

**PRESERVICE ELEMENTARY TEACHERS' INITIAL AND POST-COURSE
VIEWS OF MATHEMATICAL ARGUMENTS:
AN INTERPRETATIVE PHENOMENOLOGICAL ANALYSIS**

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by

MICHAEL PERKOWSKI

Dr. John Lannin, Dissertation Supervisor

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The undersigned, appointed by the dean of the Graduate School, have examined the dissertation entitled

PRESERVICE ELEMENTARY TEACHERS' INITIAL AND
POST-COURSE VIEWS OF MATHEMATICAL ARGUMENTS:
AN INTERPRETATIVE PHENOMENOLOGICAL ANALYSIS

presented by Michael Perkowski,

a candidate for the degree of doctor of philosophy, and hereby certify that, in their opinion, it is worthy of acceptance.

Professor John K. Lannin

Professor Fran Arbaugh – The Pennsylvania State University

Professor Kathryn Chval

Professor James Tarr

Professor Peggy Placier

DEDICATION

I dedicate this dissertation to my wife, Debbie,
my hiking companion on our life's path together,
my dancing partner and canasta rival,
my best friend always and my true love forever.

I could not have done it without you.

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ABSTRACT

Recent curriculum recommendation call for mathematical argumentation to play a significantly greater role in U. S. mathematics instruction at all grade levels, including the elementary grades. To better understand how preservice elementary teachers (PTs) enrolled in a one-semester course emphasizing mathematical argumentation might become better prepared to implement this change, I interviewed five such PTs at two points in time, near the beginning of the course and shortly after they completed it. Both interviews focused on a problem set in which nine fictional elementary school students present arguments for their solutions to mathematical problems. Interviewees compared the arguments, decided which were convincing and which were not, and gave reasons for their choices. Using an interpretative, phenomenological approach, I analyzed their responses and found that they initially preferred arguments in which they perceived the arguer as knowing what to do, getting the correct answer, using a quick way to get it, showing how with numbers, and having the right attitude. In contrast, after they had completed the course, they focused on understanding the problem, finding answers that made sense, and explaining why with diagrams. They also viewed the arguer's attitude as a more complex issue than they had at the beginning of the course. These and other findings suggest that current research on PTs' approaches to mathematical justification may: (a) overemphasize the formal aspects of mathematical arguments and undervalue their substance, (b) overemphasize the role of verification and undervalue explanation, (c) be too far removed from PTs' perspectives, and therefore (d) fail to accurately reflect significant progress in PTs' understandings.

CHAPTER 1: INTRODUCTION

Mathematical Argumentation in U. S. Classrooms

Curriculum recommendations. With the publication of *Curriculum and Evaluation Standards for School Mathematics* in 1989, the National Council of Teachers of Mathematics (NCTM) launched a major effort to reform mathematics education in the United States, an effort that continues today. In this first *Standards* document and in the others that followed, the NCTM (1989, 1991, 1995, 2000) presented a vision for the learning and teaching of mathematics that stands in stark contrast to traditional mathematics instruction. This vision demands a radical shift in what it means for students to do mathematics in school, “a shift ... toward investigating, formulating, representing, reasoning, and applying a variety of strategies to the solution of problems—then reflecting on these uses of mathematics—and away from being shown or told, memorizing, and repeating” (NCTM, 1995, p. 2). It calls for students to engage in fundamentally different types of reasoning than would normally appear in traditional classrooms, and it requires teachers to support students in their efforts to make sense of mathematics, both as individuals and as members of a larger mathematical community.

In a break with traditional instructional practices in the U. S., the NCTM (1989) called for mathematical argumentation—constructing arguments to justify conclusions and critiquing the arguments of others—to become a regular feature of classroom activity at all grade levels, including the elementary grades. Students in grades K – 4, for example, should begin to distinguish between valid and invalid arguments and engage in “the kind of informal thinking, conjecturing, and validating that helps children to see that

mathematics makes sense” (p. 29). The NCTM likewise recommended that students in grades 5 – 8 have “many experiences to develop their ability to construct valid arguments in problem settings and evaluate the arguments of others” (p. 81). The NCTM’s (2000) *Principles and Standards for School Mathematics* reaffirmed this focus on mathematical argumentation. Beginning in prekindergarten and continuing through grade 12, they encouraged teachers “to foster ways of justifying that are within the reach of students, that do not rely on authority, and that gradually incorporate mathematical properties and relationships as the basis for the argument” (p. 126), ultimately enabling all students to “develop and evaluate mathematical arguments and proofs” (p. 56).

More recently, the Common Core State Standards Initiative (CCSSI, 2012) created *Standards for Mathematical Practice*, including descriptions of the types of reasoning in which mathematically proficient students should engage. In particular, the CCSSI emphasized that students should not only manipulate abstract mathematical symbols but also be able to reason quantitatively, “creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects” (p. 6). Students should also be “able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is.” (p. 7). And like the NCTM, the CCSSI also included the elementary grades in these recommendations, noting that “students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments” (p. 7). Thus, for at least the past 24 years, curriculum reform efforts have

emphasized that all students should engage in both constructing mathematical arguments and critiquing the arguments of others as important parts of their regular classroom activities.

The recommendations in practice. Although these recommendations have been in place for almost a quarter of a century, research suggests that making mathematical argumentation a regular feature of U. S. classrooms has been and will continue to be a slow and difficult process. Mathematics instruction in the U.S. remains predominantly traditional and relatively uniform compared to classroom practices observed in other countries, and it has changed very little over the past century (Stigler & Hiebert, 1999; National Research Council, 2001). Stigler and Hiebert (1999), for example, drew on data from the Third International Mathematics and Science Study (TIMSS), in which researchers videotaped and analyzed a random sample of 8th-grade mathematics lessons from the U.S., Germany, and Japan. Based on these videotaped lessons, they characterized mathematics instruction in the U.S. as devoted almost exclusively to “learning terms and practicing procedures” (p. 27).

The typical pattern of mathematics instruction in the U.S. includes: (a) a review of previous material, (b) a teacher-directed demonstration of a particular solution method or methods, (c) seatwork in which students practice the procedures the teacher has demonstrated, and (d) the checking of some practice problems and assignment of homework (Stigler & Hiebert, 1999). The first two stages, review and demonstration, often employ a common discourse pattern known as Initiation-Response-Evaluation or IRE (Franke, Kazemi, & Battey, 2007). The teacher initiates this pattern by asking a question that requires only a quick, brief response, often just a single word. One or more

students respond, and the teacher evaluates the response, indicating whether it is correct or incorrect. Repeated use of the IRE pattern is typical of *recitation*, the predominant form of mathematics instruction in the United States (NRC, 2001). Recitation generally demands no more from the student than the rapid recall of information and one-word responses, hardly the kind of mathematical argumentation described by the NCTM (2000) or CCSSI (2012).

The second stage of typical U.S. lessons, the teacher's presentation of prescribed solution methods, exemplifies Cobb's (1988) notion of "teaching by imposition" (p.96) in traditional mathematics instruction. In contrast, student-presented solution methods, which would encourage mathematical argumentation in the classroom, occurred in only 8% of the U.S. lessons, as opposed to 14% in Germany and 42% in Japan (Stigler & Hiebert, 1999).

The ways that students spent their time during seatwork also reflected the traditional nature of U.S. mathematics instruction. In analyzing the TIMSS videos, researchers classified seatwork into three different categories, based on the type of activity in which the students were engaged: (a) practicing routine procedures, (b) applying concepts and procedures to new situations, and (c) inventing new methods or thinking about mathematical situations in new ways. In the TIMSS videos, students in the U.S. spent the vast majority of their seatwork time practicing routine procedures (95.8%), with little time left for applying concepts and procedures in new situations (3.5%) or inventing new methods or thinking about mathematical situations in new ways (0.7%). In contrast, the students from Japan showed a much greater balance in the allocation of their seatwork time (40.8%, 15.1%, and 44.1% in each of the three categories, respectively).

Finally, in the area of mathematical justification, the TIMSS videos paint a bleak picture of instruction in the U.S. Although 53 percent of Japanese lessons and 10 percent of German lessons contained proofs, not a single lesson recorded in the U.S. contained a proof (Stigler & Hiebert, 1999). Although the TIMSS video study focused only on 8th-grade lessons, the results suggest that mathematics instruction in the lower grades follows a similar pattern. A few videotaped lessons did indicate that current reform efforts may have had some effect on mathematics instruction in the U.S., but even in these rare instances, “the variations from the [traditional] theme appear in the *form* of activities, not the substance” (p. 53, italics in the original). Thus, the familiar pattern of traditional mathematics instruction in U.S. classrooms allows little room for the types of mathematical argumentation recommended by the NCTM and the CCSSI. Furthermore, an increased emphasis on mathematical argumentation may be particularly challenging to implement in the elementary grades, because mathematical arguments traditionally appear most prominently at the secondary level, specifically as proofs in secondary school geometry classes.

The persistence of traditional instruction. The current reform effort in mathematics education is not the first. Earlier attempts at reform—most notably the “new math” movement of the 1950s and 1960s—produced little lasting effect. The NRC (2001) noted that the descriptions of U.S. mathematics instruction derived from the TIMSS videos “might easily have been written to describe U.S. mathematics lessons in 1900” (p. 50). In view of the range of alternative instructional approaches used in other countries, such as Germany and Japan, why has the traditional approach used in the U.S. remained essentially unchanged for so long?

Stigler and Hiebert (1999) argued that the persistence of traditional instruction stems from the fact that mathematics teaching is a cultural activity. As such, it is based on a system of beliefs and assumptions that teachers develop by participating in traditional classrooms during their own K-12 schooling. In the U.S, this “apprenticeship of observation” (Lortie, 1975, p. 61) leads to the belief that school mathematics is a set of procedures and skills that should be learned by rote through explicit teacher-directed demonstrations and repeated practice. In contrast, most Japanese teachers believe that mathematics is a network of relationships among facts, concepts, and procedures and that an understanding of these relationships is best developed through the invention, discussion, and refinement of students’ own solution methods.

Kennedy (1999) theorized that, during their apprenticeship of observation, teachers develop a “frame of reference” (p. 55) for interpreting classroom experiences. This frame of reference continues to guide the way they interpret and act on classroom situations throughout their teaching careers. Thus, the apprenticeship of observation has a conservative effect on teaching practice, making it resistant to change. According to Kennedy and others, such as Ebby (2000) and Philipp et al. (2007), field experiences and early teaching experiences also have a conservative effect, because the form of instruction that teachers encounter in most schools tends to reinforce their initial frames of reference. In particular, teachers whose apprenticeship of observation contains little or no experience with mathematical argumentation will have no reliable frame of reference for interpreting mathematical arguments and assessing their validity and will therefore find implementing mathematical argumentation in their classrooms difficult if not impossible.

The Role of Preservice Teacher Education

I have argued, based on the NCTM's and CCSSI's recommendations, that mathematical argumentation—constructing arguments to justify mathematical conclusions and critiquing the arguments of others—should play an important role in U.S. classrooms at all grade levels. However, such argumentation rarely occurs in traditional mathematics instruction, and for a variety of reasons, traditional teaching practices are persistent and difficult to change. If mathematical argumentation is to attain the prominence that the NCTM (2000) and CCSSI (2012) suggest, where and how should we act to initiate such a change?

According to Kennedy (1999), the most reasonable starting point lies in preservice teacher education.

Preservice teacher education is ideally situated to foster such a shift in thinking. It is located squarely between teachers' past experiences as students in classrooms and their future experiences as teachers in classrooms. From their experiences, teachers develop the ideas that will guide their future practices. If these ideas are not altered during preservice teacher education, teachers' own continuing experiences will reinforce them, cementing them even more strongly into their understandings of teaching, and reducing the likelihood that these ideas might ever change. (p. 57)

Thus, preservice education has the potential to change teachers' frames of reference, allowing them to think differently about the goals of instruction, interpret classroom experiences differently, and act accordingly.

Although preservice teacher education could play a critical role in initiating this change, whether or not it will do so successfully remains an open question, one whose answer depends on the type of preservice training we provide. If we accept the view that knowledge is not simply transmitted by words or actions but constructed in the context of social interactions, we cannot simply tell or show prospective teachers how we want them

to teach. Instead, we must provide them with opportunities to construct an understanding of the kind of mathematical argumentation that NCTM and CCSSI recommend.

According to the National Research Council (1989), “Teachers themselves need experience in doing mathematics—in exploring, guessing, testing, arguing, and proving” (p. 65).

Engaging preservice elementary teachers (PTs) in mathematical argumentation addresses an important and neglected area of teacher knowledge (Grossman, 1990; Shulman, 1986; Stylianides & Ball, 2008). Grossman (1990) referred to this area as knowledge of syntactic structures. Teachers with this type of knowledge understand the ways in which knowledge claims are supported and evaluated within the discipline. Shulman (1986) also highlighted the importance of this area of teacher knowledge. He wrote, “The teacher need not only understand *that* something is so; the teacher must further understand *why* it is so, on what grounds its warrant can be asserted, and under what circumstances our belief in its justification can be weakened and even denied” (p. 9, italics in the original). Stylianides and Ball (2008) argued that teachers at all grade levels need opportunities to develop a deep understanding of mathematical justification, if they are to be successful in making this aspect of mathematics an important part of their classroom practices.

Across the U.S., many teacher education programs seek to provide preservice teachers with such opportunities, making mathematical argumentation a central feature of their courses. The design and implementation of these courses would be aided by research on preservice teachers’ knowledge of mathematical justification. Mewborn (2001), however, pointed to this area as one in which more research is badly needed, particularly at the elementary level.

The Purpose of the Study

In this study, I examined preservice elementary teachers (PTs) views of mathematical arguments at two points in time: (a) near the beginning of their participation in the Number and Operations course at the University of Missouri, a course that emphasizes mathematical argumentation, and (b) at the beginning of the following semester, after they completed the course. In particular, I sought to shed light on the frames of reference that PTs used to interpret mathematical arguments and the ways in which their interpretations—and hence their frames of reference—change during this part of their preservice training. Because I was particularly interested in their points of view, how these arguments appeared to them, I focused primarily on what they considered important in the arguments rather than what I considered important. However, I also investigated the roles played by two different aspects of the arguments, arising from theory in the research literature: (a) the form of the arguments and (b) the substance of the arguments.

Research Questions

The study focused primarily on the following question: As PTs compare and evaluate mathematical arguments, what do they initially consider important, and how do their views change by the end of a one-semester course emphasizing mathematical argumentation? In addition, I also addressed two secondary questions: (a) What role does the *form* of the argument play in PTs' evaluations, and how does this role change by the end of the semester? (b) What role does the *substance* of the argument play in PTs' evaluations, and how does this role change by the end of the semester?

Theoretical Perspective

I approached these questions from a social constructivist perspective (Ernest, 1994a, 1994b, 1994c, 1998, 2010), a perspective that offers a broad explanation for the way in which PTs' views of mathematical reasoning develop, both in their K – 12 learning experiences and within the narrower context of a one-semester course emphasizing mathematical argumentation. I do not present a full explanation of this philosophical perspective here but focus on his notion of “the social construction of subjective knowledge” (Ernest, 1998, p. 206) and how it frames this study. Ernest argues that any thinking beyond a rudimentary level—mathematical thinking in particular—is essentially *conversational* in nature and therefore relies on the use of language and symbols (1994c). The development of meaning for language and symbols therefore plays a central role in mathematical thinking and learning. He drew on Vygotsky's (1978) theory of language development, Harre's (1984) model of conceptual space, and Wittgenstein's writings on the philosophy of mathematics to support his argument for the social construction of mathematical knowledge. Ernest's theory encompasses both the reproduction of mathematical knowledge by students and teachers in schools and the creation of new mathematical knowledge by professional mathematicians.

Harre (1984) envisioned a “conceptual space” (p. 41) which locates ideas and psychological attributes in relation to their positions along two dimensions: (a) ownership—*individual* or *collective*—and (b) social location—*public* or *private*. Ernest (1998, 2010) used this model to describe the social construction of mathematical knowledge as a process that parallels the development of language. For the purposes of my study, I follow Ernest's approach but apply Harre's model more narrowly—to explain

the process through which PTs develop individual and collective ways of reasoning by engaging in mathematical argumentation. In doing so, I assume that particular ways of reasoning and ideas about reasoning can be attributed to—or “owned” by—either individuals or collective social groups. I also assume that PTs’ *intrapersonal* reasoning falls within the private social location, while public displays of reasoning occur in forms of *interpersonal* communication, such as classroom discussions or interviews.

Just as children first encounter language as adults use it collectively in public communication, PTs first encounter mathematical reasoning in the public and conventionalized mathematical explanations of teachers and textbooks in their K-12 schooling. This conventionalized form of reasoning is the basis for PTs’ initial understandings, a part of their initial frames of reference. However, throughout their semester in the Number and Operations course at the University of Missouri, PTs collectively engage in small-group and whole-class discussion of mathematical problems. Over the course of the semester, their personal understandings are increasingly influenced by the public, collective ways of reasoning that evolve within the Number and Operations classroom (see Q1 in Figure 1). Throughout the semester, public and collective reasoning plays a critical role in the development of PTs’ personal ways of reasoning.

Within the context of public, collective interaction, an individual PT may observe a particular way of reasoning and *appropriate* it, beginning to use it experimentally and imitatively in her own thinking (Q2). Through this internal conversation, she forms personal understandings of this way of reasoning, *interpreting* it and *transforming* it into her own (Q3). She may then use this transformed way of reasoning in her public mathematical arguments, thus *publishing* her own interpretation (Q4). Her public

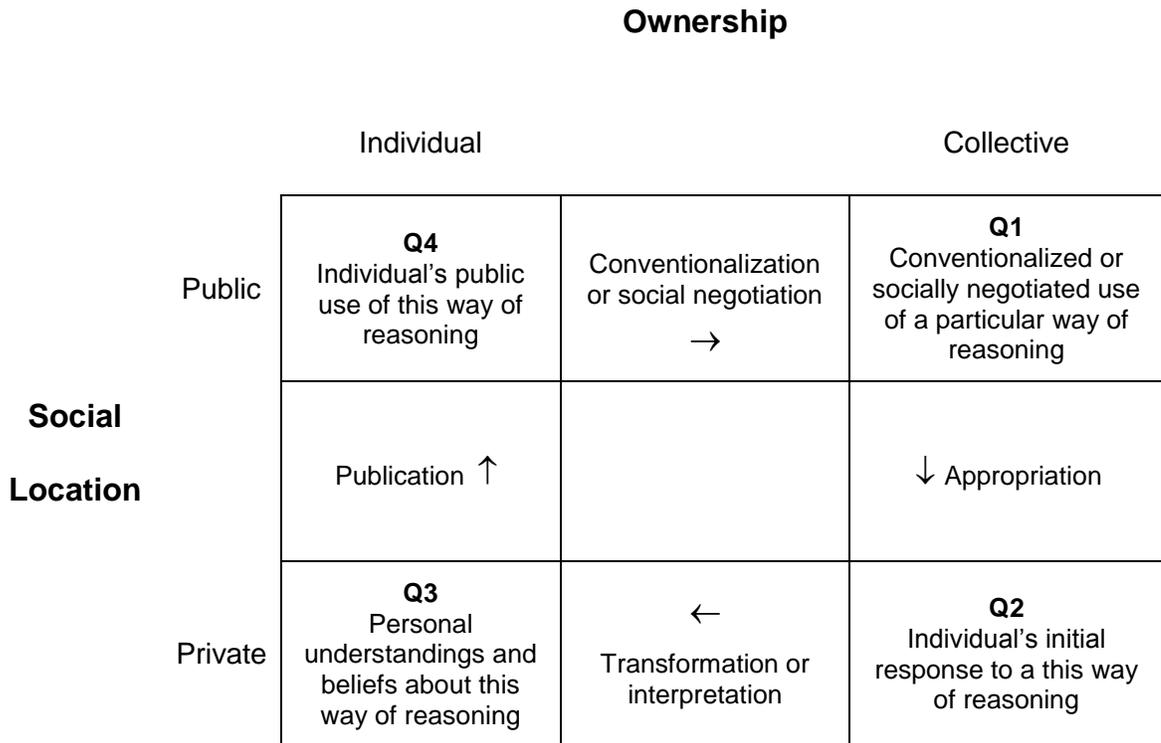


Figure 1.1. A model for the development of individual and collective ways of reasoning in social contexts (based on Ernest, 2010, p. 44).

reasoning can then be interpreted by others and either collectively accepted or critiqued and negotiated (completing the cycle by returning to Q1). Harre (1984) referred to this last stage as *conventionalization*. “The model thus describes an overall process in which individual and private meanings and collective and public expressions are mutually shaped through conversation” (Ernest, 2010, p. 45).

In this section, I used Harre’s model to explain a theoretical process by which PTs’ individual and collective understandings of mathematical reasoning evolve through engagement in mathematical argumentation. However, a single dissertation study cannot

reasonably examine this entire process in all of its details. Therefore, I have selected a part of the process as the focus of my investigation. In terms of Harre's model, my research questions focus on describing changes in Q3, PTs' personal understandings and beliefs about reasoning in mathematics. Viewed from this perspective, investigating PTs' subjective understandings does not imply treating them as isolated individuals.

Throughout their K -12 education, they have been immersed in a culture that has certain collective understandings, and their personal views are subjective interpretations of these broader cultural views. Likewise, after engaging in mathematical argumentation with other PTs over the course of a semester, changes in their subjective views reflect the commonalities in their shared experience.

Chapter Summary

According to the NCTM (1989, 2000) and the CCSSI (2012), mathematical argumentation should be a central feature of U. S. mathematics classrooms. However, research has shown that traditional mathematics instruction devotes little or no attention to this practice, and traditional instructional practices are persistent and difficult to change (Stigler & Hiebert, 1999). Based on the views of Kennedy (1999) and the NRC (1989), I contend that implementing this change should logically begin by engaging PTs in mathematical argumentation during their preservice training. To do so effectively, however, mathematics teacher educators need to understand both (a) how PTs view mathematical arguments when they begin this part of their training and (b) how their views change as a result of this experience. This study sought to provide useful information to aid in this effort.

CHAPTER 2: A REVIEW OF THE LITERATURE

In preparing to conduct this study, I reviewed the literature for two distinctly different purposes. First, I considered research with a similar focus, to determine what other researchers have found that might bear on my research questions and to gain insight into best practices for designing and conducting my study. Second, because I knew that I would need tasks to engage PTs in comparing and evaluating mathematical arguments, I reviewed the literature on mathematical justification to develop a theoretical framework for creating and selecting such tasks. I found that understanding Toulmin's model of informal reasoning was important in accomplishing both purposes, so I begin this chapter with a thorough explanation of this framework. I then discuss related research and my framework for developing mathematical argumentation tasks.

Toulmin's Model of Informal Reasoning

The research on mathematical argumentation I review here employs a theoretical framework or model that Toulmin developed for analyzing informal reasoning (Toulmin, 1958; Toulmin, Rieke, & Janik, 1979). In Toulmin's model, an argument includes up to six different elements: (a) claims, (b) grounds or data, (c) warrants, (d) backing, (e) rebuttals, and (f) qualifiers (see Figure 2.1). To understand the structure of an argument and the roles that these elements play, consider the following hypothetical conversation between Anna and her teacher.

Teacher: Is six-eighths the same as three-fourths?

Anna: Yes, it is, because three times two makes six, and four times two is eight.

Teacher: So why did you multiply?

Anna: If you multiply the numerator and denominator by the same number, you get an equivalent fraction. It's like you are splitting each part into smaller pieces. If you split each fourth into two equal parts, they would be eighths. You would have six-eighths.

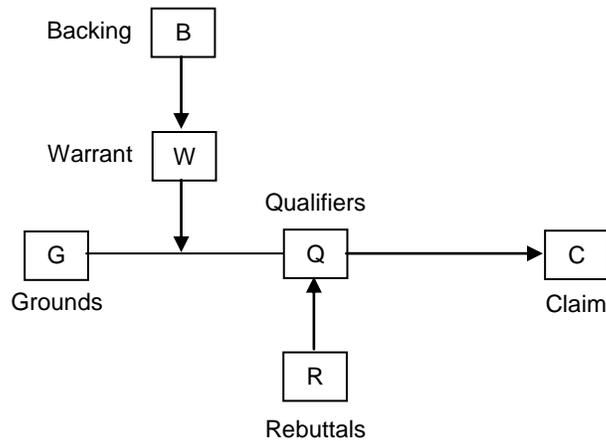


Figure 2.1. Toulmin's model of argumentation (based on Toulmin, Rieke, & Janik, 1979, p. 78).

Teacher: Does that always work?

Anna: Multiplying like that always gives an equivalent fraction—unless you multiplied by zero. Then the fraction wouldn't make any sense. Aside from that, you always get an equivalent fraction. Like I said, you are just splitting each part into smaller pieces. It's still the same amount; it has to be.

The arguer and the questioner. Anna's argument is, in some respects, typical of the way that arguments occur in everyday classroom conversation. Those engaged in the conversation play two different roles, the arguer and the questioner. In the conversation between Anna and the teacher, Anna plays the role of the arguer, and the teacher is the questioner. The *arguer* makes a claim but might provide no argument to support it if her claim is not challenged or questioned. The *questioner* challenges the arguer to explain her thinking, that is, to specify the elements that make up her argument, or at least those elements that are of interest to the questioner. In written arguments, the same individual typically plays both roles as he or she anticipates the challenges that might be made by a hypothetical questioner.

Claims and grounds. In Toulmin's model, the *claim* is a specific statement that the argument is constructed to support. *Grounds* or *data* are specific statements or facts that are presented as evidence to support the claim. In Anna's argument, she claims that three-fourths is equal to six-eighths. As grounds or data to support this claim, she offers the specific facts, $3 \times 2 = 6$ and $4 \times 2 = 8$.

Warrants, backing, and the argument's foundation. Toulmin et al. (1979) argued that specific statements are usually not sufficient to justify a claim. There must be a general rule or principle, either stated or implied, which provides a reason why the specific statements lead to the desired conclusion. This general rule is the *warrant* for the argument. Warrants are not usually accepted on faith alone; they also require support or *backing*. For example, as a warrant for her argument, Anna offers the general principle that multiplying the numerator and denominator of a fraction by the same number results in an equivalent fraction. She provides backing for her warrant with the explanation that, in general, multiplying in this way is equivalent to splitting each fractional part into smaller, equal-sized pieces. In this study, I refer to the backing and warrant together as the *foundation* of the argument.

Rebuttals. A general rule or principle sometimes has exceptions, special cases in which the rule does not apply. The questioner may point out such a special case, as a *rebuttal*, or arguer may anticipate potential rebuttals. In Anna's argument, she points out that the multiplication of numerator and denominator by zero is such an exception, identifying a possible rebuttal to her warrant.

Qualifiers. In informal arguments, even when all of the elements have been presented, the conclusion may not *necessarily* follow. In such cases, the arguer is

expected to qualify the conclusion appropriately, by using words or phrases such as *probably*, *apparently*, or *in most cases*. Such words or phrases are called *qualifiers*. In a *proof*, the conclusion follows necessarily from the argument, so no qualifiers are needed. However, in less formal mathematical argumentation, the arguer may be less than certain of the conclusion and may indicate this uncertainty by using a qualifier. In Anna's argument, she does not qualify her conclusion. She apparently considers it to be a proof. Figure 2.2 shows the full structure of Anna's argument as a Toulmin diagram.

Specific and general lines of reasoning. Arguments following Toulmin's model more contain two lines of reasoning, one general and the other specific. The first line of reasoning is that the grounds imply the truth of the claim ($G \rightarrow C$). In Anna's argument, this line of reasoning leads her to conclude that three-fourths is equivalent to six-eighths

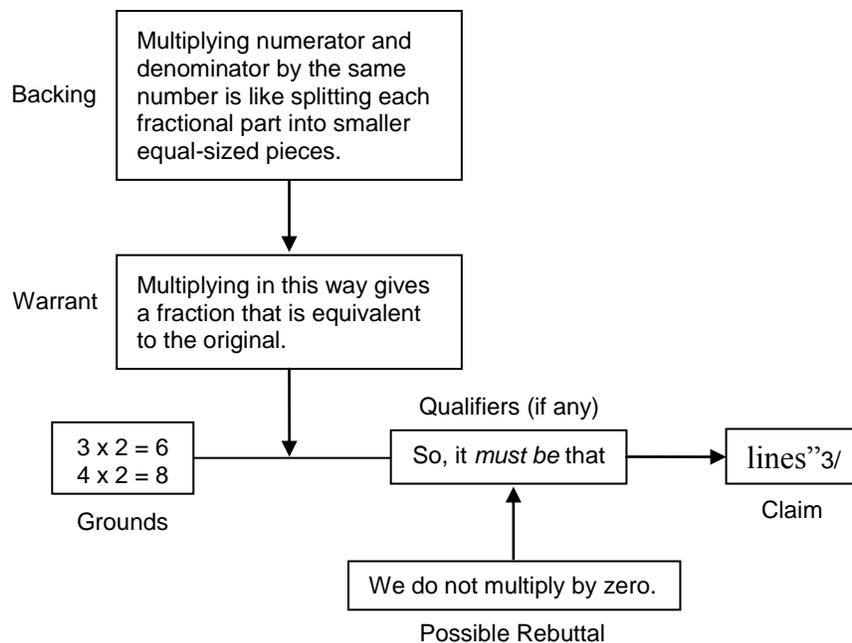


Figure 2.2. A Toulmin diagram of Anna's argument.

on the grounds that $3 \times 2 = 6$ and $4 \times 2 = 8$. Note that this line of reasoning is quite specific; it applies to just a single case. A second line of reasoning forms the foundation of the argument; the backing implies that the warrant holds ($B \rightarrow W$). This line of reasoning allows Anna to conclude that multiplying the numerator and denominator of a fraction by the same (non-zero) number always results in an equivalent fraction, based on her understanding of how fractional parts can be split into smaller, equal-sized pieces. This second line of reasoning ($B \rightarrow W$) is more general than the first ($G \rightarrow C$). In fact, $B \rightarrow W$ will always be general in arguments that follow Toulmin's model, since the warrant is "a general, step-authorizing statement" (Toulmin et al., 1979, p. 44) that includes $G \rightarrow C$ as a special case; symbolically, $W \rightarrow (G \rightarrow C)$. Therefore, when viewed through Toulmin's model, arguments about specific cases always depend upon more general lines of reasoning, if one considers warrants and backing and does not limit the discussion to grounds and claims.

In Toulmin's model of informal reasoning, claims are typically specific statements, whereas warrants are general rules or principles. In mathematical arguments, however, claims are often general rather than specific. In fact, a common image of *proof* in mathematics is that of an argument that establishes the truth of a general claim. In this study, I view such a line of reasoning to be at the general level ($B \rightarrow W$) in Toulmin's model. That is, such an argument seeks to establish a general warrant that can then be applied in specific instances in the future. Furthermore, the assertor may use a previously established warrant *implicitly*, assuming that such a warrant will be inferred and endorsed

by his (or her) intended audience, without explicitly stating it (Yackel, 2001). In effect, the backing for the warrant becomes, “We have already established this principle.”

As I discuss in the next section, researchers have used Toulmin’s framework to examine the role of classroom argumentation in students’ mathematical learning. In addition, as I discuss later in this chapter, it also plays an important role in my framework for creating and selecting argument evaluation tasks.

Research with a Similar Focus

In this section, I consider prior research that examined: (a) links between mathematical argumentation and student learning, (b) secondary school students’ views of mathematical justification, (c) PTs’ approaches to mathematical justification, and (d) the potential for mathematics methods courses to influence PTs’ mathematical beliefs. I review research in each of these areas, discussing how my study fits within the context of this prior research and the ways in which this research informed the design of the study.

Mathematical argumentation and student learning. Krummheuer (1995) appears to be the first researcher to use Toulmin’s framework to examine argumentation in mathematics classrooms. Drawing on data from a second grade classroom, he applied concepts from ethnomethodology, interactionism, and constructivism to argue for a theoretical link between classroom argumentation and students’ mathematical learning. He begins with a fundamental idea from interactionism and ethnomethodology—that individuals engage in social interaction as part of their ongoing efforts to both understand the actions of others and to make their own actions similarly understandable. The role of argumentation in this process is “to convince oneself as well as the other participants of

the [correctness] of one's own reasoning and to win over the other participants to this special kind of rational enterprise" (p. 247).

Krummheuer's argument relies on the concept of *framing*, in which individuals interpret the meaning of new experiences through *frames*, collections of cognitive routines that they have developed as a result of their prior experiences (Goffman, 1974). He posited that the *core* of an argument—composed of its warrant, grounds, and claim—is generally independent of framing. That is, if one accepts the appropriateness of the warrant, the truth of the claim follows from the grounds ($W \rightarrow (G \rightarrow C)$); its acceptance does not depend on the frames that individuals have previously constructed. Therefore, differences among individuals' acceptance of a claim occur primarily as the result of frame-dependent differences at the general level ($B \rightarrow W$). Krummheuer argued that, in their efforts to understand each other, individuals identify *structural similarities* between different arguments and modify their existing frames by developing new cognitive routines for dealing with forms of argument that have structural commonalities, specifically those that share the same warrant. He referred to such arguments as having the same *format*. The adaptive process he described is compatible with a constructivist view of learning. Thus, Krummheuer (1995) has drawn from a variety of theoretical sources to explain the mechanisms through which mathematical argumentation advances student learning.

Krummheuer (1995) illustrated this process with an episode in which second graders and their teacher discussed the problems shown in Figure 2.3. One student, Peter, used the result of the first problem to explain his solution to the second, saying, "All the

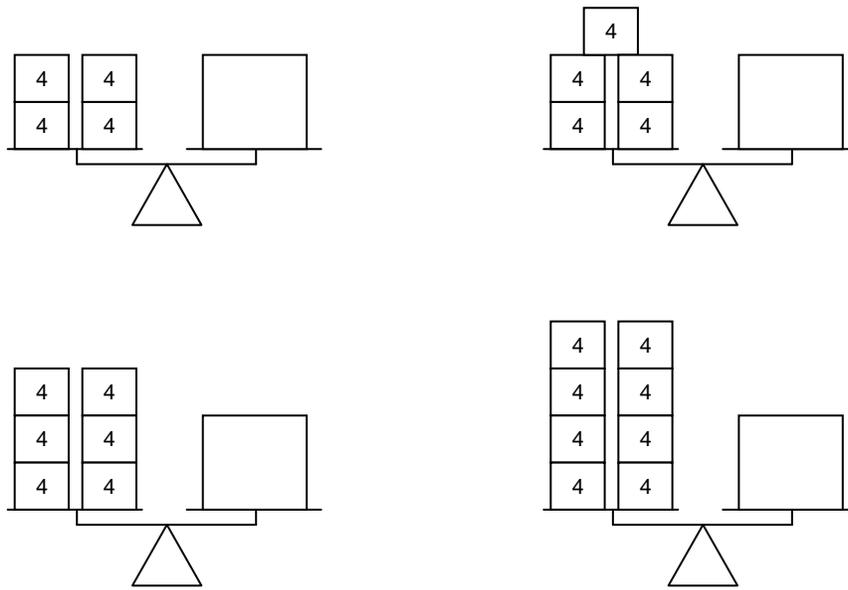


Figure 2.3. Balance problems for whole-class discussion (from Krummheuer, 1995, p. 256).

problems, you just add more fours. They just put more fours than the first problem” (p. 257). This statement provides the warrant and backing for his argument that 20 is the answer to the second problem, on the grounds that $16 + 4 = 20$. As other students reported their solutions to the third and fourth problems, they used the same warrant, producing arguments that followed the same format.

In terms of Krummheuer’s argument, Peter constructed an understanding of this situation that allowed him to obtain the solution to each problem from the previous one. In explaining his solution to the second problem, he was motivated by the basic human desire to be seen as rational and have the reasons for his actions understood by others. As the other children listened to Peter’s explanation and presented their own solutions, they

were also motivated to both understand Peter's actions and have their subsequent actions understood by others. They demonstrated their understandings of Peter's approach by producing similarly formatted arguments, thus providing evidence that each had constructed an understanding that is compatible with Peter's. If other children had reached conclusions that were incompatible with Peter's, Krummheuer would view this conflict at the specific level as the probable result of differing understandings at the general level, i.e. conflicting understandings of Peter's warrant and the backing that supports it. The absence of such conflicts can therefore be taken as evidence of shared understanding among the participants. Overall, Krummheuer's (1995) introduction of Toulmin's framework to mathematics education allowed him and others to use the structure of mathematical arguments—including their commonalities, differences, and varying levels of complexity—as a way to examine student learning.

Both Cobb (1999) and McClain (2009) cited Krummheuer (1995) and used Toulmin's framework to analyze arguments and document student learning in a classroom teaching experiment with 29 seventh-graders. Cobb (1999) described how a general scheme for data-based argumentation evolved during the course of the experiment (see Figure 2.4). He noted that, in their early arguments, students provided backing that the class later judged to be insufficient. A sociomathematical norm developed, requiring students to explain their warrants—decisions about structuring and interpreting data—by justifying them in relation to the question under consideration. Furthermore, a student's interpretation of a data set as a distribution—rather than a set of data points—initially required justification. However, as the teaching experiment

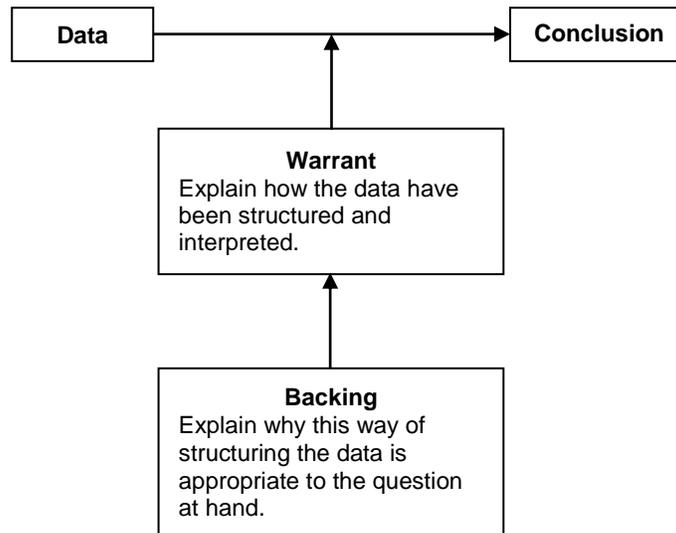


Figure 2.4. A scheme for data-based argumentation (from Cobb, 1999, p. 29).

progressed, such an interpretation became a matter of course and was no longer questioned. Cobb noted these changes, not only as indications of the development of taken-as-shared understandings by the members of the class, but also as examples illustrating that, in investigating the development of this type of understanding, researchers must focus not only on what students say and do but also on what they *cease* to say and do.

McClain (2009) analyzed data from the same teaching experiment, also using Toulmin's framework. Her analysis led her to hypothesize four phases in the development of students' arguments: (a) argument for defending, (b) argument for disagreeing, (c) argument for justification, and (d) argument for refinement. She illustrated these phases of argumentation with two classroom episodes in which she served as the teacher.

The first episode dealt with a data set on the longevity of batteries from two different brands, *Always Ready* and *Tough Cell*. The students were asked to decide which of these brands to recommend for use in the school's classroom calculators. In this episode, Juan questioned Carol about her reasons for splitting the data set into two equal groups. She responded by saying simply, "I wanted to go with half" (p. 233). Because Carol responded in a way that cut off further discussion rather than providing explicit backing for her decision, McClain classified her argument as an *argument for defending*.

Later in the discussion of the same problem, Brad proposed a solution that differed from Carol's. When questioned about his choice to use 80 hours as a cut point, he was able to supply backing for this decision, indicating that the battery they chose ought to be able to last at least 80 hours. Because the backing for his argument related directly to the question under consideration, his *argument for disagreement* carried more weight than Carol's. In the next phase, *argument for justification*, students routinely supplied backing to justify their decisions about how data should be displayed and interpreted.

The final phase, *argument for refinement*, emerged during a discussion of AIDS data that required students to compare two treatment groups of different sizes. The students collaborated to reach a consensus on which treatment was better. They then directed their efforts toward developing the most effective way to use the data to support their decision. For both Cobb (1999) and McClain (2009), analyses of students' arguments using Toulmin's model allowed them to focus on changes in the way that students structured their arguments and to point to these changes as evidence of students' deepening understanding.

Krummheuer (1995), Cobb (1999), and McClain (2009) used Toulmin's framework as a way to expose evidence of students' learning, autonomy, and independent thinking. This study sought to achieve similar goals but in a different context. The research reviewed in this section investigated argumentation in first, second, and seventh grade classrooms, while the current study focused on the argumentation of PTs. However, nothing in the research suggested that the application of Toulmin's framework should be limited to the argumentation of children. In fact, the work of Toulmin and his colleagues (Toulmin, 1958; Toulmin et al., 1979) suggests the opposite. Toulmin et al. (1979), for example, focused on argumentation—not in classrooms—but in the adult realms of law, science, the arts, business, and ethics.

Perhaps a more significant difference between this study and those described by Krummheuer (1995), Cobb (1999), and McClain (2009) lies in the type of understanding I sought to document. This study focuses on *individuals'* understanding of mathematical argumentation, whereas they highlighted collective argumentation and taken-as-shared understandings. Such a change in focus seemed quite feasible, however, because Krummheuer's (1995) theoretical link between argumentation and student learning applies to learning at the individual level, as well as collective understanding. However, this shift in focus necessitated a different approach to data collection. A greater focus on the individual required more emphasis on in-depth interviews, in order to collect evidence of PTs' subjective views of mathematical arguments.

Despite the differences I noted, Krummheuer (1995), Cobb (1999), and McClain (2009) offered valuable insights for the current study. Within the context of their methods coursework, PTs are learners, and the work of these researchers illustrates how the

analysis of learners' mathematical arguments can shed light on the learning process. In particular, the nature of the warrants and backing in learners' arguments provides critical information about their current state of understanding. Commonalities in the warrants and backing used by different individuals suggest that they have reached compatible understandings of the ideas under consideration (Krummheuer, 1995). In contrast, a change in the types of warrants and backing that an individual uses in her arguments points to her attainment of new and qualitatively different levels of understanding (Cobb, 1999; McClain, 2009). In addition, as Cobb (1999) noted, an important question that requires careful interpretation is whether the omission of explicit backing in an individual's argument indicates a lack of support for the warrant or the individual's implicit assumption that the warrant and its backing are already well-understood by the intended audience.

Secondary school students' views of mathematical justification. In this section, I briefly discuss research on secondary school students' beliefs about mathematical justification. On the surface, studies of this distinctly different population might appear to have little relevance for a study focusing on the views of PTs. However, from a social constructivist perspective, PTs' views of mathematical arguments are not narrowly restricted to this group, but reflect broader cultural beliefs. Furthermore, Kennedy's (1999) analysis suggests that the frames of reference PTs use to evaluate mathematical arguments develop primarily through their experiences as K – 12 students. Therefore, research examining the views of secondary school students could provide valuable insights—or at least some interesting points of comparison—for this study.

Most research in this area has focused on beliefs about mathematical proof. For example, Chazan (1993) investigated high school students in a technology-enhanced geometry course, focusing on two sets of student beliefs: (a) the belief that evidence is proof and (b) the belief that proof is evidence. Students who believe that evidence is proof see the truth of a mathematical statement in a few cases as sufficient to establish its truth in all such cases. Students who believe that proof is evidence may follow and accept a deductive argument for a general mathematical statement but still think that additional evidence is necessary to establish the truth of the statement, or they might believe that exceptional cases may still exist despite the validity of the general argument. Through interviews, Chazan found a significant number of students possessed these kinds of beliefs and that they were generally quite good at articulating the reasons for their beliefs.

Two of his findings are noteworthy. First, he observed that textbooks often presented proofs in the form of figure with the words “Given” and “To Prove” followed by statements pertaining to the given figure. Chazan found that many students interpreted this textbook version of proof as simply a proof about this particular figure, not the verification of a general statement. Second, he found that the students he interviewed showed a strong preference for the explanatory function of proof over the power of proof to establish certainty. He suggested that the use of proof to provide insight into why a statement is true should be emphasized in introducing students to deductive proof.

In another study, Healy and Hoyles (2000) used surveys and interviews to examine the beliefs about proof held by high-attaining 14- and 15-year-old students, including their beliefs about: (a) the meaning and purpose of proof, (b) the kinds of arguments students found most convincing for themselves, (c) the kinds of arguments

they thought would receive the highest marks from teachers, (d) the correctness and generality of different kinds of arguments, and (e) the explanatory power of different kinds of arguments. On the purpose of proof, they found that, contrary to Chazan's (1993) findings, the students in their sample tended to list verification before explanation. One possible explanation for this difference may be that the sample selected by Healy and Hoyles (2000) came only from the top 20-25% of students. Healy and Hoyles also found that, although students frequently used empirical arguments when constructing their own proofs, most were aware that these arguments were not general. Given a choice between several types of arguments presented to them, they preferred narrative arguments, viewing them as both general and explanatory. In contrast, they generally selected formal algebraic arguments involving complicated formulas as most likely to get high scores from their teachers.

Likewise, using surveys and interviews with high school students, McCrone and Martin (2009) found many of the same beliefs identified by Chazan (1993) and Healy and Hoyles (2000). However, they also found students who expressed the view that, although proofs are *intended* to convince someone or explain why a statement is true, they actually do so for only a small circle of insiders, such as teachers, who have special knowledge and understanding of the language and methods of proof. Those without this special knowledge would find other approaches, such as diagrams and empirical evidence, more convincing. Some students viewed proof primarily as a means for students to demonstrate their knowledge for the teacher, showing that they knew the right theorems to use and how to apply them to particular figures.

Taken together, the work of Chazan (1993), Healy and Hoyles (2000), and McCrone and Martin (2009) suggests that secondary school students' beliefs about the nature and purpose of proof are quite complicated. Their own proof-attempts tend to be empirical; those they find most convincing tend to be narrative; and those they think would get the best scores from their teachers are algebraic. Conflicting evidence exists about whether they view the primary purpose of proof as justification (Healy & Hoyles, 2000) or explanation (Chazan, 1993). Some believe that proof really serves neither purpose, viewing it simply as a way for students to demonstrate their knowledge for the teacher (McCrone & Martin, 2009).

PTs' approaches to mathematical justification. In this section, I review research that focused on PTs' approaches to mathematical justification, including: (a) studies that examined specific types of reasoning and (b) studies that considered mathematical justification more broadly.

PTs' approaches to specific types of reasoning. In their efforts to investigate the reasoning methods of preservice elementary teachers, some researchers focused narrowly on specific types of reasoning or proof. Gholamazad, Liljedahl, and Zazkis (2003), for example, examined the written work that 116 PTs submitted in response to two questions asking whether the set of perfect squares and the set of odd numbers are closed under multiplication. They classified PTs' responses based on whether or not they: (a) recognized the need for proof, (b) recognized the need for a useful representation, (c) chose such a representation, (d) manipulated the representation appropriately, and (e) correctly interpreted the results. They found that, although most of the PTs in their study seemed to both understand the concept of closure and recognize the need for proof, many

failed to choose an appropriate representation or were unable to manipulate the representation appropriately.

Stylianides, Stylianides, & Philippou (2004, 2005, 2007) compared preservice elementary and secondary mathematics teachers to investigate commonalities and differences in their understandings of particular aspects of mathematical reasoning and proof. In the first of these studies (Stylianides et al., 2004), they focused on prospective teachers' understandings of the equivalence of a conditional statement and its contrapositive, in both verbal and symbolic contexts. They found that the preservice secondary teachers performed equally well in both contexts. The prospective elementary teachers, however, were far more likely to respond incorrectly in a symbolic context than in a verbal context. Data from interviews suggested that the preservice elementary teachers tended to evaluate a given symbolic proof using superficial criteria—for example, saying that it was too short to be a valid proof.

Using the same comparison groups, Stylianides et al. (2005, 2007) also investigated prospective teachers' understandings of three ideas related to mathematical induction: (a) the meaning of the base step, showing that $P(n)$ is true for some initial value $n_0 \in N$; (b) the meaning of the inductive step, showing that $P(n) \Rightarrow P(n+1)$ for all $n \in N$ with $n \geq n_0$; and (c) the possibility of a difference between the truth set U for a given statement $P(n)$ and the domain of discourse D over which a given proof is valid. They found that, when given a fallacious inductive proof that omitted the base step, a significant proportion of both groups believed that this proof still implied the existence of some yet to be specified subset of N for which the conclusion held true. Furthermore, when given an inductive proof of a statement for which the domain of discourse differed

from the statement's truth set, a significant proportion of both groups accepted the proof as valid but believed, on the basis of this proof, that the statement only holds true over its domain of discourse, suggesting more generally that PTs may have difficulty understanding the domain to which a particular argument applies.

Lo, Grant, and Flowers (2008) investigated the mathematical reasoning of PTs specifically within the context of multi-digit multiplication. The authors identified several challenges that PTs faced in developing and justifying their problem-solving strategies in this context. Some PTs seemed to transfer ideas related to addition inappropriately to multiplication. For example, rather than thinking of 17×36 as 17 groups of 36 or a 17 by 36 array, they seemed to view it as a way of combining 17 items and 36 items. Others invented erroneous strategies, such as $(18 \times 26) = (10 \times 20) + (8 \times 6)$, by analogy to strategies that work for addition, where $(18 + 26) = (10 + 20) + (8 + 6)$. Another challenge involved understanding that, in the context of the methods course, the goal was not getting the correct answer, but justifying a reasoning strategy. Some PTs struggled with the distinction between explaining what they did and explaining why their strategy works. Others seemed to consider a picture alone to be sufficient justification. PTs also struggled with coordinating their explanations with the area/array and equal-groups interpretations of multiplication. For example, some explanations interchanged the number of groups and the number in each group without offering any justification for doing so.

Each of these studies (Gholamazad et al., 2003; Stylianides et al., 2004, 2005, 2007; Lo, Grant, & Flowers, 2008) focused on PTs' understanding of mathematical reasoning within a rather narrow context—proofs about the concept of closure, the

relationship between a conditional statement and its contrapositive, proofs by mathematical induction, and justifications of multi-digit multiplication strategies. Across these differing contexts, the researchers found that PTs face serious challenges in reaching a sophisticated understanding of mathematical reasoning, challenges that are also relevant to more general studies of PTs' conceptions of reasoning, such as the study I propose. For example, Gholamazad et al. (2003), Stylianides et al. (2004), and Lo et al. (2008) all found that PTs encountered a variety of difficulties in creating, manipulating, or interpreting symbolic representations. Some of these difficulties may be symptomatic of reasoning about mathematical symbols as objects rather than reasoning about the mathematical objects that the symbols signify, bearing directly on this study's investigation of the substance of PTs' mathematical arguments. In addition, PTs' difficulties with interpreting the contrapositive of a given statement (Stylianides et al., 2004) can have serious consequences for their understanding of the role of counterexamples in a broad range of contexts, including those in this study. Likewise, the findings of Stylianides et al. (2005, 2007) in the context of mathematical induction point to the potential for PTs to encounter similar difficulties in interpreting the domain of discourse for other types of arguments. Lastly, Lo et al. (2008) found that some PTs initially believed that a correct answer, a picture, or a description of their solution procedure would be an adequate justification of their solutions of multi-digit multiplication problems. The potential for intensive experience in mathematical argumentation to change such beliefs across the broader context of number and operation is a major focus of this investigation.

Broader studies of PTs' approaches to mathematical justification. Some researchers have taken a broader view in looking at PTs' methods of justification and proof. Martin and Harel (1989), for example, presented 101 preservice elementary teachers with a familiar mathematical generalization—one that had been proved and discussed in their mathematics class—and a similar unfamiliar generalization. For each generalization, they included a variety of justifications: (a) two particular instances using small numbers, (b) a table showing the systematic checking of a sequence of 12 particular instances (for the familiar generalization only), (c) one particular instance using a very large number that would appear to have been selected arbitrarily, (d) two examples, one that meets both the conditions and the conclusion and another that satisfies neither, (e) a general deductive proof, (f) a nonsensical argument that has the superficial appearance of a general deductive proof, and (g) a particular proof, following the same form as the general deductive proof but using specific numbers instead of variables. They asked the PTs to rate each justification on a scale from 1 to 4, where 4 indicated that they considered it a valid proof and 1 indicated that they did not consider it a valid proof. They found that a high proportion of the PTs accepted both inductive and deductive arguments as proof of a mathematical generalization, regardless of whether the generalization was familiar or not. In addition, many accepted the nonsensical argument as proof, suggesting that their evaluations relied on appearance rather than a meaningful assessment of the reasoning involved. Overall, the results demonstrated that a significant proportion of PTs evaluated the validity of the arguments presented to them in a way that is not compatible with the views of validity in mathematics as a discipline, despite the fact that their

mathematics course provided “extensive and explicit instruction about the nature of proof and verification in mathematics” (p. 50).

Taking a very different approach from Martin and Harel (1989), Simon and Blume (1996) conducted a teaching experiment in the context of a mathematics course for preservice elementary teachers in which Simon was the instructor. They sought to understand the negotiation of classroom norms related to justification and validity in this context and to identify key issues that affected the establishment of these classroom norms. In whole-class discussions, Simon pressed PTs to explain their solution methods, why their methods worked, and whether they would always work. PTs’ initial responses showed a preference for pragmatic explanations of motivation, appeals to authority, and empirical arguments. When one PT gave what seemed to Simon to be a valid explanation, it had little impact on the rest of the class. PTs also initially resisted Simon’s efforts to establish classroom norms requiring more sophisticated forms of justification. They failed to see the validity of their current methods as problematic and therefore had difficulty understanding why he questioned these methods.

Simon and Blume (1996) described the challenges of renegotiating classroom norms for justification. They noted the “tension between attempting to establish a new vision for mathematics in the classroom community and responding sensitively to students to help them develop a sense of competence and empowerment” (p. 29). They also identified a critical relationship between methods of justification and mathematical understanding, both for individuals and for the classroom community as a whole. For the community, taken-as-shared knowledge needs no justification. This explains PTs’ initial resistance to changing norms for justification; such a change calls into question their

unexamined assumptions about the security of their own mathematical knowledge. In addition, PTs' current conceptual understandings limit what they can accept as valid justifications; the presentation a logical deductive argument does not guarantee its acceptance by the members of the class. The authors also noted some hopeful signs that, as PTs gained experience in justifying their solution methods, they began to view the purpose of justification to be—not the establishment of secure solutions to individual problems—but the development of a broader, more conceptual understanding of mathematics.

A study by Morris (2007) is unique among those I have examined in that it investigated PTs' evaluations of mathematical arguments that children had provided in real classroom situations. For the first part of her interviews with 34 PTs, Morris created transcripts by combining minimally edited excerpts from two lessons in which a third grade class attempts to justify their conjecture that the sum of two odd numbers is always even. She purposefully selected these lessons, because they offered a wide variety of different arguments, including one student's valid, general, deductive proof. Because Morris was interested in the consistency of PTs' evaluations under different conditions, she created two different transcripts, one that contained the valid deductive proof and one that omitted it but was otherwise identical. To further investigate how PTs evaluated mathematical arguments, she also presented PTs with five arguments that she had written, concerning the conjecture that $n^2 + n$ would always be even, for any counting number n .

Morris's (2007) conceptualization of validity differs significantly from my own. In the justification of true general statements, I consider explanatory methods of reasoning to be valid, regardless of whether they rely on a specific case (a generic

example) or provide a general deductive argument. Perhaps because Morris focused specifically on *proof*, rather than argumentation, she only considered general deductive arguments to be valid and not generic examples, which she called *single-case, key idea arguments*, based on Raman's (2003) view of the importance of key ideas in mathematical proof.

Morris (2007) found that the PTs generally evaluated students' arguments using one of two different approaches. Some valued explanation and gave high ratings to single-case, key idea arguments. Others valued empirical approaches and gave high ratings to arguments that tested and confirmed multiple instances in which the generalization held true, even though most PTs acknowledged that these arguments failed to explain *why* the generalization is true. "When preservice teachers were asked to identify the 'best' arguments that proved a generalization, very few pre-service teachers claimed they were looking for general, valid arguments" (p. 492). The presence or absence of the valid general proof from the transcript did appear to affect PTs evaluations. However, it did not have the effect that Morris had anticipated; only 12% of the PTs correctly identified the general deductive argument as the only valid proof presented to them. Morris concluded that "many preservice teachers did not appear to understand the relationships among mathematical proof, explaining why, and inductive arguments" (p. 510).

Lastly, in a study of 39 preservice elementary teachers, Stylianides and Stylianides (2009a) investigated PTs capabilities in both constructing proofs and evaluating their own constructions. They found that, when asked to prove a mathematical generalization, PTs often produced flawed general arguments or empirical justifications.

However, when PTs were asked whether they had actually produced a proof, half of those who had given incorrect responses demonstrated an awareness of the weaknesses in their own arguments. Other researchers have noted similar results in other populations (e.g., Healy & Hoyles, 2000; Raman, 2003). Such findings support the conceptualization of the construction and evaluation of mathematical arguments as separate but related processes.

Together, these studies (Martin & Harel, 1989; Simon & Blume, 1996; Morris, 2007; Stylianides & Stylianides, 2009a) indicate that PTs' approaches to justification include pragmatic explanations of motivation, appeals to authority, empirical arguments, and generic examples. Their ideas about justification and validity are linked to their beliefs about the nature of school mathematics and not easily changed. They may have little understanding of the relationship between proof, explanation, and inductive argument, and their evaluations of mathematical arguments can be inconsistent. In addition, the arguments they produce may not necessarily reflect their notions of validity; many are capable of recognizing the weaknesses in their own arguments.

Mathematics methods courses and PTs' beliefs. Because the ways that PTs will teach mathematics are inextricably tied to their beliefs about the nature of mathematical knowledge and how it is acquired, researchers within the mathematics education community have recognized the need for preservice teacher education programs to influence PTs' beliefs, and they investigated a variety of approaches addressed at achieving that goal. Some investigated early field experiences as a way of changing PTs' beliefs (e.g., Ambrose, 2004; Ebby, 2000; Philipp et al., 2007; Grootenboer, 2008). Others focused on the exploration of children's mathematical thinking (e.g., Philipp, Thanheiser, & Clement, 2002; Friel & Carboni, 2000; Tirosh, 2000). In this study,

however, I focus on a third approach, engaging PTs in reform-oriented mathematics experiences (Wilcox, Schram, Lappan, & Lanier, 1991; Szydlik, Szydlik, & Benson, 2003; Stylianides & Stylianides, 2009b; Taylor, 2003).

Wilcox et al. (1991) studied 23 PTs across four semesters in a preservice education program at Michigan State University. The program utilized three design principles to facilitate change in PTs' beliefs: (a) fostering a conceptual understanding of mathematics by engaging PTs in "*doing* mathematics—analyzing, abstracting, generalizing, inventing, proving, and applying" (p. 5); (b) encouraging PTs to communicate their mathematical ideas using multiple representations, including spoken and written natural language, mathematical symbols, and a variety of concrete and visual representations; and (c) developing a learning community in which PTs routinely solved problems collaboratively and reported their findings to the whole class in the spirit of mathematical inquiry. The researchers focused on how particular norms within the learning community helped to change PTs' beliefs about the teaching and learning of mathematics. They found that establishing norms requiring them to work together and share responsibility for making sense of problems and solving them led the PTs to value multiple approaches in problem solving and to gain confidence in their ability to assess the validity of mathematical arguments and collectively determine whether a correct solution had been reached.

In a similar study, Szydlik et al. (2003) examined how classroom norms governing mathematical argumentation influenced PTs' mathematical beliefs. In a mathematics class for PTs, they sought to establish two beliefs they associated with mathematical autonomy: (a) "mathematics is a logical and consistent discipline as

opposed to a collection of facts, and therefore” (b) “mathematics is something that can be figured out as opposed to something that must be handed down by authority” (p. 256). Data from interviews and questionnaires showed that, through their participation in the class, a significant number of PTs came to see mathematics as a more sensible subject than they had initially thought, one in which students can figure out their own methods to solve problems. Some also expressed a greater interest in mathematics and greater confidence in their own problem-solving abilities. Confirming the findings of Wilcox et al. (1991), PTs attributed the changes in their beliefs primarily to working collaboratively with other students as part of a mathematical community. Other class norms that PTs cited as instrumental in changing their beliefs included: (a) addressing deep and challenging problems, (b) seeing multiple solution strategies from other students, and (c) validating those solutions through class discussion rather than instructor approval.

Taylor (2003) taught an undergraduate mathematics methods class for PTs, aiming at goals similar to those of Szydlik et al. (2003). She recounted her experiences in this class along with her initial interpretations of these experiences. Like Simon and Blume (1996), she initially found that students resisted her efforts to change their conceptions of how mathematics is taught and learned. However, after re-examining certain classroom episodes and analyzing them using Bakhtin’s concept of *appropriation*, she changed her initial interpretation. In Bakhtin’s terminology, appropriation occurs when a speaker or writer uses the words of another for his own purposes. This process of appropriation often involves a struggle over the meaning of the words and also over who will determine how and for what purpose the words will be used. By analyzing one PT’s appropriation of her words in two classroom episodes, Taylor changed her initial

interpretation of student negativity, close-mindedness, and resistance to one in which the student was struggling to find meaning in an unfamiliar context and to use unfamiliar language for her own purposes. A broad view of Taylor's study suggests that particular attention should be given to the ways that PTs interpret and use not only language but also symbols and diagram as well.

Stylianides and Stylianides (2009b) investigated an instructional sequence that was designed to help PTs “realize the limitations of empirical arguments as methods for validating mathematical generalizations and see an intellectual need to learn about secure methods for validation (i.e., proofs)” (pp. 315-316). In particular, they focused on the role of *pivotal counterexamples* (Zazkis & Chernoff, 2008) and *cognitive conflict* (e.g., Hadas, Hershkowitz, & Schwarz, 2000) in creating an intellectual need for proof. A pivotal counterexample is an example that conflicts with students' current understandings and therefore creates cognitive conflict for that student. Over a 4-year period, Stylianides and Stylianides (2009b) designed, implemented, and refined an instructional sequence aimed at creating an intellectual need for proving among PTs in an undergraduate mathematics course. They included two activities—the *Circle and Spots Problem* and the *Monstrous Counterexample Illustration*—intended to function as pivotal counterexamples. Their investigation revealed evidence that, at least for some PTs, the two activities functioned as they intended. “The students ... seemed to have recognized empirical arguments as insecure methods for validating mathematical generalizations” (p. 344), thus laying the groundwork for the introduction of proof as a more secure method of justification. This study demonstrates that carefully designed mathematical activities can have a significant impact on PTs' beliefs about mathematical reasoning, and it highlights two key

concepts—cognitive conflict and pivotal counterexamples—that may be important in changing PTs’ beliefs.

Section summary and conclusions. The literature reviewed in this section had important implications for the design of this study, the analysis of data, and the interpretation of findings. Researchers using Toulmin’s framework to analyze students’ mathematical arguments have demonstrated that this type of analysis can be effective in revealing subjects’ current understandings, with changes in the warrants and backing in their arguments therefore indicating corresponding changes in their understandings. Other researchers have revealed that secondary school students, secondary school teachers, and PTs share many beliefs about mathematical justification that differ significantly from those in the mathematical community. Studies have also shown that rich mathematical experiences can provide effective opportunities to change PTs understandings and beliefs about mathematical justification.

However, in Chapter 1, I argued that, in order to develop courses and programs that effectively engage PTs in mathematical argumentation, mathematics teacher educators need to understand both (a) how PTs view mathematical arguments when they begin this part of their training and (b) how their views could change as a result of this experience. Prior research has not adequately addressed these needs. Researchers such as Krummheuer (1995), Cobb (1999), and McClain (2009) used Toulmin’s framework to study the effects of classroom argumentation, but they focused on younger students—grade 7 and below. Studies of secondary school students, such as Chazan’s (1993), Healy and Hoyles’ (2000), and McCrone and Martin’s (2009), might more closely reflect the views that PTs bring to the start of their preservice training, but they did not, of course,

address the ways in which PTs views might change a the result of their participation in a one-semester course emphasizing mathematical argumentation.

Within research on the forms of reasoning PTs use and endorse, several studies focused on very specific types of reasoning, such as proofs about the concept of closure (Gholamazad et al., 2003), the relationship between a conditional statement and its contrapositive (Stylianides et al., 2004), proofs by mathematical induction (Stylianides et al., 2005, 2007), and justifications of multi-digit multiplication strategies (Lo, Grant, & Flowers, 2008). These studies fail to provide the broader understanding of PTs views of argumentation that mathematics teacher educators require.

Other studies that examined PTs broader views of mathematical justification failed to provide the necessary focus on *changes* in PTs views. Most of these studies examined PTs views of justification at only a single point in time (e.g., Martin & Harel, 1989; Morris, 2007; Stylianides & Stylianides, 2009a). Providing the only exception, Simon & Blume (1996) took a more longitudinal view, collecting data in a teaching experiment that ran over the course of a semester. However, they cautioned against interpreting the observed differences in PTs arguments as indications of PTs' changing views of justification, because the nature of the problems they addressed varied across different episodes. This limitation suggests that studies, such as mine, that attempt to describe changes in PTs' views should examine their responses to either similar or identical problems at different points in time.

Finally, among studies that focused on the potential for mathematics methods courses to change PTs mathematical beliefs, most did not focus on PTs views of mathematical arguments per se but rather on more general beliefs about mathematics,

such as the belief that mathematics is a logical discipline (Szydlik, Szydlik, & Benson, 2003). Stylianides & Stylianides (2009b), however, focused on how particular instructional design features affected PTs' views of empirical arguments, leading them to see such arguments as insecure and therefore priming them for the idea of mathematical proof. By focusing narrowly on PTs' beliefs about the security of empirical arguments, this study fails to capture the full extent of the change I sought to document. Therefore, although the research I reviewed here provided valuable insights for designing and conducting this study, it did not fully address the needs I identified in Chapter 1.

Mathematical Argumentation Tasks

In this section, I build on ideas in the research literature to present a framework for classifying mathematical argumentation tasks. Mathematical argumentation involves two different but related aspects, creating mathematical arguments and evaluating the arguments of others. Because arguments must first exist before anyone can evaluate them, it makes sense to first explore the various possibilities for argument construction before proceeding to argument evaluation. In presenting this theoretical framework, I also draw on ideas from the literature to describe my conceptions of the *form* and *substance* of mathematical arguments.

Argument construction tasks. To create and select a variety of tasks that elicit the construction of mathematical arguments, I needed to consider fundamental similarities and differences that such tasks might have. As a starting point, I used a framework that Stylianides and Ball (2008) developed for classifying *proving tasks*. Their classification scheme takes into account two aspects of these tasks: (a) the *number of*

cases involved—one, multiple but finitely many, or infinitely many—and (b) the *purpose* of the task—to either verify or refute a mathematical statement (see Figure 2.5).

Level of generality. In adapting Stylianides and Ball’s (2008) framework for use in this study, I needed to address two issues. The first was how their framework fits with Toulmin’s. In Toulmin’s model, one line of reasoning applies to only a *single case*, while another line of reasoning is more general. In Anna’s argument, which I discussed earlier in this chapter, this second line of reasoning includes *infinitely many cases*. Thus, Toulmin’s model recognizes only two categories based on the number of cases. To reconcile the two frameworks, I needed to decide how to deal with tasks that involve *multiple but finitely many cases*.

Consider, for example, a task that involves verifying the truth of n different statements. If n is reasonably small, the task might be approached as a collection of n distinct single-case verifications. Conversely, if n is unreasonably large, finding a general

Purpose of a proving task	Number of cases involved in a proving task		
	A single case	Multiple but finitely many cases	Infinitely many cases
Verification of a statement	Prove that 186 plus 243 is a multiple of 3.	Prove that the sum of any two multiples of 3 between 30 and 50 is a multiple of 3.	Prove that the sum of any two multiples of 3 is a multiple of 3.
Refutation of a statement	Disprove that 186 plus 243 is a multiple of 6.	Disprove that the sum of any two multiples of 3 between 30 and 50 is a multiple of 6.	Disprove that the sum of any two multiples of 3 is a multiple of 6.

Figure 2.5. Types of proving tasks with illustrative examples (from Stylianides & Ball, 2008, p. 312).

argument that applies to all cases at once would seem preferable. Either way, it may not be necessary to consider these multiple-case tasks as a separate category, since they can be approached in the same manner as either single-case tasks or those with infinitely many cases. Therefore, I omitted such tasks from this study. I considered only two categories related to the number of cases involved, separating tasks according to their level of generality, *specific* (those involving a single case) or *general* (those with infinitely many cases).

Purpose, truth value, and orientation. The second issue involved the way in which tasks are presented. A superficial reading of the examples in Figure 2.5 might suggest that Stylianides and Ball (2008) considered only tasks in which the purpose aligns with the truth value of the statement in question, verification for true statements and refutation for false statements. Reading their illustrative examples, one could easily get the impression that the purpose of the task—to verify or refute (or to prove or disprove)—is always explicitly stated in the task as it is presented to students. Elsewhere, however, they addressed the complications that can result when various members of the classroom community have conflicting ideas about the truth value of the statement under consideration and therefore approach the task with different purposes in mind. In this study, I allowed for the possibility—even the likelihood—that some PTs would attempt to verify false statements or refute true ones. To do so, I replaced what Stylianides and Ball call *purpose* with *orientation*, a characteristic of the argument, rather than a characteristic of the task. I classified construction tasks by the *truth value* of the statement in question, leaving the orientation of the argument—verification or

refutation—to the individual, rather than considering it part of the task as presented to them (see Figure 2.6).

As Figure 2.6 illustrates, argument construction tasks can be classified into four types—TS, TG, FS, and FG—depending on the truth value and level of generality of the statement under consideration. For example, TS indicates that the task is based on a *true specific* statement. It is also important to understand that the examples in Figure 2.6—and argument construction tasks in general—should not be thought of as “test items.” They are problems designed to engage PTs in discussion of mathematical arguments, not to quantify their abilities in constructing such arguments.

Argument evaluation tasks. An argument evaluation task is designed to prompt PTs to analyze a given argument, assess its strengths and weaknesses, and decide whether, in their view, it is valid or invalid. Each such task includes, embedded within it,

Generality	Specific Statement	General Statement
Truth		
True Statement	(TS) Bill had two brownies of the same size. He ate $\frac{3}{4}$ of one and $\frac{2}{4}$ of the other. Bill says he ate $\frac{5}{8}$ of the two brownies. Is Bill correct or incorrect? Justify your answer.	(TG) True or false? If you add two even counting numbers, the result is always even. Justify your answer.
False Statement	(FS) Jill had two brownies of the same size. She ate $\frac{3}{4}$ of one and $\frac{2}{4}$ of the other. Jill says she ate $\frac{5}{4}$ of the two brownies. Is Jill correct or incorrect? Justify your answer.	(FG) True or false? If you add two whole numbers, the result is always larger than both of the original numbers. Justify your answer.

Figure 2.6. A framework for classifying argument construction tasks into four types, using the truth value and generality of the statement in question.

both an argument construction task and an argument to be evaluated (see Figures 2.7a and 2.7b).

In classifying argument construction tasks, I noted that they can be grouped into four categories (TS, TG, FS, and FG), based on the truth value and generality of the statement under consideration. Likewise, argument evaluation tasks can be grouped into the same four main categories, based on the classification of the embedded construction task. Within each of these categories, however, three different argument-types might be presented to PTs, based on the *validity* and the *orientation* of the argument. For example, an argument based on a true, specific statement can result in a valid verification (VV), an invalid verification (IV), or an invalid refutation (IR). A valid refutation (VR) is impossible in this situation because the statement under consideration is true.

Similarly, in situations in which the conjectured statement is false, a valid verification (VV) is impossible. Thus, using a four-letter string, an argument evaluation task can be classified as belonging to one of twelve different types. FSIR, for example, indicates that the task is based on a false specific statement and that the argument directed at this statement is an invalid refutation.

Forms of reasoning in mathematical arguments. Researchers studying mathematical proof have focused attention on different types of reasoning that students use to justify true general statements. Simon and Blume (1996), for example, employed a five-level framework to categorize PTs' methods of justification in such cases (see Table 2.1). In this framework, Levels 3 and 4, generic examples and general deductive arguments, are considered valid; other reasoning methods, such as appeals to authority and empirical demonstrations, are judged to be invalid.

Generality	Specific Statement	General Statement
Truth		
True Statement	<p>Valid Verification (VV)</p> <p>Problem (TS): Bill had two brownies of the same size. He ate $\frac{3}{4}$ of one and $\frac{2}{4}$ of the other. Bill says he ate $\frac{5}{8}$ of the two brownies. Is Bill correct or incorrect? Explain your answer.</p> <p>Amy's Argument (VV): Bill is correct. If each brownie is cut into 4 equal pieces, then the two brownies combined are cut into eight equal pieces. Bill ate 5 of the 8 equal pieces, so he ate $\frac{5}{8}$ of the two brownies combined.</p> <p>Invalid Verification (IV)</p> <p>Problem (TS): True or false? $\frac{5}{6} = \frac{10}{12}$. Explain your answer.</p> <p>Georgia's Argument (IV): That is true. $\frac{5}{6}$ is the same number as $\frac{10}{12}$. They are the same amount, so they are equal numbers.</p> <p>Invalid Refutation (IR)</p> <p>Problem (TS): Bill had two brownies of the same size. He ate $\frac{3}{4}$ of one and $\frac{2}{4}$ of the other. Bill says he ate $\frac{5}{8}$ of the two brownies. Is Bill correct or incorrect? Explain your answer.</p> <p>Dwight's Argument (IR): Bill is incorrect. That's not how you add fractions. You are supposed to keep the denominator the same and add the numerators. Bill added both the numerators and the denominators.</p>	<p>Valid Verification (VV)</p> <p>Problem (TG): True or false? If you add two even counting numbers, the result is always even. Explain your answer.</p> <p>Ira's Argument (VV): It's true. When you have an even number of things, they can be paired together so that each has a partner and none are left over. If you have two groups of things and both groups are paired in this way, when you combine the two groups together, each of them can still have the same partner with none left over.</p> <p>Invalid Verification (IV)</p> <p>Problem (TG): True or false? Two counting numbers can be multiplied in either order, and the result will always be the same both ways. Explain your answer.</p> <p>Hannah's Argument (IV): That is true. It works for addition, and multiplication is just repeated addition, so it should work there too, and it does.</p> <p>Invalid Refutation (IR)</p> <p>Problem (TG): True or false? If you subtract one even number from another even number, the result is always even. Explain your answer.</p> <p>John's Argument (IR): It is false. I tried several of these, and I was beginning to think that it was true, but then I thought, what if both numbers were the same? You would get zero, which is not even, so it doesn't work then. It is not always true.</p>

Figure 2.7a. A framework for classifying argument evaluation tasks by the generality and truth of the statement in question, the orientation of the argument (verification or refutation), and the validity of the reasoning—Part A: Arguments based on true statements.

Generality	Specific Statement	General Statement
Truth		
False Statement	<p>Valid Refutation (VR)</p> <p>Problem (FS): Jill had two brownies of the same size. She ate $\frac{3}{4}$ of one and $\frac{2}{4}$ of the other. Jill says she ate $\frac{5}{4}$ of the two brownies. Is Jill correct or incorrect? Explain your answer.</p> <p>Carla's Argument (VR): Jill is incorrect. $\frac{5}{4}$ is more than 1 whole. If she ate $\frac{5}{4}$ of two brownies, she would have eaten more than two brownies.</p> <p>Invalid Refutation (IR)</p> <p>Problem (FS): True or false? $\frac{6}{10} > \frac{5}{8}$. Explain your answer.</p> <p>Emily's Argument (IR): That is false. In fractions, the larger the numbers, the smaller the fraction. For example, $3 > 2$, so $\frac{1}{3} < \frac{1}{2}$. In this problem, $6 > 5$ and $10 > 8$, so $\frac{6}{10}$ must be less than $\frac{5}{8}$, not more.</p> <p>Invalid Verification (IV)</p> <p>Problem (FS): Jill had two brownies of the same size. She ate $\frac{3}{4}$ of one and $\frac{2}{4}$ of the other. Jill says she ate $\frac{5}{4}$ of the two brownies. Is Jill correct or incorrect? Explain your answer.</p> <p>Becky's Argument (IV): Jill is correct. She ate parts of two different brownies. She ate $\frac{3}{4}$ of one and $\frac{2}{4}$ of another. $\frac{3}{4} + \frac{2}{4} = \frac{5}{4}$. So she ate $\frac{5}{4}$ of the two brownies.</p>	<p>Valid Refutation (VR)</p> <p>Problem (FG): True or false? If you add two whole numbers, the result is always larger than both of the original numbers. Explain your answer.</p> <p>Kelley's Argument (VR): That is false. What about zero? If you add zero, that doesn't make the number larger; it stays the same.</p> <p>Invalid Refutation (IR)</p> <p>Problem (FG): True or false? If you multiply two whole numbers, the result is always larger than both of the original numbers. Explain your answer.</p> <p>Mona's Argument (IR): That's false. If you take 6 times $\frac{1}{2}$, you get 3 and that's smaller than 6, not larger.</p> <p>Invalid Verification (IV)</p> <p>Problem (FG): True or false? If you add 1 to both the numerator and the denominator of a fraction, the new fraction is always larger than the original. Explain your answer.</p> <p>Lana's Argument (IV): It's true. If you start with $\frac{1}{2}$ and add 1 to both the top and the bottom, you get $\frac{2}{3}$, which is larger than $\frac{1}{2}$. Or if you start with $\frac{3}{4}$, you get $\frac{4}{5}$, which is larger. It has to get bigger, because you are adding more, and adding more makes it bigger.</p>

Figure 2.7b. A framework for classifying argument evaluation tasks by the generality and truth of the statement in question, the orientation of the argument (verification or refutation), and the validity of the reasoning—Part B: Arguments based on false statements.

Table 2.1

Methods of Justification (from Simon & Blume, 1996, p. 17).

Level	Description
0	Responses identifying motivations that do <i>not</i> address justification
1	Appeals to external authority
2	Empirical demonstrations
3	Deductive justification that is expressed in terms of a particular instance (generic example)
4	Deductive justification that is independent of particular instances

Explanatory arguments. My goal was to develop and use a similar classification system that can be applied—not just to arguments based on true general statements—but to arguments based on statements with each of the possible combinations of truth value and generality that make up the four types of argument construction tasks (TS, TG, FS, and FG). To do so, I draw on de Villiers’ (1999) conception of the role of proof in the technology-rich context of Geometer’s Sketchpad. He wrote:

Although it is possible to achieve quite a high level of confidence in the validity of a conjecture by means of quasi-empirical verification (for example, accurate constructions and measurement, numerical substitution, and so on), this generally provides *no satisfactory explanation* [italics added] why the conjecture may be true. (p. 5)

I contend that the same is true for appeal to authority. It seems reasonable that a student who is struggling to even understand what a conjecture means would find a higher level of confidence in the word of an authority—a teacher or a textbook, for example—than his own attempted proof, especially if his recent scores have supplied him with ample empirical evidence that his current ability to construct such proofs is shaky at best. But,

like empirical demonstrations, an authority's endorsement alone fails to provide a satisfactory explanation of *why* a statement is true.

Level of explanation. Both empirical evidence and appeal to authority can determine, to a reasonably high degree of confidence, whether a given statement is true or false. However, neither explains why it should be so. In contrast, both generic examples and deductive proof are capable of supplying just such an explanation. Therefore, I classified mathematical arguments according to their *level of explanation*, i.e. whether they seek only to *determine that* a claim is true or to further *explain why* the claim is true. This distinction is especially important for preservice teachers, in view of Shulman's (1986) assertion that "the teacher need not only understand *that* something is so; the teacher must further understand *why* it is so" (p. 9, italics in the original).

The importance of this explanatory aspect of proof has been recognized by others in mathematics education (e.g., Chazan, 1993; Hanna & Jahnke, 1996) and in mathematics as well (Thurston, 1994). Thurston noted the unsatisfying nature, at least for some mathematicians, of Appel and Haken's computer-generated proof of the four-color theorem. Thurston attributed the controversy surrounding this proof, not to doubts about the truth of its conclusion, but to the failure of this "massive automatic computation" (p. 2) to provide any insight into why its conclusion is true. He contended that what mathematicians (and others) "really want is usually not some collection of 'answers'—what they want is *understanding*" (p. 2, italics in the original).

Forms of reasoning. The two levels of explanation, distinguishing an argument that attempts to explain why a claim is true from one that merely seeks to determine its truth, combined with the level of generality of the statements that are used to support the

claim, are sufficient to distinguish between the four methods of justification in Simon and Blume's (1996) framework (see Figure 2.8). Viewed in this way, an argument's *method of reasoning* can be described by a two-letter string, indicating whether it is explanatory or non-explanatory (E or N) and whether the key statements that support it are specific or general (S or G).

True general statements. In relation to Toulmin's framework, arguments supporting true general statements are at the general level ($B \rightarrow W$). At this level, the issue is whether the assertor has provided sufficient reasons to support a general warrant. The rejection of appeals to authority and empirical demonstrations as valid reasoning methods in mathematics implies that these *non-explanatory* approaches are not sufficient to support the acceptance of a general warrant by the mathematical community. However, *explanatory* methods using either specific statements (generic examples) or general

Generality	Specific Statements	General Statements
Explanation		
Determining Truth	(NS) Empirical demonstrations	(NG) Appeals to external authority
Explaining Why	(ES) Deductive justification that is expressed in terms of a particular instance (generic example)	(EG) Deductive justification that is independent of particular instances

Figure 2.8. A framework for classifying methods of verifying true general statements, based on their level of explanation and the generality of key statements that support the argument.

statements (general deductive arguments) can supply sufficient reasons for the community to accept such a warrant.

False general statements. In legal arguments, the evidence required for a conviction must meet a higher standard than evidence for acquittal. Likewise, mathematical arguments leading to the acceptance of a general warrant are held to a higher standard than those leading to the rejection of a warrant. I have argued that, in mathematics, the acceptance of a general statement requires an explanatory argument. However, the rejection of a general statement requires just a single counterexample, a conflicting specific statement. Suppose, for example, that the statement under consideration is, *The sum of two odd numbers must be odd*. An argument based on the grounds that $3+5=8$ and 8 is even would be sufficient to refute it. However, so would an explanatory argument—either by generic example or by deductive reasoning—that the sum of two odd numbers must be even. Thus, a false general statement can be refuted by either an argument supporting a conflicting specific statement (a counterexample) or an explanatory argument supporting a conflicting general statement. In terms of Toulmin's framework, we can reject a false warrant either at the specific level, by showing there are sufficient grounds to believe a conflicting claim, or at the general level, by showing there is sufficient backing to believe a conflicting warrant.

False specific statements. The situation for false specific statements is similar. For example, the statement $345 \times 15 = 5170$ can be refuted by an argument supporting a *counterclaim*, a conflicting specific statement, such as $345 \times 15 = 5175$. It could also be refuted by an argument at the general level, such as one demonstrating that the product of

two odd numbers must be odd. However, an argument at the general level must be explanatory, while one made at the specific level need not satisfy this requirement.

True specific statements. To support a true specific statement, an argument can be made at the specific level ($G \rightarrow C$), if the warrant for this argument is already established. Otherwise, it requires a general argument ($B \rightarrow W$) as well. For example, an argument that $121 - 59 = 62$ can be made on the grounds that $121 - 59 = 122 - 60$ and $122 - 60 = 62$, if this kind of compensation is already established in the community. If not, the assertor would need to provide a more explanatory argument for this procedure. In general, mathematical arguments for specific statements can remain on the specific level and may be either explanatory or non-explanatory, provided the warrants for those arguments are firmly established, but general statements require explanatory arguments for support. Figure 2.9 summarizes the valid and invalid methods of reasoning for the four categories of argument construction tasks.

Established warrants and appeals to authority. The distinction between using an established warrant and appealing to authority deserves special emphasis. Appealing to authority refers to using an officially sanctioned rule or a procedure *without understanding why it works*. Such officially endorsed rules are typical of mathematics instruction in U.S. classrooms. Using an established warrant, however, means using a warrant for which no additional explanation is necessary, because *it is already well-understood* by the intended audience (the members of the classroom community).

Valid methods and valid arguments. It is also important to note that methods of reasoning are rather broad characterizations of arguments. To say that a certain method of reasoning is valid implies that it is possible to construct a valid argument using that

Generality	Specific Statement	General Statement
Truth		
True Statement	<p>Invalid Methods of Reasoning</p> <ul style="list-style-type: none"> • (NG) Non-explanatory argument based on insufficiently supported general statements (appeals to authority) <p>Valid Methods of Reasoning</p> <ul style="list-style-type: none"> • (NS) Non-explanatory argument based on established warrants • (ES) Explanatory argument based primarily on specific statements (generic example) • (EG) Explanatory argument based primarily on a general statements (general deductive argument) 	<p>Invalid Methods of Reasoning</p> <ul style="list-style-type: none"> • (NG) Non-explanatory argument based on insufficiently supported general statements (appeals to authority) • (NS) Non-explanatory argument based on specific statements (empirical demonstrations) <p>Valid Methods of Reasoning</p> <ul style="list-style-type: none"> • (ES) Explanatory argument based primarily on specific statements (generic example) • (EG) Explanatory argument based primarily on general statements (general deductive argument)
False Statement	<p>Invalid Methods of Reasoning</p> <ul style="list-style-type: none"> • (NG) Non-explanatory argument based on insufficiently supported general statements (appeals to authority) <p>Valid Methods of Reasoning</p> <ul style="list-style-type: none"> • (NS) Non-explanatory argument for a conflicting specific statement, based on established warrants (counterclaim) • (ES) Explanatory argument for a conflicting specific statement (generic counterclaim) • (EG) Explanatory argument based primarily on conflicting general statements (general deductive argument) 	<p>Invalid Methods of Reasoning</p> <ul style="list-style-type: none"> • (NG) Non-explanatory argument based on insufficiently supported general statements (appeals to authority) <p>Valid Methods of Reasoning</p> <ul style="list-style-type: none"> • (NS) Non-explanatory argument for a conflicting specific statement (counterexample) • (ES) Explanatory argument for a conflicting specific statement (generic counterexample) • (EG) Explanatory argument based primarily on conflicting general statements (general deductive argument)

Figure 2.9. A framework for classifying methods of reasoning as valid or invalid.

approach. It does not imply that any argument so constructed would therefore be valid.

Explanatory, general, deductive argument, for example, is a valid method of reasoning, regardless of the truth value and generality of the statement under consideration.

However, a particular example of such an argument may fail to be valid for a variety of reasons. For example, it may be based on false assumptions or jump to unwarranted conclusions. Categorizing these methods of reasoning as valid or invalid provides one useful tool for considering the validity of PTs' arguments, but this categorization alone does not provide a complete explanation for why some arguments succeed and others fail.

An argument's intent. Another useful pairing of ideas is the combination of *orientation* (to verify or refute) and *level of explanation* (to determine the truth or to explain why it should be so). These two together describe the *intent* of the assertor in a particular argument. As before, a pair of letters can be used to designate each of four possible intents: (NV) a non-explanatory verification, (NR) a non-explanatory refutation, (EV) an explanatory verification, and (ER) an explanatory refutation. By combining the intent of an argument with the truth value and level of generality for both the statement in question and the argument's key supporting statements, a six-letter string can provide an overall description of an entire argument. For example, TGNVTS would indicate a task based on a true general statement, in which the assertor sought a non-explanatory verification by supplying true specific statements, a description typical of empirical arguments. Figure 2.10 shows the four possible intents for each of the four categories of argument construction tasks. If this figure was expanded to show all possibilities for the truth value and generality of supporting statements, there would be four possibilities for each cell, making 64 possible six-letter descriptions altogether.

The substance of mathematical arguments. In the study of logic, students commonly construct diagrams of an argument's structure. For example:

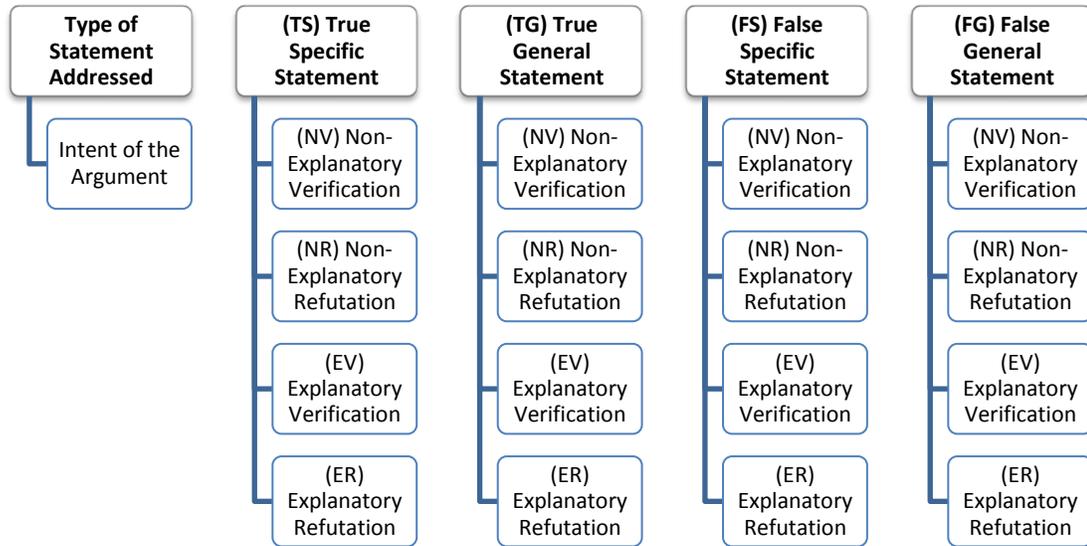


Figure 2.10. The four possible intents for each type of argument construction task.

$$\begin{array}{l}
 p \rightarrow q \\
 p \\
 \therefore q
 \end{array}$$

Such diagrams focus attention on an argument's form by eliminating information about its substance. Likewise, classifying an argument's method of reasoning according to its levels of explanation and generality—as discussed in the previous section—addresses only the argument's form. Knowing, for example, that a particular argument seeks to establish a general claim by using specific, explanatory statements provides no information about its substance—the mathematical meanings that lie at the heart of the argument.

Toulmin (1958) pointed out the limitations of using form alone to assess the validity of an argument. “To call ... an argument formally valid is to say only something about the manner in which it is phrased, and tells us nothing about the *reasons* for its validity. These reasons are to be understood only when we turn to consider the *backing* of

the warrant invoked” (p. 132, italics in the original). He drew a sharp distinction between arguments that provide backing to establish warrants and those that employ previously established warrants, comparing the first type to building a railway and the second to traveling by train. Except in rare instances that he called *analytic arguments*, he contended that any reasonable set of standards for assessing the validity of warrant-establishing arguments must consider their substance as well as their form. Such standards have evolved differently—and continue to evolve—in different fields of human endeavor, such as law, science, and mathematics. I looked for PTs’ notions of validity to likewise evolve as they gained experience with mathematical argumentation. To fully understand the nature of this evolution, I needed to consider the *substance* of their arguments as well as their forms of reasoning.

For example, consider two different arguments supporting the claim that the sum of two even whole numbers must be even. One argument begins by noting that a whole number is even if and only if its final digit is 0, 2, 4, 6, or 8. Using the standard addition algorithm with two even addends, the issue of whether the sum is even depends only on the sum of the final digits. Listing all 25 possibilities for the final digits and their sums demonstrates conclusively that the sum of two even addends must be even.

A second argument starts with the idea that an even number of objects can be paired together so that each object has a partner, with no unpaired objects remaining. If two groups so paired are combined together, each object can retain its original partner, leaving no unpaired objects. Hence the sum of two even numbers must be even.

Both arguments use essentially the same form of reasoning; they support a general claim in a way that is both general and explanatory. However, they are about very

different things, and this difference in substance reflects the nature of traditional school mathematics and inquiry mathematics. According to Cobb et al. (1992), traditional instruction often reduces mathematics “to an activity that involves constructing associations between signifiers that do not necessarily signify anything beyond themselves” (p. 587). Traditional school mathematics leads students to follow procedural instructions for performing operations on symbols without necessarily attaching meaning to the symbols involved. To the extent that traditional mathematics can be considered a field of argument, it is a field in which warrants are established by appealing to conventional procedures for manipulating symbols. Thus, the first of the two arguments reflects an understanding of traditional school mathematics. It is essentially about the possible results of manipulating mathematical symbols in a conventional way.

In contrast, inquiry mathematics focuses not on mathematical symbols per se, but on the *mathematical objects* they signify (Cobb et al., 1992; Gregg, 1995). These mathematical objects exist separately from the symbols representing them, in a mutually negotiated mathematical reality that is taken-as-shared by the members of the mathematical community. In a classroom with a firmly established inquiry mathematics tradition, warrants are established, not by appealing to conventional procedures for operating on symbols, but by examining relationships among mathematical objects or concepts. The second argument draws on an understanding of numbers as quantities, rather than symbols, and considers addition as the result of combining these quantities, not a procedure for operating on symbols. Hence, the second argument focuses on relationships among mathematical objects and therefore reflects an understanding of inquiry mathematics.

Skemp (1987) used the term “surface structure” to refer to relationships among symbols and “deep structure” (p. 177) to refer to relationships among the underlying concepts that the symbols represent. As PTs gained experience with mathematical argumentation, I expected their focus to shift from the surface structure of mathematics to its deep structure, and I looked for the substance of their arguments to reflect this deeper understanding. That is, rather than basing their arguments on relationships among mathematical symbols, I expected them to focus their arguments on the concepts the symbols represent.

Section summary. In this section, I drew on ideas from the literature to develop ways of classifying tasks that elicit mathematical arguments, as well as the various types of arguments that might result from these tasks and would therefore be available for critique. I also described my conception of the form and substance of mathematical arguments and argued that both of these aspects should be considered in assessing the validity of mathematical arguments. In the next chapter, I describe how I used this framework in selecting tasks for this study.

CHAPTER 3: METHODOLOGY AND METHODS

In this chapter, I present my methodology and methods for conducting the study, addressing: (a) the philosophical foundations, aims, and limitations of interpretative phenomenological analysis, (b) the setting and data sources for the study, (c) the tasks and arguments I created for the study, (d) the selection of interviewees, and (e) analysis of the interview data.

Interpretative Phenomenological Analysis (IPA)

Smith, Flowers, and Larkin (2009) described interpretative phenomenological analysis (IPA) as a research methodology focused on understanding “personal meaning and sense-making in a particular context, for people who share a particular experience” (p. 45), a reasonable description for this study. The PTs involved shared a common experience in which they struggled to find meaning and make sense of mathematical arguments, and it is this struggle that is the focus of my investigation. As a further argument that IPA is an appropriate choice, I discuss philosophical foundations, aims, and limitations of IPA and how they apply to this study.

Philosophical foundations of IPA. IPA’s philosophical roots lie in three areas: (a) phenomenology, (b) hermeneutics, and (c) idiography.

Phenomenology. Phenomenology focuses on human experience and its perception. The phenomenological view of perception involves not just receiving information from our senses but making sense of experience and giving meaning to it. The perception of experience is in some ways personal and subjective, but it is also influenced by the social world, which provides the language and cultural understandings individuals use to make sense of their experiences. Thus, the main focus of this study was

essentially phenomenological in nature. It concerned the ways in which PTs perceived mathematical arguments, initially making sense of them based on the cultural experience of their K – 12 schooling and later in light of their shared experience in a course emphasizing mathematical argumentation.

A key aspect of phenomenology is the need for the researcher to *bracket* or set aside his or her taken-for-granted ways of interpreting experience in order to focus on the ways that participants perceive and experience the world from their points of view. For example, I had ways of making sense of mathematical arguments that were not available to the PTs in the study, theoretical considerations about the structure, form, and substance of the arguments. However, in order to address the primary research question, I needed to temporarily set aside these concerns and focus on what PTs saw as important in the arguments.

Hermeneutics. Historically, hermeneutics originated from attempts to develop systematic and reliable ways to interpret the meaning of biblical texts. It has since evolved into a more general theory, applicable not only to interpreting meaning in texts but also to finding meaning in experience. It is in this sense that Smith et al. (2009) wrote of IPA as involving a “double hermeneutic” (p. 35), in which the researcher interprets the meaning of the participants’ conversation, as they strive to interpret the meaning of a particular experience. These multiple layers of interpretation are particularly apparent in this study, in which the participants attempted to make sense of arguments, which were texts written by me. Interviews with the participants were recorded and transcribed, producing texts to be interpreted by me. Thus I interpreted texts about the participants’

interpretation of texts that I wrote, ultimately leading me to write the text that you, the reader, are now interpreting.

An important concept from hermeneutics is the hermeneutic circle, which concerns the mutually dependent relationship between the individual parts of a text and the larger structures that contain them. For example, the meaning of a sentence depends on the meanings of the words that make up the sentence. However, a particular word may have a range of meanings, so its meaning is not specific and fixed but depends on the context—the sentence, for example—in which it appears. The same principle applies to larger structures. The meaning of a paragraph depends on the meaning of the sentences that comprise it, but the meaning of those sentences also depends on their context. The interpretation of meaning is therefore not linear but cyclic and iterative, with each rereading of a text providing potentially greater understanding by allowing the reader to examine the relationships between the whole and its various parts.

The consideration of hermeneutics along with phenomenology leads to a revised conception of bracketing. From the standpoint of hermeneutics, the researcher can never completely set aside his preconceptions and capture a participant's intended meaning. The analysis of a text therefore produces not the true meaning of the text but always an interpretation. Therefore, Smith et al. (2009) suggested that bracketing not be viewed as a step taken prior to data analysis, but bracketing and data analysis instead be undertaken together, as a cyclical and iterative process in which researchers engage in a dialogue with their data, allowing it to challenge and question their preconceptions, as well as providing answers to their questions.

Idiography. Smith et al. (2009) contrasted the goals of idiography with the procedures of the quantitative approaches prevalent in psychological research. Quantitative research uses information specific to individuals to produce information about the sample as a whole, summary measures such as means and standard deviations. These summary measures are then used to draw conclusions about the population from which the sample was selected. This process is unidirectional in the sense that, once the summary measures are obtained, the original data can essentially be ignored. The information about individuals that the data contained is no longer needed to obtain the desired conclusion. This approach therefore leads to a loss of focus on—and a concurrent loss of understanding for—the concerns and viewpoints of individuals, which are subsumed in conclusions about the sample and population.

Stake (1995) made a similar point, characterizing the search for general explanations and the understanding of particular contexts and cases as contrasting aims of quantitative and qualitative research in the social sciences:

Most social-science-oriented researchers ... try to eliminate the merely situational, letting contextual effects “balance each other out.” They try to nullify context in order to find the most general and pervasive explanatory relationships. Generalization is the important aim, with relevance to other cases hoped for. Quantitative researchers regularly treat uniqueness of cases as “error,” outside the system of explained science. Qualitative researchers [conversely] treat the uniqueness of individual cases and contexts as important to understanding. ...

To sharpen the search for explanation, quantitative researchers perceive what is happening in terms of descriptive variables [and] represent happenings with scales and measurements (i.e., numbers). To sharpen the search for understanding, qualitative researchers perceive what is happening in key episodes or testimonies [and] represent happenings with their own direct interpretation and stories (i.e., narratives). Qualitative research uses these narratives to optimize the opportunity of the reader to gain an experiential understanding of the case. (pp. 39-40)

For IPA, idiography emphasizes the need to maintain a focus on individual cases and the role they play in reaching general conclusions. Thus, “IPA adopts analytic procedures for moving from single cases to more general statements, but which still allow one to retrieve particular claims for any of the individuals involved” (Smith et al., 2009, p. 32).

The aims of IPA. The philosophical foundations of IPA provide some basic aims that guide the methods of IPA. Addressing phenomenological concerns, for example, IPA focuses on subjective experience. However, as Smith et al. (2009) point out, “that is always the subjective experience of ‘something’” (p. 33) and the ‘something’ should be of enough importance to the participants that they devote time to reflecting on their experience and trying to make sense of it. In the case of this study, it is PTs’ subjective experience of mathematical argumentation.

From hermeneutics, IPA derives a cyclic, iterative approach to data analysis and also a view of the researcher as an interpreter—rather than a revealer—of participants’ perceptions. Thus, although IPA researchers focus on participants’ perceptions of experience, they also aim to “take a look at them from a different angle, ask questions and puzzle over things they are saying” (p. 36). In this role as interpreter, IPA researchers sometimes consider “secondary or theory-driven research questions” (p. 48) to compare and contrast the views of participants with those in the research literature. Thus, in this study, I consider secondary questions about the roles that the form and substance of mathematical arguments play in PTs’ evaluations, in addition to examining PTs’ subjective views.

Idiography also has implications for both data collection and analysis. IPA studies are typically multi-case studies, in which analysis

begins with the detailed examination of each case, but then cautiously moves to an examination of similarities and differences across cases, so producing fine-grained accounts of patterns of meaning for participants reflecting upon a shared experience. In a good IPA study, it should be possible to parse the account both for shared themes and for the distinctive voices and variations on those shared themes. (p. 38)

In this study, I focused on five PTs, analyzing data from each individual separately and writing a separate case report for each. I shared each case report with the subject, prior to a final member-checking interview in which they were invited to question, comment on, or amend the report. Only one PT suggested amending the report, adding a brief description of his physical actions to clarify the meaning of one remark. After completing the member-checking interviews, I conducted a cross-case analysis, using the case reports as a reduced data set and ultimately producing the results described in Chapter 4. Thus, the study reflects the idiographic aims of IPA in its multi-case structure and in the ways that I analyzed the data from each interviewee, wrote up my interpretations, and subjected those interpretations to the scrutiny of the interviewees. Additional details on IPA's approach to data analysis and its use in this study appear later in this chapter.

Limitations of IPA. Researchers using IPA typically employ small, purposefully selected samples. They do not formulate a priori hypotheses or conduct any statistical tests. Therefore, they no make no explicit claims of confidence that the findings generalize to any pre-identified population. In addition, as this chapter's discussion of hermeneutics suggests, what IPA researchers claim to present is not the actual views of the participants but the researchers' interpretation of those views. However, a good IPA study offers sufficient supporting data along with the findings to allow the reader to judge whether the interpretation is reasonable and in what ways it might apply to other people

in similar situations. In this way, I invite the reader to judge whether the themes I identify represent accurate interpretations of the participants' views of the mathematical arguments we discussed, whether they would respond to other similar arguments in the same ways, and to what extent we should expect other PTs to respond similarly.

Setting and Data Sources

The setting. The University of Missouri is a large, public, research university, located in the city of Columbia, approximately midway between the large population centers of St. Louis and Kansas City. Within the university's Department of Learning, Teaching, and Curriculum (LTC), students can pursue state teacher certification in any of 16 fields of study ranging from Early Childhood Education to specific subject areas in Secondary Education. Those who major in Elementary Education (K-6) take a minimum of 120 credit hours, including *Learning and Teaching Number and Operation in the Elementary School*, henceforth referred to as *Number and Operation*. Students majoring in Special Education (K-12) also take this course, but these students constitute a relatively small proportion of the class and are not the focus of this study.

Before enrolling in *Number and Operation*, PTs must satisfactorily complete more than 40 credit hours of required college coursework, including *College Algebra* (unless waved from this requirement by examination) and a basic statistics course. They must also maintain an overall GPA of 2.75 and a GPA of 2.50 in LTC courses. While enrolled in *Number and Operation*, they generally take four other teacher-preparation courses, including a field experience course at the elementary school level. After completing *Number and Operation*, they also take *Learning and Teaching Geometry in the Elementary School*.

Number and Operation is intended to: (a) promote a deeper understanding of number and operation, focusing particular attention on developing a quantitative understanding of fractions and place value, and (b) connecting this deeper understanding to the learning and teaching of mathematics in the elementary grades. Dr. John Lannin, the instructor of the class, joined the University of Missouri faculty in the Fall Semester of 2001 and has since been nominated for and received several awards, for both teaching and research. He played the lead role in designing the course, and he continues to teach it and supervise the graduate students and adjunct faculty members who teach other sections of the class.

Prior to conducting this study, I taught a section of *Number and Operation* under Dr. Lannin's supervision. I also spent more than 60 hours observing and interacting with Dr. Lannin's students, both in class and in interviews I conducted as a part of a small pilot study. The class followed an inquiry mathematics approach. Dr. Lannin posed challenging problems and established classroom norms that required all PTs to: (a) explain their solutions in small-group and whole-class discussions, (b) listen thoughtfully to the explanations of others, and (c) pose questions to classmates if their explanations required clarification. The problems often led to conflicting solutions, and Dr. Lannin neither ratified correct solutions nor dismissed incorrect ones. Instead, he encouraged class members to resolve their differences by carefully considering the explanations of others and determining which explanations make sense. Although he and his students referred to "explanations," these explanations were generally mathematical arguments in the sense that each sought to justify a particular solution to a mathematical problem. He

monitored PTs' understandings and posed new problems that continue to challenge them and push them toward deeper levels of understanding. Dr. Lannin's emphasis on presenting, questioning, and evaluating mathematical arguments made *Number and Operation* an ideal setting for this study.

Data sources. Data for this study came from a larger data collection effort in which I video-recorded small-group and whole-class discussion in almost every class meeting, conducted and audio-recorded a series of five interviews with selected PTs, and photocopied samples of PTs' work. The study presented here, however, is based on only a small portion of this larger data set, consisting of: (a) PTs' written responses to a preliminary survey and (b) initial and post-course interviews, each lasting approximately 50 minutes, with five selected PTs. I conducted the initial interviews early in September and the post-course interviews in January, after the PTs had completed the course and returned for the spring semester. I mention the video recordings, additional interviews, and work samples here for two reasons. First, for the purpose of full disclosure, I want to acknowledge that my contact with class members was not limited to the initial and post-course interviews. Second, the larger data collection effort allowed me to conclude that semi-structured one-on-one interviews served the purposes of this study more effectively than either PTs' written work or video-recorded classroom observations. As Smith et al. (2009) wrote, "Interviewing allows the researcher and participant to engage in a dialogue, whereby initial questions are modified in the light of participants' responses and the investigator is able to enquire after any other interesting areas which arise" (p. 57). Neither PTs' written work nor recordings of classroom discussion provided the type of dialogue that I needed to adequately explore PTs views of mathematical arguments.

Focal Tasks and Arguments

The preliminary survey, the initial interview, and the post-course interview focused on “*Thinking about Students’ Explanations*,” a set of problems that I created prior to the first day of class. The instructions to this problem set read as follows:

In the situations described below, elementary school students answer questions and explain their answers. As you read each explanation, consider whether it convinces you that the answer must be correct; is it a valid explanation? Then write your responses to the questions below.

In creating this problem set, I selected a range of arguments that represented a variety of different categories within the framework I presented in Chapter 2. By using this problem set in the initial survey, I obtained an overview of the types of arguments PTs found convincing, which helped me to identify issues that merited further investigation in the interviews and to select interviewees who could shed light on those issues. By returning to the same problems in the initial interview, I was able to investigate PTs views of these arguments more deeply, and by revisiting them again in the post-course interview, I could explore how their views had changed over the course of the semester. I discuss these problems in this section, but for the reader’s convenience and future reference, I also include them in Appendix A. Because the time needed to discuss *Thinking about Students’ Explanations* varied from one interviewee to another, I also prepared additional problems and questions to address if time allowed, but I do not discuss those problems here.

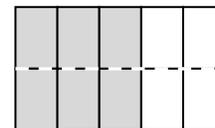
Andy and Beth’s arguments. Figure 3.1 shows the first situation and the two arguments that accompany it. The problem centers on a true specific statement, $\frac{3}{5} = \frac{6}{10}$,

Problem 1. A teacher gave her class the following problem: *True or false:* $\frac{3}{5} = \frac{6}{10}$.

Explain your answer. Two students gave the responses below.

Andy: It's true. I cross-multiplied and got $3 \times 10 = 30$ and $5 \times 6 = 30$. If you get the same number when you cross-multiply, the fractions are equal, so $\frac{3}{5} = \frac{6}{10}$.

Beth: It's true; they are equal. I drew a picture and shaded three-fifths (see the picture at right). If you cut each of the fifths into two parts, you get ten parts altogether, so each fifth is two tenths. Three of the fifths are shaded, and six of the tenths are shaded. So six-tenths is the same amount as three-fifths.



Choose one of the following:

- (a) I think Andy's explanation is more convincing than Beth's.
- (b) I think Beth's explanation is more convincing than Andy's.
- (c) I think both explanations are equally convincing.
- (d) I think neither explanation is convincing.

Explain why your choice makes sense to you.

Figure 3.1. Problem 1 from Thinking about Students' Explanations.

that both Andy and Beth attempted to verify. Andy provided an argument based on the general warrant, "If you get the same number when you cross-multiply, the fractions are equal." However, he provided no backing for this warrant. In some contexts, Andy's argument would be considered an appeal to authority, implicitly obtaining its support from the word of a teacher or textbook. In other contexts where this warrant is firmly established, that is, where the backing is already understood, Andy's argument would be valid. Here, I view the acceptance of Andy's argument skeptically. I consider it invalid to accept his warrant without explanatory backing. Within the context of subsequent interviews, PTs might supply this backing. However, until that point, I consider his warrant to have insufficient support. In addition, Andy's argument focuses only on

relationships among symbols, and not the underlying quantities those symbols represent. Therefore, it reflects only the surface structure of the mathematics and not its deep structure.

In contrast, Beth's argument explores the deeper structure—the concepts and quantitative relationships—underlying the symbolic representation. She demonstrated that three-fifths and six-tenths are equal quantities, by explaining why both represent the same portion of her diagram. I consider this a valid explanatory argument. It explains why three-fifths and six-tenths must be equal in a way that Andy's argument does not.

Caitlyn, Dawn, and Evan's arguments. Figure 3.2 shows Problem 2 from *Thinking about Students' Explanations*, in which Caitlyn, Dawn, and Evan all attempted to verify the true general statement, "Every multiplication problem can be done in either order; the results will always be the same both ways." However, their arguments differ distinctly, in form and substance. Caitlyn's argument is a clear appeal to authority, relying on the support of the "commutative property of multiplication" that she read in the textbook. Dawn presented an empirical argument, supported by data from the multiplication table. Neither Caitlyn nor Dawn's argument is explanatory. Neither explains why multiplication should be commutative. In contrast, Evan's argument provides such an explanation, based on a model for the meaning of multiplication as counting the number of objects in a rectangular array. Because Evan used his example to explain why multiplication must be commutative, his argument is a generic example.

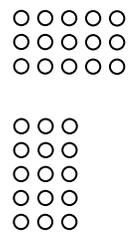
Caitlyn, Dawn, and Evan's arguments differ in substance, as well as in form. Caitlyn and Dawn relied only on surface features, focusing on the changing positions of the symbols. By considering the meaning of multiplication and the implications of

Problem 2. A fourth grade class has noticed that 45×32 and 32×45 both have the same answer, 1440. The teacher asks them if every multiplication problem can be done in either order. Will the results always be the same both ways? If so, explain why. Three students gave the responses below.

Caitlyn: Yes, they will always be the same. I saw it in a math book. It's called the commutative property of multiplication. It says $a \times b = b \times a$. That means you can multiply in either order, and the answers will be equal.

Dawn: Yes, that will always work. Look at our multiplication table. When you switch the order there, you get the same answer every time. $2 \times 3 = 6$ and $3 \times 2 = 6$. $5 \times 7 = 35$ and $7 \times 5 = 35$. I looked at every one, and it always works. For any multiplication problem, you get the same answer if you switch the numbers.

Evan: Yes, multiplying both ways will always give the same result. Look at 3×5 , for example. That's like counting three rows of dots with five in each row (see pictures at right). If you make it 5×3 , that's five rows with three dots in each row. The second is just like the first, but turned sideways. Both are going to have the same number of dots altogether. That will always happen when you change the order of the numbers you multiply. It's just flipping the picture sideways. It won't change the answer.



Choose one of the following:

- (a) I think _____'s explanation is the most convincing.
- (b) I think _____'s and _____'s explanations are equally convincing.
- (c) I think all three explanations are equally convincing.
- (d) I think none of the three explanations are convincing.

Explain why your choice makes sense to you.

Figure 3.2. Problem 2 from *Thinking about Students' Explanations*.

changing the order of the factors in light of that interpretation, Evan goes beyond the surface structure of the situation to examine its deeper structure.

Flavia and Georgia's arguments. In both Problem 1 and Problem 2, the competing arguments supported the same conclusion but in different ways. However, in

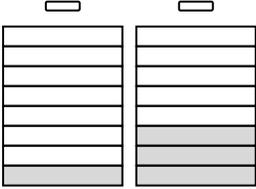
Problem 3 (shown in Figure 3.3), the arguers reach conflicting conclusions, Flavia arguing that the mixture will be one-half grape juice and Georgia arguing that it will be four-sixteenths grape juice. Each attempted to verify a specific statement, a false one in Flavia’s case and a true one in Georgia’s case.

Based on the fact that “combined means added together,” Flavia proceeded as if any problem containing the word “combined” requires adding the numbers in the problem. This is a false warrant or at least one that needs further qualification. However, it reflects a “key word” approach that is sometimes taught in U.S. mathematics classrooms. In addition, Flavia’s argument emphasizes only the surface structure of the

Problem 3. The teacher gave the following problem to a fifth-grade class: *Two one-gallon jugs are filled with liquids that are a mix of grape-juice and water. The first is one-eighth grape juice. The second is three-eighths grape juice. If the two gallons are combined, what fraction of the combined mixture will be grape juice? Explain your answer.* Two students gave the responses below.

Flavia: Combined means added together. So I added $\frac{1}{8} + \frac{3}{8} = \frac{4}{8}$. I divided by 4 and reduced it to $\frac{1}{2}$. So the combined mix is one-half grape juice.

Georgia: I drew pictures of two gallons, and I shaded one-eighth of one and three-eighths of the other (see picture at right). When the two pictures are combined together, there are 16 equal parts and 4 are shaded. So the mix would be $\frac{4}{16}$ grape juice.



Choose one of the following:

- (a) I think Flavia’s explanation is more convincing than Georgia’s.
- (b) I think Georgia’s explanation is more convincing than Flavia’s.
- (c) I think both explanations are equally convincing.
- (d) I think neither explanation is convincing.

Explain why your choice makes sense to you.

Figure 3.3. Problem 3 from *Thinking about Students’ Explanations*.

situation, focusing on accurately following the accepted procedure for adding fractions, operating on the symbols involved to obtain the “correct” answer. It is not an explanatory argument, because it does not explain why the amount of grape juice in the mix must be one-half of the total mixture.

Georgia’s argument, however, is very similar to Beth’s and Evan’s. She analyzed the actions on the quantities represented by the “ $1/8$ ” and “ $3/8$ ” in this situation and accurately concluded that the combined amount would represent four sixteenths of the total mixture. She thereby provided a valid explanatory argument to support her answer.

Heather and Ivy’s arguments. Like Problem 2, Problem 4 addresses general conclusions. (See Figure 3.4.) However, it differs from Problem 2 in four ways. First, its conjectures, “Adding makes the numbers bigger, and subtracting makes them smaller,” are stated rather ambiguously. When considering addition, for example, PTs could interpret either or both of the addends as “getting bigger.” In subtraction, however, only the first term involved can legitimately be interpreted as “getting smaller.” This ambiguity was intentional; I wanted to see how PTs would interpret the problem. Second, unlike in Problem 2, the arguers here present opposing views, Heather supporting the conjecture and Ivy casting doubt on it. Third, whereas the conclusion in Problem 2 was true, those in Problem 4 are false, with adding and subtracting zero as clear counterexamples to the twin conjectures.

Finally, unlike Caitlyn, Dawn, and Evan, neither Heather nor Ivy provided a completely satisfactory argument to support her conclusion. Heather provided an empirical argument, similar to Dawn’s, to support the conjecture, and like Dawn, she

Problem 4. A second-grade class has been talking about what happens when you add or subtract whole numbers. Several students have said that adding makes the numbers bigger, and subtracting makes them smaller. They have found lots of examples that illustrate these ideas. The teacher asks them if they think this will always happen when you add or subtract whole numbers. She also asks them to explain why they think so.

Heather: Yes, that always works. I can show you a hundred problems—or even a thousand—where adding makes the numbers bigger and subtracting makes them smaller. It always works.

Ivy: I don't think it *always* works. Just because we tried it and it worked for some problems, how do we know it will work for the next problem we try? Maybe it works for some numbers and not for others.

Choose one of the following:

- (a) I think Heather's explanation is more convincing than Ivy's.
- (b) I think Ivy's explanation is more convincing than Heather's.
- (c) I think both explanations are equally convincing.
- (d) I think neither explanation is convincing.

Explain why your choice makes sense to you.

Figure 3.4. Problem 4 from Thinking about Students' Explanations.

fails to explain why the conjecture is true. Ivy argued against empirical reasoning in a broad, general way, but she failed to provide any reason, such as a counterexample, to conclude that these particular conjectures are false. Again, this choice was intentional. I wanted to see whether PTs would construct counterexamples to these conjectures. Furthermore, this problem offers the additional opportunity for PTs to construct an explanatory argument that supports the conjectures on a more restricted domain, the positive whole numbers, for example.

Section summary. Table 3.1 summarizes the characteristics of the arguments discussed in this section. As a whole, they offer rich possibilities for engaging PTs in mathematical argumentation in a variety of ways. In the next section, I present the results

Table 3.1

Characteristics of Arguments in Thinking about Students' Explanations

Arguer	Statement Addressed	Form of the Argument	Substance of the Argument
Andy	True specific statement	Invalid verification, warrant with no backing	Surface structure
Beth	True specific statement	Valid explanatory verification	Deep structure
Caitlyn	True general statement	Invalid verification, appeal to authority	Surface structure
Dawn	True general statement	Invalid verification, empirical argument	Surface structure
Evan	True general statement	Valid verification, generic example	Deep structure
Flavia	False specific statement	Invalid verification, using a false warrant	Surface structure
Georgia	True specific statement	Valid explanatory verification	Deep structure
Heather	False general statement	Invalid verification, empirical argument	Surface structure
Ivy	False general statement	Incomplete refutation, no counterexample	Surface structure

of the preliminary survey and the way I used them to select interviewees. In Chapter 4, I present results I obtained from using them as the focus of in-depth, one-to-one interviews.

Selection of Interviewees

Participants' responses to the preliminary survey. The *Number and Operation* class had a total of 29 students, including 22 PTs and 7 preservice special education teachers. Among the 22 PTs, the group I focused on, one declined to participate in the study, and another never completed the preliminary survey, leaving 20 PTs whose responses I summarize in Table 3.2. The reader may wish to refer to the previous section or Appendix A, to recall the arguments addressed here.

Three findings from these data merited further investigation in the interviews. First, when given contrasting arguments that reached the same conclusion, most PTs accepted more than one as equally convincing, 55% for Question 1 and 75% for Question 2. For both questions, endorsing all arguments was the most frequent choice (55% of responses for Question 1 and 35% for Question 2). In contrast, when two arguments reached conflicting conclusions, only a small proportion of PTs endorsed both (10% each for Questions 3 and 4). This marked difference suggests that the PTs placed considerable importance on the conclusion of the argument, endorsing arguments whose conclusions they deemed acceptable. However, this is only one possible explanation for PTs' preferences, highlighting an issue to be explored more deeply in the interviews.

Second, in response to the grape juice mixture problem in Question 3, the largest percentage of PTs (40%) found Flavia's argument more convincing than Georgia's. However, Flavia decided to add the given fractions, based on the presence of the word "combined," and therefore obtained an incorrect answer. Because Flavia's argument focuses on surface-level relationships and inappropriately applies the standard fraction addition algorithm, the high proportion of endorsements here also deserved further

Table 3.2

Results of the Initial Survey from All PTs Responding (n = 20)

Question 1			Question 3		
Preference	Frequency	Percent	Preference	Frequency	Percent
Andy	4	20%	Flavia	8	40%
Beth	5	25%	Georgia	4	20%
Both	11	55%	Both	2	10%
Neither	0	0%	Neither	6	30%

Question 2			Question 4		
Preference	Frequency	Percent	Preference	Frequency	Percent
Caitlyn	3	15%	Heather	4	20%
Dawn	0	0%	Ivy	12*	60%
Evan	2	10%	Both	2	10%
C & D	3	15%	Neither	2	10%
C & E	3	15%	* Only one PT (5%) identified zero as a counterexample.		
D & E	2	10%			
All	7	35%			
None	0	0%			

investigation in the interviews.

Finally, in Question 4, Heather provided an empirical argument, concluding that adding whole numbers makes them bigger and subtracting makes them smaller. Ivy questioned the validity of Heathers' approach but offers no reasons to support a

conflicting conclusion. Thus, neither Heather nor Ivy provided a strong justification for her conclusions about whole number addition and subtraction. However, only two PTs (10%) found neither argument convincing. The majority (60%) found Ivy's argument more convincing than Heather's, a reasonable conclusion if they recognized zero as a counterexample. However, only one PT mentioned zero in her response. This preference for Ivy's argument could have several possible explanations. Some PTs may have known zero was a counterexample without explicitly noting it in their responses. Others may have considered counterexamples involving negative numbers, thinking these were included within the domain of Heather's claim. Still others may have thought exceptions to Heather's conjecture possible within the positive whole numbers. Thus, further investigation in the interviews was needed to explore PTs' reasons for endorsing Ivy's argument.

The interviewees. Based on their written responses, I selected seven PTs for the initial interviews, choosing PTs who represented a variety of viewpoints that could shed light on the three issues identified above. For subsequent interviews, I retained only the five whose initial interviews most effectively addressed the research questions: Corey, Diana, Ermida, Grace, and Linda. Corey was one of only two males in the class, and Ermida was the only member of an ethnic minority group. Table 3.3 categorizes the written responses of these five interviewees. For each question, the most frequent response is represented by at least three interviewees. In addition, each interviewee gave one or two responses that no other interviewee provided. By using their written responses to select interviewees in this way, I hoped to represent the prevailing views from the initial survey yet also capture some of the divergent views of individuals as well.

Table 3.3

Initial Preferences of the Five Interviewees

Interviewee	Question 1	Question 2	Question 3	Question 4
Corey	Both ¹	All ¹	Flavia's ¹	Heather's ²
Diana	Both ¹	D & E ²	Flavia's ¹	Ivy's ¹
Ermida	Both ¹	C & D ²	Flavia's ¹	Ivy's ¹
Grace	Beth's ²	All ¹	Neither ²	Ivy's ¹
Linda	Andy's ²	All ¹	Flavia's ¹	Both ²

¹ The most frequent response

² A response no other interviewee provided

Analysis of the Interview Data

Smith et al. (2009) provided step-by-step advice for analyzing data in IPA studies, noting that IPA provides room for flexibility in this process. They emphasized that, foremost in IPA, analysis involves focusing attention on participants' efforts to make sense of their experiences. The steps of their analysis process include: (a) reading and rereading, (b) initial noting, (c) developing emergent themes, (d) searching for connections across emergent themes, (e) moving to the next case, and (f) looking for patterns across cases. Because I decided to write and share a case report with each interviewee before proceeding to the next, I completed these steps approximately in the order shown in Table 3.4 but with some stages overlapping and occurring simultaneously.

Table 3.4

Steps in Data Analysis

1. Analyzing the initial interview	a. Reading and rereading b. Initial noting c. Developing emergent themes d. Searching for connections across emergent themes
2. Analyzing the post-course interview	a. Reading and rereading b. Initial noting c. Developing emergent themes d. Searching for connections across emergent themes
3. Writing the case report	
4. Sharing the case report with interviewee	
5. Conducting the member-checking interview	
6. Repeating steps 1 – 5 for each interviewee	
7. Searching the case reports for patterns across initial interviews	
8. Searching the case reports for patterns across post-course interviews	
9. Writing the final report (Chapter 4)	

For example, reading and rereading began before initial noting, but typically continued throughout the other stages of analysis. Likewise, writing the case report often began shortly after the completing the analysis of the first interview, but of course, it was not completed until after the analysis of the post-course interview. In the remainder of this section, I describe some aspects of this data analysis process.

Reading, noting comments, and initial coding. After an assistant transcribed each of the interviews, I checked and edited each transcript, listening to each recording repeatedly and focusing on what the interviewees considered important. As I read and

reread the transcripts, I added notes and comments about passages that related to the research questions, that is, those that bore on the interviewees' reasons for preferring one argument to another. Smith et al. (2009) suggest printing the transcript with wide margins to allow space for these notes to be added by hand. I experimented with this procedure but found that I generally preferred to work directly in a computer file containing the transcript, adding notes and comments in brackets and boldface font, to set them off from the original text. Sometimes these comments consisted of particular words or phrases the interviewee used, such as "time consuming." In other instances, they described what the interviewee was doing, "preferring explanations with numbers," for example. At this stage, I saved the file in rich text format, to allow me to analyze the data in HyperResearch, coding it with themes, such as "knowing automatically," and "judging by the answer."

Finding connections across emergent themes. The process of finding connections between emergent themes was closely associated with and motivated by the need to write a case report for each interviewee. To do so, I needed to eliminate some emergent themes from consideration and combine others under a manageable number of super-ordinate themes for each interview. For example, to describe what Corey considered important in his initial interview, I condensed his list of themes to: (a) explained using numbers rather than words or pictures, (b) followed routine procedures in an organized, step-by-step way, and (c) showed confidence in arriving at correct answers. Conversely, for Grace's initial interview, I used the following themes: (a) finding correct answers and explaining how they were found, (b) remembering correct procedures, (c) deciding what makes sense, and (d) expressing mathematical ideas correctly. In each

instance, the need to write a case report that would be understandable to and supported by the interviewee ensured that the themes highlighted in the case reports were firmly grounded in the data.

Moving to the next case. Smith et al. (2009) emphasize the need to bracket the themes that emerged from one case as the researcher proceeds to the next. In my study, this bracketing procedure was somewhat aided by the fact that, as I moved to examine a new interviewee's initial interview, my most recent focus in the previous interviewee's data was the post-course interview. The contrast between the two generally helped to separate the perspectives of the two interviewees.

Finding patterns across cases. Cross-case analysis was in many ways the most difficult part of the analytic process. Finding commonalities across themes from the same interviewee was much easier than finding commonalities across interviewees. Adding to the complexity was the rhetorical need not only to combine themes across interviewees but to also facilitate comparisons and contrasts between the initial interviews and the post-course interviews.

One idea from Smith et al. (2009) that was surprisingly helpful was "polarization" (p. 97), in which opposing rather than similar themes might be grouped together under a superordinate theme. For example, Corey's theme of "confidence in the correct answer" led him to support one argument, whereas Diana's theme of "open-mindedness" led her to support the opposing argument. Upon reflection, however, I realized that by emphasizing either confidence or open-mindedness, both interviewees focused on the perceived attitude of the arguer as an important consideration. Similarly, both Corey's certainty in "knowing automatically" and Grace's view of herself as "not remembering

the rules” appeared quite different on the surface. However, both emphasized the underlying idea that, when given a mathematical problem, students should “know what to do,” and pursue a course of action that leads to the correct answer.

Throughout this process, I continued to revise the codes, eliminating some, combining others, and adding new ones. To clarify the final results, I constructed a table showing the main or superordinate themes for the final report with relevant codes or subthemes listed under each. For each code or theme, I gave a description of the type of data to which it applied and a brief excerpt from the data to illustrate it. Table 3.5 shows the part of this table that relates to the superordinate theme, “Getting the Correct Answer.” Interested readers can find the entire table in Appendix B. In the next chapter, I provide the results of this analysis, describing the main themes and illustrating them with excerpts from the data.

Table 3.5

Codes, Descriptions, and Examples for the Superordinate Theme, “Getting the Correct Answer.”

Theme/Code	Description of data to which the code or theme applied	Example
Superordinate Theme: Getting the Correct Answer	Interviewees cited the correct answer as an important criterion for assessing mathematical arguments.	“First, she is correct” (Corey, Interview 1).
<ul style="list-style-type: none"> • Correct therefore convincing 	Interviewees cited the correct answer as a reason for endorsing a particular argument.	“I chose Flavia’s response, because she explained each of her steps, reduced fractions, and got the correct answer” (Corey, Preliminary Survey).
<ul style="list-style-type: none"> • All are correct 	Interviewees cited the correct answer as a reason for endorsing two or more arguments that shared the same conclusion.	“I liked all their explanations, because they backed them up with resources and really explained where they got them [their answers].” (Grace, Interview 1).
<ul style="list-style-type: none"> • Not best but still correct 	Interviewees stated that some arguments they did not prefer were still correct.	“I didn’t like [Evan’s argument] as much, but it is correct. I just preferred Caitlyn’s and Dawn’s” (Ermida, Interview 1).
<ul style="list-style-type: none"> • Correct in context 	Interviewees indicated that conflicting answers should be considered correct, due to the context in which they appear	“At this point, Heather is right. From what she’s given and what she has learned, she is correct” (Ermida, Interview 1).

CHAPTER 4: RESULTS OF THE INTERVIEWS

What PTs Initially Considered Important

In this section, I address my first research question, examining what PTs initially considered important as they compared and evaluated mathematical arguments. I examine five themes that emerged from their initial interviews, presenting evidence that they generally based their evaluations not on the argument per se but on their interpretations of the arguer's thoughts and actions. They initially tended to favor arguments in which they perceived the arguer as: (a) knowing what to do, (b) getting the correct answer, (b) using a quick way to get it, (d) showing how with numbers, and (e) having the right attitude. Although clearly not every theme was of equal importance to each of the interviewees, each is supported with data from at least three of them.

Knowing what to do. In *Thinking about Students' Explanations*, PTs encountered four situations in which hypothetical students responded to problems posed by their teachers. As they talked about these situations, all interviewees demonstrated a shared initial belief that, when presented with such problems, students should ideally *know what to do* and immediately pursue a course of action that would lead to the correct answer. The most explicit statement of this belief came from Corey's first interview, recounting his initial reaction to Andy's approach:

Cross multiplication—I *knew that*. It was in my mind. It was *drilled in*, so I *knew what to do*. Right off the bat, when I saw that, I was [thinking], “Okay, three times ten equals thirty. Five times six equals thirty. They're equal, because that's *how it's supposed to be*.” ... In elementary school, when we were taught multiplication tables [for example], multiplication facts, you get it *drilled in your brain*, so *when you see it, you know it*. *Rather than having to figure it out for [ourselves], we knew these things*. We had to *know automatically*, so [we] could build off that base, on to further math. (Corey, Interview 1, italics mine here and in all excerpts)

In Corey's experience, learning mathematics meant drilling until knowing what to do became automatic. Thus, he preferred arguments like Andy's, where the arguer seems to "know automatically" what to do, as opposed to others like Beth's, where the arguer might be perceived as trying to "figure it out."

Grace expressed a similar belief in her first interview, but from a different viewpoint. Whereas Corey saw himself as knowing what to do, Grace repeatedly criticized herself for failing to do so. For example, when asked about Flavia's argument, she showed frustration with her inability to remember the correct procedure for adding fractions:

M: What about the math here [in Flavia's argument]. One-eighth plus three-eighths equals four-eighths. Is that correct?

G: I mean—[sigh]. *The rules of fractions have escaped my mind.* But if you're keeping the denominator the same, *if that's what you're supposed to do* in adding fractions, then yes, it is. And she [Flavia] reduced it correctly. But *I think you're supposed to add the denominators in fractions.*

M: You think you're supposed to add the denominators?

G: I think so, but not in all of it. Not in [sigh]—*I don't know.*

M: Not all the time? Some of the time you should?

G: *I should have the rules of fractions in front of me.* Maybe *I should have studied those.* I don't think it's correct math, but it very well could be. It's not like she [Flavia] made all these mistakes.

M: Are there times when you keep the denominators the same?

G: Yeah, there are. I think there are. I know there are; I don't know when. [She laughs.] *I vaguely remember the denominator having to stay the same.*
(Grace, Interview 1)

From Grace's viewpoint, correct answers were not obtained by understanding mathematical concepts and relationships but from studying rules and procedures and remembering "what you are supposed to do."

Ermida and Diana expressed similar views in describing their prior learning experiences. For example, when noting a “mistake” in Georgia’s argument, Ermida described how she was taught to add and subtract fractions:

A lot of us made that mistake [adding the denominators] when we were learning addition and subtraction of fractions. I know I did. But I don’t think the teacher actually explained why we had to have the same denominator. I think *we were just told that we were supposed to, and I guess that’s always stuck with me. Just do it*—same denominator, then you can add it. (Ermida, Interview 1)

Likewise, Diana endorsed Andy’s use of cross multiplication to determine whether two fractions are equivalent, saying that she was “taught that this was *the process you use to get it and to do it.*” Both described learning mathematics as being taught what to do and remembering it, so that when the situation calls for a particular process, such as adding fractions or determining whether two fractions are equivalent, they would know what to do. They therefore endorsed Flavia and Andy’s arguments, because the arguers appeared to remember and execute these processes correctly.

In Linda’s initial interview, she gave Andy’s argument a strong endorsement. In particular, she noted that it was easier to explain what to do, using Andy’s approach, rather than simplifying by dividing by a common factor, as she was taught:

Because they could see exactly where all the numbers were coming from, and they’re all laid out on the paper for them, as opposed to my method [simplifying], where *I just know that you can divide six by two and get three and ten by two and get five. But they would be confused as to where I pulled out the two*, because you can’t always rely on that, because it could be a multiple of three. . . . Instead of six over ten, it could have been a different number, and I’d have divided by three maybe instead of just by two, *so it would have been harder to explain to them.* I feel that if he explained this to another one of the students, because he used all the numbers that were already given—he wasn’t using anything else that he already knew just from previous knowledge, besides the fact that you can cross multiply—that he would have been able to explain to them easily. So, I find that that’s a good one, and I would be able to use that. I feel like it would be a good one to use if I was teaching a class as well. (Linda, Interview 1)

In comparing these two approaches, division by a common factor and cross multiplication, Linda saw knowing what to do as the central issue. She knew to divide by two in this situation but that other situations would call for different divisors. She therefore preferred cross multiplication, because it allowed her explain *what to do* without addressing the possibly confusing issue of identifying the correct divisor. Broadly speaking, she preferred cross multiplication, because it made knowing what to do easier for students.

These excerpts illustrate the various ways in which the interviewees emphasized the importance of knowing and remembering what to do when faced with mathematical problems. Whether they saw themselves as successfully able to do so (Corey) or struggling (Grace), recalled their own learning experiences (Ermida and Diana) or looked ahead to teaching in their future classrooms (Linda), they emphasized the view that students should remember and execute mathematical procedures correctly. Consequently, they endorsed Andy and Flavia's arguments, in which the arguer's actions reflected this viewpoint.

Getting the correct answer. Although it was important to the interviewees' evaluations of mathematical arguments, knowing what to do was ultimately only a means to an end, and the end they had in mind was getting the correct answer. Throughout their initial interviews, an emphasis on getting correct answers emerged in several ways. First, when comparing arguments that reached conflicting conclusions, interviewees stressed the correctness or incorrectness of an answer as a reason for choosing one argument over another. (See Table 4.1.)

Second, the PTs frequently endorsed all arguments that shared the same

Table 4.1

Correct or Incorrect Answers as Justification for Preferring One Argument Over Another

Interviewee	Response
Grace	I liked Georgia's <i>answer</i> . So I didn't think [Flavia's argument] was convincing, <i>because I didn't think it was correct</i> . (Interview 1)
Corey	I chose Flavia's response, because she explained each of her steps, reduced fractions, and <i>got the correct answer</i> . (Preliminary Survey)
Corey	I believe Heather's <i>answer</i> is more convincing, <i>because first she is correct</i> , and because of the way she states her answer, she seems very confident she is correct. (Preliminary Survey)

conclusion. In such cases, they generally recognized clear differences among the arguments but maintained an open, uncritical stance in evaluating them, considering it sufficient that each presented a reasonable explanation of the process for obtaining the correct answer. For example, Linda defended Caitlyn, Dawn, and Evan's arguments as equally convincing:

In my opinion, *they each obviously did very different methods, but they each explain themselves very well*. And I feel that if I had to ask them *how they got this* again, they'd be able to explain it to me, because just from reading what they put down, I understood exactly where they were coming from, so at least *they understood the process they were doing*. And ... I didn't recognize the name, "the commutative property," but I do remember in school that A times B equals B times A. So I've used that before. And Dawn, she looked at the multiplication table, and she gave some examples, and I mean it is true, especially if you did have the multiplication table. I know it only goes up to twelve or thirteen, so she couldn't have seen every number; there's a limited amount. But she has at least enough evidence supporting where her decision came from. And then the diagrams that he [Evan] drew, they both make sense. And they're just kind of flip-flopped. ... One's horizontal, and one's vertical, but they're exactly the same. (Linda, Interview 1)

Grace expressed a similar view of these arguments:

I liked all their explanations, because they backed them up with resources and really *explained where they got them [their answers]*. Caitlyn says she looked in a book, and—well, I don't know if it's actually called that property, but that's what she said. And she gave a formula, which I thought was [how] she used her resources, a book. And she got this formula, which definitely made sense to me, because $A \times B$ equals $B \times A$ is exactly what that would be up there [referring to 45×32 and 32×45]. You could plug in those numbers for that, so I thought that was very convincing. And then with Dawn, she looked at another resource, her multiplication table, and it [followed] that formula, $A \times B$ equals $B \times A$. So I thought that was convincing because she said where she got it from, why she was using these numbers, and gave examples. And then Evan as well, he drew a diagram, which was nice, because he kind of showed why he thought it was just like turning it sideways, because that's 3 times 5 and that's 5 times 3. And even though the diagrams may look a little different, they get the same answer. So *all three of them explained where they got it from and why they did it that way*. So that's why I thought *they were all equally convincing*. (Grace, Interview 1)

Both Grace and Linda emphasized the different “methods” or “processes” the arguers used to obtain their answers. Because the arguers arrived at the correct answer and “explained where they got it,” the PTs considered all three arguments equally convincing.

When an interviewee expressed the view that two or more arguments were equally convincing and gave reasons to support this position, I generally followed up by asking which argument they would consider best, if they were forced to choose. They typically responded and gave reasons for their preferences. However, they often followed this discussion by reiterating their original position, characterizing the choice for which was best as merely a personal preference and emphasizing that other arguments also obtained correct answers. (See Table 4.2.)

These examples show how PTs emphasized correct answers in arguments that reach the same conclusion. However, in one case, Ermida responded similarly to arguments that reached conflicting conclusions, viewing both Heather and Ivy's

Table 4.2

Minimizing Personal Preferences in Favor of Correct Answers

Interviewee	Response
Corey	I believe Andy's is an easier route to take, but <i>both answers are equally convincing</i> , because <i>both prove the problem to be true</i> . They just approach <i>the answer</i> in two different manners. (Preliminary Survey)
Diana	I would hesitate to say [Beth's argument] is better. Like I said, <i>they're both equally convincing</i> , because <i>they're both true</i> . (Interview 1)
Ermida	I didn't like [Evan's argument] as much, but <i>it is correct</i> . I just preferred Caitlyn's and Dawn's. (Interview 1)
Grace	I do think maybe [Beth's] answer is more convincing, but <i>they both are correct</i> , I believe. (Interview 1)

arguments as correct in different ways:

At this point, *Heather is right*. From what she's given and what she has learned, *she is correct*. But I would just say that *Ivy is more convincing*, because she's thinking about what's going to happen later on, whenever she learns higher math. But I do think *both of them are right*. (Ermida, Interview 1)

As a second-grader who has not yet learned about negative numbers, Ermida believed Heather was correct in stating that adding whole numbers makes them bigger. However, she viewed this position as incorrect later, after negative numbers were introduced, and credited Ivy with somehow foreseeing this issue. Therefore, she saw both positions as correct in different contexts.

These excerpts illustrate several ways in which the interviewees emphasized getting correct answers as an important criterion for evaluating mathematical arguments. When faced with arguments that reached conflicting conclusions, they often focused on a correct or incorrect answer as their reason for choosing one argument over another. When comparing arguments that reached identical conclusions, they avoided choosing one

arguments as superior to another, maintaining an open, uncritical stance and endorsing all arguments that provided an acceptable explanation for how the answer was obtained. When pressed to choose among such arguments, they characterized their choice as merely a personal preference and reiterated their initial view that the arguments were equally correct. Finally, in the case of Ermida cited above, an interviewee endorsed two arguments that reached conflicting conclusions, viewing each as correct in different contexts.

Using a quick way. Four interviewees—Corey, Ermida, Diana, and Linda—expressed some degree of preference for arguments that were short and to the point, getting to the correct answer quickly. Corey’s response to Andy’s argument, for example, conveys a sense of urgency, a need for not just getting the correct answer but getting it almost immediately:

Cross multiplication—I knew that. It was in my mind. It was drilled in, so I knew what to do. *Right off the bat, when I saw that, I was [thinking], “Okay, three times ten equals thirty. Five times six equals thirty. They’re equal, because that’s how it’s supposed to be.”* ... In elementary school, when we were taught multiplication tables [for example], multiplication facts, you get it drilled in your brain, so *when you see it, you know it*. Rather than having to figure it out for [ourselves], we knew these things. We had to *know automatically* so [we] could build off that base, on to further math. (Corey, Interview 1)

Simply knowing what to do was not enough for Corey and his classmates. They had to “know automatically,” so “when you see it, you know it,” and they could get started “right off the bat.”

Ermida addressed the need for obtaining answers quickly as she compared Caitlin, Dawn, and Evan’s arguments about the commutativity of multiplication. She had previously described herself as “a visual learner,” but in comparing these arguments, she preferred those without diagrams—Caitlyn’s appeal to authority and Dawn’s empirical

argument—to Evan’s generic example, in which a diagram played a central role. When I asked her about this apparent contradiction, she criticized Evan’s argument as overly time-consuming:

E: I didn’t like [Evan’s argument] as much, but it is correct. I just preferred Caitlyn’s and Dawn’s. ... Unless he was an absolute visual learner, it would be *time consuming* to sit there and draw all these circles—three times five—and then try to flip the picture down. ... He could probably do this [only] a few times before *it took a lot more time*, compared to Dawn’s. She just wrote the problems out. Evan would actually have to sit there and write all of these circles in one direction, and then he would try to flip the picture over and write all those circles again. He would do trial and error as well as Dawn, but I think his would be *more time consuming*. ... It is correct, though. He had the right idea.

M: All right. So basically you see Evan’s as the same as Dawn’s but taking more time.

E: Yes.

M: I think it’s interesting [that] I’ve heard you describe yourself on more than one occasion as a visual learner.

E: I am.

M: But you didn’t like Evan’s picture on this one. ...

E: It’s just *time consuming*. I preferred Caitlyn and Dawn’s, but Evan’s was correct as well. I think that it would be *really time consuming*, if he had to sit there and actually write out eight times seven [or] nine times eight—all those circles or x’s or whatever he’s going to use as a picture. And then he would have to flip the picture over. It would be *time consuming* compared to Dawn’s.

M: And I think you could say some of the same things about your response on number one. ... You liked both Andy’s and Beth’s, but you said that, if I asked you to do it, you would have done it Andy’s way. Right?

E: Right.

M: And does that have something to do with time also, the amount of time it takes?

E: *In math class, it seemed like time always went by really fast, so you would try and do all of the problems that you could possibly do and try to retain all that information in the shortest amount of time possible, because before you knew*

it, the bell was ready to ring and you had to go to the next class or get ready for the next subject.

M: So speed is important.

E: Yes, it is. (Ermida, Interview 1)

Although Ermida saw herself as a visual learner, her experience in school taught her to view diagrams as excessively time consuming. The importance she placed on solving problems “in the shortest amount of time possible” led her to favor Caitlyn and Dawn’s quicker solutions over the visual appeal of Evan’s argument.

Likewise, Diana defended the use of shortcuts, endorsing Andy’s argument as a more advanced and quicker alternative to Beth’s more time-consuming visual explanation:

If [Andy] doesn’t need the diagram, and the cross-multiplication works just fine for him, and he understands it, then that’s fine. But I think *Beth's is more basic*. It’s like the step before the process of the multiplication. It’s like *understanding why, and then you can get to the shortcut, which is Andy’s cross-multiplication*. But *Beth is kind of the step before him*, where you’re figuring out why. So I think *Andy is just kind of farther along than Beth*. He can do the process more in his head; *he can kind of figure it out instead of just drawing a picture*. (Diana, Interview 1)

Diana saw Beth’s argument as basic, easy to explain and understand, the type of argument a student might provide when still trying to make sense of new ideas. In contrast, she described Andy’s argument as a more advanced “shortcut,” a quicker, more efficient way to arrive at the same conclusion, the type of argument a more knowledgeable student should be expected to provide. She defended the legitimacy of Andy’s shortcut, noting that “if the student can understand the process,” drawing a diagram would be pointless; he can arrive at the answer more quickly and efficiently without it.

When I asked Diana what approach she would have used in this problem, she chose Andy's:

D: I would cross-multiply it.

M: You would have cross-multiplied.

D: Yeah. Because ... I feel like that's the *more advanced [approach]*. That's the shortcut to it. *Once you understand the basics—you understand why you're doing that, you can do the shortcut. ... I'm assuming he understands why the cross multiplying will help you understand that those two fractions are equal. So once you understand why and you don't have to draw the picture anymore, then I think you should use the shortcut—if you understand the reasoning behind it.*

M: Okay. All right. So I'm still wondering a little bit about the reasoning behind it. You know, we do the cross-multiplying, and that shows that the fractions are equal? *So what is the reasoning behind that method?* Do we know what the reasoning is behind that method?

D: *We were taught it in school [laughs].*

M: So when you say you were taught it in school, you were just taught to do it in school?

D: *Taught that this was the process that you use to get it and to do it.* (Diana, Interview 1)

Diana identified Andy's "shortcut" as the approach she would use, viewing the choice of a quicker, more efficient approach as a sign of more advanced understanding. She believed that using a shortcut should be permitted or even encouraged as a natural step forward, once students understood the reasoning behind it. However, when asked about the reasoning behind Andy's approach, her response did not address why it works but instead stressed the importance of knowing what to do, saying she was "taught that that was the process that you use to get it and to do it."

Like Diana, Linda also found reasons to prefer Andy's argument to Beth's. Although she did not defend shortcuts as vigorously as Diana, she noted the brevity of

Andy's argument among the reasons for her preference, saying, "Andy's isn't very long, but I feel like his point gets across very easily." Even this short statement indicates that she valued arguments that are quick and to the point.

Unlike the other interviewees, Grace did not talk about the issue of quick solution methods. This omission might be the result of her struggles to remember mathematical rules and procedures. For example, whereas Corey, Diana, and Linda initially raised this issue in response to Andy's argument, Grace rejected Andy's approach, saying, "I can't really remember if that's the right way to do it." She was therefore not in a position to defend his shortcut.

Showing how with numbers. As the interviewees compared and evaluated arguments, they often highlighted the way arguments were expressed—with numbers or with pictures—as an important consideration. Some, like Diana, viewed the choice between the two as merely a matter of personal preference, as she noted in comparing Dawn and Evan's arguments:

Evan did a visual representation of what Dawn just used numbers for. Evan drew a diagram, [and] like the last problem [Andy and Beth's], if you need the diagram or if the diagram helps you—the visual aid helps you—then that's fine. But Dawn was able to figure it out with just using the numbers, so if that's what works better for her, then that's fine too. (Diana, Interview 1)

Others, however, viewed the use of numbers—meaning standard numerical symbols—as an indication of more advanced understanding, and therefore favored arguments with numbers over those with pictures.

Ermida, for example, identified Andy's approach as one she would use, "because for any math class that I took in high school, it was just always just do the problem, the numerical problem. You didn't have to draw a picture or anything." In comparing Andy

and Beth's arguments, she described Andy's as more appropriate for students at higher grade levels:

I think Beth's explanation would be like a precursor to Andy's explanation. ... Beth's explanation would probably be taught in ... the third or fourth grade possibly, and then Andy's later on. They would explain [Andy's method] in the fifth or sixth grade and say, "You remember in the fourth grade when you drew these pictures." And I think that's how students would understand that. Beth's explanation would be easier for the fourth-grade level and then Andy's later on. (Ermida, Interview 1)

Linda also expressed her preference for arguments with numbers while comparing Andy and Beth's arguments. She thought students would easily understand Andy's, "because *they could see exactly where all the numbers were coming from, and they're all laid out on the paper for them.*" Conversely, she found Beth's diagrammatic argument confusing:

She drew a picture, but I just felt that she didn't really understand as much of what she was doing, *because it wasn't very clear to me.* That's what I put as well [referring to her written response]; it was a little bit confusing where she came up with her method. And I guess [teachers] could be using more diagrams now, but when I remember learning fractions, especially if you were trying to find some that were equal to one another, we weren't told to use diagrams. We were told, "*Here are the numbers, and you have to find if they're equal or not,*" so *we did [things] like simplifying and cross multiplication rather than drawing pictures and shaded regions.* And I just feel [Beth's argument] wasn't explained nearly as well as Andy's. ... If one of her classmates was reading this, or a group, then they wouldn't understand where she was coming from. Whereas if Andy was trying to explain to a group, I feel that they'd be able to understand it a lot better. (Linda, Interview 1)

Her response to Flavia's argument showed a similar preference for using numbers, rather than diagrams, to explain what was done:

I think it just goes back to what I learned again. *I was always taught to do it the number way; we didn't really use all that many diagrams.* So I feel like *Flavia explained herself by just being able to add it.* She just basically took the word problem and *put it into numbers.* (Linda, Interview 1)

Linda's experience in school led her to expect solutions to be explained with numbers. So strong was this expectation that expressing solutions in standard mathematical notation was essentially synonymous with explanation. Thus, Flavia could explain her solution by just "putting it into numbers" and "being able to add it." In summarizing her views near the end of her initial interview, Linda recognized and acknowledged her preference, saying, "For whatever reason, *when I see the numbers, I feel that it's more explained.*"

Corey also favored the use of numbers as a way of presenting mathematical arguments, considering them superior to both words and pictures. For example, he wrote in his response to the initial survey, "I believe that Andy's answer was much easier to perceive at first, *because his computations are shown with numbers*, while Beth explains her process *in words.*" When I asked him about Andy and Beth's arguments in his first interview, he explained further:

The way I saw it in this particular problem was that, with *numbers*, I felt it was a lot easier to perceive, because it was *discrete*; it was *fact*; it *showed you what it was*. While with Beth's, her writing it out, I felt it was more of a *word-problem answer* to me. ... It's a lot tougher to work through word problems, because *you have to figure out what information matters and what information doesn't*. And with Andy's answer, ... I thought it was more convincing right off the bat, *because he showed every step through numbers*, which is *what I've always perceived math to be*, ... through *numbers*, as I was taught, all throughout elementary and everything. We were taught that with math, you want to start off with "this *number* times this *number* equals *this*" [referring to the equations in Andy's argument]. And when I was reading through Beth's answer, it was tough at first to perceive, because she showed a representation, but the way she explained it was more complex. (Corey, Interview 1)

In Corey's experience, statements expressed in mathematical notation were factual, precise, and unambiguous, whereas words could be slippery and more difficult to interpret correctly. Andy earned Corey's endorsement, in particular, because he not only

used numbers but “showed *every step* through numbers,” which to Corey, was what mathematics was about.

While comparing Flavia and Georgia’s arguments, Corey discussed the relative merits of using of numbers and pictures, both in these arguments and more generally:

I think that ... pictures are good things, but I think they’re *better for younger ages than older ages*, because for me—as I got older—the more I saw numbers the more I could just completely relate to a number than to a picture. But I guess that was *just my way of thinking*, because I know *with some students, pictures are better*. But for me, I believe that having more evidence like Flavia’s, where it’s *numbers*, where you’re *working through* the problem, rather than seeing a picture and trying to understand it, that’s my [approach]. ... I’m not that visual. I just have to do the problem and not actually see it, to help me understand. I don’t really like visuals. ... Visuals can be confusing at times, if you don’t understand what you’re trying to figure out. (Corey, Interview 1)

Here, Corey’s views paralleled those expressed by Ermida and Diana. Like Ermida, he believed that arguments with pictures were more appropriate for younger children and those with numbers for older, more advanced learners. And like Diana, he acknowledges that the choice between numbers and pictures may be a matter of personal preference, making it clear, however, that he preferred numbers.

Corey also favored Dawn’s argument over Evan’s, due to its focus on numbers and factual knowledge rather than visual representations:

There’s definitely a difference between Dawn’s and Evan’s, because when you look at Dawn’s, she distinctly says, “Look at our multiplication table.” So right there, she’s already learned a *fact* about the multiplication table. While with Evan, when he says three times five, he puts it in a representation, where you have three dots with five rows, or if you switch it, you have five by three. And I feel [that] with Evan, when he shows it, it is just not as distinct [clear]. ... But the way I see it is that, with Dawn, she’s distinct. ... She’s focused on the *numbers* more, and Evan is focusing on the visual representations. (Corey, Interview 1)

Near the end of his first interview, Corey returned to this issue as he described his favorite arguments, noting that Flavia and Dawn not only expressed their ideas with

numbers but also showed how they obtained their answers in a very organized step-by-step way:

I like Flavia's and Dawn's because of the way they're using the *numbers* to distinctly show their answers. As I said before, I'm not much of a visual learner. I don't like using representations to show what I want; I like to just see the numbers and work it out in my mind. And when they are doing it here, they're showing that. They're showing the steps to it. They're showing "two times three equals six," and then they're reversing it—"three times two equals six"—with Dawn. She's *showing the steps* [for] how she got it, and she's showing the answer to it and showing the property, ... just saying, "This will happen when you do this." And with Flavia, I like basically the same thing, how she's *showing the steps* of what she did. I like *organized* things—here's a step; this is what you do; this is what you do. That connects more with me, because I've always worked with that. I'm a very *organized* person, ... so I like to have things [in order]—*step, step, step*. This is what I have to do, ... and this is how I get here. That's what I liked about Flavia's and Dawn's. (Corey, Interview 1)

Having the right attitude. In her initial response to Heather and Ivy's arguments, Diana wrote, "Ivy is more convincing, because although she acknowledges that their method has worked so far, she is *keeping an open mind* and is *willing to change her perspective* on adding and subtracting." She pursued this idea further in her first interview:

There are instances that—farther along in their education, they'll realize it doesn't always work, like when you add negatives. You add two negatives together, and you're going to have a smaller number. So *it's good that she has an open mind, so she won't be as confused later on in her schooling*. So that's why I liked her answer more. ... *I like that Ivy is more open-minded about it and willing to accept that it may change later, as you learn new things*. (Diana, Interview 1)

Open-mindedness was a powerful idea for Diana. She stressed its importance not only in the context of Heather and Ivy's arguments but in relation to other arguments as well:

D: [Discussing Andy's argument] I feel that it's trial and error, that you use your method, if it's correct in that period of time, until you learn otherwise—until you learn more. ... If it keeps working, then chances are it's going to be right. But I think the key is to have an open mind that later it might change.

M: So things are true ... until you find the exception?

D: I guess—until you learn the exception. You just have to have an open mind that there can be exceptions.

M: So when people say things like this group of people in number two [Caitlin, Dawn, and Evan]. All three of those students say that this will always work ... the commutative property, that when you multiply numbers in reverse order you always get the same result. You think that maybe there are exceptions to that, and we just haven't found out about them yet?

D: I guess there could be. To my knowledge, in my schooling, I have not learned of any exceptions, but *if someone were to teach me about an exception, I wouldn't shut it down*. If someone tells me that [the commutative law has exceptions], and that just hasn't come along in my math knowledge yet, then I'd be willing to accept that. *But as of what I know right now, it's true*. (Diana, Interview 1)

Grace expressed a similar view of Heather and Ivy's arguments, but she referred to Ivy as "*curious*" rather than open-minded:

G: I cannot think of something that would make two whole numbers smaller if they were added together, so I believe Heather would be correct. But *Ivy has a good way of thinking*.

M: Ivy has a good way of thinking?

G: Yeah. She's curious.

M: Ivy doesn't seem to know. ... If she had an example like this [indicating a counterexample with negative numbers that Grace had provided earlier], I think she would just say it. She seems to think that, just because we found some that work, how do we know that there's not one like this?

G: Yeah.

M: So how do we know? Do we know that there's not one like that?

G: I mean, to our knowledge, there isn't. But I don't know. Like Ivy said, "How do we know it will always work?" We're always learning, so—[she laughs]. (Grace, Interview 1)

Corey also alludes to a difference in *attitude* between Heather and Ivy:

I believe Heather's answer is more convincing, because first she is correct, and because of the way she states her answer, *she seems very confident* she is correct. Ivy does not provide a complete answer stating yes or no and *seems unsure* of the answer she gave. (Corey, Preliminary Survey)

His interpretation of these arguments differed considerably from Diana and Grace's. Where they saw open-mindedness or curiosity, he saw uncertainty. Conversely, what he saw in Heather as confidence in a correct answer, Diana might have viewed as close-mindedness and unwillingness to acknowledge the possibility of exceptions.

As these examples illustrate, PTs sometimes perceived the arguers as having certain emotional dispositions or attitudes—open-mindedness, curiosity, confidence, or uncertainty—toward the mathematical problems and ideas addressed in their arguments, and at least three of the interviewees considered the arguer's attitude among their criteria for evaluating the arguments.

PTs' Post-course Views

In the previous section, I examined what PTs initially considered important as they compared and evaluated mathematical arguments. Supported by PTs' responses to the preliminary survey and by data from their initial interviews, I found that they generally favored arguments in which they perceived the arguer as (a) knowing what to do, (b) getting the correct answer, (c) using a quick way to get it, (d) showing how with numbers, and (e) having the right attitude. In this section, I examine the ways in which their views changed by the end of the semester, providing evidence for four themes that emerged from my analysis: (a) understanding the problem, (b) finding an answer that makes sense, (c) explaining why with diagrams, and (d) seeing attitude as complex. Overall, in contrast to the themes of the previous section, the interviewees focused less attention on the arguer's thoughts and actions and more on their own efforts to understand the problems and mathematical ideas under discussion.

Understanding the problem. In his first interview, Corey espoused the belief that, when faced with a mathematical “problem,” students should “automatically” know what to do and immediately proceed on a course of action that leads to the correct answer. Paradoxically, this notion views the problem itself as essentially unproblematic. It attributes errors to the failure of students to remember and execute procedures correctly and ignores the possibility that they could misunderstand the nature of the problem.

This issue emerged in the interviewees’ evaluations of Flavia and Georgia’s arguments. Three of the interviewees—Diana, Corey, and Ermida—initially found Flavia’s argument more convincing and criticized Georgia for failing to remember and follow the correct procedure for adding fractions. (See Table 4.3.) At some point during their interviews, however, they each began to doubt their original interpretation, and eventually became convinced that Georgia’s argument was correct.

An important factor in their changing perceptions of these arguments was an

Table 4.3

Interviewees’ Initial Perceptions of Georgia’s Argument

Interviewee	Response
Diana	Flavia got the idea that the denominator stays the same when you add it. Georgia didn’t. (Interview 1)
Corey	[Georgia] forgot to realize that, when adding fractions, the common denominator remains the same. (Preliminary Survey)
Ermida	Flavia knows that when you add fractions, one must have a common denominator, and you add the numerators, but the denominator stays the same. Georgia somewhat has the right idea, and it’s nice that she drew a picture to visualize, but she also added the denominators, which is wrong. (Preliminary Survey)

increased effort—or even a prolonged struggle—to understand the problem, ultimately leading to them to conclude that automatically following the standard procedure for adding fractions was inappropriate in this case. Diana, for example, began to reinterpret the problem during her initial interview, when I asked where Georgia might have gone wrong:

D: I like that [Georgia] drew a diagram, but she didn't draw the diagram correctly. Had she drawn the diagram correctly, then it would have been helpful, but she didn't.

M: So talk about that, drawing the diagram correctly. How would you draw the diagram correctly for this one?

Diana proceeded to draw the diagram in Figure 4.1. However, as she started to explain her drawing, she hesitated.

D: I'm having ... regrets, looking at this.

M: You think you're changing your mind?

D: I might be, yeah. Let me read through that really quick. No, it would be sixteenths, because you have two things still together.

M: It would be sixteenths?

D: [Reading] "The two one-gallon jugs are both filled with a combination of grape juice and water." So you have—because you're still going to have all of that water anyways—I don't know. It's been a really long day [laughing]. ... *I feel like this is a problem that I should easily understand, but ... for some reason it's really confusing to me right now.*

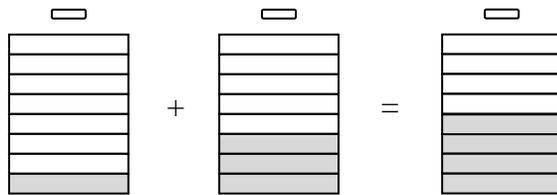


Figure 4.1. Diana's revision of Georgia's diagram.

M: Well, I think it's confusing to a lot of people.

D: Well, essentially all you're really doing is combining them, but you're still going to have the grape juice and the water mixture combined. So it's not going to be like this [her drawing in Figure 4.1]. You're pretty much just going to have this [indicating Flavia's picture] all in one big container, because you're not going to lose any of that water or any of that grape juice. So we can just imagine ... this one-eighth still and these three-eighths still but this is all one big container [both jugs combined]. So I feel that you would have four-sixteenths.

M: Four-sixteenths? Not four-eighths?

D: Well, you don't lose any of the water when you combine them. You still have all of the same liquid. So I'd imagine it would still stay. I mean you can't just take out part of the fraction because—I don't know.

M: So this idea is new compared to what you were thinking when you wrote this originally, correct?

D: Right. I feel like *I read something maybe that I didn't catch the first time around* [laughing].

M: So tell me about that. What's new now that's making you think this way that you didn't think when you first read this problem before? Or do you know?

D: I don't think it's necessarily that I learned anything new that I didn't already know. ... *I think I thought that the white part wasn't water but air*, so you were combining the three-eighths in this just with the one-eighth in this jug. If this white part was air, it wouldn't be a factor when combining it. But since the liquid is in there and it is taking space, you can't just get rid of it, because it's completely combined. So you have all of the liquid. You can't get rid of any of the liquid, so you pretty much just have to combine this into one big container, and you can't take away any of it. (Diana, Interview 1)

Diana's diagram was consistent with her initial response, but as she drew it, she realized that it did not accurately reflect the relationships among quantities in this situation. This surprised and confused her, because she thought she should have easily understood the problem and known what to do. She attributed her error to possibly misreading the problem and interpreting the white area of the diagram as air.

Alternatively, however, she may have simply interpreted the word "combined" as a signal

to add and proceeded “automatically” in accordance with what she perceived as a typical fraction addition problem. In either case, her interpretation of the problem had changed significantly, and she confirmed this in her post-course interview, specifically recognizing the importance of understanding the problem:

I think my original problem was I didn't pay close enough attention to the problem and realize that the shaded part represents grape juice and the non-shaded part represents water. I think I thought it was grape juice and nothing. But when you have the water, that liquid isn't going to disappear. You can't just get rid of it, so you have to take into account all of the parts. (Diana, Interview 1)

Corey changed his opinion of Flavia and Georgia's arguments in a similar way but over a longer time period, not reaching an epiphany until his post-course interview. Throughout his initial interview, he consistently favored Flavia's explanation and interpreted her mixture task as a standard fraction addition problem. Unlike Diana, Corey was not swayed when I asked him whether he could have used a diagram to explain Flavia's solution:

C: I would have said that each jug counts as a whole. ... I would have drawn and shown that this one-eighth that was in here [the first jug] went into this [the second jug]. I would have poured it and just completely got rid of this jug [the first], so that whole wouldn't exist.

M: So both mixes are going—

C: Into one—that's how I would have taken it. Then I would have had ended up with four-eighths in one jug, and this jug [the first] wouldn't even exist anymore. And then I would reduce the fraction. Four-eighths is the same as one-half. That's how I would have done it. ... That's what I saw [in] this picture, which was why I thought [Georgia] was wrong. (Corey, Interview 1)

After reading the same problem in his post-course interview, his views initially seemed to have remained the same:

C: I like Flavia's, because I feel like Georgia's is wrong.

M: You feel like Georgia's is wrong.

C: Because she is using two separate wholes in this picture, by drawing them apart. And one of the ... rules we learned is that they need to be combining and using the same whole when you are adding fractions together. And I feel like the answer would be four-eighths or one half of the whole, and in this situation she has two separate—hold on. Give me one second [laughing]. [He reads through the problem a second time.] Never mind. Georgia's is right.

M: Georgia's is right?

C: Georgia's is right. I'm sorry. I didn't realize that ... they are talking about two one-gallon jugs, so [the problem] already states that this [indicating the two jugs in Georgia's picture] is the whole, that both of these create the whole. I just wasn't reading it right. So both combined equal one whole. ... So *the way I read it was wrong*, because in this problem, they are combining two one-gallon jugs together, which [means], if you have one-eighth of one gallon and three-eighths of the other gallon, when combined, you would have four-sixteenths.

M: Four-sixteenths of—

C: Two gallons.

M: Oh.

C: Yes. [He pauses.] Yes. That is how it would work. *I'm confusing myself right now*, but I feel [that] Georgia's is the correct answer, because it says that two one-gallon jugs are the whole already. Yeah. [Reading the problem again] "One-eighth of one gallon and three-eighths of the other." [He pauses.] Yes. (Corey, Post-course Interview)

Like Diana, Corey decided that he had initially misinterpreted the problem and that Georgia was correct, attributing his earlier error to misreading the problem. Also like Diana, he was surprised and confused by this conclusion, pausing and rereading the problem repeatedly to check his understanding and reassure himself that this new interpretation was correct.

During her first interview, I asked Ermida about Georgia's diagram, and she struggled to reconcile her understandings of a correct representation, a correct procedure, and the correct answer:

M: You say Flavia's got the correct answer here, but I look at Georgia's picture, and it seems to make sense. Does Georgia's picture make sense—the picture she drew for this problem?

E: It does represent that one jug is one-eighth grape juice and the second jug is three-eighths grape juice.

M: So is there a way to use Georgia's picture to get the right answer, or is something strange going on here that's somehow misleading?

E: I think we could. We were talking, in Dr. Lannin's class, about defining the whole, and so I think here we would have to define what the whole is.

M: *So what is the whole?*

E: *I'm wondering if it would be actually both the jugs. ... No, because you're supposed to add the numerator and keep the same denominator. The denominator would be just one jug. The numerator would be the shaded part, and there are four of them shaded, one from one jug and three from the other jug.*

M: And then, if Georgia reasoned it out that way, she would get the correct answer?

E: She would, but I still don't see how you could—*I don't even understand how I would explain this to somebody, because there are two jugs, and I don't know if you tell them to add the shaded parts but only keep one jug, which is the whole.*

M: So what do the shaded parts represent in this case? What are the shaded parts?

E: The shaded parts are the grape juice, which is the numerator. In Flavia's we see one over eight plus three over eight. That would be the one and the three [pointing to the shaded parts of the diagram], but then the eight is just one jug, because you don't add the two jugs. And *that's what I'm having a problem explaining.*

M: So there's something strange here that needs explaining.

E: Now see, *if you would have asked me this back in when I was in secondary school, I would have had problems then, too.* (Ermida, Interview 1)

Based on her application of the standard algorithm, Ermida believed the answer to be four-eighths or one-half. She therefore rejected the idea that the whole could be both jugs combined, because it leads to a different answer, four sixteenths. Although she could

not find an adequate explanation for combining the shaded parts and not combining the two jugs to form a new whole, she viewed this as a weakness in her background and understanding and not a reason to reject Flavia's solution.

After reading through the same problem in her post-course interview, she approached the problem differently:

E: Okay, so what's the real answer here? It's four-sixteenths, right? No, it's four—[Ermida pauses, then reads aloud slowly and carefully.] "*If the two gallons are combined, what fraction of the combined mixture will be grape juice?*" "*If the two gallons are combined,*" so then it would be four-sixteenths. "I drew pictures of two gallons, and I shaded one-eighth of one and three-eighths of the other. When the two pictures are combined together, there are sixteen equal parts and four are shaded, so the mix would be four-sixteenths." I'm not understanding the wording here. "*If the two gallons are combined, what fraction of the combined mixture would be grape juice?*" *What fraction of the combined mixture?* Well, there'd be sixteen parts then. Or would it be eight? It would be sixteen because you combined them. Right?

M: Well, I can't tell you right now. I have to ask you [laughing].

E: I know. I'm sorry. I'm just thinking out loud. Okay, this is why it's so important to clarify what the whole is.

M: Okay.

E: Okay. So two gallons are combined. Then it would be sixteen parts—with four parts being grape juice. But I can see what Flavia's also saying, because that kind of confused me too. So that is one-eighth part here and three-eighths. You combine the two [sighs]. *Wording is so key.* Okay. ... In Flavia's explanation, her whole is one gallon, which is eight parts, but in Georgia's explanation, her whole is 16 parts, which is two gallons. And the question asks, "*If two gallons are combined,*" which is 16 parts, "*what fraction of the combined mixture would be grape juice?*" There are four parts from two gallons, and two gallons is sixteen parts, so Georgia's answer is the right answer.

M: Georgia's is? You're sure of that?

E: Yes.

M: Confident?

E: Yes [laughing]. (Ermida, Post-course Interview)

In contrast to her first interview, Ermida's efforts here were not directed toward justifying the result of a standard algorithm. They focused instead on understanding the problem, as evidenced by her frequent repetition of important phrases, such as "if two gallons are combined," and "what fraction of the combined mixture," as well as her comment, "Wording is so key."

Unlike Diana, Corey, and Ermida, Grace did not initially endorse Flavia's argument but instead found neither explanation convincing. However, she also showed a similar shift in focus, away from remembering correct procedures and toward understanding the problem. In her first interview, she emphasized her difficulties in remembering what to do, saying, for example, "The rules of fractions have escaped my mind." Conversely, in her post-course interview, Grace focused her efforts on understanding the problem:

G: I like Georgia's. And I could be completely wrong, because sometimes I still get confused on these things. Okay, see. It says, "Two one-gallon jugs are filled with liquids that are a mix of grape juice and water." The first one is one-eighth grape juice, and the second is three-eighths grape juice. So Flavia has one-eighth and three-eighths, which equals four-eighths. But that's only if you're combining the grape. You can't just like combine them and not [ask], "Where does the other seven-eighths go?" Do you know what I'm saying? So I like this picture [Georgia's]. If it didn't have the picture in it, I don't know if I would get this right. *If you combine these things, and you pour all this water [from the first jug] into here [the second jug], the water is going to have to double.* It doesn't just go away, and here [in Flavia's], it just kind of went away. Right?

M: In Flavia's, the water just—

G: *In Flavia's, the water—the seven-eighths—just disappeared,* because I'm assuming that she is pouring the one-eighth in here [the second container]. So these [indicating the seven non-shaded parts of Georgia's first jug] all just went away, because she says it's one-half, which would be correct if they poured the grape juice in [the second jug].

M: Okay. So if you were just combining the grape juice—

G: Well, there would still have to be [she sighs]—I don't know. Hold on a second. *But [in Flavia's] the seven-eighths disappeared!*

M: The seven-eighths in the first jug disappeared somehow?

G: Yeah, so *I think this [Georgia's] is right, because she says, "When the two pictures are combined together, there are going to be 16 equal parts."* (Grace, Post-course Interview)

Unlike her first interview, Grace attempted to make sense of the problem rather than struggling to remember whether the denominators should remain the same and faulting herself for not remembering. Some of her statements here might seem confusing or contradictory, because she referred to "water" in two different ways. She initially referred to the entire contents of the jugs as "water." Thus, when she said, "If you combine these things, and you pour *all this water* [from the first jug] into here [the second jug], *the water is going to have to double*," she indicated that, if the first mixture was poured into the second container, the amount of liquid in the second container would double. However, when she said, "In Flavia's, *the water—the seven-eighths—just disappeared*," she referred to just the part of the first mixture that is actually water. She concluded that Flavia's explanation ignored the water part of the mixture. Whereas Georgia, by saying, "When the two pictures are combined together, there are going to be 16 equal parts," accounted for both the water and the grape juice. Grace therefore rejected Flavia's argument in favor of Georgia's, because it agreed with her understanding of the problem.

Finding an answer that makes sense. At the beginning of the semester, the interviewees placed considerable importance on obtaining correct answers, often using the correct answer—or one they perceived to be correct—as a reason for endorsing one argument over another. Of course, they still considered correct answers important at the

end of the semester, but their criteria for correctness had changed. At the beginning of the semester, correct answers were those that resulted from knowing what to do, that is, from accurately applying an accepted mathematical procedure. By the end of the semester, however, they generally considered an answer correct only if it made sense to them within the context of the problem. Thus, finding a correct answer and understanding the problem were inextricably linked, and just as they struggled to understand some problems, they also struggled to determine which answers were correct. However, when faced with arguments that reached conflicting conclusions, they did not generally accept the result of a standard algorithm as correct, as they had at the beginning of the semester. Instead, they often rejected those solutions in favor of others that made more sense to them, even when some confusion remained.

Ermida, for example, continued to be troubled by the grape-juice mixture problem, despite her statement of confidence in Georgia's solution:

E: It's just confusing. Now, if we originally had [only] one gallon, and there was one-eighth in there. And then, let's say you were to fill it up with ...

M: Well, I'm thinking with another three-eighths. Couldn't you do that? If you had one gallon, couldn't you put in another three-eighths of a gallon?

E: Right. And then we could say, just adding it to this mixture with the one-eighth, "*How much grape juice do we have now in the gallon?*"

M: "In the gallon?"

E: In this gallon, one gallon, how much grape juice do we have in there now? *Then I could clearly say four-eighths, because I saw that I poured three-eighths into this one gallon.* But now that we're combining the two, I think that's what's throwing me off. *I don't know if I'm supposed to include all the parts from the other gallon—to make it 16 parts—or not.* And that's what's throwing me off.

M: So it sounds like you're still not sure.

E: No.

M: Well, five minutes ago [Ermida laughs], I swear you said you were confident that Georgia was correct.

E: [Laughs and sighs] And I know this is what I'll [need] to deal with—with my students later on, struggling and trying to understand this as well—so I need to get this now. But presently, during this interview, for some reason I'm going to go with Georgia.

M: You're going to go with Georgia?

E: Yeah. I'm going to go with Georgia because of the question it's asking me: "If the two gallons are combined, what fraction"—quote, unquote—"of the combined mixture will be grape juice?" (Ermida, Post-course Interview)

Thus, although Ermida retained some doubts, she selected Georgia's answer as correct, because it made sense to her within the context of the problem. She focused specifically on the wording of the question, contrasting it with another similar question for which she would have chosen Flavia's response.

Like Ermida, Grace was also confused and frustrated by the grape juice problem, but she found Georgia's answer more convincing than Flavia's:

M: So you're confident it's four-sixteenths, or are you still a little unsure?

G: [Groans in frustration.] *I wish someone would just tell me the answer!* I don't know, because—okay, to me *it makes sense* that it's four-sixteenths. But maybe it's just confusing me, because *if there was no word problem, and you were just [asking me to] add one-eighth and three-eighths, then ... I would come up with four-eighths.*

M: Okay. All right. That makes sense.

G: Because I don't think—you're not supposed to. Are you? Will you just tell me? Are you supposed to add the denominators when adding fractions? Please tell me.

M: When adding fractions in the usual way, people usually don't add the denominators.

G: Yeah. I didn't think so. So that's what I think!

M: Okay. So maybe that's the question. Is this a problem where you should add fractions in the usual way?

G: *No! Because it really doesn't make sense to me, if you're adding two jugs of water and just all a sudden have the same amount. (Grace, Post-course Interview)*

In her first interview, Grace repeatedly criticized herself for not remembering the “rules of fractions.” In particular, she saw this problem as one in which $1/8$ and $3/8$ should be added, and she could not remember whether the denominators should also be added or remain the same in that situation. Her support of Georgia’s answer was therefore significant here, because she now knew that the denominator should remain the same when adding $1/8$ and $3/8$, but she still rejected Flavia’s solution, because it failed to make sense to her.

Explaining why with diagrams. At the beginning of the semester, the interviewees often preferred arguments expressed in numbers, such as Andy, Dawn, and Flavia’s, to longer, more time consuming diagram-based arguments. By the end of the semester, however, they generally preferred arguments with diagrams, such as Beth, Evan, and Georgia’s, recognizing the power of diagrams to support arguments that explain why solutions work. In her first interview, for example, Diana endorsed Andy’s argument as an advanced “shortcut,” a quicker, more efficient way to reach the same conclusion as Beth’s more time-consuming approach. In her post-course interview, however, she preferred Beth’s argument to Andy’s. When I pointed out that she had initially chosen Andy’s approach as the one she would have used, she struggled to understand why she would have said that:

D: Someone tried to explain [Andy’s] method to me, and it just never clicked. I didn’t quite understand it. So I like Beth’s more, because I don’t understand Andy’s that well. I may have thought it was—I don’t know. I’m pretty sure I don’t have a good grasp of this now, so I don’t see how I did at the beginning. I may have just been confused possibly? I’m not sure.

M: Well, you said Andy's method seemed more advanced, and you really didn't need a picture to understand the procedure.

D: Yeah, I guess if that works—I don't know. . . . I never did use that [method], I don't think, when I was learning math.

M: So something has changed.

D: I don't know. Yeah. I'm trying to figure out why I thought I knew what he was doing, because I don't think that. I remember being really confused about that in class when we were talking about it, because I didn't see how . . . *I didn't know if that was just some shortcut? I didn't know the logic behind it.*

M: Well, you did use the word "shortcut," if I recall, but you said the shortcut was okay as long as you understood the procedure.

D: Right! And *I didn't understand the procedure, so I didn't try the shortcut.* [Laughs] So yeah, I guess now I like Beth's better. I just can't—I *can't understand why I would have said I understood that, if I had no idea.* (Diana, Post-course Interview)

I suggest that, in both her first interview and at the end of the semester, Diana believed that teachers should allow students to take shortcuts, provided that the students *understand* what they are doing. What changed between the two interviews was her meaning for *understanding*. At the beginning of the semester, she understood that cross multiplication was an accepted procedure—accepted in her earlier school experience—for determining whether two fractions are equivalent, and she understood how to carry out that procedure, "the process that you use to get it and to do it." She knew *what* to do and *how* to do it, and that was enough for her. But by the end of the semester, she demanded a deeper kind of understanding, an understanding of *why it works*.

Diana explicitly addressed this new conception of understanding as we examined an argument about the sum of two odd numbers:

M: In class, Katie came up with an argument. . . . To tell whether a number is even or odd, all you have to do is look at the last digit. The other digits don't matter; just the last digit is really relevant. If the last digit is one, three, five, seven, or nine, then it's an odd number; and if the last digit is zero, two, four,

six or eight, then it's an even. And so what she suggested is that you could take all the different possible combinations of one, three, five, seven, and nine and add them together and show that you always get an even number. One plus one is two, and one plus three is four. One plus five is six—just go through all of them, and that would show it. What do you think of that explanation?

D: That seems like just showing a shortcut, instead of understanding why it is like that. They'll know that, when you have two odds, you look at the last number, but they might not understand why. They might not have a really good, deep knowledge of why that is. And ... with these responses [see Figure 4.2, for example], it's really explicit and just lays it out clearly as to why. So I don't know. I just think it's better that they can understand why they're doing something, instead of just learning the shortcut to it. (Diana, Post-course Interview)

As Diana discussed the argument in Figure 4.2, she emphasized the role of the diagram in making the explanation clear and easy to understand:

I love the picture. I don't know why I liked this so much. I guess I had so much trouble coming up with a way to explain this well that, when I saw this example in class, I [thought], "It's so simple!" *You show it.* You pair them up. ... Kids at that age will know that "even" means they have partners. So if you show that they're partnered up—and you [can] even use kids in an example—you're going

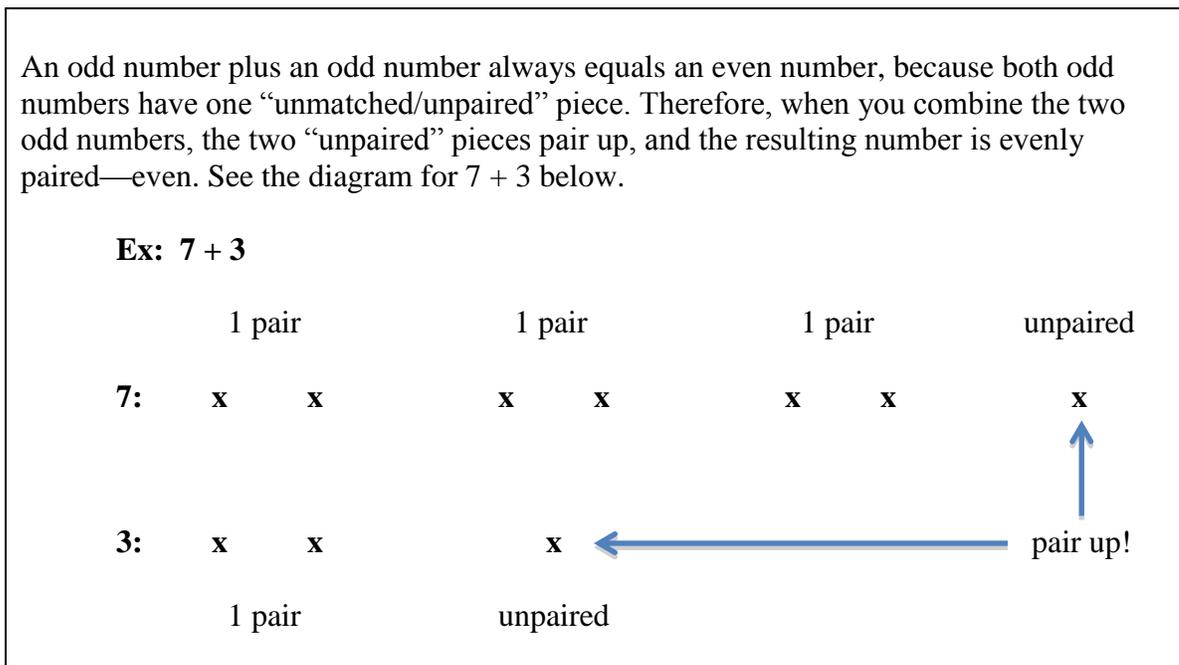


Figure 4.2. An argument explaining why the sum of two odd numbers is even.

to have one extra, if you have an odd number. And then they get together, and they'll be paired. *So it just seems like it's so simple, and it's on terms that young kids can understand, too. And the picture—it's just really easy to understand. It's very clear.* So yeah, I like that. (Diana, Post-course Interview)

At the beginning of the semester, Diana considered the choice between using numbers and using pictures as a matter of personal preference. She defended Andy's method as a more advanced shortcut, compared to Beth's diagram-based argument. In contrast, at the end of the semester, she saw the need for understanding why solutions work and recognized the power of diagrams to support that understanding. Thus, she came to view Beth's argument as not just superior to Andy's but one of her favorites overall:

D: I like Beth and Evan's the most—yes, particularly Beth's, if I had to pick one.

M: And why so?

D: Throughout this semester, *I realized how important it is to have illustrations to go along with your explanations, because it makes it so much easier to understand* and see where you're coming from. So I like that there's an illustration. And she kind of steps or walks you through each step she took; it's not just an illustration. The description of what she did, with the illustration, is detailed enough that I can follow it and I can understand what she did. *So it's a good explanation, backed up by an illustration.* So I like that one a lot. (Diana, Post-course Interview)

Among the other interviewees, Corey's shift toward the use of diagrams to explain why was the most dramatic and explicit, but Linda, Grace, and Ermida each echo different aspects of his transformation. In his post-course interview, for example, Corey began to talk about his newfound love of diagrams as he discussed Andy and Beth's arguments:

M: [In your first interview], you said that Andy's was more convincing, because "he showed every step through numbers." Does that ring a bell?

C: That does ring a bell. I actually do remember saying that, because I never thought I would use diagrams until Dr. Lannin's class, and I just love them now. It makes *it so much simpler and easier to convey the meaning of the math*, rather than just [using] numbers. ... Before I was just thinking, "Oh, it's a quick shortcut. ... Oh, that gets the answer. Why do I need to show all this?" But now I definitely think Beth's is better, because I feel *Andy's is just a shortcut, and it doesn't really explain why cross-multiplying works. It's just saying it works. But ... I like [Beth's], because it shows the exact procedure of what is happening, how three-fifths can equal six-tenths*. The only thing I would like better about Beth's is if she would label it. She says what each is, but it's not labeled on here [on her picture]. So if she labeled each piece—the smaller piece as a tenth [and] the bigger piece as a fifth—I feel that would be perfect, because it could show the students what is happening and how, by cutting in half, you're creating tenths. (Corey, Post-course Interview)

Like Corey, Linda also initially chose Andy's argument as one of her favorites but became critical of his argument by the end of the semester for its failure to explain why:

I don't think Andy's is very valid. Because he says he cross-multiplied, and then he says, "If you get the same number when you cross-multiply, then the fractions are equal." ... But he doesn't explain if he was taught that way. *He doesn't explain why he's multiplying*, so I guess it's just one of those things in an equation that he just learned, and it's kind of like plug and chug, because *he doesn't explain why you would multiply three and ten and five and six just because they're across from each other*. ... I mean Andy obviously got it right. But I just think *he needs more of an explanation as to why he's doing that, because he probably was just taught to cross-multiply and not really why*. (Linda, Post-course Interview)

Instead, Beth's argument, one that she initially found confusing, joined Evan's as one she thought best overall:

I do like Beth's though, because she draws it out initially with the fifths, and then she has three of them. And when she cuts them in half, you can see that the six are shaded out of the ten. ... So I think that hers is much more convincing and valid than Andy's. (Linda, Post-course Interview)

When I told her that she had preferred Andy's argument at the beginning of the semester, she responded:

Well, that was a long time ago. After last semester, [when] we had all those drawings and diagrams that we did in class, the more we talked about it, the more it made sense, and the more it related, I felt, to the problems. ... I'm more of a diagram person now. (Linda, Post-course Interview)

Thus, at the end of the semester, both Corey and Linda saw in Beth’s diagram-based argument an explanation of why her solution worked, something they found lacking in Andy’s. Grace emphasized the same idea using Evan’s argument:

Evan’s actually shows them. It’s not just using numbers [but] also using symbols [the circles he drew], so they can count them out and see for themselves. ... You can actually count for yourself and see that it’s going work, instead of [someone] just saying, “Well, this works,” [or] showing a bunch of examples [and saying], “They all work, so it obviously works.” With [Evan’s], you can [say], “*This is why it works,*” because the columns are just changing [to rows], but it’s all the same. (Grace, Post-course Interview)

Corey and Ermida’s discussion of Evan’s argument emphasized its generalizability and how the diagram supports this. (See Table 4.4.) They both included this argument among their favorites at the end of the semester.

Table 4.4

Responses Noting the Generality of Evan’s Argument

Interviewee	Response
Corey	I like Evan’s best, ... because it has a representation. Now, after the [Number and Operation] class, I just love representations and drawings, because I feel it’s a visual aid to show students that is very easy to connect to their knowledge. And I like the way [Evan] does it. He shows three times five is the same as three rows with five in each row, and then five times three is the same as creating five rows with three in each row. It’s just flip-flopping it, and <i>it shows that it will work in any situation</i> , visually connecting to the students by showing dots to represent each number. (Corey, Post-course Interview)
Ermida	Seeing what Evan has learned from this example, I think his thinking order would move on to the next level, ... because he sees that if you switch the two numbers, this would be three by five and the other one is five by three. <i>If he completely understands this concept, he would apply it to all of the other multiplication factors.</i> ... I know I [originally] said it was time-consuming, but <i>if he understands this, he wouldn’t have to prove this point every single time he did a math problem.</i> (Ermida, Post-course Interview)

Near the end of his post-course interview, when I reminded Corey that he had initially chosen arguments without diagrams—Flavia’s and Dawn’s—as his favorites, he summed up what had changed for him:

I used to like just numbers, and I used to think that was the best way to explain the answers to students, to show them distinct numbers [to explain] why this is working correctly. But now I definitely believe that showing representations and connecting it to the real world and *explaining each specific part—why this is this or this piece is this*—definitely is a much stronger way to connect to the students and visually *show them why each process works*. So that’s why I chose Beth’s and Evan’s and even Georgia’s—though I was confused by that one—because *it has a diagram representing what she is actually doing*. I feel like that makes a stronger argument. (Corey, Post-course Interview)

Thus, by the end of the semester, all of the interviewees expressed a greater appreciation of diagrams for their role in supporting explanatory arguments, including those that explained why a particular solution worked and those that explained why the same approach would work in other similar situations.

Seeing attitude as complex. Of the themes that emerged from the initial interviews, *having the right attitude* was the least prevalent across interviewees, emerging only in data from Diana, Grace, and Corey. Diana and Grace praised Ivy for her open-mindedness and curiosity, whereas Corey saw Heather as both correct and confident in her answer. Each of these attitudes seemed clearly praiseworthy. After all, would any teacher want her students to be close-minded or to lack curiosity or confidence? If an arguer showed such an admirable attitude, it seemed natural for them to consider this attitude positively when assessing the argument. In their post-course interviews, however, their assessments of these arguments and the attitudes behind them seemed less straightforward and more complex.

Grace, for example, in her post-course interview, found something lacking in both Heather and Ivy's arguments:

I don't know if either of them is really convincing, because Heather just says, "*I can show you a bunch of problems where it works, and it always will work.*" *She doesn't really say why.* And Ivy's right to think that, for a lot of rules, especially in math or English, there's always those exceptions, where sometimes it just doesn't work. But I don't think [her argument] is very convincing either, because *she [just] says she doesn't think it will always work. She doesn't say why she doesn't think so.* She just says that it's basically because some things don't work all the time. *So I don't like either explanation.* (Grace, Post-course Interview)

Rather than praising Ivy for her curiosity, she criticized both Heather and Ivy for failing to *explain why*—Heather for failing to explain why it works and Ivy for failing to explaining why it does not.

In Corey post-course interview, he again described Ivy as unconfident but now seemed to see some justification for her lack of confidence:

[Groaning] I don't like that Ivy doesn't seem confident with it, but I'm not going to say, ... because ... *how do we know that it's not going to work or that it will work?* And I [also] like Ivy's, because math is so different that there are some things that you need to work through, to actually find it. It's not just one set answer always. But I feel [that] Heather's is correct. When adding whole numbers, they have to be getting bigger. And subtracting, they need to be getting smaller. (Corey, Post-course Interview)

The ambivalence that Corey showed here was absent at the beginning of the semester. He originally viewed Heather as both confident and correct and saw Ivy as neither. At the end of the semester, however, he saw some justification for Ivy's skepticism, though he still agreed with Heather's conclusion.

Diana spoke significantly more about attitude than any other interviewee. Her post-course discussion of Heather and Ivy's arguments emphasized the implications of open-mindedness for learning and teaching:

I still like Ivy's more. I just think that it shows that *Ivy has a more open mind about it*. When you have set in your mind that things are a certain way, it's harder to change those ways. *So as a teacher, I would rather have a student who has an open mind about it and is more willing to hear how there can be exceptions to certain things*. Whereas Heather is just dead set in her mind that it works. "I've done it a million times. It has always worked; it is always going to work." That, I feel, would be harder to change—not change but introduce more information to—if it might conflict with their old information. Ivy would be more willing to take in that information. . . . *It's not so much the explanation, it's more the attitude behind the explanation that I like more. I feel like Ivy would just be easier to work with than Heather*. (Diana, Post-course Interview)

Although Diana clearly argued in support of open-mindedness, she also recounted some episodes in which she failed to demonstrate an open-minded attitude. The first occurred early in the semester, as the class discussed problems in which the same diagram could be used to represent different fractions, depending on what unit was considered as the whole. Looking back at the end of the semester, she viewed this as something she originally found confusing but now understood well. To illustrate this idea, she drew a circle, divided it into four equal parts, and shaded one of them:

D: If you have this picture with this shaded in, you're going to think this is a fourth. . . . This could be an eighth; this could be a tenth; this could be one one-hundredth.

M: Explain how it could be an eighth. What would make that an eighth?

D: Well, you'd have to signify what the whole is. If you say this is a fourth, you're assuming that this one circle is the whole. It could be an eighth. It could be another whole [circle] here, that's just not shaded in. It's kind of like . . . a pizza. If you have two pizzas and you have one box that's completely empty, and then you have one piece left in one [indicating the shaded part of her drawing]. If you maybe don't see that other box, you would say, "Oh this is a fourth of the pizza," but there was that entire other pizza there. You haven't signified that it was one fourth of that pizza or one-eighth of the entire thing. You have to say what the whole is that you're referring to.

M: Okay.

D: *I was so—so against that the first day. I was one of those people [who said], "No! This is a fourth! I don't know what you're talking about!" [Laughs]*

But now I really understand that a lot better. It can be whatever, as long as you make it clear what [whole] you're referring to. (Diana, Post-course Interview)

Diana's previous experience with fractions had led her to conclude that one-fourth of a circle could only represent one-fourth, and she initially resisted the idea that it could be anything else. Only by overcoming her initial resistance—her close-mindedness—could she learn to interpret this representation in other legitimate ways.

In another instance, when comparing arguments about the sum of two odd numbers, Diana characterized her own argument as close-minded:

D: I remember, [when] we had to do this homework, I could not for the life of me think of a good way to explain it, so I think I came up with some answer like [the argument in Figure 4.3]. But when I saw the other explanations, I found ones that I just liked so much more.

M: So if that's what you came up with, what's the matter with it?

D: Oh, because it's almost exactly like the explanation that I didn't like in the examples before. ... It's like Heather's. ... It says, "*I've tried it a lot, and since it's worked all of those times, it must always work,*" and that's just very *closed-minded*. Like I said, it was kind of out of desperation. I needed to finish my homework, and I couldn't think of a reason, so I came up with something like that myself. But ... *it's very closed-minded*. (Diana, Post-course Interview)

Diana clearly valued open-mindedness but provided examples in which she failed to demonstrate this trait. Although her concern for open-mindedness had not diminished by the end of the semester, these examples suggest that she now recognized the difficulty of *maintaining* an open mind in a way that she had not at the beginning of the semester.

If you add an odd and an odd, you get an even. We tried this for $3 + 5$, $7 + 1$, and $9 + 11$. All of these gave us even numbers. Since we can't find an example where it doesn't work, it must always work.

Figure 4.3. An empirical argument about the sum of two odd numbers.

Overall, the attitude of the arguer continued to be a factor that some of the interviewees considered when evaluating mathematical arguments. However, they now viewed the issue of attitude as more complex and less straightforward than they had at the beginning of the semester. It was no longer simply a matter of seeing an admirable attitude in the arguer, such as confidence or open-mindedness. Instead they seemed to recognize that, as students struggle to understand new ideas, lack of confidence is sometimes justified and open-mindedness is difficult to maintain.

Section summary. In this section, I examined the post-course concerns of interviewees while comparing and evaluating mathematical arguments. In contrast to the beginning of the semester, they generally devoted considerable time and effort to understanding the problem rather than assuming they would know what to do automatically. Consequently, they no longer simply accepted the result of a standard procedure as the correct answer, sometimes rejecting these solutions in favor of others that made more sense in the problem's context. Rather than favoring methods that led to quick answers, they now questioned the validity of arguments that used shortcuts, criticizing them for failing to explain why these procedures worked. They now preferred arguments that used diagrams to those that presented solutions in standard mathematical notation, appreciating the usefulness of diagrams in explaining why solutions work. Finally, although some still considered the "attitude behind the explanation" in the arguments they examined, they now considered the issue of attitude as more complex than they had at the beginning of the semester.

The Role of the Argument's Form in PTs' Evaluations

In this section, I consider five forms of argument: (a) appeals to authority, (b) rules without backing, (c) empirical arguments, (d) generic examples, and (e) counterexamples. Based on interviewees' responses to arguments of each type, I consider the role that the arguments' form played in their evaluations and how this changed by the end of the semester.

Appeals to authority. An argument uses an *appeal to authority* if it refers to some symbol of mathematical expertise, such as a teacher or textbook, to support the use of a rule or procedure. Thus PTs' views of appeal to authority are reflected in their responses to Caitlyn's argument, with its reference to finding this rule in a textbook. In the initial survey, 16 (80%) of the PTs included Caitlyn's argument among those they found convincing, and four of the interviewees—Corey, Ermida, Grace, and Linda—likewise initially accepted Caitlyn's argument. As noted earlier, the interviewees often placed considerable weight on obtaining the correct answer and therefore often endorsed arguments as equally convincing, if the arguments reached the conclusion they viewed as correct. This would explain their initial acceptance of Caitlyn's argument.

However, when pressed to distinguish among these arguments, to separate those they found more convincing from those they liked less, Corey and Grace joined Diana in criticizing Caitlyn's argument, providing similar reasons for doing so. (See Table 4.5.) All three accepted Caitlyn's argument as essentially correct, but faulted her for showing a lack of independent thinking and failing to provide evidence that she really understood what she said, rather than just parroting the textbook. If she would have tested it in some way, shown an example, or related it to something familiar, such as the multiplication

Table 4.5

Initial Criticisms of Caitlyn's Appeal to Authority

Interviewee	Response
Diana	I didn't like Caitlyn's explanation as much, because <i>she didn't figure it out on her own; she didn't test it. She didn't mess with it or experiment with it to have it ingrained more in her memory. She just saw it and remembered it.</i> In my opinion, Dawn and Evan's explanations will stick with them longer, because they took the time to test it and figure it out. I think that when kids discover things on their own or at least partially on their own, it will stick with them better. They'll have a better memory of it instead of just reading it or being told and completely taking the book's word for it. <i>So it's true that what she said is correct, but I just don't think it's as convincing as the others,' because she's just reading it instead of trying it and testing it for herself.</i> (Interview 1)
Corey	[Caitlyn] knows this is a property; this is true. But <i>until you do something, you don't really understand it.</i> So I feel <i>she could have said that, and it would have been correct, but if she would have shown an example, I would have felt that she knew it.</i> ... I like seeing the student's thought process rather than having to look at ... "this is what I learned in the book," [or] "the book states this." I would like to have seen a problem worked out, like what Dawn did. If [Caitlyn] would have done something similar to Dawn and stated, "This is why it's true," that would have been excellent. I would have loved that, because that would have shown her work, and she would have related it back to the property. But <i>since she just states the property, and there's no explanation, or there's no worked out example of it, I feel that she could have just read that the second before I gave her the question.</i> (Interview 1)
Grace	Caitlyn's would be my least favorite, ... because there was no application to something in her life. It was just [saying], "I read this from the book, and this is why it's that." There was [nothing] like, "I think it's like this because of that," or like [what] Dawn said from the multiplication tables. So that would be my least favorite. Although she is correct, I like hers the least. (Interview 1)

table, they would have been more inclined to believe that she understood the meaning of the commutative property, rather than just repeating it.

When pressed to distinguish among these arguments, Linda also chose Dawn and Evan's over Caitlyn's, but for different reasons than the others, considering the arguments as a teacher might present them, rather than a student:

The first one [Caitlin's] makes the most sense to me, because I think that's what I learned in school. But *if I had to explain it to students*, I feel that Dawn and Evan's, you can see—especially Evan's with the diagrams. It's so much easier to see and to explain to kids, instead of just giving them “this is the method you should use” or “this is the equation” and not really explaining as much. If they have a diagram like Evan drew, then I feel that it would make more sense to them. *So I would use [Evan's] if I was teaching, but the first one is what I used in school.* (Linda, Interview 1)

Among the interviewees, only Ermida endorsed Caitlyn's argument both initially and at the end of the semester. At the beginning of the semester, she considered Caitlyn to have a higher level of understanding than Dawn or Evan, possibly because Caitlyn's argument used abstract symbolic notation and more technical language than the others:

Caitlyn took it to *the next advanced level* by explaining the *commutative property of multiplication*. No matter how you look at it $a \cdot b = b \cdot a$, always. ... *She took the initiative to look in the book* and actually try to find this property of multiplication there. (Ermida, Interview 1)

Unlike other interviewees who saw Caitlyn's reliance on the textbook as an absence of independent thinking, Ermida considered Caitlyn's actions a sign of initiative. She expressed similar views at the end of the semester:

Caitlyn's is convincing, because that's a mathematical property she understands, and that would be a higher [level] than what Evan was thinking, ... because if she didn't understand the basics down here [in Evan's argument], ... then she wouldn't have understood [what she read in the textbook]. (Ermida, Post-course Interview)

Rather than casting doubts on Caitlyn's understanding of the commutative property, as Diana, Corey, and Grace did, Ermida credits her with a higher level of understanding than Evan, believing that the understanding exhibited in Evan's argument is prerequisite to the kind of understanding that Caitlyn showed.

Surprisingly, Grace's position shifted by the end of the semester and became somewhat more like Ermida's. Although Evan's argument was Grace's clear favorite, she saw little difference between Caitlyn and Dawn's:

G: I like Caitlyn's. *She's reading the math book; props to her.* And she said the property, [and] who can really remember all of them? But *she understands what it means.*

M: The part you're pointing at is "A times B equals B times A."

G: The commutative property. Yeah. So she read it, and she understands that it applies to this.

M: To the "45 times 32 equals 32 times 45." Okay.

G: But—I don't know—maybe they're tied. I don't know.

M: Caitlyn and Dawn are tied?

G: Maybe. (Grace, Interview 4)

She no longer criticized Caitlyn for failing to connect the commutative property to something familiar to her, like the multiplication table. Instead, like Ermida, she credited Caitlyn for taking the initiative to read the textbook and for understanding and remembering what she read. Grace's increased appreciation of Caitlyn's argument may have stemmed from her own struggles to remember mathematical rules and procedures and the resulting frustration she felt with the uncertainty of her conclusions. Shortly after this episode, while considering Flavia and Georgia's arguments, she exclaimed, "I wish someone would just tell me the answer!" which certainly suggests a willingness to accept the word of authority.

In their post-course interviews, the other interviewees showed little change in their assessment of Caitlyn's argument, expressing views similar to those they held at the beginning of the semester. (See Table 4.6.) Diana again criticized Caitlyn for not

Table 4.6

Post-Course Criticisms of Caitlyn's Appeal to Authority

Interviewee	Response
Diana	Caitlin doesn't walk you through how she knows. She says, "I saw it in a book," and that's how she knows it, but ... with Beth's, the reason I like it so much is that she gives you a step-by-step. Caitlin doesn't really do that. <i>She doesn't say anything about her mental process on it and how she relates to it, how she's able to comprehend the information.</i> She just says I saw it in a book and that's why. So <i>I would like it if I could see her spin on it and how she understands it.</i> Just because you see something in a book doesn't mean that you really have that good of an understanding of it. She's just able to recall the information. I don't know. ... Judging from her explanation, <i>I can't tell if she's able to work with the information or not.</i> (Post-course Interview)
Corey	<i>Just because she can say, "A times B equals B times A," doesn't mean she actually understands why it works.</i> It's just saying, "Oh, this is a rule; it works." ... <i>It doesn't have any reasoning behind why it works.</i> It's just saying, "I saw this once; it works." (Post-course Interview)
Linda	I think that she's accurate, but I just feel that <i>she wasn't using any of her own thinking.</i> She used the commutative property from the book, she said. So it doesn't say anything about what she's actually learned in her class. ... I feel that <i>it just didn't really explain all that much.</i> (Post-course Interview)

providing evidence of her understanding. Corey emphasized that he now expected an explanation for "why it works" that merely quoting the textbook failed to provide. Linda also criticized Caitlyn's explanation as inadequate and noted her lack of independent thinking.

Overall, most interviewees changed very little in their acceptance of Caitlyn's appeal to authority. Ermida viewed Caitlyn as having a higher level of understanding than Dawn or Evan, both at the beginning of the semester and at its end. Diana, Corey, and Linda continued to find fault with Caitlyn's argument, each listing it among those they found least convincing. Only Grace's evaluation of Caitlyn's argument seemed to have

changed significantly, rising somewhat in comparison to Dawn's, which she had initially considered superior for relating the commutative property to something familiar in her daily life, namely the multiplication table. Perhaps the greatest change was that, at the end of the semester, I no longer needed to prompt the interviewees to distinguish differences among the arguments. They generally expressed their preferences without my encouragement and did not accept arguments that reached the same conclusion as equally convincing.

Rules without backing. I consider Andy's argument to use a rule without backing, since he gives no support for his cross-multiplication approach. Other researchers, however, might place Andy's argument in essentially the same category as Caitlyn's, viewing it as using an implicit appeal to authority to justify his use of cross multiplication. Indeed, PTs responded similarly to both arguments on the preliminary survey. Overall, 75% of the PTs included Andy's argument among those they deemed convincing, compared to 80% for Caitlyn's. In addition, for both arguments, a high percentage endorsed all other arguments that reached the same conclusion (55% for Andy and Beth's arguments and 35% for Caitlyn, Dawn, and Evan's). I treated both arguments similarly in selecting the interviewees, choosing four who found Andy's argument convincing and four who found Caitlyn's argument convincing. It is therefore interesting that, during the interviews, PTs responded quite differently to Andy's argument than they did to Caitlyn's.

In their first interviews, when encouraged to distinguish among arguments rather than considering them equally convincing, 3 of 4 who originally endorsed Caitlyn's argument turned to criticize it, all three listing Caitlyn's argument among those they

found least convincing overall. In contrast, all four of Andy's original supporters continued to defend his argument throughout their first interview. By the end of the semester, however, they changed their position, now viewing Beth's argument as superior to Andy's.

If Andy and Caitlyn's arguments each rely on an appeal to authority, one implicit and the other explicit, why did the interviewees respond to them so differently? I suggest that the reason lies in "showing understanding" and what this meant to the interviewees at different times during the semester. Early in the semester, their views of understanding focused on knowing what to do, or as Diana put it in her first interview, "the process that you use to get it and to do it." Showing understanding of mathematical rules and procedures therefore involved *doing* something with them. Thus, the initial criticisms of Caitlyn's argument (see Table 4.7) focus largely on her failure to do anything with the commutative property, other than read it and remember it. As Corey noted, "Until you *do* something, you don't really understand it."

In contrast, although Andy's argument may implicitly rely on appeal to authority to justify his cross multiplication, it is not subject to the same criticism that interviewees leveled at Caitlyn's, because Andy demonstrated his understanding in a way that Caitlyn did not. He *did* something with it; he applied cross multiplication appropriately and correctly to show that $3/5$ is equal to $6/10$. In view of the importance the interviewees placed on knowing what to do, getting the correct answer, using a quick way, and showing how with numbers, it is not surprising that Andy's argument was initially endorsed by all interviewees except Grace, who said, "I can't really remember if that's the right way to do it."

By the end of the semester, however, changing views of understanding led all interviewees to reject Andy's argument in favor of Beth's. For example, an increased focus on why solutions work left Diana surprised that she once claimed to understand Andy's argument, saying, "I'm pretty sure I don't have a good grasp of this now, so I don't see how I did at the beginning." In the same way that their initial concerns led most interviewees to endorse Andy's argument, their post-course concerns for understanding the problem, finding answers that made sense, and explaining why with diagrams led them to prefer Beth's argument to Andy's. Thus, at the beginning of the semester, the interviewees generally accepted the use of a rule without backing, because they considered using the rule correctly to be evidence of understanding. However, at the end of the semester, they rejected the same argument for failing to demonstrate an understanding of *why* the rule worked. As Corey put it, "Andy's is just a shortcut, and it doesn't really explain why cross-multiplying works. It's just saying it works."

Empirical arguments. Rather than relying on the endorsement of some mathematical authority, *empirical arguments* support a general rule by providing one or more specific cases in which the rule works, that is, cases in which both the conditions of the rule and its conclusion are true. Dawn's argument, based on the results shown in the multiplication table, and Heather's argument, referring to the examples they had checked in class, both represent empirical arguments. Such arguments rely on the principle that truth in specific cases implies truth for a broader class to which those cases belong. Those who follow an empirical approach would seek to strengthen an argument by finding confirmation in additional cases, believing that every confirming case provides further evidence that the rule is true.

On the initial survey, a significant proportion of PTs seemed to consider empirical arguments valid. Sixty percent included Dawn's argument among those they deemed convincing, and 30% included Heather's argument about sums and differences of whole numbers. This latter percentage may actually underestimate the impact of empirical arguments, because some PTs considered negative integers as counterexamples, believing them to be included by the term "whole numbers." Among the interviewees, all initially found Dawn's convincing, and two found Heather's convincing.

With regard to empirical arguments, two different points of view emerged during the initial interviews: (a) unquestioning acceptance of empirical evidence as sufficient to establish mathematical truths and (b) provisional acceptance of empirical evidence, establishing mathematical truths only as working hypotheses that could still potentially be shown false, at least in some cases. Addressing this second viewpoint introduces some of the interviewees' ideas about counterexamples and exceptions to mathematical rules. I will address these ideas in more detail in another section. Here I focus only on evidence of these two views of empirical arguments.

Exemplifying the first viewpoint in his initial response, Corey's appeared to accept both Heather and Dawn's arguments without questioning their validity. In discussing Heather's argument, he focused instead on the correctness of her conclusion and her confident attitude:

I believe Heather's answer is more convincing, because first *she is correct*, and because of the way she states her answer, *she seems very confident she is correct*. Ivy does not provide a complete answer stating yes or no and seems unsure of the answer she gave. (Corey, Preliminary Survey)

He chose Dawn's argument, along with Flavia's, as his favorites overall, again without questioning the validity of her reasoning but instead focusing on how she presented her argument, using numbers in an organized, step-by-step way:

I like Flavia's and Dawn's *because of the way they're using the numbers to distinctly show their answers*. As I said before, I'm not much of a visual learner. I don't like using representations to show what I want; I like to just see the numbers and work it out in my mind. And when they are doing it here [in their explanations], they're showing that. *They're showing the steps to it*. They're showing "two times three equals six," and then they're reversing it—"three times two equals six"—with Dawn. She's *showing the steps* [for] how she got it, and she's showing the answer to it and showing the property, ... just saying, "This will happen when you do this." ... I like *organized* things—here's a step; this is what you do; this is what you do. That connects more with me, because I've always worked with that. I'm a very *organized* person, ... so I like to have things [in order]—*step, step, step*. (Corey, Interview 1)

Although Ermida did not consider Dawn's argument as advanced as Caitlyn's, she did accept Dawn's empirical approach as valid, viewing it as a successful trial-and-error process:

[Dawn] looked at the multiplication table, and she looked at multiple actual problems there, and she said that every single one worked every single time. *So if she did enough trial-and-errors and they all worked out, I think she could conclude that this works out*. (Ermida, Interview 1)

In her first interview and part of her post-course interview, Ermida's discussion of Heather's argument focused on counterexamples within the integers, so she viewed Ivy, rather than Heather, as correct. However, at one point in her post-course interview, I limited the context of Heather's conjecture to positive whole numbers, and Ermida again appeared to accept the validity of empirical evidence in establishing mathematical truths:

M: So now we're working with not just whole numbers but positive whole numbers, what we might call counting numbers—one, two, three, four, five, etc. Those are the only numbers we're working with. And let's even forget about subtracting. Let's just talk about the addition. If you add those numbers, you're always going to get a bigger number than what you started with. Do you think that's true?

E: [Long pause] Yes.

M: You're confident of that?

E: Yes. Last time I checked [laughing].

M: Can you explain why? Why are you so confident of that? There is this idea that there are exceptions out there. Negative numbers are exceptions. Zero is an exception. So can you explain why now you're confident that there aren't any other exceptions? You seem confident. So, why? Why are you confident?

E: I've been doing addition for a long time [laughing], and ... for whole counting numbers, one and up, when I added two numbers, it was always larger than the larger number in that problem.

M: Well, the number—let's think of it like a starting number, and then you add something else.

E: It was always larger.

M: So it was always larger than the starting number.

E: Yes, say a hundred plus twenty-five. It's always going to be more than a hundred.

M: All right. You're confident of that.

E: Yeah. So just like Heather and her class did lots of examples and illustrated their ideas, based on my lots of examples through the years, two whole numbers added always [grew] bigger. So I'm confident in Heather's answer in this case. Yes.

M: Just because of your past experience adding whole numbers.

E: Yes. (Ermida, Post-course Interview)

Thus, both Corey and Ermida appeared to accept the validity of empirical arguments in mathematics, and in Ermida's case, this view persisted through her post-course interview.

The second viewpoint emerged during Linda's first interview, in which she seemed to see mathematical truths as working hypotheses, accepted as provisionally true based on empirical evidence but potentially falsifiable by counterexamples yet to be discovered:

L: [Ivy] has a good explanation, because she's correct in saying that they've only tested a small amount of numbers, and how did they know that it would [continue to] be like that? *But after working more problems and going through it more, then she would find that the class is correct in saying that.*

M: Well, how do we know that? How do we know that Ivy isn't right, that there's some number that the class hasn't thought of and we haven't thought of?

L: It's the rules of math.

M: The rules of math? Are these rules in a book somewhere?

L: No [laughing].

M: Like the commutative—

L: [Laughing] Yeah, whatever that was, the commutative property.

M: Yeah, the commutative law. Are they rules like that somewhere?

L: No, no.

M: No, they're not. So how do we know? ... You said that this always works.

L: Well, no. I'm just saying that *from what I've tested, it would always work.* That's what I'm saying. *Ivy's correct in saying that, because there's no way, unless you're some brilliant mathematician, adding and subtracting numbers all day. Unless you tested billions and billions of numbers, there's no way to be definitively yes or no on that, correct or incorrect.*

M: But then, if we tested billions of numbers, we could be definitively—

L: [Interrupting] *Hypothetically.*

M: Hypothetically. So when we have things like in number two, where Caitlyn talks about this commutative law. She read that in a book, right?

L: Yeah.

M: So how does something get to be a law that we put in a book? What do you think happens to make that a law that somebody decides, "This always works?"

L: A long time ago people tested it and retested it and kept testing it and they found it was true as many times as they tested it, so ...

M: So they tried a lot of numbers? Is that what they tried?

L: I'm sure. You would have to, for it to be a property or a law, and especially for it to be published. You can't just throw anything out there.

M: Okay. So what if we had a skeptic like Ivy, who thinks, "Well, it doesn't matter how many you try, there might be an exception out there that we don't know about." Do you think there are exceptions like that, to some laws in mathematics?

L: I'm sure there are. Mathematical laws have exceptions.

M: There are exceptions? So like that commutative law. You think that might have exceptions?

L: There's no way to say for sure. There could be. There's no way to be sure. Nothing that I've ever found—I haven't worked with it all that much. *And obviously people have probably tested it a lot, with different numbers. But as far as we've found, there's nothing to contradict it.* (Linda, Interview 1)

A similar viewpoint appeared, in less detail, in Grace's description of Ivy's curiosity:

G: I cannot think of something that would make two whole numbers smaller if they were added together, so I believe Heather would be correct. But Ivy has a good way of thinking.

M: Ivy has a good way of thinking?

G: Yeah. She's curious.

M: Ivy doesn't seem to know. ... If she had an example like this [indicating a counterexample with negative numbers that Grace had provided earlier], I think she would just say it. She seems to think that, just because we found some that work, how do we know that there's not one like this?

G: Yeah.

M: So how do we know? Do we know that there's not one like that?

G: I mean, to our knowledge, there isn't. But I don't know. Like Ivy said, "How do we know it will always work?" We're always learning, so—[she laughs]. (Grace, Interview 1)

Finally, in Diana's discussion of open-mindedness, she seemed to view the discovery of exceptions or counterexamples as almost inevitable, as we learn more advanced mathematics:

D: [Discussing Andy's argument] I feel that it's trial and error, that *you use your method, if it's correct in that period of time, until you learn otherwise—until you learn more.* ... If it keeps working, then chances are it's going to be right. But I think the key is to have an open mind that later it might change.

M: So things are true ... until you find the exception?

D: I guess—*until you learn the exception.* You just have to have an open mind that there can be exceptions. (Diana, Interview 1)

The view of mathematical truths as falsifiable hypotheses would seem reasonable if the only justification for believing them was empirical evidence. This is, in fact, akin to the Popperian view of natural laws in science. However, the interviewees' preferences shifted away from empirical arguments by the end of the semester, moving toward a more mathematically acceptable form of argument. This does not mean that they completely ceased to accept empirical arguments, but they tended to prefer generic examples when such an alternative was available to them.

Generic examples. Like empirical arguments, *generic examples* also consider specific cases, but they do so in a different way and with a different purpose in mind. Rather than accumulating instances in which a general rule is true, a generic example looks at a single case or cases more analytically, seeking to explain *why the rule works* in that case and therefore why it should be expected to work in a broader class of cases. Evan's argument follows this approach, using a model for the meaning of multiplication to explain why five times three and three times five must represent the same number and doing so in a way that can transfer to other cases, at least those cases involving multiplication of positive whole numbers.

On the initial survey, 70% of the PTs included Evan's argument among those they found convincing. However, only a small minority (10%) deemed his argument more convincing than both Caitlyn and Dawn's, and a larger proportion (30%) preferred at least one of these other arguments to his. In selecting the interviewees, I included three—Corey, Grace, and Linda—who selected the most frequent response, viewing all three arguments as equally convincing. Of the remaining interviewees, Ermida preferred both Caitlyn and Dawn's arguments to Evan's, and Diana selected Dawn and Evan's as equally convincing.

In this section, I present evidence on the extent to which the interviewees shifted their preferences to Evan's argument and away from Caitlyn and Dawn's. Moreover, I argue that this change represented more than a mere preference for arguments with diagrams but was in fact a shift toward generic examples, that is, arguments that use specific examples to explain why general conclusions hold true.

In her initial interview, Ermida acknowledged that Evan obtained the correct answer but criticized his argument as excessively time consuming:

I didn't like [Evan's argument] as much, *but it is correct*. I just preferred Caitlyn's and Dawn's. ... Unless he was an absolute visual learner, it would be *time consuming* to sit there and draw all these circles—three times five—and then try to flip the picture down. ... He could probably do this [only] a few times before *it took a lot more time*, compared to Dawn's. She just wrote the problems out. Evan would actually have to sit there and write all of these circles in one direction, and then he would try to flip the picture over and write all those circles again. *He would do trial and error as well as Dawn, but I think his would be more time-consuming. ... It is correct, though. He had the right idea.* (Ermida, Interview 1)

She clearly preferred Dawn's less time-consuming argument to Evan's. She also viewed Caitlyn's, with its abstract notation and technical language, as showing a more advanced level of understanding than Evan, even at the end of the semester.

Caitlyn's is convincing, because that's a mathematical property she understands, and that would be a higher [level] than what Evan was thinking, ... because if she didn't understand the basics down here [in Evan's argument], ... then she wouldn't have understood [what she read in the textbook]. (Ermida, Interview 1)

In her post-course interview, however, Ermida recognized the generalizability of Evan's argument:

If he completely understands this concept, he would apply it to all of the other multiplication factors. ... I know I [originally] said it was time-consuming, but if he understands this, he wouldn't have to prove this point every single time he did a math problem. (Ermida, Post-course Interview)

She also selected Evan's argument as the best of the arguments in *Thinking About*

Students' Explanations:

Even though his was very basic, trying to understand the commutative property of multiplication, I guess he understood the basics here. He explained himself well. *He had a diagram to go along with his explanation, and he sees the concept. If you're just flipping between numbers or, he says here, "flipping the picture sideways," he sees that the answer is going to be the same, so I think he had the best explanation. Out of all these four problems, Evan was able to explain himself thoroughly enough.* (Ermida, Post-course Interview)

With her focus on Evan's picture, one might conclude that it was the picture itself that attracted her to Evan's argument. However, when I asked her, in general, what made a good explanation, she spoke again about Evan's "proof through example," emphasizing what it accomplished and how difficult it was for her and her classmates to emulate:

We knew how to do the mathematical [process], the numbers and everything. We knew the calculations, but *we didn't know how to explain what we were doing, why we were doing it, and how it actually worked—the mechanics of ... what we're doing in math.* So I think that if the student is going to completely understand and give a good explanation, that's how they would do it. And that's really hard. (Ermida, Post-course Interview)

Thus, she emphasized more than simply including a picture. A good mathematical argument required explaining what they did and why and how it worked.

Corey initially preferred Dawn's argument to Evan's, uncritically accepting her use of empirical evidence and emphasizing her use of numbers to show what she did in an organized, step-by-step way. In his post-course interview, however, he also chose Evan's argument among his favorites. His reasons for this preference focused on Evan's representation but also on what using this representation accomplished. Noting that, "It shows that it will work in any situation," he described what he looked for in a mathematical argument:

Now, I definitely believe that showing representations *and connecting it to the real world and explaining each specific part—why this is this or this piece is this*—definitely is a much stronger way to connect to the students and visually show them *why each process works*. (Corey, Post-course Interview)

Thus, although Corey certainly came to appreciate visual representations, it was not the mere presence of a diagram that was important but rather using diagrams to explain ideas in relation to the real world and show why mathematical procedures work.

Not only did Grace prefer Evan's argument, she suggested that, because of the understanding his example conveyed, she would have found his convincing without the picture:

I think Evan's is the best, because he drew it out. *But even if he didn't, he still understands. He gave an example and showed why it worked*—because you're just switching the order or doing three rows of five and not five rows of three. But it's all going to be the same amount. (Grace, Post-course Interview)

Thus, by the end of the semester, most interviewees provided additional reasons for preferring Evan's argument, beyond the presence of a diagram. However, not all interviewees shared this view. In Diana's post-course interview, although she selected Evan's argument as her favorite, she did not consider it significantly different from Dawn's:

Evan's is the best way to explain it to people and introduce it to people, but *Dawn's also works, because it's just a more advanced version of Evan's*. ... I'd say I like Evan's the best, just because through the whole semester there's been such a big emphasis placed on diagrams, and I can see why. It's easiest to explain something when you use a diagram, because that's another way of getting information through to people. So I like Evan's the best, but I still think *Dawn's is just pretty much the same as Evan's, just without the pictures*. (Diana, Post-course Interview)

With some interviewees, I also investigated how the mere inclusion of a picture might influence their preferences. I asked Linda whether the diagram in Figure 4.4 would improve Andy's argument.

L: I would say no.

M: You'd say no? What don't you like about that picture?

L: Well. If you did a dotted line here—[She draws a line separating the second fraction into two equal parts—three and three over five and five. She laughs at this.]

M: You want to put a dotted line—oh.

L: I was just saying you could do a little dotted line and show that. But see, it just doesn't. When you write it out like this, it doesn't look like it would be equal to one another. ... And if you show that to students, I feel like it would be extremely confusing. (Linda, Interview 1)

Grace responded similarly to this diagram:

He is showing a diagram, but it's basically just using the numbers. It is using a diagram, but I don't know. ... To go back to your original question, would it have helped me if that diagram was on there? I don't really think so. (Grace, Interview 1)

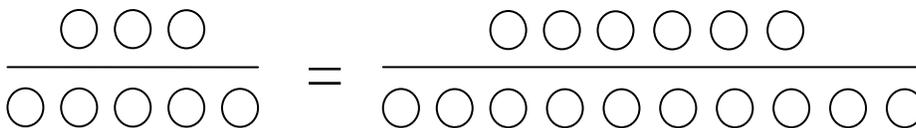


Figure 4.4. A potential diagram to show that $\frac{3}{5} = \frac{6}{10}$.

As additional evidence of the interviewees' increased appreciation of generic examples, I also note that most of the interviewees created their own generic examples to justify Heather's conjecture about whole numbers. Linda, for example, provided the following argument in her post-course interview:

M: This seems to be like a rule that they're coming up with here. When you add whole numbers, the result is bigger than what you started with. ... So are there rules like that we can be sure of? You can't test every number that there is. ... So is there some way we can be sure that certain rules are true, without always having Ivy's nagging doubts?

L: Well, if you think about it, if you're adding like 10 and 5, you already have 10 pencils [for example], and you're getting five more, *there's no way that it's going to be less than 10*. So it's going to be greater than 10. And *there's no way that it's going to be less than 5*, because you're adding five and you already had 10. *So there's no way that it can be less than those numbers. If you have a number and you're adding something onto it, how can it be less?* (Linda, Post-course Interview)

This represented quite a change for Linda, who originally said, "Unless you tested billions and billions of numbers, there's no way to be definitively yes or no on that, correct or incorrect." Corey, Grace, and Ermida also provided similar arguments, although Ermida did not do so until her member-checking interview.

Counterexamples. Appeals to authority, empirical arguments, and generic examples are distinctly different approaches to justifying general statements in mathematics. Yet to be addressed, however, is how PTs might seek to refute such general statements. In mathematics, for a general statement to be true, its conclusion must hold true in every case that satisfies the specified conditions. Therefore, one can refute a general claim in mathematics by finding a *counterexample*, a single case in which the conditions are satisfied but the conclusion is false. This type of refutation does not necessarily apply outside mathematics. For example, one can find numerous exceptions

to general statements such as, “Men are taller than women,” or “I comes before E, except after C.” However, even acknowledging the exceptions, these statements are still accepted as true. Therefore, based on their experience with general statements outside of mathematics, students may fail to understand the role of counterexamples in mathematics, believing that general statements in mathematics can have exceptions and still remain true.

As I noted early in this chapter, 60% of the PTs responding to the preliminary survey preferred Ivey’s argument to Heather’s, which would have certainly been reasonable if they recognized zero as a counterexample. However, only one (5%) mentioned zero in her response. Among those I selected to interview, only Ermida discovered the existence of this counterexample.

In their first interviews, Linda, Grace, and Diana each saw some degree of weakness in relying on empirical evidence to support mathematical generalizations, but they responded to this weakness by treating mathematical rules as only provisionally true, based on what we know so far. They maintained that, as we learn more, we might discover exceptions, such as negative numbers, for which the rules we previously accepted no longer work. Early in the semester, Ermida also expressed a similar view of Heather’s conjecture, pointing out exceptions among the negative integers and saying, “Ivy is more convincing, because she’s thinking about what’s going to happen later on, whenever she learns higher math.”

To further explore Ermida’s thinking in her fourth interview, however, I steered the discussion away from negative numbers:

M: Maybe we know about negative numbers, but say we’re not talking about negative numbers in this problem. We’re talking about whole numbers.

Usually when mathematicians use the phrase, “whole numbers,” they’re not talking about negative numbers. They don’t apply. So, if we don’t include negative numbers, [then] like Ivy, do we still think that there may be some exception, if we’re not talking about negative numbers [but] just talking about whole numbers with no negatives?

E: Is that including zero—or no? That’s just including one and up?

M: As a matter of fact, whole numbers would include zero. So what do you think now—with zero included?

E: Well, the answer—if we added a number plus zero—wouldn’t go along with what this is saying, because that number wouldn’t get bigger; it would stay the same. So that could be actually what Ivy is talking about, because Heather was saying that every time you add two numbers, it always gets bigger. When adding any number to zero, that number remains the same. (Ermida, Post-course Interview)

Thus Ermida identified zero as a counterexample to Heather’s conjecture. In contrast, when I excluded negative numbers from consideration in Corey’s post-course interview, he did not consider zero as a potential counterexample but instead used generic examples to argue in favor of Heather’s conjecture:

If you have 10 pieces and you add two to it, ... you’re *always* going to have more pieces, when combined. *You’re not going to have a smaller number of pieces when you’re adding more to it.* The same with subtracting—if you have ten pieces and you take two away, *you’re always going to have a distinctly smaller number of pieces total, because you’re taking them away.* (Corey, Post-course Interview)

However, when I brought up zero as a possibility, he immediately grasped its implications:

M: What if I told you—or what if Ivy came in and said—that even though negative numbers aren’t considered to be whole numbers, zero is considered to be a whole number.

C: Oh—plus zero?

M: So does that change your opinion?

C: Oh, that definitely does, because ten plus zero ... doesn’t change what you have. It keeps it the exact same. Therefore, Ivy could be correct, because it would be like

taking ten—like we just did—and adding nothing to it. You’re going to have the same thing. It’s not going to get bigger, and it’s not going to get smaller. It’s just going to stay the same, which would mean that Ivy’s would work. If zero is taken into account as a whole number, then Ivy is correct, because she could show that you’re not making it bigger, because you’re not adding anything to it. So it would stay the same.

I pressed Corey further, to see if he treated zero as an exception or a counterexample.

M: Heather says I could show you a hundred problems—a hundred problems or a thousand problems—where it does make it bigger. ... What if Heather said something like, “Well, I’ve got a thousand. Ivy’s only got one example.”

C: The thing that throws me off is that she says it *always* works, and “always” is a very distinct word, and I feel that the fact that there’s one exception to it shows that that’s not a rule. *It shows that there is something that can change it always working.* So that’s the way I would see it, because—just as I said—there’s an *exception* to it. *It shows that this is not a scientific law or anything like that.* (Corey, Post-course Interview)

Thus, although Corey did not think of zero until I mentioned it as a possibility, he seemed to understand of the role of counterexamples in mathematics.

In Grace’s post-course interview, our conversation about Heather and Ivy’s arguments proceeded similarly to Corey’s. However, because Grace had been among those who initially believed that mathematical laws might always have undiscovered exceptions, I asked her whether, assuming zero and negative numbers were omitted from consideration, she now thought adding whole numbers would always lead to a larger result:

G: Yes. I think so. Adding is always making it bigger if we leave out zero.

M: You sound pretty sure.

G: I am sure.

M: You're sure? What convinces you? Does this argument [indicating her generic example]?

G: Yeah, that and zero is the smallest, ... without going negative. And if you add one, that still makes it bigger. (Grace, Post-course Interview)

Grace's ideas about exceptions and counterexamples were closely linked to other ideas about mathematical justification, such as empirical arguments and generic examples. When, as in her first interview, she only had empirical evidence to believe Heather's conjecture, she had no reason to explain why it should always work. Therefore, she could not discount the possibility of undiscovered exceptions. Conversely, in her post-course interview, after she provided an argument that explained why it should always work, she was more certain of her conclusion. Although she did not think of zero as a potential counterexample, she immediately recognized it as one and was comfortable adjusting the domain to exclude zero without shaking her belief in the validity of her argument.

The Role of the Argument's Substance in PTs' Evaluations

Some arguments, such as Andy's cross-multiplication argument, deal primarily with the surface structure; they are about relationships among mathematical symbols. Andy chooses to multiply 3 times 10 and 5 times 6 because of the relative positions of the mathematical symbols—3, 10, 5, and 6. Other arguments, such as Beth's, concern relationships among underlying mathematical concepts. Beth's argument is not about symbols, such as " $\frac{3}{5}$ " and " $\frac{6}{10}$," but the quantities that those symbols represent. When she states that each fifth is two tenths, for example, she points out a relationship between these quantities, the underlying concepts, rather than a relationship between the symbols. Likewise, Flavia's argument also focuses primarily on the surface structure of mathematics, following an accepted procedure for manipulating mathematical symbols to

obtain an answer. Georgia's argument, however, like Beth's, focuses on underlying mathematical concepts, examining relationships among the quantities the symbols represent.

I previously noted the high proportion of PTs (75%) who initially accepted Andy's argument, partially due to PTs' tendency to endorse all responses that arrived at correct results. In comparing Flavia and Georgia's arguments, however, in which the two arguments reached conflicting conclusions, only 20% chose Georgia's quantitative argument over Flavia's symbol-centered one. In selecting the interviewees, I chose four who preferred Flavia's and one—Grace—who found neither argument convincing, writing that she was “bad at fractions.”

In this section, I present evidence for three different viewpoints that interviewees showed during their interviews: (a) a symbol-centered view, (b) a picto-symbolic view, in which PTs represented quantities as pictures, rather than standard mathematical symbols, but also thought it important to maintain symbol-centered relationships among the pictures, and (c) a quantitative view. One could consider the picto-symbolic view as a transitional stage between a symbol-centered view and a quantitative one, and the overall pattern generally reflected that interpretation. However, the progression from one view to another did not always follow a clear path over time. Grace, for example, showed evidence of all three viewpoints during her post-course interview. Indeed, much of the data I present in this section comes from Grace, who struggled with this issue more than any other interviewee.

The symbol-centered view. In my analysis of what PTs initially considered important, three themes reflect a symbol-centered viewpoint. In particular, their

preference for *showing how with numbers* certainly reflects symbol-centered focus. *Knowing what to do* and *using a quick way to do it* also tend to support a symbol-centered viewpoint, because the procedures that the interviewees saw as automatic, quick, and efficient, such as Andy's cross multiplication, typically involved manipulating symbols with little regard for what the symbols represented.

Some PTs initially viewed Georgia as following symbol-centered procedures, albeit incorrectly, rather than using her diagram to reason about quantities. Table 4.3, for example, shows excerpts in which Diana, Corey, and Ermida each claim that Georgia added the denominators, yet nowhere in Georgia's argument does she appear to perform that symbolic operation.

When the interviewees began to doubt Flavia's solution and suspect that Georgia's was correct, they often sought to rationalize a symbolic operation that would lead to Georgia's solution, as Linda did during her first interview, questioning her memory of the correct procedure for adding fractions:

M: So now you think Georgia's is right, and there's something wrong with Flavia's?

L: Because ... it didn't combine. Oh. ... I guess eight plus eight is not eight. It should be sixteen, right there [in Flavia's sum]. [*Flavia*] *added the numerators and not the denominators. Right?*

M: But Flavia says combined means added, so how do you add fractions? *Do you add the denominators, when you add fractions?*

L: No. I haven't done this in so long, I don't know [laughing]. I mean, you don't. But I don't know, because you can't just get rid of one of the jugs. Just because you're pouring this small portion into it there, you can't just throw the jug away. (Linda, Interview 1)

Diana and Grace also responded similarly, focusing on a symbolic operation that would lead Flavia to the correct answer but struggling to rationalize this with their

remembered procedure for adding fractions. (See Table 4.7.) The point here is that they focus on remembering or creating a rule for operating on symbols—adding the denominators—that would lead to the correct answer.

A symbol-centered viewpoint emerged in other contexts besides the grape juice mixture problem. In her first interview, for example, Grace questioned whether $14/16$ and $7/8$ were really “the same,” or merely “equivalent.” (See Figure 4.5):

G: If this [the entire figure] is the whole, then *it's not seven-eighths*.

M: It's not seven-eighths?

G: But if you specify the middle two lines ... specify that as a whole—then yes, it is. ... [Or] if you specify [that] two blocks equals one-eighth, then that would be seven-eighths. You have to explain it a little bit more, because otherwise, people looking at it would get confused. ... You have to specify ... what the whole is, because otherwise those lines confuse people, and they think that *this whole thing [the entire diagram] is the whole and that each of these [the sixteen small squares] are one-sixteenth*.

M: Okay. So if we said that it is the whole square—the whole big square—that is the whole?

Table 4.7

PTs' Struggles to Create a Symbolic Procedure That Produces Georgia's Answer

Interviewee	Response
Diana	I guess [Flavia] just— <i>she needs to add the denominators together</i> . I don't know. I'm awful with fractions. I don't remember half the things I learned about them, but just from the picture I can't figure it. You're always going to have these eight pieces with this jug and these eight pieces with this jug. You can't get rid of any of them, because they are something, they're part of the mixture. (Interview 1)
Grace	The rules of fractions have escaped my mind. But if you're keeping the denominator the same, if that's what you're supposed to do in adding fractions, then yes, it is. And she [Flavia] reduced it correctly. But <i>I think you're supposed to add the denominators in fractions</i> . (Interview 1)

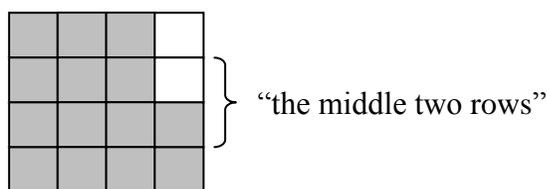


Figure 4.5. A representation of fourteen-sixteenths.

G: Well, there are 16 blocks, right? And 14 are shaded in, so *it could be equivalent to seven-eighths. If it's reduced, it's seven-eighths.*

M: But it's not seven-eighths? It's just equivalent to seven-eighths?

G: Yeah, unless you specify. Yes.

M: Say that again in your own words. I feel like I'm putting words in your mouth.

G: If we specify that the big thing is the whole, the big square, then *it's fourteen-sixteenths, which can be reduced to seven-eighths. But it's not seven-eighths, because each of those blocks, if the big thing is a whole, is one-sixteenth, not one-eighth. So it can be equivalent-slash-reduced to seven-eighths, but it's fourteen-sixteenths.*

M: Okay, so it's really fourteen-sixteenths, but you can reduce it, or it is equivalent to [seven-eighths].

G: Yeah. *So if "equivalent" equals "same," then it's the same. But I feel [that] "equivalent" and "the same" may be a little bit different.* (Grace, Interview 1)

In this episode, Grace struggled to describe the relationship between the fractions $14/16$ and $7/8$, applying a symbol-centered meaning to these fractions. That is, she viewed $14/16$ and $7/8$ as the symbols “ $14/16$ ” and “ $7/8$,” which are clearly not the same. However, using a familiar and accepted procedure, “ $14/16$ ” can be “reduced,” transforming it into “ $7/8$.” To express this idea correctly, Grace believed that we should call the two fractions “equivalent,” rather than calling them the same, when they—as symbols—are clearly not.

The picto-symbolic view. As PTs discussed their solutions to mathematical problems in the Number and Operations class, the instructor encouraged them to use diagrams to help explain their thinking. Of course, PTs often drew contrasting diagrams for the same problem, so the topic of “correct” representations naturally arose in this context. A common pattern emerged, in which PTs sometimes preferred diagrams that reflected relationships in the symbolic representations they found more familiar. Grace, for example, criticized the picture in Georgia’s argument:

G: With Georgia's—like I said—I *liked her answer*, and I liked her pictures, but *I didn't think it was truly convincing, because I thought maybe she should have drawn an “equals to” [sign] and drawn another jug to show her actual answer.*

M: Can you draw me what you're talking about there?

G: I can. (See Figure 4.6.) Okay. So here's the jug [drawing and talking to herself quietly]. They were [adding] that, so she would have to have—I don't know. [Counting parts of her drawing] *Okay, so there were sixteen. This should be a jug the same size as hers, but—*

M: When you say, “the same size as hers,” you mean this jug [on the right side of the equal sign] is the same size as those [on the left side]?

G: Well—[She reads the problem to herself]. Well, *I guess it would probably be bigger. It would have to be—maybe.* So I think she should have drawn like an “equal to” picture to show her answer.

M: Okay, so something like this [Grace’s drawing].

G: Yeah.

M: So this—[indicating the jug on the right side of the equal sign]

G: It’s a bigger jug, because it has sixteenths in it. ... Well, it should be. Let me think about this. [Sigh] Well, if you're *adding one-eighth and three-eighths—* I’m trying to think. I don't really know. Let’s just say we're going to forget about the number, because I don't really know. *But she should have drawn an “equal to” picture to be convincing.*

M: All right.

G: Fair enough? [She laughs.]

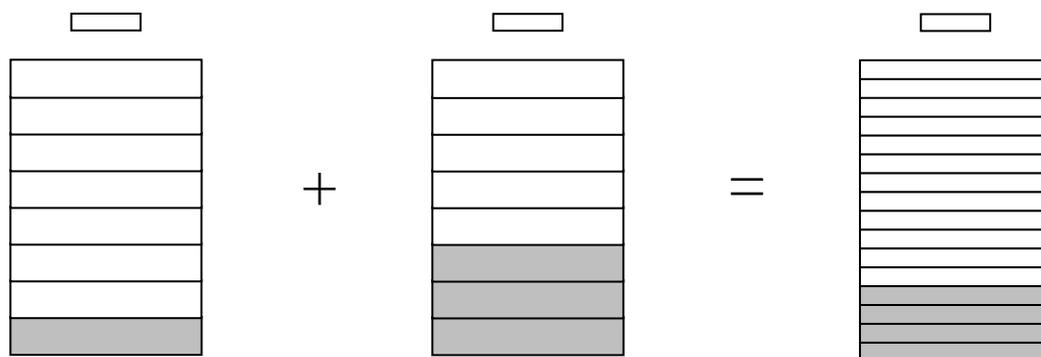


Figure 4.6. Grace's revision of Georgia's diagram.

M: Yeah, then *maybe this jug is bigger, and maybe it's not?*

G: *Maybe. We don't really know* [laughing]. (Grace, Interview 1)

Several points need to be made about this episode. First, Grace believed that Georgia, not Flavia, found the correct answer. Two jugs, each containing eight-eighths, would combine to yield sixteen parts altogether—sixteen-sixteenths—four of which would be grape juice. However, as indicated by Grace's phrase, "adding one-eighth and three-eighths," she also believed—in agreement with Flavia—that adding the fractions was appropriate in this situation. She therefore thought that Georgia's drawing should have a structure ($a + b = c$) similar to the equation in Flavia's argument $\left(\frac{1}{8} + \frac{3}{8} = \frac{4}{8}\right)$.

Her revised diagram, incorporating Flavia's equation structure with Georgia's answer, seemed to indicate that $\frac{1}{8} + \frac{3}{8} = \frac{4}{16}$, conflicting with a partially remembered rule for adding fractions with like denominators. It is also noteworthy that in Grace's picto-symbolic representation of the grape juice problem, reflecting relationships among symbols took priority over accurately portraying quantitative relationships, such as how

the amount of liquid in the original jugs compares to the amount in the combined mixture. She therefore replaced each fraction symbol in the equation with a picture that represented each symbol but ignored or distorted relationships among the underlying quantities.

Diana and Linda also initially drew picto-symbolic diagrams similar to Grace's. (See Figure 4.7.) However, in both cases, they rather quickly abandoned this diagram and turned to focus on the quantities involved in this problem. The picto-symbolic view emerged in other contexts as well, especially in the Grace's case. In discussing Heather and Ivy's arguments in her post-course interview, for example, she drew a picto-symbolic diagram to support her argument for Heather's conjecture:

M: Do you think that there could be an exception [to Heather's conjecture]?

G: To this rule? No.

M: Why not?

G: Because for whole numbers, if you're adding, you're always going to be—I don't know how to describe adding without using “adding.”

M: Could you draw pictures?

G: Yeah. [She begins drawing Figure 4.8(a).] You're *always* going to be *combining objects*, so you're *always* going to end up with more objects. It's like four plus two. If you add four plus two, you're going to get six. So *because* these are *always* whole numbers, you are *always* going to be *combining objects*. So the number is *always* going to be bigger, *because* you can't combine objects and get smaller numbers. But if you're *subtracting*, that means you're *taking away, separating objects*.

M: Want to draw a picture of that? [She begins drawing Figure 4.8(b).] How do you draw a picture of subtracting? Oh, you're doing that. I see. You're crossing out.

G: And then you're just taking away two, so that only leaves you with two. (Grace, Post-course Interview)

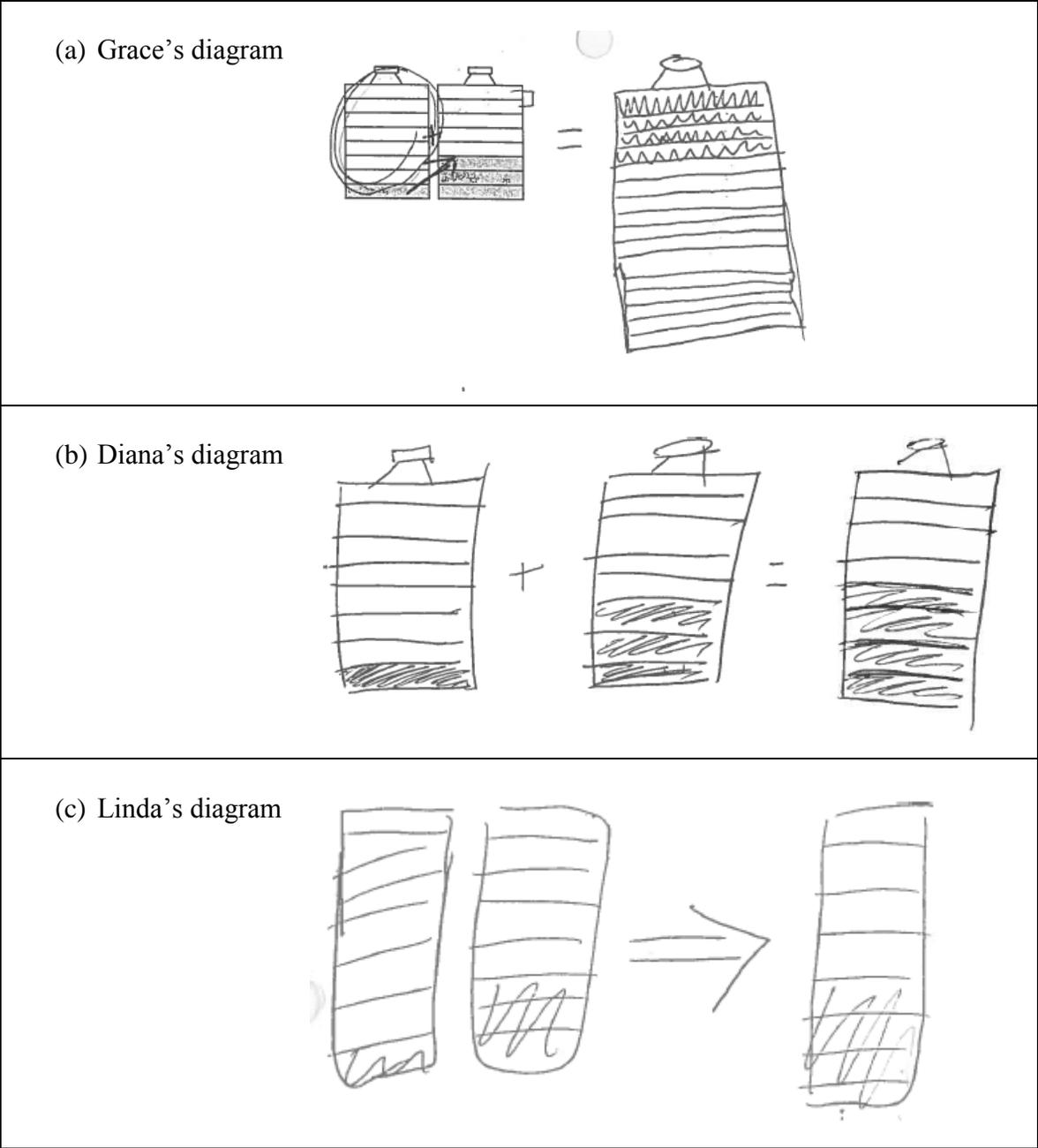


Figure 4.7. Grace, Diana, and Linda's revisions of Georgia's diagram.

Here Grace used $4 + 2$ as a generic example, arguing that addition of whole numbers can be thought of as combining objects. Therefore, adding will always result in

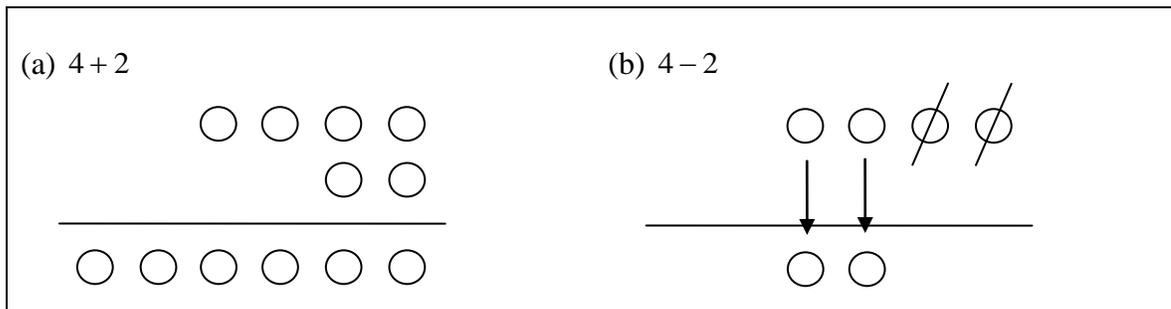


Figure 4.8. Grace’s diagrams showing addition and subtraction of whole numbers as combining or separating objects.

a larger number, because it produces more objects. She sketched a similar idea for subtraction, based on separating objects, but did not complete it in as much detail. Her use of logical conjunctions, such as “because” and “so,” indicates that she formed a chain of reasoning, based on this example, and her repeated use of “always” indicates that she recognized the generality of her argument beyond this particular case. This argument does not reflect a symbol-centered viewpoint, viewing addition as a procedure for combining and manipulating mathematical symbols. Instead, it is primarily quantitative in nature, viewing addition and subtraction as combining and separating quantities. However, note the similarity between Grace’s diagrams in Figure 4.8 and the symbolic representations of addition and subtraction in Figure 4.9, continuing to suggest her effort to reflect relationships among symbols as well as the quantities they represent.

The quantitative view. By the end of the semester, the interviewees tended to focus less on procedures for manipulating symbols and more on the quantities that the symbols represented. Corey, for example, had initially favored Andy’s cross-

(a) $\begin{array}{r} 4 \\ + 2 \\ \hline 6 \end{array}$	(b) $\begin{array}{r} 4 \\ - 2 \\ \hline 6 \end{array}$
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Figure 4.9. Symbolic representations for adding and subtracting whole numbers.

multiplication approach to show that fractions were equivalent. At the end of the semester, however, he gave a much different explanation for equivalent fractions:

C: I was thinking of an example [showing that] three-fourths is the same as six-eighths and how multiplying by two over two, which is equivalent to one, you're actually cutting the pieces in half to create more pieces that are smaller in size.

M: Do you want to draw a picture to explain what you mean?

C: Yeah. I'm just going to do an area model. So you have three-fourths of the whole filled. This is the whole [drawing a rectangle divided into four equal parts], and this is one-fourth, one-fourth, and one-fourth [shading three of the four parts]. Therefore, you have three-fourths of the whole filled. And the concept that I feel much more confident in is this algorithm. Three-fourths times two over two equals six-eighths. I'm showing that, on a diagram, ... [drawing a dotted line to split each fourth into two equal parts] what this is actually representing is that you're cutting each piece in half, to double the total amount of pieces. And then each one-fourth becomes two-eighths, because there's two pieces of one-eighth that will fit into one-fourth. So by doing this, you're doubling the total amount of pieces by cutting the pieces in half, creating smaller pieces to take up the whole. So therefore, one-fourth is the same as two-eighths. So if you cut it in half like this, the shaded region would still be equivalent, because there's six one-eighth pieces of the whole filled in.

M: And that's the same as multiplying by two over two then? So if I was to ask you or if a student was to ask you, "So what's going on with the two in the numerator?" What is that doing, in terms of the picture, versus the two in the denominator?

C: The two in the numerator is doubling the total amount of pieces, and the denominator is basically cutting the pieces in half. That's the way I perceive it. The two in the numerator is saying that there's going to be double the total

amount of pieces that will be shaded in. And the two in the denominator is saying that each piece of the whole will be cut in half. Therefore, you'll have two-eighths to fit one-fourth. And then, since three-fourths were shaded, that would be equivalent to six-eighths.

M: So if somebody had asked you to explain this before?

C: I would have had no clue how to explain it. I just would have said that you are multiplying two over two, because *it's what I learned in school*. ... I just had a very basic understanding that this is the procedure you have to follow, but *I couldn't explain why*. (Corey, Post-course Interview)

Corey's explanation does not focus on the symbols " $\frac{3}{4}$ " and " $\frac{6}{8}$," but rather on the quantities those symbols represent. Likewise, he describes doubling the numerator and denominator in terms of splitting, an action that preserve the shaded amount but changes the number of pieces.

Not all interviewees were as successful as Corey in providing quantitative explanations, however, they were generally much more successful than they were at the beginning of the semester. Grace, for example, still struggled with the idea of equivalent fractions in her post-course interview. As we discussed Andy and Beth's arguments that three-fifths is equal to six-tenths, she objected to the word, "equal," saying, "They're not *equal*. They're *equivalent* to each other." Our conversation returned to this issue:

G: I don't know if "equal" is the right word. That's confusing.

M: You don't like "equal." But don't we write that often?

G: Yeah, we do.

M: Don't we say that?

G: We say it all the time.

M: Are people wrong when they say that?

G: No, because that's just how it is.

M: [Laughs.] That's just how it is.

G: [Laughs also.] Yeah.

M: But really you're uncomfortable with it. You really think they should say “equivalent” or something.

G: Yeah, because [three-fifths] is *equivalent* to [six-tenths], because it's the same whole, but the fifths are way bigger than tenths are.

M: Fifths are way bigger than tenths.

G: Yeah.

M: So because fifths are bigger than tenths, ... you're saying *one-fifth* is bigger than *one-tenth*.

G: Yeah.

M: Right. *One-fifth* ... is bigger than *one-tenth*, but is it ...

G: But *one-fifth is equal to two-tenths*.

M: One-fifth is—you said “equal” there.

G: Yeah.

M: And you're comfortable with “equal” there?

G: Well, I guess so, because—[sigh]—I mean, okay.

M: I'm just trying to be clear about what you think, because sometimes—

G: Yeah, because *one-fifth is equal to two-tenths*. Yeah, I think “equals” is right there, because here's a fifth [pointing it out in Beth's diagram] and *it's equal to two-tenths* [indicating the two rectangles that make up the fifth]. *Oh, yeah. There you go.*

M: Okay. All right. So if one-fifth is two-tenths, then—

G: Yeah, I think *three-fifths is equal to six-tenths*.

M: Okay?

G: I agree. I agree with that. (Grace, Post-course Interview)

The same issue arose in both interviews, suggesting that Grace continued to struggle between a symbol-centered interpretation and a quantitative interpretation of fractions. However, this episode also suggests that she made some progress over the

course of the semester. She left her first interview saying that $14/16$ was not the same as $7/8$, but here in her fourth interview, she pointed out $1/5$ and $2/10$ as the same area in Beth's diagram and then readily accepted that $3/5$ was equal to $6/10$. Thus, although she did not consistently apply a quantitative meaning to fractions, she appeared to do so more readily at the end of the semester.

Evidence of a picto-symbolic viewpoint appeared in Grace's initial critique of Georgia's diagram. In her post-course interview, when I reminded Grace that she had previously considered Georgia's picture incorrect, she seemed surprised:

G: I did? Oh, gosh!

M: Do you remember that?

G: I think I do. I remember drawing a diagram, but I don't remember why.

M: Well, I think what you were saying in the first interview—and I think it relates to what we're talking about here—is that, if you're going to combine these together, you need a bigger container. You can't pour this into here, because that's already full, so you made a big jug—

G: [Laughs] Oh, so I put [an] equals [sign and] what she should have drawn? [She sketches a third jug in Georgia's picture]. Is that what I did?

M: Yeah, that's what you did. So what do you think about that idea? This jug [the one she just drew] is a somehow a bigger jug than these.

G: Bigger, yeah. Hmm. *I don't know if it's necessary now, because I think with these two pictures she kind of gets her point across* [erasing the jug she had added].

M: Okay.

G: But *maybe it's because I have a better understanding now*. (Grace, Post-course Interview)

Unlike her first interview, Grace no longer thought that Georgia's diagram needed the symbolic structure of an equation ($a + b = c$). In fact, she seems somewhat surprised and amused by this idea. This changed perspective should not be interpreted as a lack of

concern for representing mathematical ideas correctly. Instead, it was due to her improved understanding of the quantities represented in Georgia's diagram, and how they should be interpreted. This new understanding was particularly apparent when she noted that this problem differed from typical fraction addition problems by not referencing the same whole throughout:

M: Do you know whether the denominator should stay the same or not?

G: You mean in a typical addition problem or in this problem?

M: This problem.

G: No, because the whole is changing, so the denominator can change. You are [now] referencing a different whole than you were referencing [originally].

M: So when we look at this little shaded part here, when we call that one-eighth, what are we calling it one-eighth of?

G: One-eighth of the first jug.

M: But later on, we're talking about that as one-sixteenth. So what are we saying it's one-sixteenth of?

G: Two jugs ... combined.

M: The two jugs combined?

G: Yeah. So it's a different whole.

M: And can you do that? Can they both make sense?

G: Yeah. Yeah, you can do that. You can change the whole. (Grace, Post-course Interview)

The discussion of Flavia and Georgia's arguments reached quite a different conclusion here than it had in our first interview. Earlier, she had been unable to determine how the eighths on one side of her diagram compared to the sixteenths on the other. In contrast, at the end of the semester, she moves flexibly between viewing the same section of the diagram as either one-eighth of one gallon or one-sixteenth of two

gallons. Over the course of the semester, she changed her understanding of how equal quantities were represented in Georgia's diagram and how the diagram should be interpreted.

Thus, in their interviews, PTs showed evidence of three different viewpoints, regarding the relationships they considered in mathematical arguments. Initially, they focused on relationships between mathematical symbols and ways of manipulating these symbols, "adding the denominators," for example. As they were encouraged to use diagrams to explain their solutions to mathematical problems, they sometimes produced picto-symbolic representations, diagrams that preserved relationships among mathematical symbols, but ignored or distorted relationships among quantities. By the end of the semester, however, they tended to focus on relationships among quantities. They were also generally more successful in explaining mathematical operations in terms of actions on these quantities, such as splitting shaded areas into smaller pieces.

Chapter Summary

In this chapter, I presented data from the initial survey and from interviews with five selected PTs, focusing on three research questions: (a) As PTs compare and evaluate mathematical arguments, what do they initially consider important, and how do their views change by the end of a one-semester course emphasizing mathematical argumentation? (b) What role does the *form* of the argument initially play in PTs' evaluations, and how does this role change by the end of the semester? (c) What role does the *substance* of the argument initially play in PTs' evaluations, and how does this role change by the end of the semester?

In addressing the first question, I presented evidence to suggest that PTs initially valued mathematical arguments in which the arguer thought or acted in accordance with the PTs' image of a superior or advanced mathematics student. In particular, when responding to a mathematical problem, they expected such a student to: (a) know what to do, (b) get the correct answer, (c) use a quick way to get it, (d) use numbers—standard mathematical symbols—to show how they obtained their answer, and (e) have the right attitude, such as confidence in the correctness of their result or curiosity and willingness to learn more.

I presented further evidence from their post-course interviews, indicating that they changed the way they approached these arguments by the end of the semester. Rather than considering a student's response to a problem as automatic, they often devoted considerable time and effort to understanding the problem. They defended the results of these sense-making efforts, knowing that they conflicted with the results of the standard algorithm they used earlier. Rather than value quick solutions presented using standard mathematical symbols, they criticized such solutions as "shortcuts." Instead, they valued arguments in which the arguer used diagrams to explain why a solution worked.

I also addressed the second research question, examining the role that the form of the argument played in PTs' evaluations, considering the role of appeals to authority, rules without backing, empirical arguments, generic examples, and counterexamples. Most of the interviewees took a somewhat skeptical view of Caitlyn's appeal to authority, even at the beginning of the semester. However, those who initially preferred Caitlyn's argument failed to change this evaluation, and evidence suggests that Grace was more willing to accept this argument at the end of the semester than at its start. Their initial

responses to Andy's argument, which some would consider an *implicit* appeal to authority, did not coincide with their response to Caitlyn's. Instead, most interviewees initially endorsed his argument, expressing the view that he demonstrated his understanding of cross multiplication by using it appropriately and correctly. By the end of the semester, however, they rejected Andy's argument for failing to explain why his approach worked. Their heightened expectation that arguments should use diagrams to explain why solutions work also led them to endorse Evan's generic example by the end of the semester, a change from the initial view that his argument was merely a more time-consuming version of Dawn's. PTs also expressed some interesting views about the potential existence of counterexamples, and only one interviewee identified zero as a counterexample to Heather's conjecture about whole numbers, even at the end of the semester. However, when I suggested zero as a possibility in this situation, they seemed to understand its implications immediately, without any further explanation.

Finally, PTs showed evidence of three ways of viewing the substance of an argument. As suggested by their preference for arguments that showed how with numbers, they initially tended to take a symbol-centered view, accepting arguments that focused primarily on mathematical symbols and procedures for manipulating them to obtain answers without exploring the meaning of the symbols involved. In response to the instructor's expectation that PTs use diagrams to explain their solutions, some interviewees showed evidence of a picto-symbolic viewpoint, in which they drew diagrams that preserved the relationships among the symbols in standard mathematical notation but replaced some symbols with diagrams. By the end of the semester, the

interviewees generally approached arguments from a quantitative viewpoint, attempting to make sense of the arguments by analyzing relationships among the quantities involved.

CHAPTER 5: DISCUSSION AND IMPLICATIONS

In this chapter, I first discuss the study's findings in relation to prior research. I then present the study's implications for mathematics teacher education and future research.

Discussion of Findings

This study addressed three research questions: (a) As PTs compare and evaluate mathematical arguments, what do they initially consider important, and how do their views change by the end of a one-semester course emphasizing mathematical argumentation? (b) What role does the *form* of the argument initially play in PTs' evaluations, and how does this role change by the end of the semester? (c) What role does the *substance* of the argument initially play in PTs' evaluations, and how does this role change by the end of the semester? Rather than discuss each research question in sequence, I discuss the findings in two groups, starting with those that describe PTs initial views and then shifting to those that describe changes that emerged in the post-course interviews.

What PTs initially considered important. The themes presented in Chapter 4 are consistent with Kennedy's (1999) theory that, during their apprenticeship of observation, teachers develop a frame of reference for interpreting classroom experiences. I suggest that PTs' frames of reference include images of "advanced" mathematics students and the ways that these students think or act when presented with mathematical problems. An argument earned a PT's endorsement if he or she perceived it as the type of argument such a student would provide. They therefore endorsed

arguments in which they saw the arguer as: (a) knowing what to do, (b) getting the correct answer, (c) using a quick way to do it, (d) showing how with numbers, and (e) having the right attitude. These ways of thinking and acting reflect the values that PTs acquired in traditional mathematics classrooms, the values that define “good” or “advanced” performance in that context. This finding calls attention to different aspects of PTs’ understandings than most prior research on mathematical justification within this group, focusing on how PTs perceive mathematical arguments from their points of view.

The findings of this study may therefore appear to conflict with other studies of mathematical justification among PTs, but these apparent conflicts may reflect a difference in focus. For example, the finding that PTs initially preferred arguments that showed procedures with numbers, rather than words or diagrams contrasts with Stylianides, Stylianides, & Philippou’s (2004) finding that, when given proofs involving the contrapositive of a given statement, PTs were more likely to respond incorrectly to symbolic proofs than to verbal ones. However, the focus of this study is on the types of arguments PTs prefer, not on those they evaluate most accurately. Furthermore, their symbolic proofs were algebraic and the arguments in this study were primarily numerical, which could also explain the contrasting results.

The interviewees’ preference for arguments using numbers also contrasts with studies of other populations. Healy and Hoyles (2000), for example, found that, when choosing for themselves, secondary school students generally preferred arguments in narrative or visual form to those that focused on symbolic manipulations, but reversed this order of preference when asked which would receive a higher score from their teachers. In comparison to Healy and Hoyles’ findings, the results of this study suggest

that PTs' initial preferences aligned more closely to students' notions of what teachers would or should prefer, rather than what they considered personally convincing.

The initial role of an argument's form. The initial role of an argument's form was constrained by the concerns that the PTs brought to the arguments. In particular, the extent to which PTs valued correct answers often led them to endorse multiple arguments that differed in form, provided that each reached a correct answer and provided a reasonable explanation for how it was obtained. In such cases, the form of the argument played an essentially insignificant role. This finding confirms those of other researchers, such as Martin and Harel's (1989), who found that PTs endorsed a variety of contrasting arguments that led to the same conclusion, including empirical arguments, generic examples, and general deductive arguments, as well as nonsensical explanations with the superficial appearance of deductive arguments.

Explicit and implicit appeals to authority. When PTs expressed a preference for one argument over another, their evaluations contained some nuances that, to my knowledge, have not been documented in the literature. For instance, most interviewees selected Caitlin's argument, which justified the commutativity of multiplication with an explicit appeal to the textbook, as the one that they liked least. However, with the exception of Grace, all interviewees initially endorsed Andy's argument, one which implicitly relies on a similar appeal to justifying cross multiplication as a way of determining the equivalence or nonequivalence of fractions. They criticized Caitlyn for failing to do anything that demonstrated her understanding of the general conclusion she sought to support. Conversely, they viewed Andy's use of cross multiplication as a demonstration of his deeper understanding. This observation suggests that PTs consider

demonstrating understanding to be an important aspect of mathematical arguments, a finding similar to McCrone and Martin's (2009), who found that some secondary school students saw mathematical proof primarily as a way for students to demonstrate their knowledge for the teacher. The distinction that PTs drew between Caitlyn and Andy's arguments could be viewed as attention to superficial features of the situation, rather than the form of the argument. However, as I discuss later in this chapter, we should not so easily dismiss PTs' attention to students' understandings, but rather seek to develop it in productive ways.

Empirical arguments and generic examples. Also noteworthy are PTs' reasons for choosing Dawn's empirical argument for commutativity, based on examples from the multiplication table, over Evan's generic example, which used an array model to explain this property of multiplication. Those who chose Dawn's argument did not seem to do so for reasons that would universally lead them to prefer empirical arguments to generic examples, but rather because they did not consider the additional explanation in Evan's argument sufficient to compensate for its perceived disadvantages. Corey and Linda, for example, preferred Dawn's argument for using numbers to show how she reached her conclusion, and Ermida failed to see the generality in Evan's example, viewing it as merely a single step in a series of repeated, time-consuming tests that he would need to complete before reaching a conclusion.

Ermida's interpretation illustrates the same issue that Chazan (1993) raised in his investigation of secondary school geometry students. He found instances in which they saw empirical evidence as proof and others in which they saw proof as merely evidence. That is, rather than seeing a proof as generalizing to all instances satisfying the given

conditions, students saw it as only applying to a single instance, the particular geometric figure under consideration. In the same way, Ermida viewed Evan's diagram as only justifying commutativity in a single instance: $3 \times 5 = 5 \times 3$. She needed further evidence to conclude that multiplication is commutative in other instances. As her response illustrates, the problem with PTs' acceptance of empirical arguments is not only that they see these arguments as general, but also that they sometimes fail to see the generality in more explanatory arguments, such as Evan's. The generality of such arguments is not transparent. Instead, both the forms of representation and the forms of reasoning in these arguments may require time, effort, and experience for PTs to understand and appreciate them.

Counterexamples. In conjunction with their views of empirical arguments, PTs—particularly Linda, but also Diana and Grace—expressed some interesting ideas about the possible existence of counterexamples to mathematical laws. Linda initially viewed empirical evidence as ultimately the only basis for these generalizations, envisioning mathematicians testing “billions and billions of numbers.” She therefore viewed mathematical laws as working hypotheses, considered true until we had evidence to the contrary but inherently susceptible to the possibility of undiscovered counterexamples. Moreover, just as general truths within the whole number system may become false when students learn about integers and consider these statements within this wider context, Diana viewed the eventual appearance of counterexamples as almost inevitable, as students gained experience with a wider range of mathematical ideas. Thus, she emphasized that mathematical generalizations should be considered true only “until you *learn* the exception.” These beliefs about potential undiscovered exceptions also parallel

Chazan's (1993) finding that some students were skeptical that geometric proofs could guard against the possibility of undiscovered counterexamples.

The initial role of an argument's substance. In a variety of ways, the aspects of arguments that PTs valued led them to initially reject arguments that probed more deeply into mathematical concepts and relationships and favor those that relied solely on relationships among mathematical symbols. In addition, as I noted when discussing generic examples, a lack of prior experience with arguments that explore deeper concepts and relationships made arguments that depend on such relationships difficult for PTs to interpret. Here again as in my discussion of generic examples, the representations, concepts, and forms of reasoning in these arguments are not transparent but rather require time, effort, and experience to understand. In particular, PTs need experiences that focus on the meaning of the mathematical symbols they use, rather than emphasizing procedures for manipulating these symbols.

Changes in what PTs considered important. As described in Chapter 4, PTs made a significant shift in the aspects of mathematical arguments that they considered important. In part, that shift involved a change in focus, away from assessing whether the arguer's thoughts and actions reflected those of an "advanced" student and toward understanding the mathematical concepts and relationships the problems involved. Thus, they shifted: (a) away from assessing whether the arguer knew what to do and toward investing time and effort to understand the problem, (b) away from finding correct answers by applying standard mathematical procedures and toward finding answers that made sense, even when they conflicted with those obtained from standard procedures, and (c) away from arguments that showed how solutions can be obtained quickly and

toward arguments that explained why solutions work. They also (d) viewed the arguer's attitude as a more complex issue than they had initially thought.

The changes I was able to recognize and describe support the broad usefulness of Toulmin's (1958) model as a theoretical tool for exposing evidence of fundamental changes in reasoning. Previous studies that have used this approach have focused primarily on the collective argumentation of children in classrooms. Krummheuer (1995), for example, focused on second-graders, and Cobb (1999) and McClain (2009) analyzed arguments in a seventh-grade classroom. In contrast, this study considered Toulmin's model as part of the theoretical framework for constructing focal arguments for interviews with preservice teachers. Across these differing contexts, changes in the warrants and backing in the arguments the participants accepted indicated corresponding changes in thinking. Four interviewees in this study, for example, initially accepted Andy's argument, which contained no explicit backing for his use of cross multiplication. By the end of the semester, however, all four rejected this argument in favor of Beth's, generally giving reasons that focused on their failure to understand—or his failure to explain—why this method worked. Understanding why solution methods worked had become part of their criteria for accepting the general rule that Andy applied or, in terms of Toulmin's model, the backing they now expected for his warrant. Significantly, none of the interviewees expressed doubt in the truth of Andy's warrant; understanding of his rule was lacking, not confidence in his conclusion.

Considered among studies of courses that engaged PTs in reform-oriented mathematical experiences, including a focus on mathematical argumentation (e.g., Wilcox et al., 1991; Szydlik, et al., 2003; Stylianides & Stylianides, 2009b), this study

confirms the potential for such instructional experiences to change PTs' beliefs.

However, because this study focused on identifying and finding evidence of changes in what PTs valued in mathematical arguments and did not investigate classroom norms or practices that may have supported these changes, I cannot attribute these changes to particular characteristics of instruction, as these studies did.

Implications for Mathematics Teacher Education

In Chapter 1, I argued for the importance of engaging future elementary teachers in mathematical argumentation during their preservice training. Although the PTs in this study made significant progress in deepening their understanding of mathematical argumentation during their semester in the *Number and Operation* course, we can hardly conclude that they have learned all they need from this area of mathematics. Engaging teachers in mathematical argumentation is equally important for both preservice and in-service teachers, and doing so effectively requires well-designed tasks. In Chapter 2, I presented a theoretical framework for classifying such tasks. Using this framework, I created a variety of tasks which were successful in eliciting PTs' initial views of mathematical arguments and revealing aspects of their thinking that had changed by the end of the semester. In this section, I suggest three ways in which mathematics teacher educators could apply this framework, along with findings from the study, in both preservice and in-service teacher education: (a) selecting and using argument construction tasks, (b) selecting and using argument evaluation tasks, and (c) assessing teachers' understanding of mathematical argumentation.

Selecting and using argument construction tasks. Like Stylianides and Ball (2008), I contend that, to develop a thorough understanding of mathematical

argumentation, teachers should engage in constructing a variety of mathematical arguments. In particular, the tasks they encounter should vary in both the truth value and generality of the statements involved. The difference between their framework and mine is their inclusion of tasks that involve multiple but finitely many cases, a category I chose to omit. They argue that such tasks can engage teachers in types of proving activity that are less likely to occur in response to tasks with single or infinitely many cases—for example, systematic enumeration to ensure that every case has been considered. Although such tasks have some potential benefits, I believe that two points should be made regarding their use in teacher education.

First, although Stylianides and Ball (2008) do not take this position, some might consider tasks with multiple but finitely many cases as a stepping stone between single-case tasks and those with infinitely many cases, envisioning a learning trajectory that proceeds from single-case tasks to those with infinitely many cases by way of this intermediate category. I would argue against this view, based on both theory and empirical evidence. Toulmin's framework, for example, suggests that single-case arguments typically depend on more general lines of reasoning, often those that involve infinitely many cases. Perhaps for this reason, the interviewees in this study did not consider single-case arguments, such as Beth or Georgia's, significantly different from those like Evan's, which involve infinitely many cases. Based on interviewees' responses to these arguments, the insertion of multiple-case arguments as an intermediate stage between the other types seems unjustified.

Second, the number systems and geometries that we want teachers to understand typically involve properties that hold over infinite sets of mathematical objects, and as

Toulmin's framework suggests, solving problems and justifying the results often involves applying these general properties to specific cases. Therefore, I consider tasks involving single cases and infinitely many cases ultimately more important than those involving multiple but finitely many cases. If mathematics teacher educators must omit one type or treat it less thoroughly than the others, I recommend that it be those with multiple but finitely many cases.

Selecting and using argument evaluation tasks. If preservice or in-service teachers regularly engage in mathematical argumentation, both constructing arguments and critiquing the arguments of their peers, they will eventually consider arguments from several or possibly all of the categories I discussed in Chapter 2. However, I suggest that teachers should also engage in comparing and evaluating arguments purposefully selected to represent various categories within this framework, just as the PTs did in this study. Doing so could allow them to directly compare, for example, arguments that explain why solutions work and arguments that merely show how answers were obtained or, likewise, arguments that focus on relationships among symbols and arguments that focus on the meanings underlying the symbols. Juxtaposing such arguments and discussing their relative merits could focus teachers' attention on important issues and facilitate a deeper understanding of mathematical argumentation.

Assessing teachers' understandings of mathematical argumentation. The results of this study can assist mathematics teacher educators in assessing teachers' understandings of mathematical argumentation, by sensitizing them to the ways that teachers may initially view mathematical arguments and the ways they might change their views as they gain experience in this area. In particular, teacher educators can

examine PTs' arguments and critiques of arguments, looking for evidence of changes in focus: (a) away from unthinking application of standard mathematical procedures and shortcuts and toward deeper understanding of mathematical problems and concepts, (b) away from the assumption that correctly executed procedures result in correct answers and toward a thoughtful effort to ensure that answers make sense, (c) away showing how solutions were obtained by manipulating mathematical symbols and toward exploring underlying meanings and explaining why solutions work.

Implications for Future Research

In this section, I first argue against some beliefs that seem to underlie much of the recent research on preservice elementary teachers' views of mathematical justification. I then suggest some specific directions for future research studies in this area.

Beliefs underlying recent research. Compared to other studies that considered PTs' evaluations of different forms of reasoning, this study generally reaches a significantly different and more favorable view of PTs' understandings of mathematical justification than most. I suggest that this difference reflects some widely held views among mathematics education researchers, beliefs that generally: (a) overemphasize the formal aspects of arguments and undervalue the role of substance, (b) overemphasize the role of verification and undervalue explanation, (c) are too far removed from PTs' perspectives, and therefore (d) fail to accurately reflect significant progress in PTs' understandings of mathematical justification.

As a starting point, I compare my study to another that, while similar in many respects, also has significant differences, Morris's (2007) study of 34 PTs. Her participants were undergraduate juniors and seniors who had completed three

mathematics content courses and were currently enrolled in a mathematics methods course. Two of the content courses focused on number and operation, and in these courses, “preservice teachers were frequently asked to explain why mathematical ideas, algorithms, and generalizations were valid or true” (p. 485). The interviewees in my study were all juniors enrolled in a course on number and operation, and this was likewise a course that frequently engaged them in justifying their mathematical ideas and solutions to mathematical problems. Overall, the two groups of participants seem roughly comparable, with the distinct possibility that hers were more advanced.

Morris (2007) addressed several research questions, but I focus here on her third question, “What is the relationship between preservice teachers’ understandings about logical and nonlogical arguments, and their evaluations of the validity of students’ mathematical arguments?” (p. 82) and the theoretical framework underlying it. She described a logical argument as follows:

In a logical or deductive argument, ... the form of the argument is inherently valid; the conclusion follows necessarily from the premises (e.g., X is p or q ; X is not p ; therefore X is q), and the conclusion must be true if the premises are true. (p. 480)

Consider, for example, a standard example of a logically valid argument: $A \rightarrow B$; A ; therefore B . If the premises $A \rightarrow B$ and A are true, then surely B is true. The conclusion follows from the premises, satisfying Morris’s criteria. Now suppose that A stands for, “The cross products are equal,” and B stands for “The fractions are equivalent.” Does this mean that Andy’s argument is valid? I claim it does not, and the reason lies in the superficial nature of Andy’s argument and what, in practice, constitutes a premise. The PTs in my study rejected Andy’s argument, not because they doubted that the equality of the cross products necessitated the equivalence of the fractions but because they did not

understand why it must work that way. They came to value the role of understanding in mathematical arguments. The *A* in Andy's argument is not, in practice, a premise, because it requires explanation that he does not provide.

Andy's argument does not convey understanding, in part, because it lacks backing, but also because it lacks substance. It fails to address the meaning of the symbols involved. Beth's argument provides both, arguing that $3/5$ and $6/10$ represent the same quantity, and are therefore equal. However, many would consider Beth's argument to be a "single-case key idea argument," a type that Morris considered logically invalid. She therefore finds that:

When preservice teachers were asked to identify the 'best' arguments that proved a generalization, very few preservice teachers claimed they were looking for general, valid arguments. Instead preservice teachers tended to look for key ideas in the students' arguments that *explained why* a generalization was true. (p. 492, italics mine)

The explanatory role of proof is recognized by mathematics educators (Hanna & Jahnke, 1996; de Viliers, 1999) and by mathematicians (Thurston, 1994), but many researchers, like Morris, seem to ignore it or devalue it in favor of proof's role in verification.

Stylianides and Stylianides (2009b) provide another case in point. They described "an instructional sequence aimed to help students begin to realize the limitations of empirical arguments as methods of validating mathematical generalizations and see an intellectual need to learn about secure methods for validation (i.e., proofs)" (p. 314). The instructional sequence culminates in a "monstrous counterexample" (p. 330) to a conjecture that holds true for more than the first 3×10^{25} natural numbers. I do not doubt that knowledge of such an example would convince many students that empirical evidence provides no protection against undiscovered counterexamples. However, as I

noted in Chapter 4, some PTs—Diana, Grace, and Linda, for example—came to the course with that understanding.

Diana, in particular, seemed to believe that finding such counterexamples was almost inevitable, as we learned more advanced mathematics. However, it did not dissuade her from using empirical evidence to make generalizations. She simply viewed them as working hypotheses, considered true until shown false. Her shift away from empirical arguments and toward generic examples was not motivated by a need for certainty that the conclusions would hold true, but rather by the need for understanding that empirical arguments failed to provide. Rather than finding fault with PTs who use focus on key ideas in single cases to explain why generalizations hold true, as Morris (2007) did, we should encourage this type of justification, because it is the desire to understand why and not the need for certainty that will motivate many teachers to change the way they view mathematical arguments and the nature of mathematics in general.

Overall, I believe that recent research on PTs' approaches to mathematical justification overemphasizes logically formal and abstract reasoning and the verification aspect of proof. The studies by Morris (2007) and Stylianides and Stylianides (2009b) are only two examples. Other recent studies have focused on PTs' understanding of mathematical induction (Stylianides et al., 2005; 2007) and proofs about the closure of certain sets under binary operations (Gholamazed, Liljedahl, and Zazkis, 2003). Judging from the results of this study, the concerns of these researchers seem far removed from those of PTs, and as a result, their findings tend to cast PTs in a negative light. I suggest that the field would benefit from a shift in focus toward more concrete forms of

reasoning, the consideration of the substance of mathematical arguments as well as their form, and an increased emphasis on the explanatory aspects of proof.

Directions for future research. As noted in Chapter 3, data for this study came from a larger data collection effort that included (a) two intermediate interviews with each of the five PTs, conducted at approximately equal intervals, between the initial and post-course interview, as well as a final member-checking interview, (b) video recordings of small-group and whole-class discussion from almost every class period, and (c) samples of student work from the five interviewees. This additional data—or similar data from other sources—could be analyzed to answer questions that were beyond the scope of this study: (a) How do PTs views of mathematical arguments develop over time? (b) What role do particular instructional practices or classroom norms play in PTs views of mathematical arguments? (c) What are the important milestones in the development of PTs’ understandings of invalid and valid forms of mathematical reasoning? (d) What obstacles need to be overcome in facilitating PTs’ understandings of mathematical arguments?

Studies could also investigate other issues I raised in the previous section, for example: (a) What differences can be observed between classes emphasizing the role of verification in mathematical arguments and those emphasizing the role of explanation? and (b) What differences occur in classes emphasizing the substance of mathematical arguments in addition to various forms of reasoning? Answers to these questions would inform mathematics teacher educators and help them to design instructional experiences and programs that will prepare future elementary teachers to engage their students in the

type of mathematical argumentation envisioned in the NCTM *Standards* (1989, 2000) and the more recent *Standards for Mathematical Practice* (CCSSI, 2010).

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APPENDIX A: THE PRELIMINARY SURVEY

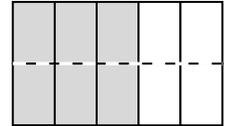
Thinking about Students' Explanations

In the situations described below, elementary students answer questions and explain their answers. As you read each explanation, consider whether it convinces you that the answer must be correct; is it a valid explanation? Then write your responses to the questions below.

1. A teacher gave her class the following problem: *True or false: $\frac{3}{5} = \frac{6}{10}$. Explain your answer.* Two students gave the responses below.

Andy: It's true. I cross-multiplied and got $3 \times 10 = 30$ and $5 \times 6 = 30$. If you get the same number when you cross-multiply, the fractions are equal, so $\frac{3}{5} = \frac{6}{10}$.

Beth: It's true, they are equal. I drew a picture and shaded three-fifths (see the picture at right). If you cut each of the fifths into two parts, you get ten parts altogether, so each fifth is two tenths. Three of the fifths are shaded, and six of the tenths are shaded. So six-tenths is the same amount as three-fifths.



Choose one of the following:

- (e) I think Andy's explanation is more convincing than Beth's.
- (f) I think Beth's explanation is more convincing than Andy's.
- (g) I think both explanations are equally convincing.
- (h) I think neither explanation is convincing.

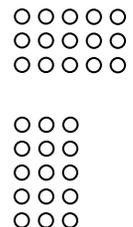
Explain why your choice makes sense to you.

2. A fourth grade class has noticed that 45×32 and 32×45 both have the same answer, 1440. The teacher asks them if every multiplication problem can be done in either order. Will the results always be the same both ways? If so, explain why. Three students gave the responses below.

Caitlyn: Yes, they will always be the same. I saw it in a math book. It's called the commutative property of multiplication. It says $a \times b = b \times a$. That means you can multiply in either order, and the answers will be equal.

Dawn: Yes, that will always work. Look at our multiplication table. When you switch the order there, you get the same answer every time. $2 \times 3 = 6$ and $3 \times 2 = 6$. $5 \times 7 = 35$ and $7 \times 5 = 35$. I looked at every one, and it always works. For any multiplication problem, you get the same answer if you switch the numbers.

Evan: Yes, multiplying both ways will always give the same result. Look at 3×5 , for example. That's like counting three rows of dots with five in each row (see pictures at right). If you make it 5×3 , that's five rows with three dots in each row. The



second is just like the first, but turned sideways. Both are going to have the same number of dots altogether. That will always happen when you change the order of the numbers you multiply. It's just flipping the picture sideways. It won't change the answer.

Choose one of the following:

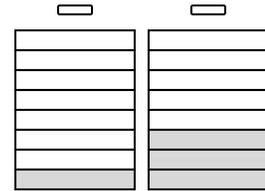
- (e) I think _____'s explanation is the most convincing.
- (f) I think _____'s and _____'s explanations are equally convincing.
- (g) I think all three explanations are equally convincing.
- (h) I think none of the three explanations are convincing.

Explain why your choice makes sense to you.

3. The teacher gave the following problem to a fifth-grade class: *Two one-gallon jugs are filled with liquids that are a mix of grape-juice and water. The first is one-eighth grape juice. The second is three-eighths grape juice. If the two gallons are combined, what fraction of the combined mixture will be grape juice? Explain your answer.* Two students gave the responses below.

Flavia: Combined means added together. So I added $\frac{1}{8} + \frac{3}{8} = \frac{4}{8}$. I divided by 4 and reduced it to $\frac{1}{2}$. So the combined mix is one-half grape juice.

Georgia: I drew pictures of two gallons, and I shaded one-eighth of one and three-eighths of the other (see picture at right). When the two pictures are combined together, there are 16 equal parts and 4 are shaded. So the mix would be $\frac{4}{16}$ grape juice.



Choose one of the following:

- (a) I think Flavia's explanation is more convincing than Georgia's.
- (b) I think Georgia's explanation is more convincing than Flavia's.
- (c) I think both explanations are equally convincing.
- (d) I think neither explanation is convincing.

Explain why your choice makes sense to you.

4. A second-grade class has been talking about what happens when you add or subtract whole numbers. Several students have said that adding makes the numbers bigger, and subtracting makes them smaller. They have found lots of examples that illustrate these ideas. The teacher asks them if they think this will always happen when you add or subtract whole numbers. She also asks them to explain why they think so.

Heather: Yes, that always works. I can show you a hundred problems—or even a thousand—where adding makes the numbers bigger and subtracting makes them smaller. It always works.

Ivy: I don't think it *always* works. Just because we tried it and it worked for some problems, how do we know it will work for the next problem we try? Maybe it works for some numbers and not for others.

Choose one of the following:

- (a) I think Heather's explanation is more convincing than Ivy's.
- (b) I think Ivy's explanation is more convincing than Heather's.
- (c) I think both explanations are equally convincing.
- (d) I think neither explanation is convincing.

Explain why your choice makes sense to you.

**APPENDIX B: TABLE OF THEMES, DESCRIPTIONS,
AND ILLUSTRATIVE EXCERPTS**

Theme/Code	Description of data to which the code or theme applied	Example
Superordinate Theme: Knowing What to Do	Interviewees indicated a belief that, when given a problem, students should know what to do and follow a procedure that would lead to the correct answer.	“It was drilled in, so I knew what to do” (Corey, Interview 1).
<ul style="list-style-type: none"> • Knowing automatically 	Interviewees indicated that, when presented with mathematical problems, students should know what to do immediately and without hesitation.	Right off the bat, when I saw that, I was [thinking], “Okay, three times ten equals thirty. Five times six equals thirty. They’re equal, because that’s how it’s supposed to be” (Corey, Interview 1).
<ul style="list-style-type: none"> • Remembering facts 	Interviewees described their own actions or those of the arguer in terms of recalling factual information.	“When we were taught ... multiplication facts, you get it drilled in your brain, so when you see it, you know it” (Corey, Interview 1).
<ul style="list-style-type: none"> • Remembering rules 	Interviewees described their own actions or those of the arguer in terms of remembering mathematical rules.	“The rules of fractions have escaped my mind” (Grace, Interview 1).
<ul style="list-style-type: none"> • Remembering procedures 	Interviewees described their own actions or those of the arguer in terms of remembering mathematical procedures.	“I vaguely remember the denominator having to stay the same” (Grace, Interview 1).

<ul style="list-style-type: none"> • Following procedures 	Interviewees described their own actions or those of the arguer in terms of a student following a mathematical procedure.	“With Flavia, I like basically the same thing, how she’s showing the steps of what she did” (Corey, Interview 1).
<ul style="list-style-type: none"> • Teaching procedures 	Interviewees described their own actions or those of the arguer in terms of a teacher demonstrating a mathematical procedure.	“I feel like [Andy’s argument] would be a good one to use if I was teaching a class as well” (Linda, Interview 1).
<hr/>		
Superordinate Theme: Getting the Correct Answer	Interviewees cited the correct answer as an important criterion for assessing mathematical arguments.	“First, she is correct” (Corey, Interview 1).
<ul style="list-style-type: none"> • Correct therefore convincing 	Interviewees cited the correct answer as a reason for endorsing a particular argument.	“I chose Flavia’s response, because she explained each of her steps, reduced fractions, and got the correct answer” (Corey, Preliminary Survey).
<ul style="list-style-type: none"> • All are correct 	Interviewees cited the correct answer as a reason for endorsing two or more arguments that shared the same conclusion.	“I liked all their explanations, because they backed them up with resources and really explained where they got them [their answers].” (Grace, Interview 1).
<ul style="list-style-type: none"> • Not best but still correct 	Interviewees stated that some arguments they did not prefer were still correct.	“I didn’t like [Evan’s argument] as much, but it is correct. I just preferred Caitlyn’s and Dawn’s” (Ermida, Interview 1).
<ul style="list-style-type: none"> • Correct in context 	Interviewees indicated that conflicting answers should be considered correct, due to the context in which they appear	“At this point, Heather is right. From what she’s given and what she has learned, she is correct” (Ermida, Interview 1).
<hr/>		
Superordinate Theme: Using a Quick Way	Interviewees indicated a preference for arguments that were brief or obtained answers quickly.	“In math class, ... you would try and do all of the problems that you could possibly do and try to retain all that information in the shortest amount of time possible” (Ermida, Interview 1).

<ul style="list-style-type: none"> • Knowing automatically 	Interviewees indicated that, when presented with mathematical problems, students should know what to do immediately and without hesitation.	Right off the bat, when I saw that, I was [thinking], “Okay, three times ten equals thirty. Five times six equals thirty. They’re equal, because that’s how it’s supposed to be” (Corey, Interview 1).
<ul style="list-style-type: none"> • Saving time 	Interviewees indicated that solutions to mathematical problems should not be excessively time consuming.	“It would be time consuming to sit there and draw all these circles—three times five—and then try to flip the picture down” (Ermida, Interview 1).
<ul style="list-style-type: none"> • Defending shortcuts 	Interviewees defended the legitimacy of shortcuts.	“Once you understand the basics—you understand why you’re doing that, you can do the shortcut” (Diana, Interview 1).
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Superordinate Theme: Showing How with Numbers	Interviewees indicated their preference for arguments that used standard mathematical symbols, “numbers” rather than words or diagrams, to show how solutions were obtained.	“I believe that Andy’s answer was much easier to perceive at first, because his computations are shown with numbers, while Beth explains her process in words” (Corey, Preliminary Survey).
<ul style="list-style-type: none"> • Numbers as a personal preference 	Interviewees viewed the choice between using numbers and using words or diagrams as a personal preference.	“Dawn was able to figure it out with just using the numbers, so if that’s what works better for her, then that’s fine too” (Diana, Interview 1).
<ul style="list-style-type: none"> • Numbers as familiar 	Interviewees indicated that using numbers was more familiar to them than using words or diagrams.	“I think it just goes back to what I learned again. I was always taught to do it the number way” (Linda, Interview 1).
<ul style="list-style-type: none"> • Numbers as advanced 	Interviewees indicated that using numbers was more appropriate for older or more advanced students than using words or diagrams.	“Pictures are good things, but I think they’re better for younger ages than older ages” (Corey, Interview 1).

<ul style="list-style-type: none"> • Numbers as easy to understand 	Interviewees indicated that numbers were easier to understand than words or diagrams.	“She drew a picture, but I just felt that she didn’t really understand as much of what she was doing, because it wasn’t very clear to me” (Linda, Interview 1).
<ul style="list-style-type: none"> • Numbers as explanation 	Interviewees indicated that numbers provided better or more direct explanations than words or diagrams.	“Flavia explained herself by just being able to add it. She just basically took the word problem and put it into numbers” (Linda, Interview 1).
<ul style="list-style-type: none"> • Numbers as precise 	Interviewees indicated that numbers were precise, unambiguous, and less subject to interpretation than words or diagrams.	“With numbers, I felt it was a lot easier to perceive, because it was discrete; it was fact; it showed you what it was” (Corey, Interview 1).
<ul style="list-style-type: none"> • Numbers as a way to show how 	Interviewees indicated that numbers were better for showing procedures than words or diagrams.	“They’re showing the steps to it. They’re showing ‘two times three equals six,’ and then they’re reversing it—‘three times two equals six’—with Dawn” (Corey, Interview 1).
<ul style="list-style-type: none"> • Numbers as the essence of math 	Interviewees indicated that using numbers was the primary focus of mathematics.	He showed every step through numbers, which is what I’ve always perceived math to be” (Corey, Interview 1).
<p>Superordinate Theme: Having the Right Attitude</p>	Interviewees considered the attitude of the arguer when evaluating mathematical arguments.	“It’s not so much the explanation, it’s more the attitude behind the explanation that I like more” (Diana, Post-course Interview).
<ul style="list-style-type: none"> • Open-mindedness 	Interviewees considered “open-mindedness” or willingness to accept new ideas, when evaluating mathematical arguments.	“I like that Ivy is more open-minded about it and willing to accept that it may change later” (Diana, Interview 1).

<ul style="list-style-type: none"> • Curiosity 	Interviewees considered “curiosity” or interest in exploring new ideas, when evaluating mathematical arguments.	“Ivy has a good way of thinking. ... She’s curious” (Grace, Interview 1).
<ul style="list-style-type: none"> • Confidence 	Interviewees considered the confidence or uncertainty of the arguer, when evaluating mathematical arguments.	I believe Heather’s answer is more convincing, ... because the way she states her answer, she seems very confident she is correct” (Corey, Preliminary Survey).
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Superordinate Theme: Understanding the Problem	Interviewees devoted time and effort to understanding the problem, in a way they had not when responding to the preliminary survey.	“I know this is what I’ll [need] to deal with—with my students later on, struggling and trying to understand this as well—so I need to get this now” (Ermidia, Post-course Interview).
<ul style="list-style-type: none"> • Expressing doubts or confusion 	Interviewees expressed doubts or confusion about their original interpretation of the problem.	“I’m having ... regrets, looking at this” (Diana, Interview 1).
<ul style="list-style-type: none"> • Rereading the problem 	Interviewees reread the problem or part of the problem.	“Let me read through that really quick” (Diana, Interview 1).
<ul style="list-style-type: none"> • Clarifying meanings and relationships 	Interviewees examined symbols, words, or diagrams, attempting to clarify meanings and relationships.	“Okay, this is why it’s so important to clarify what the whole is” (Ermidia, Post-course Interview).
<ul style="list-style-type: none"> • Struggling with meanings 	Interviewees indicated difficulty in understanding the meaning of symbols, words, or diagrams.	“I’m not understanding the wording here” (Ermidia, Post-course Interview).
<ul style="list-style-type: none"> • Reinterpreting the problem 	Interviewees decided that their original interpretations of the problem were incorrect.	“I just wasn’t reading it right. So both combined equal one whole” (Corey, Post-course Interview).

Superordinate Theme: Finding an Answer that Makes Sense	Interviewees required answers to make sense within the context of the problem.	“That’s only if you’re combining the grape. You can’t just combine them and not [ask], ‘Where does the other seven-eighths go?’ (Grace, Post-course Interview).
<ul style="list-style-type: none"> • Linking answers to problems 	Interviewees focused on links between problems and potential answers.	“[If the question was], ‘In this gallon, one gallon, how much grape juice do we have in there now?’ then I could clearly say four-eighths” (Ermida, Post-course Interview).
<ul style="list-style-type: none"> • Justifying answers in context 	Interviewees gave reasons in the context of the problem to justify their answers.	“Georgia’s is the correct answer, because it says that two one-gallon jugs are the whole already” (Corey, Post-course Interview)
<ul style="list-style-type: none"> • Rejecting standard procedures 	Interviewees rejected answers resulting from standard procedures, because they failed to make sense in the context of the problem.	M: Is this a problem where you should add fractions in the usual way? G: No! Because it really doesn’t make sense to me, if you’re adding two jugs ... and just all a sudden have the same amount (Grace, Post-course Interview)
Superordinate Theme: Explaining Why with Diagrams	Interviewees indicated a shift in their preferences, away from using numbers to get quick answers and toward using diagrams to support arguments that explain why solutions work.	“Showing representations and connecting it to the real world and explaining each specific part ... is a much stronger way to connect to the students and visually show them why each process works” (Corey, Post-course Interview).
<ul style="list-style-type: none"> • Changing views of understanding 	Interviewees indicated a change in their interpretation of “understanding.”	“I’m pretty sure I don’t have a good grasp of this now, so I don’t see how I did at the beginning” (Diana, Post-course Interview)

<ul style="list-style-type: none"> Rejecting shortcuts 	Interviewees rejected shortcuts in favor of more explanatory arguments.	“Now I definitely think Beth’s is better, because I feel Andy’s is just a shortcut” (Corey, Post-course Interview).
<ul style="list-style-type: none"> Requiring arguers to explain why 	Interviewees rejected arguments for failing to explain why solutions worked.	“He doesn't explain why you would multiply three and ten and five and six just because they're across from each other” (Linda, Post-course Interview).
<ul style="list-style-type: none"> Valuing diagrams 	Interviewees expressed a greater appreciation of diagrams than they had in their initial responses.	“I love the picture. I don’t know why I liked this so much” (Diana, Post-course Interview).
<ul style="list-style-type: none"> Diagrams conveying meaning or understanding 	Interviewees indicated that diagrams helped convey meaning or understanding in a way that numbers did not.	“It makes it so much simpler and easier to convey the meaning of the math, rather than just [using] numbers” (Corey, Post-course interview).
<ul style="list-style-type: none"> Diagrams supporting explanation 	Interviewees focused on the role of diagrams in supporting explanation.	“It’s a good explanation, backed up by an illustration” (Diana, post-course interview).
<ul style="list-style-type: none"> Seeing why in diagrams 	Interviewees focused on the role of diagrams in explaining why particular solutions work.	“I like [Beth’s], because it shows the exact procedure of what is happening, how three-fifths can equal six-tenths” (Corey, Post-course Interview).
<ul style="list-style-type: none"> Seeing generality in diagrams 	Interviewees focused on the role of diagrams in supporting generalization.	“If [Evan] understands this, he wouldn’t have to prove this point every single time he did a math problem” (Ermida, Post-course Interview).
<hr/> Superordinate Theme: Viewing Attitude as Complex	Interviewees focused on the attitude of the arguer as a complex issue.	“I was one of those people [who said], ‘No! This is a fourth! I don’t know what you’re talking about!’” (Diana, Post-course Interview).

<ul style="list-style-type: none"> • The right attitude is not enough 	Interviewees criticized arguers whose attitudes they previously endorsed.	“I don’t think [Ivy’s argument] is very convincing. ... She just says that it’s basically because some things don’t work all the time” (Grace, Post-course Interview).
<ul style="list-style-type: none"> • Seeing both sides of attitude 	Interviewees viewed the arguer’s attitude as having both positive and negative aspects.	“I don’t like that Ivy doesn’t seem confident with it, but ... I [also] like Ivy’s [argument], because math is so different that there are some things that you need to work through to actually find it” (Corey, Post-course Interview).
<ul style="list-style-type: none"> • Failing to maintain the right attitude 	Interviewees identified cases in which they failed to maintain an attitude they admired.	“I couldn’t think of a reason, so I came up with something like that myself. But ... it’s very closed-minded” (Diana, Post-course Interview).
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Theory-Driven Theme: Appeals to Authority	Interviewees provided reasons to support their evaluations of arguments that appealed to authority.	“She knows this is a property; this is true. But until you do something, you don’t really understand it” (Corey, Interview 1).
<ul style="list-style-type: none"> • Evidence of initiative 	Interviewees described Caitlyn as either showing or lacking initiative.	“She took the initiative to look in the book” (Ermida, Interview 1).
<ul style="list-style-type: none"> • Evidence of understanding 	Interviewees described Caitlyn as either providing evidence of understanding or failing to do so.	“Just because she can say, “A times B equals B times A,” doesn’t mean she actually understands why it works” (Corey, Post-course Interview).
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Theory-Driven Theme: Rules Without Backing	Interviewees provided reasons to support their evaluations of arguments that used rules without providing backing for them.	“That’s the more advanced [approach]. That’s the shortcut to it” (Diana, Interview 1).

<ul style="list-style-type: none"> Defending shortcuts 	Interviewees defended the legitimacy of shortcuts.	“Once you understand the basics—you understand why you’re doing that, you can do the shortcut” (Diana, Interview 1).
<ul style="list-style-type: none"> Using rules as demonstrating understanding 	Interviewees viewed using a rule correctly as evidence of understanding.	“I’m assuming he understands why the cross multiplying will help you understand that those two fractions are equal” (Diana, Interview 1).
<ul style="list-style-type: none"> Rejecting shortcuts 	Interviewees rejected shortcuts for failing to provide sufficient explanation.	“Now I definitely think Beth’s is better, because I feel Andy’s is just a shortcut” (Corey, Post-course Interview).
<ul style="list-style-type: none"> Requiring arguers to explain why 	Interviewees rejected arguments that failed to explain why solutions worked.	“He doesn’t explain why you would multiply three and ten and five and six just because they’re across from each other” (Linda, Post-course Interview).
<p>Theory-Driven Theme: Empirical Arguments</p>	Interviewees provided reasons to support their evaluations of empirical arguments.	“If it keeps working, then chances are it’s going to be right” (Diana, Interview 1).
<ul style="list-style-type: none"> Unquestioning acceptance 	Interviewees indicated unquestioning acceptance of empirical arguments, viewing their conclusions as universal truths.	“If she did enough trial-and-errors and they all worked out, I think she could conclude that this works out” (Ermida, Interview 1).
<ul style="list-style-type: none"> Provisional acceptance 	Interviewees indicated provisional acceptance of empirical arguments, treating their conclusions as true but potentially falsifiable by additional evidence.	“There’s no way to say for sure. ... But as far as we’ve found, there’s nothing to contradict it” (Linda, Interview 1).

Theory-Driven Theme: Generic Examples	Interviewees provided reasons to support their evaluations of generic examples or used generic examples in their own arguments.	“Out of all these four problems, Evan was able to explain himself thoroughly enough” (Ermida, Post-course Interview).
<ul style="list-style-type: none"> Emphasizing why 	Interviewees emphasized the importance of explaining why solutions worked.	“We knew the calculations, but we didn’t know how to explain what we were doing, why we were doing it, and how it actually worked” (Ermida, Post-course Interview).
<ul style="list-style-type: none"> Explanation vs. diagrams 	Interviewees focused on the relative importance of explaining why and including a diagram.	“He drew it out. But even if he didn’t, he still understands. He gave an example and showed why it worked” (Grace, Post-course Interview).
<ul style="list-style-type: none"> Comparing generic examples and empirical arguments 	Interviewees focused on similarities or differences between generic examples and empirical arguments.	“Dawn’s is just pretty much the same as Evan’s, just without the pictures” (Diana, Post-course Interview).
<ul style="list-style-type: none"> Creating generic examples 	Interviewees created generic examples to support their conclusions.	“If you’re adding like 10 and 5, you already have 10 pencils [for example], and you’re getting five more, there’s no way that it’s going to be less than 10” (Linda, Post-course Interview).
Theory-Driven Theme: Counterexamples	Interviewees provided or discussed counterexamples to false generalizations.	“I feel that the fact that there’s one exception to it shows that that’s not a rule” (Corey, Post-course Interview).
<ul style="list-style-type: none"> Providing counterexamples 	Interviewees provided counterexamples to false generalizations.	“Is that including zero? ... The answer—if we added a number plus zero—wouldn’t go along with what this is saying” (Ermida, Post-course Interview).

<ul style="list-style-type: none"> • Discussing counterexamples 	<p>The interviewees discussed the implications of counterexamples I suggested.</p>	<p>“If zero is taken into account as a whole number, then Ivy is correct, because she could show that you’re not making it bigger” (Corey, Post-course Interview).</p>
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<p>Theory-Driven Theme: The Role of the Argument’s Substance</p>	<p>Interviewees focused on relationships among symbols or relationships among the quantities the symbols represented.</p>	<p>“The whole is changing, so the denominator can change. You are [now] referencing a different whole than you were referencing [originally]” (Grace, Post-course Interview).</p>
<ul style="list-style-type: none"> • Focusing on symbols 	<p>Interviewees focused on relationships among symbols.</p>	<p>“[Georgia] forgot to realize that, when adding fractions, the common denominator remains the same” (Corey, Preliminary Survey).</p>
<ul style="list-style-type: none"> • Replacing symbols with diagrams 	<p>Interviewees replaced symbols with diagrams but preserved relationships among the original symbols.</p>	<p>“I thought maybe she should have drawn an “equals to” [sign] and drawn another jug to show her actual answer” (Grace, Interview 1).</p>
<ul style="list-style-type: none"> • Focusing on quantities 	<p>Interviewees interpreted operations on mathematical symbols in terms of actions on the quantities the symbols represent.</p>	<p>By doing this, you’re doubling the total amount of pieces by cutting the pieces in half, creating smaller pieces to take up the whole. So therefore, one-fourth is the same as two-eighths” (Corey, Post-course Interview).</p>

VITA

Born in 1954, the son of Sigismund and Catherine Perkowski, Michael grew up in the western suburbs of Chicago, near Schaumburg, Illinois. He attended Culver-Stockton College in Canton, Missouri, graduating summa cum laude in 1976 with a B.A. in mathematics and a B.S. in music education. He began graduate work in music theory at the University of Iowa, but due to the uncertain job outlook in that field, he left for the University of Missouri, earning an M.A. in mathematics there in 1982. While a student at MU, he met the love of his life, Debbie Anderson, also a graduate student in mathematics. They married in 1983 and have one son, Theron, who shares his parents' devotion to teaching but directs it toward music rather than mathematics.

With the exception of one year as an instructor at Kapiolani Community College in Honolulu, Michael has spent his working life at the University of Missouri, primarily as a tutoring coordinator at the university's Learning Center but also sometimes teaching classes in mathematics, statistics, or mathematics education. Reluctant to leave his work at Iowa unfinished, he completed and submitted his master's thesis, *Using a Metric as a Mathematical Model of Pitch-Set Relatedness*, to earn an M.A. in music theory in 1986, while continuing to work full-time at MU. Likewise, in 2005, he began working toward a Ph.D. in mathematics education at MU, completing it in 2013. For now, he continues in his position as General Tutoring Coordinator at MU's Learning Center.