

Increasing Outer Risk*

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Abstract

Recent empirical research has established that the distributions of a wide range of economic variables are kurtotic in that they have higher peak(s) in the neighborhood of the mean and greater elongation in the tails than the normal distribution. This paper provides a formal characterization of the empirically significant notions of kurtotic distributions by formulating the concept of outer risk. An increase in outer risk corresponds to a dispersion transfer from the center of a distribution to its tails. In terms of the relocation of probability mass, such a dispersion transfer accentuates the peak(s) of the distribution and elongates its tails. It is shown that ordering distributions by outer risk is equivalent to the ordering of distributions resulting from unanimous choice by all individuals whose utility function has a negative fourth derivative.

Key Words: Outer risk, outer risk aversion

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1. Introduction

In recent years a large empirical literature has investigated the distributional characteristics of a variety of financial and other economic variables. This literature has found that a wide range of distributions are kurtotic, that is, they have higher peaks in the neighborhood of the mean and greater elongation in the tails than the normal distribution.¹ For example, Mills (1995) reports that the returns on three London Stock Exchange FT-SE indexes over the period 1986-1992 are “characterized by highly non-Gaussian behaviour, being both skewed and extremely kurtotic ...”, while Aggarwal et al (1989) find significant and persistent kurtosis in the distribution of equity returns on the Tokyo Stock Exchange. Similar findings have been reported for U.S. stock return distributions by Badrinath and Chatterjee (1988), Campbell and Hentschel (1992), and others. Highly kurtotic distributions pertain not only to stock returns in developed markets, but are also characteristic of emerging equity markets as well as a wide range of other financial and economic data.² In response to these empirical findings, a number of recent econometric studies have used higher-moment distributions to incorporate skewness and kurtosis.³

In this paper we provide a formal characterization of the intuitive and empirically significant notion that one distribution has higher peaks and longer tails (i.e., more kurtotic) than another distribution. We do this by formulating the notion of outer risk to rank distributions with shapes described

¹ In the literature cited below, the normalized fourth central moment is used as a summary measure of kurtosis. While the fourth central moment is widely used, several other measures of kurtosis have been proposed in the statistical literature. Several of them embody the notion that kurtosis is increased by particular movements of probability mass from one portion of the support of a distribution to another portion (see Groeneveld and Meeden, 1984). For a review of the literature concerning the relationship between measures of kurtosis and distributional shape, see Balanda and MacGillivray (1988).

² According to Harris and Kucukozmen (2001), stock returns in the Istanbul Stock Exchange exhibit very significant leptokurtosis. Bekaert et al (1998) provide detailed documentation of the distributions of equity returns in nineteen emerging markets and found that “all but a single country has excess kurtosis in the 1990’s”. High levels of kurtosis have been identified for price change distributions of CPI and PPI in the U.S. (Bryan et al, 1997), for U.S. manufacturing sectoral investment/capital ratios (Caballero and Engel, 1994), for currency returns in ten Asian-Pacific countries (Tang, 1998).

³ For example, Harris and Kucukozmen (2001) employ the exponential generalized beta and the skewed generalized t distributions to study stock returns in the Istanbul Stock Exchange; Hwang and Satchell (1999) use generalized method of moments (GMM) to estimate the incremental value of higher moments in modeling capital asset pricing models (CAPMs) of emerging markets; Corrado and Su (1996) adapt a Gram-Charlier series expansion of the normal density function to provide skewness and kurtosis adjustment terms for the Black-Scholes formula for option prices and find significant skewness and kurtosis in S&P 500 stock index returns.

as kurtotic in the literature. Distribution $G(x)$ has more outer risk than distribution $F(x)$ if $G(x)$ can be obtained from $F(x)$ by transferring dispersion (actuarially neutral noise) from the center of F to its tails without altering its mean, variance and skewness. In terms of the relocation of probability mass, the movement of dispersion from the center to the tails of a distribution accentuates the peak(s) of the distribution and elongates its tails. To provide a choice theoretic foundation for outer risk, we characterize the group of individuals that we would expect to be averse to outer risk. We call an individual outer risk averse if he dislikes greater outer risk and show that an individual is outer risk averse if and only if the fourth derivative of his von Neumann-Morgenstern utility function ($v^{(4)}$) is negative. Ranking distributions in terms of unanimous choice by this group of individuals is shown to be equivalent to ranking them in terms of increasing outer risk.

It is generally assumed in the decision theory literature that individuals are temperate, i.e., $v^{(4)}$ is negative. Temperance has been shown to be important in comparative static analyses. For example, it is a necessary condition for Pratt and Zeckhauser's (1987) proper risk aversion, for Kimball's (1993) decreasing absolute prudence, and for Gollier and Pratt's (1996) risk vulnerability. Our analysis provides a new interpretation for the sign of $v^{(4)}$ in terms of choice between pairs of risky prospects. We show that temperance can be interpreted as aversion to outer risk; temperate individuals dislike relocations of dispersion from the center of a distribution to its tails.

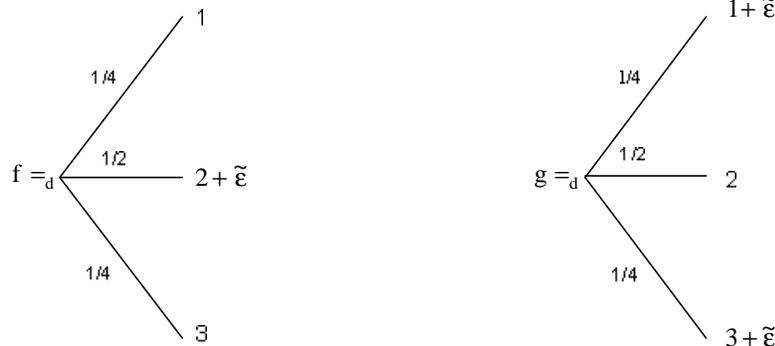
Our analysis also provides insight about the work of Ekern (1980), who derives integral conditions relating fourth-degree risk to the sign of the fourth derivative of the von Neumann-Morgenstern utility function. We establish that ranking distributions by outer risk is equivalent to Ekern's integral conditions, thereby providing an interpretation for his integral conditions in terms of preference between pairs of risks that have the property that one is obtained from the other by a transfer of dispersion from the center of the distribution to its tails. Finally, we show how the ordering of distributions by outer risk is related to the ordering of distributions by fourth-degree stochastic dominance.

2. An Example of Increasing Outer Risk

In this section, we present the following pair of risks, $f(x)$ and $g(x)$, whose comparative structure is an example of an increase in outer risk.

$f(x)$	$g(x)$
$\Pr\{X = 1\} = 0.5$	$\Pr\{X = 0\} = 0.125$
$\Pr\{X = 3\} = 0.5$	$\Pr\{X = 2\} = 0.75$
	$\Pr\{X = 4\} = 0.125$

The first three moments of f and g are the same, but the fourth moment of g is bigger than that of f . The intuition for saying that g has more outer risk than f is best illustrated by presenting this pair of risks in the following distributionally equivalent tree form, where $\tilde{\varepsilon}$ is an actuarially neutral random variable with $\Pr\{\varepsilon = 1\} = \Pr\{\varepsilon = -1\} = 1/2$.



In this form, it can be seen that g can be obtained from f by moving dispersion (actuarially neutral noise $\tilde{\varepsilon}$) from the center of f to its tails. Thus, g has more dispersion in its tails than does f . In terms of the relocation of probability mass, the movement of dispersion from the center of f to its tails results in distribution g which has higher probability mass at the mean and some probability mass on the outside of either end of the support of f (i.e., longer tails).

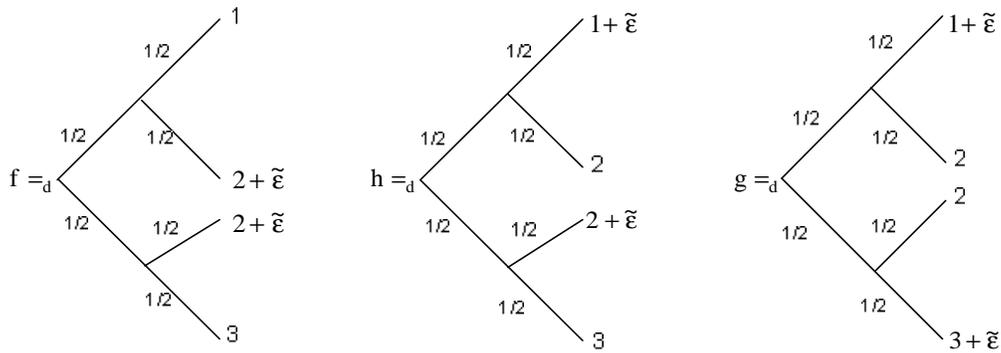
The movement of dispersion $\tilde{\epsilon}$ from the center of f to its tails corresponds to an appropriate pairing of mean-variance-preserving probability transfer functions. In the following table, the functions $\ell(x)$ and $\mu(x)$ are mean-variance-preserving transformations (MVPT). Adding $\ell(x)$ to $f(x)$ gives the risk $h(x)$ which has the same mean and the same variance as $f(x)$ but has more downside risk.⁴ Accordingly, $\ell(x)$ is a downward MVPT (henceforth denoted as MVPT_d). Similarly, adding $\mu(x)$ to $h(x)$ gives the risk $g(x)$ which has the same mean and the same variance as $h(x)$ but $h(x)$ has more downside risk than $g(x)$, i.e., $g(x)$ has more upside dispersion than $h(x)$. Accordingly, $\mu(x)$ is an upward MVPT (henceforth denoted as MVPT_u).

$X=x$	0	1	2	3	4
$f(x)$	0	0.5	0	0.5	0
$\ell(x)$	0.125	-0.375	0.375	-0.125	0
$h(x)$	0.125	0.125	0.375	0.375	0
$\mu(x)$	0	-0.125	0.375	-0.375	0.125
$g(x)$	0.125	0	0.75	0	0.125

It is apparent that $g(x)$ can be obtained directly from $f(x)$ by combining the two MVPTs since $g(x) = f(x) + [\ell(x) + \mu(x)]$. Let $o(x) = \ell(x) + \mu(x)$. We call $o(x)$ an outer dispersion transformation. It is an MVPT that preserves mean, variance as well as skewness and transfers dispersion from the center of f to its tails.

The movements of dispersion effected by $\ell(x)$ and $\mu(x)$ can be seen from the following tree representations of f , h and g .

⁴ See Menezes et al (1980) for the definition of an increase in downside risk.

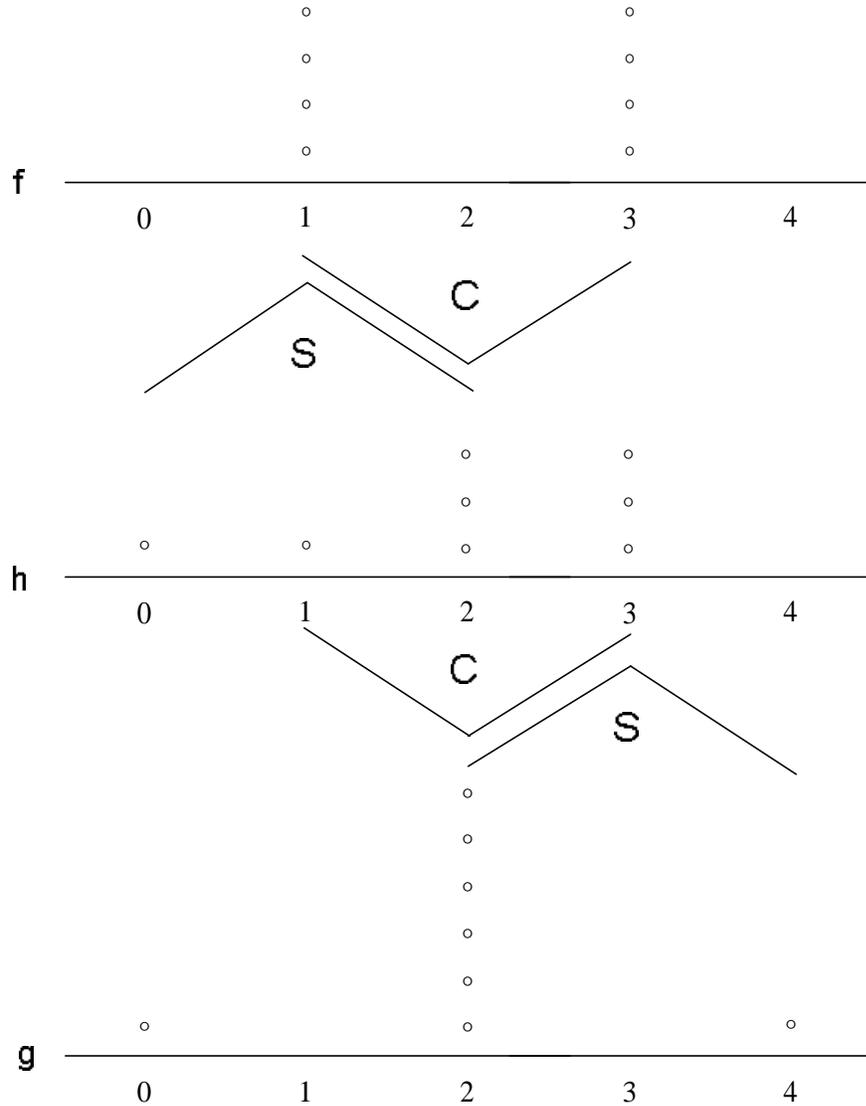


$h(x)$ is obtained from $f(x)$ by transferring dispersion $\tilde{\epsilon}$ from 2 to 1.⁵ This dispersion transfer is effected by the MVPT $\ell(x)$ and results in an increase in downside risk. $g(x)$ is obtained from $h(x)$ by transferring dispersion $\tilde{\epsilon}$ from 2 to 3. This upward dispersion transfer is effected by the MVPT $\mu(x)$. The combined effect of the two MVPTs is to move dispersion from the center of $f(x)$ to its tails and produces an increase in outer risk.⁶

We further illustrate this example of an increase in outer risk in the following distribution diagrams using the underlying spread-contraction combinations. In the following figure, each dot represents one-eighth probability, the distribution of f is shown at the top, the distribution of h is in the middle, and the distribution of g is at the bottom. Moving from f to h there is a spread-contraction pair (marked S and C) which increases downside risk. Moving from h to g is a contraction-spread pair (marked C and S) which increases upside dispersion. The combination of these two spread-contraction pairs gives an increase in outer risk.

⁵ Removing dispersion $\tilde{\epsilon}$ from a branch corresponds to a mean-preserving contraction, adding $\tilde{\epsilon}$ to a branch corresponds to a mean-preserving spread. The relocation of $\tilde{\epsilon}$ corresponds to an MVPT.

⁶ Our example here uses symmetric distributions and transfers the same dispersion $\tilde{\epsilon}$ outward. Our analysis does not require either of these properties, as shown by the example in appendix A. We thank an anonymous referee for suggesting the need for such an example.



3. Main Results

To formally characterize the notion of an increase in outer risk, we begin with the definitions and properties of downward and upward MVPTs. Let $f(x)$ and $g(x)$ be probability density functions on the unit interval $[0, 1]$. Let $\ell(x)$ be a function on $[0, 1]$ and $L(x) = \int_0^x \ell(y)dy$. $l(x)$ is an $MVPT_d$ if it satisfies the following conditions:

(D1) There exist $a_1 < a_2 < a_3$ such that

$$\ell(x) \begin{cases} \geq 0 & \text{for } x \in [0, a_1] \\ \leq 0 & \text{for } x \in [a_1, a_2] \\ \geq 0 & \text{for } x \in [a_2, a_3] \\ \leq 0 & \text{for } x \in [a_3, 1] \end{cases}$$

with strict inequality holding at at least one point within each subinterval.

$$(D2) \quad f(x) + \ell(x) \geq 0 \text{ for all } x \text{ in } [0, 1]$$

$$(D3) \quad \int_0^1 L(x) dx = 0$$

$$(D4) \quad \int_0^1 \int_0^x L(y) dy dx = 0$$

$$(D5) \quad \int_0^z \int_0^x L(y) dy dx \geq 0 \text{ for all } z \text{ in } [0, 1] \text{ and strictly greater for some } z.$$

Conditions (D3) and (D4) guarantee that adding $\ell(x)$ to $f(x)$ does not alter the mean or the variance of $f(x)$. Condition (D5) ensures that $\ell(x)$ transfers dispersion downward.

Let $\mu(x)$ be a function on $[0, 1]$ and $U(x) = \int_0^x \mu(y) dy$. $\mu(x)$ is an $MVPT_u$ if it satisfies the

following conditions:

$$(U1) \quad \text{There exist } b_1 < b_2 < b_3 \text{ such that}$$

$$\mu(x) \begin{cases} \leq 0 & \text{for } x \in [0, b_1] \\ \geq 0 & \text{for } x \in [b_1, b_2] \\ \leq 0 & \text{for } x \in [b_2, b_3] \\ \geq 0 & \text{for } x \in [b_3, 1] \end{cases}$$

with strict inequality holding at at least one point within each subinterval.

$$(U2) \quad f(x) + \mu(x) \geq 0 \text{ for all } x \text{ in } [0, 1]$$

$$(U3) \quad \int_0^1 U(x) dx = 0$$

$$(U4) \quad \int_0^1 \int_0^x U(y) dy dx = 0$$

$$(U5) \quad \int_0^z \int_0^x U(y) dy dx \leq 0 \text{ for all } z \text{ in } [0, 1] \text{ and strictly greater for some } z.$$

Conditions (U3) and (U4) guarantee that adding $\mu(x)$ to $f(x)$ does not alter the mean or the variance of $f(x)$. Condition (U5) ensures that $\mu(x)$ transfers dispersion upward.

An outer dispersion transformation requires an appropriate pairing of an $MVPT_d$ and an $MVPT_u$.

Let $o(x) = \ell(x) + \mu(x)$ and $O(x) = \int_0^x o(y) dy$. $o(x)$ is an outer dispersion transformation (ODT) if it

satisfies the following conditions:

$$(O1) \quad f(x) + o(x) \geq 0 \text{ for all } x \text{ in } [0, 1]$$

$$(O2) \quad \int_0^1 \int_0^x \int_0^y O(z) dz dy dx = 0$$

$$(O3) \quad \int_0^t \int_0^x \int_0^y O(z) dz dy dx \geq 0 \text{ for all } t \text{ in } [0, 1] \text{ and strictly greater for some } t.$$

Note that (D3) and (U3) guarantee that adding $o(x)$ to $f(x)$ does not alter the mean of $f(x)$. Similarly (D4) and (U4) guarantee that adding $o(x)$ to $f(x)$ does not alter the variance of $f(x)$. Condition (O2) ensures that the pairing of $\ell(x)$ and $\mu(x)$ preserves the skewness of $f(x)$, while (O3) guarantees that the pairing of $l(x)$ and $\mu(x)$ transfers dispersion from the center of $f(x)$ to its tails.⁷ We now formally define increasing outer risk.

Definition 1. $g(x)$ has more *outer risk* than $f(x)$ if $g(x) = f(x) + \sum_i o_i(x)$, where each $o_i(x)$ is an outer dispersion transformation.

The definition of increasing outer risk in terms of a sequence of ODTs captures the intuitive notion that more outer risk corresponds to greater dispersion in the tails of a distribution. Since it is often difficult to determine whether one distribution has more outer risk than another from the definition of

⁷ These conditions imply that $g(x) = f(x) + o(x)$ has a higher fourth central moment than $f(x)$.

increasing outer risk, it is valuable to have an equivalent criterion to order risks using only simple properties of the distribution functions. Theorem 1 shows that ordering distributions by Ekern's (1980) integral conditions for fourth-degree risk is equivalent to ordering distributions by outer risk. In Theorem 1, F and G are the distribution functions of f and g , and E_f and E_g denote the means of f and g .

Theorem 1. $g(x)$ has more outer risk than $f(x)$, i.e., $g(x) = f(x) + \sum_i o_i(x)$, if and only if

(i) $E_g = E_f$

(ii) $\int_0^1 \int_0^y [G(z) - F(z)] dz dy = 0$

(iii) $\int_0^1 \int_0^z \int_0^y [G(z) - F(z)] dz dy dx = 0$

(iv) $\int_0^s \int_0^x \int_0^y [G(z) - F(z)] dz dy dx \geq 0$ for all s in $[0, 1]$ and > 0 for some s in $(0, 1)$.

Proof. In appendix B.

We now relate the ordering of distributions in terms of more outer risk to properties of the utility function.

Definition 2. A utility function $v(x)$ exhibits *outer risk aversion* if $E_f v \geq E_g v$ for all (f, g) such that g has more outer risk than f .

Let $\mathbf{V}^* = \{v(x): v^{(4)} < 0\}$ be the set of von Neumann-Morgenstern utility functions which have negative fourth derivative. The following theorem shows that g has more outer risk than f is equivalent to that all individuals whose utility function belongs to \mathbf{V}^* prefer f to g .⁸

⁸ The result in Theorem 2 is consistent with the finding by Eeckhoudt et al (1995) that shifting a dispersion downwards by k affect expected utility in a concave way if and only if $v^{(4)} < 0$.

Theorem 2. $E_f v(x) > E_g v(x)$ for all $v(x)$ in \mathbf{V}^* if and only if $g(x)$ has more outer risk than f .

Proof. In appendix B.

4. Relationship to 4th Degree Stochastic Dominance

Fourth-degree stochastic dominance (4SD) gives an ordering of risks in terms of integral conditions on distribution functions that is equivalent to the ordering based on unanimous choice by individuals whose utility functions have the property that the first four derivatives alternate in sign. Let

$$\mathbf{V} = \{v(x): v' > 0, v'' < 0, v''' > 0, v^{(4)} < 0\},$$

and $F^1(x) = F(x)$, $G^1(x) = G(x)$, $F^k(x) = \int_0^x F^{k-1}(y)dy$, $G^k(x) = \int_0^x G^{k-1}(y)dy$, for $k = 2$ and 3 . F

dominates G by 4SD if F is preferred to G for all $v \in \mathbf{V}$. From Ekern (1980),

$$F \text{ 4SD } G \text{ if and only if } F^k(1) \leq G^k(1) \text{ for } k = 1, 2, 3.$$

Recall that the class of outer risk averters is $\mathbf{V}^* = \{v(x): v^{(4)} < 0\}$. Since \mathbf{V}^* is larger than \mathbf{V} , outer risk orders a smaller set of distributions than does 4SD. From a comparison of our Theorems 1 and 2 and the definition of 4SD, the following relationships are immediate:

- A. If $g(x)$ can be obtained from $f(x)$ by a sequence of ODTs, then $f(x)$ dominates $g(x)$ by 4SD.
- B. Let $g(x)$ and $f(x)$ be distributions with the same mean, variance and skewness. If $f(x)$ dominates $g(x)$ by 4SD then $g(x)$ can be obtained from $f(x)$ by a sequence of ODTs.

5. Summary and Conclusions

In this paper we have formulated the notion of increasing outer risk in order to formally capture the kurtotic shape of distributions identified in the recent empirical literature as characteristic of a wide range of economic variables. Three characterizations of outer risk are provided and shown to be equivalent. Distribution G has more outer risk than distribution F if G can be obtained from F by a transfer of dispersion from the center of F to its tails. This characterization analytically captures the

intuitive and empirically relevant notion that more kurtotic distributions have higher peaks in the neighborhood of the mean and greater elongation in the tails. Our second characterization ranks distributions in terms of unanimous choice by all individuals whose utility function has a negative fourth derivative. The third characterization of increasing outer risk is in terms of integral conditions on the distribution functions, originally developed by Ekern (1980).

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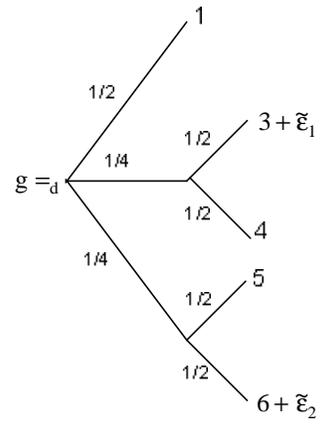
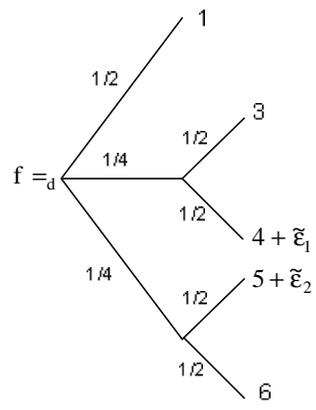
Appendix A

In this appendix, we present an example of an increase in outer risk involving a pair of asymmetric distributions f and g for which dispersion transfers differ according to direction.

In the following table, the functions $\ell(x)$ and $\mu(x)$ are MVPTs. Adding $\ell(x)$ to $f(x)$ gives the risk $h(x)$ which has more downside risk than $f(x)$. Adding $\mu(x)$ to $h(x)$ gives the risk $g(x)$ which has more upside dispersion than $h(x)$. Hence, $g(x) = f(x) + [\ell(x) + \mu(x)]$. These moves preserve mean, variance as well as skewness and transfers dispersion from the center of $f(x)$ to its tails and represent an increase in outer risk. It can also be easily verified that $o(x) = \ell(x) + \mu(x)$ satisfies conditions (O1)-(O3) and therefore is an outer dispersion transformation.

X=x	0	1	2	3	4	5	6	7	8	9
$f(x)$	0	17/32	0	4/32	3/32	3/32	4/32	0	1/32	0
$\ell(x)$	1/32	-1/32	0	-4/32	7/32	-3/32	0	0	0	0
$h(x)$	1/32	16/32	0	0	10/32	0	4/32	0	1/32	0
$\mu(x)$	0	0	0	0	-3/32	7/32	-4/32	0	-1/32	1/32
$g(x)$	1/32	16/32	0	0	7/32	7/32	0	0	0	1/32

These movements of dispersion are illustrated in the following tree diagram, in which $\tilde{\varepsilon}_1$ and $\tilde{\varepsilon}_2$ are actuarially neutral random variables with $\Pr\{\varepsilon_1 = 1\} = 3/4$ and $\Pr\{\varepsilon_1 = -3\} = 1/4$, and $\Pr\{\varepsilon_2 = 3\} = 1/4$ and $\Pr\{\varepsilon_2 = -1\} = 3/4$. $g(x)$ is obtained from $f(x)$ by transferring dispersion $\tilde{\varepsilon}_1$ from 4 to 3 and dispersion $\tilde{\varepsilon}_2$ from 5 to 6.



Appendix B

Proof of Theorem 1: Suppose $g(x) = f(x) + \sum_i o_i(x)$. Then, (i)-(iv) follow immediately from the properties of $q(x)$. We establish the converse by construction. From (O1)-(O3), an ODT $O(x)$ has the property that the function $k(x) = \int_0^x \int_0^y O(z) dz dy$ satisfies $k(0) = k(1) = 0$, is non-negative for $x \in [0, x^*]$

for some x^* and non-positive for $x \in [x^*, 1]$ and $\int_0^1 k(x) dx = 0$. Our construction isolates particular ODTs

and subtracts them from $\varphi(x) = \int_0^x \int_0^y [G(z) - F(z)] dz dy$ until it is exhausted.

Obviously, $\varphi(0) = \varphi(1) = 0$. From properties (iii) and (iv) in Theorem 1, $\int_0^x \varphi(t) dt$ is positive on $(0, 1)$. Hence, φ must be equal to zero at at least one point in $(0, 1)$. Let $0 = x_0 < x_1 < \dots < x_n = 1$ be the finite number of points at which $\varphi(x) = 0$,⁹ and let A_i denote the area between $\varphi(x)$ and the x -axis on interval $I_i = [x_{i-1}, x_i]$, $i = 1, \dots, n$. From properties (iii) and (iv) in Theorem 1, $\varphi(x)$ is positive on the interval (x_0, x_1) and alternates in sign thereafter on successive intervals and n is an even number. Furthermore,

$$A_1 \geq A_2, \tag{a1}$$

$$A_1 + A_3 \geq A_2 + A_4, \tag{a2}$$

$$A_1 + A_3 + A_5 \geq A_2 + A_4 + A_6, \tag{a3}$$

.....

$$A_1 + A_3 + \dots + A_{n-1} = A_2 + A_4 + \dots + A_n, \tag{a4}$$

where the last equality is by property (iii).

Define

⁹ For our purpose, if φ is equal to zero on any closed sub-interval of $[0, 1]$ we can treat such sub-interval as one point. Hence, it is without loss of generality to assume that zero is a regular value of φ , which implies that $\varphi^{-1}(0)$ is a finite set (see Debreu, 1970).

$$\varphi_1(x) = \begin{cases} \alpha_1 \varphi(x) & \text{for } x \in I_1 \\ \varphi(x) & \text{for } x \in I_2 \\ 0 & \text{for } x \notin I_1 \cup I_2 \end{cases}$$

By (a1), there exists an $\alpha_1 \in (0, 1]$ such that $\alpha_1 A_1 = A_2$. Hence, $\varphi_1(x)$ corresponds to an ODT.¹⁰

Now define

$$\varphi_2(x) = \begin{cases} \alpha_2 (1 - \alpha_1) \varphi(x) & \text{for } x \in I_1 \\ \alpha_2 \varphi(x) & \text{for } x \in I_3 \\ \varphi(x) & \text{for } x \in I_4 \\ 0 & \text{for } x \notin I_1 \cup I_3 \cup I_4 \end{cases}$$

By (a2), $(1 - \alpha_1)A_1 + A_3 \geq A_4$. Hence, there exists an $\alpha_2 \in (0, 1]$ such that $\alpha_2[(1 - \alpha_1)A_1 + A_3] = A_4$. Hence, $\varphi_2(x)$ corresponds to an ODT.

The pattern of construction should now be obvious. We constructed $\varphi_1(x)$ to exhaust A_2 , $\varphi_2(x)$ to exhaust A_4 . Similarly, $\varphi_3(x), \dots, \varphi_{n/2}(x)$ can be constructed to exhaust A_6, \dots, A_n . The proof is completed

by noting that (a4) guarantees that $\sum_{i=1}^{n/2} \varphi_i(x) = \varphi(x)$ for all $x \in [0, 1]$.

Proof of Theorem 2: For sufficiency, integrating by parts yields

$$\begin{aligned} E_f v - E_g v &= v'(1) \int_0^1 [G(z) - F(z)] dz - v''(1) \int_0^1 \int_0^y [G(z) - F(z)] dz dy + \\ &v'''(1) \int_0^1 \int_0^x \int_0^y [G(z) - F(z)] dz dy dx - \int_0^1 v^{(4)}(s) \int_0^s \int_0^x \int_0^y [G(z) - F(z)] dz dy dx ds. \end{aligned} \quad (\text{a5})$$

If g has more outer risk than f , it follows immediately from Theorem 1 that $E_f v > E_g v$ for any $v \in \mathbf{V}^*$.

For necessity, we prove by construction that if $E_f v > E_g v$ for any $v \in \mathbf{V}^*$ then $f(x)$ and $g(x)$ satisfy conditions (i)-(iv) in Theorem 1. Consider the following pair of utility functions in \mathbf{V}^* : $v_1 = \theta x^4 + x$ and $v_2 = \theta x^4 - x$, where θ is a negative constant. Since $E_f v - E_g v = \int_0^1 v'(x)[G(x) - F(x)] dx$,

¹⁰ It is easy to verify that the transformation function associated with φ_1 preserves mean, variance and skewness.

$E_f v_1 - E_g v_1 = \int_0^1 (4\theta x^3 + 1)[G(x) - F(x)]dx$ and $E_f v_2 - E_g v_2 = \int_0^1 (4\theta x^3 - 1)[G(x) - F(x)]dx$. Let $\theta \rightarrow$

0, it is implied that $\int_0^1 [G(x) - F(x)]dx \geq 0$ and $-\int_0^1 [G(x) - F(x)]dx \geq 0$, which together imply condition

(i). Apply this procedure to the functions $v_3 = \theta x^4 + x^2$ and $v_4 = \theta x^4 - x^2$ will imply condition (ii)

and to the functions $v_5 = \theta x^4 + x^3$ and $v_6 = \theta x^4 - x^3$ will imply condition (iii). Finally, suppose (iv) is

false at some $s_0 \in (0, 1)$. By continuity, there exists an interval (s_1, s_2) containing s_0 such that

$\int_0^s \int_0^x \int_0^y [G(z) - F(z)]dzdydx < 0$ for all s in (s_1, s_2) . Consider the following utility function:

$$v_7(x) = \begin{cases} -x^4 & \text{for } x \in [s_1, s_2] \\ \theta x^4 & \text{otherwise} \end{cases}$$

Applying conditions (i)-(iii) to (a5) implies

$$\lim_{\theta \rightarrow 0} [E_f v_7 - E_g v_7] = 24 \int_{s_1}^{s_2} \int_0^s \int_0^x \int_0^y [G(z) - F(z)]dzdydx < 0.$$

This contradicts the assumption that $E_f v > E_g v$ for any $v \in \mathbf{V}^*$, and thus establishes condition (iv).