# NONLINEAR EQUATIONS WITH NATURAL GROWTH TERMS

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# ABSTRACT

This thesis concerns the study of a class of second order quasilinear elliptic differential operators. For $1 < p < \infty$, the model equation we consider is:

$$\mathcal{L}(u) = -\Delta_p u - \sigma |u|^{p-2} u. \tag{0.0.1}$$

Here the *potential* $\sigma$ is a function (or distribution), and the differential operator $\Delta_p u$ is the *p-Laplacian*. Such operators are said to have '*natural growth*' terms, and when $p = 2$, the operator reduces to the classical time independent Schrödinger operator.

We will study the operator under minimal conditions on $\sigma$, where classical regularity theory for the operator $\mathcal{L}$ breaks down. Our focus will be on two heavily studied problems:

1. An existence and regularity theory for positive solutions of $\mathcal{L}(u) = 0$, under the sole condition of *form boundedness* on the real-valued potential $\sigma$:

$$|\langle |h|^p, \sigma \rangle| \leq C \int_\Omega |\nabla h|^p \, dx, \qquad \text{for all } h \in C_0^\infty(\Omega). \tag{0.0.2}$$

Here $\sigma$ is assumed to lie in the *local* dual Sobolev space $L^{-1,p'}_{\text{loc}}(\Omega)$, and the pairing in display (0.0.2) is the natural dual pairing.

2. The *pointwise behavior* of fundamental solutions of the operator $\mathcal{L}$. Here we will be concerned with positive solutions of $\mathcal{L}(u) = 0$ with a prescribed isolated singularity.

The techniques developed to attack these two related problems will be quite different in nature. The first problem relies on a study of the doubling properties of weak reverse Hölder inequalities along with certain weak convergence arguments. The second problem is approached via certain nonlinear integral equations involving Wolff's potential, and makes use of tools from non-homogeneous harmonic analysis.

# Chapter 1

# Introduction

Let $\Omega \subset \mathbf{R}^n$ be an open set, with $n \geq 2$. For $1 < p < \infty$, consider the following operator $\mathcal{L}$, defined by:

$$\mathcal{L}(u) = -\Delta_p u - \sigma |u|^{p-2} u \quad \text{in } \Omega, \tag{1.0.1}$$

for a real valued *potential* $\sigma$. Here $\Delta_p u$ is the $p$-Laplacian, defined by:

$$\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u). \tag{1.0.2}$$

Our methods extend to treat more general second order quasilinear operators, but for ease of exposition in this introduction we will consider the model operator $\mathcal{L}$. The operator (1.0.1) is one of the simplest operators which arises in the study of quasilinear elliptic partial differential equations, and has attracted the attention of many authors, it arrises as the Euler-Lagrange equation corresponding to a very simple energy functional, and it a quasiilinear generalization of the time independent Schrödinger operator. It formed the model operator in the studies of the local behavior of quasilinear equations by Ladyzhenskaya and Uraltseva [LU68], Serrin [Ser64, Ser65] and Trudinger [Tru67].

By adapting the techniques developed by De Giorgi (in [LU68]) and Moser (in [Ser64, Tru67]) in their seminal works on linear elliptic equations, these papers established the Harnack inequality for positive weak solutions of $\mathcal{L}(u) = 0$ in $\Omega$ under suitable $L^q$ conditions on $\sigma$. More precisely, it is shown that if $\sigma \in L^q_{\text{loc}}(\Omega)$ for $q > n/p$, then for each ball $B(x, r)$ so that $B(x, 2r) \subset \Omega$, it follows that:

$$\sup_{z \in B(x,r)} u(z) \leq C \inf_{z \in B(x,r)} u(z), \tag{1.0.3}$$

for a constant $C = C(n, p, ||\sigma||_{L^q(B(x,2r))})$.

The local integrability assumption imposed on $\sigma$ in the works [LU68, Ser64, Tru67] serves as a compactness condition, from which it follows that one can treat the perturbation term as negligible for the purposes of establishing Hölder continuity and local boundedness of solutions. It is well known that the condition here on $\sigma$ is optimal (on the scale of Lebesgue spaces) in order to obtain locally bounded solutions. One can also recover the Harnack inequality by working with more refined notions of local compactness on $\sigma$. An example here is the quasilinear Kato class (see [Bir01]):

$$\lim_{\rho \to 0^+} \sup_{x \in \mathbf{R}^n} \int_0^\rho \left( \frac{|\sigma|(B(x,r))}{r^{n-p}} \right)^{1/(p-1)} \frac{dr}{r} = 0. \tag{1.0.4}$$

Here $|\sigma|$ is the total variation of $\sigma$. The use of conditions such as (1.0.4) in establishing local regularity goes back to the fundamental papers on linear equations by Trudinger [Tru73] and Aizenman and Simon [AS82] (see also [CFG86]). The reader should consult the monograph [MZ97] for more information in this regard.

The central theme of this thesis is the study of the operator $\mathcal{L}$ in the absence of any compactness conditions on the potential $\sigma$. In particular, we will discuss two related problems. The first problem we consider is to develop a suitable theory of the homogeneous problem $\mathcal{L}(u) = 0$ under minimal assumptions on $\sigma$. Here we will be concerned with the existence of positive weak solutions of $\mathcal{L}(u) = 0$, along with the optimal local regularity that solutions can possess. Uniqueness for such equations is known to fail in general. The results proved here are in fact new in the linear case $p = 2$, where the operator $\mathcal{L}$ reduces to the time independent Schrödinger operator. The second problem is the pointwise behavior of fundamental solutions of the operator $\mathcal{L}$ in $\mathbf{R}^n$. By definition, fundamental solutions are positive solutions $u(x, x_0)$ of the equation:

$$\mathcal{L}(u) = \delta_{x_0}, \quad \inf_{x \in \mathbf{R}^n} u(x, x_0) = 0.$$

Here $\delta_{x_0}$ is the Dirac delta measure with pole at $x_0 \in \mathbf{R}^n$. Loosely speaking[1], fundamental solutions of $\mathcal{L}$ are the positive solutions of $\mathcal{L}(u) = 0$ in $\mathbf{R}^n \backslash \{x_0\}$ with a non-removable isolated singularity at $x_0$. We now turn to making a few motivational remarks regarding the second problem, before moving on to describe what is done in more detail.

The study of solutions with singularities is of basic importance in partial differential equations. For the operator $\mathcal{L}$, the first comprehensive study of singular solutions of $\mathcal{L}(u) = 0$ in the puctured space with a prescribed isolated singularity was carried out by Serrin [Ser64, Ser65]. Of primary interest in this regard are:

(i). Estimating the growth of a positive solution of $\mathcal{L}(u) = 0$ near an isolated singularity.

(ii). Establishing the existence of positive singular solutions of $\mathcal{L}(u) = 0$ with a prescribed isolated singularity and precisely the pointwise behavior from (i).

For a given operator, it is usual that establishing (ii) is more subtle than (i). By using the Harnack inequality, Serrin found a solution to both of these problems under suitable local compactness assumptions on $\sigma$. This generalized to nonlinear operators work of Royden [Roy62] and Littman, Stamppachia and Weinberger [LSW63] on linear operators with bounded measurable coefficients. The studies of such singular

[1]This is in fact precise, as long as non-removable is understood correctly, see Section 3.2.5 below.

solutions have subsequently amassed a significant literature, see the book of Véron [Veron96].

Here we offer an approach to singular solutions which does not require the validity of Harnack's inequality, nor do we require any radial symmetry in the lower order term $\sigma$. Our results here describe precisely the effect of the lower order term $\sigma|u|^{p-2}u$ on the $p$-Laplacian, and show that the operator $\mathcal{L}$ is highly non-symmetric.

Finally we remark that in all our results, the $p$-Laplacian operator may be replaced by a more general second order operator, for instance the $\mathcal{A}$ Laplacian operator (see [HKM06]).

### 1.0.1 Chapter 2: the homogeneous problem

In the first part of this thesis, we will consider positive solutions of the *homogeneous problem*, that is, positive solutions of the equation:

$$-\Delta_p u = \sigma|u|^{p-2}u \text{ in } \Omega. \tag{1.0.5}$$

Throughout this work the potential $\sigma$ is a distribution belonging to the local dual Sobolev space $L^{-1,p'}_{\text{loc}}(\Omega)$ (see the end of this introduction for notation). We will therefore admit very rough and highly oscillating potentials. This portion of the thesis forms part of some joint work by the author in collaboration with Professors V. G. Maz'ya and I. E. Verbitsky [JMV11, JMV11b].

The primary result of Chapter 2 is a two way correspondence between the existence of a suitable class of positive[2] weak solutions of (1.0.5) with the validity of the following weighted Sobolev-Poincaré inequality:

$$|\langle\sigma, |h|^p\rangle| \le C \int_\Omega |\nabla h|^p dx \text{ for all } h \in C_0^\infty(\Omega), \tag{1.0.6}$$

[2]Here positive means positive except on a set of $p$-capacity zero.

where $\sigma \in L^{-1,p'}_{\text{loc}}(\Omega)$. Furthermore, our class of weak solutions enjoy the *optimal local Sobolev regularity* for potentials $\sigma$ under the condition (1.0.6).

Note that even in the case when $\sigma \in L^1_{\text{loc}}(\Omega)$, one cannot work with the condition (1.0.6) by separating between the positive and negative parts of $\sigma$ due to the possible interaction between them.

In the case $p = 2$, when the operator $\mathcal{L}$ reduces to the time independent Schrödinger operator, potentials $\sigma$ satisfying (1.0.6) are known as *form bounded*, see [Maz85, RSS94]. In fact our results are new even in the linear case, and settle an unsolved question in the theory of the Schrödinger operator of how to associate a theory of positive weak solutions to:

$$-\Delta u = \sigma u \text{ in } \Omega,$$

for distributional, or highly singular oscillatory potentials $\sigma$.

Simultaneously to studying positive solutions to the equation (1.0.5), we will deduce new existence results for the related equation:

$$-\Delta_p v = (p-1)|\nabla v|^p + \sigma. \tag{1.0.7}$$

Equation (1.0.7) is of interest physically, as it is the stationary part of certain Hamilton-Jacobi type equations used in optimal control theory, see e.g. [ADP06]. The equation (1.0.7) is related to (1.0.5) by a logarithmic substitution $v = \log u$ which has its origins in the study of Sturm-Liouville problems (see [Hi48]). This substitution is known to be delicate as singular measures can arise when going from the equation (1.0.7) to (1.0.5), as was noticed by Ferone and Murat [FM00]. Furthermore, due to the nonlinearity in the gradient on the right hand side of (1.0.7), deducing *a priori* estimates direct from which one can deduce the existence of solutions is nontrivial, see e.g. [Ev90, FM00]. For a number of related results for equations of the type (1.0.7), the reader can consult [FM00, AHBV09, Por02, MP02, GT03, ADP06, PS06].

The relationship between the validity of (1.0.6) with the existence of solutions of (1.0.5) and (1.0.7), along with the quantitative properties of these solutions, is a topic which has attracted the interest of many authors. For results in a variety of special cases or under additional compactness assumptions on $\sigma$, see e.g. [BK79, Ag83, AS82, An86, BNV94, BM97, MZ97, Sme99, FM00, MS00, Sha00, AFT04, PT07, Pin07, Lin08, AHBV09, LSS] and references therein. The contribution of the present work is that no additional assumption will be imposed on $\sigma$ beyond the condition (1.0.6). We will momentarily expand on the difficulties caused by working with this condition alone, after we present an application.

We will see that a consequence of our main results regarding the equations (1.0.5) and (1.0.7) is a characterization of the inequality (1.0.6). This had remained an unsolved question in the theory of Sobolev spaces [MSh09], and is an extension of the work of Maz'ya and Verbitsky [MV02a] in the case $p = 2$.

Indeed, if $\Omega = \mathbf{R}^n$, we will see that *for any potential $\sigma$ in a local dual Sobolev space, then $\sigma$ satisfies (1.0.6) if and only if there exists a vector field $\vec{\Gamma}$ along with a constant $C_1 > 0$ so that:*

$$\sigma = \mathrm{div}(\vec{\Gamma}), \text{ and}$$

$$\int_E |\vec{\Gamma}|^{p'} dx \leq C_1 \mathrm{cap}_p(E) \text{ for all compact sets } E \subset \mathbf{R}^n.$$

The set function $\mathrm{cap}_p(E)$ is the $p$-capacity associated to the homogeneous Sobolev space $L^{1,p}(\mathbf{R}^n)$ (see (2.0.12) for definitions).

Let us now remark on the technical novelties of working with the equations (1.0.5) and (1.0.7) under the sole condition (1.0.6). They arise from the following essential characteristics of $\sigma$ satisfying (1.0.6):

- $\sigma$ does not in general lie *globally* in any dual Sobolev space, i.e. $\sigma \notin L^{-1,s}(\Omega)$ for any $s > 0$;
- there are no local compactness conditions on $\sigma$.

From the first item in the list, it is clear that one cannot follow standard methods to achieve global estimates which would yields the existence of solutions of (1.0.5). Indeed, there are simple examples of $\sigma$ so that a solution $u$ of (1.0.5) need not lie in $L^1(\Omega)$. On the other hand, from the second item it follows that finding the correct quantity to work with locally is a subtle issue, and one cannot verify standard compactness conditions from the calculus of variations.

Like the classical works [LU68, Ser64, Tru67], our principal results make crucial use of Caccioppoli inequalities in order to deduce local gradient estimates. However, in our set-up, it is not possible to iterate these estimates using the fundamental techniques of De Giorgi or Moser. Instead, for an appropriate sequence of approximate solutions, we (somewhat loosely speaking) interpolate between a Caccioppoli type inequality and an estimate in the BMO[3] norm of the logarithm of the approximate sequence. This yields uniform local doubling properties on the approximate sequence, from which one can obtain suitable local estimates.

In order to conclude the local doubling properties, we obtain the following characterization of weak reverse Hölder weights which are doubling (see Proposition 2.1.4 in Section 2.1.3 or Appendix A for definitions and a complete discussion):

*Suppose $w$ is a nonnegative function which satisfies a weak reverse Hölder inequality in an open set $U$. Then $w$ is doubling in $U$ if and only if* $\log(w) \in BMO(U)$.

Once these local uniform estimates are proved, the passage to the limit resembles the general scheme of treating elliptic equations with measure data which was spelled out in the important papers [BBGPV, DMMOP]. In particular we are required to prove a convergence in measure result for the gradients of our approximating sequence. The distributional nature of our underlying potential $\sigma$ means such an estimate is somewhat involved.

The method of using a uniform doubling condition in order to compensate for a

[3]BMO = bounded mean oscillation

lack of compactness could prove useful in deducing local estimates for a variety of PDEs where one has to work under minimal assumptions on the underlying coefficients and data.

### 1.0.2 Chapter 3: the fundamental solution

Fix a point $x_0 \in \mathbf{R}^n$. The second part of this thesis is concerned with the *pointwise* behavior of a positive solution of $\mathcal{L}(u) = 0$ in the punctured space $\mathbb{R}^n \backslash \{x_0\}$, with an isolated singularity at $x_0$. This work was carried out in collaboration with Professor I. E. Verbitsky [JV10].

Here we are interested in pointwise estimates, as in the previously cited work of Serrin [Ser64, Ser65]. We will continue to work with the condition (1.0.6), however, in the absence of the Harnack inequality, we will make the assumption that $\sigma$ *is a nonnegative measure* in order to make the problem tractable[4].

Following the ideas of Bôcher, we recast the problem as studying *the fundamental solution* of the operator $\mathcal{L}$, i.e. positive solutions $u(x, x_0)$ of the equation:

$$-\Delta_p u = \sigma |u|^{p-2} u + \delta_{x_0}, \qquad \inf_{x \in \mathbb{R}^n} u(x) = 0. \tag{1.0.8}$$

where $\delta_{x_0}$ is the Dirac delta measure concentrated at $x_0$.

It is well known ([Ser64], [Ser65], [Veron96]) that, under suitable compactness assumptions on $\sigma$, there exists a positive constant $c$ so that

$$\frac{1}{c}\, G(x, x_0) \le u(x, x_0) \le c\, G(x, x_0), \tag{1.0.9}$$

if $|x - x_0| < R$ for some $R > 0$, where $G(x, x_0)$ is the fundamental solution of $\Delta_p$ on

[4] It is a substantial open problem to extend what is done here to the full generality of distributional potentials $\sigma$ satisfying (1.0.6), even when $p = 2$.

$\mathbb{R}^n$:

$$G(x,x_0) = \gamma_{p,n}\,|\,x - x_0|^{\frac{p-n}{p-1}}\,, \qquad \text{when } 1 < p < n. \tag{1.0.10}$$

Here $\gamma_{p,n} = \frac{p-1}{n-p}\big(n\omega_{n-1}\big)^{-\frac{1}{p-1}}$ and $\omega_{n-1}$ is the surface area of the $n-1$ dimensional sphere in $\mathbb{R}^n$. Moreover, it was shown recently by L. Verón (see [PT07], Lemma 5.1) that $\lim_{x\to x_0} u(x,x_0)/G(x,x_0) = c$ if $\sigma \in L^\infty_{\rm loc}(\mathbf{R}^n)$. These arguments make crucial use of the Harnack inequality, and build on the classical constructions of fundamental solutions to linear operators [LSW63, Roy62]. However, as we will see below, if one allows for rough potentials $\sigma$, $u(x,x_0)$ may behave very differently in comparison to $G(x,x_0)$, both locally and globally.

In Chapter 3, we obtain sharp global estimates for the behavior of fundamental solutions, a typical estimate is the following:

*Suppose $1 < p < n$. Then any fundamental solution $u(x,x_0)$ with pole at $x_0$ satisfies the following lower bound:*

$$\begin{aligned} u(x,x_0) \geq c\,|x-x_0|^{\frac{p-n}{p-1}} \exp\Big(c\int_0^{|x-x_0|}\Big(\frac{\sigma(B(x,r))}{r^{n-p}}\Big)^{1/(p-1)}\frac{dr}{r}\Big) \\ \cdot\exp\Big(c\int_0^{|x-x_0|}\frac{\sigma(B(x_0,r))}{r^{n-p}}\frac{dr}{r}\Big), \end{aligned} \tag{1.0.11}$$

for any $x, x_0 \in \mathbb{R}^n$ under necessary conditions on the measure $\sigma$. Here $c$ is a positive constant depending on $n$ and $p$.

The sharpness of this lower bound is illustrated explicitly by our primary result: *Under a natural assumption on $\sigma$, there exists a fundamental solution $u(x,x_0)$ of $\mathcal{L}$ satisfying the corresponding upper bound, i.e. for another positive constant $c$, depending on $n, p$ and $\sigma$, it holds that:*

$$\begin{aligned} u(x,x_0) \leq c\,|x-x_0|^{\frac{p-n}{p-1}} \exp\Big(c\int_0^{|x-x_0|}\Big(\frac{\sigma(B(x,r))}{r^{n-p}}\Big)^{1/(p-1)}\frac{dr}{r}\Big) \\ \cdot\exp\Big(c\int_0^{|x-x_0|}\frac{\sigma(B(x_0,r))}{r^{n-p}}\frac{dr}{r}\Big). \end{aligned} \tag{1.0.12}$$

These theorems extend to nonlinear operators very recent results [FV10], [FNV10], [GH08] regarding the behavior of the Green function of the time independent Schrödinger operator $-\Delta u - \sigma u$.

It is known that, for singular $\sigma$, fundamental solutions of the operator $\mathcal{L}$ are not unique even when $p = 2$. However, we offer a natural substitute for such a result by proving that whenever there exists a fundamental solution, then there also exists *a (unique) minimal fundamental solution.* In addition to the pointwise bounds presented above, the regularity of the constructed fundamental solution $u(x, x_0)$ away from the pole $x_0$ will be considered.

It is somewhat surprising that expressions involving both the linear potential:

$$\mathbf{I}_p^\rho \sigma(x_0) = \int_0^\rho \frac{\sigma(B(x_0, r))}{r^{n-p}} \frac{dr}{r},$$

of fractional order $p$, and the nonlinear Wolff's potential, introduced in [HW83],

$$\mathbf{W}_{1,p}^\rho \sigma(x) = \int_0^\rho \left( \frac{\sigma(B(x, r))}{r^{n-p}} \right)^{1/(p-1)} \frac{dr}{r},$$

should appear, in the exponential form, in global bounds of fundamental solutions of the operator $\mathcal{L}$.

In a recent paper of Liskevich and Skrypnik [LS08], an indication of this behavior involving the linear potential $\mathbf{I}_p(\sigma)$ when $1 < p \leq 2$ appeared for the first time. They studied isolated singularities of operators of the type $\mathcal{L}(u) = -\Delta_p u - \sigma\, |u|^{p-2}\, u$, under the assumption that $\sigma$ is in the quasilinear Kato class (1.0.4). However, the precise pointwise behavior of fundamental solutions in terms of these two local potentials appears here for the first time, and explicitly shows the non-self adjoint nature of the operator $\mathcal{L}$.

A simple corollary of our results gives necessary and sufficient conditions on $\sigma$ which ensure that $u(x, x_0)$ and $G(x, x_0)$ are pointwise comparable globally. This requires the uniform boundedness of the Riesz potential $\mathbf{I}_p\sigma$ when $1 < p \leq 2$ and the Wolff potential $\mathbf{W}_{1,p}\sigma$ when $p > 2$ (see Corollary 3.6.2):

*Suppose there is a constant $c > 0$ so that (1.0.9) holds for all $x, x_0 \in \mathbb{R}^n$. Then necessarily,*

$$\sup_{x\in\mathbb{R}^n} \int_0^\infty \frac{\sigma(B(x,r))}{r^{n-p}} \frac{dr}{r} < \infty \quad \text{if} \quad 1 < \mathrm{p} \leq 2, \tag{1.0.13}$$

$$\sup_{x\in\mathbb{R}^n} \int_0^\infty \left(\frac{\sigma(B(x,r))}{r^{n-p}}\right)^{1/(p-1)} \frac{dr}{r} < \infty \quad \text{if} \quad \mathrm{p} > 2. \tag{1.0.14}$$

*Conversely, (1.0.13)–(1.0.14) are sufficient for (1.0.9) to hold for all $x, x_0 \in \mathbb{R}^n$, under a natural smallness assumption on $\sigma$ discussed in Chapter 3.*

The methods employed in this section are based on nonlinear integral equations. That these integral equations arise in the study of such quasilinear equations follows from the fundamental estimates of Kilpeläinen and Maly [KM92, KM94]. For a positive measure $\mu$, these estimates describe the pointwise behavior of positive solutions of the equation:

$$-\Delta_p u = \mu$$

in terms of the nonlinear Wolff's potential.

Recently there have been additional developments in this regard, with the development of gradient estimates for quasilinear equations in terms of nonlinear potentials, due to Duzaar and Mingione [DM10, DM11].

With the increase in activity in studying quasilinear equations from the point of view of integral equations, the techniques of this work are likely to have application in a host of related problems.

To show the flexibility in our approach, in Chapter 3 we will simultaneously achieve sharp pointwise bounds for fundamental solutions of the analogous equations with the fully nonlinear $k$-Hessian operator $F_k(u)$ replacing the $p$-Laplacian operator. Indeed, if we let $1 \leq k \leq n$ be an integer, then the second operator we consider, denoted by

$\mathcal{G}$, is the fully nonlinear operator defined by:

$$\mathcal{G}(u) = F_k(-u) - \sigma\,|u|^{k-1}\,u.$$

Here $\sigma$ is again a nonnegative Borel measure, and $F_k$ is the $k$-Hessian operator, introduced by Caffarelli, Nirenberg and Spruck [CNS85], and defined for smooth functions $u$ by:

$$F_k(u) = \sum_{1\leq i_1 < \cdots < i_k \leq n} \lambda_{i_1}\ldots\lambda_{i_k}$$

with $\lambda_1, \ldots \lambda_n$ denoting the eigenvalues of the Hessian matrix $D^2u$. The theory of the $k$-Hessian with measure data was developed by Trudinger and Wang [TW99], and Labutin [Lab02].

### 1.0.3 Notation

Here we record some (standard) notation which will be common throughout this thesis. For an open set $\Omega \subset \mathbf{R}^n$, we say that an open subset $U$ is compactly supported in $\Omega$, denoted by $U \subset\subset \Omega$, if there is a compact set $K \subset \Omega$ so that $U \subset K \subset \Omega$.

For $1 < p < \infty$, we denote $p' = p/(p-1)$, the Hölder conjugate exponent.

Let us now move onto introducing function spaces, all functions and distributions in this thesis will be real valued. For $0 < p < \infty$, and an open set $\Omega \subset \mathbf{R}^n$, define the Lebesgue space $L^p(\Omega)$ to be the space of Borel measurable functions $f$ so that:

$$||f||_{L^p(\Omega)} = \Big(\int_\Omega |f|^p dx\Big)^{1/p} < \infty.$$

We then define the local space $L^p_{\text{loc}}(\Omega)$ as the space of Borel measurable functions $f$ so that for each compact set $K \subset U$, it follows that $||f||_{L^p(K)} < \infty$. $L^\infty(\Omega)$ is then defined as the space of functions which are essentially bounded, with $L^\infty_{\text{loc}}(\Omega)$ the space of functions which are locally essentially bounded.

Let us next introduce the homogeneous Sobolev space. First, let $C_0^\infty(\Omega)$ to be the set of infinitely differentiable functions on $\Omega$ with compact support. For $1 < p < \infty$, we then let $L_0^{1,p}(\Omega)$ to be the closure of the norm $||\nabla h||_{L^p(\Omega)}$ in $C_0^\infty(\Omega)$. When viewed as a factor space, $L_0^{1,p}(\Omega)$ is a complete Banach space, see [Maz85]. The dual space of $L_0^{1,p}(\Omega)$ is denoted by $L^{-1,p'}(\Omega)$.

We will usually use the localized version of the homogeneous Sobolev space and its dual. We say that $f \in L^{1,p}_{\text{loc}}(\Omega)$ if $\psi f \in L_0^{1,p}(\Omega)$ for all $\psi \in C_0^\infty(\Omega)$. Similarly, we say a distribution $\sigma \in L^{-1,p'}_{\text{loc}}(\Omega)$, if $\psi\sigma \in L^{-1,p'}(\Omega)$ for all $\psi \in C_0^\infty(\Omega)$. By standard arguments (see e.g. [Maz85]), it therefore follows that $\sigma \in L^{-1,p'}_{\text{loc}}(\Omega)$ if and only if for each open set $U \subset\subset \Omega$, there exists $\vec{\Gamma}_U \in (L^{p'}(U))^n$ such that $\sigma = \text{div}(\vec{\Gamma}_U)$ in $\mathcal{D}'(U)$ (i.e. in the distributional sense).

In the second part of this thesis, weighted function spaces will crop up. For a measure $\sigma$, we denote by $L^p_{\text{loc}}(\Omega, d\sigma)$ the space of functions which are locally integrable to the $p$-th power with respect to $\sigma$ measure.

Finally, on occasion we will use the symbol $A \lesssim B$ to mean that $A \leq CB$ with the constant $C > 0$ depending on the allowed parameters of the particular result being proved.

# Chapter 2

# The homogeneous problem

Let $\Omega \subset \mathbf{R}^n$ be an open set, with $n \geq 2$, and let $1 < p < \infty$. For a real valued distribution $\sigma \in L^{-1,p'}_{\text{loc}}(\Omega)$, we will be concerned with the connections betweeen the following classical Sobolev inequality:

$$|\langle |h|^p, \sigma \rangle| \leq C \int_\Omega |\nabla h|^p dx, \text{ for all } h \in C_0^\infty(\Omega), \tag{2.0.1}$$

with the existence of positive weak solutions to the quasilinear equation:

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \sigma |u|^{p-2} u \text{ in } \Omega, \tag{2.0.2}$$

and (possibly sign changing) weak solutions of:

$$-\operatorname{div}(|\nabla v|^{p-2}\nabla v) = (p-1)|\nabla v|^p + \sigma \text{ in } \Omega. \tag{2.0.3}$$

Since it is the effect of the lower order term which is of most interest here, for ease of exposition we will only consider the $p$-Laplacian operator in equations (2.0.2) and (2.0.3). However our methods continue to work for operators with more general structure.

Note that it is not obvious how one makes sense of equation (2.0.2) under the sole condition (2.0.1), so let us make a couple of comments in how we interpret (2.0.2).

**Definition 2.0.1.** *A function $u$ is a weak solution of (2.0.2) if both $u$ and $|u|^{p-2}u$ lie in the local homogeneous Sobolev space $L^{1,p}_{loc}(\Omega)$, and (2.0.2) holds in the sense of distributions.*

If one uses weak solutions of (2.0.2) in the sense of Definition 2.0.1 then all terms in (2.0.2) are well defined as distributions. Throughout this chapter, positive will mean in the sense of Sobolev functions in $L^{1,p}_{\text{loc}}(\Omega)$, i.e. positive except on a set of null $p$-capacity (see (2.0.12)). We are now in a position to state our principle theorem.

**Theorem 2.0.2.** *Let $n \geq 1$ and let $\Omega \subset \mathbf{R}^n$ be an open set, then for $1 < p < \infty$ the following statements hold:*

*(i). Suppose that $\sigma \in L^{-1,p'}_{loc}(\Omega)$ satisfies:*

$$\langle |h|^p, \sigma \rangle \leq \lambda \int_\Omega |\nabla h|^p dx, \text{ for all } h \in C_0^\infty(\Omega), \tag{2.0.4}$$

*with the parameter $\lambda$ in the range $0 < \lambda < (p-1)^{2-p}$ if $p \geq 2$, and $0 < \lambda < 1$ if $1 < p < 2$. In addition suppose:*

$$-\langle |h|^p, \sigma \rangle \leq \Lambda \int_\Omega |\nabla h|^p dx, \text{ for all } h \in C_0^\infty(\Omega), \tag{2.0.5}$$

*for some $\Lambda > 0$. Then, there exists a positive weak solution $u$ of (2.0.2) (see Definition 2.0.1) satisfying:*

$$\int_\Omega \frac{|\nabla u|^p}{u^p} |h|^p dx \leq C_0(\Lambda, p) \int_\Omega |\nabla h|^p dx, \quad \text{for all } h \in C_0^\infty(\Omega). \tag{2.0.6}$$

*Furthermore, if we define $v = \log(u)$, then $v \in L^{1,p}_{loc}(\Omega)$ is a weak solution of (2.0.3)* *satisfying:*

$$\int_{\Omega} |\nabla v|^p |h|^p dx \leq C_0(\Lambda, p) \int_{\Omega} |\nabla h|^p dx, \quad \textit{for all } h \in C_0^\infty(\Omega). \tag{2.0.7}$$

*(ii). Conversely, if there is a solution* $v \in L^{1,p}_{loc}(\Omega)$ *of (2.0.3) so that (2.0.7) holds for a constant* $C_0$*, then the inequality (2.0.4) holds with* $\lambda = 1$ *and (2.0.5) holds for a constant* $\Lambda = \Lambda(C_0)$*.*

In statement (i) of Theorem 2.0.2, the local regularity of the solution $u$, $u^{p-1} \in L^{1,p}_{\mathrm{loc}}(\Omega)$ to (2.0.2) is optimal (i.e. cannot be replaced by $L^{1,q}_{\mathrm{loc}}(\Omega)$ for any $q > p$). This is the case even when $p = 2$. Indeed:

**Remark 2.0.3.** The conditions $0 < \lambda < (p-1)^{2-p}$ if $p \geq 2$, and $0 < \lambda < 1$ if $1 < p < 2$, are sharp in order to obtain the condition that both $u$ and $u^{p-1} \in L^{1,p}_{\mathrm{loc}}(\Omega)$. This can be seen from working with the potential:

$$\sigma = t \cdot c_0 |x|^{-p}, \text{ for } c_0 = \left(\frac{n-p}{p}\right)^p, \text{ and } 0 < t \leq 1.$$

If $t = 1$, then (2.0.4) holds with best constant $\lambda = 1$ by the classical multidimensional variant of Hardy's inequality. An elementary calculation shows that the equation (2.0.2) has a positive solution $u(x) = |x|^\gamma$, for $\gamma = \gamma(t, n, p)$, so that, when $p \geq 2$:

$$\gamma = \frac{p-n}{p(p-1)}, \text{ if } t = (p-1)^{2-p},$$

and, for all $p > 1$:

$$\gamma = \frac{p-n}{p}, \text{ if } t = 1.$$

Furthermore, in the case when $p \geq 2$ and $t = (p-1)^{2-p}$, the solution $u(x) = |x|^\gamma$ is the unique (up to constant multiple) positive solution of (2.0.2) in $L^{1,p}_{\mathrm{loc}}(\mathbf{R}^n)$. In the case $t = 1$, the solution $u$ is unique up to constant multiple for all $p > 1$. See

[Pol03, PS05] for these assertions. This shows that one cannot relax the hypothesis of Theorem 2.0.2.

This example also shows that in the generality of our set-up, one cannot expect global integrability properties (at least in unweighted Sobolev spaces).

By considering the same example in the case $p = 2$, one can also see that there exist positive solutions of (2.0.2) so that $u \notin L^{1,p}_{\mathrm{loc}}(\Omega)$, see [JMV11] where further examples are discussed.

In regard to higher integrability of the solution itself, one can obtain improved integrability properties if one is allowed more control in the parameter $\lambda > 0$, see Section 2.3 for a precise statement. This is based on a lemma by Brezis and Kato [BK79] in the classical case $p = 2$.

Regarding the proof of Theorem 2.0.2, it is statement (i) which requires work. The ideas behind the proof have been sketched in the introduction - we will obtain uniform local doubling properties from which local gradient estimates will follow.

Along the way, we obtain a characterization of when a nonnegative weight function satisfying a weak reverse Hölder inequality is doubling (see Sec. 2.1.3 for definitions). Our main hard analysis tool here is Proposition 2.1.4 which may be of independent interest.

### 2.0.4 A characterization of the inequality (2.0.1)

The second result of this chapter is a characterization of the Sobolev inequality (2.0.1). This is a generalization to the $L^p$-case of the main result of Maz'ya and Verbitsky in [MV02a]. We will focus on the case when $\Omega = \mathbf{R}^n$, since one can obtain similar inequalities for a wide class of domains by the method spelled out in [MV02a]. Furthermore, we will be concerned with $n \geq 2$, since the one dimensional case was studied in [MV02b].

**Theorem 2.0.4.** *Let $n \geq 2$, and suppose $\sigma \in L^{-1,p'}_{loc}(\Omega)$, then:*

$$|\langle |h|^p, \sigma\rangle| \leq C \int_{\mathbf{R}^n} |\nabla h|^p dx, \text{ for all } h \in C_0^\infty(\mathbf{R}^n), \tag{2.0.8}$$

*if and only if:*

*(i) in the case $1 < p < n$:*

$$\sigma = \operatorname{div}(\vec{\Gamma}), \text{ with:} \tag{2.0.9}$$

$$\int_{\mathbf{R}^n} |h|^p |\vec{\Gamma}|^{p'} dx \leq C \int_{\mathbf{R}^n} |\nabla h|^p dx \text{ for all } h \in C_0^\infty(\mathbf{R}^n). \tag{2.0.10}$$

*(ii) in the case $p \geq n$, $\sigma \equiv 0$.*

The strength of Theorem 2.0.8 lies in casting the inequality (2.0.8) with indefinite weight $\sigma$, in terms of the inequality (2.0.10) with positive weight $|\Gamma|^{p'}$. The latter inequality has a rich history in its own right, and was characterized as a result of the work of V. G. Maz'ya, D. R. Adams, and B. E. J. Dahlberg in the late 70's/early 80's, see [Maz85, AH96]. Combining this work with Theorem 2.0.4, one obtains the following:

**Remark 2.0.5.** Let $1 < p < n$, then the inequality (2.0.8) holds if and only if $\sigma = \operatorname{div}(\vec{\Gamma})$ with a constant $C > 0$ such that:

$$\int_E |\vec{\Gamma}|^{p'} dx \leq C \operatorname{cap}_p(E) \text{ for all compact sets } E \subset \mathbf{R}^n. \tag{2.0.11}$$

Here $\operatorname{cap}_p(E)$ is the capacity associated with the homogeneous Sobolev space $L^{1,p}(\mathbf{R}^n)$, defined (for a compact set $E$) by:

$$\operatorname{cap}_p(E) = \inf\{||\nabla h||^p_{L^p(\mathbf{R}^n)} : h \geq 1 \text{ on } E, h \in C_0^\infty(\mathbf{R}^n)\}. \tag{2.0.12}$$

It is well known that for such inequalities, that the capacity is the correct quantity to work with, since one needs to work with an *energy* condition. As such, there is an alternative *Sawyer type* testing condition that one may use, due to Kerman and Sawyer (see [KS86] for the $L^2$-case). The equivalence between this testing condition and the capacity condition in the $L^p$-case may be found in (for instance) [Ver99], see also Section 3.3 of Chapter 3 below.

### 2.0.5 The plan of the chapter

The plan of the chapter is as follows. In Section 2.1 we develop the required preliminaries. Section 2.2 is the heart of the chapter, and Theorem 2.0.2 is proved there. The majority of the section is devoted to the proof of statement (i) of Theorem 2.0.2, which is proved first. In Section 2.3, we remark on additional integrability properties the solution constructed can possess if one strengthens certain assumptions on the potential. Subsequently, Section 2.4 is then concerned with deducing Theorem 2.0.4 from Theorem 2.0.2.

## 2.1 Preliminaries

### 2.1.1 Local properties of $\sigma$

Our first lemma, regarding local mollification, is completely elementary. Throughout this paper we will fix a smooth nonnegative symmetric approximate identity $\phi$ so that $\phi \in C_0^\infty(B_1(0))$, and:

$$\int_{B_1(0)} \phi(x)\, dx = 1.$$

**Lemma 2.1.1.** *Let $1 < p < \infty$, and $K \subset\subset \Omega$. Suppose $\epsilon > 0$ such that $\epsilon <$ dist$(K, \partial\Omega)$. Let:*

$$\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon).$$

*If $\sigma$ satisfies:*

$$\langle |h|^p, \sigma \rangle \leq \lambda \int |\nabla h|^p dx, \text{ for all } h \in C_0^\infty(\Omega). \tag{2.1.1}$$

*Then, with $\sigma_\epsilon = \phi_\epsilon * \sigma$, it follows:*

$$\int |h|^p d\sigma_\epsilon \leq \lambda \int |\nabla h|^p dx \text{ for all } h \in C_0^\infty(K). \tag{2.1.2}$$

*Here $d\sigma_\epsilon = \sigma_\epsilon dx$.*

*Proof.* Let $h \in C_0^\infty(V)$. We first note that by the interchange of mollification and the distribution (see Lemma 6.8 of [LL01]):

$$\langle \sigma, \phi_\epsilon * |h|^p \rangle = \int_{B(0,\epsilon)} \phi_\epsilon(t) \langle \sigma, |h(\cdot - t)|^p \rangle dt.$$

By elementary geometry, $h(\,\cdot - t) \in C_0^\infty(\Omega)$ for all $t \in B_\epsilon(0)$, and hence:

$$\langle \sigma_\epsilon, |h|^p \rangle \leq \lambda \int_{B(0,\epsilon)} \phi_\epsilon(t) \Big( \int_\Omega |\nabla h(x-t)|^p dx \Big) dt = \int_\Omega |\nabla h(x)|^p dx,$$

which proves the lemma. □

### 2.1.2 A local existence result.

Let us next state a local existence result which we will use to produce a sequence of approximate solutions:

**Lemma 2.1.2.** *Let $1 < p < \infty$, and suppose that $V$ be a connected open set with smooth boundary, and fix a ball $B \subset\subset V$. Let $\tilde{\sigma} \in C^\infty(\overline{V})$ satisfy, for $0 < \lambda < 1$:*

$$\int_V |h|^p d\tilde{\sigma} \leq \lambda \int_V |\nabla h|^p dx, \text{ for all } h \in C_0^\infty(V). \tag{2.1.3}$$

*Then, there exists a positive solution* $v \in C^{1,\alpha}_{loc}(V) \cap L^{1,p}(V)$ *of:*

$$-\mathrm{div}(|\nabla v|^{p-2}\nabla v) = \tilde{\sigma} v^{p-1},$$

*so that, with* $q = \max((p-1), 1)$*:*

$$\int_B v^{qp} dx = 1.$$

*Furthermore,* $v$ *satisfies the Harnack inequality in* $V$*.*

*Proof.* The existence part follows from the classical theory of monotone operators using the smoothness of $\tilde{\sigma}$, see e.g. Chapter 6 of [MZ97], or [Li69]. The Harnack inequality along with the Hölder continuity (for all $1 < p < \infty$) is contained in Serrin's paper [Ser64]. The Hölder continuity of the gradient can be found in (for instance) [DiB83]. □

### 2.1.3 Weak reverse Hölder inequalities and BMO

In this section, deduce a characterization of when a nonnegative function satisfying a weak reverse Hölder inequality is doubling. First we introduce some notation:

For an open set $U$, we say $u \in BMO(U)$[1] if there is a positive constant $D_U$ so that:

$$⨍_{B(x,r)} |u(y) - ⨍_{B(x,r)} u(z)\, dz|^p dy \le D_U, \text{ for any ball } B(x, 2r) \subset U. \tag{2.1.4}$$

A well known consequence of the John-Nirenberg inequality (see e.g. [St93]) is that one can replace the exponent $p$ in (2.1.4) with any $0 < q < \infty$, and one will obtain a comparable definition of BMO. In addition, $u \in BMO_{\mathrm{loc}}(\Omega)$ if for each compactly

[1] BMO stands for bounded mean oscillation

supported open set $U \subset\subset \Omega$, there is a positive constant $D_U > 0$ so that (2.1.4) holds.

**Definition 2.1.3.** *Let $U \subset \mathbf{R}^n$ be an open set, and let $w$ be a nonnegative measurable function. Then $w$ is said to be doubling in $U$ if there exists a constant $A_U > 0$ so that,*

$$\fint_{B(x,2r)} w\,dx \leq A_U \fint_{B(x,r)} w\,dx, \text{ for all balls } B(x,4r) \subset U. \tag{2.1.5}$$

*Let $w$ be a nonnegative measurable function. Then $w$ is said to satisfy a weak reverse Hölder inequality in $U$ if there exists constants $q > 1$ and $B_U > 0$ so that:*

$$\Big(\fint_{B(x,r)} w^q dx\Big)^{1/q} \leq B_U \fint_{B(x,2r)} w\,dx, \text{ for all balls } B(x,2r) \subset U. \tag{2.1.6}$$

Our argument hinges on the following result:

**Proposition 2.1.4.** *Let $U$ be an open set, and suppose $w$ satisfies the weak reverse Hölder inequality (2.1.6) in $U$. Then $w$ is doubling in $U$, i.e. (2.1.5) holds, if and only if $\log(w) \in BMO(U)$ (see (2.1.4)).*

*In particular, if $w$ satisfies (2.1.6) and*

$$\fint_{B(x,s)} |\log w(y) - \fint_{B(x,s)} \log w(z)dz|^p dy \leq D_U, \text{ for all balls } B(x,2s) \subset U. \tag{2.1.7}$$

*Then there is a constant $C(A_U, D_U) > 0$, so that for any ball $B(x,4r) \subset U$:*

$$\fint_{B(x,2r)} w\,dx \leq C(A_U, D_U) \fint_{B(x,r)} w\,dx \tag{2.1.8}$$

Only the the sufficiency direction is required in what follows; however, since this characterization does not seem to appear explicitly in the literature we prove the full statement in Appendix A.

## 2.2 Proof of the main result

*Proof of Theorem 2.0.2, statement (i).* This will be proved in several steps. Let us assume that $\Omega$ is a connected open set. This is without loss of generality since we can apply the argument below in each connected component. The assumption of connectedness is used in a Harnack chain argument. □

### 2.2.1 Construction of an approximating sequence

Let $\Omega_j$ be an exhaustion of $\Omega$ by smooth connected domains, i.e. $\Omega_j \subset\subset \Omega_{j+1}$ and $\cup_j \Omega_j = \Omega$ (see e.g. [EE87]). In addition, let $B$ be a fixed ball so that $8B \subset \Omega_1$.

Let $\epsilon_j = \min(2^{-j}, d(\Omega_j, \partial\Omega_{j+1}))$, and denote by $\sigma_j = \phi_{\epsilon_j} * \sigma$, here $\phi$ is as in Lemma 2.1.1. Applying Lemma 2.1.1, it follows that (2.1.3) holds with $\tilde{\sigma} = \sigma_{\epsilon_j}$ and $V = \Omega_j$. Hence we may apply Lemma 2.1.2 to deduce the existence a sequence of positive solutions $u_j$ of:

$$\begin{cases} -\mathrm{div}(|\nabla u_j|^{p-2}\nabla u_j) = \sigma_j u_j^{p-1} \text{ in } \Omega, \\ \int_B u_j^{qp} dx = 1. \end{cases} \tag{2.2.1}$$

Here $q = \max(p-1, 1)$, as before. In addition, in each $\Omega_j$ we have that $u_j \in C^{1,\alpha}(\Omega_j)$, and that $u_j$ satisfies the Harnack inequality. Our principle task will be to prove the following estimate, which is a local gradient estimates for the tail of the sequence $\{u_k\}_{k>j}$, inside $\Omega_j$.

**Proposition 2.2.1.** *Whenever $B(x, 4r) \subset\subset \Omega_j$ and $k > j$ it follows that:*

$$\int_{B(x,r)} |\nabla u_k|^p dx \leq C(\Omega_j, B(x,r), B, \Lambda, \lambda, p), \tag{2.2.2}$$

*and:*

$$\int_{B(x,r)} |\nabla(u_k^{p-1})|^p dx \leq C(\Omega_j, B(x,r), B, \Lambda, \lambda, p). \tag{2.2.3}$$

The key in estimates (2.2.2) and (2.2.3) is that the bound is independent of $k$ for $k > j$.

### 2.2.2 Caccioppoli estimates on the approximating sequence.

In order to prove Proposition 2.2.1, we work with the following three lemmas, the first and second of which are analogous to the relevant estimates in the linear case [JMV11]. The third estimate is due to the nonlinearity, since we require $u_j^{p-1} \in L^{1,p}_{\text{loc}}(\Omega)$, and it is where the assumption on $\sigma$ that $\lambda < (p-1)^{p-2}$ comes in when $p \geq 2$. Through this section we will use the notation from (2.2.1).

**Lemma 2.2.2.** *Suppose that (2.0.4) holds for* $0 < \lambda < 1$. *For all* $k > j$ *it follows that:*

$$\int_{\Omega_j} |\nabla u_k|^p h^p dx \leq C(\lambda, p) \int_{\Omega_j} u_k^p |\nabla h|^p dx, \text{ for all } h \in C_0^\infty(\Omega_j),\ h \geq 0 \tag{2.2.4}$$

*Proof.* Let us fix $k$ and $j$ as in the statement of the lemma, and let $v = u_k$. With $h \in C_0^\infty(\Omega_j)$, $h \geq 0$, test the weak formulation of (2.2.1) with $vh^p \in L_0^{1,p}(\Omega_j)$. Using (2.0.4), it follows:

$$\begin{aligned}
\int_{\Omega_j} |\nabla v|^p h^p \, dx &= \int_{\Omega_j} (|\nabla v|^{p-2} \nabla v \cdot \nabla(vh^p) \, dx - \int_{\Omega_j} v \, |\nabla v|^{p-2} \nabla v \cdot \nabla(h^p) \, dx \\
&\leq \langle \sigma_k, h^p v^p \rangle + p \int_{\Omega_j} v \, |h|^{p-1} \, |\nabla v|^{p-1} \, |\nabla \psi| \, dx \qquad (2.2.5) \\
&\leq \lambda \int_{\Omega_j} |\nabla(hv)|^p \, dx + p \int_{\Omega_j} v \, |h|^{p-1} \, |\nabla v|^{p-1} \, |\nabla h| \, dx,
\end{aligned}$$

here we have used Lemma 2.1.1 in the last inequality.

Recall Young's inequality with $\epsilon$: for any $\epsilon > 0$, and for $a, b \geq 0$:

$$ab \leq \epsilon a^p + (p\epsilon)^{-1/(p-1)} \frac{(p-1)}{p} b^{p'}. \tag{2.2.6}$$

It follows from (2.2.6) that for any $\epsilon > 0$ there exists a constant $C_\epsilon$, depending on $\epsilon$ and $p$, so that:

$$p \int_{\Omega_j} v\,|h|^{p-1}\,|\nabla v|^{p-1}\,|\nabla h|\,dx \le \epsilon \int_{\Omega_j} |\nabla v|^p h^p\,dx + C_\epsilon \int_{\Omega_j} v^p |\nabla h|^p\,dx.$$

Therefore, raising (2.2.5) to the power $1/p$ and using Minkowski's inequality along with the elementary inequality:

$$(a+b)^{1/p} \le a^{1/p} + b^{1/p} \text{ for } a, b > 0, \tag{2.2.7}$$

we arrive at:

$$\Big(\int_{\Omega_j} |\nabla v|^p h^p\,dx\Big)^{1/p} \le (\lambda^{1/p} + \epsilon^{1/p})\Big(\int_{\Omega_j} |\nabla v|^p h^p\,dx\Big)^{1/p} + \Big(C_\epsilon \int_{\Omega_j} v^p |\nabla h|^p\,dx\Big)^{1/p}.$$

Choosing $\epsilon < (1-\lambda^{1/p})^p$ and rearranging, we recover (2.2.4). $\square$

**Lemma 2.2.3.** *Suppose that (2.0.5) holds for some $\Lambda > 0$. Then, for all $k > j$ it follows that:*

$$\int_{\Omega_j} \frac{|\nabla u_k|^p}{u_k^p} h^p dx \le C(\Lambda, p) \int_{\Omega_j} |\nabla h|^p dx, \text{ for all } h \in C_0^\infty(\Omega_j),\ h \ge 0. \tag{2.2.8}$$

*Proof.* We will test the weak formulation with (2.2.1) with $h^p u_k^{1-p}$, with $h \in C_0^\infty(\Omega_j)$, $h \ge 0$. To this end, note that since $u_j$ satisfies the Harnack inequality in $\Omega_j$, there exists a constant $c > 0$ so that $u_k > c$ on the support of $h$. It follows that $h^p u_k^{1-p} \in \mathrm{L}_0^{1,p}(\Omega_j)$ is a valid test function. This yields:

$$-\int_{\Omega_j} |\nabla u_k|^{p-2} \nabla u_k \cdot \nabla\Big(\frac{h^p}{u_k^{p-1}}\Big) dx = -\langle \sigma_k, h^p \rangle. \tag{2.2.9}$$

On the other hand:

$$(p-1)\int_{\Omega_j} \frac{|\nabla u_k|^p}{u_k^p} h^p dx \leq -\int_{\Omega_j} |\nabla u_k|^{p-2}\nabla u_k \cdot \nabla\Big(\frac{h^p}{u_k^{p-1}}\Big) dx + \\ + p\int_{\Omega_j} \frac{|\nabla u_k|^{p-1}}{u_k^{p-1}} |\nabla h| h^{p-1} dx. \tag{2.2.10}$$

By Young's inequality (2.2.6), for any $\epsilon > 0$ we estimate the second term on the right by:

$$p\int_{\Omega_j} \frac{|\nabla u_k|^{p-1}}{u_k^{p-1}} |\nabla h| h^{p-1} dx \leq \epsilon \int_{\Omega_j} \frac{|\nabla u_k|^p}{u_k^p} h^p dx + C_\epsilon \int_{\Omega_j} |\nabla h|^p dx. \tag{2.2.11}$$

Here $C_\epsilon$ depends on $p$ and $\epsilon$. Applying (2.2.9) and (2.2.11) into (2.2.10), we estimate:

$$(p-1-\epsilon)\int_{\Omega_j} \frac{|\nabla u_k|^p}{u_k^p} \psi^p dx \leq -\langle \sigma_k, h^p\rangle + C_\epsilon \int_{\Omega_j} |\nabla h|^p dx. \tag{2.2.12}$$

Now, combining Lemma 2.1.1 with the lower form bound (2.0.5), it follows (since $h \in C_0^\infty(\Omega_j)$):

$$-\langle \sigma_k, h^p\rangle \leq \Lambda \int_{\Omega_j} |\nabla h|^p dx. \tag{2.2.13}$$

Combining (2.2.13) and (2.2.12), we deduce (2.2.8). □

The third lemma will only be used in the case $p \geq 2$.

**Lemma 2.2.4.** *Suppose that (2.0.4) holds with $0 < \lambda < (p-1)^{2-p}$. Then, for all $k > j$ it follows that, for all $h \in C_0^\infty(\Omega_j)$, $h \geq 0$:*

$$\int_{\Omega_j} |\nabla (u_k)^{p-1}|^p h^p dx \leq C(\lambda, p)\int_{\Omega_j} |(u_k)^{p-1}|^p |\nabla h|^p dx. \tag{2.2.14}$$

*Proof.* Fix $k \geq j$ and $h$ as in (2.2.14), let $v = u_k$. Note that:

$$\int_\Omega |\nabla v|^p v^{(p-2)p} h^p dx = \frac{1}{(p-1)^2} \int_\Omega |\nabla v|^{p-2} \nabla v \cdot \nabla \Big( v^{(p-1)^2} h^p \Big) dx - \frac{p}{(p-1)^2} \int_\Omega |\nabla v|^{p-2} \nabla v \cdot \nabla h v^{(p-1)^2} h^{p-1} dx. \tag{2.2.15}$$

Note that by the properties of $v$ (see Lemma 3.6), it follows that $v^{(p-1)^2} h^p$ is a valid test function for all $p > 1$. Therefore:

$$\int_\Omega |\nabla v|^{p-2} \nabla v \cdot \nabla \Big( v^{(p-1)^2} h^p \Big) dx = \int_\Omega v^{p(p-1)} h^p d\sigma_k \leq \lambda \int_\Omega |\nabla (v^{p-1} h)|^p dx, \tag{2.2.16}$$

where Lemma 2.1.1 has been applied in the second inequality. By substituting (2.2.16) into (2.2.15), we derive from Minkowski's inequality:

$$\begin{aligned} \Big( \int_\Omega |\nabla v|^p v^{(p-2)p} h^p dx \Big)^{1/p} &\leq \lambda^{(1/p)} (p-1)^{1-2/p} \Big( \int_\Omega |\nabla v| v^{(p-2)p} h^p dx \Big)^{1/p} \\ &+ \lambda^{(1/p)} \Big( \int_\Omega v^{p(p-1)} |\nabla h|^p dx \Big)^{1/p} \\ &+ \Big( \frac{p}{(p-1)^2} \int_\Omega |\nabla v|^{p-1} v^{(p-2)p+1} |\nabla h| h^{p-1} dx \Big)^{1/p}. \end{aligned} \tag{2.2.17}$$

On the other hand, Young's inequality together with (2.2.7) yield the following estimate the third term in the right hand side of (2.2.17): for any $\epsilon > 0$,

$$\begin{aligned} &\Big( \frac{p}{(p-1)^2} \int_\Omega |\nabla v|^{p-1} v^{(p-2)p+1} |\nabla h| h^{p-1} dx \Big)^{1/p} \\ &\leq \epsilon \Big( \int_{\Omega_j} |\nabla v|^p v^{(p-2)p} h^p dx \Big)^{1/p} + C_\epsilon \Big( \int_{\Omega_j} v^{p(p-1)} |\nabla h|^p dx \Big)^{1/p}. \end{aligned} \tag{2.2.18}$$

By assumption on $\lambda$, we have $\lambda^{(1/p)} (p-1)^{1-2/p} < 1$. Hence we can choose $\epsilon > 0$ sufficiently small in (2.2.18) (in terms of $\lambda$ and $p$), so that:

$$\Big( \int_\Omega |\nabla v|^p v^{(p-2)p} h^p dx \Big)^{1/p} \leq C(\lambda, p) \Big( \int_\Omega v^{p(p-1)} |\nabla h|^p dx \Big)^{1/p},$$

as required. $\square$

### 2.2.3 A uniform gradient estimate: the proof of Proposition 2.2.1

Having established the required Caccioppoli inequalities, we move onto proving Proposition 2.2.1. This follows from using Proposition 2.1.4.

*The proof of Proposition 2.2.1.* Let us fix $k > j$, and let $v = u_k^q$ with $q = \max(p - 1, 1)$.

To prove (2.2.2) and (2.2.3), we will employ Proposition 2.1.4 in $U = \Omega_j$ to show that $v^p$ is doubling in $\Omega_j$, with constants independent of $k$. To verify the hypothesis of Lemma 2.1.4, we first show that $v^p$ satisfies a weak reverse Hölder inequality, i.e. that (2.1.6) holds in $\Omega_j$. To this end, let us fix $B(z, 2s) \subset\subset \Omega_j$, and let us first suppose $1 < p < n$. By Sobolev's inequality, for any $\psi \in C_0^\infty(\Omega_j)$:

$$\Big(\int_{\Omega_j} v^{\frac{pn}{n-p}} |\psi|^{\frac{pn}{n-p}} dx\Big)^{\frac{n-p}{pn}} \leq C\Big(\int_{\Omega_j} |\nabla v|^p |\psi|^p \, dx\Big)^{1/p} + C\Big(\int_{\Omega_j} v^p \, |\nabla \psi|^p \, dx\Big)^{1/p}. \quad (2.2.19)$$

Applying Lemma 2.2.2 (if $p \leq 2$) or Lemma 2.2.4 (if $p \geq 2$) in the first term on the right hand side of (2.2.19), we deduce:

$$\Big(\int_{\Omega_j} v^{\frac{pn}{n-p}} |\psi|^{\frac{pn}{n-p}} dx\Big)^{\frac{n-p}{n}} \leq C \int_{\Omega_j} v^p |\nabla \psi|^p dx. \quad (2.2.20)$$

Specialising (2.2.20) to the case $\psi \in C_0^\infty(B(z, 2s))$, with $\psi \equiv 1$ in $B(z, s)$, and $|\nabla \psi| \leq C/s$, it follows:

$$\Big(⨍_{B(z,s)} (v^p)^{\frac{n}{n-p}} dx\Big)^{\frac{n-p}{n}} \leq C ⨍_{B(z,2s)} v^p \, dx. \quad (2.2.21)$$

The constant in $C > 0$ in (2.2.21) depends on $p$, and $\lambda$. Hence (2.1.6) holds in $U = \Omega_j$, with $w = v^p$ and $q = n/(n-p)$. In the case when $p = n$, we instead use the Sobolev inequality (see e.g. [MZ97], Corollary 1.57): for each $m < \infty$, and for all $f \in C_0^\infty(B(z,2s))$,

$$\Big(\fint_{B(z,2s)} |f(y)|^m \, dy\Big)^{1/m} \leq C(m)\Big(\int_{B(z,2s)} |\nabla f(y)|^p \, dy\Big)^{1/p}. \tag{2.2.22}$$

Using (2.2.22) as in (2.2.19) and following the argument through display (2.2.21), it follows in the case $n = p$ that (2.1.6) holds in $U = \Omega_j$, with $w = v^p$ for any choice $q < \infty$. When $p > n$, it follows in the same fashion from standard Sobolev inequalities that (2.1.6) continues to hold in $U = \Omega_j$, and $w = v^p$ and any $q \leq \infty$.

To apply Proposition 2.1.4, it remains to show $\log(v) \in BMO(\Omega_j)$. For this, note that it follows from Poincaré's inequality that whenever $B(z,2s) \subset \Omega_j$:

$$\fint_{B(z,s)} |\log v - \fint_{B(z,s)} \log v|^p \, dx \leq Cs^{p-n} \int_{B(z,s)} \frac{|\nabla u_k|^p}{u_k^p} \, dx. \tag{2.2.23}$$

Now, note that from Lemma 2.2.3, it readily follows that:

$$\int_{B(z,s)} \frac{|\nabla u_k|^p}{u_k^p} dx \leq Cs^{n-p}. \tag{2.2.24}$$

Indeed, to prove display (2.2.24) one picks $h \in C_0^\infty(B(z,2s)$ so that $h \equiv 1$ on $B(z,s)$ and $|\nabla h| \leq C/s$ in display (2.2.8).

Applying (2.2.24) into (2.2.23), we immediately arrive at:

$$\fint_{B(z,s)} |\log v - \fint_{B(z,s)} \log v|^p dx \leq C(p,\Lambda). \tag{2.2.25}$$

From (2.2.25), we conclude that $\log v \in BMO(\Omega_j)$, with $BMO$-norm depending only on $p, \Lambda$ (see (2.1.4)). In particular, $v^p$ satisfies both (2.1.6) and (2.1.7) in $\Omega_j$. From Proposition 2.1.4, it follows that $v^p$ is doubling in $\Omega_j$, with constants depending on $p$, $\lambda$ and $\Lambda$, see (2.1.8).

Since $\Omega_j$ is a smooth connected set, one can find a Harnack chain from $B(x,2r)$ to the fixed ball $B \subset\subset \Omega_1$. In other words, one can three positive constants $c_0$, $c_1$ and $N > 0$, depending on the smooth parameterization of $\Omega_j$, along with points $x_0, \dots x_N$ and balls $B(x_i, 4r_i) \subset \Omega_j$ so that:

1. $B(x_0, r_0) = B(x, 2r)$, and $B(x_N, r_N) = B$;

2. $r_i \geq c_0 \min(r_0, r_N)$, and $|B(x_i, r_i) \cap B(x_{i+1}, r_{i+1})| \geq c_1 \min(r_0, r_N)^n$ for all $i = 0 \dots N-1$.

Since $v^p$ is doubling in $\Omega_j$, it follows from the chain construction above, along with a Harnack chain argument that:

$$\fint_{B(x,2r)} v^p dx \leq C(B(x,r), p, \Omega_j, B, \lambda, \Lambda) \fint_B v^p dx.$$

It therefore follows from the normalization on $v^p$ that:

$$\fint_{B(x,2r)} v^p dx \leq C(B(x,r), m, M, \Omega_j, B, \lambda, \Lambda). \tag{2.2.26}$$

To complete the proof, first suppose $p \geq 2$. In this case, we combine the Caccioppoli inequalities (Lemmas 2.2.2 and 2.2.3) with the estimate (2.2.26) to conclude that the following two estimates hold:

$$\int_{B(x,r)} |\nabla u_k|^p dx \leq \frac{C}{r^{p-n}} \Big( \fint_{B(x,2r)} v^p dx \Big)^{1/q} \leq C,$$

and:

$$\int_{B(x,r)} |\nabla u_k^{p-1}|^p dx \leq \frac{C}{r^{p-n}} \Big( \fint_{B(x,2r)} v^p dx \Big)^{1/q} \leq C,$$

for a constant $C > 0$, depending on $n$, $B$, $\Lambda$, $\lambda$, $\Omega_j$ and $B(x,r)$. Here we have also used Hölder's inequality in the first of the two displays above. Hence both estimates (2.2.2) and (2.2.3) are proved for a constant independent of $k$.

In the case $1 < p < 2$, note that combining Lemma 2.2.2 with (2.2.26), we conclude that the estimate

$$\int_{B(x,r)} |\nabla u_k|^p dx \leq \frac{C}{r^{p-n}} \fint_{B(x,2r)} v^p dx \leq C,$$

holds, for a positive constant $C > 0$ depending on $n$, $B$, $\Lambda$, $\lambda$, $\Omega_j$ and $B(x,r)$. On the other hand, as in display (2.2.24), as a consequence of Lemma 2.2.3 it follows that:

$$\int_{B(x,r)} \frac{|\nabla u_k|^p}{u_k^p} dx \leq C,$$

(making the constant $C > 0$ larger if necessary). But these estimates readily combine to yield (2.2.3). Indeed:

$$\begin{aligned}\int_{B(x,r)} |\nabla u_k|^p u_k^{p(p-2)} dx &\leq \int_{B(x,r)\cap\{u_k \geq 1\}} |\nabla u_k|^p u_k^{p(p-2)} dx \\ &+ \int_{B(x,r)\cap\{u_k \leq 1\}} |\nabla u_k|^p u_k^{p(p-2)} dx \\ &\leq \int_{B(x,r)} |\nabla u_k|^p dx + \int_{B(x,r)} \frac{|\nabla u_k|^p}{u_k^p} dx \leq C\end{aligned} \tag{2.2.27}$$

Here we have used that $-1 < p - 2 < 0$. This completes the proposition in the case $1 < p < 2$. □

### 2.2.4 Convergence to a solution

Our first goal is to deduce the existence of a solution $u^{(j)}$ of (2.0.2) in each $\Omega_j$. We will concentrate on the argument in $\Omega_1$.

From (2.2.2) and (2.2.3), it follows by choosing a suitable covering of $\Omega_1$ that:

$$\int_{\Omega_1} |\nabla u_k|^p dx \leq C, \text{ and } \int_{\Omega_1} |\nabla (u_k)^{p-1}|^p dx \leq C, \tag{2.2.28}$$

for $k \geq 2$, for a constant $C = C(\lambda, \Lambda, n, \Omega_1, B, p)$. By weak compactness in Sobolev spaces, we deduce that there is a subsequence $u_{j,1}$ of $u_j$, and $u^{(1)}$ such that:

1. $u_{j,1} \to u^{(1)}$ weakly in $L^{1,p}(\Omega_1)$;

2. $u_{j,1}^{p-1} \to (u^{(1)})^{p-1}$ weakly in $L^{1,p}(\Omega_1)$;

3. $u_{j,1} \to u^{(1)}$ a.e. in $\Omega_1$.

Indeed, (1) and (3) follow from (2.2.2) and weak compactness and Rellich's theorem. But then $u_{j,1}^{p-1}$ converges almost everywhere to $(u^{(1)})^{p-1}$ in $\Omega_1$. Since $u_{j,1}^{p-1}$ is uniformly bounded in $L^{1,p}(\Omega_1)$, it follows from standard Sobolev space theory (see Theorem 1.31 of [HKM06]) that (2) holds.

Let $h \in C_0^\infty(\Omega_1)$, and let $U \subset\subset \Omega_1$ be an open set so that $U \supset \text{supp}(h)$. Recall that $\sigma \in L^{-1,p'}(U)$, and hence from property (2) it follows:

$$\langle \sigma_{j,1}, u_{j,1}^{p-1} h\rangle \to \langle \sigma, (u^{(1)})^{p-1} h\rangle, \text{ as } j \to \infty. \tag{2.2.29}$$

Indeed, by the triangle inequality we write:

$$|\langle \sigma_{j,1}, u_{j,1}^{p-1} h\rangle - \langle \sigma, (u^{(1)})^{p-1} h\rangle| \leq |\langle \sigma, (u_{j,1}^{p-1} - (u^{(1)})^{p-1}) h\rangle| + |\langle (\sigma_{j,1} - \sigma), u_{j,1}^{p-1} h\rangle|$$

The first term on the right converges to zero on account of the weak convergence property (2). For the second term, we only need to estimate:

$$|\langle (\sigma_{j,1} - \sigma), u_{j,1}^{p-1} h\rangle| \leq ||\nabla (u_{j,1}^{p-1} h)||_{L^p(U)} ||\sigma_{j,1} - \sigma||_{L^{-1,2}(U)}.$$

The right hand side here convergences to zero due to standard properties of the mollification, along with the uniform bound (2.2.3). This establishes (2.2.29).

We next claim that there is another subsequence of $u_{j,1}$ (again denoted by $u_{j,1}$) such that:

$$|\nabla u_{j,1}|^{p-1} \to |\nabla u^{(1)}|^{p-1} \text{ in } L^1_{\text{loc}}(\Omega_1). \tag{2.2.30}$$

The proof of (2.2.30) will be quite involved. For this reason we postpone the proof to Section 2.2.5, and complete the rest of the argument.

From (2.2.29) and (2.2.30), it follows that:

$$-\mathrm{div}(|\nabla u^{(1)}|^{p-2}\nabla u^{(1)}) = \sigma(u^{(1)})^{p-1} \text{ in } \mathcal{D}'(\Omega_1). \tag{2.2.31}$$

Here the dominated convergence theorem has been used on the left hand side, in conjunction with (2.2.30). On the right hand side, we have applied the estimate (2.2.29). In addition, from the normalization of the sequence $\{u_j\}_j$ in (2.2.1), it follows from Rellich's theorem that:

$$\int_B (u^{(1)})^{qp} dx = 1, \text{ with } q = \max(p-1, 1).$$

Repeating this argument in each $\Omega_k$, by choosing a consecutive subsequence $u_{j,k}$ of $u_{j,k-1}$, we arrive at functions $u^k$ so that:

$$-\mathrm{div}(|\nabla u^{(k)}|^{p-2}\nabla u^{(k)}) = \sigma(u^{(k)})^{p-1} \text{ in } \mathcal{D}'(\Omega_k), \tag{2.2.32}$$

with:

$$\int_B (u^{(k)})^{qp} dx = 1, \text{ with } q = \max(p-1, 1). \tag{2.2.33}$$

Clearly $u_k = u_{k-1}$ in $\Omega_{k-1}$ (equality here holding in the sense of Sobolev functions).

In conclusion, if one defines a function $u$ by the formula $u = u_k$ in $\Omega_k$, then $u$ is well defined and:

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \sigma u^{p-1} \text{ in } \Omega.$$

From (2.2.33) it follows that $u$ is not the zero function. Furthermore, from construction and properties of $u_k$, it follows that $u$ is locally doubling in $\Omega$. Therefore $u > 0$ a.e. in $\Omega$, and hence $\log(u)$ is well defined almost everywhere.

Let us next show that $u$ satisfies (2.0.6). Note that first $\log(u_{j,k}) \to \log(u)$ a.e. in

$\Omega_k$. Hence from Lemma 2.2.3 and weak compactness, it follows that:

$$\int_\Omega \frac{|\nabla u|^p}{u^p}|h|^p dx \le C(\Lambda, p)\int_\Omega |\nabla h|^p \text{ for all } h \in \Omega_k. \tag{2.2.34}$$

Since there is no dependence on $k$ in constant appearing in (2.2.34), we let $k \to \infty$ to deduce (2.0.6).

Save for the estimate (2.2.30) (which will be proved in Section 2.2.5), the remaining part of statement (i) of Theorem 2.0.2 is to show that $v = \log(u)$ is a solution of (2.0.3). This is the content of the following lemma:

**Lemma 2.2.5.** *Let $\Omega$ be an open set, and suppose that $\sigma \in L^{-1,p'}_{loc}(\Omega)$. If there exists a positive solution $u$ of (2.0.2) satisfying (2.0.6), then $v = \log(u) \in L^{1,p}_{loc}(\Omega)$ is a solution of (2.0.3) so that (2.0.7) holds.*

*Proof.* Let $\epsilon > 0$. Then for $h \in C_0^\infty(\Omega)$, test the weak formulation of (2.0.2) with $\psi = h(u+\epsilon)^{1-p} \in L^{1,p}_c(\Omega)$. This yields:

$$\int_\Omega \frac{|\nabla u|^{p-2}\nabla u}{(u+\epsilon)^{p-1}}\cdot \nabla h\, dx = (p-1)\int_\Omega \frac{|\nabla u|^{p-2}\nabla u\cdot\nabla u}{(u+\epsilon)^p} h dx + \langle \sigma \frac{u^{p-1}}{(u+\epsilon)^{p-1}}, h\rangle. \tag{2.2.35}$$

It follows from the condition (2.0.6) and dominated convergence that as $\epsilon \to 0$:

$$\int_\Omega \frac{|\nabla u|^{p-2}\nabla u}{(u+\epsilon)^{p-1}}\cdot \nabla h\, dx \to \int_\Omega \frac{|\nabla u|^{p-2}\nabla u}{u^{p-1}}\cdot\nabla h\, dx, \text{ and}$$

$$\int_\Omega \frac{|\nabla u|^p}{(u+\epsilon)^p} h dx \to \int_\Omega \frac{|\nabla u|^p}{u^p} h dx.$$

To handle the last term in (2.2.35), note that from (2.0.6) and the dominated convergence theorem:

$$\nabla\Big(\frac{u}{u+\epsilon}\Big)^{p-1} = (p-1)\frac{\nabla u}{u}\Big(\frac{\epsilon u^{p-1}}{(u+\epsilon)^p}\Big) \to 0 \text{ in } L^p_{\mathrm{loc}}(\Omega) \text{ as } \epsilon \to 0,$$

on the other hand, it is clear that:

$$\frac{u}{u+\epsilon} \to 1 \text{ in } L^p_{\mathrm{loc}}(\Omega) \text{ as } \epsilon \to 0.$$

Thus it follows:

$$\frac{u}{u+\epsilon} \to 1 \text{ in } L^{1,p}_{\mathrm{loc}}(\Omega), \text{ as } \epsilon \to 0.$$

But since $\sigma \in L^{-1,p'}(V)$ for any $V \subset\subset \Omega$, we conclude:

$$\langle \sigma \Big(\frac{u}{u+\epsilon}\Big)^{p-1}, h\rangle \to \langle \sigma, h\rangle, \text{ as } \epsilon \to 0.$$

It follows that $v = \log(u)$ is a solution of (2.0.3). □

### 2.2.5 Convergence in measure

It remains to prove (2.2.30). Let us follow a well known reduction - from Vitali's convergence theorem and display (2.2.2); (2.2.30) will follow once we show that $\nabla u_{j,1}$ converges locally in measure to $\nabla u^{(1)}$ in $\Omega_1$. Since the proof of this latter fact will be slightly lengthy, let us state it as a lemma.

**Lemma 2.2.6.** *For any ball $B_r = B(x, r)$ so that $B_{2r} = B(x, 2r) \subset \Omega_1$, it follows that, for any $\delta$:*

$$|\{x \in B_r \,:\, |\nabla u_{j,1} - \nabla u_{k,1}| > \delta\}| \to 0 \text{ as } j, k \to \infty,$$

*Proof.* Our proof follows the standard method of [BBGPV], but is rather involved due to the distributional nature of $\sigma$. The proof hinges on the local dual Sobolev character of $\sigma$. Let $\delta > 0$, and let $\epsilon > 0$. First let us fix some notation: let $u_{j,1} = v_j$, and $u^{(1)} = v$. We write:

$$|\{x \in B_r \,:\, |\nabla u_{j,1} - \nabla u_{k,1}| > \delta\}| \leq I + II + III + IV$$

with, for $A, \mu > 0$:

$$I = |\{x \in B_r \,:\, |\nabla v_j| > A\}| + |\{x \in B_r \,:\, |\nabla v_k| > A\}|;$$

$$II = |\{x \in B_r \,:\, v_j > A\}| + |\{x \in B_r \,:\, v_k > A\}|;$$

$$III = |\{x \in B_r \,:\, |v_j - v_k| > \mu\}|;$$

and finally, $IV = |E|$, with:

$$\begin{aligned} E = \{x \in B_r \,: &|\nabla v_j - \nabla v_k| > \delta,\, |v_j - v_k| \leq \mu;\, |\nabla v_j| \leq A; \\ &|\nabla v_k| \leq A;\, v_j < A,\, v_k < A\}. \end{aligned} \tag{2.2.36}$$

It is the estimate for $IV$ which is not immediate. Our goal is the estimate:

$$IV \leq C(A, \delta) \cdot \mu^{\min(1,p-1)} + o(1), \text{ as } j, k \to \infty. \tag{2.2.37}$$

Let us show how this estimate allows us to conclude the lemma. First, let us pick $A$ such that:

$$I + II \leq \epsilon/4,$$

one can clearly do this by the uniform integrability properties of the relevant functions from estimates (2.2.2) and (2.2.3). Next (with $A > 0$ fixed), let us pick $\mu > 0$ and $N_1$ so that if $j, k > N_1$ then:

$$IV \leq \epsilon/4,$$

this estimate follows from (2.2.37). Now, with $\mu > 0$ fixed, it follows from the almost everywhere convergence of $v_j$ to $v$ one can choose $N \geq N_1$ so that for $j, k > N$ so that:

$$III \leq \epsilon/2.$$

We conclude that for $j, k > N$:

$$|\{x \in B_r \,:\, |\nabla u_{j,1} - \nabla u_{k,1}| > \delta\}| \leq \epsilon,$$

as required.

It therefore requires to prove (2.2.37). To this end, note the following elementary vector inequalities, if $p \geq 2$:

$$c\,|a-b|^p \leq (|a|^{p-2}\,a - |b|^{p-2}\,b)\cdot(a-b), \tag{2.2.38}$$

and, if $1 < p < 2$;

$$c\frac{|a-b|^2}{(|a|+|b|)^{2-p}} \leq (|a|^{p-2}\,a - |b|^{p-2}\,b)\cdot(a-b), \tag{2.2.39}$$

here $a, b \in \mathbf{R}^n \backslash \{0\}$, and $c > 0$ is a positive constant depending on $p$. Now, use the properties of $E$, along with the two inequalities above, to deduce that for all $x \in E$:

$$c(\delta, A) \leq \Big(|\nabla v_j|^{p-2}\nabla v_j - |\nabla v_k|^{p-2}\nabla v_k\Big)\cdot\nabla(v_j - v_k)(x)$$

Let $h \equiv 1$ on $B_r$ such that $h \in C_0^\infty(B_{2r})$. It follows from the above inequality, and since $u_j < A$ and $u_k < A$ in $E$, that:

$$IV \leq \frac{c(\delta, A)}{A}\int_E \Big((|\nabla v_j|^{p-2}\nabla v_j - |\nabla v_k|^{p-2}\nabla v_k)\cdot\nabla(v_j - v_k)\Big)(2A - \max(u_j, u_k))_+ h^p dx$$

Next, let $E_1 = E \cap \{v_j \geq v_k\}$, and $E_2 = E\backslash E_1$. We will concentrate on the previous integral with $E$ replaced by $E_1$, and the estimate for $E_2$ will follow by the same method. Let us define:

$$f = (\mu - (v_j - v_k)_+)_+, \text{ and } g = (2A - \max(v_k, v_j))_+. \tag{2.2.40}$$

Notice the following properties of $f$ and $g$:

$$\begin{gathered} 0 \le f \le \mu, \ \operatorname{supp}(\nabla f) \subset \{0 \le v_k \le v_j;\, v_j - v_k \le \mu\}, \text{ and:} \\ \nabla f = -\nabla v_j + \nabla v_k, \text{ on } \operatorname{supp}(\nabla f). \end{gathered} \tag{2.2.41}$$

$$\operatorname{supp}(\nabla g) \subset \{v_j, v_k \le 2A\}, \text{ and } g \le 2A. \tag{2.2.42}$$

Also, it is immediate that the product $fgh^p \in L^\infty(B_{2r}) \cap L_0^{1,p}(B_{2r})$ (recall $h \in C_0^\infty(B_{2r})$), and hence is a valid test function for (2.2.1). In what follows we will often make use of estimates (2.2.2) and (2.2.3) from Lemma 2.2.1.

It suffices to estimate:

$$V = -\int_\Omega \Big( (|\nabla v_j|^{p-2}\nabla v_j - |\nabla v_k|^{p-2}\nabla v_k) \cdot \nabla f \Big) gh^p dx,$$

since $IV \le (c(\delta, A)/A) \cdot V$. Here we are using that: $(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) \cdot (\xi - \eta) \ge 0$ for all $\xi$, $\eta \in \mathbb{R}^n$ (see (2.2.38) and (2.2.39)).

By using the test function $fgh^p$ in (2.2.1), we deduce:

$$V = -VI - VII + VIII, \text{ with:}$$

$$\begin{aligned} VI &= \int_\Omega \Big( (|\nabla v_j|^{p-2}\nabla v_j - |\nabla v_k|^{p-2}\nabla v_k) \cdot \nabla g \Big) fh^p dx, \\ VII &= p\int_\Omega \Big( (|\nabla v_j|^{p-2}\nabla v_j - |\nabla v_k|^{p-2}\nabla v_k) \cdot \nabla h \Big) fgh^{p-1} dx, \\ VIII &= \int_\Omega fgh^p (v_j^{p-1}\sigma_j - v_k^{p-1}\sigma_k) dx, \end{aligned}$$

where the equation (2.2.1) has been used in $VIII$. It is now the term $VIII$ which requires care. Indeed, from (2.2.2), Hölder's inequality and the properties of $f$ and $g$ in (2.2.41) and (2.2.42), it follows:

$$|VI| \leq C\mu\Big(\int_\Omega |\nabla v_j|^p h^p dx + \int_\Omega |\nabla v_k|^p h^p dx\Big) \leq C\mu, \text{ and :}$$

$$|VII| \leq C\mu A\Big(\int_\Omega |\nabla v_j|^p h^p dx + \int_\Omega |\nabla v_k|^p h^p dx\Big) \leq CA\mu.$$

Both these estimates are good, when compared to (2.2.37).

Now, to handle the remaining term we need to make use of the local dual Sobolev property of $\sigma$. Indeed, there exists $\vec{T} \in L^{p'}(B_{2r})^n$ so that: $\sigma = \text{ div } \vec{T}$ in $\mathcal{D}'(B_{2r})$. It follows that: $\sigma_j = \text{div}(\vec{T}_j)$ with $\vec{T}_j = \phi_{\epsilon_j} * T$. Note that from Minkowski's integral inequality:

$$||T_j||_{L^{p'}(B_{2r})} \leq ||T||_{L^{p'}(B_{2r})}.$$

Let us proceed by writing $VI$ as:

$$VIII = IX + X + XI,$$

with:

$$IX = \int_\Omega (\nabla(v_j^{p-1}) \cdot \vec{T}_j - \nabla(v_k^{p-1}) \cdot \vec{T}_k) f g h^p dx;$$

$$X = \int_\Omega (v_j^{p-1}\vec{T}_j - v_k^{p-1}\vec{T}_k) \cdot (\nabla g) f h^p dx + p\int_\Omega (v_j^{p-1}\vec{T}_j - v_k^{p-1}\vec{T}_k) \cdot (\nabla h) h^{p-1} f g dx;$$

and:

$$XI = \int_\Omega (v_j^{p-1}\vec{T}_j - v_k^{p-1}\vec{T}_k) \cdot (\nabla f) g h^p dx.$$

The estimate for $XI$ will be the most delicate (when the gradient falls on $f$). For $IX$ and $X$, note that $f \leq \mu$, and if both $\nabla g \neq 0$ and $f \neq 0$, then $\max(v_j, v_k) < 2A$.

Therefore, from (2.2.3) and (2.2.2), it follows:

$$|IX| + |X| \leq C||\vec{T}||_{L^{p'}(B_{2r})}(A\mu + A^{p-1}\mu).$$

It remains to estimate $XI$. To this end, let:

$$F = \{x \in B_{2r} \,:\, 0 \leq v_j - v_k \leq \mu,\, v_j \leq 2A\} \supset \mathrm{supp}(\nabla f) \cap \mathrm{supp}(g) \cap B_{2r},$$

and note that, by definitions and the triangle inequality:

$$\begin{aligned} |XI| \leq & \Big| \int_F \nabla(v_j - v_k) \cdot \vec{T}_j (v_j^{p-1} - v_k^{p-1})(2A - v_j) h^p dx \Big| \\ & + \Big| \int_F \nabla(v_j - v_k) \cdot (\vec{T}_j - \vec{T}_k) v_k^{p-1} (2A - v_j) h^p dx \Big|. \end{aligned} \tag{2.2.43}$$

Now, estimate:

$$\begin{aligned} & \Big| \int_F \nabla(u_j - u_k) \cdot (\vec{T}_j - \vec{T}_k) v_k^{p-1} (2A - v_j) h^p dx \Big| \\ & \quad \leq CA^p(||\nabla(v_j h^p)||_p + ||\nabla(v_k h^p)||_p)||(\vec{T}_j - \vec{T}_k)h^p||_{p'} \\ & \quad \leq C(\mathrm{supp}(h))||(\vec{T}_j - \vec{T}_k)h^p||_{p'} \end{aligned} \tag{2.2.44}$$

where the third inequality follows from (2.2.2). From standard properties of the mollification, we therefore deduce that:

$$\Big| \int_F \nabla(u_j - u_k) \cdot (\vec{T}_j - \vec{T}_k) v_k^{p-1} (2A - v_j) h^p dx \Big| \leq C(A) o(1).$$

We have reduced matters to estimating:

$$XII = \Big| \int_F \nabla(v_j - v_k) \cdot \vec{T}_j (v_j^{p-1} - v_k^{p-1})(2A - v_j) h^p dx \Big|.$$

To this end, note that for $x \in F$, it follows for $1 < p < 2$:

$$v_j^{p-1} - v_k^{p-1} \leq (v_j - v_k)^{p-1} \leq \mu^{p-1},$$

and, if $p \geq 2$, it follows:

$$v_j^{p-1} - v_k^{p-1} \leq (p-1)(v_j - v_k) \cdot (v_j^{p-2} + v_k^{p-2}) \leq C(p-1)A^{p-2}\mu.$$

It therefore follows that:

$$\begin{aligned} XII \leq & CA^{1+\max(p-2,0)}\mu^{\min(p-1,1)} \int_F |\nabla(v_j - v_k)||\vec{T}_j||h^p|dx \\ & \leq CA^{1+\max(p-2,0)}\mu^{\min(p-1,1)}, \end{aligned} \tag{2.2.45}$$

here the second inequality follows from (2.2.2). Putting our estimates together, (2.2.37) follows. □

### 2.2.6 Proof of Theorem 2.0.2, statement (ii)

Let us move onto to the straightforward task of proving statement (ii) of Theorem 2.0.2.

*Proof of Theorem 2.0.2, statement (ii).* This part of the proof is completed by an application of Young's inequality. Indeed, it there exists a solution $v \in L^{1,p}_{\text{loc}}(\Omega)$ of (2.0.3), then testing the weak formulation of (2.0.3) with $|h|^p$ for $h \in C_0^\infty(\Omega)$,

$$\langle \sigma, |h|^p \rangle \leq p \int_\Omega |\nabla v|^{p-1} |\nabla h| |h|^{p-1} dx - \int_\Omega |\nabla v|^p |h|^p dx. \tag{2.2.46}$$

From the inequality:

$$ab \leq \frac{1}{p} a^p + \frac{1}{p'} b^{pl}. \tag{2.2.47}$$

with $a = |\nabla h|$ and $b = p|\nabla v|^{p-1}|h|^{p-2}$, it follows:

$$\langle \sigma, |h|^p \rangle \leq \int_\Omega |\nabla h|^p dx,$$

i.e. (2.0.4) holds with $\lambda = 1$.

Now, let us in addition suppose that $v$ satisfies (2.0.7) with a constant $C_0 > 0$. Then, by testing (2.0.3) again with $|h|^p$ for $h \in C_0^\infty(\Omega)$, we can estimate:

$$\begin{aligned} \langle \sigma, |h|^p \rangle &\geq -p \int_\Omega |\nabla v|^{p-1} |\nabla h| |h|^{p-1} dx - \int_\Omega |\nabla v|^p |h|^p dx \\ &\geq -2 \int_\Omega |\nabla v|^p h^p dx - \int_\Omega |\nabla h|^p dx. \end{aligned} \tag{2.2.48}$$

Where in the second inequality we have used (2.2.47) as above. Now, applying (2.0.7) we conclude:

$$\langle \sigma, |h|^p \rangle \geq -(2C_0 + 1) \int_\Omega |\nabla h|^p dx.$$

Hence (2.0.5) holds with $\Lambda = 2C_0 + 1$. □

## 2.3 A remark on higher integrability

In this section we remark on higher integrability of positive solutions of (2.0.2). This observation is essentially due to Brézis and Kato [BK79], and is a restricted variant of the iterative technique of Moser [Mos60].

**Theorem 2.3.1.** *Suppose that $\Omega \subset \mathbf{R}^n$ is an open set, and suppose $1 < p < \infty$. In addition suppose that $\sigma \in L^{-1,p}_{loc}(\Omega)$ satisfies the upper bound (2.0.4) with constant $\lambda > 0$ and the lower bound (2.0.4) for $\Lambda > 0$. Then, for each $q < \infty$, there exists $\lambda(q) > 0$ so that if $\lambda < \lambda(q)$, then there exists a positive solution $u \in L^{1,p}_{loc}(\Omega) \cap L^q_{loc}(\Omega)$ of (2.0.2).*

We will prove Theorem 2.3.1 in the case $n \geq 3$, since the result follows from standard Sobolev inequalities in dimensions $n = 1, 2$. We will continue to use the notation from the proof of Theorem 2.0.2 from Section 2.2. In particular, we will assume without loss of generality that $\Omega$ is connected, and we will use the approximate sequence of solutions constructed from (2.2.1). The result is based on an iterative use of the following lemma:

**Lemma 2.3.2.** *Let $s > p$, and suppose that:*

$$\lambda < \lambda(s) = (s - p + 1)\left(\frac{p}{s}\right)^p. \tag{2.3.1}$$

*Then, for all $k > j$, it follows:*

$$\int_{\Omega_j} |\nabla (u_k)^{s/p}|^p |h|^p dx \leq C(\lambda, p, s) \int_{\Omega_j} u_k^s |\nabla h|^p dx, \text{ for all } h \in C_0^\infty(\Omega_j). \tag{2.3.2}$$

*Proof.* The proof follows exactly as in Lemma 2.2.4, so we leave the details to the reader. □

*Proof of Theorem 2.3.1.* Let us fix $q > 0$, and choose

$$s_j = \left(\frac{n-p}{n}\right)^j q,$$

for $j = 0, \ldots, N$. Here $N$ is chosen to be the largest integer so that $s_N > p$. Note $s_N \leq np/(n-p)$. Let us suppose that $\lambda < \lambda(s_1)$ as defined in (2.3.1). Since (2.3.1) is monotone decreasing for $q > p$, if $\lambda < \lambda(s_1)$, then $\lambda < \lambda(s_j)$ for all $1 \leq j \leq N$.

Notice that using the Sobolev inequality in (2.3.2), we obtain for for all $k > j$ and for each $\ell = 0, \ldots N-1$:

$$\left(\int_{\Omega_j} u_k^{s_\ell} |h|^p dx\right)^{\frac{n-p}{n}} \leq \int_{\Omega_j} u_k^{s_{\ell+1}} |\nabla h|^p dx, \text{ for any } h \in C_0^\infty(\Omega). \tag{2.3.3}$$

Let us now fix a ball $B(x, 8r) \subset \Omega_j$, and define $N$ functions $h_\ell$, for $\ell = 0 \ldots N-1$, so that:

$$h_\ell \in C_0^\infty(B(x,(1+\frac{\ell+1}{N})r)),\ h_\ell \equiv 1 \text{ on } B(x,(1+\frac{\ell}{N})r),\ |\nabla h_\ell| \le \frac{CN}{r}.$$

Applying these test functions in (2.3.3), we obtain, for $\ell = 0, \dots N-1.$:

$$\Big(\int_{B(x,(1+\ell/N)r)} u_k^{s_\ell} dx\Big)^{\frac{n-p}{n}} \le C\Big(\frac{N}{r}\Big)^p \int_{B(x,(1+(\ell+1)/N)r)} u_k^{s_{\ell+1}} dx.$$

After an $N$-fold application of the preceeding display, we conclude:

$$\int_{B(x,r)} u_k^{s_0} dx \le C(N,q,\lambda,r)\Big(\int_{B(x,2r)} u_k^{s_N} dx\Big)^{\frac{nN}{n-p}}.$$

Combining this with the estimate (2.2.2) (by way of Sobolev's inequality, recalling that $N < np/(n-p)$), we arrive at:

$$\int_{B(x,r)} u_k^q dx \le C(N,q,p,B(x,r),\lambda), \text{ for all } k > j. \tag{2.3.4}$$

Now, when we pass to the limit as in Section 2.2.4, we achieve from (2.3.4) a positive solution $u$ of (2.0.2) with the additional property that $u \in L^q_{\text{loc}}(\Omega)$. The theorem is proved. □

One can show that the restriction on $\lambda$ used in the proof of Theorem 2.3.1 is sharp to obtain the integrability $u \in L^q_{\text{loc}}(\Omega)$. This follows by examining the same example as in Remark 2.0.3.

## 2.4 The proof of Theorem 2.0.4

Before the proof of Theorem 2.0.4, it will be useful to introduce a couple of integral operators. We denote by $\mathbf{I}_\alpha(\mu)$ the Riesz potental of order $\alpha$ of a positive measure $\mu$, defined by:

$$\mathbf{I}_\alpha(\mu)(x) = c_n \int_{\mathbf{R}^n} \frac{d\mu(y)}{|x-y|^{n-\alpha}}.$$

For $n \geq 3$, the constant $c_n > 0$ has been chosen so that:

$$-\Delta_x c_n |x-y|^{2-n} = \delta_y \qquad \text{in } \mathcal{D}'(\mathbf{R}^n),$$

where $\Delta_x$ is the Laplace operator in the $x$-variables, and $\delta_y$ is the Dirac delta measure concentrated at the point $y$. If $n = 2$ we let $c_n = 1/(2\pi)$. Let us denote by $(-\Delta)^{-1}$ the Green's operator in $\mathbf{R}^n$ i.e.:

$$(-\Delta)^{-1}(\mu)(x) = \begin{cases} \dfrac{1}{2\pi} \displaystyle\int_{\mathbf{R}^n} \log|x-y| d\mu(y), & \text{if } n = 2, \\ \mathbf{I}_2(\mu)(x), & \text{if } n \geq 3. \end{cases} \tag{2.4.1}$$

*Proof of Theorem 2.0.4.* Let us first prove statement (i). The sufficiency of conditions (2.0.9) and (2.0.10) for the inequality (2.0.8) follows from Hölder's inequality. On the other hand, let $\sigma$ satisfy (2.0.1) with a constant $C > 0$. Then, note that:

$$\tilde{\sigma} = \frac{(p-1)^{2-p}}{2C} \sigma$$

satisfies the hypothesis of Theorem 2.0.2. Therefore there exists $v \in L^{1,p}_{\text{loc}}(\Omega)$ such that:

$$-\text{div}(|\nabla v|^{p-2} \nabla v) = |\nabla v|^p + \tilde{\sigma},$$

with:

$$\int_{\mathbf{R}^n} |\nabla v|^p h^p dx \leq C_1 \int_\Omega |\nabla h|^p, \tag{2.4.2}$$

for a constant $C_1$ depending on $C, p$. Now denote $d\mu = |\nabla v|^p dx$, then $\mu$ is a nonnegative measure, and satisfies the property:

$$\int_{\mathbf{R}^n} |h|^p d\mu \leq C_1 \int_{\mathbf{R}^n} |\nabla h|^p dx, \text{ for all } h \in C_0^\infty(\mathbf{R}^n).$$

It follows by [MV95] (see also Theorem 1.7 of [Ver99]), that there exists a constant $C_2 = C_2(C_1, p)$ such that:

$$\int_E (\mathbf{I}_1(\mu))^{p'} dx \leq C_2 \mathrm{cap}_p(E), \text{ for all compact sets } E \subset \mathbf{R}^n. \tag{2.4.3}$$

We claim that there exists a solution $w$ of:

$$-\Delta w = \Big(\frac{2C}{(p-1)^{2-p}}\Big)\mu = \Big(\frac{2C}{(p-1)^{2-p}}\Big)|\nabla v|^p \text{ in } \mathbf{R}^n, \tag{2.4.4}$$

so that furthermore there exists a constant $C_3 = C_3(C, C_2, n, p)$ with:

$$\int_E |\nabla w|^{p'} dx \leq C_3 \mathrm{cap}_p(E), \text{ for all compact sets } E \subset \mathbf{R}^n. \tag{2.4.5}$$

Indeed, let $\mu_N = |\nabla v|^p \chi_{B(0,2^N)} dx$. Then (2.4.3) is satisfied with $\mu$ replaced by $\mu_N$. Let:

$$w_N = \Delta^{-1}\mu_N - c_N.$$

Where $c_N$ is chosen so that:

$$|\int_{B(0,1)} w_N dx| = 1. \tag{2.4.6}$$

Using the identity that $|\nabla \Delta^{-1}\mu_N| \leq c(n)\mathbf{I}_1(\mu_N)$, we see that:

$$\int_E |\nabla w_N|^{p'} dx \leq C_3 \mathrm{cap}_p(E), \text{ for all compact sets } E \subset \mathbf{R}^n. \tag{2.4.7}$$

Therefore $\{w_N\}_N$ are uniformly bounded in $L^{1,p'}_{\mathrm{loc}}(\mathbf{R}^n)$. By weak compactness and a diagonal argument, there is a subsequence of $w_N$ (still denoted by $w_N$), so that

$w_N \to w$ in $L^{1,p'}_{\text{loc}}(\mathbf{R}^n)$, for some $w \in L^{1,p}_{\text{loc}}(\mathbf{R}^n)$. From (2.4.6), $w$ is not infinite. This limit function $w$ is easily seen to be our desired solution of (2.4.4).

Now, from the capacitary strong type inequality (see e.g. [Maz85, AH96]), the display (2.4.5) is equivalent to:

$$\int_{\mathbf{R}^n} |\nabla w|^{p'} |h|^p \leq C_4 \int_{\mathbf{R}^n} |\nabla h|^p dx. \tag{2.4.8}$$

With $C_4 = (p')^p C_3$. Let:

$$\vec{\Gamma} = -\Big(\frac{2C}{(p-1)^{2-p}}\Big)|\nabla v|^{p-2}\nabla v + \nabla w.$$

From displays (2.4.2) and (2.4.8), we see that $\vec{\Gamma}$ satisfies the conclusion of the theorem.

Let us now turn to statement (ii), which is more straightforward. We suppose $p \geq n$. As in statement (i), we can reduce matters to when $C < (p-1)^{2-p}$ in (2.0.8). Then, applying Theorem 2.0.2, we deduce the exists of $v \in L^{1,p}_{\text{loc}}(\mathbf{R}^n)$, so that:

$$-\text{div}(|\nabla v|^{p-2}\nabla v) = |\nabla v|^p + \sigma \qquad \text{in } \mathbf{R}^n, \tag{2.4.9}$$

with (2.4.2) holding. It is immediate from (2.4.2) and from the definition of capacity (2.0.12) that:

$$\int_E |\nabla v|^p dx \leq C \text{cap}_p(E), \text{ for all compact sets } E \subset \mathbf{R}^n.$$

However, with $p \geq n$, it is well known (see [AH96, Maz85]) that:

$$\text{cap}_p(E) = 0 \text{ for all compact sets } E \subset \mathbf{R}^n.$$

Therefore $|\nabla v| \equiv 0$, and so it clearly follows from (2.4.9) that $\sigma \equiv 0$. □

# Chapter 3

# The fundamental solution

In this chapter we study the fundamental solution associated with certain nonlinear operators perturbed by natural growth terms. Recall the operator $\mathcal{L}$:

$$\mathcal{L}(u) = \mathcal{L}^{(p)}(u) = -\Delta_p u - \sigma\, |u|^{p-2}\, u, \tag{3.0.1}$$

where $\Delta_p u = \mathrm{div}(\nabla u\, |\nabla u|^{p-2})$ is the $p$-Laplacian operator. Throughout this chapter the potential $\sigma$ is a *nonnegative Borel measure* on $\mathbf{R}^n$.

Our main goal is to investigate the interaction between the differential operator $-\Delta_p u$, and the lower order term $\sigma\, |u|^{p-2}\, u$ *pointwise*, under necessary conditions on $\sigma$. This interaction between the differential operator and the lower order term turns out to be highly nontrivial. We will also study the corresponding problem when the $p$-Laplacian is replaced by a more general quasilinear operator, or a fully nonlinear operator of Hessian type.

As described in the introduction, our primary concern will be the equation:

$$\mathcal{L}(u) = \delta_{x_0} \quad \text{in } \mathbf{R}^n, \qquad \inf_{x \in \mathbf{R}^n} u(x) = 0, \tag{3.0.2}$$

where $\delta_{x_0}$ is the Dirac delta measure concentrated at $x_0$. A solution $u(x, x_0)$ of (3.0.2) understood in a suitable weak, or potential theoretic sense (e.g. renormalized, viscosity, or approximate solutions), is called a *fundamental solution* of the operator $\mathcal{L}$, with pole at $x_0$.

In this paper we will assume that $\sigma$ is a positive Borel measure satisfying the following capacity condition:

$$\sigma(E) \leq C \, \mathrm{cap_p(E)} \;\text{ for any compact set } \mathrm{E} \subset \mathbf{R}^{\mathrm{n}}, \tag{3.0.3}$$

where $\mathrm{cap}_p$ is the standard $p$-capacity:

$$\mathrm{cap}_p(E) = \inf\{ \, \|\nabla f\|^p_{\mathrm{L}^p} \;:\; f \geq 1 \text{ on } E, \; f \in C_0^\infty(\mathbf{R}^n) \, \}. \tag{3.0.4}$$

This intrinsic condition, which originated in the work of Maz'ya in the context of linear problems (see [Maz85]), is less stringent than the quasilinear Kato condition (1.0.4). However, when working in this generality, we cannot expect solutions to be continuous or satisfy a Harnack inequality. In particular, our theorems are applicable to the 'Hardy potential' $\sigma(x) = c|x|^{-p}$, see Example 3.1.10 below.

It is easy to see that (3.0.3) with constant $C = 1$ is necessary in order that $u(\cdot, x_0)$ be finite a.e., which is an immediate consequence of the inequality

$$\int_{\mathbf{R}^n} |h|^p \, d\sigma \leq \int_{\mathbf{R}^n} |\nabla h|^p \, dx, \qquad h \in C_0^\infty(\mathbf{R}^n). \tag{3.0.5}$$

The preceding inequality holds whenever there exists a positive supersolution $u$ so that $-\Delta_p u \geq \sigma u^{p-1}$ (see Section 3.3). We observe that, in its turn, (3.0.3) with $C = (p-1)^p/p^p$ yields (3.0.5) (see [Maz85]).

Recall that the fundamental solution of the Laplacian operator plays an important role in the theory of harmonic functions not only because of the principle of super-position, but also because of its importance in understanding how solutions near an isolated singularity can behave, see e.g. Theorem 1.3.7 of [AG01]. The latter theory carries over to the theory of the quasilinear and fully nonlinear operators considered here (see Section 3.2.5), and hence from the bounds for the fundamental solution we deduce a rather complete analysis of the behavior of solutions of $\mathcal{L}(u) = 0$, and the analogue for the $k$-Hessian operator, in the punctured space. For the quasilinear operator, this has been considered under a variety of assumptions on $\sigma$ in [LS08, NSS03, Ser65, Ser64, Veron96]. Isolated singularities of nonlinear operators have been studied recently in [Lab01, Li06].

### 3.0.1 Structure of the chapter

The content of this chapter will be as follows. In Section 3.1 we precisely state our main results regarding the fundamental solution of (3.0.1) and its fully nonlinear analogue.

In Section 3.2, we rapidly review some elements of the theory of nonlinear PDE from a potential theoretic perspective. We are essentially interested in two aspects of this theory: potential estimates for solutions, and weak continuity of the elliptic operators. In this section we also collect a few facts about capacities, and discuss minimal fundamental solutions. After this, in Section 3.3, we discuss how the potential estimates reduce matters to the study of certain nonlinear integral inequalities. In this section we also discuss the necessary capacity conditions on the measure $\sigma$ in order for positive solutions of the differential inequalities $\mathcal{L}u \geq 0$ or $\mathcal{G}u \geq 0$ to exist.

Section 3.4 is concerned with finding a lower bound for any positive solution of a certain nonlinear integral inequality. This bound is proved by estimating successive iterations of the inequality by induction. From this bound Theorems 3.1.2 and 3.1.12 are deduced, and their proofs conclude Section 3.4.

In Section 3.5, we consider the problem of constructing a positive solution to the integral inequality of Section 3.4. This construction forms the main technical step in the arguments asserting Theorems 3.1.5 and 3.1.13, which we prove in Section 3.6. In this section we also discuss criteria for the fundamental solutions of $\mathcal{L}$ and $\mathcal{G}$ to be pointwise equivalent to the fundamental solutions of the unperturbed differential operators.

Finally, in Section 3.7, we consider the Sobolev regularity of the fundamental solution away from its pole. This is the content of Theorem 3.1.8 below.

## 3.1 Main results

We need to introduce some notation before we can state our results. The global bounds will involve two local potentials, a nonlinear Wolff potential, and a linear Riesz potential. If $s > 1, \alpha > 0$ with $0 < \alpha s < n$, we define the local Wolff potential of a measure $\sigma$, for $\rho > 0$, by:

$$\mathbf{W}^{\rho}_{\alpha,s}\sigma(x) = \int_0^{\rho} \left( \frac{\sigma(B(x,r))}{r^{n-\alpha s}} \right)^{1/(s-1)} \frac{dr}{r}. \tag{3.1.1}$$

For $0 < \alpha < n$ the local Riesz potential of $\sigma$ is defined by:

$$\mathbf{I}^{\rho}_{\alpha}\sigma(x) = \int_0^{\rho} \frac{\sigma(B(x,r))}{r^{n-\alpha}} \frac{dr}{r}. \tag{3.1.2}$$

We make the convention that when $\rho = +\infty$, then we write $\mathbf{W}_{\alpha,s}\sigma$ and $\mathbf{I}_{\alpha}\sigma$ for $\mathbf{W}^{\infty}_{\alpha,s}\sigma$ and $\mathbf{I}^{\infty}_{\alpha}\sigma$ respectively. In particular,

$$\mathbf{I}_{\alpha}\sigma(x) = \int_0^{+\infty} \frac{\sigma(B(x,r))}{r^{n-\alpha}} \frac{dr}{r} = (n-\alpha)^{-1} \int_{\mathbf{R}^n} \frac{d\sigma(y)}{|x-y|^{n-\alpha}}. \tag{3.1.3}$$

When $d\sigma = f(x)\,dx$ where $f \in L^1_{loc}(dx)$, we will denote the corresponding potentials by $\mathbf{W}_{\alpha,s}f$ and $\mathbf{I}_{\alpha}f$ respectively.

### 3.1.1 Global bounds for the quasilinear operator

Let us first state our main result for the quasilinear operator $\mathcal{L}$ defined by (3.0.1). We choose to work with solutions in the potential theoretic sense, see Section 3.2 below. The reader should note that these solutions are by definition lower semicontinuous, and hence defined everywhere, and so it makes sense to talk about pointwise bounds. We could have alternatively worked with solutions in the *renormalized sense*, see [DMMOP] for a thorough introduction.

**Definition 3.1.1.** *A fundamental solution (with pole at $x_0$) of the operator $\mathcal{L}$ defined by (3.0.1), is a positive p-superharmonic function $u(\,\cdot\,, x_0)$, such that $u \in L^{p-1}_{\mathrm{loc}}(\sigma)$, satisfying equation (3.0.2). The equality in (3.0.2) is understood in the p-superharmonic sense. See Section 3.2 below for more details.*

When we write $u(x, x_0)$ *is a fundamental solution of* $\mathcal{L}$, with no mention of the pole, we tacitly assume that it has pole at $x_0$.

The first theorem concerns the lower bound for fundamental solutions. Throughout this paper, unless stated otherwise, we will make the assumption that the measure $\sigma$ is not identically 0.

**Theorem 3.1.2.** *a) Let $1 < p < n$, $x_0 \in \mathbf{R}^n$, and suppose $u(\,\cdot\,, x_0)$ is a fundamental solution of $\mathcal{L}$ with pole at $x_0$. Then (3.0.3) holds with $C = 1$. In addition, there is a constant $c > 0$, depending on $n, p$ such that the bound (1.0.11) holds. In other words, for all $x \in \mathbf{R}^n$*

$$u(x, x_0) \geq c\,|x - x_0|^{\frac{p-n}{p-1}} \exp\Big(c\mathbf{W}^{|x-x_0|}_{1,p}(\sigma)(x) + c\mathbf{I}^{|x-x_0|}_{p}(\sigma)(x_0)\Big).$$

*b) If $p \geq n$, and $u$ is a nonnegative p-superharmonic function satisfying the differential inequality:*

$$\mathcal{L}u \geq 0, \quad \textit{in } \mathbf{R}^n$$

*then* $u \equiv 0$.

**Remark 3.1.3.** Part b) of Theorem 3.1.2 is a Liouville theorem, and when $p > n$ it is related to the important recent works of Serrin and Zou (see [SZ02], Theorem II'), and Bidaut-Véron and Pohozaev [BVP01]. When $p = n$ the result is a straightforward consequence of well known local estimates of the Riesz measure of a $p$-superharmonic function, for instance one may use Lemma 3.5 in [KM92]. For several special cases the result follows from those in [BVP01].

**Remark 3.1.4.** As we shall see below (in Lemma 3.3.3), the condition (3.0.3) is in fact necessary for the existence of a positive $p$-superharmonic function satisfying the inequality $\mathcal{L}u \geq 0$ in the $p$-superharmonic sense.

In the case when $1 < p \leq n$, it is a nontrivial fact that when $\sigma \equiv 0$ that the fundamental solution is in fact unique; this was proved in [KV86]. An alternative method is outlined in [TW02a], where uniqueness of the fundamental solution to the fully nonlinear $k$-Hessian operators when $1 \leq k \leq n/2$ is treated. However, when $\sigma$ is not trivial, it is known even in the linear case ($p = 2$, or $k = 1$) that solutions of $\mathcal{L}$ are not necessarily unique for a general measure $\sigma$ (see [Mur86]). It is therefore desirable to single out a distinguished class of fundamental solutions. We are interested in fundamental solutions of $\mathcal{L}$ which behave like the lower bound (1.0.11). The existence of such fundamental solutions, called *minimal fundamental solutions*, is the content of the next theorem.

**Theorem 3.1.5.** *Let* $1 < p < n$, $x_0 \in \mathbf{R}^n$ *and suppose* $\sigma$ *is a nonnegative Borel measure so that (3.0.3) holds. There is a constant* $C_0 = C_0(n,p) > 0$ *such that if (3.0.3) holds with constant* $C < C_0$*, then there exists a fundamental solution* $u(\,\cdot, x_0)$ *of* $\mathcal{L}$ *with pole at* $x_0$*, together with a constant* $c = c(n,p,C) > 0$*, so that the upper bound (1.0.12) holds for all* $x \in \mathbf{R}^n$*, i.e.*

$$u(x, x_0) \leq c\,|x - x_0|^{\frac{p-n}{p-1}} \exp\Big(c\mathbf{W}_{1,p}^{|x-x_0|}(\sigma)(x) + c\mathbf{I}_p^{|x-x_0|}(\sigma)(x_0)\Big).$$

**Remark 3.1.6.** As a corollary of Proposition 3.2.7 - which states that *whenever there exists a fundamental solution of $\mathcal{L}$ with pole at $x_0$, then there exists a unique minimal fundamental solution of $\mathcal{L}$ with pole at $x_0$* - we assert the existence of a unique minimal fundamental solution of (3.0.1) obeying the bounds (1.0.11) and (1.0.12). See Corollary 3.2.9 below.

When $p = 2$, the $p$-Laplacian reduces to the Laplacian operator and Theorems 3.1.2 and 3.1.5 are contained in some very recent work of M. Frazier and I. E. Verbitsky [FV10]. In fact when $p = 2$ the lower bound, Theorem 3.1.2, has been known for some time, under various restrictions on $\sigma$ (see [GH08]). The corresponding upper bound seems to be much deeper. In [FV10], [FNV10] such bounds for the Green function of Schrödinger type equations with the fractional Laplacian operator are discussed.

**Remark 3.1.7.** From our method it is clear that Theorems 3.1.2 and 3.1.5 continue to hold if we replace the $p$-Laplacian operator by the general quasilinear $\mathcal{A}$-Laplacian operator $\operatorname{div}\mathcal{A}(\mathrm{x}, \nabla\mathrm{u})$ (see, e.g., [HKM06], and Section 3.2 below). The constants appearing in the theorems will then in addition depend on the structural constants of $\mathcal{A}$.

Having constructed a fundamental solution, we now turn to considering how regular it is away from the pole $x_0$. This is the content of the next theorem.

**Theorem 3.1.8.** *Suppose the hypothesis of Theorem 3.1.5 are satisfied, and that $u(x, x_0) \not\equiv \infty$, with $u(x, x_0)$ the fundamental solution constructed in Theorem 3.1.5. Then, there exists $C_0 = C_0(n, p) > 0$ so that if (3.0.3) holds with $C < C_0$, then:*

$$u(\,\cdot, x_0) \in L^{1,p}_{loc}(\mathbf{R}^n \backslash \{x_0\}).$$

**Remark 3.1.9.** The local Sobolev regularity $L^{1,p}_{\mathrm{loc}}(\mathbf{R}^n \backslash \{x_0\})$ is optimal for solutions of $\mathcal{L}(u) = 0$ under the assumption (3.0.3) on $\sigma$, see [JMV11]. Theorem 3.1.8 seems to be new in the linear case $p = 2$. In this case the proof, given in Section 3.7, can clearly be easily adapted to deduce the local regularity of the minimal Green's function of the Schrödinger operator in a bounded domain $\Omega$, as was constructed recently in [FV10, FNV10].

Let us now state an example in the case of the 'Hardy' potential $\sigma(x) = \dfrac{c}{|x|^p}$, for $c > 0$.

**Example 3.1.10.** Let $\sigma(x) = \dfrac{c}{|x|^p}$. Then, there exists $c_0 = c_0(n, p) > 0$ so that if $c < c_0$, and $x_0 \neq 0$, there exists positive constants $a_0$, $a_1$ and $a_2$, depending on $n$ and $p$, together with a unique minimal solution of:

$$-\Delta_p u = \frac{c}{|x|^p} u^{p-1} + \delta_{x_0}$$

such that:

$$\frac{1}{a_0} \frac{\left(\max\left\{\frac{|x|}{|x_0|}, \frac{|x_0|}{|x|}\right\}\right)^{a_1 c}}{|x - x_0|^{\frac{p-n}{p-1}}} \leq u(x) \leq a_0 \frac{\left(\max\left\{\frac{|x|}{|x_0|}, \frac{|x_0|}{|x|}\right\}\right)^{a_2 c}}{|x - x_0|^{\frac{p-n}{p-1}}}$$

The simple calculations required to verify this example are analogous to those carried out in [FV10], and so we omit the details here.

### 3.1.2 Global bounds for the fully nonlinear operator

We now move onto a fully nonlinear analogue of Theorems 3.1.2 and 3.1.5. Let $1 \leq k \leq n$ be an integer. Then the second operator we consider, denoted by $\mathcal{G}$, is the fully nonlinear operator defined by:

$$\mathcal{G}(u) = F_k(-u) - \sigma\, |u|^{k-1}\, u. \tag{3.1.4}$$

Here $\sigma$ is again a nonnegative Borel measure, and $F_k$ is the $k$-Hessian operator, introduced by Caffarelli, Nirenberg and Spruck [CNS85], and defined for smooth functions $u$ by:

$$F_k(u) = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \ldots \lambda_{i_k}$$

with $\lambda_1, \ldots \lambda_n$ denoting the eigenvalues of the Hessian matrix $D^2 u$. We will use the notion of $k$-convex functions, introduced by Trudinger and Wang [TW99], to state our results. See Section 3.2 for a brief discussion and definitions.

**Definition 3.1.11.** *A fundamental solution (with pole at $x_0$) $u(\,\cdot\,, x_0)$ of $\mathcal{G}$ is a function such that $-u(\,\cdot, x_0)$ is a k-convex function so that $u(\,\cdot, x_0) \in L^k_{loc}(\sigma)$ satisfying $\mathcal{G}u(\,\cdot, x_0) = \delta_{x_0}$ in the viscosity sense, and $\inf_{x \in \mathbf{R}^n} u(x, x_0) = 0$.*

The necessary condition on $\sigma$ is now considered in terms of the $k$-Hessian capacity, introduced in [TW02b];

$$\mathrm{cap}_k(E) = \sup\{\ \mu_k[u](E)\ :\ u \text{ is } k\text{-convex in } \mathbf{R}^n,\ -1 < u < 0\ \}, \tag{3.1.5}$$

for a compact set $E$. Here $\mu_k[u]$ is the $k$-Hessian measure of $u$; see Theorem 3.2.5 below.

**Theorem 3.1.12.** *a) Let $1 \le k < n/2$, and let $x_0 \in \mathbf{R}^n$. If $u(\,\cdot, x_0)$ is a fundamental solution of $\mathcal{G}$, then there is a constant $C > 0$, $C = C(n, k)$, such that*

$$\sigma(E) \le C\,\mathrm{cap_k(E)} \quad \textit{for all compact sets } \mathrm{E} \subset \mathbf{R}^{\mathrm{n}}. \tag{3.1.6}$$

*In addition, there is a constant $c > 0$, $c = c(n, k, C)$, such that*

$$\begin{aligned} u(x, x_0) \ge c\,|x - x_0|^{2-\frac{n}{k}} \exp\Big(c \int_0^{|x-x_0|} \Big(\frac{\sigma(B(x,r))}{r^{n-2k}}\Big)^{1/k} \frac{dr}{r}\Big) \\ \cdot \exp\Big(c \int_0^{|x-x_0|} \frac{\sigma(B(x_0,r))}{r^{n-2k}} \frac{dr}{r}\Big). \end{aligned} \tag{3.1.7}$$

*b) Let $k \geq n/2$. Then if $u$ is a nonnegative function so that $-u$ is a $k$-convex function satisfying the inequality:*

$$\mathcal{G}(u) \geq 0 \quad \textit{in } \mathbf{R}^n$$

*then $u \equiv 0$.*

**Theorem 3.1.13.** *Let $1 \leq k < n/2$, and suppose $\sigma$ is a nonnegative Borel measure satisfying (3.1.6). There is a constant $C_0 = C_0(n,k)$, such that if $C < C_0$ and (3.1.6) holds with constant $C$, then there exists a fundamental solution $u(\,\cdot, x_0)$ of $\mathcal{G}$, together with a constant $c = c(n,k,C)$ so that*

$$u(x,x_0) \leq c\,|x-x_0|^{2-\frac{n}{k}} \exp\Big(c\int_0^{|x-x_0|}\Big(\frac{\sigma(B(x,r))}{r^{n-2k}}\Big)^{1/k}\frac{dr}{r}\Big) \cdot \exp\Big(c\int_0^{|x-x_0|}\frac{\sigma(B(x_0,r))}{r^{n-2k}}\frac{dr}{r}\Big). \tag{3.1.8}$$

**Remark 3.1.14.** Part b) of Theorem 3.1.12 is easy to see using well known local estimates. For instance, one can readily deduce the result from [TW99], Theorem 3.1, along with a routine approximation argument using weak convergence of Hessian measures.

## 3.2 Preliminaries

### 3.2.1 Nonlinear potential theory for quasilinear operators

In this section we will introduce some fundamental results from the potential theory of nonlinear elliptic equations. Two results will be key to our study: a potential estimate; and a weak continuity result. The potential which the estimates will involve is called the Wolff potential [HW83]. For $s > 1$ and $0 < \alpha s < n$, we define the Wolff potential of a nonnegative Borel measure $\mu$ by:

$$\mathbf{W}_{\alpha,s}\mu(x) = \int_0^\infty \Big(\frac{\mu(B(x,r))}{r^{n-\alpha s}}\Big)^{1/(s-1)} \frac{dr}{r} \tag{3.2.1}$$

We first will discuss quasilinear equations. The material regarding these equations is drawn from [HKM06, KM92, KM94, PV08, PV09, TW02b, MZ97].

Let us assume that $\mathcal{A} : \mathbf{R}^n \,\mathrm{x}\, \mathbf{R}^n \to \mathbf{R}^n$ satisfies:

$$x \to \mathcal{A}(x,\xi) \text{ is measurable for all } \xi \in \mathbf{R}^n, \text{ and}$$

$$\xi \to \mathcal{A}(x,\xi) \text{ is continuous for a.e. } x \in \mathbf{R}^n.$$

In addition suppose that there are constants $0 < \alpha \leq \beta < \infty$ so that for a.e. $x \in \mathbf{R}^n$:

$$\alpha\,|\xi|^p \leq \mathcal{A}(x,\xi)\cdot\xi, \text{ and } |\mathcal{A}(x,\xi)| \leq \beta\,|\xi|^{p-1}.$$

We will also assume that:

$$(\mathcal{A}(x,\xi_1) - \mathcal{A}(x,\xi_2))\cdot(\xi_1 - \xi_2) > 0$$

whenever $\xi_1 \neq \xi_2$.

Now, let $\Omega$ be an open subset of $\mathbf{R}^n$, (we will be most interested in the case $\Omega = \mathbf{R}^n$). Whenever $u \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)$, we define the divergence of $\mathcal{A}(x,\nabla u)$ in the distributional sense. As follows from the classical regularity theory of De Giorgi, Nash and Moser, any $u \in L^{1,p}_{\mathrm{loc}}(\Omega)$ solution of $-\mathrm{div}\mathcal{A}(x,\nabla u) = 0$ in the distributional sense has a locally Hölder continuous representative, and we call these representatives $\mathcal{A}$-harmonic functions. Here and in the following the $p$-Laplacian operator corresponds to the choice of $\mathcal{A}(x,\xi) = |\xi|^{p-2}\,\xi$, in this case $\mathcal{A}$-harmonic functions are called $p$-harmonic functions, and similarly $p$-superharmonic functions are $\mathcal{A}$-superharmonic functions (as defined below) in this special case.

In analogy with classical superharmonic functions, we define the $\mathcal{A}$-superharmonic functions via a comparison principle. We say that $u : \Omega \to (-\infty, \infty]$ is $\mathcal{A}$-superharmonic if $u$ is lower semicontinuous, is not identically infinite in any component of $\Omega$, and satisfies the following comparison principle: Whenever $D \subset\subset \Omega$ and $h \in C(\bar{D})$ is $\mathcal{A}$-harmonic in $D$, with $h \leq u$ on $\partial D$, then $h \leq u$ in $D$.

An $\mathcal{A}$-superharmonic function $u$ does not necessarily have to belong to $\mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)$, but its truncates $T_k(u) = \min(u, k) \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)$ for all $k > 0$. In addition $T_k(u)$ are supersolutions, i.e. $-\mathrm{div}\mathcal{A}(\,\cdot, \nabla T_k(u)) \geq 0$, in the distributional sense (see [HKM06]).

The above paragraph leads us to the definition of the generalized gradient of an $\mathcal{A}$-superharmonic function $u$ as:

$$Du = \lim_{k \to \infty} \nabla(T_k(u)).$$

**Remark 3.2.1.** There are alternative notions of solutions which we could have introduced to obtain our results, for instance either *renormalized solutions* or *supersolutions up to all levels*, see [DMMOP] and [MZ97] respectively. We chose to use the language of $\mathcal{A}$-superharmonic functions because Theorems 3.2.3 and 3.2.4 were developed in this framework.

Let $u$ be $\mathcal{A}$-superharmonic and let $1 \leq q < n/(n-1)$. Then it is proved in [KM92] that $|Du|^{p-1}$ and $\mathcal{A}(\cdot, Du)$ belong to $\mathrm{L}^q_{loc}(\Omega)$. This allows us to define a nonnegative distribution for each $\mathcal{A}$-superharmonic function $u$ by:

$$-\mathrm{div}\mathcal{A}(x, \nabla u)(\psi) = \int_\Omega \mathcal{A}(x, Du) \cdot \nabla\psi \, dx \qquad (3.2.2)$$

for $\psi \in C_0^\infty(\Omega)$. So, the Riesz representation theorem yields the existence of a unique nonnegative Borel measure $\mu[u]$ so that $-\mathrm{div}\mathcal{A}(x, \nabla u) = \mu[u]$. Furthermore, by the integrability of the gradient, it follows that for any $r > n$:

$$\int_\Omega \mathcal{A}(\,\cdot, Du) \cdot \nabla\phi dx = \int_\Omega \phi d\mu, \text{ for all } \phi \in W^{1,r}(\Omega) \text{ with compact support.} \quad (3.2.3)$$

For a nonnegative measure $\omega$ we will say that $-\text{div}\mathcal{A}(\cdot, \nabla u) = \omega$ in the $p$-superharmonic sense if $u$ is $p$-superharmonic, and $\mu[u] = \omega$. Thus $\mathcal{L}(u) = \omega$ in the $p$-superharmonic sense if $\mu[u] = \sigma u^{p-1} + \omega$.

We now state a very useful convergence result, contained in Kileplainen and Maly [KM92], Theorem 1.17.

**Theorem 3.2.2.** *[KM92] Suppose $\{u_j\}_j$ is a sequence of nonnegative $\mathcal{A}$-superharmonic functions in an open set $\Omega$. Then there is a subsequence $\{u_{j_k}\}_k$ which converges almost everywhere to a nonnegative function $u$ which is either $p$-superharmonic or identically infinite in each component of $\Omega$.*

The next result, first stated explicitly in [TW02b], shows that $\mathcal{A}$-Laplace operator is weakly continuous.

**Theorem 3.2.3.** *[TW02b] Suppose $\{u_j\}_j$ is a sequence of nonnegative $\mathcal{A}$-superharmonic functions which converge almost everywhere to an $\mathcal{A}$-superharmonic function $u$. Then $\mu[u_j]$ converges weakly to $\mu[u]$.*

The second major result we need is the Wolff's potential estimates of Kilpeläinen and Maly [KM94] (see also [MZ97], [PV08]).

**Theorem 3.2.4.** *[KM94] Let $u$ be a nonnegative $\mathcal{A}$-superharmonic function in $\mathbf{R}^n$ so that $\inf_{x\in\mathbf{R}^n} u(x) = 0$. If $\mu = -\text{div}\,\mathcal{A}(\cdot\,, \nabla \text{u})$, then there is a constant $K = K(n, p, \alpha, \beta)$, so that for all $x \in \mathbf{R}^n$,*

$$\frac{1}{K}\,\mathbf{W}_{1,p}\mu(x) \le u(x) \le K\,\mathbf{W}_{1,p}\mu(x). \quad (3.2.4)$$

### 3.2.2 Nonlinear potential theory for fully nonlinear operators

We now turn to the fully nonlinear counterpart of these results. A very recent and comprehensive account of the $k$-Hessian equation is [Wan09]. Here $k$-convex functions associated to the $k$-Hessian operator, introduced by Trudinger and Wang [TW99], will play the role of $\mathcal{A}$-superharmonic functions in the quasilinear theory above. Let $\Omega \subset \mathbf{R}^n$ be an open set, let $k = 1, \dots, n$ and $u \in C^2(\Omega)$, then the $k$-Hessian operator is:

$$F_k(u) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \dots \lambda_{i_k}$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the matrix $D^2 u$. We will then say that $u$ is $k$-convex in $\Omega$ if $u : \Omega \to [-\infty, \infty)$ is upper semicontinuous and satisfies $F_k(u) \geq 0$ in the viscosity sense, i.e. for any $x \in \Omega$, $F_k(q)(x) \geq 0$ for any quadratic polynomial $q$ so that $u - q$ has a local finite maximum at $x$. Equivalently (see [TW99]), we may define $k$-convex functions by a comparison principle: an upper semicontinuous function $u : \Omega \to [-\infty, \infty)$ is $k$-convex in $\Omega$ if for every open set $D \subset\subset \Omega$, and $v \in C^2_{\text{loc}}(D) \cap C(\bar{D})$ with $F_k(v) \geq 0$ in $D$, then

$$u \leq v \text{ on } \partial D \implies u \leq v \text{ in } D.$$

Let $\Phi^k(\Omega)$ be the set of $k$-convex functions such that $u$ is not identically infinite in each component of $\Omega$. The following weak continuity result is key to us.

**Theorem 3.2.5.** *[TW99] Let $u \in \Phi^k(\Omega)$. Then there is a nonnegative Borel measure $\mu_k[u]$ in $\Omega$ such that*

- *$\mu_k[u] = F_k(u)$ whenever $u \in C^2(\Omega)$, and*
- *If $\{u_m\}_m$ is a sequence in $\Phi^k(\Omega)$ converging in $L^1_{loc}(\Omega)$ to a function $u$, then the sequence of measures $\{\mu_k[u_m]\}_m$ converges weakly to $\mu_k[u]$.*

The measure $\mu_k[u]$ associated to $u \in \Phi^k(\Omega)$ is called the *Hessian measure* of $u$. Hessian measures were used by Labutin [Lab02] to deduce Wolff's potential estimates for a $k$-convex function in terms of its Hessian measure. The following global version of Labutin's estimate is deduced from his result in [PV08]:

**Theorem 3.2.6.** *[PV08] Let $1 \leq k \leq n$, and suppose that $u \geq 0$ is such that $-u \in \Phi^k(\Omega)$ and $\inf_{x\in\mathbf{R}^n} u(x) = 0$. Then, if $\mu = \mu_k[u]$, there is a positive constant $K$, depending on $n$ and $k$, such that:*

$$c_1 \mathbf{W}_{\frac{2k}{k+1},k+1}\mu(x) \leq u(x) \leq c_2 \mathbf{W}_{\frac{2k}{k+1},k+1}\mu(x), \qquad x \in \mathbf{R}^n.$$

### 3.2.3 Minimal fundamental solutions

This subsection is concerned with minimality of fundamental solutions. A *minimal fundamental solution* $u(x, x_0)$ of $\mathcal{L}$ defined by (3.0.1), is a fundamental solution of $\mathcal{L}$ as in Definition 3.1.1, so that $u(x, x_0) \leq v(x, x_0)$ whenever $v(x, x_0)$ is a fundamental solution of $\mathcal{L}$. Our aim is to prove the following proposition.

**Proposition 3.2.7.** *Let $1 < p < n$ and $\sigma$ be a nonnegative measure. Suppose that there exists a fundamental solution $v(x, x_0)$ of $\mathcal{L}$ with pole at $x_0$. Then there exists a unique minimal fundamental solution $u(x, x_0)$ of $\mathcal{L}$.*

We will need the following simple lemma, and as we could not locate a reference we will provide a proof.

**Lemma 3.2.8.** *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain, and suppose that $v$ is a positive $p$-superharmonic in $\Omega$ so that $T_k(u) \in L^{1,p}(\Omega)$ for all $k > 0$, and $-\Delta_p v = \nu$. Let $\mu \leq \nu$, be a compactly supported measure in $\Omega$, then there is a nonnegative $p$-superharmonc fuction $w$, such that $w \leq v$ and:*

$$-\Delta_p w = \mu \text{ in } \Omega, \;\; w = 0 \text{ continuously on } \partial\Omega. \tag{3.2.5}$$

*Proof.* Let $T_k(v) = \min(v, k)$, and let $\nu_k$ be the Riesz measure of $T_k(v)$. Then $\nu_k \in W^{-1,p'}(\Omega)$, and $\nu_k \to \nu$ weakly. Let $\mu_k$ be a sequence in $W^{-1,p'}(\Omega)$ so that $\mu_k \leq \nu_k$ and $\mu_k \to \mu$ weakly. By the compact support of $\mu$ we may also assume that there is a compactly supported set $K \subset \Omega$, which contains the support of $\mu_k$, for each $k$ (otherwise we just multiply $\mu_k$ by a smooth bump function $\phi \in C_0^\infty(K)$ such that $\phi \equiv 1$ on the support of $\mu$). Let $w_k \in L^{1,p}(\Omega)$ be the solution of:

$$-\Delta_p w_k = \mu_k \text{ in } \Omega, \quad w_k = 0 \text{ on } \partial\Omega.$$

Such a unique solution exists by the theory of monotone operators, see e.g. [Li69]. In addition, $w_k \leq v_k \leq v$ in $\Omega$ by the classical comparison principle. Therefore, by [KM92], Theorem 1.17, we see that by a relabeling of the sequence, we may assert that there is a $p$-superharmonic function $w = \lim_{k\to\infty} w_k$ almost everywhere, with $w \leq v$ and $-\Delta_p w = \mu$.

It remains to prove that $w$ is zero at the boundary and attains its boundary value continuously. First note that each $w_k$ is $p$-harmonic in $\Omega \backslash K$. Since $\Omega$ is Lipschitz, there exists $M \geq 2$, $c > 0$ and $0 < r_0 < d(K, \partial\Omega)/4$, such that for all $z \in \partial\Omega$ and $0 < r < r_0$: $\sup_{B(z,r/c)\cap\Omega} w_k \leq c\, w_k(a(z))$, here $a(z)$ is a point such that $M^{-1}r \leq |a(z) - z| \leq Mr$. This is a well known boundary estimate, see e.g. [BVBV06, LN07]. Combined with the boundary regularity of $p$-harmonic functions, [Maz70] (see also [MZ97, HKM06]), we see that each $w_k$ is locally Hölder continuous in a neighbourhood of each boundary point with constants independent of $k$. Indeed, there exists constants $c, \theta > 0$ depending on $n$ and $p$, such that if $0 < r < r_0$, then for each $z \in \partial\Omega$ and $x, y \in B(z, r/c) \cap \Omega$:

$$\begin{aligned} |w_k(x) - w_k(y)| &\leq c \max_{B(w,r/c)\cap\Omega} w_k \cdot |x-y|^\theta \leq c\, w_k(a(z)) \cdot |x-y|^\theta \\ &\leq c \inf_{B(a(z),r/2M)} w_k \cdot |x-y|^\theta \leq c \inf_{B(a(z),r/2M)} v \cdot |x-y|^\theta . \end{aligned} \tag{3.2.6}$$

The third inequality in display (3.2.6) follows from the second by Harnack's inequality. That $w = 0$ continuously on $\partial\Omega$ follows from (3.2.6). $\square$

By Theorem 3.1.2, we may assume that $\sigma$ satisfies (3.0.3) (see Lemma 3.3.3 below), in proving Proposition 3.2.7. This assumption is the key for the construction, as we will apply uniqueness results. For general measure data, the uniqueness of solutions in a suitable sense is an open problem for the $p$-Laplacian.

*Proof of Proposition 3.2.7.* Let $w$ be any fundamental solution of the operator $\mathcal{L}$ defined by (3.0.1) with pole at $x_0$. We will construct a fundamental solution $u$ so that $u \le w$. This construction will be independent of choice of $w$ and hence will prove the proposition. Our first goal is to show $w \ge u_0 := G(\cdot, x_0)$, with $G(x, x_0)$ defined as in (1.0.10). By using Lemma 3.2.8 repeatedly in a sequence of concentric balls, along with Theorems 3.2.2 and 3.2.3, we assert the existence of a solution $w_0$ of $-\Delta_p w_0 = \delta_{x_0}$ in $\mathbf{R}^n$, with $w_0 \le w$, and hence $\inf_{x\in\mathbf{R}^n} w_0(x) = 0$. Since $G(x, x_0)$ is unique (see [KV86]), it follows that $w_0 = u_0$. Thus $w \ge u_0$.

Now suppose that $w \ge u_{m-1}$. Then, for each $j$ and $k > j$, we see by Lemma 3.2.8 there is a positive $p$-superharmonic function $u_m^{j,k}$ solving:

$$-\Delta_p u_m^{j,k} = (\sigma u_{m-1}^{p-1})\chi_{B(x_0,2^j)} + \delta_{x_0} \quad \text{in } B(x_0, 2^k), \quad u_m^{j,k} = 0 \text{ on } \partial B(x_0, 2^k)$$

with $u_m^{j,k} \le w$. But using Theorem 4.2 of [TW09] (which applies as a simple consequence of (3.0.3), and that $u_m^{k,j}$ being $p$-harmonic near $\partial B(x_0, 2^k)$), we see that $u_m^{j,k}$ is unique (and hence independent of $w$). By combining Theorems 3.2.2 and 3.2.3, we conclude that there exists a $p$-superharmonic function $u_m^j$ such that $-\Delta_p u_m^j = (\sigma u_{m-1}^{p-1})\chi_{B(x_0,2^j)} + \delta_{x_0}$ in $\mathbf{R}^n$. Furthermore $u_m^j \le w$, and hence $\inf_{x\in\mathbf{R}^n} u_m^j(x) = 0$. We remark here that there are other uniqueness results, (for instance see [DMMOP]) which could very probably be used, but the cited theorem above is quickest to verify with our notion of solution.

Again by Theorem 3.2.2, and weak continuity (Theorem 3.2.3), there exists a $p$-superharmonic function $u_m$ such that: $-\Delta_p u_m = \sigma u_{m-1}^{p-1} + \delta_{x_0}$ in $\mathbf{R}^n$ and $u_m \leq w$. Therefore $\inf_{x\in\mathbf{R}^n} u_m(x) = 0$. Appealing to Theorem 3.2.2 and weak continuity a final time, we find a $p$- superharmonic function $u$ such that $-\Delta_p u = \sigma u^{p-1} + \delta_{x_0}$ in $\mathbf{R}^n$ and $u \leq w$, thus $\inf_{x\in\mathbf{R}^n} u(x) = 0$ and $u$ is a fundamental solution of $\mathcal{L}$.

The proposition is proved, since whenever $w$ is a fundamental solution of $\mathcal{L}$, then iteratively we see that $w \geq u_m$ for all $m$ and hence $w \geq u$. □

With this proposition the following Corollary is an immediate consequence of Theorems 3.1.2 and 3.1.5.

**Corollary 3.2.9.** *Suppose that $\sigma$ is a nonnegative measure satisfying (3.0.3) with constant $C > 0$. Then there exists a positive constant $C_0$ depending on $n$ and $p$, so that if $C < C_0$, there exists a unique minimal fundamental solution $u(x, x_0)$ of $\mathcal{L}$ defined by (3.0.1). Furthermore $u(x, x_0)$ satisfies global bilateral bounds (1.0.11) and (1.0.12), with a different constant $c = c(n, p) > 0$ in each direction.*

The existence of a minimal fundamental solution for the $k$-Hessian operators can be shown in a similar way to the quasilinear case presented above, adapting techniques in [TW02a].

### 3.2.4 Capacity

We finish this section with a brief discussion of capacity. In the range of exponents we are interested in, both the $p$-capacity and the $k$-Hessian capacities are equivalent, for compact sets, with certain Riesz capacities.

Let $s > 1$ and $0 < \alpha < n$. For $E \subset \mathbf{R}^n$, we define the Riesz capacity of $E$ by the following:

$$\text{cap}_{\alpha,s}(E) = \inf\{ \|f\|_{L^s}^s \ : \ f \in L^s(\mathbf{R}^n), \ f \geq 0, \ \mathbf{I}_\alpha f \geq 1 \text{ on } E \ \}. \tag{3.2.7}$$

See (3.1.3) for the definition of the Riesz potential $\mathbf{I}_\alpha$.

Recall the $p$-capacity defined in (3.0.4). Then we have the following equivalence.

**Lemma 3.2.10.** *Let $1 < p < n$. Then there is a positive constant $C = C(n,p)$ so that, for all compact sets $E \subset \mathbf{R}^n$:*

$$\frac{1}{C}\mathrm{cap}_{1,\mathrm{p}}(\mathrm{E}) \le \mathrm{cap}_{\mathrm{p}}(\mathrm{E}) \le \mathrm{C}\,\mathrm{cap}_{1,\mathrm{p}}(\mathrm{E}).$$

For a proof of this Lemma, see, e.g., [Maz85] or [MZ97].

Now, recall the $k$-Hessian capacity (3.1.5). Then the following equivalence holds (see Theorem 2.20 in [PV08]).

**Lemma 3.2.11.** *Let $1 \le k < n/2$. Then there is a positive constant $C = C(n,k) > 0$ so that for all compact sets $E \subset \mathbf{R}^n$:*

$$\frac{1}{C}\mathrm{cap}_{\frac{2\mathrm{k}}{\mathrm{k}+1},\mathrm{k}+1}(\mathrm{E}) \le \mathrm{cap}_{\mathrm{k}}(\mathrm{E}) \le \mathrm{C}\,\mathrm{cap}_{\frac{2\mathrm{k}}{\mathrm{k}+1},\mathrm{k}+1}(\mathrm{E}).$$

### 3.2.5 Fundamental solutions and isolated singularities

The purpose of this section is to show that positive $\mathcal{A}$-superharmonic solutions of $\mathcal{L}(u) = 0$ in $\mathbf{R}^n\backslash\{x_0\}$ with a non-removable singularity at $\{x_0\}$ are (up to constants) fundamental solutions of $\mathcal{L}$. Such a principle goes back to Bôcher in the case that $\mathcal{L}$ is the Laplacian operator. For nonlinear equations, similar statements are often referred to as a Brezis–Lions type Lemma, or a Bidaut-Veron type Lemma (see [Veron96]). Similar statements are available for the $k$-Hessian operator, but we omit the details here.

**Lemma 3.2.12.** *Suppose that $\sigma$ is a Borel measure so $\sigma(\{x_0\}) = 0$, and suppose $u$ is an $p$-superharmonic function in $\mathbb{R}^n\backslash\{x_0\}$ so that $\mathcal{L}(u) = 0$ in $\mathbb{R}^n\backslash\{x_0\}$. Then $u$ can be extended to an $\mathcal{A}$-superharmonic in $\mathbb{R}^n$, and there is a constant $c \ge 0$ so that*

$\mathcal{L}(u) = c\delta_0$.

*Proof.* We will follow [PV08]. For $\mathcal{A}$-superharmonic functions since single points are removable (see [HKM06]), so we may conclude that $u$ is $p$-superharmonic in the whole space. Hence $u$ has a Riesz measure $\mu$ in $\mathbb{R}^n$. Let $\psi \in C_0^\infty(\mathbb{R}^n)$ and let $\phi_j$ be a sequence of smooth functions so that $0 \leq \phi_j \leq 1$, $\phi_j(x_0) = 1$, and $\phi_j(x) \to 0$ for all $x \in \mathbf{R}^n \backslash \{x_0\}$. Then, by Fatou's lemma,

$$\begin{aligned}\int_{\mathbb{R}^n} u^{p-1}\psi\, d\sigma &\leq \liminf_{j\to\infty} \int_{\mathbb{R}^n} u^{p-1}\psi(1-\phi_j)\, d\sigma \\ &= \liminf_{j\to\infty} \int_{\mathbb{R}^n} \psi(1-\phi_j)\, d\mu \\ &\leq \int_{\mathbb{R}^n} \psi\, d\mu.\end{aligned}$$

Thus, $\mathcal{L}u \geq 0$, and so there is a measure $\omega$ so that $\mathcal{L}u = \omega$. But, $\omega$ is clearly supported in $\{x_0\}$, and so $\mu = c\delta_{x_0}$ for some $c \geq 0$. □

## 3.3 Reduction to integral inequalities and necessary conditions on $\sigma$

### 3.3.1

In this section we will show how our study of the fundamental solutions of $\mathcal{L}$ and $\mathcal{G}$ can be rephrased into a question of nonlinear integral operators. The Wolff potential estimate will be the key to this idea, recall the definition from (3.2.1).

Let us introduce two nonlinear integral operators, $\mathcal{N}_1$ and $\mathcal{N}_2$, acting on non-negative functions $f \geq 0$ by:

$$\mathcal{N}_1(f)(x) := \mathbf{W}_{1,p}(f^{p-1}d\sigma)(x), \quad \text{and:} \tag{3.3.1}$$

$$\mathcal{N}_2(f)(x) := \mathbf{W}_{\frac{2k}{k+1},k+1}(f^k d\sigma)(x) \tag{3.3.2}$$

see also (3.3.5) below. These operators appear naturally in studying the equations $\mathcal{L}(u) = \omega$ and $\mathcal{G}(u) = \omega$ for a nonnegative Borel measure $\omega$. Indeed, if $1 < p < n$ and $u$ is a nonnegative $p$-superharmonic function such that $\mathcal{L}(u) = \omega$, then by the Wolff potential estimate, Theorem 3.2.4, there is a constant $C = C(n,p) > 0$ such that

$$u(x) \geq C\mathbf{W}_{1,p}(u^{p-1}d\sigma)(x) + C\mathbf{W}_{1,p}(\omega)(x).$$

Note that from this it follows that $u \in L^{p-1}_{\text{loc}}(\sigma)$. Hence, if $u$ is a fundamental solution of $\mathcal{L}$, then it follows:

$$u(x) \geq C\mathcal{N}_1(u)(x) + C\,|x - x_0|^{\frac{p-n}{p-1}} \tag{3.3.3}$$

since $\mathbf{W}_{1,p}(\delta_{x_0})(x) = c(n,p)\,|x - x_0|^{\frac{p-n}{p-1}}$ when $1 < p < n$. Here $C$ is a positive constant depending on $n, p$.

In much the same way, if $1 \leq k < n/2$ and $u$ is a nonnegative function so that $-u$ is a $k$-convex solution of $\mathcal{G}(u) = \omega$ in the sense of $k$-Hessian measures, then by the Wolff potential estimate, Theorem 3.2.6, there is a constant $C = C(n,k) > 0$ such that

$$u(x) \geq C\mathcal{N}_2(u)(x) + C\mathbf{W}_{\frac{2k}{k+1},k+1}(\omega)(x).$$

Thus $u \in L^k_{\text{loc}}(\sigma)$, and hence if $u$ is a fundamental solution of $\mathcal{G}$, then there is a constant $C = C(n,k)$ so that

$$u(x) \geq C\mathcal{N}_2(u)(x) + C\,|x - x_0|^{2-n/k}\,. \tag{3.3.4}$$

With the aid of the Wolff potential, by introducing the $\mathcal{N}_1$ and $\mathcal{N}_2$, we have rephrased the problem of finding lower bounds for the fundamental solutions to finding lower bounds of solutions of the nonlinear integral inequalities (3.3.3) and (3.3.4).

In addition, we will see in Section 3.5 that explicitly constructing solutions of (3.3.3) and (3.3.4) will be the main technical step in proving existence of minimal fundamental solutions of the differential operators $\mathcal{L}$ and $\mathcal{G}$.

As a result of this discussion it makes sense to introduce a more general nonlinear operator which generalizes both $\mathcal{N}_1$ and $\mathcal{N}_2$. To this end, recall that the Wolff potential acting on a measure $\omega$ is given by (3.2.1).

Let $s > 1$, $\alpha > 0$ so that $0 < \alpha s < n$, then we define the nonlinear operator $\mathcal{N}$, for a Borel measurable function $f \geq 0$, by:

$$\begin{aligned} \mathcal{N}(f)(x) &= \mathbf{W}_{\alpha,s}(f^{s-1}d\sigma)(x) \\ &= \int_0^\infty \Big( \frac{1}{r^{n-\alpha s}} \int_{B(x,r)} f^{s-1}(z) d\sigma(z) \Big)^{1/(s-1)} \frac{dr}{r} \end{aligned} \tag{3.3.5}$$

The operators $\mathcal{N}_1$ and $\mathcal{N}_2$ are clearly special cases of $\mathcal{N}$ for certain choices of $\alpha$ and $s$.

### 3.3.2

Fix $s > 1$ and $\alpha$ so that, $0 < \alpha s < n$. For the remainder of this section we will be concerned with positive solutions $u$ of the integral inequality:

$$u(x) \geq C_0 \mathcal{N} u(x) \tag{3.3.6}$$

where $C_0$ is a positive constant. Our first goal will be to prove some necessary conditions on the measure $\sigma$ for there to exist positive solutions of (3.3.6). In particular, we will prove the following theorem. Recall the definition of the capacity in (3.2.7).

**Theorem 3.3.1.** *Suppose that $u$ is a positive solution of the inequality (3.3.6) with constant $C_0 > 0$. Then, there is a positive constant $C$, depending on $\alpha, s, n$ and $C_0$,* *so that for every compact set $E \subset \mathbf{R}^n$*

$$\sigma(E) \le C\,\mathrm{cap}_{\alpha,s}(\mathrm{E}). \tag{3.3.7}$$

**Remark 3.3.2.** Theorem 3.3.1 implies the capacity estimates which appear in Theorems 3.1.2 and 3.1.12.

*Proof of Remark 3.3.2.* Suppose first that $u$ is a fundamental solution of $\mathcal{L}$. Then $u$ satisfies (3.3.3), and hence $u$ satisfies (3.3.6) with $\mathcal{N} = \mathcal{N}_1$. This corresponds to taking $\alpha = 1$ and $s = p$ in the definition of $\mathcal{N}$. Hence Theorem 3.3.1 implies that there is a constant $C > 0$ so that $\sigma(E) \le C\,\mathrm{cap}_{1,\mathrm{p}}(\mathrm{E})$ for all compact sets $E$. By Lemma 3.2.10, this is equivalent to the required capacity estimate in Theorem 3.1.2.

Similarly, if $u$ is a fundamental solution of $\mathcal{G}$, then $u$ satisfies (3.3.4), which is the same as (3.3.6) with $\alpha = \frac{2k}{k+1}$ and $s = k+1$. Hence Theorem 3.3.1 asserts the existence of a constant $C > 0$ so that $\sigma(E) \le C\,\mathrm{cap}_{\frac{2\mathrm{k}}{\mathrm{k}+1},\mathrm{k}+1}(\mathrm{E})$ for all compact sets $E$. Appealing to Lemma 3.2.11, we see that this is equivalent to the capacity condition appearing in Theorem 3.1.12. □

The same proof shows that Theorem 3.3.1 in fact implies the same capacity estimates for any positive solutions of the differential inequalities $\mathcal{L}u \ge 0$ and $\mathcal{G}(u) \ge 0$.

### 3.3.3

We will now briefly discuss an alternative approach to the capacity estimate (3.0.3) in the case of the $p$-Laplacian operator.

**Lemma 3.3.3.** *Let $\Omega$ be an open set in $\mathbf{R}^n$, and let $\sigma$ be a nonnegative Borel measure absolutely continuous with respect to p-capacity. Suppose that $u$ is a positive p-superharmonic function such that $-\Delta_p u \ge \sigma u^{p-1}$ in $\Omega$. Then then following em-*

*bedding inequality holds:*

$$\int_{\Omega} h^p \, d\sigma \le \int_{\Omega} |\nabla h|^p \, dx, \text{ for all } h \in C_0^\infty(\Omega),\ h \ge 0, \tag{3.3.8}$$

*Proof.* Let $h \ge 0$, $h \in C_0^\infty(\Omega)$, Let $\mu[u]$ be the Riesz measure of $u$ (see Section 3.2), and $\mu_k$ be the Riesz measure of $T_k(u) = \min(u, k) \in L^{1,p}_{\mathrm{loc}}(\Omega)$. It follows that $\mu_k \in L^{-1,p'}_{\mathrm{loc}}(\Omega)$. Let us decompose $\mu_k$ as:

$$d\mu_k = u^{p-1} d\nu_k + d\omega_k,$$

with $d\nu_k = u^{1-p}\chi_{\{u<k\}} d\mu_k$, and $d\omega_k = \chi_{\{u \ge k\}} d\mu_k$. This decomposition follows from the minimum principle, since for any compact set $K \subset\subset \Omega$, there exists a constant $c > 0$ such that $u \ge c > 0$ on $K$. Since $\mu_k$ lies locally in the dual Sobolev space $L^{-1,p'}_{\mathrm{loc}}(\Omega)$, and $h^p T_k(u)^{1-p} \in L^{1,p}(\Omega)$ has compact support, the following manipulations are valid:

$$\begin{aligned} \int h^p \, d\nu_k &\le \int h^p T_k(u)^{1-p} d\mu_k = \int \nabla T_k(u)^{p-2} \nabla T_k(u) \cdot \nabla \Big( \frac{h^p}{T_k(u)^{p-1}} \Big) dx \\ &\le \Big( p \int \frac{h^{p-1}}{T_k(u)^{p-1}} \nabla T_k(u)^{p-2} \nabla T_k(u) \cdot \nabla h \\ &\quad - (p-1) \int h^p \frac{|\nabla T_k(u)|^p}{T_k(u)^p} dx \Big) \le \int |\nabla h|^p \, dx, \end{aligned} \tag{3.3.9}$$

where we have used Young's inequality in the last line. To prove the Lemma, we claim that:

$$u^{p-1} \chi_{\{u<k\}} d\sigma \le u^{p-1} d\nu_k \text{ on } \mathrm{supp}(h). \tag{3.3.10}$$

This will follow by an adaptation of a similar argument in [DMM97]. Indeed, since $T_{2k}(u) \in L^{1,p}_{\mathrm{loc}}(\Omega)$, it follows that the set $\{u < k\}$ is quasi-open, see e.g. [MZ97, DMM97]. Therefore, there exists an increasing sequence $\phi_j \in W^{1,\infty}(\Omega)$, so that $\phi_j$ converges to $\chi_{\{u<k\}}$ q.e.. This is a simple adaptation of the proof of Lemma 2.1 in [DMG94], since the functions $u_k$ considered in the proof of Lemma 2.1 of [DMG94]

can be chosen to be smooth. It follows (see (3.2.3)) that for any $\psi \in C_0^\infty(\mathrm{supp}(h))$, that:

$$\int_{\{u<k\}} \psi\phi_j u^{p-1} d\nu_k = \int |\nabla T_k(u)|^{p-2}\nabla T_k(u)\cdot\nabla(\psi\phi_j)dx$$
$$= \int |\nabla u|^{p-2}|\nabla u\cdot\nabla(\psi\phi_j)dx \geq \int \phi_j\psi u^{p-1}d\sigma,$$

the second equality here follows since $\phi_j$ is supported in $\{u < k\}$, and last inequality is by hypothesis. Allowing $j \to \infty$, (3.3.10) follows. Combining (3.3.10) with (3.3.9) we conclude:

$$\int_{\{u<k\}} h^p d\sigma \leq \int |\nabla h|^p dx.$$

Letting $k \to \infty$ with the aid of the monotone convergence theorem proves the lemma.

□

**Remark 3.3.4.** The reader may find it constructive to compare the proof of Lemma 3.3.3 with statement (ii) of Theorem 2.0.2 of Chapter 2. Note in particular that the above lemma is somewhat more difficult to prove than the result of the previous chapter since we are only assuming superharmonicity in the supersolution.

It is easy to see by the definition of $p$-capacity that inequality (3.3.8) implies the capacity inequality (3.0.3) with constant $C = 1$. As was mentioned in the introduction, the converse is also true: if (3.0.3) holds with constant $C = ((p-1)/p)^p$, then (3.3.8) holds (see [Maz85]). Under the assumption that $\sigma \in L^\infty_{\mathrm{loc}}$, (3.3.8) is known to be equivalent to the existence of a solution to the inequality $\mathcal{L}(u) \geq 0$; see Theorem 2.3 in [PT07].

### 3.3.4

Let us now prove Theorem 3.3.1, we will do so by verifying an equivalent characterization of (3.3.7).

**Lemma 3.3.5.** *There is a constant $C$ so that (3.3.7) holds for all compact sets $E$ if and only if there is a constant $C_1 > 0$ so that:*

$$\int_E \mathbf{W}_{\alpha,s}(\chi_E d\sigma)\, d\sigma \leq C_1\, \sigma(E) \tag{3.3.11}$$

*for all compact sets $E \subset \mathbf{R}^n$. Furthermore, if (3.3.11) holds, then there is a positive constant $A > 0$, depending on $\alpha, s$ and $n$, such that*

$$A^{-1}C_1 \leq C \leq AC_1.$$

Lemma 3.3.5 is well known, for instance a proof can be found in [AH96], Theorem 7.2.1.

We will verify that the equivalent statement in Lemma 3.3.5 holds by first showing it holds for a dyadic analogue of the Wolff potential, and then using a standard shifting argument which goes back at least to Fefferman and Stein [FS71]; see also Garnett and Jones [GJ82].

To this end, we define the dyadic mesh at level $k$ for $k \in \mathbb{Z}$, denoted by $\mathcal{D}_k$, as the collection of cubes in $\mathbf{R}^n$ which are the translations by $2^k\lambda$ for $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbf{Z}^n$ of the cube $[0, 2^k)^n$. Then the dyadic lattice $\mathcal{D}$ is the collection of dyadic meshes $\mathcal{D}_k$, $k \in \mathbb{Z}$.

With this notation, we define the discrete Wolff potentials $\mathcal{W}^t_{\alpha,s}$ (see [COV04] for an in depth discussion) by

$$\mathcal{W}^t_{\alpha,s}(f d\sigma)(x) = \sum_{Q\in\mathcal{D}:\, x\in Q+t} c_Q \Big(\int_{Q+t} f(z) d\sigma(z)\Big)^{1/(s-1)} \tag{3.3.12}$$

where $c_Q = \ell(Q)^{\frac{\alpha s - n}{s-1}}$ and $t \in \mathbf{R}^n$. Note that there is a constant $C$, depending only on $n, \alpha$ and $s$ (but not the shift $t$) so that for any nonnegative function $f$

$$\mathcal{W}^t_{\alpha,s}(f d\sigma) \leq C \mathbf{W}_{\alpha,s}(f d\sigma). \tag{3.3.13}$$

We will use the following definition of the discrete Carleson measure.

**Definition 3.3.6.** *Let $1 < s < \infty$, and let $\sigma$ be a Borel measure on $\mathbf{R}^n$. Then $\sigma$ is said to be a discrete Carleson measure if there is a positive constant $C = C(n, s)$ such that for each dyadic cube $P \in \mathcal{D}$ and every $t \in \mathbf{R}^n$*

$$\sum_{Q \subset P,\, Q \in \mathcal{D}} c_Q \, |Q + t|_\sigma^{s'} \leq C \, |P + t|_\sigma \, . \tag{3.3.14}$$

**Remark 3.3.7.** It is well known that the inequality

$$\sum_{Q \in \mathcal{D}} c_Q \left| \int_{Q+t} f d\sigma \right|^{s'} \leq C \, ||f||^{s'}_{L^{s'}(d\sigma)} \tag{3.3.15}$$

holds for every $f \in L^{s'}(d\sigma)$ if and only if $\sigma$ is a discrete Carleson measure, and the constants in (3.3.14) and (3.3.15) are equivalent. For completeness we provide a proof of this inequality in Appendix C. From this it is immediate that if $\sigma$ is a Carleson measure then $\chi_E \, d\sigma$ is also a Carleson measure, for every measurable $E \subset \mathbf{R}^n$.

We now formulate a discrete analogue of the characterization in Lemma 3.3.5 which will be sufficient for our purposes, where we make use of Definition 3.3.6 and Remark 3.3.7.

**Lemma 3.3.8.** *Suppose there is a positive solution $u$ to the integral inequality (3.3.6). Then the measure $\sigma$ is a discrete Carleson measure, that is there is a positive constant $C = C(n, s, C_0)$ such that for each dyadic cube $P \in \mathcal{D}$ and every compact set $E \subset \mathbf{R}^n$,*

$$\sum_{\substack{Q \subset P \\ Q \in \mathcal{D}}} c_Q \, |(Q + t) \cap E|_\sigma^{s'} \leq C \, |(P + t) \cap E|_\sigma \, . \tag{3.3.16}$$

*Furthermore, we have that*

$$\sum_{Q\in\mathcal{D}} c_Q \left|(Q+t)\cap E\right|_\sigma^{s'} \le C\left|E\right|_\sigma . \tag{3.3.17}$$

*Proof.* We will prove (3.3.16). The proof of (3.3.17) follows by the same reasoning. The proof is rather reminiscent of the classical Schur's Lemma. First note that by hypothesis and (3.3.13) there is a positive function $u$ together with a constant $C > 0$ so that

$$u(x) \ge C\mathcal{W}^t_{\alpha,s}(u^{s-1}d\sigma)(x)$$

and hence, using Hölder's inequality, we see that:

$$\begin{aligned}\sum_{\substack{Q\subset P\\ Q\in\mathcal{D}}} c_Q \left|(Q+t)\cap E\right|_\sigma^{s'} &= \sum_{Q\subset P} c_Q\Big\{\int_{(Q+t)\cap E} u^{-\frac{s-1}{s}}\cdot u^{\frac{s-1}{s}}d\sigma\Big\}^{s'}\\ &\le \sum_{Q\subset P} c_Q \int_{(Q+t)\cap E} u^{-1}\,d\sigma\cdot\Big\{\int_{(Q+t)\cap E)} u^{s-1}d\sigma\Big\}^{\frac{1}{s-1}}.\end{aligned}$$

By interchanging summation and integration, which is permitted by the monotone convergence theorem, we see that the last line is equal to:

$$\begin{aligned}&\int_{(P+t)\cap E} u^{-1}\sum_{Q\subset P} c_Q\Big\{\int_{(Q+t)\cap E} u^{s-1}d\sigma\Big\}^{\frac{1}{s-1}}\chi_{Q+t}(x)d\sigma\\ &\le \int_{(P+t)\cap E} u^{-1}\cdot\mathcal{W}^t_{\alpha,s}(u^{s-1}d\sigma)d\sigma\\ &\le C\int_{(P+t)\cap E} u^{-1}\cdot u\,d\sigma = C\left|(P+t)\cap E\right|_\sigma .\end{aligned}$$

□

We now state a suitable version of the dyadic averaging result which will be sufficient for our purposes.

**Lemma 3.3.9.** *There is a positive integer $j_0\in\mathbb{N}$ so that for any $j\in\mathbb{Z}$ there is a* *constant $C = C(n,\alpha,s)$, not depending on $j$, so that*

$$\mathbf{W}^{2^j}_{\alpha,s}(f d\sigma)(x) \le C \fint_{B(0,2^{j+j_0})} \mathcal{W}^t_{\alpha,s}(f d\sigma)(x)\, dt$$

*where $\mathbf{W}^{2^j}_{\alpha,s}$ is the local Wolff potential defined in (3.1.1).*

A proof of this lemma can be found, for instance, in [COV04].

We will next use the dyadic shifting argument to prove the following lemma:

**Lemma 3.3.10.** *Suppose $u$ is a positive solution of (3.3.6) with constant $C_0$. Then there is a constant $C = C(n,\alpha,s)$ so that for any compact set $E \subset \mathbf{R}^n$, and each $m \in \mathbb{N}$ the measure $\sigma$ satisfies:*

$$\int_E \Big(\mathbf{W}_{\alpha,s}(\chi_E d\sigma)\Big)^m d\sigma \le C^m\, m!\, \sigma(E).$$

**Remark 3.3.11.** This Lemma in the case $m = 1$ shows that Lemma 3.3.5 is satisfied, and hence proves Theorem 3.3.1. We prove the Lemma in the form stated as it gives us an exponential integrability result, which will be very useful in the sequel (see Corollary 3.3.12 below).

*Proof.* Let $E$ be a compact set. Then first we note that by Fatou's lemma,

$$\int_E \Big(\mathbf{W}_{\alpha,s}(\chi_E d\sigma)\Big)^m d\sigma \le \liminf_{k\to\infty} \int_E \Big(\mathbf{W}^{2^k}_{\alpha,s}(\chi_E d\sigma)\Big)^m d\sigma$$

where $\mathbf{W}^{2^k}_{\alpha,s}(\chi_E d\sigma)(x) = \displaystyle\int_0^{2^k} \Big(\frac{\sigma(B(x,r)\cap E)}{r^{n-\alpha s}}\Big)^{1/(s-1)} \frac{dr}{r}$.

It therefore suffices to find a bound on the right hand side of the preceding inequality which is independent of $k$. Lemma 3.3.9 yields:

$$\begin{aligned}
&\int_E \Big(\mathbf{W}^{2^k}_{\alpha,s}(\chi_E d\sigma)\Big)^m d\sigma \\
&\qquad\qquad \leq C^m \int_E \Big(\fint_{B(0,2^{k+j_0})} \mathcal{W}^t_{\alpha,s}(\chi_E d\sigma) dt\Big)^m d\sigma \\
&\qquad\qquad \leq C^m \Big(\fint_{B(0,2^{k+j_0})} \Big(\int_E \Big(\mathcal{W}^t_{\alpha,s}(\chi_E d\sigma)\Big)^m d\sigma\Big)^{\frac{1}{m}} dt\Big)^m,
\end{aligned}$$

where the second inequality follows from Minkowski's integral inequality.

We will need the elementary summation by parts inequality:

$$\Big(\sum_{j=1}^{\infty} \lambda_j\Big)^m \leq m \sum_{j=1}^{\infty} \lambda_j \Big(\sum_{k=1}^{j} \lambda_k\Big)^{m-1} \tag{3.3.18}$$

which holds for any nonnegative sequence $\{\lambda_j\}_j$ and $m \geq 1$. We apply Lemma 3.3.8 to the dyadic Wolff potential, after an $m$ fold application of (3.3.18). Indeed, considering the inner integral in the right hand side of the last line above, we obtain:

$$\begin{aligned}
&\int_E \Big(\mathcal{W}^t_{\alpha,s}(\chi_E d\sigma)\Big)^m d\sigma \\
&\quad = \int_E \Big(\sum_{Q\in\mathcal{D}} c_Q \, |Q+t\cap E|_\sigma^{\frac{1}{s-1}} \chi_{Q+t}\Big)^m d\sigma \\
&\quad \leq m! \int_E \sum_{Q_1\in\mathcal{D}} c_{Q_1} \, |Q_1+t\cap E|_\sigma^{\frac{1}{s-1}} \dots \sum_{Q_m\subset Q_{m-1}} c_{Q_m} \, |Q_m+t\cap E|_\sigma^{\frac{1}{s-1}} \chi_{Q_m+t} \, d\sigma \\
&\quad = m! \sum_{Q_1\in\mathcal{D}} c_{Q_1} \, |Q_1+t\cap E|_\sigma^{\frac{1}{s-1}} \dots \sum_{Q_m\subset Q_{m-1}} c_{Q_m} \, |Q_m+t\cap E|_\sigma^{\frac{s}{s-1}} \\
&\quad \leq m! C^m \sigma(E).
\end{aligned} \tag{3.3.19}$$

In the last line we have used (3.3.16) $m-1$ times and then (3.3.17) once. Bringing together our estimates proves the lemma. □

The following exponential integrability result easily follows from Lemma 3.3.10, the power series representation of the exponential, and the monotone convergence theorem.

**Corollary 3.3.12.** *Suppose $u$ is a positive solution of (3.3.6). If we let $\beta > 0$ so that $C\beta < 1$, where $C$ is the constant appearing in Lemma 3.3.10, then we have the following:*

$$\int_E e^{\,\beta \mathbf{W}_{\alpha,s}(\chi_E d\sigma)(y)} d\sigma(y) \leq \frac{1}{1 - C\beta}\sigma(E) \tag{3.3.20}$$

*whenever $E$ is a compact set.*

In our next result, we specialize (3.3.7) to when the set $E$ is a ball. By a standard formula for the capacity of a ball (see [AH96], Chapter 5),

$$\sigma(B(x,r)) \leq C_1 \operatorname{cap}_{\alpha,s}(B(x,r)) = C_2 r^{n-\alpha s} \tag{3.3.21}$$

for all balls $B(x,r)$, where $C_2 = A\,C_1$, and $A$ depends only on $n, \alpha$ and $s$. However, as is well known, (3.3.21) does not imply (3.3.7) for all compact sets $E$.

Our next lemma shows that the tail of the Wolff potential is nearly constant, which is a key estimate to our construction of the supersolution.

**Lemma 3.3.13.** *Let $\sigma$ be a Borel measure satisfying (3.3.21). Then there is a positive constant $C = C(n, \alpha, s, C_2) > 0$, so that for all $x \in \mathbb{R}^n$ and $y \in B(x,t)$, $t > 0$, it follows:*

$$\left| \int_t^\infty \left[ \Big( \frac{\sigma(B(x,r))}{r^{n-\alpha s}} \Big)^{\frac{1}{s-1}} - \Big( \frac{\sigma(B(y,r))}{r^{n-\alpha s}} \Big)^{\frac{1}{s-1}} \right] \frac{dr}{r} \right| \leq C. \tag{3.3.22}$$

We defer the proof of Lemma 3.3.13 to Appendix B.

## 3.4 Lower bounds for nonlinear integral equations, the proof of Theorems 3.1.2 and 3.1.12

In this section, we will prove Theorems 3.1.2 and 3.1.12. Recall the operator

$$\mathcal{N}(f)(x) = \mathbf{W}_{\alpha,s}(f^{s-1}d\sigma)(x).$$

We will begin this section by proving a lower bound for solutions of the inequality:

$$u(x) \geq C_0\mathcal{N}(u)(x) + C_0\,|x-x_0|^{\frac{\alpha s-n}{s-1}}\,. \tag{3.4.1}$$

We will show the following theorem:

**Theorem 3.4.1.** *Suppose that $u$ satisfies (3.4.1) with constant $C_0$. Then there is a constant $c = c(n, \alpha, s, C_0) > 0$ such that:*

$$\begin{aligned} u(x) \geq c\,|x-x_0|^{\frac{\alpha s-n}{s-1}} \exp\Big(c\int_0^{|x-x_0|}\Big(\frac{\sigma(B(x,r))}{r^{n-\alpha s}}\Big)^{1/(s-1)}\frac{dr}{r}\Big) \\ \cdot \exp\Big(c\int_0^{|x-x_0|}\frac{\sigma(B(x_0,r))}{r^{n-\alpha s}}\frac{dr}{r}\Big). \end{aligned} \tag{3.4.2}$$

Theorems 3.1.2 and 3.1.12 will follow quickly from this theorem, as we shall show once it is proved.

We shall prove Theorem 3.4.1 by iterating (3.4.1). To illustrate the iteration, suppose that $T$ is a homogeneous superlinear operator acting on nonnegative functions, i.e. that $T(cf) = cT(f)$ for $c > 0$ and $T(f+g) \geq T(f) + T(g)$ whenever $f$ and $g$ are nonnegative measurable functions. In addition suppose that $u$ satisfies the inequality:

$$u \geq T(u) + f \tag{3.4.3}$$

where $f \geq 0$. Now we define the $j$-th iterate of $T$ by $T^j(f) = T(T^{j-1}(f))$, for all

$j \geq 2$. Iterating (3.4.3) $m$ times yields:

$$u \geq T(T(\dots T(T(u) + f) + f \cdots) + f) + f$$
$$\geq T^m(f) + T^{m-1}(f) + \cdots + T(f) + f,$$

and since $m$ here was arbitrary,

$$u \geq \sum_{j=1}^{\infty} T^j(f) + f.$$

Now, if $1 < s \leq 2$, it is clear from Minkowski's inequality that $\mathcal{N}$ is a superlinear homogeneous operator and hence if $u$ is a solution of (3.4.1), then:

$$u \geq \sum_{j=1}^{\infty} C_0^j \mathcal{N}^j(|\cdot - x_0|^{\frac{\alpha s - n}{s-1}}) + C_0 |x - x_0|^{\frac{\alpha s - n}{s-1}}.$$

However, if $2 < s < n$, the operator $\mathcal{N}$ does not fall within this framework. In this case we consider an operator $\mathcal{T}(f) = \mathcal{N}(f^{1/(s-1)})^{s-1} = (\mathbf{W}_{\alpha,s}(f))^{s-1}$. Then by Minkowski's inequality, $\mathcal{T}$ is superlinear, and it is homogenous, and so we may apply the above discussion. If $u$ satisfies (3.4.1), then we have that:

$$u^{s-1}(x) \geq C\mathcal{T}^j(u^{s-1})(x) + C |x - x_0|^{\alpha s - n}$$

where $C$ is a positive constant depending on $n, \alpha, s$ and $C_0$. Hence, we see that

$$u^{s-1}(x) \geq \sum_{j=1}^{\infty} C^j \mathcal{T}^j(|\cdot - x_0|^{\alpha s - n})(x) + C |x - x_0|^{\alpha s - n}.$$

By comparing iterates of $\mathcal{T}$ with the iterates of $\mathcal{N}$, we obtain

$$u(x) \geq \Big(\sum_{j=1}^{\infty} C^j \mathcal{N}^j(|\cdot - x_0|^{\frac{\alpha s - n}{s-1}})(x)^{s-1}\Big)^{1/(s-1)} + C |x - x_0|^{\frac{\alpha s - n}{s-1}}.$$

Thus, by Jensen's (or Hölder's) inequality, we have that for any $q > 1$,

$$u \geq C \sum_{j=1}^{\infty} j^{(q\frac{2-s}{s-1})} C^j \mathcal{N}_1^j(|\cdot - x_0|^{\frac{\alpha s - n}{s-1}})(x) + C\,|x - x_0|^{\frac{\alpha s - n}{s-1}}$$

where $C$ is a positive constant depending on $q, n, s, \alpha$ and $C_0$.

We summarize this discussion as follows:

**Lemma 3.4.2.** *Suppose $u$ is a solution of (3.4.1) with constant $C_0$. Then there is a constant $C = C(n, s, \alpha, C_0) > 0$ so that if $1 < s \leq 2$, it follows:*

$$u \geq \sum_{j=1}^{\infty} C^j \mathcal{N}^j(|\cdot - x_0|^{\frac{\alpha s - n}{s-1}}) + C\,|x - x_0|^{\frac{\alpha s - n}{s-1}}\,. \tag{3.4.4}$$

*If $2 < s < n$, then for any $q > 1$,*

$$u \geq C(q) \sum_{j=1}^{\infty} j^{(q\frac{2-s}{s-1})} C^j \mathcal{N}_1^j(|\cdot - x_0|^{\frac{\alpha s - n}{s-1}})(x) + C\,|x - x_0|^{\frac{\alpha s - n}{s-1}} \tag{3.4.5}$$

*where $C(s) = C(q, n, \alpha, s, C_0) > 0$.*

### 3.4.1 Proof of Theorem 3.4.1

Suppose that $u$ is a solution of (3.4.1). Then clearly $u$ also satisfies (3.3.6), and hence by Theorem 3.3.1, (3.3.7) holds for all balls compact sets $E$. Hence there is a constant $C(\sigma) > 0$ so that:

$$C(\sigma) = \sup_E \frac{\sigma(E)}{\mathrm{cap}_{\alpha,s}(E)} < \infty.$$

where the supremum is taken over compact sets $E$ so that $\mathrm{cap}_{\alpha,s}(E) > 0$. Note that this implies $\sigma(B(x,r)) \leq AC(\sigma)r^{n-\alpha s}$ for all balls $B(x,r)$, where $A$ is a positive constant depending on $n, \alpha$ and $s$. To prove Theorem 3.4.1, we estimate the iterates $\mathcal{N}^j(|\cdot - x_0|^{\frac{\alpha s - n}{s-1}})$. We will do this in two lemmas, giving us two bounds. We then average the two bounds to conclude the theorem.

**Lemma 3.4.3.** *For a given $x \in \mathbf{R}^n$, define $j_x$ to be the integer so that*

$$2^{j_x} \le |x - x_0| < 2^{j_x+1}.$$

*Then, with $B_k = B(x_0, 2^k)$, for any $m \ge 1$,*

$$\begin{aligned} \mathcal{N}^m(|\cdot - x_0|^{\frac{\alpha s - n}{s-1}})(x) \ge \Big(\frac{s-1}{n-\alpha s} 8^{\frac{\alpha s-n}{s-1}}\Big)^m |x - x_0|^{\frac{\alpha s - n}{s-1}} \\ \cdot \Big(\frac{1}{m!}\Big\{\sum_{k=-\infty}^{j_x} 2^{k(\alpha s - n)} \sigma(B_{k+1} \backslash B_k)\Big\}^m\Big)^{1/(s-1)}. \end{aligned} \tag{3.4.6}$$

*Proof.* We will prove this lemma by induction. Let us recall the definition of the operator $\mathcal{N}$:

$$\mathcal{N}(|\cdot - x_0|^{\frac{\alpha s-n}{s-1}})(x) = \int_0^\infty \Big(\frac{1}{r^{n-\alpha s}} \int_{B(x,r)} |y - x_0|^{\alpha s - n}\, d\sigma(y)\Big)^{1/(s-1)} \frac{dr}{r}.$$

First, restrict the integration in the variable $r$ to $r > 4\,|x - x_0|$. Then, observe that as $r > 4\,|x - x_0|$: $B(x_0, 2\,|x - x_0|) \subset B(x, r)$. This results in the bound:

$$\begin{aligned} \mathcal{N}(|\cdot - x_0|^{\frac{\alpha s-n}{s-1}})(x) \ge \int_{4|x-x_0|}^\infty r^{\frac{\alpha s - n}{s-1}} \frac{dr}{r} \\ \cdot \Big(\int_{B(x_0, 2|x-x_0|)} |y - x_0|^{\alpha s - n}\, d\sigma(y)\Big)^{1/(s-1)}. \end{aligned} \tag{3.4.7}$$

Now, recalling the definition of $j_x$, we have:

$$\int_{B(x_0,2|x-x_0|)} |y - x_0|^{\alpha s - n}\, d\sigma(y) \ge \sum_{k=-\infty}^{j_x} 2^{(k+1)(\alpha s - n)} \sigma(B_{k+1} \backslash B_k).$$

Using this and evaluating the integral in (3.4.7) yields the case where $k = 1$.

Now suppose (3.4.6) holds for some $m$. Then by the induction hypothesis, and the observation above:

$$\mathcal{N}^{m+1}(|\cdot - x_0|^{\frac{\alpha s-n}{s-1}})(x) \geq \Big(\frac{s-1}{n-\alpha s}8^{\frac{\alpha s-n}{s-1}}\Big)^m \frac{s-1}{n-\alpha s}4^{\frac{\alpha s-n}{s-1}}\,|x-x_0|^{\frac{\alpha s-n}{s-1}}$$
$$\cdot\Big(\frac{1}{m!}\int_{B(x_0,2|x-x_0|)}|z-x_0|^{\alpha s-n}\Big(\sum_{\ell=-\infty}^{j_y}2^{\ell(\alpha s-n)}\sigma(B_{\ell+1}\backslash B_\ell)\Big)^m d\sigma(y)\Big)^{1/(s-1)}.$$

We now consider the integral

$$\int_{B(x_0,2|x-x_0|)}|z-x_0|^{\alpha s-n}\Big(\sum_{\ell=-\infty}^{j_y}2^{\ell(\alpha s-n)}\sigma(B_{\ell+1}\backslash B_\ell)\Big)^m d\sigma(y). \tag{3.4.8}$$

To complete the inductive step and hence prove the lemma it suffices to show that (3.4.8) is greater than

$$\frac{2^{\alpha s-n}}{m+1}\Big(\sum_{\ell=-\infty}^{j_x}2^{\ell(\alpha s-n)}\sigma(B_{\ell+1}\backslash B_\ell)\Big)^{m+1}. \tag{3.4.9}$$

To this end, note that by the definition of $j_x$, (3.4.8) is greater than

$$\sum_{k=-\infty}^{j_x}2^{(k+1)(\alpha s-n)}\int_{B_{k+1}\backslash B_k}\Big(\sum_{\ell=-\infty}^{j_y}2^{\ell(\alpha s-n)}\sigma(B_{\ell+1}\backslash B_\ell)\Big)^m d\sigma(y). \tag{3.4.10}$$

We next remark that for all $y \in B_{k+1}\backslash B_k$, we have by definition $j_y = k$, and so (3.4.10) equals:

$$2^{\alpha s-n}\sum_{k=-\infty}^{j_x}2^{k(\alpha s-n)}\sigma(B_{k+1}\backslash B_k)\Big(\sum_{\ell=-\infty}^{k}2^{\ell(\alpha s-n)}\sigma(B_{\ell+1}\backslash B_\ell)\Big)^m. \tag{3.4.11}$$

But an application of the elementary summation by parts inequality (3.3.18) now gives that (3.4.11) is greater than (3.4.9). This concludes the proof of the Lemma. □

By using Jensen's (or Hölder's) inequality, inserting Lemma 3.4.3 into the bounds (3.4.4) and (3.4.5) in Lemma 3.4.2 yields the existence of positive constants $c_1$ and

$c_2$, depending on $n, \alpha, s$ and $C_0$, so that:

$$u(x) \geq c_1 \, |x - x_0|^{\frac{\alpha s - n}{s-1}} \exp\Big( c_2 \sum_{\ell=-\infty}^{j_x} 2^{\ell(\alpha s - n)} \sigma(B_{\ell+1} \backslash B_\ell) \Big).$$

But, since $\sigma$ satisfies (3.3.21), we may further estimate the sum. Indeed,

$$\sum_{\ell=-\infty}^{j_x} 2^{\ell(\alpha s - n)} \sigma(B_{\ell+1} \backslash B_\ell) \geq C \int_0^{|x-x_0|} \frac{\sigma(B(x,r))}{r^{n-\alpha s}} \frac{dr}{r},$$

where $C = C(n, \alpha, s) > 0$. Hence we may conclude that there are positive constants $c_1$ and $c_2$, depending on $n, \alpha, s, C_0$ and $C(\sigma)$, so that:

$$u(x) \geq c_1 \, |x - x_0|^{\frac{\alpha s - n}{s-1}} \exp\Big( c_2 \int_0^{|x-x_0|} \frac{\sigma(B(x,r))}{r^{n-\alpha s}} \frac{dr}{r} \Big).$$

The second part of the exponential build up in Theorem 3.4.1 is accounted for in the following lemma:

**Lemma 3.4.4.** *For any $m \geq 1$,*

$$\begin{aligned} \mathcal{N}^m(|\cdot - x_0|^{\frac{\alpha s - n}{s-1}})(x) \geq & (3/2)^{\frac{\alpha s - n}{s-1}} \frac{1}{m!} \, |x - x_0|^{\frac{\alpha s - n}{s-1}} \\ & \cdot \Big( \int_0^{|x-x_0|} \Big( \frac{\sigma(B(x, r/2))}{r^{n-\alpha s}} \Big)^{1/(s-1)} \frac{dr}{r} \Big)^m. \end{aligned} \tag{3.4.12}$$

*Proof.* We will prove Lemma 3.4.4 when $m = 3$, as the case of general $m$ is completely similar. The proof is based on the following claim:

For any locally finite Borel measures $\sigma$ and $\omega$, and $x, x_0 \in \mathbf{R}^n$:

$$\begin{aligned} &\int_0^{|x-x_0|} \Big( \frac{1}{r^{n-\alpha s}} \int_{B(x,r/2)} \Big\{ \int_r^\infty \Big( \frac{1}{u^{n-\alpha s}} \omega(B(y,u)) \Big)^{1/(s-1)} \frac{du}{u} \Big\}^{s-1} d\sigma(y) \Big)^{1/(s-1)} \frac{dr}{r} \\ &\qquad \geq \int_0^{|x-x_0|} \Big( \frac{\sigma(B(x, r/2)}{r^{n-\alpha s}} \Big)^{1/(s-1)} \int_r^\infty \Big( \frac{1}{u^{n-\alpha s}} \omega(B(x, u/2)) \Big)^{1/(s-1)} \frac{du}{u} \frac{dr}{r}. \end{aligned} \tag{3.4.13}$$

The claim is just the triangle inequality. Suppose that $|y-x| < r/2$ and $r < u$, then whenever $z \in B(x,u/2)$: $B(x,u/2) \subset B(y,u)$. Thus, $\omega(B(x,u/2)) \leq \omega(B(y,u))$. The claim (3.4.13) then follows by using this estimate in the left hand side and noting that the inner integrand no longer depends on $y$.

The Lemma will follow from repeated use of the claim. First, by using definition and restricting domains of integration:

$$\mathcal{N}^3(|\cdot - x_0|^{\frac{\alpha s-n}{s-1}})(x) \geq \int_0^{|x-x_0|} \Big(\frac{1}{r^{\alpha s-n}} \int_{B(x,r/2)} \cdot \Big\{ \int_r^\infty \Big(\frac{1}{u^{n-\alpha s}} \omega(B(y,u))\Big)^{1/(s-1)} \frac{du}{u} \Big\}^{s-1} d\sigma(y)\Big)^{1/(s-1)} \frac{dr}{r} \tag{3.4.14}$$

where:

$$\omega(B(y,u)) = \int_{B(y,u)} \Big\{ \int_0^\infty \Big(\frac{1}{t^{n-\alpha s}} \int_{B(z,t)} |w-x_0|^{\alpha s-n}\, d\sigma(w)\Big)^{1/(s-1)} \frac{dt}{t} \Big\}^{s-1} d\sigma(z).$$

Applying the claim (3.4.13) to (3.4.14), we have that (3.4.14) is greater than:

$$\int_0^{|x-x_0|} \Big(\frac{\sigma(B(x,r/2)}{r^{n-\alpha s}}\Big)^{1/(s-1)} \int_r^\infty \Big(\frac{1}{u^{n-\alpha s}} \omega(B(x,u/2))\Big)^{1/(s-1)} \frac{du}{u} \frac{dr}{r}.$$

Let's now consider the integral:

$$\int_r^\infty \Big(\frac{1}{s^{n-\alpha s}} \omega(B(x,u/2))\Big)^{1/(s-1)} \frac{du}{u} \geq \int_r^{|x-x_0|} \Big(\frac{1}{u^{n-\alpha s}} \omega(B(x,u/2))\Big)^{1/(s-1)} \frac{du}{u}.$$

Then we may rewrite the right hand side of this last line as:

$$\int_r^{|x-x_0|} \Big(\frac{1}{u^{n-\alpha s}} \int_{B(x,u/2)} \Big\{ \int_0^\infty \Big(\frac{1}{t^{n-\alpha s}} \mu(B(z,t))\Big)^{1/(s-1)} \frac{dt}{t} \Big\}^{s-1} d\sigma(z)\Big)^{1/(s-1)} \frac{du}{u} \tag{3.4.15}$$

where

$$\mu(B(z,t)) = \int_{B(z,t)} |w - x_0|^{\alpha s - n} \, d\sigma(w).$$

Now, restricting the integral over $t$ to $t > u$, and applying the claim (3.4.13) with $\omega = \mu$, we see that (3.4.15) is greater than

$$\int_r^{|x-x_0|} \Big( \frac{1}{u^{n-\alpha s}} \sigma(B(x,u/2)) \Big)^{1/(s-1)} \int_u^{|x-x_0|} \Big( \frac{1}{t^{n-\alpha s}} \mu(B(x,t/2)) \Big)^{1/(s-1)} \frac{dt}{t} \frac{du}{u}$$

where we have also restricted the integration over $t$ to $t < |x - x_0|$. Now, let us consider:

$$\begin{aligned} &\int_u^{|x-x_0|} \Big( \frac{1}{t^{n-\alpha s}} \mu(B(x,t)) \Big)^{1/(s-1)} \frac{dt}{t} \\ &\qquad = \int_u^{|x-x_0|} \Big( \frac{1}{t^{n-\alpha s}} \int_{B(x,t/2)} |w - x_0|^{\alpha s - n} \, d\sigma(w) \Big)^{1/(s-1)} \frac{dt}{t}. \end{aligned}$$

But, for $w \in B(x, t/2)$, note that: $|w - x_0| < 3/2 \, |x - x_0|$. Thus,

$$\begin{aligned} &\int_u^{|x-x_0|} \Big( \frac{1}{t^{n-\alpha s}} \mu(B(x,t/2)) \Big)^{1/(s-1)} \frac{dt}{t} \\ &\qquad \geq (3/2)^{\frac{\alpha s - n}{s-1}} \, |x - x_0|^{\frac{\alpha s - n}{s-1}} \int_u^{|x-x_0|} \Big( \frac{1}{t^{n-\alpha s}} \sigma(B(x,t/2)) \Big)^{1/(s-1)} \frac{dt}{t}. \end{aligned}$$

Putting together what we have so far,

$$\begin{aligned} \mathcal{N}^3(|\cdot - x_0|^{\frac{\alpha s - n}{s-1}})(x) &\geq (3/2)^{\frac{\alpha s - n}{s-1}} \, |x - x_0|^{\frac{\alpha s - n}{s-1}} \int_{r=0}^{|x-x_0|} \Big( \frac{\sigma(B(x,r/2))}{r^{n-\alpha s}} \Big)^{1/(s-1)} \\ &\cdot \int_{u=r}^{|x-x_0|} \Big( \frac{\sigma(B(x,u/2))}{u^{n-\alpha s}} \Big)^{1/(s-1)} \int_{t=u}^{|x-x_0|} \Big( \frac{\sigma(B(x,t/2))}{t^{n-\alpha s}} \Big)^{1/(s-1)} \frac{dt}{t} \frac{du}{u} \frac{dr}{r}. \end{aligned}$$

Integration by parts now yields the Lemma in the case $m = 3$. It is easy to see that a completely similar argument works for general $m$, using the claim (3.4.13) $m - 1$ times as we have done twice in the above argument. Thus the Lemma is proved. □

As with Lemma 3.4.3, we readily see that applying Lemma 3.4.4 to the iterates in the bounds (3.4.4) and (3.4.5) of Lemma 3.4.2 yields the existence of positive constants $c_1$ and $c_2$, depending on $n, \alpha, s$ and $C_0$, so that

$$u(x) \geq c_1 \, |x - x_0|^{\frac{\alpha s - n}{s-1}} \exp\Big( c_2 \int_0^{|x-x_0|} \Big( \frac{\sigma(B(x, r/2))}{r^{n-\alpha s}} \Big)^{1/(s-1)} \frac{dr}{r} \Big). \tag{3.4.16}$$

But, since $C(\sigma) < \infty$, we can replace $\sigma(B(x, r/2))$ by $\sigma(B(x, r))$ in the integral in (3.4.16). Indeed, by change of variables:

$$\int_0^{|x-x_0|} \Big( \frac{\sigma(B(x, r/2))}{r^{n-\alpha s}} \Big)^{1/(s-1)} \frac{dr}{r} = 2^{\frac{\alpha s - n}{s-1}} \int_0^{|x-x_0|/2} \Big( \frac{\sigma(B(x, r))}{r^{n-\alpha s}} \Big)^{1/(s-1)} \frac{dr}{r},$$

and by (3.3.21):

$$\int_{|x-x_0|/2}^{|x-x_0|} \Big( \frac{\sigma(B(x, r))}{r^{n-\alpha s}} \Big)^{1/(s-1)} \frac{dr}{r} \leq C(n, \alpha, s, C(\sigma)).$$

Thus we conclude that there are positive constants $c_1$ and $c_2$ depending on $n, \alpha, s, C(\sigma)$ and $C_0$ so that

$$u(x) \geq c_1 \, |x - x_0|^{\frac{\alpha s - n}{s-1}} \exp\Big( c_2 \int_0^{|x-x_0|} \Big( \frac{\sigma(B(x, r))}{r^{n-\alpha s}} \Big)^{1/(s-1)} \frac{dr}{r} \Big).$$

*Proof of Theorem 3.4.1.* We have showed that if $u$ is a solution of (3.4.1) with constant $C_0$, then there are constants $c_1$ and $c_2$, depending on $n, \alpha, s, C_0$ and $C(\sigma)$, so that the following two inequalities hold:

$$u(x) \geq c_1 \, |x - x_0|^{\frac{\alpha s - n}{s-1}} \exp\Big( c_2 \int_0^{|x-x_0|} \frac{\sigma(B(x, r))}{r^{n-\alpha s}} \frac{dr}{r} \Big), \tag{3.4.17}$$

$$u(x) \geq c_1 \, |x - x_0|^{\frac{\alpha s - n}{s-1}} \exp\Big( c_2 \int_0^{|x-x_0|} \Big( \frac{\sigma(B(x, r))}{r^{n-\alpha s}} \Big)^{1/(s-1)} \frac{dr}{r} \Big). \tag{3.4.18}$$

Averaging (3.4.17) and (3.4.18) with the inequality of the arithmetic mean and geometric mean, $a/2 + b/2 \geq \sqrt{ab}$, yields the required lower bound for solutions of

(3.4.1), and hence completes the proof of Theorem 3.4.1. □

*Proof of Theorems 3.1.2 and 3.1.12.* The capacity estimates have been proven in Remark 3.3.2 so it remains to prove the bounds on the fundamental solutions. Suppose first that $u$ is a fundamental solution of $\mathcal{L}$. Then, as a result of the Wolff potential estimate, $u$ satisfies the inequality (3.3.3), which is (3.4.1) in the case when $\alpha = 1$ and $s = p$. Applying Theorem 3.4.1 when specialized to this case is precisely the bound (1.0.11) of Theorem 3.1.2.

Similarly, if $u$ is a fundamental solution of $\mathcal{G}$, then $u$ satisfies (3.3.4), which is just (3.4.1) when $\alpha = \frac{2k}{k+1}$ and $s = k+1$ and so we may apply Theorem 3.4.1. We again see that the bound (3.4.2) in Theorem 3.4.1 with this choice of $\alpha$ and $s$ is exactly the required bound (3.1.7) in Theorem 3.1.12. □

## 3.5 Construction of a supersolution

In this section we will construct a solution corresponding to the integral inequality (3.5.1) below, which as we have already seen is closely related to the fundamental solutions of $\mathcal{L}$ and $\mathcal{G}$. Suppose that $v$ is a solution of the integral inequality:

$$v(x) \geq C_0 \mathcal{N}(v)(x) + |x - x_0|^{\frac{\alpha s - n}{s-1}} \tag{3.5.1}$$

where

$$\mathcal{N}(f)(x) = \mathbf{W}_{\alpha,s}(f^{s-1} d\sigma)(x)$$

for any positive constant $C_0 > 0$. Then by Theorem 3.3.1 there is a constant $C(\sigma) > 0$ such that $\sigma$ satisfies:

$$\sigma(E) \leq C(\sigma) \mathrm{cap}_{\alpha,s}(E) \tag{3.5.2}$$

for all compact sets $E \subset \mathbf{R}^n$. By Corollary 3.3.12, a consequence of this is that there is a positive constant $A = A(s, \alpha, n)$ so that:

$$\int_{B(x,r)} e^{\beta \mathbf{W}_{\alpha,s}(\chi_{B(x,r)} d\sigma)} \, d\sigma \leq \frac{1}{1 - \beta A C(\sigma)} \sigma(B(x,r)), \tag{3.5.3}$$

provided $\beta A C(\sigma) < 1$. In addition note that by standard capacity estimates we may also assume that

$$AC(\sigma) \geq \sup_{x \in \mathbf{R}^n, \, r > 0} \frac{\sigma(B(x,r))}{r^{n-\alpha s}}$$

and hence the hypothesis of Lemma 3.3.13 are satisfied.

To solve the inequality (3.5.1) it suffices to find a function $u$ so that $v \geq |x - x_0|^{\frac{\alpha s - n}{s-1}}$ and $v \geq C\mathcal{N}(v)$. With this in mind the following theorem will be enough for our purposes. Recall that $B_k = B(x_0, 2^k)$ and $j_x$ is defined to be the integer so that $2^{j_x} \leq |x - x_0| < 2^{j_x+1}$.

**Theorem 3.5.1.** *Let $\sigma$ be a measure satisfying (3.5.2) (and hence (3.5.3)). In addition suppose that*

$$\int_0^1 \frac{\sigma(B(x_0, r))}{r^{n-\alpha s}} \frac{dr}{r} < \infty. \tag{3.5.4}$$

*Define a function $v$ by the following:*

$$\begin{aligned} v(x) = |x - x_0|^{\frac{\alpha s - n}{s-1}} \exp\Big(\beta \sum_{\ell=-\infty}^{j_x} 2^{\ell(\alpha s - n)} \sigma(B_{\ell+1})\Big) \\ \cdot \exp\Big(\beta \int_0^{|x-x_0|} \Big(\frac{\sigma(B(x,r))}{r^{n-\alpha s}}\Big)^{1/(s-1)} \frac{dr}{r}\Big). \end{aligned} \tag{3.5.5}$$

*Then, if $C(\sigma)$ is sufficiently small, there exists a positive $\beta = \beta(C(\sigma), n, \alpha, s)$, along with a positive constant $C_0 = C_0(\beta, n, \alpha, s, \sigma)$ so that*

$$v \geq C_0 \mathcal{N}(v), \text{ and in addition } \inf_{x \in \mathbf{R}^n} v(x) = 0.$$

**Remark 3.5.2.** The condition (3.5.4) is only used to ensure that $v$ is not identically infinite. By inspection of the bound in Theorem 3.4.1 it is clear that if it is not satisfied then any fundamental solution is identically infinite.

*Proof.* We let $\mathcal{N}(v) = I + II$, where $I$ is defined by

$$I = \int_0^{|x-x_0|/2} \Big( \frac{1}{r^{n-\alpha s}} \int_{B(x,r)} v^{s-1}(y)\, d\sigma(y) \Big)^{1/(s-1)} \frac{dr}{r}. \tag{3.5.6}$$

First note that for any $y \in B(x,r)$ with $r \leq |x - x_0|/2$, we have that $|y - x_0| \leq (3/2)|x - x_0|$ and $j_y \leq j_x + 1$. In addition note that for such $y$,

$$|y - x_0| \geq |x - x_0| - |x - y| \geq |x - x_0|/2.$$

These two observations, when plugged into $I$, yield:

$$\begin{aligned} I \leq 2^{\frac{n-\alpha s}{s-1}} |x - x_0|^{\frac{\alpha s - n}{s-1}} \exp\Big(\beta \sum_{\ell=-\infty}^{j_x+1} 2^{\ell(\alpha s - n)} \sigma(B_{\ell+1})\Big) \int_0^{\frac{1}{2}|x-x_0|} \Big( \frac{1}{r^{n-\alpha s}} \\ \cdot \int_{B(x,r)} \exp\Big((s-1)\beta \int_0^{\frac{3}{2}|x-x_0|} \Big(\frac{\sigma(B(y,t))}{t^{n-\alpha s}}\Big)^{\frac{1}{s-1}} \frac{dt}{t}\Big)\, d\sigma(y)\Big)^{\frac{1}{s-1}} \frac{dr}{r}. \end{aligned}$$

We now pay attention to the integral

$$\int_{B(x,r)} \exp\Big((s-1)\beta \int_0^{\frac{3}{2}|x-x_0|} (t^{\alpha s - n} \sigma(B(y,t))^{1/(s-1)} \frac{dt}{t}\Big) d\sigma(y). \tag{3.5.7}$$

Note that we may rewrite (3.5.7) as

$$\begin{aligned} &\int_{B(x,r)} \exp\Big((s-1)\beta \int_0^r \Big(\frac{\sigma(B(y,t)}{t^{n-\alpha s}}\Big)^{1/(s-1)} \frac{dt}{t}\Big) \\ &\cdot \exp\Big((s-1)\beta \int_r^{\frac{3}{2}|x-x_0|} \Big[\Big(\frac{\sigma(B(y,t)}{t^{n-\alpha s}}\Big)^{1/(s-1)} - \Big(\frac{\sigma(B(x,t)}{t^{n-\alpha s}}\Big)^{1/(s-1)}\Big] \frac{dt}{t}\Big)\, d\sigma(y) \\ &\cdot \exp\Big((s-1)\beta \int_r^{\frac{3}{2}|x-x_0|} \Big(\frac{\sigma(B(x,t)}{t^{n-\alpha s}}\Big)^{1/(s-1)} \frac{dt}{t}\Big). \end{aligned} \tag{3.5.8}$$

By the Wolff potential tail estimate, Lemma 3.3.13, it follows:

$$\left| \int_r^{\frac{3}{2}|x-x_0|} \left[ \Big(\frac{\sigma(B(y,t)}{t^{n-\alpha s}}\Big)^{1/(s-1)} - \Big(\frac{\sigma(B(x,t)}{t^{n-\alpha s}}\Big)^{1/(s-1)} \right] \frac{dt}{t} \right| \leq C(n,\alpha,s,C(\sigma)).$$

Thus (3.5.8) is less than a constant multiple of:

$$\begin{aligned} &\int_{B(x,r)} \exp\Big((s-1)\beta \int_0^r \Big(\frac{\sigma(B(y,t)}{t^{n-\alpha s}}\Big)^{1/(s-1)} \frac{dt}{t}\Big)\, d\sigma(y) \\ &\qquad \cdot \exp\Big((s-1)\beta \int_r^{\frac{3}{2}|x-x_0|} \Big(\frac{\sigma(B(x,t)}{t^{n-\alpha s}}\Big)^{1/(s-1)} \frac{dt}{t}\Big). \end{aligned} \tag{3.5.9}$$

Now, provided $\beta C(\sigma)$ is small enough we may apply (3.5.3), and hence we may estimate the integral in (3.5.9) by:

$$\begin{aligned} &\int_{B(x,r)} \exp\Big((s-1)\beta \int_0^r \Big(\frac{\sigma(B(y,t)}{t^{n-\alpha s}}\Big)^{1/(s-1)} \frac{dt}{t}\Big)\, d\sigma(y) \\ &\qquad \leq \int_{B(x,2r)} \exp\Big((p-1)\beta W^{\sigma}_{\alpha,s}(\chi_{B(x,2r)})(y)\Big)\, d\sigma(y) \leq C\sigma(B(x,2r)). \end{aligned}$$

Putting these estimates together, there is a constant $C = C(n,\alpha,s,C(\sigma))$ so that:

$$\begin{aligned} I &\leq C\,|x-x_0|^{\frac{\alpha s-n}{s-1}} \exp\Big(\beta \sum_{\ell=-\infty}^{j_x+1} 2^{\ell(\alpha s-n)}\sigma(B_{\ell+1})\Big) \\ &\quad \cdot \int_0^{|x-x_0|} \Big(\frac{\sigma(B(x,2r))}{r^{n-\alpha s}}\Big)^{1/(p-1)} \cdot \exp\Big(\beta \int_r^{\frac{3}{2}|x-x_0|} \Big(\frac{\sigma(B(x,t)}{t^{n-\alpha s}}\Big)^{1/(s-1)} \frac{dt}{t}\Big) \frac{dr}{r}. \end{aligned}$$

But now note since $\sigma$ satisfies (3.5.2), we have, for any $\rho > 0$:

$$\int_\rho^{2\rho} \Big(\frac{\sigma(B(x,t)}{t^{n-\alpha s}}\Big)^{1/(s-1)} \frac{dt}{t} \leq C, \quad \text{and} \quad 2^{(j_x+1)(\alpha s-n)}\sigma(B_{j_x+2}) \leq C, \tag{3.5.10}$$

where in this last display the constant depends on $n, \alpha, s$ and $C(\sigma)$, but is independent of $\rho$. By a change of variables and (3.5.10), we see there is a positive constant

$C = C(n, \alpha, s, C(\sigma))$, so that:

$$I \leq C\,|x - x_0|^{\frac{\alpha s - n}{s-1}} \exp\Big(\beta \sum_{\ell=-\infty}^{j_x} 2^{\ell(\alpha s - n)} \sigma(B_{\ell+1})\Big) \cdot \int_0^{|x-x_0|} \Big(\frac{\sigma(B(x,r))}{r^{n-\alpha s}}\Big)^{1/(s-1)} \cdot \exp\Big(\beta \int_r^{|x-x_0|} \Big(\frac{\sigma(B(x,t))}{t^{n-\alpha s}}\Big)^{1/(s-1)} \frac{dt}{t}\Big) \frac{dr}{r}.$$

An application of integration by parts now yields $I \leq C\,v$ for a positive constant $C = C(n, \alpha, s, C(\sigma))$.

We next consider the remainder of the Wolff potential $II$. By writing the integral as a sum over dyadic annuli, it is not difficult to see that there exists a constant $C > 0$, depending on $n, s$ and $\alpha$, so that:

$$II \leq C \sum_{k=j_x}^{\infty} 2^{k\frac{\alpha s - n}{s-1}} \Big(\int_{B(x,2^k)} v^{s-1}\, d\sigma\Big)^{1/(s-1)}. \tag{3.5.11}$$

Let us first consider a single integral in the sum. Since $k \geq j_x$, it follows that $B(x, 2^k) \subset B(x_0, 2^{k+2})$. Thus,

$$\int_{B(x,2^k)} v^{s-1}\, d\sigma \leq \int_{B(x_0,2^{k+2})} v^{s-1}\, d\sigma = \sum_{\ell=-\infty}^{k+2} \int_{B_\ell \backslash B_{\ell-1}} v^{s-1}\, d\sigma. \tag{3.5.12}$$

We now concentrate on one term in the sum on the right hand side of (3.5.12). Observe that for $z \in B_\ell \backslash B_{\ell-1}$, we have $2^\ell \geq |z - x_0| \geq 2^{\ell-1}$ and $j_z = \ell - 1$. This yields:

$$\int_{B_\ell \backslash B_{\ell-1}} v^{s-1}(z) d\sigma(z) \leq 2^{(\ell-1)(p-n)} \exp\Big(\beta(s-1) \sum_{m=-\infty}^{\ell-1} 2^{m(\alpha s - n)} \sigma(B_{m+1})\Big) \cdot \int_{B_\ell} \exp\Big((s-1)\beta \int_0^{2^\ell} \Big(\frac{\sigma(B(y,t))}{t^{n-\alpha s}}\Big)^{1/(s-1)} \frac{dt}{t}\Big) d\sigma(y).$$

But, again, if we suppose that $\beta C(\sigma)$ is small, then by the exponential integration result (3.5.3), there is a constant $C = C(n, p, s, C(\sigma)) > 0$ so that:

$$\int_{B_\ell} \exp\Big((s-1)\beta \int_0^{2^\ell} \Big(\frac{\sigma(B(y,t))}{t^{n-\alpha s}}\Big)^{1/(s-1)} \frac{dt}{t}\Big)\, d\sigma(y) \le C\sigma(B(x, 2^{\ell+1})).$$

Thus, plugging this into (3.5.12), we find that there is a constant $C = C(n, p, s, C(\sigma)) >$ 0 so that:

$$\begin{aligned}\int_{B(x,2^k)} v^{s-1} d\sigma(z) \le C \sum_{\ell=-\infty}^{k+2} 2^{\ell(\alpha s-n)} \sigma(B(x, 2^{\ell+1}))& \\ \cdot \exp\Big(\beta(s-1) \sum_{m=-\infty}^{\ell-1} 2^{m(\alpha s-n)} \sigma(B_{m+1})\Big).&\end{aligned} \tag{3.5.13}$$

Next, consider the following summation by parts estimate (see [FV09]). Suppose that $\{\lambda_j\}_j$ is a nonnegative sequence such that $0 \le \lambda_j \le 1$. Then:

$$\sum_{j=0}^{\infty} \lambda_j e^{\sum_{k=j}^{\infty} \lambda_k} \le 2\, e^{\sum_{j=0}^{\infty} \lambda_j}. \tag{3.5.14}$$

Provided $C(\sigma) \le 1$, we may apply (3.5.14) to see that the right hand side of (3.5.13) is less than a constant (depending on $n, \alpha, s, C(\sigma)$) multiple of:

$$\exp\Big((s-1)\beta \sum_{\ell=-\infty}^{k+2} 2^{\ell(\alpha s-n)} \sigma(B_{\ell+1})\Big).$$

Hence (as we may bound two top terms in the above sum using the $C(\sigma)$ condition),

$$II \le C \sum_{k=j_x}^{\infty} 2^{k\frac{\alpha s-n}{s-1}} \exp\Big(\beta \sum_{\ell=-\infty}^{k} 2^{\ell(\alpha s-n)} \sigma(B_{\ell+1})\Big). \tag{3.5.15}$$

This is less than a constant multiple of $u$ provided $C(\sigma)$ is small enough. Indeed, note that the right hand side of (3.5.15) is a constant multiple of:

$$2^{j_x \frac{\alpha s-n}{s-1}} \exp\Big(\beta \sum_{\ell=-\infty}^{j_x} 2^{\ell(\alpha s-n)}\sigma(B_{\ell+1})\Big)\cdot \sum_{k=0}^{\infty} 2^{k\frac{\alpha s-n}{s-1}} \exp(\beta \sum_{\ell=1}^{k} 2^{\ell(\alpha s-n)}\sigma(B_{\ell+1})\Big). \quad (3.5.16)$$

Now, using the definitions of $j_x$, $v$ and also (3.5.2), it is immediate that (3.5.16) is less than

$$C\, v(x) \sum_{k=0}^{\infty} 2^{k\frac{\alpha s-n}{s-1}} \exp\Big(\beta A C(\sigma)^{s-1} k\Big) \quad (3.5.17)$$

where $C = C(n, \alpha, s, C(\sigma))$ and $A = A(n, s, \alpha)$. Now, with $C(\sigma)$ small enough, this series converges and so $v \geq CII$ for a positive constant $C > 0$ depending on $n, s, \alpha, C(\sigma)$.

It is left to see that $\inf_{x\in \mathbf{R}^n} v(x) = 0$. To this end, first note that we can chose $C(\sigma)$ sufficiently small so that:

$$\begin{aligned} &|x - x_0|^{\frac{\alpha s-n}{s-1}} \exp\Big(\beta \sum_{\ell=-\infty}^{j_x} 2^{\ell(\alpha s-n)}\sigma(B_{\ell+1})\Big) \\ &\qquad \cdot \exp\Big(\beta \int_1^{|x-x_0|} \Big(\frac{\sigma(B(x,r)}{r^{n-\alpha s}}\Big)^{1/(s-1)} \frac{dr}{r}\Big) \to 0, \ \text{ as } |x| \to \infty. \end{aligned} \quad (3.5.18)$$

Indeed, this follows from the argument in (3.5.17), using the condition (3.5.2), along with noting that:

$$\sum_{\ell=-\infty}^{1} 2^{\ell(\alpha s-n)}\sigma(B_{\ell+1}) \leq C \int_0^1 \frac{\sigma(B(x_0, r))}{r^{n-\alpha s}} \frac{dr}{r} < \infty.$$

Let us define a sequence $a_j$ by:

$$a_j = \inf_{x\in B(0,2^j)\setminus B(0,2^{j-1})} \int_0^1 \Big(\frac{\sigma(B(x,r))}{r^{n-\alpha s}}\Big)^{1/(s-1)} \frac{dr}{r}.$$

To finish the proof it therefore suffices to show that $a_j$ tends to zero as $j \to \infty$. First suppose $s \geq 2$, then consider:

$$b_R = \frac{1}{R^n} \int_{B(0,R)} \int_0^1 \Big( \frac{\sigma(B(x,r))}{r^{n-\alpha s}} \Big)^{1/(s-1)} \frac{dr}{r} dx.$$

By Fubini and Hölder's inequality,

$$b_R \leq C \int_0^1 \frac{1}{r^{\frac{n-\alpha s}{s-1}+1}} \Big( \frac{1}{R^n} \int_{B(0,R)} \sigma(B(x,r)) dx \Big)^{1/(s-1)} dr. \tag{3.5.19}$$

Then by Fubini once again, $\int_{B(0,R)} \sigma(B(x,r))dx \leq Cr^n \sigma(B(0,2R)) \leq Cr^n R^{n-p}$, where we have used (3.5.2) in this last line. Plugging this estimate into (3.5.19) we find that $b_R \to 0$ as $R \to \infty$. This clearly implies that $a_j$ is a null sequence, since $a_j \leq Cb_{2^j}$ for a positive constant independent of $j$.

Now let $1 < s < 2$ and note that for any integer $k$:

$$\begin{aligned} (\int_0^{2^k} \Big( \frac{\sigma(B(x,r))}{r^{n-\alpha s}} \Big)^{1/(s-1)} \frac{dr}{r} \Big)^{s-1} &\leq C \Big( \sum_{j=-\infty}^{k} \Big( \frac{\sigma(B(x,2^j))}{2^{j(n-\alpha s)}} \Big)^{1/(s-1)} \Big)^{s-1} \\ &\leq C \sum_{j=-\infty}^{k} \frac{\sigma(B(x,2^j))}{2^{j(n-\alpha s)}} \leq C \int_0^{2^k} \frac{\sigma(B(x,r))}{r^{n-\alpha s}} \frac{dr}{r}. \end{aligned} \tag{3.5.20}$$

Since the previous argument shows that:

$$\frac{1}{R^n} \int_{B(0,R)} \int_0^1 \frac{\sigma(B(x_0,r))}{r^{n-\alpha s}} \frac{dr}{r} dx \quad \to 0, \text{ as } R \to \infty,$$

we conclude that $a_j \to 0$ as $j \to \infty$ when $1 < s < 2$. Thus $\inf_{x \in \mathbf{R}^n} v(x) = 0$. □

## 3.6 Proofs of Theorems 3.1.5 and 3.1.13

### 3.6.1 The quasilinear existence theorem

In this section we will prove Theorems 3.1.5 and 3.1.13. We make use of the construction in Section 3.5. Combined with a simple iteration scheme based on weak continuity, which is similar to those in [PV08, PV09]. Let us first consider the quasilinear case.

*Proof of Theorem 3.1.5.* Recall that we denote by $C(\sigma)$ the positive (and by assumption finite) constant:

$$C(\sigma) = \sup_E \frac{\sigma(E)}{\text{cap}_{1,p}(E)},$$

where the supremum is taken over all compact sets $E \subset \mathbf{R}^n$ of positive capacity. Note that by Lemma 3.2.10;

$$C(\sigma) \geq C \sup_E \frac{\sigma(E)}{\text{cap}_p(E)}$$

where $\text{cap}_p$ is the $p$-capacity, and $C = C(n,p) > 0$. Suppose first that:

$$\int_0^1 \frac{\sigma(B(x_0,r))}{r^{n-p}} \frac{dr}{r} = \infty. \tag{3.6.1}$$

Then, we see that by Theorem 3.1.2 any fundamental solution $u(x, x_0) \equiv \infty$, and there is nothing to prove. Hence we may assume that the integral in (3.6.1) is finite, and so we may apply Theorem 3.5.1. This implies that if $C(\sigma)$ is sufficiently small, in terms of $n$ and $p$, then there is a function $v \in L^{p-1}_{\text{loc}}(\sigma)$ and a constant $C_0 > 0$, depending on $n$ and $p$ such that:

$$v(x) \geq C_0 \mathbf{W}^{\sigma}_{1,p}(v^{p-1})(x) + \tilde{K} \, |x - x_0|^{\frac{p-n}{p-1}}, \tag{3.6.2}$$

and $\inf_{x \in \mathbf{R}^n} v(x) = 0$. Here $\tilde{K} = \frac{p-1}{n-p} K$, with $K = K(n,p) > 0$ the same constant that appears in the Wolff potential estimate, Theorem 3.2.4. Indeed, recalling that $j_x$ is the integer such that $2^{j_x} \le |x - x_0| \le 2^{j_x+1}$, we can let

$$v(x) = 2\tilde{K}\,|x - x_0|^{\frac{p-n}{p-1}} \exp\Big(\beta \sum_{\ell=-\infty}^{j_x} 2^{\ell(p-n)} \sigma(B(x_0, 2^{\ell+1}))\Big) \cdot \exp\Big(\beta \int_{r=0}^{|x-x_0|} \Big(\frac{\sigma(B(x,r))}{r^{n-p}}\Big)^{1/(p-1)} \frac{dr}{r}\Big)$$

for a suitable choice of $\beta = \beta(n,p) > 0$. Let $u_0 = G(\,\cdot, x_0)$ where $G(x, x_0)$ is defined in (1.0.10). Then $u_0$ is $p$-superharmonic in $\mathbf{R}^n$ and $-\Delta_p u_0(x) = \delta_{x_0}$ (in fact $u_0$ is the unique such solution, see, e.g. [KV86]). By choice of $\tilde{K}$ (assuming $K > 1$) we have that $u_0 \le v$, and hence $u_0 \in L^{p-1}_{\text{loc}}(\sigma)$. Let $\epsilon > 0$ be such that $\epsilon K \le C_0$, then we claim that there exists a sequence $\{u_m\}_{m\ge 0}$ of functions which are $p$-superharmonic in $\mathbf{R}^n$, $u_m \in L^{p-1}_{\text{loc}}(\sigma)$:

$$-\Delta_p u_m = \epsilon\sigma(u_{m-1})^{p-1} + \delta_{x_0}, \quad \text{and} \inf_{x\in\mathbf{R}^n} u_m(x) = 0, \tag{3.6.3}$$

and in addition $u_m(x) \le v(x)$. The existence of this sequence can be shown by the techniques of [PV09], using the notion of *renormalized solutions*. However, as we are dealing exclusively with $p$-superharmonic functions this detour would be somewhat artificial and so we prove the claim directly. Indeed, suppose that $u_1, \ldots, u_{m-1}$ have been constructed. Then, $\epsilon\sigma(u_{m-1})^{p-1} + \delta_{x_0}$ is a locally finite Borel measure. For each $j \in \mathbb{N}$, let $u_m^j$ be a positive $p$-superharmonic function such that

$$-\Delta_p u_m^j = \epsilon\sigma(u_{m-1})^{p-1}\chi_{B(x_0,2^j)} + \delta_{x_0} \;\text{ in }\; \mathbf{R}^n.$$

The existence of such a $p$-superharmonic function is guaranteed by [Kil99], Theorem 2.10. By subtracting a positive constant, we may assume that $\inf_{x\in\mathbf{R}^n} u_m^j = 0$.

Now, by the global Wolff potential estimate and since $\mathbf{W}_{1,p}(\delta_{x_0}) = \frac{p-1}{n-p}\,|x - x_0|^{\frac{p-n}{p-1}}$, we find that

$$u_m^j(x) \le K\epsilon \mathbf{W}^{\sigma}_{1,p} u_{m-1}^{p-1}(x) + \tilde{K}\,|x-x_0|^{\frac{p-n}{p-1}}\,.$$

But since $u_{m-1} \le v$,

$$u_m^j(x) \le K\epsilon \mathbf{W}^{\sigma}_{1,p} v^{p-1}(x) + \tilde{K}\,|x-x_0|^{\frac{p-n}{p-1}}\,.$$

By choice of $\epsilon > 0$ so that $K\epsilon \le C_0$, we conclude that $u_m^j(x) \le v(x)$.

Appealing now to Theorem 3.2.2 ([KM92], Theorem 1.17), we find a subsequence $u_m^{j_k}$ and an $p$-superharmonic function $u_m$ such that $u_m^{j_k}(x) \to u_m(x)$ for almost every $x \in \mathbf{R}^n$. Thus $u_m(x) \le v(x)$ and hence $\inf_{x\in\mathbf{R}^n} u_m(x) = 0$. The claim is then completed by appealing to Theorem 3.2.3 to see that

$$-\Delta_p u_m = \epsilon\sigma(u_{m-1})^{p-1} + \delta_{x_0} \quad \text{in } \mathbf{R}^n.$$

Now, since $u_m(x) \le v(x)$, for all $m$, we may again find a subsequence $\{u_{m_k}\}_k$ and a positive $p$-superharmonic function $u$ so that $u_{m_k}(x) \to u(x)$ almost everywhere. Since it follows that $u(x) \le v(x)$, we have that $\inf_{x\in\mathbf{R}} u(x) = 0$. Finally, by Theorem 3.2.3, we may conclude that:

$$-\Delta_p u = \epsilon\sigma u^{p-1} + \delta_{x_0} \quad \text{in } \mathbf{R}^n.$$

This completes the proof of Theorem 3.1.5 with the potential $\tilde{\sigma} = \epsilon\sigma$, once we notice that:

$$\sum_{\ell=-\infty}^{j_x} 2^{\ell(p-n)}\sigma(B(x_0, 2^{\ell+1})) \le C\int_0^{|x-x_0|} \frac{\sigma(B(x_0,r))}{r^{n-p}}\frac{dr}{r} + C$$

for a positive constant $C$ depending on $n$ and $p$. □

### 3.6.2 The $k$-Hessian existence theorem

For the Hessian existence theorem, we may state the following Lemma, contained in [PV09], Lemma 4.7.

**Lemma 3.6.1.** *[PV09] Let* $\mu$ *and* $\nu$ *be nonnegative locally finite Borel measures in* $\mathbf{R}^n$*, so that* $\mu \leq \nu$ *and* $\mathbf{W}_{\frac{2k}{k+1},k+1}\nu < \infty$ *almost everywhere. Suppose that* $u \geq 0$ *satisfies* $-u \in \Phi^k(\mathbf{R}^n)$*,* $\mu_k[-u] = \mu$*, and* $u$ *is a pointwise a.e. limit of a subsequence of the sequence* $\{u_m\}_m$*, with* $-u_m \in \Phi^k(B(x_0, 2^{m+1}))$ *and*

$$\begin{cases} \mu_k[-u_m] = \mu\chi_{B(x_0,2^m)} & \text{in } B(x_0, 2^{m+1}) \\ u_m = 0 \ \text{ on } & \partial B(x_0, 2^{m+1}). \end{cases}$$

*Then there is a nonnegative function so that* $-w \in \Phi^k(\mathbf{R}^n)$*,* $w \geq u$*, and*

$$\mu_k[-w] = \nu \quad \text{and} \quad \inf_{x\in\mathbf{R}^n} v(x) = 0.$$

*Moreover,* $w$ *is a pointwise a.e. limit of a sequence* $\{w_m\}_m$*, so that* $-w_m \in \Phi^k(B(x_0, 2^{m+1}))$ *and*

$$\begin{cases} \mu_k[-w_m] = \nu\chi_{B(x_0,2^m)} & \text{in } B(x_0, 2^{m+1}) \\ w_m = 0 \ \text{ on } & \partial B(x_0, 2^{m+1}). \end{cases}$$

*Proof of Theorem 3.1.13.* This is very similar to the previous proof so we will be slightly brief to avoid repetition. As in the previous proof, the theorem is only nontrivial in the case when,

$$\int_0^1 \frac{\sigma(B(x_0, r))}{r^{n-2k}} \frac{dr}{r} < \infty.$$

Hence if $C(\sigma)$ small enough, where now

$$C(\sigma) = \sup_{E\,\text{compact}} \frac{\sigma(E)}{\text{cap}_{2k/(k+1),k+1}(E)},$$

then we may apply Theorem 3.5.1 to find a positive function $v$ such that $\inf_{x\in\mathbf{R}^n} v(x) = 0$ and

$$v(x) \geq C_0 \mathbf{W}^{\sigma}_{\frac{2k}{k+1},k+1}(v^k)(x) + \tilde{K}\,|x - x_0|^{2/k-n}$$

with $\tilde{K} = \frac{k}{n-2k}K$. Here $K$ is a constant appearing in the global Wolff potential bound Theorem 3.2.6.

Let $\epsilon > 0$ be such that $\epsilon K \leq C_0$. Let $u_0 = c(n,k)\,|x - x_0|^{2/k-n}$, where $c(n,k) = (\frac{k}{n-2k}) \cdot (\binom{n}{k}\omega_{n-1})^{-1/k}$. Then $u_0$ is the (unique) fundamental solution of the $k$-Hessian operator in $\mathbf{R}^n$, see [TW02a]. By a repeated application of Lemma 3.6.1, we find a sequence $\{u_m\}_m$ of nonnegative functions so that $-u_m \in \Phi^k(\mathbf{R}^n)$, $\inf_{x\in\mathbf{R}^n} u_m(x) = 0$, $u_m \in L^k_{\text{loc}}(\sigma)$ and

$$\mu_k[-u_m] = \epsilon\sigma(u_{m-1})^{p-1} + \delta_{x_0}.$$

Furthermore, as in the previous proof, we see that by choice of $\tilde{K}$ that $u_m \leq v$. Now, appealing to the weak continuity of the $k$-Hessian operator (Theorem 3.2.5), we assert the existence of a nonnegative $u$ such that $-u \in \Phi^k(\mathbf{R}^n)$,

$$\mu_k[-u] = \epsilon\sigma u^k + \delta_{x_0},$$

and $u \leq v$. Hence $\inf_{x\in\mathbf{R}^n} u(x) = 0$. Thus, noting Lemma 3.2.11, we see that Theorem 3.1.13 is proved with potential $\tilde{\sigma} = \epsilon\sigma$, once we make the easy observations that $v$ is comparable to the right hand side of the bound (3.1.8). □

### 3.6.3 Criteria for equivalence of perturbed and unperturbed fundamental solutions

In this short section we consider necessary and sufficient conditions for fundamental solutions of $\mathcal{L}$, defined by (3.0.1), to be equivalent to the fundamental solutions of the $p$-Laplacian. Similar results also holds for the $k$-Hessian operator. Recall the fundamental solution of $-\Delta_p$, which we denoted by $G(x, x_0)$ in (1.0.10), and the Wolff and Riesz potentials from (3.1.1) and (3.1.2) respectively.

**Corollary 3.6.2.** *Suppose that there is a positive constant $c > 0$ such that for all $x_0 \in \mathbf{R}^n$ (1.0.9) holds whenever $u(x, x_0)$ is a fundamental solution of $\mathcal{L}$. Then $\sigma(E) \leq \mathrm{cap_p(E)}$ for all compact sets $E$, and furthermore (1.0.13) and (1.0.14) hold.*

*Conversely, suppose that (1.0.13) holds if $1 < p \leq 2$, or (1.0.14) holds if $p \geq 2$. Then there exists a positive constant $C$, depending on $n$ and $p$, such that if $\sigma(E) \leq C\mathrm{cap_p(E)}$ for all compact sets $E$, then for any $x_0 \in \mathbf{R}^n$ there is a fundamental solution $u(x, x_0)$ of $\mathcal{L}$ with pole at $x_0$ satisfying (1.0.9) for a constant $c = c(n, p) > 0$.*

The Corollary is an immediate consequence of Theorems 3.1.2 and 3.1.5 once we notice that if $1 < p < 2$ then there is a constant $C = C(n, p) > 0$ such that:

$$\big(\mathbf{W}_{1,p}(\sigma)(x)\big)^{p-1} \leq C\mathbf{I}_p(\sigma)(x)$$

for all $x \in \mathbf{R}^n$. This inequality has been proved in (3.5.20). The opposite inequality holds if $p > 2$, this is clear from (3.5.20), as the sequence space imbeddings are reversed.

## 3.7 Regularity away from the pole: the proof of Theorem 3.1.8

In this section we will turn to considering the regularity of fundamental solutions, and in particular we will prove Theorem 3.1.8. Throughout this section we will assume the hypothesis of Theorem 3.1.5 hold, and that the fundamental solution $u$ constructed there is not identically infinite. It therefore follows from Theorem 3.1.2 that:

$$\int_0^1 \frac{\sigma(B(x_0,r))}{r^{n-p}}\frac{dr}{r} = B < \infty. \tag{3.7.1}$$

By hypothesis, the constant $C(\sigma)$, defined by:

$$C(\sigma) = \sup_{E \text{ compact}} \frac{\sigma(E)}{\mathrm{cap}_{1,p}(E)}, \tag{3.7.2}$$

is finite, this is nothing more than a restatement of the condition (3.0.3). Thus we will assume that $C(\sigma) < C_0$, for a constant $C_0 = C_0(n,p) > 0$. The first step will be to perform some auxiliary calculations for the function $v(x)$, defined by:

$$v(x) = B(n,p)\,|x-x_0|^{\frac{p-n}{p-1}} \exp\Big(c\mathbf{W}_{1,p}^{|x-x_0|}(\sigma)(x) + c\mathbf{I}_p^{|x-x_0|}(\sigma)(x_0)\Big), \tag{3.7.3}$$

for a positive constant $B(n,p) > 0$ to be chosen later. In particular, we will need to show that $v \in L^p_{\mathrm{loc}}(\mathbf{R}^n\backslash\{x_0\})$. We will see that this is true assuming only that:

$$\sigma(B(x,r)) \lesssim C(\sigma) r^{n-p} \text{ for all balls } B(x,r) \subset \mathbf{R}^n, \tag{3.7.4}$$

with the implicit constant depending on $n$ and $p$. Display (3.7.4) is a special case of (3.7.2), using (3.3.21).

**Lemma 3.7.1.** *There exists a constant so that if $\sigma(B(x,r)) \leq C_1 r^{n-p}$ for all balls*

*$B(x,r) \subset \mathbf{R}^n$. Then for any ball $B(x,r) \subset \mathbf{R}^n$, it follows:*

$$\int_{B(x,r)} e^{a\mathbf{W}_{1,p}(\chi_{B(x,r)}d\sigma)}dx \leq C(r,p,C_1), \tag{3.7.5}$$

*for a constant $a \leq A/(C_1)^{1/(p-1)}$ with $A > 0$ depending on $n$ and $p$.*

There are several ways one can prove this lemma, for instance one can adopt the proof of Lemma 3.3.10, leading to Corollary 3.3.12, which requires some lengthy estimates of sums of dyadic cubes. We shall avoid this by instead offering a more elegant proof, employing a regularity result from [Min07].

*Proof.* Fix a ball $B(x,r)$. Then under the present assumption on $\sigma$, we may apply Theorem 1.12 of [Min07], to find a $p$-superharmonic solution $w$ of:

$$\begin{cases} -\Delta_p u = \sigma \text{ in } B(x,10r), \\ u = 0 \text{ on } \partial B(x,10r). \end{cases} \tag{3.7.6}$$

so that $w \in BMO(B(x,5r))$, and furthermore:

$$\sup_{B(z,s)\subset B(x,5r)} \fint_{B(z,s)} \Big| w(y) - \fint_{B(z,s)} w(y)dy \Big| dy \lesssim C_1^{1/(p-1)}.$$

Therefore, by the John Nirenberg lemma, it follows that there exists a constant $c \lesssim C_1^{-1/(p-1)}$ so that:

$$\fint_{B(x,r)} e^{cw(y)}dy \leq \exp\Big(c\fint_{B(x,r)} w(y)dy\Big) < \infty \tag{3.7.7}$$

Employing the local Wolff potential estimate, Theorem 3.1 in [KM92], it follows, for

$y \in B(x,r)$ that:

$$\begin{aligned} w(y) &\geq C \int_0^{4r} \Big(\frac{\sigma(B(y,s))}{s^{n-p}}\Big)^{1/(p-1)} \frac{ds}{s} \\ &\geq C\mathbf{W}_{1,p}(\chi_{B(x,r)} d\sigma)(y). \end{aligned} \tag{3.7.8}$$

Substituting (3.7.8) into (3.7.7), the lemma follows. □

With this lemma proved, we may now prove that $v \in L^p_{\text{loc}}(\mathbf{R}^n)\backslash\{x_0\}$.

**Lemma 3.7.2.** *There exists $C_0$ so that if $C(\sigma) < C_0$, then:*

$$v \in L^p_{loc}(\mathbf{R}^n)\backslash\{x_0\}.$$

*Proof.* Let $K \subset \mathbf{R}^n\backslash\{x_0\}$ be a compact set, and let $B(x_j, r_j)$ be a finite cover of $K$. Then, note that by crude estimates:

$$\begin{aligned} \int_K v^p dx &\lesssim d(K, x_0)^{p(n-p)/(p-1)} \exp\Big(cp \int_0^{|x_0|+\text{diam}(K)} \frac{\sigma(B(x_0,r)}{r^{n-p}} \frac{dr}{r}\Big) \\ &\cdot \sum_j \int_{B(x_j,r_j)} \exp\Big(pc \int_0^{x-x_0} \Big(\frac{\sigma(B(z,r)\backslash B(x_j,2r_j))}{r^{n-p}}\Big)^{1/(p-1)} \frac{dr}{r}\Big) \\ &\cdot e^{pc\mathbf{W}_{1,p}(\chi_{B(x_j,2r_j)} d\sigma)} dx. \end{aligned} \tag{3.7.9}$$

Employing the estimate (3.7.4), and recalling the definition of the constant $B$ from (3.7.1), we readily derive:

$$\int_0^{|x_0|+\text{diam}(K)} \frac{\sigma(B(x_0,r)}{r^{n-p}} \frac{dr}{r} \lesssim B + C(\sigma)(\log(|x_0| + \text{diam(K)})),$$

and using the same estimate on $\sigma$, we similarly see for all $z \in B(x_j, r_j)$, that:

$$\begin{aligned}\int_0^{x-x_0} \Big(\frac{\sigma(B(z,r)\backslash B(x_j,2r_j))}{r^{n-p}}\Big)^{1/(p-1)} \frac{dr}{r} \\ \leq \int_{r_j}^{\operatorname{diam}(K)+|x_0|} \Big(\frac{\sigma(B(x_j,r))}{r^{n-p}}\Big)^{1/(p-1)} \frac{dr}{r} \\ \lesssim C(\sigma)^{1/(p-1)} \log\Big(\frac{\operatorname{diam}(K)+|x_0|}{r_j}\Big).\end{aligned}$$

Substituting these two displays into (3.7.9), it follows:

$$\int_K v^p dx \leq \sum_j C(n,p,C(\sigma),r_j,K) \int_{B(x_j,r_j)} e^{pc\mathbf{W}_{1,p}(\chi_{B(x_j,2r_j)}d\sigma)} dx. \tag{3.7.10}$$

Note that under the current assumptions, we may choose $C_1 \lesssim C(\sigma)$, with $C_1$ as in Lemma 3.7.1. This is just a restatement of (3.7.4). It follows that if $C_0$ is chosen small enough in terms on $n$ and $p$, then (3.7.5) will be valid, and therefore:

$$\int_{B(x_j,2r_j)} e^{pc\mathbf{W}_{1,p}(\chi_{B(x_j,2r_j)}d\sigma)} dx < \infty, \text{ for each } j.$$

This completes the proof of the lemma. □

Note that in a similar way, using Corollary 3.3.12 instead of Lemma 3.7.1, we deduce the following lemma:

**Lemma 3.7.3.** *There exists $C_0$ so that if $C(\sigma) < C_0$, then:*

$$v \in L^p_{loc}(\mathbf{R}^n\backslash\{x_0\}, d\sigma).$$

We are now in a position to prove Theorem 3.1.8.

*Proof of Theorem 3.1.8.* Let us assume that $C_0$ has been chosen so that Lemmas 3.7.2 and 3.7.3 are both valid. To prove the theorem, we will aim to construct the sequence $\{u_m\}_m$ as in (3.6.3) from the proof of Theorem 3.1.5 with the additional property that

$u_m \in L^{1,p}_{\text{loc}}(\mathbf{R}^n \backslash \{x_0\})$, with constants independent on $m$. We will do this inductively, as in the proof of Theorem 3.1.5. Let $u_0 = G(\,\cdot, x_0)$, with $G(x, x_0)$ as in (1.0.10). Note $G(\cdot, x_0) \in C^\infty_{\text{loc}}(\mathbf{R}^n \backslash \{x_0\})$. Suppose that we have constructed $u_1, \ldots, u_{m-1}$ so that:

$$-\Delta_p u_j = \epsilon \sigma u_{j-1}^{p-1} + \delta_{x_0},$$

with $u_j \leq v$, and $u_{j-1} \in L^{1,p}_{\text{loc}}(\mathbf{R}^n \backslash \{x_0\})$. Let $K$ be a compact subset of $\mathbf{R}^n \backslash \{0\}$, then we claim that $u_{m-1}^{p-1} d\sigma \in W^{-1,p'}(K)$. This will follow from the capacity strong type inequality. Indeed, since $\sigma$ satisfies (3.0.3) with constant $C(\sigma) < C_0$, it follows [Maz85], that:

$$\int |h|^p d\sigma \leq C(\sigma) \Big( \frac{p}{p-1} \Big)^p \int |\nabla h|^p dx, \text{ for all } h \in C_0^\infty(\mathbf{R}^n),$$

and this can be extended by continuity to functions $h \in L^{1,p}_0(\mathbf{R}^n)$. Now, let $h \in C_0^\infty(K)$, and $K'$ be a subset $K \subset\subset K' \subset\subset \mathbf{R}^n \backslash \{x_0\}$ along with a function $g \in C_0^\infty(K')$, $g \equiv 1$ on $K$, $g \geq 0$. Then:

$$\int h u_{m-1}^{p-1} d\sigma = \int h u_{m-1}^{p-1} g^{p-1} d\sigma \leq \Big( \int |h|^p d\sigma \Big)^{1/p} \Big( \int u_{m-1}^p g^p d\sigma \Big)^{\frac{p-1}{p}}$$
$$\lesssim ||\nabla h||_p ||\nabla (u_{m-1} g)||_p^{p-1} \leq C_K ||\nabla h||_p,$$

and hence $u_{m-1}^{p-1} d\sigma \in W^{-1,p'}(K)$, as claimed. Now let $\nu_j$ be the measure:

$$\nu_j = \frac{\chi_{B(x_0, 2^{-j})}}{|B(x_0, 2^{-j})|},$$

from Poincaré's inequality it follows that $\nu_j \in L^{-1,p'}(B(x_0, 2^j))$. Note in addition that $\nu_j \to \delta_{x_0}$ weakly as measures. Invoking the theory of monotone operators, see e.g.

[Li69], we assert the existence of a unique solution $w_m^j \in L_0^{1,p}(B(x_0, 2^j))$ of:

$$\begin{cases} -\Delta_p w_m^j = \epsilon\sigma u_{m-1}^{p-1}\chi_{B(x_0,2^j)\backslash B(x_0,2^{-j})} + \nu_j \text{ in } B(x_0, 2^j), \\ w_m^j \in L_0^{1,p}(B(x_0, 2^j)). \end{cases} \tag{3.7.11}$$

Furthermore, by the global potential estimate for renormalized solutions, Theorem 2.1 of [PV08], it follows:

$$w_m^j(x) \leq K\epsilon\mathbf{W}_{1,p}(u_{m-1}^{p-1}d\sigma)(x) + K\mathbf{W}_{1,p}(\nu_k)(x),$$

where the constant $K > 0$ can be assumed to be the same as the constant appearing in Theorem 3.2.4. But, for $x \notin B(x_0, 2 \cdot 2^{-j})$, a simple computation yields:

$$\mathbf{W}_{1,p}(\nu_k)(x) \leq \frac{n-p}{p-1}2^{\frac{n-p}{p-1}}|x - x_0|^{\frac{p-n}{p-1}}. \tag{3.7.12}$$

Using the hypothesis $u_{m-1} \leq v$, it follows for $x \in B(x_0, 2^j)\backslash B(x_0, 2^{1-j})$ that:

$$w_m^j(x) \leq K\epsilon\mathbf{W}_{1,p}(v^{p-1}d\sigma)(x) + K\frac{n-p}{p-1}2^{\frac{n-p}{p-1}}|x - x_0|^{\frac{p-n}{p-1}}.$$

Let us now choose the constant $B(n, p)$ appearing in (3.7.3) as $B(n, p) = 2K(n - p)/(p - 1)2^{\frac{n-p}{p-1}}$. Then, by construction of $v$, it follows as in the argument around display (3.6.2), that we can choose $\epsilon > 0$ and $C_0 > 0$ so that if $C(\sigma) < C_0$, then:

$$K\epsilon\mathbf{W}_{1,p}(v^{p-1}d\sigma)(x) + K\frac{n-p}{p-1}2^{\frac{n-p}{p-1}}|x - x_0|^{\frac{p-n}{p-1}} \leq v(x),$$

and hence,

$$w_m^j(x) \leq v(x), \text{ for all } x \in B(x_0, 2^j)\backslash B(x_0, 2 \cdot 2^{-j}). \tag{3.7.13}$$

We are now in a position to derive the uniform gradient estimate. Let $\phi \in C_0^\infty(B(x_0, 2^j)\backslash B(x_0, 2\cdot$

$2^{-j}$). Then test the weak formulation of $w_m^j$ with the valid test function function $\phi^p \cdot w_m^j \in L_0^{1,p}(B(x_0, 2^j))$. It follows:

$$\int |\nabla w_m^j|^p \phi^p dx = -p \int |\nabla w_m^j|^{p-2} \nabla w_m^j \cdot \nabla \phi w_m^j \phi^{p-1} + \int \phi^p w_m^j u_{m-1}^{p-1} d\sigma$$

Using Young's inequality in the first term, and utilizing the bounds (3.7.13) and $u_{m-1} \leq v$, we find that:

$$\frac{1}{p} \int |\nabla w_m^j|^p dx \leq \int v^p \phi^p d\sigma + \frac{1}{p} \int v^p |\nabla \phi|^p dx = C(n, p, C(\sigma), \mathrm{supp}(\phi)) < \infty,$$

where Lemmas 3.7.2 and 3.7.3 have been used. Using Theorems 3.2.2 and 3.2.3, we let $j \to \infty$ to find a solution $u_m$ of (3.6.3). Furthermore, by weak compactness in $L^{1,p}$, we deduce that $u_m \in L_{\mathrm{loc}}^{1,p}(\mathbf{R}^n \backslash \{x_0\})$ with the local bound on the gradient independent of $m$. We now follow the rest of the proof of Theorem 3.1.5 from display (3.6.3), using weak compactness again to deduce a fundamental solution $u \in L_{\mathrm{loc}}^{1,p}(\mathbf{R}^n \backslash \{x_0\})$, so that $u \leq v$. □

# Appendix A

# On the weak reverse Hölder inequality

In this first appendix, we prove Proposition 2.1.4 from Chapter 2, which is a characterization of when a nonnegative function satisfying a weak reverse Hölder inequality is doubling. First let us recall some notation:

For an open set $U$, we say $u \in BMO(U)$ if there is a positive constant $D_U$ so that:

$$\fint_{B(x,r)} |u(y) - \fint_{B(x,r)} u(z)\, dz| dy \leq D_U, \text{ for any ball } B(x,2r) \subset U. \tag{A.0.1}$$

In addition, $u \in BMO_{\text{loc}}(\Omega)$ if for each compactly supported open set $U \subset\subset \Omega$, there is a positive constant $D_U > 0$ so that (A.0.1) holds.

Here we have adopted a slightly different definition of BMO in (A.0.1) than in display (2.1.4) of Chapter 2. However, the definition above is well known to be equivalent to our previous definition in (2.1.4). This follows as a standard consequence of the John-Nirenberg inequality (see e.g. [St93]).

**Definition A.0.4.** *Let $U \subset \mathbf{R}^n$ be an open set, and let $w$ be a nonnegative measurable function. Then $w$ is said to be doubling in $U$ if there exists a constant $A_U > 0$ so* *that,*

$$\fint_{B(x,2r)} w\,dx \leq A_U \fint_{B(x,r)} w\,dx, \text{ for all balls } B(x,4r) \subset U. \tag{A.0.2}$$

*Let $w$ be a nonnegative measurable function. Then $w$ is said to satisfy a weak reverse Hölder inequality in $U$ if there exists constants $q > 1$ and $B_U > 0$ so that:*

$$\Big(\fint_{B(x,r)} w^q dx\Big)^{1/q} \leq B_U \fint_{B(x,2r)} w\,dx, \text{ for all balls } B(x,2r) \subset U. \tag{A.0.3}$$

**Proposition A.0.5.** *Let $U$ be an open set, and suppose $w$ satisfies the weak reverse Hölder inequality (2.1.6) in $U$. Then $w$ is doubling in $U$, i.e. (2.1.5) holds, if and only if $\log(w) \in BMO(U)$ (see (2.1.4)).*

*In particular, if $w$ satisfies (A.0.3) and*

$$\fint_{B(x,s)} |\log w(y) - \fint_{B(x,s)} \log w(z) dz| dy \leq D_U, \text{ for all balls } B(x,2s) \subset U. \tag{A.0.4}$$

*Then there is a constant $C(A_U, D_U) > 0$, so that for any ball $B(x,4r) \subset U$:*

$$\fint_{B(x,2r)} w\,dx \leq C(A_U, D_U) \fint_{B(x,r)} w\,dx \tag{A.0.5}$$

To prove Proposition A.0.5, we use the following lemma:

**Lemma A.0.6.** *Let $U \subset \mathbf{R}^n$ be an open set. Suppose that there exist $q > 1$ and $w \geq 0$, so that $u$ satisfies (A.0.3). Then, for any $t > 0$, there exists a constant $C_t = C(t, B_U) > 0$ so that:*

$$\Big(\fint_{B(x,r)} w^q\,dx\Big)^{1/q} \leq B_U \Big(\fint_{B(x,2r)} w^t\,dx\Big)^{1/t}, \quad \text{whenever } B(x,2r) \subset U.$$

This well known lemma had been used in proving estimates for quasilinear equations by G. Mingione [Min07]. For the benefit of the reader we provide a proof based on Remark 6.12 of [Giu03].

*Proof.* Let $\tau > 1$. The first step will be to prove that, for each ball $B(x,r)$ so that $B(x,\tau r) \subset U$, then there is a constant $C = C(n,q,B_U) > 0$ (independent of $\tau$) so that:

$$\int_{B(x,r)} w^q dx \leq C \frac{1}{[(\tau-1)r]^{n(q-1)}} \Big( \int_{B(x,\tau r)} w dx \Big)^q \tag{A.0.6}$$

The proof of (A.0.6) is a consequence of a rather standard covering argument. Indeed, note that for each ball $B(z,(\tau-1)r) \subset U$, from (A.0.3):

$$\int_{B(z,(\tau-1)r/2)} w^q dx \leq \frac{C(n,q,B_U)}{[(\tau-1)r]^{n(q-1)}} \Big( \int_{B(z,(\tau-1)r)} w dx \Big)^q . \tag{A.0.7}$$

We now note that $B(x,r)$ can be covered by balls $B(z_j,(\tau-1)r/2)$ so that, for a dimensional constant $C(n)$:

$$\sum_j \chi_{B(z_j,(\tau-1)r)} \leq C(n). \tag{A.0.8}$$

As a consequence of these observations we derive:

$$\int_{B(x,r)} w^q dx \leq \sum_j \int_{B(z_j,(\tau-1)r/2)} w^q dx \leq \frac{C(n,qB_U)}{[(\tau-1)r]^{n(q-1)}} \sum_j \Big( \int_{B(z_j,(\tau-1)r)} w dx \Big)^q .$$

However, since $|| \cdot ||_{\ell^q} \leq || \cdot ||_{\ell^1}$, we have that:

$$\int_{B(x,r)} w^q dx \leq \frac{C(n,qB_U)}{[(\tau-1)r]^{n(q-1)}} \Big( \sum_j \int_{B(z_j,(\tau-1)r)} w dx \Big)^q .$$

Finally, as a consequence of (A.0.8) and elementary geometry, we obtain (A.0.6).

Next, let us note that by Hölder's inequality:

$$\Big( \int_{B(x,\tau r)} w dx \Big)^q \leq \Big( \int_{B(x,\tau r)} w^q dx \Big)^{q\frac{1-t}{q-t}} \Big( \int_{B(x,\tau r)} w^t dx \Big)^{q\frac{q-1}{q-t}}$$

Hence by Young's inequality (2.2.6) with suitable exponents:

$$\frac{C(n,qB_U)}{[(\tau-1)r]^{n(q-1)}}\Big(\int_{B(x,\tau r)} w dx\Big)^q \le \frac{1}{2}\Big(\int_{B(x,\tau r)} w^q dx\Big) + \frac{C(q,t,B_U,n)}{[(\tau-1)r]^\alpha}\Big(\int_{B(x,\tau r)} w^t dx\Big)^{q/t}, \tag{A.0.9}$$

here

$$\alpha = \frac{n}{t}(q-t) > 0.$$

Let us denote:

$$\Phi(s) = \int_{B(x,s)} w^q dx.$$

Then, as a result of (A.0.9) and (A.0.6), it follows, for any $r < s_1 < s_2 < 2r$:

$$\Phi(s_1) \le \frac{1}{2}\Phi(s_2) + \frac{C}{(s_2-s_1)^\alpha}\Big(\int_{B(x,2r)} w^t dx\Big)^{q/t}. \tag{A.0.10}$$

Let us now pick $0 < \lambda < 1$ so that $\lambda^\alpha > 1/2$. In addition, define $s_j$ so that $s_0 = r$ and $s_{j+1} - s_j = (1-\lambda)\lambda^j r$. Then, by using (A.0.10) inductively:

$$\Phi(r) \le \frac{1}{2^j}\Phi(s_j) + \Big[\sum_{k=0}^{j}\frac{1}{(2\lambda^\alpha)^j}\Big] C\frac{1}{r^\alpha}\Big(\int_{B(x,2r)} w^t dx\Big)^{q/t}.$$

By choice of $\lambda$ the geometric series converges. Recalling the definition of $\alpha$, the lemma is proved by letting $j \to \infty$. □

Let us now turn to proving the proposition.

*Proof of Proposition A.0.5.* Let us first prove the necessity, suppose that $w$ satisfies the weak reverse Hölder inequality (A.0.3), and in addition that $w$ is doubling in $U$. Then, for each ball $B(x,4r) \subset U$:

$$\Big(⨍_{B(x,r)} w^q dx\Big)^{1/q} \le B_U ⨍_{B(x,2r)} w\, dx \le A_U B_U ⨍_{B(x,r)} w\, dx.$$

It follows that $w$ satisfies a reverse Hölder inequality in $U$, and is therefore a Muckenhoupt $A_\infty$-weight. It follows (see Chapter 5 of [St93]) that $\log(u) \in BMO(U)$.

Let us now turn to the converse statement. Suppose $w$ satisfies (A.0.4) and (A.0.3). From (A.0.4), it is a well known consequence of the John-Nirenberg inequality that there exists a constant $0 < t \leq 1$ so that $w^t$ is an $A_2$-weight in $U$, i.e. there exists a positive constant $A > 0$ (depending on $D_U$ in (A.0.4)) so that for all balls $B(z, 2s) \subset U$:

$$\fint_{B(z,s)} w^t \, dx \leq A\Big(\fint_{B(z,s)} w^{-t} \, dx\Big)^{-1}. \tag{A.0.11}$$

Indeed, from the John-Nirenberg inequality (see [St93]), there exists a constant $t = t(D_U)$ so that for any ball so that $B(z, 2s) \subset U$:

$$\fint_{B(z,s)} \exp\Big(t\Big|\log(w)(y') - \fint_{B(z,s)} \log(w(y))dy\Big|\Big)dy' \leq C(D_U). \tag{A.0.12}$$

Inequality (A.0.12) clearly implies the two inequalities:

$$\fint_{B(z,s)} \exp\Big(\log(w^t(y')) - \fint_{B(z,s)} \log(w^t(y))dy\Big)dy' \leq C(D_U), \quad \text{and:}$$

$$\fint_{B(z,s)} \exp\Big(\log(w^{-t}(y')) + \fint_{B(z,s)} \log(w^t(y))dy\Big)dy' \leq C(D_U).$$

Multiplying these two inequalities together, one obtains (A.0.11).

It follows from (A.0.11) and Jensen's inequality that, if $B(z, 4s) \subset U$:

$$\fint_{B(z,2s)} w^t \, dx \leq A2^n (\fint_{B(z,s)} w^{-t} \, dx\Big)^{-1} \leq A2^n \fint_{B(z,s)} w^t \, dx. \tag{A.0.13}$$

Let $B(z, 8s) \subset U$, then, applying Lemma A.0.6 with this choice of $t$:

$$\fint_{B(z,2s)} w \, dx \leq C_{U,t}\Big(\fint_{B(z,4s)} w^t \, dx\Big)^{1/t} \leq \tilde{C}_{U,t}\Big(\fint_{B(z,s)} w^t \, dx\Big)^{1/t} \leq \tilde{C}_{U,t} \fint_{B(z,s)} w \, dx.$$

The second inequality in the chain follows from the doubling of $w^t$, and the last inequality follows from Hölder's inequality. By a standard covering argument (as in the proof of Lemma A.0.6), the factor of 8 in the enlargement of the ball can be replaced by 4, which yields (A.0.5). This completes the proposition. $\square$

# Appendix B

# A tail estimate for nonlinear potentials

In this appendix we include a proof of Lemma 3.3.13.

**Lemma B.0.7.** *Let $\sigma$ be a Borel measure satisfying:*

$$\sigma(B(x,r)) \leq Cr^{n-\alpha s}, \text{ for all balls } B(x,r). \tag{B.0.1}$$

*Then there is a positive constant $C = C(n, \alpha, s, \sigma) > 0$ so that for all $x \in \mathbb{R}^n$ and $y \in B(x,t)$, $t > 0$, it follows:*

$$\left| \int_t^\infty \left[ \Big( \frac{\sigma(B(x,r))}{r^{n-\alpha s}} \Big)^{\frac{1}{s-1}} - \Big( \frac{\sigma(B(y,r))}{r^{n-\alpha s}} \Big)^{\frac{1}{s-1}} \right] \frac{dr}{r} \right| \leq C. \tag{B.0.2}$$

*Proof.* Without loss of generality, suppose that:

$$\int_t^\infty \left[ \Big( \frac{\sigma(B(x,r))}{r^{n-\alpha s}} \Big)^{\frac{1}{s-1}} - \Big( \frac{\sigma(B(y,r))}{r^{n-\alpha s}} \Big)^{\frac{1}{s-1}} \right] \frac{dr}{r} > 0.$$

We want to rearrange the integrand so it is nonnegative. To this end, we define two sets:

$$A = \{z \in \mathbf{R}^n : |x - z| \leq |y - z|\}, \quad \text{and} \quad B = \{z \in \mathbf{R}^n : |y - z| < |x - z|\}.$$

Then if $z \in B$ and $|z - x| < r$, we have that $|y - z| < r$, and thus $B(x, r) \cap B \subset B(y, r) \cap B$ so that:

$$\sigma(B(x, r) \cap B) \leq \sigma(B(y, r) \cap B), \text{ and:} \tag{B.0.3}$$

$$\sigma(B(y, r) \cap A) \leq \sigma(B(x, r) \cap A). \tag{B.0.4}$$

Using (B.0.3) gives:

$$\begin{aligned} &\int_t^\infty \left[ \Big( \frac{\sigma(B(x,r))}{r^{n-\alpha s}} \Big)^{\frac{1}{s-1}} - \Big( \frac{\sigma(B(y,r))}{r^{n-\alpha s}} \Big)^{\frac{1}{s-1}} \right] \frac{dr}{r} \\ &\quad \leq \int_t^\infty \Big( \frac{\sigma(B(x,r) \cap A) + \sigma(B(y,r) \cap B)}{r^{n-\alpha s}} \Big)^{\frac{1}{s-1}} \\ &\quad - \Big( \frac{\sigma(B(y,r) \cap A) + \sigma(B(x,r) \cap B)}{r^{n-\alpha s}} \Big)^{\frac{1}{s-1}} \frac{dr}{r} = \int_t^\infty \left[ I^{\frac{1}{s-1}} - II^{\frac{1}{s-1}} \right] \frac{dr}{r}. \end{aligned}$$

From (B.0.3) and (B.0.4) it immediately follows that the integrand in nonnegative, i.e. that $I \geq II$.

The proof now splits into two cases, when $1 < s < 2$ and when $s \geq 2$. First suppose $1 < s < 2$, then note the following elementary inequality; for $a, b \in (0, \infty)$ with $a > b$, and $\gamma \geq 1$:

$$a^\gamma - b^\gamma \leq \gamma a^{\gamma - 1}(a - b) \tag{B.0.5}$$

Plugging $I$ and $II$ into (B.0.5) yields: $I^{\frac{1}{s-1}} - II^{\frac{1}{s-1}} \leq \frac{1}{s-1}(I - II) I^{\frac{2-s}{s-1}} \leq C(I - II)$. Here we have used the estimate (B.0.1) in the last inequality, noting that $2 - s > 0$.

Thus, if $1 < s < 2$, the Lemma will follow from the following inequality:

$$\int_t^\infty \frac{\sigma(B(x,r)\cap A) + \sigma(B(y,r)\cap B)}{r^{n-\alpha s}} - \frac{\sigma(B(y,r)\cap A) + \sigma(B(x,r)\cap B)}{r^{n-\alpha s}} \frac{dr}{r} \leq C. \tag{B.0.6}$$

Let us now split $\sigma$ into $\sigma_1 = \sigma \cdot \chi_{\mathbf{R}^n \setminus B(x,2t)}$ and $\sigma_2 = \sigma \cdot \chi_{B(x,2t)}$ and if we can control the left hand side of (B.0.6) with either $\sigma_1$ or $\sigma_2$ in place of $\sigma$ then we are done. The estimate for $\sigma_2$ is a straightforward application of (B.0.1):

$$\int_t^\infty \frac{\sigma_2(B(x,r)\cap A) + \sigma_2(B(y,r)\cap B)}{r^{n-\alpha s}} - \frac{\sigma_2(B(y,r)\cap A) + \sigma_2(B(x,r)\cap B)}{r^{n-\alpha s}} \frac{dr}{r}$$
$$\leq C\sigma(B(x,2t)) \int_t^\infty \frac{1}{r^{n-\alpha s}} \frac{dr}{r} \leq C \frac{\sigma(B(x,2t))}{(2t)^{n-\alpha s}} \leq C$$

where (B.0.1) has been used in this last inequality.

We now move onto the estimate for $\sigma_1$. First note that if $r < t$ and $y \in B(x,t)$, then $B(y,r) \subset B(x,2t)$ and so $\sigma_1(B(y,r)) = 0$. This allows us to extend the integration to over the half line:

$$\int_t^\infty \frac{\sigma_1(B(x,r)\cap A) + \sigma_1(B(y,r)\cap B)}{r^{n-\alpha s}} - \frac{\sigma_1(B(y,r)\cap A) + \sigma_1(B(x,r)\cap B)}{r^{n-\alpha s}} \frac{dr}{r}$$
$$= \frac{1}{n-\alpha s} \int_{\mathbf{R}^n} \left[ \frac{\chi_A(z)}{|x-z|^{n-\alpha s}} - \frac{\chi_A(z)}{|y-z|^{n-\alpha s}} + \frac{\chi_B(z)}{|y-z|^{n-p}} - \frac{\chi_B(z)}{|x-z|^{n-\alpha s}} \right] d\sigma_1(z)$$
$$= \frac{1}{n-\alpha s} \int_{\mathbb{R}^n \setminus B(x,2t)} \left| \frac{1}{|x-z|^{n-\alpha s}} - \frac{1}{|y-z|^{n-\alpha s}} \right| d\sigma(z)$$

Let $z \notin B(x,2t)$, then whenever $y \in B(x,t)$, it is easy to see that:

$$\frac{1}{2} |y-z| \leq |x-z| \leq 2|y-z|. \tag{B.0.7}$$

Note the following elementary inequality. For $a, b \in (0,\infty)$ with $a > b$, and $\gamma \geq 0$:

$$a^\gamma - b^\gamma \leq \gamma(a^{\gamma-1} + b^{\gamma-1})(a-b). \tag{B.0.8}$$

Due to (B.0.7) and (B.0.8), and that $y \in B(x,t)$, it follows:

$$\begin{aligned}\left|\frac{1}{|x-z|^{n-\alpha s}} - \frac{1}{|y-z|^{n-\alpha s}}\right| &\leq C\frac{\left||x-z|^{n-\alpha s} - |y-z|^{n-\alpha s}\right|}{|x-z|^{2(n-\alpha s)}} \\ &\leq C\frac{|x-y|}{|x-z|^{n-\alpha s+1}} \leq C\frac{t}{|x-z|^{n-\alpha s+1}}.\end{aligned}$$

Plugging this in we get:

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B(x,2t)}\left|\frac{1}{|x-z|^{n-\alpha s}} - \frac{1}{|y-z|^{n-\alpha s}}\right| d\sigma(z) &\leq C\int_{\mathbb{R}^n\setminus B(x,2t)}\frac{t}{|x-z|^{n-\alpha s+1}}d\sigma(z) \\ &\leq Ct\int_{2t}^{\infty}\frac{\sigma(B_r(x))}{r^{n-\alpha s}}\frac{dr}{r^2} \leq Ct\int_{2t}^{\infty}\frac{dr}{r^2} \leq C.\end{aligned}$$

Combining our two estimates prove the lemma in the case $1 < s \leq 2$.

We now move onto the $s \geq 2$ case. First recall that with $I$ and $II$ as before, we have $I \geq II$, and hence $I^{\frac{1}{s-1}} - II^{\frac{1}{s-1}} \leq (I-II)^{\frac{1}{s-1}}$. This implies that:

$$\begin{aligned}&\int_t^\infty \left[\Big(\frac{\sigma(B(x,r))}{r^{n-\alpha s}}\Big)^{\frac{1}{s-1}} - \Big(\frac{\sigma(B(y,r))}{r^{n-\alpha s}}\Big)^{\frac{1}{s-1}}\right]\frac{dr}{r} \\ &\quad\leq \int_t^\infty\Big(\frac{\sigma(B(x,r)\cap A)+\sigma(B(y,r)\cap B)}{r^{n-\alpha s}} - \frac{\sigma(B(y,r)\cap A)+\sigma(B(x,r)\cap B)}{r^{n-\alpha s}}\Big)^{\frac{1}{s-1}}\frac{dr}{r}.\end{aligned}$$

Let $\epsilon > 0$ small enough so that $\epsilon(s-2) < \min(n-\alpha s, 1)$ . Then, by Hölder's inequality:

$$\begin{aligned}\int_t^\infty (I-II)^{\frac{1}{s-1}}\frac{dr}{r} &\leq Ct^{\epsilon(\frac{1}{s-1}-1)}\Big(\int_t^\infty (I-II)r^{\epsilon(s-1)}\frac{dr}{r^{1+\epsilon}}\Big)^{\frac{1}{s-1}} \\ &= Ct^{\epsilon(\frac{1}{s-1}-1)}\Big(\int_t^\infty\Big(\frac{\sigma(B(x,r)\cap A)+\sigma(B(y,r)\cap B)}{r^{n-\alpha s}} \\ &\quad - \frac{\sigma(B(y,r)\cap A)+\sigma(B(x,r)\cap B)}{r^{n-\alpha s}}\Big)r^{\epsilon(s-1)}\frac{dr}{r^{1+\epsilon}}\Big)^{\frac{1}{s-1}}.\end{aligned}$$

We wish to bound the right hand side by a constant. To this end we will split the measure $\sigma$ as before into $\sigma_1$ and $\sigma_2$. The following estimate for $\sigma_2$ follows easily using

(B.0.1):

$$t^{\epsilon(\frac{1}{s-1}-1)}\Big(\int_t^\infty\Big(\frac{\sigma_2(B(x,r)\cap A)+\sigma_2(B(y,r)\cap B)}{r^{n-\alpha s}} - \frac{\sigma_2(B(y,r)\cap A)+\sigma_2(B(x,r)\cap B)}{r^{n-\alpha s}}\Big)r^{\epsilon(s-1)}\frac{dr}{r^{1+\epsilon}}\Big)^{\frac{1}{s-1}} \leq C,$$

We now concentrate on the $\sigma_1$ estimate. First we note that we may extend the domain of integration over the whole half line and use Fubini's theorem as in the $1 < s \leq 2$ case to find that

$$\begin{aligned}
t^{\epsilon(\frac{1}{s-1}-1)}&\Big(\int_t^\infty\Big(\frac{\sigma(B(x,r)\cap A)+\sigma(B(y,r)\cap B)}{r^{n-\alpha s}} \\
&- \frac{\sigma(B(y,r)\cap A)+\sigma(B(x,r)\cap B)}{r^{n-\alpha s}}\Big)r^{\epsilon(s-1)}\frac{dr}{r^{1+\epsilon}}\Big)^{\frac{1}{s-1}} \\
&\leq Ct^{\epsilon(\frac{1}{s-1}-1)}\Big(\int_{\mathbb{R}^n\setminus B(x,2t)}\Big|\frac{1}{|x-z|^{n-\alpha s-\epsilon(s-1)+\epsilon}} - \frac{1}{|y-z|^{n-\alpha s-\epsilon(s-1)+\epsilon}}\Big|\,d\sigma(z)\Big)^{\frac{1}{s-1}} \\
&= III.
\end{aligned}$$

Now by adapting the previous argument in the $s \leq 2$ case, we have:

$$\Big|\frac{1}{|x-z|^{n-\alpha s-\epsilon(s-1)+\epsilon}} - \frac{1}{|y-z|^{n-\alpha s-\epsilon(s-1)+\epsilon}}\Big| \leq C\frac{t}{|x-z|^{n-\alpha s-\epsilon(s-1)+\epsilon+1}}.$$

Hence

$$\begin{aligned}
III &\leq Ct^{\epsilon(\frac{1}{s-1}-1)}\Big(\int_{\mathbb{R}^n\setminus B(x,2t)}\frac{t}{|x-z|^{n-\alpha s-\epsilon(s-1)+\epsilon+1}}d\sigma(z)\Big)^{\frac{1}{s-1}} \\
&\leq Ct^{\epsilon(\frac{1}{s-1}-1)}\Big(\int_{2t}^\infty\frac{t\,\sigma(B(x,r))}{r^{n-\alpha s-\epsilon(s-1)+\epsilon+1}}\frac{dr}{r}\Big)^{\frac{1}{s-1}} \leq C,
\end{aligned}$$

where in the last inequality we have used (B.0.1), then we are left with a convergent integral by choice of $\epsilon$. This completes the proof in the case $s \geq 2$. □

# Appendix C

# The dyadic Carleson measure theorem

In this chapter we prove the following dyadic Carleson measure theorem, which we made use of in Chapter 3. Recall that $\mathcal{D}$ is the lattice of dyadic cubes in $\mathbf{R}^n$.

**Theorem C.0.8.** *Suppose $\sigma$ is a nonnegative measure, and $\{a_Q\}_{Q\in\mathcal{D}}$ is a nonnegative sequence such that there is a constant $C(\sigma) > 0$ so that:*

$$\sum_{Q\subset P} a_Q \leq C(\sigma)|P|_\sigma, \text{ for all dyadic cubes } P \in \mathcal{D}. \tag{C.0.1}$$

*Then, for all $f \in L^p(\sigma)$, it follows:*

$$\sum_{Q\in\mathcal{D}} a_Q \Big(\frac{1}{|Q|_\sigma}\int_Q f d\sigma\Big)^p \leq C(\sigma)\Big(\frac{p}{p-1}\Big)^p \int_{\mathbf{R}^n} f^p d\sigma \tag{C.0.2}$$

This coincides with the sharp bound of 4 found in [NTV02] when $p = 2$, and a deep result of Melas [Me05] shows that this constant cannot be improved.

We will see that Theorem C.0.8 can be deduced from a recent paper of Cascante and Ortega [CO09]. Indeed, from Doob's estimate of the norm of the dyadic maximal operator:

$$\int_{\mathbf{R}^n} \Big( \sup_{x\in Q\in\mathcal{D}} \frac{1}{|Q|} \int_Q |f(z)|dz \Big)^p dx \leq \Big(\frac{p}{p-1}\Big)^p \int_{\mathbf{R}^n} |f(x)|^p dx,$$

we see that Theorem C.0.8 follows immediately from the following proposition:

**Proposition C.0.9.** *Let $\sigma$ be a locally finite Borel measure, and let $\{a_Q\}_Q$ be a sequence such that there is a constant $C(\sigma) > 0$ with:*

$$\sum_{Q\subset P} a_Q \leq C(\sigma)|P|_\sigma, \text{ for all dyadic cubes } P \in \mathcal{D}. \tag{C.0.3}$$

*Then for every nonnegative sequence $\{\lambda_Q\}_Q$, so that $\lambda_Q = 0$ whenever $|Q|_\sigma = 0$, it follows:*

$$\sum_{Q\subset P} a_Q\lambda_Q \leq C(\sigma) \int_P \sup_{Q\subset P} \lambda_Q \chi_Q(x) d\sigma(x) \tag{C.0.4}$$

*for all cubes $P$.*

*Proof.* This proposition is a special case of Theorem 2.5 of [CO09], but let us repeat the elegant proof. Fix $P$, and first, suppose that the sequence $\lambda_Q$ has a finite number a non-zero terms in $P$, say in the first $m$ levels of dyadic cubes contained in P. Consider an enumeration of these cubes:

Let $P_0 = P$, and we write $P_{j_1,\ldots,j_{k-1},j_k}$, for $k \leq m$ and $j_k \in \{1,\ldots,2^n\}$, to be the $2^n$ cubes at level $k$ contained in $P_{j_1,\ldots,j_{k-1}}$. I.e. we have that $P_{j_1,\ldots,j_{k-1},\ell_1} \cap P_{j_1,\ldots,j_{k-1},\ell_2} = \emptyset$ and $P_{j_1,\ldots,j_{k-1},\ell} \subset P_{j_1,\ldots,j_{k-1}}$ for any $\ell, \ell_1, \ell_2 \in \{1,\ldots,2^n\}$.

In addition we suppose the sequence $\{\lambda_Q\}$ has a monotonicity property:

$$\lambda_{P_{j_1,\ldots,j_{k-1},\ell}} \geq \lambda_{P_{j_1,\ldots,j_{k-1}}}, \tag{C.0.5}$$

whenever $k = 0,\ldots m$, $\ell = 1,\ldots 2^n$. In other words, $\lambda_{Q'} \geq \lambda_Q$ whenever $Q' \subset Q \subset P$, and $Q'$ is one of the dyadic cubes in the first $m$ generations of $P$. We now prove the theorem in this special case. The left hand side of (C.0.4) is:

$$\sum_{j_1,\ldots,j_{k-1},j_k} a_{P_{j_1,\ldots,j_{k-1},j_k}} \lambda_{P_{j_1,\ldots,j_{k-1},j_k}} + \cdots + a_{P_0}\lambda_{P_0}. \tag{C.0.6}$$

By a simple telescoping sum argument the previous display is equal to:

$$\begin{aligned}\sum_{j_1,\ldots,j_{m-1},j_m} & a_{P_{j_1,\ldots,j_{m-1},j_m}} (\lambda_{P_{j_1,\ldots,j_{m-1},j_m}} - \lambda_{P_{j_1,\ldots,j_{m-1}}}) \\ & + \sum_{j_1,\ldots,j_{m-1}} (\lambda_{P_{j_1,\ldots,j_{m-1}}} - \lambda_{P_{j_1,\ldots,j_{m-2}}})(a_{P_{j_1,\ldots,j_{m-1}}} + \sum_{j_m} a_{P_{j_1,\ldots,j_{m-1},j_m}}) \\ & \cdots + \lambda_{P_0}\Big(a_{P_0} + \sum_{j_1} a_{P_{j_1}} + \cdots \sum_{j_1,\ldots,j_{m-1},j_m} a_{P_{j_1,\ldots,j_{m-1},j_m}}\Big).\end{aligned} \tag{C.0.7}$$

Now, by the monotonicity condition, each difference in the $\lambda_Q$ in the above display is nonnegative. Applying the Carleson condition to each sum over the $a_Q$ we get that (C.0.7) is bounded by:

$$\begin{aligned}C(\sigma)\Big\{ & \sum_{j_1,\ldots,j_{m-1},j_m} \sigma(P_{j_1,\ldots,j_{m-1},j_m})(\lambda_{P_{j_1,\ldots,j_{m-1},j_m}} - \lambda_{P_{j_1,\ldots,j_{m-1}}}) \\ & + \sum_{j_1,\ldots,j_{m-1}} (\lambda_{P_{j_1,\ldots,j_{m-1}}} - \lambda_{P_{j_1,\ldots,j_{m-2}}})(\sigma(P_{j_1,\ldots,j_{m-1}})) \\ & \cdots + \lambda_{P_0}\sigma(P_0)\Big\}.\end{aligned} \tag{C.0.8}$$

But note that

$$\sum_{j_1,\ldots,j_{m-1},j_m} \sigma(P_{j_1,\ldots,j_{m-1},j_m})\lambda_{P_{j_1,\ldots,j_{m-1}}} = \sum_{j_1,\ldots,j_{m-1}} \sigma(P_{j_1,\ldots,j_{m-1}})\lambda_{P_{j_1,\ldots,j_{m-1}}},$$

and hence (C.0.8) is equal to:

$$C(\sigma) \sum_{j_1,\cdots j_m} \sigma(P_{j_1,\ldots,j_{m-1},j_m})\lambda_{P_{j_1,\ldots,j_{m-1},j_m}}, \tag{C.0.9}$$

which is precisely the right hand side of (C.0.4) in this special case.

It remains to remove the monotonicity condition. Note that if $\lambda_Q > \lambda_{Q'}$ for $Q' \subset Q \subset P$, and $Q', Q$ in the enumeration of cubes above, then by replacing $\lambda_{Q'}$ by $\lambda_Q$ we see that the supremum on the right hand side of (C.0.4) is left unchanged as $Q' \subset Q$, but the left hand side is increased. Thus the monotonicity assumption is removed and by appealing to monotone convergence the proposition follows. □

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# VITA

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