STUDY OF THE CONVERGENCE OF SERIES IN CERTAIN ORTHOGONAL FUNCTIONS

by

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Bibliography.


4. de la Vallée Poussin, "Note sur l'approximation par un polynôme d'une fonction dont la dérivée est à variation bornée". Bulletins de l'Académie royale de Belgique, Classe des Sciences, 1908, Pages 403 -410.


and trigonometric sums.


12. Stieltjes, "Table des valeurs des sommes \( S_k = \sum_{n}^{\infty} n^{-k} \)". Acta Mathematica, Vol. X, pages 299 - 302.

13. Byerly, "An elementary treatise on Fourier's series and Spherical, cylindrical and ellipsoidal harmonics".

14. Whittaker, "Modern Analysis".

Introduction.

The question of the accuracy with which a given function, say \( f(x) \), can be expressed by means of a known function or in a series of known functions is a very important one. Such developments are continually arising in problems in applied mathematics such as in mathematical physics and astronomy. The value of any development depends largely on the accuracy with which one can judge its error at any point.

Reviewing hastily some of the work that has been done along this line, we see that Weierstrass was the first to give the proof for the following theorem:

If in the interval \( a \leq x \leq b \) \( f(x) \) is a single valued, defined, continuous and real function of the real variable \( x \), and if \( \delta \) is an arbitrarily chosen positive quantity, then there is an integral rational function \( P(x) \) such that throughout the interval \( a \leq x \leq b \),

\[
|f(x) - P(x)| < \delta.
\]

It is clear that in general the degree of the polynomial \( P(x) \) is dependent upon \( \delta \). As \( \delta \) is chosen smaller and smaller the degree of \( P(x) \) will in general increase. Lesbesgue has considered the question of what the relation is between \( \delta \) and the degree of the polynomial of lowest degree which satisfies the above inequality throughout the interval for any given function, \( f(x) \).

For any polynomial \( P(x) \) of the \( n \)th degree, the expression \( |f(x) - P(x)| \) has an upper limit in the interval \( a \leq x \leq b \). This upper limit is a function of the coefficients of \( P(x) \). If we consider several polynomials of the \( n \)th degree or lower, this function has a definite lower limit which for a given \( a \), \( b \), and
f(x) depends only on n. Let us call this lower limit \( \varphi(n) \).

It is in general positive. It becomes zero only when \( f(x) \) is itself a polynomial of the \( n \)th degree or lower. The theorem by Weierstrass states that for a continuous function \( f(x) \), the

\[
\lim_{n \to \infty} \varphi(n) = 0.
\]

Introducing the notation \( \varphi(n) = O(\psi(n)) \) and \( \varphi(n) = o(\psi(n)) \), by which we shall mean that for \( n \) sufficiently large \( |\varphi(n)| < A \psi(n) \), where \( A \) is a constant independent of \( n \), or that \( \lim_{n \to \infty} f(n)/\psi(n) = 0 \) respectively, we can state Weierstrass' result by saying that \( \varphi(n) = o(1) \) for a continuous function. Lesbesgue \(^2\) gives a result which says that in case a function satisfies a Lipschitz condition, i.e.

\[
|f(x_2) - f(x_1)| \leq \lambda |x_2 - x_1|,
\]

where \( \lambda \) is a constant, \( \varphi(n) = O(\sqrt{\log n}) \).

Assuming practically the same conditions de la Vallee Poussin \(^3\) finds that \( \varphi(n) = O(\frac{1}{\sqrt{n}}) \). Later \(^4\) he finds that when \( f(x) \) possesses a derivative of limited variation, \( \varphi(n) = O(\frac{1}{n}) \). Lesbesgue \(^5\) has considered the case where \( f(x) \) satisfies a Lipschitz-Dini condition, i.e. \( \lim_{\delta \to 0} \omega(\delta) \log \delta = 0 \) where \( \omega(\delta) \) represents the upper limit of the difference between the maximum and minimum values assumed by \( f(x) \) in an interval of length \( \delta \). He finds that in such a case \( \varphi(n) = o(\frac{1}{\log n}) \).

More in line with this paper is the work done on approximations by means of trigonometric sums. There is a very close relation between trigonometric and polynomial approximations. From a result by Lesbesgue \(^6\) about the convergence of a Fourier's series one can get, assuming the Lipschitz condition, that \( \varphi(n) = O(\frac{\log n}{n}) \). Jackson \(^7\) gives a somewhat better result finding that \( \varphi(n) = O(\frac{1}{n}) \). He also shows that there exist func-
tions satisfying a Lipschitz condition for which \( \varphi(n) \) is not \( o\left(\frac{1}{n}\right) \). In the case of trigonometric approximations we will write \( \tau(n) \) in place of \( \varphi(n) \) to distinguish the two cases. Lesbesgue has found that in case a function satisfies a Lipschitz condition \( \tau(n) = O\left(\frac{1}{n}\right) \) and later he finds that \( \tau(n) = O\left(\frac{\log n}{n}\right) \).

Jackson's result is that \( \tau(n) = O\left(\frac{1}{n}\right) \). In a later paper he proves the following theorem:

"Theorem I. There exists an absolute numerical constant \( K \) having the following property: If \( f(x) \) is a real function of the real variable \( x \), of period \( 2\pi \), which everywhere satisfies the Lipschitz condition

\[ |f(x_2) - f(x_1)| \leq \lambda|x_2 - x_1|, \]

\( \lambda \) being a constant, then there exists for every positive integral value of \( n \) a trigonometric sum \( T_n(x) \), of the \( n^{th} \) order at the most, such that for all values of \( x \)

\[ |f(x) - T_n(x)| \leq \frac{K\lambda}{n}. \]

He finds that \( \frac{1}{2} < K \leq 2 \). A generalization of this theorem is his Theorem III of the same paper.

"For each positive integral value of \( k \) there exists a constant \( K_k \) having the following property: If \( f(x) \) is a function of period \( 2\pi \) possessing a \((k - 1)^{th}\) derivative which everywhere satisfies the Lipschitz condition

\[ |f(x_2) - f(x_1)| \leq \lambda|x_2 - x_1|, \]

\( \lambda \) being a constant, then there exist for every positive integral value of \( n \) a trigonometric sum \( T_n(x) \), of the \( n^{th} \) order at most, such that for all values of \( x \)

\[ |f(x) - T_n(x)| \leq \frac{K_k\lambda}{n^k}. \]

Similar theorems are given for polynomial approximations.

Jackson has also given results for the degree of convergence of certain series. For instance in the case of a Fourier's series he gets, in the paper just quoted, the follow-
ing theorem:

"Theorem X. If \( f(x) \) is a function of period \( 2\pi \) satisfying the Lipschitz condition, it is represented by the partial sum of its Fourier's series to terms of the \( n \)th order, provided \( n \geq 5 \), with an error not exceeding \( \frac{6\lambda (\log n)}{n} \)."

In a later paper \(^8\) he has a generalization of this result, viz.,

"Theorem X. If \( f(x) \) is a function of period \( 2\pi \) possessing a \((k - 1)\)th derivative which everywhere satisfies the Lipschitz condition

\[
|f^{k-1}(x_2) - f^{k-1}(x_1)| \leq \lambda |x_2 - x_1|
\]

where \( \lambda \) is a constant, then \( f(x) \) is everywhere approximately represented by the partial sum of its Fourier's series to terms of the \( n \)th order, \( n \geq 5 \), with an error not exceeding \( 36\lambda \frac{\log n}{n} \). If \( k \) is odd, the factor 36 may be replaced by 12."

For the case of the development of a function in series of Legendre's polynomials Jackson \(^9\) has also given a result. Denoting by \( r_n(x) \) the difference \( \sum a_m P_m(x) - f(x) \), where \( P_m(x) \) are the polynomials of Legendre, and \( a_m = \frac{2m + 1}{2} \int f(x) P_m(x) \, dx \) he gets:

"Theorem II. If \( f(x) \) is a function possessing a \((k - 1)\)th derivative which satisfies a Lipschitz condition throughout the closed interval \((1, -1)\), then, for this function \( f(x) \),

\[
|r_n(x)| = O\left(\frac{\log n}{n^k}\right).
\]

Gronwall \(^{10}\) gives a somewhat more general result in discussing the degree of convergence of a Laplace's series of which a Legendre's series is but a special case.

In this present paper we will develop some theorems concerning the degree of convergence of certain series, in
particular a Fourier's series, a Legendre's series, and a series of Bessel's functions, getting results which are indeed very much similar to the ones already mentioned but which are derived in a much simpler manner. Before proceeding directly to the proof of these theorems, however, we will give some theorems showing the conditions under which the Fourier coefficients in the development of a function will converge to zero.
Chapter I.

1. Total variation finite. Let $f(x)$ be a real function of the real variable $x$ and let us impose the condition that the total variation of $f(x)$ shall remain finite in an interval from $x = 0$ to $x = 2\pi$. This condition forces $f(x)$ to be bounded in the interval under consideration i.e., it is possible to find a positive quantity $N$ such that $|f(x)| \leq N$ when $0 \leq x \leq 2\pi$.

We have then the Fourier development:

$$ f(x) = b_0 + \sum_{k=1}^{\infty} b_k \cos kx + \sum_{k=1}^{\infty} a_k \sin kx $$

where

$$ b_k = \frac{1}{\pi} \int_{0}^{2\pi} f(x) \cos kx \, dx, \quad a_k = \frac{1}{\pi} \int_{0}^{2\pi} f(x) \sin kx \, dx. $$

Since the absolute value of the $\sin kx$ and $\cos kx$ can never exceed unity we can study the degree of convergence of this development by studying the degree of convergence of the coefficients, $b_k$ and $a_k$. These coefficients may indeed approach zero without the series converging but we shall study this convergence to zero of the coefficients before imposing further conditions.

We can write:

$$ b_k = \frac{1}{\pi} \int_{0}^{2\pi} f(x) \cos kx \, dx = \frac{1}{\pi} \left( \int_{0}^{\pi} f(x) \cos kx \, dx + \int_{\pi}^{2\pi} f(x) \cos kx \, dx \right) $$

(2)

$$ + \int_{\frac{\pi}{k}}^{\frac{2\pi}{k}} f(x) \cos kx \, dx + + \int_{\frac{(2k-1)\pi}{k}}^{\frac{2k\pi}{k}} f(x) \cos kx \, dx $$

And similarly:
\[ a_k = \frac{1}{\pi} \left\{ \int_{0}^{k} f(x) \sin \frac{kx}{\pi} \, dx + \int_{\frac{2\pi}{k}}^{\pi} f(x) \sin \frac{kx}{\pi} \, dx \right\} \]

(3)

\[ + \int_{\frac{2\pi}{k}}^{\frac{3\pi}{k}} f(x) \sin \frac{kx}{\pi} \, dx + \cdots + \int_{\frac{(2n-1)\pi}{k}}^{\frac{2n\pi}{k}} f(x) \sin \frac{kx}{\pi} \, dx \right\} \]

By the law of the mean we know that

\[ \int_{a}^{b} f(x) \varphi(x) \, dx = \mu \int_{a}^{b} \varphi(x) \, dx, \]

where \( m < \mu < M \), \( m \) and \( M \) being the minimum and maximum values assumed by the function, \( f(x) \), in the interval \( a \leq x \leq b \). There is no restriction on \( f(x) \) other than that it be bounded.

Also we have that

\[ \int_{\frac{(2k-n-1)\pi}{k}}^{\frac{(2k-n)\pi}{k}} \cos \frac{kx}{\pi} \, dx = \pm \frac{1}{k} \text{ or } -\frac{1}{k}; \text{ and } \int_{\frac{(2k-n)\pi}{k}}^{\frac{(2k-n-2)\pi}{k}} \sin \frac{kx}{\pi} \, dx = \pm \frac{1}{k} \text{ or } -\frac{1}{k}, \]

where \( n = 1, 2, 3, \ldots, 4k \).

We will then have, after substitution in (2),

\[ b_k = \frac{1}{\pi} \left\{ \frac{1}{k} \left( \mu_1 - \mu_2 - \mu_3 + \mu_4 + \cdots - \mu_{2n-2} - \mu_{2n-1} + \mu_{2n} \right) \right\}, \]

where \( \mu_1, \mu_2, \mu_3, \mu_4, \ldots, \text{ etc.}, \) lie between the minimum and maximum values assumed by \( f(x) \) in the intervals

\[ 0 \leq x \leq \frac{\pi}{2k}, \frac{\pi}{2k} \leq x \leq \frac{\pi}{k}, \frac{\pi}{k} \leq x \leq \frac{3\pi}{2k}, \ldots \text{ respectively.} \]

Then

\[ |b_k| \leq \left\{ \frac{1}{\pi} \left[ \frac{1}{k} \left( |\mu_1 - \mu_2| + |\mu_3 - \mu_4| + |\mu_5 - \mu_6| + \cdots + |\mu_{2n-1} - \mu_{2n}| \right) \right] \right\} \]

(5)

Similarly after substitution in (3) we have,
(6) \[ a_k = \frac{1}{\pi} \left\{ \frac{1}{k} \left( \mu_0 + \mu_1 - \mu_x - \mu_2 + \mu_3 - \cdots - \mu_{4k-2} + \mu_{4k-1} - \mu_{4k} \right) \right\} \]

where \( \mu_0, \mu_1, \mu_2, \mu_3, \ldots \) etc., lie between the minimum and maximum values assumed by \( f(x) \) in the intervals

\[ 0 \leq x \leq \frac{\pi}{2k}, \quad \frac{\pi}{2k} \leq x \leq \frac{\pi}{k}, \quad \frac{\pi}{k} \leq x \leq \frac{3\pi}{2k}, \ldots \] respectively.

Then \[ |a_k| \leq \frac{1}{\pi} \left\{ \frac{1}{k} \left( |\mu_0 - \mu_{4k-1}| + |\mu_1 - \mu_x| + |\mu_2 - \mu_2| + \cdots + |\mu_{4k-2} - \mu_{4k-1}| \right) \right\} \]

But in equation (5) each of the quantities \( |\mu_1 - \mu_x|, \quad |\mu_2 - \mu_x|, \ldots \) is evidently less than the total variation of \( f(x) \) in the intervals

\[ 0 \leq x \leq \frac{\pi}{k}, \quad \frac{\pi}{k} \leq x \leq \frac{2\pi}{k}, \quad \frac{2\pi}{k} \leq x \leq \frac{3\pi}{k}, \ldots \] respectively.

For denoting the total variation by \( \omega \), we have \( \omega \geq M_1 - m_1 \), where \( M_1 \) and \( m_1 \) are the maximum and minimum values of \( f(x) \) in the interval \( 0 \leq x \leq \pi/k \). But \( \mu_1 > m_1 \) and \( \mu_x < M_1 \), since the minimum of a function in an interval is always less than or at the most equal to the minimum of the function in a subinterval, and the maximum of a function in an interval is always greater than or at the most equal to the maximum of the function in a subinterval. Then after substitution we will have \( \omega \geq \mu_x - \mu_1 \), or we can write \( \omega \geq |\mu_x - \mu_1| \). Similarly \( \omega \geq |\mu_2 - \mu_2|, \quad \omega \geq |\mu_{4k-2} - \mu_{4k-1}|, \ldots \) where \( \omega_1, \omega_x, \omega_2, \ldots \) represent the total variation of \( f(x) \) in the intervals

\[ 0 \leq x \leq \frac{\pi}{k}, \quad \frac{\pi}{k} \leq x \leq \frac{2\pi}{k}, \quad \frac{2\pi}{k} \leq x \leq \frac{3\pi}{k}, \ldots \] respectively.

But the total variation, \( \Theta \), of the function \( f(x) \) in
the interval \( 0 \leq x \leq 2\pi \) is

\[ \Omega = \omega_1 + \omega_2 + \omega_3 + \cdots + \omega_k. \]

Thus we have

\[
|b_k| \leq \frac{1}{\pi} \frac{1}{k} \Omega.
\]

Following the same method for equation (7) we get

\[
\omega_0 \geq |\mu_0 - \mu_{4k+1}|, \quad \omega_2 \geq |\mu_2 - \mu_2|, \quad \omega_3 \geq |\mu_3 - \mu_3|, \quad \cdots
\]

where \( \omega_0, \omega_2, \omega_3, \cdots \) represent the total variation of \( f(x) \) in the intervals,

\[
\frac{\pi}{2k} \leq x \leq \frac{3\pi}{2k}, \quad \frac{3\pi}{2k} \leq x \leq \frac{5\pi}{2k}, \quad \frac{5\pi}{2k} \leq x \leq \frac{7\pi}{2k}, \quad \cdots
\]

and \( \omega_1 \) represents the total variation in the two intervals

\[
0 \leq x \leq \frac{\pi}{2k} \quad \text{and} \quad \frac{(4k - 1)\pi}{2k} \leq x \leq \frac{2\pi}{2k}.
\]

Then we have as before:

\[
|a_k| \leq \frac{1}{\pi} \frac{1}{k} \Omega.
\]

Thus if \( \Omega \) remains finite in the interval \( 0 \leq x \leq 2\pi \) the quantities \( |b_k| \) and \( |a_k| \) will be defined as well as finite and the

\[
\lim_{k \to \infty} |b_k| = 0 \quad \text{and} \quad \lim_{k \to \infty} |a_k| = 0.
\]

Hence we can conclude:

**If the total variation of a function remains finite the coefficients of its Fourier development converge to zero.**

2. **Integrable in the Riemann sense.** Now let us assume that the function, \( f(x) \), is integrable in the Riemann sense.
Then \[ \int_a^b f(x) \, dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(\xi_i) \Delta x, \quad \text{where} \quad m_i \leq f(\xi_i) \leq M_i. \]

We could have written equation (4) in the following form, considering \( 1/k = \Delta x \):

\[ b_k = \frac{1}{\pi} \left\{ \left( \mu_1 \Delta x + \mu_2 \Delta x + \mu_3 \Delta x + \ldots + \mu_{n-1} \Delta x \right) \right. \]

\[ - \left( \mu_{n-1} \Delta x + \mu_n \Delta x + \mu_1 \Delta x + \ldots + \mu_{n-2} \Delta x \right) \}\]  

And we have \( m_i \leq \mu_i \leq M_i \).

Similarly, equation (6) can be written in the form:

\[ a_k = \frac{1}{\pi} \left\{ \left( \mu_0 \Delta x + \mu_1 \Delta x + \mu_2 \Delta x + \mu_3 \Delta x + \ldots + \mu_{n-2} \Delta x \right) \right. \]

\[ - \left( \mu_{n-2} \Delta x + \mu_{n-1} \Delta x + \mu_0 \Delta x + \ldots + \mu_{n-3} \Delta x \right) \}\]  

(10) \[ \lim_{k \to \infty} b_k = \frac{1}{\pi} \left\{ \frac{1}{2} \int_a^b f(x) \, dx - \frac{1}{2} \int_a^b f(x) \, dx \right\} = 0. \]

Thus we can conclude:

**If a function is integrable in the Riemann sense the coefficients of its Fourier development converge to zero.**

3. **Generalized Lipschitz condition.** Now let us assume that the function, \( f(x) \), satisfies the following condition:

\[ |\Delta f| < A |\Delta x|^a \]

where \( A \) is any constant and \( a \) satisfies the condition \( 0 < a < 1 \).
This restriction on \( f(x) \) is not nearly as great as that imposed by the so-called Lipschitz condition which is that

\[
|\Delta f| \leq A |\Delta x|.
\]

Writing \( \Delta x \) as \( \frac{1}{k} \), we will then have

\[
|\Delta f| < A \frac{1}{k^a}.
\]

From (5) we have, setting \( |\mu_k - \mu_x| \leq |\Delta f_1|, \ |\mu_k - \mu_3| \leq |\Delta f_2| \)

\[
|k b_k| \leq \frac{1}{\pi} \left\{ \sum_{i=1}^{k} |\Delta f_i| \right\}.
\]

Then \( |k \cdot b_k| \leq \frac{kA}{\pi} \frac{1}{k^a} = \frac{C}{k^a} \), where \( C \) is a constant.

Then \( |b_k| = \frac{c}{k^a} \) and \( \lim_{k \to \infty} |b_k| = 0 \). \hspace{1cm} (12)

Similarly: \( |a_k| = \frac{c}{k^a} \) and \( \lim_{k \to \infty} |a_k| = 0 \). \hspace{1cm} (13)

Hence:

If a function satisfies a generalized Lipschitz condition the coefficients of its Fourier development converge to zero.
Chapter II.

1. **Remainder in a Fourier's series.** We will now proceed to the consideration of the accuracy of a Fourier development taken for a finite number of terms and will develop an expression for the remainder of a Fourier's series after \( n \) terms.

Let us consider the case where the function of \( x \), \( f(x) \), is a periodic function of period \( 2\pi \), having an infinite number of derivatives of the same period.

The Fourier development of \( f(x) \) will then be of the form:

\[
    f(x) = b_0 + b_1 \cos x + b_2 \cos 2x + b_3 \cos 3x + \ldots
\]

\[
    + a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \ldots
\]

We will investigate the coefficients:

\[
    a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx \, dx \quad \text{and} \quad b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx \, dx.
\]

Integrating the first by parts we have:

\[
    a_k = \frac{1}{k\pi} \left[ f(x) \cos kx \right]_0^{2\pi} + \frac{1}{k\pi} \int_0^{2\pi} f'(x) \cos kx \, dx.
\]

\( k \) being an integer, we have \( \left[ f(x) \cos kx \right]_0^{2\pi} = 0 \).

Repeating the integration by parts we get:

\[
    a_k = \frac{1}{\pi k^2} \left[ f'(x) \sin kx \right]_0^{2\pi} - \frac{1}{\pi k^2} \int_0^{2\pi} f''(x) \sin kx \, dx.
\]

Again:

\[
    \left[ f'(x) \sin kx \right]_0^{2\pi} = 0.
\]
Repeating the integration again:

\[ a_k = \frac{1}{\pi k^2} \left[ f''(x) \cos kx \int_0^{2\pi} \frac{1}{\pi k^2} f'''(x) \cos kx \, dx \right] \]

And again:

\[ a_k = -\frac{1}{\pi k^4} \left[ f'''(x) \cos kx \int_0^{2\pi} \frac{1}{\pi k^4} f''''(x) \sin kx \, dx \right] \]

This process could be kept up indefinitely since \[ f''(x) \cos kx \int_0^{2\pi} f'''(x) \cos kx \, dx \] and similar factors all drop out, being equal to zero, and since \( f(x) \) has an infinite number of derivatives by hypothesis.

Carrying through the same process for \( b_k \) we get:

\[ b_k = \frac{1}{\pi k^2} \left[ f(x) \sin kx \int_0^{2\pi} f'(x) \sin kx \, dx \right] \]

But \[ f(x) \sin kx \int_0^{2\pi} = 0 \]

Integrating again:

\[ b_k = \frac{1}{\pi k^2} \left[ f'(x) \cos kx \int_0^{2\pi} f''(x) \cos kx \, dx \right] \]

Repeating again we have since \[ f'(x) \cos kx \int_0^{2\pi} = 0 \]

\[ b_k = -\frac{1}{\pi k^3} \left[ f''(x) \sin kx \int_0^{2\pi} f'''(x) \sin kx \, dx \right] \]

And again:

\[ b_k = -\frac{1}{\pi k^4} \left[ f'''(x) \cos kx \int_0^{2\pi} f''''(x) \cos kx \, dx \right] \]

since \[ f''(x) \sin kx \int_0^{2\pi} \] and \[ f'''(x) \cos kx \int_0^{2\pi} = 0 \]
Then the absolute value of the $n^{th}$ Fourier coefficient will be:

$$|a_n| \leq \frac{1}{\pi n^k} \int_{\pi}^{2\pi} |f^k(x)||\cos nx| \, dx,$$

when $k$ is odd or

(2)

$$|a_n| \leq \frac{1}{\pi n^k} \int_{\pi}^{2\pi} |f^k(x)||\sin nx| \, dx,$$

when $k$ is even.

Similarly:

$$|b_n| \leq \frac{1}{\pi n^k} \int_{\pi}^{2\pi} |f^k(x)||\sin nx| \, dx,$$

when $k$ is odd or

(3)

$$|b_n| \leq \frac{1}{\pi n^k} \int_{\pi}^{2\pi} |f^k(x)||\cos nx| \, dx,$$

when $k$ is even.

By the law of the mean we have:

$$|a_n| \leq \frac{1}{\pi n^k} |f^k(\bar{x})| \int_{0}^{\pi} |\cos nx| \, dx,$$

where $0 < \bar{x} < 2\pi$.

But

$$\int_{\pi}^{2\pi} |\cos nx| \, dx = \frac{1}{n} \cdot 4n = 4.$$

(4) \therefore |a_n| = \frac{4}{\pi n^k} |f^k(\bar{x})|.$$

And similarly $|b_n| = \frac{4}{\pi n^k} |f^k(\bar{x})|.$

Then in order to have a good representation of $f(x)$ by its Fourier development the remainder after $n$ terms of the series must approach zero. From (1), since $|\sin nx| \leq 1$ and $|\cos nx| \leq 1$, we can write the remainder:

$$R_n \leq |a_n| + |b_n| + |a_{n-1}| + |b_{n-1}| + \cdots$$

Substituting the values determined for $|a_n|$, $|b_n|$, etc.,

(5)

$$R_n \leq |f^k(\bar{x})| \frac{4}{\pi} \left\{ \frac{1}{n^k} + \frac{1}{n^k} + \frac{1}{(n+1)^k} + \frac{1}{(n+1)^k} + \cdots \right\},$$

$$R_n \leq |f^k(\bar{x})| \frac{6}{\pi} \left\{ \frac{1}{n^k} + \frac{1}{(n+1)^k} + \frac{1}{(n+2)^k} + \cdots \right\}.$$
Let
\[ F(n,k) = \frac{1}{n^k} + \frac{1}{(n+1)^k} + \frac{1}{(n+2)^k} + \cdots \] 
(6)

Then
\[ R_n(x) = \theta \left[ \frac{8}{\pi} \left| f^k(x) \right| F(n,k) \right], \text{ where } 0 < \theta < 1, \text{ or} \]
(7)
\[ R_n(x) \leq \left[ \frac{8}{\pi} M_k F(n,k) \right], \]
where \( M_k \) is the maximum value of \( |f^k(x)| \).

We can then judge the character of our remainder by computing \( |f^k(x)| \) for a number of values of \( k \) and noting what choice of \( k \) will make \( R_n(x) \) the smallest. For this purpose we will tabulate \( F(n,k) \) for a number of values of \( n \) and \( k \).

2. Range of \( k^{th} \) derivative of \( f(x) \). We shall at this point prove a rather striking theorem:

If a function of \( x \), \( f(x) \), is a periodic function possessing an infinite number of derivatives having the same period as \( f(x) \), and if \( f(x) \) cannot be developed in a finite Fourier's series, then after a certain point the maximum value of the \( k^{th} \) derivative of \( f(x) \) will increase faster than an exponential function of the form \( n^k \).

Let \( M_k = \text{maximum value of } |f^k(x)| \).

Then from equations (4):
\[ |a_n| \leq \frac{4}{\pi n^k} M_k \quad \text{and} \quad |b_n| \leq \frac{4}{\pi n^k} M_k. \]

Rewriting:
\[ |a_n| \leq \frac{4 M_k}{\pi n^k} \quad \text{and} \quad |b_n| \leq \frac{4 M_k}{\pi n^k}. \]

We see at once that if \( |a_n| = 0 \) or \( |b_n| = 0 \) we would know nothing definite about the conduct of \( M_k \). It might increase as fast as \( n^k \) or it might not, just as the case happened
But our hypothesis was that \(|a_n| > 0\) or \(|b_n| > 0\), or both, for an infinite number of values of \(n\). This will then force \(M_k\) to increase faster than \(n^k\) increases for any value of \(n\) or otherwise the fraction \(M_k/n^k\) will approach zero and hence violate the hypothesis of our theorem, since if \(M_k/n^k\) approaches zero \(|a_n|\) and \(|b_n|\) must also approach zero, each being less than \(M_k/n^k\).

Thus if we have a function that satisfies the conditions of our theorem and we assign integral values to \(k, 1, 2, 3, \ldots\), then after a certain value of \(k\) is reached, \(M_k\) will increase faster than an exponential function of the type \(n^k\).

As a corollary to the above theorem we can state the following:

If \(M_k\) is bounded by a function of the type \(m^k\), \(f(x)\) is a trigonometric polynomial of order at most equal to \(m\).

For the reasoning shows that then \(a_n = 0\) and \(b_n = 0\) for \(n > m\) and the function would thus be a trigonometric polynomial of order at most equal \(m\).

A simple example of such a function is the function

\[ f(x) = \sin 2x. \]

Then \(f'(x) = 2 \cos 2x\), \(f''(x) = -4 \sin 2x\), \(f'''(x) = -8 \cos 2x\), etc.,

and thus we have \(|f^k(x)| \leq 2^k\). Therefore \(M_k = 2^k\), which verifies our corollary.

3. **Table of values of \(F(n,k)\).** We will now tabulate the function \(F(n,k)\) for values of \(n\) ranging from 1 to 20 and for \(k\) ranging from 1 to 20. In computing this table I have made use of some results given by Stieltjes\(^\text{12}\).
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4. **Comparison of results.** As has already been mentioned in the introduction, Jackson has found that in the case of a periodic function of period $2\pi$ whose $(k - 1)$th derivative satisfies a Lipschitz condition the remainder after $n$ terms of its Fourier's series is:

$$R_n(x) \leq 36 \lambda \frac{\log n}{n^k}.$$ 

If $k$ is odd the factor 36 can be replaced by 12.

We will compare this with the result which we have just derived, namely,

$$R_n(x) \leq \frac{\pi}{n} M_k F(n,k),$$

$M_k$ being the maximum value assumed by the $k$th derivative of $f(x)$ in the interval $0$ to $2\pi$.

We see at once that $\pi/n < 12$ and $M_k \leq \lambda$, $\lambda$ being the constant of the Lipschitz condition. If we let

$$\frac{\log n}{n^k} = J(n,k),$$

we have left to compare only $J(n,k)$ and $F(n,k)$. When $F(n,k) < J(n,k)$ our result gives a more accurate expression for $R_n$ and when $F(n,k) > J(n,k)$ Jackson's result is the better.

Since we have already constructed a table of values for $F(n,k)$ we have but to make a corresponding table for $J(n,k)$ and compare the two. In order to compare the functions accurately for large values of $n$ and $k$ it would be necessary to compute $F(n,k)$ to a much greater number of decimal places than has been done in our table. However, in any particular problem in which the development of a function arises, the development, to be of any value, must be such that it will be necessary to use only a few terms of it. Thus it will not be necessary to consider large values of $n$. However it will be advantageous to
let \( k \) be as large as possible. In making the following comparison
I have considered values of \( n \) from 3 to 8 and of \( k \) from 2 to 20
inclusive, this being the extent of \( k \) in our table of \( F(n,k) \).

\[
\begin{align*}
n & \quad F(n,k) > J(n,k) \text{ for } k = 2 \text{ to } k = 8, \quad F < J \text{ for } k = 9 \text{ to } k = 20 \\
3 & \quad F(n,k) > J(n,k) \text{ for } k = 2 \text{ to } k = 6, \quad F < J \text{ for } k = 7 \text{ to } k = 20 \\
4 & \quad F(n,k) > J(n,k) \text{ for } k = 2 \text{ to } k = 5, \quad F < J \text{ for } k = 6 \text{ to } k = 20 \\
5 & \quad F(n,k) > J(n,k) \text{ for } k = 2 \text{ to } k = 5, \quad F < J \text{ for } k = 6 \text{ to } k = 20 \\
6 & \quad F(n,k) > J(n,k) \text{ for } k = 2 \text{ to } k = 5, \quad F < J \text{ for } k = 6 \text{ to } k = 20 \\
7 & \quad F(n,k) > J(n,k) \text{ for } k = 2 \text{ to } k = 6, \quad F < J \text{ for } k = 6 \text{ to } k = 20 \\
8 & \quad F(n,k) > J(n,k) \text{ for } k = 2 \text{ to } k = 6, \quad F < J \text{ for } k = 6 \text{ to } k = 20 \\
\end{align*}
\]

As far as I can judge, as \( k \) is increased further
\( F(n,k) \) still remains less than \( J(n,k) \). Since in general \( k \) will
be chosen quite large we see by the above table that our result
has some advantage over that of Jackson's for this range of \( n \).
Chapter III.

1. **Remainder in a Legendre's series.** We will now consider the development of a function of \( x, f(x) \), in a series of Zonal Harmonics or what are usually called Legendre's polynomials, in the interval from \( x = -1 \) to \( x = +1 \).

We have the development:

\[
(1) \quad f(x) = A_0 P(x) + A_1 P_1(x) + A_2 P_2(x) + \cdots + A_m P_m(x) + \cdots
\]

where

\[
(2) \quad A_m = \frac{2m+1}{2} \int_{-1}^{1} f(x) P_m(x) \, dx, \quad m = 1, 2, 3, \ldots
\]

If the series (1) converges it will be sufficient to study the convergence of the series of coefficients in investigating this development since we know that \( |P_m(x)| \leq 1 \).

In doing this we will use a method somewhat similar to the one used in the case of a Fourier development.

Integrating (2) by parts we will have:

\[
(3) \quad A_m = \frac{2m+1}{2} \left\{ \int_{-1}^{x} P_m(x) \, dx \cdot f(x) \right\}^{x}_{-1} - \int_{-1}^{x} \left( \int_{-1}^{x} P_m(x) \, dx \right) f'(x) \, dx
\]

From Legendre's equation,

\[
\frac{d}{dx} (1 - x^2) \frac{dP_m(x)}{dx} + m(m+1) P_m(x) = 0,
\]

we have

\[
\int_{-1}^{1} P_m(x) \, dx = \frac{1}{m(m+1)} (1 - x^2) \frac{dP_m(x)}{dx}, \quad m \neq 0.
\]

Hence (3) becomes, the first term being equal to zero,
(4) \[ A_m = \frac{2m + 1}{2m(m+1)} \int_{-1}^{1} \frac{d}{dx} P_m(x) \left[ (1 - x^2) f'(x) \right] dx. \]

The work can be very much simplified by making the following substitutions: Let

\[ F_0(x) = f(x), \quad F_1(x) = (1 - x^2) \frac{d}{dx} f(x), \]
\[ F_2(x) = \frac{d}{dx} F_1(x), \quad F_2(x) = (1 - x^2) \frac{d}{dx} F_2(x), \]
\[ F_4(x) = \frac{d}{dx} F_3(x), \quad F_5(x) = (1 - x^2) \frac{d}{dx} F_4(x), \]

etc.

Equation (4) then becomes:

\[ A_m = \frac{2m + 1}{2m(m+1)} \int_{-1}^{1} \frac{d}{dx} P_m(x) F_1(x) dx. \]

Integrating by parts again:

\[ A_m = \frac{2m + 1}{2m(m+1)} \left\{ \left[ P_m(x) F_1(x) \right]_{-1}^{1} - \int_{-1}^{1} P_m(x) \frac{d}{dx} F_1(x) dx \right\}. \]

Again the first term drops out and we have:

\[ A_m = - \frac{2m + 1}{2m(m+1)} \int_{-1}^{1} P_m(x) F_2(x) dx. \]

Proceeding as before we get:

\[ A_m = - \frac{2m + 1}{2m(m+1)} \left\{ \left[ \int_{-1}^{1} P_m(x) dx \right] F_2(x) \right\}_{-1}^{1} - \int_{-1}^{1} \left( \int_{-1}^{1} P_m(x) dx \right) \frac{d}{dx} F_2(x) dx \]

\[ = \frac{2m + 1}{2m(m+1)^2} \int_{-1}^{1} \frac{d}{dx} P_m(x) \left[ (1 - x^2) \frac{d}{dx} F_2(x) \right] dx. \]
In the same way we get:

\[ A_m = \frac{2m + 1}{2m^2 (m+1)^2} \int_{-1}^{+1} \frac{d}{dx} P_m(x) F_3(x) \, dx \]

Hence after integration by parts \(2k-1\) times we will have:

\[ A_m = \frac{2m + 1}{2m^2 (m+1)^2k} \int_{-1}^{+1} \frac{d}{dx} P_m(x) F_{2k-1}(x) \, dx, \quad k = 1, 2, 3, \ldots \]

and after integration by parts \(2k\) times,

\[ A_m = \frac{2m + 1}{2m^2 (m+1)^2k} \int_{-1}^{+1} P_m(x) F_{2k}(x) \, dx, \quad k = 0, 1, 2, \ldots \]

We have however that

\[ F_1(x) = (1 - x^2) f'(x), \quad F_2(x) = (1 - x^2) f''(x) - 2x f'(x), \]
\[ F_3(x) = (1 - x^2)^2 f'''(x) - 4x(1 - x^2) f''(x) - 2(1 - x^2) f'(x). \]

And similarly:

\[ F_k(x) = c_1(x) f^k(x) + c_2(x) f^{k-1}(x) + c_3(x) f^{k-2}(x) + \ldots + c_k f'(x), \]

where \(c_i(x)\) are integral rational functions of \(x\).

Now suppose that the \(k\)th derivative of \(f(x)\) is bounded, i.e.,

\[ |f^k(x)| < N \]

where \(N\) is a positive quantity. Then \(F_k(x)\) will also be
bounded. Throughout the discussion it must be remembered that we are considering only the values of \( x \) in the interval \(-1\) to \(+1\).

Call the maximum value of \(|P_k(x)|\) in the interval, \( M_k \). Then if \( k \) is an even number equal, say, to \( 2n \) we will have:

\[
|A_m| \leq \frac{2m + 1}{2m^{n+1}} M_k \int_{-1}^{+1} |P_m(x)| \, dx
\]

or if \( k \) is an odd number, \( k = 2n - 1 \),

\[
|A_m| \geq \frac{1}{2m^n(m+1)} M_k \int_{-1}^{+1} \left| \frac{d}{dx} P_m(x) \right| \, dx.
\]

Thus we have an upper bound for \(|A_m|\). We need only consider one of these namely the first. To determine the value of the integral \( \int_{-1}^{+1} |P_m(x)| \, dx \) we will use Schwarz's inequality which states that

\[
\left( \int_{a}^{b} f(x) \cdot \varphi(x) \, dx \right)^2 \leq \int_{a}^{b} (f(x))^2 \, dx \cdot \int_{a}^{b} (\varphi(x))^2 \, dx,
\]

where \( f(x) \) and \( \varphi(x) \) are two real and continuous functions.

Now if we set \( f(x) = 1 \) and \( \varphi(x) = |P_m(x)| \) we have:

\[
\left( \int_{-1}^{+1} |P_m(x)| \, dx \right)^2 \leq 2 \int_{-1}^{+1} (P_m(x))^2 \, dx \quad \text{since} \quad \left( |P_m(x)|^2 \right)^2 = (P_m(x))^2.
\]

But \( \int_{-1}^{+1} (P_m(x))^2 \, dx = \frac{2}{2m+1} \). Hence \( \int_{-1}^{+1} |P_m(x)| \, dx < \frac{2}{\sqrt{2m+1}} \).
Since \( |P_m(x)| > (P_m(x))^2 \) throughout the interval -1 to +1, we can say that
\[
\frac{2}{2m+1} \int_{-1}^{+1} |P_m(x)| \, dx \leq \frac{2}{\sqrt{2m+1}}.
\]
We have that the remainder after \( m \) terms of our series is
\[
R_m \leq |A_m| + |A_{m+1}| + |A_{m+2}| + \ldots + |A_{m+p}| + \ldots
\]
This will then become:
\[
R_m \leq M \left\{ \frac{(2m + 1)^{\frac{1}{2}}}{m^n(m+1)^n} + \frac{(2m + 3)^{\frac{1}{2}}}{(m+1)^n(m+3)^n} + \frac{(2m + 5)^{\frac{1}{2}}}{(m+2)^n(m+3)^n} + \ldots \right\}
\]
where \( 2n \) is again equal to \( k \).

The series
\[
\frac{(2m + 1)^{\frac{1}{2}}}{m^n(m+1)^n} + \frac{(2m + 3)^{\frac{1}{2}}}{(m+1)^n(m+2)^n} + \frac{(2m + 5)^{\frac{1}{2}}}{(m+2)^n(m+3)^n} + \ldots
\]
is a convergent series for values of \( n \geq 1 \) and \( m \geq 1 \). It converges very rapidly as \( n \) is taken larger and larger. Since it will be in general possible to assign a moderately large value to \( n \) it will only be necessary to compute a few terms of the series to obtain a result accurate enough for all purposes. \( M_k \) then being known or computed we will have an upper bound for the remainder after \( n \) terms of a Legendre's series.
Chapter IV.

1. Remainder in a series of Bessel's functions. Let us now consider the development of a function in a series of Bessel's functions of the zeroth order, between the limits 0 and 1.

We have:

\[ f(x) = A_1 J_0(\mu_1 x) + A_2 J_0(\mu_2 x) + A_3 J_0(\mu_3 x) + \ldots \]

where \( \mu_1, \mu_2, \mu_3, \ldots \) are the successive roots of the transcendental equation \( J_0(\mu x) = 0 \) and

\[ A_k = \frac{2}{J_1^2(\mu_k)} \int_0^1 x f(x) J_0(\mu_k x) \, dx . \]

We will use the same method that we have used in discussing the other series.

Integrating \( A_k \) by parts we have:

\[ A_k = \frac{2}{J_1^2(\mu_k)} \left\{ \left[ \frac{x f(x) J_1(\mu_k x)}{\mu_k} \right]_0^1 - \frac{1}{\mu_k} \int_0^1 x f'(x) J_1(\mu_k x) \, dx \right\} . \]

We will at this point simplify our work a great deal by setting up the following auxiliary functions:

\[ F_0(x) = f(x) , \quad F_1(x) = \frac{f'(x)}{x} , \quad F_2(x) = \frac{F'_1(x)}{x} , \]

\[ \ldots \quad F_{n-1}(x) = \frac{F'_{n-2}(x)}{x} , \quad F_n(x) = \frac{F'_{n-1}(x)}{x} . \]

It will be necessary to impose the following conditions upon these functions:

\[ F_0(1) = F_1(1) = F_2(1) = \ldots = F_{n-1}(1) = 0 . \]

With these conditions we will then have:
\[ \left[ \frac{x f(x) J_1(\mu_k x)}{\mu_k} \right]_0^1 = 0 \quad \text{and consequently} \]

\[ A_k = \frac{e^{-2}}{J_1(\mu_k)} \frac{1}{\mu_k} \int_0^1 x^2 F_1(x) J_1(\mu_k x) \, dx . \]

Integrating by parts again:

\[ A_k = \frac{-2}{J_1(\mu_k) \mu_k} \frac{1}{\mu_k} \left\{ \left[ \int_0^1 x^2 J_1(\mu_k x) \, dx \right] F_1(x) \bigg|_0^1 - \int_0^1 \left[ \int_0^1 x^2 J_1(\mu_k x) \, dx \right] F_1'(x) \, dx \right\} \]

\[ = \frac{-2}{J_1(\mu_k) \mu_k} \frac{1}{\mu_k} \int_0^1 x^2 J_2(\mu_k x) F_1(x) \, dx \]

\[ = \frac{2}{J_1(\mu_k) \mu_k} \int_0^1 F_2(x) x^3 J_2(\mu_k x) \, dx . \]

Repeating the integration by parts:

\[ A_k = \frac{2}{J_1(\mu_k) \mu_k} \frac{1}{\mu_k} \left\{ \left[ F_2(x) \int_0^1 x^3 J_2(\mu_k x) \, dx \right] \bigg|_0^1 - \int_0^1 \left( \int_0^1 x^3 J_2(\mu_k x) \, dx \right) F_2'(x) \, dx \right\} \]

\[ = \frac{2}{J_1(\mu_k) \mu_k} \frac{1}{\mu_k} \left\{ \left[ F_2(x) x^3 J_2(\mu_k x) \right] \bigg|_0^1 - \frac{1}{\mu_k} \int_0^1 F_2'(x) x^3 J_2(\mu_k x) \, dx \right\} \]

\[ = \frac{-2}{J_1(\mu_k) \mu_k} \frac{1}{\mu_k} \int_0^1 F_2(x) x^4 J_2(\mu_k x) \, dx . \]
Repeating this process m times we will finally get:

\[ A_k = \frac{(-1)^m}{\mu_k} \int_{0}^{1} F_m(x) x^{m+1} J_m(\mu_k x) \, dx \]

By the law of the mean we can say:

\[ A_k = \frac{(-1)^m}{\mu_k} \int_{0}^{1} F_m(x) J_m(\mu_k x) x^{m+1} \, dx \]

where \( 0 \leq x \leq 1 \).

Then

\[ A_k = \frac{2}{J_1(\mu_k)} \frac{1}{\mu_k} \frac{M F_m(x) \cdot M J_m(x)}{m + 2} \]

where by \( MF_m(x) \) and \( MJ_m(x) \) we mean the maximum values assumed by \( F_m(x) \) and \( J_m(x) \) respectively in the interval \( x = 0 \) to \( x = 1 \).

We have however that \( |J_m(x)| \leq 1 \) for all values of \( x \).

For Whittaker's Modern Analysis, article 153, gives us an expression for the Bessel functions in the form of an integral. The following formula is developed:

\[ J_m(x) = \frac{1}{\pi} \int_{0}^{\pi} \cos(m \theta - x \sin \theta) \, d\theta \]

But since the value of the integrand can never exceed unity we can conclude that \( |J_m(x)| \leq 1 \) for all values of \( x \). It can also be shown that the first maximum of \( J_m(x) \) is the largest one.

Then in considering the convergence of the series

\[ f(x) = A_1 J_0(\mu_k x) + A_2 J_0(\mu_k x) + A_3 J_0(\mu_k x) + \ldots \]

we can consider instead the series

\[ f(x) \leq |A_1| + |A_2| + |A_3| + \ldots + |A_n| + \ldots \]
Hence we have:

\[ R_n(x) \leq |A_n| + |A_{n-1}| + |A_{n-2}| + \ldots \]

\[ R_n(x) \leq \frac{2}{m+2} \sum_{m} \left\{ \frac{M_1^m(\mu_n x)}{\mu_n J_1^2(\mu_n)} + \frac{M_1^m(\mu_n x)}{\mu_n J_1^2(\mu_n)} + \frac{M_1^m(\mu_n x)}{\mu_n J_1^2(\mu_n)} \right\} \]

Or we could write:

\[ R_n(x) \leq \frac{2}{m+2} \sum_{m} \left\{ \frac{1}{\mu_n J_1^2(\mu_n)} + \frac{1}{\mu_n J_1^2(\mu_n)} + \frac{1}{\mu_n J_1^2(\mu_n)} \right\} \]

Since \( \mu_n, \mu_{n+1}, \mu_{n+2}, \ldots \) are successive roots of the equation \( J_0(x) = 0 \), \( J_1^2(\mu_n) \), \( J_1^2(\mu_{n+1}) \), \( J_1^2(\mu_{n+2}) \), \ldots are all greater than zero.

If then the series

\[ \frac{1}{\mu_n J_1^2(\mu_n)} + \frac{1}{\mu_n J_1^2(\mu_n)} + \frac{1}{\mu_n J_1^2(\mu_n)} + \ldots \]

is a convergent series it will be possible to compute an upper bound for the values of \( R_n(x) \).

We will now proceed to show that the above series is convergent for values of \( m > 2 \), and to obtain an upper limit for its value.

We have the following asymptotic formula\(^{15}\):

(1) \[ J_0(x) = \frac{2 \cos(\pi/4 - x)}{\sqrt{2\pi x}} + \theta 2 \sqrt{\frac{2}{\pi}} \exp(-\frac{x^2}{4}) \]

where \(-1 < \theta < 1\) and \( g < 3.5 \).
Then

\[ x J_0(x) = \frac{2^\frac{3}{2}}{\sqrt{2\pi}} \cos(\frac{\pi}{4} - x) + \theta 2 \sqrt{\frac{\pi}{2}} g^2 e^{-1} x^{-\frac{3}{2}} \]

and

\[ x J_0(x) = \frac{d}{dx}(x J_1(x)) \]

by a well known formula.

Integrating between two indefinite limits, \(a\) and \(\lambda\) we will have:

\[
\int_a^\lambda x J_0(x) dx = \lambda J_1(\lambda) - a J_1(a).
\]

And then

\[
(2) \quad \lambda J_1(\lambda) - a J_1(a) = \frac{\phi}{2\sqrt{2\pi}} \int_a^\lambda x^{\frac{1}{2}} \cos(\frac{\pi}{4} - x) dx - \left[ \theta 4 \sqrt{\frac{\pi}{2}} g^2 e^{-1} x^{-\frac{3}{2}} \right]_a^\lambda.
\]

Let us call the second member of the right hand side of this equation \(R\). \(R\) then tends toward zero as \(\lambda\) and \(a\) increase.

If now we could choose \(a\) as a root of \(J_1(x) = 0\) we would have an expression for \(\lambda J_1(\lambda)\). However if we choose \(a\) not equal to such a root it amounts only to adding a constant to \(\lambda J_1(\lambda)\).

Integrating the first member of the right hand side of equation (2) by parts we get, leaving off for the moment the constant term:

\[
\int x^\frac{1}{2} \cos(\frac{\pi}{4} - x) dx = - \sin(\frac{\pi}{4} - x) x^\frac{1}{2} + \frac{1}{2} \int x^{-\frac{3}{2}} \sin(\frac{\pi}{4} - x) dx
\]

\[
= - \sin(\frac{\pi}{4} - x) x^\frac{1}{2} + \frac{1}{2} \left\{ \cos(\frac{\pi}{4} - x) x^{-\frac{3}{2}} + \frac{1}{2} \int x^{-\frac{3}{2}} \cos(\frac{\pi}{4} - x) dx \right\}
\]

\[
(2) \quad = - \sin(\frac{\pi}{4} - x) x^\frac{1}{2} + \frac{1}{2} \cos(\frac{\pi}{4} - x) x^{-\frac{3}{2}} + \frac{1}{4} \int x^{-\frac{3}{2}} \cos(\frac{\pi}{4} - x) dx
\]

But in the last term of the above equation we have,
by the law of the mean,

\[ \int_{\alpha}^{\lambda} x^{-\frac{3}{2}} \cos\left( \frac{\pi}{4} - x \right) \, dx = \cos\left( \frac{\pi}{4} - \bar{x} \right) \int x^{-\frac{3}{2}} \, dx \]

\[ = -2 \cos\left( \frac{\pi}{4} - \bar{x} \right) x^{-\frac{3}{2}}, \]

where \( \bar{x} \) lies between \( \alpha \) and \( \lambda \). This term is of the order \( x^{-\frac{3}{2}} \) and can be included in the \( R \) of equation (2). The same is true of the second term of equation (3).

Hence, we will now have:

\[ \lambda J_1(\lambda) - a J_1(a) = \frac{2}{\sqrt{2\pi}} \left\{ -\sin\left( \frac{\pi}{4} - x \right) x^{-\frac{1}{2}} \right\}_{\alpha}^{\lambda} + R_1 \]

where

\[ R_1 = \left[ -\theta \sqrt{\frac{2}{\pi}} e^{-\frac{1}{2}x^{-\frac{1}{2}}} + \frac{1}{2} \cos\left( \frac{\pi}{4} - x \right) x^{-\frac{3}{2}} - \frac{1}{2} \cos\left( \frac{\pi}{4} - \bar{x} \right) x^{-\frac{3}{2}} \right]_{\alpha}^{\lambda} \]

Then

\[ \lambda J_1(\lambda) - a J_1(a) = \frac{2}{\sqrt{2\pi}} \left\{ \sin(\lambda - \frac{\pi}{4}) \lambda^{-\frac{1}{2}} - \sin(a - \frac{\pi}{4}) a^{-\frac{1}{2}} \right\} + R_1 \]

and

\[ \lambda J_1(\lambda) - \frac{2}{\sqrt{2\pi}} \sin(\lambda - \frac{\pi}{4}) \lambda^{-\frac{1}{2}} - R_1(\lambda) \]

\[ = a J_1(a) - \frac{2}{\sqrt{2\pi}} \sin(a - \frac{\pi}{4}) a^{-\frac{1}{2}} - R_1(a). \]

Hence if \( a \) is a constant

\[ \lambda J_1(\lambda) - \frac{2}{\sqrt{2\pi}} \sin(\lambda - \frac{\pi}{4}) \lambda^{-\frac{1}{2}} - R_1(\lambda) = K, \text{ a constant.} \]

We will now prove that this constant is zero.

It is a well known fact that in any range \( 2\pi \) of the
variable, \( J_1(\lambda) \) has at least one root and the same is true of
\( \lambda J_1(\lambda) \). The function \( \frac{2}{\sqrt{2\pi}} \sin(\lambda - \frac{\pi}{4}) \) also has roots
lying at intervals of \( \pi \) apart. Hence the two curves must cross.
Suppose they cross at \( \lambda = \lambda^* \). For such a value, \( R_1(\lambda^*) = -K \).

But such an interval can be found so that \( \lambda^* \) is as
big as you please and since \( R_1(\lambda) \) is of order \( \frac{1}{\sqrt{\lambda}} \), \( R_1(\lambda^*) \)
can be made as small as we please.

Hence \( R_1(\lambda^*) = 0 \) and consequently \( K = 0 \).

Thus we have:

\[
\lambda J_1(\lambda) = \frac{2}{\sqrt{2\pi}} \sin(\lambda - \frac{\pi}{4}) \frac{\lambda}{4} + R_1(\lambda), \text{ where } |R_1(\lambda)| < \frac{C}{\sqrt{\lambda}}
\]

\( C \) being a constant.

Now the maxima of this function are not less in absolute
value than \( \frac{2}{\sqrt{2\pi}} |\mu_i - \frac{C}{\sqrt{\mu_i}}| \), where \( \mu_i \) are the roots of \( J_0(x) = 0 \).

The general term of the series which we are investigating
can be written:

\[
\frac{1}{\mu_i m^2 (\mu_i J_1(\mu_i))^2}
\]

This now becomes:

\[
\frac{1}{\mu_i m^2 (\mu_i J_1(\mu_i))^2} < \frac{1}{\mu_i m^2 (4 \pi / \mu_i - 4C / \mu_i)} < \frac{1}{\mu_i m^2 (4 \pi / \mu_i - 4C / \mu_i)}
\]

Or if \( \frac{2\mu_i}{\pi} > 4C \) our general term is less than \( \frac{\pi}{m!^2} \).

But no roots of \( J_0(x) = 0 \) are closer spaced than \( \frac{\pi}{2} \).

Hence \( \mu_i > \pi / 2 \) and the \( i \)th term of our series becomes
less than \( \frac{\pi}{m!^2} \) or \( \frac{2 m^{-1}}{\pi^{m-1}} \frac{1}{i^{m-1}} \), \( i = 1, 2, 3, \ldots \)

Hence our series converges and converges at least as
fast as $N \sum_{h=1}^{\infty} \frac{1}{n^{m+1}}$, where $N$ is a constant. It is of course clear that we will have to choose $m > 2$. 
This thesis is never to leave this room. Neither is it to be checked out overnight.