GENERALIZED LOCAL TB THEOREM AND APPLICATIONS

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The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

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presented by Ana Grau de la Herrán,

a candidate for the degree of Doctor of Philosophy

and hereby certify that in their opinion it is worthy of acceptance.

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In memory of my beloved friends

Emilio Oliva and Javier Terrón.

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ABSTRACT

The Tb theorem, like its predecessor, the T1 Theorem, is an L^2 boundedness criterion, originally established by McIntosh and Meyer, and by David, Journé and Semmes in the context of singular integrals, but later extended by Semmes to the setting of "square functions". The latter arise in many applications in complex function theory and in PDE, and may be viewed as singular integrals taking values in a Hilbert space. The essential idea of Tb and T1 type theorems, is that they reduce the question of L^2 boundedness to verifying the behavior of an operator on a single test function b (or even the constant function 1). The point is that sometimes particular properties of the operator may be exploited to verify the appropriate testing criterion. In particular, it would be presented some results for "square functions" with non-pointwise bounded kernels as well as the motivation that leads us to study such case.

We apply such result to give a proof of the Kato problem and also to prove that the single layer potential associated to a divergence form, t-independent elliptic operator or system in the half-space $\mathbb{R}^{n+1}_+$ is an L^2 bounded operator, more precisely that $t\partial_t^2 S_t : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^{n+1}_+, \frac{dxdt}{t})$, assuming some appropriate solvability result for the Dirichlet problem $(D)_q$ and the Regularity problem $(R)_p$.

Chapter 1

Introduction

1.1 Introduction, statement of results, history

The Tb theorem, like its predecessor, the T1 Theorem, is an L^2 boundedness criterion, originally established by McIntosh and Meyer [McM], and by David, Journé and Semmes [DJS] in the context of singular integrals, but later extended by Semmes to the setting of "square functions". The latter arise in many applications in complex function theory and in PDE, and may be viewed as singular integrals taking values in a Hilbert space.

The "local" versions that we have obtained are related to previous work of M. Christ [Ch](who proved the first local Tb theorem in the singular integral setting.) The term "local" in this context, refers to the fact that, instead of one globally defined testing function b, one is allowed to test the operator locally, say on each dyadic cube, with a local testing function that is adapted to that cube. The advantage here, in applications, is the additional flexibility that one gains: it may be easier to verify "good" behavior of the operator locally, when the testing functions are allowed to vary.

An extension of Christ result to the non-doubling setting is due to Nazarov, Treil and Volberg [NTV] and Hytönen and Martikainen [HyM]. For doubling measures, one can also consider more general L^p type testing conditions introduced by Aucher, Hofmann, Muscalu, Tao and Thiele [AHMTT], and further studied by Hofmann [H3], Auscher and Yan [AY], Aucher and Routin [AR], Hytönen and Martikainen [HyM] and Tan and Yan [TY].

In fact, this sort of "local Tb" criterion lies at the heart of the solution of the Kato square root problem. The essential idea of Tb and T1 type theorems, is that they reduce the question of L^2 boundedness to verifying the behavior of an operator on a single test function b (or even the constant function 1). The point is that sometimes particular properties of the operator may be exploited to verify the appropriate testing criterion.

With this aim we consider the local Tb Theorems for Square functions. In the Euclidean setting the result was presented in [H2], but was already implicit in the solution of the Kato problem [HMc],[HLMc],[AHLMcT], (see also [AT] and [Se] for related results). For domains with Ahlfors-David regular boundaries the result is presented by A. Grau de la Herrán and M. Mourgoglou [GM] with applications to problems that connect the behavior of the harmonic measure for domains with uniformly rectifiable boundaries (see [HMar] and [HMarUT]).

1.2 Notation

- We shall use the letters c, C to denote positive constants, not necessarily the same at each occurrence, which depend only on dimension and the constants appearing in the hypotheses of the theorems. We shall also write $A \lesssim B$ and $A \approx B$ to mean, respectively, that $A \leq CB$ and $0 < c \leq A/B \leq C$, where the constants c and C are as above, unless explicitly noted. Moreover if we want to specify any particular dependency of the constant we will denote it by $C(n)$ or C_n which means that the constant depends on n.

- We are going to work in $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ and we are going to denote the points belonging to such a set as $(x, t) \in \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$ (we use the notational convention that $x_{n+1} = t$), or $X \in \mathbb{R}^{n+1}$ to convenience where $(x, t) = X$.

- We denote $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, +\infty)$ and $\partial\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times \{0\}$.

- For a Borel set $A \subset \mathbb{R}^{n+1}$, we let $\mathbb{1}_A$ denote the usual indicator function of A, i.e. $\mathbb{1}_A(x) = 1$ if $x \in A$, and $\mathbb{1}_A(x) = 0$ if $x \notin A$.

- Let $A = Q_R \setminus Q_r$, where Q_r and Q_R are centred at the same point,$0 < r, R < \infty$ and the side length of Q_r (respect. Q_R) is r (respect. R).Then $\rho(A) := R - r$.

- We decompose $\mathbb{R}^{n+1}$ in a Dyadic grid of open cubes. We denote $Q(x,t)$ the cube of side length 2^k on the Dyadic grid such that $x \in Q(x,t)$ and $2^k < t < 2^{k+1}$, for some $k \in \mathbb{Z}$.

- For a Borel set $A \subset \mathbb{R}^n$, we define $\fint_A dx = \frac{1}{|A|}\int_A dx$, and for a Borel set $B \subset \mathbb{R}^{n+1}$, we define $⨏⨏_B dX = \frac{1}{|B|}\iint_B dxdt$.

- Let $q \in (1,\infty)$, we denote by $q' \in (1,\infty)$ the number such that we have $\frac{1}{q} + \frac{1}{q'} = 1$.

1.3 Definitions

Definition 1.3.1. We say that P_t is a nice approximate identity, if P_t is an operator of convolution type, with a smooth, compactly supported kernel Φ. That means that for a function $f : \mathbb{R}^n \to \mathbb{C}$

$$P_t f = \Phi_t * f \; with \; \Phi_t = t^{-n}\Phi\left(\frac{x}{t}\right), \int \Phi(x)dx = 1 \;\; \Phi_t \in C_0^\infty(\mathbb{R}^n),$$

$$(P_t f)(x) \leq C\mathcal{M}f(x),$$

$$P_t 1 = 1.$$

Definition 1.3.2. Let $f \in L^1_{loc}(\mathbb{R}^n)$ we define the Hardy Littlewood Maximal operator of and denote it by $\mathcal{M}$ by

$$\mathcal{M}(f)(x) := \sup_{r>0} \frac{1}{|B_r(x)|}\int_{B_r(x)} f(y)dy$$

where $B_r(x)$ is the ball centered at x and radius r.

Definition 1.3.3. Let $s \in \mathbb{R}$, the homogeneous Sobolev space $\dot{L}^2_s$ is the completion of C_0^∞ with respect to the norm $\|f\|_{\dot{L}^2_s} := \|(-\Delta)^{s/2})f\|_{L^2}$, where Δ is the usual Laplacian.

Definition 1.3.4. [St2] If $0 < \alpha < n$, then the Riesz potential $I_\alpha f$ of a locally integrable function f on $\mathbb{R}^n$ is the function defined by

$$I_\alpha f(x) = \frac{1}{C_{\alpha,n}} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy$$

where the constant is given by $C_{\alpha,n} = \pi^{n/2} 2^\alpha \frac{\Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)}$.

This singular integral is well-defined provided f decays sufficiently rapidly at infinity, specifically if $f \in L^p(\mathbb{R}^n)$ with $1 \le p < \frac{n}{\alpha}$.

1.4 Auxiliary Results

Remark 1.4.1. ([CR],[GR])

Since we are going to be working with weights let's remind ourselves of some useful facts.

(1) $\mu = (\mathcal{M}g)^\lambda, 0 \le \lambda \le 1 \Rightarrow \mu \in A_1-$weight, provided that $\mathcal{M}g$ is finite a.e.

(2) $\mu \in A_1 \Rightarrow \eta := \frac{1}{\mu} \in A_2$.

(3) $\eta \in A_2 \Rightarrow \int_{\mathbb{R}^n} ((\mathcal{M}g)(x))^2 \eta(x) dx \le C \int_{\mathbb{R}^n} (g(x))^2 \eta(x) dx.$

(4) $1 < p < 2 \Rightarrow 0 < 2 - p < 1$ so $\omega \in A_1-$weight, $\nu \in A_2-$weight.

(5) If $1 < r < 2$, $\nu \in A_{\frac{2}{r}} \Rightarrow \int_{\mathbb{R}^n} (\mathcal{M}(g^r)(x))^{\frac{2}{r}} \nu(x) dx \le \int_{\mathbb{R}^n} (g(x))^2\, dx.$

Proposition 1.4.2. *(Caccioppoli on horizontal slices [AAAHK])*

Suppose that the matrix A is t-independent, i.e, $A = A(x)$. Then there is a uniform constant $\epsilon > 0$ depending only on n and ellipticity, and for every $p \in [2, 2+\epsilon)$, a uniform constant $C = C(p,\delta)$ such that, for each fixed cube $Q \subset \mathbb{R}^n$, and $t \in \mathbb{R}$, if $Lu = 0$ in the box $I_Q := 4Q \times (t - \ell(Q), t + \ell(Q))$, then we have the following estimates

$$\left(\frac{1}{|Q|} \int_Q |\nabla u(x,t)|^p dx \right)^{\frac{1}{p}} \le C_p \left(\frac{1}{|Q^*|} \iint_{Q^*} |\nabla u(x,\tau)|^2 dx d\tau \right)^{\frac{1}{2}},$$

$$\left(\frac{1}{|Q|}\int_Q |\nabla u(x,t)|^p dx\right)^{\frac{1}{p}} \leq C_p\left(\frac{1}{\ell(Q)^2}\frac{1}{|Q^{**}|}\iint_{Q^{**}}|u(x,\tau)|^2 dx d\tau\right)^{\frac{1}{2}},$$

$$\left(\frac{1}{|T_R|}\int_{2Q\setminus Q}|\nabla u(x,t)|^2 dx\right)^{\frac{1}{2}} \leq C\left(\frac{1}{|T_R^*|}\iint_{T_R^*}|\nabla u(x,\tau)|^2 dx d\tau\right)^{\frac{1}{2}},$$

$$\left(\frac{1}{|T_R|}\int_{T_R}|\nabla u(x,t)|^2 dx\right)^{\frac{1}{2}} \leq C\left(\frac{1}{\rho(T_R)^2}\frac{1}{|T_R^{**}|}\iint_{T_R^{**}}|u(x,\tau)|^2 dx d\tau\right)^{\frac{1}{2}},$$

where $Q^* := (1+\delta)Q \times (t-\delta\ell(Q), t+\delta\ell(Q))$ *for any fixed* $\delta > 0$. *We also define:*

$Q^{l)*} = (1 + l\cdot\delta)Q \times (t - l\cdot\delta\ell(Q), t + l\cdot\delta\ell(Q))$;

$T_R = 2Q \setminus Q$;

$T_R^* = ((1+\delta)Q \setminus (1-\delta)Q) \times (t - \frac{\ell(Q)}{m}, t + \frac{\ell(Q)}{m})$;

$T_R^{l)*} = ((1+l\delta)2Q \setminus (1-l\delta)Q) \times (t - l\cdot\delta\ell(Q), t + l\cdot\delta\ell(Q))$.

Remark 1.4.3. We are going to consider $\delta = \frac{1}{m}$ and $t = R = \ell(Q)$ which will be more convenient for our application in chapter 5. We are going to iterate the Caccioppoli estimates m-times, but by our choice of δ we are not leaving the box I_Q. Also $Q^{**} := Q^{2)*}, T_R^{**} := T_R^{2)*}$. Note that the constants in the inequalities also depend upon δ.

Lemma 1.4.4. *Let's consider*

$$L = -\nabla\cdot(A\nabla u) \equiv -\sum_{i,j=1}^{n+1}\frac{\partial}{\partial x_i}\left(A_{i,j}\frac{\partial}{\partial x_j}\right) \tag{1.4.1}$$

is defined in $\mathbb{R}^{n+1} = \{(x,t)\in\mathbb{R}^n\times\mathbb{R}\}, n \geq 2$, *and where* $A = A(x)$ *is an* $(n+1)\times(n+1)$ *matrix of* L^∞ *complex-valued coefficients, defined on* $\mathbb{R}^n$ *(i.e. independent of the t variable) and satisfying the uniform ellipticity condition*

$$\lambda|\xi|^2 \leq \Re e\langle A(x)\xi,\xi\rangle \equiv \Re e\sum_{i,j=1}^{n+1}A_{ij}(x)\xi_j\bar{\xi}_i,\ \|A\|_{L^\infty(\mathbb{R}^n)} \leq \Lambda \tag{1.4.2}$$

for some $\lambda > 0$, $\Lambda < \infty$, and for all $\xi \in \mathbb{C}^{n+1}$, $x \in \mathbb{R}^n$ (the divergence form equation is interpreted in the weak sense).

More generally, we may consider elliptic systems defined as follows:

$$L\vec{u} := -D_\alpha \cdot (A_{\alpha\beta} D_\beta \vec{u}) \tag{1.4.3}$$

is defined on $\mathbb{R}^{n+1} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R}\}, n \geq 2$, $\vec{u}$ are N-dimensional vector valued functions, where $D_\alpha = \frac{\partial}{\partial x_\alpha}$ is the partial derivative with respect the variable x_α, $1 \leq \alpha \leq n+1$, and where $A_{\alpha\beta} = A_{\alpha\beta}(x)$, $1 \leq \alpha, \beta \leq n+1$, are $N \times N$ matrices of L^∞ complex-valued coefficients, defined on $\mathbb{R}^n$ (i.e. independent of the t variable) and satisfying the uniform ellipticity condition

$$\lambda \sum_{i=1}^{N} \sum_{\alpha=1}^{n+1} |\xi_\alpha^i|^2 \leq A_{\alpha\beta}^{ij} \xi_\beta^j \bar{\xi}_\alpha^i, \; ||A||_{L^\infty(\mathbb{R}^n)} \leq \Lambda \tag{1.4.4}$$

for some $\lambda > 0$, $\Lambda < \infty$, and for all $\xi \in \mathbb{C}^N$, $x \in \mathbb{R}^n$ (the divergence form operator L is interpreted in the weak sense via a sesquilinear form).

Let's define the operator $D^\alpha = \displaystyle\sum_{i=1}^{n+1} \frac{\partial^{\alpha_i}}{\partial x_i}$, $\alpha_i \in \{0,1\}$ $\forall i$.

We define D as the operator D^α where $\alpha_i = 1 \forall i$.

Then we have

$DL^{-1}D : L^2(\mathbb{R}^{n+1}) \to L^2(\mathbb{R}^{n+1})$,

$L^{-1}D : L^2(\mathbb{R}^{n+1}) \to \dot{L}_1^2(\mathbb{R}^{n+1})$,

$L^{-1}D : L^2(\mathbb{R}^{n+1}) \to \dot{L}_{\frac{1}{2}}^2(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ *where* $q = 2\left(\frac{n}{n-1}\right)$.

sketch of proof. The first two facts are standard: the first follows by ellipticity of L and integration by parts, while the second follows from the first by definition. The third follows from the second, by trace theory. □

Lemma 1.4.5. *(Lemma 3.11 on [AAAHK]) Suppose that θ_t is an operator satisfying*

$$||\theta(f\mathbb{1}_{2^{k+1}Q\setminus 2^kQ})||^2_{L^2(Q)} \leq C2^{-(n+2)k}\left(\frac{|t|}{2^k\ell(Q)}\right)^\beta ||f||_{L^2(2^{k+1}Q\setminus 2^kQ)}$$

for some $\beta > 0$, whenever $0 < t \leq C\ell(Q)$ and that $||\theta_t||_{2->2} \leq C$. Let $b \in L^\infty(\mathbb{R}^n)$, and let $\mathcal{A}_t$ denote a self-adjoint averaging whose kernel $\varphi_t(x)$ satisfies $|\varphi_t(x)| \leq Ct^{-n}\mathbb{1}_{\{|x|<Ct\}}$, $\varphi_t \geq 0, \int \varphi_t(x)dx = 1$. Then

$$\sup_{t>0} ||(\theta_t b)\mathcal{A}_t f||_{L^2(\mathbb{R}^n)} \leq C||b||_{L^\infty(\mathbb{R}^n)}||f||_{L^2(\mathbb{R}^n)}.$$

Chapter 2

Theorem 1

Definition 2.0.1. Let $\mathbb{M}^N$ denote the $N \times N$ matrices with complex entries. Suppose that $\Psi_t : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{M}^N$ satisfies the following properties for some exponent $\alpha > 0$

$$|\Psi_t(x,y)| \leq C \frac{t^\alpha}{(t + |x-y|)^{n+\alpha}}, \tag{2.0.2}$$

$$|\Psi_t(x, y+h) - \Psi_t(x,y)| + |\Psi_t(x+h, y) - \Psi_t(x,y)| \leq C \frac{|h|^\alpha}{(t+|x-y|)^{n+\alpha}}, \tag{2.0.3}$$

whenever $|h| \leq \frac{1}{2}|x-y|$ or $|h| \leq \frac{|t|}{2}$.

Then we define for $f : \mathbb{R}^n \to \mathbb{C}^N$ the operator

$$\Theta_t \cdot f(x) = \int_{\mathbb{R}^n} \Psi_t(x,y) \cdot f(y) dy := \left(\sum_{j=1}^N \int_{\mathbb{R}^n} (\Psi_t)_{ij}(x,y) f_j(y)\, dy \right)_{1 \leq i \leq N}. \tag{2.0.4}$$

Theorem 2.0.6. *We define Θ_t as above and suppose that there exists a constant $C_0 < \infty$, and exponent $p > 1$, $\delta > 0$ and a system $\{b_Q\}$ of functions indexed by dyadic cubes $Q \subset \mathbb{R}^n$, such that for each cube Q.*

$$\int_{\mathbb{R}^n} |b_Q(x)|^p dx \leq C_0 |Q|, \tag{2.0.5}$$

$$\int_Q \left(\int_0^{\ell(Q)} |\Theta_t b_Q(x)|^2 \frac{dt}{t} \right)^{\frac{p}{2}} dx \leq C_0 |Q|, \tag{2.0.6}$$

$$\delta|\xi|^2 \leq \mathcal{R}e\left(\xi \cdot \fint_Q b_Q(x)dx \cdot \bar{\xi}\right), \ \forall \xi \in \mathbb{C}^N \tag{2.0.7}$$

where the action of Θ_t *on the matrix valued function* b_Q *is defined in the obvious way as in* (2.0.4) *by viewing the kernel* $\Psi_t(x,y)$ *as a* $1 \times N$ *matrix which multiplies the* $N \times N$ *matrix* b_Q *and by* $\fint_Q b_Q(x)dx$ *we mean the average integral* $\frac{1}{|Q|}\int_Q b_Q(x)dx$.

Then

$$\iint_{\mathbb{R}^{n+1}_+} |\Theta_t \cdot f(x)|^2 \frac{dxdt}{t} \leq C\|f\|_2^2. \tag{2.0.8}$$

The outline of the proof goes as follows, by T1 Theorem 2.0.7, we are reduced to prove that our operator satisfies the Carleson measure estimate 2.0.9. Then the proof has three steps: 1) the conditions of the theorem 2.0.6 imply the conditions of the lemma 2.0.8; 2) the conditions of the lemma 2.0.8 imply the conditions of the sublemma 2.0.9. Finally the sublemma 2.0.9 would prove the Carleson measure estimate 2.0.9, which by the T1 theorem would lead to our conclusion.

Let's introduce them and then we would start with the proofs.

Theorem 2.0.7. *T1 Theorem [CJ]*

Let $\Theta_t f(x) \equiv \int_{\mathbb{R}^n} \Psi_t(x,y)f(y)dy$ *be an square function where it's kernel* $\Psi_t(x,y)$ *satisfies conditions (2.0.2) and (2.0.3) as above.*

Suppose that we have the Carleson measure estimate

$$\sup_Q \frac{1}{|Q|}\int_0^{\ell(Q)}\int_Q |\Theta_t 1(x)|^2 \frac{dxdt}{t} \leq C. \tag{2.0.9}$$

Then we have the square function estimate

$$\iint_{\mathbb{R}^{n+1}_+} |\Theta_t \cdot f(x)|^2 \frac{dxdt}{t} \leq C\|f\|_2^2 \tag{2.0.10}$$

Lemma 2.0.8. *Suppose that there exists $\eta \in (0,1)$, $\epsilon > 0$ small and $C_1 < +\infty$ such that for every dyadic cube $Q \in \mathbb{R}^n$, there is a family $\{Q_j\}$ of non-overlapping dyadic sub-cubes of Q, satisfying*

$$\sum_j |Q_j| \leq (1-\eta)|Q| \tag{2.0.11}$$

and

$$\int_Q \left(\int_{\tau_Q(x)}^{\ell(Q)} |\Theta_t 1(x)|^2 \mathbb{1}_{\Gamma_k^{2\epsilon}}(\Theta_t 1(x)) \frac{dt}{t} \right)^{\frac{p}{2}} dx \leq C_1 |Q| \,, \tag{2.0.12}$$

where $\tau_Q(x) = \sum_j \ell(Q_j) \mathbb{1}_{Q_j}(x)$. Then we have the Carleson Measure estimate (2.0.9). Here, ϵ is small, but fixed (to be made precise in 2.1.1 below). We cover $\mathbb{C}^n$ by cones of aperture ϵ, enumerating these cones as $\Gamma_1^\epsilon, ..., \Gamma_k^\epsilon$, where $k = k(\epsilon, N)$. For the previous estimate we are doubling the cones, the doubled cone has the same vector direction as the original ones but we double its aperture.

Sublemma 2.0.9. *Suppose that $\exists N < +\infty$ and $\beta \in (0,1)$ such that for all dyadic cube Q, and for all cones Γ^ϵ we have*

$$|\{x \in Q : g_Q(x) > N\}| \leq (1-\beta)|Q|, \tag{2.0.13}$$

where

$$g_Q(x) := \left(\int_0^{\ell(Q)} |\Theta_t 1(x)|^2 \mathbb{1}_{\Gamma^{2\epsilon}}(\Theta_t 1(x)) \frac{dt}{t} \right)^{\frac{1}{2}} . \tag{2.0.14}$$

Then we have the Carleson Measure estimate (2.0.9).

Remark 2.0.10. Every g_Q also depends on the cone of definition but since we are choosing a generic cone we avoid complicating the notation by adding more indices.

2.1 Conditions of Theorem 2.0.6 imply conditions of Lemma 2.0.8

We may assume without loss of generality that $1 < p < 2$, as the case $p > 2$ may be reduced to the known case $p = 2$ by Hölder's inequality which is proven in [H1].

Proof. First of all let's construct such a family of non-overlapping dyadic subcubes. To do that we are going to use a stopping time argument.

Without loss of generality (by renormalizing), we may assume $\delta \equiv 1$ on (2.0.7). Fix a cube Q, and then fix a cone $\Gamma^{2\epsilon}$. Now we subdivide Q dyadically and select a family $\{Q_j\}, Q_j \subset Q$ which are maximal with respect to the condition that at least one of the following conditions hold:

$$\frac{1}{|Q_j|}\int_{Q_j} |b_Q(x)|dx \geq \frac{1}{8\epsilon} \ \ (type\ I)$$

$$\mathcal{R}e\left(\nu \cdot \fint_{Q_j} b_Q(x)dx \cdot \bar{\nu}\right) \leq \frac{3}{4} \ \ (type\ II)$$

where ν is the unit normal in the direction of the central axis of the cone $\Gamma^{2\epsilon}$ and ϵ is half the aperture of the cone.

Once that we constructed the family we need to verify it satisfies the required conditions (2.0.11) and (2.0.12).

Let's start with condition (2.0.11).

Define $E := Q \setminus \{\bigcup_j Q_j\}$. Then from condition (2.0.7) since $\delta \equiv 1$ and taking $\xi = \nu$ where ν is the unit normal in the direction of the central axis of $\Gamma^{2\epsilon}$ we get

$$
\begin{aligned}
|Q| \leq & \mathcal{R}e \sum_{i,j} \int_Q (b_Q)_{ij}(x) \nu_j \bar{\nu}_i dx \\
&= \mathcal{R}e \sum_{i,j} \int_E (b_Q)_{ij}(x) \nu_j \bar{\nu}_i dx + \mathcal{R}e \sum_k \sum_{i,j} \int_{Q_k} (b_Q)_{ij}(x) \nu_j \bar{\nu}_i dx \\
&:= I + II.
\end{aligned}
$$

For the first part using condition (2.0.5) and Hölder's inequality we get

$$
\begin{aligned}
I :=& \mathcal{R}e \sum_{i,j} \int_E (b_Q)_{ij}(x) \nu_j \bar{\nu}_i dx \\
&\leq \left| \nu \cdot \int_E b_Q(x) dx \cdot \bar{\nu} \right| \\
&= \left| \int_E b_Q(x) dx \right| \\
&\leq |E|^{\frac{1}{p'}} \left(\int_Q |b_Q(x)|^p dx \right)^{\frac{1}{p}} \\
&\leq C|E|^{\frac{1}{p'}} |Q|^{\frac{1}{p}}.
\end{aligned}
$$

For the second part we are working with the family of subcubes and to be able to use their properties we need to separate them in two cases: the ones that satisfy the type I condition and the ones that satisfy the type II condition (the same subcube can satisfy both conditions at the same time; in this case we arbitrarily assign them to be of type I).

$$
\begin{aligned}
II \ \leq \ & \left| \mathcal{R}e \sum_{k, type\ I} \sum_{i,j} \int_{Q_k} (b_Q)_{ij}(x) \nu_j \bar{\nu}_i dx \right| + \left| \mathcal{R}e \sum_{k, type\ II} \sum_{i,j} \int_{Q_k} (b_Q)_{ij}(x) \nu_j \bar{\nu}_i dx \right| \\
& := II_1 + II_2.
\end{aligned}
$$

For the part of type I subcubes we are going to apply Hölder's inequality, the property of being type I, and condition (2.0.5).

$$
\begin{aligned}
II_1 &:= \left| \mathcal{R}e \sum_{k,typeI} \sum_{i,j} \int_{Q_k} (b_Q)_{ij}(x) \nu_j \bar{\nu}_i dx \right| \\
&\leq \sum_{k,typeI} \int_{Q_k} |b_Q(x)| dx \\
&= \int_{\bigcup_{k,typeI} Q_k} |b_Q(x)| dx \\
&\leq \left(\int_Q |b_Q(x)|^p dx \right)^{\frac{1}{p}} \left| \bigcup_{k,typeI} Q_k \right|^{\frac{1}{p'}} .
\end{aligned}
$$

For the measure of the set just note that

$$
\begin{aligned}
\left| \bigcup_{k,typeI} Q_k \right| &\leq 8\epsilon \int_{\bigcup_{k,typeI} Q_k} |b_Q(x)| dx \\
&\leq 8\epsilon \left| \bigcup_{k,typeI} Q_k \right|^{\frac{1}{p'}} \left(\int_Q |b_Q(x)|^p dx \right)^{\frac{1}{p}}
\end{aligned}
$$

$$
\left| \bigcup_{k,typeI} Q_k \right|^{\frac{1}{p}} \leq 8\epsilon \left(\int_Q |b_Q(x)|^p dx \right)^{\frac{1}{p}} \leq C\epsilon |Q|^{\frac{1}{p}}
$$

$$
\left| \bigcup_{k,typeI} Q_k \right|^{\frac{1}{p'}} \leq C\epsilon^{\frac{p}{p'}} |Q|^{\frac{1}{p'}} .
$$

Adding this to the previous computation we get

$$
II_1 \leq \left| \mathcal{R}e \sum_{k,type\ I} \sum_{i,j} \int_{Q_k} (b_Q)_{ij}(x) \nu_j \bar{\nu}_i dx \right| \leq \left(\int_Q |b_Q(x)|^p dx \right)^{\frac{1}{p}} \left| \bigcup_{k,typeI} Q_k \right|^{\frac{1}{p'}} \leq C\epsilon^{\frac{p}{p'}} |Q|^{\frac{1}{p'}} |Q|^{\frac{1}{p}} = C\epsilon^{\frac{p}{p'}} |Q|.
$$

We choose ϵ small enough so

$$C\epsilon^{\frac{q}{p}} \leq \frac{1}{8}. \tag{2.1.1}$$

For the type II subcubes just using the property of being type II we get
$II_2 := \left|\mathcal{R}e \sum_{k,typeII} \sum_{i,j} \int_{Q_k} (b_Q)_{ij}(x)\nu_j \bar{\nu}_i dx\right| \leq \frac{3}{4}\left|\bigcup_{k,typeII} Q_k\right| \leq \frac{3}{4}|Q|.$
Finally we can conclude that

$$\begin{aligned} |Q| &\leq I + II \leq C|E|^{\frac{1}{p'}}|Q|^{\frac{1}{p}} + \frac{1}{8}|Q| + \frac{3}{4}|Q| \\ &\leq 8C|E|^{\frac{1}{p'}}|Q|^{\frac{1}{p}} \\ &\leq C|E|. \end{aligned}$$

We take $0 < \eta \leq \frac{1}{C} \leq 1$ if $C > 1$ so

$$\sum_k |Q_k| = |Q \setminus E| = |Q| - |E| \leq |Q| - \eta|Q| = (1-\eta)|Q|.$$

This concludes that the family that we have constructed satisfy the measure condition. Now let's proceed to verify the condition (2.0.12)

Claim 2.1.1.

$$\int_Q \left(\int_{\tau_Q(x)}^{\ell(Q)} |\Theta_t 1(x)|^2 \mathbb{1}_{\Gamma^{2\epsilon}}(\Theta 1(x)) \frac{dxdt}{t}\right)^{\frac{p}{2}} \leq C \int_Q \left(\int_0^{\ell(Q)} |\Theta_t 1(x) \cdot A_t b_Q(x)\bar{\nu}|^2 \frac{dt}{t}\right)^{\frac{p}{2}} dx \tag{2.1.2}$$

where $\nu \in \mathbb{C}^n$ is the unit normal in the direction of the central axis of $\Gamma^{2\epsilon} := \{z \in \mathbb{C}^N : |\frac{z}{|z|} - \nu| < 2\epsilon\}$, and $A_t f(x) := |Q(x,t)|^{-1} \int_{Q(x,t)} f(y)dy$ with $Q(x,t)$ the minimal dyadic cube containing x with side length at least t.

proof of claim 2.1.2. Let's first introduce some notation as follows:

$(x,t) \in E_Q^* \equiv R_Q \setminus \left(\bigcup_j R_{Q_j} \right)$ where $R_Q \equiv Q \times (0, \ell(Q))$ and

$$\Gamma^{2\epsilon} = \{z \in \mathbb{C}^N : |\frac{z}{|z|} - \nu| < 2\epsilon\}.$$

We are going to prove that if $z \in \Gamma^{2\epsilon}$ and $(x,t) \in E_Q^*$ then $|z \cdot A_t b_Q(x)\bar{\nu}| \geq \frac{1}{2}|z|$.

Since $(x,t) \in E_Q^*$ we have that $Q(x,t)$ is not of type I neither type II, therefore by the triangle inequality

$$|\omega \cdot A_t b_Q(x)\bar{\nu}| \geq |\nu \cdot A_t b_Q(x)\bar{\nu}| - |(\omega - \nu)A_t b_Q(x)\bar{\nu}| \geq \frac{3}{4} - |(\omega - \nu)|\frac{1}{8\epsilon}, \quad \forall \omega \in \mathbb{C}^N.$$

If we choose $\omega = \frac{z}{|z|}$ then $|\omega - \nu| < 2\epsilon$ so we get that

$$\left| \frac{z}{|z|} \cdot A_t b_Q(x)\bar{\nu} \right| \geq \frac{3}{4} - \frac{2\epsilon}{8\epsilon} = \frac{1}{2},$$

which implies $|z \cdot A_t b_Q(x)\bar{\nu}| \geq \frac{1}{2}|z|$.

We are integrating where $\Theta_t 1(x) \in \Gamma^{2\epsilon}$; moreover, $x \in Q$, $\tau_Q(x) \leq t \leq \ell(Q) \Rightarrow (x,t) \in E_Q^*$; thus $|\Theta_t 1(x)|^2 \leq 4|\Theta_t 1(x) \cdot A_t b_Q(x)\bar{\nu}|^2$ in our domain of integration and the claim is true.

□

We are reduced to proving that,

$$\int_Q \left(\int_{\tau_Q(x)}^{\ell(Q)} |\Theta_t 1(x) \cdot A_t b_Q(x)\bar{\nu}|^2 \frac{dt}{t} \right)^{\frac{p}{2}} dx \leq C|Q|.$$

To finish the proof of condition (2.0.12) we use the Coifman-Meyer method as follows

$\Theta_t 1 A_t = (\Theta_t 1)(A_t - P_t) + (\Theta_t 1 P_t - \Theta_t) + \Theta_t := R_t^{(1)} + R_t^{(2)} + \Theta_t$

where P_t is a nice approximate identity as in definition 1.3.1.

By (2.0.6), the contribution of $\Theta_t b_Q$ is controlled by $C|Q|$ as desired. Moreover $R_t^{(2)}1 = 0$, and its kernel satisfies (2.0.2) and (2.0.3). Thus, by standard Littlewood-Paley/Vector-valued Calderón-Zygmund Theory, we have that

$$\int_Q \left(\int_0^{\ell(Q)} |R_t^{(2)} b_Q(x)|^2 \frac{dt}{t} \right)^{\frac{p}{2}} dx \leq C_p \|b_Q\|_p^p \leq C|Q|.$$

Furthermore, the same L^p bound holds for $R_t^{(1)}$ by interpolation arguments and that finishes the proof. □

2.2 Conditions of the Lemma 2.0.8 imply conditions of the Sublemma 2.0.9

Proof. For a large, but fixed N to be chosen momentarily, let

$$\Omega_N := \{x \in Q : g_Q(x) > N\}.$$

If conditions of lemma hold and we define $E := Q \setminus \left(\bigcup_j Q_j \right)$ then by Chebyshev's inequality we have

$$\begin{aligned}
|\Omega_N| &\leq \sum_j |Q_j| + |\{x \in E : g_Q(x) > N\}| \\
&\leq (1-\eta)|Q| + \left| \{x \in Q : \left(\int_{\tau_Q(x)}^{\ell(Q)} |\Theta_t 1(x)|^2 \mathbb{1}_{\Gamma^{2\epsilon}}(\Theta_t 1(x)) | \frac{dt}{t} \right)^{\frac{1}{2}} > N\} \right| \\
&\leq (1-\eta)|Q| + \left| \{x \in Q : \left(\int_{\tau_Q(x)}^{\ell(Q)} |\Theta_t 1(x)|^2 \mathbb{1}_{\Gamma^{2\epsilon}}(\Theta_t 1(x)) | \frac{dt}{t} \right)^{\frac{p}{2}} > N^p\} \right| \\
&\leq (1-\eta)|Q| + \frac{1}{N^p} \int_Q \left(\int_{\tau_Q(x)}^{\ell(Q)} |\Theta_t 1(x)|^2 \mathbb{1}_{\Gamma^{2\epsilon}}(\Theta_t 1(x)) | \frac{dt}{t} \right)^{\frac{p}{2}} dx \\
&\leq (1-\eta)|Q| + \frac{C_1}{N^p} |Q|.
\end{aligned}$$

Choose N large enough so $\frac{C_1}{N^p} \leq \frac{\eta}{2} \equiv \beta \Rightarrow |\Omega_N| \leq (1-\beta)|Q|$. □

2.3 Proof of Sublemma 2.0.9

Proof. Let N, β be as in the hypothesis. Fix $\gamma \in (0,1)$ a dyadic cube Q, and a cone Γ^ϵ. First of all let's add some notation by setting:

$$g_{Q,\gamma}(x) := \left(\int_{\gamma}^{min(\ell(Q),\frac{1}{\gamma})} |\Theta_t 1(x)|^2 \chi_\epsilon(\Theta_t 1(x)) \frac{dt}{t} \right)^{\frac{1}{2}}, \tag{2.3.1}$$

where we set this term to be 0 if $\ell(Q) \leq \gamma$, and where χ is a smooth function such that

$$\chi_\epsilon(\Theta_t 1(x)) = \begin{cases} 1 & if \quad \mathbb{1}_{\Gamma^{\frac{3}{2}\epsilon}}(\Theta_t 1(x)) = 1 \\ 0 & if \quad \mathbb{1}_{\Gamma^{2\epsilon}}(\Theta_t 1(x)) = 0 \\ (0,1) & otherwise \end{cases}$$

We also define $k(\gamma) := \sup_Q \frac{1}{|Q|} \int_Q \int_{\gamma}^{min(\ell(Q),\frac{1}{\gamma})} |\Theta_t 1(x)|^2 \mathbb{1}_{\Gamma^\epsilon}(\Theta_t 1(x)) \frac{dtdx}{t}$

where the supremum runs over all dyadic cubes Q.

Finally let's define the set $\Omega_{N,\gamma} := \{x \in Q : g_{Q,\gamma}(x) > N\}$ which is open.

By the truncation $k(\gamma)$ is finite for each fixed γ, and our goal is to show that $\sup_{0<\gamma<1} k(\gamma) < \infty$.

Once that all the notation is introduced let's start with the proof.

$\Omega_{N,\gamma}$ is open so we can make a Whitney decomposition of it such that $\Omega_{N,\gamma} = \bigcup_j Q_j$ and $F_{N,\gamma} = Q \setminus \Omega_{N,\gamma}$.

$$\begin{aligned}
&\int_Q \int_{\gamma}^{min(\ell(Q),\frac{1}{\gamma})} |\Theta_t 1(x)|^2 \mathbb{1}_{\Gamma^\epsilon}(\Theta_t 1(x)) \frac{dtdx}{t} \\
&= \int_{F_{N,\gamma}} \int_{\gamma}^{min(\ell(Q),\frac{1}{\gamma})} |\Theta_t 1(x)|^2 \mathbb{1}_{\Gamma^\epsilon}(\Theta_t 1(x)) \frac{dtdx}{t} \\
&+ \sum_j \int_{Q_j} \int_{\gamma}^{min(\ell(Q),\frac{1}{\gamma})} |\Theta_t 1(x)|^2 \mathbb{1}_{\Gamma^\epsilon}(\Theta_t 1(x)) \frac{dtdx}{t} \\
&\leq \int_{F_{N,\gamma}} \int_{\gamma}^{min(\ell(Q),\frac{1}{\gamma})} |\Theta_t 1(x)|^2 \chi_\epsilon(\Theta_t 1(x)) \frac{dtdx}{t} \\
&+ \sum_j \int_{Q_j} \int_{\gamma}^{min(\ell(Q_j),\frac{1}{\gamma})} |\Theta_t 1(x)|^2 \mathbb{1}_{\Gamma^\epsilon}(\Theta_t 1(x)) \frac{dtdx}{t} \\
&+ \sum_j \int_{Q_j} \int_{max(\gamma,\ell(Q_j))}^{min(\ell(Q),\frac{1}{\gamma})} |\Theta_t 1(x)|^2 \mathbb{1}_{\Gamma^\epsilon}(\Theta_t 1(x)) \frac{dtdx}{t} \\
&\leq N^2|Q| + k(\gamma) \sum_j |Q_j| + \sum_j \int_{Q_j} \int_{max(\gamma,\ell(Q_j))}^{min(\ell(Q),\frac{1}{\gamma})} |\Theta_t 1(x)|^2 \mathbb{1}_{\Gamma^\epsilon}(\Theta_t 1(x)) \frac{dtdx}{t} \\
&\leq N^2|Q| + k(\gamma)(1-\beta)|Q| + \sum_j \int_{Q_j} \int_{max(\gamma,\ell(Q_j))}^{min(\ell(Q),\frac{1}{\gamma})} |\Theta_t 1(x)|^2 \mathbb{1}_{\Gamma^\epsilon}(\Theta_t 1(x)) \frac{dtdx}{t}.
\end{aligned}$$

Claim 2.3.1.

$$L := \int_{Q_j} \int_{max(\ell(Q_j),\gamma)}^{min(\ell(Q),\frac{1}{\gamma})} |\Theta_t 1(x)|^2 \mathbb{1}_{\Gamma^\epsilon}(\Theta_t 1(x)) \frac{dtdx}{t} \leq C|Q_j|. \tag{2.3.2}$$

If the claim is true then $k(\gamma)$ is bounded uniformly in γ since

$$\frac{1}{|Q|} \int_Q \int_\gamma^{min(\ell(Q),\frac{1}{\gamma})} |\Theta_t 1(x)|^2 \mathbb{1}_{\Gamma^\epsilon}(\Theta_t 1(x)) \frac{dtdx}{t} \leq C + k(\gamma)(1-\beta) \quad \Rightarrow \quad k(\gamma) \leq \frac{C}{\beta}.$$

Therefore letting γ approach zero we obtain that

$$\int_Q \int_0^{\ell(Q)} |\Theta 1(x)|^2 1_{\Gamma^\epsilon}(\Theta_t 1(x)) \frac{dtdx}{t} \leq C|Q|.$$

Summarizing we can conclude that

$$\begin{aligned}
\int_Q \int_0^{\ell(Q)} |\Theta_t 1(x)|^2 \frac{dt}{t} dx &= \int_Q \int_0^{\ell(Q)} \sum_j |\Theta_t 1(x)|^2 \mathbb{1}_{\Gamma_j^\epsilon}(\Theta_t 1(x)) \frac{dt}{t} dx \\
&\leq \sum_j \int_Q \int_0^{\ell(Q)} |\Theta_t 1(x)|^2 \mathbb{1}_{\Gamma_j^\epsilon}(\Theta_t 1(x)) \frac{dt}{t} dx \\
&\leq k(\epsilon, N) \cdot C_1 |Q|,
\end{aligned}$$

where $k(\epsilon, N)$ is the number of cones in which we have subdivide $\mathbb{C}^n$. □

Proof of Claim 2.3.1. Let's have $0 < \beta < \alpha$ where α is the exponent of the pointwise estimates of our kernel in (2.0.2) and (2.0.3) and define the following sets

$$\begin{aligned}
Q_j^{(1)} &:= \left\{ x \in Q_j : \ |\Theta_t 1(x)| \leq \left(\frac{\ell(Q_j)}{t} \right)^\beta \frac{1}{\epsilon} \right\}; \\
Q_j^{(2)} &:= \{ x \in Q_j : \ \mathbb{1}_{\Gamma^\epsilon}(\Theta_t 1(x)) = 0 \}; \\
Q_j^{(3)} &:= Q_j \setminus (Q_j^{(1)} \cup Q_j^{(2)}).
\end{aligned}$$

Then $L \leq L_1 + L_2 + L_3$ where $L_i = \int_{Q_j^{(i)}} \int_{max(\ell(Q_j),\gamma)}^{min(\ell(Q),\frac{1}{\gamma})} |\Theta_t 1(x)|^2 \mathbb{1}_{\Gamma^\epsilon}(\Theta_t 1(x)) \frac{dtdx}{t},\ i = 1, 2, 3.$

Trivially $L_2 = 0$ and $L_1 \leq C|Q_j|$ since

$$\begin{aligned} L_1 :=& \int_{Q_j^{(1)}} \int_{max(\ell(Q_j),\gamma)}^{min(\ell(Q),\frac{1}{\gamma})} |\Theta_t 1(x)|^2 \mathbb{1}_{\Gamma^\epsilon}(\Theta_t 1(x)) \frac{dtdx}{t} \\ \leq& \int_{Q_j} \int_{\ell(Q_j)}^{\ell(Q)} \left(\frac{\ell(Q_j)}{t} \right)^{\beta} \frac{1}{\epsilon} \frac{dxdt}{t} \\ \leq& C(n,\beta,\epsilon)|Q_j|. \end{aligned}$$

For L_3 let's note that since we have done a Whitney decomposition $\exists x_j \in F_{N,\gamma}$ such that $dist(x_j, Q_j) \leq C\ell(Q_j)$. Take such x_j and decompose L_3 as follows

$$\begin{aligned} L_3 \lesssim& \int_{Q_j^{(3)}} \int_{\ell(Q_j)}^{C_n\ell(Q_j)} |\Theta_t 1(x)|^2 \mathbb{1}_{\Gamma^\epsilon}(\Theta_t 1(x)) \frac{dtdx}{t} \\ &+ \int_{Q_j^{(3)}} \int_{C_n l(Q_j)}^{\ell(Q)} |\Theta_t 1(x) \mathbb{1}_{\Gamma^\epsilon}(\Theta_t 1(x)) - \Theta_t 1(x_j) \chi_\epsilon(\Theta_t 1(x_j))|^2 \frac{dtdx}{t} \\ &+ \int_{Q_j^{(3)}} \int_{\gamma}^{min(\ell(Q),\frac{1}{\gamma})} |\Theta_t 1(x_j)|^2 \chi_\epsilon \Theta_t 1(x_j)) \frac{dtdx}{t} \\ &:= I + II + III. \end{aligned}$$

For I we use that $||\Theta_t 1||_\infty < \infty$, and for III that $x_j \in F_{N,\gamma}$, so $I + III \leq C(n, N)|Q_j|$.

For II we have two cases

<u>Case 1</u> : $\mathbb{1}_{\Gamma^\epsilon}(\Theta_t 1(x_j)) = 1$. Then for C_n large enough we have

$$|\Theta_t 1(x) - \Theta_t 1(x_j)| \leq C \left(\frac{\ell(Q_j)}{t} \right)^{\alpha}$$

for every $x \in Q_j \Rightarrow II \leq \int_{Q_j} \int_{C_n\ell(Q_j)}^{\infty} \left(\frac{\ell(Q_j)}{t} \right)^{2\alpha} \frac{dtdx}{t} \leq C|Q_j|$

<u>Case 2</u> : $\mathbb{1}_{\Gamma^\epsilon}(\Theta_t 1(x_j)) = 0$. Then $|\nu - \frac{\Theta_t 1(x_j)}{|\Theta_t 1(x_j)|}| > \epsilon$. We also have $|\nu - \frac{\Theta_t 1(x)}{|\Theta_t 1(x)|}| \leq \epsilon$, and that $x \in Q_j^{(3)}$, which implies that $|\Theta_t 1(x)| > \left(\frac{\ell(Q_j)}{t} \right)^{\beta} \frac{1}{\epsilon} \Rightarrow \frac{1}{|\Theta_t 1(x)|} < \left(\frac{t}{\ell(Q_j)} \right)^{\beta} \epsilon$. Thus (using the elementary inequality in Remark

2.3.2 below),

$$\left|\frac{\Theta_t 1(x)}{|\Theta_t 1(x)|} - \frac{\Theta_t 1(x_j)}{|\Theta_t 1(x_j)|}\right| \leq 2|\Theta_t 1(x) - \Theta_t 1(x_j)| \cdot \frac{1}{|\Theta_t 1(x)|}$$

$$\leq (2C)\left(\frac{\ell(Q_j)}{t}\right)^{\alpha-\beta} \epsilon \leq C\left(\frac{1}{C_n}\right)^{\alpha-\beta} \epsilon \leq \frac{\epsilon}{2},$$

if C_n is large enough, so that

$$\left|\nu - \frac{\Theta_t 1(x_j)}{|\Theta_t 1(x_j)|}\right| \leq \left|\frac{\Theta_t 1(x)}{|\Theta_t 1(x)|} - \frac{\Theta_t 1(x_j)}{|\Theta_t 1(x_j)|}\right| + \left|\nu - \frac{\Theta_t 1(x)}{|\Theta_t 1(x)|}\right| \leq \frac{\epsilon}{2} + \epsilon \leq \frac{3}{2}\epsilon$$

$$\Rightarrow II \leq \int_{Q_j} \int_{C_n \ell(Q_j)}^{\ell(Q)} |\Theta_t 1(x) - \Theta_t 1(x_j)| \mathbb{1}_{\Gamma^\epsilon}(\Theta_t 1(x)) \frac{dtdx}{t} \leq C|Q_j|,$$

as in case 1.

□

Remark 2.3.2. Observe that

$$|(x|y|) - (y|x|)| \leq |(x|y|) - (y|y|)| + |(y|y|) - (y|x|)| \leq 2|y| \cdot |x - y|,$$

so that

$$\frac{|x|y| - y|x||}{|x||y|} \leq 2\frac{|x-y|}{|x|} \quad \Rightarrow \left|\frac{x}{|x|} - \frac{y}{|y|}\right| \leq 2\frac{|x-y|}{|x|}.$$

Chapter 3

Theorem 2

Definition 3.0.1. We define for $f : \mathbb{R}^n \to \mathbb{C}$, $f \in L^2(\mathbb{R}^n)$ an operator $\theta_t f(x)$ satisfying the following properties

(a) (Uniform L^2 bounds and off-diagonal decay in L^2.)

$$\sup_{t>0} \|\theta_t f\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)}, \tag{3.0.2}$$

$$\|\theta_t f_j\|_{L^2(Q)} \leq C2^{-\frac{n+2+\beta}{2}j}\|f_j\|_{L^2(2^{j+1}Q\setminus 2^jQ)}, \qquad \ell(Q) \leq t \leq 2\ell(Q), \tag{3.0.3}$$

for some $\beta > 0$, where $f_j := f\mathbb{1}_{2^{j+1}Q\setminus 2^jQ}$.

(b) (Quasi-orthogonality in L^2.) There is some $\beta > 0$ such that for $s < t$,

$$\|\theta_t Q_s f\|_{L^2(\mathbb{R}^n)} \leq C\left(\frac{s}{t}\right)^{\beta}\|f\|_{L^2(\mathbb{R}^n)}, \tag{3.0.4}$$

where $\{Q_s\}_{0\leq s\leq\infty}$ is some family of operators with $\int_{\mathbb{R}^n}\int_0^\infty |Q_s f(x)|^2 \frac{dsdx}{s} \leq C\|f\|^2_{L^2(\mathbb{R}^n)}$, $\|\nabla Q_s f\|_{L^2(\mathbb{R}^n)} \leq C\frac{1}{s}\|f\|_{L^2(\mathbb{R}^n)}$, and $\int_0^\infty Q_s^2 \frac{ds}{s} = I$.

(c) ("Hypercontractive" off-diagonal decay.) There is some $1 < r < 2$, and some $\nu > \frac{n}{r}$ ($\nu = \frac{n}{r} + \epsilon, \epsilon > 0$), such that

$$\left(\int_{Q^*} |\theta_t(f\mathbb{1}_{S_j}(Q))(y)|^2 dy\right)^{\frac{1}{2}} \leq C2^{-j\nu}t^{-n(\frac{1}{r}-\frac{1}{2})}\left(\int_{S_j(Q)} |f(y)|^r dy\right)^{\frac{1}{r}},$$

$$\forall j \geq 0,\ \ell(Q) < t \leq 2\ell(Q), \tag{3.0.5}$$

where $S_0(Q) = 16Q$, $S_j(Q) = 2^{j+4}Q \setminus 2^{j+3}Q$, $j \geq 1$, and $Q^* \equiv 8Q$.

(d) (Improved integrability.)

$$\sup_{t>0} \|\theta_t f\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^q(\mathbb{R}^n)}, \quad for\ some\ q > 2. \tag{3.0.6}$$

Remark 3.0.3. We observe that, for example, (b), (c), and (d) hold, with $\theta_t = t\partial_t P_t$, where $P_t = e^{-t\sqrt{-\Delta}}$ is the usual Poisson semigroup, and that (a) holds with $\beta = 0$, for the same operator. We may obtain a positive value of β in (a), by considering higher order derivatives of P_t. As a practical matter, when considering square functions arising in PDE applications, it is often a fairly routine matter to pass to higher order derivatives (cf. Chapter 5 below). The advantage of the present formulations of our conditions, is that they continue to hold in the absence of pointwise kernel bounds. In PDE applications, (d) is typically obtained as a consequence of higher integrability estimates of "N. Meyers" type (cf. [Me2]).

Theorem 3.0.4. *Let's define the square function operator θ_t as above and suppose that there exists a constant $0 < C_0 < \infty$, $0 < C_1 < 1$, an exponent $p > r$, $\delta > 0$, $\eta > 0$ a system $\{b_Q\}$ of functions indexed by cubes $Q \subset \mathbb{R}^n$ and a system of Lipschitz functions $\{\Phi_Q\}$ also indexed by cubes, such that for each cube Q,*

$$\int_{\mathbb{R}^n} |b_Q(x)|^p dx \leq C_0|Q|, \tag{3.0.7}$$

$$\int_Q \left(\iint_{|x-y|<t<\ell(Q)} |\theta_t b_Q(y)|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{p}{2}} dx \leq C_0|Q|, \tag{3.0.8}$$

$$\|\nabla \Phi_Q\|_\infty \leq C_0\, \ell(Q)^{-1},\ C_1 \leq \Phi_Q(x) \leq 1\ on\ Q \tag{3.0.9}$$

$$\delta\, m_Q(Q) \leq \left| \int_Q b_Q(x) dm_Q(x) \right|,\ where\ dm_Q(x) = \Phi_Q(x)dx. \tag{3.0.10}$$

Then

$$\iint_{\mathbb{R}^{n+1}_+} |\theta_t f(x)|^2 \frac{dxdt}{t} \leq C\|f\|^2_{L^2(\mathbb{R}^n)}. \tag{3.0.11}$$

Remark 3.0.5. Frequently, one may simply take $\Phi_Q \equiv 1$, but in some applications, it is useful to have the extra flexibility inherent in (3.0.10) (cf. Chapter 5).

Once the point-wise estimates are dropped from the Square function, we need to redo all the previous techniques that we had before. We can not use standard Littlewood Paley Theory or Vector-valued Calderón-Zygmund Theory, so we needed to look for new techniques to approach this problem.

The previous results' proofs are based on proving that our conditions lead to the T1 Theorem for Square functions of Christ-Journé [CJ], which means that we are reduced to prove that $d\mu = |\theta_t 1(x)|^2 dxdt/t$ is a Carleson measure. This new result scheme is not going to be different so lets redefine the T1 Theorem, Lemma and Sublemma in this new context.

Theorem 3.0.6. *(T1 Theorem) Let $\theta_t f$ satisfying conditions* (3.0.2)*,* (3.0.3) *and* (3.0.4) *and the Carleson measure estimate*

$$\sup_Q \frac{1}{|Q|} \int_0^{\ell(Q)} \int_Q |\theta_t 1(x)|^2 \frac{dxdt}{t} \leq C. \tag{3.0.12}$$

Then we have the Square function estimate (3.0.11)*.*

Lemma 3.0.7. *Suppose that $\exists \alpha \in (0, \frac{1}{2}], \eta \in (0,1)$ and $C < \infty$ such that for every cube $Q \in \mathbb{R}^n$, there exists a family $\{\tilde{Q}_j\}_j$ of non-overlapping subcubes of Q, dyadic with respect to the grid induced by Q, with the properties*

$$\sum_j (1+\alpha)^n |\tilde{Q}_j| \leq (1-\eta)|Q|\,, \tag{3.0.13}$$

and

$$\int_E \left(\iint_{|x-y|<\frac{\alpha}{100}t<\frac{\alpha}{100}\ell(Q)} |\theta_t 1(y)|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{p}{2}} dx \leq C|Q|\,, \tag{3.0.14}$$

where $E = (1-\alpha)Q \setminus \{\bigcup_j Q_j\}$, and $Q_j := (1+\alpha)\tilde{Q}_j$.

Then the Carleson measure estimate (3.0.12) *holds.*

Sublemma 3.0.8. *Suppose that* $\exists N < \infty$ *and* $\alpha \in (0, \frac{1}{2}], \beta \in (0,1)$ *such that for every cube* Q

$$|\{x \in Q : G_Q(x) > N\}| \leq (1-\beta)|Q|,$$

where $G_Q(x) := \left(\iint_{|x-y|<\frac{\alpha}{100}t<\frac{\alpha}{100}\ell(Q)} |\theta_t 1(y)|^2 \frac{dtdy}{t^{n+1}} \right)^{\frac{1}{2}}$

Then the Carleson measure estimate (3.0.12) *holds.*

3.1 Proof of T1 Theorem

To prove this new version of the T1 Theorem for Square functions, we have used a convenient modification of a similar lemma that appears in [AAAHK] (Lemma 3.5). The modified lemma reads as follows.

Lemma 3.1.1. *(i) Suppose that* $\{\theta_t\}_{t\in\mathbb{R}}$ *is a family of operators satisfying*

(a) For some $\beta > 0$*, and for all* $|t| \approx \ell(Q)$,

$$\|\theta_t(f \mathbb{1}_{2^{k+1}Q\setminus 2^k Q})\|^2_{L^2(Q)} \leq C2^{-(n+2)k} \left(\frac{|t|}{2^k \ell(Q)} \right)^{\beta} \|f\|_{L^2(2^{k+1}Q\setminus 2^k Q)}, \tag{3.1.1}$$

(b)

$$\sup_{t\geq 0} \|\theta_t f\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)} \tag{3.1.2}$$

(c)

$$\theta_t 1 = 0, \forall t \in \mathbb{R} \tag{3.1.3}$$

Then for $h \in \dot{L}^2_1(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} |\theta_t h(x)|^2 dx \leq Ct^2 \int_{\mathbb{R}^n} |\nabla_x h(x)|^2 dx$$

(ii) If in addition, there exists a family $\{Q_s\}_{s>0}$ *such that for all* $f \in L^2(\mathbb{R}^n)$

($\tilde{a}$*)*

$$\|\theta_t Q_s f\|_{L^2(\mathbb{R}^n)} \leq c \left(\frac{s}{t}\right)^{\beta} \|f\|_{L^2(\mathbb{R}^n)}, \qquad \forall s \leq t, \tag{3.1.4}$$

for some family $\{Q_s\}_{0<s<\infty}$ *satisfying*

$(\tilde{b})$

$$\|\nabla Q_s f\|_{L^2(\mathbb{R}^n)} = \|Q_s f\|_{\dot{L}^2_1(\mathbb{R}^n)} \leq C\frac{1}{s}\|f\|_{L^2(\mathbb{R}^n)}\,,$$

$$\|Q_s f\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)}\,, \tag{3.1.5}$$

$$\int_0^\infty Q_s^2 \frac{ds}{s} = I\,.$$

Then

$$\iint_{\mathbb{R}^{n+1}_+} |\theta_t f(x)|^2 \,\frac{dxdt}{t} \leq C\|f\|^2_{L^2(\mathbb{R}^n)}\,, \qquad \forall f \in L^2(\mathbb{R}^n).$$

Proof. First, let's prove (*ii*), assuming that (*i*) holds. Take $f \in L^2(\mathbb{R}^n)$, and $\{Q_s\}$ the family of operators assumed in (*ii*). For $s \leq t$ we have (3.1.4). For $t < s$ we apply (i) to $h(x) = Q_s f(x)$ so that

$$\|\theta_t Q_s f\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\theta_t Q_s f(x)|^2 dx \leq Ct^2 \int_{\mathbb{R}^n} |\nabla_x Q_s f(x)|^2 dx \leq \frac{t^2}{s^2}\|f\|^2_{L^2(\mathbb{R}^n)}\,,$$

by the first condition in (3.1.5). Consequently,

$$\|\Theta_t Q_s f\|_{L^2(\mathbb{R}^n)} \leq C \min\left(\frac{s}{t}, \frac{t}{s}\right)^{\tilde{\beta}} \|f\|_{L^2(\mathbb{R}^n)},$$

for some $\tilde{\beta} > 0$. The result follows from the last inequality by a standard orthogonality argument.

So let's prove (i). Let $\mathbb{D}(t)$ denote the grid of dyadic cubes with $\ell(Q) \leq |t| \leq 2\ell(Q)$. For convenience of notation we set $m_Q h \equiv \fint_Q h(x)dx$ then

$$\begin{aligned}
\left(\int_{\mathbb{R}^n} |\theta_t h(x)|^2 dx\right)^{\frac{1}{2}} &= \left(\sum_{Q\in\mathbb{D}(t)} \int_Q |\theta_t h(x)|^2 dx\right)^{\frac{1}{2}} \\
&= \left(\sum_{Q\in\mathbb{D}(t)} \int_Q |\theta_t (h - m_{2Q}h)(x)|^2 dx\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{Q\in\mathbb{D}(t)} \int_Q |\theta_t (h - m_{2Q}h)(x)\mathbb{1}_{2Q}(x)|^2 dx\right)^{\frac{1}{2}} \\
&\quad + \left(\sum_{Q\in\mathbb{D}(t)} |\theta_t (h - m_{2Q}h)(x)\mathbb{1}_{(2Q)^c}(x)|^2 dx\right)^{\frac{1}{2}} := I + II\,,
\end{aligned}$$

where we have used that $\theta_t 1 = 0$ and it's a linear operator $\Rightarrow \theta_t(m_{2Q}h) = 0$.

Since $\theta_t : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, we have by Poincaré's inequality that

$$I \leq C \left(\sum_{Q \in \mathbb{D}(t)} \int_{2Q} |(h - m_{2Q}h)(x)|^2 dx \right)^{\frac{1}{2}} \leq C|t| \left(\sum_{Q \in \mathbb{D}(t)} \int_{2Q} |\nabla_x h(x)|^2 dx \right)^{\frac{1}{2}} \leq C|t| \, \|\nabla_x h\|_{L^2(\mathbb{R}^n)} .$$

For the second part by condition (3.1.1)

$$\begin{aligned}
II &\leq \sum_{k=1}^{\infty} \left(\sum_{Q \in \mathbb{D}(t)} \int_Q \left| \theta_t[(h - m_{2Q}h)(x) \mathbb{1}_{2^{k+1}Q \setminus 2^k Q}(x)] \right|^2 dx \right)^{\frac{1}{2}} \\
&\leq C \sum_{k=1}^{\infty} \left(\sum_{Q \in \mathbb{D}(t)} 2^{-(n+2+\beta)k} \int_Q |(h - m_{2Q}h)(x)|^2 \mathbb{1}_{2^{k+1}Q \setminus 2^k Q}(x) dx \right)^{\frac{1}{2}} \\
&\leq C \sum_{k=1}^{\infty} \left(\sum_{Q \mathbb{D}(t)} 2^{-(n+2+\beta)k} \int_{2^{k+1}Q} |(h - m_{2Q}h)(x)|^2 dx \right)^{\frac{1}{2}} \\
&\leq C \sum_{k=1}^{\infty} \sum_{j=1}^{k} \left(\sum_{Q \in \mathbb{D}(t)} 2^{-k(\beta+2)} 2^{-jn} \int_{2^{j+1}Q} |(h - m_{2^{j+1}Q}h)(x)|^2 dx \right)^{\frac{1}{2}} ,
\end{aligned}$$

where in the last step we have used a telescoping argument.

By Poincaré's inequality, since $j \leq k$ we obtain in turn the bound

$$\begin{aligned}
II &\leq C \sum_{k=1}^{\infty} \sum_{j=1}^{k} \left(\sum_{Q \in \mathbb{D}(t)} 2^{-k(\beta+2)} 2^{-jn} \left(\operatorname{diam}(2^{j+1}Q) \right)^2 \int_{2^{j+1}Q} |\nabla_x h(x)|^2 dx \right)^{\frac{1}{2}} \\
&\leq C|t| \sum_{k=1}^{\infty} 2^{-\frac{\beta}{2}k} \sum_{j=1}^{k} \left(\sum_{Q \in \mathbb{D}(t)} 2^{-jn} \int_{2^{j+1}Q} |\nabla_x h(x)|^2 dx \right)^{\frac{1}{2}} \\
&\leq C|t| \sum_{k=1}^{\infty} 2^{-\frac{\beta}{2}k} \sum_{j=1}^{k} \left(\sum_{Q \in \mathbb{D}(t)} \int_Q ⨍_{2^{j+1}Q} |\nabla_x h(x)|^2 dx dy \right)^{\frac{1}{2}} \\
&\leq C|t| \sum_{k=1}^{\infty} 2^{-\frac{\beta}{2}k} \sum_{j=1}^{k} \left(\int_{\mathbb{R}^n} ⨍_{|x-y| \leq C2^j|t|} |\nabla_x h(x)|^2 dx dy \right)^{\frac{1}{2}} \\
&\leq C|t| \cdot \|\nabla_x h\|_{L^2(\mathbb{R}^n)}
\end{aligned}$$

□

In order to be able to use this lemma to prove our T1 Theorem, as in the regular T1 Theorem, we want to reduce the problem to $\tilde{\theta}_t 1 = 0$.

Proof. By the Coifman-Meyer method we can rewrite our operator as follows:

$$\theta_t = \theta_t - (\theta_t 1)P_t + (\theta 1)P_t =: \tilde{\theta}_t + (\theta_t 1)P_t\,,$$

where P_t is a nice approximate identity as in Definition 1.3.1. Observe that with this definition $\tilde{\theta}_t 1 = 0$ but we need to verify that $\tilde{\theta}_t$ also satisfies(3.0.2), (3.0.3) and (3.0.4). This is not immediate, because, in the absence of pointwise kernel bounds, it may be that $\theta_t 1$ is not uniformly bounded. We proceed as follows.

(1) $\sup_{t>0} \|\tilde{\theta}_t f\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)}$

By Lemma 1.4.5, choosing $b(x) = 1$ and $\mathcal{A}_t = P_t$ we get that

$$\|(\theta_t 1)P_t f\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)} \Rightarrow \|\tilde{\theta}_t\|_{L^2(\mathbb{R}^n)} \leq \|\theta_t f\|_{L^2(\mathbb{R}^n)} + \|(\theta_t 1)P_t f\|_{L^2(\mathbb{R}^n)}$$
$$\leq C\|f\|_{L^2(\mathbb{R}^n)}.$$

(2) $\|\tilde{\theta}_t f_j\|_{L^2(Q)} \leq C2^{-\frac{n+2+\beta}{2}j}\|f_j\|_{L^2(2^{j+1}Q\setminus 2^j Q)}$, for $f_j = f\mathbb{1}_{2^{j+1}Q\setminus 2^j Q}, \ell(Q) \leq |t| \leq 2\ell(Q)$.

For $j > 2$, there is no contribution from $(\theta_t 1)P_t$, because the kernel of P_t has compact support. Therefore, (2) follows from the analogous bound for θ_t, namely, (3.0.3). The case $j = 1, 2$ reduces to (1) above.

(3) $\|\tilde{\theta}_t Q_s f\|_{L^2(\mathbb{R}^n)} \leq C(\frac{s}{t})^\beta \|f\|_{L^2(\mathbb{R}^n)}, \forall f \in H$, and $s < t$.

We may choose our approximate identity P_t to be of the form $P_t = (\widetilde{P}_t)^2$, where $\widetilde{P}_t$ is of the same type. We then have

$$\|\tilde{\theta}_t Q_s f\|_{L^2(\mathbb{R}^n)} \leq \|\theta_t Q_s f\|_{L^2(\mathbb{R}^n)} + \|(\theta_t 1(x))\widetilde{P}_t\widetilde{P}_t Q_s f\|_{L^2(\mathbb{R}^n)}$$
$$\leq C\left(\frac{s}{t}\right)^\beta \|f\|_{L^2(\mathbb{R}^n)} + C\|\widetilde{P}_t Q_s f\|_{L^2(\mathbb{R}^n)} \leq C\left(\frac{s}{t}\right)^\beta \|f\|_{L^2(\mathbb{R}^n)}\,,$$

where in the second inequality, we have applied Lemma 1.4.5 to $(\theta_t 1)\widetilde{P}_t$.

Then by the previous lemma, $\tilde{\theta}_t$ satisfies (3.0.11).

To finish reducing our problem we need that if we have the Square function estimate (3.0.11) for $\tilde{\theta}_t$, then it also holds for the original operator θ_t:

$$
\begin{aligned}
\iint_{\mathbb{R}^{n+1}_+} |\theta_t f(x)|^2 \frac{dxdt}{t} &\leq \iint_{\mathbb{R}^{n+1}_+} |\tilde{\theta}_t f|^2 \frac{dxdt}{t} + \iint_{\mathbb{R}^{n+1}_+} |\theta_t 1(x)|^2 |P_t f(x)|^2 \frac{dxdt}{t} \\
&\leq C||f||^2_{L^2(\mathbb{R}^n)} + \int_0^\infty \int_{\mathbb{R}^n} |P_t f(x)|^2 |\theta_t 1(x)|^2 \frac{dxdt}{t} \\
&\leq C||f||^2_{L^2(\mathbb{R}^n)} + C||\mu||_C ||\mathcal{N}_*(P_t f)||^2_{L^2(\mathbb{R}^{n+1})} \\
&\leq C||f||^2_{L^2(\mathbb{R}^n)}.
\end{aligned}
$$

where $d\mu = |\theta_t 1(x)|^2 dx$ which is a Carleson measure by hypothesis and $\mathcal{N}_*$ is the nontangential maximal function.

□

3.2 Conditions of Theorem 3.0.4 imply conditions of the Lemma 3.0.7

Proof. As in the similar proof on the previous section, condition (3.0.13) is a consequence of the choice of the family of subcubes of Q . We choose the family in similar way as in [H2], that means that WLOG (by renormalizing) we may suppose that $\frac{1}{m_Q(Q)} \int_Q b_Q(x) dm_Q(x) = 1$ and we sub-divide Q dyadically and select a family of non-overlapping cubes $\{\tilde{Q}_j\}$ which are maximal with respect to the property that $\mathcal{R}e \frac{1}{m_Q(\tilde{Q}_j)} \int_{\tilde{Q}_j} b_Q(x) dm_Q(x) \leq \frac{C_1}{2}$.

First, from the definition of the measure $m_Q(x)$ we have that for every subset A of Q, $C_1|A| \leq m_Q(A) \leq |A|$. Define $\tilde{E} := Q \setminus \{\bigcup_j \tilde{Q}_j\}$, $Q_j := (1+\alpha)\tilde{Q}_j$ and $E := (1-\alpha)Q \setminus \{\bigcup_j Q_j\}$, $\alpha \in (0, \frac{1}{2}]$ to be determined later.

Then to prove the first condition (3.0.13), by Hölder and condition (3.0.7) we have

$$
\begin{aligned}
|Q| &\le \frac{1}{C_1} m_Q(Q) = \frac{1}{C_1}\int_Q b_Q(x) dm_Q(x) \\
&= \frac{1}{C_1}\left[\mathcal{R}e\int_{\tilde{E}} b_Q(x)dx + \mathcal{R}e\sum_j \int_{\tilde{Q}_j} b_Q(x)dm_Q(x)\right] \\
&\le \frac{1}{C_1}\left[|\tilde{E}|^{\frac{1}{p'}}\left(\int_Q |b_Q(x)|^p dx\right)^{\frac{1}{p}} + \frac{C_1}{2}\sum_j m_Q(\tilde{Q}_j)\right] \\
&\le C|\tilde{E}|^{\frac{1}{p'}}|Q|^{\frac{1}{p}} + \frac{1}{2}|Q| \\
&\le C|\tilde{E}|^{\frac{1}{p'}}|Q|^{\frac{1}{p}} + \frac{1}{2}|Q| \\
\Rightarrow& |\tilde{E}| > \tilde{\eta}|Q|, \text{ with } 0 < \tilde{\eta} = \left(\frac{1}{2C}\right)^{p'} < 1 .
\end{aligned}
$$

Therefore,

$$
\sum_j |\tilde{Q}_j| \le (1-\tilde{\eta})|Q| .
$$

Thus,

$$
\sum_j |Q_j| \le \sum_j (1+\alpha)^n |\tilde{Q}_j| \le (1+\alpha)^n (1-\tilde{\eta})\,|Q| \le (1-\eta)\,|Q|
$$

provided that we choose η and α sufficiently small, depending upon $\tilde{\eta}$.

Finally we need to prove the second condition (3.0.14) and we want to reduce ourselves to

$$
\int_{\mathbb{R}^n}\left(\iint_{\Gamma_Q(x)} |\theta_t 1(y) A_{m,t} b_Q(y)|^2 \frac{dydt}{t^{n+1}}\right)^{\frac{p}{2}} dx \le C_p \|b_Q\|^p_{L^p(\mathbb{R}^n)} + C|Q| \lesssim |Q| ,
$$

where in the last step we have used hypothesis (3.0.7). As above, $A_{m,t}f(x) := \frac{1}{m_Q(Q(x,t))}\int_{Q(x,t)} b_Q(y)dm_Q(y)$ and $Q(x,t)$ is the minimal dyadic cube (with respect to the grid induced by Q), that contains x, with length at least t. Here, Γ_Q is the truncated cone with height $\ell(Q)$.

To reduce ourselves to the previous inequality let's note that if

$$
(y,t) \in R_Q \setminus \bigcup_j R_{\tilde{Q}_j}
$$

where $R_Q = Q \times (0, \ell(Q))$, then by the maximality of the family choice

$$\mathcal{R}e\, A_{m,t} b_Q(y) := \mathcal{R}e \frac{1}{m_Q(Q(y,t))} \int_{Q(y,t)} b_Q(z) dm_Q(z) \geq \frac{C_1}{2}$$

Moreover, if $x \in E,\ 0 < t < \ell(Q),\ |x-y| < \frac{\alpha}{100} t \Rightarrow (y,t) \in R_Q \setminus \bigcup_j R_{\tilde{Q}_j}$ which can be seen clearly by contradiction:

$$(y,t) \in R_{\tilde{Q}_j} \Rightarrow\ |x-y| \leq \frac{\alpha}{100} \ell(Q_j) \Rightarrow\ x \in Q_j$$

$$y \in \mathbb{R}^n \setminus Q \text{ and } |\mathrm{x}-\mathrm{y}| \leq \frac{\alpha}{100} \ell(\mathrm{Q}) \Rightarrow\ \mathrm{x} \in \mathbb{R}^{\mathrm{n}} \setminus (1-\alpha)\mathrm{Q}\,.$$

We therefore have

$$\begin{aligned}
\int_E \left(\iint_{|x-y|<\frac{\alpha}{100} t \leq \frac{\alpha}{100}\ell(Q)} |\theta_t 1(y)|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{p}{2}} dx \\
\lesssim \int_E \left(\iint_{|x-y|<\frac{\alpha}{100} t \leq \frac{\alpha}{100}\ell(Q)} |\theta_t 1(y) A_{m,t} b_Q(y)|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{p}{2}} dx \\
\lesssim \int_E \left(\iint_{|x-y|<\frac{\alpha}{100} t < \frac{\alpha}{100}\ell(Q)} |\theta_t 1(y) A_{m,t} b_Q(y) - \theta_t b_Q(y)|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{p}{2}} dx \\
+ \int_E \left(\iint_{|x-y|<\frac{\alpha}{100} t \leq \frac{\alpha}{100}\ell(Q)} |\theta_t b_Q(y)|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{p}{2}} dx \\
\leq \int_{\mathbb{R}^n} |S(b_Q)(x)|^p dx + \int_Q \left(\iint_{|x-y|<t<\ell(Q)} |\theta_t b_Q(y)|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{p}{2}} dx \\
\leq \int_{\mathbb{R}^n} |S(b_Q)(x)|^p dx + C_0 |Q|.
\end{aligned}$$

where in the last step we have used hypothesis (3.0.8), and where we define

$$S(f)(x) := \left(\iint_{\Gamma_Q(x)} |R_t b_Q(y)|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}},$$

with $\Gamma_Q(x) := \{(y,t) \in \mathbb{R}^{n+1} : |x-y| < t < \ell(Q)\}$,

and

$$R_t f(x) := \theta_t 1(x) A_{m,t} f(x) - \theta_t f(x)\,.$$

Let us recall that by definition, the measure $m = m_Q$, and the "dyadic grid", and thus also $A_{m,t}$, R_t and S, depend implicitly on Q.

Our goal at this point is to show that, with Q (and hence the truncation of the cones, and the definition of $A_{m,t}$) fixed, we have

$$\|S(f)\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}\,, \tag{3.2.1}$$

but with bounds that are uniform in Q. Once we have established the latter estimate, we may apply it with $f = b_Q$, and then invoke hypothesis (3.0.7), to obtain (3.0.14).

To prove (3.2.1), we first recall the following.

Proposition 3.2.1. *[C-UMP] Let T be a sublinear operator satisfying*

$$T : L^2(\mathbb{R}^n, \omega) \to L^2(\mathbb{R}^n, \omega)\,, \qquad \forall \omega \in A_{\frac{2}{r}}\,.$$

Then $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, $for\ p > r$.

By the Proposition, we are left to prove that

$$\int_{\mathbb{R}^n} |S(f)(x)|^2 v(x)dx \leq C \int_{\mathbb{R}^n} |f(x)|^2 v(x)dx, \text{ where } v(x) \in A_{\frac{2}{r}}\,.$$

This follows by a standard orthogonality argument, once we show that

$$\int_{\mathbb{R}^n} \fint_{|x-y|<t<\ell(Q)} |R_t Q_s h(y)|^2 dy\, v(x)dx \leq C \min\left(\frac{s}{t}, \frac{t}{s}\right)^{\beta} \int_{\mathbb{R}^n} |h(x)|^2 v(x)dx\,, \tag{3.2.2}$$

for some $\beta > 0$, with $h(x) = Q_s f(x)$.

To establish (3.2.2), we begin by observing that, by the same arguments used to prove Lemma 3.1.1, with R_t in place of θ_t, we have

$$\int_{\mathbb{R}^n} \fint_{|x-y|<t<\ell(Q)} |R_t Q_s h(y)|^2 dydx \leq Cmin\left(\frac{s}{t}, \frac{t}{s}\right)^{\tilde{\beta}} \int_{\mathbb{R}^n} |h(x)|^2 dx\,.$$

Indeed, checking the conditions of Lemma 3.1.1, (a) is immediate by our hypotheses on θ_t, and the compact support of the kernel of $A_{m,t}$; (b) follows immediately by hypothesis, and [AAAHK, Lemma 3.11]; (c) holds for R_t by definition; and $(\tilde{a})$ holds for the present θ_t by hypothesis. Finally, we may verify $(\tilde{a})$ for $(\theta_t 1)A_{m,t}$ by observing that $A_{m,t}$ is a projection, so that $(\theta_t 1)A_{m,t} = (\theta_t 1)A_{m,t}A_{m,t}$, whence it follows that

$$\|(\theta_t 1)A_{m,t}A_{m,t}Q_s f\|_2 \lesssim \|A_{m,t}Q_s f\|_2 \lesssim \left(\frac{s}{t}\right)^{\beta} \|f\|_2 , \qquad \forall s \le t \le \ell(Q),$$

by (b), and the construction of $A_{m,t}$ (in the last step, we have used that $t \le \ell(Q)$).

Next, we need the following claim:

Claim 3.2.2.

$$\int_{\mathbb{R}^n} ⨍_{|x-y|<t<\ell(Q)} |R_t Q_s h(y)|^2 dy\, \tilde{v}(x) dx \le C \int_{\mathbb{R}^n} |h(x)|^2 \tilde{v}(x) dx, \forall \tilde{v} \in A_{\frac{2}{r}} \tag{3.2.3}$$

Interpolating with change of measure ([SW]) we get (3.2.2). Indeed, for each $v \in A_{\frac{2}{r}}$, there exist $\lambda_0 > 0$ such that $v^{1+\lambda_0} \in A_{\frac{2}{r}}$ so we choose $\tilde{v}(x) = v^{1+\lambda_0}(x)$.

The idea of using the interpolation with change of measure in this way first appeared in the paper [DRdeF].

□

Proof of Claim 3.2.2. Define $\tilde{h}(x) = Q_s h(x)$. By the properties of Q_s (it is controlled by the maximal operator), $\|\tilde{h}\|_{L^2_{\tilde{v}}(\mathbb{R}^n)} \le C\|h\|_{L^2_{\tilde{v}}(\mathbb{R}^n)}$. So it's enough to show

$$\left(\int_{\mathbb{R}^n} ⨍_{|x-y|<t<\ell(Q)} |R_t \tilde{h}(y)|^2 dy\, \tilde{v}(x) dx\right)^{\frac{1}{2}} \le C \left(\int_{\mathbb{R}^n} \tilde{h}(x)^2 \tilde{v}(x) dx\right)^{\frac{1}{2}}.$$

By hypothesis (3.0.5), and since $\tilde{v} \in A_{\frac{2}{r}}$ (which gives us an $L^{2/r}_{\tilde{v}}$ bound for the maximal function), we have

$$\left(\int_{\mathbb{R}^n} \fint_{|x-y|<t<\ell(Q)} |R_t\tilde{h}(y)|^2 dy\, \tilde{v}(x) dx\right)^{\frac{1}{2}}$$

$$= \left(\sum_{P\in\mathbb{D}(t)} \int_P \fint_{|x-y|<t<\ell(Q)} |R_t\tilde{h}(y)|^2 dy\, \tilde{v}(x) dx\right)^{\frac{1}{2}}$$

$$\leq C\left(\sum_{P\in\mathbb{D}(t)} \int_P \fint_{P^*} |R_t\tilde{h}(y)|^2 dy\, \tilde{v}(x) dx\right)^{\frac{1}{2}}$$

$$\leq C\sum_{j=0}^{\infty}\left(\sum_{P\in\mathbb{D}(t)} \int_P \fint_{P^*} |R_t(\tilde{h}\mathbb{1}_{S_j(P)})(y)|^2 dy \tilde{v}(x) dx\right)^{\frac{1}{2}}$$

$$\leq C\sum_{j=0}^{\infty}\left(\sum_{P\in\mathbb{D}(t)} \int_P \frac{1}{|P^*|} 2^{-2j\upsilon} t^{-n(\frac{2}{r}-1)} \left(\int_{S_j(P)} |\tilde{h}(y)|^r dy\right)^{\frac{2}{r}} \tilde{v}(x) dx\right)^{\frac{1}{2}}. \quad (3.2.4)$$

In the last step, we have used that we may apply (3.0.5) to $(\theta_t 1)A_{m,t}$, since $A_{m,t}$ is a projection, with a compactly supported kernel. Thus, for $t \approx \ell(P)$, we have

$$\|(\theta_t 1)A_{m,t} f\|_{L^2(P^*)} = \|(\theta_t 1)A_{m,t}A_{m,t}(f 1_{5P^*})\|_{L^2(P^*)}$$

$$\lesssim \|A_{m,t}(f 1_{5P^*})\|_{L^2(\mathbb{R}^n)} \lesssim t^{-n(1/r-1/2)}\|f\|_{L^2(5P^*)},$$

for every $r \in [1,2]$, where in the first inequality we have used [AAAHK, Lemma 3.11]. In turn, the last expression in (3.2.4) is comparable to

$$\sum_{j=0}^{\infty}\left(\sum_{P\in\mathbb{D}(t)} \int_P 2^{-2j\upsilon} t^{-\frac{2n}{r}} \tilde{v}(x) \left(\int_{S_j(P)} |\tilde{h}(y)|^r dy\right)^{\frac{2}{r}} dx\right)^{\frac{1}{2}}$$

$$\approx \sum_{j=0}^{\infty}\left(\sum_{P\in\mathbb{D}(t)} \int_P 2^{-2j\epsilon} 2^{-j\frac{2}{r}n} t^{-\frac{2}{r}n} \tilde{v}(x) \left(\int_{S_j(P)} |\tilde{h}(y)|^r dy\right)^{\frac{2}{r}} dx\right)^{\frac{1}{2}}$$

$$\approx \sum_{j=0}^{\infty}\left(\sum_{P\in\mathbb{D}(t)} \int_P 2^{-2j\epsilon} \tilde{v}(x) \left(\fint_{S_j(P)} |\tilde{h}(y)|^r dy\right)^{\frac{2}{r}} dx\right)^{\frac{1}{2}}$$

$$\leq C\sum_{j=0}^{\infty}\left(\sum_{P\in\mathbb{D}(t)} 2^{-2j\epsilon} \int_P (\mathcal{M}(|\tilde{h}(x)|^r))^{\frac{2}{r}} \tilde{v}(x) dx\right)^{\frac{1}{2}}$$

$$\leq C\sum_{j=0}^{\infty} 2^{-j\epsilon}\left(\int_{\mathbb{R}^n} |\tilde{h}(x)|^2 \tilde{v}(x) dx\right)^{\frac{1}{2}} \leq C\left(\int_{\mathbb{R}^n} |\tilde{h}(x)|^2 \tilde{v}(x) dx\right)^{\frac{1}{2}}.$$

The proof of Claim 3.2.2 is now complete, and as noted above, (3.0.14) follows. We have therefore established that the hypotheses of Lemma 3.0.7 hold, given the conditions of Theorem 3.0.4, so our next step is to prove Lemma 3.0.7.

□

3.3 Conditions of Lemma 3.0.7 imply conditions of Sublemma 3.0.8

Define

$$g_Q(x) := \left(\int_0^{\ell(Q)} |\theta_t 1(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}}$$

$$G_Q(x) = \left(\iint_{|x-y|< \frac{\alpha}{100} t < \frac{\alpha}{100} \ell(Q)} |\theta_t 1(y)|^2 \frac{dtdy}{t^{n+1}} \right)^{\frac{1}{2}}$$

Our goal is to prove (3.0.12) which with this notation is equal to prove that $\sup_Q \frac{1}{|Q|} \int_Q \left(g_Q(x)\right)^2 dx < C$.

Claim 3.3.1. $\sup_Q \frac{1}{|Q|} \int_Q \left(g_Q(x)\right)^2 dx \approx \sup_Q \frac{1}{|Q|} \int_Q \left(G_Q(x)\right)^2 dx$

Proof. of Claim

$$\begin{aligned}
\sup_Q \frac{1}{|Q|} \int_Q \left(G_Q(x)\right)^2 dx &\leq C \sup_Q \frac{1}{|Q|} \int_Q \int_0^{\ell(Q)} \fint_{|x-y|< \frac{\alpha}{100} t} |\theta_t 1(y)|^2 dy \frac{dt}{t} dx \\
&\leq C \sup_Q \int_{kQ} \int_0^{\ell(Q)} |\theta_t 1(y)|^2 dy \frac{dt}{t} \\
&\leq Ck \sup_Q \int_Q g_Q^2(x) dx
\end{aligned}$$

$$\begin{aligned}
\int_Q g_Q^2(y) dy &= \int_Q \int_0^{\ell(Q)} |\theta_t 1(y)|^2 \frac{dt}{t} dy \\
&= \int_Q \int_0^{\ell(Q)} |\theta_t 1(y)|^2 \fint_{|x-y|< \frac{\alpha}{100} t} dx \frac{dt}{t} dy \\
&\leq C \int_{kQ} \int_0^{\ell(Q)} \int_{|x-y|< \frac{\alpha}{100} t} |\theta_t 1(y)|^2 \frac{dt}{t^{n+1}} dy dx \\
&\leq C \int_{kQ} \int_0^{k\ell(Q)} \int_{|x-y|< \frac{\alpha}{100} t} |\theta_t 1(y)|^2 \frac{dt}{t^{n+1}} dy dx
\end{aligned}$$

By taking supremum on both sides the proof is finished.

□

This claim allows us to work in a conical setting instead of the vertical setting.

Proof. For a large but fixed N (to be chosen momentarily) let

$$\Omega_N := \{x \in Q : G_Q(x) > N\}\,.$$

If the conditions of the lemma hold with $E = (1-\alpha)Q \setminus \bigcup_j Q_j$ we have

$$\begin{aligned}
|\Omega_N| &\le |Q \setminus (1-\alpha)Q| + \sum_j |Q_j| + |\{x \in E : G_Q(x) > N\}| \\
&\le C\alpha|Q| + (1-\eta)\,|Q| + \left|\{x \in E : \left(\iint_{|x-y|<\frac{\alpha}{100}t<\frac{\alpha}{100}\ell(Q)} |\theta_t 1(y)|^2 \frac{dtdy}{t^{n+1}}\right)^{\frac{p}{2}} > N^p\}\right| \\
&\le C\alpha|Q| + (1-\eta)\,|Q| + \frac{1}{N^p}\int_E \left(\iint_{|x-y|<\frac{\alpha}{100}t<\frac{\alpha}{100}\ell(Q)} |\theta_t 1(y)|^2 \frac{dtdy}{t^{n+1}}\right)^{\frac{p}{2}} dx \\
&\le C\alpha|Q| + (1-\eta)\,|Q| + \frac{C}{N^p}|Q| \le (1-\beta)\,|Q|\,,
\end{aligned}$$

for some $\beta > 0$, where we obtain the last estimate by choosing α sufficiently small, depending on η, and then N large enough, depending on α and η. □

3.4 Proof of Sublemma 3.0.8

Proof. Fix $\gamma \in (0,1)$ and let N, α, β be as in the hypothesis. For a dyadic cube Q set

$$G_{Q,\gamma}(x) := \left(\int_\gamma^{\min(\ell(Q),\frac{1}{\gamma})} \int_{|x-y|<\frac{\alpha}{100}t} |\theta_t 1(y)|^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}$$

(term to be 0 if $\ell(Q) < \gamma$), and

$$K(\gamma) = \sup_Q \frac{1}{|Q|}\int_Q \left(G_{Q,\gamma}(x)\right)^2 dx\,.$$

By the truncation, $K(\gamma)$ is finite and our goal is to show that $\sup\limits_{0\le\gamma\le 1} K(\gamma) < \infty$ since

$$\sup_Q \frac{1}{|Q|}\int_Q \int_0^{\ell(Q)} |\theta_t 1(x)|^2 dx \frac{dt}{t} \le C \sup_Q \frac{1}{|Q|}\int_Q |G_Q(x)|^2 dx \le C \sup_{0\le\gamma\le 1} K(\gamma)$$

Now we fix a cube Q and define

$\Omega_{N,\gamma} := \{x \in Q : G_{Q,\gamma}(x) > N\}$. This set is open so we can make a Whitney decomposition for it $\Omega_{N,\gamma} = \bigcup\limits_j Q_j$. We also define

$$F_{N,\gamma} := Q \setminus \Omega_{N,\gamma}.$$

$$\begin{aligned}
\int_Q \left(G_{Q,\gamma}(x)\right)^2 dx &\le \int_{F_{N,\gamma}} \left(G_{Q,\gamma}(x)\right)^2 + \sum_j \int_{Q_j} \left(G_{Q,\gamma}(x)\right)^2 dx \\
&\le N^2|Q| + \sum_j |Q_j| \frac{1}{|Q_j|}\int_{Q_j}\int_{\gamma}^{min(\ell(Q_j),\frac{1}{\gamma}))}\int_{|x-y|<\frac{\alpha}{100}t} |\theta_t 1(y)|^2 \frac{dydtdx}{t^{n+1}} \\
&\quad + \sum_j |Q_j| \frac{1}{|Q_j|}\int_{Q_j}\int_{max(\gamma,\ell(Q_j))}^{min(\ell(Q),\frac{1}{\gamma})}\int_{|x-y|<\frac{\alpha}{100}t} |\theta_t 1(y)|^2 \frac{dydtdx}{t^{n+1}} \\
&\le N^2\,|Q| + K(\gamma)(1-\beta)\,|Q| + III\,.
\end{aligned}$$

Claim 3.4.1.

$$L := \int_{Q_j}\int_{max(\gamma,\ell(Q_j))}^{min(\ell(Q),\frac{1}{\gamma})}\int_{|x-y|<\frac{\alpha}{100}t} |\theta_t 1(y)|^2 \frac{dydt}{t^{n+1}} dx < C|Q_j|. \tag{3.4.1}$$

Assuming the claim, we have

$$\begin{aligned}
K(\gamma) &\le N^2 + K(\gamma)(1-\beta) + C \\
&\Rightarrow K(\gamma) \le \frac{N^2 + C}{\beta} \text{ uniformly in } \gamma \Rightarrow \sup_{0<\gamma<1} K(\gamma) \le C \\
&\Rightarrow \int_Q \int_0^{\ell(Q)} \int_{|x-y|<\frac{\alpha}{100}t} |\theta_t 1(y)|^2 \frac{dydt}{t^{n+1}} dx \le C|Q| \\
&\Rightarrow \int_Q \int_0^{\ell(Q)} |\theta_t 1(x)|^2 \frac{dt}{t} dx \le C|Q|.
\end{aligned}$$

□

Proof. of Claim 3.4.1

Since we have done a Whitney decomposition, for every cube Q, there exists $x_j \in F_{N,\gamma}$ satisfying:

(a) $|x - x_j| \sim \ell(Q_j)\ \forall x \in Q_j$.

(b) $\exists C_n < \infty$ such that $\{y \in \mathbb{R}^n : |x - y| < C\ell(Q_j)\} \cap \{y \in \mathbb{R}^n : |x_j - y| < C_n\ell(Q_j)\} \neq \emptyset\ \forall x \in Q_j$.

(c) For $t < C_n\ell(Q_j)$, $\{y \in \mathbb{R}^n : |x - y| < t\} \subset \{y \in \mathbb{R}^n : |x_j - y| < t + C\ell(Q_j)$ and $|x_j - y| > t\}$.

Define

$$\rho := C_n\ell(Q_j).$$

$$\delta := \frac{q-2}{2},\ q\ from\ condition\ (3.0.6).$$

$$\Delta(x,t) :=\ cube\ of\ center\ x\ and\ radius\ t.$$

$$S_0(t) := \Delta(x_j, t + 2\rho) \setminus \Delta(x_j, t).$$

$$S_k(t) := \Delta(x_j, t + 2^{k+1}\rho) \setminus \Delta(x_j, t + 2^k\rho),\ for\ k \geq 1.$$

$$D_1 := \{y \in \mathbb{R}^n : |x - y| < t\} \subset \{y \in \mathbb{R}^n : |x_j - y| < t + C\ell(Q_j)\ and\ |x_j - y| > t\}.$$

$$D_2 := \Delta(x_j, t + C_n\ell(Q_j)) \setminus \Delta(x_j, t).$$

$$\begin{aligned}
L &:= \int_{Q_j} \int_{max(\gamma,\ell(Q_j))}^{min(\ell(Q),\frac{1}{\gamma})} \int_{|x-y|<\frac{\alpha}{100}t} |\theta_t 1(y)|^2 \frac{dydt}{t^{n+1}} dx \\
&\leq \int_{Q_j} \int_{max(\gamma,\ell(Q_j))}^{min(\ell(Q),\frac{1}{\gamma})} \int_{|x-y|<t} |\theta_t 1(y)|^2 \frac{dydt}{t^{n+1}} dx \\
&\leq \int_{Q_j} \int_{\ell(Q_j)}^{C_n\ell(Q_j)} \int_{D_1} |\theta_t \mathbb{1}_{2Q_j}(y)|^2 \frac{dydt}{t^{n+1}} dx + \sum_{k=0}^{\infty} \int_{Q_j} \int_{C_n\ell(Q_j)}^{\infty} \int_{|x-y|<t} |\theta_t \mathbb{1}_{S_k(t)}(y)|^2 \frac{dydt}{t^{n+1}} dx \\
&:= L_{-1} + \sum_{k=0}^{\infty} L_k.
\end{aligned}$$

From condition (3.0.2) $\int_{D_1} |\theta_t \mathbb{1}_{2Q_j}(y)|^2 \frac{dy}{t^{n+1}} \leq \frac{C\ell(Q_k)}{t^2}$ which implies that $L_{-1} \leq C|Q_j| \leq C|Q_j|$.

Take $1 > \epsilon = \frac{1}{2}(1 - \frac{n}{n+\delta}) > 0$ then for each $k \geq 0$ we are going to subdivide L_k them in two as follows

$$
\begin{aligned}
L_k &= \int_{Q_j} \int_{C_n \ell(Q_j)}^{2^{(1-\epsilon)k}\rho} \int_{|x-y|<t} |\theta_t(\mathbb{1}_{S_k(t)}(y))|^2 dy \frac{dt}{t^{n+1}} dx + \int_{Q_j} \int_{2^{(1-\epsilon)k}\rho}^{\infty} \int_{|x-y|<t} |\theta_t(\mathbb{1}_{S_k(t)})|^2 dy \frac{dt}{t^{n+1}} dx \\
&:= L_k^1 + L_k^2.
\end{aligned}
$$

<u>Case 1:</u>

$\exists l \in (0, (1-\epsilon)k)$, such that $2^l = \frac{t}{\rho}$, which implies that $|\Delta(x_j, 2t)| \approx |Q|$.

$$
\begin{aligned}
\int_{D_2} |\theta_t \mathbb{1}_{S_k(t)}(y)|^2 dy &\leq \int_{\Delta(x_j,2t)} |\theta_t \mathbb{1}_{S_k(t)}(y)|^2 dy \\
&\leq C|S_k(t)| 2^{-(n+2+\beta)(k-l)} \\
&\leq C 2^{-(n+2+\beta)k} \left(\frac{t}{\rho}\right)^{n+2+\beta} (2^k\rho)^n \\
&\leq C 2^{-(2+\beta)k} t^n \left(\frac{t}{\rho}\right)^{2+\beta}.
\end{aligned}
$$

$$
\begin{aligned}
\int_{C_n\ell(Q_j)}^{2^{(1-\epsilon)k}} \int_{|x-y|<t} |\theta_t(\mathbb{1}_{S_k(t)}(y))|^2 dy \frac{dt}{t^{n+1}} dx &\leq \int_0^{2^{(1-\epsilon)k}\rho} \int_{D_2} |\theta_t \mathbb{1}_{S_k(t)}(y)|^2 dy \frac{dt}{t^{n+1}} \\
&\leq C 2^{-(2+\beta)k} \int_0^{2^{(1-\epsilon)k}\rho} \left(\frac{t}{\rho}\right)^{2+\beta} \frac{dt}{t} \\
&= C 2^{-(2+\beta)k} \left(\frac{2^{(1-\epsilon)k}\rho}{\rho}\right)^{2+\beta} = C 2^{-\epsilon k(2+\beta)}.
\end{aligned}
$$

$$
\sum_k \int_{C_n\ell(Q_j)}^{2^{(1-\epsilon)k}} \int_{|x-y|<t} |\theta_t(\mathbb{1}_{S_k(t)}(y))|^2 dy \frac{dt}{t^{n+1}} dx \leq C \Rightarrow \sum_k L_k^1 \leq C|Q_j|.
$$

<u>Case 2</u>

By (3.0.6) $\theta_t : L^q \to L^2$ for $q > 2$,

$$\int_{D_2} |\theta_t(1_{S_k(t)})(y)|^2 dy \leq |D_2(t)|^\delta \left(\int_{\mathbb{R}^n} |\theta_t \mathbb{1}_{S_k(t)}|^q \right)^{\frac{2}{q}}$$

$$\leq \left(\rho t^{n-1}\right)^\delta |S_k(t)|^{\frac{2}{q}}$$

$$\leq \left(\frac{\rho}{t}\right)^\delta (t^n)^\delta max\left(t^n, (2^k\rho)^n\right)^{\frac{2}{q}}$$

$$\leq C \left(\frac{\rho}{t}\right)^\delta max\left(t^n, (2^k\rho)^n\right).$$

$$\int_{2^{(1-\epsilon)k}\rho}^{\infty} \int_{D_2} |\theta_t(1_{S_k(t)})(y)|^2 dy \leq C \int_{2^{(1-\epsilon)k}\rho}^{\infty} \left(\frac{\rho}{t}\right)^\delta max(t^n, (2^k\rho)^n) \frac{dt}{t^{n+1}}$$

$$\leq \int_{2^{(1-\epsilon)k}\rho}^{2^k\rho} \left(\frac{\rho}{t}\right)^\delta \frac{(2^k\rho)^n}{t^{n+1}} dt + \int_{2^k\rho}^{\infty} \left(\frac{\rho}{t}\right)^\delta \frac{dt}{t}$$

$$\leq \rho^\delta (2^k\rho)^n \int_{2^{(1-\epsilon)k}}^{2^k\rho} \frac{dt}{t^{n+1+\delta}} + 2^{-k\delta}$$

$$= \rho^\delta (2^k\rho)^n [2^{-k(n+\delta)} - 2^{-k(1-\epsilon)(n+\delta)}]\rho^{-n-\delta} + 2^{-k\delta}$$

$$= 2^{n-n-\delta} - 2^{k(n-n-\delta+\epsilon n+\epsilon\delta)} + 2^{-k\delta}$$

$$\leq 2^{-k\delta} + 2^{-k[(1-\epsilon)(n+\delta)-n]} + 2^{-k\delta}.$$

By our choice of ϵ one can easily see that $(1-\epsilon)(n+\delta) - n > 0$

$$\sum_k \int_{2^{(1-\epsilon)k}\rho}^{\infty} \int_{D_2} |\theta_t(\mathbb{1}_{S_k(t)})(y)|^2 dy < \sum_k 2^{-k\delta} + 2^{-k[(1-\epsilon)(n+\delta)-n]} + 2^{-k\delta} < C$$

$$\Rightarrow \sum_k L_k^2 \leq C|Q_j|.$$

□

Chapter 4

Theorem 3

Definition 4.0.1. We define for $f : \mathbb{R}^n \to \mathbb{C}^N$, $f \in L^2(\mathbb{R}^n)$, an $N \times N$ matrix valued operator $f \to \Theta_t \cdot f$ satisfying the following properties:

(a) (Uniform L^2 bounds and off-diagonal decay in L^2.)

$$\sup_{t>0} ||\Theta_t \cdot f||_{L^2(\mathbb{R}^n)} \leq C||f||_{L^2(\mathbb{R}^n)}, \tag{4.0.2}$$

$$||\Theta_t \cdot f_j||_{L^2(Q)} \leq C2^{-\frac{n+2+\beta}{2}j}||f_j||_{L^2(2^{j+1}Q\setminus 2^jQ)},\ \ell(Q) \leq t \leq 2\ell(Q), \tag{4.0.3}$$

for some $\beta > 0$, where $f_j := f\mathbb{1}_{2^{j+1}Q\setminus 2^jQ}$.

(b) (Quasi-orthogonality in L^2.)There exists $\beta > 0$ and H a subspace of $L^2(\mathbb{R}^n)$ such that

$$||\Theta_t \cdot Q_s h||_{L^2(\mathbb{R}^n)} \leq C\left(\frac{s}{t}\right)^{\beta}||h||_{L^2(\mathbb{R}^n)},\ where\ h \in H, \tag{4.0.4}$$

where $\{Q_s\}_{0\leq s\leq\infty}$ is some family of operators with $\int_{\mathbb{R}^n}\int_0^\infty |Q_s f(x)|^2 \frac{dsdx}{s} \leq C||f||^2_{L^2(\mathbb{R}^n)}$, $||\nabla Q_s f||_{L^2(\mathbb{R}^n)} \leq \frac{1}{s}||f||_{L^2(\mathbb{R}^n)}$, $\int_0^\infty Q_s^2 \frac{ds}{s} = I$.

(c) ("Hypercontractive" off-diagonal decay.) There is some $1 < r < 2$, and some $\nu > \frac{n}{r}$ ($\nu = \frac{n}{r} + \epsilon, \epsilon > 0$), such that

$$\left(\int_{Q^*} |\Theta_t \cdot (f\mathbb{1}_{S_j}(Q))(y)|^2 dy\right)^{\frac{1}{2}} \leq C2^{-j\nu}t^{-n(\frac{1}{r}-\frac{1}{2})}\left(\int_{S_j(Q)} |f(y)|^r dy\right)^{\frac{1}{r}}, \qquad \forall j \geq 0\,,\ \ell(Q) < t \leq 2\ell(Q)\,, \tag{4.0.5}$$

where $S_0(Q) = 16Q$ and $S_j(Q) = 2^{j+4}Q \setminus 2^{j+3}Q$, $j \geq 1$. $Q^* \equiv 8Q$.

(d) (Improved integrability)

$$\sup_{t>0} ||\Theta_t \cdot f||_{L^q(\mathbb{R}^n)} \leq C||f||_{L^q(\mathbb{R}^n)}, \quad for\ some\ q > 2. \tag{4.0.6}$$

Theorem 4.0.2. *Let's define the square function operator* Θ *as above and suppose that there exists some constants* $0 < C_0 < \infty$, $0 < C_1 < 1$ *and exponent* $p > r$, $\delta > 0$ *and a system* $\{b_Q\}$ *of functions indexed by cubes* $Q \subset \mathbb{R}^n$, *and a system of Lipschitz functions* $\{\Phi_Q\}$ *also indexed by cubes, such that for each cube Q:*

$$\int_{\mathbb{R}^n} |b_Q(x)|^p dx \leq C_0|Q|, \tag{4.0.7}$$

$$\int_Q \left(\int_0^{\ell(Q)} \int_{|x-y|<t} |\Theta_t \cdot b_Q(y)|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{p}{2}} dx \leq C_0|Q|, \tag{4.0.8}$$

$$||\nabla \Phi_Q||_\infty \leq C_0\, \ell(Q)^{-1},\ C_1 \leq \Phi_Q(x) \leq 1 \text{ on } Q, \tag{4.0.9}$$

$$\delta|\xi|^2 m_Q(Q) \leq Re \left(\xi \cdot \int_Q b_Q(x) dm_Q(x) \cdot \bar{\xi} \right),\ \forall \xi \in \mathbb{C}^n \text{ where } dm_Q(x) = \Phi_Q(x)dx, \tag{4.0.10}$$

where the action of Θ_t *on the matrix valued function* b_Q *is defined in the obvious way as in definition 4.0.1.*

Then

$$\iint_{\mathbb{R}^{n+1}_+} |\Theta_t \cdot f(x)|^2 \frac{dxdt}{t} \leq C||f||^2_{L^2(\mathbb{R}^n)}, \forall f \in H. \tag{4.0.11}$$

Remark 4.0.3. In Theorem 4.0.2, H may be all of $L^2(\mathbb{R}^n)$, or it may be a proper subspace. For example, the case that H is the space of L^2 gradient fields, arises naturally in some applications (as in the case of the Kato "square root" problem for divergence form elliptic operators).

Theorem 4.0.4. *(T1 Theorem)*

If $\Theta_t f(x)$ satisfies conditions (4.0.2), (4.0.3) *and* (4.0.4) *as above and the Carleson measure estimate*

$$\sup_Q \frac{1}{|Q|} \int_Q \int_0^{\ell(Q)} |\Theta_t 1(x)|^2 \frac{dxdt}{t} \leq C. \tag{4.0.12}$$

Then we have the square function estimate

$$\iint_{\mathbb{R}^{n+1}_+} |\Theta_t f(x)|^2 \frac{dxdt}{t} \leq C\|f\|^2_{L^2(\mathbb{R}^n)}, \forall f \in H. \tag{4.0.13}$$

Remark 4.0.5. Here, the constant function 1 should be interpreted in the matrix-valued sense, i.e., as the $N \times N$ identity matrix.

Remark 4.0.6. The proof of this T1 Theorem is the same as the proof of the T1 Theorem in the previous section since same properties apply and we can replicate the proof the lemma 3.0.7 by restricting it to the subspace H.

Lemma 4.0.7. *Suppose that there exists $\eta \in (0,1)$, $\alpha \in (0, \frac{1}{2}]$, $\epsilon > 0$ small and $C < \infty$ such that for every cube $Q \in \mathbb{R}^n$, there is a family $\{\tilde{Q}_j\}_j$ of non-overlapping sub-cubes of Q, dyadic with respect to the grid induced by Q, with*

$$\sum_j (1+\alpha)^n |\tilde{Q}_j| \leq (1-\eta)\,|Q| \tag{4.0.14}$$

and

$$\int_E \left(\int_0^{\ell(Q)} \int_{|x-y|<\frac{\alpha}{100}t} |\Theta_t 1(y)|^2 \mathbb{1}_{\Gamma_k^{2\epsilon}}(\Theta_t 1(y)) \frac{dydt}{t^{n+1}} \right)^{\frac{p}{2}} dx \leq C|Q| \tag{4.0.15}$$

where $E := (1-\alpha)Q \setminus \{\bigcup_j Q_j\}$ for every cone of aperture 2ϵ, and $Q_j := (1+\alpha)\tilde{Q}_j$.

Then

$$\sup_Q \frac{1}{|Q|} \int_0^{\ell(Q)} \int_Q |\Theta_t 1(x)|^2 \frac{dxdt}{t} \leq C$$

Remark 4.0.8. ϵ small but fixed, we cover $\mathbb{C}^N$ by cones of aperture ϵ. The constant then depend on $k = k(\epsilon, N)$ the number of cones.

Sublemma 4.0.9. *Suppose that* $\exists N < +\infty, \alpha \in (0, \frac{1}{2}]$ *and* $\beta \in (0, 1)$ *such that for every cube Q and for all* $\Gamma^{2\epsilon}$

$$|\{x \in Q : G_Q^j(x) > N\}| \leq (1 - \beta)\,|Q| \tag{4.0.16}$$

for all k, where

$$G_Q(x) = \left(\iint_{|x-y|<\frac{\alpha}{100}t<\frac{\alpha}{100}\ell(Q)} |\Theta_t 1(y)|^2 \mathbb{1}_{\Gamma^{2\epsilon}}(\Theta_t 1(y)) \frac{dtdy}{t^{n+1}} \right)^{\frac{1}{2}} \tag{4.0.17}$$

Then

$$\sup_Q \frac{1}{|Q|} \int_Q \left(G_Q(x)\right)^2 dx \leq C. \tag{4.0.18}$$

Remark 4.0.10. In the previous section we have seen that (4.0.18) implies (4.0.12).

4.1 Conditions of Theorem 4.0.2 imply conditions of the Lemma 4.0.7

Proof. Let $\upsilon \in \mathbb{C}^N$ be the unit normal in the direction of the central axis of a cone of aperture 2ϵ $\Gamma^{2\epsilon} = \{z \in \mathbb{C}^N : |\frac{z}{|z|} - \upsilon| < 2\epsilon\}$ and let $A_{m,t} f(x) := \left(m_Q(x,t)\right)^{-1} \int_{Q(x,t)} f(y) dm_Q(y)$ with $Q(x,t)$ = minimal dyadic cube containing x with side length at least t.

Let's first construct such a family using a stopping time argument as we did in Section 2, Theorem 1, such that the first condition of the lemma is satisfied as well as $|\Theta_t 1(x)|^2 \leq 4|\Theta_t 1(x) A_{m,t} b_Q(x) \bar{\upsilon}|^2$.

Without loss of generality (by renormalizing), assume $\delta \equiv 1$ on (4.0.10). Fix a cube Q, and then fix a cone $\Gamma^{2\epsilon}$. Now we subdivide Q dyadically and select a family $\{\tilde{Q}_j\}$, $\tilde{Q}_j \subset Q$ which are maximal with respect to the condition that at least one of the following conditions hold:

$$\frac{1}{m_Q(\tilde{Q}_j)} \int_{\tilde{Q}_j} |b_Q(x)| dm_Q(x) \geq \frac{C_1}{8\epsilon} \quad (\textit{type I})$$

$$Re\left(\nu \cdot \int_{\tilde{Q}_j} b_Q(x)dm_Q(x)\cdot \bar{\nu}\right) \le \frac{3C_1}{4} m_Q(\tilde{Q}_j) \ \ (type\ II)$$

where ν is the unit normal vector in the direction of the central axis of the cone $\Gamma^{2\epsilon}$ and ϵ is half the aperture of the cone.

Define $\tilde{E} := Q \setminus \{\bigcup_j \tilde{Q}_j\}$. Then from condition (4.0.10) since $\delta \equiv 1$ and taking $\xi = \nu$ where ν is the unit normal in the direction of the central axis of $\Gamma^{2\epsilon}$ we get

$$\begin{aligned}
|Q| \le \frac{1}{C_1} m_Q(Q) \le & \mathcal{R}e \sum_{i,j} \int_Q (b_Q)_{ij}(x)\nu_j \bar{\nu}_i dm_Q(x) \\
& = \frac{1}{C_1}\left(\mathcal{R}e \sum_{i,j} \int_{\tilde{E}} (b_Q)_{ij}(x)\nu_j \bar{\nu}_i dm_Q(x) + \mathcal{R}e \sum_k \sum_{i,j} \int_{\tilde{Q}_k} (b_Q)_{ij}(x)\nu_j \bar{\nu}_i dm_Q(x)\right) \\
& := \frac{1}{C_1}(I + II).
\end{aligned}$$

For the first part using condition (4.0.7) and Hölder we get

$$\begin{aligned}
I := & \mathcal{R}e \sum_{i,j} \int_{\tilde{E}} (b_Q)_{ij}(x)\nu_j \bar{\nu}_i dm_Q(x) \\
& \le |\nu \cdot \int_{\tilde{E}} b_Q(x)dm_Q(x) \cdot \bar{\nu}| \\
& = \left| \int_{\tilde{E}} b_Q(x)dm_Q(x) \right| \\
& \le \left| m_Q(\tilde{E}) \right|^{\frac{1}{p'}} \left(\int_Q |b_Q(x)|^p dm_Q(x) \right)^{\frac{1}{p}} \\
& \le \left| \tilde{E} \right|^{\frac{1}{p'}} \left(\int_Q |b_Q(x)|^p dx \right)^{\frac{1}{p}} \\
& \le C|\tilde{E}|^{\frac{1}{p'}} |Q|^{\frac{1}{p}}.
\end{aligned}$$

For the second part we are working with the family of subcubes and to be able to use their properties we need to separate them in two cases: the ones that satisfy the type I condition and the ones that satisfy the type II condition (the same subcube can satisfy both conditions at the same time; in this case we arbitrarily assign them to be of type I).

$$II \leq \left| \mathcal{R}e \sum_{k, type\ I} \sum_{i,j} \int_{\tilde{Q}_k} (b_Q)_{ij}(x) v_j \bar{v}_i dm_Q(x) \right| + \left| \mathcal{R}e \sum_{k, type\ II} \sum_{i,j} \int_{\tilde{Q}_k} (b_Q)_{ij}(x) v_j \bar{v}_i dm_Q(x) \right| := II_1 + II_2.$$

For the part of type I subcubes we are going to apply Hölder, the property of being type I, and condition (4.0.7).

$$\begin{aligned} II_1 &:= \left| \mathcal{R}e \sum_{k, typeI} \sum_{i,j} \int_{\tilde{Q}_k} (b_Q)_{ij}(x) v_j \bar{v}_i dm_Q(x) \right| \\ &\leq \sum_{k, typeI} \int_{\tilde{Q}_k} |b_Q(x)| dm_Q(x) \\ &= \int_{\bigcup_{k, typeI} \tilde{Q}_k} |b_Q(x)| dm_Q(x) \\ &\leq \left(\int_Q |b_Q(x)|^p dm_Q(x) \right)^{\frac{1}{p}} \left(m_Q \left(\bigcup_{k, typeI} \tilde{Q}_k \right) \right)^{\frac{1}{p'}}. \end{aligned}$$

For the measure of the set just note that

$$m_Q\left(\bigcup_{k,typeI}\tilde{Q}_k\right) \leq \frac{8\epsilon}{C_1}\int_{\bigcup_{k,typeI}\tilde{Q}_k} |b_Q(x)| dm_Q(x)$$

$$\leq \frac{8\epsilon}{C_1}\left(m_Q\left(\bigcup_{k,typeI}\tilde{Q}_k\right)\right)^{\frac{1}{p'}}\left(\int_Q |b_Q(x)|^p dm_Q(x)\right)^{\frac{1}{p}}$$

$$\left(m_Q\left(\bigcup_{k,typeI}\tilde{Q}_k\right)\right)^{\frac{1}{p}} \leq \frac{8\epsilon}{C_1}\left(\int_Q |b_Q(x)|^p dx\right)^{\frac{1}{p}} \leq C\epsilon|Q|^{\frac{1}{p}}$$

$$\left(m_Q\left(\bigcup_{k,typeI}\tilde{Q}_k\right)\right)^{\frac{1}{p'}} \leq C\left(\frac{\epsilon}{C_1}\right)^{\frac{p}{p'}}|Q|^{\frac{1}{p'}}.$$

Adding this to the previous computation we get

$$II_1 \leq \left|\mathcal{R}e\sum_{k,type\ I}\sum_{i,j}\int_{\tilde{Q}_k}(b_Q)_{ij}(x)\nu_j\bar{\nu}_i dm_Q(x)\right| \leq \left(\int_Q |b_Q(x)|^p dm_Q(x)\right)^{\frac{1}{p}}\left(m_Q\left(\bigcup_{k,typeI}\tilde{Q}_k\right)\right)^{\frac{1}{p'}}$$
$$\leq C\left(\frac{\epsilon}{C_1}\right)^{\frac{p}{p'}}|Q|^{\frac{1}{p'}}|Q|^{\frac{1}{p}} = C\left(\frac{\epsilon}{C_1}\right)^{\frac{p}{p'}}|Q|.$$

We choose ϵ small enough so

$$\frac{C\epsilon^{\frac{p}{p'}}}{C_1^p} \leq \frac{1}{8}. \tag{4.1.1}$$

For the type II subcubes just using the property of being type II and $0 < C_1 \leq 1$ we get

$$II_2 := \left|\mathcal{R}e\sum_{k,typeII}\sum_{i,j}\int_{\tilde{Q}_k}(b_Q)_{ij}(x)\nu_j\bar{\nu}_i dm_Q(x)\right| \leq \tfrac{3}{4C_1}m_Q\left(\bigcup_{k,typeII}\tilde{Q}_k\right) \leq \tfrac{3C_1}{4}|Q| \leq \tfrac{3}{4}|Q|.$$

Finally we can conclude that

$$
\begin{aligned}
|Q| &\leq \frac{1}{C_1}(I + II) \leq C|\tilde{E}|^{\frac{1}{p'}}|Q|^{\frac{1}{p}} + \frac{1}{8}|Q| + \frac{3}{4}|Q| \\
&\leq 8C|\tilde{E}|^{\frac{1}{p'}}|Q|^{\frac{1}{p}} \\
&\leq C|\tilde{E}| \\
&\Rightarrow |\tilde{E}| > \tilde{\eta}|Q|, \ with\ 0 < \tilde{\eta} = \frac{1}{C} < 1
\end{aligned}
$$

Therefore,

$$\sum_j |\tilde{Q}_j| \leq (1 - \tilde{\eta})|Q|.$$

Thus,

$$\sum_j |\tilde{Q}_j| \leq \sum_j (1+\alpha)^n |\tilde{Q}_j| \leq (1+\alpha)^n (1-\tilde{\eta})|Q| \leq (1-\eta)|Q|,$$

provided that we choose η and α sufficiently small.

We define $Q_j := (1+\alpha)\tilde{Q}_j$ and $E := (1-\alpha)Q \setminus \bigcup_j Q_j$.

Then as in the previous section we can choose $\alpha \in (0, \frac{1}{2}]$ and $\eta \in (0,1)$ such that

$$\sum_j |Q_j| \leq \sum_j (1+\alpha)^n |\tilde{Q}_j| \leq (1+\alpha)^n (1-\bar{\eta}) \left|\frac{1}{2}Q\right| \leq (1-\eta)\left|\frac{1}{2}Q\right|.$$

This concludes that the family that we have constructed satisfy the measure condition. Now let's proceed to verify the condition (4.0.15).

Claim 4.1.1. If $x \in E$, $|x-y| < \frac{\alpha}{100}t$ and $0 < t < \ell(Q)$ then

$$|\Theta_t 1(y)| \mathbb{1}_{\Gamma^{2\epsilon}}(\Theta_t 1(y)) \leq C|\Theta_t 1(y) \cdot A_{m,t} b_Q(y) \bar{\nu}| \tag{4.1.2}$$

where $\nu \in \mathbb{C}^n$ is the unit normal vector in the direction of the central axis of $\Gamma^{2\epsilon} := \{z \in \mathbb{C}^N : |\frac{z}{|z|} - \nu| < 2\epsilon\}$, and $A_{m,t} f(y) := (m_Q(y,t))^{-1} \int_{Q(y,t)} f(w) dm_Q(w)$ with $Q(y,t)$ the minimal dyadic cube containing y with side length at least t.

Proof. of claim 4.1.1

Let's introduce some notation first

$(y,t) \in E^*_Q \equiv R_Q \setminus (\bigcup_j R_{Q_j})$ where $R_Q \equiv Q \times (0, \ell(Q))$ and $\Gamma^{2\epsilon} = \{z \in \mathbb{C}^N : |\frac{z}{|z|} - \nu| < 2\epsilon\}$.

We are going to prove that if $z \in \Gamma^{2\epsilon}$ and $(y,t) \in E^*_Q$ then $|z \cdot A_{m,t} b_Q(y)\bar{\nu}| \geq \frac{C_1^2}{2}|z|$ since, as in the previous section, if $x \in E$, $|x-y| < \frac{\alpha}{100}t$ and $0 < t < \ell(Q)$ then $(y,t) \in E^*_Q$.

Since $(y,t) \in E^*_Q$ we have that $Q(y,t)$ is not of type I neither type II, therefore by the triangle inequality

$|\omega \cdot A_{m,t} b_Q(y)\bar{\nu}| \geq |\nu \cdot A_{m,t} b_Q(y)\bar{\nu}| - |(\omega - \nu) A_{m,t} b_Q(y)\bar{\nu}| \geq \frac{3C_1^2}{4} - |(\omega-\nu)|\frac{C_1^2}{8\epsilon}\ \forall \omega \in \mathbb{C}^N$.

If we choose $\omega = \frac{z}{|z|}$ then $|\omega - \nu| < 2\epsilon$ so we get that $|\frac{z}{|z|} \cdot A_{m,t} b_Q(y)\bar{\nu}| \geq \frac{3C_1^2}{4} - \frac{C_1^2 2\epsilon}{8\epsilon} = \frac{C_1^2}{2}$

$\Rightarrow |z \cdot A_t b_Q(y)\bar{\nu}| \geq \frac{C_1^2}{2}|z|$.

Since we are integrating when $\Theta_t 1(y) \in \Gamma^{2\epsilon}$ and $(y,t) \in E^*_Q$ then $|\Theta_t 1(y)|^2 \leq \frac{4}{C_1^4}|\Theta_t 1(y) \cdot A_{m,t} b_Q(y)\bar{\nu}|^2$ in our domain of integration and the claim is true.

□

Let's prove now the second condition of the lemma (4.0.15)

$$\int_E \left(\int_0^{\ell(Q)} \int_{|x-y|<\frac{\alpha}{100}t} |\Theta_t 1(y)|^2 \mathbb{1}_{\Gamma^{2\epsilon}}(\Theta_t 1(y)) \frac{dydt}{t^{n+1}} \right)^{\frac{p}{2}} dx$$
$$\leq C \int_E \left(\int_0^{\ell(Q)} \int_{|x-y|<\frac{\alpha}{100}t} |\Theta_t 1(y) \cdot A_{m,t} b_Q(y)\bar{\upsilon}|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{p}{2}} dx$$
$$\leq C(p) \int_Q \left(\int_0^{\ell(Q)} \int_{|x-y|<t} |\Theta_t 1(y) \cdot A_{m,t} b_Q(y)\bar{\upsilon}|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{p}{2}} dx.$$

Then the proof goes as in section 3. □

4.2 Conditions lemma 4.0.7 imply conditions sublemma 4.0.9

Proof. For a large, but fixed N (to be chosen momentarily) let $\Omega_N := \{x \in Q : G_Q(x) > N\}$.

If conditions of lemma hold with $E = Q \setminus \bigcup_j Q_j$.

We have

$$
\begin{aligned}
|\Omega_N| &\le |Q \setminus (1-\alpha)Q| + \sum_j |Q_j| + |\{x \in E : G_Q(x) > N\}| \\
&\le C\alpha|Q| + (1-\eta)\,|Q| + \left|\{x \in E : \left(\iint_{|x-y|<\frac{\alpha}{100}t<\ell(Q)} |\Theta_t 1(y)|^2 \mathbb{1}_{\Gamma^{2\epsilon}}\left(\Theta_t 1(y)\right) \frac{dtdy}{t^{n+1}} \right)^{\frac{p}{2}} > N^p\}\right| \\
&\le C\alpha|Q| + (1-\eta)\,|Q| + \frac{1}{N^p} \int_E \left(\iint_{|x-y|<\frac{\alpha}{100}t<\ell(Q)} |\Theta_t 1(y)|^2 \mathbb{1}_{\Gamma^{2\epsilon}}\left(\Theta_t 1(y)\right) \frac{dtdy}{t^{n+1}} \right)^{\frac{p}{2}} dx \\
&\le \left[C\alpha + (1-\eta) + \frac{C}{N^p} \right] |Q|
\end{aligned}
$$

for some $\beta > 0$, where we obtain the last estimate by choosing α sufficiently small, depending on η, and then N large enough, depending on α and η.

□

4.3 Proof of Sublemma 4.0.9

Proof. Fix $\gamma \in (0,1)$ and let N, β be as in the hypothesis. For a cube Q set

$$
G_{Q,\gamma}(x) := \left(\int_{\gamma}^{min(\ell(Q),\frac{1}{\gamma})} \int_{|x-y|<\frac{\alpha}{100}t} |\Theta_t(y)|^2 \chi_{(\epsilon,\gamma)}(\Theta_t 1(y)) \frac{dtdy}{t^{n+1}} \right)^{\frac{1}{2}}
$$

term to be 0 if $\ell(Q) < \gamma$ and where

$$
\chi_{(\epsilon,\gamma)}(y) = \begin{cases} 1 & \text{if } \mathbb{1}_{\Gamma^{\frac{3}{2}\epsilon}}(\Theta_t 1)(y) = 1 \\ 0 & \text{if } \mathbb{1}_{\Gamma^{2\epsilon}}(\Theta_t 1)(y) = 0 \\ (0,1) & \text{otherwise} \end{cases}
$$

$$
K(\gamma) := \sup_Q \frac{1}{|Q|} \int_Q \int_{\gamma}^{min(\ell(Q),\frac{1}{\gamma})} \int_{|x-y|<\frac{\alpha}{100}t} |\Theta_t 1(y)|^2 \mathbb{1}_{\Gamma^{\epsilon}}(\Theta_t 1(y)) \frac{dydt}{t^{n+1}} dx.
$$

By the truncation $K(\gamma)$ is finite, and our goal is to show that $\sup_{0<\gamma<1} K(\gamma) < C < \infty$ since

$$
\sup_Q \frac{1}{|Q|} \int_Q \int_0^{\ell(Q)} \int_{|x-y|<\frac{\alpha}{100}t} |\Theta_t 1(y)|^2 \mathbb{1}_{\Gamma^{\epsilon}}(\Theta_t 1(y)) \frac{dydt}{t^{n+1}} dx \le \sup_{0<\gamma<1} K(\gamma).
$$

Fix a cube Q and define $\Omega_{N,\gamma} := \{x \in Q : G_\gamma(x) > N\}$. This set is open so we can make a Whitney decomposition for it $\Omega_{N,\gamma} = \bigcup_j Q_j =$. We also define

$$F_{N,\gamma} := Q \setminus \Omega_{N,\gamma}.$$

$$\begin{aligned}
\int_Q \int_\gamma^{min(\ell(Q),\frac{1}{\gamma})} \int_{|x-y|<\frac{\alpha}{100}t} |\Theta_t 1(y)|^2 \mathbb{1}_{\Gamma^\epsilon}(\Theta_t 1(y)) \frac{dydt}{t^{n+1}} \leq & \int_{F_{N,\gamma}} \int_\gamma^{min(\ell(Q),\frac{1}{\gamma})} \int_{|x-y|<\frac{\alpha}{100}t} |\Theta_t 1(y)|^2 \mathbb{1}_{\Gamma^\epsilon}(\Theta_t 1(y)) \frac{dydt}{t^{n+1}} dx \\
& + \sum_j \frac{|Q_j|}{|Q_j|} \int_{Q_j} \int_\gamma^{min(\ell(Q_j),\frac{1}{\gamma})} \int_{|x-y|<\frac{\alpha}{100}t} |\Theta_t 1(y)|^2 \mathbb{1}_{\Gamma^\epsilon}(\Theta_t 1(y)) \frac{dydt}{t^{n+1}} dx \\
& + \sum_j \frac{|Q_j|}{|Q_j|} \int_{Q_j} \int_{max(\gamma,\ell(Q_j))}^{min(\ell(Q),\frac{1}{\gamma})} \int_{|x-y|<\frac{\alpha}{100}t} |\Theta_t 1(y)|^2 \mathbb{1}_{\Gamma^\epsilon}(\Theta_t 1(y)) \frac{dydt}{t^{n+1}} dx \\
& \leq N^2|Q| + K(\gamma)(1-\beta)\,|Q| \\
& + \sum_j \int_{Q_j} \int_{max(\gamma,\ell(Q_j))}^{min(\ell(Q),\frac{1}{\gamma})} \int_{|x-y|<\frac{\alpha}{100}t} |\Theta_t 1(y)|^2 \mathbb{1}_{\Gamma^\epsilon}(\Theta_t 1(y)) \frac{dydt}{t^{n+1}} dx.
\end{aligned}$$

Claim 4.3.1.

$$\int_{Q_j} \int_{max(\gamma,\ell(Q_j))}^{min(\ell(Q),\frac{1}{\gamma})} \int_{|x-y|<\frac{\alpha}{100}t} |\Theta_t 1(y)|^2 \mathbb{1}_{\Gamma^\epsilon}(\Theta_t 1(y)) \frac{dydt}{t^{n+1}} dx < C|Q_j|. \tag{4.3.1}$$

Assuming the claim, we have

$$\begin{aligned}
K(\gamma) \leq N^2 + & K(\gamma)(1-\beta) + C \\
& \Rightarrow K(\gamma) \leq \frac{N^2 + C}{\beta} \; uniformly \; in \; \gamma \Rightarrow \sup_{0<\gamma<1} K(\gamma) \leq C \\
& \Rightarrow \int_Q \int_0^{\ell(Q)} \int_{|x-y|<\frac{\alpha}{100}t} |\Theta_t 1(y)|^2 \mathbb{1}_{\Gamma^\epsilon}(\Theta_t 1(y)) \frac{dydt}{t^{n+1}} dx \leq C|Q| \\
& \Rightarrow \int_Q \int_0^{\ell(Q)} |\Theta_t 1(x)|^2 \mathbb{1}_{\Gamma^\epsilon}(\Theta_t 1(x)) \frac{dt}{t} dx \leq C|Q| \\
& \Rightarrow \int_Q \int_0^{\ell(Q)} |\Theta_t 1(x)|^2 \frac{dt}{t} dx \leq C|Q|.
\end{aligned}$$

□

Proof. of Claim 4.3.1

Take $x_j \in F_{N,\gamma}$,

$|x - x_j| \sim \ell(Q_j)\ \forall x \in Q_j \Rightarrow \exists C_n < \infty$ such that $\{y \in \mathbb{R}^n : |x-y| < C\ell(Q_j)\} \cap \{y \in \mathbb{R}^n : |x_j - y| < C_n\ell(Q_j)\} \neq \emptyset$

$\forall x \in Q_j$.

For $t < C_n\ell(Q_j)$,

$\{y \in \mathbb{R}^n : |x-y| < t\} \subset \{y \in \mathbb{R}^n : |x_j - y| < t + C\ell(Q_j)$ and $|x_j - y| > t\} \equiv D_1$.

$$\begin{aligned}
\int_{Q_j}\int_{max(\gamma,\ell(Q_j))}^{min(\ell(Q),\frac{1}{\gamma})}\int_{|x-y|<\frac{\alpha}{100}t} |\Theta_t 1(y)|^2 \mathbb{1}_{\Gamma^\epsilon}(\Theta_t 1(y))\frac{dydt}{t^{n+1}}dx &\leq \int_{Q_j}\int_{max(\gamma,\ell(Q_j))}^{min(\ell(Q),\frac{1}{\gamma})}\int_{|x-y|<t} |\Theta_t 1(y)|^2 \mathbb{1}_{\Gamma^\epsilon}(\Theta_t 1(y))\frac{dydt}{t^{n+1}}dx \\
&\leq \int_{Q_j}\int_{\ell(Q_j)}^{C_n\ell(Q_j)}\int_{D_1} |\Theta_t 1(y)|^2 \mathbb{1}_{\Gamma^\epsilon}(\Theta_t 1(y))\frac{dydt}{t^{n+1}}dx \\
&+ \int_{Q_j}\int_{C_n\ell(Q_j)}^{\infty}\int_{|x-y|<t} |\Theta_t 1(y)|^2 \mathbb{1}_{\Gamma^\epsilon}(\Theta_t 1(y))\frac{dydt}{t^{n+1}}dx \\
&:= I + II.
\end{aligned}$$

[I]

$$\sup_{t>0}\|\Theta_t f\|_2 \leq C\|f\|_2 \Rightarrow \int_{D_1} |\Theta_t \mathbb{1}_{2Q_j}(y)|^2 \mathbb{1}_{\Gamma^{2\epsilon}}(\Theta_t \mathbb{1}_{2Q_j}(y))\frac{dy}{t^{n+1}} \leq \frac{C\ell(Q_k)}{t^2}$$

$$\Rightarrow I \leq C|Q_j|.$$

[II]Define $S_k(t) = \Delta(x_j, t+2^{k+1}\rho) \setminus \Delta(x_j, t+2^k\rho)$, where $\rho = C_n\ell(Q_j)$, for $k \geq 1$, $S_0(t) = \Delta(x_j, t+2\rho) \setminus \Delta(x_j, t)$.

Take $1 > \epsilon = \frac{1}{2}(1 - \frac{n}{n+\delta}) > 0$.

Then

$$\begin{aligned}
II_1 &:= \sum_k \int_{Q_j}\int_{2^{(1-\epsilon)k}\rho}^{\infty}\int_{|x-y|<t} |\Theta_t(\mathbb{1}_{S_k(t)}(y))|^2 dy\frac{dt}{t^{n+1}}dx, \\
II_2 &:= \sum_k \int_{Q_j}\int_{C_n\ell(Q_j)}^{2^{(1-\epsilon)k}}\int_{|x-y|<t} |\Theta_t(\mathbb{1}_{S_k(t)}(y))|^2 dy\frac{dt}{t^{n+1}}dx,
\end{aligned}$$

$D_2 = \Delta(x_j, t + C\ell(Q_j)) \setminus \Delta(x_j, t)$ and $II \leq II_1 + II_2$.

From here the proof goes exactly as the one in Section 3. □

Chapter 5

Application 1

Let's consider

$$L\vec{u} := -D_\alpha \cdot (A_{\alpha\beta} D_\beta \vec{u}) \tag{5.0.1}$$

is defined on $\mathbb{R}^{n+1} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R}\}, n \geq 2$, $\vec{u}$ are N-dimensional vector valued functions,where $D_\alpha = \frac{\partial}{\partial x_\alpha}$ is the partial derivative with respect the variable x_α, $1 \leq \alpha \leq n+1$, and where $A_{\alpha\beta} = A_{\alpha\beta}(x)$, $1 \leq \alpha, \beta \leq n+1$, are $N \times N$ matrices of L^∞ complex-valued coefficients, defined on $\mathbb{R}^n$ (i.e. independent of the t variable) and satisfying the uniform ellipticity condition

$$\lambda \sum_{i=1}^{N} \sum_{\alpha=1}^{n+1} |\xi_\alpha^i|^2 \leq A_{\alpha\beta}^{ij} \xi_\beta^j \bar{\xi}_\alpha^i, \ ||A||_{L^\infty(\mathbb{R}^n)} \leq \Lambda \tag{5.0.2}$$

for some $\lambda > 0$, $\Lambda < \infty$, and for all $\xi \in \mathbb{C}^N$, $x \in \mathbb{R}^n$ (the divergence form operator L is interpreted in the weak sense via a sesquilinear form).

Let's note that $\nabla\vec{u}$ is a $(n+1)$-dimensional vector of N dimensional vectors (i.e. an $(n+1) \times N$ matrix).

We say that $L\vec{u} = 0$ in a domain Ω if $\vec{u} \in W^{1,2}_{loc}(\Omega)$ and

$$\int_\Omega A(X)\nabla\vec{u}(X) \cdot \overline{\nabla\vec{\psi}}(X)dX := \int_\Omega A^{ij}_{\alpha\beta} D_\beta u_j \overline{D_\alpha \psi_i} = 0\,,$$

for every $\mathbb{C}^N$-valued $\vec{\psi} \in C_0^\infty(\Omega)$.

We define the *Dirichlet problem* in the upper half-space

$$L\vec{u} = 0 \text{ in } \mathbb{R}^{n+1}_+, \quad \vec{u}(x,t) \xrightarrow{t\to 0} \vec{f}(x) \in L^p(\mathbb{R}^n), \quad \sup_{t>0} \left\|\vec{u}(\cdot,t)\right\|_{L^p(\mathbb{R}^n)} \leq C\|\vec{f}\|_{L^p(\mathbb{R}^n)},$$

and we denote it by $(D)_p$.

We define the *Regularity problem* in the upper half-space

$$L\vec{u} = 0 \text{ in } \mathbb{R}^{n+1}_+, \quad \vec{u}(x,t) \xrightarrow{t\to 0} \vec{f}(x) \in \dot{L}^p_1(\mathbb{R}^n), \quad \sup_{t>0} \left\|\nabla\vec{u}(\cdot,t)\right\|_{L^p(\mathbb{R}^n)} \leq C\|\nabla\vec{f}\|_{L^p(\mathbb{R}^n)},$$

and we denote it by $(R)_p$.

For L as above, there exist the fundamental solutions E, E^* associated with L and L^*, respectively, in $\mathbb{R}^{n+1}$, so that $L_{x,t}\, E(x,t,y,s) = \delta_{(y,s)}$, and $L^*_{y,s}\, E^*(y,s,x,t) \equiv L^*_{y,s}\, \overline{E(x,t,y,s)} = \delta_{(x,t)}$, where δ_X denotes the Dirac mass at the point X, and L^* is the hermitian adjoint of L (cf. [HK]). We define the Single layer potential operator, by

$$\mathcal{S}_t\vec{f}(x) \equiv \int_{\mathbb{R}^n} E(x,t,y,0)\,\vec{f}(y)\,dy, \quad t \in \mathbb{R}.$$

Remark 5.0.2. For simplicity of notation we won't carry the vectorial notation through the remainder of the section.

Theorem 5.0.3. *If $(R)_q$ and $(D)_{p'}$ are uniquely solvable boundary problems, for L^*, in the lower half-space $\mathbb{R}^{n+1}_-$, with $\frac{2n}{n+2} < p < \frac{2n}{n-2}$, and some $q > 1$, then*

$$\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} |t\partial_t^2 S_t f(x)|^2 \frac{dxdt}{t} \leq C\|f\|^2_{L^2(\mathbb{R}^n)}, \tag{5.0.3}$$

or

$$\int_{\mathbb{R}^n}\int_0^{\infty} |t\partial_t^2 S_t f(x)|^2 \frac{dtdx}{t} \leq C\|f\|^2_{L^2(\mathbb{R}^n)}, \tag{5.0.4}$$

where S_t is the Single layer potential for L.

5.1 Reduction of Problem

Claim 5.1.1. In order to prove the Theorem 5.0.3 it's enough to prove that

$$\int_{\mathbb{R}^n}\int_0^\infty |t^m \partial_t^{m+1} S_t f(x)|^2 \frac{dt}{t} dx \leq C_m \|f\|^2_{L^2(\mathbb{R}^n)} \; for\ some\ m \geq 1. \tag{5.1.1}$$

Proof. Let's prove that (5.1.1) ⇒ (5.0.4).

Fix $\epsilon > 0$, $R >> \epsilon$, $m \geq 2$, $0 < \eta \ll R$. Define $S_t^\eta = \int_0^\infty \varphi_\eta(t-s) S_s ds$, where $\varphi \in C_c^\infty(\mathbb{R})$, $\varphi_\eta(t) = \eta^{-1}\varphi\left(\frac{t}{\eta}\right)$. It is enough to verify that that (5.1.1) ⇒ (5.0.4), for $f \in C_0^\infty$, and with S_t replaced by S_t^η, as long as we obtain bounds that are independent of η. To simplify the notation, we suppress the dependence on η, and just write S_t, except when the dependence on η is relevant. Consider now

$$\int_{|x|<R}\int_\epsilon^R |t^{m-1}\partial_t^m S_t f(x)|^2 \frac{dt}{t} dx = \int_{|x|<R}\int_\epsilon^R (\partial_t^m S_t f(x)) \cdot (\partial_t^m S_t f(x)) \cdot t^{2m-3} dt dx$$

$$= C_m \int_{|x|<R} \left(|t^{m-1}\partial_t^m S_t f(x)|^2 dx\right]_{t=\epsilon}^{t=R} - \frac{1}{2m-2}\int_{|x|<R}\int_\epsilon^R \frac{\partial}{\partial t}\left(\partial_t^m S_t f(x) \cdot \overline{\partial_t^m S_t f(x)}\right) t^{2m-2} dt dx$$

$$\leq (Boundary\ Term) + \frac{1}{2m-2}\int_{\mathbb{R}^n}\int_\epsilon^R |\partial_t^{m+1} S_t f(x)| \cdot |\partial_t^m S_t f(x)| t^{2m-2} dt dx$$

$$\leq (BT) + \frac{1}{2m-2}\left[\frac{1}{2}\left(\int_{|x|<R}\int_\epsilon^R |t^m \partial_t^{m+1} S_t f(x)|^2 \frac{dtdx}{t}\right) + \frac{1}{2}\left(\int_{|x|<R}\int_\epsilon^R |t^{m-1}\partial_t^m S_t f(x)|^2 \frac{dtdx}{t}\right)\right],$$

and therefore, hiding the last term, we have

$$\int_{|x|<R}\int_\epsilon^R |t^{m-1}\partial_t^m S_t f(x)|^2 \frac{dt}{t} dx \leq C(BT) + C_m \int_{|x|<R}\int_\epsilon^R |t^m \partial_t^{m+1} S_t f(x)|^2 \frac{dtdx}{t}.$$

Claim 5.1.2.

(a)

$$\lim_{\epsilon \to 0}\int_{|x|<R} \left(|t^{m-1}\partial_t^m S_t^\eta f(x)|^2\right]_{t=\epsilon} dx = 0.$$

(b)

$$\limsup_{R\to\infty}\int_{|x|<R} |t^{m-1}\partial_t^m S_t f(x)|^2\Big]_{t=R} dx \leq C\|f\|^2_{L^2(\mathbb{R}^n)}.$$

Assuming this claim means that if it's true for case m, is also true for the previous case m-1, and by induction would be true for $m = 2$ that it's (5.0.4).

□

Proof. of Claim 5.1.2

(a) $\lim_{\epsilon\to 0}\int_{|x|<R}\left(|t^{m-1}\partial_t^m S_t^\eta f(x)|^2\right]_{t=\epsilon} dx = 0.$

Recall that $S_t^\eta = \int_0^\infty \varphi_\eta(t-s)S_s ds$, where $\varphi \in C_c^\infty(\mathbb{R})$, $\varphi_\eta(t) = \eta^{-1}\varphi\left(\frac{t}{\eta}\right)$. Note that if for every $R \gg \eta > 0$ fixed, we have $\lim_{\epsilon\to 0}\int_{|x|<R}\left(|t^{m-1}\partial_t^m S_t^\eta f(x)|^2\right]_{t=\epsilon} dx = 0$ (note also that R and η don't depend on ϵ), then we get (5.1.2) (a). We write

$$\begin{aligned}
\partial_t(\partial_t^{m-1}S_t^\eta) &= \partial_t\int_0^\infty \partial_t^{m-1}\varphi_\eta S_s ds\\
&= \partial_t\int_0^\infty\left(\frac{1}{\eta}\right)^{m-1}\psi_\eta(t-s)S_s ds\\
&= \frac{\partial}{\partial t}\left(\frac{1}{\eta}\right)^{m-1}\int_0^\infty S_{t-s}\psi_\eta(s)ds,
\end{aligned}$$

for some $\psi \in C_c^\infty(\mathbb{R})$.

$$\begin{aligned}
(\partial_t^m S_t^\eta f) &= \partial_t(\partial_t^{m-1}S_t^\eta f)\\
&= \left(\frac{1}{\eta}\right)^{m-1}\frac{\partial}{\partial t}L^{-1}(f\psi_\eta)\\
&= \left(\frac{1}{\eta}\right)^{m-1}L^{-1}D(f\psi_\eta).
\end{aligned}$$

So

$$\begin{aligned}
\int_{|x|<R}|t^{m-1}\partial_t^m S_t^\eta f(x)|^2 dx|_{t=\epsilon} &= \left(\frac{\epsilon}{\eta}\right)^{2(m-1)}\int_{|x|<R}|L^{-1}D(f\psi_\eta)|^2 dx\\
&\le \left(\frac{\epsilon}{\eta}\right)^{2(m-1)}(R^n)^{\frac{1}{n}}\left(\int_{|x|<R}|L^{-1}D(f\psi_\eta)(x)|^{2(\frac{n}{n-1})}dx\right)^{\frac{n-1}{n}}\\
&\le C\left(\frac{\epsilon}{\eta}\right)^{2m-2}\cdot R\cdot\iint_{\mathbb{R}^{n+1}}|f(x)|^2|\psi_\eta(s)|^2 dxds \longrightarrow 0 \; as \; \epsilon\to 0.
\end{aligned}$$

Then $\int_{|x|<R}|t^m\partial_t^{m+1}S_t^\eta f(x)|^2 dx]_{t=\epsilon} \longrightarrow 0.$

(b) $\left(\int_{|x|<R}\left(|t^m\partial_t^{m+1}S_tf(x)|^2dx\right]_{t=R}\right)^{\frac{1}{2}} \leq C\|f\|_{L^2(\mathbb{R}^n)}$ (uniformly in R). Set

$$Q := \Delta_R = \{x : |x| < R\}, \qquad I_R = \{(x,t) : |x| < \frac{3}{2}R, \frac{R}{2} < t < \frac{3R}{2}\}. \tag{5.1.2}$$

Claim 5.1.3. It's enough to prove that

$$\sup_{R<\infty} R\cdot \iint_{I_R} |\partial_t^2 S_tf(x)|^2dxdt \leq C\|f\|^2_{L^2(\mathbb{R}^n)}. \tag{5.1.3}$$

Proof. Claim 5.1.3 Let's prove it by induction using Caccioppoli in slices (proposition 1.4.2) to reduce the derivatives in each step, bearing in mind that $t = R$.

$m \mapsto m+1,\ m > 1,$

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |t^m\partial_t^{m+1}S_tf(x)|^2dx &\leq CR^{2m}\frac{1}{|Q|}\int_Q |\partial_t^{m+1}S_tf(x)|^2dx \\
&\leq C\frac{R^{2m}}{(\ell(Q))^2}\frac{1}{|Q^*|}\iint_{Q^*}|\partial_t^m S_tf(x)|^2dxdt \\
&\leq C\frac{R^{2m}}{(\ell(Q))^4}\frac{1}{|Q|^{**}}\iint_{Q^{**}}|\partial_t^{m-1}S_tf|^2dxdt \\
&\leq (applying\ Caccioppoli\ (m-3)\ times) \\
&\leq C\frac{R^{2m}}{(\ell(Q))^{2(m-1)}}\frac{1}{|Q^{m-1}*|}\iint_{Q^{m-1}*}|\partial_t^2S_tf(x)|^2dxdt \\
&\leq CR\frac{1}{|Q|}\iint_{I_R}|\partial_t^2S_tf(x)|^2dxdt \\
&\Rightarrow \int_{\Delta_R}|t^m\partial_t^{m+1}S_tf(x)|^2dx \\
&\leq C\cdot R\iint_{I_R}|\partial_t^2S_tf(x)|^2dxdt.
\end{aligned}$$

For $m = 1$,

$\int_{\Delta_R}|t\partial_t^2S_tf(x)|^2dx \leq C\left(\frac{1}{|Q^*|}\iint_{Q^*}|t\partial_t^2S_tf(x)|^2dxdt\right) \leq C\cdot R\iint_{I_R}|\partial_t^2S_tf(x)|^2dxdt.$ □

To bound $\left(R\iint_{I_R}|\partial_t^2S_tf(x)|^2dxdt\right)^{\frac{1}{2}}$ we dualize with $h \in L^2(I_R)$ such that $\|h\|_{L^2(I_R)} \leq 1$.

$$\left(R\iint_{I_R}|\partial_t^2 S_t f(x)|^2 dxdt\right)^{\frac{1}{2}} \le CR^{\frac{1}{2}}\iint_{I_R} h\cdot\partial_t^2 S_t f dxdt$$
$$= -CR^{\frac{1}{2}}\int_{\Delta_R} f\partial_{n+1}(L^*)^{-1}\partial_{n+1}h dx$$
$$\le CR^{\frac{1}{2}}\|f\|_{L^2(\mathbb{R}^n)}\left(\int_{\Delta_R}|\partial_{n+1}(L^*)^{-1}\partial_{n+1}h|^2 dx\right)^{\frac{1}{2}} \le C\|f\|_{L^2(\mathbb{R}^n)}.$$

For the last inequality we have used the following claim.

Claim 5.1.4.

$$\left(\int_{\Delta_R}|\partial_{n+1}(L^*)^{-1}\partial_{n+1}h|^2 dx\right)^{\frac{1}{2}} \le C\cdot R^{\frac{-1}{2}}. \tag{5.1.4}$$

□

Proof. of Claim 5.1.4

$$\partial_{n+1}(L^*)^{-1}\partial_{n+1}h(x,s) = \iint \frac{\partial}{\partial t}\frac{\partial}{\partial s}E^*(x,s,y,t)h(y,t)dydt$$
$$= \iint \frac{\partial}{\partial t}\frac{\partial}{\partial s}E^*(x,s-t,y,0)h(y,t)dydt.$$

We know $\nabla(L^*)^{-1}div : L^2(\mathbb{R}^{n+1}) \to L^2(\mathbb{R}^{n+1})$. In our case, $s = 0$ (or more precisely, $|s| < \eta \ll R$, when working with S_t^η), so we are away from the pole. Therefore, as in the proof of Cacciopoli on slices, we have

$$\left(\int_{\Delta_R}|\partial_{n+1}(L^*)^{-1}\partial_{n+1}h(x)|^2 dx\right)^{\frac{1}{2}} \le C\left(\frac{1}{R}\iint_{\{(x,t):|x|<R,\frac{-R}{4}<t<\frac{R}{4}\}}|\partial_{n+1}(L^*)^{-1}\partial_{n+1}h(x)|^2 dxdt\right)^{\frac{1}{2}}$$
$$\le C\left(\frac{1}{R}\iint_{I_R}|h(x)|^2 dxdt\right)^{\frac{1}{2}}$$
$$\le CR^{\frac{-1}{2}}.$$

□

.

5.2 $t^m\partial_t^{m+1}S_t$ satisfies the Square function conditions as in 4.0.1 for some m big enough ($m > \frac{n+2}{2}$).

(a)(i) $\sup_{t>0}\|t^m\partial_t^{m+1}S_t f\|_{L^2(\mathbb{R}^n)} \le C_m\|f\|_{L^2(\mathbb{R}^{n+1})}, \quad \forall m \ge 1$. We may assume $f \in C_0^\infty$. We fix $t > 0$, and set $t = R$. Let $\mathbb{D}(t)$ denote the dyadic grid of cubes such that $t/2 < \ell(Q) < t$, for $Q \in \mathbb{D}(t)$. Given $Q \in \mathbb{D}(t) = \mathbb{D}(R)$, let $I_R := Q\times(R/2, 3R/2)$ denote the "Whitney box" above Q, and set $S_0(Q) := 2Q$, $S_j(Q) := 2^{j+1}Q \setminus 2^jQ,\ j \ge 1$. Finally, set $f_j := f1_{S_j(Q)},\ j \ge 0$. Then, as in the proofs of Claim 5.1.3, and of estimate (5.1.3), we have

$$\begin{aligned}\left(\int_{\mathbb{R}^n}|t^m\partial_t^{m+1}S_t f(x)|^2dx\right)^{1/2} &= \left(\sum_{Q\in\mathbb{D}(t)}\int_Q|t^m\partial_t^{m+1}S_t f(x)|^2dx\right)^{1/2}\\ &\lesssim \sum_{j=0}^{\infty}\left(\sum_{Q\in\mathbb{D}(t)}\int_Q|t^m\partial_t^{m+1}S_t f_j(x)|^2dx\right)^{1/2}\\ &\lesssim \left(\sum_{Q\in\mathbb{D}(t)}R\iint_{I_R}|\partial_t^2S_t f_0(x)|^2dxdt\right)^{1/2} + \sum_{j=1}^{\infty}\left(\sum_{Q\in\mathbb{D}(t)}\int_Q|t^m\partial_t^{m+1}S_t f_j(x)|^2dx\right)^{1/2}\\ &\lesssim \left(\sum_{Q\in\mathbb{D}(t)}\int_{\Delta_{2R}}|f(x)|^2dx\right)^{1/2} + \sum_{j=1}^{\infty}\left(\sum_{Q\in\mathbb{D}(t)}\int_Q|t^m\partial_t^{m+1}S_t f_j(x)|^2dx\right)^{1/2}\\ &\le C\|f\|_{L^2(\mathbb{R}^n)} + \sum_{j=1}^{\infty}\left(\sum_{Q\in\mathbb{D}(t)}\int_Q|t^m\partial_t^{m+1}S_t f_j(x)|^2dx\right)^{1/2}.\end{aligned}$$

In turn, by the case $j \ge 1$ of (5.2.3) below (whose proof does not depend upon the present estimate (a)(i)), we have

$$\sum_{Q\in\mathbb{D}(t)}\int_Q|t^m\partial_t^{m+1}S_t f_j(x)|^2dx \lesssim 2^{-\varepsilon j}\int_{\mathbb{R}^n}\left(\mathcal{M}(|f|^r)(x)\right)^{2/r}dx\,,$$

for some $r < 2$, and some $\varepsilon > 0$, whence the desired bound follows.

(a)(ii) $\|t^m\partial_t^{m+1}S_t(f\cdot\mathbb{1}_{2^{j+1}Q\setminus2^jQ})\|_{L^2(Q)} \le C\cdot2^{-\frac{(n+2+\beta)}{2}j}\|f_j\|_{L^2(2^{j+1}Q\setminus2^jQ)}$.

Remember that since $adj(S_t^L) = (S_{-t}^{L^*})$ (and both are of same type) so it's equivalent to prove (by duality)

$$\|t^m\partial_t^{m+1}S_t(f\cdot\mathbb{1}_Q)\|_{L^2(2^{j+1}Q\setminus2^jQ)} \le C2^{-j(m-1)}\|f\|_{L^2(Q)}. \tag{5.2.1}$$

Set $\tilde{f} = f\cdot1_Q$, $R_j = 2^jQ$ and $R = \ell(Q)$ (note that $R \le t \le 2R$ by (4.0.3)). By induction, for $m \ge 1$,

$$
\begin{aligned}
\frac{1}{|T_{R_j}|}\int_{T_{R_j}} |t^m \partial^{m+1} S_t \tilde{f}(x)|^2 dx &\le CR^{2m}\frac{1}{|T_{R_j}|}\int_{T_{R_j}} |\partial_t^{m+1}\partial_t^{m+1} S_t \tilde{f}(x)|^2 dx \\
&\le C\frac{R^{2m}}{\ell(R_j)^2}\frac{1}{|T^*_{R_j}|}\iint_{T^*_{R_j}} |\partial_t^m S_t \tilde{f}(x)|^2 dxdt \\
&\le C\frac{R^{2m}}{\ell(R_j)^{2(m-1)}}\frac{1}{|T^{m-1)*}_{R_j}|}\iint_{T^{m-1)^*}_{R_j}} |\partial_t^2 S_t \tilde{f}(x)|^2 dxdt \\
&\le CR^2 2^{-j(m-1)2}\frac{1}{|T^{m-1)*}_{R_j}|}\iint_{I_{R_j}} |\partial_t^2 S_t \tilde{f}(x)|^2 dxdt \\
&\le C\frac{R2^{-j[2(m-1)+1]}}{|T_{R_j}|}\iint_{I_{R_j}} |\partial_t^2 S_t \tilde{f}(x)|^2 dxdt
\end{aligned}
$$

where $I_{R_j} := 4R_j \times (t - \ell(R_j), t + \ell(R_j))$ and $T_{R_j} := 2R_j \setminus R_j$ as defined in Proposition 1.4.2.

So

$$
\int_{T_{R_j}} |t^m \partial_t^{m+1} S_t \tilde{f}(x)|^2 dx \le C \cdot R \cdot 2^{-j[2(m-1)+1]} \iint_{I_{R_j}} |\partial_t^2 S_t \tilde{f}(x)|^2 dxdt.
$$

For $m = 1$,

$$
\begin{aligned}
\frac{1}{|T_{R_j}|}\int_{T_{R_j}} |t\partial_t^2 S_t \tilde{f}(x)|^2 dx &\le C\frac{R^2}{|I_{R_j}|}\iint_{I_{R_j}} |\partial_t^2 S_t \tilde{f}(x)|^2 dxdt \\
&\le \frac{C \cdot R \cdot 2^{-j}}{|T_{R_j}|}\iint_{I_{R_j}} |\partial_t^2 S_t \tilde{f}(x)|^2 dxdt.
\end{aligned}
$$

From a variant of (5.1.3) (which is proved in the same way), and using that $\tilde{f} = f \cdot 1_Q$,

$$
2^{-j}R\iint_{I_{R_j}} |\partial_t^2 S_t \tilde{f}(x)|^2 dxdt \le C\int |\tilde{f}(x)|^2 dx \le C\int_Q |f(x)|^2 dx\,.
$$

We choose m such that $2(m-1) = n + \beta \rightarrow\ 2m = n + \beta + 2\ so\ m > \frac{n+2}{2}$.

(b)

For $s \le t$,

$$
\|t^m \partial_t^{m+1} S_t Q_s f\|_{L^2(\mathbb{R}^n)} \le C\|Q_s f\|_{L^2(\mathbb{R}^n)} \le C\|f\|_{L^2(\mathbb{R}^n)} \le \left(\frac{s}{t}\right)\|f\|_{L^2(\mathbb{R}^n)}\,.
$$

Claim 5.2.1.

$$
\|t^m \partial_t^{m+1}(S_t \nabla) f\|_{L^2(\mathbb{R}^n)} \le C\frac{1}{t}\|f\|_{L^2(\mathbb{R}^n)}. \tag{5.2.2}
$$

Remark 5.2.2. $S_t^L \sim S_{-t}^{L^*}$ so we have the claim for L^* in order to move ∇ to the front.

Proof. Let $\mathbb{D}(t)$ denote the dyadic grid of cubes such that $t/2 < \ell(Q) < t$, then for $Q \in \mathbb{D}(t)$

$$
\begin{aligned}
\|\nabla t^m \partial_t^{m+1} S_t f\|_{L^2(\mathbb{R}^n)}^2 &= \sum_{Q \in \mathbb{D}(t)} |Q| \frac{1}{|Q|} \int_Q |\nabla t^m \partial_t^{m+1} S_t f|^2 dx \\
&\leq C \sum_{Q \in \mathbb{D}(t)} \frac{|Q|}{t^2} \frac{1}{|Q^{**}|} \iint_{Q^{**}} |s^m \partial_s^{m+1} S_s f|^2 dx ds \\
&\leq C \sum_{Q \in \mathbb{D}(t)} \frac{1}{t^2} \cdot \sup_{t>0} \|t^m \partial_t^{m+1} S_t f\|_{L^2(2Q)}^2 \\
&\leq C \sum_{Q \in \mathbb{D}(t)} \frac{1}{t^2} \|f\|_{L^2(2Q)}^2 \\
&\leq C \frac{1}{t^2} \|f\|_{L^2(\mathbb{R}^n)}^2
\end{aligned}
$$

$\Rightarrow \|\nabla t^m \partial_t^{m+1} S_t f\|_{L^2(\mathbb{R}^n)} \leq C \frac{1}{t} \|f\|_{L^2(\mathbb{R}^n)}$.

In the proof we have used Caccioppoli in slices to get the second inequality on the previous reasoning and denoted $Q^{**} := 2Q \times (t/2, 2t)$ so that the constant is independent of m. □

If we choose $Q_s = s^2 \operatorname{div} \nabla e^{s^2 \Delta}$ $(= s^2 \Delta e^{s^2 \Delta})$ and substitute ∇f by $Q_s f$ in the claim we obtain the desired inequality for the case $s \leq t$. In the appendix, proposition .1.4 we have that Q_s satisfy L^2 off-diagonal estimates, this is due to be able to pull out one of the "s" and the fact that $s \nabla e^{s^2 \Delta} : L^2 \to L^2$. In [A] is proven that this operator Q_s satisfy the required conditions which proves (b) with $\beta = 1$.

(c) For $j = 0, 1, 2, ...,$ and $t \approx \ell(Q)$, we have

$$
\left(\int_{8Q} |t^m \partial_t^{m+1} S_t (g \cdot 1_{S_j(Q)})(y)|^2 dy \right)^{\frac{1}{2}} \leq C \cdot 2^{-j\nu} \cdot t^{-n(\frac{1}{r} - \frac{1}{2})} \left(\int_{S_j(Q)} |g(y)|^r dy \right)^{\frac{1}{r}}, \tag{5.2.3}
$$

$1 < r < 2,\ \nu > \frac{n}{r},\ S_0(Q) = 16Q,\ S_j(Q) = 2^{j+4} Q \setminus 2^{j+3} Q.$

Note that by duality, it's equivalent to proving $L^r \to L^2 \leftrightarrow L^2 \to L^{r'}$.

So we are going to prove

$$
\left(\int_{S_j(Q)} |t^m \partial_t^{m+1} S_t^{L^*} (f(x) \cdot 1_{8Q}(x))|^{r'} dx \right)^{\frac{1}{r'}} \leq C \cdot 2^{-j\nu} |t|^{-n(\frac{1}{r} - \frac{1}{2})} \left(\int_{8Q} |f(x)|^2 dx \right)^{\frac{1}{2}}
$$

with $r' = \frac{2n}{n-2}$, $r = \frac{2n}{n+2}$, $t < 0$.

Define $u_m(x,t) := t^m \partial_t^{m+1} S_t^{L^*}(f \cdot 1_{8Q}(x))$.

$$\left(\int_{S_j(Q)} |u_m(x,t)|^{r'} dx \right)^{\frac{1}{r'}} \leq \left(\sum_{k=1}^{M} \int_{Q_k^j} |u_m(x,t) - \fint_{Q_k^j} u_m(y,t) dy|^{r'} dx \right)^{\frac{1}{r'}} + \sum_{k=1}^{M} \left(\int_{Q_k^j} \left(\fint_{Q_k^j} u_m(y,t) dy \right)^{r'} dx \right)^{\frac{1}{r'}} := I + II,$$

where we have decomposed the annuli $S_j(Q)$ into cubes of length $2^{j+3}\ell(Q)$, so that $\cup_{k=1}^M Q_k^j \supseteq S_j(Q)$, with $M \leq C_n$.

Term II. By the Cauchy-Schwartz inequality and the dual version of part (a)(ii) above,

$$\begin{aligned} II = \sum_k \left(\int_{Q_k^j} \left(\fint_{Q_k^j} u_m(y,t) dy \right)^{r'} dx \right)^{\frac{1}{r'}} &\leq \sum_k |Q_k^j|^{\frac{1}{r'}} \cdot \left(\fint_{Q_k^j} u_m(y,t)^2 dy \right)^{\frac{1}{2}} \\ &\leq C \left(2^{jn} |Q| \right)^{\frac{1}{r'} - \frac{1}{2}} \left(\int_{S_j(Q)} |u_m(y,t)|^2 dy \right)^{\frac{1}{2}} \\ &\leq C \left(2^{jn} |Q| \right)^{\frac{1}{r'} - \frac{1}{2}} \cdot 2^{-j(\frac{n+2+\beta}{2})} \cdot \|f\|_{L^2(8Q)} \\ &\leq C t^{-n(\frac{1}{r} - \frac{1}{2})} \cdot 2^{jn(\frac{1}{r'} - \frac{1}{2})} 2^{-j(\frac{n+2+\beta}{2})} \cdot \|f\|_{L^2(8Q)} \\ &= C t^{-n(\frac{1}{r} - \frac{1}{2})} \cdot 2^{-j\nu} \cdot \|f\|_{L^2(8Q)}, \end{aligned}$$

in this case with $\nu = \frac{n}{r} + 1 + \frac{\beta}{2} > \frac{n}{r}$.

Term I. By Sobolev's inequality,

$$\left(\fint_{Q_k^j} |u_m(x,t) - \fint_{Q_k^j} u_m(y,t) dy|^{r'} dx \right)^{\frac{1}{r'}} \lesssim 2^j \ell(Q) \left(\fint_{Q_k^j} |\nabla u_m(x,t)|^2 dx \right)^{\frac{1}{2}}.$$

By Caccioppoli on horizontal slices, and the dual version of part (a)(ii) above,

$$\begin{aligned} \frac{1}{|Q_k^j|} \int_{Q_k^j} |\nabla u_m(x,t)|^2 dx &\lesssim \left(2^j \ell(Q) \right)^{-2} \frac{1}{|(Q_k^j)^*|} \iint_{(Q_k^j)^*} |u_m(y,s)|^2 dy ds \\ &\lesssim \left(2^j \ell(Q) \right)^{-2} \frac{1}{|Q_k^j|} 2^{-(n+2+\beta)j} \int_{8Q} |f(x)|^2 dx. \end{aligned}$$

Hence,

$$\left(\fint_{Q_k^j} |\nabla u_m(x,t)|^2 dx\right)^{\frac{1}{2}} \lesssim \left(2^j \ell(Q)\right)^{-1-n/2} 2^{-(\frac{n+2+\beta}{2})j} \left(\int_{8Q} |f(x)|^2 dx\right)^{\frac{1}{2}}.$$

Since $r^{-1} = (n+2)/2n$, setting $\nu = \frac{n+2}{2} + 1 + \frac{\beta}{2} = \frac{n}{r} + 1 + \frac{\beta}{2} > \frac{n}{r}$, we conclude that

$$\begin{aligned}\left(\int_{S_j(Q)} |t^m \partial_t^{m+1} S_t^{L^*}(f(x)\cdot 1_{8Q}(x))|^{r'} dx\right)^{\frac{1}{r'}} \\ \lesssim \left(2^j \ell(Q)\right)^{\frac{n}{r'}-\frac{n}{2}} 2^{-j(\frac{n+2+\beta}{2})} \left(\int_{8Q} |f(x)|^2 dx\right)^{\frac{1}{2}} \\ \lesssim 2^{-j\nu} |t|^{-n(\frac{1}{r}-\frac{1}{2})} \left(\int_{8Q} |f(x)|^2 dx\right)^{\frac{1}{2}}.\end{aligned}$$

(d)

$$\sup_{t>0} \|t^m \partial_t^{m+1} S_t f\|_{L^s(\mathbb{R}^n)} \le C\|f\|_{L^s(\mathbb{R}^n)}, \quad \text{for } 2n/(n+2) \le s \le 2n/(n-2). \tag{5.2.4}$$

Remark 5.2.3. For $n = 2$, we have that (5.2.4) holds for $1 \le s \le \infty$, because in that case the kernel of $t^m \partial_t^{m+1} S_t$, $m \ge 1$, enjoys appropriate pointwise bounds; see [AAAHK]. Therefore, we shall assume that $n \ge 3$.

By interpolation with (a)(i) above, and duality, it is enough to treat the case $s = \frac{2n}{n-2} > \frac{2n-4}{n-2} = 2$. We fix $t > 0$, proving the inequality with a constant independent of t. We claim that, for $\ell(Q) \approx t$,

$$\left(\fint_Q |t^m \partial_t^{m+1} S_t(f\cdot 1_{S_j(Q)})(x)|^s dx\right)^{\frac{1}{s}} \lesssim 2^{-j\varepsilon} \left(\fint_{S_j(Q)} |f(x)|^2 dx\right)^{\frac{1}{2}}, \tag{5.2.5}$$

for some uniform $\varepsilon > 0$. Taking the claim for granted momentarily, we write

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |t^m \partial_t^{m+1} S_t f(x)|^s dx\right)^{1/s} &= \left(\sum_{Q\in\mathbb{D}(t)} \int_Q |t^m \partial_t^{m+1} S_t f(x))|^s dx\right)^{1/s} \\ &\le \sum_{j=0}^{\infty} \left(\sum_{Q\in\mathbb{D}(t)} |Q| \fint_Q |t^m \partial_t^{m+1} S_t(f\cdot 1_{S_j(Q)})(x)|^s dx\right)^{1/s} \\ &\lesssim \sum_{j=0}^{\infty} 2^{-j\varepsilon} \left(\sum_{Q\in\mathbb{D}(t)} |Q| \operatorname{essinf}_Q \left(\mathcal{M}\left(|f|^2\right)\right)^{s/2}\right)^{1/s} \lesssim \left(\int_{\mathbb{R}^n} \left(\mathcal{M}\left(|f|^2\right)\right)^{s/2}\right)^{1/s},\end{aligned}$$

where in the next to last inequality, we have used (5.2.5); (d) now follows.

It remains to prove (5.2.5). To this end, fix $j \geq 0$ and set $u_m^j(x,t) := t^m \partial_t^{m+1} S_t(f \cdot 1_{S_j(Q)}(x))$. Then

$$\left(\fint_Q |u_m^j(x,t)|^s dx\right)^{\frac{1}{s}} \leq \left(\fint_Q |u_m^j(x,t) - \fint_Q u_m^j(y,t)dy|^s dx\right)^{\frac{1}{s}} + \fint_Q |u_m^j(y,t)|dy := I + II.$$

Term II. By the Cauchy-Schwartz inequality and (a)(ii) above,

$$\begin{aligned} II &\leq \left(\fint_Q u_m^j(y,t)^2 dy\right)^{\frac{1}{2}} \\ &= |Q|^{-\frac{1}{2}} \left(\int_Q |u_m^j(y,t)|^2 dy\right)^{\frac{1}{2}} \\ &\lesssim |Q|^{-\frac{1}{2}} \cdot 2^{-j(\frac{n+2+\beta}{2})} \|f\|_{L^2(S_j(Q))} \\ &\approx 2^{-j(1+\frac{\beta}{2})} \left(\fint_{S_j(Q)} |f|^2\right)^{1/2} \end{aligned}$$

which yields (5.2.5) for term II with $\varepsilon = 1 + \frac{\beta}{2} > 0$.

Term I. By Sobolev's inequality,

$$\left(\fint_Q |u_m^j(x,t) - \fint_Q u_m^j(y,t)dy|^s dx\right)^{\frac{1}{s}} \lesssim \ell(Q) \left(\fint_Q |\nabla u_m^j(x,t)|^2 dx\right)^{\frac{1}{2}}.$$

By Caccioppoli on horizontal slices, and (a)(ii) above,

$$\begin{aligned} \fint_Q |\nabla u_m^j(x,t)|^2 dx &\lesssim (\ell(Q))^{-2} \frac{1}{|Q^*|} \iint_{Q^*} |u_m^j(y,s)|^2 dy ds \\ &\lesssim (\ell(Q))^{-2} \frac{1}{|Q|} 2^{-(n+2+\beta)j} \int_{S_j(Q)} |f(x)|^2 dx. \end{aligned}$$

Hence,

$$\ell(Q) \left(\int_Q |\nabla u_m(x,t)|^2 dx\right)^{\frac{1}{2}} \lesssim |Q|^{-1} 2^{-(\frac{n+2+\beta}{2})j} \left(\int_{S_j(Q)} |f(x)|^2 dx\right)^{\frac{1}{2}},$$

so that

$$I \lesssim 2^{-(1+\frac{\beta}{2})j} \left(\fint_{S_j(Q)} |f(x)|^2 dx\right)^{\frac{1}{2}},$$

as desired.

5.3 Construction of b_Q

In order to construct the family $\{b_Q\}$ we need to construct a Poisson kernel type of function.

Claim 5.3.1. If $(D)_{p'}$ is solvable for L^* in the lower half-space then there exists a function k_Q such that

$$\fint\!\!\fint_{R_Q} u(y,s)dyds = \int_{\mathbb{R}^n} k_Q(z)f(z)\,dz,$$

where $R_Q = \frac{1}{2}Q \times \left[-\gamma\ell(Q), \frac{-\gamma\ell(Q)}{2}\right]$, and $u(x,t)$ is the solution of the Dirichlet problem $(D)_{p'}$ for L^* with data f (and where $\gamma > 0$ small to be fixed later in 5.3.2). Moreover k_Q is an L^p function and

$$\int_{\mathbb{R}^n} |k_Q(z)|^p dz \leq C|Q|^{1-p}$$

Proof. of claim 5.3.1 Applying Hölder and that u(x,t) solves the Dirichlet problem we get

$$\begin{aligned}
\left|\fint\!\!\fint_{R_Q} u(y,s)dyds\right| &\leq \left(\fint\!\!\fint_{R_Q} |u(y,s)|^{p'} dyds\right)^{\frac{1}{p'}} \\
&= \left(\frac{1}{|R_Q|}\int_{-\gamma\ell(Q)}^{\frac{-\gamma\ell(Q)}{2}}\int_{\frac{1}{2}Q} |u(y,s)|^{p'} dyds\right)^{\frac{1}{p'}} \\
&\leq C\frac{1}{\ell(Q)^{\frac{n}{p'}}}\left(\fint_{-\gamma\ell(Q)}^{\frac{-\gamma\ell(Q)}{2}}\int_{\mathbb{R}^n} |u(y,s)|^{p'} dyds\right)^{\frac{1}{p'}} \\
&\leq C\frac{1}{|Q|^{\frac{1}{p'}}}\left(\fint_{-\gamma\ell(Q)}^{\frac{-\gamma\ell(Q)}{2}} \sup_{t<0} \|u(y,s)\|^{p'}_{L^{p'}(\mathbb{R}^n)} ds\right)^{\frac{1}{p'}} \leq |Q|^{\frac{-1}{p'}}\|f\|_{L^{p'}(\mathbb{R}^n)}.
\end{aligned}$$

Taking the supremum over all functions in $L^{p'}(\mathbb{R}^n)$ with unit norm and applying the Riesz Representation theorem to the linear functional $T : f \to \fint\!\!\fint_{R_Q} u(y,s)dyds$. □

For every cube Q we define $b_Q(x) := |Q|k_Q(x)$. So let's verify that such a family satisfy all the required conditions.

(i)

$$\int_Q |b_Q(x)|^p dx \leq C|Q|$$

$$\int_Q |b_Q(x)|^p dx = |Q|^p \int_Q |k_Q(x)|^p dx \leq C|Q|^p|Q|^{\frac{-p}{p'}}$$

$$\leq C|Q|^{p(1-\frac{1}{p'})} \leq C|Q|$$

(ii)

$$\int_Q \left(\int_0^{\ell(Q)} \int_{|x-y|<t} |t^m \partial_t^{m+1} S_t b_Q(y)|^2 \frac{dydt}{t^{n+1}}\right)^{\frac{p}{2}} dx \leq C|Q|$$

By (5.2.4), the unique solvability of $D_{p'}$, the fact that $V(y,s) := E(x,t,y,s)$ is an adjoint solution in the lower half-space, with (x,t) fixed in $\mathbb{R}^{n+1}_+$, and the fact that $k_Q \in L^p$ with $2n/(n+2) < p < 2n/(n-2)$, we get

$$|t^m \partial_t^{m+1} S_t b_Q(x)| = |Q| t^m \partial_t^{m+1} \int_{\mathbb{R}^n} E(x,t,y,0) k_Q(y) dy$$

$$= |Q| t^m \partial_t^{m+1} \fint\!\!\fint_{R_Q} E(x,t,y,s) dyds$$

Then,

$$\int_Q \left(\int_0^{\ell(Q)} \int_{|x-y|<t} |t^m \partial_t^{m+1} S_t b_Q(y)|^2 \frac{dydt}{t}\right)^{\frac{p}{2}} dx$$

$$= \int_Q \left(\int_0^{\ell(Q)} \int_{|x-y|<t} \left| \fint\!\!\fint_{R_Q} |Q| t^m \partial_t^{m+1} E(y,t,z,r) dzdr \right|^2 \frac{dydt}{t}\right)^{\frac{p}{2}} dx$$

$$\leq C \int_Q \left(\int_0^{\ell(Q)} \int_{|x-y|<t} \left| \frac{1}{\ell Q} t^m \iint_{R_Q} \partial_t^{m+1} E(y,t,z,r) \mathbb{1}_{R_Q}(z,r) dxdr \right|^2 \frac{dydt}{t}\right)^{\frac{p}{2}} dx$$

$$\leq C|Q| \left(\fint_Q \int_0^{\ell(Q)} \int_{|x-y|<t} \left| \frac{1}{\ell Q} t^m \iint_{R_Q} \partial_t^{m+1} E(y,t,z,r) \mathbb{1}_{R_Q}(z,r) dxdr \right|^2 \frac{dydt}{t} dx\right)^{\frac{p}{2}}$$

$$\leq C|Q| \left(t_Q^{2m-3} \fint_Q \int_0^{\ell(Q)} \int_{|x-y|<t} |\partial_t^{m+1} u(y,t)|^2 dydtdx\right)^{\frac{p}{2}}$$

$$\leq C|Q| \left(\ell(Q)^{-1-n} \int_Q \int_0^{\ell(Q)} \int_{|x-y|<t} |\partial_t^2 L^{-1} \mathbb{1}_{R_Q}(y,t)|^2 dydtdx\right)^{\frac{p}{2}}$$

$$\leq C|Q| \left(t_Q^{-1-n} \int_{5Q} \int_0^{\ell(Q)} |\partial_t^2 L^{-1} \mathbb{1}_{R_Q}(x,t)|^2 dtdx\right)^{\frac{p}{2}}$$

$$\leq C|Q| \left(\ell(Q)^{-1-n} \iint_{\mathbb{R}^{n+1}} |\mathbb{1}_{R_Q}(x)|^2 dxdt\right)^{\frac{p}{2}}$$

$$\leq C|Q|$$

Let's note that the constant depends on γ.

(iii)

$$\|\nabla \Phi_Q\|_{L^\infty(\mathbb{R}^n)} \le C_0\, \ell(Q)^{-1}, \quad 0 < C_1 \le \Phi_Q(x) \le 1 \textit{ on } Q$$

Let $\Phi_Q(x)$ be the constant function 1.

(iv)

$$\delta |\xi|^2 \le \mathcal{R}e \left(\xi \cdot (|Q|)^{-1} \int_Q b_Q(x) dx \right) \cdot \bar{\xi}$$

To help us with this last condition let's define some auxiliary functions. Fix a cube Q and an unit vector ξ.

$$\tilde{\Phi}_Q(x) := \begin{cases} 1 & \text{if } x \in Q \\ 0 & \text{if} x \notin (1+\epsilon)Q \\ (0,1) & \text{otherwise} \end{cases}$$

$\epsilon > 0$ small to be determined later in 5.3.3. $\tilde{\Phi}_Q$ is a Lipschitz function with $\|\nabla \tilde{\Phi}_Q\|_{L^\infty(\mathbb{R}^n)} \le \frac{1}{\epsilon \ell(Q)}$.

Also define W_Q as the solution of

$$(R)_q^{L^*} := \begin{cases} L^* W_Q = 0 & \text{in } \mathbb{R}^{n+1}_- \\ W_Q(\cdot, 0) = \tilde{\Phi}_Q \cdot \bar{\xi} & \\ \sup_{t<0} \|\nabla W_Q(\cdot, s)\|_{L^q(\mathbb{R}^n)} \le C \|\nabla \tilde{\Phi}_Q\|_{L^q(\mathbb{R}^n)} \le C \frac{1}{\epsilon \ell(Q)} |Q|^{\frac{1}{q}} & \end{cases}$$

Claim 5.3.2.

$$\frac{1}{|Q|} \xi \int_{\mathbb{R}^n} b_Q(z) \tilde{\Phi}_Q(z) \bar{\xi} dx = |\xi|^2 (1 + O(\epsilon)).$$

Proof. of claim 5.3.2 Let's note two facts. By construction we have

$$\iint_{R_Q} W_Q(y,s) dy ds = \int_{\mathbb{R}^n} k_Q(z) \tilde{\Phi}_Q(z) dz, \tag{5.3.1}$$

and by the Fundamental Theorem of Calculus

$$-\int_s^0 \frac{\partial}{\partial \tau} W_Q(y,\tau) d\tau = W_Q(y,s) - W_Q(y,0) = W_Q(y,s) - \tilde{\Phi}_Q(y) \bar{\xi}.$$

Therefore,

$$\begin{aligned}
\xi \int_{\mathbb{R}^n} b_Q(x)\tilde{\Phi}_Q(x)\bar{\xi}dx &= |Q|\xi \iint\!\!\!\!\!\!\!-\!\!-_{R_Q} W_Q(y,s)dyds \\
&= |Q|\left[\left(\xi \fint_{-\gamma\ell(Q)}^{\frac{-\gamma\ell(Q)}{2}} \fint_{\frac{1}{2}Q} W_Q(y,s) - \tilde{\Phi}_Q(y)\bar{\xi}dyds\right) + \left(\xi \fint_{-\gamma\ell(Q)}^{\frac{-\gamma\ell(Q)}{2}} \fint_{\frac{1}{2}Q} \Phi_Q(y)\bar{\xi}dyds\right)\right] \\
&= |Q|(I + II).
\end{aligned}$$

Using Hölder's inequality and the fact that W_Q is a solution of $(R)_q$ we bound the part I as follows

$$\begin{aligned}
\mathcal{R}eI \le |I| &\le \left|\xi \fint_{-\gamma\ell(Q)}^{\frac{-\gamma\ell(Q)}{2}} \fint_{\frac{1}{2}Q} W_Q(y,s) - \tilde{\Phi}_Q(y)\bar{\xi}dyds\right| \\
&\le \fint_{-\gamma\ell(Q)}^{\frac{-\gamma\ell(Q)}{2}} \fint_{\frac{1}{2}Q} \int_s^0 \left|\frac{\partial}{\partial\tau}W_Q(y,\tau)\right| d\tau dyds \\
&\le \gamma\ell(Q) \fint_{\frac{1}{2}Q} \fint_{-\gamma\ell(Q)}^0 \left|\frac{\partial}{\partial\tau}W_Q(y,\tau)\right| d\tau dy \\
&\le \gamma\ell(Q) \left(\fint_{\frac{1}{2}Q} \fint_{-\gamma\ell(Q)}^0 \left|\frac{\partial}{\partial\tau}W_Q(y,\tau)\right|^q d\tau dy\right)^{\frac{1}{q}} \\
&\le C\gamma\ell(Q)\frac{1}{|Q|^q}\|\nabla\tilde{\Phi}_Q\|_{L^q(\mathbb{R}^n)} \\
&\le C\frac{\gamma\ell(Q)}{\epsilon\ell(Q)} \\
&\le C\frac{\gamma}{\epsilon} \\
&= O(\epsilon),
\end{aligned}$$

where in the last step we have fixed

$$\gamma \approx \epsilon^2. \tag{5.3.2}$$

Also, we get that $II = |\xi|^2 = 1$ from $\tilde{\Phi}_Q = 1$ in $\frac{1}{2}Q$.

□

By Hölder's inequality

$$\mathcal{R}e\left(\xi\int_{\mathbb{R}^n} b_Q(x)(\mathbb{1}_Q(x)-\tilde{\Phi}_Q(x))\bar{\xi}dx\right) \leq |Q|\, \|b_Q\|_{L^p(\mathbb{R}^n)}\, |(1+\epsilon)Q\setminus Q|^{\frac{1}{p'}}$$

$$\leq C|Q|\,|Q|^{\frac{1}{p}}(\epsilon|Q|)^{\frac{1}{p'}}$$

$$\leq \epsilon^{\frac{1}{p'}}|Q|$$

Finally putting all the computations together and choosing epsilon small so that

$$C\epsilon - C\epsilon^{\frac{1}{p'}} \leq \frac{1}{2} \tag{5.3.3}$$

$$\mathcal{R}e\xi ⨍_Q b_Q(x)\bar{\xi}dx = \frac{1}{|Q|}\mathcal{R}e\left(\xi\int_{\mathbb{R}^n} b_Q(x)\tilde{\Phi}_Q(x)dx + \xi\int_{\mathbb{R}^n} b_Q(x)(\mathbb{1}_Q(x)-\tilde{\Phi}_Q(x))\bar{\xi}dx\right)$$

$$\geq 1 - C\epsilon - C\epsilon^{\frac{1}{p'}} \geq \frac{1}{2}.$$

Chapter 6

Application 2

Let A be an $n \times n$ matrix of complex, L^∞ coefficients, defined on $\mathbb{R}^n$, and satisfying the ellipticity (or "accretivity") condition

$$\lambda|\xi|^2 \le \mathcal{R}e < A\xi, \xi > \equiv \sum_{i,j} A_{ij}(x)\xi_j\bar{\xi}_i, \ ||A||_\infty \le \Lambda,$$

for $\xi \in \mathbb{C}^n$ and for some λ, Λ such that $0 < \lambda \le \Lambda < \infty$. We define the divergence form operator

$$Lu \equiv -div\,(A(x)\nabla u)\,,$$

which we interpret in the usual weak sense via a sesquilinear form.

The accretivity condition above enables one to define an accretive square root $\sqrt{L} \equiv L^{\frac{1}{2}}$.

Theorem 6.0.3. *[AHLMcT]*

Let L be a divergence form operator defined as above. Then for all $h \in \dot{L}^2_1(\mathbb{R}^n)$, we have

$$||\sqrt{L}h||_{L^2(\mathbb{R}^n)} \le C||\nabla h||_{L^2(\mathbb{R}^n)},$$

with C depending only on n, λ and Λ.

Proof.

Proposition 6.0.4. *([A], [D])*

$$L^{\frac{1}{2}} f(x) = c \int_0^\infty e^{-t^2 L} L f(x) dt.$$

In [AT], it is shown that the conclusion of Theorem 6.0.3 is equivalent to the square function estimate

$$\iint_{\mathbb{R}^{n+1}_+} |\Theta_t \nabla h|^2 \frac{dxdt}{t} \leq C_{n,\lambda,\Lambda} \int_{\mathbb{R}^n} |\nabla h|^2 dx\,,$$

where $\Theta_t \equiv te^{-t^2L}divA$. Thus, to prove this theorem, it is enough to verify the conditions of the Local Tb Theorem (Theorem 4.0.2), for the operator $\Theta_t \equiv te^{-t^2L}divA$, with $N = n$, and with $H := \{\nabla h :\ h \in \dot{L}^2_1(\mathbb{R}^n, \mathbb{C}^n)\}$, a subspace of $L^2(\mathbb{R}^n, \mathbb{C}^n)$.

(a) Since the family of operators $\{\sqrt{t}\nabla e^{-tL}\}_{t>0}$ satisfy L^2 off-diagonal estimates (Proposition .1.6) and the fact that $A \in L^\infty(\mathbb{R}^n)$ we get condition (a).

(b) If we choose $\{Q_s\}_{s>0}$ of convolution type satisfying the required conditions.

$Q_s\nabla F = \nabla Q_s F$ since Q_s of convolution type.

Let $\nabla F \in H$, then (by definition) $F \in \dot{L}^2_1(\mathbb{R}^n, \mathbb{C}^N) \iff F = I_1 f$ where $f \in L^2(\mathbb{R}^n, \mathbb{C}^N)$ and I_1 is the Riesz potential for $\alpha = 1$ as defined in 1.3.4.

$$\begin{aligned} te^{-t^2L}divAQ_s\nabla F &= te^{-t^2L}divA\nabla Q_s F \\ &= -tLe^{-t^2L}Q_s F \\ &= -\frac{1}{t}t^2Le^{-t^2L}Q_s I_1 f \\ &= \frac{-s}{t}t^2Le^{-t^2L}\left(\frac{1}{s}Q_s I_1\right)f. \end{aligned}$$

Using the fact that $t^2Le^{-t^2L} : L^2 \to L^2$ and $\frac{1}{s}Q_sI_1 : L^2 \to L^2$ we finish condition (b).

(c),(d) As in (a) in [A] we have that the family of operators $\{\sqrt{t}\nabla e^{-tL}\}_{t>0}$ satisfy such conditions joint with the fact that $A \in L^\infty(\mathbb{R}^n)$ we get conditions (c) and (d).

Finally, we need to find a the family of b_Q indexed by cubes Q satisfying the required conditions. In [HMc], [HLMc] and [AHLMcT] is proven that such a family exists and satisfy such conditions.

□

.1 Appendix A

Definition .1.1. Let $\mathcal{T} = (T_t)_{t>0}$ be a family of uniformly bounded operators on $L^2(\mathbb{R}^d)$. We say that $\mathcal{T}$ is $L^p - L^q$ bounded for some $p, q \in [1, \infty]$ with $p \leq q$ if for some constant C, for all $t > 0$ and all $h \in L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$

$$||T_t h||_{L^q(\mathbb{R}^d)} \leq C t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} ||h||_{L^p(\mathbb{R}^d)}.$$

We say that $\mathcal{T}$ satisfies $L^p - L^q$ off-diagonal estimates for some p,q$\in [1, \infty]$ with $p \leq q$ if for some constants $C, c > 0$, for all closed sets E and F, all $h \in L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ with support in E and all $t > 0$ we have

$$||T_t h||_{L^q(F)} \leq C t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} e^{-\frac{dist^2(E,F)}{C\cdot t}} ||h||_{L^p(\mathbb{R}^d)}.$$

We say that $\mathcal{T}$ satisfies L^2 off-diagonal estimates if for some constant $C \geq 0$ and for all closed sets E and F, all $h \in L^2(\mathbb{R}^d)$ and all $t > 0$ we have

$$||T_t h||_{L^2(F)} \leq C e^{-\frac{dist^2(E,F)}{C\cdot t}} ||h||_{L^2(\mathbb{R}^d)}.$$

Proposition .1.1. *If $(T_t)_{t>0}$ is a family of operators that satisfies $L^p - L^q$ boundedness (resp. off-diagonal estimates) and $(S_t)_{t>0}$ is a family of operators that satisfies $L^q - L^r$ boundedness (resp. off-diagonal estimates) then $(S_t T_t)_{t>0}$ satisfies $L^p - L^r$ boundedness (resp. off-diagonal estimates).*

Proposition .1.2. *Let $p \in [1, 2)$ and $n \geq 1$. Let $\mathcal{S} = (e^{-tL})_{t>0}$ where L is an elliptic operator of divergence form.*

(1) If $\mathcal{S}$ is $L^p(\mathbb{R}^d)$ bounded then it is $L^p - L^2$ bounded.

(2) If $\mathcal{S}$ is $L^p - L^2$ bounded, then for all $q \in (p, 2)$ it satisfies $L^q - L^2$ off-diagonal estimates.

(3) If $\mathcal{S}$ satisfies $L^p - L^2$ off-diagonal estimates then it is $L^p(\mathbb{R}^d)$ bounded.

Remark .1.3. The result applies when $2 < p \leq \infty$ by duality: replace $L^p - L^2$ by $L^2 - L^p$ everywhere. $L^2(\mathbb{R}^d)$ could be replaced by $L^q(\mathbb{R}^d)$ for q larger than 2 if necessary.

Proposition .1.4. *The families* $(e^{-tL})_{t>0}$, $(tLe^{-tL})_{t>0}$ *and* $(\sqrt{t}\nabla e^{-tL})_{t>0}$ *satisfy* L^2 *off-diagonal estimates.*

Corollary .1.5. *Let I be a cube in* $\mathbb{R}^{n+1}$, $f_k(X) = f(X)\mathbb{1}_{S_k(I)}(X)$ *and* $S_k(I) = 2^{k+1}I \setminus 2^k I$ *then for some constant* $C > 0$,

$$\left(\fint\!\!\fint_I |e^{-\tau L} f_k(X)|^2 dX \right)^{\frac{1}{2}} \leq Ce^{-\frac{dist^2(I,S_k(I))}{C\cdot\tau}} \left(\fint\!\!\fint_{S_k(I)} |f(X)|^2 dX \right)^{\frac{1}{2}};$$

$$\left(\fint\!\!\fint_I |\sqrt{\tau}\nabla_x e^{-\tau L} f_k(X)|^2 dX \right)^{\frac{1}{2}} \leq Ce^{-\frac{dist^2(I,S_k(I))}{C\cdot\tau}} \left(\fint\!\!\fint_{S_k(I)} |f(X)|^2 dX \right)^{\frac{1}{2}};$$

$$\left(\fint\!\!\fint_I |\tau\partial_t e^{-\tau L} f_k(X)|^2 dX \right)^{\frac{1}{2}} \leq Ce^{-\frac{dist^2(I,S_k(I))}{C\cdot\tau}} \left(\fint\!\!\fint_{S_k(I)} |f(X)|^2 dX \right)^{\frac{1}{2}}.$$

Proposition .1.6. *(Gaffney estimates [A],[D],[AMcT],[BLP],[IS]) Set* $d = n + 1$*,L be t-independent as described in section 5, then* $\exists\epsilon_0 > 0$ *such that* $\forall q$, $2 \leq q < \frac{2n}{n-2} + \epsilon_0$, *we have*

$$\|e^{-\tau L} f\|_{L^q(A)} \leq C\tau^{\frac{-d}{2}(\frac{1}{q'}-\frac{1}{q})} e^{\{\frac{-dist(A,B)^2}{C\tau}\}} \|f\|_{L^{q'}(B)},$$

where $supp(f) \subseteq B$, $C := C(d, q, \lambda, \Lambda)$, $\epsilon_0 := \epsilon_0(d, \lambda, \Lambda)$.

Corollary .1.7. *If we consider* $A = B = \mathbb{R}^d$

$$\|e^{-\tau L} f\|_{L^q(\mathbb{R}^d)} \leq C\tau^{-\frac{d}{2}(\frac{1}{q'}-\frac{1}{q})} \|f\|_{L^{q'}(\mathbb{R}^d)}.$$

Remark .1.8. As stated in [A], if we prove that our operator is L^s bounded, then $\forall q' \in (s', 2)$ our operator satisfies $L^{q'} \to L^q$ off-diagonal estimates, which is exactly the previous proposition.

Proof. Proposition .1.6

Let's define $\mathbb{D}_\tau(\mathbb{R}^{n+1})$ the dyadic grid such that for $I \in \mathbb{D}_\tau(\mathbb{R}^{n+1})$, $\tau \approx \ell^2(I)$,

$f_k = f \cdot \mathbb{1}_{S_k(I)}$, with $S_0(I) = 2I$, and $S_k(I) = 2^{k+1}I \setminus 2^k I$.

Claim .1.9.

$$\left(\fint\!\!\fint_I |e^{-\tau L} f_k|^q(X) dX \right)^{\frac{1}{q}} \leq Ce^{-\frac{dist^2(I,S_k(I))}{C\cdot\tau}} \left(\fint\!\!\fint_{S_k(I)} |f|^2(X) dX \right)^{\frac{1}{2}}.$$

Claim .1.10. If previous claim is true, then

$$\left(\fint\!\!\fint_I |e^{-\tau L} f|^q(X) dX \right)^{\frac{1}{q}} \leq C \inf_{X \ni I} \left(\mathcal{M}(|f|^2) \right)^{\frac{1}{2}} (X).$$

Assume both claims .1.9 and .1.10 are true.

$$\begin{aligned}
\iint_{\mathbb{R}^{n+1}} |e^{-\tau L} f(X)|^q dX &= \sum_{I \in \mathbb{D}_\tau} \fint\!\!\fint_I |e^{-\tau L} f(X)|^q dX \cdot |I| \\
&\leq C \sum_{I \in \mathbb{D}_\tau} \inf_{X \ni I} \left(\mathcal{M}(|f|^2)(X) \right)^{\frac{q}{2}} \cdot |I| \\
&\leq C \sum_{I \in \mathbb{D}_\tau} \iint_I \left(\mathcal{M}(|f|^2)(X) \right)^{\frac{q}{2}} dX \\
&\leq C \iint_{\mathbb{R}^{n+1}} \left(\mathcal{M}(|f|^2)(X) \right)^{\frac{q}{2}} dX \\
&\leq C \iint_{\mathbb{R}^{n+1}} |f|^q(X) dX.
\end{aligned}$$

□

Proof. Claim .1.10

$$\begin{aligned}
\left(\fint\!\!\fint_I |e^{-\tau L} f(X)|^q dX \right)^{\frac{1}{q}} &\leq \sum_K \left(\fint\!\!\fint_I |e^{-\tau L} f_k(X)|^q dX \right)^{\frac{1}{q}} \\
&\leq C \sum_k e^{-\frac{dist^2(I, S_k(I))}{C \cdot \tau}} \left(\fint\!\!\fint_{S_k(I)} |f|^2(X) dX \right)^{\frac{1}{2}} \\
&\leq C \inf_{X \in I} \left(\mathcal{M}(|f|^2) \right)^{\frac{1}{2}} (X),
\end{aligned}$$

in the last inequality we have used that

$dist(I, S_k(I)) \approx (2^k \ell(I)) \Rightarrow dist^2(I, S_k(I)) \approx (2^k \ell(I))^2, \tau \approx \ell(I)^2 \Rightarrow \frac{dist^2(I, S_k(I))}{C \cdot \tau} \approx 2^{2k}$

$\sum_k e^{-2^{2k}} \leq C$

$\fint\!\!\fint_{S_k(I)} |f|^2(X) dX \leq C \fint\!\!\fint_{2^{k+1} I} |f|^2(X) dX \leq C \sup_{J \ni X} \fint\!\!\fint_J |f|^2(X) dX, \ \forall X \in I.$ □

Proof. Claim .1.9 We are going to prove it with exponent $s = 2^* = \frac{2n}{n-2}$.

We know that the following results are true:

$$\left(\fint\!\!\fint_I |e^{-\tau L} f_k(X)|^2 dX\right)^{\frac{1}{2}} \leq e^{-\frac{dist^2(I,S_k(I))}{C\cdot\tau}} \left(\fint\!\!\fint_{S_k(I)} |f|^2(X) dX\right)^{\frac{1}{2}};$$

$$\left(\fint\!\!\fint_I |\sqrt{\tau}\nabla_x e^{-\tau L} f_k(X)|^2 dX\right)^{\frac{1}{2}} \leq e^{-\frac{dist^2(I,S_k(I))}{C\cdot\tau}} \left(\fint\!\!\fint_{S_k(I)} |f|^2(X) dX\right)^{\frac{1}{2}};$$

$$\left(\fint\!\!\fint_I |\tau\partial_t e^{-\tau L} f_k(X)|^2 dX\right)^{\frac{1}{2}} \leq e^{-\frac{dist^2(I,S_k(I))}{C\cdot\tau}} \left(\fint\!\!\fint_{S_k(I)} |f|^2(X) dX\right)^{\frac{1}{2}}.$$

□

Define $h := \ell(I) = \ell(Q)$.

$$\begin{aligned}
\left(\fint\!\!\fint_I |e^{-\tau L} f_k(x,t)|^{2^*} dxdt\right)^{\frac{1}{2^*}} &\leq \left(\fint\!\!\fint_I |e^{-\tau L} f_k(x,t) - C_t|^{2^*} dxdt\right)^{\frac{1}{2^*}} + \left(\fint_a^{a+h} |C_t|^{2^*} dt\right)^{\frac{1}{2^*}} \\
&= \left(\fint_a^{a+h} \fint_Q |e^{-\tau L} f_k(x,t) - C_t|^{2^*} dxdt\right)^{\frac{1}{2^*}} + \left(\fint_a^{a+h} |C_t|^{2^*} dt\right)^{\frac{1}{2^*}} \\
&\leq C \sup_{t\in[a,a+h]} \left(\fint_Q |e^{-\tau L} f_k(x,t) - C_t|^{2^*} dx\right)^{\frac{1}{2^*}} + \sup_{t\in[a,a+h]} |C_t| \\
&:= I + II.
\end{aligned}$$

Choose $C_t = \left(\fint_Q |2^{\tau L} f_k(x,t)|^2 dx\right)^{\frac{1}{2}}$.

$$\begin{aligned}
|I| &= \sup_{t\in[a,a+h]} \left(\fint_Q |e^{-\tau L} f_k(x,t) - C_t|^{2^*} dx\right)^{\frac{1}{2^*}} \\
&\leq C \sup_{t\in[a,a+h]} \left(\fint_Q |\sqrt{\tau}\nabla_x e^{\tau L} f_k(x,t)|^2 dx\right)^{\frac{1}{2}} \\
&\leq C \sup_{t\in[a,a+h]} \left(\fint_Q \tau |\nabla_x e^{-\tau L} f_k(x,t) - \fint_a^{a+h} \nabla_x e^{-\tau L} f_k(x,t') dt'|^2 dx\right)^{\frac{1}{2}} \\
&+ C \left(\fint_Q \tau |\fint_a^{a+h} \nabla_x e^{-\tau L} f_k(x,t') dt'|^2 dx\right)^{\frac{1}{2}} \\
&:= I_1 + I_2
\end{aligned}$$

by Poincaré.

$$
\begin{aligned}
|I_2| &= \left(\fint_Q \tau \left| \fint_a^{a+h} \nabla_x e^{-\tau L} f_k(x,t')dt' \right|^2 dx \right)^{\frac{1}{2}} \\
&\le C \left(\tau \fint_Q \fint_a^{a+h} |\nabla e^{-\tau L} f_k(x,t')|^2 dt' dx \right)^{\frac{1}{2}} \\
&\le C e^{-\frac{dist^2(I, dist(S_k(I))}{C\cdot\tau}} \left(\iint_{S_k(I)} |f|^2(X) dX \right)^{\frac{1}{2}}.
\end{aligned}
$$

By Gaffney,

$$
\begin{aligned}
|I_1| &= \sup_{t\in[a,a+h]} \left(\fint_Q \tau \left| \nabla_x e^{-\tau L} f_k(x,t) - \fint_a^{a+h} \nabla_x e^{-\tau L} f_k(x,t')dt' \right|^2 dx \right)^{\frac{1}{2}} \\
&\le \sup_{t\in[a,a+h]} \left(\fint_Q \tau \left| \fint_a^{a+h} (\nabla_x e^{-\tau L} f_k(x,t) - \nabla_x e^{-\tau L} f_k(x,t'))dt' \right|^2 dx \right)^{\frac{1}{2}} \\
&\le \sup_{t\in[a,a+h]} \left(\fint_Q \tau \left| \fint_a^{a+h} \int_{t'}^{t} \nabla_x \partial_{t''} e^{-\tau L} f_k(x,t'')dt''dt' \right|^2 dx \right)^{\frac{1}{2}} \\
&\le C \left(\fint_Q \tau^2 \fint_a^{a+h} |\nabla_x \partial_{t''} e^{-\tau L} f_k(x,t'')|^2 dt'' dx \right)^{\frac{1}{2}} \\
&\le C \left(\frac{\tau^2}{|I|} \iint_I |\nabla_x u(X)|^2 dX \right)^{\frac{1}{2}} \\
&\le C \left(\frac{\tau^2}{|I|} \iint_{\mathbb{R}^{n+1}} |\nabla_x u|^2(x) \eta_I^2(X) dX \right)^{\frac{1}{2}}.
\end{aligned}
$$

$u(X) := u(x,t'') = \partial_{t''} e^{-\tau L} f_k(x,t'')$.

$\eta_I \in C_0^\infty, \eta_I \equiv 1 \; on \; I, supp(\eta_I) \subseteq 2I, \|\nabla \eta_I\|_\infty \le \frac{C}{\ell(I)}$.

Also note that $Lu = -\partial_{t''} u$ since L is t''-independent. For simplification we make a change of variable $t = t''$.

$$
\begin{aligned}
\left(\frac{\tau^2}{|I|} \iint_{\mathbb{R}^{n+1}} |\nabla_x u|^2(x) \eta_I^2(X) dX \right)^{\frac{1}{2}} &\approx \left(Re \frac{\tau^2}{|I|} \iint_{\mathbb{R}^{n+1}} A \nabla_x u(X) \cdot \overline{\nabla_x u}(X) \eta_I^2(X) dX \right)^{\frac{1}{2}} \\
&\le C \left(\frac{\tau^2}{|I|} \iint_{\mathbb{R}^{n+1}} |\partial_t u(X)| \cdot |u(X)| \eta_I^2(X) dX \right)^{\frac{1}{2}} \\
&+ C \left(\frac{\tau^2}{|I|} \iint_{\mathbb{R}^{n+1}} |\nabla_x u(X)| \cdot |u(X)| \cdot |\nabla \eta_I(X)| \eta_I(X) dX \right)^{\frac{1}{2}} \\
&:= I_1' + I_1''.
\end{aligned}
$$

Above we have used integration by parts.

$$
\begin{aligned}
|I_1''| &= \left(\frac{\tau^2}{|I|}\iint_{\mathbb{R}^{n+1}} |\nabla_x u(X)|\cdot|u(X)|\cdot|\nabla\eta_I(X)|\eta_I(X)dX\right)^{\frac{1}{2}} \\
&\le \epsilon\left(\frac{\tau^2}{|I|}\iint_{\mathbb{R}^{n+1}} |\nabla_x(u)(X)|^2\eta_I^2(X)dX\right)^{\frac{1}{2}} + \frac{1}{\epsilon}\left(\frac{\tau}{|I|}\iint_{2I}|u(X)|^2dX\right)^{\frac{1}{2}} \\
&\le \epsilon|I_1| + \frac{1}{\epsilon}e^{-\frac{dist^2(I,S_k(I))}{C\cdot\tau}}\left(\fint\!\!\fint_{S_k(I)} |f|^2(X)dX\right)^{\frac{1}{2}}.
\end{aligned}
$$

$$
\begin{aligned}
|I_1'| &= \left(\frac{\tau^2}{|I|}\iint_{\mathbb{R}^{n+1}} |\partial_t u(X)|\cdot|u(X)|\eta_I^2(X)dX\right)^{\frac{1}{2}} \\
&\le C\left(\frac{\tau^3}{|I|}\iint_{2I}|\partial_t u(X)|^2\eta_I^2(X)dX\right)^{\frac{1}{2}} + \left(\frac{\tau}{|I|}\iint_{2I}|u(X)|^2dX\right)^{\frac{1}{2}}.
\end{aligned}
$$

$\left(\frac{\tau}{|I|}\iint_{2I}|u(X)|^2dX\right)^{\frac{1}{2}}$ is resolved as before by Gaffney.

$$
\begin{aligned}
\left(\frac{\tau^3}{|I|}\iint_{2I}|\partial_t u(X)|^2\eta_I^2(X)dX\right)^{\frac{1}{2}} &= \left(\frac{\tau^3}{|I|}\iint_{2I}|\partial_t\partial_t e^{-\tau L}f_k(x,t)|^2dxdt\right)^{\frac{1}{2}} \\
&= \left(\frac{\tau^3}{|I|}\iint_{2I}|\partial_t L e^{-\tau L}f_k(x,t)|^2dxdt\right)^{\frac{1}{2}} \\
&= \left(\frac{1}{|I|}\iint_{2I}|\sqrt{\tau}\partial_t e^{-\frac{\tau}{2}L}\tau L e^{-\frac{\tau}{2}L}f_k(x,t)|^2dxdt\right)^{\frac{1}{2}}.
\end{aligned}
$$

The composition of 2 operators that satisfy the Gaffney estimates, also satisfy the Gaffney estimates so this finishes the part I.

$$
\begin{aligned}
|II| &= \sup_{t\in[a,a+h]}\left(\fint_Q |e^{-\tau L}f_k(x,t)|^2dx\right)^{\frac{1}{2}} \\
&\le \sup_{t\in[a,a+h]}\left(\fint_Q |e^{-\tau L}f_k(x,t) - \fint_a^{a+h} e^{-\tau L}f_k(x,t')dt'|^2dx\right)^{\frac{1}{2}} + \left(\fint_Q |\fint_a^{a+h} e^{-\tau L}f_k(x,t')dt'|^2dx\right)^{\frac{1}{2}} \\
&:= II_1 + II_2.
\end{aligned}
$$

$$
\begin{aligned}
|II_2| &= \left(\fint_Q | \fint_a^{a+h} e^{-\tau L} f_k(x,t')dt'|^2 dx \right)^{\frac{1}{2}} \\
&\leq \left(\fint\!\!\fint_I |e^{-\tau L} f_k(x,t')dt'|^2 dx \right)^{\frac{1}{2}} \\
&\leq Ce^{-\frac{dist^2(I,S_k(I))}{C\cdot\tau}} \left(\fint\!\!\fint_{S_k(I)} |f|^2(X)dX \right)^{\frac{1}{2}}.
\end{aligned}
$$

$$
\begin{aligned}
|II_1| &= \sup_{t\in[a,a+h]} \left(\fint_Q |e^{-\tau L} f_k(x,t) - \fint_a^{a+h} e^{-\tau L} f_k(x,t')dt'|^2 dx \right)^{\frac{1}{2}} \\
&\leq \sup_{t\in[a,a+h]} \left(\fint_Q | \fint_a^{a+h} \int_{t'}^{t} \partial_{t''} e^{-\tau L} f_k(x,t'')dt''dt'|^2 dx \right)^{\frac{1}{2}} \\
&\leq \left(\fint_Q \tau \fint_a^{a+h} |\partial_{t''} e^{-\tau L} f_k(x,t'')|^2 dt''dx \right)^{\frac{1}{2}} \\
&\leq Ce^{-\frac{dist^2(I,S_k(I))}{C\cdot\tau}} \left(\iint_{S_k(I)} |f|^2(X)dX \right)^{\frac{1}{2}}.
\end{aligned}
$$

.2 Appendix B

In theorem 2 we needed to find a family of functions indexed over all the cubes Q. With a little modification we can be reduced to work only with dyadic cubes. In order to do so let's introduce some notation first.

Definition .2.1. Fix $x \in \partial\mathbb{R}^{n+1}_+$. Then we define the dyadic cone $\tilde{\Gamma}(x)$ with vertex x to be

$$\tilde{\Gamma}(x) := \bigcup_{Q \ni x} \mathcal{U}_Q$$

where

$$\mathcal{U}_Q := Q \times \left(\frac{\ell(Q)}{2}, \ell(Q) \right).$$

The Q-truncated dyadic cone will be denoted by

$$\tilde{\Gamma}_Q(x) := \bigcup_{\substack{Q' \ni x \\ Q' \subseteq Q}} \mathcal{U}_{Q'}.$$

Then Theorem 3.0.4 is also satisfied if the system of $\{b_Q\}$ functions are reduced to a family of functions indexed by dyadic cubes satisfying the same conditions as before by changing condition 3.0.8 by

$$\int_Q \left(\iint_{\tilde{\Gamma}_Q(x)} |\theta_t b_Q(y)|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{p}{2}} dx \leq C_0 |Q|. \tag{.2.2}$$

Let's detail how this modify the proof of the Theorem starting for the modifications on the lemma and sublemma.

Lemma .2.1. *Suppose that* $\exists \eta \in (0,1)$ *and* $C < \infty$ *such that for every dyadic cube* $Q \in \mathbb{R}^n$*, there exists a family* $\{Q_j\}$ *of non-overlapping dyadic subcubes of Q, with the properties*

$$\sum_j |Q_j| \leq (1-\eta)|Q| \tag{.2.3}$$

and

$$\int_Q \left(\iint_{\tilde{\gamma}_Q(x)} |\theta_t 1(y)|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{p}{2}} dx \leq C|Q| \tag{.2.4}$$

where

$$\tilde{\gamma}_Q(x) := \bigcup_{\substack{Q' \ni x \\ Q' \in Good(Q)}} \mathcal{U}_{Q'}$$

and

$$Good(Q) := \{Q' \subseteq Q : either\ Q' \cap Q_k = \emptyset,\ for\ every\ k,$$

$$or\ if\ Q_k \cap Q' \neq \emptyset,\ for\ some\ k,\ \ell(Q') > \ell(Q_k)\}$$

Then the Carleson measure estimate (3.0.12) *holds.*

Sublemma .2.2. *Suppose that* $\exists N < \infty$ *and* $\alpha \in (0, \frac{1}{2}]$, $\beta \in (0,1)$ *such that for every cube* Q

$$|\{x \in Q : G_Q(x) > N\}| \leq (1-\beta)|Q|,$$

where $G_Q(x) := \left(\iint_{\tilde{\Gamma}_Q(x)} |\theta_t 1(y)|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}}$.

Then the Carleson measure estimate (3.0.12) *holds.*

Regarding the proofs, first we have that the **conditions of the theorem implies the conditions of the lemma**. Our family $\{Q_j\}$ is going to be equal the family $\{\tilde{Q}_j\}$ on the original proof and the function $A_{m,t}$ identical to the one in the original proof.

First part of this proof was reducing ourselves from

$$\int_Q \left(\iint_{\tilde{\gamma}_Q(x)} |\theta_t 1(y)|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{p}{2}} dx$$

to work with

$$\int_Q \left(\iint_{\tilde{\Gamma}_Q(x)} |R_t b_Q(y)|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{p}{2}} dx$$

where $R_t b_Q(y) = \theta 1(y) A_{m,t} b_Q(y) - \theta_t b_Q(y)$.

The proof continues identical to the original just substituting the projection of the cone $|x - y| < t < \ell(Q)$ by $Q(x,t)$ where $Q(x,t)$ as before is the minimal dyadic cube that contains x and have side length at least t.

Regarding **Conditions of the lemma implies the conditions of the sublemma** is standard by Chebychev as before just noting that if $x \in Q \setminus \bigcup Q_j$ then $\tilde{\Gamma}_Q(x) = \tilde{\gamma}_Q(x)$.

Finally we are concerned about **the proof of the sublemma**. In order to proof this we need to add some definitions in order to "truncate" the dyadic cones by below. We define

$$K(\epsilon) := \sup_Q \frac{1}{|Q|} \int_Q \iint_{\tilde{\Gamma}_{Q,\epsilon}(x)} |\theta_t 1(y)|^2 \frac{dydt}{t^{n+1}} dx$$

where

$$\tilde{\Gamma}_{Q,\epsilon} := \bigcup_{\substack{Q' \ni x \\ Q' \subseteq Q \\ \epsilon < \ell(Q') < \frac{1}{\epsilon}}} \mathcal{U}_{Q'} \text{ and } \tilde{\gamma}_{Q,\epsilon}(x) := \bigcup_{\substack{Q' \ni x \\ Q' \in Good(Q) \\ \epsilon < \ell(Q') < \frac{1}{\epsilon}}} \mathcal{U}_{Q'}$$

With this definition $\Omega_{N,\epsilon}$ is not an open set any more but by outer regularity, we may choose $O_{N,\epsilon}$, open subset of Q, such that $O_{N,\epsilon} \supseteq \Omega_{N,\epsilon}$, with $|O_{N,\epsilon} < (1 - \frac{\beta}{2})|Q|$. Since $O_{N,\epsilon}$ is open we can make a Whitney decomposition of such set such that $O_{N,\epsilon} = \bigcup_j Q_j$ and we set $F_{N,\epsilon} := Q \setminus O_{N,\epsilon}$.

The end of the proof is identical to the proof of lemma 4.1 on [GM] which use the following facts:

$$\int_Q \iint_{\tilde{\Gamma}_{Q,\epsilon}(x)} |\theta_t 1(y)|^2 \frac{dydt}{t^{n+1}} dx = \int_{F_{N,\epsilon}} \iint_{\tilde{\Gamma}_{Q,\epsilon}(x)} |\theta_t 1(y)|^2 \frac{dydt}{t^{n+1}} dx + \int_{O_{N,\epsilon}} \iint_{\tilde{\Gamma}_{Q,\epsilon}(x)} |\theta_t 1(y)|^2 \frac{dydt}{t^{n+1}} dx$$

$$= \int_{F_{N,\epsilon}} \iint_{\tilde{\Gamma}_{Q,\epsilon}(x)} |\theta_t 1(y)|^2 \frac{dydt}{t^{n+1}} dx + \sum_j \int_{Q_j} \iint_{\tilde{\Gamma}_{Q_j,\epsilon}(x)} |\theta_t 1(y)|^2 \frac{dydt}{t^{n+1}} dx + \sum_j \int_{Q_j} \iint_{\tilde{\gamma}_{Q,\epsilon}(x)} |\theta_t 1(y)|^2 \frac{dydt}{t^{n+1}} dx$$

$$= I + II + III.$$

By the Whitney decomposition $I \leq C_{N,p}|Q|$ and $II \leq (1 - \frac{\beta}{2})K(\epsilon)|Q|$. Regarding III we fix a cube Q_j and $x \in Q_j$ then by the definition of our cones and by the maximality of $Q_j's$, there exists a point x_j in the dyadic father of Q_j, say Q_j^* such that $x_j \in F_{N,\epsilon}$. Therefore, since $\tilde{\gamma}_{Q,\epsilon}(x) \subseteq \tilde{\gamma}_{Q,\epsilon}(x_j)$ we obtain $III \leq C_{N,p}(1 - \frac{\beta}{2})|Q|$.

Our conclusion follows from the fact that $K(\epsilon) \leq C_{N,p,\beta}$ $\forall \epsilon \in (0,1)$ and letting $\epsilon \searrow 0$.

.3 Appendix C

For theorem 3 we can make modifications as in the previous appendix in order to work only with dyadic cubes instead of working with all cubes, i.e. theorem 4.0.2 is also satisfied if the family of functions $\{b_Q\}$ are reduced to a family of functions indexed by the dyadic cubes Q satisfying the same conditions as before changing condition 4.0.8 by

$$\int_Q \left(\iint_{\tilde{\Gamma}_Q(x)} |\Theta_t \cdot b_Q(y)|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{p}{2}} dx \leq C_0 |Q|. \tag{.3.1}$$

Let's state now the new Lemma and Sublemma used to prove this modified version of the theorem.

Lemma .3.1. *Suppose that there exists $\eta \in (0,1)$, $\epsilon > 0$ small and $C < \infty$ such that for every dyadic cube $Q \in \mathbb{R}^n$, there is a family Q_j of non-overlapping dyadic subcubes of Q verifying:*

$$\sum_j |Q_j| \leq (1-\eta)|Q|,$$

and

$$\int_Q \left(\iint_{\tilde{\gamma}_Q(x)} |\Theta_t 1(y)|^2 \mathbb{1}_{\Gamma_k^{2\epsilon}}(\Theta_t 1(y)) \frac{dydt}{t^{n+1}} \right)^{\frac{p}{2}} dx \leq C|Q|,$$

for every cone of aperture 2ϵ.

Then the Carlesson measure estimate 4.0.12 holds.

Sublemma .3.2. *Suppose that $\exists N < \infty$ and $\beta \in (0,1)$ such that for every cube Q and*

$$|\{x \in Q : G_Q^k(x) > N\}| \leq (1-\beta)|Q|,$$

for all k, where

$$G^k_Q(x) = \left(\iint_{\tilde{\Gamma}_Q(x)} |\Theta_t 1(y)|^2 \mathbb{1}_{\Gamma^{2\epsilon}_k}(\Theta_t 1(y)) \frac{dtdy}{t^{n+1}} \right)^{\frac{1}{2}}.$$

Then the Carleson measure estimate 4.0.12 holds.

Once that we have all the "ingredients" we start making the comments on the proofs starting with **the conditions of the theorem implies the conditions of the lemma**

Our family $\{Q_j\}$ is going to be equal the family $\{\tilde{Q}_j\}$ on the original proof.

Note that $\sum_{Q' \subseteq Q} \mathcal{U}_{Q'} = R_Q$ and $\sum_{\substack{Q' \subseteq Q \\ Q' \in Good(Q)}} \mathcal{U}_{Q'} = R_Q \setminus \bigcup_j R_{Q_j}$ where $R_Q := Q \times (0, \ell(Q))$. This implies that for every $x \in Q$ if $(y,t) \in \tilde{\gamma}_Q(x)$ then $(y,t) \in R_Q \setminus \bigcup_j R_{Q_j}$.

Then

$$\int_Q \left(\iint_{\tilde{\gamma}_Q(x)} |\Theta_t 1(y)|^2 \mathbb{1}_{\Gamma^{2\epsilon}_k}(\Theta_t 1(y)) \tfrac{dydt}{t^{n+1}} \right)^{\frac{p}{2}} dx \leq \int_Q \left(\iint_{\tilde{\Gamma}_Q(x)} |\Theta_t 1(y) \cdot A_{m,t} b_Q(y) \bar{v}|^2 \tfrac{dydt}{t^{n+1}} \right)^{\frac{p}{2}}$$

And then as in the previous appendix the proof follows from the original changing the projection of the cone $|x-y| < t < \ell(Q)$ for $Q(x,t)$.

Conditions of the lemma imply the conditions of the sublemma and the proof of the sublemma are as in the previous appendix with

$$K(\lambda) := \sup_Q \frac{1}{|Q|} \int_Q \iint_{\tilde{\Gamma}_{Q,\lambda}} |\Theta_t 1(y)|^2 \mathbb{1}_{\Gamma^\epsilon}(\Theta_t 1(y)) \frac{dydt}{t^{n+1}} dx.$$

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