented by Matt Switzer

A Candidate for the degree of Doctor of Philosophy

And hereby certify that in their opinion it is worthy of acceptance.

Professor John K. Lannin

Professor Douglas A. Grouws

Professor Kathryn B. Chval

Professor Fran Arbaugh - Penn State University

Professor David H. Jonassen

Professor David A. Bergin

DEDICATION

I dedicate this dissertation to my wife, Amy. Through so many events in our lives, you have always been there. As I have said so many times, it was love at first sight and that love has grown, changed, matured, and enriched my life in far too many ways to count. I could not have done it without you. I love you.

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Far too many people to mention have influenced the work contained in this dissertation. I am sure I will leave out many and to these individuals I apologize. Several have played a central role in my being at this point in my career and to these I owe special thanks.

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ABSTRACT

This study reports how 24 grade 4-6 students in one elementary and middle school interpreted formal and informal representations of variables. While interpretations for variables represented as letters (e.g., x and y) have been well established for students in algebra classes and beyond, little research into elementary school students‘ initial interpretations of variables exists. This study examined student interpretations of formal (e.g., $x + y = 12$) and informal representations of variables (e.g., $\square + \triangle = 12$).

The students in this study were consistent in their meaning of various representations of variables presented in equations, but did not parallel normative algebraic solutions. For example, students treated the representation of the variables as different variables even if they were the same (e.g., $y + y = 12$). Student also consistently produced multiple solutions for each variable. For example, they supplied the ordered pair solutions such as (6,6), (5,7), (4,8), regardless of the representation of the variable (e.g., $y + y = 12$; $a + b = 12$; and $\square + \triangle = 12$).

Further, these students did not exhibit many of the misconceptions exhibited by students in algebra classes and beyond. For example, the common misconception that different variables can only take on different values was not a typical response for these students (Fujii, 2003).

However, when these same tasks were presented as word problems, students treated variables in an algebraically normative way. In other words, the students were more –successful” solving the word problems (Koedinger & Nathan, 2004). Students attended to the syntactic and semantic structure of the word problems to determine meanings for the variables that were not evident in the equations.

CHAPTER 1: INTRODUCTION

Rationale for the Study

Successful completion of an *algebra* course, or equivalent, serves as a gatekeeper for students' future educational, professional, and economic opportunities (e.g., Ball, 2004; National Academy of Science, 2007). Those who successfully complete an algebra course are at a considerable advantage over their peers who have not. This critical role of algebra has also been a focus in several recent high profile reports. *Trends in Mathematics and Science Study* (TIMMS), *Rising Above the Gathering Storm* (National Academy of Science, 2007), *Mathematical Proficiency for All Students* (Ball, 2004), and *Foundations for Success* (National Mathematics Advisory Panel, 2008) have each placed an increased emphasis on two specific, but related, aspects of learning and teaching algebra. First, students' early mathematics education must prepare them for success in algebra (Kilpatrick & Izsak, 2008). Second, in order to increase students' proficiency in algebra their experiences in algebra and beyond must also support *all* students in attaining this goal.

In addition to the increased emphasis on the importance of students' success in algebra, is the corresponding emphasis on *algebra for all.* From a social justice perspective Robert Moses has argued for and worked toward this goal through the *Algebra Project* (Moses, 2011). The stance that all students can be successful in mathematics in general, and algebra in particular, is also consistent with the Equity Principle in NCTM's *Principles and Standards for School Mathematics* (NCTM, 2000). Likewise, Achieve has taken the position that everyone can do algebra (American

Diploma Project (ADP), 2004). Further, numerous state legislatures have taken the position that students must complete the equivalent of at least an algebra course in order to receive a high school diploma.

Considerable knowledge exists regarding the teaching and learning of *algebra* (e.g., Blanton, et al., 2007; Booth, 1984; Brizuela & Schliemann, 2004; Carpenter, Levi, Berman, & Pligge, 2005; Drijvers, 2003; Kaput, 2008a; Kieran, 2007; Kuchemann, 1981; Lee, 2006; NCTM, 2000; Radford, Bardino, & Sabena, 2007). This includes the relatively new and evolving domain of research on *early algebra*, which has begun to develop a knowledge base addressing the relationships between arithmetic and algebra (e.g., Blanton & Kaput, 2001; Carpenter, Franke, & Levi, 2003; Carraher & Schliemann, 2007; Kieran, 2007; Schliemann, et al., 2003; Van Amerom, 2003; Warren & Cooper, 2008b). One finding consistent across these areas of research is that many students at all levels demonstrate difficulties with the meaning and use of conventional mathematical symbols (e.g., Cooper & Warren, 2008; Fujii, 2003; Kuchemann, 1981).

A subset of the research on students‘ meaning and use of conventional mathematical symbols addresses students‘ meanings for and subsequent use of variables. In fact, the research (cf., Booth, 1984; Carraher, Brizuela, & Schliemann, 2000; Carraher, Schielmann, & Brizuela, 2001; Ellis, 2007; Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005; Kuchemann, 1981; Lannin, Barker, & Townsend, 2006; MacGregor & Stacey, 1997; Swafford & Langrall, 2000; Warren & Cooper, 2008b) has established a great deal about students‘ meaning and use of *conventional letter-symbolic* representations of variables. This research has demonstrated that many students appear to have different meanings and strategies for dealing with variables and these strategies may vary across task types (e.g., word problems, word equations and equations (Koedinger & Nathan, 2004)). However, the setting for the majority of this research has been at the middle school level and beyond and does not include students' meanings and use of *informal* representations of variables, or why or how difficulties arise.

Before progressing further, I briefly discuss the evolution of the definition of variable. Philipp (1992) stated that Gottfried Wilhelm Leibnitz (1646-1716) and Sir Isaac Newton (1643-1727) first introduced the notion of variables representing varying quantities which was closely tied to the notion of function. By 1718 Johann Bernoulli regarded a function as any expression consisting of a variable and constants, and Euler later regarded function as any equation or formula consisting of a variable and constants (Eves, 1983). However, Philipp (1992) applied a definition of variable as ―consisting of a symbol standing as a referent for a set consisting of at least two elements," which is often used today (p. 557).

When using this definition

> [e]ven the literal symbol *x* in the statement *x* +3 = 7 is a variable, because *x* represents any of the elements of the set in the unstated but implicitly assumed domain, be it the real numbers, the rational numbers, the integers, the natural numbers, as so forth (Philipp, 1992, p. 557).

Throughout the remainder of this dissertation, I use this definition of variable.

Purpose of the Study

Therefore, the purpose of this study was to examine the meaning grade 4-6 students develop for variables across representations of these variables, tasks with equivalent mathematical structures, and task types. Typically, mathematics curriculums introduce students to conventional letter-symbolic representations of variables around fifth grade. Including grades 4-6 allowed a glimpse into students' meanings before, during, and immediately after this introduction.

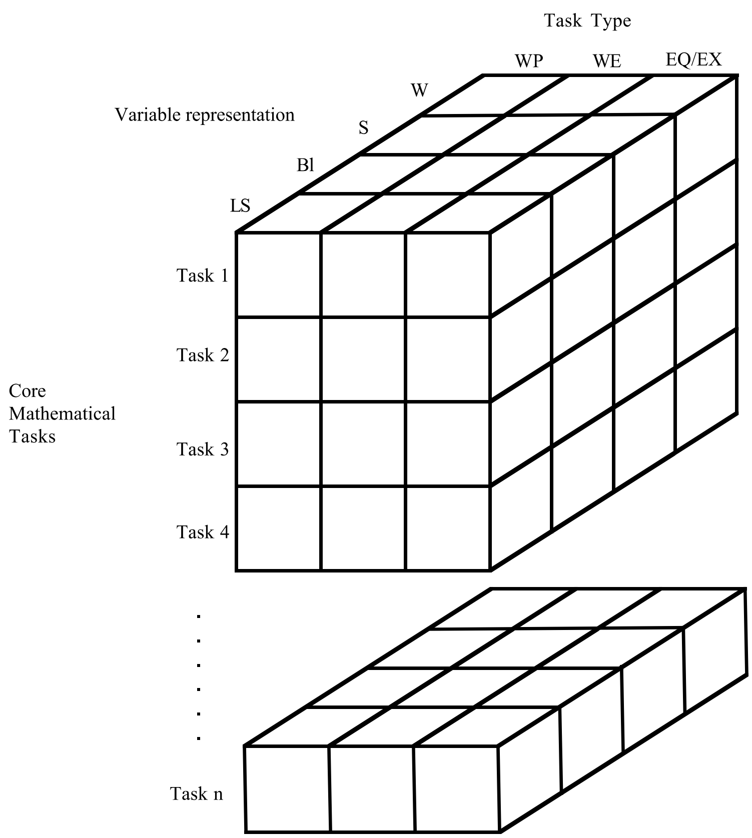
Research Questions

In this study, I provided an initial answer to the following research question: How do grade 4-6 students interpret various representations of variables when presented in different forms and different task types? To answer the aforementioned question, I addressed the following subquestions:

a. What solution sets do grade 4 – 6 students generate for tasks with equivalent mathematical structures across representations of variables (i.e., blanks, letters, shapes and words) *and* different task types (i.e., word problem or equation)?
b. How do grade 4-6 students interpret variables across various representations of the variable (i.e., place holder or letter-symbolic)?
c. How do grade 4-6 students interpret variables across different task types (word problem or equation)?
d. How do grade 4 - 6 students interpret variables across various representations of a variable (i.e., place holder or letter-symbolic) *and* different task types (word problem or equation)?

Design Framework

I focused this study on representations of variables, task types, and various core tasks with common mathematical structures. Based on a review of the literature and elementary textbooks I developed the task design framework (see Figure 1) to guide the development of tasks used in this study. This framework includes four representations of variables: blanks, letters, shapes, and words. It also includes three task types: word problems, word equations, and equations/expressions. I combined the equations and expressions based on their common reliance on mathematical symbolism. The final dimension consists of the individual core mathematical tasks. I was then able to write each of the core mathematical tasks to include the given representation of the variable and task type producing twelve tasks with a common mathematical structure.



Task design framework definitions

Problem representation	**Variable representation**
WP – Word Problem	W – Words (Literal)
WE – Word Equation	Bl – Blank (Placeholder)
EQ/EX – Equation/Expression	S – Shape (Placeholder)
	LS – Letter-symbolic

Mathematically equivalent tasks: Each task is modified to include a representation of the variable (W, Bl, V) and a problem type (WP, WE, EQ/NS) resulting in twelve tasks, each with mathematically equivalent structures.

Figure 1. Task design framework.

For this study, I focused primarily on three core mathematical tasks. From these three core mathematical tasks, I developed a set of tasks as the primary measures used this study. Each task was written in four different formats (1) a word problem with the variable represented in words, (2) an equation with the variable represented with shape(s), (3) an equation with the variable represented with blank(s), and (4) an equation with the variable represented with letter(s) representing a subgroup of the task design framework, see Figure 2. These tasks are the tasks used throughout interview one of the interview protocol and guided the task design of the second interview.

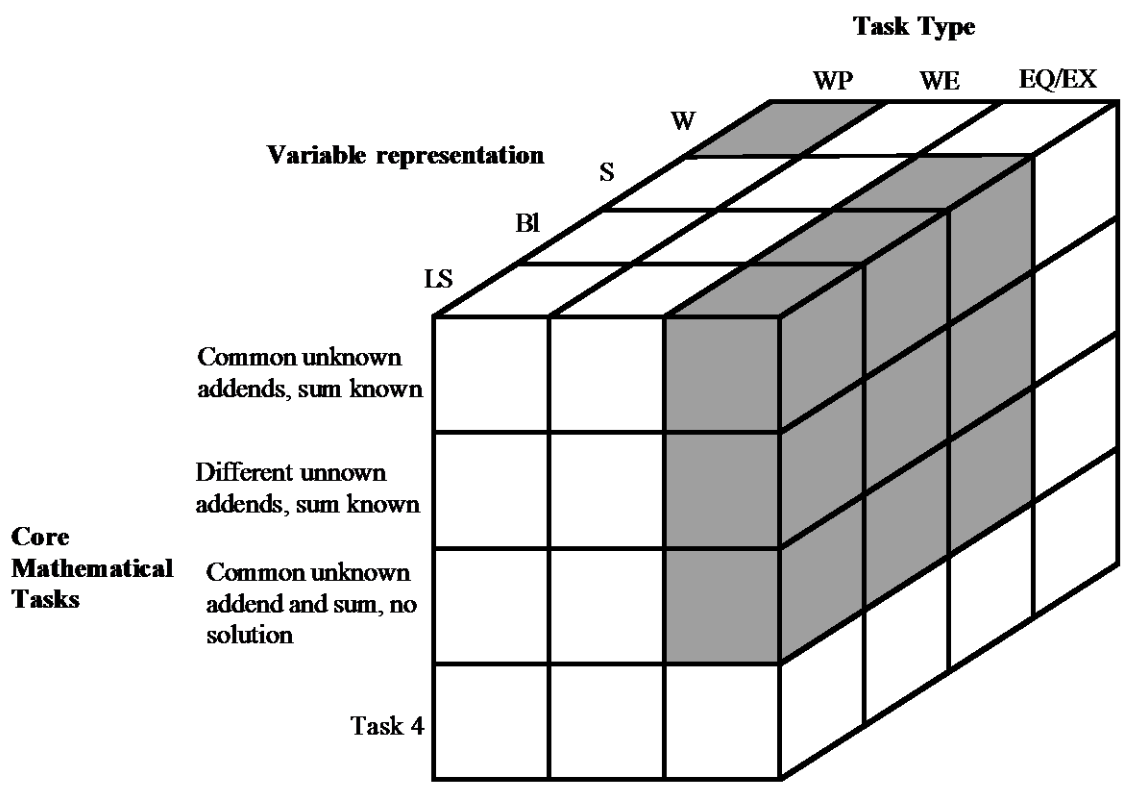


Figure 2. Task design of study.

CHAPTER 2: LITERATURE REVIEW

In the following sections, I discuss the conceptual framework including the literature I drew upon to develop this framework. I then review the extant literature related to the teaching and learning of algebra, and early algebra. I provide an overview of this research in general by reviewing the research on teaching and learning of algebra and then early algebra describing the relationship of this literature to students meaning for mathematical symbols, specifically variables. I then expand on the problem area this study addresses, namely the section on the teaching and learning of algebra focuses on three categories of research: generational, transformational, and global/meta-level activities (Kieran, 2007). Each of the three categories addresses a specific aspect of this study as students must use variables to generate equations and expressions, transform these equations and expressions, and the role their experiences in and outside of school play in their solving, justifying, and modeling in algebraic activities.

In the section on research findings from early algebra, I describe three current perspectives employed for examining the relationships between arithmetic and algebra: numerical reasoning, quantitative reasoning, and functions (Kieran, 2007). In the following sections, I discuss how each perspective provides important ideas related to and in support of current and future research on students‘ meanings for variables. Finally, in light of the review of the extant literature I describe the problem area under consideration.

While much of this research does not directly address students‘ meanings for variables, it does provide evidence of the importance of students‘ understanding and meaning for mathematical symbols in general and variables specifically. If students do not understand these symbols then the corpus of research on algebra and early algebra would be difficult if not impossible to examine. Therefore, students understanding of these symbols is critical to their further mathematical development.

Conceptual Background

To guide this study related to students' meaning and use of variables, I drew on the extant literature related to algebra and early algebra in developing the conceptual framework described in this section (e.g., Booth, 1984; e.g., Carraher & Schliemann, 2007; Kieran, 2007; Koedinger & Nathan, 2004; Kuchemann, 1981; MacGregor & Stacey, 1997). Specifically, I focused on difficulties and misconceptions that students demonstrate for the meaning and use of variables, and the types of tasks in which students engage with variables. From this review of the literature, I identified three interrelated factors that may influence students' construction of meanings for variables: (a) various representations of variables, (b) task types, and (c) tasks with common mathematical structures. I elaborate on each of these in the following sections.

Representations of variables. Within the extant research on students' understanding of mathematical symbols, conventional letter-symbolic representation of variables has been the primary focus of research on variables (e.g.,Fujii & Stephens, 2008; Kaput, 2008a; MacGregor & Stacey, 1997; Radford, 2000; Sfard & Linchevski, 1994). In order to examine elementary students' meanings for variables, I extended these representations of variables to include both informal and formal representations of variables including but not limited to blanks, shapes, words, and letters. While there has been an ongoing debate in the field regarding the inclusion or exclusion of unknown single values as variables, I agree with Carraher and Schliemann's (2007) suggestion that there are good reasons for treating unknowns as variables or indeterminates for which a single value will satisfy an equation or inequality. For instance, in situations commonly seen by elementary students, such as solving $3 + x = 8$ for x, treating x as a variable that can stand for *any* numerical value of which *only one* satisfies the equation can be an important and powerful position that can assist students transitioning to variable usage in algebra (Saul, 2001).

Extending the definition of variable to include informal representations of variables increases the potential issues associated with variable use. Carpenter, Franke, and Levi (2003) noted the difficulty that common notations used in elementary grades like ―*Find the different numbers you can put in the boxes:* $\square + \square = 9$” can produce (p. 75). They noted that this notation could be confusing to students as well as contribute to the development of misconceptions about the use of variables. They suggest that it would be preferable to use the number sentence $\square + \triangle = 9$. However, this change in notation assumes that students attend to and recognize the square and triangle as different variables.

For instance, in the first number sentence, $\square + \square = 9$, do the boxes have to represent the same number or can they represent different numbers? While conventions exist for the treatment of x in $x + x = 9$, no conventions exist, or at least have not been clearly agreed upon, for ―boxes” unless we retrospectively apply the conventions for conventional letter-symbolic variables to these informal representations. In addition, in $\square + \triangle = 9$ we do not know if elementary students interpret this notation as requiring the square and triangle to be different values, (i.e., they cannot be the same value) a common misconception in algebra courses and beyond, or if they differentiate between the square and triangle. The phrasing of the prompt for ☐ + ☐ = 9 also imposes a specific meaning on the representation of the boxes that the students may not hold. By asking the student what numbers they *can put in the box*, the box takes on the role of a placeholder that they are to fill in as opposed to a representation that can *stand for* or *represent* numbers.

From my review of elementary mathematics textbooks, I found that another common representation for unknowns, or variables, in the elementary grades is the use of a blank as a placeholder (Pearson Education, 2011). It is very difficult if not impossible to distinguish between two different variables using blanks. For instance, the equation ☐ + ☐ = 9 could be written as ___ + ___ = 9. However, rewriting the second equation, ☐ + △ = 9, using blanks is problematic if we apply the conventions for formal letter-symbolic variables that the same variable in the same equations must be the same value and different variables can have different values. We know little about how students view these various representations. Therefore, unless researchers address the semantic and syntactic issues associated with the use of these symbols, ongoing confusion and misconceptions could result.

Task Type. In addition to the potential influence of various representations of the variable on the corresponding meaning elementary students construct for these representations, the type of task in which the variable is used also appears to be a potentially relevant factor. Elementary students regularly see tasks with varying quantities presented in equations, number sentences, and word problems. As students progress in their mathematical studies these task types extend to include, but are not limited to, inequalities, tables, graphs, and combinations of these task types.

Koedinger and Nathan (2004) reported on the common conception among teachers and researchers that word problems are the most difficult for students to solve at both the algebraic and arithmetic levels. They note that apart from their own study they –found no prior experimental data that compared students‘ solution correctness for matched algebra story problems and equations” (p. 131).

Koedinger and Nathan (2004) specifically explored what they referred to as the *representational effect* for students in early algebra, in two studies. This effect refers to the impact that using different representations of problems (i.e., story problem, word equation, and symbolic equation) have on students‘ performance.

The first study consisted of seventy-six students from an urban high school (58 enrolled a mainstream Algebra course and 18 ninth graders enrolled in a geometry course). The second study consisted of 171 students enrolled in a first-year Algebra I course at three urban high schools. Using four different core mathematical tasks, Koedinger and Nathan (2004) generated 96 problems by varying difficulty factors, problem representation, or task type, (i.e., word problem, word equation, or symbol equation); unknown position (i.e., result or start); and number types (whole numbers or decimal numbers) per cover story.

For example, one of the problems, a decimal word problem with starting unknown position stated,

> After buying donuts at Wholey Donuts, Laura multiplies the number of donuts she bought by their price of $0.37 per donut. Then she adds the $0.22 charges for the box they came in and gets $2.81. How many donuts did she buy? (p. 132).

The corresponding word equation, with the referents included in the algebraic word problem removed, stated,

> “Starting with some number, if I multiply it by .37 and then add .22, I get 2.81. What number did I start with?” (p. 132). Finally, the corresponding symbol equation stated, “Solve for *x*: *x* * .37 + 22 = 2.81” (p. 132).

In their first study, Koedinger and Nathan (2004) found main effects for each of the three factors. For task type, 76 students performed better on the word problems (66%) and the word equations (62%) than they did for the corresponding symbol equations. In addition, students performed better on the result-unknown problems in comparison to the start-unknown problems, and better on whole number problems than on decimal number problems. They note that, “the differences between verbal and symbolic representation, and not the difference between situational context and abstract description, accounts for the observed performance differences” (p. 143).

In addition to differences in students‘ success across problem representations, students also demonstrated differences in the solution strategies they employed across different task types. For instance, 50% of students employed an *unwind* strategy, reversing the processes described in the problem, for word problems. For word equations, only 22% of the students used the unwind strategy and 23% used a guess and test strategy. For the equations, 32% of students gave no response, in comparison to 19% for word equations, and 18 % for word problems, and 22% manipulated symbols to solve the equation.

Of the strategies employed by the students in the study, the “guess and test” strategy resulted in the highest likelihood of leading to a correct answer (71% of the time), and the unwind strategy resulting in a correct answer 69% of the time. Also, the unwind strategy was employed most often, 335 times out of 819 responses, followed by an answer with no discernable strategy occurring 161 times.

Therefore, not only did the task type result in differences in student performance data but also in the solution methods the students used to solve the problem. Since the strategy the students used was dependent to some degree on the task type, it is not possible to determine if one strategy results in a higher percentage of correct answers in general. However, the influence that the task type had on students' solution strategies is an important result.

Further, the task type appears to play an import role in student performance data. Koedinger and Nathan (2004) found that the task type, or problem representation, changed beginning algebra students' performance on the task and their underlying cognitive processes. Specifically, they found that the common conception by researchers and teachers that the word problems would be most difficult for students to solve was not necessarily the case. They note that,

> "contrary to some views of situated cognition, this result is not simply a consequence of situated world knowledge facilitating problem-solving performance, but rather a consequence of student difficulties with comprehending the formal symbolic representation of quantitative relationships" (p. 129).

However, I argue that the *story problem* shown above is not in the form that a traditional story problem would tend to take. Instead, it takes the same form as their *word equation* but includes a context, in this case buying of donuts, in that it describes a calculation. I suggest that if the story problem above were found in a current mathematics textbook it would take a form similar to, Wholey Donuts sells donuts for $0.37 each plus $0.22 for a box to hold the donuts. If Laura buys some donuts, how much would she spend? Previous studies, (e.g., Hudson, 1983) have found that even small differences in the way problems are phrased can result in differences in student strategy selection and performance. Therefore, this difference in the phrasing of the word problem, describing a context in general versus strictly supplying a description of the *calculation* within a context, may result in differences in performance and solution strategy selection by students.

Based on the research findings that students often have difficulty with mathematical symbols, including variables, I believe it is important to note that Koedinger and Nathan (2004) did not take into consideration the differences in the representation of the variables across the task types in their findings. For instance, in the examples provided, the variable in the word problem was in words with a specific referent, donuts; the variable in the word equation was in words with no referent; and the variable in the symbol equation was a conventional letter-symbolic representation, *x*. Since Koedinger and Nathan found that the differences in performance were due to students‘ difficulties comprehending formal symbolic representations of quantitative relationships, differences in the symbolic representation of variable and potential differences in the meanings students have across these representations may also play a contributing factor in the students‘ performance, as I argued in the previous section.

Therefore, the type of task in which the representation of the variable occurs likely plays an important role in students‘ selection of solution strategy, performance, and the meaning they assign to the given representation of the variable. While Koedinger and Nathan (2004) have provided preliminary insights into the relationship between task type, solution strategy, and student performance, questions remain regarding the phrasing of the task types including the representation of the variable used in and across task types.

Tasks with common mathematical structures. As noted earlier, Koedinger and Nathan (2004) developed the tasks used in their study from four core mathematical tasks generating 96 problems by varying difficulty factors, problem representation, or task type, (i.e., word problem, word equation, or symbol equation); unknown position (i.e., result or start); and number types (whole numbers or decimal numbers). This structure resulted in eight subsets of the tasks, each including either integer or decimal values with each having what they referred to as a common mathematical structure (e.g., each of the 12 forms of the decimal story problem noted earlier are based on the equation $7 \times 0.37 + .22 = 2.81$).

In their analysis, Koedinger and Nathan (2004) used the problem type (word problem, word equation, and symbol equation) as the unit of analysis when comparing student performance and solution strategies. However, no such analysis occurred across tasks with common mathematical structures. This was reasonable as the purpose of their study was to examine student performance across each of the three task types. Given their claim that the differences in student performance were due to student difficulties comprehending formal symbolic representations of quantitative relationships, examining how students' difficulties comprehending formal symbolic representations of quantitative relationships were manifested across task types with a common mathematical structure is a natural extension of the study.

Such an extension is reasonable when extending the factors under consideration to include both formal and informal representations of variables used across these tasks, a consideration that is noticeably absent from the existing research. As noted earlier, the representation of the variable used across their task types was limited, varying from contextualized words with a referent in the word problems to decontextualized words without a referent in the word equations, to letters in the symbol equations. Since students often see and use representations of variables other than words and letters, including these in this extension of the study would provide a better sense of the role these various representation play in student performance, solution strategy selection, and meaning for the representation of the variable.

Conceptual Framework

As noted in the previous section, in my review of the extant literature I identified three intertwined categories related to variable: (a) various representations of variables, (b) task types, and (c) tasks with common mathematical structures. In a general sense, people construct new knowledge and understanding from their previous knowledge and beliefs (Bransford, Brown, & Cocking, 2000). This includes the prior knowledge students have constructed inside and outside their school experiences. This prior knowledge becomes a basis from which students construct their conceptions of variables. In the ongoing process of constructing their meaning of variables as they engage with and in new experiences, these new experiences then becomes the prior knowledge in which the students engage in their worlds both within and outside of school in a continuously recursive cycle as shown in Figure 3.

As demonstrated in the review of the literature above, how students develop meaning for representations of variables and what factors contribute to their meanings is a domain lacking important research findings. Therefore, I focused this study on the portion of Figure 1 contained within the dashed box. I acknowledge that the students' prior experiences, including prior experiences both within and outside of mathematics, their classroom, and school play an important role in their construction of knowledge in general. However, the inclusion of these was beyond the scope of this study.

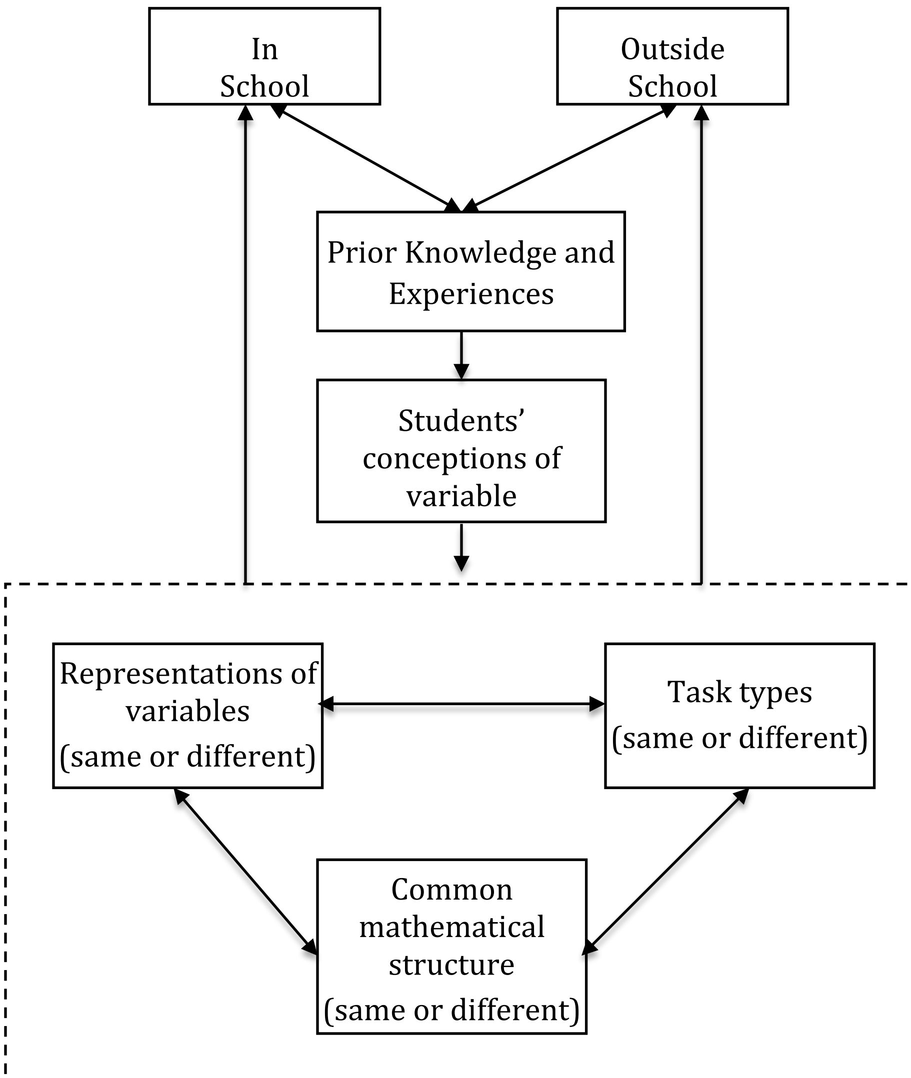


Figure 3. Development of student meaning for representations of variable.

The dashed box in Figure 3 illustrates the three interrelated factors related to students' meanings for representations of variables. In the elementary grades, textbooks often represent variables and unknowns in multiple ways (e.g., blanks, shapes, words, and letters), which may play an important role in the difficulties, and misconceptions students develop for conventional letter symbolic representations of variables. Research has also demonstrated that the task type in which the representation of the variable occurs influences student performance as well as their selection of solution strategy. Finally, while we know that differences in task type influence student performance, research has not established if or how these differences occur across various task types with common mathematical structures.

By reviewing the literature, it is evident that researchers, curriculum developers, and teachers assume that students naturally progress from these multiple representations, which often stand for a single solution, to a single conventional representation that can stand for multiple values. It is also possible that they have not recognized the potential relationship between student meanings between informal and formal representations, as these connections are not present in the extant research reviewed. In either case, research has not adequately examined these assumptions.

With the various potential combinations of representation of variables and task types across common or different mathematical structures, students may develop or generalize meanings for these representations of variables that are dependent upon each of these three factors. Therefore, in addition to coordinating and synthesizing their meanings for various representations of variables, they may need to coordinate different meanings for the same representation as determined by its use. Unless we know more about the meanings students have for informal and formal representations of variables we cannot begin to explore how they use this prior knowledge to construct meaning for conventional letter-symbolic representations of variables or how to assist them in overcoming or preventing the common misconceptions held by so many students in algebra and beyond.

Research Findings on Student Learning of Algebra

In the following sections, I review the relevant extant literature related to K-college level students learning of algebra. In this review, I demonstrate that the vast majority of the existing research has been conducted at the middle school through college levels in algebra courses and beyond. Therefore, our current knowledge base for early algebra is in dire need of further research. At the middle school through college level, Kieran (2007) identified and synthesized three areas of research addressing the teaching and learning of algebra; (a) generational activities, (b) transformational activities, and (c) global/meta-level activities. I organize the following synthesis of the research for middle school through college levels around these three activity types.

Generational Activities. Kieran (2007) described generational activities as ―the area where, according to Radford (2001), the role of algebra is that of a language to express meaning, and where the habit of mind of those who are in ‗algebra mode‘ can find expression (Cuoco, Goldenberg, & Mark, 1996)" (p. 714). This includes:

- Equations containing an unknown that represents problem situations (c.f., Bell 1995);
- Expressions of generality arising from geometric patterns or numerical sequences (c.f., Mason, 1996); and
- Expressions of the rules governing numerical relationships (c.f., Lee & Wheeler, 1987) (p. 713).

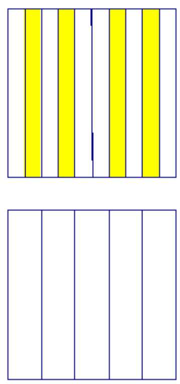
Kieran (2007) noted that researchers have found that many students exhibit difficulties within such activities. This includes, but is not limited to difficulties and misconceptions for letter-symbolic notation representations (c.f., Arcavi, 1994; Booth, 1984; Carraher, et al., 2001; Fujii & Stephens, 2008; Kuchemann, 1981; Philipp, 1992), multiple representations (c.f., Koedinger & Nathan, 2004; Kuchemann, 1981; Lobato, Ellis, & Munoz, 2003), and working within the context of word problems (c.f., Koedinger & Nathan, 2004; Kuchemann, 1981).

Within the use of conventional letter-symbolic representations of variables, Fujii and Stephens (2008) found that ―students who are aware of the proposition that the same letter stands for the same number … tend to think that the converse of this proposition is also correct" (p. 1-51). In other words, they believe that when two or more variables are present in an equation the variables must take on *different* values and cannot be the same value (e.g., for $a + b = 12$, a and b cannot both be 6). While this misconception has been well documented with students‘ use of conventional letter-symbolic representations of variables, the existing research does not provide adequate evidence to determine if the same case holds for student meanings for informal representation of variables. Therefore, it is unclear if this misconception is independent of the representation of the variable, if it is present during elementary grades, or arises later.

In addition, Carraher, et al., (2001) found that students were often not comfortable with the introduction of letters as representations of variables. When introduced to the idea of using a letter to stand for a person‘s height the students in their study were hesitant to accept the notation. After several lessons using this teacher-introduced notation, the students began to use the language that the letter could stand for any number. However, it is unclear from the study if students adopted these meanings, or simply parroted the teacher‘s language. Further complicating the issue of student meanings for variables, Kuchemann (1981) reported ―the continuing tendency to regard letters as symbols for objects rather than numbers appears to be a significant stumbling block in learning algebra" (p. 136). Therefore, the issue of students acceptance and understanding of letters as representations for variables goes beyond their developing the concept of variable as standing for a number(s) to include their understanding of these representations as objects that can be operated on within the conventions of algebraic manipulation, which I will discuss further in the section on transformational activities.

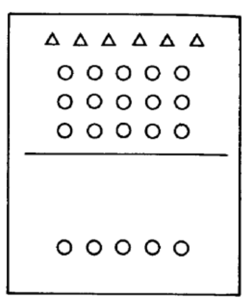
Although the current research findings show that students often have difficulties with multiple representations (i.e., the relationships between tables, graphs, and equations), a broader understanding of representations should extend to different representations of variables, especially when considering the developing research base for early algebra and the prominence of such representations in the elementary curriculum. This position is supported by Bell, Costello, and Kuchemann's (1983) finding that for students, ―structurally equivalent symbolic and conceptual tasks are not necessarily recognized as the same" (p. 89). In other words, even though the tasks are structurally equivalent, the differences in the ways that the tasks are represented, including but not limited to differences in the representation of the variable, results in students not recognizing them as the same. They reported that the following two pairs of tasks (see Figure 4) had a low correlation (about 0.37) indicating that students saw them as substantially different, possibly pointing to how the structurally equivalent tasks' different representations resulted in students viewing them differently.

1a. Look at the square in the top picture. Four of the ten equal parts are shaded. Now look at the bottom picture. This square must have the same amount shaded. How many of the five equal parts should be shaded so that the same amount will be shaded in both squares? The squares are unit squares.



1b. $\frac{4}{10} = \frac{\ }{5}$

2a. If there are six triangles for every fifteen circles, how many triangles would there be for five circles?



2b. $\frac{6}{15} = \frac{\ }{5}$

Figure 4. Different representations of mathematically equivalent tasks (Bell, et al., 1983, p. 90).

In addition to student difficulties across representations, researchers have found that students may also develop misconceptions based on the classroom implementation of multiple representations. Lobato and Ellis (2002) examined students generalizations for slope and linear functions in a reform curriculum that regularly uses real-world settings to develop concepts. All of the students interviewed developed the unintended generalization of $y = mx + b$ as a difference. While this points to the importance of the teachers‘ beliefs (i.e., *personal mathematics*), actions and expectations (Blanton & Kaput, 2005; Cai, 2004; Zbiek, Peters, Boone, Johnson, & Foletta, 2009), it also demonstrates how students interpret and construct their own *personal mathematics* as a result of engaging in instructional tasks. Lobato and Ellis described how four focusing phenomena (–goes up by” language, well ordered tables, graphing calculator, and uncoordinated sequences and differences) contributed to students‘ generalization of the slope-intercept form of a linear equation as a difference.

This highlights the importance of attending to how and what knowledge students construct as well as the influence, potentially unintended and erroneous, instructional practices can have on these constructs. As students engage in and work with informal representations of variables, the conventions of use, if they exist, and the potential misconceptions that may arise, as mentioned earlier, need monitored. However, the lack of existing research on student meanings for variables, including the conventions of use that they infer from the contexts in which the variables are used, make instructional decisions difficult.

While Lobato and Ellis (2002) focused their research on students using curricula that employed real-world settings to introduce and develop concepts, students regularly see a variety of task types including, but not limited to, word problems and equations. Koedinger and Nathan (2004) found that a common conception among mathematics teachers and researchers—word problems are the most difficult type of task for students to solve is not necessarily the case (see Figure 5 for an example of the tasks used in their study). They found that students performed better on algebra story problems than word equations (i.e., a description of the computation employed), or symbolic equations.

However, the *algebra story problem* shown in Figure 5 is not in the form that a traditional story problem would tend to take. Instead, it takes the same form as their *word equation* shown in Figure 5 but includes a context, in this case the buying of donuts. I hold that if the *algebra story problem* above were found in a current mathematics textbook it would take a form similar to, Wholey Donuts sells donuts for \$0.37 each plus \$0.22 for a box to hold the donuts. If Laura buys some donuts, how much would she spend? Therefore, the differences found between the algebra story problems and word equations may not reflect the reality of the task types in which students would generally engage in mathematics classrooms. However, these findings do point to the influence that task type plays in students' solutions and solution methods as reported in chapter 1.

Task Type	**Task**
Algebra story problem	After buying donuts at Wholey Donuts, Laura multiplies the number of donuts she bought by their price of \$0.37 per donut. Then she adds the \$0.22 charge for the box they came in and gets \$2.81. How many donuts did she buy?
Word Equation	Starting with some number, if I multiply it by .37 and then add .22 I get 2.81. What number did I start with?
Equation	Solve for *x*: $x * .37 + .22 = 2.81$

Figure 5. Example of task types (Koedinger & Nathan, 2004, p. 132)

Two other issues with Koedinger and Nathan's (2004) findings reflect the limited research on students' meanings for various representations of variables. First, they did not take into account how the different representations used in the task types (i.e., words and letter-symbolic) may have resulted in differences in student solutions or solutions methods. Second, the solution sets for different task types (i.e., values of the variable) differ even for tasks with a common mathematical structure and differ in the solutions referents. For the algebra story problem in Figure 5, the potential solution set is limited to whole numbers of donuts. For their corresponding word equation and equation, they place no limitation on the solution set due to the absence of a referent. Therefore, while these tasks have a common mathematical structure, these differences, and students‘ attention to or lack of attention to them, may result in differences in how difficult each problem is to solve. This, along with the difficulties with variables reported thus far, raises the need for further research including the inclusion of other variable representations.

Thus, generational activities, an integral part of algebra courses, including their use of contextual situations, are problematic for students on several dimensions including student meaning-making for variables. As noted, much of the literature on generational activities has occurred in algebra classes and beyond. Included in these findings are misconceptions that often arise with the introduction and use of variables. When introduced to formal algebraic symbols, students often develop misconceptions of the possible domain for which the variables represent, leading to potential difficulties with their future work with variables.

Transformational activities. Kieran (2007) characterized transformational activities as including,

> collecting like terms, factoring, expanding, substituting one expression for another, adding and multiplying polynomial expressions, exponentiation with polynomials, solving equations and inequalities, simplifying expressions, substituting numerical values into expressions, working with equivalent expressions and equations, and so on.” (p. 714).

The focus of this type of activity is on the use of mathematics as a tool, on the discipline of algebra, and the rules governing its use. In this sense, it extends generational activities to the solving of the generated tasks.

Research in these areas provides several important findings related to the meaning that students have for mathematical symbols, pointing to the importance of attending to students meaning-making as they engage with these symbols. Kieran (2007) reported that within transformational activities, research has found that students often develop misconceptions about the equal sign (c.f., Behr, 1976), treating it as a sign to ―do” something (e.g., students may agree that 3 + 5 = 8 + 2 is true). Linchevski and Livneh (1999) found that, overall, students who had interpretive difficulties with equations containing several unknowns and numerical terms replicated the same errors in purely numerical contexts, demonstrating the consistency of student transformational errors across arithmetic and algebraic tasks. This may point to the possibility that the errors students make in numerical contexts predict the types of errors they make in transformational activities. This is but one illustrative example of the potential connection between students‘ arithmetic activity and algebraic activity.

Kieran (2007) discussed Graham and Thomas‘s (2000) teaching experiment with 13- and 14 year old students‘ use of the letter-store function key (often seen as the STO button) of the graphing calculator as a model for variable to evaluate different expressions for a variety of values. In using the letter-store function on the graphing calculator, students were able to evaluate expressions by assigning values to letter-symbolic variables. In the study, students ―came to see that different expressions were being used to represent the same process” (p. 723). In addition, students‘ views of expressions and variable changed suggesting –that a task that is transformational in nature can simultaneously be related to generational activity, if it leads to an evolution of students' conceptions of the objects of algebra" (p. 723). The dynamic nature of this assigning of values points to the importance of examining multiple situations from which students can generate meanings for variables. However, as noted previously, little such research exists for representations of variables other than the letter-symbolic.

Kieran (2007) reported other findings on transformational activities included bracket expansion errors (Booth, 1984; Bazzini, Boero, and Garuti, 2001a; and Ayers, 2000), equations errors (e.g., Herscovics & Linchevski, 1994; Kieran 1984b; and Vlassis, 2001), errors in checking equation solutions (Perrent & Wolters, 1994; Pawley, 1999), and difficulties with systems of equation (Filloy, Rojano, & Solares, 2003, 2004; Drijvers, 2003). Booth reported that students often do not see the need for using brackets. For example, some students perform the first written operation (e.g., multiplying $a + m$ by p resulted in $pa + m$). Further, if the student knows which operation they should be performed first then they do not see a need to record this fact with brackets to ensure the correct order of operations.

These studies demonstrate that approximately half of the students in middle school and lower secondary grades fail to correctly solve equations that include various numerical operations on one or both sides of the equal sign. These errors include students only attending to the first term after the equal sign, and errors with computation (e.g., $115 - n + 9 = 61$ transformed to $106 - n = 61$, referred to as *jumping off with the posterior operation* by Linchevski and Herscovics (1996)). When students make errors solving equations they often repeat these same errors when checking their solutions, thereby confirming their erroneous solution as correct. Kieran (2007) reported that Filloy, Rojano, and Solares (2003) found that students also tend to make more sense of comparison approaches to solving systems of equations then using substitution, and tend to not accept expressions as valid solutions. Kieran further noted “the extension of the notion of transitivity of equality from the numeric to the algebraic domain, as well as the idea of substituting one expression with another, was not at all obvious to students” (p. 724).

Kieran (2007) described conflicting findings on the use of concrete manipulatives for transformational activities for algebra. For example, studies by Filloy and Rojano (1989), and Boulton-Lewis, et al. (1997) found that balance scales, and concrete manipulatives (cups, counters, and sticks) were not helpful for students in learning about equations and equation solving. In contrast, Brown, Eade, and Wilson (1999), Linchevski and Williams (1996), and Radford and Grenier (1996) “have argued that the balance scale facilitates the understanding of the operation of eliminating the same term from both sides of an equation” (p. 724). It is unclear from the research how much errors such as those reported in this section are limited to algebraic problems as opposed to arithmetic. Researchers have established that students often have difficulties with the conventions of use for letter-symbolic variables, which may manifest themselves as such errors as these.

Therefore, the errors students make in transformational activities are often the same as those they make when working in numerical situations, however we know less about whether the converse is true. Students in the studies showed numerous difficulties with manipulating algebraic equations and expressions. Compounding this issue are the difficulties students have with generational activities. When a student has misconceptions about the variable representation, especially letter-symbolic, then they will correspondingly have difficulties with transformational activities involving these symbols. Therefore, their chances of success with algebra can be seriously at-risk.

Global meta/level activities. Kieran (2007) reported that in global/meta-level activities algebra is used as a tool, but this is not exclusive to algebra. This domain includes students' outside experiences and conceptions that influence their algebraic activities. Global/Meta-level activities include: problem solving, modeling, working with generalizable patterns, justifying and proving, making predictions and conjectures, studying change in functional situations, looking for relationships or structure, and so on. (p. 714).

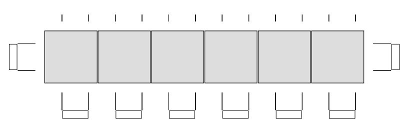
An important aspect of each of these global/meta-level activities is the critical role played by variables. Modeling, generalizing patterns, justification and proof, conjecturing, changes in functional relations, and finding relationships and structures all require the use of variables (Kaput, 2008a).

Kieran (2007) posited that the roots of generalizing in algebra involve the use of algebraic notation (e.g., variables) for expressing proof (e.g., Bell, 1976; Fischbein & Kedem, 1982; Mason & Pimm, 1984). More recently Mason, Graham, Pimm, and Gowar (1985; see also Mason, Graham, Johnson-Wilder, 2005) took the position that generalization is a route to algebra (p. 725). However, even with the important role that these activities play, Kieran reported that Lee (1987),

> found that few students use algebra or appreciate its role in justifying a general statement about numbers" when used as a tool for expressing general rules for both numeric and geometric patterns, or for providing justifications for equivalence of differing forms for these generalizations (p. 725).

Healy and Hoyles (1999), and Warren (2000) have shown that while students *can* use symbolic forms to generalize patterns, the use of formal symbols does not come easily to students. Ainley, Wilson, and Bills (2003) expanded on this idea when they reported on the distinction between students‘ *generalization of the context* and *generalization of the calculation*, finding the former to be insufficient for students to accomplish the latter. Ainley, et al., (2003) reported that student responses for the task shown in Figure 6 fell into three categories. Some of the students only generalized the context giving a ―general description of the arrangement of furniture, but did not describe the calculation” (p. 12). These students treated the middle and end tables separately. The resulting ―calculation which arises from this description is more complex than the ‗two chairs for each tables and two for the ends‘ image, and this may have been a factor which inhibited these pairs from being able to express a calculation” (12).

Tables and chairs are arranged like this in the school dining room:



Can you explain how to work out how many chairs are needed for each long table? Can you write it down?

Figure 6. The 'tables and chairs' question.

Some students who generalized the context then described a calculation, the initial description of the table arrangement was dynamic. These students focused on *what* occurred as more tables were added. These descriptions of the calculation were ―algebra-like expressions…though these were not always written down” (Ainley, et al., 2003, p. 12)

The final group of students provided a description of the calculation directly. These students ―were usually able to move easily to a written version of the calculation expressed in algebra-like notation, even though they sometimes struggled with syntax”

(Ainley, et al., 2003, p. 13). Ainley, et al. found that the students ―who were unable to articulate an algebra-like version of the rule had previously generalized the calculation required; generalizing the context did not seem to be sufficient to support pupils in moving to a symbolic version of the rule" (p. 14). This raises the question of the role of a well-formed understanding of variable in students' ability to model a generalization with an algebra-like rule.

Kieran (2007) reported that students use of tabular representations in generalization activities often leads to a disconnect between numerical and geometric relationships. The students use the tabular representation to derive a closed-form formula for which a few examples are checked. Kieran argues that such use of tabular representations ―shortcircuits all the richness of the process of generalization" (p. 725). In order to make the break from informal representations to conventional symbolic algebra to express generalizations, Radford (2003) stated that this break ―involves two ruptures, one with the sensual geometry of the patterns and the other with their numerical features such as rank" (p. 725). In other words, students must make a break between the extension of and drawing of the geometric patterns themselves (i.e., extending the geometric patterns) and numerical identification of the patterns such as the step number in the pattern. This requires a way to generalize and prove an algebraic generalization, which often requires the use of variables.

Research has demonstrated both student success and difficulties with generalization and proof. Kieran reported that studies by Edwards (1998), Miyakawa (2002), and Dreyfus, Hershkowitz, and Schwartz (2001) found that students with limited backgrounds in proof have difficulty using conventional algebraic notation, including variables, to express generalizations, and prove that these symbolic representations are generalizations. Specifically, Miyakawa found that *both* algebraic competence and general mathematical competence are related to student difficulty with constructing algebraic proofs, pointing to the inter-related nature of these domains of mathematics. However, Dreyfus, Hershkowitz, and Schwartz reported the success of seventh grade students in collaboratively constructing an algebraic proof that the difference between the products of the diagonals of rectangles of the following type is 12, see Figure 7.

7	**13**
9	**15**

3	**9**
5	**11**

Figure 7. Rectangles where differences in products of diagonals is 12.

These students first *generalized the process*, and then justified it using an algebraic representation. The rectangles were generated using a spreadsheet where entering any value in the upper left quadrant generated the other three quadrants. The students generalized the values of the rectangle and the subsequent products and differences and then expressed this generalization algebraically.

In discussing a study by Healy and Hoyles (2000) involving students with prior experience in formulating and testing conjectures, Kieran (2007) noted that ―when students were asked to pick the argument that they thought would get the best mark, the majority of students chose the ones based on algebraic form” (726). However, when they examined students‘ own arguments they found that students were ―unlikely to base their own arguments on similar algebraic constructions, feeling that these arguments neither communicated nor illuminated the mathematics involved” (p. 726). Instead, students were more likely to provide numerical examples and then provide a narrative explaining the generalization. This difference between selecting and generating arguments may be due in part to the difficulties students have with algebraic symbols. In other words, how students understand these algebraic forms and symbolizations and the role they assign to them in communicating and justifying may help explain this difference.

These findings demonstrate that students have numerous difficulties and misconceptions with algebra in a variety of areas and for a variety of reasons. These include using algebra as a language to generate meaning for and modeling of situations, transforming algebraic equations and expressions, and the influence of outside experiences and conceptions on students' algebraic understanding.

Summary. Generalization, justification, and proof are essential aspects of mathematics, including algebra. The use of symbolic forms to generalize patterns, as well as other situations, is an important element of algebra. If students exhibit the difficulties reported from the literature with generational and transformational activities, then the use of these symbolic representations to generalize both contexts and calculations may be impacted. Additionally, the role that students' outside experiences and conceptions, including but not limited to their experiences in elementary mathematics classes, bear on how they come to understand algebra and points to the importance of not only taking them into account in an algebra class but also in the types of experiences that students have during their elementary mathematics experiences. With the well established findings on students' difficulties with mathematical symbols, including variables, extending the meanings students have for variables to informal representations in classes prior to their introduction and use in an algebra class is likely to provide a more well rounded understanding of their meanings for letter-symbolic representations of variables and the misconceptions that often accompany their use.

Research Findings on Early Algebra (EA)

As previously noted, research on student learning of algebra has predominantly occurred at the middle school through college levels. With the increased emphasis on algebra noted in the opening section of this study, a corresponding increased interest in the role of algebra at the elementary grades has arisen. However, as Carraher and Schliemann (2007) noted, while *some* agreement exists for algebra within the elementary curriculum ―the research basis needed for integrating algebra into the early mathematics curriculum is still emerging, little, known, and far from consolidated" (p. 671). The study reported in this dissertation builds on what we know about research on EA as well as some of the gaps in this existing research. Therefore, in the following section I provide a synthesis of the existing research on EA and how EA may provide students with a foundation for future work in algebra.

Carraher and Schliemann (2007, p. 678) cite Bass' (1998) view of school algebra.

> School algebra – and *the root of all algebra* – is about the following:
>
> - The basic *number systems* – the *integers* and the *real numbers* and those derived from them, such as the rational and complex numbers.
> - The *arithmetic operations* (+, -, ×, /) on these number systems.
> - The *linear ordering* and resulting *geometric structure* defined on the real line. By these I [Bass] mean the notions of size (whether one number is larger or smaller than another) and of distance between numbers.
> - The study of *algebra equations* that arise naturally in these systems. (p. 8, emphasis in original).

Based on this view, Carraher and Schliemann examined how conceptions of algebra impact early algebra. Specifically, they examined arithmetic as understood as the science of (1) numbers (2) numbers, quantities and magnitudes (3) numbers, quantities, magnitudes and functions (p. 678).

Arithmetic, numerical reasoning, and EA. In examining arithmetic and numerical reasoning as an entry point to EA, Carraher and Schliemann (2007) noted that the field axioms, (i.e., commutativity, associativity, distributivity, identity, and inverse axioms for addition and multiplication), ―highlight the connections between arithmetic and algebra for those *who are already comfortable with algebraic ideas and notations*" (p. 679, emphasis added).

Researchers have examined student generalizations of classes of number sentences, including but not limited to, the field axioms (e.g., Carpenter, et al., 2003; Fujii & Stephens, 2008). Carpenter, et al. found that students could generalize that adding zero to a number results in the number they started with. Likewise, if you subtract a number from itself you get zero. Fujii and Stephens (2008) found that students can engage in *quasi-variable* thinking, ―*general* explanations of why number sentences like 78 – 49 + 49 = 78 are true and their ability to generate specific instances of what they will later see as a general relationship (78 – a + a = 78)" (p. 128). They note that students often impose ―boundary values" that are not valid when applied to formal algebraic equations where the variable can be unbounded. For instance, in the previous example students often limit the values of a to be between 0 and 78 since a is subtracted from 78. Therefore, part of the students' assigning of boundary values may be related to their understanding of numbers and operations (e.g., in this case students believe that it is not permissible to subtract a greater value from a lesser value reflecting their understanding of the operation of subtraction and/or negative values).

Carraher and Schliemann (2007) summarized the relationship of numerical reasoning and EA as follows.

1. Arithmetic has an inherently algebraic character and can be usefully regarded as a part of algebra rather than as a domain distinct from algebra;
2. Young students sometimes make algebraic generalizations without using algebraic notation (although natural language is often poorly suited for expressing algebraic relations);
3. Studies of arithmetic as an entry point into algebra are promising, but most of what needs to be known has yet to be investigated (p. 681).

While this indicates that arithmetic may have an algebraic character, the question becomes whether students engaging in arithmetic are by default engaging in algebra. Further, if Carraher and Schliemann‘s suggestion is correct, what do we do with the students who are *not* already comfortable with algebraic ideas and notations?

Arithmetic, quantitative reasoning, and EA. While research has discovered much about students‘ understanding of measured and unmeasured physical quantities, Carraher and Schliemann (2007) noted two reasons why this research has had limited influence in the development of and research on EA. First, –given the sparse theoretical groundwork about quantitative thinking, with noteworthy exceptions (e.g., Schwartz, 1996a; Thompson, 1988; Smith & Thompson, in press), it continues to be challenging to relate investigations about quantities, *particularly unmeasured and indeterminate quantities*, to studies associated with arithmetic, broadly conceived" (p. 683, emphasis added). Second, mathematics educators tend to avoid studies from developmental psychology –that seek out universals and downplay the roles of teaching and particular representational forms on students‘ thinking" (p. 683).

For example, the number line is a staple of elementary mathematics classrooms as children learn about counting, natural numbers, and integers. However, students often struggle with whether to consider a number as a point on the number line or a displacement (distance). In addition, Carraher and Schliemann (2007) correctly stated that ―as students‘ concept of number develops, it becomes increasingly difficult for them to reconcile operations on numbers with displacements, intervals, and points on the number line; something that has implications for using number lines to discuss scalar quantities" (p. 683). For instance, students can model 8 – 3 on the number line in several ways. First, they can express the answer, 5, as a point on the number line. Second, they can represent the operation as a displacement or distance from zero after moving eight units to the right and then three units to the left. Finally, they can represent the operation as the interval from 3 to 8. Each representation carries a unique, yet related, conception of the operation on the numbers. This demonstrates how the use of representational forms that may initially seem simple and straightforward can result in difficulties and misconceptions for students.

Quantities are often used to assist students in making connections between mathematics and real-world situations as well as to move from solving tasks presented in an algebraic form to word problems, often found at the end of problem sets. When using quantities, the referent becomes important as it introduces issues with exploring operations. For instance, multiplying two quantities, *a* and *b* means different things if both are lengths of line segments, often modeled as an array or area, versus one referring to a line segment and the other as a number. Similarly, Carraher and Schliemann (2007) noted that the relatively simple problem of purchasing 8 juices that cost $.75 each becomes somewhat problematic when solved using multiplication while attending to the referent. Students can interpret the multiplication as

> (a) multiply .75 by 8, obtaining 6.00 and then attaching the unit of measure, $; (b) multiply $0.75 by 8, obtaining $6.00 straightaway; or (c) multiply two quantities, the intensive quantity, [a quantity independent of the size of the system, in this case a unit value] $.75/can, by the extensive quantity [a quantity proportional to the system size, in this case the number of cans], 8 cans, obtaining $6.00 as the result (p. 684).

This points to potential difficulties, and differences, in the meanings students develop for variables depending on the context of the problem, the task type, and the representation of the variable and invariant quantities.

One line of research addressing these issues has been to not assign specific values to magnitudes but to leave them indeterminate and variable. Such an approach is consistent with the work of Davydov (1990). Carraher and Schliemann (2007) noted that it is easy to appreciate –the relevance of indeterminates for introducing variables: only a slight adjustment in thinking *would appear* to be needed to shift from treating a letter representing a single, indeterminate value to each and every value in the domain” (p. 685, emphasis added). While this may appear to be a simple shift, as viewed from the perspective of someone who already is aware of this understanding of domain, it is not as clear if it is the case for students.

Arithmetic, functions, and EA. Function has come to play a central role in middle school and high school mathematics. Carraher and Schliemann (2007) noted that several researchers have suggested that function be the main focus of study in algebra. One of the main ways that teachers and researchers have implemented functions in EA is through the study of numerical and geometric growing patterns (e.g., see Cooper & Warren, 2008; Papic & Mulligan, 2007; Radford, 2000; Swafford & Langrall, 2000;

Warren & Cooper, 2008a; Yeap & Kaur, 2008). Through the study of these patterns, students identified, extended, and generalized patterns, often using informal language and methods. The findings were that students might generate recursive and/or explicit rules.

Another extension of the focus on function in EA is the potential use of multiple representations of the functions. For example, Carraher and Schliemann (2007) incorporated natural language, line segments, function tables, Cartesian graphs, and algebra notation in representing operations as functions in elementary classrooms. They found that:

> - Reasoning about variable quantities and their interrelations would constitute a natural setting for the discussions about variable and functions.
> - When we [Carraher, Schliemann, and Brizuela] introduced function tables relating number of items and price, we found that although the children could correctly fill in the tables, they did not attend to the invariant relationship between the values in the first and second columns.
> - The introduction of letters to denote any value for the first variable in a function table proved to be a powerful tool for children to focus on the general rule of the two variables. (pp. 691-692)

The second finding listed above is consistent with Lobato and Ellis‘ (2002) findings from a study in a secondary mathematics class examining ―the nature of the generalizations students formed about slope and linear functions as a result of their interactions with a reform curriculum that regularly develops concepts in real-world settings" (p. 2). They found that all students who were interviewed developed the unintended generalization of $y = mx + b$ as a difference. They describe how four focusing phenomena (―goes up by" language, well ordered tables, graphing calculator, and uncoordinated sequences and differences) contributed to students‘ generalization of the slope-intercept form of a linear equation as a difference. Similar to the findings reported in the previous sections, the language, representations, and symbolism teachers and researchers use in instruction play an important, and sometimes unintended, role in the meaning that students make.

Summary. While the research noted in this section holds promise for identifying what EA may include and its implementation in the elementary grades, it is also clear that students may not always recognize the ―algebraic" nature of the mathematics in which they are engaging, including relationships between variant and invariant quantities and symbolism used to express relationships and generalizations. EA does not *require* introducing new ideas into the elementary mathematics program, although it may. It does require both the teacher and student to reconceptualize the nature of the activities they currently use. Therefore, it seems evident that student engagement in arithmetic activities is insufficient for their engagement in algebraic ideas. Instead, they need opportunities to explore relationships and generalizations within and by using arithmetic (e.g., the field axioms, and functional relationships). However, they must *also* have the opportunity to develop their understanding of the symbolism needed to express, describe, and justify these relationships and generalizations beyond listing specific cases.

The Problem: Student Meaning for Variable

While there are multiple ways for researchers to define algebra (c.f., Carraher & Schliemann, 2007; Kaput, 2008b; Kuchemann, 1981; Radford, 2006; Sfard & Linchevski, 1994), a common element involves the use of symbolism. Kaput (2008a) took a broad view of symbolization to include both conventional mathematical notation as well as informal written and verbal notation. Others used a narrower view to only include conventional mathematical notation (Fujii, 2003; Kuchemann, 1981). Included in the broad range of definitions of algebra, and more specifically symbolization, are the use of letters as variables.

A critical aspect of every facet of the previously discussed research involves student use and meaning for the formal and informal representations of variables. Students‘ understanding of and use of variables is a crucial aspect of algebra regardless of how one defines algebra. Carraher and Schliemann (2007) recognized this when they stated,

> Given the importance of variables in algebra, there are good reasons for mathematics educators to treat unknown values as indeterminates or variables. This holds even for equations. For instance, in $8 = 5 + x$, x is profitably conceived as a variable (and is thus free to vary) despite the fact that there is only one value, namely 3, for which $8 = 5 + x$ is true. Any other substitution for x is legitimate even though it results in a false statement. Such a framing can help students become familiar with variables from early on rather than having to radically overhaul the meaning of symbols that originally had constricted meanings. (p. 685)

We can make the same argument for other commonly used representations of unknowns, such as blanks, words, and shapes used as placeholders. Carraher and Schliemann further noted, ―one can imagine a line of research to investigate the conditions under which numerical assignment facilitates or interferes with learning, something from which a learning progression in number line understanding and EA might eventually develop" (p. 685). For this study, I view and refer to unknowns as variables, and therefore representations of unknowns as representations of variables.

Research studies have shown that the majority of students demonstrate difficulties with interpreting letters as specific unknowns or generalized numbers (Booth, 1984; Carraher, et al., 2000; Cooper & Warren, 2008; Fujii, 2003; Knuth, et al., 2005). As noted previously, researchers conducted the majority of these studies at the middle school through college levels. Research studies at the elementary school level involved situations where the researcher *provided* letters as variables to students, often resulting in initial student confusion or rejection of the notation (e.g., Booth, 1984; e.g., Carraher, et al., 2000; Stacey, 1989).

My review of the literature showed that little research exists on how students conceptualize, generalize, and understand variables, represented as both formal conventional, and informal symbolizations, before and during the transition to formal symbols. In addition, when researchers did introduce variables in the studies (e.g., Booth, 1984; Carraher, et al., 2000; Carraher, et al., 2001; Cooper & Warren, 2008; Koedinger & Nathan, 2004; MacGregor & Stacey, 1997), an implicit assumption often existed that students would quickly adopt and understand their use. In doing so, the researchers did not attend to the semantic and syntactic issues associated with the use of the variables (Kaput, 1999, 2008a).

Research has demonstrated the crucial role that variables play in students' success in algebra. Booth (1984) found that student difficulties in algebra appear to stem from three main issues, ―namely [a] the meaning children attached to letters, [b] the process of operating with letters and [c] question of notation and convention in algebra" (p. 12). These issues correspond to the three types of activities discussed in the review of the literature on algebra, generational, transformational, and global/meta-level respectively. Booth reported that students bring a broad range of meanings to letters including, but not limited to, their understanding and attention to the referent when presented in a context, the idea that they do not stand for numbers, and that the letters will stand for a unique number instead of a range of numbers. When operating with letters, student errors included not representing a solution with variables that they could solve with numbers, ambiguity between $4m$ and 4 m‘s, not recognizing that $4m$ and $m + m + m + m$ were equivalent, and use of sequence of letters (e.g., $x + y = z$ because of the sequence of the letters in the alphabet. Notation and conventional symbolization when using variables was another area in which students showed difficulty. These included not using brackets, or parentheses, when appropriate, and replacing open algebraic sums, such as $a + b$, with ab.

An example of the confusion with formal algebraic symbols that can occur emerged in a study by Carraher, et al., (2000) with third grade students who were learning in an “algebrafied” arithmetical setting. As students solved a problem related to people‘s heights Carraher introduced T as a variable. The researchers asked two children who had solved the general case what T stood for. They provided responses of “tall” and “ten”. When the researcher explained that T could stand for whatever Tom‘s height might be, the students were reluctant to accept this explanation. Carraher, et al. report that over the next several classes students began to use the words “whatever” and “any” to explain letters, variables that represented measures. It is not clear from the research how students interpreted these letters or whether they were parroting the terminology of the researcher.

As noted earlier, Fujii (2003) found that students who understood that the same letter (variable) stood for the same number in a given equation tended to believe that different letters could *not* stand for the same number. Fujii also reported that some students do not realize that the same letter in the same expression or equation must stand for the same numerical value at the same time. This supports Booth‘s (1984) finding that students have difficulty operating on letters and bring a range of meanings to the use of conventional formal symbols.

Knuth, et al., (2005) suggested that middle school students‘ understanding of literal symbols, particularly when the literal symbols can have multiple-values, may be fragile, particularly for students in grade 6. They link this to the types of tasks in which students engage in elementary school where literal symbols often have single-value solutions. While the above findings focused on conventional letter-symbolic representations of variables, I posit that similar findings may be found for other representations of variables. Currently little is known about how students understand and use other representations of variables.

Knuth, et al. (2005) further reported that,

> students often encounter literal symbols during their elementary school education (e.g., 8 + 3 = □, 3 + ? = 7). However, the nature of such exposure may lead students to consider literal symbols in less sophisticated and mathematically powerful ways (e.g., as specific numbers)" (p. 75).

While Knuth correctly points to issues with the use of literal symbols in elementary school, another issue arises. Teachers, researchers, and curricular materials do not always represent variables using letters, especially in early elementary school, as the examples above demonstrate. From a review of commonly used curricular materials (Pearson Education, 2011) I found that representations of variables included blanks, boxes, words, question marks, and other non-conventional symbolizations. While conventions exist for the use of letter-symbolic representations of variables, no such clearly established and accepted conventions exist for literal symbols. Instead, it appears teachers and researchers impose the conventions they learned for letter-symbolic representations on these informal representations of variables. Yet, the research is lacking on student conceptions and generalization of these informal notations, and how they build upon this when introduced to conventional symbolization.

In addition to the representations of the variables, the representations of the tasks in which the variables are used also play an important role. As noted previously, Koedinger and Nathan (2004) reported on the widely held belief that algebra story problems are more difficult for students to solve than word equations, or equations. They found that,

> Contrary to beliefs held by practitioners and researchers in mathematics education, students were more successful solving simple algebra story problems than solving mathematically equivalent equations. Contrary to some views of situated cognition, this result is not simply a consequence of situated world knowledge facilitating problem-solving performance, but rather a consequence of student difficulties with comprehending the formal symbolic representation of quantitative relations (p. 129).

It is interesting to note that within the task types used by Koedinger and Nathan, the representation of the variable changes from donuts in the algebra story problem, to number in the word equation, and *x* in the equation. However, the researchers did not take into account the distinction between these different representations in the research findings. This raises the question of whether the differences reported by Koedinger and Nathan were due to the representation of the task, as claimed, the representation of the variable, or both.

In summary, students experience multiple representations for unknowns in the elementary program and when these unknowns are considered as variables, as Carraher and Schliemann (2007) suggested, the connections between these representations, or lack thereof, becomes an important aspect of the mathematical experiences of elementary students. Yet, little is know about how students view these representations for variables and if or how they make connections across representations of the variable. In addition, the findings of Koedinger and Nathan (2004) point to the role that the representation of the task may play in students understanding of and success with tasks even when these tasks are mathematically equivalent.

In addition to addressing a gap in the existing research, this study also addresses issues and questions raised by teachers as important to their practice. In *Linking Research & Practice* (NCTM, 2010), 25 questions were constructed from a synthesis of 350 questions submitted from mathematics education practitioners at the NCTM Research Agenda Conference. A subset of these questions was ―chosen, along with accompanying text, to present to the community" (p. 6). The findings from this study will help to answer and inform further research on four of these questions:

- What are coherent frameworks for characterizing the development of student thinking about specific mathematical concepts or processes? (p. 15)
- What are the mathematical concepts and reasoning processes that prepare and enable students to learn and use algebra? (p. 16)
- What ―interventions" help teachers reach students who they perceive have difficulty developing mathematical proficiency? (p. 26)
- How can teachers engage students in productive struggle to support the development of mathematical proficiency? (p. 28)

Summary. Much of the reviewed literature depends on students understanding of mathematical symbols, including variables. However, as I discussed in the summary of each section, students‘ understanding of these symbols is a crucial component embedded within each of the areas of research. While much research has been conducted on students understanding, or misconceptions, of these symbols little has been done specifically on students‘ meaning of formal and informal representations of variables prior to their introduction and use at the algebra level and beyond. While we know much about misconceptions that students have for variables in algebra and beyond we do not know if these same misconceptions are present prior to their introduction to conventions of use for letters. Therefore, we also do not know if these misconceptions are present as students begin their study of algebra, or if these misconceptions arise as teachers introduced to these conventions. Furthermore, this gap in the research results in our not knowing what prior knowledge and experiences students build upon as they construct their meanings for variables and the conventions for their use. This study addresses this gap in the research.

Chapter 3: Methodology

As noted above, early algebra in general, and elementary grade student understanding and utilization of variable representations specifically, are in need of further research. Therefore, researching how elementary students interpret representations of variables and solve tasks involving these representations across representations of variables and representations of tasks is the focus of this study. These results provide evidence needed to answer the research questions under investigation.

Theoretical Framework

I draw on two theoretical frameworks to frame this study. In the following sections I describe Kaput's (Kaput, 2008a, 2008b) conception of algebra and the process of symbolization. I then describe Sfard and Linchevski's (1994) conception of algebra and the process of symbolization. I describe how each of these perspectives provides valuable insight into how students develop meaning for mathematical symbols. I then describe how a merging of the ideas behind each of these conceptions provides a more comprehensive framework to describe and understand students meaning making for mathematical symbols than either conception alone. It is from this more comprehensive framework that I view students' meaning making for representations of variables.

Algebra from a symbolization perspective. Kaput (2008b) understands algebra as consisting of two core aspects that are embedded in three strands (see Figure 8). At the core of this perspective are the relationships between generalization and the process of symbolization.

The Two Core Aspects
(A) Algebra as systematically symbolizing generalizations of regularities and constraints. (B) Algebra as syntactically guided reasoning and actions on generalizations expressed in conventional symbol systems.
Core Aspects A & B Are Embedded in Three Strands
1. Algebra as the study of structures and systems abstracted from computations and relations, including those arising in arithmetic (algebra as generalized arithmetic) and in quantitative reasoning. 2. Algebra as the study of functions, relations, and joint variation. 3. Algebra as the application of a cluster of modeling languages both inside and outside of mathematics.

Note. From Kaput, J. J. (2008b, p. 11).
Figure 8. Core Aspects and Strands of Algebra.

Kaput (2008b) theorized a process to describe students meaning making for algebraic symbols and noted that the

> only way a person can make a single statement that applies to multiple instances (i.e., a generalization), without making a repetitive statement about each instance, is to refer to multiple instances through some sort of unifying expression that refers to all of them in some unitary way, in a single statement (p. 20).

In this sense, generalizing and symbolizing are inextricably related and necessary for student development of the two core aspects of algebra noted in Figure 8.

Several researchers have argued that developing students' ability to generalize in the elementary grades is an important aspect of early algebra (e.g., Carraher, Martinez, & Schliemann, 2008; Davydov, 1990; Dorfler, 1991; Kaput, 1999, 2008b; Warren & Cooper, 2008a; Yeap & Kaur, 2008). Taking Kaput's (2008b) perspective on algebra, symbolization and the processes that students engage in as they generate both informal and formal conventional algebraic symbolizations are also important aspects of algebra at all levels.

The types of symbolic objects and the meanings students develop for expressing generalizations are broader than conventional notation systems and include private, or individual, student symbolism that may be, and is often, informal oral or written descriptions that do not fully capture generalizations, variables, mathematical symbols (e.g., +, =, $\surd$, $\sum$, π, and $\int$), graphs, or tables, to name a few. This symbolism represents a referent from which the student generalizes. Students may then attend to either the symbol or the referent. As Kaput (2008a) stated,

> This perceptual difference is analogous to a deep epistemological distinction between mathematics as an object of study in its own right versus mathematics as an intellectual tool, as a means of seeing, organizing, and reasoning about experience, including the highly structured experience that takes the form of science and the ever-widening areas of human endeavor where mathematics is applied (p. 25).

Kaput (2008a) notes that an individual may *look at* or *look through* symbols while engaging in the process of symbolization. At any particular time, the symbol system can be the students‘ focus or the student can attend to and take action on the referent by looking through the symbols. On the other hand, the student may attend to and take action on the symbols themselves. By action, Kaput refers to both mental and physical actions.

As they develop meanings for mathematical symbols, students are confronted with and engage in mediated experiences in the world of mathematics. As they engage in the world of mathematics, they begin to build ―oral, written, and drawn descriptions of the situation – records of those aspects of the situation that are accessible to them at the time.” (p. 28). Then, ―the physical symbolization [this world of mathematics] is built from and tested against observations about [the mediated experiences] in the social context of discussion and interaction” (p. 29). This new conceptualization is a result of the symbolization process emphasizing, “the constructive, additive nature of the process of symbolization, as opposed to the view often taken that it is abstractive and subtractive" (p. 31).

Kaput‘s (2008a) process of symbolization provides a perspective through which students‘ views of symbols, in general, and variables in particular, can be studied. Based on this perspective, it seems reasonable to hypothesize that student conceptions of informal representations of variables, through mediated experiences, will be foundational to their generalizations of conventional letter-symbolic variable symbolization. If so, this points to the importance of understanding the generalizations of the informal representations.

Reification. Sfard and Linchevski (1994) take a slightly different perspective on algebra, and mathematics, in terms of the progressions that must occur. This is evident in their statement that “mathematics is a hierarchical structure in which some strata *cannot* be built before another has been completed" (emphasis added, p. 195). This hierarchical structure is based, in part, on the claim that

> the same representation, the same mathematical concepts, may sometimes be interpreted as processes and at other times as objects: or, to use the language introduced elsewhere (Sfard, 1991), they may be conceived both operationally and structurally" (p. 193).

This is similar to Kaput‘s (2008a) referents and symbols.

Viewing algebra operationally includes students‘ verbal descriptions, which are often initially informal and lack the specificity necessary to adequately describe the generalization under consideration. Their theoretical perspective of the development of algebra begins with operational generalized arithmetic and moves to structural generalized arithmetic. Following this development of generalized arithmetic comes operational abstract algebra and structural abstract algebra.

Through this perspective, Sfard and Linchevski (1994) viewed algebra through the lens of the theory of reification, ―our mind‘s eye‘s ability to envision the result of processes as permanent entities in their own right" (p. 194). They noted that while what one sees in algebraic symbols depends on the specific problem and context, it also depends on ―what one is *prepared* to notice and *able* to perceive" (p. 192). Based on this theoretical perspective they claimed ―algebra is a hierarchical structure in which what is conceived operationally at one level must be perceived structurally at a higher level" (p. 202). Reification, therefore, progresses from the operational (process-oriented) to the structural (mathematical objects)

The reification process, from the operational (process-oriented) to the structural (mathematical objects), requires the introduction of symbolic notation. However, the introduction of symbolic notation is insufficient for reification to occur. Elaborating on the distinction between the operational and the structural, Sfard and Linchevski (1994) noted:

> The operational way of thinking dictates the actual actions to be taken to solve the problem at hand, while the structural approach condenses the information and broadens the view. The abstract objects serve as landmarks with the help of which the problem-solving process may be navigated. Since a jump from operational to structural mode of thinking means a transition from detailed and diffuse to general and concise – from the foot of a mountain to its top – it is only natural that it is accompanied by an increase in student‘s ability to cope with the task at hand (p. 203). [1]

[1] A more thorough explanation of these perspectives see (Kaput, 1999, 2008a, 2008b; Sfard & Linchevski, 1994)

Sfard and Linchevski's (1994) theory of reification extends Kaput's view of generalization and symbolization. While Kaput provides a perspective on how the process of building from informal to formal conventional symbolizations may occur, reification goes further to examine *how* this development occurs. As students move from treating these symbols, and their use, as processes and reify these objects, it becomes even more important to understand how the students conceive of these informal representations.

If students reify representations of variables of which they have generalized incomplete conceptions or misconceptions then the students may carry these errors on as they progress from informal to formal representations. For example, if they conceive of informal representations of variables as representing single values, as often used in the elementary grades, then the student may generalize this same conception to formal conventional representations of variables. This raises the question of the extent to which the common errors and misconceptions for letter-symbolic variable representations are a result of the conceptions students carry from their generalizations of the informal representations versus misconceptions that they generate through the introduction of the formal letter-symbolic letter representations.

Reified generalizations. Critical to the consideration of the development of meaning for algebraic symbols, Davydov (1990) distinguished the *process* of generalization from the *result* of this process noting both phenomena are often linked to generalization. A key aspect of this process involves students' transition from informal generalizations to formal mathematically expressed generalizations using conventional mathematical symbols. Understanding the process through which students progress as they generalize patterns and come to understand these symbols may provide educators and researchers with a generalization trajectory. Such a trajectory may shed light on how to assist students in making sense of variables and other mathematical symbols, an area that has been well documented as being difficult for students (Arcavi, 1994; Clement, Lochhead, & Monk, 1981; Hiebert, 1988).

Thorpe (1999) stated that students —should see algebra as an aid for thinking rather than a bag of tricks" (p. 31). Referring to Whitney (1985), he noted that,

> students should grow in their natural powers of seeing the mathematical elements in a situation, reasoning with these elements to come to relevant conclusions, and carrying out the process with confidence and responsibility" (p. 31).

To reach this goal requires a rethinking of the process to include both the process of symbolization and reification discussed above as well as the sequencing of tasks.

As noted above, similarities between the perspectives of students‘ algebraic reasoning, put forth by Sfard and Linchevski (1994) and Kaput (1999, 2008a, 2008b) exist in the way that students build their conceptions from their prior knowledge and experiences. Differences exist in what the students generalize based largely on how each conceptualizes algebra. Taking Kaput‘s (2008b) position that —the heart of algebraic reasoning is comprised of complex symbolization processes that serve purposeful generalization and reasoning with generalization" (p. 9), reification provides a lens through which to view how students view new conceptualizations.

Kaput‘s model of the symbolization process provides one way to examine how students iteratively revisit their conceptions based on the interaction between mediated experiences and representations. This model provides a lens for examining generalizations, and the symbolizations that students develop to express these generalizations. However, It does not provide a way of examining *how* the students view these symbolizations (e.g., as an object or a process).

As noted earlier, Lobato, et al. (2002) examined ―the nature of the generalizations students formed about slope and linear functions as a result of their interactions with a reform curriculum that regularly develops concepts in real-world settings” (p. 2). They found that the students they interviewed developed the unintended generalization of $y = mx + b$ as a difference (i.e., the difference in y-values is the same for consecutive values of x). They describe how four focusing phenomena (―goes up by” language, well ordered tables, graphing calculator, and uncoordinated sequences and differences) contributed to students‘ generalization of the slope-intercept form of a linear equation as a difference. These students developed meaning for this equation in an unintended manner. Kaput‘s (2008a) model provides a way to study how students came to this generalization. Sfard and Linchevski‘s (1994) reification model provides a way to examine how the students view the symbolization of the generalization and how this symbolization itself becomes an object upon which students generalize, as opposed to strictly a process.

A combination of the representations in which students engaged (e.g., well structured tables with consecutive values for x) along with the mediated activity of the focusing phenomena appears to have contributed to the unintended student generalizations. Sfard and Linchevski‘s (1994) reification model provides a way to examine how the students view the symbolization of the generalization and how this symbolization itself becomes an object upon which students generalize, as opposed to strictly a process.

Sfard and Linchevski (1994) cited Freudenthal (1978) who stated that ―If a learning process is to be observed, the moments that count are its *discontinuities*, the jumps in the learning process (p. 78)" (p. 195). Kaput described how students construct generalizations through an iterative process of communicating and analyzing the relationships between mediated representations and experiences as they move toward more or less conventional symbols. Using Sfard and Linchevski‘s idea of reification, we can examine the discontinuities that arise to investigate how students view these objects and processes as they develop meaning for various algebraic symbols. Identifying these objects and processes during moments of discontinuity provides researchers with an opportunity to identify a *chain of significance* (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997) for the development of student‘s algebraic reasoning.

Task sequencing. A crucial aspect of this perspective involves sequencing experiences and representations provided to students. The traditional sequencing of topics and tasks in U. S. mathematics classrooms has not changed significantly in the last hundred years. Often, teachers and researchers take this sequence, beginning with the operational and then moving to the structural, for granted. This may produce unintended and erroneous student conceptualizations. Beginning with operational stages (i.e., numeric computations in generalized arithmetic, and processes on symbols in abstract algebra) can result in erroneous generalizations based on incomplete understanding and generalizations of the operations.

For instance, researchers have documented students‘ erroneous conceptions of the equals sign. As Sfard and Linchevski noted, ―when algebraic expressions are seen as processes rather than objects, the equality sign is interpreted as a ‗do something signal‘

(Behr, et al., 1976; Kieran, 1981)” (p. 208). They note how the use of the equal sign in calculators may reinforce this idea. However, for the majority of students in the U. S. this equal sign is often represented as a ―do something signal” and therefore, not surprisingly, they develop this conceptualization.

Davydov (1990) argued for the ―ascent from the abstract to the concrete” (pp. 129-139) and has developed a curriculum that employs this perspective. Students begin with the abstract, generalities and symbolizations, and move to the concrete. Therefore, researchers must reconsider the assumption, as the National Mathematics Advisory Panel (NMAP) suggests, that the teaching of mathematics *requires* the sequencing of ―major topics (from whole numbers to fractions, from positive numbers to negative numbers, and from the arithmetic of rational numbers to algebra) and an increasingly complex progression from specific number computations to symbolic computations. The structural reasons for this sequence and its increasing complexity dictate what must be taught and learned before students take course work in Algebra” (p. 17).

In examining student thinking and conceptions of variables, it seems relatively obvious that their progression toward the correct use of conventional mathematical symbols, including variables, may follow the symbolization process described by Kaput (2008a). However, as I argue above, this view of symbolization does not take into account the process of *how* students move from informal symbolizations (in the case of variables this includes but is not limited to words, blanks, and shapes) to the conventional letter-symbolic representations used in algebra courses. This is where Sfard and Linchevski‘s (1994) idea of reification plays an important role. As students move from these informal symbolizations to the conventional letter-symbolic representations we need to understand how, or if, they reify these informal representations. In Kaput‘s process of symbolization, this occurs during the analysis and communication between the representations and the mediated experience in the world, which yields a new conceptualization. Currently the research literature does not provide insight into how students view these informal representations or how the representation of the task in which they are presented influence the students‘ understanding or use of these variable representations.

I posit that students reify and/or transfer their generalizations, understandings, and conceptions of the common representations of variables in which they engage in elementary grades (e.g., blanks, words, and shapes as placeholders) to conventional letter-symbolic representations of variables through their experiences in and outside of the classroom. It is interesting to note that the majority of the research cited on student difficulties with, and misconceptions of, variables are descriptive in nature but provide little insight into the roots of these difficulties and misconceptions. If Kaput‘s process of symbolization provides a meaningful framework for describing students‘ meaning making for mathematical symbols then students likely generalize and/or reify their conceptions and use of the informal literal symbols for variables that they commonly engage with in elementary grades to the formal conventional letter-symbolic representation of variable that is used in algebra courses and beyond. Yet, we know very little about students‘ understanding and use of these informal representations. Therefore, in this study I propose to examine both how students use a variety of informal representations of variables along with the conventional letter-symbolic representations of variable across different representations of tasks.

Participants

The students included in this study consisted of fourth and fifth graders from a Midwestern K-5 elementary school and sixth graders from a Midwestern middle school composed of sixth and seventh graders. Table 1 displays the demographic data for each school population in 2010.

Table 1

School Demographic Data

	Elementary	Middle School
Total Enrollment	759	791
Asian	9.9%	4.0%
African American	7.1%	23.5%
Hispanic	2.9%	5.6%
Indian	0.7%	0.6%
White	79.4%	66.2%
Free/Reduced Lunch	11.4%	39.3%

I sent consent forms to each of the 150 fourth graders, 131 fifth graders, and 161 sixth graders enrolled in each grade. Each consent form included four separate consent statements; participation in the study, videotaping of interviews, use of videos at professional conferences and for professional development, and release of demographic data. The parent or guardian could also choose to exclude their student from inclusion in the study. Of the consent forms sent, 61 (41.7%), 61 (46.6%), and 43 (37.1%) respectively were returned. Of those returned, 30 (20.0%), 32 (24.4%), and 15 (12.9%) parents or guardians gave full consent (i.e., consented to each of the four consent statements) for participation in the study, see Table 2. Some students returned forms with partial consent (i.e., consent was given for some, but not all, of the four consent statements). Four consent forms were returned with all four consent statements and the statement indicating they wished to be excluded from the study marked, see **All Marked** for fifth grade in Table 2.

Table 2

Consent Data

Grade (N)	Full Consent (N, %)	Partial Consent (N, %)	Opt Out (N, %)	All marked (N, %)	Total (N, %)
4 (150)	30 20.0%	7 4.7%	24 16.0%	0 0.0%	61 40.7%
5 (131)	32 24.4%	4 3.1%	21 16.0%	4 3.1%	61 46.6%
6 (161)	15 12.9%	9 7.8%	19 16.4%	0 0.0%	43 37.1%

The students' classroom teachers identified each of the students for whom full consent or partial consent was given as low-, typical- or high-achieving in mathematics. Each student was assigned a number indicating the grade in which they were enrolled and a roster number (e.g., 42 was the second student listed for fourth grade). Each of the students who the teachers identified as low-, typical- or high-achieving in mathematics were listed in a spreadsheet by grade and the level their teacher indicated. I randomized the order of each of these lists and selected the first three students in each list for participation in the study.

None of the sixth grade students for whom consent was received was identified as low-achieving. Therefore, only six sixth grade students, three typical- and three high-achieving, were included in the study (N=24). One fourth grade teacher did not return the categorization information resulting in the total number of students with full consent from which the final nine included in the study to be reduced by three students, see Table 3.

Table 3

Teacher Achievement Feedback Data

Grade	Low-achieving	Typical-achieving	High-achieving	Total
4	4	14	9	27
5	7	16	9	32
6	0	20	7	17

The students in the study consisted of 12 male and 12 female students. The grade 4-5 students were all from the same Midwestern elementary school, and the grade 6 students were all from the same Midwestern middle school. All but one student was Caucasian, the other was Asian. This latter student spoke both English and Chinese. One student also qualified for free/reduced lunch.

A review of the elementary mathematics textbook series, *EnVisions (Pearson Education, 2011)* the students in this study have used over the last two years demonstrates the variety of variable representations under consideration in this study. This textbook series includes blanks, letters, shapes and words, among other symbols, as representations of variables. Nearly all of the tasks containing a variable consisted of a single variable that had a single value satisfying the task. This included tasks where students were to find the missing value as well as tasks where students were expected to substitute a given value for a variable into an expression and then evaluate the expression.

For example, shapes are a common staple throughout much of the series, often using a light blue shaded square to represent a variable. In addition, the text uses squares, like those included in this study, along with letters. For instance, the following task is included on a unit assessment for 5th grade.

$$y \times 9 = 72$$

$$y \times 9 \div \square = 72 \div \square$$

$$y = \square$$

This use of ☐ is problematic on two fronts. First, the square in the second equation could take on any non-zero value and make the equation true. However, the intention is that this square is 9 in the second equation and 8 in the final equation. The intention of the authors therefore appears to use the ☐ as a placeholder for the student to fill in as opposed to a symbol to represent a number. However, as outlined in this study, it is not clear if students will make this subtle distinction.

The textbook also used blanks in the earlier grades. In third grade, students saw problems such as 18 – 9 = ____ , and 3 + ____ = 7. The textbook appears to discontinue the use of blanks after third grade as no further use of them was found.

In the teachers' guide for third grade, a tip is provided to teachers for helping students solve the problem of the day (Pearson Education, 2011, p. 48A). In this problem students are to write a number sentence and then solve the problem, ―Hope had 14 dolls in her collection. She received 2 more as gifts. How many dolls did Hope have then?" The tip states, ―Have students use a question mark, a box, or a variable (such as *n* or *x*) to represent the unknown quantity in the number sentence" (Pearson Education, 2011). This tip implies that only letters, conventional representations of variables, are in fact variables. Question marks or boxes are not considered variables if used in the same task. This use of variable also parallels the definition of variable cited earlier (Philipp, 1992)

Instrumentation

I conducted two semi-structured task-based interviews per student of approximately 30 minutes each (i.e., forty-eight total interviews). I developed the particular tasks used, follow-up questions, and order of the tasks (see Appendix A), in two phases. In the initial phase, I generated sets of tasks designed to gather data to answer the research questions for this study. In the second phase, I piloted the tasks and the interview protocol and made revisions based on the results.

The first interview for this phase consisted of tasks similar to those in the final form. The second interview consisted of presenting students with fictional student responses to each of the tasks from the first interview. The fictional student responses represented the common errors students make with letter-symbolic variables of assigning different values to multiple instances of the same variable within an equation, and believing that different variables within the same equation cannot be the same value. I developed the tasks in the final version of the second interview based on the results of the pilot, reported below, and initial analysis of the first interviews.

Task development. In generating the tasks for this study, I focused on three aspects of students' meanings for variables. This arose initially from Fujii and Stephens' (2008) findings that ―students who are aware of the proposition that the same letter stands for the same number … tend to think that the converse of this proposition is also correct" (p. 1-51). In other words, if the variables are different then they *cannot* take on the same value. For example, for equations such as $x + y = 12$ students stated that the variables cannot both be 6 since they are different letters. There is no research to determine if students in elementary grades also exhibit this same meaning for variables or if this arises sometime later. If this same meaning exists for students in elementary grades then this finding would have implications for instruction at the elementary grades. If the students do not exhibit this meaning then it arises at some point later, possibly during the introduction of conventional letters as variables. Either way, the findings will be important for future research, curriculum developers, and future research.

In order to gather evidence to determine if students in the study have this meaning for variables, I identified three important meanings for pairs of variables for which I needed to gather data. These consisted of determining if the students treated two variables as the same or different within a task (i.e., have the same referent), if the students assigned multiple, or single values to the variables, and if they assigned the same and/or different values to both variables.

To determine if the students treated the pairs of variables as the same or different I decided that it would be important to gather data for pairs of variables that were the same (e.g., $y + y = 12$) and different (e.g., $a + b = 12$) within each task. Such tasks would provide evidence of whether the students have developed an intuitive sense of the conventions used for letters as variables or if they treat different variables as the same.

I also wanted to develop a series of tasks where the normative algebraic solutions would produce a single solution (e.g., $y + y = 12$), multiple solutions (e.g., $a + b = 12$), and no solution (e.g., $x + 6 = x$). While this study does *not necessarily* expect students to have constructed a normative algebraic solution strategy for the tasks, these tasks provide the opportunity to compare students' solutions across different core mathematical tasks, as viewed from a normative algebraic perspective.

Finally, in order to determine if the students in this study exhibited Fujii and Stephens' (2008) findings that "students who are aware of the proposition that the same letter stands for the same number … tend to think that the converse of this proposition is also correct" (p. 1-51), I needed to gather evidence to determine if the students assigned the same and/or different values for each variable. Students who do hold this misconception would not assign the solution of 6 for each y in $y + y = 12$. The tasks used needed to provide this opportunity along with the opportunity to demonstrate the converse, that $a + b = 12$ could have a solution of $a = 6$ and $b = 6$, as well as other solutions.

In order to develop tasks that would provide this evidence, the tasks had to consist of pairs of variables. Further, the tasks needed to consist of at least three core mathematical tasks. I decided to use relatively simple addition tasks in an effort to put the focus of the students' solutions and solution strategies on the meaning of the variables and minimize the computation. Therefore, I decided to use two core addition tasks with sums of 12; $y + y = 12$ and $a + b = 12$. Since these two tasks, when viewed from an algebraically normative perspective, had a single and multiple solution respectively, I also decided to use a core task with no solution when viewed from an algebraically normative perspective, $x + 6 = x$.

Interview protocol development. During May and June of 2010, I piloted the initial tasks with four students, two fourth graders, one fifth grader, and one sixth grader. This resulted in several subsequent modifications to the tasks. In the remainder of this section, I describe these changes and the episodes from the pilots that prompted these modifications. I only provide general descriptions of the tasks in the final version of the protocol in this section and provide a more detailed description of the tasks in the following section.

During the pilot, I found that the information I intended the tasks from the second interview to gather was available from student responses to the tasks in the first interview. In addition, I found that the tasks in the two interviews did not provide evidence of students‘ recognition, or lack thereof, between tasks with a common mathematical structure. Based on the results of the interviews, I modified the tasks for the first interview slightly to improve clarity of tasks. I included a word problem whose solution set included non-integer solutions (i.e., originally the word problem for the addition tasks with different addends unknown referred to numbers of pencils each person had and was changed to a continuous problem for sharing a length of ribbon).

In order to gather information on how students attended to similarities and differences across tasks with common mathematical structures, I eliminated the fictional student response tasks in the second interview. I replaced these with sorting and comparing tasks where I asked students to sort or compare sets of tasks with common mathematical structures as well as for tasks of the same task type. Due to the short time needed for the first interviews, I included the sorting tasks in the first interview.

Based on the results of the pilot, I redesigned the second interview to include confirmatory, modeling (i.e., generational activities), and manipulation (i.e., transformational activities) tasks. The confirmatory tasks consist of tasks similar to the tasks from the first interview to gather evidence of the consistency of student responses since the first interview.

The final sets of tasks for the second interview focused on comparing expressions (e.g., $a + a$ and $a + 5$), asking which of two expressions would be greater, when, and for what values. While students would still need to apply a meaning to the variables in the expressions, they would no longer have a given sum from which to generate pairs of addends. The students would also need to decide, if they selected one of the expressions, if it would *always* be greater. For the second type of manipulation task, I showed the students two equations modeling an algebraic tautology (e.g., $\square + \triangle - \triangle = \square$) and asked them if it would be true always, sometimes, or never.

Following the pilot phase of the study, in June and July 2010, I revised the initial tasks. I designed these tasks to provide students with an opportunity to solve sets of tasks with mathematically equivalent structures. Within each set, problems have different task types (i.e., word problems and equations) and different representations of the variable(s) (i.e., words, blanks, shapes, and letters). The addition tasks with common unknown addends contain unknowns, as opposed to variables. However, as I argued previously, we can consider these representations of variables as any value of which only one of these values will satisfy the task. Further, students regularly did not attend to or distinguish between the same or different representations of the variables during the pilot. Therefore, one of the purposes of this particular set of tasks is to gather evidence from which I may infer their distinction between the use and representation of unknown and variable, even if not familiar with this terminology.

I conducted the interviews from October through December 2010. The first interview consisted of two types of tasks. For the first type, students completed individual tasks (see Figures 9 - 11) presented in the following order, tasks 1, 2, 8, 11, (3, 7), 6, 10, 9, 4, 12, 5. Task 3 and 7 are identical since textbooks do not traditionally use different types of blanks to represent different values, as done with letters and shapes. I ordered the tasks by staggering the order of the word problems for each core mathematical task. I used the word problem for the addition task with common addends before any of the corresponding equations. I used the word problem for the addition task with different addends unknown after each of the corresponding equations. Finally, I used the equations for the addition task with no solution both before and after the corresponding word problem.

The students solved each task after I showed them a printed version of the task and read it to them. Once a student finished the task, I asked them follow-up questions to gather additional evidence of their thinking (see the interview protocol in Appendix A for specific questions for each task). These questions included but not limited to:

- Tell me what you thought about as you tried to solve this problem.
- What does the [representation of the variable] in this problem mean? (The purpose of this question is to examine (a) the referent of variable (e.g., number of gummy bears for the word problems), and (b) the domain that the student assigns to the variable. This will most likely require follow-up probing questions based on the student response.
- Could the [representation of the variable] be any number? Why or why not?
- Give me an example of what numbers the [representation of the variable] can be.

1. Shakira and Tim have the same number of gummy bears. Together they have 12 gummy bears. How many gummy bears could Shakira have? How many gummy bears could Tim have?
2. Show: $\square + \square = 12$ Ask: What numbers can the $\square$ be?
3. Show: ___ + ___ = 12 Ask: What numbers can go in to blanks?
4. Show: y + y = 12 Ask: What numbers can *y* be?

Figure 9. Individual tasks: Addition - common unknown addends.

5. Together Tom and Anne have 12 feet of ribbon. How long could Tom‘s ribbon be? How long could Anne‘s ribbon be?
6. Show: $\square + \triangle = 12$ Ask: What numbers can the $\square$ and $\triangle$ be?
7. Show: ___ + ___ = 12 Ask: What numbers can the blanks be?
8. Show: $a + b = 12$ Ask: What numbers can a and b be?

Figure 10. Individual tasks: Addition - different unknown addends.

9. I start with some number then add 6 and get the same number I started with. What is the number I started with?
10. Show: $\triangle + 6 = \triangle$ Ask: What numbers can the $\triangle$ be?
11. Show: ___ + 6 = ___ Ask: What numbers can the blanks be?
12. Show: $x + 6 = x$ Ask: What numbers can x be?

Figure 11. Individual tasks: Addition - Common unknown addend and sum.

For the next part of the interview, I had students engage in sorting and comparing tasks. I showed the students each of the task types for the addition tasks with common addends unknown (problems 1-4) and the addition tasks with different addends unknown problems 5, 6, and 8 omitting problem 7 which is the same as problem 3). I then asked them to put together problems that mean the same thing. Once they completed sorting the tasks I asked them why they put the tasks together as they did and what was different about the their groups.

Next, I showed the student each of the tasks for the addition task with one addend and sum unknown with no solution (problems 9-11) and asked them tell me how they were the same and different. I then followed up by stating that in the first interview they had told me what numbers would work in each problem and asked them if there were any of these problems that have the same numbers that would work for them and why. I then showed them the word problems (problems 1 and 5) and equations with the variable represented with shapes (problems 2 and 6), and letters (problems 4 and 8) respectively for the addition tasks with common addends unknown and different addends unknown and asked them the same follow-up questions as for the previous set of tasks.

The second interview consisted of four types of tasks, confirmatory, modeling, algebraic property equations, and comparing expressions. I designed the confirmatory tasks after the initial tasks the students completed in the first interview (see Figure 12). I designed these tasks for the sole purpose of establishing if the students' responses form the initial set of tasks in the first interview had changed in the period between the two interviews. For these tasks, I also included odd sums to see if student answers were different then for even sums (e.g., would they still give a double such as three-and-one-half for $y + y = 7$).

1. Juan and Alexa each have a piece of string. Together they have 16 inches of string. How long can Alexa's string be? How long can Juan's string be?
2. Show: □ + □ = 8 Ask: What numbers can the blanks be?
3. Show: ___ + ___ = 5 Ask: What numbers can the blanks be?
4. Show: □ + △ = 12 Ask: What numbers can □ and △ be?
5. Show: y + y = 7 Ask: What numbers can *y* be?
8. Show and say: I am thinking of two numbers. When I add these two numbers together I get 7. What can my numbers be?
9. Show: ___ + ___ = 7 Ask: What can the blanks be?
12. Show: △ + □ = 7 Ask: What can the shapes be?

Figure 12. Confirmatory tasks.

To address the issues noted earlier of students not attending to differences between the same or different variable representations in the same equation I designed a modeling task (see Figure 13) where I first gave the student the same word problem from the first interview where Shakira and Tim have the same number of gummy bears. I then provide two fictional student equations (□ + □ = 12 and □ + △ = 12) and asked if either of these number sentences meant the same thing as the word problem. If the student responded that either or both equation did mean the same thing as the word problem I asked them to explain why, this included asking them what the representations of the variables in each equation meant.

6. **Show:** Shakira and Tim have the same number of gummy bears. Together they have 12 gummy bears. How many gummy bears could Shakira have? How many gummy bears could Tim have?

Say: For this problem, I saw two students write these two number sentences.

$\square + \square = 12$ $\square + \triangle = 12$

Does either of these number sentences mean the same thing as this word problem? Why or why not?

If the say neither works ask: How would you write a number sentence for this problem?

Figure 13. Modeling task.

If the student stated that neither of the two fictional student equations meant the same thing as the word problem I asked them, “how would you write a number sentence for this problem?” If they generated an equation I asked them to explain why it meant the same thing as the word problem, this included asking them what the representations of the variable(s) meant.

The next set of tasks consisted of two equations modeling an algebraic property. Students were asked if these equations were true: always, sometimes, or never. I developed these two tasks to gain further evidence for how the students thought about and used the representations of the variables. These two tasks were unique in that they did not have any numeric parts, only variables (see Figure 14). Therefore, the students would not be able to rely on basic fact knowledge to generate a list of solution pairs as they had before. Instead, they would have to make some type of generalization about the equation.

7. **Show:** $\square + \triangle - \triangle = \square$ **Ask:** Will $\square + \triangle - \triangle = \square$ be true always, sometimes, or never? Why?

15. **Show:** $x + y - y = x$ **Ask:** Will $x + y - y = x$ be true always, sometimes, or never? Why?

Figure 14. Algebraic property tasks.

The final set of tasks in the second interview consisted of pairs of expressions that the students were asked to compare (see Figure 15). These four tasks all had equivalent mathematical structures in that one expression was the sum of two values represented with the same variable and the other expression was the sum of the same representation used in the first expression and five. Students were asked which of the two expressions was more.

10. **Say and Show:** $\triangle + \triangle$ $5 + \triangle$. **Ask:** Which is more, $\triangle + \triangle$ or $5 + \triangle$?
11. **Say and Show:** ___ + ___ 5 + ___. **Ask:** Which is more, ___ + ___ or 5 + ___?
13. **Say and Show:** $a + a$ $5 + a$. **Ask:** Which is more, $a + a$ or $5 + a$?
14. **Say and Show:** I am thinking of a number. Which is more, my number added to itself or five plus my number?

Figure 15. Expression comparison tasks.

Analysis

I video recorded and analyzed each interview (see appendix B for a copy of the analysis form). The analysis involved aspects of grounded theory methodology. I used the existing research on student difficulties with letter-symbolic variables described in the literature review to establish initial codes/themes. However, as I anticipated, I modified, replaced, and developed further codes during the analysis. I began by examining the values that *individual students* assigned to the variables, how they interpreted the various variable representations, and their solution strategies to determine if they,

1. Assigned the same or different values to the same variable in a single equation/problem,
2. Assigned the same or different values to different variables in a single equation/problem,
3. Interpreted different representations of variables across tasks with mathematically equivalent structures the same or different, including solution sets for each variable (I will establish descriptive codes for each of the differences if they arise),
4. Treated two blanks within an equation as representing the same or different variable,
5. Solved equations/problems with equivalent mathematical structures but different task types and different representations of the variables the same or different (I will establish descriptive codes for each of the differences if they arise).

Second, I used the data analysis spiral, see Figure 16, (Creswell, 2007) as a framework for analyzing the data collected. However, the case(s) will now reflect the commonalities and differences *across students* at both the entire sample and grade levels.

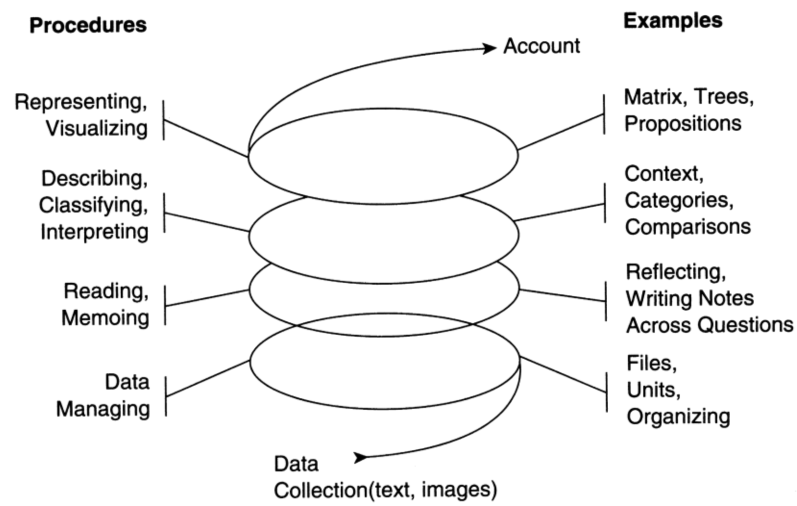


Figure 16. Data analysis spiral (Creswell, 2007, p. 151).

Once I completed the interviews, I used the data analysis spiral (Creswell, 2007) as a framework for analyzing the data collected. I worked with IRB to establish the data management procedures in order to ensure student confidentiality while still being able to identify students in case I needed to follow-up with any of the students. The initial codings used reflected the types of variable interpretations and solutions I expected to find based on the review of the literature. However, as I reviewed the data, as I anticipated, further codes and/or sub codes emerged during the reading and memoing portion of the analysis. I compared, refined, and recoded the data based on these new codes and sub codes, revisiting the data until no new codes emerged.

Codes. Based on the review of the extent research, I developed codes for three categories: variable comparison, solution type, and variable value. For the variable comparison category, I coded student responses as either ***Same*** or ***Diff.*** I coded student responses as ***Same*** if they demonstrated evidence that the student treated the two variables as having the same referent. Typically, responses coded as ***Same*** arose when a student stated that the variable quantities were the same and/or by only assigning the same value or referent to both variables. Student responses coded as ***Diff*** demonstrated evidence that the student treated the two variables as having different referents. Typically, responses coded as ***Diff*** arose when a student stated that the variable quantities were different and/or assigned different values to each.

For the solution type category, I coded student responses as ***SOL_Var***, ***SOL_SingVal*** or ***SOL_None.*** I coded student responses as ***SOL_Var*** if they provided multiple numerical values for each representation of the variable. I coded student responses as ***SOL_SingVal*** if they provided a single numerical value for the representation of the variable. Finally, I coded student responses as ***SOL_None*** if they indicated there was no solution, or no number that the representation of the variable could have that would work for the problem.

For the variable value category, I coded student responses as either ***VAL_Same*** or ***VAL_Diff.*** I coded student responses to indicate if the numerical values given by the student were the same for both variables, ***VAL_Same***, *and/or* if the numerical values given by the student were different, ***VAL_Diff*** for both variables. Unlike the former two categories where I only assigned one code per category, I assigned one or both codes to student responses.

Using the data analysis spiral, I developed codes for the sorting tasks and for the comparison tasks, see Figures 17 and 18. I developed these codes by sorting and comparing student responses and establishing initial codes. I then recoded the responses and edited codes until no further codes or clarifications emerged.

Code	Description
WP/EQ	Student sorted the tasks into two groups, one with the word problems and the other with the equations.
All Same	Student placed all of the tasks into a single group indicating that they were all the same.
VarRep	Student sorted the tasks into groups based on the representation of the variable (i.e., letters, shapes, words, and blanks).
SameDiff	Student sorted the tasks into groups where the two representations of the variable were the same (e.g., *y* and *y*) and the two representations of the variable were different (e.g., *a* and *b*).
Other	Student sorted the tasks into groups not characterized by one of the above codes.

Figure 17. Sorting codes.

Code	Description
Same	Two tasks compared mean the same thing (same #s work for both)
SS/DD	Same representation means same number, and/or different representation means different number (e.g., 6,6 not for sum of 12)
Rep#	Representation alone determines number (e.g., square big number, triangle small number)
SS/DD L	Same letter means same #/Different letter means diff # (does not hold for shapes)

Figure 18. Comparison codes.

During the describing, classifying, and interpreting phase of the analysis, I used the codings from the reading and memo part of the analysis to identify commonalities and differences across cases. In the course of this part of the analysis, I worked toward a rich description of how the meanings students displayed for representations of variables, (e.g., treating them the same and/or differently from other representations of the variable), and similarities and/or differences in their solutions. This required multiple passes through the data and codings.

Finally, based on the results of the prior analysis, I generated a rich in-depth description of students‘ meanings for representations of variables in the tasks and their solution methods. In the next chapter, I report the findings in detail.

Reliability. Finally, in order to check the reliability of the codes, two researchers viewed the videos and coded the student responses for three randomly chosen student interviews for each of interviews one and two (12.5% of all interviews). I compared these sample codes against my original codes, resulting in 91.9% agreement.

Chapter 4: Analysis and Findings

In this chapter, I summarize how grade 4-6 students interpret various representations of variables when presented in different representations and different task types. I present these results in six sections. I begin by providing an overview of the results of the coding of student responses. I then organize and discuss the results in relation to each of the four research subquestions. To answer each of the research questions, I sorted the data from the general results to highlight the appropriate attributes of the tasks (e.g., representation of the variable or task type). Finally, I synthesize these results to address the main research question under investigation.

General Results

The following section provides an overview of the initial coding results, using the codes described in chapter three, for the tasks from both interviews in three subsections corresponding to the three criteria I used to develop the tasks: variable comparison, solution type, and variable value categories. I begin by providing a description of these tasks.

Tasks. Initial analysis of the first set of interviews consisted of coding each student response for each of the first eleven tasks where students were asked what values could be applied for variables in the equations. These eleven tasks consisted of three core mathematical tasks written as an equation with the variable represented as a blank, letter, or shape, and a word problem with the variable represented as a word(s) (see Figure 19). I coded the student responses for these eleven tasks across three categories: variable comparison, solution type, and variable values.

Each task consisted of two variables of the same representation of the variable (i.e., words, shapes, blanks, or letters) that were either the same representations (e.g., *y* and *y*, or ___ and ___), or different representations (e.g., *a* and *b*, or □ and △) per task. Therefore, it was important to determine if each student interpreted the representation of the variable as the same or different values, the type of solution(s) provided (i.e., single value, variable values, or no solution) for the equation, and the potential values they assigned to the variables. As noted in chapter three, the equation with blanks as the representation of the variables was used for the equation with blanks for the common unknown addends *and* different unknown addends core mathematical task since blanks are not generally represented differently to distinguish between them.

In the first interview, the students also engaged in sorting the common unknown addends (e.g., $y + y = 12$), different unknown addends tasks (e.g., $a + b = 12$), and then the no solution tasks (e.g., $x + 6 = x$). The students then compared tasks for the common unknown addends and different unknown addends tasks with the same task type (i.e., equation or word problem) with different representations of the variables. These results are discussed later in the section on students meaning of variables across different representations of variables.

The confirmatory tasks for the second interview (see Figure 20) were coded using the same codes for the initial eleven tasks from the first interview. These tasks were designed to determine if students continued to interpret the tasks, and the meanings for the representations of variables, similarly for interviews one and two.

Core Mathematical Task	Equation (**E**) Blank (**b**)	Letter (**l**)	Shape (**s**)	Word Problem (**W**) Word (**w**)
Common unknown addends, Sum known (**C**)	___ + ___ = 12 (**CEb**)	y + y = 12 (**CEl**)	$\square + \square = 12$ (**CEs**)	Shakira and Tim have the same number of gummy bears. Together they have 12 gummy bears. How many gummy bears could Shakira have? How many gummy bears could Tim have? (**CWw**)
Different unknown addends, Sum known (**D**)	___ + ___ = 12 (**DEb**)	$a + b = 12$ (**DEl**)	$\square + \triangle = 12$ (**DEs**)	Together Tom and Anne have 12 feet of ribbon. How long could Tom‘s ribbon be? How long could Anne‘s ribbon be? (**DWw**)
Common unknown addend and sum, addend known, no solution (**N**)	___ + 6 = ___ (**Neb**)	$x + 6 = x$ (**NEl**)	$\triangle + 6 = \triangle$ (**NEs**)	I start with some number then add 6 and get the same number that I started with. What is the number? (**NWw**)

C – common unknown addends task; D – different unknown addends task; N – No solution task; E – equation; W – word problem; b – blanks; l - letters; s – shapes; w –words;

Figure 19. Interview one initial eleven tasks.

Two explicit differences exist between the initial eleven tasks and confirmatory tasks. First, all of the sums in the first interview were even. This could result in a false positive for the ***VAL_Same*** code (i.e., assigning the same value to both variables) if students assigned the same value to both addends in the common unknown addends and different unknown addends tasks due to their knowledge that only even sums have common addends for what they have had described to them as their basic math facts (i.e., whole numbers versus rational number solutions). For example, a student may state that a sum of 12 could be 6 + 6 but a sum of 7 could not have common addends because the basic math facts do not include such a solution.

If this were the reason for the ***VAL_Same*** responses, then the student would not necessarily be determining these values based on their meaning for the representation of the variable but instead on the particular task. This seemed like a plausible scenario to explore based on the quasi-variable (Fujii & Stephens, 2008) discussion from interview one included in the section on students' solutions later in this chapter.

Core Mathematical Task	Equation (**E**) Blank (**b**)	Letter (**l**)	Shape (**s**)	Word Problem (**W**) Word (**w**)
Common unknown addends, Sum known (**C**)	___ + ___ = 5 (**CEb**) ___ + ___ = 7 (**CEb**)	y + y = 7 (**CEl**)	8 = □ + □ (**CEs**)	
Different unknown addends, Sum known (**S**)	___ + ___ = 5 (**DEb**) ___ + ___ = 7 (**DEb**)		12 = □ + △ (**DEs**) △ + □ = 7 (**DEs**)	Juan and Alexa each have a piece of string. Together they have 16 inches of string. How can Juan's string be? How long can Alexa's string be? (**DWw**) I am thinking of two numbers. When I add these two numbers together, I get 7. What can my numbers be? (**DWw**)

C – common unknown addends task; D – different unknown addends task; N – No solution task; E – equation; W – word problem; b – blanks; l - letters; s – shapes; w –words;

Figure 20. Interview two confirmatory tasks.

The second change for tasks in the second interview included the use of equations of the form *sum* = *addend* + *addend*. This was to attempt to avoid the common use of the equal sign as an operation (Booth, 1984) and to put the equations in a form that is different from the form that students regularly see their “basic math facts” in an effort to distance these from their rote application of these facts. Figure 20 shows the eight confirmatory tasks categorized by core mathematical task, task type, and representation of variable.

Tasks 7 ($\square + \triangle - \triangle = \square$) and task 15 ($x + y - y = x$) from the second interview both model a generalization, or algebrafication, of an arithmetic property: adding a value to a starting value and then subtracting the same value, resulting in the original starting value. Task 7 uses shapes and task 15 uses letters as representations of variables. In each case, I asked students if the equation would be true always, sometimes true, or never true. I then asked the students follow up questions to determine why they answered as they did and to provide examples to support their responses. I then coded the student responses using the same three categories used for the other sets of tasks: variable comparison, solution type, and variable values. I discuss details of the results for these tasks in the section on students‘ meaning for variables across representations of variables and task type.

The final set of tasks for the second interview consisted of the core mathematical task of deciding which of two expressions (the sum of a number and itself, and the sum of the number and 5) are greater and for what values. This core mathematical task was written with shapes (task 10: $\triangle + \triangle$ and $5 + \triangle$), blanks (task 11: ___ + ___ and ___ + 5), letters (task 13: $a + a$ and $a + 5$), and words (task 14: I am thinking of a number.

Which is more, my number added to itself or five plus my number?). These tasks were coded for the three categories described for the eleven tasks from interview one and the confirmatory tasks from interview two: variable comparison, solution type, and variable value. I discuss details of the results for these tasks in the section on students' meaning for variables across representations of variables and task type.

Variable comparison. Each student's response was coded as ***Same*** or ***Diff*** for the variable comparison category, see Table 4. Student responses coded as ***Same*** demonstrated evidence that the student treated the two variables as having the same referent. Typically, responses coded as ***Same*** arose when a student stated that the variable quantities were the same and/or by only assigning the same value to both variables. Student responses coded as ***Diff*** demonstrated evidence that the student treated the two variables as having different referents. Typically, responses coded as ***Diff*** arose when a student stated that the variable quantities were different and/or assigned different values to each. This category provided evidence for how, or if, students differentiated between various representations of variables, the same or different, across various task types and tasks with common and different mathematical structures.

For students in algebra classes, and beyond, the convention that the same variable in the same equation must take on the same value is a crucial element of learning algebra. However, as noted in chapters one and two, we know little of how elementary grade students interpret variables, either the same or different, within the same problem. This category provides insights into the meanings students in grades four through six have for these situations.

Table 4 shows the total number of codes, and percentages, for each task for the variable comparison category for each grade and for the total sample.

Table 4

Interview One Variable Comparison Results

	4th grade		**5th grade**		**6th grade**		**Total**	
	Same	***Diff***	***Same***	***Diff***	***Same***	***Diff***	***Same***	***Diff***
1 CWw	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)	0 (0%)
2 CEs	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
3 DEl	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
4 NEb	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
5 CEb/DEb	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
6 DEs	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
7 NEs	1 (11.1%)	8 (88.9%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	1 (4.2%)	23 (95.8%)
8 NWw	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)	0 (0%)
9 CEl	2 (22.2%)	7 (77.8%)	0 (0%)	9 (100%)	1 (16.7%	5 (83.3%)	3 (12.5%)	21 (87.5%)
10 NEl	1 (11.1%)	8 (88.9%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	1 (4.2%)	23 (95.8%)
11 DWw	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)

C – common unknown addends task; D – different unknown addends task; N – No solution task; E – equation; W – word problem; b – blanks; l - letters; s – shapes; w –words;

Student responses were consistent for problems 1(CWw), 2(CEs), 3(DEl), 4(NEb), 5(CEb/DEb), 6(Des), 8(NWw), and 11(NEl) treating each pair of variables as different from each other. The only task where students consistently treated the two variables as the same was for the no solution word problem (i.e., I start with some number then add 6 and get the same number that I started with).

Table 5 shows the total number of codes, and percentages, for each confirmatory task for the variable comparison category for each grade and the entire sample population.

For the variable comparison category students consistently treated the two variables as different, ***Diff***, regardless of the representation of the variable or task type. In addition, only tasks 1(SWw), 4(DEs), 8(DWw), and 12(DEs) would be coded as ***Diff*** if applying normative algebraic conventions of use. Tasks 3(CEb/DEb) and 9(CEb/DEb) were excluded since the blank is problematic, as described before, and tasks 2(CEs) and 5(CEl) would be coded as ***Same***. Further, of the six student responses coded as ***Same***, treating the two variables as the same for the confirmatory tasks in the second interview, four occurred at the fourth grade level and two at the fifth grade level. Interestingly, task 5, the common unknown addend task presented as an equation with a letter as the representation of the variable (i.e., $y + y = 7$) would be coded as ***Same*** if the conventional uses of algebra were employed. The only two responses coded as ***Same***, treating the two variables as the same, were at the fourth grade level.

Student responses were consistent for the variable comparison codings from interview one to interview two. This was the case when comparing the codes by variable representation and core mathematical task as well as by task type and core mathematical task. These results confirmed the results for the variable comparison category from interview one to interview two. I included the results for each interview and a table comparing percentage point change from interview one to interview two in Appendix C.

Table 5

Interview Two Variable Comparison Results

	4th grade		5th grade		6th grade		Total	
	Same	***Diff***	***Same***	***Diff***	***Same***	***Diff***	***Same***	***Diff***
1 DWw	0 (0%)	9 (100%)	1 (11.1%)	8 (88.9%)	0 (0%)	6 (100%)	1 (4.2%)	23 (95.8%)
2 CEs	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
3 CEb/SEb	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
4 DEs	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
5 CEl	3 (33.3%)	6 (66.7%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	3 (12.5%)	21 (87.5%)
8 DWw	0 (11.1%)	9 (88.9%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (4.2%)	24 (95.8%)
9 CEb/DEb	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
12 DEs	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)

C – common unknown addends task; D – different unknown addends task; N – No solution task; E – equation; W – word problem; b – blanks; l - letters; s – shapes; w –words;

Solution type. I also coded student responses to reflect the types of solution sets they gave for the variables, see Table 6. I coded student responses as ***SOL_Var*** if they provided multiple numerical values for each representation of the variable, ***SOL_SingVal*** if they provided a single numerical value for the representation of the variable, and ***SOL_None*** if they indicated that there was no solution or no number that was an appropriate solution to the task. These codes provided an overview of the types of solution sets (multiple values, single values, or no solutions) the students used for various representations of variables across task types and problems with common and different mathematical structures.

Table 6

Solution Type Category Results

	4th grade			**5th grade**			**6th grade**			**Total**		
	SOL_ Var	***SOL_ SingVal***	***SOL_ None***	***SOL_ Var***	***SOL_ SingVal***	***SOL_ None***	***SOL_ Var***	***SOL_ SingVal***	***SOL_ None***	***SOL_ Var***	***SOL_ SingVal***	***SOL_ None***
1	0	9	0	0	9	0	0	6	0	0	24	0
CWw	(0%)	(100%)	(0%)	(0%)	(100%)	(0%)	(0%)	(100%)	(0%)	(0%)	(100%)	(0%)
2	9	0	0	9	0	0	6	0	0	24	0	0
CEs	(100%)	(0%)	(0%)	(100%)	(0%)	(0%)	(100%)	(0%)	(0%)	(100%)	(0%)	(0%)
3	9	0	0	9	0	0	6	0	0	24	0	0
DEl	(100%)	(0%)	(0%)	(100%)	(0%)	(0%)	(100%)	(0%)	(0%)	(100%)	(0%)	(0%)
4	7	2	0	9	0	0	6	0	0	22	2	0
NEb	(77.8%)	(22.2%)	(0%)	(100%)	(0%)	(0%)	(100%)	(0%)	(0%)	(91.7%)	(8.3%)	(0%)
5	9	0	0	9	0	0	6	0	0	24	0	0
CEb/DEb	(100%)	(0%)	(0%)	(100%)	(0%)	(0%)	(100%)	(0%)	(0%)	(100%)	(0%)	(0%)
6	9	0	0	9	0	0	6	0	0	24	0	0
DEs	(100%)	(0%)	(0%)	(100%)	(0%)	(0%)	(100%)	(0%)	(0%)	(100%)	(0%)	(0%)
7	7	1	1	9	0	0	6	0	0	21	1	1
NEs	(77.8%)	(11.1%)	(11.1%)	(100%)	(0%)	(0%)	(100%)	(0%)	(0%)	(87.5%)	(4.2%)	(4.2%)
8	0	2	7	0	0	8	0	0	4	0	2	19
NWw	(0%)	(22.2%)	(77.8%)	(0%)	(0%)	(88.9%)	(0%)	(0%)	(66.7%)	(0%)	(8.3%)	(79.2%)
9	7	2	0	9	0	0	5	1	0	21	3	0
CEl	(77.8%)	(22.2%)	(0%)	(100%)	(0%)	(0%)	(83.3%)	(16.7%)	(0%)	(87.5%)	(12.5%)	(0%)
10	7	1	1	9	0	0	6	0	0	22	1	1
NEl	(77.8%)	(11.1%)	(11.1%)	(100%)	(0%)	(0%)	(100%)	(0%)	(0%)	(91.7%)	(4.2%)	(4.2%)
11	8	1	0	8	1	0	6	0	0	22	2	0
DWw	(88.9%)	(11.1%)	(0%)	(88.9%)	(11.1%)	(0%)	(100%)	(0%)	(0%)	(91.7%)	(8.3%)	(0%)

C – common unknown addends task; D – different unknown addends task; N – No solution task; E – equation; W – word problem; b – blanks; l - letters; s – shapes; w –words;

For example, Sally, who treated the two variables as the same, ***Same***, was also coded as ***SOL_SingVal*** since she stated that only six and six, a single value for each variable, were the only values that made the equation true. Jill, whose response I coded as ***Diff***, I also coded as ***SOL_Var*** since she provided multiple pairs of values that would work for the variables, (i.e., 5 and 7, 6 and 6, 9 and 3, and 2 and 10).

These codes do not address the types of numbers (e.g., whole, integer, rational) that students stated the variables could represent or the completeness of their solutions. For instance, Jill, above coded as ***SOL_Var***, did not give a complete solution set for the whole numbers to which she was referring. I further discuss the types of numbers that students stated for the various representations of the variables (e.g., whole numbers, integers, and rational) in the section on quasi-variables.

All students in the study gave responses resulting in consistent codings for problems 1(CWw), 2(CEs), 3(DEl), 5(CEb/DEb), and 6(DEs). The majority of the differences in the remaining problems occurred at the fourth grade (problems 4(NEb), 7(NEs), 8(NWw), 9(CEl), 10(NEl), and 11(DWw)) as opposed to fifth grade (tasks 8(NWw), and 11(DWw)) and sixth grade (problems 8(NWw), and 9(CEl)). Of the six problems where differences in coding arose, four occurred for the no solution tasks (problems 4(NEb), 7(NEs), 8(NWw), and 10(NEl)), which appear to have been more challenging for the fourth graders than the fifth or sixth graders. However, the common coding of students‘ responses did not demonstrate a meaning that corresponded with the conventional algebraic usage of variables. For the three equation formats of the problem, students primarily provided multiple solutions, treating the two variables as different, even though they represented them with the same symbol.

I primarily coded the fourth grade students who I coded differently from the rest of the students for the no solution problems as ***SOL_SingVal*** (*n* = 6) compared to those coded as ***SOL_None*** (*n* = 2). The one student whose solution I coded as ***SOL_None*** also demonstrated evidence of treating the variables as the same values, an algebraically correct interpretation and solution of the problems. However, this student only applied this meaning to the equations with the variables represented with shapes (e.g., $\square + \square = 12$) and letters (e.g., $y + y = 12$), not the equation where the representation of the variables were blanks (e.g., ___ + ___ = 12). For this latter equation, this same student provided multiple pairs of values that would work treating the two blanks as different.

Of the six student responses coded as ***SOL_SingVal*** for the no solution task, four were from, Brett, who interpreted each of the four formats of the problem the missing addend as the same as the given addend, six, resulting in a sum of twelve. Therefore, this student treated the two variables as different but due to his assumption about the problem believed that there was only a single solution. The other two student codes were single instances for a single problem, the equation with blanks as the representation of the variable and the other the word problem.

The other student whose solution was coded as ***SOL_SingVal*** for the equation with blanks as the representation of the variable assumed that the sum for this problem had to be twelve, resulting in the missing addend being six. When asked her if there were other numbers that worked she indicated that there would not be because then you would not get twelve as an answer. When I asked her what would happen if the answer were not twelve she stated, ―then you wouldn‘t really know what you were doing.” It is unclear from the interview if this student believed that the addends had to be the same or if the sum had to be twelve. The other students whose solution I coded as ***SOL_SingVal*** for the word problem switched the order of the addends giving an answer of 0 + 6 = 6.

Table 7 shows the total number of codes, and percentages, for each confirmatory task for the solution type category for each grade and the entire sample population. I consistently coded students‘ responses for the solution type category as ***SOL_Var,*** regardless of the representation of the variable or task type. Of 192 total codes, ***SOL_Var*** was coded 186 times, ***SOL_SingVal*** was coded four times, and ***SOL_None*** twice. The four ***SOL_SingVal*** codes arose in tasks 1 (DWw), 2 (CEs), and 5 (CEl). The two instances of the ***SOL_None*** coding only occurred in task 5 (CEl). Further, of the non-***SOL_Var*** coding that occurred, all but one occurred at fourth grade. The other single instance occurred with a fifth grade student.

While the majority of responses were coded as ***SOL_Var,*** this was only the appropriate coding for tasks 1(DWw), 4(SEs), 8(DWw), and 12(DEs) when applying algebraic conventions to the tasks. ***SOL_SingVal*** was the appropriate coding for tasks 2(CEs) and 5(CEl), each a common unknown addend task, when applying algebraic conventions to the tasks. None of the tasks would have ―no solutions" when applying the usual algebraic conventions to the tasks. I excluded tasks 3 and 9 due to the difficulties of distinguishing between blanks discussed before. I elaborate on this idea further in the next section on solutions.

Table 7

Solution Type Results

	4th grade			5th grade			6th grade			Total		
	SOL_ Var	***SOL_ SingVal***	***SOL_ None***	***SOL_ Var***	***SOL_ SingVal***	***SOL_ None***	***SOL_ Var***	***SOL_ SingVal***	***SOL_ None***	***SOL_ Var***	***SOL_ SingVal***	***SOL_ None***
1	8	1	0	8	1	0	6	0	0	22	2	0
SWw	(88.9%)	(11.1%)	(0%)	(88.9%)	(11.1%)	(0%)	(100%)	(0%)	(0%)	(91.7%)	(8.3%)	(0%)
2	8	1	0	9	0	0	6	0	0	23	1	0
CEs	(88.9%)	(11.1%)	(0%)	(100%)	(0%)	(0%)	(100%)	(0%)	(0%)	(95.8%)	(4.2%)	(0%)
3	9	0	0	9	0	0	6	0	0	24	0	0
CEb/SEb	(100%)	(0%)	(0%)	(100%)	(0%)	(0%)	(100%)	(0%)	(0%)	(100%)	(0%)	(0%)
4	9	0	0	9	0	0	6	0	0	24	0	0
SEs	(100%)	(0%)	(0%)	(100%)	(0%)	(0%)	(100%)	(0%)	(0%)	(100%)	(0%)	(0%)
5	6	1	2	9	0	0	6	0	0	21	1	2
CEl	(66.7%)	(11.1%)	(22.2%)	(100%)	(0%)	(0%)	(100%)	(0%)	(0%)	(87.5%)	(4.2%)	(8.3%)
8	9	0	0	9	0	0	6	0	0	24	0	0
SWw	(100%)	(0%)	(0%)	(100%)	(0%)	(0%)	(100%)	(0%)	(0%)	(100%)	(0%)	(0%)
9	9	0	0	9	0	0	6	0	0	24	0	0
DEb/SEb	(100%)	(0%)	(0%)	(100%)	(0%)	(0%)	(100%)	(0%)	(0%)	(100%)	(0%)	(0%)
12	9	0	0	9	0	0	6	0	0	24	0	0
SEs	(100%)	(0%)	(0%)	(100%)	(0%)	(0%)	(100%)	(0%)	(0%)	(100%)	(0%)	(0%)

C – common unknown addends task; D – different unknown addends task; N – No solution task; E – equation; W – word problem; b – blanks; l - letters; s – shapes; w –words;

Of the four ***SOL_SingVal*** codings, only two were appropriate when applying algebraic conventions to the tasks and both occurred with fourth grade students. However, the two instances occurred with two different tasks, 2(CEs), 8 = $\square + \square$, and 5(CEl), $y + y = 7$, and by two different students, Brett and George respectively. Therefore, neither student consistently applied this meaning across tasks or representations of variables.

Student responses were consistent for the solution type codings from interview one to interview two. This was the case when comparing the codes by variable representation and core mathematical task as well as by task type and core mathematical task. Since these results confirm the results for the variable comparison category from interview one to interview two, I provide no further discussion here. I included the results for each interview and a table comparing percentage point change from interview one to interview two in appendix C**.**

Variable values. Last, I coded student responses to indicate if the numerical values given by the student were the same for both variables, ***VAL_Same***, and/or if the numerical values given by the student were different, ***VAL_Diff*** for both variables, see Table 8. For the first student response discussed above, her response was coded as ***VAL-_Same*** since the only solution she provided, six and six, was the same value for each of the two variables. The latter student response was coded as both ***VAL_Same***, since she also provided the solution six and six, *and* ***VAL_Diff*** since she also provided the solutions five and seven, nine and three, and two and ten which assigned different values to each of the two variables.

Table 8

Variable Value Category Results

	4th grade		5th grade		6grade		Total	
	VAL_ Diff	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***
1 CWw	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
2 CEs	9 (100%)	9 (100%)	9 (100%)	9 (100%)	6 (100%)	6 (100%)	24 (100%)	24 (100%)
3 DEl	9 (100%)	9 (100%)	9 (100%)	9 (100%)	6 (100%)	6 (100%)	24 (100%)	24 (100%)
4 NEb	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)	0 (0%)
5 CEb/ DEb	9 (100%)	9 (100%)	9 (100%)	9 (100%)	6 (100%)	6 (100%)	24 (100%)	24 (100%)
6 DEs	9 (100%)	8 (88.9%)	9 (100%)	9 (100%)	6 (100%)	5 (83.3%)	24 (100%)	22 (91.7%)
7 NEs	8 (88.9%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	23 (95.8%)	0 (0%)
8 NWw	0 (0%)	1 (11.1%)	0 (0%)	0 (0%)	0 (0%)	0 (0%)	0 (0%)	1 (4.2%)
9 CEl	7 (77.8%)	9 (100%)	9 (100%)	9 (100%)	5 (83.3%)	6 (100%)	21 (87.5%)	24 (100%)
10 NEl	8 (88.9%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	23 (95.8%)	0 (0%)
11 DWw	8 (88.8%)	9 (100%)	8 (88.9%)	9 (100%)	6 (100%)	6 (100%)	22 (91.7%)	24 (100%)

C – common unknown addends task; D – different unknown addends task; N – No solution task; E – equation; W – word problem; b – blanks; l - letters; s – shapes; w –words;

Unlike the previous categories, which required each student response to have a single code for the category, student responses for this category could be either of the two codes or both. This code provides evidence of, specifically for tasks coded as ***Diff*** for the variable comparison category, the degree to which students in grades four through six recognize the convention that the same variable in the same problem must be the same value, and the common algebraic misconception that different variables cannot be the same value. As noted in chapter two, this misconception has been well documented for students in algebra and beyond but we do not know if this same misconception arises with students before taking an algebra course. I discuss this in detail below.

I coded two students, Sally and Jill, as ***Same, SOL_SingVal,*** and ***VAL_Same***. and ***Diff, SOL_Var***, ***VAL_Same***, and ***VAL-Different*** respectively. While it may appear the codings for ***Diff/Same*** and ***VAL_Diff/VAL_Same*** are redundant, these codes provide the data necessary to identify students who recognize that the two variables are different and apply the common misconception that the values must also be different versus being able to be different and the same. (e.g., the solution to $x + y = 12$ is only 6 and 6). It is interesting to note that this misconception did not occur for any of the students in the course of solving the first eleven tasks. However, it did arise during the comparison tasks when the students compared pairs of equations from the first eleven tasks, comparing two equations with the common mathematical structures but different representations of the variables, or equations with different mathematical structures but the same representations of the variables. I discuss this further in the section describing the findings for the comparison tasks.

The fifth grade students gave consistent responses for every problem except problem 11 (word problem for the core mathematical task $y + y = 12$). In this problem, I coded one student‘s response differently from the rest of the sixth graders. This student stated that the two people had to have the same length of ribbon, even though the problem did not indicate this.

The sixth grade students provided consistent responses for every problem except for problem 9(CEl) where I coded one student‘s response differently. This student stated that only 6 and 6 would work for the equation $y + y =12$. However, this same student did not apply this same idea to either the equation with same shape as the representation of the variables or for the equation with blanks as the representation of the variables. They also did not apply this when comparing the equations $y + y = 12$ and $a + b = 12$ where they sated that these two equations were the same and that the same numbers worked for both equations.

Table 9 shows the total number of codes, and percentages, for each confirmatory task for the solution type category for each grade and the entire sample population. In contrast to the results for the variable comparison and solution type category codings, the coding of the variable value category for the interview two confirmatory tasks were inconsistent with those from interview one. In interview one, with the exception of the no solution tasks and the word problem for the common unknown addend problem, I consistently coded student responses as both ***VAL_Diff*** and ***VAL_Same*** for each task. This occurred in the second interview for tasks 1 (DWw), 2 (CEs), and 4 (DEs).

However, as noted in the opening section describing interview two, one change that was made from the first eleven tasks in interview one and the confirmatory tasks in the second interview was the inclusion of tasks with odd sums to test for false positives in the data from the first interview. The confirmatory tasks that had an even sum were tasks 1 (i.e., Juan and Alexa each have a piece of string. Together they have 16 inches of string.), 2 (i.e., $8 = \square + \square$), and 4 (i.e., $12 = \square + \triangle$), the same tasks that had a common coding pattern with the tasks from interview one. The remaining confirmatory tasks, where there were a high percentage of responses coded as ***VAL_Diff*** and a varying percentage of student responses coded as ***VAL_Same*** ranging from 25% to 62.5%, were tasks with odd sums.

Table 9

Variable Value Codings for Interview Two Confirmatory Tasks

	4th grade		**5th grade**		**6th grade**		**Total**	
	VAL_ Diff	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***
1 DWw	8 (88.9%)	9 (100%)	8 (88.9%)	9 (100%)	6 (100%)	6 (100%)	22 (91.7%)	24 (100%)
2 CEs	8 (88.9%)	9 (100%)	9 (100%)	9 (100%)	6 (100%)	6 (100%)	23 (95.8%)	24 (100%)
3 CEb/DEb	9 (100%)	3 (33.3%)	9 (100%)	1 (11.1%)	6 (100%)	5 (83.3%)	24 (100%)	9 (37.5%)
4 SEs	9 (100%)	7 (77.8%)	9 (100%)	9 (100%)	6 (100%)	6 (100%)	24 (100%)	22 (91.7%)
5 CEl	6 (66.6%)	5 (55.6%)	9 (100%)	5 (55.5%)	6 (100%)	5 (83.3%)	21 (87.5%)	15 (62.5%)
8 DWw	9 (100%)	1 (11.1%)	9 (100%)	2 (22.2%)	6 (100%)	3 (37.5%)	24 (100%)	6 (25%)
9 CEb/DEb	9 (100%)	2 (22.2%)	9 (100%)	2 (22.2%)	6 (100%)	5 (83.3%)	24 (100%)	9 (37.5%)
12 DEs	9 (100%)	3 (33.3%)	9 (100%)	4 (44.4%)	6 (100%)	5 (83.3%)	24 (100%)	12 (50%)

C – common unknown addends task; D – different unknown addends task; N – No solution task; E – equation; W – word problem; b – blanks; l - letters; s – shapes; w –words;

Summary. From the initial coding of student responses, I can make two observations. First, student responses were very consistent across the three categories for the eleven tasks from the first interview. Even where there were differences in codes the number of differences were small. Second, the change from even sums in the first interview to the inclusion of odd sums in the second interview appears to have made a difference in the number and consistency of student coded responses. I elaborate on each of these ideas in the following chapter.

Students solutions across various representations of variables, task types and core mathematical tasks (research question a)

In the following section, I compare each student's response codes to the algebraically normative coding for each task in interview one and two (see Figure 18). A normative algebraic response is a response that treats the variables, in terms of the three coding categories, as they would if conventional algebraic meanings and uses were employed. This provides an overview of how students solved the tasks and how closely their solution methods and meanings aligned with or paralleled an algebraic normative response to the tasks. As noted in Chapter 1, the purpose of this study is not to determine if students, prior to instruction, solve problems using an algebraically normative solution process itself, a topic lacking a research base. Instead, the purpose is to begin research on the prior knowledge related to meaning of variables with which elementary students enter algebra courses.

	Variable comparison	**Solution Type**	**Variable value**
Common unknown addends	***Same*** – treated the two variables as the same.	***SOL_SingVal*** – provided a single value solution for each variable.	***VAL_Same*** – provided the same value for both variables.
Different unknown addends	***Diff*** – treated the two variables as different.	***SOL_Var*** – provided multiple value solutions for each variable.	***VAL_Same*** ***VAL_Diff*** provided the same *and* different value for both variables.
No Solution	***Same*** – treated the two variables as the same.	***SOL_None*** - indicated that there was no solution.	***VAL_Same**** - provided the same value for both variables.

*While these tasks have no solution, this is due to the representation of the variables having the same value.

Figure 21: Normative algebraic responses for each core mathematical task.

Results by task. Table 10 shows the number, and percent, of student codes for each of the eleven tasks from interview one by grade that paralleled a normative algebraic interpretation of the task across the three categories: variable comparison, solution type, and variable value.

Since differentiating between blanks is problematic and there is no existing research on students' meanings for these representations of variables, I included two entries for task 5 (___ + ___ = 12), task 5a(CEb) and task 5b(DEb). 5a(CEb) reports the number and percent of students whose solutions paralleled a normative algebraic interpretation of the problem where the student treated the two blanks as the same variable (i.e., common unknown addends). 5b(DEb) reports the number and percent of students whose solutions paralleled a normative algebraic interpretation of the problem where the student treated the two blanks as different variable (i.e., different unknown addends).

The percentages of students who paralleled a normative algebraic interpretation of the tasks varied from 0% in tasks 2(CEs), 4(NEb), and 5a(CEb) to 100% for tasks 1(CWw), 3(DEl), and 5b(DEb). Further, students' solutions did not parallel a normative algebraic solution for tasks 7 (4.2%), 9 (12.5%) and 10 (4.2%). Students' solutions also paralleled a normative algebraic solution for tasks 6 (95.8%), 8 (79.2%), and 11 (91.7%).

Table 11 shows the number, and percent, of student codes for each of the eight confirmatory tasks from interview two by grade that paralleled a normative algebraic interpretation of the task across the three categories: variable comparison, solution type, and variable value, see figure 21.

As in the tasks with blanks from interview one, I included two entries for tasks 3 (___ + ___ = 5), and 9 (___ + ___ =7). Task 3a and 9a display the number and percent of students whose solutions paralleled a normative algebraic interpretation of the problem where they treated the two blanks as the same variable (i.e., common unknown addends). Task 3b and 9b display the number and percent of students whose solutions paralleled a normative algebraic interpretation of the problem where they treated the two blanks as different variable (i.e., different unknown addends).

Fewer tasks in interview two reflected student responses paralleling a normative algebraic solution than in interview one, see Tables 10 and 11. Task 1 had a large number of responses, 91.7%, paralleling a normative algebraic response. The next highest percent of these responses was task 12 with 50% of student responses paralleling a normative algebraic solution. The remaining tasks varied from 0% to 37.5% of student responses paralleling a normative algebraic solution. However, the purpose of these tasks was only to determine if student responses from interview one to interview two had changed. Therefore, the number and variety of tasks in these confirmatory tasks are not as extensive as those from interview one. Therefore, any comparison of the results must consider this.

Table 10

Student Normative Algebraic Responses From Interview One

	1 CWw	2 CEs	3 DEl	4 NEb	5a CEb	5b DEb	6 SEs	7 NEs	8 NWw	9 CEl	10 NEl	11 DWw
4th grade	9	0	9	0	0	9	8	1	7	2	1	8
	100.0%	0.0%	100.0%	0.0%	0.0%	100.0%	88.9%	11.1%	77.8%	22.2%	11.1%	88.9%
5th grade	9	0	9	0	0	9	9	0	8	0	0	8
	100.0%	0.0%	100.0%	0.0%	0.0%	100.0%	100.0%	0.0%	88.9%	0.0%	0.0%	88.9%
6th grade	6	0	6	0	0	6	6	0	4	1	0	6
	100.0%	0.0%	100.0%	0.0%	0.0%	100.0%	100.0%	0.0%	66.7%	16.7%	0.0%	100.0%
Total	24	0	24	0	0	24	23	1	19	3	1	22
	100.0%	0.0%	100.0%	0.0%	0.0%	100.0%	95.8%	4.2%	79.2%	12.5%	4.2%	91.7%

C – common unknown addends task; D – different unknown addends task; N – No solution task; E – equation; W – word problem; b – blanks; l - letters; s – shapes; w –words;

Table 11

Student Normative Algebraic Responses From Interview Two

	1 DWw	2 CEs	3a CEb	3b DEb	4 DEs	5 CEl	8 DWw	9a CEb	9b DEb	12 DEs
4^{th} grade	8 88.9%	0 0.0%	0 0.0%	3 33.3%	7 77.8%	1 11.1%	1 11.1%	0 0.0%	2 22.2%	3 33.3%
5^{th} grade	8 88.9%	0 0.0%	0 0.0%	1 11.1%	9 100.0%	0 0.0%	2 22.2%	0 0.0%	2 22.2%	4 44.4%
6^{th} grade	6 100.0%	0 0.0%	0 0.0%	5 83.3%	6 100.0%	0 0.0%	3 50.0%	0 0.0%	5 83.3%	5 83.3%
Total	22 91.7%	0 0.0%	0 0.0%	9 37.5%	22 91.7%	1 4.2%	6 25.0%	0 0.0%	9 37.5%	12 50.0%

C – common unknown addends task; D – different unknown addends task; N – No solution task; E – equation; W – word problem; b – blanks; l - letters; s – shapes; w –words;

The differences between the two interviews may have arisen from the inclusion of odd sums in interview two. For instance, the examples cited in the general results for Paul, Tricia, and Mark for the task $y + y = 7$ demonstrated the confusion that arose from the inclusion of the odd sums. Paul indicated that the same value could not be assigned to each y as he only drew from whole numbers. Mary, on the other hand, provided the solution 3½ and 3½ drawing on, at least positive, rational values. Greg appears to draw a distinction between the types of values he could use to answer the task. When I asked him if the y's could be the same, he indicated that they could be *if* you used fractions. He was unsure about when fractional responses would be appropriate. He also stated that if you used negatives then the other y could be greater than the sum, but again he was unsure about when this would be appropriate.

Results across representations of variables. Table 12 displays the data from interview one by the representation of the variable used in the task then by the core mathematical task. From this table a similar pattern emerges. Students, in the sample as a whole, were inconsistent in providing solution that paralleled a normative algebraic solution. None of the students' solutions paralleled a normative algebraic solution when the representation of the variable was a blank. This appears to highlight the difficulty with interpreting blanks, which are indistinguishable from each other.

Table 12

Interview One Normative Algebraic Responses by Representation of Variable and Core Mathematical Task

		4th grade	**5th grade**	**6th grade**	**Total**
b	C	0 (0.0%)	0 (0.0%)	0 (0.0%)	0 (0.0%)
	D	0 (0.0%)	0 (0.0%)	0 (0.0%)	0 (0.0%)
	N	0 (0.0%)	0 (0.0%)	0 (0.0%)	0 (0.0%)
l	C	2 (22.2%)	0 (0.0%)	1 (16.7%)	3 (12.5%)
	D	9 (100.0%)	9 (100.0%)	6 (100.0%)	24 (100.0%)
	N	1 (11.1%)	0 (0.0%)	0 (0.0%)	1 (4.2%)
s	C	0 (0.0%)	0 (0.0%)	0 (0.0%)	0 (0.0%)
	D	8 (88.9%)	9 (100.0%)	6 (100.0%)	23 (95.8%)
	N	1 (11.1%)	0 (0.0%)	0 (0.0%)	1 (4.2%)
w	C	9 (100.0%)	9 (100.0%)	6 (100.0%)	24 (100.0%)
	D	8 (88.9%)	8 (88.9%)	6 (100.0%)	22 (91.7%)
	N	7 (77.8%)	8 (88.9%)	4 (66.7%)	19 (79.2%)

b – blanks; l letters; s – shapes; w –words; C – common unknown addends task; D – different unknown addends task; N – No solution task

As shown in the letter section of Table 12, when the representation of the variable was a letter all students' solutions paralleled a normative algebraic solution for the different unknown addends tasks (e.g., $a + b = 12$). As shown in the shapes section of

Table 4.9, when the representation of the variable was shapes, all but one student's response paralleled a normative algebraic solution (95.8%) for the different unknown addend tasks (e.g., $\square + \triangle = 12$). However, when the representation of the variable was words in a word problem, see word section of Table 12, the percentage of student responses paralleling a normative algebraic solution for the common unknown addend, different unknown addend, and no solution tasks were 100%, 91.7%, and 79.2% respectively. Students appeared to draw on the referent and descriptive nature of the word problems, which were not in the equations, to generate meanings that were algebraically implicit in the equations.

Table 13 displays these results by the representation of the variable used in the task by the core mathematical task. From this table a similar pattern emerges. As in the previous table, fewer student responses in the sample paralleled a normative algebraic solution. As demonstrated in the shape section of Table 13**,** shapes used as the representation of the variable for different unknown addends tasks had the most student responses paralleling a normative algebraic solution with 70.8%. The next most common task paralleling a normative algebraic solution were word problems with different unknown addends and words as the representation of the variable with 58.3% of student responses paralleling a normative algebraic solution.

Table 13

Interview Two Confirmatory Task Normative Algebraic Responses by Representation of Variable and Core Mathematical Task

		4th grade	**5th grade**	**6th grade**	**Total**
b	C	0 (0.0%)	0 (0.0%)	0 (0.0%)	0 (0.0%)
	D	5 (27.8%)	3 (16.7%)	10 (83.3%)	18 (37.5%)
l	C	1 (11.1%)	0 (0.0%)	0 (0.0%)	1 (4.2%)
s	C	0 (0.0%)	0 (0.0%)	0 (0.0%)	0 (0.0%)
	D	10 (55.6%)	13 (72.2%)	11 (91.7%)	34 (70.8%)
w	D	9 (50.0%)	10 (55.6%)	9 (75%)	28 (58.3%)

b – blanks; l letters; s – shapes; w –words; C – common unknown addends task; D – different unknown addends task; N – No solution task

Results by task type. Table 14 displays the number and percent of students whose responses paralleled a normative algebraic solution by task type (Equation, or Word problem), and then by core mathematical task. When I presented the tasks as an equation, (see the equation section of Table 14) all but one student response (71 of 72) paralleled a normative algebraic solution for the different unknown addend tasks (98.6%). In contrast, the equations for the core mathematical tasks common unknown addends, and no solution tasks did not parallel a normative algebraic solution with 4.2% and 2.8% respectively. However, when the representation of the variable involved words (e.g., I am thinking of two numbers. When I add these two numbers together, I get 7), in a word problem, the percentage of student responses paralleling a normative algebraic solution for the common unknown addend, different unknown addend, and no solution tasks were 100%, 91.7%, and 79.2% respectively.

Table 14

Interview One Normative Algebraic Responses by Task Type and Core Mathematical Tasks

		4th grade	**5th grade**	**6th grade**	**Total**
E	C	2 (7.4%)	0 (0.0%)	1 (5.6%)	3 (4.2%)
	D	26 (96.3%)	27 (100.0%)	18 (100.0%)	71 (98.6%)
	N	2 (7.4%)	0 (0.0%)	0 (0.0%)	2 (2.8%)
W	C	9 (100.0%	9 (100.0%	6 (100.0%	24 (100.0%)
	D	8 (88.9%)	8 (88.9%)	6 (100.0%)	22 (91.7%)
	N	7 (77.8%)	8 (88.9%)	4 (66.7%)	19 (79.2%)

E – equation; W – word problem; C – common unknown addends task (e.g., $y + y = 12$); D – different unknown addends task (e.g., $a + b = 12$); N – No solution task (e.g., $x + 6 = x$)

Table 15 displays these results by the task type used in the task then by the core mathematical task. From this table a similar pattern emerges. As in the previous tables, fewer student responses in the sample as a whole paralleled a normative algebraic solution. Word problems for different unknown addends tasks (e.g., Together Tom and Anne have 12 feet of ribbon) had the most student responses paralleling a normative algebraic solution with 58.3%. The tasks with the next highest percent of responses paralleling a normative algebraic solution were equations with different unknown addends with 54.2% of student responses paralleling a normative algebraic solution. This latter result appears to be more of a result of the students consistently treating the two variables as different regardless of whether the two variables were the same or different which parallels the normative algebraic solution for different unknown addend tasks.

Table 15

Interview Two Confirmatory Task Normative Algebraic Responses by Task Type and Core Mathematical Task

		4th grade	5th grade	6th grade	Total
E	C	1 (2.8%)	0 (0.0%)	0 (0.0%)	1 (1.0%)
E	D	15 (41.7%)	16 (44.4%)	21 (87.5%)	52 (54.2)
W	D	9 (50.0%)	10 (55.6%)	9 (75%)	28 (58.3%)

E – equation; W – word problem; C – common unknown addends task (e.g., $y + y = 12$); D – different unknown addends task (e.g., $a + b = 12$); N – No solution task (e.g., $x + 6 = x$)

Quasi-variables. Every student in the study incorporated the idea of limiting boundary values to the sums in each of the first eleven tasks reported above. Students indicated that missing addends could not be greater than the given sum because adding resulted in the number you are adding increasing in magnitude. This was exemplified in the following exchange with Kate, a sixth grader, where she was asked if there were any numbers that the first square could not be in the equation $\square + \square = 12$.

> M: Are there any numbers that this first square cannot be?
> K: 13 and up
> M: Why couldn‘t it be thirteen and up?
> K: Because you can‘t add anything to thirteen or above to get twelve.

During the first interview, the focus of the tasks and the follow-up questions did not lend themselves to determining if the students would assign negative values to the variables. Initially I assumed that if a student gave a response such as the one supplied by

Kate above, then I could infer that they did not believe they could use negative values. However, I called this assumption into question during the following two exchanges.

George, a fourth grader, supplied solutions for the equation $a + b = 12$. When I asked him about numbers that would not work, he used the sum as a boundary value but then also brought up the possibility of using negative numbers.

M: So are there any numbers the *a* could not be, that wouldn‘t work?
G: Thirteen or over.
M: Okay, so kind of like the last one?
G: [*nods his head affirmatively*] unless you had negative.
M: If I had a negative? So, tell me about that. What did you mean by that?
G: Because if it had, well, if you were doing, well, negative one plus.
M: Would that work?
G: [*nods his head affirmatively*]
M: So it could be thirteen?
G: yes, if you were doing negatives.
M: what about fourteen?
G: mhm because then you could do negative two.
M: What about one hundred?
G: [*nods his head affirmatively*]
M: One thousand?
G: [*nods his head affirmatively*]
M: One million?
G: [*nods his head affirmatively*]

In this exchange, George invoked the idea of the sum being a boundary number, which it would be for whole numbers or non-negative rational numbers, but then acknowledged that there was an exception to this: negative numbers. Another fourth grader, Tia, provided a similar response to that of George for the same task.

M: So, what numbers could those be?
T: 12 and under, and no negatives.
M: And why can‘t they be negatives.
T: Because it gets a little too confusing and negatives it‘s really hard to add.

Tia‘s response differed from George‘s in that she did not directly acknowledge that negative values could be used but did acknowledge that they exist and that the primary reason she would not use them is that they are difficult to use, not that they *cannot* be used. Therefore, in both cases the students used the sum as a boundary value which limited the upper end of the solution set for the variables even though they acknowledged the existence of the negative values and, at least in Georges case, further noted that their use was an exception to this boundary value.

In addition to the commonality of the use of the sum as a boundary value, students‘ responses were also consistent in that initially every student supplied *only* whole number solutions to the tasks. Much like the two discussions reported above, it was only through follow up questions that the students‘ beliefs about non-integer rational solutions began to come to the surface. Through specifically asking students if values such as the mixed number 2½ would work in the equation, initial views of their meaning and use of fractions began to come into focus. For instance, of the 24 students, 15 indicated that variables in use could take on a fractional value of at least one number that was specifically asked of them. This should not be taken as the students holding the belief that the variables could be *any* rational number. That would be beyond the scope of the evidence collected for this interview.

Of the remaining nine students, eight indicated that 2½ would not work: the one other student did not supply enough evidence upon which to make any inferences about this topic. While most students did not provide justification for the exclusion of the prompted mixed number beyond stating that it was unacceptable, some students appeared to hold the belief that these were not ―even numbers.” For instance, the following conversation with Carry, a fifth grader, occurred when she was asked if 3½ could be used for one of the squares in $\square + \square = 12$.

> M: Could I put like three and a half in that first square?
> C: Um, no.
> M: Why wouldn‘t that work?
> C: Because that‘s a like fraction of a number so, it‘s a fraction. So you would have to have another fraction I guess.

Carrie‘s statement that 3½ is a ―fraction of a number" was interesting. It was not clear from this exchange or others during this interview, if she made a distinction between numbers and fractions of a number. It may be that if she made such a distinction then they would not be valid because she did not view fractions as numbers. Either way, this exchange demonstrated Carry‘s confusion with the use of fractions, which she continued to state would not be valid solutions in the other ten tasks.

A more definitive example of this confusion occurred with Julie, a fourth grader, when she attempted to find solutions for the equation $a + b = 12$. After providing a set of whole number solutions, (e.g., 6 and 6, 5 and 7, and 10 and 2) I asked her if 2½ would work as shown in the exchange below.

> M: Would two-and-a-half work for this one [points to the a in $a + b = 12$].
> J: Maybe.
> M: What do you mean, maybe?
> J: Like there could be, okay, probably not.
> M: Why not? Tell me what you were thinking about.
> J: A half is not a number and I thought numbers could get answers like, it might have to be twelve-and-a-half.

Julie‘s statement that, ―a half is not a number" provides a glimpse into her meaning for fractions. Her statement that using 2½ would result in an answer that also had a half, which she rejected as a possible acceptable answer, points to a narrow interpretation of what is meant by the word ―number."

While the purpose of this study was not to explore the students' number sense or knowledge of the number sequence, including the rational numbers, these arose during the questioning intended to explore how the students thought about the tasks posed and how they went about solving them. Therefore, a subset of the tasks in the second interview were specifically designed to gather further evidence of students use of integers and rational numbers as possible solutions to the equations, which are reported on further in the quasi-variable section for interview two.

Summary. The results of this section illustrate that coded student responses were inconsistent with normative algebraic solutions across tasks with equivalent mathematical structures. While none of the coded student responses paralleled a normative algebraic solution when I presented the task with blanks, students' response codes for the word problems were much higher for the word problems. Coded student responses also paralleled a normative algebraic solution for the different unknown addend tasks except when I presented them as an equation with blanks as the representation of the variable.

When I presented the tasks as equations coded student responses did not parallel a normative algebraic solution for the common unknown addend or the no solution tasks. However, the coded student responses did parallel a normative algebraic response for the different unknown addends tasks, although the confirmatory tasks were less so.

The inclusion of odd sums in the confirmatory tasks may have resulted in fewer coded student responses paralleling a normative algebraic solution. For those tasks, where both an even and an odd sum were present in the confirmatory tasks, student responses mirrored those from the initial tasks from the first interview. However, the odd sums did not and had a lower percentage of coded student responses paralleling a normative algebraic response.

Finally, the boundary values that students employed as they solved the tasks may have also played a role in the differences in the coding results between the even and odd sums. In addition to the majority of students limiting the solution values to those equal to or less than the sum but greater than or equal to zero, students demonstrated a fragile understanding of number and their properties. As demonstrated in the section on quasi-variables, some students showed evidence of viewing fractions differently than they viewed whole numbers, often referring to the latter as numbers and the prior as something else. Further, their confusion regarding the whole number properties of even numbers (i.e., only even numbers have two equal whole number addends) and the extension of this property to fractions reduced the number of students who believed that the addends of an odd sum could be the same value.

Student meaning of variable across representations of variables (research question b)

In the following section, I examine the data to determine students‘ meaning of variables across representations of variables. I examined the coded student responses for each of the three categories: variable comparison, solution type, and variable value, across the representations of the variables (i.e., blanks, letters, shapes, and words) for the initial eleven tasks from interview one, and the confirmatory tasks from interview two. For each of these sets of tasks I sorted the codes by the representation of the variable and then by the core mathematical task.

Interview 1 variable comparison across representations of variables. In order to determine how students interpreted variables across various representations of the variable for tasks with common mathematical structures, the initial eleven tasks from interview one were resorted by the representation of the variable used (i.e., blank, letter, shape, or word) and the core mathematical task (i.e., common unknown addends, different unknown addends, or no solution). I then analyzed each of the three coding categories: variable comparison, solution type, and variable value to determine initial inferences for students‘ meaning of the representations of the variables.

Table 16 shows the results for the variable comparison category codings by representation of the variable and then core mathematical task for the variable comparison category. As demonstrated in the first section of the table, all students across all three grade levels (100%) treated the two blanks in each equation ___ + ___ = 12 as different variables (i.e., providing different values for the two blanks) regardless of the task type. As noted in the opening section of this chapter, the equation ___ + ___ = 12 was used as the equation for both the common unknown addends task as well as the different unknown addends task since blanks are not generally modified to be distinguishable.

Table 16

Variable Comparison Across Representations of Variables

		4th grade		5th grade		6th grade		Total	
		Same	***Diff***	***Same***	***Diff***	***Same***	***Diff***	***Same***	***Diff***
b	C	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
	D	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
	N	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
l	C	2 (22.2%)	7 (77.8%)	0 (0%)	9 (100%)	1 (16.7%)	5 (83.3%)	3 (12.5%)	21 (87.5%)
	D	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
	N	1 (11.1%)	8 (88.9%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	1 (4.2%)	23 (95.8%)
s	C	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
	D	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
	N	1 (11.1%)	8 (88.9%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	1 (4.2%)	23 (95.8%)
w	C	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)	0 (0%)
	D	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
	N	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)	0 (0%)

b – blanks; l letters; s – shapes; w –words; C – common unknown addends task (e.g., $y + y = 12$); D – different unknown addends task (e.g., $a + b = 12$); N – No solution task (e.g., $x + 6 = x$)

The second section of Table 16 shows the coding results for variables represented as letters (e.g., $a + b = 12$ and $y + y = 12$). These results show that when letters were used as the representation of the variable, the majority of student responses (n=68) demonstrated that students treated the two letters as different, supplying different values for each letter. As shown in the last column of Table 16**,** only four students treated the letters as the same by supplying only the same value to both letters, three for $y + y = 12$

and one for $x + 6 = x$. In addition, each of these four student responses occurred in either the common unknown addends task $y + y = 12$ or no solution equation $x + 6 = x$ where, algebraically speaking, this would be a correct interpretation and use of the variables. However, the three students who demonstrated this meaning for the common unknown addends task inconsistently employed this meaning when they solved the no solution task $x + 6 = x$. Of these three students, only a single fourth grader, Sally, treated the *x*'s as the same, resulting in her concluding that there was no solution for this task since it was not possible to add 6 to a number and get that same number.

In problem 9(CEl), $y + y = 12$, an equation with common addends and a sum of twelve with the same letter as the representation of the variables, three students, two fourth grade students and one fifth grade student, indicated that the *y*'s in $y + y = 12$ were the same. For example, one fourth grade students I will call her Jill, stated, ―Earlier we were talking about *a* and *b,* and since these are the same letters . . . I'm thinking it's like six plus six equals twelve because it's the same letter and it's the same number." The following discussion occurred for the same task with another fourth grader.

> M: So what do the *y*'s mean in this one?
> S: They could mean a number.
> M: So what numbers would work?
> S: Five, seven and a six and a six, nine and a three, two and a ten.

Unlike Jill, this student gave multiple pairs of values for the *y*'s, which included pairs of different numbers, implying that the *y*'s were not the same.

When shapes were used as the representation of the variable (e.g., □ + □ = 12 and □ + △ = 12), every student but one treated the shapes as different variables by providing different values for each shape regardless of the core mathematical task. Sally, who treated the letters as the same variable in both the common unknown addends and no solution task as reported above, did *not* treat the shapes in the common unknown addends task, □ + □ = 12, as the same variable. Instead, she provided multiple different number pair answers such as 10 and 2, 8 and 4, 11 and 1, as well as 6 and 6. She did treat the shapes in the no solution task △ + 6 = △ as the same variable just as she did for the letters. Therefore, she did not consistently treat the same representation of the variable as the same in all cases.

Finally, when words were used as the representation of the variable in word problems, the last section of Table 16, the codes for the students' responses were not only consistent, but also paralleled an algebraically correct interpretation that the variables were the same for the common unknown addends task (i.e., Shakira and Tim have the same number of gummy bears. Together they have 12 gummy bears) and no solution task (i.e., I start with some number then add 6 and get the same number that I started with) and different for the different unknown addends task (i.e., Together Tom and Anne have 12 feet of ribbon). Therefore, it appears that the more descriptive nature of the word problems, as compared to the equations with the variables represented as blanks, letters, and shape, assisted students in interpreting when two representations of variables were the same or different variables.

Table 17 shows the results of the variable comparison codings for the confirmatory tasks by representation of the variable and then core mathematical task. As demonstrated in the final column of the table, students consistently treated the variables as different across all confirmatory tasks. Only four instances of students treating the two representations of the variable as the same occurred, three fourth graders for the common unknown addends tasks where the variable was represented with letters (i.e., $y + y = 12$)

and one fifth grader for the different unknown addends task where the variable was represented with words.

Therefore, the students in the study, other than the four instances noted above, treated the two representations of the variable as different variables regardless whether they were the same or different representation (e.g., $y + y = 12$ or $a + b = 12$), the representation of the variable (i.e., blanks, letters, shapes, or words), or core mathematical task.

Table 17

Confirmatory Task Variable Comparison Coding Across Representations of Variables

		4th grade		5th grade		6th grade		Total	
		Same	***Diff***	***Same***	***Diff***	***Same***	***Diff***	***Same***	***Diff***
b	C	0 (0%)	18 (100%)	0 (0%)	18 (100%)	0 (0%)	12 (100%)	0 (0)%	48 (100%)
	S	0 (0%)	18 (100%)	0 (0%)	18 (100%)	0 (0%)	12 (100%)	0 (0%)	48 (100%)
l	C	3 (33.3%)	6 (66.0)%	0 (0%)	9 (100%)	0 (0%)	6 (100%)	3 (12.5%)	21 (87.5%)
s	C	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
	S	0 (0%)	18 (100%)	0 (0%)	18 (100%)	0 (0%)	12 (100%)	0 (0%)	48 (100%)
w	S	0 (0%)	18 (100%)	1 (5.6%)	17 (94.4%)	0 (0%)	12 (100%)	1 (2.1%)	47 (97.9%)

b – blanks; l letters; s – shapes; w –words; C – common unknown addends task (e.g., $y + y = 12$); D – different unknown addends task (e.g., $a + b = 12$); N – No solution task (e.g., $x + 6 = x$)

Since the purpose of the confirmatory tasks was to determine if the student responses changed from the first interview to the second I determined the percentage point change in coding results by subtracting the first percentage from interview one from the corresponding percentage in interview two, see Table 18**.** From this table it is apparent that coded student responses for the variable comparison category were consistent from interview one to interview two. The only change for the entire population was a 2.1 percentage point increase in students treating the variables as the same and a 2.1 percentage point decrease in students treating the variables as different when the variable was represented in words. There was an 11.1 percentage point increase in students treating the variables as the same and an 11.1 percentage point decrease in fourth grade students treating the letters as different in the common unknown addends task. There was also a 16.7 percentage point decrease in students treating the variables as the same and a 16.7 percentage point increase in sixth grade students treating the letters as different in the common unknown addends task. These were due to a switch of one student response for each.

Table 18

Percentage Point Change in Variable Comparison Coding From Interview One to Interview Two

		4th grade		**5th grade**		**6th grade**		**Total**	
		Same	***Diff***	***Same***	***Diff***	***Same***	***Diff***	***Same***	***Diff***
b	C	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
	D	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
l	C	11.1%	-11.1%	0.0%	0.0%	-16.7%	16.7%	0.0%	0.0%
s	C	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
	D	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
w	D	0.0%	0.0%	5.6%	-5.6%	0.0%	0.0%	2.1%	-2.1%

b – blanks; l letters; s – shapes; w –words; C – common unknown addends task (e.g., $y + y = 12$); D – different unknown addends task (e.g., $a + b = 12$); N – No solution task (e.g., $x + 6 = x$)

Solution type across representations of variables. In this section, I report on students' solution types, whether they provided multiple solutions, single solutions, or no solution for each variable for each task. While the previous section examined whether the students treated the two variables in each task as the same or different variables it was also important to determine if the students provided multiple, single, or no solutions for each variable for each task.

Table 19 shows the results for the solution type responses by representation of the variable and core mathematical task. As demonstrated in the first section of the table, students across all three grade levels treated the two blanks in each equation (i.e., ___ + ___ = 5 and ___ + ___ = 7) as having multiple possible values except for the two fourth graders discussed in the previous section. Therefore, other than these two students apparent interpretation of the problems, students consistently provided multiple values for the blanks across all core mathematical tasks.

The results from the section of the table, showing the representation of the variable as a letter, demonstrate that when I represented the variable with letters, students' solution types were highly consistent. Sixty–seven of all students providing multiple values for each letter, four providing single values for each variable**,** and one student stating that the problem did not work (i.e., no solution). Students did not appear to distinguish between equations where the same letter was used (i.e., $y + y$ =12) and different letters were used (i.e., $a + b$ =12) in that they supplied the solutions 6 and 6 as well as pairs of different numbers such as 10 and 2. Only three students stated that the former equation had a single solution and none stated that the latter had a single solution.

Table 19

Solution Type Across Representation of Variables

		4th grade			5th grade			6th grade			Total		
		Multiple Values	***Single Value***	***No Solution***	***Multiple Values***	***Single Value***	***No Solution***	***Multiple Values***	***Single Value***	***No Solution***	***Multiple Values***	***Single Value***	***No Solution***
b	C	9 (100%)	0 (0%)	0 (0%)	9 (100%)	0 (0%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	24 (100%)	0 (0%)	0 (0%)
	D	9 (100%)	0 (0%)	0 (0%)	9 (100%)	0 (0%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	24 (100%)	0 (0%)	0 (0%)
	N	7 (77.8%)	2 (22.2%)	0 (0%)	9 (100%)	0 (0%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	22 (91.7%)	2 (8.3%)	0 (0%)
l	C	7 (77.8%)	2 (22.2%)	0 (0%)	9 (100%)	0 (0%)	0 (0%)	5 (83.3%)	1 (16.7%)	0 (0%)	21 (87.5%)	3 (12.5%)	0 (0%)
	D	9 (100%)	0 (0%)	0 (0%)	9 (100%)	0 (0%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	24 (100%)	0 (0%)	0 (0%)
	N	7 (77.8%)	1 (11.1%)	1 (11.1%)	9 (100%)	0 (0%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	22 (91.7%)	1 (4.2%)	1 (4.2%)
s	C	9 (100%)	0 (0%)	0 (0%)	9 (100%)	0 (0%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	24 (100%)	0 (0%)	0 (0%)
	D	9 (100%)	0 (0%)	0 (0%)	9 (100%)	0 (0%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	24 (100%)	0 (0%)	0 (0%)
	N	7 (77.8%)	1 (11.1%)	1 (11.1%)	9 (100%)	0 (0%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	21 (87.5%)	1 (4.2%)	1 (4.2%)
w	C	0 (0%)	9 (100%)	0 (0%)	0 (0%)	9 (100%)	0 (0%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	24 (100%)	0 (0%)
	D	8 (88.9%)	1 (11.1%)	0 (0%)	8 (88.9%)	1 (11.1%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	22 (91.7%)	2 (8.3%)	0 (0%)
	N	0 (0%)	2 (22.2%)	7 (77.8%)	0 (0%)	0 (0%)	8 (88.9%)	0 (0%)	0 (0%)	4 (66.7%)	0 (0%)	2 (8.3%)	19 (79.2%)

b – blanks; l letters; s – shapes; w –words; C – common unknown addends task (e.g., $y + y = 12$); D – different unknown addends task (e.g., $a + b = 12$); N – No solution task (e.g., $x + 6 = x$)

As shown in the section of Table 19 showing the results for the representation of the variable as a shape, when shapes were used as the representation of the variables (e.g., □ + △ = 12), the results were more homogeneous with all but two students supplying multiple values for each variable in each task. Two fourth graders' responses produced the lone exceptions. Sally, stated that the no solution task △ + 6 = △ had no solution applying the same meaning to letters, as reported earlier, as she did for shapes. The other fourth grader, Brett, interpreted each of the four formats of the problem with the missing addend (e.g., □ + △ = 12) as having the same value, six in this case.

Finally, as seen in the section for the use of words as variables in Table 19, all students gave single value solutions for the common unknown addends word problem (i.e., Shakira and Tim have the same number of gummy bears. Together they have 12 gummy bears.). Of students providing a codeable response, all but two students provided multiple solutions for the different unknown addends problem (i.e., Together Tom and Anne have 12 feet of ribbon) and no solution for the no-solution problem (i.e., I start with some number then add 6 and get the same number that I started with). In each instance, the students provided a single value for each variable.

Table 20 shows the results of the solution type responses for the confirmatory tasks broke down by representation of the variable and then core mathematical task. Student consistently provided multiple values for each variable across all of the confirmatory tasks. Of the 240 total coded responses, only four students provided single values for the variables, one each for the common unknown addends tasks when the variable was represented with a letter, $y + y = 7$ with a solution of 3½ and 3½, and a shape, 8 = □ + □ with a solution of 4 and 4. These occurred only at the fourth grade level. The other two instances occurred for the different unknown addends tasks when the variable was represented with words (i.e., I am thinking of two numbers. When I add these two numbers together, I get 7), one each at fourth and fifth grade.

Table 20

Confirmatory Task Solution Type Coding Across Representations of Variables

		4th grade			5th grade			6th grade			Total		
		Multiple Values	***Single Value***	***No Solution***	***Multiple Values***	***Single Value***	***No Solution***	***Multiple Values***	***Single Value***	***No Solution***	***Multiple Values***	***Single Value***	***No Solution***
b	D	18 (100%)	0 (0%)	0 (0%)	18 (100%)	0 (0%)	0 (0%)	12 (100%)	0 (0%)	0 (0%)	48 (100%)	0 (0%)	0 (0%)
	S	18 (100%)	0 (0%)	0 (0%)	18 (100%)	0 (0%)	0 (0%)	12 (100%)	0 (0%)	0 (0%)	48 (100%)	0 (0%)	0 (0%)
l	D	6 (66.7%)	1 (11.1%)	2 (22.2)%	9 (100%)	0 (0%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	21 (87.5%)	1 (4.2%)	2 (8.3%)
s	D	8 (88.9%)	1 (11.1%)	0 (0%)	9 (100%)	0 (0%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	23 (95.8%)	1 (4.2%)	0 (0%)
	S	18 (100%)	0 (0%)	0 (0%)	18 (100%)	0 (0%)	0 (0%)	12 (100%)	0 (0%)	0 (0%)	48 (100%)	0 (0%)	0 (0%)
w	S	17 (94.4%)	1 (5.6%)	0 (0%)	17 (94.4%)	1 (5.6%)	0 (0%)	12 (100%)	0 (0%)	0 (0%)	46 (95.8%)	2 (4.2%)	0 (0%)

b – blanks; l letters; s – shapes; w –words; C – common unknown addends task (e.g., $y + y = 12$); D – different unknown addends task (e.g., $a + b = 12$); N – No solution task (e.g., $x + 6 = x$)

Since the purpose of the confirmatory tasks was to determine if the student responses changed from the first interview to the second, I determined the percent change in coding results from the first interview to the second, see Table 21**.** From this table it is apparent that coded student responses for the solution type category were consistent from interview one to interview two. A small change occurred in the common unknown addends tasks with the variable represented with letters, $y + y = 7$, with an 8.3 % decrease in single value solutions and an 8.3 % increase in students responding that there was no solution. Further, when the variable was represented with shapes, a 4.2 percentage point decrease was found in multiple value solutions (e.g., $y + y = 7$ has solutions of 7 and 0, 6 and 1, 5 and 2, and 3 and 4) and an 4.2 percentage point increase in students responding that there was no solution. The only other change occurred for the different unknown addend task was where I represented the variable with words, a 4.1 percentage point decrease in single value solutions and a 4.1 percentage point increase in students providing multiple solutions.

Table 21

Percentage Point Change in Solution Type Coding Across Representations of Variables from Interview One to Interview Two

		4th grade			**5th grade**			**6th grade**			**Total**		
		Multiple Values	***Single Value***	***No Solution***	***Multiple Values***	***Single Value***	***No Solution***	***Multiple Values***	***Single Value***	***No Solution***	***Multiple Values***	***Single Value***	***No Solution***
b	C	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
	D	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
l	C	-11.1%	-11.1%	22.2%	0.0%	0.0%	0.0%	16.7%	-16.7%	0.0%	0.0%	-8.3%	8.3%
s	C	-11.1%	11.1%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	-4.2%	4.2%	0.0%
	D	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
w	D	5.5%	-5.5%	0.0%	5.5%	-5.5%	0.0%	0.0%	0.0%	0.0%	4.1%	-4.1%	0.0%

b – blanks; l letters; s – shapes; w –words; C – common unknown addends task (e.g., $y + y = 12$); D – different unknown addends task (e.g., $a + b = 12$); N – No solution task (e.g., $x + 6 = x$)

Variable values across representations of variables. This final category examined if the student provided the same and/or different values for the two variables in each task. In contrast to the previous categories, student responses were coded with both codes if they provided the same value to each variable *and* different values for each variable. This category, along with the variable comparison category, provided the necessary information to determine if students who treated the variables as different believed that they *must* also have different values or if they could have the same value.

Table 22 displays the results for the variable value responses by representation of the variable and core mathematical task. Students across all three grade levels treated the two blanks in each equation as being able to take on different values. However, the equation ___ + ___ = 12, used for the common unknown addends and different unknown addends tasks resulted in all student providing single values for each blank (i.e., 6 and 6). Thus, every student in the study produced solutions where the two blanks took on the same and different values from each other for the common unknown addends and different unknown addends tasks when a blank was the representation of the variable.

As shown in the first section of Table 22, the no solution version of the equation with blanks, ____ + 6 = ____, resulted in all 24 students providing only different values for the two blanks. No students provided the same value for the two blanks, which may be a result of the structure of the problem rather than the students meaning for the blanks since it is not possible to add six to a number and get the same number even if the blanks were interpreted as having to be the same number.

As shown in the second section of Table 22, when letters were used as the representation of the variable, student responses were consistent for both the different unknown addends task $a + b = 12$ and no solution tasks $x + 6 = x$. For the different unknown addends task, $a + b = 12$, every student provided the same value (i.e., 6 and 6) for a and b as well as different values for a and b (e.g., 10 and 2). For the no solution task, $x + 6 = x$, all students provided only different values for the two letter representations. As was the case with blanks, this may be more a result of the structure of the problem than the students meaning for the letters since it is not possible to add six to a number and obtain the same number even if the letters are interpreted as representing the same number.

Table 22

Value Comparison Across Representation of Variables

		4th grade		5th grade		6th grade		Total	
		VAL_ Diff	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***
b	C	9 (100%)	9 (100%)	9 (100%)	9 (100%)	6 (100%)	6 (100%)	24 (100%)	24 (100%)
	S	9 (100%)	9 (100%)	9 (100%)	9 (100%)	6 (100%)	6 (100%)	24 (100%)	24 (100%)
	N	9 (100%)	9 (100%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)	0 (0%)
l	C	7 (77.8%)	9 (100%)	9 (100%)	9 (100%)	5 (83.3%)	6 (100%)	21 (87.5%)	24 (100%)
	S	9 (100%)	9 (100%)	9 (100%)	9 (100%)	6 (100%)	6 (100%)	24 (100%)	24 (100%)
	N	8 (88.9%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	23 (95.8%)	0 (0%)
s	C	9 (100%)	9 (100%)	9 (100%)	9 (100%)	6 (100%)	6 (100%)	24 (100%)	24 (100%)
	S	9 (100%)	8 (88.9%)	9 (100%)	9 (100%)	6 (100%)	5 (83.3%)	24 (100%)	22 (91.7%)
	N	8 (88.9%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	23 (95.8%)	0 (0%)
w	C	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
	S	8 (88.9%)	9 (100%)	8 (88.9%)	9 (100%)	6 (100%)	6 (100%)	22 (91.7%)	24 (100%)
	N	0 (0%)	1 (11.1%)	0 (0%)	0 (0%)	0 (0%)	0 (0%)	0 (0%)	1 (4.2%)

b – blanks; l letters; s – shapes; w –words; C – common unknown addends task (e.g., $y + y = 12$); D – different unknown addends task (e.g., $a + b = 12$); N – No solution task (e.g., $x + 6 = x$)

From the third section of Table 22, when letters were used as the representation of the variables for the common unknown addends tasks, $y + y = 12$, every student provided the same value, 6 and 6, and all but three students provided different values for the two *y*s. Three students provided the same value for the two *y*s, two from fourth grade and one from sixth. I identified that these same three students treated the *y*s as the same variable for the variable comparison category and provided a single solution in the solution type category as reported above. All other students treated the *y*s as different variables, providing different values for each, for the variable comparison category and provided multiple solutions as reported above. Therefore, of the 24 coded student responses for $y + y = 12$, three paralleled a normative algebraic solution.

In the third section of Table 22, when shapes were used as the representation of the variable, student responses were consistent for both the common unknown addends task, $\square + \square = 12$. All students provided the same and different values for the two squares. For the no solution task, $\triangle + 6 = \triangle$, all students provided different values for the triangles. One student, Sally, indicated that no numbers worked for this task and therefore she did not assign a value to the variables. This resulted in eight of the nine fourth graders providing different values for the triangles. Even though their responses were consistent, they did no parallel a normative algebraic solution in that they did not correspond to the algebraic convention that the same variable in the same equation represented the same value. A normative response for the common unknown addends task would have the same value for the squares and the no solution task would be not be coded as no numerical values for the triangles make the equation true.

In the final section of Table 22**,** when words were used as the representation of the variable in a word problem, student responses were consistent for the common unknown addends problem (i.e., Shakira and Tim have the same number of gummy bears. Together they have 12 gummy bears) with every student providing only the same value for both variables (i.e., 6 and 6). For the different unknown addends task (i.e., Together Tom and Anne have 12 feet of ribbon), all 6 sixth graders provided both the same and different values for the variable, see Table 22. For the fourth and fifth graders‘ responses, eight of nine students provided different values for the variables and nine of nine provided the same value for both variables, see Table 22. In each case, the difference between the numbers of responses providing different values and the same value was due to one student at each grade providing a single value solution. In other words, these two students believed that the two addends had to be the same even though this was not stated or implied in the task. It was not clear from the interviews why the students interpreted the task this way. It is possible that they inferred this from other tasks.

Table 23 displays the results for the variable value responses for the confirmatory tasks by representation of the variable and core mathematical task. In addition, Table 4.20 displays the percentage point differences from interview one to interview two for the overlapping categories.

Table 23

Variable Value Codings by Representation of the Variable for Interview Two Confirmatory Tasks

		4th grade		**5th grade**		**6th grade**		**Total**	
		VAL_ Diff	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***
b	C	18 (100%)	5 (27.8%)	18 (100%)	3 (16.7%)	12 (100%)	10 (83.3%)	48 (100%)	18 (37.5%)
	S	18 (100%)	5 (27.8%)	18 (100%)	3 (16.7%)	12 (100%)	10 (83.3%)	48 (100%)	18 (37.5%)
l	C	6 (66.7%)	5 (55.6%)	9 (100%)	5 (55.6%)	6 (100%)	5 (41.7%)	21 (87.5%)	15 (62.5%)
s	C	8 (88.9%)	9 100%)	9 (100%)	9 (100%)	6 (100%)	6 (100%)	23 (95.8%)	24 (100%)
	S	18 (100%)	10 (55.6%)	18 (100%)	13 (72.2%)	12 (100%)	11 (91.7%)	48 (100%)	34 (70.8%)
w	S	17 (94.4%)	10 (55.6%)	17 (94.4%)	11 (61.1%)	12 (100%)	9 (75.0%)	46 (95.8%)	30 (62.5%)

b – blanks; l letters; s – shapes; w –words; C – common unknown addends task (e.g., $y + y$ =12); D – different unknown addends task (e.g., $a + b$ =12); N – No solution task (e.g., $x + 6 = x$)

It is evident that a decrease occurred in the number of students providing the same value for both variables for nearly all tasks. The two tasks where the representation of the variable was a shape change 0% for the common unknown addend (e.g., $y + y = 7$) and decreased 20.9% for the different unknown addends tasks (e.g., 12 = □ + △) from interview one to interview two. The common unknown addend confirmatory task with shapes as the representation of the variable was an even sum task (i.e., 8 = □ + □), as were the corresponding tasks from interview one. For these similar tasks, no change occurred in the percent of students responses providing the same value for both variables, and a slight decrease of 4.7 percentage points occurred for responses providing different values for both variables.

In examining the student responses it appears that the change in the sum from an even to an odd sum influenced the values students assigned to missing addends. For example, when Mark, a fourth grader, was asked to solve the equation $y + y = 7$ he stated, ―Actually, you can‘t do this because the *y'*s, like the seven is odd and two of the same numbers can‘t equal seven." Two important points need to be made regarding Mark‘s response. First, from his response we can infer that he believes that the *y*‘s must be the same value, a meaning he did not consistently apply to other representations. Second, his interpretation of what it means for a number to be odd influenced his response.

Other students provided similar responses but did not state that the *y*‘s had to be the same value. For example, Paul, a sixth grader, provided multiple solution pairs for $y + y = 7$ including 3 and 4, 6 and 1, and 7 and 0. When asked if the two numbers could be the same value he stated that they could not because, ―seven isn‘t an even number." Paul provided a similar response to this task in terms of his view of the two *y*‘s taking on the same value but had different meanings for the variables.

Tricia, another sixth grader, demonstrated a similar interpretation of the variables as Paul, supplying multiple solutions to $y + y = 7$. However, her response to whether the two *y's* could be the same value differed from Paul and Mark as she provided 3½ and 3½ as a solution. Therefore, her interpretation for the two *y*‘s was that they could be the same or different and her view of values for the domain of the variables were not limited to the set of whole numbers as they were for Mark and Paul.

Finally, George, a fourth grader, held a similar meaning for the variables in $y + y = 7$ as Mark, and a similar meaning for odd as Tricia. However, he was unsure when solutions could be whole numbers, integers, or negative numbers.

M: What do the *y*'s mean in that one.
G: They can represent a number and they cannot be the same number unless you are using fractions.
M: But if I was using fractions, they could be the same numbers?
G: Yeah, well you can't have 3 and 3, that's 6. Can't have 4 and 4, that's 8 but there's no two [values] between that. So you can use fractions you could do three-and-a-half and then plus three-and-a-half, or three and two-fourths and three-and-two-fourths.
M: So are there any numbers that first *y* could not be?
G: If you're not using negatives then they couldn't be any number under zero, and any number over seven.
M: What if I could use negatives?
G: Then you could put any negative you want, but after that you would have to have a certain other number.
M: To get the other one?
G: Yeah.
M: So how do you decide in a problem whether or not you can use negatives or fractions?
G: Well, it depends on what kind of problems you're doing, what you're learning at that point. You could use it pretty much any time unless you are supposed to do something else.

Based on these illustrative examples it appears that the inclusion of odd sums impacted the differences seen for the variable value responses from interview one to interview two.

Table 24 shows the percentage point change from interview one to interview two for the variable value responses for tasks with a common variable representation, task type, and core mathematical task across both interviews. Differences arose for responses indicating that the variables would have the same value for all but the common unknown addends tasks with the representation of the variable as a shape, see Table 24.

Table 24

Confirmatory Task Variable Value Coding Across Representation of Variables

Percentage Point Difference from Interview One to Interview One

		4th grade		5th grade		6th grade		Total	
		VAL_ Diff	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***
b	C	0.0%	-72.2%	0.0%	-83.3%	0.0%	-16.7%	0.0%	-62.5%
	D	0.0%	-72.2%	0.0%	-83.3%	0.0%	-16.7%	0.0%	-62.5%
L	D	-11.1%	-44.4%	0.0%	-44.4%	16.7%	-58.3%	0.0%	-37.5%
s	C	-11.1%	0.0%	0.0%	0.0%	0.0%	0.0%	-4.2%	0.0%
	D	0.0%	-33.3%	0.0%	-27.8%	0.0%	8.4%	0.0%	-20.9%
w	D	5.5%	-44.4%	5.5%	-38.9%	0.0%	-25.0%	4.1%	-37.5%

b – blanks; l letters; s – shapes; w –words; C – common unknown addends task (e.g., $y + y = 12$); D – different unknown addends task (e.g., $a + b = 12$); N – No solution task (e.g., $x + 6 = x$)

The different unknown addends with the variables represented, as shapes had no change in the percentage of students' responses providing different values for both variables. However, a decrease of 20.9 percentage points occurred in the students providing the same value for both variables**.** While all of the different unknown addend tasks with shapes as the representation of the variables had even sums for interview one, interview two consisted of one odd and one even sum task. When examined separately, task 4 ($12 = \square + \triangle$) resulted in different values provided for every student response and the same value by 22 of the 24 student responses (91.7%). However, tasks 12, $\triangle + \square = 7$, with an odd sum, resulted in different values provided for every student response and the same value provided by 12 of the 24 student responses (50%). Therefore, the smaller change from interview one to interview two for this category was partly the result of having both an odd and an even sum task in interview two.

Likewise, while all of the different unknown addend word problems in the first interview had even sums, the two different unknown addend word problems in the confirmatory tasks for interview two consisted of one with an odd sum and one with an even sum. When examined separately, task 1 (i.e., Juan and Alexa each have a piece of string. Together they have 16 inches of string), with an even sum, resulted in different values provided by 22 of the 24 student responses (91.7%) and the same value provided by 24 of the 24 student responses (100%). However, all student responses for tasks with an odd sum involved different values and six of the 24 student responses (25%) involved the same value. Therefore, the smaller change from interview one to interview two for this category was partly a result of having both an odd and an even sum task in interview two.

The common unknown addends with letters as the representation of the variable (i.e., $y + y = 7$) also had a greater decrease in the number of students who provided the same value for both variables than the tasks with shapes as the representation of the variables and the same as the different unknown addends word problems. However, only a single task involved an odd sum. All but one of the students (23 of 24) included different values as these students viewed the two *y*'s as being different variables (i.e., representing different referents). Of the 23 students providing different values for both variables, all also provided multiple solutions. However, of these 23 students, nine provided different values for the two *y*'s (e.g., 5 and 2, and 3 and 4). The remaining fourteen students provided the same *and* different values for the two variables, indicating that the *y*'s could take on different values or the same value. Therefore, of the 15 student responses the same value for both variables, fourteen provided different values for both variables Only one student, the fourth grader George, treated the two *y*'s as *only* having the same value.

Summary. In this section I reviewed the data for each of the three categories: variable comparison, solution type, and variable value, for the initial eleven tasks from interview one and the confirmatory tasks from interview two across representations of the variables. When the representation of the variable was blanks, letters, and shapes for the initial eleven tasks from interview one, students consistently treated the representation of the variable as different variables for the initial eleven tasks from interview one and the confirmatory tasks from interview two regardless of the core mathematical task being modeled. In interview one, only five of 216 students treated the two variables as the same, providing a single common value for both variables. Of these five, three were for the common unknown addends tasks with the variable represented with letters (e.g., $y + y = 12$). The other two were both for the common unknown addend task, one when I represented the variable with letters (e.g., $a + b = 12$) and the other represented with shapes (e.g., $\square + \triangle = 12$). I further demonstrated that this coding pattern was consistent from interview one to interview two.

However, when I represented the variables with words in a word equation, all students treated the variable as the same for the common unknown addend and no solution tasks. Further, all student treated the variables as different for the different unknown addends tasks (e.g., Together Tom and Anne have 12 feet of ribbon.). However, since words as representations of the variable only occurred in word problems and the blanks, letters, and shapes only occurred in equations the task type may also play a contributing role in this distinction. This is explored further in the section on students meaning of variables across task types.

When I represented the variable with blanks, letters, and shapes, students consistently produced multiple solutions for both the initial eleven tasks from interview one and the confirmatory tasks from interview two regardless of the core mathematical task modeled. In interview one, only seven of 216 student responses involved a single solution. Of these seven, four were for the no solution tasks (e.g., $x + 6 = x$) with the other three all for the common unknown addend task with the variable represented with letters (e.g., $a + b = 12$). Of these, only one student stated there was no solution, which was for the no solution task with the variable represented with shapes. I further demonstrated that this coding pattern was consistent from interview one to interview two.

However, when the representation of the variable was presented in words students consistently provided single solutions for the common unknown addends task, multiple solutions for the different unknown addend tasks, and no solution for the no solution tasks. Therefore, the representation of the variable appears to impact the type of solutions students provided (i.e., multiple solutions, single solution, or no solution). However, since words as representations of the variable occurred only in word problems and the blanks, letters, and shapes only occurred in equations, the task type may also contribute to this distinction. This is explored further in the section on student meaning of variables across task types. I further demonstrate that student responses were consistent from interview one to interview two.

In contrast to the prior two categories, student responses for the variable value category were inconsistent across representations of the variables from interview one to two. In interview one, when the representation of the variable was blanks, letters, or shapes coded students were very consistent for the same core mathematical tasks. For these representations of the variables, students consistently provided the same and different values for the two variables for the common unknown addends and the different unknown addends tasks. They also produced only different values for the no solution core mathematical tasks (e.g., $x + 6 = x$).

However, student interpretations changed when the representation of the variable involved words. All students provided the same values for both variables in the common unknown addends tasks. All students provided the same values for both variables *and* all but two provided the different values for both variables in the different unknown addends tasks (e.g., $a + b = 12$). Finally, for the no solution core mathematical task only one student provided a solution where the values were the same for both variables.

In interview two, I found no change in the percent of student responses providing different values for the two variables when the representation of the variable was a blank or a letter. However, a decrease of 62.5 percentage points was found for the common unknown addends and different unknown addends tasks when the variable was presented as blanks. In addition, a 37.5 percentage point decrease occurred for the different unknown addends tasks when the variable was presented as letters. Less change occurred when the representation of the variable was letters with a 4.2 percentage point decrease in different solutions provided for both variables for the common unknown addends tasks and a 20.9 percentage point decrease in the different unknown addends tasks. Finally, when the representation of the variables were presented in words a 4.1 percentage point increase was found in different solutions provided for both variables and a 37.5

percentage point increase in the same solutions provided for both variables responses for the different unknown addends tasks. As noted earlier, when I presented the even sum confirmatory tasks, the coding results were consistent from interview one to interview two for the variable value category. The differences occurred for the odd sum confirmatory tasks.

However, not every combination of representation of variable, task type, and core mathematical were included in the confirmatory tasks. Therefore, the only comparisons made were for blanks and shapes for both the common unknown addends (e.g., ___ + ___ = 12 and □ + □ = 12) and different unknown addends task (e.g. $a + b = 12$ and □ + △ = 12), and words for the different unknown addends tasks (e.g., Together Tom and Anne have 12 feet of ribbon).

Student meaning of representations of variables across task types (research question c)

In the following section, I examine the data to determine students' meaning of variables across task types. I provide the coded student responses for each of the three categories: variable comparison, solution type, and variable value, across the task types (i.e., equations and word problems) for the initial eleven tasks from interview one, and the confirmatory tasks from interview two. For each of these sets of tasks I sorted the codes by the task type and then by the core mathematical task.

Variable comparison. In order to determine how students interpreted variables across various task types for tasks with common mathematical structures, the initial eleven tasks from interview one were resorted by the task type (equation or word problem) and then the core mathematical task (common unknown addends, different unknown addends, or no solution). These were then analyzed for each of the three coding categories: variable comparison, solution type, and variable value.

Table 25 shows the results for the variable comparison student responses broke down by task type and then core mathematical task. When totaling the results for the total column in Table 25, the student responses was highly consistent for tasks posed as equations regardless of the core mathematical task with 211 responses treating the two representations of the variables as different and only five treating them as the same**.** All student responses treated the two variables as different for the different unknown addends with sum known task (e.g., $a + b = 12$) when presented as an equation, which parallels a normative algebraic responses. However, the student responses were similar for the common unknown addends tasks (e.g., y + y = 12) and no solution task (e.g., $x + 6 = x$) when presented as an equation. For these two core mathematical tasks, student responses did not parallel an algebraically normative solution but were highly consistent with the responses for the different unknown addends tasks. This provides evidence that, in general, students did not view two variables in the same equation as the same when presented with the same representation. Further, they did not distinguish between equations where the variables were the same from equations where the variables were different.

As discussed previously, a single fourth grade student, Brett, attributed to the four student responses that involved treating the variables as the same. He responded this way for, two each for the common unknown addends and no solution tasks. The single instance of treating the two variables as the same occurred at the sixth grade level with

Maggy's response that the letters in $y + y = 12$ were the same, which she did not extend to other representations of the variable for the same core task.

Table 25

Variable Comparison Across Task Type

		4th grade		5th grade		6th grade		Total	
		Same	***Diff***	***Same***	***Diff***	***Same***	***Diff***	***Same***	***Diff***
E	C	2 (7.4%)	25 (92.6%)	0 (0%)	27 (100%)	1 (5.6%)	17 (94.4%)	3 (4.2%)	69 (95.8%)
	S	0 (0%)	27 (100%)	0 (0%)	27 (100%)	0 (0%)	18 (100%)	0 (0%)	72 (100%)
	N	2 (7.4%)	25 (92.6%)	0 (0%)	27 (100%)	0 (0%)	18 (100%)	2 (2.8%)	70 (97.2%)
W	C	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)	0 (0%)
	S	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
	N	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)	0 (0%)

E- equation; W – word problem; C – common unknown addends task (e.g., $y + y = 12$); D – different unknown addends task (e.g., $a + b = 12$); N – No solution task (e.g., $x + 6 = x$)

The results for the word problems differed from those of the equations in two primary ways. First, student responses were consistent for tasks posed as word problems across each of the three core mathematical tasks. Second, the percent of student responses that involved treating the two variables as the same and different for the common unknown addend (i.e., Shakira and Tim have the same number of gummy bears. Together they have 12 gummy bears) and no solution tasks (i.e., I start with some number then add 6 and get the same number that I started with) differed for the equations as compared to the word problems. The percent of student responses that involved treating the representations of the variables as the same and different for the common unknown addend when presented as an equation and word problem, were 4.2% and 95.8%, and 100% and 0% respectively. Similarly, the percent of responses that involved treating the representations of the variables as the same and different for the no solution task when presented as an equation and word problem were 2.8% and 97.2%, and 100% and 0% respectively.

Table 26 shows the results of the task type student responses for the confirmatory tasks broke down by task type and then core mathematical task. These responses demonstrate that students consistently treated the two variables as different across the confirmatory tasks. Only four instances of students treating the variables as the same arose in the analysis, three for the common unknown addend task and one for the word problem, see Table 26. Three student responses, all fourth grade, involved treating the two representations of the variables as the same for the common unknown addends tasks where the task was presented as an equation, see first row of Table 26. The other instance occurred for one fifth grader who treated the two representations of the variables as the same for the different unknown addends task presented as a word problem (e.g., I am thinking of two numbers. When I add these two numbers together, I get seven).

Table 26

Confirmatory Task Variable Comparison Coding Across Task Types

		4th grade		**5th grade**		**6th grade**		**Total**	
		Same	***Diff***	***Same***	***Diff***	***Same***	***Diff***	***Same***	***Diff***
E	C	3 (8.3%)	33 (91.7%)	0 (0%)	36 (100%)	0 (0%)	24 (100%)	3 (3.1%)	93 (96.9%)
	D	0 (0%)	36 1(00%)	0 (0%)	36 (100%)	0 (0%)	24 (100%)	0 (0%)	96 (100%)
W	D	0 (0%)	18 (100%)	1 (5.6%)	17 (94.4%)	0 (0%)	12 (100%)	1 (2.1%)	47 (97.9%)

b – blanks; l letters; s – shapes; w –words; C – common unknown addends task (e.g., $y + y = 12$); D – different unknown addends task (e.g., $a + b = 12$); N – No solution task (e.g., $x + 6 = x$)

Since the purpose of the confirmatory tasks was to determine if the student responses changed from the first interview to the second, I calculated the percentage point change in coding results from the first interview to the second, see Table 27**.** From this table it is apparent that student responses for the variable comparison category were consistent from interview one to interview two. The only changes were a 1.1 percentage point increase in students viewing the two representations of the variable as the same and a 1.1 percentage point decrease in those viewing them as different. Further the common unknown addends equation and a 2.1 percentage point increase in responses treating the two representations of the variable as the same and a 2.1 percentage point decrease in those treating them as different for the different unknown addends word problem.

Table 27

Percentage Point Change in Task Type Coding from Interview One to Interview Two

		4th grade		**5th grade**		**6th grade**		**Total**	
		Same	***Diff***	***Same***	***Diff***	***Same***	***Diff***	***Same***	***Diff***
E	C	-0.9%	0.9%	0.0%	0.0%	5.6%	-5.6%	1.1%	-1.1%
	D	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
W	D	0.0%	0.0%	5.6%	-5.6%	0.0%	0.0%	2.1%	-2.1%

E – equation; W – word problem; C – common unknown addends task (e.g., $y + y = 12$); D – different unknown addends task (e.g., $a + b = 12$); N – No solution task (e.g., $x + 6 = x$)

Solution type. To determine the types of solutions students provided across various task types with common mathematical structures, the initial eleven tasks were resorted by the task type (equation or word problem) and the core mathematical task (common unknown addends, different unknown addends, or no solution), see Table 28.

Nearly all variation in student responses occurred at the fourth grade level. A single difference occurred at the fifth grade level where Tim‘s response of a single value for the different unknown addends equations (e.g., 12 = □ + △) as described earlier. A single difference at the sixth grade level was Maggy‘s response supplying multiple solutions for the common unknown addends equations.

The remaining differences occurred at the fourth grade level. The only categories where all fourth grade responses were coded the same was the different unknown addends equation where the 27 responses provided multiple solutions for each variable, and the word problem for the common unknown addends where the nine responses provided a single solution for the variables.

In problem 7, an equation with no solution with shapes as the representation of the variables, one of the fourth grade students, Sally, interpreted the two triangles in △ + 6 = △ as identifying the same value, as demonstrated in the exchange below.

> M: What do those triangles mean?
> S: The same number.
> M: So what numbers would work for that?
> *(long pause)*
> M: Tell me what you‘re thinking.
> S: Well, I was thinking one like maybe zero but zero wouldn‘t work either. I couldn‘t put six there cause then the answer would be twelve. So, I‘m trying to think of one that will fit together. I don‘t think that can work.
> M: Why couldn‘t it work?
> S: Because any number you put here [first triangle] cannot be the same here [second triangle] because you are adding six to it.

Sally applied the same meaning for the variables in problem 10 ($x + 6 = x$) which is the same problem type but with the representation of the variables involved letters instead of shapes △ + 6 = △. In addition, she also stated that the corresponding word problem had no solutions. Therefore, she applied a consistent interpretation for these three problems. However, she did not employ the same meaning when the representation of the variables involved blanks, ___ + 6 = ___. Instead, she treated the blanks as having different values (e.g., <u>6</u> + 6 = <u>*12*</u>). Interestingly, she was the only student across all three grades to apply this meaning to each of these forms of the problem. However, all but two of the students in the study stated that the corresponding word problem had no solution. Therefore, it appears that students across the grades did not interpret the word problem in the same way as the corresponding equations.

The two discrepant interpretations for the common unknown addend equation (e.g., ___ + ___ = 12 and $y + y = 12$) occurred when the representation of the variable was a letter. Both students treated the common letter in the equation $y + y = 12$ as the same value and determined that the only solution for this equation involved assigning a value of 6. It is worth noting that these two students were the only ones who applied this interpretation across all of common unknown addends equations regardless of the representation of the variable. Even Sally and Maggy did not carry the meaning they had for the letters to the equivalent equations with the representation of the variables as blanks or shapes. However, all fourth graders, and in fact all students in the study, viewed the two variables as the same and assigning the same value when the common unknown addend task was presented as a word problem. In addition, the consistent response by all students for the common unknown addend word problem was normative while the consistent response for the equivalent equations were not normative. In fact, only 4.2% of all students gave a normative response to the common unknown addends equations (e.g., $y + y = 12$).

Table 28

Solution Type Across Representation of Variables

		4th grade			5th grade			6th grade			Total		
		Multiple Values	***Single Value***	***No Solution***	***Multiple Values***	***Single Value***	***No Solution***	***Multiple Values***	***Single Value***	***No Solution***	***Multiple Values***	***Single Value***	***No Solution***
E	C	25 (92.6%)	2 (7.4%)	0 (0%)	27 (100%)	0 (0%)	0 (0%)	17 (94.4%)	1 (5.6%)	0 (0%)	69 (95.8%)	3 (4.2%)	0 (0%)
	D	27 (100%)	0 (0%)	0 (0%)	27 (100%)	0 (0%)	0 (0%)	18 (100%)	0 (0%)	0 (0%)	72 (100%)	0 (0%)	0 (0%)
	N	20 (74.1%)	4 (14.8%)	2 (7.4%)	27 (100%)	0 (0%)	0 (0%)	18 (100%)	0 (0%)	0 (0%)	65 (90.3%)	4 (5.6%)	2 (2.8%)
W	C	0 (0%)	9 (100%)	0 (0%)	0 (0%)	9 (100%)	0 (0%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	24 (100%)	0 (0%)
	D	8 (88.9%)	1 (11.1%)	0 (0%)	8 (88.9%)	1 (11.1%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	22 (91.7%)	2 (8.3%)	0 (0%)
	N	0 (0%)	2 (22.2%)	7 (77.8%)	0 (0%)	0 (0%)	8 (88.9%)	0 (0%)	0 (0%)	4 (66.7%)	0 (0%)	2 (8.3%)	19 (79.2%)

E – equation; W - word problem; C – common unknown addends task (e.g., $y + y = 12$); D – different unknown addends task (e.g., $a + b = 12$); N – No solution task (e.g., $x + 6 = x$)

An examination of the totals from Table 28 demonstrates that responses across the grade levels for the different unknown addends tasks (e.g., $a + b = 12$ and Together Tom and Anne have 12 feet of ribbon) were also consistent regardless of the task type, but in contrast to the common unknown addends equations they were also normative. Every student‘s response for the equation task types, and all but two for the word problem, provided multiple solutions for the variables, see Table 28**.** The two anomalies for the word problem occurred with one fourth grader, Julie, and one fifth grader, Tim, who each provided a single value solution for the variables.

When comparing the responses for the equations with common unknown addends and different unknown addends tasks in Table 28 nearly all of the students provided multiple solutions for the variables, 95.8% and 100% respectively. This indicates that students did not distinguish between equations where the representations of the two variables were the same and when they were different, resulting in normative responses for the different unknown addends (e.g., $a + b = 12$) where this type of solution is normative.

The largest discrepancy in responses for solution type occurred for the no solution tasks. The response percentages for multiple solutions, single solutions, and no solutions were 90.3%, 5.6%, and 2.8% respectively for the equations and 0%, 8.3%, and 79.2% respectively for the word problems, see Table 28. In a similar pattern to the reversal of coding percentages between students providing multiple solution values and single solutions for the common unknown addends equations (i.e., ___ + ___ = 12 and $y + y = 12$) and word problem (i.e., Shakira and Tim have the same number of gummy bears. Together they have 12 gummy bears), the no solution equations (i.e., ___ + 6 = ____ and

$x + 6 = x$) and word problem (i.e., I start with some number then add 6 and get the same number that I started with) percentages switched between students providing multiple solutions and no solution, 90.3% and 79.2% respectively when switching from the equations to the word problems, see Table 28.

Differences arose for each of the forms of this task for the fourth grade students but only for the word problem version for fifth grade and sixth grade. The single fifth grade student who was responded differently, Tim, was unsure about how to interpret the word problem (i.e., I start with some number then add 6 and get the same number that I started with), as is demonstrated in the exchange below.

T: I don‘t know.
M: So what are you thinking about?
T: I don‘t know. I don‘t really get it.
M: What?
T: I don‘t get it.
M: So you don‘t get the question?
T: Well, I don‘t know the answer.
M: So, what were you thinking about?
T: I thought maybe zero but that wouldn‘t be the same, or twelve cause six plus six equals twelve.
M: So, what are you thinking about there. What is kind of hard there?
T: Because you can‘t have like a number then plus six and get that same number.
M: So you can‘t do that?
T: [*Shakes head no.*]
M: So are there any numbers that will work?
T: zero and zero.
M: So if you take zero plus six you get zero?
T: No.
M: So you just told me that you can‘t take a number and add six and get that same number, right?
T: [*Shakes head yes.*]
M: So would there be any numbers that work for this?
T: three and three.
M: So if I take three plus six do I get three?
T: [*Shakes head no.*]

Both sixth grade students who I did not code with one of the solution codes were unsure about the solution to the problem and did not provide a solution. One sixth grade student, Maggy, originally thought that zero was a solution but realized that she did not get the same number as the sum. When asked if she thought there would be any solutions for this task she said, ―maybe." The other student, Paul, was also unsure about the solution to this problem as demonstrated in the exchange below.

> M: So this one says I start with some number then add six and get the same number I started with. What can that number be?
> P: Zero.
> M: I start with zero and add six.
> P: Oh, I start with one number then add six and get.
> M: Tell me what you are thinking about.
> P: I don't know. I don't really know a number that you can add six and get the same number.
> M: Why not? What happens when you add six to a number?
> P: It becomes, if you added six to zero it'd be six and if you add six to one it would just . . . you just do six, and you add that number so six plus two would equal eight.
> M: So are there any numbers you can add six to and get the same number?
> P: (No response.)
> M: Not sure?
> P: No.

In each of these cases, the student did not provide a numerical solution and did not state that there would be no solution.

Therefore, for this set of tasks students' responses were highly consistent across each of the three core mathematical tasks when these tasks were presented as equations. In other words, the core mathematical tasks did not affect responses for the majority of the students. This resulted in only the different unknown addends equation (e.g., $a + b = 12$) responses as normative while all of the word problems for each of the three core mathematical tasks; common unknown addends (e.g., Shakira and Tim have the same number of gummy bears. Together they have 12 gummy bears), different unknown addends (e.g., Together Tom and Anne have 12 feet of ribbon), and no solution (e.g., I start with some number then add 6 and get the same number that I started with) being normative with student responses comprising 100%, 91.7%, and 79.2% of responses respectively, see Table 28. Therefore, the task type appeared to influence the solutions that students provided.

Table 29 shows the results of the solution type responses for the confirmatory tasks broke down by task type and then core mathematical task. Student responses consistently provided multiple solutions across all of the confirmatory tasks. Of the 150 responses from Table 29, four student responses provided single solutions for each variable, two each for the common unknown addends equations and different unknown addends word problems. Both of these instances occurred at fourth grade. Two student responses indicated that there were no solutions for the common unknown addends equations.

Table 29

Confirmatory Task Solution Type Coding Across Task Types

		4th grade			**5th grade**			**6th grade**			**Total**		
		Multiple Values	***Single Value***	***No Solution***	***Multiple Values***	***Single Value***	***No Solution***	***Multiple Values***	***Single Value***	***No Solution***	***Multiple Values***	***Single Value***	***No Solution***
E	C	32 (88.9%)	2 (5.6%)	2 (5.6%)	36 (100%)	0 (0%)	0 (0%)	24 (100%)	0 (0%)	0 (0%)	92 (95.8%)	2 (2.1%)	2 (2.1%)
	D	36 (100%)	0 (0%)	0 (0%)	36 (100%)	0 (0%)	0 (0%)	24 (100%)	0 (0%)	0 (0%)	96 (100%)	0 (0%)	0 (0%)
W	D	17 (94.4%)	1 (5.6%)	0 (0%)	17 (94.4%)	1 (5.6%)	0 (0%)	12 (100%)	0 (0%)	0 (0%)	46 (95.8%)	2 (4.2%)	0 (0%)

E – equation; W – word problem; C – common unknown addends task (e.g., $y + y = 12$); D – different unknown addends task (e.g., $a + b = 12$); N – No solution task (e.g., $x + 6 = x$)

Table 30

Percentage Point Change in Solution Type Coding Across Representations of Variables from Interview One to Interview Two

		4th grade			**5th grade**			**6th grade**			**Total**		
		Multiple Values	***Single Value***	***No Solution***	***Multiple Values***	***Single Value***	***No Solution***	***Multiple Values***	***Single Value***	***No Solution***	***Multiple Values***	***Single Value***	***No Solution***
E	C	3.7%	1.8%	-5.6%	0.0%	0.0%	0.0%	-5.6%	5.6%	0.0%	0.0%	2.1%	-2.1%
	D	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
W	D	-5.5%	5.5%	0.0%	-5.5%	5.5%	0.0%	0.0%	0.0%	0.0%	-4.1%	4.1%	0.0%

E – equation; W – word problem; C – common unknown addends task (e.g., $y + y = 12$); D – different unknown addends task (e.g., $a + b = 12$); N – No solution task (e.g., $x + 6 = x$)

Since the purpose of the confirmatory tasks was to determine if the student responses changed from the first interview to the second, I calculated the percentage point change in coding results from the first interview to the second, see Table 30**.** From this Table it is apparent that student responses for the solution type category were consistent from interview one to interview two. A small change occurred in the common unknown addends equation, a 2.1 percentage point decrease in students stating there was no solution and a 2.1 percentage point increase in students providing single values for both variables. The only other change occurred for the different unknown addends word problem, a 4.1 percentage point decrease in students providing multiple solutions and an 4.1 percentage point increase in students providing single value solutions.

Variable value responses across task types. Table 31 shows the results for the variable value responses by task type (i.e., equation or word problem) and then core mathematical task. When the tasks were presented as equations, student responses were consistent. The equations for the no solution core task resulted in all students providing different values for both variables, a non-normative response since no values will make these equations true when applying algebraic conventions to the representations of the variables. Nearly all student responses included both the same and different values for the two variables for the common unknown addends and different addends tasks: 95.8% and 100%, and 100% and 97.2% respectively.

The difference in the percentages for the student responses providing both the same and different values for both variables for the common unknown addends were due to the three students reported earlier: Jill, Sally, and Maggy who treated the variables as the same and assigned a single value to them when the equation was presented with letter representations of the variables. Other than these three students, all students assigned the same and different values to the two variables regardless of whether or not the two representations of the variables were the same or different.

Table 31

Variable Value Across Task Types

		4th grade		**5th grade**		**6th grade**		**Total**	
		VAL_ Diff	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***
E	C	25 (92.6%)	27 (100%)	27 (100%)	27 (100%)	17 (94.4%)	18 (100%)	69 (95.8%)	72 (100%)
E	D	27 (100%)	26 (96.3%)	27 (100%)	27 (100%)	18 (100%)	17 (94.4%)	72 (100%)	70 (97.2%)
E	N	25 (92.6%)	0 (0%)	27 (100%)	0 (0%)	18 (100%)	0 (0%)	70 (97.2%)	0 (0%)
W	C	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
W	D	8 (88.9%)	9 (100%)	8 (88.9%)	9 (100%)	6 (100%)	6 (100%)	22 (91.7%)	24 (100%)
W	N	0 (0%)	1 (11.1%)	0 (0%)	0 (0%)	0 (0%)	0 (0%)	0 (0%)	1 (4.2%)

E – equation; W – word problem; C – common unknown addends task (e.g., $y + y = 12$); D – different unknown addends task (e.g., $a + b = 12$); N – No solution task (e.g., $x + 6 = x$)

The difference in the percentages for students providing both the same and different values for the two variables for the different unknown addends were due to two student responses for the equation were shapes were used as the representation of the variable.

Sally, a fourth grader, viewed the two shapes in $\square + \triangle = 12$ as taking on different values and stated that these could not be the same values as demonstrated in the following exchange.

M: So what do the shapes mean for this one?
S: Different numbers.
M: So what numbers can I put in there that would make that work?

S: Different numbers, eight and four. I can't do six and six because they represent different numbers. Ten and twelve, eleven and one, five and six, wait, five and seven.

Likewise, Tricia, a sixth grader, treated the two shapes in $\square + \triangle = 12$ as different when comparing the variables and believed that they could not be the same values as demonstrated in the following exchange.

T: For these I think they have to be different numbers but they can still be any other number as long as they equal twelve. So it could be like nine and three, would be 12. So, you could put any number that was. I think maybe they are both just blanks and it doesn't matter if they are the same number or not but I know they are both just symbols you would use.
M: So at first you said something about they can't be the same number?
T: I don't know. I can't remember if they can't be the same number. Actually, I think they can be the same number, they are just different symbols. Yeah, they can be the same number because of the *a* and *b* one. They're like, it's kind of, maybe they can't be the same number because they are different and they probably would have used the same one. I think they might need to be different numbers but they can be any other number any other number it didn't really matter what they were as long as they equal twelve together.

While Sally and Tricia applied a normative algebraic convention to the shapes in terms of treating them as different values, they also exhibited the common misconception that different variables cannot be the same value. While they applied this meaning for the equation when the variables were shapes, they did not extend this meaning to the same core problem when the variables were letters.

When the same core tasks were presented as word problems the results for the different unknown addends were close to those for the equations with 91.7% and 100%, and 100% and 97.2% For providing different *and* the same values for the two variables respectively. Two students who believed that the answer to the word problem for the different unknown addends were the same (i.e., 6 and 6 alone) resulting in the difference in the percentages between students providing different values for the two variables, 91.7%, and those providing the same values for both variables, 100%.

However, the results for the common unknown addend (e.g., $y + y = 12$) and the no solution tasks (e.g., $x + 6 = x$) were very different when the task type is considered. As noted above, all student responses for the common unknown addend equation except two provided the same and different values for both variables. When I presented this same task as a word problem, no student provided different values for both variables and all student responses provided the same value for both variables**.** For the no solution equations, all coded responses, 97.2% of all responses, provided different values for both variables and none provided the same value for both variables. When I presented this same task as a word problem, only one student's response, Brett, provided the same value for both variables. This single response resulted from him switching the order of the addends implied in the problem to get the equation $6 + 0 = 6$. The rest of the students indicated that no solution for this task existed.

Table 32 shows the results for the variable value responses for the confirmatory tasks broke down by task type and then core mathematical task. In addition, Table 33 shows the percentage point differences from interview one to interview two for the overlapping categories.

Table 32

Variable Value Codings by Task Type for Interview Two Confirmatory Tasks

		4th grade		5th grade		6th grade		Total	
		VAL_ Diff	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***
E	C	32 (88.9%)	19 (52.8%)	36 (100%)	17 (47.2%)	24 (100%)	21 (87.5%)	92 (95.8%)	57 (59.4%)
	D	36 (100%)	15 (41.7%)	36 (100%)	16 (44.4%)	24 (100%)	21 (87.5%)	96 (100%)	52 (54.2%)
W	D	17 (94.4%)	10 (55.6%)	17 (94.4%)	11 (61.1%)	12 (100%)	9 (75.0%)	46 (95.8%)	30 (62.5%)

E – equation; W – word problem; C – common unknown addends task (e.g., $y + y = 12$); D – different unknown addends task (e.g., $a + b = 12$); N – No solution task (e.g., $x + 6 = x$)

The same pattern of a decrease in the percentage of responses providing the same value for both variables and little to no change in the percentage of student responses providing different values**,** as noted in the discussion thus far for the variable value category, is also evident when viewed by task type and core mathematical task. The examples and explanation for these differences in the prior sections are also applicable and relevant to the results when viewed by task type and core mathematical task.

Table 33

Variable Value Across Task Type Percentage Point Difference from Interview One to Interview Two

		4th grade		5th grade		6th grade		Total	
		VAL_ Diff	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***
E	C	-3.7%	-47.2%	0.0%	-52.8%	5.6%	-12.5%	0.0%	-40.6%
	D	0.0%	-54.6%	0.0%	-55.6%	0.0%	-6.9%	0.0%	-43.0%
W	D	5.5%	-44.4%	5.5%	-38.9%	0.0%	-25.0%	4.1%	-37.5%

E – equation; W – word problem; C – common unknown addends task (e.g., $y + y = 12$); D – different unknown addends task (e.g., $a + b = 12$); N – No solution task (e.g., $x + 6 = x$)

Summary. In the previous section I reviewed the data for each of the three categories: variable comparison, solution type, and variable value, for the initial eleven tasks from interview one and the confirmatory tasks from interview two across task types. In addition, I determined the percentage point change from interview one to interview two for each category across task types. In this section, I summarize these results.

When the task type involved equations, student responses indicated that they consistently treated the representation of the variables as taking on different variables for both the initial eleven tasks from interview one and the confirmatory tasks from interview two regardless of the core mathematical task modeled. In the first interview, only five of 216 student responses involved students viewing the two variables as the same, see Table 32. Of these five responses, three were for the common unknown addends equation. The other two were both for the no solution equations. I further demonstrated that this response pattern was consistent from interview one to interview two.

However, when I presented the task as a word problem, all student responses treated the two variables as the same for the common unknown addends and no solution word problems, see Table 32. Further, all student responses treated the two variables as different for the different unknown addends word problems. However, since words as representations of the variable only occurred in word problems and the blanks, letters, and shapes only occurred in equations, the representation of the variable might also play a contributing role in these results, see Table 32. This is explored further in the section on students meaning of variables across task types. I further demonstrate that student responses were consistent from interview one to interview two.

When I presented the tasks as an equation, student responses consistently included multiple solutions for both the initial eleven tasks from interview one and the confirmatory tasks from interview two regardless of the representation of the variable and the core mathematical task being modeled. In interview one, only four of 216 student responses involved a single solution. Of these four, two were for the common unknown addends equation (e.g., $y + y = 12$), both at fourth grade, and the other two for the different unknown addends word problem (e.g., Together Tom and Anne have 12 feet of ribbon), one each at fourth and fifth grade. Further, two student responses stated there was no solution for the common unknown addends equation, both at fourth grade, stating there was no solution. I further demonstrated that this coding pattern was consistent from interview one to interview two.

In contrast to the prior two categories, coded student responses for the variable value category were inconsistent across task types from interview one to two, see Table 33. In interview one, when the task involved an equation, student responses were consistent for the common unknown addends and different unknown addends tasks with nearly all students providing both the same and different values for the two variables 100% and 95.8%, and 97.2% and 100% respectively. For the no solution equation from interview one, no students provided different values for the two variables and nearly all (97.2%) provided different values. However, when I presented the task as a word problem in interview one, the student responses changed. All students provided the same values for both variables in the common unknown addends tasks. All students provided the same values for both variables *and* all but two provided the different values for both variables in the different unknown addends tasks. Finally, for the no solution core mathematical task only one student provided a solution where the values were the same for both variables.

In interview two, no change occurred in the percent of student responses provided different values for both variables when I presented the task as an equation. However, a decrease of 40.6 percentage points occurred for the common unknown addends equation and a 43 percentage point decrease in the different unknown addends equations. In addition, a 37.5 percentage point decrease occurred for the different unknown addends word problems. A smaller change occurred in students responding with different values for both variables when I presented the task as a word problem. As noted earlier, when I presented even sum confirmatory tasks, the coding results were consistent from interview one to interview two for the variable value category. The differences occurred for the odd sum confirmatory tasks.

However, as noted in the summary of the previous section, not every combination of representation of variable, task type, and core mathematical were included in the confirmatory tasks. Therefore, the only comparisons made were for blanks and shapes for both the common unknown addends and different unknown addends task, letters for the common unknown addends tasks, and words for the different unknown addends tasks.

Student meaning for variables across representation of variables and task type (research question d)

In the following section, I discuss students' meaning of variables across representations of variables *and* task types. While the previous two sections examined the data for each of these separately, this section involves data that explore students' meanings for variables when considering both the representation of the variable and the task type. In this section, I share coded student responses for the comparison and sorting tasks for the initial eleven tasks from interview one, and the algebraic property tasks from interview two. Finally, I summarize the findings from this section along with the results from the prior two sections.

Sorting tasks. The students completed two sorting tasks. For the first sorting task, the students sorted the seven tasks for the common unknown addends (e.g., $y + y = 12$ and $\square + \square = 12$) and different unknown addends tasks (e.g., $a + b = 12$ and $\square + \triangle = 12$). The second sorting task included the four no solution tasks. I instructed students to sort the tasks into groups that *meant the same thing*. I then asked the students to explain how they grouped the problems and how the groups they generated were different from each other.

The student responses were coded using one of five codes to determine the criteria the students used to sort the tasks into groups of tasks that mean the same thing, see Figure 21**.** I developed these codes using a constant comparison method to capture the essence of the commonalities across students' responses and the strategies employed for sorting the tasks. As codes were developed, the codes were applied, modified, and additional codes were added until no further codes emerged.

Code	Description
WP/EQ	Student sorted the tasks into two groups, one with the word problems and the other with the equations.
All Same	Student placed all of the tasks into a single group indicating that they were all the same.
VarRep	Student sorted the tasks into groups based on the representation of the variable (i.e., letters, shapes, words, and blanks).
SameDiff	Student sorted the tasks into groups where the two representations of the variable were the same (e.g., *y* and *y*) and the two representations of the variable were different (e.g., *a* and *b*).
Other	Student sorted the tasks into groups not characterized by one of the above codes.

Figure 22. Sorting Codes

Table 34 shows the number of responses for each of the two sorting tasks, and corresponding percentages, for each grade as well as for all students included in the study. A single fourth grade student did not complete the sorting or comparison tasks for the first interview resulting in 23 total responses.

For both sorting tasks, the number of students sorting the tasks by task type (i.e., sorting into two groups, one with the word problems and the other with the equations) increased across grades four to six. For the first sorting task, the percent of students applying this sorting strategy from fourth to sixth grade was 22.2%, 33.3%, and 83.3% respectively. For the second sorting task, the percent of students applying this sorting strategy from fourth to sixth grade was 33.3%, 55.6%, and 100% respectively. As shown in the table, this strategy was the most often employed for the two sorting tasks, 43.5% and 60.9% respectively.

Table 34:

Sorting Code Results

Core Tasks	Grouping strategy	4th grade	5th grade	6th grade	Total
C	Two groups, one with the word problems and the other with the equations.	2 (25%)	3 (33.3%)	5 (83.3%)	10 (43.5%)
	Placed all of the tasks into a single group indicating that they were all the same	1 (12.5%)	0 (0%)	0 (0%)	1 (4.3%)
	Sorted the tasks into groups based on the representation of the variable (i.e., letters, shapes, words, and blanks)	1 (12.5%)	3 (33.3%)	0 (0%)	4 (17.4%)
	Sorted the tasks into groups based on the representation of the variable (i.e., letters, shapes, words, and blanks)	1 (12.5%)	0 (0%)	1 (16.7%)	2 (8.7%)
	Sorted the tasks into groups based on the representation of the variable (i.e., letters, shapes, words, and blanks)	3 (37.5%)	3 (33.3%)	0 (0%)	6 (26.1%)
N	Two groups, one with the word problems and the other with the equations.	3 (37.5%)	5 (55.5%)	6 (100%)	14 (60.9%)
	Placed all of the tasks into a single group indicating that they were all the same	3 (37.5%)	0 (0%)	0 (0%)	3 (13.0%)
	Sorted the tasks into groups based on the representation of the variable (i.e., letters, shapes, words, and blanks)	1 (12.5%)	1 (11.1%)	0 (0%)	2 (8.7%)
	Sorted the tasks into groups based on the representation of the variable (i.e., letters, shapes, words, and blanks)	0 (0%)	1 (11.1%)	0 (0%)	1 (4.3%)
	Sorted the tasks into groups based on the representation of the variable (i.e., letters, shapes, words, and blanks)	1 (12.5%)	2 (22.2%)	0 (0%)	3 (13.0%)

C – common unknown addends task (e.g., $y + y = 12$); N – No solution task (e.g., $x + 6 = x$)

For the first sorting task, no student sorted the tasks into groups having mathematically equivalent structures (e.g., grouping $y + y = 12$ and $\square + \square = 12$). The closest any of the students came to such a grouping were the two students who sorted the tasks into groups where the two representations of the variable were the same (e.g., $y + y = 12$ and $\square + \square = 12$) and the two representations of the variable were different (e.g., $a + b = 12$, and $\square + \triangle = 12$). These students grouped *equations* together that had mathematically equivalent structures but also put the word problems together into a separate group not recognizing the relationship between the word problems and the equations.

For example, Sally generated four groups. One groups consisted of the two word problems, another contained the equations with the same variables (i.e., $y + y = 12$ and $\square + \square = 12$), another contained equations with different representations of the variables (i.e., $a + b = 12$, and $\square + \triangle = 12$). Her final group contained the equation with blanks, which she referred to as an ―oddball" and treated differently from the other equations. Sally stated that the group containing the equations with the same variable ($y + y = 12$, and $\square + \triangle = 12$) only had solution sets of 6 and 6. She stated that the group containing the equations with the different variables ($a + b = 12$, and $\square + \square = 12$) could not be 6 and 6 since the representations of the variables were different. However, she stated that the numbers that could go in the blanks for ___ + ___ = 12 could be the same *or* different. Therefore, she held a different meaning for the blanks then she did for the other representations of the variables.

For the second sorting task, only three fourth graders identified all equations and the word problem as the same. However, only one student identified a mathematically valid reason for them being the same. One other student, Beth, initially stated that they were all the same because they were all adding six. However, when she was asked what the solutions were for the word problem she stated, ―I don‘t know.” When asked what numbers would work for the equations she initially stated that any number would work but later changed her answer to any number over 6, assuming that the sum had to be greater than the greatest addend. When asked if that would also work for the word problem she said it would not.

The other student, Mark, said they could all equal the same number, ―like four” and they all had sixes. He appears to have believed that the two representations of the variables could be different (e.g., 4 + 6 = 10).

Comparison tasks. For the comparison tasks, each student was presented with a pair of tasks based on the representations of the variables (word, shapes, and letters). For one of the problems the variables were the same and for the other they were different (e.g., $y + y = 12$ and $a + b = 12$, and $\square + \square = 12$ and $\square + \triangle = 12$). The purpose of these tasks was to gather evidence of any differences or similarities students had for these pairs of tasks. I coded each comparison task using the codes shown in Figure 22. I developed these codes based on the student responses using a constant comparison method.

Code	Description
Same	Two tasks compared mean the same thing (same #s work for both).
SS/DD	Same representation means same number, and/or different representation means different number (e.g., 6 and 6 a solution for $a + b = 12$).
Rep#	Representation alone determines number (e.g., square big number, triangle small number).
SS/DD L	Same letter means same #/Different letter means diff # (does not hold for shapes).

Figure 23. Comparison codes

Table 35 shows the number of student responses for each of the comparison tasks, and corresponding percentages, for each grade as well as for all students included in the study. A single fourth grade student did not complete the sorting or comparison tasks for the first interview resulting in a final $n = 23$.

Table 35

Comparison Results

Code	**4th grade**	**5th grade**	**6th grade**	**Total**
Two tasks compared mean the same thing (same #s work for both).	5 (21.7%)	7 (30.4%)	4 (17.4%)	16 (69.6%)
Same representation means same number, and/or different representation means different number (e.g., 6 and 6 a solution for $a + b = 12$).	2 (8.7%)	0 (0%)	2 (8.7%)	4 (17.4%)
Representation alone determines number (e.g., square big number, triangle small number).	1 (4.3%)	1 (4.3%)	0 (0%)	2 (8.7%)
Same letter means same #/Different letter means diff # (does not hold for shapes).	0 (0%)	1 (4.3%)	0 (0%)	1 (4.3%)

When comparing the two sets of equations the students responded consistently (i.e., they did not change their responses based on whether the representation of the variables were shapes or letters). Sixteen students (69.6%) stated in each case that the equations were the same, (i.e., it did not matter if the representation of the variables were the same or different). The common misconception held by students in algebra and beyond “who are aware of the proposition that the same letter stands for the same number … tend to think that the converse of this proposition is also correct” (Fujii & Stephens, 2008), occurred with four of the students, two fourth and two sixth graders.

Two students, one fourth and one fifth grader, held that the representation of the variables determines the numbers that could be used in the equations. For example, Julie, a fourth grader, initially stated that the solutions for □ + □ = 12 would also work for □ + △ = 12. When asked if the solutions for □ + △ = 12 would also work for □ + □ = 12 she said, “no.” She then said that she was not sure. She was also uncertain if 5 and 7 could be solutions for both equations. She then stated that maybe 6 and 6 would work for □ + □ = 12 and 5 and 7 would work for □ + △ = 12. When asked if 6 and 6 would also work for □ + △ = 12 she said she was unsure. However, she did not demonstrate same uncertainty for $y + y = 12$ and $a + b =12$ where she stated that 6 and 6 was a solution as were 9 and 3 and 10 and 2. Therefore, the representation of the variables, shapes versus letters, resulted in different interpretations for her.

The other student, Larry, a fifth grader, stated that he thought the squares in □ + □ = 12 and □ + △ = 12 meant “big” numbers and the triangle meant “small” numbers. Therefore, Larry stated that □ + □ = 12 meant two “big” numbers added together to equal twelve and □ + △ = twelve meant a “big” number plus a “small” number equaled twelve. He did not indicate that the two “big” numbers had to be the same. However, when presented with the equations $y + y = 12$ and $a + b =12$, he indicated that the same values were solutions for both variables in both equations. Therefore, similar to Julie, different representations of the variables, shapes versus letters, resulted in different meanings for the variables.

Finally, one fifth grader, Lisa sorted the tasks based on the belief that the same letter means same number and different letters mean different numbers for all cases except when the representation of the variables were shapes. She held that the same letter meant the same number and different letters meant different number in $y + y = 12$ and $a + b = 12$. However, she did not extend this meaning to shapes where she said that she did not think that there was anything different about $\square + \square = 12$ and $\square + \triangle = 12$. When prompted further for any differences she stated that the shapes were different, “but that doesn‘t matter.”

Algebraic property. Tasks 7 and 15 both model a generalization, or algebrafication, of an arithmetic property: if you add a value to a starting value and then subtract this same value, you get the starting value. Task 7 ($\square + \triangle - \triangle = \square$) uses shapes and task 15 ($x + y - y = x$) uses letters as representations of the variables. In each case, the students were asked if the equation would be true always, sometimes, or never. Students were then asked follow up questions to determine why they answered as they did, and for examples of their thinking. The student responses were then coded using the same three categories used for the other sets of tasks: variable comparison, solution type, and variable values. Tables 36 and 37 display these results for each task.

Table 36

Task 7 (□ + △ – △ = □) Student Responses

	Variable comparison		Solution type			Variable Value		When equation is true		
	Same	***Diff***	***Multiple Values***	***Single Value***	***No Solution***	***VAL_ Diff***	***VAL_ Same***	***Always***	***Sometimes***	***Never***
4th grade	9 (100%)	0 (0%)	9 (100%)	0 (0%)	0 (0%)	0 (0%)	9 (100%)	9 (100%)	0 (0%)	0 (0%)
5th grade	6 (66.7%)	3 (33.3%)	9 (100%)	0 (0%)	0 (0%)	4 (44.4%)	9 (100%)	1 (11.1%)	8 (88.8%)	0 (0%)
6th grade	0 (0%)	5 (83.3%)	5 (83.3%)	0 (0%)	0 (0%)	5 (83.3%)	3 (50%)	0 (0%)	6 (100%)	0 (0%)
Total	16 (66.7%)	8 (33.3%)	23 (95.8%)	0 (0%)	0 (0%)	9 (37.5%)	21 (87.5%)	10 (41.7%)	14 (58.3%)	0 (0%)

Table 37

Task 15 (x + y – y = x) Student Responses

	Variable comparison		**Solution type**			**Variable Value**		**When equation is true**		
	Same	***Diff***	***Multiple Values***	***Single Value***	***No Solution***	***VAL_ Diff***	***VAL_ Same***	***Always***	***Sometimes***	***Never***
4th grade	6 (66.7%)	3 (33.3%)	9 (100%)	0 (0%)	0 (0%)	3 (33.3%)	9 (100%)	5 (55.6%)	4 (44.4%)	0 (0%)
5th grade	6 (66.7%)	3 (33.3%)	8 (88.9%)	0 (0%)	0 (0%)	3 (33.3%)	8 (88.9%)	4 (44.4%)	3 (33.3%)	1 (11.1%)
6th grade	0 (0%)	4 (66.7%)	4 (66.7%)	0 (0%)	0 (0%)	3 (50%)	3 (50%)	0 (0%)	4 (66.7%)	1 (16.7%)
Total	12 (50%)	10 (41.7%)	21 (87.5%)	0 (0%)	0 (0%)	9 (37.5%)	20 (83.3%)	9 (37.5%)	11 (45.8%)	2 (8.3%)

Variable comparison. The student responses for the equation $\square + \triangle - \triangle = \square$ provides an interesting pattern in the variable comparison category where students treated the variables as referring to the same or different values. At the fourth grade level, every student viewed the same representation of the variable as the same variable, paralleling a normative algebraic interpretation. At the fifth grade level, 6 of 9 student responses involved interpreting the two pairs of representations of the variable (i.e., $\square$s and $\triangle$s) as the same and 3 of 9 treated the two pairs of variables as referring to different variables, assigning different values to each $\square$ and/or $\triangle$**.** At the sixth grade level, no student viewed the two pairs of representations of the variables as the same and five of six student responses demonstrated that that they viewed the two pairs of representations as different values with one student unsure whether to treat same representation of the variable as the same or different. Therefore, for this task the fourth graders paralleled a normative algebraic meaning of the variable. This normative response decreased at the fifth grade level and no sixth grade students displayed such a meaning.

For $x + y - y = x$ the coding responses for fifth and sixth grade were consistent with those of $\square + \triangle - \triangle = \square$ with the exception of one fewer coded response at the sixth grade level. Two sixth grade students were unsure whether they should treat the letters as the same or different. However the codes for the fourth grade responses shifted from all treating the two representations of the variables as the same for task seven, to 66.7% treating the two representations of the variables as the same and 33.3% treating the two representations of the variables as different for $x + y - y = x$. Therefore, it appears that fourth graders, while paralleling a normative algebraic meaning for comparing variables, were less stable in their comparison across this tasks where the core mathematical task stayed the same and the representation of the variable changed.

Solution Type. Across both tasks at each grade level, students assigned multiple values to each of the variables. As noted before, this does not address the types of values that they assigned. Instead, I only intended this coding scheme to establish the trichotomous distinction between multiple, single, or no assignment of values to variables. These results show that, for this population and these two tasks, the solution type was highly stable.

Variable value. For task seven, the student responses for the variable value category parallel those of the variable comparison category. The fourth grade responses all paralleled a normative algebraic response in that all of the fourth grade response assigned the same value to the same representation of the variable, in this case shapes. While all fifth grade responses also assigned the same value to the same representation of the variable, four of nine student responses *also* assigned different values to the same representation of the variable. At the sixth grade, only half of the students gave responses where they assigned the same value to the same representation of the variable while five of the six students *also* assigned different values to the same representation of the variable.

For $x + y - y = x$, now representing the variables with letters instead of shapes, the fourth grade responses all paralleled a normative algebraic response in that all of the fourth grade response assigned the same value to the same representation of the variable, but 3 of 9 assigned different values to the same representation of the variable. At the fifth grade level, the number of responses for each code decreased by one resulting in eight of the nine responses assigning the same value to the same representation of the variable and three of the nine student responses assigned different values to the same representation of the variable. At the sixth grade level the number of student responses for $x + y - y = x$ were the same as those for task seven.

Therefore it appears that fourth graders, while paralleling a normative algebraic meaning for solution type for $\square + \triangle - \triangle = \square$, were less stable in their comparison across these tasks where the core mathematical task stayed the same and the representation of the variable changed. The fifth and sixth grade responses were more consistent from task to task but less algebraically normative than those of the fourth grade responses.

Always, sometimes, never. Finally, when asked if the equations would be true always sometimes or never, for task seven the coding results for when the equation would be true somewhat parallel those of the variable comparison and solution type category. The fourth grade responses all paralleled a normative algebraic response in that all of the fourth grade responses stated that the equation would be always true. At the fifth grade level 1 of 9 students stated that the equation would be true always and the remaining eight all said it would be true sometimes. All sixth graders responded that the equation would be true sometimes.

For $x + y - y = x$, as in the other categories except for solution strategy, the results were inconsistent with those of task seven. Five fourth graders responded that the equation would always be true with the remaining students stating that they equation would be true sometimes. Four fifth graders responded that the equation would always be true; three stated it would be true sometimes, and a single fifth grader stated that it would never be true. No sixth graders responded that the equation would always be true; four stated it would be true sometimes, and a single sixth grader stated that it would never be true.

Therefore it appears that fourth graders, while paralleling a normative algebraic meaning for when the equation would be true for task seven, were less stable in their comparison across these tasks where the core mathematical task stayed the same and the representation of the variable changed. The fifth and sixth grade responses were more consistent from task to task but less algebraically normative than those of the fourth grade responses.

Comparing expressions. The final set of tasks for the second interview consisted of the core mathematical task of deciding which of two expressions (the sum of a number and itself, and the sum of the number and 5) are greater and for what values. This core mathematical task was written with shapes, blanks, letters, and words, see Figure 23. For each pair of expressions the student responses were coded for the three core categories: variable comparison, solution type, and variable value. In addition, student responses were coded to identify which of the expression they stated would be greater, or if it depended which would be more. If the student stated that it depended then follow-up questions were posed, to determine what they meant by this and for which values each would be greater. If a student stated that one of the expressions was greater, they were then asked if that expression would *always* be greater than the other.

Task #: Representation of variables	**Task**
10: shapes	Which is more, △ + △ or 5 + △?
11: blanks	Which is more, ___ + ___ or 5 + ___?
13: letters	Which is more, *a* + *a* or *a* + 5?
14: words	I am thinking of a number. Which is more, my number added to itself or five plus my number?

Figure 24: Expression comparison tasks

Variable comparison. The results of coded student responses for the variable comparison category are shown in Table 38. Student responses consistently involved different values for the non-word representation of the variables (i.e., shapes, blanks and letters) as shown in the total column in Table 38. However, when I presented the representation of the variable in words, more student responses (79.2%) involved the same value to both variables**.** The expressions with the non-word representations of the variables (i.e., blanks, letters, and shapes) resulted in different values being assigned by students, 83.3%, 87.5%, and 75% respectively, and providing the same value to both variables, 20.8%, 8.3%, and 20.8% respectively. In contrast, the expressions where the representation of the variable was words, in a word problem, students more often provided the same value for both variable, 79.2%, than different values for the two variables, 16.7%.

Table 38

Variable Comparison Category for Expression Comparison Tasks

	4th grade		**5th grade**		**6th grade**		**Total**	
	Same	***Diff***	***Same***	***Diff***	***Same***	***Diff***	***Same***	***Diff***
s	3 (33.3%)	6 (66.7%)	1 (11.1%)	8 (88.9%)	1 (16.7%)	6 (100%)	5 (20.8%)	20 (83.3%)
b	2 (22.2%)	6 (66.7%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	2 (8.3%)	21 (87.5%)
l	3 (33.3%)	5 (55.6%)	1 (11.1%)	8 (88.9%)	1 (16.7%)	5 (83.3%)	5 (20.8%)	18 (75%)
w	6 (66.7%)	2 (22.2%)	8 (88.9%)	1 (11.1%)	5 (83.3%)	1 (16.7%)	19 (79.2%)	4 (16.7%)

b – blanks; l letters; s – shapes; w –words

Solution Type. Table 39 displays the results of the solution type responses for the expression comparison tasks. Student responses were consistent across all of the tasks. With the exception of a single instance in task 13 (i.e., Which is more, $a + a$ or $a + 5$?), where one fifth grade student supplied a single value solution, every other response for every the tasks provided multiple values for the variables regardless of the representation of the variables. This was the same student reported earlier who believed that both letters had to be the same value.

Table 39

Solution Types for Expression Comparison Tasks

	4th grade			**5th grade**			**6th grade**			**Total**		
	SOL_ Var	***SOL_ SingVal***	***SOL_ None***	***SOL_ Var***	***SOL_ SingVal***	***SOL_ None***	***SOL_ Var***	***SOL_ SingVal***	***SOL_ None***	***SOL_ Var***	***SOL_ SingVal***	***SOL_ None***
s	9 (100%)	0 (0%)	0 (0%)	9 (100%)	0 (0%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	24 (100%)	0 (0%)	0 (0%)
b	9 (100%)	0 (0%)	0 (0%)	9 (100%)	0 (0%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	24 (100%)	0 (0%)	0 (0%)
l	9 (100%)	0 (0%)	0 (0%)	8 (88.9%)	1 (11.1%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	23 (95.8%)	1 (4.2%)	0 (0%)
W	8 (88.9%)	0 (0%)	0 (0%)	9 (100%)	0 (0%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	23 (95.8%)	0 (0%)	0 (0%)

b – blanks; l letters; s – shapes; w –words

Variable value. The results of coding student responses for the variable value category, see total column in Table 40**,** show that the expressions with the non-word representations of the variables (i.e., blanks, letters and shapes) all had a high percentage of responses that included different values for the two variables 87.5%, 95.8%, and 75% respectively. Further, these responses provided the same value to both variables, 91.7%, 95.8%, and 91.7% respectively. For these tasks, students tended to assign both the same values to the same representation of the variable *and* different values to the same representation of the variable. In contrast, the expressions where the representation of the variable was words, in a word problem, a higher percentage of responses involved providing the same value for both variables, 91.7%, than responses with different values for both variables, 20.8%. In these latter tasks, students viewed ―my number” as always being the same value whereas they viewed the same letter, shape, and blank in the others as being able to take on both the same and different values.

Table 40

Variable Values for Expression Comparison Tasks

	4th grade		**5th grade**		**6th grade**		**Total**	
	VAL_ Diff	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***
s	7 (77.8%)	8 (88.9%)	8 (88.9%)	9 (100%)	6 (100%)	5 (83.3%)	21 (87.5%)	22 (91.7%)
b	8 (88.9%)	8 (88.9%)	9 (100%)	9 (100%)	6 (100%)	6 (100%)	23 (95.8%)	23 (95.8%)
l	6 (66.7%)	8 (88.9%)	7 (77.8%)	8 (88.9%)	5 (83.3%)	6 (100%)	18 (75%)	22 (91.7%)
w	2 (22.2%)	8 (88.9%)	1 (11.1%)	9 (100%)	2 (33.3%)	5 (83.3%)	5 (20.8%)	22 (91.7%)

b – blanks; l letters; s – shapes; w –words

Summary. The results of the sorting task demonstrate that students attended to the task type in determining tasks that meant the same for the common and different unknown addend, and no solution tasks, 43.5% and 60.9% respectively. This was supported in the previous sections were student coded responses were consistent across the three coding categories when the representation of the variable was blanks, letters, and shapes which were all presented in equations but different from those represented with words which were in word problems. The only differences in these results were the odd sum confirmatory tasks discussed previously.

The conclusion that students focus on task type versus representation of the variable was also supported in the comparison tasks were students compared the same variable representation and task type for the common and different unknown addend tasks. Students‘ responses demonstrated that they treated the two variables in each task as the same for 69.6% of coded responses. Further, only four students (17.4%) viewed the same representation of the variable as the same value and/or different representations of the variable meant only different values. One student held the same meaning for representations of the variables for letters but not for blanks or shapes. The remaining students inferred values for the representation of the variable based solely on attributes of the representation of the variable itself such as the square means a bigger number than the triangle because the square was bigger.

Student responses for the algebraic property tasks also pointed to the impact of the students‘ fragile and inconsistent meanings for variables in interpreting equations. For the equation $\square + \triangle - \triangle = \square$ every student response for the solution type provided multiple values for the variables (95.8%) and 66.7% as treated the same representation of the variable (e.g., the two squares in $\square + \triangle - \triangle = \square$) showing they held that the same representation meant the same number. However, only 10 of 24 students responses (41.7%) involved stating that the equation would always be true with the remaining 14 stating the equation would be true sometimes. In contrast, when the equation was $x + y - y = x$ all student responses for the solution type provided multiple values for the two variables (87.5%) but fewer student responses demonstrated that they treated the same representation of the variable as the same (50%) showing they held that the same representation meant the same number. Further, only nine of 24 students' responses (37.5%) stated that the equation would always be true, 11 stating the equation would be true sometimes (45.8%), and two stating it would never be true (8.3%). While there was some consistency in student coding between these two tasks the differences in coding appear to be due to the representation of the variable, although the influence of prior tasks may have also played a part.

Finally, the coding results for the expression comparison tasks were consistent with the coding results found for the eleven equations from interview one and the confirmatory tasks from interview two. Students consistently, although to a lesser degree, treated the same representations of the variables in the expressions as different when the representations were shapes, blanks, and letters, 83.3%, 87.5%, a and 75% respectively which were all presented symbolically (i.e., non-word). However, when I presented the representation of the variable in words in a word problem only 16.7% of student responses treated the variables as different with 19.2% treating the representation of the variables as the same.

A similar difference occurred for the variable value category where students consistently assigned both the same and different values to the same representations of the variables in the expressions when the representations were shapes, blanks, and letters, 91.7% and 87.5%, 95.8% and 95.8%, and 91.7% and 75% respectively which were all presented symbolically (i.e., non-word). However, when the representation of the variable was presented in words in a word problem only 20.8% of student responses provided different values for the two variables with 91.7%% assigning the same value to the representation of the variables.

Therefore, students' meanings across representations of variables and task types were fragile for the meaning of the variable and more consistent for the task type. The results appear to point to students attending more to the type of task in which the problem is posed than to the representations of the variables. Further, as discussed in the section on quasi-variables, the values presented in the tasks also influence the values that students assign to the variables. Students consistently assigned boundary values to the variables believing that addends had to be less than or equal to the sum, extending a property of whole numbers to integers and rational numbers. Further, differences between coding results for the even and odd sums also produced boundary values where the majority of students believed that odd sums had to have different addend values, extending a property of whole numbers to rational numbers.

Summary. In the previous section I reviewed the data for each of the three categories: variable comparison, solution type, and variable value, for the initial eleven tasks from interview one and the confirmatory tasks from interview two across task types.

In addition, I determined the percentage point change from interview one to interview two for each category across task types. In this section, I summarize these results.

When the task type was equations, student responses consistently treated the representation of the variables as different variables for both the initial eleven tasks from interview one and the confirmatory tasks from interview two regardless of the core mathematical task being modeled. In interview one, only five of 216 student responses involved treating the two variables as taking on the same values. Of these five, three were for the common unknown addends equation. The other two were both for the no solution equations. I further demonstrated that this coding pattern was consistent from interview one to interview two.

However, when I presented the task as a word equation all student responses involved viewing the two variables as the same for the common unknown addend and no solution word problems. Further, all student responses involved viewing the two variables as different for the different unknown addends word problems. However, since words as representations of the variable only occurred in word problems and the blanks, letters, and shapes only occurred in equations the representation of the variable may also play a contributing role in these results. This is explored further in the section on students meaning of variables across task types. I further demonstrated that this coding pattern was consistent from interview one to interview two.

When I presented the tasks as an equation, student responses consistently produced multiple solutions for both the initial eleven tasks from interview one and the confirmatory tasks from interview two regardless of the representation of the variable and the core mathematical task being modeled. In interview one, only four of 216 students responses involved a single solution. Of these four, two were for the common unknown addends equation, both at fourth grade, and the other two for the different unknown addends word problem, one each at fourth and fifth grade. Further, two student responses stated there was no solution for the common unknown addends equation, both at fourth grade, stating there was no solution. I further demonstrated that this coding pattern was consistent from interview one to interview two.

In contrast to the prior two categories, student responses for the variable value category were inconsistent across task types from interview one to two. In interview one, when the task was presented as an equation coded student responses were very consistent for the common unknown addends and different unknown addends tasks with nearly all students providing both the same and different values for the two variables 100% and 95.8%, and 97.2% and 100% respectively. For the no solution equation from interview one, no students gave different values for the two variables and nearly all (97.2%) gave different values. However, when I presented the task as a word problem in interview one, the student responses changed. All students provided the same values for both variables in the common unknown addends tasks. All students provided the same values for both variables *and* all but two provided the different values for both variables in the different unknown addends tasks. Finally, for the no solution core mathematical task only one student provided a solution where the values were the same for both variables.

In interview two, there was no change in the percent of student responses provided different values for both variables when I presented the task as an equation. However, a decrease of 40.6 percentage points occurred for the common unknown addends equation and a 43 percentage point decrease in the different unknown addends equations. In addition, a 37.5 percentage point decrease was found for the different unknown addends word problems. A smaller change occurred in students responding with different values for both variables when the task was presented as a word problem As noted earlier, when even sum confirmatory tasks were presented the coding results were consistent from interview one to interview two for the variable value category. The differences occurred for the odd sum confirmatory tasks.

However, as noted in the summary of the previous section, not every combination of representation of variable, task type, and core mathematical were included in the confirmatory tasks. Therefore, the only comparisons made were for blanks and shapes for both the common unknown addends and different unknown addends task, letters for the common unknown addends tasks, and words for the different unknown addends tasks.

CHAPTER 5: DISCUSSION OF KEY FINDINGS, IMPLICATIONS, LIMITATIONS, FUTURE RESEARCH

In this study, I sought to answer the question: How do grade 4-6 students interpret various representations of variables when presented in different forms and different task types? To answer the aforementioned question, I addressed the following subquestions.

a. What solution sets do grade 4 – 6 students generate for tasks with equivalent mathematical structures across representations of variables (i.e., blanks, letters, shapes and words) *and* different task types (i.e., word problem or equation)?

b. How do grade 4-6 students interpret variables across various representations of the variable (i.e., place holder or letter-symbolic)?

c. How do grade 4-6 students interpret variables across different task types (word problem or equation)?

d. How do grade 4 - 6 students interpret variables across various representations of a variable (i.e., place holder or letter-symbolic) *and* different task types (word problem or equation)?

In this chapter, I synthesize the results reported in the previous chapters in light of the theoretical and conceptual frameworks discussed in chapter one. I divided this chapter into four main sections: discussion of key findings, implications, limitations, and recommendations for future research.

Discussion of Key Findings

As noted in the first two chapters, the primary focus of research on students‘ meaning for and use of variables has focused almost solely on conventional letter-symbolic representation of variables (e.g., Fujii & Stephens, 2008; Kaput, 2008a; MacGregor & Stacey, 1997; Radford, 2000; Sfard & Linchevski, 1994). Further, nearly all of this research has been conducted at the middle school level and beyond with students in algebra classes and beyond (cf., Booth, 1984; Carraher, et al., 2000; Carraher, et al., 2001; Ellis, 2007; Knuth, et al., 2005; Kuchemann, 1981; Lannin, et al., 2006; MacGregor & Stacey, 1997; Swafford & Langrall, 2000; Warren & Cooper, 2008b). In other words, research on students' meanings for and use of conventional and informal representations of variables in the elementary grades is anemic. In this study, I extended the findings of the extant literature to include grade 4- 6 students meanings for formal and informal representations across various representations of the variable (i.e., blanks, letters, shapes, and words), task types (i.e., equations and word problems), and core mathematical tasks.

Multiple Variable Tasks. I found that the students in this study did not struggle with tasks where they dealt with one or two pairs of variables. In fact, even though they did not generally employ a normative algebraic meaning (i.e., the same variables take on the same value and different variables take on different and the same values) and use of the variables, they drew on their knowledge of addition to reason about both single and multiple solutions for variables (e.g., $y + y = 12$ and $\square + \triangle = 12$). Students tended to provide the same solutions for both of these examples (e.g., 6 and 6, 5 and 7, 4 and 8, etc.).

From the review of the elementary grades textbooks that the students in this study used, *enVisionMath* (Pearson Education, 2011), I determined that generally the students dealt with tasks in which a single variable is used. It appears that such an approach is common among curricular materials and may be due to the assumption that students are not developmentally ready to deal with multiple variables. Therefore, curricular materials often introduce single variable tasks [e.g., 18 – 9 = ____, ―Hope had 14 dolls in her collection. She received 2 more as gifts. How many dolls did Hope have then?” (Pearson Education, 2011)] that allow for single value solutions and then progress to multiple variable tasks and variables with multiple solutions.

These findings support Carpenter, Franke, and Levi‘s (2003) concern that the inclusion of informal representations of variables increases the potential issues associated with variable use, such as the difficulty that common notations used in elementary grades like ―*Find the different numbers you can put in the boxes:* □ + □ = 9” can produce (p. 75). They noted that this notation could be confusing to students as well as contribute to the development of misconceptions about the use of variables. They suggest that it would be preferable to use the number sentence □ + △ = 9. However, this change in notation assumes that students recognize the square and triangle as different variables. Based on the results of this study, I argue that this same concern arises with formal algebraic symbols of variables as well.

Further, these results support the research findings (e.g., Booth, 1984) that a common difficulty that students have in algebra and beyond with using formal algebraic symbols involves viewing these symbols as ―objects.” However, these findings generally did not support the findings that students may interpret formal algebraic symbols as taking on specific unknown numbers when they should interpret them as generalized numbers. The students in this study regularly provided multiple values that satisfy the tasks, as per their interpretation of the task, which did not always parallel a normative algebraic response.

However, I cautiously conclude that students viewed their solutions as sets of values, or generalized numbers, that satisfy the tasks. I found it difficult to determine if the students viewed each variable in this way or if they were instead interpreting each pair of values that satisfied the task as different and disconnected from the others. In other words, it is unclear if the students generalized their list of solutions as a set of solutions or as independent unrelated solutions.

If the students are not viewing the variables as generalized numbers then the introduction of variables in situations that support the generalization of variables as objects or specific numbers, as is common in *enVisionMath* (Pearson Education, 2011), may contribute to these common misconceptions. Lobato and Ellis (Lobato & Ellis, 2002) found that students generalized the slope-intercept form for linear equations, $y = mx + b$, as a difference and how four focusing phenomena (i.e., ―goes up by” language, well ordered tables, graphing calculator, and uncoordinated sequences and differences) contributed to this generalizations. In a similar fashion, the primary use of single variables with single value solutions may inadvertently lead students to make the generalization that variables represent single value solutions. Further, McNeil, et al., (2010) found that the use of letters as mnemonic symbols (e.g., *h* for height, *d* for the number of dogs) hindered students developing of an algebraically normative meaning for variables. Instead of viewing the letters as representing numbers, the students viewed them as objects (e.g., *h* meant height, *d* meant the number of dogs).

These findings are also at variance with Fujii and Stephens‘ (2008) findings that “students who are aware of the proposition that the same letter stands for the same number … tend to think that the converse of this proposition is also correct” (p. 1-51). In other words, the students in Fujii and Stephens‘ study often believe that when two or more variables are present in an equation the variables must take on *different* values and cannot be the same value (e.g., for $a + b = 12$, a and b cannot both be 6). While this misconception arose for students in algebra and beyond in the research for formal algebraic symbols of variables, it arose for only a few students in this study and these students did not consistently apply this misconception across various representations of the variables or task types.

Representations of Variables. Students in this study consistently treated multiple representations of variables in the same task as different regardless of whether the representations were the same (e.g., $y + y =12$) or different ($\square + \triangle = 12$) in equations. The few students who differentiated between the same and different variables inconsistently applied in this meaning across core mathematical tasks presented as equations with letters and shapes as the representation of the variables. When the representation of the variable was blanks, these students‘ responses differed from those of the letters and shapes. Students consistently treated the blanks as different even though the blanks were the same representation. In this way, they appear to have a different meaning for the same representation of the variable when presented with blanks than they did for the letters and shapes. However, when presented as words in word problems the students drew on the contextual situation to differentiate between two variables with the same or different referents.

In this regard, these findings are at variance with the potential difficulties with tasks such as ―*Find the different numbers you can put in the boxes:* □ + □ = 9" noted by Carpenter, Franke, and Levi (2003). The reason for this is likely due to students needing to recognize that the same variable carry the same value and different variables involve the same and different values. For instance, in the first number sentence, □ + □ = 9, The students in this study did not treat the two boxes as having the same value. While conventions exist for the treatment of x in $x + x = 9$, no conventions exist, or at least have not been clearly agreed upon, for ―boxes" unless we retrospectively apply the conventions for conventional letter-symbolic variables to these informal representations. In addition, for the equation □ + △ = 9 students must interpret this notation as requiring the square and triangle to be different values (i.e., they cannot be the same value), a common misconception in algebra courses and beyond. However, the students in this study did not distinguish between tasks such as □ + □ = 9 and □ + △ = 9.

The phrasing of the prompt for □ + □ = 9, ―*Find the different numbers you can put in the boxes"* also imposes a specific meaning on the representation of the boxes that the students in this study did not appear to always hold. By asking the student what numbers they *can put in the box*, the box takes on the role of a placeholder that they are to fill in as opposed to a representation that can *stand for* or *represent* numbers. However, students in this study often stated that the square could be or represent a certain number or numbers instead of putting the number(s) *in* the boxes. While this was not an explicit focus of this study, it is worthy of further consideration.

Since the students in this study did not make such distinctions, the suggestion that □ + △ = 9 would be preferable would not, for the students in this study, produce differences in the solutions they provided or how they interpreted the variables. However, this distinction is one that we would like students to make and the inclusion of such tasks provides the opportunity for this to occur as well as address the misconception that variables take on single values and/or represent objects.

Task Type. As mentioned in the previous section, students consistently treated the multiple representations of variables in the same task as different regardless of whether the representations were the same (e.g., $y + y = 12$) or different ($\square + \triangle = 12$) in equations. However, this was not true when the representation of the variable involved words in word problems.

The students in this study paralleled a normative algebraic understanding of the variables in the word problems as they drew on the explicit meaning of the variables provided in the word problems. The language in the word problems supplied information not explicitly present in the equations providing the students the needed context to determine a normative algebraic solution. As noted in the previous two sections, students treated the multiple representations of variables in the same task as different regardless of whether the representations were the same (e.g., $y + y = 12$) or different (e.g., $\square + \triangle = 12$) in equations. However, the students' solutions for the corresponding word problems with a common mathematical structure did not exhibit this same meaning.

These results are similar to Koedinger and Nathan's (2004), although they studied algebra students solutions across task types using only words and letters as variables. While Koedinger and Nathan's (2004) study did not show the drastic difference in how students interpreted equations and word problems or include the potential differences that representations of variables may have had in students' solutions, the results of this study supports their findings that word problems were not necessarily more difficult for students to solve than symbolic equations. While Koedinger and Nathan (2004) explored the *representational effect* for students early in algebra courses, the impact that using different representations of problems (i.e., story problem, word equation, and symbolic equation) have on students‘ performance (see chapters two and three), this study focused on students‘ meanings for various representations of variables, something not considered by Koedinger and Nathan. For the students in this study, the consistent use of an algebraically non-normative solution appears related to the equations‘ lack of a referent, which was not the case with the word problems.

Quasi-Variable Thinking. Nearly all students in this study consistently demonstrated quasi-variable thinking (Fujii & Stephens, 2008) when interpreting representations of the variable regardless of the representation of the variable or task type, and all students did so for at least some of the tasks. For example, student initial responses were often limited to whole number solutions. When probed further some students included positive rational values and/or negative values. Further, students limited solutions to values that were less than the sum of the task, introducing boundary values (Fujii & Stephens, 2008) that the addends could not be greater than the sum. This was true even for those students who stated that negative numbers would be acceptable values for one of the addends. Students‘ use of boundary values was also evident in the large number of students who stated that the two variables in a task with an odd sum could not be the same value because only even numbers could have the same addends. In this case, the students erroneously extended a property of whole numbers to rational numbers.

Students who stated that rational and/or negative values could be included in the solution nearly always did so only after I asked if these values could be included. While many students stated that one or both of these sets of values would not serve as solutions, the students who did state they could be solutions often stated that they would work *if* you included negatives or fractions. For these students, it appears that they needed some type of permission or explicit prompt to include these values. In other words, they tended to default to whole number solutions and were uncertain whether fractions or negative values should be included without prompting.

This supports and extends Fujii and Stephens (2008) findings to include informal representations of variables for students prior to taking an algebra class. Even though the above findings point to the differences between student meanings for variables and paralleling of normative algebraic solutions for equations and word problems, the solutions the students provided did not include all possible solutions. Fujii and Stephens (2008) found that students can engage in *quasi-variable* thinking, “*general* explanations of why number sentences like $78 - 49 + 49 = 78$ are true and their ability to generate specific instances of what they will later see as a general relationship ($78 - a + a = 78$)” (p. 128). They also noted that students often impose “boundary values” that artificially bound the solution set and are not valid when applied to formal algebraic equations where the variable can be unbounded. For instance, in the previous example students often limit the values of a to be between 0 and 78 since a is subtracted from 78.

Implications

In this section, I provide implications primarily for researchers as they expand on and add to this study. However, the findings of this study also have implications for instruction and curriculum developers. The latter implications are made with the acknowledgement of the limitations noted in the following section. Therefore, these implications necessitate further research in this area.

Based on the findings of this study, it is apparent that these students' meanings for variables were fragile and ill formed. Students' meanings for variables varied across representations of variables and task types. Therefore, if students in this study are typical of grade 4-6 students across the US, they need opportunities to explore the meanings of various representations of variables across various task types and common core mathematical tasks.

As reported earlier, students in elementary grades typically engage in tasks where they only work with single variables in an equation or expression [i.e., generational activities (Kieran, 2007)]. Often these tasks required students to determine a single value solution or substitute a single value into an expression and evaluate the result focusing on numerical and quantitative reasoning (Carraher & Schlieman, 2007). As reported in the extant research, this can result in students generalizing that variables represent a single value instead of a set of values (Booth, 1984). To address these issues I provide the following four implications from this research that, in addition to transformational activities, also include generational activities and global meta/level activities (Kieran, 2007) as well as building on students numerical and quantitative reasoning and use of functions.

Meaning of variables. Students in this study consistently demonstrated *quasi-variable* thinking (Fujii & Stephens, 2008), focusing on whole number solutions less than or equal to the sums of the addition tasks. As students are introduced to variables, and unknowns, the definition of variable needs to be extended beyond the current idea that the variable *only* takes on values satisfying the equation to all values, some of which satisfy the equation and others that do not. In doing so, the opportunity for discussing with students what sets of values do and do not satisfy the task will become a normative part of evaluating the use of variables by incorporating global meta/level activities (Kieran, 2007) and students' numeric and quantitative reasoning (Carraher & Schlieman, 2007).

Through this interpretation of variables, as students move from working with whole numbers, to rational numbers, to integers and beyond, they will be better positioned to consider the viability of these values for solution sets. For instance, when students move from working with whole numbers to fractions they can be engaged in a conversation focused on whether these values should be included in the set of values that satisfy the task or the set of values that do not satisfy the task. These discussions can address the finding of this study that students tended to only consider whole number solutions, the set of numbers that they begin learning about in school, by providing a framework through which to consider other sets of numbers as they are introduced.

Extending the definition of variable as proposed could encourage students to view variables as varying quantities that can take of a variety of values, as opposed to the common misconception of a single value. Students can consider both the sets of values that make the task true, false, or not possible (e.g., x cannot take on the value of 0 for $y = \frac{8}{x}$) incorporating the use of functions as an entry point to early algebra. As students progress to higher mathematics, such experiences provide students the foundation to draw upon in understanding the trichotomous nature of inequalities (i.e., dividing the Cartesian plane into three regions)

Multiple variable tasks. Students in this study were inconsistent with the meanings they assigned to variables across tasks with a common mathematical structure. While we may consider this inconsistency as a problem, we can also view it as an opportunity for students to compare how these tasks are similar and different. In order to accomplish this, students need to have experiences working with and comparing a variety of formal and informal representations of variables in various task types for tasks with a common core mathematical task, see figure 25 for an example. The results of this study demonstrated that students did not recognize such tasks as structurally equivalent.

For example, throughout this study students consistently interpreted word problems differently from equations with a common mathematical structure employing a normative algebraic solution for the former and not for the latter. If students do not recognize that a given equation has an equivalent mathematical structure as a given word problem then it seems highly probable that they will also have difficulties writing an equivalent equation for a word problem [i.e., generational activities (Kieran, 2007)].

1. Shakira and Tim have the same number of gummy bears. Together they have 12 gummy bears. How many gummy bears could Shakira have? How many gummy bears could Tim have?
2. Show: $\square + \square = 12$ Ask: What numbers can the $\square$ be?
3. Show: ___ + ___ = 12 Ask: What numbers can go in to blanks?
4. Show: $y + y = 12$ Ask: What numbers can y be?

Figure 25. Tasks with various representations of the variable and a common mathematical structure.

This ability to model word problems using mathematical symbolism is an important aspect of school mathematics. Therefore, providing students the opportunity to solve and compare tasks with a common mathematical structure but various task types and representations of variables will make the inconsistencies seen in this study explicit and then may be addressed.

Extension of symbolization. Each of the prior implications is an important foundational component of students‘ early algebra experiences. In viewing these results through Kaput‘s (2008b) process of symbolization, it becomes evident that students‘ conceptions of informal representations of variables, through mediated experiences, are foundational to their generalizations of conventional letter-symbolic variable symbolization. Kaput‘s model of the symbolization process provides one way to examine how students iteratively revisit their conceptions based on the interaction between mediated experiences and representations.

This model provides a lens for examining generalizations, and the symbolizations that students develop to express these generalizations. However, it does not provide a means for examining *how* the students view these symbolizations (e.g., as an object or a process) or how students‘ meanings for representations of variables contribute to the misconceptions and normative understanding of their meaning and use. If researchers, educators, and curriculum developers take the idea that students prior knowledge and experiences are to consider in students‘ development of meanings for variable then their meanings for these various representations of variables must be taken into account.

Therefore, understanding students‘ generalizations of the informal representations is of great importance to researchers, educators, and curriculum developers. Educators, researchers, and curriculum developers need to consider the prior implications in providing the students opportunities to make their understandings explicit. As these meanings become evident, further tasks and mediated experiences can then be knowledgably developed to build on and address students‘ misconceptions to guide them toward a normative understanding of variable and recognize the similarities across variables, task types, and mathematical tasks.

Limitations

As noted in the opening chapters of this study, little research into students‘ meanings for variables exists prior to middles school algebra classes. Therefore, this study was exploratory in nature, and limitations were inevitable.

Generalizability. This study involved a small sample of twenty-three students. Due to the four consent statements required for participation in the study, the resulting sample may be more homogeneous, and less representative, than the student populations of the schools. Furthermore, the students in this study all used the same elementary mathematics textbook. In addition, no teacher-identified low-achieving sixth grade students were included in the study, making the sixth grade sample less representative of the population in comparison to the fourth and fifth grade samples. Thus, the results of this study cannot be generalized to different populations of students from different mathematical backgrounds. Thus, further research is necessary to determine the extent to which other students hold similar or different interpretations of variable representations.

Extension of tasks. The tasks included in this study only addressed the use of various representations of variables for equations, expressions, and word problems. The tasks used for this study did not include the same representations of variables for different task types (i.e., blanks, letters, and shapes were only used in equations and expressions and words were only used in word problems). Therefore, comparisons of students' meanings for these specific representations of the variables across various task types were not possible.

In addition, the equations and word problems included in this study consisted only of relatively simple addition situations. The inclusion of mathematical tasks with other operations would further expand these findings. Further, during the course of the study it became evident that probing students more on what values work *and* do not work for variables across a variety of representations and task types may have provided worthwhile results that could result in greater detail into the values students assigned to variables.

Generational activities. Finally, providing students with opportunities to generate different representations of a task (e.g., writing a symbolic equation for a word problem or writing a word problem for a symbolic equation) could have expanded on the findings from the comparison tasks from interview one. For the common unknown addend word problems, having the students attempt to write a symbolic equation would have made the need for a way to represent the same value with two variables evident.

Recommendations for Future Research

This study was exploratory in that the extant literature does not provide insight into the variety of meanings that students have for various formal and informal representations of variables across task types and tasks with common mathematical structures. Therefore, these results need replicated in order to determine the generalizability of the findings reported herein.

Expanded studies. Future research needs to expand upon this study to include tasks with structures other than the addition problems used in this study, other grade levels, and a larger more generalizable sample of students in order to scale up the findings and implications for instruction, and curricular materials (Battista & Clements, 2000; Clements, 2007). This should include further research that involves the same representations of variables across task types for tasks with common mathematical structures. The mathematical tasks also need to be extended beyond the addition tasks used in this study to include subtraction, multiplication, division, and part-part-whole tasks.

Trajectories. As the research base on students' meanings for variables evolves and becomes more cohesive; research can extend to generating learning trajectories for the development of students' meanings for variables. One way to accomplish this would be to examine the development of students' meanings for representations of variables through the lens of Kaput's (2008b) process of symbolization by identifying key stages along the continuum of students' construction of meaning for various representations of variables.

Mediated experiences. In addition, further research is needed to understand the mediated experiences that provide students with opportunities to develop deeper meanings for various representations of variables. These mediated experiences, in interaction with the meanings students have developed for the various representations of variables, result in new meanings that, as Kaput argued, move students toward the formal normative understanding of variables. However, little research into how different mediated experiences influence this development of meaning exists.

Common misconceptions from algebra. Finally, the results of this study demonstrate that students in grades four through six did not exhibit many of the misconceptions for variables that students in algebra classes and beyond typically exhibit [e.g., different variables in an equation must take on different values from each other (Fujii, 2003)]. It appears that these misconceptions may arise somewhere between sixth grade and when students begin to demonstrate these misconceptions in algebra. Therefore, further research needs to be undertaken to identify when and why these misconceptions arise. Such research has the potential to provide important implications for instruction and curricular materials.

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Appendix A: Interview Protocol

Grade 4-6 Student Conceptions and Utilization of Informal and Formal Variable Representation across Mathematically Equivalent Tasks

Pilot Interview Protocol 3.0
11/22/10
Matt Switzer

<u>To be read prior to each interview:</u>

______ (*Insert child's name),* I would like you to take part in a research project. This will help me understand how students think about ideas in math. You will get to solve different kinds of math problems with me like ones you solve in your math class. There is no risk beyond what you do in class every day. You should know that you can choose to stop at any time. Would you be willing to work with me today?

Interview 1: Solving problems

1. Shakira and Tim have the same number of gummy bears. Together they have 12 gummy bears. How many gummy bears could Shakira have? How many gummy bears could Tim have? • Could [Shakira and Tim] have any other number of Gummy bears? Why or why not? • Are there any numbers of Gummy bears they cannot have? Why or why not?
2. **Show:** $\square + \square = 12$ **Ask:** What number(s) can $\square$ be? • What do the squares in this problem mean? • Could the square be any number? Why or why not? Give me an example of what numbers the square can be. • Are there any numbers the first square cannot be?
3. **Show:** $a + b = 12$ **Ask:** What numbers can a and b be? • What do the letters in this problem mean? • Could the letters be any number? Why or why not? Give me an example of what numbers the letters can be. • Are there any numbers a cannot be? • Are there any numbers that b cannot be?
4. **Show:** ___ + 6 = ___ **Ask:** What numbers can go in the blanks? • What do the blanks in this problem mean? • Could the blanks be any number? Why or why not? Give me an example of what numbers the blanks can be. • Are there any numbers the first blank cannot be? • Are there any numbers the second blank cannot be?

5. **Show:** ___ + ___ = 12 **Ask:** What numbers can go in to blanks?
    - What do the blanks in this problem mean?
    - Could the blanks be any number? Why or why not? Give me an example of what numbers the blanks can be.
    - Are there any numbers the first blank cannot be?
    - Are there any numbers the second blank cannot be?

6. **Show:** □ + △ = 12 **Ask:** What numbers can □ and △ be?
    - What do the shapes in this problem mean?
    - Could the shapes be any number? Why or why not? Give me an example of what numbers the shapes can be.
    - Are there any numbers the square cannot be?
    - Are there any numbers the triangle cannot be?

7. **Show:** △ + 6 = △ **Ask:** What numbers can the △ be?
    - What do the shapes in this problem mean?
    - Could the shapes be any number? Why or why not? Give me an example of what numbers the shapes can be.
    - Are there any numbers the first triangle cannot be?
    - Are there any numbers the second triangle cannot be?

8. I start with some number then add 6 and get the same number that I started with. What is the number?
    - Could the starting number be any number? Why or why not?
    - Are there any numbers the starting number cannot be? Why or why not?

9. **Show:** y + y = 12 **Ask:** What numbers can *y* be?
    - What do the letters in this problem mean?
    - Could the letters be any number? Why or why not? Give me an example of what numbers the letters can be.
    - Are there any numbers the first *y* cannot be?
    - Are there any numbers the second *y* cannot be?

10. **Show:** $x + 6 = x$ **Ask:** What numbers can x be?
    - What do the letters in this problem mean?

• Could the letters be any number? Why or why not? Give me an example of what numbers the letters can be. • Are there any numbers the first *x* cannot be? • Are there any numbers the second *x* cannot be?
11. Together Tom and Anne have 12 feet of ribbon. How long could Tom‘s ribbon be? How long could Anne‘s ribbon be? • Could [Tim or Anne] have any other length of ribbon? Why or why not? • Are there any lengths of ribbon they cannot have? Why or why not?

Comparison tasks
Show the following 8 tasks.

Say: I want you to put together the problems that mean the same thing.

After they are done, for each grouping ask:

Why did you put these together?

Do these have the same numbers that make them true?

For all the groups ask:

What is different about the groups?

How is this group (select each group) different from the others?

Shakira and Tim have the same number of gummy bears. Together they have 12 gummy bears. How many gummy bears does Shakira have? Tim?

$\square + \square = 12$

___ + ___ = 12

---

y + y = 12

Together Tom and Anne have 12 pencils. How many pencils could Tom have? Anne?

$\square + \triangle = 12$

$\_\_\_ + \_\_\_ = 12$

$a + b = 12$

Say: I am going to show you problems that you did last time and I want you to tell me how they are the same and how they are different.

Show sets of tasks

After the student has compared the tasks:

Ask: When you did these last time you told me what numbers would work for each problem. Are there any problems that have the same numbers that work for them? Which ones?

I start with some number then add 6 and get the same number that I started with. What is the number?

$\triangle + 6 = \triangle$

$\_\_\_ + 6 = \_\_\_$

$x + 6 = x$

Shakira and Tim have the same number of gummy bears. Together they have 12 gummy bears. How many gummy bears does Shakira have? Tim?

Together Tom and Anne have 12 pencils. How many pencils could Tom have? Anne?

$\square + \square = 12$
$\square + \triangle = 12$

y + y = 12
$a + b = 12$

Interview 2: Confirmatory and modeling tasks.

<u>Confirmatory task</u>

1. **Show:** Juan and Alexa each have a piece of string. Together they have 16 inches of string. How can Juan's string be? How long can Alexa's string be? • Could [Juan and Alexa] have any other length of string? Why or why not? • Are there any lengths of string they cannot have? Why or why not?
2. **Show:** 8 = □ + □ **Ask:** What number(s) can □ be? • What do the squares in this problem mean? • Could the square be any number? Why or why not? Give me an example of what numbers the square can be. • Are there any numbers the first square cannot be? • Are there any numbers the second square cannot be?
3. **Show:** ___ + ___ = 5 **Ask:** What numbers can go in to blanks? • What do the squares in this problem mean? • Could the square be any number? Why or why not? Give me an example of what numbers the square can be. • Are there any numbers the first blank cannot be? • Are there any numbers the second blank cannot be?
4. **Show:** 12 = □ + △ **Ask:** What numbers can □ and △ be? • What do the shapes in this problem mean? • Could the shapes be any number? Why or why not? Give me an example of what numbers they can be. • Are there any numbers the square cannot be? • Are there any numbers the triangle cannot be?
5. **Show:** y + y = 7 **Ask:** What numbers can *y* be? • What do the letters in this problem mean? • Could the *y's* be any number? Why or why not? Give me an example of what numbers the square can be. • Are there any numbers the first blank cannot be? • Are there any numbers the second blank cannot be? • Can my two numbers be the same? • One of my numbers is 9. What is the other number? • One of my numbers is 2 ½. What is my other number?

Modeling

Show: Shakira and Tim have the same number of gummy bears. Together they have 12 gummy bears. How many gummy bears could Shakira have? How many gummy bears could Tim have?

Say: For this problem, I saw two students write these two number sentences.

□ + □ = 12 □ + △ = 12

Does either of these number sentences mean the same thing as this word problem? Why or why not?

If the say neither works ask: How would you write a number sentence for this problem?

Manipulation

Show: □ + △ – △ = □

Ask: Will □ + △ – △ = □ be true always, sometimes, or never? Why?

Show: I am thinking of two numbers. When I add these two numbers together, I get 7. What can my numbers be?

- Can my two numbers be the same?
- One of my numbers is 9. What is the other number?
- One of my numbers is 2 ½. What is my other number?

Show: __ + __ = 7 **Ask:** What can the blanks be?

- Can the two blanks be the same number?
- Can one of the blanks be 9? **Or** One of the blanks is 9. What is the other blank?
- One of the blanks is 2 ½. What is the other blank?

Say and Show: △ + △ 5 + △. **Ask:** Which is more, △ + △ or 5 + △.

Ask: Is [student response] always more?

- *Yes*: How do you know?
- *No*:
    - When is [student response] more?
    - When is [student response] less?

Ask: Can △ + △ and 5 + △. equal each other?

- No: Why?
- Yes: When will they be the equal to each other?

Say and Show: ___ + ___ 5 + ___. Which is more, ___ + ___ or 5 + ___?

- **Ask:** Is [student response] always more?
    - *Yes*: How do you know?
    - *No*:
        - When is [student response] more?
        - When is [student response] less?
- **Ask:** Can ___ + ___ and 5 + ___equal each other?
    - No: Why?
    - Yes: When will they be the equal to each other?

Show: △ + □ = 7. **Ask:** What can the shapes be?

- Can the two shapes be the same number?
- One of the shapes is 9. What is the other shape?
- One of the shapes is 2 ½. What is the other shape?

Say and Show: $a + a$ $5 + a$. Which is more, $a + a$ or $5 + a$?

Ask: Is [student response] always more?

- *Yes*: How do you know?
- *No*:
    - When is [student response] more?
    - When is [student response] less?

Ask: Can $a + a$ and $5 + a$ equal each other?

- No: Why?
- Yes: When will they be the equal to each other?

Say and Show: I am thinking of a number. Which is more, my number added to itself or five plus my number?

- **Ask:** Is [student response] always more?
    - *Yes*: How do you know?
    - *No*:
        - When is [student response] more?
        - When is [student response] less?
- **Ask:** Can my number added to itself and five plus my number equal each other?
    - No: Why?

Yes: When will they be the equal to each other?

Appendix B: Analysis template

Student:______________________________ **Interview Date:**

Interview 1: Solving problems

Time: Shakira and Tim have the same number of gummy bears. Together they have 12 gummy bears. How many gummy bears could Shakira have? How many gummy bears could Tim have?		
Distinguish_Different_No Distinguish_Different_Yes	SOL_Variable SOL_SingleValue	VAL_Different VAL_Same

Time: **Show:** $\square + \square = 12$ **Ask:** What number(s) can $\square$ be?			
Distinguish_Same_No Distinguish_Same_Yes	SOL_Variable SOL_SingleValue	VAL_Different VAL_Same	

Time: **Show:** $a + b = 12$ **Ask:** What numbers can *a* and *b* be?			
Distinguish_Different_No Distinguish_Different_Yes	SOL_Variable SOL_SingleValue	VAL_Different VAL_Same	
Time: **Show:** ___ + 6 = ___ **Ask:** What numbers can go in the blanks?			
Distinguish_Same_No Distinguish_Same_Yes	SOL_Variable SOL_SingleValue	VAL_Different VAL_Same	
Time: **Show:** ___ + ___ = 12 **Ask:** What numbers can go in to blanks?			
Distinguish_Same_No Distinguish_Same_Yes	SOL_Variable SOL_SingleValue	VAL_Different VAL_Same	

Time:
Show: □ + △ = 12 **Ask:** What numbers can □ and △ be?

Distinguish_Different_No Distinguish_Different_Yes	SOL_Variable SOL_SingleValue	VAL_Different VAL_Same	

Time:
Show: △ + 6 = △ **Ask:** What numbers can the △ be?

Distinguish_Same_No Distinguish_Same_Yes	SOL_Variable SOL_SingleValue	VAL_Different VAL_Same	

Time:
I start with some number then add 6 and get the same number that I started with. What is the number?

Distinguish_Same_No Distinguish_Same_Yes	SOL_Variable SOL_SingleValue	VAL_Different VAL_Same	

Time:
Show: y + y = 12 **Ask:** What numbers can *y* be?

Distinguish_Same_No Distinguish_Same_Yes	SOL_Variable SOL_SingleValue	VAL_Different VAL_Same	

Time:
Show: $x + 6 = x$ **Ask:** What numbers can *x* be?

Distinguish_Same_No Distinguish_Same_Yes	SOL_Variable SOL_SingleValue	VAL_Different VAL_Same	

Time:
Together Tom and Anne have 12 feet of ribbon. How long could Tom‘s ribbon be? How long could Anne‘s ribbon be?

Distinguish_Same_No Distinguish_Same_Yes	SOL_Variable SOL_SingleValue	VAL_Different VAL_Same	

Comparison tasks

Time:
Shakira and Tim have the same number of gummy bears. Together they have 12 gummy bears. How many gummy bears does Shakira have? Tim?
Together Tom and Anne have 12 feet of ribbon. How long could Tom's ribbon be? How long could Anne's ribbon be?
$\square + \triangle = 12$; ___ + ___ = 12 ; $a + b = 12$; $\square + \square = 12$; ___ + ___ = 12; y + y = 12

Distinguish_Same_No Distinguish_Same_Yes	VAL_Different VAL_Same	Compare_AllSame Compare_WP_EQ

Time:
I start with some number then add 6 and get the same number that I started with. What is the number? $\triangle + 6 = \triangle$; ___ + 6 = ___; $x + 6 = x$

Distinguish_Same_No Distinguish_Same_Yes	VAL_Different VAL_Same	Compare_AllSame Compare_WP_EQ

Time:
Shakira and Tim have the same number of gummy bears. Together they have 12 gummy bears. How many gummy bears does Shakira have? Tim?
Together Tom and Anne have 12 pencils. How many pencils could Tom have? Anne?

Distinguish_Different_No Distinguish_Different_Yes	Distinguish_Same_No Distinguish_Same_Yes	VAL_Different VAL_Same	Compare_Same Compare_Different

Time:
$\square + \square = 12$ $\square + \triangle = 12$

Distinguish_Different_No Distinguish_Different_Yes	Distinguish_Same_No Distinguish_Same_Yes	VAL_Different VAL_Same	Compare_Same Compare_Different

Time:
$y + y = 12$ $a + b = 12$

Distinguish_Different_No Distinguish_Different_Yes	Distinguish_Same_No Distinguish_Same_Yes	VAL_Different VAL_Same	Compare_Same Compare_Different

Student:______________________________ **Interview Date:**

Interview 2: Confirmatory

Time:

Juan and Alexa each have a piece of string. Together they have 16 inches of string. How long can Juan's string be? How long can Alexa's string be?

Distinguish_Different_No Distinguish_Different_Yes	SOL_Variable SOL_SingleValue	VAL_Different VAL_Same

Could [Juan and Alexa] have any other length of string? Why or why not?

Are there any lengths of string they cannot have? Why or why not?

Time:

Show: 8 = $\square + \square$ **Ask:** What number(s) can $\square$ be?

Distinguish_Same_No Distinguish_Same_Yes	SOL_Variable SOL_SingleValue	VAL_Different VAL_Same	VM_Fill VM_NotSure VM_Replace VM_Represent

What do the squares in this problem mean? Could the square be any number? Why or why not?

Give me an example of what numbers the square can be. Are there any numbers the first square cannot be?

Are there any numbers the second square cannot be?

Time:
Show: ___ + ___ = 5 **Ask:** What numbers can go in to blanks?

Distinguish_Different_No Distinguish_Different_Yes	SOL_Variable SOL_SingleValue	VAL_Different VAL_Same	VM_Fill VM_Replace	VM_NotSure VM_Represent

What do the squares in this problem mean?

Could the square be any number? Why or why not?

Give me an example of what numbers the square can be.

Are there any numbers the first blank cannot be?

Are there any numbers the second blank cannot be?

Time:
Show: 12 = □ + △ **Ask:** What numbers can □ and △ be?

Distinguish_Same_No Distinguish_Same_Yes	SOL_Variable SOL_SingleValue	VAL_Different VAL_Same	VM_Fill VM_Replace	VM_NotSure VM_Represent

What do the shapes in this problem mean?

Could the shapes be any number? Why or why not?

Give me an example of what numbers they can be.

Are there any numbers the square cannot be?

Are there any numbers the triangle cannot be?

Time:
Show: y + y = 7 **Ask:** What numbers can *y* be?

Distinguish_Same_No	SOL_Variable	VAL_Different	VM_Fill	VM_NotSure
Distinguish_Same_Yes	SOL_SingleValue	VAL_Same	VM_Replace	VM_Represent

What do the letters in this problem mean? Could the *y's* be any number? Why or why not?

Give me an example of what numbers the *y* can be. Are there any numbers the first *y* cannot be?

Are there any numbers the second *y* cannot be? Can the two numbers be the same?

If one of the numbers is 9. What is the other number? One of the numbers is 2 ½. What is the other number?

Modeling

Time:
Show: Shakira and Tim have the same number of gummy bears. Together they have 12 gummy bears. How many gummy bears could Shakira have? How many gummy bears could Tim have?
Say: For this problem, I saw two students write these two number sentences.

$\square + \square = 12$ $\square + \triangle = 12$

Distinguish_Same_No Distinguish_Same_Yes	Model_BothSame Model_DblSame Model_NeithSame Model_SumSame	SOL_Variable SOL_SingleValue	VAL_Different VAL_Same

Time:
Show: $\square + \triangle - \triangle = \square$

Distinguish_Same_No Distinguish_Same_Yes	SOL_Variable SOL_SingleValue	VAL_Different VAL_Same	Always Sometimes Never

Time: I am thinking of two numbers. When I add these two numbers together, I get 7. What can my numbers be?			
Distinguish_Same_No Distinguish_Same_Yes	SOL_Variable SOL_SingleValue	VAL_Different VAL_Same	VM_Fill VM_NotSure VM_Replace VM_Represent
Can my two numbers be the same? If one of the numbers is 9. What is the other number? One of the numbers is 2 ½. What is the other number?			

Time: **Show:** __ + __ = 7 **Ask:** What can the blanks be?			
Distinguish_Same_No Distinguish_Same_Yes	SOL_Variable SOL_SingleValue	VAL_Different VAL_Same	VM_Fill VM_NotSure VM_Replace VM_Represent
Can the two blanks be the same number? If one of the numbers is 9. What is the other number? One of the numbers is 2 ½. What is the other number?			

Time:

Say and Show: △ + △ 5 + △. **Ask:** Which is more, △ + △ or 5 + △.

Distinguish_Same_No Distinguish_Same_Yes	SOL_Variable SOL_SingleValue	VAL_Different VAL_Same	△ + △ or 5 + △

Ask: Is [student response] always more?

- *Yes*: How do you know?

No:

- When is [student response] more?
- When is [student response] less?

Ask: Can △ + △ and 5 + △ equal each other?

- No: Why?
- Yes: When will they be equal to each other?

Time:

Say and Show: ___ + ___ 5 + ___. Which is more, ___ + ___ or 5 + ___?

Distinguish_Same_No Distinguish_Same_Yes	SOL_Variable SOL_SingleValue	VAL_Different VAL_Same	___ + ___ or 5 + ___

Ask: Is [student response] always more?

- *Yes*: How do you know?

No:

- When is [student response] more?
- When is [student response] less?

Ask: Can △ + △ and 5 + △ equal each other?

- No: Why?
- Yes: When will they be equal to each other?

<table>
<tr><td colspan="4">Time:
Show: △ + □ = 7. Ask: What can the shapes be?</td></tr>
<tr><td>Distinguish_Different_No
Distinguish_Different_Yes</td><td>SOL_Variable
SOL_SingleValue</td><td>VAL_Different
VAL_Same</td><td></td></tr>
<tr><td colspan="4">Can the two shapes be the same number? If one of the shapes is 9. What is the other shape?

If one of the shapes is 2 ½. What is the other shape?</td></tr>
<tr><td colspan="4">Time:
Say and Show: $a + a$ $5 + a$. Which is more, $a + a$ or $5 + a$?</td></tr>
<tr><td>Distinguish_Same_No
Distinguish_Same_Yes</td><td>SOL_Variable
SOL_SingleValue</td><td>VAL_Different
VAL_Same</td><td>$a + a$ or $5 + a$</td></tr>
<tr><td colspan="4">Ask: Is [student response] always more?
• Yes: How do you know?
No:
▪ When is [student response] more?
▪ When is [student response] less?

Ask: Can △ + △ and 5 + △ equal each other?
• No: Why?
• Yes: When will they be equal to each other?</td></tr>
</table>

Time:

Say and Show: I am thinking of a number. Which is more, my number added to itself or five plus my number?

Distinguish_Different_No Distinguish_Different_Yes	SOL_Variable SOL_SingleValue	VAL_Different VAL_Same

Ask: Is [student response] always more?

- *Yes*: How do you know?
- *No*:
  - When is [student response] more?
    When is [student response] less?

Ask: Can △ + △ and 5 + △ equal each other?

- No: Why?
- Yes: When will they be equal to each other?

Appendix C: Data tables

Table 41

Variable Comparison Results, Interview Two

	4		5		6		T	
	Same	***Diff***	***Same***	***Diff***	***Same***	***Diff***	***Same***	***Diff***
1 SWw	0 (0%)	9 (100%)	1 (11.1%)	8 (88.9%)	0 (0%)	6 (100%)	1 (4.2%)	23 (95.8%)
2 CEs	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
3 CEb/SEb	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
4 SEs	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
5 CEl	3 (33.3%)	6 (66.7%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	3 (12.5%)	21 (87.5%)
8 SWw	0 (11.1%)	9 (88.9%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (4.2%)	24 (95.8%)
9 Deb/SEb	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
12 SEs	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)

Table 42

Solution Type Results, Interview Two

	4			5			6			T		
	SOL_ Var	***SOL_ SingVal***	***SOL_ None***	***SOL_ Var***	***SOL_ SingVal***	***SOL_ None***	***SOL_ Var***	***SOL_ SingVal***	***SOL_ None***	***SOL_ Var***	***SOL_ SingVal***	***SOL_ None***
1 SWw	8 (88.9%)	1 (11.1%)	0 (0%)	8 (88.9%)	1 (11.1%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	22 (91.7%)	2 (8.3%)	0 (0%)
2 CEs	8 (88.9%)	1 (11.1%)	0 (0%)	9 (100%)	0 (0%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	23 (95.8%)	1 (4.2%)	0 (0%)
3 CEb/SEb	9 (100%)	0 (0%)	0 (0%)	9 (100%)	0 (0%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	24 (100%)	0 (0%)	0 (0%)
4 SEs	9 (100%)	0 (0%)	0 (0%)	9 (100%)	0 (0%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	24 (100%)	0 (0%)	0 (0%)
5 CEl	6 (66.7%)	1 (11.1%)	2 (22.2%)	9 (100%)	0 (0%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	21 (87.5%)	1 (4.2%)	2 (8.3%)
8 SWw	9 (100%)	0 (0%)	0 (0%)	9 (100%)	0 (0%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	24 (100%)	0 (0%)	0 (0%)
9 Deb/SEb	9 (100%)	0 (0%)	0 (0%)	9 (100%)	0 (0%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	24 (100%)	0 (0%)	0 (0%)
12 SEs	9 (100%)	0 (0%)	0 (0%)	9 (100%)	0 (0%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	24 (100%)	0 (0%)	0 (0%)

Table 43

Variable Value Results, Interview Two

	4		**5**		**6**		**T**	
	VAL_ Diff	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***
1 SWw	8 (88.9%)	9 (100%)	8 (88.9%)	9 (100%)	6 (100%)	6 (100%)	22 (91.7%)	24 (100%)
2 CEs	8 (88.9%)	9 (100%)	9 (100%)	9 (100%)	6 (100%)	6 (100%)	23 (95.8%)	24 (100%)
3 CEb/ SEb	9 (100%)	3 (33.3%)	9 (100%)	1 (11.1%)	6 (100%)	5 (83.3%)	24 (100%)	9 (37.5%)
4 SEs	9 (100%)	7 (77.8%)	9 (100%)	9 (100%)	6 (100%)	6 (100%)	24 (100%)	22 (91.7%)
5 CEl	6 (66.6%)	5 (55.6%)	9 (100%)	5 (55.5%)	6 (100%)	5 (83.3%)	21 (87.5%)	15 (62.5%)
8 SWw	9 (100%)	1 (11.1%)	9 (100%)	2 (22.2%)	6 (100%)	3 (37.5%)	24 (100%)	6 (25%)
9 Deb/ SEb	9 (100%)	2 (22.2%)	9 (100%)	2 (22.2%)	6 (100%)	5 (83.3%)	24 (100%)	9 (37.5%)
12 SEs	9 (100%)	3 (33.3%)	9 (100%)	4 (44.4%)	6 (100%)	5 (83.3%)	24 (100%)	12 (50%)

Table 44

Variable Comparison Across Representations of Variables, Interview One

		4		5		6		Total	
		Same	***Diff***	***Same***	***Diff***	***Same***	***Diff***	***Same***	***Diff***
B	D	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
	S	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
	N	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
L	D	2 (22.2%)	7 (77.8%)	0 (0%)	9 (100%)	1 (16.7%)	5 (83.3%)	3 (12.5%)	21 (87.5%)
	S	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
	N	1 (11.1%)	8 (88.9%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	1 (4.2%)	23 (95.8%)
S	D	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
	S	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
	N	1 (11.1%)	8 (88.9%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	1 (4.2%)	23 (95.8%)
W	D	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)	0 (0%)
	S	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
	N	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)	0 (0%)

Table 45

Variable comparison across representation of variables, Interview two

		4		5		6		Total	
		Same	***Diff***	***Same***	***Diff***	***Same***	***Diff***	***Same***	***Diff***
B	D	0 (0%)	18 (100%)	0 (0%)	18 (100%)	0 (0%)	12 (100%)	0 (0%)	48 (100%)
	S	0 (0%)	18 (100%)	0 (0%)	18 (100%)	0 (0%)	12 (100%)	0 (0%)	48 (100%)
L	D	3 (33.3%)	6 (66.7%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	3 (12.5%)	21 (87.5%)
S	D	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
	S	0 (0%)	18 (100%)	0 (0%)	18 (100%)	0 (0%)	12 (100%)	0 (0%)	48 (100%)
W	S	0 (0%)	18 (100%)%)	1 (5.6%)	17 (94.4%)	0 (0%)	12 (100%)	1 (2.1%)	47 (97.9%)

Table 46

Variable Comparison Across Representation of Variables Percentage Point Difference from interview one to interview two

		4		5		6		Total	
		Same	***Diff***	***Same***	***Diff***	***Same***	***Diff***	***Same***	***Diff***
B	D	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
	S	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
L	D	-11.1%	11.1%	0.0%	0.0%	16.7%	-16.7%	0.0%	0.0%
S	D	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
	S	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
W	S	0.0%	0.0%	-5.6%	5.6%	0.0%	0.0%	-2.1%	2.1%

Table 47

Variable Comparison Across Task Type, Interview One

		4		**5**		**6**		**Total**	
		Same	***Diff***	***Same***	***Diff***	***Same***	***Diff***	***Same***	***Diff***
EQ	D	2 (7.4%)	25 (92.6%)	0 (0%)	27 (100%)	1 (5.6%)	17 (94.4%)	3 (4.2%)	69 (95.8%)
	S	0 (0%)	27 (100%)	0 (0%)	27 (100%)	0 (0%)	18 (100%)	0 (0%)	72 (100%)
	N	2 (7.4%)	25 (92.6%)	0 (0%)	27 (100%)	0 (0%)	18 (100%)	2 (2.8%)	70 (97.2%)
WP	D	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)	0 (0%)
	S	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
	N	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)	0 (0%)

Table 48

Variable Comparison Across Task Types, Interview Two

		4		5		6		Total	
		Same	***Diff***	***Same***	***Diff***	***Same***	***Diff***	***Same***	***Diff***
EQ	D	3 (8.3%)	33 (91.7%)	0 (0%)	36 (100%)	0 (0%)	24 (100%)	3 (3.1%)	93 (96.9%)
	S	0 (0%)	36 (100%)	0 (0%)	36 (100%)	0 (0%)	24 (100%)	0 (0%)	96 (100%)
WP	S	0 (0%)	18 (100%)	1 (5.6%)	17 (94.4%)	0 (0%)	12 (100%)	1 (2.1%)	47 (97.9%)

Table 49

Variable Comparison Across Task Type Percentage Point Difference from Interview One to Interview Two

		4		5		6		Total	
		Same	***Diff***	***Same***	***Diff***	***Same***	***Diff***	***Same***	***Diff***
EQ	D	0.9%	-0.9%	0.0%	0.0%	-5.6%	5.6%	-1.1%	1.1%
	S	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
WP	S	0.0%	0.0%	-5.6%	5.6%	0.0%	0.0%	-2.1%	2.1%

Table 50

Solution Type Across Representation of Variables, Interview One

		4			5			6			Total		
		SOL_ Var	***SOL_ SingVal***	***SOL_ None***	***SOL_ Var***	***SOL_ SingVal***	***SOL_ None***	***SOL_ Var***	***SOL_ SingVal***	***SOL_ None***	***SOL_ Var***	***SOL_ SingVal***	***SOL_ None***
Blank	D	9 (100%)	0 (0%)	0 (0%)	9 (100%)	0 (0%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	24 (100%)	0 (0%)	0 (0%)
	S	9 (100%)	0 (0%)	0 (0%)	9 (100%)	0 (0%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	24 (100%)	0 (0%)	0 (0%)
	N	7 (77.8%)	2 (22.2%)	0 (0%)	9 (100%)	0 (0%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	22 (91.7%)	2 (8.3%)	0 (0%)
Letter	D	7 (77.8%)	2 (22.2%)	0 (0%)	9 (100%)	0 (0%)	0 (0%)	5 (83.3%)	1 (16.7%)	0 (0%)	21 (87.5%)	3 (12.5%)	0 (0%)
	S	9 (100%)	0 (0%)	0 (0%)	9 (100%)	0 (0%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	24 (100%)	0 (0%)	0 (0%)
	N	7 (77.8%)	1 (11.1%)	1 (11.1%)	9 (100%)	0 (0%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	22 (91.7%)	1 (4.2%)	1 (4.2%)
Shape	D	9 (100%)	0 (0%)	0 (0%)	9 (100%)	0 (0%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	24 (100%)	0 (0%)	0 (0%)
	S	9 (100%)	0 (0%)	0 (0%)	9 (100%)	0 (0%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	24 (100%)	0 (0%)	0 (0%)
	N	7 (77.8%)	1 (11.1%)	1 (11.1%)	9 (100%)	0 (0%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	21 (87.5%)	1 (4.2%)	1 (4.2%)
Word	D	0 (0%)	9 (100%)	0 (0%)	0 (0%)	9 (100%)	0 (0%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	24 (100%)	0 (0%)
	S	8 (88.9%)	1 (11.1%)	0 (0%)	8 (88.9%)	1 (11.1%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	22 (91.7%)	2 (8.3%)	0 (0%)
	N	0 (0%)	2 (22.2%)	7 (77.8%)	0 (0%)	0 (0%)	8 (88.9%)	0 (0%)	0 (0%)	4 (66.7%)	0 (0%)	2 (8.3%)	19 (79.2%)

Table 51

Solution Type Across Representation of Variables, Interview Two

		4			**5**			**6**			**Total**		
		SOL_ Var	***SOL_ SingVal***	***SOL_ None***	***SOL_ Var***	***SOL_ SingVal***	***SOL_ None***	***SOL_ Var***	***SOL_ SingVal***	***SOL_ None***	***SOL_ Var***	***SOL_ SingVal***	***SOL_ None***
Blank	D	18 (100%)	0 (0%)	0 (0%)	18 (100%)	0 (0%)	0 (0%)	12 (100%)	0 (0%)	0 (0%)	48 (100%)	0 (0%)	0 (0%)
	S	18 (100%)	0 (0%)	0 (0%)	18 (100%)	0 (0%)	0 (0%)	12 (100%)	0 (0%)	0 (0%)	48 (100%)	0 (0%)	0 (0%)
Letter	D	6 (66.7%)	1 (11.1%)	2 (22.2%)	9 (100%)	0 (0%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	21 (87.5%)	1 (4.2%)	2 (8.3%)
Shape	D	8 (88.9%)	1 (11.1%)	0 (0%)	9 (100%)	0 (0%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	23 (95.8%)	1 (4.2%)	0 (0%)
	S	18 (100%)	0 (0%)	0 (0%)	18 (100%)	0 (0%)	0 (0%)	12 (100%)	0 (0%)	0 (0%)	48 (100%)	0 (0%)	0 (0%)
Words	S	17 (94.4%)	1 (5.6%)	0 (0%)	17 (94.4%)	1 (5.6%)	0 (0%)	12 (100%)	0 (0%)	0 (0%)	46 (95.8%)	2 (4.2%)	0 (0%)

Table 52

Solution Type Across Representation of Variables Percentage Point Difference from Interview one to Interview Two

		4			**5**			**6**			**Total**		
		SOL_ Var	***SOL_ SingVal***	***SOL_ None***	***SOL_ Var***	***SOL_ SingVal***	***SOL_ None***	***SOL_ Var***	***SOL_ SingVal***	***SOL_ None***	***SOL_ Var***	***SOL_ SingVal***	***SOL_ None***
Blank	D	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
	S	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Letter	D	11.1%	11.1%	-22.2%	0.0%	0.0%	0.0%	-16.7%	16.7%	0.0%	0.0%	8.3%	-8.3%
Shape	D	11.1%	-11.1%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	4.2%	-4.2%	0.0%
	S	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Words	S	-5.5%	5.5%	0.0%	-5.5%	5.5%	0.0%	0.0%	0.0%	0.0%	-4.1%	4.1%	0.0%

Table 53

Solution type across representation of variables, interview one

		4			5			6			Total		
		SOL_ Var	***SOL_ SingVal***	***SOL_ None***	***SOL_ Var***	***SOL_ SingVal***	***SOL_ None***	***SOL_ Var***	***SOL_ SingVal***	***SOL_ None***	***SOL_ Var***	***SOL_ SingVal***	***SOL_ None***
E	D	25 (92.6%)	2 (7.4%)	0 (0%)	27 (100%)	0 (0%)	0 (0%)	17 (94.4%)	1 (5.6%)	0 (0%)	69 (95.8%)	3 (4.2%)	0 (0%)
	S	27 (100%)	0 (0%)	0 (0%)	27 (100%)	0 (0%)	0 (0%)	18 (100%)	0 (0%)	0 (0%)	72 (100%)	0 (0%)	0 (0%)
	N	20 (74.1%)	4 (14.8%)	2 (7.4%)	27 (100%)	0 (0%)	0 (0%)	18 (100%)	0 (0%)	0 (0%)	65 (90.3%)	4 (5.6%)	2 (2.8%)
WP	D	0 (0%)	9 (100%)	0 (0%)	0 (0%)	9 (100%)	0 (0%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	24 (100%)	0 (0%)
	S	8 (88.9%)	1 (11.1%)	0 (0%)	8 (88.9%)	1 (11.1%)	0 (0%)	6 (100%)	0 (0%)	0 (0%)	22 (91.7%)	2 (8.3%)	0 (0%)
	N	0 (0%)	2 (22.2%)	7 (77.8%)	0 (0%)	0 (0%)	8 (88.9%)	0 (0%)	0 (0%)	4 (66.7%)	0 (0%)	2 (8.3%)	19 (79.2%)

Table 54

Solution Type Across Representation of Variables, Interview Two

		4			**5**			**6**			**Total**		
		SOL_ Var	***SOL_ SingVal***	***SOL_ None***	***SOL_ Var***	***SOL_ SingVal***	***SOL_ None***	***SOL_ Var***	***SOL_ SingVal***	***SOL_ None***	***SOL_ Var***	***SOL_ SingVal***	***SOL_ None***
EQ	D	32 (88.9%)	2 (5.6%)	2 5.6%	36 (100.0%)	0 (0.0%)	0 (0.0%)	24 (100.0%)	0 (0.0%)	0 (0.0%)	92 (95.8%)	2 (2.1%)	2 (2.1%)
	S	36 (100.0%)	0 (0.0%)	0 (0.0%)	36 (100.0%)	0 (0.0%)	0 (0.0%)	24 (100.0%)	0 (0.0%)	0 (0.0%)	96 (100.0%)	0 (0.0%)	0 (0.0%)
WP	S	17 (94.4%)	1 (5.6%)	0 (0.0%)	17 (94.4%)	1 (5.6%)	0 (0.0%)	12 (100.0%)	0 (0.0%)	0 (0.0%)	46 (95.8%)	2 (4.2%)	0 (0.0%)

Table 55

Task Type Across Task Type Percentage Point Difference From Interview One to Interview Two

		4			5			6			Total		
		SOL_ Var	***SOL_ SingVal***	***SOL_ None***	***SOL_ Var***	***SOL_ SingVal***	***SOL_ None***	***SOL_ Var***	***SOL_ SingVal***	***SOL_ None***	***SOL_ Var***	***SOL_ SingVal***	***SOL_ None***
EQ	D	3.7%	1.8%	-5.6%	0.0%	0.0%	0.0%	-5.6%	5.6%	0.0%	0.0%	2.1%	-2.1%
	S	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
WP	S	-5.5%	5.5%	0.0%	-5.5%	5.5%	0.0%	0.0%	0.0%	0.0%	-4.1%	4.1%	0.0%

Table 56

Value Comparison Across Representation of Variables, Interview One

		4		5		6		Total	
		VAL_ Diff	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***
B	D	9 (100%)	9 (100%)	9 (100%)	9 (100%)	6 (100%)	6 (100%)	24 (100%)	24 (100%)
	S	9 (100%)	9 (100%)	9 (100%)	9 (100%)	6 (100%)	6 (100%)	24 (100%)	24 (100%)
	N	9 (100%)	9 (100%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)	0 (0%)
L	D	7 (77.8%)	9 (100%)	9 (100%)	9 (100%)	5 (83.3%)	6 (100%)	21 (87.5%)	24 (100%)
	S	9 (100%)	9 (100%)	9 (100%)	9 (100%)	6 (100%)	6 (100%)	24 (100%)	24 (100%)
	N	8 (88.9%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	23 (95.8%)	0 (0%)
S	D	9 (100%)	9 (100%)	9 (100%)	9 (100%)	6 (100%)	6 (100%)	24 (100%)	24 (100%)
	S	9 (100%)	8 (88.9%)	9 (100%)	9 (100%)	6 (100%)	5 (83.3%)	24 (100%)	22 (91.7%)
	N	8 (88.9%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	23 (95.8%)	0 (0%)
W	D	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
	S	8 (88.9%)	9 (100%)	8 (88.9%)	9 (100%)	6 (100%)	6 (100%)	22 (91.7%)	24 (100%)
	N	0 (0%)	1 (11.1%)	0 (0%)	0 (0%)	0 (0%)	0 (0%)	0 (0%)	1 (4.2%)

Table 57

Interview Two Variable Comparison Across Representation of Variables

		4		5		6		Total	
		Same	***Diff***	***Same***	***Diff***	***Same***	***Diff***	***Same***	***Diff***
B	D	0 (0%)	18 (100%)	0 (0%)	18 (100%)	0 (0%)	12 (100%)	0 (0%)	48 (100%)
	S	0 (0%)	18 (100%)	0 (0%)	18 (100%)	0 (0%)	12 (100%)	0 (0%)	48 (100%)
L	D	3 (33.3%)	6 (66.7%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	3 (12.5%)	21 (87.5%)
S	D	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
	S	0 (0%)	18 (100%)	0 (0%)	18 (100%)	0 (0%)	12 (100%)	0 (0%)	48 (100%)
W	S	0 (0%)	18 (100%)%)	1 (5.6%)	17 (94.4%)	0 (0%)	12 (100%)	1 (2.1%)	47 (97.9%)

Table 58

Variable Value Across Representation of Variables Percentage Point Difference from Interview One to Interview One

		4		**5**		**6**		**T**	
		VAL_ Diff	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***
Bl	D	0.0%	72.2%	0.0%	83.3%	0.0%	16.7%	0.0%	62.5%
	S	0.0%	72.2%	0.0%	83.3%	0.0%	16.7%	0.0%	62.5%
L	D	11.1%	44.4%	0.0%	44.4%	-16.7%	58.3%	0.0%	37.5%
S	D	11.1%	0.0%	0.0%	0.0%	0.0%	0.0%	4.2%	0.0%
	S	0.0%	33.3%	0.0%	27.8%	0.0%	-8.4%	0.0%	20.9%
W	S	-5.5%	44.4%	-5.5%	38.9%	0.0%	25.0%	-4.1%	37.5%

Table 59

Variable Value Across Task Types, Interview One

		4		5		6		Total	
		VAL_ Diff	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***
E	D	25 (92.6%)	27 (100%)	27 (100%)	27 (100%)	17 (94.4%)	18 (100%)	69 (95.8%)	72 (100%)
	S	27 (100%)	26 (96.3%)	27 (100%)	27 (100%)	18 (100%)	17 (94.4%)	72 (100%)	70 (97.2%)
	N	25 (92.6%)	0 (0%)	27 (100%)	0 (0%)	18 (100%)	0 (0%)	70 (97.2%)	0 (0%)
W	D	0 (0%)	9 (100%)	0 (0%)	9 (100%)	0 (0%)	6 (100%)	0 (0%)	24 (100%)
	S	8 (88.9%)	9 (100%)	8 (88.9%)	9 (100%)	6 (100%)	6 (100%)	22 (91.7%)	24 (100%)
	N	0 (0%)	1 (11.1%)	0 (0%)	0 (0%)	0 (0%)	0 (0%)	0 (0%)	1 (4.2%)

Table 60

Interview Two Variable Comparison Across Task Types, Interview Two

		4		5		6		Total	
		Same	***Diff***	***Same***	***Diff***	***Same***	***Diff***	***Same***	***Diff***
E	D	3 (8.3%)	33 (91.7%)	0 (0%)	36 (100%)	0 (0%)	24 (100%)	3 (3.1%)	93 (96.9%)
	S	0 (0%)	36 (100%)	0 (0%)	36 (100%)	0 (0%)	24 (100%)	0 (0%)	96 (100%)
W	S	0 (0%)	18 (100%)	1 (5.6%)	17 (94.4%)	0 (0%)	12 (100%)	1 (2.1%)	47 (97.9%)

Table 61

Variable Value Across Task Type Percentage Point Difference from Interview One to Interview Two

		4		5		6		T	
		VAL_ Diff	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***	***VAL_ Diff***	***VAL_ Same***
E	D	3.7%	47.2%	0.0%	52.8%	-5.6%	12.5%	0.0%	40.6%
	S	0.0%	54.6%	0.0%	55.6%	0.0%	6.9%	0.0%	43.0%
W	S	-5.5%	44.4%	-5.5%	38.9%	0.0%	25.0%	-4.1%	37.5%

VITA

J. Matt Switzer was born on June 19, 1967 in Terre Haute, Indiana, the son of John and Vicki Switzer of Centerpoint, Indiana. He attended public schools in Brazil, Indiana, graduating from Northview High School as a member of the class of 1985. He has earned the following degrees: B.S. in Mathematics Education from Indiana State University (1989); M.A. in Mathematics with a Teaching Emphasis from the University of Northern Colorado (2007); and a Ph.D. in Curriculum and Instruction from the University of Missouri, Columbia (2011).

Prior to his doctoral studies, he was a secondary mathematics teacher for 15 years. He taught at Santa Fe High School in Santa Fe Springs, California and at various schools in the Poudre School District in Fort Collins, Colorado. He was also the K-12 Mathematics Curriculum Facilitator for Poudre School District for two years. During his time in the classroom, he taught various mathematics courses including Algebra I, Geometry, Integrated Mathematics I, II, and III, Precalculus, and remedial mathematics courses. Working with fellow teachers, district, and state level colleagues, he worked in a variety of professional development projects to implement state and district level mathematics standards, textbook adoption committees, Lesson Study projects, and intervention programs for students struggling in mathematics.

Matt is currently a member of the Andrews Institute of Mathematics and Science Education in the College of Education at Texas Christian University.