

# Herding and Bank Runs

Chao Gu<sup>1</sup>

August 27, 2007

## Abstract

Traditional models of bank runs do not allow for herding effects, because in these models withdrawal decisions are assumed to be made simultaneously. I extend the banking model to allow a depositor to choose his withdrawal time. When he withdraws depends on his liquidity type (patient or impatient), his private, noisy signal about the quality of the bank's portfolio, and the withdrawal histories of the other depositors. In some cases, the optimal banking contract permits herding runs. Some of these "runs" are efficient in that the bank is liquidated before the portfolio worsens. Others are not efficient; these are cases in which the herd is misled.

JEL Classification Numbers: C73, D82, E59, G21.

Keywords: Bank runs, herding, imperfect information, perfect Bayesian equilibrium, optimal bank contract, sequential-move game, fundamental-based bank runs.

## 1 Introduction

In the classic bank-runs model of Diamond and Dybvig (1983), individual withdrawal decisions are made simultaneously. The lack of detailed dynamics of withdrawals makes it difficult to explain some observed features of bank runs. In reality, at least some withdrawals are based on the information about the previous withdrawals of others<sup>2</sup>.

---

<sup>1</sup>I would like to thank Levon Barseghyan, Pablo Becker, David Easley, Edward Green, Ani Guerdjikova, Todd Keister, Tapan Mitra, Assaf Razin, Fernando Vega-Redondo, Tao Zhu and seminar participants at the Cornell Macro Workshop and Cornell-Penn State Macro Workshop for insightful comments. I am especially grateful to Karl Shell for numerous discussions and helpful guidance. All remaining errors are my own. Correspondence: Department of Economics, University of Missouri, Columbia, MO 65293, USA. Email: guc@missouri.edu.

<sup>2</sup>Brunnermeier (2001) says that "...withdrawals by deposit holders occur sequentially in reality, [while] the literature typically models bank runs as a simultaneous move game".

During the 1994-1995 Argentine banking crisis, large depositors were responsible for most of the deposit outflows at the beginning of the crisis. Small depositors began to make substantial withdrawals two months later<sup>3</sup>. In their analysis on the runs on Turkish special finance houses (SFHs)<sup>4</sup> in 2001, Starr and Yilmaz (2007) find that depositors made sequential withdrawals influenced by the history of the withdrawals of others. The authors argue that the “increased withdrawals by moderate-size accountholders tended to boost withdrawals by [their] small counterparts, suggesting that the latter viewed the former as informative with respect to the SFH’s financial condition”.

In the present paper, I build a model in which the timing of individual withdrawals is determined by the depositor’s information about his consumption type (patient, which means he does not need to consume immediately, or impatient, which means he needs to consume immediately), his noisy signal about the quality of the bank’s portfolio, and the observed withdrawal history of other depositors. In my model, the signals are received in an exogenously determined sequence, but the timing of withdrawal is endogenously determined<sup>5</sup>. Because one’s simple withdraw-or-not action does not reveal perfectly to others the pair of private signals that the depositor receives, other depositors can only imperfectly extract the depositor’s private signals from his action. They update their beliefs about the quality of the bank’s portfolio accordingly.

This paper does not focus on the panic-based bank runs of Diamond and Dybvig (1983). (See also Peck and Shell (2003).) I focus instead on bank runs which occur as a result of depositors trying to extract information about bank portfolio quality from the withdrawal histories of others. Because signals about the fundamentals are imperfect, and because signal extraction from the observed withdrawal history is also imperfect, a bank run can occur when the bank fundamentals are strong. In particular, it can occur when “too many” depositors receive early liquidity shocks. A bank run due to imperfect signal extraction is unique to the model with non-simultaneous withdrawal decisions. Bank runs in this sense are not purely fundamental-based. Compare my model with Allen and Gale

---

<sup>3</sup>See Schumacher (2005).

<sup>4</sup>Special financial houses are like commercial banks, but their deposits are not insured.

<sup>5</sup>Chari and Kehoe (2003) are the first to introduce a model of herding in investment decisions with endogenous timing.

(1994) and Goldstein and Pauzner (2005), etc.

I show that there is a perfect Bayesian equilibrium in which a depositor withdraws if his expected utility is below his threshold level, and otherwise he waits. A depositor's expected utility depends upon his beliefs about the quality of the bank's portfolio, which are recursively updated by the observed withdrawal history of the other depositors. Before a depositor's beliefs become sufficiently favorable, he follows his private signals: If he is impatient or the portfolio signal is unfavorable, he withdraws; otherwise he waits. A bank run occurs as a result of a herd of withdrawals when all depositors withdraw because of unfavorable signals and/or unfavorable observations on withdrawals. If his belief is sufficiently favorable, the private signal received by the depositor will not be decisive: the depositor always waits to withdraw unless he is impatient. In this case, his private signal will not be revealed through his withdrawal behavior, so his withdrawal behavior does not affect others' beliefs nor their expected utilities. A "no-bank-run" regime thus takes place as a result of a "herd of non-withdrawals".

Compared with herding in investment decisions (Banerjee (1992), Bikhchandani et al (1992), and more recently Chari and Kehoe (2003, 2004)), herding in bank runs has some special features that complicate the model and lead to interesting results. The most important difference lies in the payment inter-dependence and uncertainty. In the banking setup, a depositor's payoff depends not only on his own actions, but also on the actions of others. The uncertainty in future payoffs – in particular, whether a bank run occurs or not – adds additional risk to the depositor's decision-making. This uncertainty is not necessarily bad, because a run can force the bank to liquidate assets before low productivity is actually realized, i.e., before a higher welfare cost is incurred.

An interesting result due to payment inter-dependence and uncertainty is the possibility that the expected utility is not monotone in the depositor's beliefs and the possibility that his threshold beliefs are not unique. If a bank run takes place when depositors' aggregate expected utility, or social welfare, would be lower if there would be no bank run due to the low probability of having a high return, then the bank run serves as a lower bound on social welfare. Information about production is valuable in this situation.

Though a more favorable level of beliefs makes a depositor more confident in the quality of bank's portfolio, it is also more likely to lead to a herd of non-withdrawals where no more information will be made available in the future. Hence, expected utilities might not be increasing in the probability that the portfolio is good. As a result, the uniqueness of the threshold beliefs in the traditional herding literature is not guaranteed.

Computed examples show that in some economies a run-admitting contract is optimal because it not only provides more liquidity to the depositors to insure against liquidity shocks, but it also encourages depositors to reveal the signals they receive. In other economies, a run-proof contract is optimal as it protects the economy from costly undesirable bank runs. Herding runs are equilibrium phenomena when the risk of bank runs or cost is sufficiently small. Compare my herding runs with the somewhat similar results of Peck and Shell's (2003) on panic-based bank runs.

This remainder of the paper is organized as follows: The model is introduced in Section 2. In Section 3, I describe the equilibrium for an arbitrary demand-deposit contract. A perfect Bayesian Nash equilibrium is shown to exist. In section 4, I calculate some examples of optimal demand-deposit contract. I offer in the final section some concluding observations.

## 2 Model Set-up

**Time:** There are three periods,  $t = 0, 1, 2$  (period 0, 1 and 2, respectively).  $t = 0$  is a planning period, which is called *ex ante*.  $t = 1$  and  $t = 2$  are *ex post*. Period 1 is divided into  $N + 1$  stages.  $N$  is a finite integer.

**Depositors:** There is a measure 1 of depositors in the economy. Each depositor is endowed with 1 unit of the consumption good in period 0. Depositors are identical at  $t = 0$ , but they face consumption shocks at  $t = 1$ . If a depositor receives a consumption shock, he is called impatient and has to consume immediately. An impatient depositor's utility is given by  $u(c^1)$ , where  $c^1$  is the consumption received at  $t = 1$ . If a depositor does not receive a consumption shock, his consumption type is patient. Patient depositors derive

utility from the consumption in the last period. If a patient depositor receives consumption at  $t = 1$ , he can reinvest it in a storage technology privately and consume it at  $t = 2$ . Thus, a patient depositor's utility is described by  $u(c^1 + c^2)$ , where  $c^2$  is the consumption received at  $t = 2$ .  $u(x)$  is strictly increasing, strictly concave, and twice differentiable. The coefficient of relative risk aversion of the utility function,  $-xu''(x)/u'(x)$ , is greater than 1 for  $x \geq 1$ . The utility function is normalized to 0 at  $x = 0$ , i.e.,  $u(0) = 0$ . Each depositor has probability  $\alpha$  ( $0 < \alpha < 1$ ) to be impatient and probability  $1 - \alpha$  to be patient. By the "law of large numbers", a proportion  $\alpha$  of the depositors is impatient.

**Storage:** Depositors can store the consumption good at no cost.

**The bank and its technology:** The bank takes deposits from depositors and invests in a production project. Production is risky and rigid. The investment in production can only be made in the initial period. One unit of consumption good invested at  $t = 0$  yields  $R$  units at  $t = 2$ .  $R = \bar{R} > 1$  with probability  $p_0$ , and  $R = \underline{R} \leq 1$  with probability  $1 - p_0$ . The production asset can be liquidated at  $t = 1$ . Either all or none must be liquidated. The project can therefore be treated as an "indivisible good" after it is started. I assume an individual depositor cannot invest in production on his own.

**The contract:** For convenience, I assume that if a depositor decides to deposit at the bank, the minimum amount of the deposit is 1 unit of consumption good. A competitive bank offers a simple demand-deposit contract that describes the amount of consumption goods paid to the depositors who withdraw in periods 1 and 2,  $c^1$  and  $c^2$  respectively.  $c^1$  is independent of the productivity state.  $c^2$  is state contingent. The bank pays  $c^1$  to the depositors at  $t = 1$  until it is out of funds. If the amount of consumption good in storage cannot meet the withdrawal demand, the bank has to liquidate assets. The bank distributes the remaining resource plus or minus the return on the portfolio equally among the depositors who wait until the last period. Denote the fraction of deposits that the bank keeps in storage by  $\lambda$ , and the fraction of depositors who withdraw deposits in

period 1 by  $\beta$  ( $0 \leq \beta \leq 1$ ). The payment to the depositors who withdraw in period 2 is

$$c^2 = \begin{cases} \frac{\lambda - \beta c^1 + (1 - \lambda)R}{1 - \beta} & \text{if } \beta c^1 \leq \lambda; \\ \frac{1 - \beta c^1}{1 - \beta} & \text{if } \lambda < \beta c^1 \leq 1; \\ 0 & \text{if } \beta c^1 > 1. \end{cases}$$

Because at least a fraction  $\alpha$  of the depositors need to consume at  $t = 1$ ,  $\lambda$  must at least be  $\alpha c^1$ . In the situation that the bank cannot meet payment requirements at  $t = 1$ , the bank fails. Because  $c^2$  is dependent on the choice of  $c^1$  and  $\lambda$ , the demand-deposit contract can therefore be described by  $(c^1, \lambda)$ .

**Withdrawal stages and Information:** In each of the first  $N$  stages of  $t = 1$ , only one depositor is informed of his consumption type. Information about consumption is precise. He also receives a signal about the productivity of the bank portfolio. The signal about production status is accurate with probability  $q$ ,  $q > 0.5$ . That is,

$$\Pr(S_n = H | R = \bar{R}) = \Pr(S_n = L | R = \underline{R}) = q.$$

$S_n$  denotes the signal about productivity obtained by the depositor who is informed at stage  $n$ . Given productivity status, the probability of receiving a correct signal is  $q$ . Receiving a signal, a depositor updates his belief about productivity by Bayes rule. The common initial prior is  $p_0$ . At stage  $N + 1$ , all depositors who have not received signals are informed of their consumption types, but no signal about productivity. An impatient depositor has to consume at the stage when he receives the consumption shock.

Depositors have equal opportunity to be informed at each stage. Because  $N$  is tiny compared with the infinite number of depositors, the probability of getting informed in the first  $N$  stages is zero. Depositors do not communicate with each other about the signals they receive. However, a depositor's withdrawal action is observed by all others. Once a depositor withdraws, he cannot reverse his decision. But if a depositor chooses to wait, he can withdraw at a later stage. The final deadline for depositors to withdraw at  $t = 1$  is stage  $N + 1$ . Depositors are not allowed to change decisions after observing other

depositors' decisions at stage  $N + 1$ .

There are four types of depositors at each of the first  $N$  stages. The first type are those who have already withdrawn their deposits from the bank. Those are inactive depositors who have no more decisions to make. The second type is the newly informed depositor who receives signals at current stage. The third type are those who were previously informed but have been waiting. The rest are the uninformed depositors.

The rigidity in liquidation of long-term assets imposes difficulty for the bank to adjust its portfolio at  $t = 1$  by varying the fraction of assets in production. The bank does not have private information about productivity. It is in the same position as an uninformed depositor in terms of information. The bank does not liquidate the assets unless it is forced to do so when a bank run occurs.

A finite number of stages is necessary because it imposes a deadline to the depositors to make decisions at  $t = 1$ , so the expected utility can be calculated by backward induction. The specification of a continuum of depositors tremendously simplifies calculation. Consider a model that has a finite number of depositors. Each depositor has a non-atomic share at the bank. Seeing a depositor withdraw his funds, the rest need to re-calculate their payoffs in different productivity states as the amount of remaining resource at the bank has changed significantly. The description of the equilibrium will be dependent on the parameters of the economy, and there will be many more cases to discuss. In the appendix, I present a simple example of a two-stage, two-depositor economy. Similar results are obtained in the example.

The sequence of timing of the banking game is as follows.

$t = 0$  :

Bank announces the contract;

Depositors make deposit decision.

$t = 1$  :

Stage 1:

One depositor receives signals about his consumption type and about productivity.

He decides whether to withdraw or not.

Other depositors decide whether to withdraw or not.

(repeat for  $N$  stages)

Stage  $N + 1$  :

Consumption types are revealed to those who are not informed.

Depositors decide whether to withdraw or not.

$t = 2$  :

Bank allocates the remaining resource to the rest of the depositors.

The post-deposit game starts after depositors make deposits at the bank. An individual depositor decides when to withdraw from the bank. Knowing what depositors will be doing in the post-deposit game, the bank offers a competitive contract that maximizes the *ex-ante* expected utility of the depositors at  $t = 0$ . Depositors determine whether to deposit at the bank or stay in autarky. Starting at  $t = 0$ , the entire game is called the pre-deposit game. I start with the analysis in the post-deposit game. I first prove that in the post-deposit game, there exists a perfect Bayesian equilibrium given a contract. Then I will calculate some examples of the optimal contract that the bank offers in the pre-deposit game given the equilibrium strategies in the post-deposit game.

### 3 Post-Deposit Game

In Diamond and Dybvig (1983), a demand-deposit banking contract allows for a panic-based bank run in the post-deposit game given  $c^1 > 1$ . For convenience, the panic-based run is not considered in the present paper. A bank run occurs in my model solely due to the information about the productivity or the imperfect extraction of the information from the actions of other depositors.

Let  $X_n$  denote the total number of withdrawals at stage  $n$ . The history of withdrawals,  $h_n = (X_1, X_2, \dots, X_n)$ , publicly records the number of withdrawals at each stage up to stage  $n$ . The history of depositor  $i$  who receives a signal at stage  $r$  is  $h_{i,n} = (h_n, s_r, r)$ . The history of an uninformed depositor is  $h_{i,n} = (h_n, \emptyset, \emptyset)$ .

The strategies  $x_{i,n}(h_{i,n})$  is a sequence of functions that map the history of depositor  $i$

into zero-one withdrawal decisions at stage  $n$ . Let  $x_{i,n}(h_{i,n}) = 1$  represent the decision to withdraw, and let  $x_{i,n}(h_{i,n}) = 0$  represent the decision to wait. The beliefs  $p_{i,n}(h_{i,n})$  is a sequence of functions that map the history of depositor  $i$  into the probabilities that the productivity is high at stage  $n$ . At stage  $n$ , the newly informed depositor's belief is the belief as an uninformed depositor at stage  $n - 1$  updated by the signal he receives. The belief of an uninformed or of a previously informed depositor is based on his observation on the number of withdrawals in each stage up to  $n$ . Uninformed depositors have the same history. Their beliefs are the same.

To simplify the notation, let  $x_n^U$  and  $p_n^U$  denote the strategy and belief of an uninformed depositor at stage  $n$ , respectively. Let  $x_n^{S_r}$  and  $p_n^{S_r}$  denote the strategy and belief of a depositor who is informed at stage  $r$  of a productivity signal  $S_r$ , respectively. If  $r = n$ , the depositor is newly informed. Otherwise, he is previously informed.

In order to show how withdrawals by some depositors affect the beliefs and actions of the others, I am interested in finding an equilibrium in which the newly informed depositors are willing to make decisions according to the signals that they receive under some conditions. I consider symmetric pure strategies. That is, depositors with the same history adopt the same pure strategy at each stage. I will discuss equilibrium according to whether  $c^1$  is greater than 1 or not. For a contract that offers  $c^1 < 1$ , there does not exist a symmetric pure strategy equilibrium in which all depositors withdraw in period 1, because if all other depositors withdraw, an individual depositor can expect to obtain an infinite amount of consumption goods in period 2 by choosing to wait.

### 3.1 Equilibrium Given $c^1 \geq 1$

#### 3.1.1 Bayesian Updates

A newly informed depositor at stage  $n$  Bayesian updates his belief by the productivity signal that he receives. His prior at stage  $n$  is his posterior at the stage  $n - 1$  when he

was an uninformed depositor.

$$p_n^{S_n} = \begin{cases} P_H(p_{n-1}^U) = \frac{p_{n-1}^U q}{p_{n-1}^U q + (1 - p_{n-1}^U)(1 - q)}, & \text{if } S_n = H; \\ P_L(p_{n-1}^U) = \frac{p_{n-1}^U (1 - q)}{p_{n-1}^U (1 - q) + (1 - p_{n-1}^U)q}, & \text{if } S_n = L. \end{cases}$$

$P_H$  and  $P_L$  denote the rules of Bayesian updates when a high or a low signal is received, respectively.  $p \leq P_H(p) \leq 1$  and  $0 \leq P_L(p) \leq p$  for  $p \in [0, 1]$ .  $P_H(p)$  and  $P_L(p)$  are strictly increasing in  $p$ .

The uninformed and previously informed depositors update their belief about the productivity being high by observing the decision made by the newly informed depositor. Suppose that the newly informed depositor waits if and only if a high signal is received and he is patient. The uninformed depositors then update their beliefs by

$$p_n^U = \begin{cases} P_H(p_{n-1}^U) = \frac{p_{n-1}^U q}{p_{n-1}^U q + (1 - p_{n-1}^U)(1 - q)}, & \text{if the newly informed waits;} \\ P_{\tilde{L}}(p_{n-1}^U) = \frac{p_{n-1}^U (1 - q + \alpha q)}{\alpha + (1 - \alpha) [p_{n-1}^U (1 - q) + (1 - p_{n-1}^U)q]}, & \text{if the newly informed withdraws.} \end{cases}$$

$P_{\tilde{L}}$  denotes the Bayesian update where the probability of observing an impatient depositor is taken into account.  $0 \leq P_L(p) \leq P_{\tilde{L}}(p) \leq p$  for  $p \in [0, 1]$ . Note that  $P_H^{n_1} \left( P_{\tilde{L}}^{n_2}(p) \right) = P_{\tilde{L}}^{n_2}(P_H^{n_1}(p))$ , where the power on  $P_{\tilde{L}}$  (or  $P_H$ ) denotes the number of updates by  $P_{\tilde{L}}$  (or  $P_H$ ), given the prior. So long as depositors update their beliefs by the same numbers of  $P_H$  and  $P_{\tilde{L}}$ , their beliefs are the same, no matter at which stages these updates have occurred. A previously informed depositor updates his prior in the same way.

Suppose the newly informed does not make decisions according to his signal about productivity. In this case, the uninformed and the previously informed depositors do not change their beliefs because the decision of the newly informed carries no information about the productivity. Therefore,  $p_n^U = p_{n-1}^U$ , and  $p_n^{S_r} = p_{n-1}^{S_r}$  for  $r < n$ .

### 3.1.2 Expected Utility at Stage $n$

By the assumption of symmetric pure strategies, either all uninformed depositors withdraw or none of them withdraws at stages before  $N + 1$ . If a depositor withdraws while all other uninformed depositors do not, he will get  $c^1$  definitely. If all uninformed depositors withdraw, an individual depositor who also withdraws has a chance of  $\frac{1}{c^1}$  to receive  $c^1$ . In this case, the expected utility from withdrawing immediately is  $\frac{1}{c^1}u(c^1)$ .

The expected utility of a depositor who does not withdraw at stage  $n$  is more complicated. The expected utility obviously depends on his current belief. Furthermore, it depends on how other depositors behave at future stages. I start with stage  $N$  to illustrate this. The expected utility here refers to the expected utility from optimal decisions at each stage.

Let  $u_1 = u(c^1)$ ,  $\bar{u}_2 = u\left(\frac{\lambda - \alpha c^1 + (1 - \lambda)\bar{R}}{1 - \alpha}\right)$ , and  $\underline{u}_2 = u\left(\frac{\lambda - \alpha c^1 + (1 - \lambda)\underline{R}}{1 - \alpha}\right)$ .  $\bar{u}_2$  and  $\underline{u}_2$  represent a patient depositor's utility in  $t = 2$ , depending on the realization of production, if there is no bank run  $t = 1$ . I suppress  $(c^1, \lambda)$  because  $c^1$  and  $\lambda$  are given in the post-deposit game. Define the cutoff belief,  $\hat{p}$ , as follows:

$$u_1 = \hat{p}\bar{u}_2 + (1 - \hat{p})\underline{u}_2. \quad (1)$$

$\hat{p}$  is a function of  $(c^1, \lambda)$ .  $\hat{p}$  is the cutoff belief with which a patient depositor is indifferent between withdrawing immediately and waiting until the last period if no information about productivity is available. Note that given  $c^1 \geq 1$  and  $\underline{R} \leq 1$ , we always have  $\hat{p} \geq 0$ .  $\hat{p} = 0$  if and only if  $c^1 = \underline{R} = 1$  or  $c^1 = \lambda = 1$ . Let  $\bar{p}$  denote  $P_H(\hat{p})$ , and  $\underline{p}$  denote  $P_L(\hat{p})$ .

Suppose that the uninformed depositors have the posterior belief  $p_N^U$  at the end of stage  $N$ . They will not get information about productivity at stage  $N + 1$ . Therefore,  $p_N^U$  is an uninformed depositor's finalized belief. If  $p_N^U \geq \hat{p}$ , an uninformed depositors will wait for period 2 unless he is told to be impatient at stage  $N + 1$ . Otherwise, he will withdraw regardless of the actions of the other depositors. By symmetric strategies, each depositor has a chance of  $\frac{1}{c^1}$  to get paid given  $c^1 \geq 1$ . The expected utility of an

uninformed depositor at the end of stage  $N$  is

$$w_N^U(p_N^U) = \begin{cases} \alpha u_1 + (1 - \alpha) [p_N^U \bar{u}_2 + (1 - p_N^U) \underline{u}_2], & \text{if } p_N^U \geq \hat{p}; \\ \frac{1}{c^1} u_1, & \text{otherwise.} \end{cases} \quad (2)$$

How about an uninformed depositor's expected utility at an arbitrary stage  $n$ ? Suppose the newly informed depositor follows a simple rule: he withdraws if and only if his posterior at stage  $n$  is below the cutoff level  $\hat{p}$ , or he is impatient. A newly informed depositor was an uninformed depositor the stage before. So the uninformed depositors and the newly informed depositor share the same prior at current stage. Knowing the newly informed depositor's decision rule, an uninformed can update his belief according to the newly informed depositor's actions. His expected utility is also updated with his beliefs accordingly. Define the expected utility of an uninformed depositor at stage  $n < N$  in a recursive way:

$$w_n^U(p_n^U) = \begin{cases} \alpha u_1 + (1 - \alpha) [p_n^U \bar{u}_2 + (1 - p_n^U) \underline{u}_2], & \text{if } p_n^U \geq \bar{p}; \\ \pi(p_n^U) w_{n+1}^U(P_H(p_n^U)) + & \text{if } \underline{p} \leq p_n^U < \bar{p} \text{ and} \\ +(1 - \pi(p_n^U)) w_{n+1}^U(P_{\tilde{L}}(p_n^U)), & \pi(p_n^U) w_{n+1}^U(P_H(p_n^U)) + (1 - \pi(p_n^U)) \cdot \\ & w_{n+1}^U(P_{\tilde{L}}(p_n^U)) \geq u_1; \\ \frac{1}{c^1} u_1, & \text{otherwise,} \end{cases} \quad (3)$$

where

$$\pi(p) = (1 - \alpha) [(1 - p)(1 - q) + pq] \quad (4)$$

is the probability that the depositor informed at next stage receives a high signal and is also patient, given the posterior of  $p$  at the current stage.

If the prior at stage  $n + 1$  is very high (very low), i.e.,  $p_n^U \geq \bar{p}$  ( $p_n^U < \underline{p}$ ), even though a low (high) signal is received, the newly informed depositor's posterior belief at stage  $n + 1$  is still above (below) the critical level of  $\hat{p}$ . So the newly informed depositor will not withdraw (wait). The newly informed depositor's action does not carry information about his signal, so the beliefs of the uninformed depositors will not change. From then on, no

more information can be inferred from the decisions by the newly informed depositors at future stages. According to their current belief, the expected utility of an uninformed depositor in the last period is  $\alpha u_1 + (1 - \alpha) [p_n^U \bar{u}_2 + (1 - p_n^U) \underline{u}_2]$ , which is greater (lower) than  $u_1$  as  $p_n^U \geq \bar{p}$  ( $p_n^U < \underline{p}$ ).

Suppose the newly informed depositor's prior is moderately high. If a low signal is received, the posterior belief falls below  $\hat{p}$ ; while if a high signal is received, the posterior belief is above  $\hat{p}$ . When the newly informed waits, his decision fully reveals that he gets a high signal. The belief of the uninformed depositors will be updated to the same level as the newly informed depositor. While if a withdrawal is observed, an uninformed depositor's belief will be updated by  $P_{\tilde{L}}$ . The expected utility of an uninformed depositor at the current stage is the weighted average of the possible expected utilities at next stage, where the weights are the probabilities that his current belief will be updated by either  $P_H$  or  $P_{\tilde{L}}$  at next stage. Whether an uninformed depositor decides to withdraw depends on whether the weighted average exceeds  $u_1$ .

### 3.1.3 A Perfect Bayesian Equilibrium

The expected utility of an uninformed depositor defined by (2) – (4) depends on the conjecture that the newly informed depositor withdraws if his posterior is lower than  $\hat{p}$ , and waits otherwise. In this section, I will show that the conjecture is part of the equilibrium. I will also illustrate the equilibrium strategies and belief update rules of all active depositors.

Define a previously informed patient depositor's expected utility as

$$w_N^{S_r}(p_N^{S_r}) = \begin{cases} \max \{ p_N^{S_r} \bar{u}_2 + (1 - p_N^{S_r}) \underline{u}_2, u_1 \}, & \text{if } w_N^U(p_N^U) \geq u_1; \\ \frac{1}{c^1} u_1, & \text{otherwise.} \end{cases} \quad (5)$$

$$w_n^{S_r}(p_n^{S_r}) = \begin{cases} p_n^{S_r} \bar{u}_2 + (1 - p_n^{S_r}) \underline{u}_2, & \text{if } p_n^U \geq \bar{p}; \\ \max \{ \pi(p_n^{S_r}) w_{n+1}^{S_r}(P_H(p_n^{S_r})) + & \text{if } \underline{p} \leq p_n^U < \bar{p} \text{ and} \\ + (1 - \pi(p_n^{S_r})) w_{n+1}^{S_r}(P_{\tilde{L}}(p_n^{S_r})), u_1 \}, & w_n^U(p_n^U) \geq u_1; \\ \frac{1}{c^1} u_1, & \text{otherwise.} \end{cases} \quad (6)$$

for  $1 \leq n < N$ ,  $r < n$ . The expected utility of a previously informed depositor is defined in the same way as that of an uninformed depositor. A previously informed depositor is patient, otherwise he should have withdrawn already. He knows the beliefs of the uninformed depositors, and he can predict whether the uninformed depositors will withdraw or not. As the uninformed are of measure 1, when they withdraw, a previously informed should also do so, otherwise he will be left unpaid. Therefore, the expected utility of a previously informed depositor is conditional on whether the uninformed depositors withdraw or not. Also note that the expected utility of a previously informed depositor only depends on his current belief. His private history path does not matter. If  $r = n$ , (5)–(6) defines a newly informed depositor's expected utility if he is patient.

For  $1 \leq n \leq N$ , a newly informed depositor's strategy is

$$x_n^{S_n} = \begin{cases} 1, & \text{if impatient or } p_n^{S_n} < \hat{p}. \\ 0, & \text{otherwise.} \end{cases} \quad (7)$$

For  $1 \leq n \leq N$ , an uninformed depositor's strategy is

$$x_n^U = \begin{cases} 1, & \text{if } w_n^U(p_n^U) < u_1. \\ 0, & \text{otherwise.} \end{cases} \quad (8)$$

For  $1 \leq n \leq N$ , a previously informed depositor's strategy is ( $r < n$ )

$$x_n^{S_r} = \begin{cases} 1, & \text{if } w_n^{S_r}(p_n^{S_r}) < u_1. \\ 0, & \text{otherwise.} \end{cases} \quad (9)$$

For  $n = N + 1$ , an active depositor's strategy is

$$x_{N+1} = \begin{cases} 1, & \text{if impatient or } p_{N+1} < \hat{p}. \\ 0, & \text{otherwise.} \end{cases} \quad (10)$$

If no one else makes a withdrawal, the belief of a newly informed depositor at stage  $n$

( $1 \leq n \leq N$ ) is updated by the signal he receives

$$p_n^{S_n} = \begin{cases} P_L(p_{n-1}^U), & \text{if } S_n = L; \\ P_H(p_{n-1}^U), & \text{if } S_n = H, \end{cases} \quad (11)$$

with  $p_0^U = p_0$ . If anyone else makes a withdrawal,  $p_n^{S_n} = 0$ .

The belief of an uninformed depositor at stage  $1 \leq n \leq N$  is updated by

$$p_n^U = \begin{cases} 0, & \text{if } X_n > 1, \text{ or } (X_n = 0 \text{ and } p_{n-1}^U < \underline{p}); \\ P_L(p_{n-1}^U), & \text{if } X_n = 1, \underline{p} \leq p_{n-1}^U < \bar{p}; \\ P_H(p_{n-1}^U), & \text{if } X_n = 0, \underline{p} \leq p_{n-1}^U < \bar{p}; \\ p_{n-1}^U, & \text{otherwise.} \end{cases} \quad (12)$$

with  $p_0^U = p_0$ .

The belief of a previously informed depositor at stage  $1 \leq n \leq N$  is updated by ( $r < n$ )

$$p_n^{S_r} = \begin{cases} 0, & \text{if } X_n > 1, \text{ or } (X_n = 0 \text{ and } p_{n-1}^U < \underline{p}); \\ P_L(p_{n-1}^{S_r}), & \text{if } X_n = 1, \underline{p} \leq p_{n-1}^U < \bar{p}; \\ P_H(p_{n-1}^{S_r}), & \text{if } X_n = 0, \underline{p} \leq p_{n-1}^U < \bar{p}; \\ p_{n-1}^{S_r}, & \text{otherwise.} \end{cases} \quad (13)$$

At stage  $N + 1$ , an active depositor's belief is equal to his belief at stage  $N$ . That is,  $p_{N+1} = p_N$ .

A newly informed depositor updates his belief by  $P_H$  or  $P_L$  if no other withdrawals are observed. If other depositors withdraw, his belief drops to 0, and he will withdraw if  $\hat{p} > 0$ . Thus, at least two withdrawals occur at the current stage. The beliefs of the uninformed depositors also drop to 0, and they also withdraw. If  $\hat{p} = 0$ , depositors always prefer to wait even though deviations are detected. If  $p_{n-1}^U < \underline{p}$ , the newly informed at stage  $n$  is supposed to withdraw even if he receives a high signal (although in equilibrium, there is no active depositor with beliefs lower than  $\underline{p}$ ). If he does not withdraw, the uninformed depositors detect the deviation, and their beliefs become 0.

Before the equilibrium is proved, I first introduce the definitions of a herd of with-

drawals and a herd of non-withdrawals.

**Definition 1** *A herd of non-withdrawals begins when (1) the newly informed depositor does not withdraw deposits unless he is impatient even though a low signal on productivity is received, and (2) all other depositors wait until their consumption types are revealed to be impatient.*

**Definition 2** *A herd of withdrawals begins when all depositors withdraw deposits.*

The logic behind the proof of the equilibrium is similar to Chari and Kehoe (2003). However, due to the fact that the payoffs of the depositors are dependent on each other's action, and that the liquidity type is private, the following lemmas are needed to establish the properties of an active depositor's expected utility function at any stage. I will discuss the properties of an uninformed depositor's expected utility function according to whether the contract satisfies the "high cutoff probability" condition or the "low cutoff probability" condition. The meaning of the conditions will become clear at the end of this section.

**Definition 3** *Define a cutoff probability of  $w_n^U(p)$  as follows:  $\tilde{p}_n$  is a cutoff probability if there exist  $\varepsilon_1, \varepsilon_2 > 0$  such that  $w_n^U(p) \geq u_1$  for  $p \in [\tilde{p}_n, \tilde{p}_n + \varepsilon_1]$ , and  $w_n^U(p) < u_1$  for  $p \in [\tilde{p}_n - \varepsilon_2, \tilde{p}_n)$ .*

**"High Cutoff Probability" Condition:**  $\alpha u_1 + (1 - \alpha) [P_{\tilde{L}}(\hat{p}) \bar{u}_2 + (1 - P_{\tilde{L}}(\hat{p})) \underline{u}_2] > \frac{1}{c^1} u_1.$

**"Low Cutoff Probability" Condition:**  $\alpha u_1 + (1 - \alpha) [P_{\tilde{L}}(\hat{p}) \bar{u}_2 + (1 - P_{\tilde{L}}(\hat{p})) \underline{u}_2] \leq \frac{1}{c^1} u_1.$

The left-hand side of the "high/low cutoff probability" condition is an uninformed depositor's expected utility with belief  $P_{\tilde{L}}(\hat{p})$  at stage  $N$  if no bank run occurs. The right-hand side is his expected utility when a bank run occurs.

**Lemma 1** *Consider a contract that pays  $c^1 \geq 1$  and satisfies the "high cutoff probability" condition. For each stage  $0 \leq n \leq N$ ,  $w_n^U(p)$  is increasing in  $p$ . There exists a unique cutoff probability  $\tilde{p}_n$  such that  $w_n^U(p) \geq u_1$  for  $p \in [\tilde{p}_n, 1]$ , and  $w_n^U(p) = \frac{1}{c^1} u_1$  for  $p \in [0, \tilde{p}_n)$ .  $\tilde{p}_n$  is decreasing in  $n$ .  $w_n^U(p) \leq \alpha u_1 + (1 - \alpha) [p \bar{u}_2 + (1 - p) \underline{u}_2]$  for  $p \in [\tilde{p}_n, 1]$ .*

**Proof.** Prove by induction. See appendix. ■

Lemma 1 says if the defined  $w_n^U(p)$  is the expected utility of an uninformed depositor in the equilibrium and the “high cutoff probability” condition is satisfied, there is a unique cutoff belief at each stage above which the uninformed depositors are willing to wait, below which they will withdraw.

**Lemma 2** Consider a contract that pays  $c^1 \geq 1$  and satisfies the “low cutoff probability” condition.  $w_n^U(p) \geq u_1$  on  $[\hat{p}, 1]$ .

**Proof.** See appendix. ■

Lemma 2 says that if  $w_n^U(p)$  is the expected utility of the uninformed depositors in the equilibrium and the “low cutoff probability” condition holds, depositors are willing to wait if their beliefs are above  $\hat{p}$ . In other words, the cutoff probabilities of  $\tilde{p}_n$  are lower than  $\hat{p}$  for stages before  $N$ .

**Corollary 1** Consider a contract that pays  $c^1 \geq 1$ . Given a posterior of  $p$  at stage  $n$ , if  $w_n^U(p) \geq u_1$ , then  $w_{n+1}^U(P_H(p)) \geq u_1$ .

**Proof.** See appendix. ■

Corollary 1 has the following implication: Given  $c^1 \geq 1$ , assume an uninformed depositor is willing to wait the stage before. He is also willing to wait at the current stage assuming a high signal is inferred. If a newly informed depositor’s decision of waiting conveys a high signal to the uninformed depositors, his decision will not trigger a bank run.

**Example 1:**

Figure 1<sup>6</sup> shows an example of  $w_n^U(p)$  where the “high cutoff probability” condition holds.

$u(c) = \frac{(c+b)^{1-\gamma} - b^{1-\gamma}}{1-\gamma}$ ,  $b = 0.001$ ,  $\gamma = 1.01$ .  $\bar{R} = 1.5$ ,  $\underline{R} = 1$ ,  $p_0 = 0.9$ .  $q = 0.999$ .  $\alpha = 0.01$ .

Let  $c^1 = 1.04$ , and  $\lambda = \alpha c^1 = 0.0104$ .  $\bar{u}_2 = 7.5568$ ,  $\underline{u}_2 = 7.1525$ ,  $u_1 = 7.1921$ .

---

<sup>6</sup>In all figures in this paper, solid thin line represents  $\alpha u_1 + (1 - \alpha) [p_n \bar{u}_2 + (1 - p_n) \underline{u}_2]$ , solid thick line represents  $w_n^U$ , and dash line represents  $u_1$ .

In this example,  $\tilde{p}_N = \hat{p} = 0.0978$ ,  $\tilde{p}_n = 0.4383$  for  $n = N - 1, N - 2, \dots, 1$ .

**Example 2:**

Figure 2 shows an example of  $w_n^U(p)$  where the “low cutoff probability” condition holds.

$u(c) = \frac{(c+b)^{1-\gamma} - b^{1-\gamma}}{1-\gamma}$ ,  $b = 0.001$ ,  $\gamma = 1.01$ .  $\bar{R} = 1.5$ ,  $\underline{R} = 0.8$ ,  $p_0 = 0.9$ .  $q = 0.9$ .  $\alpha = 0.01$ .

Let  $c^1 = 1.011$ ,  $\lambda = \alpha c^1 = 0.0101$ .  $\bar{u}_2 = 7.5571$ ,  $\underline{u}_2 = 6.9297$ ,  $u_1 = 7.1629$ .

In this example, there exist unique cutoff probabilities at stages  $N$ ,  $N - 1$ ,  $N - 2$ , and  $N - 100$ , above which  $w_n^U(p)$  is greater than  $u_1$ , below which  $w_n^U(p)$  is less than  $u_1$ .  $\tilde{p}_N = \hat{p} = 0.3716$ ,  $\tilde{p}_{N-1} = 0.2032$ ,  $\tilde{p}_{N-2} = 0.1971$ ,  $\tilde{p}_{N-100} = 0.1783$ . However, the cutoff probability is not always unique. We will see an example of non-uniqueness later. Also note that  $w_n^U(p)$  is not necessarily increasing in  $p$ .

**Lemma 3** *If  $p_n^U = p_n^{Sr}$ , and  $w_n^U(p_n^U) \geq u_1$ , then  $w_n^{Sr}(p_n^{Sr}) \geq u_1$ .*

**Proof.** See appendix. ■

Lemma 3 states the following: if a previously informed and an uninformed depositor share the same belief, and the uninformed depositor is willing to wait, then the previously informed depositor is also willing to wait. The intuition behind the lemma is the following. Conditional on being impatient, a depositor prefers to withdraw immediately. If an uninformed depositor is willing to wait, it must be true that conditional on being patient, the expected utility from waiting is higher than that from withdrawing immediately.

**Proposition 1** *Given  $c^1 \geq 1$ , the beliefs and strategies in (1) – (13) constitute a perfect Bayesian equilibrium in the post-deposit game.*

**Proof.** The proof process is divided into several steps to facilitate reading.

Step 1: Check the beliefs.

By construction, beliefs are updated by the Bayes' rule whenever possible.

Step 2: Check the strategies of an uninformed depositor with no detectable deviation.

By construction, the expected utility of an uninformed depositor at stage  $n$  is  $w_n^U(p_n^U)$ .

If it is lower than or equal to  $u_1$ , he should withdraw. Otherwise, he should not.

Step 3: Check the strategies of a newly informed depositor with no detectable deviation.

For a newly informed depositor at stage  $n$ , it must be true that  $w_{n-1}^U(p_{n-1}^U) \geq u_1$  at the stage before. That is,  $p_{n-1}^U \geq \underline{p}$ . If a herd of non-withdrawals has begun already, that is,  $p_n^U = p_{n-1}^U \geq \bar{p}$ , the newly informed depositor's action does not change other depositors' beliefs, and he will not be able to infer any information in future. Even though he receives a low signal, his private belief is still above  $\hat{p}$ , so he will be waiting. If a herd of non-withdrawals has not begun yet, that is,  $\underline{p} \leq p_{n-1}^U < \bar{p}$ , the uninformed depositors' belief will be updated by either  $P_H$  or  $P_L$ . Let us discuss cases by the signal that the newly informed gets at stage  $n$ .

(1) The newly informed depositor gets a high signal. His belief is  $p_n^{S_n=H} = P_H(p_{n-1}^U) \geq \hat{p}$ . If he waits, an uninformed depositor's belief will also be  $p_n^U = P_H(p_{n-1}^U)$ . By corollary 1, the uninformed depositors will be waiting too. If the newly informed depositor waits, he will become a previously informed depositor and share the same belief with the uninformed depositors. By lemma 3, he will wait.

(2) The newly informed depositor gets a low signal. His belief is now  $p_n^{S_n=L} = P_L(p_{n-1}^U) < \hat{p}$ . According to the strategies, he should withdraw and get  $u_1$ . Suppose he waits. Then the belief of an uninformed depositor is misled to be updated to  $p_n^U = P_H(p_{n-1}^U)$ . From then on, the belief of an uninformed depositor is always two signals above that of the depositor informed at  $n$ , that is,  $p_m^{S_n=L} = P_L^2(p_m^U)$  for  $m \geq n$ . By choosing to wait, the best outcome that the newly informed depositor can anticipate is a herd of non-withdrawals. (If he anticipates a herd of withdrawals to occur, he should withdraw immediately.) Suppose a herd of non-withdrawals occurs at a later stage  $m < N$ . The posterior of an uninformed depositor at stage  $m$  satisfies  $p_m^U \geq \bar{p}$ . It also

must be true that  $p_{m-1}^U < \bar{p}$ . Otherwise, the herd of non-withdrawals could have begun earlier. As  $p_{m-1}^U < p_m^U$ , we have  $p_m^U = P_H(p_{m-1}^U)$ . Updating both sides by  $P_L^2$ , we have  $p_m^{S_n} = P_L^2(p_m^U) = P_L^2 P_H(p_{m-1}^U) = P_L(p_{m-1}^U) < \hat{p}$ . Thus, at the stage that the herd of non-withdrawals begins, the expected utility of the depositor informed at stage  $n$  is still lower than  $u_1$ . In the case when neither a herd of withdrawals nor a herd of non-withdrawals occurs before stage  $N$ , it must be true that the uninformed depositors' belief satisfies  $p_{N-1}^U < \bar{p}$ , which implies the deviator's belief at stage  $N$  is below  $\hat{p}$ . Therefore, the depositor informed at stage  $n$  does not benefit from deviation. A newly informed depositor weakly prefers to withdraw immediately if a low productivity signal is received.

Step 4: Check a previously informed depositor's strategy with no detectable deviation.

If a previously informed depositor chose to wait before the herd of non-withdrawals begins, he must have received a high signal. By choosing to wait, he has conveyed the high signal to all other depositors. Thus, the previously informed depositors and the uninformed depositors have the same belief. By lemma 3, a previously informed depositor always waits if an uninformed depositor waits given the same belief. If a previously informed depositor waits because a herd of non-withdrawals has begun before he got the signal, then he will be waiting from then as no more updates on the belief are available and his belief is above  $\hat{p}$ .

Step 5: Check the strategies of active depositors if there is a detectable deviation.

Because the consumption types are private information, the deviations are undetectable to the uninformed depositors unless more than 1 withdrawals are observed at a stage before a herd of withdrawals begins. The newly informed depositor detects deviations at current stage, and he will withdraw if  $\hat{p} > 0$  because his belief is 0 now. In this case, the uninformed depositors' beliefs also falls to zero because at least two withdrawals at a stage are observed. Therefore, all depositors withdraw. If  $\hat{p} = 0$ , no one will withdraw even though deviations are detected as  $u_1 = \underline{u}_2$ . Waiting is the dominant strategy in this case even if all other depositors withdraw.

Another plausible detectable deviation is as follows: the newly informed depositor should withdraw regardless of the signal. If he waits, the uninformed depositors detect

the deviation. If this was the case, it must be true that at stage  $n$ ,  $p_{n-1}^U < \underline{p}$ . However, given such a belief at stage  $n - 1$ , the uninformed depositors must have all withdrawn at stage  $n - 1$  from the bank already. ■

### 3.1.4 Discussion of the Equilibrium - the “High Cutoff Probability” Condition Holds

With the “high cutoff probability” condition, the sequence of  $(\tilde{p}_0, \tilde{p}_1, \dots, \tilde{p}_{N-1}, \hat{p}, \hat{p})$  is the threshold beliefs above which the uninformed depositors wait, below which they withdraw. While  $(\hat{p}, \hat{p}, \dots, \hat{p}, \hat{p}, \hat{p})$  is the sequence of the threshold beliefs above which the newly informed depositors wait, below which they withdraw. A herd of non-withdrawals happens before stage  $n$  if  $p_n^U \geq \bar{p}$ . At stages  $N$  and  $N + 1$ , if beliefs are above  $\hat{p}$ , depositors will wait unless they are impatient. Therefore, for all depositors  $(\bar{p}, \bar{p}, \dots, \bar{p}, \hat{p}, \hat{p})$  is the sequence of beliefs above which a herd of non-withdrawals occurs at a stage.

Because  $\tilde{p}_n$  is unique and is decreasing in  $n$ , we can calculate the number of updates by  $P_{\tilde{L}}$  that are needed to trigger a bank run at stage  $n$  starting with  $p_0$ . Let  $Z_n$  solve

$$P_{\tilde{L}}^{Z_n-1}(p_0) \geq \tilde{p}_n, \text{ and } P_{\tilde{L}}^{Z_n}(p_0) < \tilde{p}_n.$$

If there have  $Z_n$  number of withdrawals up to stage  $n$ , a bank run will take place. Because  $\tilde{p}_n \geq \hat{p}$ , a non-withdrawal will trigger a herd of non-withdrawals.

What we observe in the equilibrium is as follows: A newly informed depositor follows his productivity signal if his prior at the stage is below  $\bar{p}$ . If the newly informed depositors keep lining up in front of the bank, the beliefs of the uninformed depositors will finally fall below the cutoff, and they will demand their deposits back. Before their beliefs drop below the cutoff, if one high signal can be conveyed by the non-withdrawal decision of a newly informed depositor, they will be convinced to wait. In a situation that the uninformed depositors observe consecutive withdrawals, but the number of withdrawals is not too large, the uninformed depositors watch the line closely. Their beliefs will be updated by the decisions of the newly informed depositors.

Let us try to understand why the cutoff probabilities are higher before stage  $N$  if the

“high cutoff probability” condition is satisfied. Given  $p_N^U$  in the interval of  $[P_{\tilde{L}}(\hat{p}), \hat{p})$ , a bank run takes place at stage  $N$ . The social welfare, measured by the aggregate expected utility, falls to  $\frac{1}{c^1}u_1$ . However, with the “high cutoff probability” condition, if depositors do not withdraw, the social welfare would actually be higher than that in the bank run. From the view of social welfare, bank run is undesirable. Nevertheless, it is to an individual depositor’s own interest to withdraw early. To an individual depositor, due to the costly liquidation, his expected utility also experiences a sudden drop when there presents a possibility of bank runs. Aware of the possibility of having a bank run at next stage, the depositors must be more optimistic to wait for more information at stage  $N - 1$ . Hence, the cutoff belief at stage  $N - 1$  is higher than  $\hat{p}$ . Working backwards, as the uncertainty of having a bank run gradually resolves, the cutoff beliefs are decreasing as time goes by. Depositors are becoming more and more willing to wait.

### 3.1.5 Discussion of the Equilibrium - the “Low Cutoff Probability” Condition Holds

If the “low cutoff probability” condition is satisfied, when depositors withdraw with the belief of  $P_{\tilde{L}}(\hat{p})$  at stage  $N$ , the aggregate expected utility is  $\frac{1}{c^1}u_1$ . While if they wait, the expected utility in the last period will be lower. Bank runs that happen under such a circumstance is not undesirable as they mitigate future losses. As bank runs serve as a valuable “option”, the uninformed depositors with the belief that is slightly lower than  $\hat{p}$  are still willing to wait at next stage  $N - 1$ , even though they are aware of the positive probability of bank runs. The expected utility at stage  $N - 1$  given the posterior of  $\hat{p}$  is thus raised above  $u_1$ . By backward induction, the cutoff probabilities are lower than  $\hat{p}$  for any stage before  $N$ .

Two possible and interesting results associated with the “low cutoff probability” condition are (1) non-monotonicity of the expected utility in belief, and (2) non-uniqueness of the cutoff probabilities.

#### **Non-monotonicity of the expected utility in belief:**

As early liquidation can help mitigate future losses, the economy in which information

has a chance to be revealed can do better than the economy without information. From figure 2, we can see that  $w_n^U(p)$  is above  $\alpha u_1 + (1 - \alpha)[p\bar{u}_2 + (1 - p)\underline{u}_2]$ , which is the expected utility in an economy with no information about production, for some  $p$ .

Because information about production is valuable, and a herd of non-withdrawals suppresses the inference of private information, a higher belief does not necessarily result in a higher expected utility. There are two opposite forces behind the expected utility: A higher belief brings more confidence in production. However, an economy with a higher belief also reaches a herd of non-withdrawals faster, where no information will be available since then. Whether the expected utility increases in belief depends on the strength of the two forces<sup>7</sup>.

The non-monotonicity of the expected utility function in herding has not been paid attention in the literature. In the literature, herding is usually treated as a partial equilibrium problem, in which the cutoffs are determined by the assumption of parameters. An agent's 0-1 decision either perfectly reveals the signal received, or both decisions carry the same amount of noises. Given an initial prior, only a few crucial probability levels (1 and 2 signals above and below the initial prior) are needed to prove the equilibrium. In the banking set-up with one-side signal extraction problem, the belief updated by observing a non-withdrawal is not completely offset by a withdrawal. The number of possible posteriors is increasing geometrically in each stage. Therefore, a general description of the expected utility function on the full domain of beliefs becomes necessary. Also, the cutoff probabilities vary with the contract. In order to calculate the optimal contract, the value of the expected utility given any parameters (in particular,  $c^1$  and  $\lambda$ ) needs to be determined.

Then why the expected utility function is always increasing in beliefs when the “high cutoff probability” condition holds? Note that the back-up option here is a bank run. Unlike a safe asset in an investment herding problem, a bank run is costly as some depositors are not paid. If the welfare cost is too high, a bank run is no longer a “safety net”. The “high cutoff probability” condition is a sufficient condition for a bank run to

---

<sup>7</sup>The monotonicity is guaranteed for  $w_N^U$  and  $w_{N-1}^U$ .

be too costly. The uncertainty of having a bank run lowers the expected utility. A higher belief not only stands for a higher expected return, but it also means a lower probability of having a costly bank run. Because an earlier stage faces more future history paths, and the paths are gradually ruled out throughout period 1, the uncertainty is smaller at a later stage than at an earlier stage. The cutoff belief is thus decreasing in  $n$ .

Note that the “high/low cutoff probability” condition only compares the expected utility at stage  $N$  given the belief of  $P_{\bar{L}}(\hat{p})$  if there is no run with the expected utility in a bank run. It is a condition that relies on backward induction to decide whether the cutoff probabilities at stages before  $N$  are higher or lower than  $\hat{p}$ . It is not the necessary and sufficient condition for the monotonicity of the expected utility function.

**Non-uniqueness of the cutoff probabilities:**

As the monotonicity of expected utility is not guaranteed, our next question is whether the cutoff probability  $\tilde{p}_n$  is unique. In fact, the uniqueness of the cutoff probabilities is no longer assured<sup>8</sup>. Figure 3 shows an example.

**Example 3:** An example of non-uniqueness of the cutoff probabilities:

$$u(c) = \frac{(c+b)^{1-\gamma} - b^{1-\gamma}}{1-\gamma}, \quad b = 0.01, \quad \gamma = 1.5. \quad \bar{R} = 2.07, \quad \underline{R} = 0, \quad p_0 = 0.9. \quad q = 0.7. \quad \alpha = 0.25.$$

Let  $c^1 = 1.011$ , and  $\lambda = \alpha c^1 = 0.2528$ .  $\bar{u}_2 = 18.6107$ ,  $\underline{u}_2 = 0$ ,  $u_1 = 18.0207$ .  $\hat{p} = 0.9683$ .  $\bar{p} = 0.9862$ .

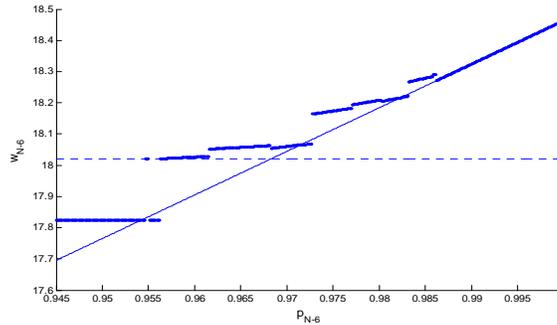


Figure 3: An Example of Non-uniqueness of the Cutoff Probabilities

<sup>8</sup>It is guaranteed for  $w_N^U$ ,  $w_{N-1}^U$ , and  $w_{N-2}^U$ .

Figure 3 shows the expected utility of an uninformed depositor at the stage of  $N - 6$ . There are two cutoffs at stage  $N - 6$ , 0.9546 and 0.9562. If the posterior at stage  $N - 6$  falls below (including) 0.9546 or between (including) 0.9551 and (excluding) 0.9562, the uninformed depositors will run on the bank.

Non-uniqueness of the cutoff beliefs results from payment inter-dependence. In an investment herding problem with no payment dependence, an investor's expected utility is always higher than the return on safe asset as the safe asset is always available and its value is constant. Therefore, the cutoff belief is the lowest level of belief given which information is still able to be revealed. It is always unique. Here in the banking setup, the value of the option to withdraw decreases when all depositors exercise it. An individual depositor compares his expected utility with  $u_1$ , while his expected utility in a bank run is actually  $\frac{1}{c}u_1$ . The cutoff level of his expected utility is higher than the realized value of his option to withdraw. When the expected utility is low, an individual depositor prefers to use his option to withdraw before all others do so (although all others do the same things) rather than wait for more information. As the expected utility does not necessarily increase in belief, there can be more than one cutoff beliefs. A bank run can happen given a relatively higher belief instead of a lower one.

Non-uniqueness of the cutoff beliefs implies the following: Given the same contract, an economy that starts with higher initial prior  $p_0$  can be more vulnerable to bank runs than the one with lower initial prior. A bank run may be triggered by fewer withdrawals in the economy with a higher level of belief than with a lower level of belief. This is because an economy with higher initial prior has higher probability to reach a herd of non-withdrawals, thus has less chance to reveal information. In example 3, uninformed depositors with belief of  $p_{N-7}^U = 0.9727$  ( $P_{\bar{L}}(0.9727) = 0.9562$ ) run on the bank if a withdrawal is observed at stage  $N - 6$ . While if their belief is  $p_{N-7}^U = 0.9717$  ( $P_{\bar{L}}(0.9717) = 0.9547$ ), they prefer to wait.

A question associated with the non-uniqueness is whether it is possible that a shorter queue can encourage a bank run more than a longer queue given the same parameters but different sequences of signals. To formalize the question, suppose  $w_n^U(p^1) \geq u_1$ , while

$w_n^U(p^2) < u_1$ , and  $\underline{p} \leq p^1 < p^2 < \hat{p}$ . Is it possible that  $p^1$  results from more observed withdrawals than  $p^2$ ? The answer is no. Suppose the economy observes  $m$  withdrawals up to stage  $n$  to reach  $p^1$ , while it takes  $m - 1$  withdrawals up to stage  $n$  to reach  $p^2$ . We have  $p^1 = P_L P_{\bar{L}}(p^2)$ . As  $p^2 < \hat{p}$ ,  $p^1 < P_L(\hat{p}) = \underline{p}$ . It contradicts the assumption that  $p^1$  is above  $\underline{p}$ . Therefore, in the equilibrium, a longer queue always implies that low productivity is more likely, and it encourages people to run on the bank.

Without the uniqueness of the cutoffs, it is difficult to describe generally the sequences that can trigger a bank run. Two non-withdrawals in a row will definitely trigger a herd of non-withdrawals. As a decision of withdrawal conveys noisy information about the signal received, it does not offset a decision of non-withdrawal completely. Hence, sequences such as  $(0, 1, 0)$ ,  $(0, 1, 1, 0)$ ,  $(0, 1, 1, 0, 1, 0)$  also can lead to a herd of non-withdrawals, depending on the parameters of the economy.

In summary, the following will be observed in the equilibrium: A newly informed depositor follows his productivity signals until his belief is above  $\bar{p}$ . If many informed depositors do not withdraw, the beliefs of the uninformed depositors will be raised above  $\bar{p}$ , and a herd of non-withdrawals will start. In the opposite case, if many people withdraw, all other depositors will demand their deposits back. In a situation that the uninformed depositors observe neither too many withdrawals nor too many non-withdrawals, they will watch the line closely. Their beliefs will be updated by the decisions of the newly informed depositor.

The equilibrium proved in proposition 1 is not unique. For example, there can be equilibria in which at the first few stages, the newly informed depositors adopt the strategies described in proposition 1. But from stage  $m$  ( $1 < m \leq N$ ) on, the newly informed depositors always wait for the last stage to make their decisions. Because  $w_n^U(p_n^U)$  changes with the strategies adopted, it is difficult to exhaust all possible equilibria. However, as the purpose of this paper is to illustrate how people make withdrawal decisions based on the observed withdrawals by others, I assume that depositors only play the equilibrium strategies in proposition 1 in the post-deposit game.

### 3.2 Equilibrium Given $c^1 < 1$

When  $c^1 < 1$ , there does not exist a symmetric pure strategy run equilibrium at any stage, because given all others withdraw from the bank, an individual depositor prefers to wait to get all the remaining resources, which is expected to be an infinite amount. Not withdrawing before stage  $N + 1$  is a patient depositor's weakly dominant strategy regardless of all other depositors' actions and signals. Therefore, given  $c^1 < 1$ , I assume all patient depositors always wait until stage  $N + 1$  to make decisions according to their belief and consumption type. Because no information can be inferred from the decisions of the newly informed depositors, and because the measure of depositors who are informed before the last stage is 0, bank run does not occur.

There exist multiple symmetric pure strategy no-run equilibria given  $c^1 < 1$ . In order to allow for a symmetric pure strategy no-run equilibrium given any possible sequence of signals, the uninformed depositors' belief should be always above  $\hat{p}$  at any stage.

Let  $Z$  solve

$$P_L^Z(p_0) \geq \hat{p}, P_L^{Z+1}(p_0) < \hat{p}.$$

A symmetric pure strategy no-run equilibrium requires that no more than  $Z$  newly informed depositors follow their signals at the first  $Z$  stages. The rest of the depositors will delay their withdrawals until stage  $N + 1$ . If  $P_L^Z(p_0) < \hat{p}$ , all informed depositors will be waiting until stage  $N + 1$  unless they are impatient.

Construct the expected utility of an uninformed depositor in a no-run equilibrium at each stage by backward induction in the similar way of constructing (2) – (3). We will find that  $w_n^U(p_0) = w_N^U(p_0)$  for any  $n$ . In terms of social welfare, the equilibrium is equivalent to the one in the economy without information about production. This is because information is never revealed to the valuable point in this equilibrium. The *ex-ante* welfare is the same in all of the symmetric pure strategy equilibria given the same contract.

## 4 Pre-deposit Game

Once the equilibrium in the post-deposit game is proved, the probability of having a bank run given a contract is determined. Questions remain are (1) Knowing the probability of bank runs in any possible situation, what is the optimal contract that a competitive bank will provide? (2) Is the optimal contract individually rational (is it better than autarky and accepted by the depositors *ex ante*)? Peck and Shell (2003) show that the *ex-ante* acceptable optimal contract can tolerate panic-based bank runs if the probability of runs is small enough, and that bank runs are equilibrium phenomena. In this section, I will follow their logic to illustrate that the optimal demand-deposit contract can permit herding runs.

In the static bank-runs model, a feasible contract should at least satisfy the participation incentive compatibility constraint, which says given all other patient depositors do not withdraw the deposits, an individual patient depositor prefers to wait. In the dynamic setup, a bank run can happen at any stage, but a feasible contract should at least give depositors the incentive to wait before anyone gets a signal. The participation incentive compatibility constraint is

$$w_0^U(p_0) \geq u_1. \quad (14)$$

The participation incentive compatibility constraint in the traditional Diamond-Dybvig model is a special case here with  $N = 0$  and  $p = 1$ .

The bank chooses a contract to offer. There are two types of contracts available to the bank: run-proof contracts and run-admitting contracts. A run-proof contract guarantees that whichever signals are sent in the post-deposit game, the expected utility of the uninformed depositors never fall below the threshold at any stage.

### 4.1 Run-Proof Contracts

A run-proof contract is in one of the three cases in my model:

**Case 1:** A contract that provides  $c^1 < 1$ .

**Case 2:**  $c^1 \geq 1$ , and

$$P_L(p_0) \bar{u}_2 + (1 - P_L(p_0)) \underline{u}_2 \geq u_1. \quad (15)$$

That is, the initial belief is already above  $\bar{p}$ . A herd of non-withdrawals has already begun before anyone gets signals. The uninformed depositors never update their beliefs by the observed actions. If (17) is satisfied, we have  $p_n^U = p_0 > \tilde{p}_0 > \tilde{p}_n$  for any  $n$ .

**Case 3:**  $c^1 \geq 1$ , and

$$P_L(p_0) \bar{u}_2 + (1 - P_L(p_0)) \underline{u}_2 < u_1, \quad (16)$$

$$w_n^U(P_{\tilde{L}}^n(p_0)) \geq u_1 \forall 0 \leq n \leq N. \quad (17)$$

That is, the newly informed depositors withdraw if low signals are received. However, because there are too few stages and/or because the probability of being impatient is relatively high, the beliefs of the uninformed depositors are still above the thresholds even though the beliefs are updated by  $P_{\tilde{L}}(\cdot)$  at every stage. Note that if (17) holds,  $w_n(p_n) = \alpha u_1 + (1 - \alpha) [p_n \bar{u}_2 + (1 - p_n) \underline{u}_2]$  for any  $0 \leq n \leq N + 1$  and for any  $p_n$  derived from  $p_0$ . Therefore, (17) can be re-written as

$$P_{\tilde{L}}^N(p_0) \geq \hat{p}. \quad (17')$$

Given a run-proof contract,  $w_0(p_0) = \alpha u_1 + (1 - \alpha) [p_0 \bar{u}_2 + (1 - p_0) \underline{u}_2]$ . The best run-proof contract solves

$$\begin{aligned} \max_{c^1, \lambda} w_0(p_0) &= \alpha u_1 + (1 - \alpha) [p_0 \bar{u}_2 + (1 - p_0) \underline{u}_2] \\ \text{s.t. } c^1 &< 1, && \text{or} \\ c^1 &\geq 1 \text{ and (14) - (15)}, && \text{or} \\ c^1 &\geq 1, \text{ (14), and (16) - (17)}. \end{aligned}$$

## 4.2 Run-Admitting Contracts ( $N = 2$ )

A run-admitting contract admits a herd of withdrawals because  $w_n^U(p_n^U) < u_1$  at at least one stage for some realization of  $p_n^U$  derived from  $p_0$ . The *ex-ante* probability of having a bank run given a contract can be calculated by checking how likely  $w_n^U(p_n^U)$  will be

lower than  $u_1$  at each stage. With the “high cutoff probability” condition, the probability of bank runs given a contract is determined by the probability of getting  $Z_n$  number of withdrawals in a row up to stage  $n$ .  $Z_n$  is determined by the cutoff probabilities, and the cutoffs are functions of  $c^1$  and  $\lambda$ . If the “high cutoff probability” condition does not hold, it is difficult to write out the general rules of calculating the probability of bank runs. The stage of which a herd of withdrawals occurs depends not only on the parameters, but also on the random process in which the signals are sent.

In this section, a pre-deposit game of  $N = 2$  is calculated. A more general case can be calculated in the same way. There are five cases for a run-admitting contract for  $N = 2$ , depending on the conditions with which a herd of withdrawals starts. The conditions for each case and the objective function of  $w_0^U(p_0)$  of each case are listed in the appendix.

**Case I:** A herd of non-withdrawals begins if the first informed depositor waits. If the first informed depositor withdraws and the second also withdraws, then a bank run occurs. If the first withdraws and the second waits, the uninformed depositors wait. The probability of bank runs is

$$\sigma_1 = (1 - \pi(p_0)) (1 - \pi(P_{\tilde{L}}(p_0))).$$

**Case II:** A herd of non-withdrawals does not occur if the first informed depositor waits. The second depositor still follows his signals, but the uninformed depositors do not withdraw regardless of the second depositor’s decision. A herd of withdrawals does not occur after the first depositor withdraws. If both the first and the second informed depositors withdraw, then a bank run occurs. The probability of bank runs is  $\sigma_1$ .

**Case III:** A herd of withdrawals begins if the first informed depositor withdraws. If the first informed depositor waits, a herd of non-withdrawals begins. The probability of bank runs is

$$\sigma_2 = 1 - \pi(p_0).$$

**Case IV:** A herd of withdrawals starts if the first informed depositor withdraws. If the first informed depositor waits, the second depositor still follows his signal. However, the uninformed depositors do not withdraw regardless of the second depositor’s decision. The

probability of bank runs is  $\sigma_2$ .

**Case V:** A herd of withdrawals starts if the first informed depositor withdraws. If the first informed depositor waits, the second depositor still follows the signal. The uninformed depositors wait if the second depositor waits, and they withdraw if the second depositor withdraws. The probability of bank runs is

$$\sigma_3 = 1 - \pi(p_0) + \pi(p_0)(1 - \pi(P_H(p_0))).$$

A competitive bank chooses the optimal contract from the classes of run-proof and run-admitting contracts. A run-proof contract is usually associated with lower  $c^1$ . The bank keeps more asset in storage so that the difference between payments in different periods and in different production state is small. A run-admitting contract usually provides higher  $c^1$ . Although  $c^2$  in a run-admitting contract varies more between different production states, when the probability of low productivity is small, investing more in production is more desirable. There are three factors concerning which type of contract to offer. First, because a run-admitting contract usually provides more liquidity to early withdrawals, and the bank invests more in production though it is risky, the contract helps smooth consumptions and allows for higher return in the last period when production is high. This is a positive side of providing a run-admitting contract. Second, a run admitting contract allows depositors to reveal their private information by their decisions. A herding run is partly fundamental driven. It is not necessarily undesirable in an economy with weak fundamentals as it mitigates future losses. It is again a positive side of a run-admitting contract. Thirdly, as the signals and the information extracted from a depositor's action are not perfect, a bank run can happen when fundamentals are strong. This is a negative side of a run-admitting contract. Which contract to provide depends on the overall effects of the three.

The choice among run-admitting contracts also depends on several factors. First, a higher  $c^1$  helps smooth consumptions across types, but it is usually associated with higher probability of bank runs and lower social welfare in bank runs. The second factor is unique to a sequential-move game. The optimal run-admitting contract should allow as much

information as possible to be sensed publicly before any type of herd begins. The first  $N$  depositors can be treated as experiments. The result of each experiment can only be read before herds begin. A careful choice of contract should prolong the effective experiment process as much as possible. High  $c^1$  and low  $c^2$ 's can encourage people to run on the bank, and a bank run can happen too soon.

I compute two examples to illustrate that in some economies a run-admitting contract is optimal, while in other economies a run-proof contract is optimal. I compute the best contract in each of the three run-proof cases and the five run-admitting cases. The optimal contract is “the best of the best”.

In the economy without signals about production, the bank chooses  $c^1$  and  $\lambda$  to maximize  $\alpha u_1 + (1 - \alpha) [p_0 \bar{u}_2 + (1 - p_0) \underline{u}_2]$ , subject to the incentive compatibility constraint  $p_0 \bar{u}_2 + (1 - p_0) \underline{u}_2 \geq u_1$ . If even given the optimal demand deposit contract herding runs are undesirable, the bank may want to use a “curtain” to prevent depositors from seeing each others’ actions. From the examples below, we will see that information can improve *ex-ante* welfare.

An individual depositor’s expected utility in autarky is  $u(1)$ . If the optimal banking contract is accepted *ex-ante*,  $w_0^U(p_0)$  must be at least equal to  $u(1)$ .

### 4.3 Computed Examples

Parameters and functions used in examples 4 and 5 are  $u(c) = \frac{(c+b)^{1-\gamma} - b^{1-\gamma}}{1-\gamma}$ ,  $b = 0.001$ ,  $\gamma = 1.01$ .  $\bar{R} = 1.5$ ,  $\underline{R} = 0.2$ ,  $p_0 = 0.99$ .  $q = 0.99$ .

**Example 4:**  $\alpha = 0.01$ .

Table 1: Optimal Contract - Example 4

	$\sigma$	$c^1$	$\lambda$	$w_0(p_0)$
Autarky	0	1.0000	1	7.1529
Banking economy without info	0	1.0001	0.0100	7.5332
Best run-proof contract in case 1	0	1.0000	0.0100	7.5332
Best run-proof contract in case 2	0	1.0000	1.0000	7.1529
Best run-proof contract in case 3	0	1.0000	1.0000	7.1529
Best run-admitting contract in case I	0.0102	1.0000	0.0100	7.5487*
Best run-admitting contract in case II	0.0102	1.0000	1.0000	7.1529
Best run-admitting contract in case III	0.0296	1.0876	0.0109	7.5263
Best run-admitting contract in case IV	0.0296	1.0000	1.0000	7.1529
Best run-admitting contract in case V	0.0490	1.4868	0.0149	7.4310

Note that the best run-proof contract in case 1 provides  $c^1$  less than, but very close to 1. A run-proof contract is not the best in this example mainly because it does not induce depositors to reveal the signals they received. The economy cannot benefit from the available information about productivity. This is also the reason why the economy with information about production can achieve higher *ex-ante* welfare than the economy without information.

**Example 5:**  $\alpha = 0.2$ .

Table 2: Optimal Contract - Example 5

	$\sigma$	$c^1$	$\lambda$	$w_0(p_0)$
Autarky	0	1	1	7.1529
Banking economy without info	0	1.0028	0.2006	7.4602
Best run-proof contract in case 1	0	1.0000	0.2000	7.4602*
Best run-proof contract in case 2	0	1.0000	1.0000	7.1529
Best run-proof contract in case 3	0	1.0028	0.2006	7.4602*
Best run-admitting contract in case I	0.0527	1.0213	0.2043	7.4523
Best run-admitting contract in case II	0.0527	1.0000	1.0000	7.1529
Best run-admitting contract in case III	0.2158	1.1047	0.2209	7.2785
Best run-admitting contract in case IV	0.2158	1.0000	1.0000	7.1529
Best run-admitting contract in case V	0.3790	1.0000	1.0000	7.1529

In this example, a run-proof contract is optimal. The increase in  $\alpha$  adds more noise to the informed depositors' withdrawal decisions. If it is a run-admitting contract, the probability of bank runs is increased because the probability of observing informed depositors withdraw is raised. In addition, as there are more impatient depositors in the economy, the payments to the depositors in period 1 are decreased due to the resource constraint,

which leaves more room for using a run-proof contract. In this example, a run-admitting contract is not desirable as bank runs happen too frequently when the fundamentals are strong.

Green and Lin (2003 (*a, b*)) provide a model in which depositors make decisions whether to withdraw in sequence, although the depositors do not observe the line nor the decisions by others. They show that there exists an optimal banking contract which completely eliminates panic-based bank runs. My paper discusses bank runs given a demand-deposit contract. It does not seek a banking mechanism that eliminates herding runs. A demand-deposit contract with sequential service is widely used in the banking industry<sup>9</sup>. It is worthwhile as the first attempt to explain the queuing process given a contract in a narrow class of banking mechanism such as a simple demand-deposit contract.

A crucial difference between Green and Lin's economy and my economy is that there is no production uncertainty in Green and Lin's economy. Green and Lin's mechanism induces the depositors to tell their private information – their consumption type – truthfully by their decisions. While in this model there are two dimensions of uncertainty. The 0-1 withdrawal decision cannot fully reveal the private information that a depositor has. Thus, there exists information asymmetry between the bank and depositors. Even if the bank is allowed to provide a contract that offers payments contingent on withdrawal history, it may not be able to eliminate bank runs. In a different paper, I show that in a two-depositor, two-stage economy with partial suspension of convertibility in the sense of Wallace (1988, 1990), a run-admitting contract can be optimal. However, those are still open questions whether a run-admitting contract can be optimal in a multi-depositor, multi-stage economy with partial suspension of convertibility, and whether there exists an optimal banking mechanism that eliminates both panic-runs and fundamental-runs.

---

<sup>9</sup>Calomiris and Kahn (1991) show that demand-deposit contract is efficient if a bank's moral hazard problem potentially exists. Since bank runs are costly, depositors are motivated to monitor the bank and the moral hazard problem will be reduced.

## 5 Conclusion

This paper provides a model for studying detailed dynamics in bank runs. In an economy with uncertainty in production, a line in front of a bank carries information about the production status. The formation of a line outside a bank can persuade others to join the line. In my model, a depositor makes withdrawal decision according to his observation of the withdrawal histories of the others as well as his private information about the bank fundamentals. Given a simple demand-deposit contract, there is a perfect Bayesian equilibrium in which depositors withdraw deposits too many withdrawals are observed, and wait otherwise. In some economies, the simple demand-deposit contract allowing for herding runs is optimal because it achieves higher risk-sharing among depositors and/or allows private information about production to be revealed.

There is some literature on bank runs that is closely related to this paper. Goldstein and Pauzner (2005) construct a model in which depositors receive i.i.d signals on fundamentals and determine whether to run on the bank simultaneously. Chen's (1999) explains contagious bank runs using information externality. Chari and Jagannathan (1988) analyze an economy with random productivity. Some depositors are informed of the productivity status and others are not. The uninformed depositors infer information on productivity by observing the aggregate withdrawals rate. There is a rational expectation equilibrium in the model which allows for bank runs. However, Chari and Jagannathan adopt a static equilibrium concept. The bank in their model does not have intrinsic role in the economy. The cost of bank runs is imposed exogenously. Long-run payments do not depend on whether bank runs occur in the short run. My paper addresses these problems, and emphasizes the welfare aspect of herding runs.

In the present paper, the bank has no information advantage over the majority, which is not quite true in reality. In a more complicated model in which the bank receives signals about productivity, there arise problems such as how to eliminate bank's moral hazard problem due to the information asymmetry between the bank and the depositors, and how the bank reduces the probability of bank runs due to the misleading signals. This can be the extensions to the paper.

Allowing payments to vary with the evolution of history will give the bank more flexibility and will achieve higher social welfare (Wallace (1988, 1990)). Is there a more general banking mechanism, for example, a mechanism induces people to report truthfully about the signals, achieving a more efficient allocation? An efficient banking mechanism should not only allow the bank to provide a contract depending on the withdrawal history, but also eliminate asymmetric information between the bank and the depositors as much as possible. To find a more efficient mechanism in the economy with uncertainties in both production and consumption is another extension of this paper, and more policy implications can be derived from the finding of such a mechanism.

## References

- [1] Allen, Franklin, and Gale, Douglas, “Limited Market Participation and Volatility of Asset Prices”, *American Economic Review*, vol. 84, no. 4 (1994): 933-955.
- [2] Banerjee, Abhijit V., “A Simple Model of Herd Behavior”, *Quarterly Journal of Economics*, 107 (1992): 797-817.
- [3] Bikhchandani, Sushil, Hirshleifer, David, and Welch, Ivo, "A Theory of Fads, Fashion, Custom, and Cultural Change as Informational Cascades", *Journal of Political Economy*, 100, 5, (1992): 992-1026.
- [4] Brunnermeier, Markus, “Asset Pricing under Asymmetric Information-Bubbles, Crashes, Technical Analysis, and Herding”, *Oxford University Press*, 2001.
- [5] Calomiris, Charles and Kahn, Charles, “The Role of Demandable Debt in Structuring Optimal Banking Arrangements”, *American Economic Review*, vol. 81, no. 3 (1991): 497-513.
- [6] Chari, V.V. and Jagannathan, Ravi, “Banking Panics, Information, and Rational Expectations Equilibrium”, *Journal of Finance*, vol. XLIII, No. 3, (1988): 749-760.

- [7] Chari, V.V. and Kehoe, Patrick, “Hot Money”, *Journal of Political Economy*, vol. 11, no. 6, (2003): 1262-1292.
- [8] ——. “Financial Crises and Herds: Overturning the Critiques”, *Journal of Economic Theory*, 119, (2004): 128-150.
- [9] Chen, Yehning, “Banking Panics: The Role of the First-Come, First-Served Rule and Information Externalities”, *Journal of Political Economy*, vol. 107, no. 5 (1999): 946-968.
- [10] Diamond, Douglas and Dybvig, Philip, “Bank Runs, Deposit Insurance, and Liquidity”, *Journal of Political Economy*, 91 (1983): 401-19.
- [11] Goldstein, Itay. and Pauzner, Ady., “Demand-Deposit Contracts and the Probability of Bank Runs.” *Journal of Finance*, Vol. LX, No.3 (2005): 1293-1327.
- [12] Green, Edward J., and Lin, Ping. “Diamond and Dybvig’s Classic Theory of Financial Intermediation: What’s Missing?” *Fed Reserve Bank Minneapolis Q. Rev.* 24 (Winter 2000): 3-13. (a)
- [13] ——. “Implementing Efficient Allocations in a Model of Financial Intermediation”, *Journal of Economic Theory*, 109 (2003): 1-23. (b)
- [14] Peck, James, and Shell, Karl, “Equilibrium Bank Runs”, *Journal of Political Economy*, vol. 111, no.1 (2003): 103-123.
- [15] Schumacher, Liliana, “Bank Runs and Currency Run in a System without a Safety Net: Argentina and the ‘Tequila’ Shock”, *Journal of Monetary Economics* 46 (2000): 257-277.
- [16] Starr, Martha, and Yilmaz, Rasim, “Bank Runs in Emerging-Market Economies: Evidence from Turkey’s Special Finance Houses”, *Southern Economic Journal*, Vol. 73 Issue 4, (April 2007): 1112-1132.

- [17] Wallace, Neil, “Another Attempt to Explain an Illiquid Banking System: The Diamond and Dybvig Model with Sequential Service Taken Seriously”, *Fed Reserve Bank Minneapolis Q. Rev.* 12 (Fall 1988): 3-16.
- [18] ——. “A Banking Model in Which Partial Suspension is Best”, *Fed Reserve Bank Minneapolis Q. Rev.* 14 (Fall 1990): 3-16.

## 6 Appendix

### 6.1 Proofs of Lemmas 1- 3 and Collarary 1

**Lemma 1** Consider a contract that pays  $c^1 \geq 1$  and satisfies the “high cutoff probability” condition. For each stage  $0 \leq n \leq N$ ,  $w_n^U(p)$  is increasing in  $p$ . There exists a unique cutoff probability  $\tilde{p}_n$  such that  $w_n^U(p) \geq u_1$  for  $p \in [\tilde{p}_n, 1]$ , and  $w_n^U(p) = \frac{1}{c^1}u_1$  for  $p \in [0, \tilde{p}_n)$ .  $\tilde{p}_n$  is decreasing in  $n$ .  $w_n^U(p) \leq \alpha u_1 + (1 - \alpha)[p\bar{u}_2 + (1 - p)\underline{u}_2]$  for  $p \in [\tilde{p}_n, 1]$ .

**Proof.** If  $\hat{p} \leq 0$ , we have  $u_1 \leq \underline{u}_2$ , and  $p\bar{u}_2 + (1 - p)\underline{u}_2 \geq u_1$  for any  $p \in [0, 1]$ . Hence,  $w_n^U(p_n) = \alpha u_1 + (1 - \alpha)[p_n\bar{u}_2 + (1 - p_n)\underline{u}_2] \geq u_1$  for  $p_n \in [0, 1]$ .  $\tilde{p}_n = 0$  for all  $n$ .

Same argument applies to  $\hat{p} \geq 1$ .  $w_n^U(p) = \frac{1}{c^1}u_1$  on  $p_n \in [0, 1]$ .  $\tilde{p}_n = 1$  for all  $n$ .

Let us discuss the case when  $\hat{p} \in (0, 1)$ .

$w_N^U(p)$  is increasing in  $p$  by its definition. It has a unique cutoff probability is  $\hat{p}$ . For  $N - 1$ ,  $w_{N-1}^U(p) = \alpha u_1 + (1 - \alpha)[p\bar{u}_2 + (1 - p)\underline{u}_2]$  for  $p \geq \bar{p}$  by definition. Check  $\pi(p)w_N^U(P_H(p)) + (1 - \pi(p))w_N^U(P_L(p))$  for  $p < \bar{p}$ .

Because  $w_N^U(p)$  is increasing in  $p$ ,  $\pi(p)w_N^U(P_H(p)) + (1 - \pi(p))w_N^U(P_L(p))$  is also increasing in  $p$  for  $p < \bar{p}$ .  $\lim_{p \rightarrow \bar{p}} \pi(p)w_N^U(P_H(p)) + (1 - \pi(p))w_N^U(P_L(p)) = \alpha u_1 + (1 - \alpha)[\bar{p}\bar{u}_2 + (1 - \bar{p})\underline{u}_2]$ . Hence  $w_{N-1}^U(p)$  is increasing on  $[0, 1]$ , and a unique cutoff probability  $\tilde{p}_{N-1}$  can be found.

Let  $P_{\tilde{H}}(p)$  be the inverse function of  $P_L(p)$ .  $w_{N-1}^U(p) = \alpha u_1 + (1 - \alpha)[p\bar{u}_2 + (1 - p)\underline{u}_2]$  for  $p \geq P_{\tilde{H}}(\hat{p})$ .

If

$$\begin{aligned} & \pi(P_{\tilde{H}}(\hat{p})) \{ \alpha u_1 + (1 - \alpha)[P_H(P_{\tilde{H}}(\hat{p}))\bar{u}_2 + (1 - P_H(P_{\tilde{H}}(\hat{p})))\underline{u}_2] \} + \\ & + (1 - \pi(P_{\tilde{H}}(\hat{p}))) \frac{1}{c^1} u_1 \\ & < u_1, \end{aligned}$$

then  $\hat{p} < \tilde{p}_{N-1} = P_{\tilde{H}}(\hat{p}) < \bar{p}$ .  $w_{N-1}^U(p) = \alpha u_1 + (1 - \alpha)[p\bar{u}_2 + (1 - p)\underline{u}_2] \geq u_1$  for  $p \geq \tilde{p}_{N-1}$ .

If the inequality does not hold, a unique cutoff  $\tilde{p}_{N-1} < P_{\tilde{H}}(\hat{p})$  can be found to solve

$$\begin{aligned} & \pi(\tilde{p}_{N-1}) \{ \alpha u_1 + (1 - \alpha) [P_H(\tilde{p}_{N-1}) \bar{u}_2 + (1 - P_H(\tilde{p}_{N-1})) \underline{u}_2] \} + \\ & + (1 - \pi(\tilde{p}_{N-1})) \frac{1}{c^1} u_1 \\ = & u_1, \end{aligned}$$

by the continuity and the monotonicity of the above function in  $p$ . By “high cutoff probability” condition,  $\hat{p} < \tilde{p}_{N-1} < P_{\tilde{H}}(\hat{p}) < \bar{p}$ . Also by the “high cutoff probability” condition,  $w_{N-1}^U(P_{\tilde{L}}(p)) = \frac{1}{c^1} u_1 < \alpha u_1 + (1 - \alpha) [P_{\tilde{L}}(p) \bar{u}_2 + (1 - P_{\tilde{L}}(p)) \underline{u}_2]$  for  $p \in [\tilde{p}_{N-1}, P_{\tilde{H}}(\hat{p})]$ . Therefore,  $w_{N-1}^U(p) \leq \alpha u_1 + (1 - \alpha) [p \bar{u}_2 + (1 - p) \underline{u}_2]$  on  $[\tilde{p}_{N-1}, 1]$ .

Prove the rest by induction.

Suppose it is true for every stage up to stage  $n + 1$  that (1)  $w_{n+1}^U(p)$  is increasing in  $p$ . (2)  $\hat{p} < \tilde{p}_{n+2} \leq \tilde{p}_{n+1} \leq \bar{p}$ . If  $w_{n+1}^U(\tilde{p}_{n+1}) > u_1$ ,  $\tilde{p}_{n+1} = \min \left\{ P_{\tilde{H}}^{N-(n+1)}(\hat{p}), \bar{p} \right\}$ ,  $w_{n+1}^U(p) = \alpha u_1 + (1 - \alpha) [p \bar{u}_2 + (1 - p) \underline{u}_2]$  for  $p \geq \tilde{p}_{n+1}$ . If  $w_{n+1}^U(\tilde{p}_{n+1}) = u_1$ ,  $\tilde{p}_{n+1} < \min \left\{ P_{\tilde{H}}^{N-(n+1)}(\hat{p}), \bar{p} \right\}$ .  $w_{n+2}^U(P_{\tilde{L}}(\tilde{p}_{n+1})) = \frac{1}{c^1} u_1$ ; (3)  $w_{n+1}^U(p) \leq \alpha u_1 + (1 - \alpha) [p \bar{u}_2 + (1 - p) \underline{u}_2]$  for  $p \in [\tilde{p}_{n+1}, 1]$ .

Check the properties of  $w_n^U(p)$ :

(i) monotonicity.

$w_n^U(p) = \alpha u_1 + (1 - \alpha) [p \bar{u}_2 + (1 - p) \underline{u}_2]$  for  $p \geq \bar{p}$ . For  $p < \bar{p}$ , as  $w_{n+1}^U(p)$  is increasing in  $p$ ,  $\pi(p) w_{n+1}^U(P_H(p)) + (1 - \pi(p)) w_{n+1}^U(P_{\tilde{L}}(p))$  is also increasing. Check  $\pi(\bar{p}) w_{n+1}^U(P_H(\bar{p})) + (1 - \pi(\bar{p})) w_{n+1}^U(P_{\tilde{L}}(\bar{p}))$ .

If  $P_{\tilde{L}}(\bar{p}) \geq \tilde{p}_{n+1}$ ,

$$u_1 \leq w_{n+1}^U(P_{\tilde{L}}(\bar{p})) \leq \alpha u_1 + (1 - \alpha) [P_{\tilde{L}}(\bar{p}) \bar{u}_2 + (1 - P_{\tilde{L}}(\bar{p})) \underline{u}_2].$$

If  $P_{\tilde{L}}(\bar{p}) < \tilde{p}_{n+1}$ ,

$$w_{n+1}^U(P_{\tilde{L}}(\bar{p})) = \frac{1}{c^1} u_1 < \alpha u_1 + (1 - \alpha) [P_{\tilde{L}}(\bar{p}) \bar{u}_2 + (1 - P_{\tilde{L}}(\bar{p})) \underline{u}_2].$$

Therefore,

$$\pi(\bar{p}) w_{n+1}^U(P_H(\bar{p})) + (1 - \pi(\bar{p})) w_{n+1}^U(P_{\tilde{L}}(\bar{p})) \leq \alpha u_1 + (1 - \alpha) [\bar{p} \bar{u}_2 + (1 - \bar{p}) \underline{u}_2].$$

Hence,  $w_n^U(p)$  is increasing on  $[0, 1]$ , and there is a unique cutoff probability  $\tilde{p}_n$ .

$$(ii) \hat{p} < \tilde{p}_{n+1} \leq \tilde{p}_n \leq \bar{p}.$$

Plug  $\tilde{p}_{n+1}$  into  $\pi(p) w_{n+1}^U(P_H(p)) + (1 - \pi(p)) w_{n+1}^U(P_{\tilde{L}}(p))$ , we have

$$\begin{aligned} & \pi(\tilde{p}_{n+1}) w_{n+1}^U(P_H(\tilde{p}_{n+1})) + (1 - \pi(\tilde{p}_{n+1})) w_{n+1}^U(P_{\tilde{L}}(\tilde{p}_{n+1})) \\ &= \pi(\tilde{p}_{n+1}) \{ \alpha u_1 + (1 - \alpha) [P_H(\tilde{p}_{n+1}) \bar{u}_2 + (1 - P_H(\tilde{p}_{n+1})) \underline{u}_2] \} + \\ & \quad (1 - \pi(\tilde{p}_{n+1})) \frac{1}{c^1} u_1. \end{aligned}$$

(a) If  $w_{n+1}^U(\tilde{p}_{n+1}) = u_1$ ,  $w_n^U(\tilde{p}_{n+1}) = u_1$  because  $\tilde{p}_{n+1}$  solves the same problem. Hence, we have  $\tilde{p}_n = \tilde{p}_{n+1} < \min \left\{ P_{\hat{H}}^{N-(n+1)}(\hat{p}), \bar{p} \right\} \leq \min \left\{ P_{\hat{H}}^{N-n}(\hat{p}), \bar{p} \right\}$ ,  $w_{n+1}^U(P_{\tilde{L}}(\tilde{p}_n)) = w_{n+2}^U(P_{\tilde{L}}(\tilde{p}_{n+1})) = \frac{1}{c^1} u_1$ .

(b) If  $w_{n+1}^U(\tilde{p}_{n+1}) > u_1$ ,  $\tilde{p}_{n+1} = \min \left\{ P_{\hat{H}}^{N-(n+1)}(\hat{p}), \bar{p} \right\}$ . However, it must be true that

$$\begin{aligned} & \pi(\tilde{p}_{n+1}) \{ \alpha u_1 + (1 - \alpha) [P_H(\tilde{p}_{n+1}) \bar{u}_2 + (1 - P_H(\tilde{p}_{n+1})) \underline{u}_2] \} + \\ & \quad (1 - \pi(\tilde{p}_{n+1})) \frac{1}{c^1} u_1 \leq u_1. \end{aligned}$$

If not, we could have found a cutoff that is less than  $\tilde{p}_{n+1}$  for stage  $n + 1$ . Therefore,

$w_n^U(\tilde{p}_{n+1}) \leq u_1$ , and  $\tilde{p}_n \geq \tilde{p}_{n+1}$  by the monotonicity of  $w_{n+1}^U(p)$ .

Discuss  $\tilde{p}_n$  in case (b). At  $p_n = \min \left\{ P_{\hat{H}}^{N-n}(\hat{p}), \bar{p} \right\}$ ,  $w_n^U(p_n) = \alpha u_1 + (1 - \alpha) [p_n \bar{u}_2 + (1 - p_n) \underline{u}_2] > u_1$ . Check

$$\pi(p_n) \{ \alpha u_1 + (1 - \alpha) [P_H(p_n) \bar{u}_2 + (1 - P_H(p_n)) \underline{u}_2] \} + (1 - \pi(p_n)) \frac{1}{c^1} u_1.$$

If it is greater than  $u_1$ , we can find a cutoff of  $\tilde{p}_n$  between  $\left( \tilde{p}_{n+1}, \min \left\{ P_{\hat{H}}^{N-n}(\hat{p}), \bar{p} \right\} \right)$  to satisfy  $w_n^U(\tilde{p}_n) = u_1$ . If it is less than or equal to  $u_1$ ,  $\tilde{p}_n = \min \left\{ P_{\hat{H}}^{N-n}(\hat{p}), \bar{p} \right\}$ .

$$(iii) w_n^U(p) \leq \alpha u_1 + (1 - \alpha) [p \bar{u}_2 + (1 - p) \underline{u}_2] \text{ for } p \in [\tilde{p}_n, 1].$$

$w_n^U(p) = \alpha u_1 + (1 - \alpha)[p\bar{u}_2 + (1 - p)\underline{u}_2]$  for  $p \in [\bar{p}, 1]$ . For  $p \in [\tilde{p}_n, \bar{p})$ ,

$$\begin{aligned}
w_n^U(p) &= \pi(p) w_{n+1}^U(P_H(p)) + (1 - \pi(\tilde{p}_n)) w_{n+1}^U(P_{\tilde{L}}(p)) \\
&= \pi(p) \{ \alpha v_1 + (1 - \alpha)[P_H(p)\bar{u}_2 + (1 - P_H(p))\underline{u}_2] \} + \\
&\quad + (1 - \pi(p_n)) w_{n+1}^U(P_{\tilde{L}}(p)) \\
&\leq \pi(p) \{ \alpha u_1 + (1 - \alpha)[P_H(p)\bar{u}_2 + (1 - P_H(p))\underline{u}_2] \} + \\
&\quad + (1 - \pi(p_n)) \{ \alpha u_1 + (1 - \alpha)[P_{\tilde{L}}(p)\bar{u}_2 + (1 - P_{\tilde{L}}(p))\underline{u}_2] \} \\
&= \alpha u_1 + (1 - \alpha)[p\bar{u}_2 + (1 - p)\underline{u}_2].
\end{aligned}$$

■

**Lemma 2** Consider a contract that satisfies “low cutoff probability” condition.  $w_n^U(p) \geq u_1$  on  $[\hat{p}, 1]$ .

**Proof.**  $w_n^U(p)$  is increasing on  $[\bar{p}, 1]$  by definition.  $w_n^U(p) > u_1$  on  $[\bar{p}, 1]$ . For  $p \in [\hat{p}, \bar{p})$ , we have  $P_H(p) \geq \bar{p}$ . Check  $\pi(p) w_{n+1}^U(P_H(p)) + (1 - \pi(\tilde{p}_n)) w_{n+1}^U(P_{\tilde{L}}(p))$ .

(I) If  $w_{n+1}^U(P_{\tilde{L}}(p)) \geq u_1$ ,

$$\pi(p) w_{n+1}^U(P_H(p)) + (1 - \pi(p)) w_{n+1}^U(P_{\tilde{L}}(p)) > u_1.$$

(II) If  $w_{n+1}^U(P_{\tilde{L}}(p)) = \frac{1}{c^1} u_1$ ,

$$\begin{aligned}
&\pi(p) w_{n+1}^U(P_H(p)) + (1 - \pi(p)) w_{n+1}^U(P_{\tilde{L}}(p)) \\
&= \pi(p) \{ \alpha u_1 + (1 - \alpha)[P_H(p)\bar{u}_2 + (1 - P_H(p))\underline{u}_2] \} + \\
&\quad + (1 - \pi(p)) \frac{1}{c^1} u_1
\end{aligned}$$

is strictly increasing in  $p$  in this case. Because

$$\begin{aligned}
& \pi(\hat{p}) \{ \alpha u_1 + (1 - \alpha) [\bar{p} \bar{u}_2 + (1 - \bar{p}) \underline{u}_2] \} + (1 - \pi(\hat{p})) \frac{1}{c^I} u_1 \\
& \geq \pi(\hat{p}) \{ \alpha u_1 + (1 - \alpha) [\bar{p} \bar{u}_2 + (1 - \bar{p}) \underline{u}_2] \} + (1 - \pi(\hat{p})) \{ \alpha u_1 + (1 - \alpha) [P_{\tilde{L}}(\hat{p}) \bar{u}_2 + (1 - P_{\tilde{L}}(\hat{p})) \underline{u}_2] \} \\
& = \alpha u_1 + (1 - \alpha) [\hat{p} \bar{u}_2 + (1 - \hat{p}) \underline{u}_2] \\
& \geq u_1
\end{aligned}$$

by the “low cutoff probability” condition,  $\pi(p) w_{n+1}^U(P_H(p)) + (1 - \pi(\tilde{p}_n)) w_{n+1}^U(P_{\tilde{L}}(p)) \geq u_1$  for  $p \in [\hat{p}, \bar{p}]$ . In both cases,  $w_n^U(p) \geq u_1$  on  $p \in [\hat{p}, 1]$ . ■

**Collarary 1** Consider a contract that pays  $c^1 \geq 1$ . Given a posterior of  $p$  at stage  $n$ , if

$$w_n^U(p) \geq u_1, \text{ then } w_{n+1}^U(P_H(p)) \geq u_1.$$

**Proof.** It is obvious that Collarary 1 is true if the “high cutoff probability condition” is satisfied. If the “low cutoff probability condition” holds,  $p$  must be greater than or equal to  $\underline{p}$  as  $w_n^U(p) \geq u_1$ .

$$\text{If } p \in [\hat{p}, 1], P_H(p) \geq \bar{p}.$$

$$w_n^U(P_H(p)) = \alpha u_1 + (1 - \alpha) [P_H(p) \bar{u}_2 + (1 - P_H(p)) \underline{u}_2] \geq u_1.$$

$$\text{If } p \in [\underline{p}, \hat{p}], \hat{p} \leq P_H(p) < \bar{p}. \text{ By lemma 2, } w_n^U(P_H(p)) \geq u_1. \quad \blacksquare$$

**Lemma 3** If  $w_n^U(p) \geq u_1$ , then  $w_n^{S_r}(p) \geq u_1$ .

**Proof.** Prove by induction. Show that at each stage, if  $w_n^U(p) \geq u_1$ ,  $w_n^U(p)$  can be written as

$$w_n^U(p) = \alpha [\rho_n(p) u_1 + (1 - \rho_n(p)) \frac{1}{c^I} u_1] + (1 - \alpha) w_n^{S_r}(p),$$

where  $\rho_n(p) \in [0, 1]$ , and  $w_n^{S_r}(p) \geq u_1$ .

Begin with stage  $N$ , if  $w_N^U(p) \geq u_1$ ,

$$\begin{aligned}
w_N^U(p) &= \alpha u_1 + (1 - \alpha) [p\bar{u}_2 + (1 - p)\underline{u}_2] \\
&= \alpha u_1 + (1 - \alpha) w_N^{S_r}(p) \\
&\geq u_1, \text{ so} \\
w_N^{S_r}(p) &= p\bar{u}_2 + (1 - p)\underline{u}_2 \geq u_1, \text{ and} \\
\rho_N &= 1.
\end{aligned}$$

Suppose it is true for every stage up to stage  $n + 1$ . If  $w_{n+1}^U(p) \geq u_1$ , we have

$$w_{n+1}^U(p) = \alpha [\rho_{n+1}(p) u_1 + (1 - \rho_{n+1}(p)) \frac{1}{c^1} u_1] + (1 - \alpha) w_{n+1}^{S_r}(p),$$

where  $\rho_{n+1}(p) \in [0, 1]$ , and  $w_{n+1}^{S_r}(p) \geq u_1$ .

At stage  $n$ , suppose  $w_n^U(p) \geq u_1$ .

If  $p \geq \bar{p}$ ,  $w_n^U(p) = \alpha u_1 + (1 - \alpha) [p\bar{u}_2 + (1 - p)\underline{u}_2] \geq u_1$ .  $w_n^{S_r}(p) = p\bar{u}_2 + (1 - p)\underline{u}_2 \geq u_1$ .  $\rho_n = 1$ .

If  $p < \bar{p}$ ,  $w_n^U(p) = \pi(p) w_{n+1}^U(P_H(p)) + (1 - \pi(p)) w_{n+1}^U(P_{\tilde{L}}(p)) \geq u_1$ . By corollary 1,  $w_{n+1}^U(P_H(p)) \geq u_1$ . Suppose  $w_{n+1}^U(P_{\tilde{L}}(p)) \geq u_1$ . By the assumption at stage  $n$ , we have  $w_{n+1}^{S_r}(P_H(p)) \geq u_1$  and  $w_{n+1}^{S_r}(P_{\tilde{L}}(p)) \geq u_1$ . So  $w_n^{S_r}(p) = \pi(p) w_{n+1}^{S_r}(P_H(p)) + (1 - \pi(p)) w_{n+1}^{S_r}(P_{\tilde{L}}(p)) \geq u_1$ .

$$\begin{aligned}
w_n^U(p) &= \pi(p) w_{n+1}^U(P_H(p)) + (1 - \pi(p)) w_{n+1}^U(P_{\tilde{L}}(p)) \\
&= \pi(p) \left\{ \alpha [\rho_{n+1}(P_H(p)) u_1 + (1 - \rho_{n+1}(P_H(p))) \frac{1}{c^1} u_1] + (1 - \alpha) w_{n+1}^{S_r}(P_H(p)) \right\} + \\
&\quad + (1 - \pi(p)) \left\{ \alpha [\rho_{n+1}(P_{\tilde{L}}(p)) u_1 + (1 - \rho_{n+1}(P_{\tilde{L}}(p))) \frac{1}{c^1} u_1] + (1 - \alpha) w_{n+1}^{S_r}(P_{\tilde{L}}(p)) \right\} \\
&= \alpha [\rho_n(p) u_1 + (1 - \rho_n(p)) \frac{1}{c^1} u_1] + (1 - \alpha) w_{n+1}^{S_r}(p)
\end{aligned}$$

and  $\rho_n = \pi(p) \rho_{n+1}(P_H(p)) + (1 - \pi(p)) \rho_{n+1}(P_{\tilde{L}}(p))$ .

Suppose  $w_{n+1}^U(P_{\tilde{L}}(p)) < u_1$ , so

$$w_{n+1}^U(P_{\tilde{L}}(p)) = \frac{1}{c^1} u_1,$$

and  $w_{n+1}^{S_r}(P_{\tilde{L}}(p)) = \frac{1}{c^1}u_1$  by definition.

$$\begin{aligned}
w_n^U(p) &= \pi(p) w_{n+1}^U(P_H(p)) + (1 - \pi(p))w_{n+1}^U(P_{\tilde{L}}(p)) \\
&= \pi(p) w_{n+1}^U(P_H(p)) + (1 - \pi(p))\frac{1}{c^1}u_1 \\
&= \pi(p) \left\{ \begin{array}{l} \alpha [\rho_{n+1}(P_H(p)) u_1 + (1 - \rho_{n+1}(P_H(p)))\frac{1}{c^1}u_1] + \\ (1 - \alpha) w_{n+1}^{S_r}(P_H(p)) \end{array} \right\} + (1 - \pi(p))\frac{1}{c^1}u_1 \\
&= \alpha [\rho_n(p) u_1 + (1 - \rho_n(p))\frac{1}{c^1}u_1] + \\
&\quad + (1 - \alpha) [\pi(p) w_{n+1}^{S_r}(P_H(p)) + (1 - \pi(p)) w_{n+1}^{S_r}(P_{\tilde{L}}(p))] \\
&\geq u_1, \text{ where } \rho_n(p) = \pi(p) \rho_{n+1}(P_H(p)) \text{ so} \\
w_n^{S_r}(p) &= \pi(p) w_{n+1}^{S_r}(P_H(p)) + (1 - \pi(p))w_{n+1}^{S_r}(P_{\tilde{L}}(p)) \\
&\geq u_1.
\end{aligned}$$

■

## 6.2 Conditions and Objective Functions of Run-Admitting Contracts ( $N = 2$ )

A run-admitting contract should at least satisfy (14) and the following:

$$P_{\tilde{L}}^2(p_0) \bar{u}_2 + (1 - P_{\tilde{L}}^2(p_0)) \underline{u}_2 \leq u_1, \quad (18)$$

$$P_H^2(p_0) \bar{u}_2 + (1 - P_H^2(p_0)) \underline{u}_2 > u_1, \quad (19)$$

(18) and (19) imply

$$\begin{aligned}
w_2^U(P_{\tilde{L}}^2(p_0)) &\leq u_1, \text{ and} \\
w_2^U(P_H^2(p_0)) &> u_1.
\end{aligned}$$

The feasible contract also implies  $w_1^U(P_H(p_0)) > u_1$  by corollary 1. I first list the conditions for all of the possible outcomes after each newly informed depositor's decision is observed.

1. If the first informed depositor waits, a herd of non-withdrawals occurs.

$$P_L P_H(p_0) \bar{u}_2 + (1 - P_L P_H(p_0)) \underline{u}_2 = p_0 \bar{u}_2 + (1 - p_0) \underline{u}_2 \geq u_1. \quad (20)$$

2. If the first informed depositor withdraws, a herd of withdrawals occurs.

$$w_1(P_{\tilde{L}}(p_0)) < u_1.$$

3. If the first informed depositor withdraws, a herd of withdrawals does not occur.

The second depositor follows the signal as  $P_L P_{\tilde{L}}(p_0) \bar{u}_2 + (1 - P_L P_{\tilde{L}}(p_0)) \underline{u}_2 < u_1$ , guaranteed by (18). The uninformed depositors withdraw if the second depositor withdraws, and they wait if the second depositor waits.

$$\begin{aligned} w_1(P_{\tilde{L}}(p_0)) &\geq u_1 \\ w_2(P_H P_{\tilde{L}}(p_0)) &= \alpha u_1 + (1 - \alpha) [P_H P_{\tilde{L}}(p_0) \bar{u}_2 + (1 - P_H P_{\tilde{L}}(p_0)) \underline{u}_2] \geq u_1 \end{aligned} \quad (21)$$

4. If the first informed depositor waits, a herd of non-withdrawals does not occur.

The second depositor follows the signal. The uninformed depositors withdraw if the second depositor withdraws, and they wait if the second depositor waits.

$$p_0 \bar{u}_2 + (1 - p_0) \underline{u}_2 < u_1, \text{ and} \quad (22)$$

$$\alpha u_1 + (1 - \alpha) [P_{\tilde{L}} P_H(p_0) \bar{u}_2 + (1 - P_{\tilde{L}} P_H(p_0)) \underline{u}_2] < u_1. \quad (23)$$

5. If the first informed depositor waits, a herd of non-withdrawals does not occur. The second depositor follows the signal. The uninformed depositors wait regardless of the second depositor's decision. i.e. (21) – (22).

The combinations of the above 5 constitute descriptions of equilibrium outcomes given the contract.

**Case I:** Combine 1 and 3. A herd of non-withdrawals begins if the first informed depositor waits. If the first informed depositor withdraws and the second also withdraws, then a

bank run occurs. If the first withdraws and the second waits, the uninformed depositors wait.

The probability of bank runs is

$$\sigma_1 = (1 - \pi(p_0)) (1 - \pi(P_{\tilde{L}}(p_0))).$$

Equations (18) – (20) are necessarily required for the outcome. The participation incentive constraint is

$$w_0^U(p_0) = \pi(p_0) w_1^U(P_H(p_0)) + (1 - \pi(p_0)) w_1^U(P_{\tilde{L}}(p_0)) \geq u_1 \quad (24)$$

where

$$\begin{aligned} w_1^U(P_{\tilde{L}}(p_0)) &= \pi(P_{\tilde{L}}(p_0)) \{ \alpha u_1 + (1 - \alpha) [P_H P_{\tilde{L}}(p_0) \bar{u}_2 + (1 - P_H P_{\tilde{L}}(p_0)) \underline{u}_2] \} + \\ &\quad (1 - \pi(P_{\tilde{L}}(p_0))) \frac{1}{c^1} u_1 \quad (25) \\ &\geq u_1, \end{aligned}$$

and

$$w_1^U(P_H(p_0)) = \alpha u_1 + (1 - \alpha) [P_H(p_0) \bar{u}_2 + (1 - P_H(p_0)) \underline{u}_2] \geq u_1, \quad (26)$$

which is guaranteed by (20).

The *ex-ante* expected utility maximization problem is

$$\begin{aligned} &\max_{c^1, \lambda} w_0^U(p_0) \\ &s.t. \quad c^1 \geq 1, \quad (18) - (20), (24) - (26). \end{aligned}$$

**Case II:** Combine 3 and 5. A herd of non-withdrawals does not occur if the first informed depositor waits. The second depositor still follows the signal, but the uninformed depositors do not withdraw regardless the second depositor's decision. A herd of withdrawals does not occur after the first depositor withdraws. If the first and the second informed depositors withdraw, then a bank run occurs.

The probability of bank runs is  $\sigma_1$ .

The conditions for the outcome are (18) – (19), (21) – (22), and (24) – (26), where (26) is guaranteed by (21) in this case.

The *ex-ante* expected utility maximization problem is

$$\begin{aligned} & \max_{c^1, \lambda} w_0^U(p_0) \\ & \text{s.t. } c^1 \geq 1, \text{ (18) – (19), (21) – (22), and (24) – (26)}. \end{aligned}$$

**Case III:** Combine 1 and 2. A herd of withdrawals begins if the first informed depositor withdraws. If the first informed depositor waits, a herd of non-withdrawals begins.

The probability of bank runs is

$$\sigma_2 = 1 - \pi(p_0).$$

Equations (18) – (20), and (24) are necessarily required for the outcome. In addition, the participation incentive constraint requires:

$$w_0^U(p_0) = \pi(p_0) w_1^U(P_H(p_0)) + (1 - \pi(p_0)) w_1^U(P_{\tilde{L}}(p_0)) \geq u_1$$

where

$$w_1^U(P_H(p_0)) = \alpha u_1 + (1 - \alpha) [P_H(p_0) \bar{u}_2 + (1 - P_H(p_0)) \underline{u}_2] \geq u_1, \quad (27)$$

is guaranteed by (20), and

$$w_1^U(P_{\tilde{L}}(p_0)) = \frac{1}{c^1} u_1,$$

requires

$$\pi(P_{\tilde{L}}(p_0)) \{ \alpha u_1 + (1 - \alpha) [P_H P_{\tilde{L}}(p_0) \bar{u}_2 + (1 - P_H P_{\tilde{L}}(p_0)) \underline{u}_2] \} + (1 - \pi(P_{\tilde{L}}(p_0))) \frac{1}{c^1} u_1 < u_1. \quad (28)$$

The *ex-ante* expected utility maximization problem is

$$\begin{aligned} & \max_{c^1, \lambda} w_0^U(p_0) \\ & s.t. c^1 \geq 1, \quad (18) - (20), \quad (24), \quad \text{and} \quad (27) - (28). \end{aligned}$$

**Case IV:** Combine 2 and 5. A herd of withdrawals starts if the first informed depositor withdraws. If the first informed depositor waits, the second depositor still follows the signal. However, the uninformed depositors do not withdraw regardless of the second depositor's decision.

The probability of bank runs is  $\sigma_2$ .

The conditions for the outcome are (18) – (19), (21) – (22), (24) and (27) – (28), where (27) is guaranteed by (21).

The *ex-ante* expected utility maximization problem is

$$\begin{aligned} & \max_{c^1, \lambda} w_0(p_0) \\ & s.t. c^1 \geq 1, \quad (18) - (19), \quad (21) - (22), \quad (24), \quad (27) - (28). \end{aligned}$$

**Case V:** Combine 2 and 4. A herd of withdrawals starts if the first informed depositor withdraws. If the first informed depositor waits, the second depositor still follows the signal. The uninformed depositors wait if the second depositor waits, and they withdraw if the second depositor withdraws.

The probability of bank runs is

$$\sigma_3 = 1 - \pi(p_0) + \pi(p_0)(1 - \pi(P_H(p_0)))$$

Equations (14), (18) – (19), and (22) – (24) are necessarily required for the outcome. The participation incentive constraint requires:

$$w_0^U(p_0) = \pi(p_0) w_1^U(P_H(p_0)) + (1 - \pi(p_0)) w_1^U(P_L(p_0)) \geq u_1$$

where

$$\begin{aligned}
w_1^U(P_H(p_0)) &= \pi(P_H(p_0)) \{ \alpha v_1 + (1 - \alpha) [P_H^2(p_0) \bar{u}_2 + (1 - P_H^2(p_0)) \underline{u}_2] \} + (29) \\
&\quad (1 - \pi(P_H(p_0))) \frac{1}{c^1} u_1 \\
&\geq u_1.
\end{aligned}$$

Also,

$$w_1^U(P_L(p_0)) = \frac{1}{c^1} u_1,$$

which is guaranteed by (23).

The *ex-ante* expected utility maximization problem is

$$\begin{aligned}
&\max_{c^1, \lambda} w_0^U(p_0) \\
&s.t. c^1 \geq 1, (18) - (19), (22) - (24), (29).
\end{aligned}$$

### 6.3 An Example of an Economy with Two Depositors

In this section, I will present the model in a 2-depositor, 2-stage version. I follow the setup in section 2 except that there are only two stages. One of the two depositors will be informed about his consumption type as well as the productivity status at the beginning of stage 1, and the other will be informed only about his consumption type at stage 2. Both depositors have equal probability to be the first informed depositor *ex ante*. The 2-depositor, 2-stage setup is the simplest case which allows for herding runs. The deadline for the decision in period 1 is the end of stage 2. Depositor 1 (the depositor who is informed at stage 1) does not have the chance to revise his decision after observing the decision by the other. But he can delay his decision until stage 2. If both depositors are active at stage 2, they will make decisions simultaneously.  $\bar{R} > 1$  and  $\underline{R} < 1$ . For convenience, the signal about production is assumed to be perfect ( $q = 1$ ). As there are only two depositors, there is no need for depositor 2 to make decision before he receives his signal about consumption.

The bank announces the demand-deposit contract which describes the payment to the

depositor who withdraws in period 1,  $c^1$ , and the amount of resource kept in storage,  $\lambda$ . The bank liquidates either all or none of the assets in production, and liquidates the assets only when it cannot meet the payment demands. If  $c^1 > 1$ , the depositor who withdraws second will not receive the full amount of  $c^1$ . So let  $c^1(1)$  and  $c^2(2)$  denote the payment received by depositors who withdraws first and second in period 1, respectively. Let  $c^2(x_1 + x_2, R)$  denote the payment in period 2 conditional on the total withdrawals in period 1 and the realization of production.

To comply with the assumption in section 3, I assume that given  $c^1 < 1$ , depositor 1 always delays his decision until stage 2, and that depositor 2 cannot obtain any information from depositor 1's action at stage 1. Depositors play a simultaneous-move game if both are active at stage 2. I first illustrate the equilibrium given  $c^1 \geq 1$ , then the one given  $c^1 < 1$ .

### 6.3.1 Equilibrium given $c^1 \geq 1$

When  $c^1 \geq 1$ , the equilibrium strategies include: (1) depositor 1's strategy when he receives signals at stage 1; (2) depositor 2's strategy contingent on depositor 1's decision at stage 2.

I begin with depositor 2's strategy at stage 2. At stage 2, depositor 2 has an updated belief  $p_2$ . If he waits, he expects  $p_2 u(c^2(x_1, \overline{R})) + (1 - p_2) u(c^2(x_1, \underline{R}))$ , while if he withdraws, he will get  $u(c^1(x_1 + 1))$ . It is easy to see that there exists a cutoff belief  $\hat{p}_2(x_1)$  above which the depositor waits, below which he withdraws.  $\hat{p}_2$  is contingent on  $x_1$ , as depositor 2's expected payoffs vary with depositor 1's decision.

If a contract specifies  $c^1 = 1$  and  $\lambda = 2$ , depositor 1 does not have the incentive to withdraw if a low signal is received. Except for such a contract, withdrawing immediately is depositor 1's best response regardless of the decision of other depositors if a low signal is received given  $c^1 > 1$ ,

Given  $c^1 \geq 1$ , an acceptable contract must satisfy the following condition: If the

productivity is known to be high, both depositors are willing to wait *ex ante*. That is,

$$\begin{aligned} & \alpha^2 (0.5u(c^1(1)) + 0.5u(c^1(2))) + (1 - \alpha)^2 u(c^2(0, \bar{R})) + \\ & + 2\alpha(1 - \alpha) [0.5u(c^1(1)) + u(0.5c^2(1, \bar{R}))] \geq u(1). \end{aligned} \quad (30)$$

If a high signal is received, depositor 1 will always have the incentive to wait if he can convey the high signal to depositor 2 because

$$\alpha u(c^2(1, \bar{R})) + (1 - \alpha) u(c^2(0, \bar{R})) \geq u(1)$$

by (30).

In this simplest setup, there is a perfect Bayesian equilibrium in the post-deposit game given any contract that provides  $c^1 \geq 1$ . That is,

1. If  $c^1 = 1$  and  $\lambda = 2$ , depositors 1 and 2 withdraw if and only if they are impatient. Depositor 1's belief is updated by the signal received. Depositor 2's belief does not change. This contract results in the same welfare level as in autarky.
2. If  $c^1 > 1$  or  $\lambda \neq 2$ , depositor 1 withdraws if he is impatient and/or a low signal is received, and does not otherwise. Depositor 2 has the updated belief  $P_L(p_0)$  ( $P_H(p_0 = 1)$ ) if depositor 1 withdraws (does not withdraw). Depositor 2 withdraws if he is impatient and/or his updated belief is below  $\hat{p}_2(x_1)$ .

### 6.3.2 Equilibrium given $c^1 < 1$

When  $c^1 < 1$  is provided ( $c^1(1) = c^1(2) = c^1$ ), depositor 1 and 2 play a simultaneous-move game at stage 2 if depositor 1 is still active (patient). Depositor 1 knows the productivity status but does not know depositor 2's type. Depositor 2 does not know the productivity status but knows depositor 1 is patient. In this game at stage 2, there exist Bayesian Nash equilibria. There are four possible equilibrium outcomes, depending on the parameters and contract.

1.  $\alpha u(c^2(1, \underline{R})) + (1 - \alpha) u(c^2(0, \underline{R})) < u(c^1)$  and  $p_0 u(c^2(0, \bar{R})) + (1 - p_0) u(c^2(1, \underline{R})) \geq$

$u(c^1)$ : Depositor 1 withdraws if he has received a low signal, and does not otherwise. Depositor 2 withdraws if he is impatient, and does not otherwise.

2.  $u(c^2(1, \underline{R})) < u(c^1)$  and  $p_0 u(c^2(0, \bar{R})) + (1 - p_0) u(c^2(1, \underline{R})) < u(c^1)$ : Depositor 1 withdraws if he has received a low signal, and does not otherwise. Depositor 2 withdraws.
3.  $u(c^2(1, \underline{R})) \geq u(c^1)$  and  $p_0 u(c^2(0, \bar{R})) + (1 - p_0) u(c^2(0, \underline{R})) < u(c^1)$ : Depositor 1 does not withdraw. Depositor 2 withdraws.
4.  $\alpha u(c^2(1, \underline{R})) + (1 - \alpha) u(c^2(0, \underline{R})) \geq u(c^1)$  and  $p_0 u(c^2(0, \bar{R})) + (1 - p_0) u(c^2(0, \underline{R})) \geq u(c^1)$ : Depositor 1 does not withdraw. Depositor 2 withdraws if he is impatient, and does not withdraws.

Note that there exists multiple equilibria given some parameter values. Also note that depositor 1 always has incentive to wait if he has received a high signal as  $c^1 < 1$  and  $c^2(1, \bar{R}) > 1$ .

At stage 1, depositor 1 withdraws if he is impatient. If depositor 1 has withdrawn, depositor 2 withdraws at stage 2 if  $p_0 u(c^2(1, \bar{R})) + (1 - p_0) u(c^2(1, \underline{R})) < u(c^1)$  and/or he is impatient, and does not otherwise.

### 6.3.3 A Numerical Example

In this example, I will employ the following utility function and parameters:  $u(c) = \frac{(c+b)^{1-\gamma} - b^{1-\gamma}}{1-\gamma}$ ,  $b = 0.001$ ,  $\gamma = 1.01$ ;  $\bar{R} = 1.25$ ,  $\underline{R} = 0.95$ ,  $p_0 = 0.95$ ;  $q = 1$ ;  $\alpha = 0.05$ .

Table 3: Optimal Contract - 2-depositor, 2-stage

	$c^1$	$\lambda$	$w_0(p_0)$
Best contract that provides $c^1 > 1$ or $\lambda \neq 2$	1	0	7.3439*
Contract that provides $c^1 = 1$ and $\lambda = 2$ (Autarky)	1	2	7.1529
Best contract that provides $c^1 < 1$	1.0000	0	7.3439

The contract that provides  $c^1 = 1$  and  $\lambda = 2$  (equivalent to autarky) yields the *ex-ante* expected utility of 7.1529. The optimal contract in this example requires  $c^1 = 1$  and  $\lambda = 0$ . As the liquidity demand is small ( $\alpha$  is small) and the production has high

probability to be successful, the bank invests all resources in production. The *ex-ante* expected utility is 7.3439. Given  $c^1 = 1$  and  $\lambda = 0$ , depositor 1 withdraws at stage 1 if and only if a low signal is received or he is impatient, depositor 2 withdraws at stage 2 if depositor 1 has withdrawn at stage 1 or he is impatient, and does not otherwise. (If depositor 1 has withdrawn, depositor 2 is indifferent between withdrawing immediately at stage 2 and waiting until  $t = 2$ .) When productivity is low, depositor 1's withdrawal forces the bank to liquidate all its assets so depositor 2 also benefits from depositor 1's private information. Of course, if either of the depositors is impatient, the bank has to interrupt production. However, the probability of having a liquidity shock is small enough to be tolerated. The best contract in the category of  $c^1 = 1$  provides  $c^1$  very close to 1, and the bank also invests all resources in production. The *ex-ante* expected utility is very close to 7.3439. Given this contract, depositor 1 withdraws in stage 1 if he is impatient, depositor 2 does not withdraw at stage 1. At stage 2, depositor 1 withdraws if he has received a low signal and wait otherwise. Depositor 2 withdraws if he is impatient and does not otherwise.