

**The Evans function, the Weyl-Titchmarsh function, and the
Birman-Schwinger operators**

A Dissertation
presented to
the Faculty of the Graduate School
University of Missouri

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

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July 2012

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The Evans function, the Weyl-Titchmarsh function, and the Birman-Schwinger operators

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I am forever indebted to my mother and brother, Azize Sukhtayeva and Selim Sukhtaiev, for their inspiration, support and unconditional love. I thank you from the bottom of my heart.

ACKNOWLEDGEMENTS

First of all, I wish to express sincere gratitude to my advisor, Professor Yuri Latushkin, for his vision and direction. Professor Latushkin was a consistent source of learning and warm support and he thoughtfully guided me through all stages of my doctoral degree. His teachings and encouragement will continue to instruct and inspire long beyond my years as a graduate student.

I would also like to thank my committee members Dr. Carmen Chicone, Dr. Fritz Gesztesy, Dr. Marius Mitrea and Dr. David Retzloff for their support and encouragement.

I sincerely thank Dr. Fritz Gesztesy and Dr. Konstantin Makarov whose participation at the initial stage of this project was crucial for the thesis.

I thank my colleagues Alin Pogan and Valerian Yurov for useful mathematical discussions and constructive feedback regarding my research work.

My special thanks to MU Mathematics department's academic support staff Amy Crews and Gwen Gilpin for their "ever ready" help and support.

TABLE OF CONTENTS

ACKNOWLEDGMENTS	iii
ABSTRACT	vi
1. INTRODUCTION	1
2. The algebraic multiplicity of eigenvalues and the Evans function	11
2.1. Abstract perturbation theory	11
2.2. Applications	23
2.2.1. The Schrödinger equation	23
2.2.2. Degenerate reaction-diffusion systems	27
2.2.3. General n -th order linear differential equations	34
3. The Evans function and the Weyl-Titchmarsh function	39
3.1. Singular Sturm-Liouville differential expressions	39
3.2. The reduced Jost function	55
3.3. The Weyl-Titchmarsh M -function and Hamiltonian systems	65
3.4. Examples	82
BIBLIOGRAPHY	87
VITA	94

The Evans function, the Weyl-Titchmarsh function, and the Birman-Schwinger
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ABSTRACT

We focus on the spectral stability of travelling wave solutions of partial differential equations. First, we use the Gohberg-Rouche Theorem to prove equality of the algebraic multiplicity of an isolated eigenvalue of an abstract operator on a Hilbert space, and the algebraic multiplicity of the eigenvalue of the corresponding Birman-Schwinger type operator pencil. Next, we apply this result to discuss three particular classes of problems: the Schrödinger operator, the operator obtained by linearizing a degenerate system of reaction diffusion equations about a pulse, and a general high order differential operator. We study relations between the algebraic multiplicity of an isolated eigenvalue for the respective operators, and the order of the eigenvalue as the zero of the Evans function for the corresponding first order system.

We also describe relations between the Evans function, a modern tool in the study of stability of traveling waves and other patterns for PDEs, and the classical Weyl-Titchmarsh function for singular Sturm-Liouville differential expressions and for matrix Hamiltonian systems. Also, for the scalar Schrödinger equation, we discuss a related issue of approximating eigenvalue problems on the whole line by that on finite segments.

Chapter 1

Introduction

The purpose of this work is to study connections between the Evans function, a modern tool in stability theory for traveling wave solutions of partial differential equations, and such classical objects in perturbation theory for differential operators as the Weyl-Titchmarsh function and the Birman-Schwinger operators. The results of this dissertation are published in [41, 42].

In the first part of this thesis, we continue the work began in [20, 21, 22, 24], and investigate further the connections of the Evans function and (modified) Fredholm determinants of the Birman-Schwinger type operators. In particular, we bring into the discussion a new element, the Gohberg-Rouche Theorem [28, Theorem XI.9.1]. Also, we study in details three important concrete cases: the Schrödinger operator, the operator obtained by linearizing a system of degenerate reaction diffusion equations about a traveling wave, and a general high order differential operator.

In the second part of this thesis, we also establish connections between the classical Weyl-Titchmarsh function for Hamiltonian ordinary differential equations, and the Evans function, a Wronskian-type determinant designed to detect point spectrum of ordinary differential operators arising in the study of stability of traveling waves and other patterns for partial differential equations. As a byproduct, for the scalar

Schrödinger equation, we give a formula relating the Evans function for problems on the full line to that on finite segments.

In Section 2.1, we deal with abstract perturbations. We first recall well-known results from [28, Ch. XI] regarding the algebraic multiplicity $m(\lambda_0; W(\cdot))$ of an isolated eigenvalue λ_0 of finite type for an operator pencil $W = W(\lambda)$. Next, following [21, 24], we consider a class of factorable non-self-adjoint perturbations, formally given by B^*A , of a given unperturbed non-self-adjoint operator H_0 in a Hilbert space by introducing a densely defined, closed linear operator H which represents an extension of $H_0 + B^*A$. Furthermore, we describe the properties of the Birman-Schwinger type operator pencil $K = K(\lambda)$ associated with H_0 and H by the formula $K(\lambda) = -\overline{A(H_0 - \lambda)^{-1}B^*}$. Under appropriate assumptions (including that λ_0 is an isolated eigenvalue of H of finite algebraic multiplicity denoted by $m(\lambda_0; H)$), and using the Gohberg-Rouche Theorem, we show the equality $m(\lambda_0; I - K(\cdot)) = m(\lambda_0; H)$ (see Theorem 2.1.19). In turn, this leads to the fact that $m(\lambda_0; H)$ is the order of λ_0 as the zero of the modified Fredholm determinant, that is, $\det_2(I - K(\lambda)) = (\lambda - \lambda_0)^{m(\lambda_0; H)}S(\lambda)$, $S(\lambda_0) \neq 0$, (see Theorem 2.1.21). We mention [29, 30] where yet another application of the Gohberg-Rouche Theorem is given.

In Section 2.2 we discuss three particular classes of problems: the Schrödinger equation, the degenerate reaction-diffusion system of equations, and a general higher order differential operator with constant leading coefficient. The main tool in our investigation is the connection of the (modified) Fredholm determinant of the Birman-Schwinger type integral operator for the linearized eigenvalue problem of a given partial differential equation, and the Evans function for the equivalent to this eigen-

value problem first order system, see [20, 34, 35]. Our strategy can be described as follows. Consider the (higher order, space dimension one) differential operator H obtained by linearizing a partial differential equation along a steady state or traveling wave solution. The operator H is a perturbation of the operator H_0 determined by the asymptotic behavior of the solution. As in Section 2.1, we associate to H and H_0 a Birman-Schwinger type operator pencil $I - K(\cdot)$. To pass to the Evans function analysis, we re-write the eigenvalue problem $Hu = \lambda u$ for H as a first order system of differential equations $dy/dx = \mathbb{M}(x, \lambda)y(x)$, $x \in \mathbb{R}$, and consider its Evans function $\mathbb{E} = \mathbb{E}(\lambda)$, see [4, 18, 44, 48, 55]. Also, we consider the corresponding first order differential operator $T(\lambda) = \partial_x - \mathbb{M}(x, \lambda)$. The operator $T(\lambda)$ is a perturbation of the first order differential operator $T_0(\lambda)$ obtained from the eigenvalue problem $H_0u = \lambda u$ for H_0 . We associate to $T(\lambda)$ and $T_0(\lambda)$ a Birman-Schwinger type operator pencil $I - \mathbb{K}(\lambda)$. For the three classes of problems considered in Section 2.2 we show that the (modified) Fredholm determinants for $I - K(\lambda)$ and $I - \mathbb{K}(\lambda)$ are equal (see Lemma 2.2.4, (2.2.52) and (2.2.72)). Now the abstract results from Section 2.1 imply that the algebraic multiplicity $m(\lambda_0; H)$ of a discrete eigenvalue λ_0 of H coincides with the multiplicity of λ_0 as the zero of the function $\det_2(I - \mathbb{K}(\cdot))$. We recall that the main result in [20] is an explicit formula relating $\det_2(I - \mathbb{K}(\cdot))$ and the Evans function $\mathbb{E} = \mathbb{E}(\lambda)$ for the equation $dy/dx = \mathbb{M}(x, \lambda)y(x)$. This leads to the equalities

$$\begin{aligned} \det_2(I - K(\lambda)) &= \det_2(I - \mathbb{K}(\lambda)) = e^{\Theta(\lambda)}\mathbb{E}(\lambda) \\ &= (\lambda - \lambda_0)^{m(\lambda_0; H)}S(\lambda), \text{ where } S(\lambda_0) \neq 0, \end{aligned} \tag{1.0.1}$$

and $\Theta(\lambda)$ is an analytic in λ function, explicitly computed in [20].

In particular, (1.0.1) shows that the algebraic multiplicity $m(\lambda_0; H)$ is equal to the multiplicity of λ_0 as a zero of the Evans function. The latter assertion is well-known

and proved in many concrete situations, see, e.g., [4, 18, 48, 44, 55] and the literature therein. To conclude this introduction, we will briefly review the main insight in the classical strategy of the proof of this assertion as it is quite different from ours. First, one remarks that λ_0 is an eigenvalue of H if and only if 0 is an eigenvalue of $T(\lambda_0)$. However, unlike H , the operator $T(\lambda_0)$ does not have isolated eigenvalues (in fact, it is easy to see that the spectrum of $T(\lambda_0)$ is invariant with respect to vertical translations, cf. [11, Prop. 2.36(b)]). Thus, the “algebraic multiplicity” of 0 as an eigenvalue of $T(\lambda_0)$ is defined via the lengths of the Jordan chains. Namely, since the higher order differential equation $(H - \lambda_0)u = 0$ generates the first order differential equation $dy/dx = \mathbb{M}(x, \lambda_0)y(x)$, one observes that a Jordan chain $\{u_j\}_{j=1}^\ell$ for λ_0 , satisfying $u_{j-1} = (H - \lambda_0)u_j$, $u_0 = 0$, $j = 1, \dots, \ell$, generates the chain of functions $\{y_j\}_{j=1}^\ell$, satisfying $T(\lambda_0)y_j = \mathbb{M}^\bullet(\cdot, \lambda_0)y_{j-1}$, $j = 1, \dots, \ell$, $y_0 = 0$. Here, \bullet denotes differentiation in λ . Differentiating in λ the first order differential equation $dy/dx = \mathbb{M}(x, \lambda_0)y(x)$, we arrive at the equation $dy^\bullet/dx = \mathbb{M}(x, \lambda_0)y^\bullet(x) + \mathbb{M}^\bullet(x, \lambda_0)y$ which, in fact, is very close to the equation for the Jordan chain. Using this main observation, the equality of the algebraic multiplicity of λ_0 and the multiplicity as the zero of the Evans function follows using some elementary but extremely clever computations with the derivative of the latter, see, e.g., [4, 18, 48, 44, 55] and the literature therein.

We will now describe the part of the thesis related to the Weyl-Titchmarsh function. In addition to the classical sources [5, 14, 15] of the Weyl-Titchmarsh theory, we refer to recent work [12, 25, 26, 27, 38] containing illuminating surveys, and, especially, to [19]. There is of course big literature on approximating the spectra of differential

operators on the line by the spectra of respective operators on finite segments, see, for instance, [7, 9, 31, 45, 49, 53, 57] and the literature cited therein. Finally, we refer to [4, 17, 18, 20, 34, 35, 44, 48, 49, 50, 54, 55] for the general discussion of the Evans function some relevant aspects of which will be briefly reviewed next.

Given a partial differential equation in one space dimension, say, a reaction diffusion equation on $(-\infty, +\infty)$, one is interested in studying stability of such special solutions as traveling waves. Passing to the moving coordinates and linearizing the partial differential equation about the wave, one considers an eigenvalue problem, $Hu = zu$, for the linearized operator H , an ordinary differential operator of some order. The instability of the wave is related to the presence of unstable eigenvalues, and thus the problem of locating the isolated eigenvalues is of importance. Rewriting the high order differential equation $Hu = zu$ as a first order matrix differential equation, one is looking for the values of the spectral parameter z for which the latter equation has solutions exponentially decaying at both plus and minus infinity. These values are the zeros of the Evans function, defined as the determinant of the matrix whose columns are the solutions of the matrix differential equations that are exponentially decaying at $+\infty$ and at $-\infty$. For instance, in the particular case when H is the Schrödinger operator with a summable real valued potential on the line, the Evans function is known to be equal, see [20, 35], to the Jost function, that is, to the rescaled Wronskian of the exponentially decaying at $+\infty$ and $-\infty$ Jost solutions of the Schrödinger equation, see, e.g., [10, Chapter XVII].

For comparison, we now briefly outline the main features of the Weyl-Titchmarsh and the Evans functions. The Weyl-Titchmarsh function can be constructed for

selfadjoint problems, and captures many of their important properties (in particular, it determines the potential of the Schrödinger operator on the half-line). The Evans function can be defined for quite general and, most notably, *non-selfadjoint* problems. The main computational advantage of the Weyl-Titchmarsh versus the Evans function is that the solutions defining the Evans function satisfy some *asymptotic* boundary conditions while the solutions used to construct the Weyl-Titchmarsh function satisfy usual boundary conditions at a finite point. The Evans function is designed to handle problems only on the full line with two singular ends, $+\infty$ and $-\infty$; both singular ends must be infinite, and both must be present in the construction. The Weyl-Titchmarsh function is constructed for problems with at least one regular end, while the other end can be either singular or regular; the singular end, if present, can be either finite or infinite. It is just heuristically plausible to expect that the Evans function can be calculated via two Weyl-Titchmarsh functions, each of them corresponding to a “simpler” problem having just one singular end. In fact, this calculation is the main result of the current paper. One of the many nice features of the Weyl-Titchmarsh function is that it is constructed by approximation, that is, using respective objects defined on regular (in particular, finite) segments. This is very much in concert with the approach of approximating the Evans function by respective objects on finite segments frequently used by numerical analysts working in stability analysis of traveling waves. And, finally, as an open direction for future research, we note that at this point we do not know of any literature that discusses direct applications of the Weyl-Titchmarsh function in stability analysis of traveling waves.

A typical result, see, e.g., Corollary 3.1.17, is that the Evans function (matching

solutions at the two singular ends) is, essentially, equal to the difference of the two Weyl-Titchmarsh functions (corresponding to each of the singular ends). The main novel result of the paper, Theorem 3.3.10, is a formula expressing the Evans function via the matrix Weyl-Titchmarsh M -function for a quite general matrix coefficient Hamiltonian equation. In particular, these results reduce the calculation of the Evans function on the line to the calculation of two Weyl-Titchmarsh functions on the half-lines, cf. Example 3.1.18.

We do not really know of any previous literature relating the Evans and the Weyl-Titchmarsh functions. The relation between the Jost and the Weyl-Titchmarsh functions, however, should look familiar to experts in the Weyl-Titchmarsh theory although we were not able to pinpoint in the literature the exact facts that we want. Our methods are totally elementary, e.g., we even do not use connections of the Weyl-Titchmarsh functions and spectral measures.

In Section 3.1 we study general Sturm-Liouville differential operators on $(a, b) \subset \mathbb{R}$ where each end can be singular (finite or infinite, either in the limit point or in the limit circle case). After a brief review of the Weyl-Titchmarsh theory, we construct the solutions of the Sturm-Liouville equation whose Wronskian is equal to zero exactly at the points of the discrete spectrum of the differential operator. This could be viewed as constructing an analogue of the Evans function in our situation. In Theorem 3.1.11 we offer formulas relating this Wronskian and the Weyl-Titchmarsh function for the values of the spectral parameter z outside of the essential spectrum. (Formulas of this type are of course well known for non-real values of z ; here, we just carefully recorded what happens on the real line, in particular, at the points of the discrete

spectrum). We then use the Weyl-Titchmarsh functions to describe the discrete spectrum, see Corollary 3.1.13. At the end of Section 3.1 we specialize to the situation of the Schrödinger operator on the line with a summable potential when the situation becomes more transparent.

In Section 3.2, still working on the model case of the scalar Schrödinger equation, we discuss how to approximate the Jost (or the Evans) function, \mathcal{J} , on the line by restricting the problem to finite intervals $[-L, L]$ and imposing some boundary conditions at $\pm L$. This is a pilot project as we suspect that similar results hold for a much more general setting. Of course, this topic belongs to a huge and well-studied area, but we are mainly inspired by important results in [49] (see also [7, 45]) obtained in a much more general (non-selfadjoint) situation. In [49] the (multiplicity of the) zeros of the full line Evans function, $D_\infty(z)$, were related to the (multiplicity of the) zeros of the Evans function, $D_L(z)$, for a boundary value problem on a long domain $[-L, L]$. Our contribution here is a “product formula”, see Theorem 3.2.3, directly relating, for the scalar Schrödinger equation, the asymptotic behavior for large L of the Jost function \mathcal{J}_L on the finite segments to that of the product of \mathcal{J} and some quantities induced by the boundary conditions, see Remark 3.2.4 for further comments. It is an interesting but still open question if the “product formula” holds for more general classes of equations.

In Section 3.3 we study the first order *matrix* self-adjoint Hamiltonian differential systems on $(a, b) \subset \mathbb{R}$ for which the matrix valued Weyl-Titchmarsh theory is available, see [12, 13, 32, 38, 39, 52]. Considering either the limit point or the limit circle case at either end, we describe square summable at each end solutions and

match them by means of a matrix valued Wronskian. Our main result, Theorem 3.3.10, gives a formula relating the Evans function for the first order system and the Wronskians of the matrix valued Weyl-Titchmarsh functions.

In Section 3.4 we give a number of examples for which our results are applicable.

Notations. We use the following notation. Let \mathcal{H} and \mathcal{K} be separable complex Hilbert spaces, $(\cdot, \cdot)_{\mathcal{H}}$ and $(\cdot, \cdot)_{\mathcal{K}}$ the scalar products in \mathcal{H} and \mathcal{K} (linear in the second factor), and $I_{\mathcal{H}}$ and $I_{\mathcal{K}}$ the identity operators in \mathcal{H} and \mathcal{K} , respectively. Next, let T be a closed linear operator from $\text{dom}(T) \subseteq \mathcal{H}$ to $\text{ran}(T) \subseteq \mathcal{K}$, with $\text{dom}(T)$ and $\text{ran}(T)$ denoting the domain and range of T . The closure of a closable operator S is denoted by \overline{S} . The kernel (null space) of T is denoted by $\ker(T)$. The spectrum and resolvent set of a closed linear operator in \mathcal{H} will be denoted by $\sigma(\cdot)$ and $\rho(\cdot)$. The Banach spaces of bounded and compact linear operators in \mathcal{H} are denoted by $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}_{\infty}(\mathcal{H})$, respectively. Similarly, the Schatten-von Neumann (trace) ideals will subsequently be denoted by $\mathcal{B}_p = \mathcal{B}_p(\mathcal{H})$, $p \in \mathbb{N}$. Analogous notation $\mathcal{B}(\mathcal{H}, \mathcal{K})$, $\mathcal{B}_{\infty}(\mathcal{H}, \mathcal{K})$, etc., will be used for bounded, compact, etc., operators between two Hilbert spaces \mathcal{H} and \mathcal{K} . In addition, $\text{tr}(T)$ denotes the trace of a trace class operator $T \in \mathcal{B}_1(\mathcal{H})$ and $\det_p(I_{\mathcal{H}} + S)$ represents the (modified) Fredholm determinant associated with an operator $S \in \mathcal{B}_p(\mathcal{H})$, $p \in \mathbb{N}$, see [51]. For a closed operator T we denote by $(\text{dom}(T), \|\cdot\|_T)$ its domain equipped with the graph norm $\|f\|_T = (\|f\|^2 + \|Tf\|^2)^{1/2}$. We denote by $L^2(\mathbb{R}; dx)^n$ and $L^2(\mathbb{R}; dx)^{n \times n}$ the space of $(n \times 1)$ vector valued functions and $(n \times n)$ matrix valued functions, respectively.

We denote by $[\alpha \ \beta]$ a row-vector or a rectangular block-matrix, so that $[\alpha \ \beta]^{\top}$ is a column vector, where \top is transposition. We denote by $e_i = [\delta_{ij}]_{j=1}^n$, $i = 1, \dots, n$,

the standard unit column $(n \times 1)$ vectors in \mathbb{C}^n ; the same symbols e_i , $i = 1, \dots, 2n$, are used to denote the standard unit vectors in \mathbb{C}^{2n} . For an operator T , we denote by $\sigma(T)$ the spectrum, by $\sigma_d(T)$ the discrete spectrum (the set of isolated eigenvalues of finite algebraic multiplicity), and by $\sigma_{ess}(T) = \sigma(T) \setminus \sigma_d(T)$ the essential spectrum. Given a measurable and almost-everywhere positive function p , for f, g and pf', pg' absolutely continuous, we denote by $W_t(f, g) = f(t)pg'(t) - pf'(t)g(t)$ the value of the Wronskian $W(f, g)$ of the functions f, g at the point t (as a rule, we prefer to write $pf'(t)$ instead of $p(t)f'(t)$). Also, we will use notations $W_a(f, g) = \lim_{c \rightarrow a} W_c(f, g)$ with $a < c < 0$ and $W_b(f, g) = \lim_{d \rightarrow b} W_d(f, g)$ with $0 < d < b$. Given a measurable and almost-everywhere positive function r , we denote by $L^2(a, b)$ the Hilbert space with the scalar product $\langle f, g \rangle_{L^2(a, b)} = \int_a^b \overline{f(t)} \cdot g(t)r(t) dt$; here and below bar stands for complex conjugation.

Chapter 2

The algebraic multiplicity of eigenvalues and the Evans function

2.1 Abstract perturbation theory

To make the exposition self-contained, we begin by reminding some known facts from [28, Chap. XI]. Let $W : \Omega \rightarrow \mathcal{B}(\mathcal{H})$ be an operator-valued function analytic in an open domain Ω of the complex plane. Assume that the operator $W(\lambda_0)$ is Fredholm of index zero for some $\lambda_0 \in \Omega$. Then there exists an operator $F : \mathcal{H} \rightarrow \mathcal{H}$ of finite rank such that $W(\lambda_0) + F$ is invertible. Since $W(\lambda)$ is continuous in λ , the operator $E(\lambda) = W(\lambda) + F$ is invertible for λ in some open disc $|\lambda - \lambda_0| < \delta_0$, and thus

$$W(\lambda) = E(\lambda) - F = E(\lambda)[I - E(\lambda)^{-1}F], \quad |\lambda - \lambda_0| < \delta_0. \quad (2.1.1)$$

Since F is the operator of finite rank, $\ker F$ has a finite dimensional complement \mathcal{H}_0 in \mathcal{H} . Let P be the projection of \mathcal{H} along $\ker F$ onto \mathcal{H}_0 . It follows that

$$I - E(\lambda)^{-1}F = [I - PE(\lambda)^{-1}FP][I - (I - P)E(\lambda)^{-1}FP]. \quad (2.1.2)$$

We put $G(\lambda) = I - (I - P)E(\lambda)^{-1}FP$ and note that G is well-defined and analytic in the disc $|\lambda - \lambda_0| < \delta_0$. Furthermore, the value of G are invertible operators on \mathcal{H} ;

in fact

$$G(\lambda)^{-1} = I + (I - P)E(\lambda)^{-1}FP, \quad |\lambda - \lambda_0| < \delta_0.$$

Combining this together, we infer:

$$W(\lambda) = E(\lambda)[I - PE(\lambda)^{-1}FP]G(\lambda), \quad |\lambda - \lambda_0| < \delta_0, \quad (2.1.3)$$

where E and G are analytic operator-valued functions on $|\lambda - \lambda_0| < \delta_0$ and their values are invertible operators.

Definition 2.1.1. Let Ω be an open set in \mathbb{C} , and let $T(\cdot)$ and $S(\cdot)$ be operator-valued functions defined on Ω . Given λ_0 in Ω , we say that $T(\cdot)$ and $S(\cdot)$ are *equivalent at λ_0* if there exists an open neighborhood \mathcal{U} of λ_0 in Ω such that

$$T(\lambda) = F(\lambda)S(\lambda)E(\lambda), \quad \lambda \in \mathcal{U}, \quad (2.1.4)$$

where $F(\cdot)$ and $E(\cdot)$ are invertible operators which depend analytically on λ in \mathcal{U} .

For convenience, we isolate a part of the proof of Theorem XI.8.1 in [28] as the following lemma.

Lemma 2.1.2. [28, pp. 200-201] *Assume that*

$$W_0(\lambda) = [a_{ij}(\lambda)]_{i,j=1}^n, \quad (2.1.5)$$

where a_{ij} are scalar-valued functions that are analytic at λ_0 . Then $W_0(\cdot)$ is equivalent at λ_0 to an analytic operator-valued function D_0 of the form

$$D_0(\lambda) = \pi_0 + (\lambda - \lambda_0)^{k_1}\pi_1 + \dots + (\lambda - \lambda_0)^{k_r}\pi_r, \quad k_1 \leq k_2 \leq \dots \leq k_r, \quad (2.1.6)$$

where $\pi_0, \pi_1, \dots, \pi_r$ are mutually disjoint projections in \mathbb{C}^n such that $\text{rank } \pi_j = 1$ for $j = 1, \dots, r$.

Proof. If all entries a_{ij} are identically zero in a neighborhood of λ_0 , then the theorem is trivially true. Therefore, assume that for at least one pair (i, j) the function a_{ij} does not vanish identically in a neighborhood of λ_0 . In that case we may write

$$a_{ij}(\lambda) = (\lambda - \lambda_0)^{l(i,j)} b_{ij}(\lambda), \quad (2.1.7)$$

where $b_{ij}(\lambda_0) \neq 0$ and $l(i, j) \in \mathbb{N} \cup \{0\}$. Choose (i_0, j_0) in such a way that the number $l(i_0, j_0)$ is minimal. By renumbering rows and columns in (2.1.5) we may assume without loss of generality that $i_0 = 1, j_0 = 1$. Furthermore, by multiplying $W_0(\lambda)$ on the left by the diagonal matrix $E(\lambda) = \text{diag}[b_{11}(\lambda)^{-1}, 1, \dots, 1]$, we may suppose that $a_{11} = (\lambda - \lambda_0)^{k_1}$ and $a_{ij} = (\lambda - \lambda_0)^{k_1} c_{ij}(\lambda)$, where c_{ij} is analytic at λ_0 . Note that the diagonal matrix $E(\lambda)$ is invertible and $E(\lambda)$ depends analytically on λ in a neighborhood of λ_0 . Thus multiplication by $E(\lambda)$ produces an equivalent at λ_0 matrix-valued function $E(\lambda)W_0(\lambda)$.

Next, in the matrix $E(\lambda)W_0(\lambda)$ we subtract c_{i1} times the first row from the i -th row, that is, multiply $E(\lambda)W_0(\lambda)$ from the left by an elementary matrix which is invertible and depends analytically on λ in a neighborhood of λ_0 . In the resulting product, we subtract c_{1j} times the first column from the j -th column, and we will do this for $1 \leq i, j \leq n$. It follows that W_0 is equivalent at λ_0 to an operator function of the form

$$\begin{bmatrix} (\lambda - \lambda_0)^{k_1} & 0 & \cdots & 0 \\ 0 & \alpha_{22}(\lambda) & \cdots & \alpha_{2n}(\lambda) \\ \vdots & \vdots & & \vdots \\ 0 & \alpha_{n2}(\lambda) & \cdots & \alpha_{nn}(\lambda) \end{bmatrix},$$

where $\alpha_{ij}(\lambda) = (\lambda - \lambda_0)^{k_1} \beta_{ij}(\lambda)$ with β_{ij} analytic at λ_0 . By applying induction by the dimension of the submatrix, the lemma is proved. \square

Remark 2.1.3. The procedure, described in the proof of Lemma 2.1.2, will either (a) exhaust the dimension of the remain submatrix, or (b) the remaining submatrix will become identically zero in a neighborhood of λ_0 . In case (a), $\pi_0 + \pi_1 + \dots + \pi_r = I$.

Theorem 2.1.4. [28, Theorem XI.8.1] *Let $W : \Omega \rightarrow \mathcal{B}(\mathcal{H})$ be an analytic operator-valued function, and assume that for some $\lambda_0 \in \Omega$ the operator $W(\lambda_0)$ is Fredholm with index zero. Then W is equivalent at λ_0 to an analytic operator-valued function D of the form*

$$D(\lambda) = P_0 + (\lambda - \lambda_0)^{k_1} P_1 + \dots + (\lambda - \lambda_0)^{k_r} P_r, \quad (2.1.8)$$

where P_0, P_1, \dots, P_r are mutually disjoint projections such that P_1, \dots, P_r have rank one, the projection $I - P_0$ has finite rank, and $k_1 \leq k_2 \leq \dots \leq k_r$. Moreover, there exist operators $E(\lambda), G(\lambda)$ on \mathcal{H} and G_0 on \mathcal{H}_0 so that

$$W(\lambda) = E(\lambda)G_1(\lambda)D(\lambda)G(\lambda), \quad (2.1.9)$$

where $G_1(\lambda) = \begin{bmatrix} G_0(\lambda) & 0 \\ 0 & I_{\ker P} \end{bmatrix} : \text{ran } P \oplus \ker P \rightarrow \text{ran } P \oplus \ker P$.

In addition, $E(\cdot), G_1(\cdot), G(\cdot)$ are analytic and invertible.

Proof. According to formula (3.1.4) the operator-valued function W is equivalent at λ_0 to an operator function of the form

$$\begin{bmatrix} W_0(\cdot) & 0 \\ 0 & I_{\ker P} \end{bmatrix} : \text{ran } P \oplus \ker P \rightarrow \text{ran } P \oplus \ker P. \quad (2.1.10)$$

Here $W_0(\cdot)$ is holomorphic on $|\lambda - \lambda_0| < \delta_0$ and $W_0(\lambda)$ acts on the finite dimensional space $\mathcal{H}_0 = \text{ran } P$. Therefore, by Lemma 2.1.2 W_0 is equivalent at λ_0 to an operator-valued function D_0 of the form

$$D_0(\lambda) = \pi_0 + (\lambda - \lambda_0)^{k_1} \pi_1 + \dots + (\lambda - \lambda_0)^{k_r} \pi_r, \quad (2.1.11)$$

where $\pi_0, \pi_1, \dots, \pi_r$ are mutually disjoint projections on $\text{ran } P$ and $\text{rank } \pi_j = 1$ for $j = 1, \dots, r$. Put $P_j = \pi_j P$ for $j = 1, \dots, r$, and let

$$P_0 = \begin{bmatrix} \pi_0 & 0 \\ 0 & I_{\ker P} \end{bmatrix} : \text{ran } P \oplus \ker P \rightarrow \text{ran } P \oplus \ker P. \quad (2.1.12)$$

Then the operator function (3.1.5) (and hence W) is equivalent at λ_0 to the function

$$D(\lambda) = P_0 + (\lambda - \lambda_0)^{k_1} P_1 + \dots + (\lambda - \lambda_0)^{k_r} P_r, \quad (2.1.13)$$

and P_0, P_1, \dots, P_r have the desired properties. \square

Corollary 2.1.5. *The operators $(G_1(\lambda) - I), (D(\lambda) - I), (G(\lambda) - I)$, see (3.1.6), are of finite rank and, therefore, belong to \mathcal{B}_p for every p .*

Definition 2.1.6. We say that $\lambda_0 \in \Omega$ is an *eigenvalue of finite type* of an analytic function $W : \Omega \rightarrow \mathcal{B}(\mathcal{H})$ if $W(\lambda_0)$ is Fredholm, $W(\lambda_0)f = 0$ for some non-zero $f \in \mathcal{H}$, and $W(\lambda)$ is invertible for all λ in some punctured disc $0 < |\lambda - \lambda_0| < \varepsilon$ around λ_0 .

If λ_0 is an eigenvalue of finite type, then $\text{ind } W(\lambda_0) = 0$, and hence, by Theorem 2.1.4, the operator-valued function W is equivalent at λ_0 to the operator-valued function of the form

$$D(\lambda) = P_0 + (\lambda - \lambda_0)^{k_1} P_1 + \dots + (\lambda - \lambda_0)^{k_r} P_r, \quad (2.1.14)$$

where P_0, P_1, \dots, P_r are as in Theorem 2.1.4 and satisfy the additional condition

$$P_0 + P_1 + \dots + P_r = I, \quad (2.1.15)$$

which follows from the fact that $D(\lambda)$ is invertible for $\lambda \neq \lambda_0$ and λ sufficiently close to λ_0 .

Definition 2.1.7. The sum $k_1 + k_2 + \dots + k_r$ of the indices in (2.1.14) is called the *algebraic multiplicity* of W at λ_0 , and is denoted by $m(\lambda_0; W(\cdot))$.

The following result is called the Gohberg-Rouche Theorem.

Theorem 2.1.8. [28, Theorem XI.9.1] *Let $W : \Omega \rightarrow \mathcal{B}(\mathcal{H})$ be an analytic operator-valued function and λ_0 be an eigenvalue of finite type of $W(\cdot)$. Then, there is a Cauchy contour Γ enclosing λ_0 such that:*

$$m(\lambda_0; W(\cdot)) = \operatorname{tr} \left(\frac{1}{2\pi i} \int_{\Gamma} W'(\lambda) W(\lambda)^{-1} d\lambda \right). \quad (2.1.16)$$

Remark 2.1.9. If $W(\lambda) = \lambda I - T$, where T is a bounded linear operator on \mathcal{H} , and λ_0 is an isolated eigenvalue of $W(\lambda)$ in the sense of Definition 2.1.6, then $m(\lambda_0; W(\cdot))$ defined in Definition 2.1.7 is equal to the algebraic multiplicity of λ_0 as an eigenvalue of T , that is, to the dimension $\operatorname{tr} P$ of the range of the Riesz spectral projection. This follows from Theorem 2.1.8 and the identity

$$\operatorname{tr} \left(\frac{1}{2\pi i} \int_{\Gamma} W'(\lambda) W(\lambda)^{-1} d\lambda \right) = \operatorname{tr} \left(\frac{1}{2\pi i} \int_{\Gamma} (\lambda I - T)^{-1} d\lambda \right),$$

where Γ is a positively oriented circle centered at λ_0 such that $\sigma(T) \cap \Gamma = \emptyset$ and λ_0 is the only point in the spectrum of T inside Γ .

We will now recall the setup used in [21], and several facts proved in that paper.

Hypothesis 2.1.10. (i) *Suppose that $H_0 : \operatorname{dom}(H_0) \rightarrow \mathcal{H}$, $\operatorname{dom}(H_0) \subseteq \mathcal{H}$ is a densely defined, closed, linear operator in \mathcal{H} with nonempty resolvent set, $\rho(H_0) \neq \emptyset$,*

$A : \operatorname{dom}(A) \rightarrow \mathcal{K}$, $\operatorname{dom}(A) \subseteq \mathcal{H}$ is a densely defined, closed, linear operator from \mathcal{H} to \mathcal{K} , and

$B : \text{dom}(B) \rightarrow \mathcal{K}, \text{dom}(B) \subseteq \mathcal{H}$ is a densely defined, closed, linear operator from \mathcal{H} to \mathcal{K} such that

$$\text{dom}(A) \supseteq \text{dom}(H_0), \text{dom}(B) \supseteq \text{dom}(H_0^*). \quad (2.1.17)$$

In the following, we denote

$$R_0(z) = (H_0 - zI_{\mathcal{H}})^{-1}, z \in \rho(H_0). \quad (2.1.18)$$

(ii) For some (and hence for all) $z \in \rho(H_0)$, the operator $-AR_0(z)B^*$, defined on $\text{dom}(B^*)$, has a bounded extension in \mathcal{K} , denoted by $K(z)$,

$$K(z) = -\overline{AR_0(z)B^*} \in \mathcal{B}(\mathcal{K}). \quad (2.1.19)$$

(iii) $1 \in \rho(K(z_0))$ for some $z_0 \in \rho(H_0)$.

Lemma 2.1.11. [21, Lem. 2.2] Let $z, z_1, z_2 \in \rho(H_0)$. Then Hypothesis 3.1.14 implies the following facts:

$$AR_0(z) \in \mathcal{B}(\mathcal{H}, \mathcal{K}), \quad \overline{R_0(z)B^*} = [B(H_0^* - \bar{z})^{-1}]^* \in \mathcal{B}(\mathcal{H}, \mathcal{K}), \quad (2.1.20)$$

$$\overline{R_0(z_1)B^*} - \overline{R_0(z_2)B^*} = (z_1 - z_2)R_0(z_1)\overline{R_0(z_2)B^*} \quad (2.1.21)$$

$$= (z_1 - z_2)R_0(z_2)\overline{R_0(z_1)B^*}, \quad (2.1.22)$$

$$K(z) = -A\overline{R_0(z)B^*}, \quad K(\bar{z})^* = -B\overline{R_0(\bar{z})^*A^*}, \quad (2.1.23)$$

$$\text{ran}(\overline{R_0(z)B^*}) \subseteq \text{dom}(A), \quad \text{ran}(\overline{R_0(\bar{z})^*A^*}) \subseteq \text{dom}(B), \quad (2.1.24)$$

$$K(z_1) - K(z_2) = (z_2 - z_1)AR_0(z_1)\overline{R_0(z_2)B^*} \quad (2.1.25)$$

$$= (z_2 - z_1)AR_0(z_2)\overline{R_0(z_1)B^*}. \quad (2.1.26)$$

Corollary 2.1.12. The operator-valued function $K(\cdot)$ is analytic on $\rho(H_0)$ and

$$K'(z) = -AR_0(z)[BR_0(z)^*]^*, \quad z \in \rho(H_0). \quad (2.1.27)$$

Next, following Kato [36], one introduces

$$R(z) = R_0(z) - \overline{R_0(z)B^*} [I_{\mathcal{K}} - K(z)]^{-1} AR_0(z), \quad z \in \{\zeta \in \rho(H_0) \mid 1 \in \rho(K(\zeta))\}. \quad (2.1.28)$$

Theorem 2.1.13. [21, Theorem 2.3] *Assume Hypothesis 3.1.14 and suppose that $z \in \{\zeta \in \rho(H_0) \mid 1 \in \rho(K(\zeta))\}$. Then, $R(z)$ given in (2.1.28) defines a densely defined, closed, linear operator H in \mathcal{H} by*

$$R(z) = (H - zI_{\mathcal{H}})^{-1}. \quad (2.1.29)$$

Moreover,

$$AR(z), BR(z)^* \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \quad (2.1.30)$$

and

$$R(z) = R_0(z) - \overline{R(z)B^*} AR_0(z) \quad (2.1.31)$$

$$= R_0(z) - \overline{R_0(z)B^*} AR(z). \quad (2.1.32)$$

Proof. Suppose $z \in \{\zeta \in \rho(H_0) \mid 1 \in \rho(K(\zeta))\}$. Since, by (2.1.28),

$$AR(z) = [I_{\mathcal{K}} - K(z)]^{-1} AR_0(z), \quad (2.1.33)$$

$$BR(z)^* = [I_{\mathcal{K}} - K(z)^*]^{-1} BR_0(z)^*, \quad (2.1.34)$$

$R(z)f = 0$ implies $AR(z)f = 0$, and hence, by (2.1.33), $AR_0(z)f = 0$. The latter implies $R_0(z)f = 0$ by (2.1.28) and thus $f = 0$. Consequently,

$$\ker(R(z)) = \{0\}. \quad (2.1.35)$$

Similarly, (2.1.34) implies

$$\ker(R(z)^*) = \{0\} \quad \text{and hence} \quad \overline{\text{ran } R(z)} = \mathcal{H}. \quad (2.1.36)$$

Next, combining (2.1.28), the resolvent equation for H_0 , (2.1.21), (2.1.22), (2.1.25) and (2.1.26) proves the resolvent equation

$$R(z_1) - R(z_2) = (z_1 - z_2)R(z_1)R(z_2), \quad z_1, z_2 \in \{\zeta \in \rho(H_0) \mid 1 \in \rho(K(\zeta))\}. \quad (2.1.37)$$

Thus, $R(z)$ is indeed the resolvent of a densely defined, closed, linear operator H in \mathcal{H} as claimed in connection with (2.1.29).

By (2.1.33) and (2.1.34), $AR(z) \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $[BR(\bar{z})^*]^* \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, proving (2.1.30). A combination of (2.1.28), (2.1.33) and (2.1.34) then proves (2.1.31) and (2.1.32). \square

Hypothesis 2.1.14. *In addition to Hypothesis 3.1.14, we impose the following assumption:*

$$(iv) \quad K(z) \in \mathcal{B}_\infty(\mathcal{K}) \text{ for all } z \in \rho(H_0).$$

Theorem 2.1.15. *[21, Theorem 3.2] Assume Hypothesis 2.1.14 and let $\lambda_0 \in \rho(H_0)$.*

Then

$$Hf = \lambda_0 f, 0 \neq f \in \text{dom}(H) \quad \text{implies} \quad K(\lambda_0)g = g \quad (2.1.38)$$

where, for fixed $z_0 \in \{\zeta \in \rho(H_0) \mid 1 \in \rho(K(\zeta))\}$, $z_0 \neq \lambda_0$,

$$0 \neq g = (I_{\mathcal{K}} - K(z_0))^{-1} AR_0(z_0)f \quad (2.1.39)$$

$$= (\lambda_0 - z_0)^{-1} Af. \quad (2.1.40)$$

Conversely,

$$K(\lambda_0)g = g, 0 \neq g \in \mathcal{K} \quad \text{implies} \quad Hf = \lambda_0 f, \quad (2.1.41)$$

where

$$0 \neq f = -\overline{R_0(\lambda_0)B^*g} \in \text{dom}(H). \quad (2.1.42)$$

Moreover,

$$\dim(\ker(H - \lambda_0 I_{\mathcal{H}})) = \dim(\ker(I_{\mathcal{K}} - K(z_0))) < \infty. \quad (2.1.43)$$

In particular, let $z \in \rho(H_0)$, then

$$z \in \rho(H) \text{ if and only if } 1 \in \rho(K(z)). \quad (2.1.44)$$

Hypothesis 2.1.16. *In addition to Hypothesis 2.1.14, we assume:*

$$\lambda_0 \in \rho(H_0) \cap \{\lambda \in \mathbb{C} \mid \lambda \text{ is an isolated eigenvalue of } H\}. \quad (2.1.45)$$

Hypothesis 3.4.4 implies that $I_{\mathcal{K}} - K(\cdot)$ is an operator-valued function analytic in some neighborhood of λ_0 . Moreover, λ_0 is an eigenvalue of finite type of $I_{\mathcal{K}} - K(\cdot)$ as described in Definition 2.1.6. Therefore, the algebraic multiplicity $m(\lambda_0; I_{\mathcal{K}} - K(\cdot))$ from Definition 2.1.7 is well-defined.

Lemma 2.1.17. *[28, Section XI.9] Let G_1 and G_2 be $\mathcal{B}(\mathcal{H})$ -valued operator functions which are finitely meromorphic at λ_0 . Then $G_1 G_2$ and $G_2 G_1$ are finitely meromorphic at λ_0 and*

$$\text{tr } \Xi(G_1 G_2)(\lambda) = \text{tr } \Xi(G_2 G_1)(\lambda), \quad \lambda \neq \lambda_0, \quad (2.1.46)$$

where $\Xi(G)(\lambda)$ stands for the principal part of G at λ_0 .

We will now apply the Gohberg-Rouche Theorem 2.1.8 in the setup of [21] to show equality of the algebraic multiplicity of the eigenvalue λ_0 of finite type of $I_{\mathcal{K}} - K(\cdot)$,

see Definition 2.1.7, and the “usual” algebraic multiplicity $m(\lambda_0; H)$ of the eigenvalue λ_0 of H . We recall that $m(\lambda_0; H)$ is defined as the dimension of the range of the Riesz spectral projection:

$$m(\lambda_0; H) = -\frac{1}{2\pi i} \operatorname{tr} \left(\int_{\Gamma} d\lambda R(\lambda) \right), \quad \text{where } R(\lambda) = (H - \lambda)^{-1}. \quad (2.1.47)$$

Remark 2.1.18. $\sigma_{ess}(H_0) = \sigma_{ess}(H)$ by Theorem 4.5 [21]. Thus, Hypothesis 3.4.4 implies that λ_0 has $m(\lambda_0; H) < \infty$.

Theorem 2.1.19. *Assume Hypothesis 3.4.4. Then*

$$m(\lambda_0; I_{\mathcal{K}} - K(\cdot)) = m(\lambda_0; H). \quad (2.1.48)$$

Proof. Since λ_0 is an eigenvalue of finite type of $I_{\mathcal{K}} - K(\cdot)$, $(I_{\mathcal{K}} - K(\lambda))'(I_{\mathcal{K}} - K(\lambda))^{-1}$ is finitely meromorphic at λ_0 and

$$m(\lambda_0; I_{\mathcal{K}} - K(\cdot)) = \frac{1}{2\pi i} \operatorname{tr} \left(\int_{\Gamma} d\lambda (I_{\mathcal{K}} - K(\lambda))'(I_{\mathcal{K}} - K(\lambda))^{-1} \right). \quad (2.1.49)$$

In (2.1.49) the integrand may be replaced by the principal part of $(I_{\mathcal{K}} - K(\lambda))'(I_{\mathcal{K}} - K(\lambda))^{-1}$ at λ_0 . Then the integral exists in the trace class norm, and thus trace and integral may be interchanged. Moreover, using (2.1.27) and (2.1.34), we infer:

$$\begin{aligned} m(\lambda_0; I_{\mathcal{K}} - K(\cdot)) &= \frac{1}{2\pi i} \operatorname{tr} \left(\int_{\Gamma} d\lambda \Xi \{ (I_{\mathcal{K}} - K(\lambda))'(I_{\mathcal{K}} - K(\lambda))^{-1} \} \right) \\ &= \frac{1}{2\pi i} \operatorname{tr} \left(\int_{\Gamma} d\lambda \Xi \{ AR_0(\lambda) [BR_0(\lambda)^*]^* (I_{\mathcal{K}} - K(\lambda))^{-1} \} \right) \\ &= \frac{1}{2\pi i} \operatorname{tr} \left(\int_{\Gamma} d\lambda \Xi \{ AR_0(\lambda) [BR(\lambda)^*]^* \} \right) \\ &= \frac{1}{2\pi i} \left(\int_{\Gamma} d\lambda \operatorname{tr} \Xi \{ AR_0(\lambda) [BR(\lambda)^*]^* \} \right). \end{aligned} \quad (2.1.50)$$

Next, we note that $G_1(\lambda) := AR_0(\lambda)$ and $G_2(\lambda) := [BR(\lambda)^*]^*$ are finitely meromorphic at λ_0 [21, (4.29)-(4.30)]. Then, using (2.1.46), we arrive at the following formula:

$$m(\lambda_0; I_{\mathcal{K}} - K(\cdot)) = \frac{1}{2\pi i} \left(\int_{\Gamma} d\lambda \operatorname{tr} \Xi \{ [BR(\lambda)^*]^* AR_0(\lambda) \} \right). \quad (2.1.51)$$

On the other hand, since according to Remark 2.1.18, $m(\lambda_0; H) < \infty$, $R(\lambda) - R_0(\lambda)$ is finitely meromorphic at λ_0 . Then, using (2.1.28) and (2.1.34), we infer:

$$\begin{aligned} m(\lambda_0; H) &= -\frac{1}{2\pi i} \operatorname{tr} \left(\int_{\Gamma} d\lambda [R(\lambda) - R_0(\lambda)] \right) \\ &= -\frac{1}{2\pi i} \operatorname{tr} \left(\int_{\Gamma} d\lambda \Xi \{ R(\lambda) - R_0(\lambda) \} \right) \\ &= \frac{1}{2\pi i} \operatorname{tr} \left(\int_{\Gamma} d\lambda \Xi \{ [BR_0(\lambda)^*]^* (I_{\mathcal{K}} - K(\lambda))^{-1} AR_0(\lambda) \} \right) \\ &= \frac{1}{2\pi i} \operatorname{tr} \left(\int_{\Gamma} d\lambda \Xi \{ [BR(\lambda)^*]^* AR_0(\lambda) \} \right) \\ &= \frac{1}{2\pi i} \left(\int_{\Gamma} d\lambda \operatorname{tr} \Xi \{ [BR(\lambda)^*]^* AR_0(\lambda) \} \right). \end{aligned} \quad (2.1.52)$$

Combining (2.1.51) and (2.1.52), we obtain the desired identity. \square

Hypothesis 2.1.20. *In addition to Hypothesis 3.4.4, we assume the following condition:*

(v) *For some $p \in \mathbb{N}$, we have $K(z) \in \mathcal{B}_p(\mathcal{K})$ for all $z \in \rho(H_0)$.*

Theorem 2.1.21. *Assume Hypothesis 2.1.20. Then, the following holds:*

$$\det_p(I_{\mathcal{K}} - K(\lambda)) = (\lambda - \lambda_0)^{m(\lambda_0; I_{\mathcal{K}} - K(\cdot))} S(\lambda) \quad (2.1.53)$$

$$= (\lambda - \lambda_0)^{m(\lambda_0; H)} S(\lambda), \quad S(\lambda_0) \neq 0. \quad (2.1.54)$$

Proof. The first equality follows directly from Theorem 2.1.4 and Corollary 2.1.5 applied to $W(\lambda) = I_{\mathcal{K}} - K(\lambda)$. Hence, from (3.1.6),

$$\det_p(I_{\mathcal{K}} - K(\lambda)) = \det_p(D(\lambda)) S_1(\lambda) \quad (2.1.55)$$

$$= (\lambda - \lambda_0)^{m(\lambda_0; I_{\mathcal{K}} - K(\cdot))} S(\lambda), \quad S(\lambda_0) \neq 0. \quad (2.1.56)$$

Equality (2.1.54) follows from Theorem 2.1.19. □

2.2 Applications

2.2.1 The Schrödinger equation

Let us consider the Schrödinger equation

$$-\phi''(x) + V(x)\phi(x) = \lambda\phi(x) \quad (2.2.1)$$

with the potential $V \in L^1(\mathbb{R}; dx)$. We introduce the closed operators in $L^1(\mathbb{R}; dx)$ defined by

$$H_0 f = -f'', \quad f \in \text{dom}(H_0), \quad (\text{dom}(H_0), \|\cdot\|_{H_0}) = W_2^2(\mathbb{R}),$$

$$H f = -f'' + V f, \quad (2.2.2)$$

$$f \in \text{dom}(H) = \{g \in L^2(\mathbb{R}; dx) \mid g, g' \in AC_{\text{loc}}(\mathbb{R}); (-g'' + Vg) \in L^2(\mathbb{R}; dx)\}.$$

Also, we introduce the factorization

$$V = u(x)v(x), \quad u(x) = \text{sign}(V(x))|V(x)|^{1/2}, \quad v(x) = |V(x)|^{1/2}, \quad x \in \mathbb{R}. \quad (2.2.3)$$

Finally, we introduce the integral operator $K(\lambda)$ in $L^2(\mathbb{R}; dx)$ with the integral kernel

$$K(\lambda, x, x') = -v(x)M^\vee(\lambda, x - x')u(x'), \quad \lambda \in \mathbb{C} \setminus \sigma(H_0), \quad (2.2.4)$$

where $M(\lambda, \eta) = (\eta^2 - \lambda)^{-1}$, and $^\vee$ denotes the inverse Fourier transform with respect to the variable $\eta \in \mathbb{R}$. The following result is, of course, well-known; we recall its proof to emphasize its similarity with the proof of Theorems 2.2.12 and 2.2.20 below.

Theorem 2.2.1. *[51, Theorem 4.1.] Suppose $V \in L^1(\mathbb{R}; dx)$ and let $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$. Then $K(\lambda) \in \mathcal{B}_2(L^2(\mathbb{R}; dx))$.*

Proof. Using [46, Theorem VI.23], the assertion in the theorem follows from the formula for the \mathcal{B}_2 -norm of an integral operator:

$$\begin{aligned} \|K(\lambda)\|_{\mathcal{B}_2(L^2(\mathbb{R};dx))} &= \int \int_{\mathbb{R} \times \mathbb{R}} dx dx' |K(\lambda, x, x')|^2 = \\ &= \int \int_{\mathbb{R} \times \mathbb{R}} dx dx' | -v(x)M^\vee(\lambda, x - x')u(x')|^2 < \infty. \end{aligned}$$

The last inequality follows from $u, v \in L^2(\mathbb{R})$ because $M^\vee(\lambda, \cdot) \in L^\infty(\mathbb{R}; dx)$ by $M(\lambda, \cdot) \in L^1(\mathbb{R}; dx)$ and the Riemann-Lebesgue lemma. \square

Remark 2.2.2. Assume $V \in L^1(\mathbb{R}; dx)$ and let $\lambda \in \mathbb{C} \setminus \sigma(H_0)$. Since $M(\lambda, \cdot) \in L^2(\mathbb{R}; d\eta) \cap L^\infty(\mathbb{R}; d\eta)$, by [47, Theorem IX.29] the operator $K(\lambda)$ coincides with $-M_v(H_0 - \lambda)^{-1}M_u$ on the domain of M_u , where M_v, M_u are the operators of multiplication by v, u with maximal domains. Hence, $K(\lambda) = -\overline{M_v(H_0 - \lambda)^{-1}M_u} \in \mathcal{B}_2(L^2(\mathbb{R}; dx))$. From now on, we will use the notation $K(\lambda)$ also for the operator $-\overline{M_v(H_0 - \lambda)^{-1}M_u}$.

The Schrödinger equation (2.2.1) is equivalent to the first order system

$$\Psi'(x) = \begin{bmatrix} 0 & 1 \\ V(x) - \lambda & 0 \end{bmatrix} \Psi(x), \quad \Psi = \begin{bmatrix} \phi' \\ \phi \end{bmatrix}. \quad (2.2.5)$$

Introduce the corresponding first order operator $T(\lambda)$ and matrices A and B :

$$\begin{aligned} T(\lambda) &= \partial_x + \begin{bmatrix} 0 & -1 \\ \lambda & 0 \end{bmatrix}, \quad (\text{dom}(T(\lambda)), \|\cdot\|_{T(\lambda)}) = W_2^1(\mathbb{R}) \oplus W_2^1(\mathbb{R}), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}_+, \\ A(\lambda) &= \begin{bmatrix} 0 & 1 \\ -\lambda & 0 \end{bmatrix}, \quad B(x) = \begin{bmatrix} 0 & 0 \\ V(x) & 0 \end{bmatrix}. \end{aligned} \quad (2.2.6)$$

Then, the first order system (2.2.5) can be rewritten as follows:

$$\Psi'(x) = (A(\lambda) + B(x))\Psi(x). \quad (2.2.7)$$

Lemma 2.2.3. *The operator $T(\lambda)$, with $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$, has a bounded inverse given by*

$$T(\lambda)^{-1} = \begin{bmatrix} -\partial_x (-\partial_{xx}^2 - \lambda)^{-1} & -(-\partial_{xx}^2 - \lambda)^{-1} \\ \lambda (-\partial_{xx}^2 - \lambda)^{-1} & -\partial_x (-\partial_{xx}^2 - \lambda)^{-1} \end{bmatrix}. \quad (2.2.8)$$

Proof. Clearly,

$$\left(i\eta I_{2 \times 2} + \begin{bmatrix} 0 & -1 \\ \lambda & 0 \end{bmatrix} \right)^{-1} = -\frac{1}{\eta^2 - \lambda} \begin{bmatrix} i\eta & 1 \\ -\lambda & i\eta \end{bmatrix}, \quad \eta \in \mathbb{R}.$$

Taking the Fourier transform in η -variable proves the assertion. \square

Introduce the matrices $\tilde{u}(x)$, $\tilde{v}(x)$ and the operator $\mathbb{K}(\lambda)$ as follows:

$$\tilde{u}(x) = \begin{bmatrix} 0 & 0 \\ u(x) & 0 \end{bmatrix}, \quad \tilde{v}(x) = \begin{bmatrix} v(x) & 0 \\ 0 & 0 \end{bmatrix}, \quad (2.2.9)$$

$$\mathbb{K}(\lambda) = -\overline{M_{\tilde{v}} T(\lambda)^{-1} M_{\tilde{u}}}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}_+. \quad (2.2.10)$$

Here, the operator $M_{\tilde{v}} T(\lambda)^{-1} M_{\tilde{u}}$ is originally defined on the (maximal) domain of the operator $M_{\tilde{u}}$ of multiplication by \tilde{u} . We will see in (2.2.13) that $\mathbb{K}(\lambda) \in \mathcal{B}(L^2(\mathbb{R}; dx)^2)$.

Also, $B(x)$ in (2.2.6) has the following representation:

$$B(x) = \tilde{u}(x)\tilde{v}(x). \quad (2.2.11)$$

Lemma 2.2.4. *Suppose that $V \in L^1(\mathbb{R}; dx)$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$. Then*

$\mathbb{K}(\lambda) \in \mathcal{B}_2(L^2(\mathbb{R}; dx)^2)$ and the following equality holds.

$$\det_2 (I_{L^2(\mathbb{R}; dx)^2} - \mathbb{K}(\lambda)) = \det_2 (I_{L^2(\mathbb{R}; dx)} + K(\lambda)). \quad (2.2.12)$$

Proof. Using (2.2.8), we arrive at the following identity:

$$\begin{aligned} \mathbb{K}(\lambda) &= \overline{\begin{bmatrix} -v & 0 \\ 0 & 0 \end{bmatrix} T(\lambda)^{-1} \begin{bmatrix} 0 & 0 \\ u & 0 \end{bmatrix}} = \\ &= \overline{\begin{bmatrix} v(x) (-\partial_{xx}^2 - \lambda)^{-1} u(x) & 0 \\ 0 & 0 \end{bmatrix}} = \begin{bmatrix} -K(\lambda) & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \quad (2.2.13)$$

The required assertion now follows from Theorem 2.2.1 and (2.2.13). \square

We now recall that the eigenvalues of H are zeros of the Evans function \mathbb{E} associated with the first order system (2.2.7), see, e.g., [4, 18, 44, 48, 55]. Our next goal is to describe the relations of the algebraic multiplicity $m(\lambda_0; H)$ of the eigenvalue λ_0 of the operator H defined in (2.2.2), the algebraic multiplicity $m(\lambda_0; I - K(\cdot))$ of the operator-valued function $K(\cdot)$ at λ_0 (as defined in Definition 2.1.7), and the multiplicity $m(\lambda_0; \mathbb{E}(\cdot))$ of λ_0 as the zero of the Evans function \mathbb{E} .

Let us first recall the definition of the Evans function, see, e.g., [48]. Consider a first order system

$$\frac{d}{d\xi}u = \mathbb{M}(\xi, \lambda)u, \quad u \in \mathbb{C}^n, \xi \in \mathbb{R}, \quad (2.2.14)$$

where \mathbb{M} is a bounded piecewise continuous function of ξ analytic in λ in some domain. We say that λ is not in the essential spectrum σ_{ess} of (2.2.14) if the operator $\partial_\xi - \mathbb{M}(\cdot, \lambda)$ is Fredholm with zero index. If this is the case then equation (2.2.14) has exponential dichotomies on \mathbb{R}^+ and \mathbb{R}^- with projections $P_+(\xi; \lambda)$ and $P_-(\xi; \lambda)$, respectively, and, moreover, the Morse indices of the dichotomies are equal, that is, $\dim \ker(P_+(0; \lambda)) = \dim \ker(P_-(0; \lambda))$, see Palmer's theorem in [48]. Let Ω be a simply-connected subset of $\mathbb{C} \setminus \sigma_{ess}$. Then the Morse index $\dim \ker(P_+(0; \lambda)) = \dim \ker(P_-(0; \lambda))$ is constant for $\lambda \in \Omega$; let us denote it by k . We choose ordered bases $[u_1(\lambda), \dots, u_k(\lambda)]$ and $[u_{k+1}(\lambda), \dots, u_n(\lambda)]$ of the subspaces $\ker(P_-(0; \lambda))$ and $\text{ran}(P_+(0; \lambda))$, respectively. We can choose the basis vectors such that they are analytic in λ .

Definition 2.2.5. The Evans function \mathbb{E} is defined by

$$\mathbb{E}(\lambda) = \det[u_1(\lambda), \dots, u_n(\lambda)]. \quad (2.2.15)$$

We note that the Evans function depends on the choice of the basis vectors $u_j(\lambda)$. As shown in [20], if the first order system (2.2.14) has an additional perturbation

structure as in (2.2.7), then the Evans function can be chosen to agree with the (modified) Fredholm determinant of the operator $I - \mathbb{K}(\lambda)$. Specifically, the basis vectors $u_j(\lambda)$ can be chosen as the columns of the generalized matrix-valued Jost solutions of the first order system (2.2.7). The definition of the generalized matrix-valued Jost solution can be found in [20]. The paper [20] contains the following formula:

$$\det_2 (I_{L^2(\mathbb{R}; dx)^2} - \mathbb{K}(\lambda)) = e^{\Theta(\lambda)} \mathbb{E}(\lambda), \quad (2.2.16)$$

where $\Theta(\lambda)$ is some explicitly computed in [20] number.

In what follows we always assume that the Evans function is selected such that (2.2.16) holds. If $\mathbb{E}(\lambda_0) = 0$ then we denote by $m(\lambda_0; \mathbb{E}(\cdot))$ the multiplicity of λ_0 as a zero of the function \mathbb{E} such that $\mathbb{E}(\lambda) = (\lambda - \lambda_0)^{m(\lambda_0; \mathbb{E}(\cdot))} S(\lambda)$, $S(\lambda_0) \neq 0$.

Hypothesis 2.2.6. *We assume that $V \in L^1(\mathbb{R})$ and*

$$\lambda_0 \in \{\mathbb{C} \setminus \mathbb{R}_+\} \cap \{\lambda \in \mathbb{C} \mid \lambda \text{ is an isolated eigenvalue of } H\}. \quad (2.2.17)$$

Theorem 2.2.7. *Assume Hypothesis 2.2.6 and let $\mathbb{E}(\cdot)$ be the Evans function for the perturbed equation (2.2.7). Then*

$$m(\lambda_0; \mathbb{E}(\cdot)) = m(\lambda_0; I_{L^2(\mathbb{R}; dx)} - K(\cdot)) = m(\lambda_0; H). \quad (2.2.18)$$

Proof. This follows from formula (2.2.16), Lemma 2.2.4, Theorem 2.1.21 and Theorem 2.1.19. □

2.2.2 Degenerate reaction-diffusion systems

Let D be a diagonal $n \times n$ matrix with the diagonal entries $d_1, \dots, d_\ell > 0$ and $d_{\ell+1} = \dots = d_n = 0$, and $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a smooth function. We consider the system of

reaction diffusion equations

$$U_t = DU_{xx} + F(U), \quad x \in \mathbb{R}, U \in \mathbb{R}^n. \quad (2.2.19)$$

In the moving coordinate frame $\xi = x - ct$, with some $c > 0$, system (2.2.19) is given by

$$U_t = DU_{\xi\xi} + cU_\xi + F(U), \quad \xi \in \mathbb{R}, U \in \mathbb{R}^n. \quad (2.2.20)$$

Suppose that $Q = Q(\xi)$ is a traveling wave for (2.2.19), that is, is a stationary solution of (2.2.20) so that

$$DQ_{\xi\xi}(\xi) + cQ_\xi(\xi) + F(Q(\xi)) = 0, \quad \xi \in \mathbb{R}. \quad (2.2.21)$$

The eigenvalue problem associated with the linearization of (2.2.20) about Q is given by

$$DU_{\xi\xi} + cU_\xi + F'(Q)U = \lambda U; \quad (2.2.22)$$

here and below $F' = \partial F$ denotes the differential of F .

We decompose $U(\xi), F'(Q(\xi))$ in the following way:

$$U = [U_1(\xi), U_2(\xi)]^\top \in \mathbb{R}^\ell \oplus \mathbb{R}^{n-\ell},$$

$$F'(Q(\xi)) = \begin{bmatrix} F'_{11}(Q(\xi)) & F'_{12}(Q(\xi)) \\ F'_{21}(Q(\xi)) & F'_{22}(Q(\xi)) \end{bmatrix} : \mathbb{R}^\ell \oplus \mathbb{R}^{n-\ell} \rightarrow \mathbb{R}^\ell \oplus \mathbb{R}^{n-\ell}.$$

Let D_ℓ be the diagonal $\ell \times \ell$ matrix with the diagonal entries d_1, \dots, d_ℓ , and denote $W_1 = \frac{d}{d\xi} U_1$. Then the eigenvalue problem (2.2.22) can be recast as follows:

$$\begin{bmatrix} \partial_\xi & 0_{\ell \times (n-\ell)} & -I_{\ell \times \ell} \\ c^{-1}F'_{21}(Q(\xi)) & c^{-1}(-\lambda + F'_{22}(Q(\xi))) + \partial_\xi & 0_{(n-\ell) \times \ell} \\ D_\ell^{-1}(-\lambda + F'_{11}(Q(\xi))) & D_\ell^{-1}F'_{12}(Q(\xi)) & cD_\ell^{-1} + \partial_\xi \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ W_1 \end{bmatrix} = 0. \quad (2.2.23)$$

Hypothesis 2.2.8. *Suppose that the travelling wave Q is a pulse, that is, there exists the limit*

$$\lim_{|\xi| \rightarrow \infty} Q(\xi) = Q(\infty) \in \mathbb{R}^n. \quad (2.2.24)$$

Assume Hypothesis 2.2.8. Denoting by W_2^k the Sobolev space of k times differentiable functions with L^2 -derivatives, let us introduce the operators $T(\lambda)$, H_0 , H , and the matrices A, B, V as follows:

$$T(\lambda) = \begin{bmatrix} \partial_\xi & 0_{\ell \times (n-\ell)} & -I_{\ell \times \ell} \\ c^{-1}F'_{21}(Q(\infty)) & c^{-1}(-\lambda + F'_{22}(Q(\infty))) + \partial_\xi & 0_{(n-\ell) \times \ell} \\ D_\ell^{-1}(-\lambda + F'_{11}(Q(\infty))) & D_\ell^{-1}F'_{12}(Q(\infty)) & cD_\ell^{-1} + \partial_\xi \end{bmatrix}, \quad (2.2.25)$$

$$\text{with the domain } (\text{dom}(T(\lambda)), \|\cdot\|_{T(\lambda)}) = W_2^1(\mathbb{R})^{n+\ell}, \quad (2.2.26)$$

$$H_0 = \begin{bmatrix} D_\ell \partial_{\xi\xi}^2 + c\partial_\xi & 0_{\ell \times (n-\ell)} \\ 0_{(n-\ell) \times \ell} & c\partial_\xi \end{bmatrix} + F'(Q(\infty)), \quad (2.2.27)$$

$$\text{with the domain } (\text{dom}(H_0), \|\cdot\|_{H_0}) = W_2^2(\mathbb{R})^\ell \oplus W_2^1(\mathbb{R})^{n-\ell}, \quad (2.2.28)$$

$$H = H_0 + V, \text{ with } V(\xi) = F'(Q(\xi)) - F'(Q(\infty)) \text{ and the domain} \quad (2.2.29)$$

$$\text{dom}(H) = \left\{ f = (g, h)^\top \in L^2(\mathbb{R}; d\xi)^\ell \oplus L^2(\mathbb{R}; d\xi)^{n-\ell} \mid \right. \quad (2.2.30)$$

$$\left. g, g' \in AC_{\text{loc}}(\mathbb{R})^\ell, h \in AC_{\text{loc}}(\mathbb{R})^{n-\ell}; H_0 f + V f \in L^2(\mathbb{R}; d\xi)^n \right\},$$

$$A(\lambda) = \begin{bmatrix} 0 & 0 & I \\ -c^{-1}F'_{21}(Q(\infty)) & c^{-1}(\lambda - F'_{22}(Q(\infty))) & 0 \\ D_\ell^{-1}(\lambda - F'_{11}(Q(\infty))) & -D_\ell^{-1}F'_{12}(Q(\infty)) & -cD_\ell^{-1} \end{bmatrix}, \quad (2.2.31)$$

$$B(\xi) = \begin{bmatrix} 0 & 0 & 0 \\ c^{-1}(-F'_{21}(Q(\xi)) + F'_{21}(Q(\infty))) & c^{-1}(-F'_{22}(Q(\xi)) + F'_{22}(Q(\infty))) & 0 \\ D_\ell^{-1}(-F'_{11}(Q(\xi)) + F'_{11}(Q(\infty))) & D_\ell^{-1}(-F'_{12}(Q(\xi)) + F'_{12}(Q(\infty))) & 0 \end{bmatrix}. \quad (2.2.32)$$

Then, the eigenvalue problem (2.2.23) can be recast as

$$\Psi'(\xi) = (A(\lambda) + B(\xi))\Psi(\xi), \quad (2.2.33)$$

where $\Psi(\xi) = [U_1(\xi), U_2(\xi), W_1(\xi)]^\top$. Passing to the Fourier transform, we obtain the following fact.

Lemma 2.2.9. *Let H_0 be as in (2.2.27). Then*

$$\rho(H_0) = \{\lambda \in \mathbb{C} \mid \inf_{\eta \in \mathbb{R}} |\det N(\lambda, \eta)| \neq 0\}, \quad (2.2.34)$$

where we denote:

$$N(\lambda, \eta) = \begin{bmatrix} -\eta^2 D_\ell + ic\eta + F'_{11}(Q(\infty)) - \lambda & F'_{12}(Q(\infty)) \\ F'_{21}(Q(\infty)) & ic\eta + F'_{22}(Q(\infty)) - \lambda \end{bmatrix}. \quad (2.2.35)$$

For $\lambda \in \rho(H_0)$ one can express $(H_0 - \lambda)^{-1}$ in terms of the multiplication operator:

$$(H_0 - \lambda)^{-1} = \mathcal{F}^{-1} M(\lambda, \cdot) \mathcal{F}, \quad (2.2.36)$$

where \mathcal{F} is the Fourier transform, and we denote

$$M(\lambda, \eta) = \begin{bmatrix} -\eta^2 + ic\eta + F'_{11}(Q(\infty)) - \lambda & F'_{12}(Q(\infty)) \\ F'_{21}(Q(\infty)) & ic\eta + F'_{22}(Q(\infty)) - \lambda \end{bmatrix}^{-1}. \quad (2.2.37)$$

Remark 2.2.10. Since $\lambda \in \rho(H_0)$, by Lemma 2.2.9 the determinant of $N(\lambda, \eta)$ is separated from zero uniformly in $\eta \in \mathbb{R}$. Moreover, each element m_{ij} of the matrix $M(\lambda, \eta)$ is of the form $p_{ij}(\lambda, \eta)/\det(N(\lambda, \eta))$, where $p_{ij}(\lambda, \eta)$ and $\det(N(\lambda, \eta))$ are polynomials in η , and $\deg(p_{ij}(\lambda, \eta)) + 1 \leq \deg(\det(N(\lambda, \eta)))$. Hence, $M(\lambda, \cdot) \in L^2(\mathbb{R}; d\eta)^{n \times n} \cap L^\infty(\mathbb{R}; d\eta)^{n \times n}$. So, we conclude that the Fourier transform of $M(\lambda, \cdot)$ is well-defined in L^2 -sense.

Hypothesis 2.2.11. *In addition to Hypothesis 2.2.8, we assume that*

$$F'(Q(\cdot)) - F'(Q(\infty)) \in L^1(\mathbb{R})^{n \times n}.$$

Let $\tilde{U}(\xi)$ and $|F'(Q(\xi)) - F'(Q(\infty))|$ denote the $n \times n$ matrices in the polar decomposition of $F'(Q(\xi)) - F'(Q(\infty))$. We introduce the matrices

$$u(\xi) = \tilde{U}(\xi) |F'(Q(\xi)) - F'(Q(\infty))|^{1/2}, \quad v(\xi) = |F'(Q(\xi)) - F'(Q(\infty))|^{1/2}. \quad (2.2.38)$$

Assume Hypothesis 2.2.11. Then $u, v \in L^2(\mathbb{R})^{n \times n}$. Next, using Remark 2.2.10, we introduce the integral operator $K(\lambda)$ in $L^2(\mathbb{R}; d\xi)^n$ with the integral kernel

$$K(\lambda, \xi, \xi') = -v(\xi) M^\vee(\lambda, \xi - \xi') u(\xi'). \quad (2.2.39)$$

Theorem 2.2.12. *Assume Hypothesis 2.2.11 and let $\lambda \in \mathbb{C} \setminus \sigma(H_0)$. Then $K(\lambda) \in \mathcal{B}_2(L^2(\mathbb{R}; d\xi)^n)$.*

Proof. First, we claim that $M^\vee(\lambda, \cdot) \in L^\infty(\mathbb{R}; d\xi)^{n \times n}$. Indeed, by Remark 2.2.10 each element m_{ij} of the matrix $M(\lambda, \eta)$ is of the form $p_{ij}(\lambda, \eta)/\det(N(\lambda, \eta))$, where $p_{ij}(\lambda, \eta)$ and $\det(N(\lambda, \eta))$ are polynomials in η . Denoting by η_k the roots of $N(\lambda, \cdot)$, we can decompose m_{ij} in the following way:

$$m_{ij}(\lambda, \eta) = \sum_{k=1}^d \sum_{l=1}^{s_k} \frac{a_{ijkl}}{(\eta - \eta_k)^l}, \quad a_{ijkl} \in \mathbb{C}, \quad \text{Im}(\eta_k) \neq 0, \quad (2.2.40)$$

where d is the number of different roots of $\det(N(\lambda, \cdot))$ and s_k is the multiplicity of η_k in $\det(N(\lambda, \cdot))$. Note that some of a_{ijkl} might be zero. If $l > 1$ then $a_{ijkl}/(\cdot - \eta_k)^l \in L^1(\mathbb{R}; d\eta)$ and $(a_{ijkl}/(\eta - \eta_k)^l)^\vee \in L^\infty(\mathbb{R}; d\xi)$ by the Riemann-Lebesgue lemma. If $l = 1$ then $(a_{ijkl}/(\eta - \eta_k)^l)^\vee \in L^\infty(\mathbb{R}; d\xi)$ since

$$\text{if } \text{Re}(\lambda) < 0 \text{ then } \left(\frac{1}{i\eta - \lambda}\right)^\vee(\xi) = \begin{cases} \sqrt{2\pi}e^{\lambda\xi}, & \xi \geq 0; \\ 0, & \xi < 0; \end{cases} \quad (2.2.41)$$

$$\text{if } \text{Re}(\lambda) > 0 \text{ then } \left(\frac{1}{i\eta - \lambda}\right)^\vee(\xi) = \begin{cases} 0, & \xi \geq 0; \\ \sqrt{2\pi}e^{\lambda\xi}, & \xi < 0. \end{cases} \quad (2.2.42)$$

Combining the cases $l > 1$ and $l = 1$, we justify the claim $M^\vee(\lambda, \cdot) \in L^\infty(\mathbb{R}; d\xi)^{n \times n}$.

The assertion in the theorem now follows from the well-known formula for the \mathcal{B}_2 -norm of an integral operator, see [8, Theorem 11.3.6]:

$$\begin{aligned} \|K(\lambda)\|_{\mathcal{B}_2(L^2(\mathbb{R}; d\xi)^n)}^2 &= \int \int_{\mathbb{R} \times \mathbb{R}} d\xi d\xi' \|K(\lambda, \xi, \xi')\|_{\mathbb{C}^{n \times n}}^2 = \\ &= \int \int_{\mathbb{R} \times \mathbb{R}} d\xi d\xi' \| -v(\xi)M^\vee(\lambda, \xi - \xi')u(\xi') \|_{\mathbb{C}^{n \times n}}^2 < \infty. \end{aligned}$$

The last inequality holds since $u, v \in L^2(\mathbb{R})^{n \times n}$ and $M^\vee(\lambda, \cdot) \in L^\infty(\mathbb{R}; d\xi)^{n \times n}$. \square

Remark 2.2.13. Assume Hypothesis 2.2.11 and let $\lambda \in \mathbb{C} \setminus \sigma(H_0)$. Since $M(\lambda, \cdot) \in L^2(\mathbb{R}; d\eta)^{n \times n} \cap L^\infty(\mathbb{R}; d\eta)^{n \times n}$, it follows by [47, Theorem IX.29] that $f \in \text{dom}(M_u)$

yields $K(\lambda)f = -M_v(H_0 - \lambda)^{-1}M_u f$; here M_v, M_u are the operators of multiplication by v, u . In other words, the integral operator $K(\lambda)$ can be also written as $K(\lambda) = -\overline{M_v(H_0 - \lambda)^{-1}M_u} \in \mathcal{B}_2(L^2(\mathbb{R}; d\xi)^n)$. From now on, we will use the notation $K(\lambda)$ also for the operator $-\overline{M_v(H_0 - \lambda)^{-1}M_u}$.

We introduce the following matrices obtained by taking the Fourier transform in (2.2.25) and (2.2.27):

$$H_{0\eta} = \begin{bmatrix} -\eta^2 D_\ell + c\mu + F'_{11}(Q(\infty)) & F'_{12}(Q(\infty)) \\ F'_{21}(Q(\infty)) & c i\eta + F'_{22}(Q(\infty)) \end{bmatrix}, \quad \eta \in \mathbb{R}, \quad (2.2.43)$$

$$T_\eta(\lambda) = \begin{bmatrix} i\eta & 0 & -I \\ c^{-1}F'_{21}(Q(\infty)) & c^{-1}(-\lambda + F'_{22}(Q(\infty))) + i\eta & 0 \\ D_\ell^{-1}(-\lambda + F'_{11}(Q(\infty))) & D_\ell^{-1}F'_{12}(Q(\infty)) & cD_\ell^{-1} + i\eta \end{bmatrix}.$$

Assume $\lambda \in \rho(H_0)$. Let us denote by $H_{ij}, \tilde{H}_{ij}, \hat{H}_{ij}$ the blocks of the block-operators $(H_{0\eta} - \lambda), (H_{0\eta} - \lambda)^{-1}, (H_0 - \lambda)^{-1}$, respectively, in the direct sum decomposition $L^2(\mathbb{R}; d\xi)^n = L^2(\mathbb{R}; d\xi)^\ell \oplus L^2(\mathbb{R}; d\xi)^{n-\ell}$ such that

$$(H_{0\eta} - \lambda) = [H_{ij}]_{i,j=1}^2, (H_{0\eta} - \lambda)^{-1} = [\tilde{H}_{ij}]_{i,j=1}^2, (H_0 - \lambda)^{-1} = [\hat{H}_{ij}]_{i,j=1}^2. \quad (2.2.44)$$

Lemma 2.2.14. *The inverse of the operator $T(\lambda)$ from (2.2.25) is given by the formula*

$$T(\lambda)^{-1} = \begin{bmatrix} (c + \partial_\xi D_\ell)\hat{H}_{11} & c\hat{H}_{12} & D_\ell\hat{H}_{11} \\ (c + \partial_\xi D_\ell)\hat{H}_{21} & c\hat{H}_{22} & D_\ell\hat{H}_{21} \\ -I + \partial_\xi(c + \partial_\xi D_\ell)\hat{H}_{11} & c\partial_\xi\hat{H}_{12} & \partial_\xi D_\ell\hat{H}_{11} \end{bmatrix}, \quad \lambda \in \rho(H_0). \quad (2.2.45)$$

Proof. Since $\lambda \in \rho(H_0)$, the invertibility of the matrix $H_{0\eta} - \lambda$ follows from Lemma 2.2.9. A direct verification shows that the inverse of the matrix $T_\eta(\lambda)$ is given by the formula

$$T_\eta^{-1}(\lambda) = \begin{bmatrix} (c + i\eta D_\ell)\tilde{H}_{11} & c\tilde{H}_{12} & D_\ell\tilde{H}_{11} \\ (c + i\eta D_\ell)\tilde{H}_{21} & c\tilde{H}_{22} & D_\ell\tilde{H}_{21} \\ -I + i\eta(c + i\eta D_\ell)\tilde{H}_{11} & c i\eta\tilde{H}_{12} & i\eta D_\ell\tilde{H}_{11} \end{bmatrix}. \quad (2.2.46)$$

Taking the Fourier transform in (2.2.46) proves the required assertion (2.2.45). \square

Next, we consider the matrices $u(\xi)$ and $v(\xi)$ defined in (2.2.38). Using the block representation $u(\xi) = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$, $v(\xi) = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$ in the direct sum decomposition $\mathbb{C}^n = \mathbb{C}^\ell \oplus \mathbb{C}^{n-\ell}$, we introduce the matrices $\tilde{u}(\xi)$, $\tilde{v}(\xi)$ and the operator $\mathbb{K}(\lambda)$ as follows:

$$\tilde{u}(\xi) = \begin{bmatrix} 0_{\ell \times \ell} & 0_{\ell \times (n-\ell)} & 0_{\ell \times \ell} \\ c^{-1}u_{21} & c^{-1}u_{22} & 0_{(n-\ell) \times \ell} \\ D_\ell^{-1}u_{11} & D_\ell^{-1}u_{12} & 0_{\ell \times \ell} \end{bmatrix}, \quad \tilde{v}(\xi) = \begin{bmatrix} v_{11} & v_{12} & 0_{\ell \times \ell} \\ v_{21} & v_{22} & 0_{(n-\ell) \times \ell} \\ 0_{\ell \times \ell} & 0_{\ell \times (n-\ell)} & 0_{\ell \times \ell} \end{bmatrix}, \quad (2.2.47)$$

$$\mathbb{K}(\lambda) = -\overline{M_{\tilde{v}}T(\lambda)^{-1}M_{\tilde{u}}}, \quad \lambda \in \rho(H_0), \quad (2.2.48)$$

where the operator $M_{\tilde{v}}T(\lambda)^{-1}M_{\tilde{u}}$ is originally defined on the (maximal) domain of $M_{\tilde{u}}$. We will see in (2.2.51) that $\mathbb{K}(\lambda) \in \mathcal{B}(L^2(\mathbb{R}; d\xi)^{n+\ell})$. Also, we remark that $B(\xi)$ in (2.2.32) can be written as $B(\xi) = \tilde{u}(\xi)\tilde{v}(\xi)$.

Hypothesis 2.2.15. *We assume:*

$$\lambda_0 \in \rho(H_0) \cap \{\lambda \in \mathbb{C} \mid \lambda \text{ is an isolated eigenvalue of } H\}. \quad (2.2.49)$$

Theorem 2.2.16. *Assume Hypothesis 2.2.15 and let $\mathbb{E}(\cdot)$ be the Evans function for the perturbed equation (2.2.33). Then*

$$m(\lambda_0; \mathbb{E}(\cdot)) = m(\lambda_0; I_{L^2(\mathbb{R})^{n+\ell}} - \mathbb{K}(\cdot)) = m(\lambda_0; I_{L^2(\mathbb{R})^n} - K(\cdot)) = m(\lambda_0; H). \quad (2.2.50)$$

Proof. The first equality follows from formula (2.2.16). The second equality follows from the identity

$$\mathbb{K}(\lambda) = \begin{bmatrix} K(\lambda) & 0_{n \times \ell} \\ 0_{\ell \times n} & 0_{\ell \times \ell} \end{bmatrix}, \quad \lambda \in \rho(H_0), \quad (2.2.51)$$

which, in turn, follows from (2.2.45) and (2.2.47):

$$\mathbb{K}(\lambda) = -\overline{M_{\tilde{v}}T(\lambda)^{-1}M_{\tilde{u}}}$$

$$\begin{aligned}
&= - \overline{\begin{bmatrix} \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \begin{bmatrix} c\widehat{H}_{12} & D_\ell\widehat{H}_{11} \\ c\widehat{H}_{22} & D_\ell\widehat{H}_{21} \end{bmatrix} \begin{bmatrix} c^{-1}u_{21} & c^{-1}u_{22} \\ D_\ell^{-1}u_{11} & D_\ell^{-1}u_{12} \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
&= - \overline{\begin{bmatrix} \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \begin{bmatrix} \widehat{H}_{12}u_{21} + \widehat{H}_{11}u_{11} & \widehat{H}_{12}u_{22} + \widehat{H}_{11}u_{12} \\ \widehat{H}_{22}u_{21} + \widehat{H}_{21}u_{11} & \widehat{H}_{22}u_{22} + \widehat{H}_{21}u_{12} \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
&= - \overline{\begin{bmatrix} \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \begin{bmatrix} \widehat{H}_{11} & \widehat{H}_{12} \\ \widehat{H}_{21} & \widehat{H}_{22} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix}} = \begin{bmatrix} K(\lambda) & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned}$$

By Theorem 2.2.12, from (2.2.51) we derive $\mathbb{K}(\lambda) \in \mathcal{B}_2(L^2(\mathbb{R}; d\xi)^{n+\ell})$ and

$$\det_2(I_{L^2(\mathbb{R}; d\xi)^{n+\ell}} - \mathbb{K}(\lambda)) = \det_2(I_{L^2(\mathbb{R}; d\xi)^n} - K(\lambda)), \quad (2.2.52)$$

yielding $m(\lambda_0; I_{L^2(\mathbb{R}; d\xi)^{n+\ell}} - \mathbb{K}(\cdot)) = m(\lambda_0; I_{L^2(\mathbb{R}; d\xi)^n} - K(\cdot))$. Finally, the last equality in (2.2.50) follows from Theorem 2.1.21 and Theorem 2.1.19. \square

2.2.3 General n -th order linear differential equations

Let us consider the eigenvalue problem associated with the general n -th order linear differential operator:

$$a_n \partial_x^n U(x) + \sum_{l=0}^{n-1} a_l(x) \partial_x^l U(x) = \lambda U(x), \quad x \in \mathbb{R}, \quad (2.2.53)$$

where a_n is a non-zero constant and $a_l(\cdot)$, $l = 0, \dots, n-1$, are given $L^1_{\text{loc}}(\mathbb{R})$ -functions.

Equation (2.2.53) is equivalent to the first order system:

$$\Psi'(x) = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & & & & 1 \\ \frac{\lambda - a_0(x)}{a_n} & -\frac{a_1(x)}{a_n} & \cdots & \cdots & -\frac{a_{n-1}(x)}{a_n} \end{bmatrix} \Psi(x), \quad \Psi = \begin{bmatrix} U \\ U' \\ \vdots \\ U^{(n-2)} \\ U^{(n-1)} \end{bmatrix}. \quad (2.2.54)$$

Hypothesis 2.2.17. *Assume that the following limits exist:*

$$a_l^\infty := \lim_{|x| \rightarrow \infty} a_l(x), \quad l = 0, \dots, n-1. \quad (2.2.55)$$

Assume Hypothesis 2.2.17. Denoting by W_2^k the Sobolev space of k times differentiable functions with L^2 -derivatives, let us introduce the operators $T(\lambda)$, H_0 , H , $K(\lambda)$, and the matrices A, B as follows:

$$T(\lambda) = \begin{bmatrix} \partial_x & -1 & \cdots & \cdots & 0 \\ 0 & \partial_x & -1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & & & \partial_x & -1 \\ \frac{a_0^\infty - \lambda}{a_n} & \frac{a_1^\infty}{a_n} & \cdots & \cdots & \partial_x + \frac{a_{n-1}^\infty}{a_n} \end{bmatrix}, \quad (2.2.56)$$

$$\text{with the domain } (\text{dom}(T(\lambda)), \|\cdot\|_{T(\lambda)}) = W_2^1(\mathbb{R})^n, \quad (2.2.57)$$

$$H_0 = a_n \partial_x^n + \sum_{l=0}^{n-1} a_l^\infty \partial_x^l, \quad (2.2.58)$$

$$\text{with the domain } (\text{dom}(H_0), \|\cdot\|_{H_0}) = W_2^n(\mathbb{R}), \quad (2.2.59)$$

$$H = H_0 + V, \text{ where } V(x) = \sum_{l=0}^{n-1} (a_l(x) - a_l^\infty(x)) \partial_x^l, \quad (2.2.60)$$

$$\text{with the domain } (\text{dom}(H), \|\cdot\|_H) = W_2^n(\mathbb{R}), \quad (2.2.61)$$

$$K(\lambda) = (H - H_0)(H_0 - \lambda)^{-1}, \lambda \in \rho(H_0), \quad (2.2.62)$$

$$A(\lambda) = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & & & & 1 \\ \frac{\lambda - a_0^\infty}{a_n} & -\frac{a_1^\infty}{a_n} & \cdots & \cdots & -\frac{a_{n-1}^\infty}{a_n} \end{bmatrix}, \quad (2.2.63)$$

$$B(x) = \begin{bmatrix} 0 & & 0 & \cdots & 0 \\ 0 & & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ \frac{-a_0(x) + a_0^\infty}{a_n} & \frac{-a_1(x) + a_1^\infty}{a_n} & \cdots & \cdots & \frac{-a_{n-1}(x) + a_{n-1}^\infty}{a_n} \end{bmatrix}. \quad (2.2.64)$$

Then the eigenvalue problem (2.2.54) can be recast as

$$\Psi'(x) = (A(\lambda) + B(x))\Psi(x). \quad (2.2.65)$$

Preparing to use the Fourier transform, we introduce the following matrices obtained by replacing ∂_x by $i\eta$ in (2.2.58) and (2.2.56):

$$H_{0\eta} = a_n (i\eta)^n + \sum_{l=0}^{n-1} a_l^\infty (i\eta)^l, \quad \eta \in \mathbb{R}, \quad (2.2.66)$$

$$T_\eta(\lambda) = \begin{bmatrix} i\eta & -1 & \cdots & \cdots & 0 \\ 0 & i\eta & -1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & & & i\eta & -1 \\ \frac{a_0^\infty - \lambda}{a_n} & \frac{a_1^\infty}{a_n} & \cdots & \cdots & i\eta + \frac{a_{n-1}^\infty}{a_n} \end{bmatrix}.$$

Lemma 2.2.18. *The inverse of the operator $T(\lambda)$ is given by the formula*

$$T(\lambda)^{-1} = \begin{bmatrix} * & \cdots & * & a_n(H_0 - \lambda)^{-1} \\ * & \cdots & * & a_n \partial_x (H_0 - \lambda)^{-1} \\ \vdots & & \vdots & \vdots \\ * & \cdots & * & a_n \partial_x^{(n-1)} (H_0 - \lambda)^{-1} \end{bmatrix}, \quad \lambda \in \rho(H_0), \quad (2.2.67)$$

where stars denote the elements which are not important in the sequel.

Proof. Since $\lambda \in \rho(H_0)$, the matrix $H_{0\eta} - \lambda$ is invertible. A direct calculation shows that the inverse of the matrix $T_\eta(\lambda)$ is given by the formula

$$T_\eta(\lambda)^{-1} = \begin{bmatrix} * & \cdots & * & a_n(H_{0\eta} - \lambda)^{-1} \\ * & \cdots & * & a_n(i\eta)(H_{0\eta} - \lambda)^{-1} \\ \vdots & & \vdots & \vdots \\ * & \cdots & * & a_n(i\eta)^{(n-1)}(H_{0\eta} - \lambda)^{-1} \end{bmatrix}. \quad (2.2.68)$$

Using the Fourier transform in (2.2.68) proves the required assertion (2.2.67). \square

Next, we introduce the operator $\mathbb{K}(\lambda)$ on $L^2(\mathbb{R}; dx)^n$ as follows:

$$\mathbb{K}(\lambda) = -B(\cdot)T(\lambda)^{-1}, \quad \lambda \in \rho(H_0). \quad (2.2.69)$$

We will see in Theorem 2.2.20 that $\mathbb{K}(\lambda) \in \mathcal{B}(L^2(\mathbb{R}; dx)^n)$.

Hypothesis 2.2.19. *In addition to Hypothesis 2.2.17, we assume that*

$$a_l(\cdot) - a_l^\infty \in L^2(\mathbb{R}), \quad l = 0, \dots, n-1.$$

Theorem 2.2.20. *Assume Hypothesis 2.2.19 and let $\lambda \in \rho(H_0)$. Then*

$$\mathbb{K}(\lambda) \in \mathcal{B}_2(L^2(\mathbb{R}; dx)^n).$$

Proof. The assertion in the theorem now follows from the well-known formula for the \mathcal{B}_2 -norm of an integral operator, see [8, Theorem 11.3.6]:

$$\begin{aligned} \|\mathbb{K}(\lambda)\|_{\mathcal{B}_2(L^2(\mathbb{R}; dx)^n)}^2 &= \int \int_{\mathbb{R} \times \mathbb{R}} dx dx' \|\mathbb{K}(\lambda, x, x')\|_{\mathbb{C}^{n \times n}}^2 \\ &= \int \int_{\mathbb{R} \times \mathbb{R}} dx dx' \| -B(x)(T_\eta^{-1})^\vee(\lambda, x - x') \|_{\mathbb{C}^{n \times n}}^2 \\ &\leq \int_{\mathbb{R}} dx \|B(x)\|_{\mathbb{C}^{n \times n}}^2 \int_{\mathbb{R}} dx \|(T_\eta^{-1})^\vee(\lambda, x - x')\|_{\mathbb{C}^{n \times n}}^2 = \|B\|_{L^2}^2 \|T_\eta(\lambda)^{-1}\|_{L^2}^2 < \infty. \end{aligned}$$

The last inequality holds since $\|B\|_{L^2}^2 < \infty$ by Hypothesis 2.2.19, and since the L^2 -norm of each entry of the matrix $T_\eta(\lambda)^{-1}$ is finite. \square

Hypothesis 2.2.21. *We assume:*

$$\lambda_0 \in \rho(H_0) \cap \{\lambda \in \mathbb{C} \mid \lambda \text{ is an isolated eigenvalue of } H\}. \quad (2.2.70)$$

Theorem 2.2.22. *Assume Hypothesis 2.2.21 and let $\mathbb{E}(\cdot)$ be the Evans function for the perturbed equation (2.2.65). Then*

$$m(\lambda_0; \mathbb{E}(\cdot)) = m(\lambda_0; I_{L^2(\mathbb{R})^n} - \mathbb{K}(\cdot)) = m(\lambda_0; I_{L^2(\mathbb{R})} - K(\cdot)) = m(\lambda_0; H). \quad (2.2.71)$$

Proof. The first equality follows from formula (2.2.16). The second equality is based on the identity

$$\mathbb{K}(\lambda) = \begin{bmatrix} 0 & 0 \\ * & (H - H_0)(H_0 - \lambda)^{-1} \end{bmatrix}, \quad (2.2.72)$$

which, in turn, follows from (2.2.64) and (2.2.67):

$$\begin{aligned} \mathbb{K}(\lambda) &= -B(x)T(\lambda)^{-1} \\ &= - \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ \frac{-a_0 + a_0^\infty}{a_n} & \frac{-a_1 + a_1^\infty}{a_n} & \cdots & \frac{-a_{n-1} + a_{n-1}^\infty}{a_n} \end{bmatrix} \begin{bmatrix} * & \cdots & * & a_n(H_0 - \lambda)^{-1} \\ * & \cdots & * & a_n \partial_x (H_0 - \lambda)^{-1} \\ \vdots & & \vdots & \vdots \\ * & \cdots & * & a_n \partial_x^{(n-1)} (H_0 - \lambda)^{-1} \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 0 & 0 \\ * & (H - H_0)(H_0 - \lambda)^{-1} \end{bmatrix}.$$

By Theorem 2.2.20, from (2.2.72) we derive

$$\det_2 \left(I_{L^2(\mathbb{R}; dx)^n} - \mathbb{K}(\lambda) \right) = \det_2 \left(I_{L^2(\mathbb{R}; dx)} - (H - H_0)(H_0 - \lambda)^{-1} \right), \quad (2.2.73)$$

yielding the second equality in (2.2.71). Finally, the last equality in (2.2.71) follows

from Theorem 2.1.21 and Theorem 2.1.19. □

Chapter 3

The Evans function and the Weyl-Titchmarsh function

3.1 Singular Sturm-Liouville differential expressions

Let us consider the general Sturm-Liouville differential expression τ ,

$$\tau y(t) = \frac{1}{r(t)} \left(- (py')'(t) + V(t)y(t) \right), \quad t \in (a, b), \quad -\infty \leq a < b \leq \infty, \quad (3.1.1)$$

and impose the following assumptions.

Hypothesis 3.1.1. *Assume that p, r, V are real-valued measurable functions on (a, b) such that $p(t), r(t) > 0$ almost everywhere in (a, b) , and $1/p, r, V$ belong to $L^1_{loc}(a, b)$, that is, belong to $L^1(c, d)$ for any $a < c < d < b$.*

First, we briefly review some basics of the Weyl-Titchmarsh theory [14, 15]. Without loss of generality we will assume that $0 \in (a, b)$. The point b is called regular if b is finite and assumptions in Hypothesis 3.1.1 hold on $(a, b]$, and singular otherwise; analogously for a . Following [14, Chapter IX], we fix an $\omega \in [0, \pi)$, and let $\theta(\cdot, z), \phi(\cdot, z)$ denote the solutions on (a, b) of the differential equation $\tau y = zy$, $z \in \mathbb{C}$, satisfying

$$\begin{aligned} \theta(0, z) &= \sin \omega, & \phi(0, z) &= \cos \omega, \\ p\theta'(0, z) &= -\cos \omega, & p\phi'(0, z) &= \sin \omega. \end{aligned} \quad (3.1.2)$$

Then clearly $\theta(\cdot, z)$, $\phi(\cdot, z)$ are linearly independent solutions, and $\theta, \theta', \phi, \phi'$ are entire functions of z and continuous in t, z . Moreover, since $W_0(\theta, \phi) = 1$ one has $W_t(\theta, \phi) = 1$ for all t . These solutions are real for real z .

Every solution χ of $\tau y = zy$ except ϕ is, up to a constant multiple, of the form

$$\chi(\cdot, z) = \theta(\cdot, z) + m\phi(\cdot, z), z \in \mathbb{C}, \quad (3.1.3)$$

for some $m = m(z)$ which will depend on $z \in \mathbb{C}$. Consider now a real boundary condition at some point d , $0 < d < b$,

$$\cos \eta y(d) + \sin \eta p y'(d) = 0, \quad 0 \leq \eta < \pi. \quad (3.1.4)$$

We want to find m so that the solution χ in (3.1.3) would satisfy (3.1.4). Clearly, m must satisfy:

$$m = -\frac{\cot \eta \theta(d, z) + p\theta'(d, z)}{\cot \eta \phi(d, z) + p\phi'(d, z)}. \quad (3.1.5)$$

As z, d, η vary, m becomes a function of these arguments, $m = m(z, d, \eta)$, and since $\theta, \theta', \phi, \phi'$ are entire in z it follows that $m(\cdot, d, \eta)$ is meromorphic in z and real for real z . For fixed (z, d) , when η changes from 0 to π , the complex numbers $m(z, d, \eta)$ form a circle, denoted by C_d , whose equation in the m -plane, by a direct verification, is given by the formula

$$W_d(\chi, \bar{\chi}) = 0, \quad \text{where } \chi(\cdot, z) = \theta(\cdot, z) + m\phi(\cdot, z), m \in \mathbb{C}. \quad (3.1.6)$$

Theorem 3.1.2. ([14, Theorem 9.2.2]) *If $z \in \mathbb{C} \setminus \mathbb{R}$ and θ, ϕ are the linearly independent solutions of $\tau y = zy$ satisfying (3.1.2), then the solution $\chi(\cdot, z) = \theta(\cdot, z) + m(z)\phi(\cdot, z)$ satisfies the real boundary condition (3.1.4) if and only if m lies on the circle C_d in the complex m -plane whose equation is given by (3.1.6)*

As $d \rightarrow b$, either $C_d \rightarrow C_b$, a limit circle, or $C_d \rightarrow m_b$, a limit point. If $z \in \mathbb{C}$ then all solutions of the differential equation $\tau y = zy$ belong to $L^2(0, b)$ in the limit circle case (lcc), and if $z \in \mathbb{C} \setminus \mathbb{R}$ then exactly one solution (up to a constant multiple) belongs to $L^2(0, b)$ in the limit point case (lpc). Moreover, in the limit-circle case, a point m is on the circle C_b if and only if $W_b(\chi, \bar{\chi}) = 0$ with $\chi = \theta + m\phi$.

Analogous assertions hold for a and c , $a < c < 0$, as $c \rightarrow a$.

In what follows, we denote by \widehat{m}_a (respectively, \widehat{m}_b) a point on the limit circle C_a (respectively, C_b) if τ is in lcc at a (respectively, at b), or the limit point m_a (respectively, m_b) if τ is in lpc at a (respectively, at b). The functions $\widehat{m}_a, \widehat{m}_b, m_a, m_b$ are called the Weyl-Titchmarsh functions. We refer to [14, Chapter 9] for more details regarding the definition and to [14, Theorem 9.2.3] (in the lpc) and [14, Theorem 9.4.1] (in the lcc) for basic properties of the Weyl-Titchmarsh functions.

Next, we define differential operators associated with τ , see [43, Chapter 6], a brief review in [57], and a systematic exposition in [56]. The maximal operator H_{\max} in $L^2(a, b)$ associated with τ is defined by

$$H_{\max}f = \tau f, \tag{3.1.7}$$

$$f \in \text{dom } H_{\max} = \{g \in L^2(a, b) \mid g, pg' \in AC_{loc}(a, b); \tau g \in L^2(a, b)\}.$$

The minimal operator H_{\min} in $L^2(a, b)$ associated with τ is defined as the closure of the operator H'_{\min} given as follows:

$$H'_{\min}f = \tau f, \tag{3.1.8}$$

$$f \in \text{dom } H'_{\min} = \{g \in \text{dom } H_{\max} \mid g \text{ has compact support in } (a, b)\}.$$

Theorem 3.1.3. ([43, Theorem 6.1], [57, Theorem 3.9]) *Assume Hypothesis 3.1.1.*

Then the adjoint of H_{\min} defined in $L^2(a, b)$ is the maximal operator H_{\max} :

$$H_{\min}^*f = H_{\max}f = \tau f, \tag{3.1.9}$$

$$f \in \text{dom } H_{\min}^* = \{g \in L^2(a, b) \mid g, pg' \in AC_{loc}(a, b); \tau g \in L^2(a, b)\}.$$

We will impose some boundary conditions in order to define self-adjoint extensions H of the operator H_{\min} in $L^2(a, b)$, see [43, Section 6.3], [57, Section 4.5], and [56]. For this purpose, we define smooth functions ρ_a and ρ_b such that

$$\rho_a \equiv 1 \text{ near } a, \quad \rho_a \equiv 0 \text{ near } b, \quad (3.1.10)$$

$$\rho_b \equiv 1 \text{ near } b, \quad \rho_b \equiv 0 \text{ near } a. \quad (3.1.11)$$

Next, we fix a $z \in \mathbb{C} \setminus \mathbb{R}$, then fix any $\widehat{m}_a(z) \in C_a$ and $\widehat{m}_b(z) \in C_b$ (in particular, $\widehat{m}_a(z) = m_a(z)$ if τ is in lpc at a and $\widehat{m}_b(z) = m_b(z)$ if τ is in lpc at b), and define $u_a(\cdot, z) \in L^2(a, b)$ and $u_b(\cdot, z) \in L^2(a, b)$ by

$$u_a(t, z) = \rho_a(t)(\theta(t, z) + \widehat{m}_a(z)\phi(t, z)), \quad (3.1.12)$$

$$u_b(t, z) = \rho_b(t)(\theta(t, z) + \widehat{m}_b(z)\phi(t, z)).$$

Let us define in $L^2(a, b)$ the differential operator H_1 by

$$H_1 f = \tau f, \quad (3.1.13)$$

$$f \in \text{dom } H_1 = \text{dom } H'_{\min} + \{u_a(\cdot, z)\} + \{u_b(\cdot, z)\},$$

where $u_a(\cdot, z)$ and $u_b(\cdot, z)$ are given by (3.1.12). In other words, the domain of H_1 consists of compactly supported in (a, b) functions from $\text{dom } H_{\max}$ together with $u_a(\cdot, z)$, $u_b(\cdot, z)$, and all linear combinations of finitely many such functions.

Theorem 3.1.4. ([43, Theorem. 6.5.], [56, Theorem 5.8]) *Assume Hypothesis 3.1.1.*

Let H_1 be as in (3.1.13). Then H_1 is an essential self-adjoint extension of the minimal operator H_{\min} given in (3.1.8), (3.1.9). Let H be the closure of H_1 . Then

$$Hf = \tau f,$$

$$f \in \text{dom } H = \{g \in L^2(a, b) \mid g, pg' \in AC_{loc}(a, b), \tau g \in L^2(a, b), \quad (3.1.14)$$

$$W_a(g, \bar{u}_a(z)) = 0, W_b(g, \bar{u}_b(z)) = 0\}.$$

We will also need the operators H_a in $L^2(a, 0)$ and H_b in $L^2(0, b)$ defined by

$$H_a f = \tau f, \quad (3.1.15a)$$

$$f \in \text{dom } H_a = \{g \in L^2(a, 0) | g, pg' \in AC_{loc}(a, 0], \tau g \in L^2(a, 0), \quad (3.1.15b)$$

$$W_a(g, \bar{u}_a(z)) = 0, W_0(g, \phi) = \sin \omega g(0) - \cos \omega pg'(0) = 0\}, \quad (3.1.15c)$$

$$H_b f = \tau f, \quad (3.1.16a)$$

$$f \in \text{dom } H_b = \{g \in L^2(0, b) | g, pg' \in AC_{loc}[0, b), \tau g \in L^2(0, b), \quad (3.1.16b)$$

$$W_0(g, \phi) = \sin \omega g(0) - \cos \omega pg'(0) = 0, W_b(g, \bar{u}_b(z)) = 0\}. \quad (3.1.16c)$$

As shown in [43, page 25] or [38, Section VII.7], the boundary conditions at a and b in (3.1.14), (3.1.15c), (3.1.16c) are z -independent.

Remark 3.1.5. If τ is in limit point case either at a or at b then the corresponding boundary condition at a or b in (3.1.14), (3.1.15c), (3.1.16c) may be omitted (that is, if τ is in lpc at a (respectively, at b) then the condition $W_a(g, \overline{u(z)}_a) = 0$ (respectively, $W_b(g, \overline{u(z)}_b) = 0$) is automatically satisfied for any $g \in \text{dom } H_{\max}$, see e.g. [56], [57, Theorem 5.7], [43, Theorem 6.5 (iv)]).

Since (see [15, Theorem XIII.7.4])

$$\sigma_{ess}(H) = \sigma_{ess}(H_a) \cup \sigma_{ess}(H_b), \quad (3.1.17)$$

the following lemma holds (see, e.g. [25, (A.25)]).

Lemma 3.1.6. *Assume Hypothesis 3.1.1. Then the Weyl-Titchmarsh functions are meromorphic in $z \in \mathbb{C} \setminus \sigma_{ess}(H)$.*

Proof. Let \widehat{m}_a be the Weyl-Titchmarsh function (that is, a point on the circle C_a if τ is in lcc at a , or the point m_a if τ is in lpc at a). Consider the operator H_a defined in $L^2(a, 0)$ by (3.1.15). Green's function of H_a for $z \in \mathbb{C} \setminus \sigma(H_a)$ is given by the formula

$$G(t, s, z) = \begin{cases} \phi(t, z)\chi_a(s, z), & t \leq s, \\ \phi(s, z)\chi_a(t, z), & s \leq t, \end{cases} \quad (3.1.18)$$

where $\chi_a(t, z) = \theta(t, z) + \widehat{m}_a(z)\phi(t, z)$ (see [14, Chapter IX, (4.10)]). In particular,

$$G(t, t, z) = \phi(t, z)(\theta(t, z) + \widehat{m}_a(z)\phi(t, z)), \quad t \leq 0.$$

Since Green's function is meromorphic in z away from the essential spectrum of H_a for any fixed t, s (cf. [15, Section XIII.5, Theorem XIII.5.18, Corollary XIII.5.30]), \widehat{m}_a is meromorphic for $z \in \mathbb{C} \setminus \sigma_{ess}(H_a)$ and, therefore, in $\mathbb{C} \setminus \sigma_{ess}(H)$ due to (3.1.17). Analogously, one can prove that \widehat{m}_b is meromorphic in $\mathbb{C} \setminus \sigma_{ess}(H)$. \square

Remark 3.1.7. If τ is in limit point case at b , if m is any point on C_d and $d \rightarrow b$, then $m \rightarrow m_b$, where m_b is the limit point, and the relation $m \rightarrow m_b$ holds independently of the choice of η in the boundary condition (3.1.4). In particular, it holds when $\eta = 0$, and thus the limit point is given by the formula

$$m_b(z) = -\lim_{d \rightarrow b} \frac{\theta(d, z)}{\phi(d, z)} \quad \text{for } z \in \mathbb{C} \setminus \mathbb{R}. \quad (3.1.19)$$

Similarly, if τ is in limit point case at a then

$$m_a(z) = -\lim_{c \rightarrow a} \frac{\theta(c, z)}{\phi(c, z)} \quad \text{for } z \in \mathbb{C} \setminus \mathbb{R}. \quad (3.1.20)$$

The purpose of the next proposition is to establish the existence and uniqueness of the solutions $f_a(\cdot, z)$ and $f_b(\cdot, z)$ of the differential equation $\tau y = zy$ that are square summable near a and b , respectively, and satisfy the respective boundary conditions

in (3.1.15c) and (3.1.16c), if any. For $z \in \mathbb{C} \setminus \mathbb{R}$ this is contained in [15, Theorem XIII.2.32] or [43, Theorem 6.2]; we emphasize that z in Proposition 3.1.8 below can be real.

Proposition 3.1.8. *Assume Hypothesis 3.1.1. If $z \in \mathbb{C} \setminus \sigma_{\text{ess}}(H_a)$ then the following assertions hold:*

(i) *If τ is in lpc at a then there is a unique (up to a constant multiple) solution $f_a(\cdot, z)$ of $\tau y = zy$ that is square integrable near a ;*

(ii) *If τ is in lcc at a then there is a unique (up to a constant multiple) solution $f_a(\cdot, z)$ of $\tau y = zy$ that is square integrable near a and satisfy the boundary condition at a in (3.1.15c).*

Moreover, in either case, if $z \in \mathbb{C} \setminus \sigma(H_a)$ then (up to a constant multiple) $f_a(\cdot, z)$ is equal to $\chi_a(\cdot, z) = \theta(\cdot, z) + \widehat{m}_a(z)\phi(\cdot, z)$, where $\widehat{m}_a(z)$ is used in (3.1.12) and (3.1.15c) to define the operator H_a if τ is in lcc at a , and $\widehat{m}_a(z) = m_a(z)$ if τ is in lpc at a . Lastly, if $z \in \sigma_d(H_a)$ then $f_a(\cdot, z)$ is the nonzero eigenfunction of the operator $H_a - z$.

Analogous assertions hold for the point b .

Proof. First, assuming $z \in \mathbb{C} \setminus \sigma(H_a)$, we let μ denote the number of linearly independent solutions of $\tau y = zy$ that are square integrable near a and satisfy the boundary condition at a in (3.1.15c) provided τ is in lcc at a , and that are just square integrable near a provided τ is in lpc at a . Also, we let ν denote the number of linearly independent solutions that are square integrable near 0 and satisfy the boundary condition at 0 in (3.1.15c). Then $\mu + \nu = 2$ by [15, Theorem XIII.3.11] or [56, Theorem 7.1]. Since 0 is a regular point, $\nu = 1$. Thus $\mu = 1$ proving (i) and (ii) for $z \in \mathbb{C} \setminus \sigma(H_a)$. Further-

more, if τ is in lpc at a then the solution $\chi_a(\cdot, z) = \theta(\cdot, z) + m_a(z)\phi(\cdot, z)$ is square integrable near a , and if τ is in lcc at a then the solution $\chi_a(\cdot, z) = \theta(\cdot, z) + \widehat{m}_a(z)\phi(\cdot, z)$ is square integrable near a and satisfies the boundary condition at a in (3.1.15c). Indeed, for nonreal z this follows from the construction of the Weyl-Titchmarsh function in Theorem 3.1.2, cf. [15, Theorem XIII.2.32] or [43, Theorem 6.2]. For $z \in \mathbb{R} \setminus \sigma(H_a)$ this follows from the first line in formula (3.1.18) for Green's function, and the fact that Green's function is square integrable and $G(\cdot, s, z)$ (for a fixed s) satisfies the boundary conditions defining H_a , see, e.g., [15, Lemma XIII.3.4, Lemma XIII.3.7].

Next, we assume that $z \in \sigma_d(H_a)$, the discrete spectrum of H_a ; in particular, $z \in \mathbb{R}$. If τ is in lpc at a , then the nonzero eigenfunction of H_a corresponding to the given z is the required solution $f_a(\cdot, z)$ since the number of linearly independent solutions that are square integrable near a is at most one. It remains to consider the case when $z \in \sigma_d(H_a)$ and τ is in lcc at a . We know that the eigenfunction corresponding to the given z is a solution of $\tau y = zy$ that is square integrable near a and satisfy the boundary condition at a in (3.1.15c). Seeking a contradiction, let us assume that there are two linearly independent solutions, $f_a^{(1)}(\cdot, z)$ and $f_a^{(2)}(\cdot, z)$, of $\tau y = zy$ that are square integrable near a and satisfy the boundary condition at a in (3.1.15c). According to [56, Theorem 5.8(v)], the domain of H_a can be rewritten in the following way:

$$\begin{aligned} \text{dom } H_a = \{ & g \in L^2(a, 0) \mid g, pg' \in AC_{loc}(a, 0], \tau g \in L^2(a, 0), \\ & W_a(v, g) = W_0(w, g) = 0 \}, \end{aligned} \tag{3.1.21}$$

where v and w are nontrivial real solutions of $\tau y = zy$. Indeed, as required in [56, Theorem 5.8(iv)], τ is in lcc at both a and 0, and the operator H_a is defined using separated boundary conditions (3.1.15c). Let us introduce the functions $\widetilde{f}_a^{(1)}$ and $\widetilde{f}_a^{(2)}$

as follows:

$$\tilde{f}_a^{(j)}(t, z) = \begin{cases} f_a^{(j)}(t, z) & \text{for } t \text{ close to } a, \\ 0 & \text{for } t \text{ close to } 0, \end{cases} \quad (3.1.22)$$

and such that $\tilde{f}_a^{(j)}(\cdot, z) \in \text{dom } H_{a, \max}$ (the latter inclusion is possible as described in [56, page 50]). Then $\tilde{f}_a^{(j)} \in \text{dom } H_a$. Consequently, $\tilde{f}_a^{(1)}(\cdot, z)$ and $\tilde{f}_a^{(2)}(\cdot, z)$ should satisfy the boundary condition at a in (3.1.21). Therefore, $f_a^{(1)}(\cdot, z)$ and $f_a^{(2)}(\cdot, z)$ should satisfy the boundary condition at a in (3.1.21) as well. Since $f_a^{(1)}(\cdot, z)$ and v are solutions of $\tau y = zy$, they are linearly dependent. Analogously, $f_a^{(2)}(\cdot, z)$ and v are linearly dependent. Thus, $f_a^{(1)}(\cdot, z)$ and $f_a^{(2)}(\cdot, z)$ are linearly dependent, a contradiction. \square

Remark 3.1.9. If $z \in \mathbb{C} \setminus \sigma_{\text{ess}}(H_a)$ and f_a is the solution described in Proposition 3.1.8 then both Wronskians $W(\theta, f_a(\cdot, z))$ and $W(f_a(\cdot, z), \phi)$ are finite, and cannot be equal to 0 simultaneously as otherwise $f_a(0, z_0) = pf_a'(0, z_0) = 0$. Analogous facts hold for the point b .

Our next objective is to relate the Weyl-Titchmarsh m -function to the Wronskian determinant of the solutions θ, ϕ, f_a and f_b . As the following lemma shows, this information can be used to characterize the discrete spectrum of the operators H_a, H_b and H .

Lemma 3.1.10. *Assume Hypothesis 3.1.1. If $z_0 \in \mathbb{C} \setminus \sigma_{\text{ess}}(H_a)$ then the following assertions are equivalent:*

- (i) $W(f_a(\cdot, z_0), \phi(\cdot, z_0)) = 0$;
- (ii) $z_0 \in \sigma_d(H_a)$;
- (iii) z_0 is a pole of $\hat{m}_a(\cdot)$,

and if the equivalent assertions hold then $f_a(\cdot, z_0)$ is an eigenfunction of H_a . Analogous facts hold for the point b . Finally, if $z_0 \in \mathbb{C} \setminus \sigma_{\text{ess}}(H)$ then the following assertions are equivalent:

$$(iv) \quad W(f_a(\cdot, z_0), f_b(\cdot, z_0)) = 0;$$

$$(v) \quad z_0 \in \sigma_d(H).$$

Proof. $(i) \Leftrightarrow (ii)$: By Proposition 3.1.8, f_a is the unique solution of $\tau y = zy$ square summable at a (when τ is in lpc at a) that satisfies the boundary condition at a in (3.1.15c) (when τ is in lcc at a). Then (i) holds if and only if f_a is proportional to ϕ if and only if f_a satisfies both boundary conditions in (3.1.15c) (when τ is in lcc at a) if and only if f_a is an eigenfunction of H_a .

$(ii) \Leftrightarrow (iii)$: Indeed, (ii) holds if and only if there exists at least one point (t, s) for which Green's function $G(t, s, \cdot)$ of H_a has a pole at z_0 as a function of z , and this is the case if and only if z_0 is a pole of $m_a(\cdot)$ due to (3.1.18).

$(iv) \Leftrightarrow (v)$: Indeed, (iv) holds if and only if f_a is proportional to f_b if and only if f_a satisfies Proposition 3.1.8 at both a and b if and only if f_a is an eigenfunction of H . □

We refer to [25, (A.36)] and [19, (5.8),(5.10)] for versions of the next theorem when $z \in \mathbb{C} \setminus \mathbb{R}$. Our main point, however, is that z can be real in (3.1.23)-(3.1.25).

Theorem 3.1.11. *Assume Hypothesis 3.1.1 and suppose that θ and ϕ satisfy (3.1.2) for some $\omega \in [0, \pi)$. Let $f_a(\cdot, z)$ and $f_b(\cdot, z)$ denote the square integrable near a and b solutions of $\tau y = zy$ described in Proposition 3.1.8. Then the following formulas*

hold:

$$m_a(z) = \frac{W_0(\theta(\cdot, z), f_a(\cdot, z))}{W_0(f_a(\cdot, z), \phi(\cdot, z))}, \quad z \in \mathbb{C} \setminus \sigma_{ess}(H_a), \quad (3.1.23)$$

$$m_b(z) = \frac{W_0(\theta(\cdot, z), f_b(\cdot, z))}{W_0(f_b(\cdot, z), \phi(\cdot, z))}, \quad z \in \mathbb{C} \setminus \sigma_{ess}(H_b), \quad (3.1.24)$$

$$m_b(z) - m_a(z) = \frac{W_0(f_a(\cdot, z), f_b(\cdot, z))}{W_0(f_a(\cdot, z), \phi(\cdot, z))W_0(f_b(\cdot, z), \phi(\cdot, z))}, \quad z \in \mathbb{C} \setminus \sigma_{ess}(H). \quad (3.1.25)$$

Proof. Let us prove (3.1.24), the proof of (3.1.23) is analogous. By Proposition 3.1.8, $f_b(\cdot, z)$ and $\chi_b(\cdot, z) = \theta(\cdot, z) + \widehat{m}_b(z)\phi(\cdot, z)$ have to be proportional whenever $m_b(z)$ is finite, that is, for $z \in \mathbb{C} \setminus \sigma(H_b)$. Hence, there exists a t -independent constant $c_b(z)$ such that $f_b(t, z) = c_b(z)(\theta(t, z) + m_b(z)\phi(t, z))$. Differentiating, letting $t = 0$, and using (3.1.2) yields the system of equations for $m_b(z)$ and $c_b(z)$,

$$c_b(z) \sin \omega + c_b(z)m_b(z) \cos \omega = f_b(0, z),$$

$$-c_b(z) \cos \omega + c_b(z)m_b(z) \sin \omega = pf'_b(0, z),$$

whose solution is given by

$$\begin{aligned} m_b(z) &= \frac{f_b(0, z) \cos \omega + pf'_b(0, z) \sin \omega}{f_b(0, z) \sin \omega - pf'_b(0, z) \cos \omega} \\ &= \frac{W_0(\theta(\cdot, z), f_b(\cdot, z))}{W_0(f_b(\cdot, z), \phi(\cdot, z))}, \end{aligned} \quad (3.1.26)$$

thus proving (3.1.24) for $z \in \mathbb{C} \setminus \sigma(H_b)$. Further, if $z \in \sigma_d(H_b)$ then m_b has a pole at z by Lemma 3.1.10. Since $f_b(\cdot, z)$ is the eigenfunction of H_b by Proposition 3.1.8, $W_0(f_b(\cdot, z), \phi(\cdot, z)) = 0$ by (3.1.16c). Also, $W_0(\theta(\cdot, z), f_b(\cdot, z)) \neq 0$ by Remark 3.1.9. This extends (3.1.24) for $z \in \mathbb{C} \setminus \sigma_{ess}(H_b)$. Using (3.1.26), its analogue for a , and (3.1.17), a short calculation yields (3.1.25). \square

Remark 3.1.12. Assume Hypothesis 3.1.1 and let $z \in \mathbb{C} \setminus (\sigma(H_a) \cup \sigma(H_b))$. If we choose $f_a(\cdot, z) = \chi_a(\cdot, z)$ and $f_b(\cdot, z) = \chi_b(\cdot, z)$ in Theorem 3.1.11, then (3.1.25) becomes

$$m_b(z) - m_a(z) = W(\chi_a(\cdot, z), \chi_b(\cdot, z)), \quad z \in \mathbb{C} \setminus (\sigma(H_a) \cup \sigma(H_b)). \quad (3.1.27)$$

This formula could be found in [25, (A. 36)] for $a = -\infty$, $b = \infty$ and $z \in \mathbb{C} \setminus \mathbb{R}$. For $\omega = \pi/2$ formulas (3.1.23), (3.1.24), (3.1.25) become:

$$m_a(z) = \frac{pf'_a(0, z)}{f_a(0, z)}, \quad m_b(z) = \frac{pf'_b(0, z)}{f_b(0, z)}, \quad m_b(z) - m_a(z) = \frac{W(f_a(\cdot, z), f_b(\cdot, z))}{f_a(0, z)f_b(0, z)}, \quad (3.1.28)$$

and, in particular, $m_b(z) = p\chi'_b(0, z)$, $m_a(z) = p\chi'_a(0, z)$. Choosing $f_a = \chi_a$, $f_b = \chi_b$ shows that the Wronskians in (3.1.23), (3.1.24), (3.1.25) are not necessarily analytic in $z \in \mathbb{C} \setminus \sigma_{\text{ess}}(H_a)$, $z \in \mathbb{C} \setminus \sigma_{\text{ess}}(H_b)$, $z \in \mathbb{C} \setminus \sigma_{\text{ess}}(H)$.

As shown in Lemma 3.1.10, the Wronskian $W(f_a(\cdot, z_0), f_b(\cdot, z_0))$ plays the role of the Evans function. Usually, the Evans function is defined by means of exponentially decaying solutions, see e.g. [4, 20, 48, 54] and the literature therein. An interesting point of our analysis is that in a different from [48] setting of the current paper the Evans function is defined using square integrable solutions. Also, combining Lemma 3.1.10 and the next corollary, one can use the Weyl-Titchmarsh function to detect the discrete spectrum of H .

Corollary 3.1.13. *If the assumptions in Theorem 3.1.11 are satisfied, then for any $z_0 \in \mathbb{C} \setminus \sigma_{\text{ess}}(H)$ the Wronskian $W(f_a(\cdot, z_0), f_b(\cdot, z_0)) = 0$ if and only if the following alternative holds: Either both $m_a(z_0)$ and $m_b(z_0)$ have poles at z_0 , or they both have finite values at z_0 and $m_a(z_0) = m_b(z_0)$.*

Proof. Since m_a is meromorphic, (3.1.23) implies that z_0 is a pole of m_a if and only if $W(f_a(\cdot, z_0), \phi) = 0$. Analogous assertion holds for $W(f_b(\cdot, z_0), \phi)$. Formula (3.1.25) yields:

$$W(f_a(\cdot, z_0), f_b(\cdot, z_0)) = (m_b(z_0) - m_a(z_0))W(f_a(\cdot, z_0), \phi)W(f_b(\cdot, z_0), \phi) \quad (3.1.29)$$

$$= m_b(z_0)W(f_b(\cdot, z_0), \phi) \cdot W(f_a(\cdot, z_0), \phi)$$

$$- m_a(z_0)W(f_a(\cdot, z_0), \phi) \cdot W(f_b(\cdot, z_0), \phi).$$

$$= W(\theta, f_b(\cdot, z_0)) \cdot W(f_a(\cdot, z_0), \phi) \quad (3.1.30)$$

$$- W(\theta, f_a(\cdot, z_0)) \cdot W(f_b(\cdot, z_0), \phi). \quad (3.1.31)$$

Thus, if both $W(f_a(\cdot, z_0), \phi)$ and $W(f_b(\cdot, z_0), \phi)$ are not zero then (3.1.29) implies that $W(f_a(\cdot, z_0), f_b(\cdot, z_0)) = 0$ if and only if $m_b(z_0) = m_a(z_0)$. If one of the expressions $W(f_a(\cdot, z_0), \phi)$ or $W(f_b(\cdot, z_0), \phi)$ is not zero and another is zero, then one of the expressions in (3.1.30), (3.1.31) is zero and another is not; thus, $W(f_a(\cdot, z_0), f_b(\cdot, z_0))$ is not zero. Finally, if $W(f_a(\cdot, z_0), \phi) = W(f_b(\cdot, z_0), \phi) = 0$ then both (3.1.30), (3.1.31) are zero, and thus $W(f_a(\cdot, z_0), f_b(\cdot, z_0)) = 0$. \square

Let us now consider the Schrodinger equation on the line:

$$-y''(t) + V(t)y(t) = k^2y(t), \quad -\infty < t < \infty. \quad (3.1.32)$$

We denote the spectral parameter by k so that $z = k^2 \in \mathbb{C}$, and assume throughout that $\text{ran}(k) \geq 0$. The Schrödinger equation (3.1.32) is equivalent to the first order system

$$Y'(t) = (A(k) + R(t))Y(t), \quad A(k) = \begin{bmatrix} 0 & 1 \\ -k^2 & 0 \end{bmatrix}, \quad R(t) = \begin{bmatrix} 0 & 0 \\ V(t) & 0 \end{bmatrix}, \quad t \in \mathbb{R}, \quad (3.1.33)$$

where $Y(t) = [y(t) \ y'(t)]^\top$ is the column-vector. As soon as the values $y(t_0)$ and $y'(t_0)$ at any point t_0 are given, one can find the corresponding solution y of (3.1.32) on \mathbb{R} such that y, y' are locally absolutely continuous functions by solving the corresponding Cauchy problem for (3.1.33).

Hypothesis 3.1.14. *Assume that $a = -\infty, b = +\infty, p(t) = r(t) = 1, t \in \mathbb{R}$, and the potential in (3.1.32) is real valued and satisfies $V \in L^1(\mathbb{R})$.*

As it is well known, see, e.g. [10], under Hypothesis 3.1.14 all solutions of the Schrödinger equation (3.1.32) can be obtained as linear combinations of the solutions $f_\pm(t, k)$ of (3.1.32) satisfying the asymptotic boundary conditions

$$\lim_{t \rightarrow \pm\infty} e^{\mp ikt} f_\pm(t, k) = 1, \text{ran}(k) > 0. \quad (3.1.34)$$

These solutions are called the Jost solutions; they are defined as solutions of the Volterra integral equations

$$f_\pm(t, k) = e^{\pm ikt} - \int_t^{\pm\infty} \frac{\sin(k(t-s))}{k} V(s) f_\pm(s, k) ds, \quad (3.1.35)$$

$$\text{ran}(k) > 0, t \in \mathbb{R}.$$

The Jost function, $\mathcal{J} = \mathcal{J}(k)$, is defined by

$$\mathcal{J}(k) = \frac{1}{2ik} W(f_-(\cdot, k), f_+(\cdot, k)), \text{ran}(k) > 0. \quad (3.1.36)$$

The isolated zeros of the Jost function are the discrete eigenvalues of the self-adjoint operator H associated with (3.1.32), see, e.g. [10, Chapter XVII].

Assuming Hypothesis 3.1.14 and $\text{ran}(k) > 0$, the first order matrix system (3.1.33) has a solution $Y_+(\cdot, k)$ exponentially decaying on \mathbb{R}_+ and a solution $Y_-(\cdot, k)$ exponentially decaying on \mathbb{R}_- ; each of them is defined up to a constant multiple. The Evans

function $D(k)$ is then defined as $D(k) = \det [Y_+(0, k) \quad Y_-(0, k)]$, see, e.g., [48]. As shown in [20], the Jost function, $J(k)$, for (3.1.32) is equal to the appropriately chosen Evans function, $D(k)$, for (3.1.33). We will now show how to calculate the Evans and the Jost functions via the Weyl-Titchmarsh functions.

Lemma 3.1.15. [14, Problem 9.4] *Assume Hypothesis 3.1.14. Then the equation $\tau y = k^2 y$ is in the limit point case at both $-\infty$ and ∞ .*

Proof. As known from [14, Problem 9.4], for $k > 0$ the first equation in (3.1.35) has a bounded solution on the positive semi-axis and, hence, $f_+(t, k) - e^{ikt} \rightarrow 0$ as $t \rightarrow \infty$. Therefore, f_+ is not square integrable at $+\infty$ and thus according to Theorem 3.1.2 the equation $\tau y = k^2 y$ is in the limit point case at $+\infty$. Analogously one can show that the equation $\tau y = k^2 y$ is in the limit point case at $-\infty$. \square

Lemma 3.1.16. *Assume Hypothesis 3.1.14. Then the Weyl-Titchmarsh functions $m_{-\infty}(\cdot)$ and $m_\infty(\cdot)$ are meromorphic in $\mathbb{C} \setminus [0, \infty)$.*

Proof. This follows from Lemma 3.1.6 and the fact that $\sigma_{\text{ess}}(H) = [0, \infty)$, see, e.g., [56, Theorem 15.3]. \square

Corollary 3.1.17. *Assume Hypothesis 3.1.14 and let $\omega = \pi/2$ in (3.1.2). Then $z_0 \in \sigma_d(H)$ if and only if either $m_\infty(z_0) = m_{-\infty}(z_0)$ or $m_{-\infty}$ and m_∞ both have a simple pole at z_0 . Moreover, the following formula holds:*

$$D(k) = \mathcal{J}(k) = \frac{1}{2ik} (f_-(0, k) f_+(0, k)) (m_\infty(k^2) - m_{-\infty}(k^2)), \quad \text{ran}(k) > 0. \quad (3.1.37)$$

Proof. The Jost solutions f_- and f_+ are the solutions f_a , $a = -\infty$, and f_b , $b = +\infty$, described in Proposition 3.1.8. Thus, the required assertions follow from Lemma

3.1.10, Theorem 3.1.11, Corollary 3.1.13, (3.1.36), and equality $D(k) = \mathcal{J}(k)$ proved in [20, Theorems 5.4(i), 9.4]. \square

Example 3.1.18. We conclude this section with an explicit example of calculation of the Weyl-Titchmarsh, and Jost and Evans functions, for the Schrödinger equation (3.1.32) with the potential $V(t) = -2 \operatorname{sech}^2 t$, $t \in \mathbb{R}$, induced by the traveling wave solution $2 \operatorname{sech}^2(x - 4t)$ of the KdV equation $u_t + 6uu_x + u_{xxx} = 0$. Clearly, $\sigma_{\text{ess}}(H) = [0, \infty)$. Letting $\omega = \pi/2$, we solve (3.1.2) by variation of constants utilizing the ansatz $\theta(t, z) = c_1 \cos kt + c_2(t) \sin kt$, $\phi(t, z) = c_3(t) \cos kt + c_4 \sin kt$, where $z = k^2$ and $\operatorname{ran}(k) > 0$, or $z \in \mathbb{C} \setminus \sigma_{\text{ess}}(H)$. This simple calculation reveals

$$\begin{aligned}\theta(t, z) &= \frac{1}{2}e^{ikt} \left(1 - \frac{1}{ik} \tanh t\right) + \frac{1}{2}e^{-ikt} \left(1 + \frac{1}{ik} \tanh t\right), \\ \phi(t, z) &= -\frac{ik}{k^2 + 1} \left(\frac{1}{2}e^{ikt} \left(1 - \frac{1}{ik} \tanh t\right) - \frac{1}{2}e^{-ikt} \left(1 + \frac{1}{ik} \tanh t\right)\right).\end{aligned}$$

Now (3.1.19), (3.1.20) yield formulas for the Weyl-Titchmarsh functions,

$$m_{\pm\infty}(k^2) = \mp \frac{k^2 + 1}{ik}, \quad \operatorname{ran}(k) > 0. \quad (3.1.38)$$

Corollary 3.1.17 immediately shows that $z_0 = -1$ is the only discrete eigenvalue.

Moreover, $\chi_{\pm\infty}(t, z) = \theta(t, z) + m_{\pm\infty}(z)\phi(t, z) = e^{\pm ikt} \left(1 \mp \frac{1}{ik} \tanh t\right)$. The Jost solutions then are given by the formulas

$$f_{\pm}(t, k) = \frac{1}{ik - 1} e^{\pm ikt} (ik \mp \tanh t), \quad \operatorname{ran}(k) > 0, \quad (3.1.39)$$

since $f_{\pm}(t, k) = c_{\pm} \chi_{\pm\infty}(t, z)$ and we have chosen the constants c_{\pm} to satisfy (3.1.34).

Using (3.1.38) and (3.1.39) in (3.1.37) we finally obtain $D(k) = \mathcal{J}(k) = (ik + 1)(ik - 1)^{-1}$ for $\operatorname{ran}(k) > 0$.

3.2 The reduced Jost function

In this section we derive an asymptotic formula relating the Jost function (3.1.36) for the Schrödinger equation (3.1.32) on the line, and the reduced Jost function for the Schrödinger equation on a large but finite segment. We assume throughout that Hypothesis 3.1.14 holds, and choose k with $\text{ran}(k) > 0$ so that $e^{ikt} \rightarrow 0$ as $t \rightarrow \infty$. We recall that the Jost solutions $f_{\pm}(\cdot, k)$ from (3.1.34), (3.1.35) are the solutions of the Schrödinger equation (3.1.32) that are asymptotic to the plane waves $e^{\pm ikt}$ which are solutions of the Schrödinger equation with zero potential. The asymptotic properties of the Jost solutions are summarized in the following elementary lemma (for a related material see, e.g. [10, Sec. XVII.1.2]).

Lemma 3.2.1. *Assume Hypothesis 3.1.14. Then, with $\text{ran}(k) > 0$, the following asymptotic relations hold:*

$$\lim_{t \rightarrow \pm\infty} f_{\pm}(t, k)e^{\mp ikt} = 1, \quad \lim_{t \rightarrow \pm\infty} \frac{1}{\pm ik} f'_{\pm}(t, k)e^{\mp ikt} = 1, \quad (3.2.1)$$

$$\lim_{t \rightarrow \pm\infty} f_{\mp}(t, k)e^{\pm ikt} = \mathcal{J}(k), \quad \lim_{t \rightarrow \pm\infty} \frac{1}{\mp ik} f'_{\mp}(t, k)e^{\pm ikt} = \mathcal{J}(k). \quad (3.2.2)$$

Proof. Formulas (3.2.1) follow from (3.1.35). To show (3.2.2), we begin by recalling the following well-known property of the asymptotically constant coefficient ODE system (3.1.33) (see, e.g., [14, Problem III.29] or [16, Chapter 1]): There exist solutions $Y_{\pm}(\cdot, k)$ and $\tilde{Y}_{\pm}(\cdot, k)$ of (3.1.33) such that

$$\lim_{t \rightarrow \pm\infty} Y_{\pm}(t, k)e^{\mp ikt} = [1 \ \pm ik]^{\top}, \quad \lim_{t \rightarrow \pm\infty} \tilde{Y}_{\pm}(t, k)e^{\pm ikt} = [1 \ \mp ik]^{\top}, \quad (3.2.3)$$

where $\pm ik$ are the eigenvalues and $[1 \ \pm ik]^{\top}$ are the corresponding eigenvectors of the matrix $A(k)$ defined in (3.1.33). By (3.2.1), we have $Y_{\pm}(\cdot, k) = [f_{\pm}(\cdot, k) \ f'_{\pm}(\cdot, k)]^{\top}$

with the Jost solutions $f_{\pm}(\cdot, k)$ of (3.1.32). Denoting by $\tilde{f}_{\pm}(\cdot, k)$ the solutions of (3.1.32) such that $\tilde{Y}_{\pm}(\cdot, k) = [\tilde{f}_{\pm}(\cdot, k) \quad \tilde{f}'_{\pm}(\cdot, k)]^{\top}$, we derive from the second equation in (3.2.3) the following asymptotic formulas:

$$\lim_{t \rightarrow \pm\infty} \tilde{f}_{\pm}(t, k)e^{\pm ikt} = 1, \quad \lim_{t \rightarrow \pm\infty} \frac{1}{\mp ik} \tilde{f}'_{\pm}(t, k)e^{\pm ikt} = 1. \quad (3.2.4)$$

By letting $t \rightarrow \pm\infty$, we note that (3.2.1), (3.2.4) also imply

$$W(f_{\pm}(\cdot, k), \tilde{f}_{\pm}(\cdot, k)) = \mp 2ik, \quad \text{ran}(k) > 0. \quad (3.2.5)$$

Therefore, $f_{+}(\cdot, k)$ and $\tilde{f}_{+}(\cdot, k)$ are linearly independent solutions of (3.1.3), and thus there are some constants c_1, c_2 such that

$$f_{-}(t, k) = c_1 f_{+}(t, k) + c_2 \tilde{f}_{+}(t, k) \quad \text{for all } t \in \mathbb{R}. \quad (3.2.6)$$

Computing the Wronskian $W(f_{-}(\cdot, k), f_{+}(\cdot, k))$ in (3.2.6) and using (3.2.5), we see that $c_2 = \mathcal{J}(k)$. Multiplying (3.2.6) by e^{ikt} , letting $t \rightarrow +\infty$, and using (3.2.1), (3.2.4) yields the first formula in (3.2.2), that is, $\lim_{t \rightarrow +\infty} f_{-}(t, k)e^{+ikt} = \mathcal{J}(k)$. Differentiating (3.2.6), in the same manner one obtains the formula in (3.2.2) for the derivative of $f_{-}(\cdot, k)$. Expressing $f_{+}(\cdot, k)$ as a linear combination

$$f_{+}(t, k) = c_1 f_{-}(t, k) + c_2 \tilde{f}_{-}(t, k) \quad \text{for all } t \in \mathbb{R}, \quad (3.2.7)$$

computing the Wronskian $W(f_{-}(\cdot, k), f_{+}(\cdot, k))$ in (3.2.7) and using (3.2.5), we see that $c_2 = \mathcal{J}(k)$. Multiplying (3.2.7) by e^{-ikt} , letting $t \rightarrow -\infty$, and using (3.2.1), (3.2.4) yields the formula $\lim_{t \rightarrow -\infty} f_{+}(t, k)e^{-ikt} = \mathcal{J}(k)$ in (3.2.2). Differentiating (3.2.7), in the same manner one obtains the formula in (3.2.2) for the derivative of $f_{+}(\cdot, k)$. □

Let us introduce the operator $Hf = -f'' + V(t)f$ on $L^2(\mathbb{R})$ with the domain

$$\text{dom } H = \left\{ f \in L^2(\mathbb{R}) \mid f, f' \in AC_{\text{loc}}(\mathbb{R}), -f'' + Vf \in L^2(\mathbb{R}) \right\}, \quad (3.2.8)$$

and remark that $k^2 \in \sigma_d(H)$ if and only if $k \in i\mathbb{R} \setminus \{0\}$ and $\mathcal{J}(k) = 0$, see, e.g., [10, Sec. XVII.1.3]. Indeed, if $\mathcal{J}(k) = 0$ then $f_+(\cdot, k)$ is proportional to $f_-(\cdot, k)$ and thus is an eigenfunction for H as it is exponentially decaying at both $+\infty$ and $-\infty$. Conversely, expressing an eigenfunction $f \in L^2(\mathbb{R})$ of H as a linear combination of $f_+(\cdot, k)$ and $f_-(\cdot, k)$, we see that the Jost solutions must be proportional yielding $\mathcal{J}(k) = 0$.

We will now construct a *reduced* Jost function, $J_L(k)$, that corresponds to the differential operator on the segment $[-L, L]$ with (large) positive L whose discrete spectrum approximates the spectrum of H . Fix an $L > 0$ and two angles, ω_- and ω_+ , in $[0, \pi)$. We will consider the following boundary conditions at the points $-L$ and $+L$,

$$f(-L) \cos \omega_- + f'(-L) \sin \omega_- = 0, \quad (3.2.9)$$

$$f(+L) \cos \omega_+ + f'(+L) \sin \omega_+ = 0, \quad (3.2.10)$$

and define the operator $H_L f = -f'' + V(t)f$ on $L^2(-L, L)$ with the domain

$$\text{dom } H_L = \left\{ f \in L^2(-L, L) \mid f, f' \in AC_{\text{loc}}[-L, L], -f'' + Vf \in L^2(-L, L), \right. \\ \left. \text{and both conditions (3.2.9), (3.2.10) hold} \right\}. \quad (3.2.11)$$

Solving the corresponding Cauchy problems for (3.1.33), we let $g_+(\cdot, k)$ and $g_-(\cdot, k)$ denote the locally absolutely continuous together with their derivatives solutions of the Schrödinger equation (3.1.32) on \mathbb{R} satisfying the following conditions at $\pm L$,

$$g_{\pm}(\pm L, k) = e^{ikL} \sin \omega_{\pm}, \quad g'_{\pm}(\pm L, k) = -e^{ikL} \cos \omega_{\pm}, \quad (3.2.12)$$

and define the *reduced* Jost function, $\mathcal{J}_L = \mathcal{J}_L(k)$, as follows:

$$\mathcal{J}_L(k) = \frac{1}{2ik} W(g_-(\cdot, k), g_+(\cdot, k)), \quad \text{ran}(k) > 0. \quad (3.2.13)$$

Clearly, $g_+(\cdot, k)$ satisfies (3.2.10) while $g_-(\cdot, k)$ satisfies (3.2.9). We claim that $k^2 \in \sigma_d(H_L)$ if and only if $\mathcal{J}_L(k) = 0$. Indeed, if $\mathcal{J}_L(k) = 0$ then $g_+(\cdot, k)$ is proportional to $g_-(\cdot, k)$ and thus is an eigenfunction for H_L as it satisfies *both* conditions (3.2.9) and (3.2.10). Conversely, if g is an eigenfunction for H_L then it satisfies both conditions (3.2.9) and (3.2.10). Evaluating the Wronskian at $t = +L$, we notice that $W(g, g_+(\cdot, k)) = 0$ and hence g is proportional to $g_+(\cdot, k)$. Evaluating the Wronskian at $t = -L$, we notice that $W(g, g_-(\cdot, k)) = 0$, and hence g is proportional to $g_-(\cdot, k)$, proving the claim.

In addition, we will consider two more operators with zero potential, $H_{L,0}^+$ and $H_{L,0}^-$, such that $H_{L,0}^\pm f = -f''$. The operator $H_{L,0}^+$ is defined in $L^2(-\infty, +L)$ with the domain

$$\text{dom } H_{L,0}^+ = \left\{ f \in L^2(-\infty, L) \mid f, f' \in AC_{\text{loc}}(-\infty, L], -f'' \in L^2(-\infty, L), \right. \\ \left. \text{and condition (3.2.10) holds} \right\}.$$

The operator $H_{L,0}^-$ is defined in $L^2(-L, +\infty)$ with the domain

$$\text{dom } H_{L,0}^- = \left\{ f \in L^2(-L, +\infty) \mid f, f' \in AC_{\text{loc}}[-L, +\infty), -f'' \in L^2(-L, +\infty), \right. \\ \left. \text{and condition (3.2.9) holds} \right\}.$$

Furthermore, we let $h_+(\cdot, k)$ and $h_-(\cdot, k)$ denote the solutions of the differential equation $-h'' = k^2 h$ on \mathbb{R} which are locally absolutely continuous together with their derivatives and satisfy the following conditions at $\pm L$,

$$h_\pm(\pm L, k) = e^{ikL} \sin \omega_\pm, \quad h'_\pm(\pm L, k) = -e^{ikL} \cos \omega_\pm, \quad (3.2.14)$$

and define the functions, $\mathcal{J}_{L,0}^\pm = \mathcal{J}_{L,0}^\pm(k)$, as follows:

$$\mathcal{J}_{L,0}^+(k) = \frac{1}{2ik} W(e^{-ikt}, h_+(\cdot, k)), \quad (3.2.15)$$

$$\mathcal{J}_{L,0}^-(k) = \frac{1}{2ik} W(h_-(\cdot, k), e^{ikt}), \quad \text{ran}(k) > 0.$$

Computing the Wronskian for $\mathcal{J}_{L,0}^\pm(k)$ at $\pm L$ yields

$$2ik \mathcal{J}_{L,0}^+(k) = -\det \begin{bmatrix} \sin \omega_+ & 1 \\ -\cos \omega_+ & -ik \end{bmatrix} = -\cos \omega_+ + ik \sin \omega_+, \quad (3.2.16)$$

$$2ik \mathcal{J}_{L,0}^-(k) = \det \begin{bmatrix} \sin \omega_- & 1 \\ -\cos \omega_- & ik \end{bmatrix} = \cos \omega_- + ik \sin \omega_-; \quad (3.2.17)$$

in particular,

$$\mathcal{J}_{L,0}^\pm(k) \text{ does not depend on } L. \quad (3.2.18)$$

Clearly, $h_+(\cdot, k)$ satisfies (3.2.10) and $h_-(\cdot, k)$ satisfies (3.2.9) while $e^{ikt} \in L^2(-L, +\infty)$ and $e^{-ikt} \in L^2(-\infty, +L)$ since $\text{ran}(k) > 0$.

Remark 3.2.2. We claim that $k^2 \in \sigma_d(H_{L,0}^\pm)$ if and only if $\mathcal{J}_{L,0}^\pm(k) = 0$. Indeed, if, say, $\mathcal{J}_{L,0}^+(k) = 0$ then $h_+(\cdot, k)$ is proportional to e^{-ikt} and thus is an eigenfunction for $H_{L,0}^+$ as it satisfies conditions (3.2.10) and $h_+(\cdot, k) \in L^2(-\infty, +L)$. Conversely, if h is an eigenfunction for $H_{L,0}^+$ then it satisfies conditions (3.2.10) and $h \in L^2(-\infty, +L)$. Evaluating the Wronskian at $t = +L$, we notice that $W(h, h_+(\cdot, k)) = 0$ and hence h is proportional to $h_+(\cdot, k)$. Expressing $h \in L^2(-\infty, +L)$ as a linear combination of the plane waves e^{-ikt} and e^{ikt} , we conclude that h is proportional to e^{-ikt} , thus proving the claim for $H_{L,0}^+$. The argument for $H_{L,0}^-$ is analogous. \diamond

We are ready to present the main result of this section (recall (3.2.18)).

Theorem 3.2.3. *Assume that $V \in L^1(\mathbb{R}; \mathbb{R})$. Then, for the Jost functions $\mathcal{J}(k)$, $\mathcal{J}_L(k)$, $\mathcal{J}_{L,0}^\pm(k)$ defined in (3.1.36), (3.2.13), (3.2.15), we have:*

$$\lim_{L \rightarrow \infty} \mathcal{J}_L(k) = \mathcal{J}(k) \mathcal{J}_{L,0}^-(k) \mathcal{J}_{L,0}^+(k), \quad \text{ran}(k) > 0. \quad (3.2.19)$$

Proof. Varying k a little, if needed, with no loss of generality we may and will assume throughout the proof that k^2 is not an isolated eigenvalue of H , that is, that $\mathcal{J}(k) \neq 0$. Then the Jost solutions $f_+(\cdot, k)$ and $f_-(\cdot, k)$ are linearly independent and hence there are exist constants c_1^-, c_2^- and c_1^+, c_2^+ such that for all $t \in \mathbb{R}$ one has the following representations:

$$\begin{aligned} g_-(t, k) &= c_1^- f_+(t, k) + c_2^- f_-(t, k), \\ g_+(t, k) &= c_1^+ f_+(t, k) + c_2^+ f_-(t, k). \end{aligned} \tag{3.2.20}$$

Computing the Wronskians, we find:

$$\begin{aligned} c_1^- &= \frac{W(f_-(\cdot, k), g_-(\cdot, k))}{W(f_-(\cdot, k), f_+(\cdot, k))}, & c_2^- &= \frac{W(g_-(\cdot, k), f_+(\cdot, k))}{W(f_-(\cdot, k), f_+(\cdot, k))}, \\ c_1^+ &= \frac{W(f_-(\cdot, k), g_+(\cdot, k))}{W(f_-(\cdot, k), f_+(\cdot, k))}, & c_2^+ &= \frac{W(g_+(\cdot, k), f_+(\cdot, k))}{W(f_-(\cdot, k), f_+(\cdot, k))}. \end{aligned} \tag{3.2.21}$$

A computation using (3.2.20), (3.2.21) reveals:

$$\begin{aligned} W(g_-(\cdot, k), g_+(\cdot, k)) &= W(c_1^- f_+(\cdot, k) + c_2^- f_-(\cdot, k), c_1^+ f_+(\cdot, k) + c_2^+ f_-(\cdot, k)) \\ &= (-c_1^- c_2^+ + c_2^- c_1^+) W(f_-(\cdot, k), f_+(\cdot, k)) = W_1 + W_2, \end{aligned} \tag{3.2.22}$$

where we temporarily introduced the following notations:

$$W_1 = -W(f_-(\cdot, k), g_-(\cdot, k)) W(g_+(\cdot, k), f_+(\cdot, k)) / W(f_-(\cdot, k), f_+(\cdot, k)), \tag{3.2.23}$$

$$W_2 = W(g_-(\cdot, k), f_+(\cdot, k)) W(f_-(\cdot, k), g_+(\cdot, k)) / W(f_-(\cdot, k), f_+(\cdot, k)). \tag{3.2.24}$$

Assertion (3.2.19) now follows from (3.2.12) and Lemma 3.2.1. Indeed, the expression

$$\begin{aligned} W(f_-(\cdot, k), g_-(\cdot, k)) &= f_-(-L, k) g'_-(-L, k) - f'_-(-L, k) g_-(-L, k) \\ &= -f_-(-L, k) e^{ikL} \cos \omega_- - f'_-(-L, k) e^{ikL} \sin \omega_- \\ &= -\left(f_-(-L, k) e^{ik(-L)}\right) e^{2ikL} \cos \omega_- \\ &\quad - (-ik) \left(\frac{1}{-ik} f'_-(-L, k) e^{ik(-L)}\right) e^{2ikL} \sin \omega_- \end{aligned} \tag{3.2.25}$$

tends to zero as $L \rightarrow \infty$ due to (3.2.1) and $\text{ran}(k) > 0$. This and a similar argument for $W(g_+(\cdot, k), f_+(\cdot, k))$ yields $W_1 \rightarrow 0$ as $L \rightarrow \infty$, and hence it suffices to handle the term W_2 in (3.2.22). If $L \rightarrow \infty$ then, using (3.2.2), the expression

$$\begin{aligned} W(g_-(\cdot, k), f_+(\cdot, k)) &= g_-(-L, k)f'_+(-L, k) - g'_-(-L, k)f_+(-L, k) \\ &= e^{ikL} \sin \omega_- f'_+(-L, k) + e^{ikL} \cos \omega_- f_+(-L, k) \\ &= \left(\frac{1}{ik} e^{-ik(-L)} f'_+(-L, k) \right) (ik) \sin \omega_- + \left(e^{-ik(-L)} f_+(-L, k) \right) \cos \omega_- \end{aligned}$$

tends to

$$\mathcal{J}(k)(ik) \sin \omega_- + \mathcal{J}(k) \cos \omega_- = 2ik \mathcal{J}(k) \mathcal{J}_{L,0}^-(k),$$

where in the last equality we used (3.2.17). Also, the expression

$$\begin{aligned} W(f_-(\cdot, k), g_+(\cdot, k)) &= f_-(L, k)g'_+(L, k) - f'_-(L, k)g_+(L, k) \\ &= -f_-(L, k)e^{ikL} \cos \omega_+ - f'_-(L, k)e^{ikL} \sin \omega_+ \\ &= -\left(f_-(L, k)e^{ikL} \right) \cos \omega_+ + \left(\frac{1}{-ik} f'_-(L, k)e^{ikL} \right) (ik) \sin \omega_+ \end{aligned}$$

tends to

$$-\mathcal{J}(k) \cos \omega_+ + \mathcal{J}(k)(ik) \sin \omega_+ = 2ik \mathcal{J}(k) \mathcal{J}_{L,0}^+(k),$$

where in the last equality we used (3.2.16). Combining this with (3.2.22) yields

$$\begin{aligned} \lim_{L \rightarrow \infty} \mathcal{J}_L(k) &= \lim_{L \rightarrow \infty} \frac{1}{2ik} W(g_-(\cdot, k), g_+(\cdot, k)) = \lim_{L \rightarrow \infty} \frac{1}{2ik} W_2 \\ &= \lim_{L \rightarrow \infty} \frac{1}{2ik} \frac{W(g_-(\cdot, k), f_+(\cdot, k))W(f_-(\cdot, k), g_+(\cdot, k))}{2ik \mathcal{J}(k)} \\ &= \mathcal{J}(k) \mathcal{J}_{L,0}^-(k) \mathcal{J}_{L,0}^+(k), \end{aligned}$$

as required. □

We conclude this section with a short discussion of the relation between Theorem 3.2.3 and certain results in [49] and [7]. In particular, as a part of a general theory relating the spectra of first order differential operators on the line and on finite segments, [49, Theorem 3] relates the multiplicity of the eigenvalues of a general first order differential operator \mathcal{T}_L in the space $L^2((-L, L); \mathbb{C}^d)$ of d -dimensional vector valued functions to that of the operator \mathcal{T} in $L^2(\mathbb{R}; \mathbb{C}^d)$, and to the multiplicity of zeros of certain functions, $D_{\pm}(z)$, of the spectral parameter z , determined by the boundary conditions at $\pm L$ used to define \mathcal{T}_L . The discussion in [49] involves the Evans functions $D_{\infty}(z)$, defined for \mathcal{T} , and $D_L(z)$, defined for \mathcal{T}_L , such that the zeros of the Evans functions are the eigenvalues of the respective operators. One of the major conclusions in [49, Theorem 3], see also the preceding analysis in [7] and a more recent paper [45], is the following eigenvalue multiplicity result: Under certain natural assumptions, for L sufficiently large, the algebraic multiplicity of an isolated eigenvalue of \mathcal{T}_L (that is, the order of a zero of D_L) is equal to the sum of the orders of zeros of the functions D_{∞} , D_- , and D_+ in a small vicinity of the eigenvalue. We emphasize that the results in [7, 45, 49] are obtained for a significantly more general systems than (3.1.33).

We will now furnish the definitions of the functions $D_{\infty}(z)$, $D_L(z)$, $D_{\pm}(z)$ of the spectral parameter $z = k^2$ for the particular case of the first order system (3.1.33) corresponding to the Schrödinger equation (3.1.32) and relate them to the Jost functions $\mathcal{J}(k)$, $\mathcal{J}_L(k)$, $\mathcal{J}_{L,0}^{\pm}(k)$ studied earlier in this section.

First, we recall that the Evans function $D_{\infty}(z)$ is the determinant of the (2×2) matrix whose first column is the initial data at $t = 0$ of the solution of (3.1.33) that

exponentially decays to zero as $t \rightarrow +\infty$ and whose second column is the initial data at $t = 0$ of the solution of (3.1.33) that exponentially decays to zero as $t \rightarrow -\infty$. It is known that if these solutions are appropriately chosen then $D_\infty(z) = \mathcal{J}(k)$, $z = k^2$, see [20, Section 9] and Corollary 3.1.17.

Next, we define $D_L(z)$ as follows. Let us fix two arbitrary unit vectors, denoted by $[\sin \omega_- \quad -\cos \omega_-]^\top$ and $[\sin \omega_+ \quad -\cos \omega_+]^\top$, and let Q_\pm denote the subspace spanned by the vector $[\sin \omega_\pm \quad -\cos \omega_\pm]^\top$. The subspaces Q_- and Q_+ determine the boundary conditions at $-L$ and $+L$ respectively in the sense that a solution f of the Schrödinger equation (3.1.32) satisfies the boundary conditions (3.2.9), (3.2.10) if and only if the corresponding solution $Y(t) = [f(t) \quad f'(t)]^\top$ of the first order system (3.1.33) satisfies the boundary conditions $Y(\pm L) \in Q_\pm$. Following [49], given two subspaces F and E of \mathbb{C}^d with $\dim E + \dim F = d$, we denote by $E \wedge F$ the determinant of the $(d \times d)$ matrix whose columns are the bases vectors of E and F put consequently. We let $\varphi(t, s; z)$ denote the propagator of the system (3.1.33). With this notations, we define $D_L(z) = \varphi(0, -L; z)Q_- \wedge \varphi(0, L; z)Q_+$ (see, e.g., [49, (4.5)] where notation D_{sep} was used instead of D_L). Of course, $D_L(z)$ depends, up to a constant factor, on the choice of the bases vectors in Q_\pm . In particular, if $g_\pm(\cdot, k)$ is the solutions of (3.1.32) that satisfy (3.2.12) then $[g_\pm(\pm L, k) \quad g'_\pm(\pm L, k)]^\top \in Q_\pm$ and thus $D_L(z) = \mathcal{J}_L(k)$ by (3.2.13).

Finally, we define $D_\pm(z)$. We recall that $\pm ik$ are the eigenvalues of the matrix $A(k)$ from (3.1.33) with the corresponding eigenvectors $[1 \quad \pm ik]^\top$. Let us denote by $E_-^s(z)$ the linear subspace spanned by the vector $[1 \quad + ik]^\top$, and by $E_+^u(z)$ the linear subspace spanned by the vector $[1 \quad - ik]^\top$. Then $E_-^s(z)$ is the set of the initial

data of the solutions of the constant coefficient differential equation $Y' = A(k)Y$ that exponentially grow to infinity as $t \rightarrow -\infty$ and $E_+^u(z)$ is the set of the initial data of the solutions of the constant coefficient differential equation $Y' = A(k)Y$ that exponentially grow to infinity as $t \rightarrow +\infty$. Using subspaces Q_\pm introduced in the previous paragraph, following [49, (4.5)], we define $D_-(z) = Q_- \wedge E_-^s(z)$ and $D_+(z) = Q_+ \wedge E_+^u(z)$. By (3.2.16), (3.2.17) we conclude that, up to a constant factor, $D_\pm(z)$ is equal to $\mathcal{J}_{L,0}^\pm(k)$.

Remark 3.2.4. Given a general system of first order linear differential equations, for the full-line Evans function one can use a normalization described in [20], which agrees with the respective (2-modified Fredholm) perturbation determinant. It is under this normalization the Evans function for (3.1.33) coincides with the Jost function. Contrary to frequently used different conventions, under this normalization the Evans function is defined *not* up to a nonzero analytic multiple, but is *unique*. A “naive” expectation which one might have is that the full-line Evans function $D_\infty(z)$ is equal to the limit as $L \rightarrow \infty$ of the sequence of the Evans functions $D_L(z)$ associated with the long finite segments. The results above show that this expectation is not quite correct already for the scalar Schrödinger equation. Indeed, Theorem 3.2.3 yields the following “product formula”:

$$\lim_{L \rightarrow \infty} D_L(z) = D_\infty(z)D_-(z)D_+(z), \quad z \in \mathbb{C} \setminus [0, \infty), \quad (3.2.26)$$

which gives an exact account of how this limit is different from the full-line Evans function. This formula is consistent with the eigenvalue multiplicity results from [7, 49] mentioned above. Remark 3.2.2 describes the following operator theoretical property of the quantities $D_\pm(z)$: The zeros of $D_+(z)$ and $D_-(z)$ are the eigenvalues of

the *unperturbed* differential operator on the respective half-lines $(-\infty, L]$ and $[-L, \infty)$ induced by the boundary conditions at $+L$ and $-L$. Finally, we suspect that formula (3.2.26) holds for much more general systems than (3.1.33), but the proof of this fact is still an ongoing project.

3.3 The Weyl-Titchmarsh M -function and Hamiltonian systems

Following [13, 32, 38, 39], we briefly review the basics of the Weyl-Titchmarsh (Kodaira-Hinton-Shaw-Krall) theory for the matrix valued Hamiltonian systems. We consider two singular endpoints problem for the $(2n \times 2n)$ Hamiltonian system

$$JY'(t) = (zA(t) + B(t))Y(t), \quad t \in (a, b), \quad -\infty \leq a < b \leq \infty, \quad (3.3.1)$$

where $Y(t)$ is a $(2n \times 1)$ -vector, z is a complex parameter, and $J = \begin{bmatrix} 0 & -I_{n \times n} \\ I_{n \times n} & 0 \end{bmatrix}$. The coefficients A, B in (3.3.1) satisfy the following assumptions.

Hypothesis 3.3.1. (i) $A(t)$ and $B(t)$ are $(2n \times 2n)$ Hermitian matrices for $t \in (a, b)$ whose entries are complex valued measurable and locally integrable functions on (a, b) ;

(ii) $A(t) \geq 0$ a.e. in the sense of quadratic forms and satisfies Atkinson's definiteness condition, i.e., if Y is a nontrivial solution of (3.3.1) then

$$\int_c^d Y(t)^* A(t) Y(t) dt > 0 \text{ for any } c, d \text{ such that } a < c < d < b.$$

This is a quite general classical setup for Hamiltonian systems that goes back to at least [37] and [5]. In particular, many spectral problems can be recast in the form (3.3.1), see Section 3.4 for examples. To illustrate Hypothesis 3.3.1, we present next two examples of the spectral problems that can be written as (3.3.1). In one of the

examples both assumptions (i) and (ii) in Hypothesis 3.3.1 hold while in another assumption (i) holds and (ii) does not.

Example 3.3.2. The (generalized) one-dimensional Swift-Hohenberg equation (see, e.g., [6] and the literature cited therein) is given by

$$u_t + (1 + \partial_x^2)^2 u + \mu u - \nu u^2 + u^3 = 0, \quad x \in \mathbb{R}, t \geq 0, \quad (3.3.2)$$

where $u = u(t, x)$ and μ, ν are positive parameters. Linearizing (3.3.2) about a real valued steady state solution $U = U(x)$ yields

$$u_t + (1 + \partial_x^2)^2 u + W(x)u = 0, \quad \text{where } W(x) = \mu - 2\nu U(x) + 3U^2(x), \quad (3.3.3)$$

which induces the eigenvalue problem $(1 + \partial_x^2)^2 u + W(x)u = zu$ for $z \in \mathbb{C}$, $u = u(x)$.

Introducing the column vector $Y(x) = [u \quad u' \quad -(u' + u''') \quad u + u'']^\top$, as suggested in [40], we rewrite the eigenvalue problem as a first order system,

$$\partial_x Y(x) = M(x, z)Y(x), \quad \text{where } M(x, z) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ -z + W(x) & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \quad (3.3.4)$$

Clearly, (3.3.4) can be rewritten as (3.3.1) with the coefficients

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -W & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

satisfying both assumptions (i) and (ii) in Hypothesis 3.3.1.

Example 3.3.3. The one-dimensional focusing nonlinear Schrödinger equation is

$$iu_t + \frac{1}{2}u_{xx} - \omega + |u|^2 u = 0, \quad x \in \mathbb{R}, t \geq 0, \quad (3.3.5)$$

where $u = u(t, x)$ and ω is a positive parameter. Linearizing (3.3.5) about the real valued steady state solution $U(x) = \sqrt{2\omega} \operatorname{sech}(\sqrt{2\omega}x)$ and denoting by $v = \bar{u}$ the

complex conjugate yields

$$\begin{aligned} iu_t + \frac{1}{2}u_{xx} - \omega u + U^2(2u + v) &= 0, \\ -iv_t + \frac{1}{2}v_{xx} - \omega v + U^2(2v + u) &= 0, \end{aligned} \tag{3.3.6}$$

which induces the eigenvalue problem,

$$\begin{aligned} i\lambda + \frac{1}{2}u_{xx} - \omega u + U^2(2u + v) &= 0, \\ -i\lambda + \frac{1}{2}v_{xx} - \omega v + U^2(2v + u) &= 0, \end{aligned} \tag{3.3.7}$$

for $\lambda \in \mathbb{C}$ and $u = u(x)$, $v = v(x)$. Introducing $Y(x) = [u \ v \ u' \ v']^\top$, we rewrite the eigenvalue problem (3.3.7) as a first order system,

$$\partial_x Y(x) = M(x, z)Y(x), \text{ where } M(x, z) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2z + W(x) & -2U^2(x) & 0 & 0 \\ -2U^2(x) & 2z + W(x) & 0 & 0 \end{bmatrix}, \tag{3.3.8}$$

and we introduced the notations $W(x) = 2(\omega - 2U^2(x))$ and $z = i\lambda$. Clearly, (3.3.8)

can be rewritten as (3.3.1) with the coefficients

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -W(x) & 2U^2(x) & 0 & 0 \\ 2U^2(x) & -W(x) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

satisfying assumption (i) but not (ii) in Hypothesis 3.3.1. We recall, however, that the NLS equation is closely related to the Zakharov-Shabat problem, see [1] or a more relevant to the current paper discussion in [34]. As discussed in Example 3.4.6 below, for the Zakharov-Shabat problem both (i) and (ii) in Hypothesis 3.3.1 hold.

In the remaining part of the paper, we assume that both assumptions (i) and (ii) in Hypothesis 3.3.1 hold. A solution of (3.3.1) is said to be A -square integrable if $\int_a^b Y^*(t)A(t)Y(t) dt < \infty$, and we denote this by $Y \in L_A^2(a, b)$. As already mentioned

in (3.3.1), we allow the endpoints a and b to be finite or infinite, and, with no loss of generality, we assume that $0 \in (a, b)$. We will now discuss boundary conditions at $c, 0, d$ for any c, d satisfying $a < c < 0 < d < b$. Let us fix three $(n \times 2n)$ matrices, $[\alpha_1 \ \alpha_2]$, $[\gamma_1 \ \gamma_2]$ and $[\beta_1 \ \beta_2]$, where $\alpha_1, \alpha_2, \gamma_1, \gamma_2, \beta_1, \beta_2$ are $(n \times n)$ matrices satisfying the conditions

$$\text{rank} [\alpha_1 \ \alpha_2] = \text{rank} [\gamma_1 \ \gamma_2] = \text{rank} [\beta_1 \ \beta_2] = n, \quad (3.3.9)$$

$$\alpha_1 \alpha_2^* - \alpha_2 \alpha_1^* = 0_{n \times n}, \quad \gamma_1 \gamma_2^* - \gamma_2 \gamma_1^* = 0_{n \times n}, \quad \beta_1 \beta_2^* - \beta_2 \beta_1^* = 0_{n \times n}. \quad (3.3.10)$$

If (3.3.9), (3.3.10) hold then, cf. [38, page 108], with no loss of generality we may and will assume that

$$\alpha_1 \alpha_1^* + \alpha_2 \alpha_2^* = I, \quad \gamma_1 \gamma_1^* + \gamma_2 \gamma_2^* = I, \quad \beta_1 \beta_1^* + \beta_2 \beta_2^* = I. \quad (3.3.11)$$

Let $\mathcal{Y}(t, z)$ be the fundamental matrix solution of the differential equation (3.3.1) normalized such that

$$\mathcal{Y}(0, z) = \begin{bmatrix} \gamma_1^* & -\gamma_2^* \\ \gamma_2^* & \gamma_1^* \end{bmatrix}. \quad (3.3.12)$$

Then $\mathcal{Y}(t, z)$ satisfies the following conditions at $t = 0$:

$$[\gamma_1 \ \gamma_2] \mathcal{Y}(0, z) = [I_{n \times n} \ 0_{n \times n}], \quad (3.3.13)$$

$$\mathcal{Y}^*(0, z) \mathcal{Y}(0, z) = I_{2n \times 2n} = \mathcal{Y}(0, z) \mathcal{Y}^*(0, z).$$

We decompose $\mathcal{Y}(\cdot, z)$ into $2n \times n$ blocks θ, ϕ , where θ and ϕ are further decomposed into $n \times n$ blocks as follows:

$$\mathcal{Y}(t, z) = [\theta(t, z) \ \phi(t, z)] = \begin{bmatrix} \theta_1(t, z) & \phi_1(t, z) \\ \theta_2(t, z) & \phi_2(t, z) \end{bmatrix}. \quad (3.3.14)$$

We note that θ, ϕ are entire functions in z and are continuous in t, z . Also, we let $\phi_1(t, z), \dots, \phi_n(t, z)$ denote the columns of the $(2n \times n)$ matrix $\phi(t, z)$ and, for future

use, derive from (3.3.12) the following identities:

$$\begin{aligned}\phi(0, z) &= J\theta(0, z), \quad \theta(0, z) = -J\phi(0, z), \\ \phi^*(0, z) &= -\theta^*(0, z)J, \quad \theta^*(0, z) = \phi^*(0, z)J.\end{aligned}\tag{3.3.15}$$

We impose a regular, self-adjoint boundary condition at c and d for solutions of (3.3.1),

$$[\alpha_1 \quad \alpha_2] Y(c) = 0_{n \times 1}, \quad [\beta_1 \quad \beta_2] Y(d) = 0_{n \times 1},\tag{3.3.16}$$

For $z \in \mathbb{C} \setminus \mathbb{R}$, one attempts to satisfy the boundary condition (3.3.16) at d for the solution $\chi_d(t, z) = \theta(t, z) + \phi(t, z)M_d(z)$ with some $(n \times n)$ matrix $M_d(z)$. Inserting χ_d into the equation $[\beta_1 \quad \beta_2] Y(d) = 0$ shows that

$$M_d(z) = -([\beta_1 \quad \beta_2] \phi(d, z))^{-1}([\beta_1 \quad \beta_2] \theta(d, z)).\tag{3.3.17}$$

A direct calculation shows that $M_d(z)$ satisfies the following circle equations:

$$\pm \chi_d(d, z)^*(J/i)\chi_d(d, z) = 0, \quad \text{where } \chi_d(d, z) = \theta(d, z) + \phi(d, z)M_d(z).\tag{3.3.18}$$

It can be shown, see [38, Section VII.3], that, as d approaches b , $M_d(z)$ approaches one of the matrices of the form

$$M_b(z) = C_b(z) + R_b(z)U_b(z)\overline{R}_b(z),\tag{3.3.19}$$

where we define

$$\begin{aligned}C_b(z) &= -\lim_{d \rightarrow b} \left(2 \operatorname{ran}(z) \int_0^d \phi^* A \phi dt \right)^{-1} \left(2 \operatorname{ran}(z) \int_0^d \phi^* A \theta dt - iI_{n \times n} \right), \\ R_b(z) &= \lim_{d \rightarrow b} \left(2 |\operatorname{ran}(z)| \int_0^d \phi^* A \phi dt \right)^{-1/2}, \quad z \in \mathbb{C} \setminus \mathbb{R},\end{aligned}\tag{3.3.20}$$

so that $\overline{R}_b(z) = R_b(\overline{z})$, and where $U_b(z)$ is a unitary matrix. It can be further shown, see [38, Section VII.4], that if $\chi_b(t, z) = \theta(t, z) + \phi(t, z)M_b(z)$, then

$$\int_0^b \chi_b^*(t, z) A(t) \chi_b(t, z) dt \leq (M_b(z) - M_b^*(z)) / (2i \operatorname{ran}(z)),\tag{3.3.21}$$

and thus by Atkinson's condition in Hypothesis 3.3.1 (ii) one has

$$\operatorname{ran}(M_b(z))/\operatorname{ran}(z) = (M_b(z) - M_b^*(z))/(2i \operatorname{ran}(z)) > 0, \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (3.3.22)$$

Similarly, one attempts to satisfy boundary conditions (3.3.16) at c for the solution $\chi_c(t, z) = \theta(t, z) + \phi(t, z)M_c(z)$ with some $n \times n$ matrix $M_c(z)$. Inserting χ_c into the equation $[\alpha_1 \quad \alpha_2] Y(c) = 0$ shows that

$$M_c(z) = -([\alpha_1 \quad \alpha_2] \phi(c, z))^{-1}([\alpha_1 \quad \alpha_2] \theta(c, z)). \quad (3.3.23)$$

The circle equations, satisfied by $M_c(z)$, are as follows:

$$\mp \chi_c(c, z)^*(J/i)\chi_c(c, z) = 0, \quad \text{where } \chi_c(c, z) = \theta(c, z) + \phi(c, z)M_c(z). \quad (3.3.24)$$

As c approaches a , $M_c(z)$ approaches

$$M_a(z) = C_a(z) + R_a(z)U_a(z)\overline{R}_a(z), \quad (3.3.25)$$

where

$$\begin{aligned} C_a(z) &= -\lim_{c \rightarrow a} \left(2 \operatorname{ran}(z) \int_c^0 \phi^* A \phi dt \right)^{-1} \left(2 \operatorname{ran}(z) \int_c^0 \phi^* A \theta dt + iI \right), \\ R_a(z) &= \lim_{c \rightarrow a} \left(2 |\operatorname{ran}(z)| \int_c^0 \phi^* A \phi dt \right)^{-1/2}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \end{aligned} \quad (3.3.26)$$

so that $\overline{R}_a(z) = R_a(\bar{z})$, and where $U_a(z)$ is a unitary matrix. If $\chi_a(t, z) = \theta(t, z) + \phi(t, z)M_a(z)$, then

$$\int_a^0 \chi_a^*(t) A(t) \chi_a(t) dt \leq (M_a^*(z) - M_a(z))/(2i \operatorname{ran}(z)), \quad (3.3.27)$$

and by Atkinson's condition in Hypothesis 3.3.1 (ii)

$$\operatorname{ran}(M_a(z))/(\operatorname{ran}(z)) = (M_a^*(z) - M_a(z))/(2i \operatorname{ran}(z)) < 0, \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (3.3.28)$$

The matrix valued functions $M_a(z)$ and $M_b(z)$ are called the Weyl-Titchmarsh M -functions.

We introduce the following spaces:

$$N(b, z) = \left\{ Y \in L_A^2(0, b) \mid JY'(t) = (zA(t) + B(t))Y(t) \text{ a.e. on } (0, b) \right\}, \quad (3.3.29)$$

$$N(a, z) = \left\{ Y \in L_A^2(a, 0) \mid JY'(t) = (zA(t) + B(t))Y(t) \text{ a.e. on } (a, 0) \right\}. \quad (3.3.30)$$

The Hamiltonian system (3.3.1) is said to be (see [38, Page 88], [32, Page 274], [39]) in the limit point case at b (respectively, at a) whenever

$$\dim_{\mathbb{C}}(N(b, z)) = n \text{ (respectively, } \dim_{\mathbb{C}}(N(a, z)) = n) \text{ for all } z \in \mathbb{C} \setminus \mathbb{R}, \quad (3.3.31)$$

and in the limit circle case at b (respectively, at a) whenever

$$\dim_{\mathbb{C}}(N(b, z)) = 2n \text{ (respectively, } \dim_{\mathbb{C}}(N(a, z)) = 2n) \text{ for all } z \in \mathbb{C}. \quad (3.3.32)$$

Let us define the maximal operator T_{\max} in $L_A^2(a, b)$ associated with (3.3.1):

$$\begin{aligned} T_{\max} f &= F, \\ f \in \text{dom } T_{\max} &= \left\{ g \in L_A^2(a, b) \mid g \in AC_{loc}((a, b); \mathbb{C}^{2n}) \text{ and there exists} \right. \\ &\quad \left. F \in L_A^2(a, b) \text{ such that } Jg'(t) - B(t)g(t) = A(t)F(t) \text{ for a.e. } t \in (a, b) \right\}. \end{aligned}$$

The minimal operator T_{\min} in $L_A^2(a, b)$ associated with (3.3.1) is defined as the closure of the operator T'_{\min} given as follows:

$$\begin{aligned} T'_{\min} f &= F, \\ f \in \text{dom } T'_{\min} &= \left\{ g \in \text{dom } T_{\max} \mid g \text{ has compact support in } (a, b) \right\}. \end{aligned} \quad (3.3.33)$$

We recall from [52, Theorem 4.1] that the deficiency indices of T_{\min} are given by

$$\dim \ker (T_{\max} - zI_{L_A^2}) = \dim_{\mathbb{C}} (N(a, z)) + \dim_{\mathbb{C}} (N(b, z)) - 2n, \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (3.3.34)$$

Also, we recall from [38, Section VII.8] Green's formula associated with (3.3.1): If $F_1, F_2 \in \text{dom } T_{\max}$ then

$$\begin{aligned} \int_a^b \left(F_2^*(JF_1' - BF_1) - (JF_2' - BF_2)^*F_1 \right) dt \\ = F_2^*JF_1 \Big|_a^b := F_2^*(b)JF_1(b) - F_2^*(a)JF_1(a). \end{aligned} \quad (3.3.35)$$

The following assumption holds provided that the entries of A, B are real valued. It is also holds for all examples discussed at the end of this section.

Hypothesis 3.3.4. *In addition to Hypothesis 3.3.1 we assume that*

$$\dim_{\mathbb{C}} (N(a, z)) = \dim_{\mathbb{C}} (N(a, \bar{z})), \quad (3.3.36)$$

$$\dim_{\mathbb{C}} (N(b, z)) = \dim_{\mathbb{C}} (N(b, \bar{z})), \text{ for all } z \in \mathbb{C} \setminus \mathbb{R}.$$

Next, we fix some $z \in \mathbb{C} \setminus \mathbb{R}$, and then fix any $M_a(z)$ and $M_b(z)$ in (3.3.25), (3.3.19).

We introduce the operator T on $L_A^2(a, b)$ associated with (3.3.1) as follows:

$$Tf = F,$$

$$f \in \text{dom } T = \{g \in \text{dom } T_{\max} \mid B_a(g) := \lim_{c \rightarrow a} \chi_a(c, \bar{z})^* Jg(c) = 0, \quad (3.3.37)$$

$$B_b(g) := \lim_{d \rightarrow b} \chi_b(d, \bar{z})^* Jg(d) = 0\}. \quad (3.3.38)$$

Theorem 3.3.5. [38, Theorem VIII.3.4] *Assume Hypothesis 3.3.4. The operator T defined in (3.3.37) is self-adjoint.*

Also, we introduce the self-adjoint operators T_a in $L_A^2(a, 0)$ and T_b in $L_A^2(0, b)$ (see e.g., [38, Theorem VII.6.4]) by

$$T_a f = F, \quad (3.3.39a)$$

$$f \in \text{dom } T_a = \{g \in \text{dom } T_{\max} \mid B_a(g) := \lim_{c \rightarrow a} \chi_a(c, \bar{z})^* Jg(c) = 0, \quad (3.3.39b)$$

$$[\gamma_1 \quad \gamma_2] g(0) = 0\}. \quad (3.3.39c)$$

$$T_b f = F, \tag{3.3.40a}$$

$$f \in \text{dom } T_b = \{g \in \text{dom } T_{\max} \mid [\gamma_1 \ \gamma_2] g(0) = 0, \tag{3.3.40b}$$

$$B_b(g) := \lim_{d \rightarrow b} \chi_b(d, \bar{z})^* J g(d) = 0\}. \tag{3.3.40c}$$

As shown in [38, Section VII.7], the domain of the operators, defined via the boundary conditions at a and b in (3.3.37), (3.3.38), (3.3.39b), (3.3.40c), is z -independent.

Remark 3.3.6. Using Green's formula (3.3.35), the deficiency index theorems in [56, Sections 4–6] can be proved for the closed symmetric operator T_{\min} , cf. [52]. Also, as in Remark 3.1.5, if (3.3.1) is in lpc at a (respectively, at b) then the boundary conditions in (3.3.37) and (3.3.39b) (respectively, in (3.3.38) and (3.3.40c)) can be omitted as they hold automatically for all $g \in \text{dom } T_{\max}$, see, e.g., [38, Section VI.10].

Since (see [56, Theorem 11.5])

$$\sigma_{ess}(T) = \sigma_{ess}(T_a) \cup \sigma_{ess}(T_b), \tag{3.3.41}$$

the following lemma holds.

Lemma 3.3.7. *Assume Hypothesis 3.3.4. Then the Weyl-Titchmarsh M -functions M_a and M_b are meromorphic in $\mathbb{C} \setminus \sigma_{ess}(T)$.*

Proof. Let M_b be the matrix-valued Weyl-Titchmarsh function. Consider the operator T_b defined in $L_A^2(0, b)$ by (3.3.40). Green's function of T_b for $z \in \mathbb{C} \setminus \sigma(T_b)$ is given by the formula

$$G(t, s, z) = \begin{cases} \chi_b(t, z) \phi^*(s, \bar{z}), & s < t, \\ \phi(t, z) \chi_b^*(s, \bar{z}), & t < s, \end{cases} \tag{3.3.42}$$

where $\chi_b(s, z) = \theta(s, z) + \phi(s, z)M_b(z)$ (see [38, Theorem VII.6.3]). Denoting the columns of ϕ by ϕ_j , for $a < c < d < b$ and $j = 1, \dots, n$ we define

$$\tilde{\phi}_j(t, z) = \begin{cases} \phi_j(t, z), & t \in (c, d), \\ 0, & \text{otherwise,} \end{cases} \quad (3.3.43)$$

and introduce the matrix valued function

$$G(z) = \int_c^d \phi^*(t, z)A(t)\phi(t, z)dt. \quad (3.3.44)$$

Clearly, $G(\cdot)$ is analytic for $z \in \mathbb{C}$. Due to Atkinson's condition in Hypothesis 3.3.1

(ii), $\langle \cdot, \cdot \rangle_{L_A^2(c, d)}$ is an inner product in the vector space of the solutions of (3.3.1).

Therefore, $G(z)$ is the Gram matrix of the linearly independent solutions ϕ_1, \dots, ϕ_n .

Consequently, $G(z)$ is non-singular for any $z \in \mathbb{C}$.

For $z \in \rho(T_b)$ the resolvent operator $(T_b - z)^{-1}$ satisfies:

$$\begin{aligned} \langle (T_b - z)^{-1} \tilde{\phi}_j(\cdot, \bar{z}), \tilde{\phi}_i(\cdot, z) \rangle_{L_A^2(a, b)} &= \int_a^b \tilde{\phi}_i^*(t, z)A(t) \int_a^b G(t, s, z)A(s)\tilde{\phi}_j(s, \bar{z})dsdt \\ &= \int_a^b \tilde{\phi}_i^*(t, z)A(t) \left(\phi(t, z) \int_t^b (\theta^*(s, \bar{z}) + M^*(\bar{z})\phi^*(s, \bar{z}))A(s)\tilde{\phi}_j(s, \bar{z})ds \right. \\ &\quad \left. + (\theta(t, z) + \phi(t, z)M(z)) \int_a^t \phi^*(s, \bar{z})A(s)\tilde{\phi}_j(s, \bar{z})ds \right) dt. \end{aligned} \quad (3.3.45)$$

Let $R(z)$ denotes the $(n \times n)$ matrix whose entries $R_{ij}(z)$ are defined as follows:

$$\begin{aligned} R_{ij}(z) &= \langle (T_b - z)^{-1} \tilde{\phi}_j(\cdot, \bar{z}), \tilde{\phi}_i(\cdot, z) \rangle_{L_A^2(a, b)} \\ &\quad - \int_a^b \tilde{\phi}_i^*(t, z)A(t) \left(\phi(t, z) \int_t^b \theta^*(s, \bar{z})A(s)\tilde{\phi}_j(s, \bar{z})ds \right. \\ &\quad \left. + \theta(t, z) \int_a^t \phi^*(s, \bar{z})A(s)\tilde{\phi}_j(s, \bar{z})ds \right) dt. \end{aligned} \quad (3.3.46)$$

Then (3.3.45) yields

$$\begin{aligned} R_{ij}(z) &= \left(\int_a^b \tilde{\phi}_i^*(t, z)A(t)\phi(t, z)dt \right) M_b(z) \left(\int_a^b \phi^*(s, \bar{z})A(s)\tilde{\phi}_j(s, \bar{z})ds \right) \\ &= \left(e_i^* \int_c^d \phi^*(t, z)A(t)\phi(t, z)dt \right) M_b(z) \left(\int_c^d \phi^*(s, \bar{z})A(s)\phi(s, \bar{z})ds e_j \right), \end{aligned}$$

where e_i are the standard unit vectors in \mathbb{C}^n . In other words,

$$R(z) = G(z)M_b(z)G(\bar{z}), \quad (3.3.47)$$

where $G(z)$ is defined in (3.3.44). It is clear from (3.3.47) that $M_b(\cdot)$ is meromorphic for $z \in \mathbb{C} \setminus \sigma_{\text{ess}}(T_b)$ and, due to (3.3.41), for $\mathbb{C} \setminus \sigma_{\text{ess}}(T)$ since $R(z)$ is meromorphic and $G(z)$ is analytic and nonsingular. Analogously, one can prove that M_a is meromorphic for $z \in \mathbb{C} \setminus \sigma_{\text{ess}}(T)$. \square

Next, we prove an analog of Proposition 3.1.8. We stress again that z can be real in Proposition 3.3.8.

Proposition 3.3.8. *Assume Hypothesis 3.3.1. If $z \in \mathbb{C} \setminus \sigma_{\text{ess}}(T_a)$ then the following assertions hold:*

(i) *If the Hamiltonian system (3.3.1) is in lpc at a then there exist exactly n linearly independent solutions of (3.3.1) that are A -square integrable near a (that is, belonging to $L_A^2(a, 0)$);*

(ii) *If the Hamiltonian system (3.3.1) is in lcc at a then there exist exactly n linearly independent solutions of (3.3.1) that are A -square integrable near a and satisfy the boundary condition at a in (3.3.39b).*

Moreover, assuming that (3.3.1) is either in lpc or lcc at a , let $z \in \mathbb{C} \setminus \sigma(T_a)$ and let $F_a(\cdot, z)$ denote the $(2n \times n)$ matrix valued function whose columns are the solutions described in part (i) or (ii). Then $F_a(\cdot, z) = \chi_a(\cdot, z)C(z)$, where $\chi_a(\cdot, z) = \theta(\cdot, z) + M_a(z)\phi(\cdot, z)$, $C(z)$ is some constant matrix, and $M_a(z)$ is the Weyl-Titchmarsh M -function used in (3.3.39b) to define the operator T_a . Analogous assertions hold for the point b .

Proof. We let μ denote the number of linearly independent solutions of (3.3.1) that are A -square integrable near a and satisfy the boundary conditions at a in (3.3.39b) provided (3.3.1) is in lcc at a , and that are just A -square integrable near a provided (3.3.1) is in lpc at a . Also, we let ν denote the number of linearly independent solutions that are A -square integrable near 0 and satisfy the boundary conditions at 0 in (3.3.39c). Assuming $z \in \mathbb{C} \setminus \sigma(T_a)$, we have $\mu + \nu = 2n$ by [56, Theorem 7.1]. Since 0 is a regular point, $\nu = n$ due to (3.3.39c) and (3.3.9). Thus $\mu = n$ proving (i) and (ii) for $z \in \mathbb{C} \setminus \sigma(T_a)$. Furthermore, if (3.3.1) is in lpc at a then the columns of $\chi_a(\cdot, z) = \theta(\cdot, z) + \phi(\cdot, z)M_a(z)$ are the solutions of (3.3.1) that are A -square integrable near a , and if (3.3.1) is in lcc at a then the columns of $\chi_a(\cdot, z) = \theta(\cdot, z) + \phi(\cdot, z)M_a(z)$ are the solutions of (3.3.1) that are A -square integrable near a and satisfy the boundary conditions at a in (3.3.39c). Indeed, this holds because $\chi_a(\cdot, z)$ enters the formula for Green's function, cf. (3.3.42), and Green's function is composed of the A -square integrable near a solutions that satisfy the boundary conditions when appropriate (see, e.g., [56, Theorem 7.6]). This proves the representation $F_a(\cdot, z) = \chi_a(\cdot, z)C(z)$ for $z \in \mathbb{C} \setminus \sigma(T_a)$.

Next, we assume that $z \in \sigma_d(T_a)$, the discrete spectrum of T_a , and stress that z is real. We claim that $\mu \geq n$ provided (3.3.1) is either in lpc or lcc at a . Starting the proof of the claim, let $F_a(\cdot, z)$ denote the matrix valued function whose columns are the solutions $Y_a^{(j)}(\cdot, z)$ of (3.3.1) with the properties described in assertions (i) or (ii). Seeking a contradiction, let us suppose that $\mu < n$. Since μ is the rank of $F_a(\cdot, z)$, by multiplying $F_a(\cdot, z)$ from the right by a constant ($n \times n$) matrix, we may and will assume that the first μ columns of $F_a(\cdot, z)$ are linearly independent and the remaining

columns are zero. Using $\mu = \text{rank } F_a(\cdot, z)$ again, let us suppose that the linearly independent rows of $F_a(\cdot, z)$ are located in the rows with the numbers k_1, \dots, k_μ , where $k_1 \geq \dots \geq k_\mu$. Let us consider the $(n \times 2n)$ matrix $\tilde{\gamma} = [e_{k_1} \ \dots \ e_{k_\mu} \ 0_{2n \times 1} \ \dots \ 0_{2n \times 1}]^\top$, where e_i are the standard unit column vectors in \mathbb{C}^{2n} . If $j = 1, \dots, \mu$, then the j -th row of the matrix $\tilde{\gamma}J$ is equal to $-e_{k_j+n}^\top$ provided $k_j \leq n$ and is equal to $e_{k_j-n}^\top$ provided $k_j > n$, and the remaining rows are zero. It follows that $\tilde{\gamma}J\tilde{\gamma}^* = 0$, and thus $\tilde{\gamma} = [\tilde{\gamma}_1 \ \tilde{\gamma}_2]$ satisfies (3.3.10). Since the first μ rows of the matrix $\tilde{\gamma}F_a(\cdot, z)$ are linearly independent, each of the first μ columns of this matrix is not zero. In other words, all columns of the product $[\tilde{\gamma}_1 \ \tilde{\gamma}_2] \cdot [Y_a^{(1)}(\cdot, z) \ \dots \ Y_a^{(\mu)}(\cdot, z)]$ are nonzero. We denote by \tilde{T}_a the operator defined as in (3.3.39) but with γ_1, γ_2 replaced by $\tilde{\gamma}_1, \tilde{\gamma}_2$, and remark that the solutions $Y_a^{(1)}(\cdot, z), \dots, Y_a^{(\mu)}(\cdot, z)$ of (3.3.1) do *not* satisfy the boundary condition $[\tilde{\gamma}_1 \ \tilde{\gamma}_2]g(0) = 0$. Thus, $\ker(\tilde{T}_a - z) = \{0\}$ and so $z \notin \sigma_d(\tilde{T}_a)$. Since \tilde{T}_a is yet another self-adjoint extension of the operator T_{\min} , one has $\sigma_{\text{ess}}(\tilde{T}_a) = \sigma_{\text{ess}}(T_a)$, and thus $z \notin \sigma_{\text{ess}}(\tilde{T}_a)$ since $z \notin \sigma_{\text{ess}}(T_a)$ by the assumption in the proposition. Therefore, $z \in \mathbb{C} \setminus \sigma(\tilde{T}_a)$. As we have seen in the previous paragraph, the latter inclusion implies $\mu = n$, a contradiction which proves the claim.

It remains to prove that $\mu \leq n$. Let us first consider the case when (3.3.1) is in lpc at a (still assuming $z \in \sigma_d(T_a)$). In this case the proof is analogous to the proof of [56, Theorem 4.8]. Indeed, since $Y_a^{(j)}(\cdot, z)$ are solutions of (3.3.1) and z is real, we have $Y_a^{(j)*}(t, z)JY_a^{(j)}(t, z) = \text{const}$ for each $i, j = 1, \dots, \mu$. By Remark 3.3.6, these constants are in fact equal to zero since $Y_a^{(j)} \in \text{dom } T_{\max}$ and the boundary condition (3.3.37) and (3.3.39b) at a are satisfied automatically for all functions from $\text{dom } T_{\max}$ because (3.3.1) is in lpc at a . So, $Y_a^{(i)*}(t, z)JY_a^{(j)}(t, z) = 0$ for all $i, j =$

$1, \dots, \mu$ and $t \in (a, 0)$. Fix any $c \in (a, 0)$ and define the linear functionals on \mathbb{C}^{2n} by $\rho_i(e) = Y_a^{(i)*}(c, z)Je$, $i = 1, \dots, \mu$. They are linearly independent since $Y_a^{(i)*}(\cdot, z)$ are linearly independent. Then $\cap_{i=1}^{j-1} \ker \rho_i \not\subseteq \ker \rho_j$ for each $j = 1, \dots, \mu$. This and $\rho_i(Y_a^{(j)}(c, z)) = 0$ for all $i, j = 1, \dots, \mu$ imply $\mu \leq \dim \cap_{i=1}^{\mu} \ker \rho_i \leq 2n - \mu$, thus proving $\mu \leq n$.

It remains to prove that $\mu \leq n$ in the case when (3.3.1) is in lcc at a (still assuming $z \in \sigma_d(T_a)$). Let us assume that $\mu > n$. Let $Y_a^{(j)}$, $j = 1, \dots, \mu$, be the linearly independent solutions of (3.3.1) that are A -square integrable near a and satisfy the boundary conditions in (3.3.39b) at a . Next, we introduce the functions $\tilde{Y}_a^{(j)}$, $j = 1, \dots, \mu$, as follows:

$$\tilde{Y}_a^{(j)}(t, z) = \begin{cases} Y_a^{(j)}(t, z) & \text{for } t \text{ close to } a, \\ 0 & \text{for } t \text{ close to } 0, \end{cases} \quad (3.3.48)$$

and such that $\tilde{Y}_a^{(j)}(\cdot, z) \in \text{dom } T_{a, \max}$ (the latter inclusion is possible as described in [56, page 50]). Then $\tilde{Y}_a^{(j)} \in \text{dom } T_a$. Moreover, $\tilde{Y}_a^{(j)}$ are linearly independent modulo $\text{dom } T_{a, \min}$ (see, e.g., [56, Theorem 5.4(a)]). We also know that $\nu = n$. Let $Y_0^{(j)}$, $j = 1, \dots, n$, be the linearly independent solutions of (3.3.1) that are A -square integrable near 0 and satisfy the boundary conditions in (3.3.39c) at 0. Now, we construct the functions $\tilde{Y}_0^{(j)}$, $j = 1, \dots, n$, as follows:

$$\tilde{Y}_0^{(j)}(t, z) = \begin{cases} Y_0^{(j)}(t, z) & \text{for } t \text{ close to } 0, \\ 0 & \text{for } t \text{ close to } a, \end{cases} \quad (3.3.49)$$

and such that $\tilde{Y}_a^{(j)} \in \text{dom } T_a$. Moreover, $\tilde{Y}_0^{(j)}$ are linearly independent modulo $\text{dom } T_{a, \min}$. Thus, there are $\mu + n$ linearly independent modulo $\text{dom } T_{a, \min}$ elements from $\text{dom } T_a$. This contradicts the fact that

$$\dim \left(\text{dom } D(T_a) / \text{dom } D(T_{a, \min}) \right) = 2n, \quad (3.3.50)$$

which holds because the deficiency index of $T_{a,\min}$ is $2n$ since (3.3.1) is in lcc at a . \square

Next, we describe the connections between the Weyl-Titchmarsh M -functions, the isolated eigenvalues of the respective differential operators, and the determinants of matrices composed of the solutions θ, ϕ, F_a and F_b ; here and below $F_a(\cdot, z)$ and $F_b(\cdot, z)$ are the matrix valued functions whose columns are the solutions of (3.3.1) with the properties described in assertions (i) or (ii) of Proposition 3.3.8.

Lemma 3.3.9. *Assume Hypothesis 3.3.1. If $z_0 \in \mathbb{C} \setminus \sigma_{ess}(T_a)$ and (3.3.1) is either in lpc or lcc at a then the following assertions are equivalent:*

- (i) $\det [F_a(t, z_0) \quad \phi(t, z_0)] = 0$ for some/all $t \in (a, 0)$;
- (ii) $z_0 \in \sigma_d(T_a)$;
- (iii) z_0 is a pole of $M_a(\cdot)$.

Analogous facts hold for the point b . Finally, if $z_0 \in \mathbb{C} \setminus \sigma_{ess}(T)$ and (3.3.1) is either in lpc or lcc at either a or b then the following assertions are equivalent:

- (iv) $\det [F_a(t, z_0) \quad F_b(t, z_0)] = 0$ for some/all $t \in (a, b)$;
- (v) $z_0 \in \sigma_d(T)$.

Proof. (i) \Leftrightarrow (ii): By Proposition 3.3.8, the columns of $F_a(\cdot, z_0)$ span the space of solutions of (3.3.1) that are A -square integrable at a (when (3.3.1) is in lpc at a) and that satisfy the boundary conditions at a in (3.1.15c) (when (3.3.1) is in lcc at a). Assertion (i) holds if and only if the columns of $F_a(\cdot, z_0)$ and $\phi(\cdot, z_0)$ are linearly dependent. Since $\text{rank } F_a(\cdot, z_0) = \text{rank } \phi(\cdot, z_0) = n$, this holds if and only if there is a solution of (3.3.1) which is equal to a linear combination of the columns of $F_a(\cdot, z_0)$ and is equal to a linear combination of the columns of $\phi(\cdot, z_0)$. Thus, this solution satisfies both boundary conditions in (3.3.39b), (3.3.39c), yielding equivalence to (ii).

(ii) \Leftrightarrow (iii): By (3.3.47), z_0 is a pole of $M_a(\cdot)$ if and only if z_0 is a pole of the resolvent $(T_a - z)^{-1}$. Since $z_0 \in \mathbb{C} \setminus \sigma_{ess}(T_a)$, the latter is equivalent to (ii).

(iv) \Leftrightarrow (v): The proof is similar to the proof of (i) \Leftrightarrow (ii). \square

Lemma 3.3.9 shows that the Evans function, $D(z)$, for the Hamiltonian system (3.3.1) can be naturally defined in terms of F_a and F_b as follows:

$$D(z) = \det \begin{bmatrix} F_a(0, z) & F_b(0, z) \end{bmatrix}, \quad z \in \mathbb{C} \setminus \sigma_{ess}(T), \quad (3.3.51)$$

so that $D(z_0) = 0$ if and only if $z_0 \in \sigma_d(T)$. We will now express the Evans function in terms of the Weyl-Titchmarsh M -functions. For this, we introduce an $(n \times n)$ matrix Wronskian, $\mathcal{W}(F, G)$, by denoting

$$\mathcal{W}(F, G) = F_1^* G_2 - F_2^* G_1 = -F^* JG \quad (3.3.52)$$

for any $2n \times n$ -matrices $F = \begin{bmatrix} F_1 & F_2 \end{bmatrix}^\top$ and $G = \begin{bmatrix} G_1 & G_2 \end{bmatrix}^\top$, and remark that

$$\mathcal{W}^*(F, G) = (-F^* JG)^* = -\mathcal{W}(G, F). \quad (3.3.53)$$

Also, we recall that $\mathcal{Y}(\cdot, z)$ is the fundamental matrix solution of (3.3.1) satisfying (3.3.12), (3.3.13).

Theorem 3.3.10. *Assume Hypothesis 3.3.4 and that (3.3.1) is either in lpc or lcc at either a or b. Let $F_a(\cdot, z)$ and $F_b(\cdot, z)$ be the matrix valued functions whose columns are the A -square integrable solutions of (3.3.1) described in assertions (i) or (ii) of Proposition 3.3.8. Then the following formulas hold:*

$$M_a(z) = \mathcal{W}(\theta(0, z), F_a(0, z)) \mathcal{W}^*(F_a(0, z), \phi(0, z))^{-1}, \quad z \in \mathbb{C} \setminus \sigma_{ess}(T_a), \quad (3.3.54)$$

$$M_b(z) = \mathcal{W}(\theta(0, z), F_b(0, z)) \mathcal{W}^*(F_b(0, z), \phi(0, z))^{-1}, \quad z \in \mathbb{C} \setminus \sigma_{ess}(T_b), \quad (3.3.55)$$

$$\begin{aligned}
M_b(z) - M_a^*(z) &= \left(\mathcal{W}(F_a(0, z), \phi(0, z)) \right)^{-1} \mathcal{W}(F_a(0, z), F_b(0, z)) \\
&\quad \times \left(\mathcal{W}^*(F_b(0, z), \phi(0, z)) \right)^{-1}, \quad z \in \mathbb{C} \setminus \sigma_{ess}(T). \tag{3.3.56}
\end{aligned}$$

Moreover, the Evans function $D(z)$ for (3.3.1) and the Weyl-Titchmarsh M -functions are related as follows:

$$\begin{aligned}
D(z) &= \det \mathcal{Y}(0, z) \det \mathcal{W}^*(F_a(0, z), \phi(0, z)) \det \mathcal{W}^*(F_b(0, z), \phi(0, z)) \\
&\quad \times \det \left(M_b(z) - M_a(z) \right). \tag{3.3.57}
\end{aligned}$$

Proof. Let us prove (3.3.55), the proof of (3.3.54) is analogous. By Proposition 3.3.8, $F_b(\cdot, z) = \chi_b(\cdot, z)C_b(z)$ for $z \in \mathbb{C} \setminus \sigma(T_b)$. Letting $t = 0$ and using (3.3.12) yield the system of equations for $M_b(z)$ and $C_b(z)$,

$$(\gamma_1^* - \gamma_2^* M_b(z))C_b(z) = F_{b1}(0, z), \quad (\gamma_2^* + \gamma_1^* M_b(z))C_b(z) = F_{b2}(0, z), \tag{3.3.58}$$

where we subdivide $F_b(0, z) = [F_{b1}^\top(0, z) \quad F_{b2}^\top(0, z)]^\top$ into two $(n \times n)$ blocks. In other words, using (3.3.13), (3.3.14), (3.3.15), (3.3.52),

$$\begin{aligned}
F_b(0, z) &= \mathcal{Y}(0, z) \begin{bmatrix} C_b(z) \\ M_b(z)C_b(z) \end{bmatrix}, \\
\begin{bmatrix} C_b(z) \\ M_b(z)C_b(z) \end{bmatrix} &= \mathcal{Y}^*(0, z)F_b(0, z) = \begin{bmatrix} \theta^*(0, z)F_b(0, z) \\ \phi^*(0, z)F_b(0, z) \end{bmatrix} = \begin{bmatrix} \phi^*(0, z)JF_b(0, z) \\ -\theta^*(0, z)JF_b(0, z) \end{bmatrix} \\
&= \begin{bmatrix} -\mathcal{W}(\phi(0, z), F_b(0, z)) \\ \mathcal{W}(\theta(0, z), F_b(0, z)) \end{bmatrix}. \tag{3.3.59}
\end{aligned}$$

By Lemma 3.3.9, $z \in \sigma_d(T_b)$ if and only if M_b has a pole at z or, equivalently, $\det [F_b(0, z) \quad \phi(0, z)] = 0$. Since, cf. (3.3.13), (3.3.15),

$$\begin{aligned}
\det \left(\mathcal{Y}^*(0, z) [F_b(0, z) \quad \phi(0, z)] \right) &= \det \left(\begin{bmatrix} \theta^*(0, z) \\ \phi^*(0, z) \end{bmatrix} \cdot [F_b(0, z) \quad \phi(0, z)] \right) \\
&= \det \begin{bmatrix} \theta^*(0, z)F_b(0, z) & 0 \\ \phi^*(0, z)F_b(0, z) & I_{n \times n} \end{bmatrix} = \det \begin{bmatrix} -\mathcal{W}(\phi(0, z), F_b(0, z)) & 0 \\ \phi^*(0, z)F_b(0, z) & I_{n \times n} \end{bmatrix},
\end{aligned}$$

the matrix $\mathcal{W}^*(F_b(0, z), \phi(0, z))$ is singular if and only if $z \in \sigma_d(T)$. Now (3.3.59), (3.3.53) imply

$$\begin{aligned} M_b(z) &= (\gamma_1 F_{b2}(0, z) - \gamma_2 F_{b1}(0, z))(\gamma_1 F_{b1}(0, z) + \gamma_2 F_{b2}(0, z))^{-1} & (3.3.60) \\ &= \mathcal{W}(\theta(0, z), F_b(0, z))(\mathcal{W}^*(F_b(0, z), \phi(0, z)))^{-1}, \end{aligned}$$

$$C_b(z) = \gamma_1 F_{b1}(0, z) + \gamma_2 F_{b2}(0, z) = \mathcal{W}^*(F_b(0, z), \phi(0, z)) \quad (3.3.61)$$

$$= -\mathcal{W}(\phi(0, z), F_b(0, z)), \quad (3.3.62)$$

thus proving (3.3.55) for $z \in \mathbb{C} \setminus \sigma(T_b)$ and therefore for $z \in \mathbb{C} \setminus \sigma_{ess}(T_b)$ as both sides of (3.3.55) are not defined at $z \in \sigma_d(T_b)$ by Lemma 3.3.9.

Using (3.3.60), its analogue for a , and (3.3.10), a short calculation yields (3.3.56). Formula (3.3.57) follows from formula (3.3.59), its analogue for a , formulae (3.3.53), (3.3.54), (3.3.55), and from the following computation:

$$\begin{aligned} \begin{bmatrix} F_a(0, z) & F_b(0, z) \end{bmatrix} &= \mathcal{Y}(0, z) \begin{bmatrix} -\mathcal{W}(\phi(0, z), F_a(0, z)) & -\mathcal{W}(\phi(0, z), F_b(0, z)) \\ \mathcal{W}(\theta(0, z), F_a(0, z)) & \mathcal{W}(\theta(0, z), F_b(0, z)) \end{bmatrix} \\ &= -\mathcal{Y}(0, z) \begin{bmatrix} I & I \\ M_a(z) & M_b(z) \end{bmatrix} \begin{bmatrix} \mathcal{W}(\phi(0, z), F_a(0, z)) & 0 \\ 0 & \mathcal{W}(\phi(0, z), F_b(0, z)) \end{bmatrix} \\ &= -\mathcal{Y}(0, z) \begin{bmatrix} I & 0 \\ M_a(z) & M_b(z) - M_a(z) \end{bmatrix} \cdot \begin{bmatrix} I & I \\ 0 & I \end{bmatrix} \\ &\quad \times \begin{bmatrix} \mathcal{W}(\phi(0, z), F_a(0, z)) & 0 \\ 0 & \mathcal{W}(\phi(0, z), F_b(0, z)) \end{bmatrix}. \end{aligned}$$

□

3.4 Examples

We collected in this section several examples of Hamiltonian systems where the assumptions in Lemma 3.3.9 and Theorem 3.3.10 are satisfied.

Example 3.4.1. *The Dirac-type system* (see, e.g., [13]) is obtained by setting $A(t) = I_{2n \times 2n}$ in (3.3.1).

Lemma 3.4.2. [13, Lemma 2.15] *The limit point case holds for Dirac-type systems at $a = -\infty$ and $b = +\infty$.*

Example 3.4.3. *The Schrödinger equation with matrix-valued potential,*

$$-y''(t) + Q(t)y(t) = k^2y(t), \quad -\infty < t < \infty; \quad (3.4.1)$$

for which we impose the following assumption.

Hypothesis 3.4.4. *Assume that $Q = Q^* \in L^1(\mathbb{R})^{n \times n}$.*

Equation (3.4.1) may be re-written as the system

$$JY'(t) = (k^2A(t) + B(t))Y(t), \quad t \in (-\infty, +\infty), \quad (3.4.2)$$

where

$$Y(t) = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}, \quad A(t) = \begin{bmatrix} I_{n \times n} & 0 \\ 0 & 0 \end{bmatrix}, \quad B(t) = \begin{bmatrix} -Q(t) & 0 \\ 0 & I_{n \times n} \end{bmatrix}. \quad (3.4.3)$$

As it is well known (see the proof of Theorem 1.4.1 [3]), equation (3.4.1) has a system of matrix valued solutions $f_+(\cdot, k)$ and $\tilde{f}_+(\cdot, k)$ satisfying the asymptotic boundary conditions

$$\lim_{t \rightarrow \infty} e^{-ikt} f_+(t, k) = I_{n \times n}, \quad \lim_{t \rightarrow \infty} e^{ikt} \tilde{f}_+(t, k) = I_{n \times n}, \quad \text{ran}(k) > 0. \quad (3.4.4)$$

If $F_+(\cdot, k) = [f_+(\cdot, k) \quad f'_+(\cdot, k)]^\top$ and $\tilde{F}_+(\cdot, k) = [\tilde{f}_+(\cdot, k) \quad \tilde{f}'_+(\cdot, k)]^\top$, then $F(\cdot, k)$ and $\tilde{F}(\cdot, k)$ form a fundamental system of solutions of (3.4.2) satisfying asymptotic boundary conditions

$$\begin{aligned} \lim_{t \rightarrow \infty} e^{-ikt} F_+(t, k) &= [I_{n \times n} \quad ikI_{n \times n}]^\top, \\ \lim_{t \rightarrow \infty} e^{ikt} \tilde{F}_+(t, k) &= [I_{n \times n} \quad -ikI_{n \times n}]^\top, \quad \text{ran}(k) > 0. \end{aligned} \quad (3.4.5)$$

Lemma 3.4.5. *Assume Hypothesis 3.4.4. Then equation (3.4.2) is in the limit point case at both $-\infty$ and $+\infty$.*

Proof. Equation (3.4.2) is in the limit point case at $+\infty$ since the columns of $F_+(\cdot, k)$ are A -square integrable solutions of (3.4.2) while the columns of $\tilde{F}_+(\cdot, k)$ are *not* A -square integrable solutions of (3.4.2) which is clear from (3.4.5) and the formula

$$\int F^*(t, k)A(t)F(t, k)dt = \int f^*(t, k)f(t, k)dt, \quad (3.4.6)$$

which holds for any solution $F(\cdot, k) = [f(\cdot, k) \quad f'(\cdot, k)]^\top$ of (3.4.2). Similarly, it can be proved that (3.4.2) is in the limit point case at $-\infty$. \square

Example 3.4.6. *The Zakharov-Shabat problem (see, e.g., [2]) is given by*

$$v' = \begin{bmatrix} -ik & r(t) \\ q(t) & ik \end{bmatrix} v, \quad -\infty < t < \infty, \quad (3.4.7)$$

where we assume that the scalar functions $r, q \in L^1(\mathbb{R})$ satisfy $r(t) = \overline{q(t)}$ for all $t \in (-\infty, \infty)$. Let us make the change of variables

$$u = \frac{1}{\sqrt{2}} \begin{bmatrix} i & i \\ 1 & -1 \end{bmatrix} v. \quad (3.4.8)$$

Then (3.4.7) becomes

$$Ju' = \left(kI_{2 \times 2} + \frac{1}{2} \begin{bmatrix} i(q-r) & r+q \\ r+q & i(r-q) \end{bmatrix} \right) u. \quad (3.4.9)$$

Since $r(t) = \overline{q(t)}$,

$$Ju'(t) = (kI_{2 \times 2} + B(t))u(t), \quad \text{where } B = B^* = \frac{1}{2} \begin{bmatrix} i(q-\bar{q}) & \bar{q}+q \\ \bar{q}+q & i(\bar{q}-q) \end{bmatrix}. \quad (3.4.10)$$

Since $q \in L^1(\mathbb{R})$ and $r(t) = \overline{q(t)}$, there exist four solutions $f_\pm(\cdot, k)$, $\tilde{f}_\pm(\cdot, k)$ of (3.4.7) (see [2, Chapter 1.3]) defined by the following boundary conditions:

$$\lim_{t \rightarrow \infty} f_+(t, k)e^{-ikt} = [0 \quad 1]^\top, \quad \lim_{t \rightarrow -\infty} f_-(t, k)e^{ikt} = [1 \quad 0]^\top, \quad (3.4.11)$$

$$\lim_{t \rightarrow \infty} \tilde{f}_+(t, k)e^{ikt} = [1 \ 0]^\top, \quad \lim_{t \rightarrow -\infty} \tilde{f}_-(t, k)e^{-ikt} = [0 \ 1]^\top. \quad (3.4.12)$$

Therefore, via (3.4.8) there exist four solutions $F_\pm(\cdot, k)$, $\tilde{F}_\pm(\cdot, k)$ of (3.4.10) defined by the following boundary conditions:

$$\lim_{t \rightarrow \infty} F_+(t, k)e^{-ikt} = \frac{1}{\sqrt{2}} [i \ -1]^\top, \quad \lim_{t \rightarrow -\infty} F_-(t, k)e^{ikt} = \frac{1}{\sqrt{2}} [i \ -1]^\top, \quad (3.4.13)$$

$$\lim_{t \rightarrow \infty} \tilde{F}_+(t, k)e^{ikt} = \frac{1}{\sqrt{2}} [i \ 1]^\top, \quad \lim_{t \rightarrow -\infty} \tilde{F}_-(t, k)e^{-ikt} = \frac{1}{\sqrt{2}} [i \ -1]^\top. \quad (3.4.14)$$

It is clear from (3.4.13) and (3.4.14) that (3.4.10) is in lpc at both $-\infty$ and $+\infty$.

Example 3.4.7. *The nearly real (complex) Ginzburg-Landau equation* (see [33]) is

$$u_t - \frac{1}{2}u_{xx} - u + |u|^2u = i\epsilon\left(\frac{1}{2}d_1u_{xx} + d_2u + d_3|u|^2u + d_4|u|^4u\right), \quad (3.4.15)$$

where $u = u(t, x)$, $t \geq 0$, $x \in \mathbb{R}$, $\epsilon > 0$ is small, and the other parameters are real and of $O(1)$ as $\epsilon \rightarrow 0$. Assuming $\epsilon = 0$ and introducing $v(t, x) = \overline{u(t, x)}$, equation (3.4.15) can be rewritten as the system

$$\begin{aligned} u_t - \frac{1}{2}u_{xx} - u + u^2v &= 0, \\ v_t - \frac{1}{2}v_{xx} - v + v^2u &= 0. \end{aligned} \quad (3.4.16)$$

Linearizing (3.4.16) around the hole solution $U(x) = \tanh x$ yields

$$\begin{aligned} u_t - \frac{1}{2}u_{xx} - u + 2U^2(x)u + U^2(x)v &= 0, \\ v_t - \frac{1}{2}v_{xx} - v + U^2(x)u + 2U^2(x)v &= 0, \end{aligned} \quad (3.4.17)$$

which induces the eigenvalue problem for $z \in \mathbb{C} \setminus \mathbb{R}_+$ and $u = u(x)$, $v = v(x)$,

$$\begin{aligned} -\frac{1}{2}u_{xx} - (1 - 2U^2(x))u + U^2(x)v &= zu, \\ -\frac{1}{2}v_{xx} - (1 - 2U^2(x))v + U^2(x)u &= zv. \end{aligned} \quad (3.4.18)$$

Letting $Y(x) = [u(x) \ v(x) \ u'(x) \ v'(x)]^\top$, we rewrite (3.4.18) as the first order system

$$\partial_x Y(x) = M(x, z)Y(x), \quad (3.4.19)$$

where

$$M(x, z) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2(-z - 1 + 2U^2(x)) & 2U^2(x) & 0 & 0 \\ 2U^2(x) & 2(-z - 1 + 2U^2(x)) & 0 & 0 \end{bmatrix}. \quad (3.4.20)$$

Note that $\lim_{x \rightarrow \pm\infty} M(x, z) = M_\infty(z)$ exponentially fast, where

$$M_\infty(z) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2(-z + 1) & 2 & 0 & 0 \\ 2 & 2(-z + 1) & 0 & 0 \end{bmatrix}. \quad (3.4.21)$$

The eigenvalues of $M(z)$ are $\mu_{1,2} = \pm i\sqrt{2z}$, $\mu_{3,4} = \pm i\sqrt{2z + 4}$ and thus nonimaginary if and only if $z \in \mathbb{C} \setminus \mathbb{R}_+$. Since $M(\cdot, z) - M_\infty(z) \in L^1(\mathbb{R})^{4 \times 4}$, by the standard asymptotic theory (see, e.g., [16, Chapter I] or [14, Chapter 3, Problem 29]), there exist four solutions $Y^{(j)}, j = 1, \dots, 4$, of (3.4.19) defined by the boundary conditions

$$\lim_{x \rightarrow +\infty} Y^{(j)}(x, z)e^{-\mu_j x} = p_j, \quad \text{Re}(\mu_j) \neq 0, \quad z \in \mathbb{C} \setminus \mathbb{R}_+, \quad (3.4.22)$$

where p_j is the eigenvector of $M_\infty(z)$ corresponding to the eigenvalue μ_j . We can rewrite (3.4.19) as

$$J(\partial_x Y)(x) = (zA + B(x))Y(x), \quad (3.4.23)$$

where

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2(1 - 2U^2) & -2U^2 & 0 & 0 \\ -2U^2 & 2(1 - 2U^2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -I_2 \\ I_2 & 0 \end{bmatrix}.$$

It is clear from (3.4.22) that (3.4.23) is in lpc at $+\infty$. Similarly, it can be shown that (3.4.23) is in lpc at $-\infty$.

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