Income and Substitution Effects of Increases in Risk when Payoffs are Linear in the Random Variable

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Abstract

Payoff functions that are linear in the random variable arise in a wide variety of decision models under uncertainty. The decomposition of the effect of increased risk on decisions into income and substitution terms for such models has received much attention in the literature. This paper provides a Hicks-Slutsky decomposition of the effect of Rothschild-Stiglitz increases in risk on the optimal decision. Two measures of aversion to additional risk are introduced. Their behavior is shown to control the signs of the income and substitution effects.

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1. Introduction

Many decision problems under uncertainty give rise to payoff functions that are linear in the random variable.[1] Consequently, there has been considerable interest in the effect of increases in the riskiness of the random variable on the optimal decision in such models and in the decomposition of this effect into income and substitution terms. While the literature has obtained necessary and sufficient conditions on preferences to sign the effect of increased risk on the optimal decision, a Hicks-Slutsky decomposition of this effect has yet to be derived. Sandmo (1971) separated this effect into "Slutsky-like" income and substitution terms. Davis (1989) provided a theoretical foundation for Sandmo's decomposition by showing that it can be derived using a random-compensation method, and Hadar and Seo (1992) extended Davis' analysis to general shifts in the riskiness of the random variable. As Davis (1989, p. 133) noted, there is no meaningful relationship between the "Slusky-like" income effect and the income derivative (which shows how the optimal decision changes when income changes but risk is unchanged). In a separate development, Pope and Chavas (1985) derived the Hicks-Slutsky equation for an increase in the *expected value* of the random variable in a production model with a payoff that is linear in the random variable. Their compensation method is the uncertainty counterpart of the Hicks-Slutsky income compensation under certainty. In this paper, we use duality to derive the Hicks-Slutsky equation for a Rosthschild-Stiglitz (RS) increase in the riskiness of the random variable. Our compensation method is similar to that used by Pope and Chavas (1985) and Chavas and Larson (1991).

For expositional convenience, and following Hadar and Seo (1992) and Pope and Chavas (1985), our analysis is presented in terms of the standard problem of a competitive firm facing price uncertainty. In section 2 we present the firm's decision problem, its dual, and derive the Hicks-Slutsky equation for an RS increase in price uncertainty. In section 3 we present two measures of aversion to additional risk, called unit-Ross and unit-endogenous risk premiums. In section 4 we relate the behavior of these risk premiums to the signs of the income and substitution effects and provide a geometric illustration of these

[1] For examples of the variety of contexts in which such payoffs arise, see Bigelow and Menezes (1995).

effects. Concluding remarks are in section 5.

2. The Model

Consider a competitive firm which chooses its output q before market price $\tilde{p}$ is known so as to maximize the expected utility of final wealth $\tilde{w}$, i.e.,

$$\text{Max}_q \;\; Eu(\tilde{w}) \equiv \int_a^b u(w_0 + pq - c(q))dF(p,\beta). \tag{1}$$

In (1), E is the expectation operator and u(•) is the firm's von Neumann-Morgenstern utility function, which is increasing and concave. The firm's initial (non-random) wealth is w_0, and its total variable cost function is c(q).[2] The distribution function of $\tilde{p}$ is F(p, β) with support [a, b] (a > 0), where β is an index of Rothschild-Stiglitz (RS) risk. The RS riskiness of $\tilde{p}$ increases as β increases.

To provide insights about the effects of increased price risk on the output decision, we transform (1) into an equivalent constrained optimization problem. Let $\bar{p}$ denote the expected price and $p^* = \tilde{p} - \bar{p}$.[3] The firm's expected wealth is $w = w_0 + \bar{p}q - c(q)$, and its total wealth is $\tilde{w} = w + w^*$, where $w^* = p^*q$ is the actuarially neutral risky component of total wealth. Since w^* depends on the choice variable q, we refer to w^* as *endogenous risk*. The firm's expected utility function Eu(•) induces a preference ordering over pairs (q, w), where q is the scale of endogenous risk.[4] This induced ordering is represented by the *derived utility function* $v(q, w, \beta)$, defined by

$$v(q, w, \beta) \equiv \int_c^d u(w + p^*q)dF(p^*, \beta).$$

From the properties of u, the derived utility function v is decreasing in q and β, increasing in w and

[2] The firm's initial wealth w_0 can be positive, negative or zero.

[3] The distribution of p^* is the same as that of $\tilde{p}$ but its support is [c, d] with c = $a - \bar{p}$ and d = $b - \bar{p}$.

[4] Changes in q induce *endogenous* changes in the riskiness of revenues and profits, while changes in β represent *exogenous* changes in risk.

concave in q and w.

Let $I^0 = \{(q, w) \mid v(q, w, \beta) = v^0\}$ denote the set of (q, w) bundles which have expected utility v^0. Consider two bundles (q_1, w_1) and (q_2, w_2) from I^0. By definition, the firm is indifferent between endogenous risk-expected wealth pairs $(p^* q_1, w_1)$ and $(p^* q_2, w_2)$. The graph of I^0 is therefore the locus of endogenous risk-expected wealth pairs among which the firm is indifferent. The slope of an indifference curve, the marginal rate of substitution between endogenous risk and expected wealth, is

$$m(q, w, \beta) = -\frac{v_q(w, q, \beta)}{v_w(w, q, \beta)} = -\frac{E[p^* u'(w + p^* q)]}{Eu'(w + p^* q)}, \tag{2}$$

where subscripts of functions denote partial derivatives.[5] The marginal rate of substitution gives the minimum amount of expected wealth required to compensate the firm for a unit increase in the scale of endogenous risk. From the properties of v, m is positive and is increasing in q and w along any indifference curve of v.

The decision problem (1) can be rewritten as a constrained maximization in which both q and w are choice variables, i.e.,

$$\begin{aligned} &\underset{\{q,\, w\}}{\text{Max}} \quad v(q, w, \beta) \\ &\text{s.t.} \quad w = w_0 + \bar{p}q - c(q). \end{aligned} \tag{3}$$

The solution to (3), $\left(q^U(\bar{p}, w_0, \beta), w^U(\bar{p}, w_0, \beta)\right)$, satisfies the necessary conditions

$$\begin{aligned} &m(q, w, \beta) = \bar{p} - c'(q), \\ &w_0 + \bar{p}q - c(q) - w = 0. \end{aligned} \tag{4}$$

Figure 1 illustrates the solution to (3). The optimal point (q^U, w^U) is where the indifference curve, v =

[5] In section 3, we interpret the marginal rate of substitution m as the risk premium per unit increase in endogenous risk.

v^0, and the opportunity constraint, w = $w_0 + \bar{p}q - c(q)$, are tangent.[6] The firm's (uncompensated) output supply is $q^U(\bar{p}, w_0, \beta)$ and its expected wealth is $w^U(\bar{p}, w_0, \beta)$.

<Insert Figure 1 here>

The dual to the firm's expected utility maximization problem (3) is to choose q and w so as to minimize the amount of initial wealth required to attain a given level of expected utility, i.e.,

$$\begin{aligned} &\underset{\{q,\, w\}}{\text{Min}} \quad w - [\bar{p}q - c(q)] \\ &\text{s.t.} \quad v(q, w, \beta) = v^0. \end{aligned} \tag{5}$$

The solution to (5), $\left(q^C(\bar{p}, v^0, \beta), w^C(\bar{p}, v^0, \beta)\right)$, satisfies the necessary conditions

$$\begin{aligned} &m(q, w, \beta) = \bar{p} - c'(q), \\ &v(q, w, \beta) = v^0. \end{aligned} \tag{6}$$

$q^C(\bar{p}, v^0, \beta)$ is the firm's compensated output supply. As in the original problem (3), the solution to the dual problem (5) occurs where the derived indifference curve is tangent to the opportunity constraint.

To relate compensated supply $q^C(\bar{p}, v^0, \beta)$ and (uncompensated) supply $q^U(\bar{p}, w_0, \beta)$, we define the *compensation function*

$$C(\bar{p}, v^0, \beta) \equiv w^C(\bar{p}, v^0, \beta) - [\bar{p}q^C(\bar{p}, v^0, \beta) - c(q^C(\bar{p}, v^0, \beta))] - w_0. \tag{7}$$

C is the minimum wealth in excess of w_0 required for the firm to attain expected utility v^0. Supply and compensated supply coincide when $C(\bar{p}, v^0, \beta)$ = 0, i.e.,

$$q^U(\bar{p}, w_0 + C(\bar{p}, v^0, \beta), \beta) \equiv q^C(\bar{p}, v^0, \beta). \tag{8}$$

This identity provides the basis for decomposing the firm's output supply into income and substitution effects.

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[6] In this and subsequent graphical illustrations, the cost function c(q) is assumed to be linear in q so that the opportunity constraint is a straight line.

If $v^0 = v\left(q^U(\bar{p}, w_0, \beta), w^U(\bar{p}, w_0, \beta), \beta\right)$, differentiating (8) and using the fact that $C(\bar{p}, v^0, \beta) = 0$ gives the Hicks-Slutsky equation for an increase in RS risk

$$\frac{\partial q^U(\bullet)}{\partial \beta} = \frac{\partial q^C(\bullet)}{\partial \beta} - \frac{\partial q^U(\bullet)}{\partial w_0} \times \frac{\partial C(\bullet)}{\partial \beta}. \tag{9}$$

In (9), $-[\partial q^U / \partial w_0] \times [\partial C / \partial \beta]$ is the *income effect*; it is the negative of the product of the income derivative $\partial q^U / \partial w_0$ and the compensation $\partial C / \partial \beta$ required to keep expected utility constant. The compensated term $\partial q^C / \partial \beta$ is the *substitution effect*.[7]

3. Unit-Risk Premiums

In this section we present two measures of aversion to additional risk whose behavior respectively controls the signs of the substitution and income effects of increases in RS risk. We first introduce the unit-Ross risk premium, which is the Ross risk premium per unit increase in RS risk. It is a measure of aversion to exogenous additions in risk. We show that the unit-Ross risk premium is the marginal rate of substitution between the RS risk parameter and wealth corresponding to the derived utility function. We then consider a measure of aversion to additions to endogenous risk, called unit-endogenous risk premium. It is the marginal rate of substitution between the scale of endogenous risk and wealth corresponding to the derived utility function.

Ross (1981) defined the counterpart of the Arrow-Pratt risk premium for situations involving unavoidable background risk $\tilde{w} = w + q p^*$. Let $\tilde{\varepsilon}$ be a random variable with $E(\tilde{\varepsilon} \mid p^*) = 0$. The Ross premium π is defined implicitly by the equation

$$E_{p^*} E_{\tilde{\varepsilon}|p^*} u(w + q(p^* + \tilde{\varepsilon})) = E_{p^*} u(w - \pi + qp^*). \tag{10}$$

π is the maximum amount that the individual is willing to pay to avoid the risk $\tilde{\varepsilon}$ when $w + q p^*$ is

[7] A Hicks-Slutsky equation can be derived for a change in any parameter in the model. For example, if β is replaced by the expected price (9) gives the Hicks-Slutsky equation derived by Pope and Chavas (1985, eq. (13)).

unavoidable. Rothschild and Stiglitz (1972) have shown the equivalence between several characterizations of increasing risk. Their work implies the existence of $\Delta\beta > 0$ such that the distribution of $p^* + \tilde{\varepsilon}$ is $F(p^*, \beta+\Delta\beta)$. [8] We can therefore rewrite (10) as

$$\int_c^d u(w + qp^*)dF(p^*, \beta + \Delta\beta) = \int_c^d u(w - \pi + qp^*)dF(p^*, \beta) . \tag{11}$$

From (11), in the limit as $\Delta\beta \to 0$, $\pi/\Delta\beta$ becomes

$$\Pi = \lim_{\Delta\beta\to 0} \frac{\pi}{\Delta\beta} = -\frac{\int_c^d u(w + qp^*)dF_\beta(p^*, \beta)}{Eu'(w + qp^*)} . \tag{12}$$

We call Π the *unit-Ross risk premium*. From (16) below, the unit-Ross risk premium is the derivative of the compensation function with respect to the RS risk parameter. Hence, the unit-Ross risk premium is the compensation in initial wealth required to keep expected utility constant when there is an infinitesimal increase in exogenous risk. In Figure 2, Π is the vertical distance between the indifference curves for RS risk parameters β and β' ($> \beta$) both having the same expected utility.[9]

<Insert Figure 2 here>

The endogenous risk premium (Δw) is the Arrow-Pratt risk premium for the increase in the scale (Δq) of endogenous risk, defined by

$$\int_c^d u(w + (q + \Delta q)p^*)dF(p^*, \beta) = \int_c^d u(w - \Delta w + qp^*)dF(p^*, \beta) . \tag{13}$$

From (13), in the limit as $\Delta q \to 0$, $\Delta w/\Delta q$ becomes

$$m = \lim_{\Delta q\to 0} \frac{\Delta w}{\Delta q} = -\frac{E[p^* u'(w + qp^*)]}{Eu'(w + qp^*)} . \tag{14}$$

We call m the *unit-endogenous risk premium*. From (2), the unit-endogenous risk premium is the marginal rate of substitution between endogenous risk and expected wealth, i.e., the minimum amount of expected wealth required to compensate for a unit increase in the scale of endogenous risk.

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[8] See Theorem 2 of Rothschild and Stiglitz (1970) and Theorem 3 of Machina and Pratt (1997).
[9] Note that any two such indifference curves have the same vertical intercept and the one with a higher RS risk parameter must lie above the other everywhere else.

It should be noted that the unit-Ross risk premium is proportional to the unit-endogenous risk premium when RS increases in risk take the often used multiplicative form, $\tilde{p} = \bar{p} + \beta p^*$. In this case,

$$\Pi = -\frac{v_\beta}{v_w} = -\frac{E[u'(w+\beta p^* q)p^* q]}{E[u'(w+\beta p^* q)]} = qm/\beta\,. \tag{15}$$

4. Income and Substitution Effects

We now show that the signs of the income and substitution effects are respectively controlled by the behavior of the unit-endogenous risk premium and the unit-Ross risk premium. We first present expressions for the income and substitution effects. Applying the envelope theorem to (7) gives

$$\frac{\partial C}{\partial \beta} = -\frac{v_\beta}{v_w} = -\frac{\int_c^d u(w+qp^*)dF_\beta(p^*,\beta)}{Eu'(w+qp^*)}\,. \tag{16}$$

Totally differentiating (4) gives

$$\frac{\partial q^U}{\partial w_0} = \frac{m_w}{J}\,, \tag{17}$$

where $J = -(m_q + m \cdot m_w) - c''(q)$ is the Jacobian determinant for (4) and is negative by the second order condition for (3). By (16) and (17),

$$\textit{Income effect} = -\frac{\partial q^U}{\partial w_0} \times \frac{\partial C}{\partial \beta} = -[\frac{m_w}{J}] \times [-\frac{v_\beta}{v_w}]. \tag{18}$$

From (18), the income effect is the negative of the product of the income derivative $\partial q^U/\partial w_0$ and the unit-Ross risk premium Π (= ∂C/∂β). Hence, the sign of the income effect is opposite to the sign of the income derivative. This is because the income effect is obtained by removing the income compensation required to keep expected utility unchanged when the RS risk parameter increases. Totally differentiating (6) gives

$$\textit{Substitution effect} = \frac{\partial q^C}{\partial \beta} = \frac{m_\beta v_w - m_w v_\beta}{v_w J}\,. \tag{19}$$

The theorem that follows relates the signs of the income and substitution effects to the behavior of the two unit risk premiums.

Theorem 1.

(i) The income effect of an RS increase in price risk is negative (positive) if and only if the unit-endogenous risk premium m is decreasing (increasing) in wealth, and this happens if and only if absolute risk aversion $-u''(w)/u'(w)$ is decreasing (increasing) in w.

(ii) The substitution effect of an RS increase in price risk is negative (positive) if and only if the unit-Ross risk premium Π is increasing (decreasing) in q along a derived indifference curve.

Proof: (i) Since J < 0, $v_\beta < 0$, $v_w > 0$, the first part of (i) is immediate from (18). Differentiating m in (14) with respect to w yields,

$$m_w = -\frac{[Eu'][E(p^* u'')] - [Eu''][E(p^* u')]}{(Eu')^2} = -\frac{E[(\tilde{p} - c'(q))u'']}{Eu'}.$$

Since, as is well-known, $E[(\tilde{p} - c'(q))u'']$ is positive (negative) if and only if absolute risk aversion is decreasing (increasing),[10] the second part of (i) follows.

(ii) From (12), (16) and (19),

$$\left.\frac{\partial \Pi}{\partial q}\right|_{v=v^0} = -\frac{1}{(v_w)^2}\left(v_w \cdot \left.\frac{\partial v_\beta}{\partial q}\right|_{v=v^0} - v_\beta \cdot \left.\frac{\partial v_w}{\partial q}\right|_{v=v^0} \right)$$

$$= -\frac{1}{(v_w)^2}\left(v_w \cdot [v_{q\beta} + m \cdot v_{w\beta}] - v_\beta \cdot [v_{qw} + m \cdot v_{ww}] \right)$$

$$= \frac{v_w m_\beta - v_\beta m_w}{v_w} = J \cdot \frac{\partial q^C}{\partial \beta}.$$

Since J < 0, (ii) follows from the above equality.

[10] See Sandmo (1971, p. 69).

Theorem 1(i) shows that, under decreasing absolute risk aversion, the unit-endogenous risk premium is decreasing in wealth and the income effect is negative. Theorem 1(ii) shows that if the unit-Ross risk premium is increasing along a derived indifference curve the substitution effect is also negative.[11] That is, the two effects work in the same direction.

Figure 3 illustrates the income and substitution effects. The solid β-indifference curve represents the firm's preferences prior to an increase in the RS riskiness of p^*. The dashed β'-indifference curves represent preferences after the increase in the riskiness of p^*. The firm is initially in equilibrium at point A on the β-indifference curve. *The movement from A to B is the substitution effect of the increase in the riskiness of* p^*.[12] It is obtained by shifting upward the opportunity constraint so that it is tangent to the β'-indifference curve (which has the same expected utility as that of the β-indifference curve). *The movement from B to C is the income effect of the increase in the riskiness of* p^*.[13] It is obtained by moving from the compensated optimal bundle (B) to the bundle (C) where a β'-indifference curve is tangent to the initial opportunity constraint. *The movement from A to C is the total effect of the increase in the riskiness of* p^*.

<Insert Figure 3 here>

5. Concluding Remarks

The approach and results obtained for the competitive firm under price uncertainty are valid for models in which the random payoff $\tilde{y}$ is linear in the random variable $\tilde{z}$. Specifically, the payoff

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[11] Since u" < 0, Π must be increasing immediately to the right of q = 0. Hence, Π is either non-monotone or it is uniformly increasing in q. From (15), we see that at least for the multiplicative form of RS increases in risk, Π is increasing in q along a derived indifference curve since m is increasing in q along a derived indifference curve.

[12] The direction of change in expected wealth (namely, whether A is above or below B in Figure 3) cannot be determined without further assumptions about preferences.

[13] Both effects work to decrease the firm's output. Specifically, the income effect is negative given decreasing absolute risk aversion and the substitution effect is negative given that the unit-Ross risk premium is increasing in output along a derived indifference curve.

function is of the form

$$\tilde{y} = y_0 + h(x) + g(x)\tilde{z} \qquad (20)$$

where y_0 is non-random initial income or wealth; g(x) and h(x) are concave in the choice variable x. Many decision problems have this structure. An example is the two-asset (one safe, one risky) portfolio model. In this model, $\tilde{z}$ is the rate of return on the risky asset, h(x) = 0, and g(x) = x denotes the amount invested in the risky asset. Theorem 1(i) implies the well-known result that an increase in y_0 increases risky investment under DARA. Theorem 1(ii) gives a new result. It implies that a compensated increase in the riskiness of the return to the risky asset reduces risky investment and therefore increases investment in the safe asset if the unit-Ross risk premium increases along a derived indifference curve.

The dual approach used in this paper can be used to analyze the income and substitution effects of changes in other parameters in (20). For example, a change in the expected value of the random variable $\tilde{z}$ simply rotates the opportunity constraint in Figure 1. The effect of this rotation can be separated into income and substitution effects. As another example, a tax on payoffs (e.g., profits) induces a rotation in the opportunity constraint and a shift in the derived indifference curves similar to that induced by an increase in RS risk.

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