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JAMESON CAHILL

Dr. Peter G. Casazza, Dissertation Supervisor

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The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled:

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presented by Jameson Cahill,

a candidate for the degree of Doctor of Philosophy and hereby certify that, in their opinion, it is worthy of acceptance.

Dr. Peter G. Casazza

Dr. Loukas Grafakos

Dr. Dan Edidin

Dr. Stephen Montgomery-Smith

Dr. David Retzloff

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ABSTRACT

In this dissertation we explore several ways in which the concept of projections arise in finite frame theory. In the first chapter we show that the Paulsen problem from frame theory is equivalent to a long standing open problem about orthogonal projections with constant diagonal. In the second chapter we introduce the idea of nonorthogonal fusion frames and derive some conditions for when tight nonorthogonal fusion frames exist. In particular, we give a classification of how to factor a self-adjoint matrix into a product of projections. The third chapter explores the idea that the cross gramian of a dual pair of frames forms a projection. We use this to give a classification of when two tight frames form a dual pair. We also introduce a notion of Naimark complement of dual pairs and derive some of its basic properties. The fourth chapter is devoted to questions that relate to applying an invertible operator to a given frame to get a new frame with some desired properties. The last chapter looks at frames as sets of rank one projections rather than as sets of vectors. In this chapter we discuss two problems: the first is the question of rescaling a given frame in order to get a tight frame, the second is known as phase retrieval.

Chapter 1

Introduction

1.1 Frames

To date, Hilbert space frame theory has broad applications in pure mathematics, see, for instance, [42, 34, 21], as well as in applied mathematics, computer science, and engineering. This includes time-frequency analysis [50], wireless communication [56, 70], image processing [61], coding theory [71], quantum measurements [47], sampling theory [46], and bioimaging [62], to name a few. Let us start by recalling some basic definitons from frame theory. For a very thorough account of the current state of finite dimensional frame theory we refer to [43]. Throughout let $\mathcal{H}_N$ denote an N-dimensional Hilbert space.

Definition 1. *A family of vectors* $\Phi = \{\varphi_i\}_{i=1}^M$ *in an* N*-dimensional Hilbert space*

$\mathcal{H}_N$ is a frame if there are constants $0 < A \leq B < \infty$ so that for all $f \in \mathcal{H}_N$ we have

$$A\|f\|^2 \leq \sum_{i=1}^{M} |\langle f, \varphi_i\rangle|^2 \leq B\|f\|^2.$$

If $A = B$, this is a tight frame and if $A = B = 1$, it is a Parseval frame. If there is a constant c so that $\|\varphi_i\| = c$, for all $i = 1, 2, \ldots, M$ it is an equal norm frame and if $c = 1$, it is a unit norm frame.

By a slight abuse of notation we use the same symbol Φ to denote the $N \times M$ matrix whose ith column is φ_i. As an operator from $\mathcal{H}_M \to \mathcal{H}_N$ this matrix is called the *synthesis operator* and is given by

$$\Phi\left(\sum_{i=1}^{M} a_i e_i\right) = \sum_{i=1}^{M} a_i \varphi_i$$

where $\{e_i\}_{i=1}^M$ is a fixed orthonormal basis of $\mathcal{H}_M$. Its adjoint $\Phi^* : \mathcal{H}_N \to \mathcal{H}_M$ is called the *analysis operator*:

$$\Phi^*(f) = \sum_{i=1}^{M} \langle f, \varphi_i\rangle e_i,$$

The *frame operator* is the positive, self-adjoint invertible operator $S = \Phi\Phi^*$ on $\mathcal{H}_N$ and satisfies

$$S(f) =: \Phi\Phi^*(f) = \sum_{i=1}^{M} \langle f, \varphi_i\rangle \varphi_i,$$

and the *Gram matrix* is $G_\Phi = \Phi^*\Phi = [\langle \varphi_i, \varphi_j\rangle]$.

If $\{\varphi_i\}_{i=1}^M$ is a frame with frame operator S having eigenvalues $\{\lambda_j\}_{j=1}^N$, then

$$\sum_{i=1}^{M} \|\varphi_i\|^2 = \mathrm{trace}(G_\Phi) = \mathrm{trace}(S) = \sum_{j=1}^{N} \lambda_j.$$

So if $\{\varphi_i\}_{i=1}^M$ is an equal norm Parseval frame then

$$\|\varphi_1\|^2 = \frac{1}{M}\sum_{i=1}^M \|f_i\|^2 = \frac{N}{M}.$$

A direct calculation shows that the frame $\{S^{-1/2}\varphi_i\}_{i=1}^M$ is a Parseval frame called the *canonical Parseval frame* for the frame. Also, $\{\varphi_i\}_{i=1}^M$ is a Parseval frame if and only if $S = I_N$. The following is known as Naimark's theorem, and will be used extensively:

Theorem 2. *A family $\{\varphi_i\}_{i=1}^M$ is a Parseval frame for $\mathcal{H}_N$ if and only if the analysis operator Φ^* for the frame is an isometry satisfying:*

$$\Phi^*\varphi_i = Pe_i, \text{ for all } i = 1, 2, \ldots, M,$$

where $\{e_i\}_{i=1}^M$ is a fixed orthonormal basis of $\mathcal{H}_M$ and P is the orthogonal projection of $\mathcal{H}_M$ onto $\Phi(\mathcal{H}_M)$.

If $\{\varphi_i\}_{i=1}^M$ is a Parseval frame for $\mathcal{H}_N$ and P is the orthogonal projection of $\mathcal{H}_M$ onto $\mathrm{im}(\Phi^*)$, the *Naimark complement* of $\{\varphi_i\}_{i=1}^M$ is the Parseval frame $\{(I-P)e_i\}_{i=1}^M$ for $\mathcal{H}_{M-N}$, where $\{e_i\}_{i=1}^M$ is a fixed orthonormal basis for $\mathcal{H}_M$. Note that the Naimark complement is only defined up to multiplication by a unitary operator.

We say that two frames $\{\varphi_i\}_{i\in I}, \{\psi_i\}_{i\in I}$ for $\mathcal{H}$ are *isomorphic* if there is an invertible operator L on $\mathcal{H}$ satisfying $L\varphi_i = \psi_i$, for all $i \in I$. It is known [35] that two frames are isomorphic if and only if their analysis operators have the same image, and two Parseval frames are isomorphic if and only if the isomorphism is a unitary operator.

Given a frame $\{\varphi_i\}_{i=1}^M \subseteq \mathcal{H}_N$, another sequence of vectors $\{\psi_i\}_{i=1}^M$ is said to be a *dual frame* if the following reproducing formula holds:

$$f = \sum_{i=1}^{M} \langle f, \varphi_i \rangle \psi_i \quad \text{for all } f \in \mathcal{H}_N. \tag{1.1}$$

If the frame $\{\varphi_i\}_{i=1}^M$ consists of more vectors than necessary for the spanning property, that is, if $M > N$, then there exist infinitely many dual frames.

In matrix notation the equation (1.1) reads $\Psi\Phi^* = I_N$, where I_N is the $N \times N$ identity matrix. Hence, a frame $\Psi = \{\psi_i\}_{i=1}^M$ is dual to Φ if and only if

$$\Psi\Phi^* = I_N,$$

or, equivalently,

$$\Phi\Psi^* = I_N.$$

Therefore, all duals of Φ are the left-inverses Ψ to Φ^* (or equivalently, right-inverse to Φ). The *canonical dual* frame is the pseudo-inverse of Φ^* which can be written as

$$\Phi^\dagger = (\Phi\Phi^*)^{-1}\Phi = S^{-1}\Phi. \tag{1.2}$$

The canonical dual has frame bounds $1/B$ and $1/A$, where A and B are frame bounds of Φ.

1.2 Projections

Definition 3. *An operator $P : \mathcal{H}_N \to \mathcal{H}_N$ is called a projection if $P^2 = P$. If in addition we have $P^* = P$ then P is called an orthogonal projection.*

We collect some basic properties of projections in the following proposition which we state without proof:

Proposition 4. *Let P be a projection and suppose $\mathrm{im}(P) = W$ and $\ker(P)^\perp = W^*$. Then $P^*, I_N - P$, and $I_N - P^*$ are all projections and*

1. $\mathrm{im}(P^*) = W^*$ *and* $\ker(P^*) = W^\perp$,
2. $\mathrm{im}(I_N - P) = (W^*)^\perp$ *and* $\ker(I_N - P) = W$,
3. $\mathrm{im}(I_N - P^*) = W^\perp$ *and* $\ker(I_N - P^*) = W^*$.

Furthermore we have that P is an invertible operator on W^* mapping W^* onto W.

Projections are defined by their range and null space. Let $\mathcal{H}_M = W \oplus (W^*)^\perp$, this is equivalent to assuming that $W \cap (W^*)^\perp = \{0\}$ and $\dim W = \dim W^* = N$, the projection $P = P_{W \| W^*}$ such that $\ker(P) = (W^*)^\perp$ and $\mathrm{im}(P) = W$ is called the projection along $(W^*)^\perp$ onto W. In the case of orthogonal projections we have $W = W^*$. Conversely, if P is a projection on $\mathcal{H}_M$ then $\mathcal{H}_M = W \oplus (W^*)^\perp$.

The standard way of constructing projections along $(W^*)^\perp$ onto W is to let $\{a_j\}_{j=1}^N$ be an orthonormal basis for W and let $\{b_j\}_{j=1}^N$ be an orthogonal basis for W^*. We collect these bases as columns of two matrices: $A = [a_j]$ and $B = [b_j]$. Then

$$P = A(B^*A)^{-1}B^*.$$

We now show that the singular value decomposition of a projection has a special property:

Proposition 5. *Suppose P is a rank N projection, and let $(a_j)_{j=1}^N$ and $(b_j)_{j=1}^N$ be left and right singular vectors corresponding to the non-zero singular values $(\sigma_i)_{i=1}^N$. Then*

$$\langle a_j, b_k \rangle = \begin{cases} \frac{1}{\sigma_j} & j = k, \\ 0 & j \neq k \end{cases} \tag{1.3}$$

for all $j, k \in \{1, \ldots, N\}$.

Proof. Let $P = U_N \Sigma_N V_N^*$ be the compact SVD of P, where $\Sigma_N = \operatorname{diag}(\sigma_1, \ldots, \sigma_N) \in GL_N(\mathbb{R}_+)$. Since $P^2 = P$, we have that

$$\Sigma_N V_N^* U_N \Sigma_N = \Sigma_N$$

hence, $V_N^* U_N = \Sigma_N^{-1}$. □

Proposition 5 has some useful consequences, but we need a definition first.

Definition 6. *Given N-dimensional subspaces W_1 and W_2 of a Hilbert space, define the N-tuple $(\sigma_1, \sigma_2, \ldots, \sigma_N)$ as follows:*

$$\gamma_1 = max\{\langle a, b \rangle : a \in W_1,\ b \in W_2, \|a\| = \|b\| = 1\} = \langle a_1, b_1 \rangle.$$

For $2 \leq i \leq N$,

$$\gamma_j = max\{\langle a, b \rangle : \|a\| = \|b\| = 1,\ \langle a_k, a \rangle = 0 = \langle b_k, b \rangle,\ for\ 1 \leq k \leq j-1\},$$

where

$$\gamma_j = \langle a_j, b_j \rangle.$$

The N-tuple $(\theta_1, \theta_2, \ldots, \theta_N)$ with $\theta_j = cos^{-1}(\gamma_j)$ is called the principle angles between W_1, W_2.

Letting $\{a_j\}_{j=1}^N, \{b_j\}_{j=1}^N$, and $(\theta_1, ..., \theta_N)$ be as above it is well known and straightforward to verify that there exists an orthonormal set $\{e_j\}_{j=1}^{2N} \subseteq \mathcal{H}_M$ such that $a_j = e_j$ for $j = 1, ..., N$ and $b_j = \cos(\theta_j)e_j + \sin(\theta_j)e_{j+N}$ for $j = 1, ..., N$. It now follows that

$$\langle a_j, b_k \rangle = \begin{cases} \cos(\theta_j) & j = k, \\ 0 & j \neq k. \end{cases} \tag{1.4}$$

Combining Proposition 5 with (1.4) yields the following useful corollaries:

Corollary 7. *Let P be a projection of rank N, $\mathrm{im}(P) = W$, and $\ker(P) = (W^*)^\perp$. Then*

$$\sigma_j = \frac{1}{\cos(\theta_j)}$$

where $(\sigma_1, ..., \sigma_N)$ are the singular values of P and $(\theta_1, ..., \theta_N)$ are the principal angles between W and W^*.

Corollary 8. *Let $W, W^* \subseteq \mathcal{H}_M$ be N-dimensional subspaces such that $W^\perp \cap W^* = \{0\}$. Let $(\theta_1, ..., \theta_N)$ be the principal angles between W and W^* and suppose that $\{a_j\}_{j=1}^N \subseteq W$ and $\{b_j\}_{j=1}^N \subseteq W^*$ are orthonormal bases which satisfy $\langle a_j, b_j \rangle = \cos(\theta_j)$. Let P be the projection onto W along $(W^*)^\perp$. Then we can decompose P as $P = \sum_{j=1}^N P_j$ where P_j is the rank 1 projection onto $\mathrm{span}\{a_j\}$ along $\{b_j\}^\perp$.*

Chapter 2

The Paulsen problem in operator theory

2.1 Introduction

The Paulsen Problem has proved to be one of the most intractable problems in frame theory. Roughly speaking, the Paulsen problem asks that if a given frame is close to a Parseval frame and simultaneously close to an equal norm frame, is it necessarily close to an equal norm Parseval frame? For a dozen years no progress at all was made on the Paulsen Problem. Recently, some progress has been made on the problem, but a completely satisfactory solution to this problem has not been given. First, Bodmann and Casazza [14] used differential equations to give an estimate. This paper leaves open the case where M, N are not relatively prime. Using gradient descent of the *frame potential*, Casazza, Fickus and Mixon [32] gave a completely different solution for the Paulsen problem which works in the case where M, N are relatively prime.

The estimates in these two papers seem to be quite far from optimal since it is on the order of $N^2M^9\epsilon$ when the frame is within ϵ of being equal norm and Parseval, and best evidence indicates the answer should be of the form $cN\epsilon$ or at worst $cM\epsilon$.

We will show why the Paulsen Problem has proved to so intractable by showing that it is equivalent to a fundamental, deep problem in operator theory. Roughky speaking, this problem asks that if an orthogonal projection has a nearly constant diagonal must it be close to another orthogonal projection whose diagonal is identically constant. The fact that there must be a connection between these two problems was first observed in [14]. In effect, we are answering a problem left open in that paper.

Analyzing the diagonal properties of projections has a long history. Kadison [57, 58] gave a complete characterization of the diagonals of projections for both the finite and infinite dimensional case. Analogous results on projections in type II_1 factors was given by Argerami and Massey [5]. For the more general problem of characterizing the diagonals of the unitary orbit of a self-adjoint operator, there is much more literature. This is equivalent in frame theory to characterizing the sequences which occur as the norms of the frame vectors with a specified frame operator. We refer the reader to [3, 4, 6, 7, 17, 40, 41, 53, 55, 61, 59, 68, 70] for a review of the work in this direction.

We will also consider the Naimark complement of nearly equal norm Parseval frames. We will show that the Paulsen function for a Parseval frame and its Naimark complement have a natural relationship. As a consequence of this, we will see that the Paulsen Problem only has to be solved for frames with a small number of elements relative to the dimension of the space. In particular, we only have to deal with the case of $M \leq 2N$.

This chapter is organized as follows. In Section 2.2 we give a formal statement of the Paulsen problem and the Projection Problem. In Section 2.3 we will prove a sequence of results which give an exact relationship between M element nearly equal norm Parseval frames for $\mathcal{H}_N$ and the distance between rank N orthogonal projections on $\mathcal{H}_M$ of rank N. In Section 2.4 we give an exact calculation relating the Paulsen Problem function and the function in the Projection Problem. Finally, in Section 2.5 we will relate the Paulsen Problem functions for a frame and its Naimark complement. The work in this chapter originally appeared in [21].

2.2 Problem statements

Before stating the Paulsen Problem we first collect several definitions:

Definition 9. *If $\Phi = \{\varphi_i\}_{i=1}^M$ and $\Psi = \{\psi_i\}_{i=1}^M$ are frames for $\mathcal{H}_N$, we define the distance between them by*

$$d(\Phi, \Psi) = \sum_{i=1}^{M} \|\varphi_i - \psi_i\|^2.$$

Definition 10. *A frame $\{\varphi_i\}_{i=1}^M$ with frame operator S is ϵ-nearly Parseval if*

$$(1-\epsilon)I \leq S \leq (1+\epsilon)I.$$

Definition 11. *A frame $\{\varphi_i\}_{i=1}^M$ is ϵ-nearly equal norm if*

$$(1-\epsilon)\frac{N}{M} \leq \|\varphi_i\|^2 \leq (1+\epsilon)\frac{N}{M}.$$

Problem 12 (Paulsen Problem)**.** *Find the function $h(\epsilon, N, M)$ so that for any ϵ-*

nearly equal norm, ϵ-nearly Parseval frame $\{\varphi_i\}_{i=1}^M$ for a N-dimensional Hilbert space $\mathcal{H}_N$, there is an equal norm Parseval frame $\{\psi_i\}_{i=1}^M$ for $\mathcal{H}_N$ satisfying:

$$\sum_{i=1}^{M} \|\varphi_i - \psi_i\|^2 \leq h(\epsilon, N, M).$$

A fundamental question here is whether the function $h(\epsilon, N, M)$ actually depends upon M. We have no examples showing this at this time, although it is known that this function must depend upon N. For all examples we know at this time, we have

$$h(\epsilon, N, M) \leq 16\epsilon N.$$

Definition 13. *If P and Q are orthogonal projections on $\mathcal{H}_M$, we define*

$$d(P, Q) = \sum_{i=1}^{M} \|Pe_i - Qe_i\|^2,$$

where $\{e_i\}_{i=1}^M$ is a fixed orthonormal basis for $\mathcal{H}_M$.

Problem 14 (Projection Problem)**.** *Let $\mathcal{H}_M$ be an M-dimensional Hilbert space with orthonormal basis $\{e_i\}_{i=1}^M$. Find the function $g(\epsilon, N, M)$ satisfying the following: If P is a projection of rank N on $\mathcal{H}_M$ satisfying*

$$(1-\epsilon)\frac{N}{M} \leq \|Pe_i\|^2 \leq (1+\epsilon)\frac{N}{M}, \text{ for all } i = 1, 2, \ldots, M,$$

then there is a projection Q with $\|Qe_i\|^2 = \frac{N}{M}$ for all $i = 1, 2, \ldots, M$ satisfying

$$d(P, Q) \leq g(\epsilon, N, M).$$

A reduction of the Paulsen problem to the Parseval case is done in [14].

Proposition 15. *If $\Phi = \{\varphi_i\}_{i=1}^M$ is an ϵ-nearly Parseval frame for $\mathcal{H}_N$ then the Parseval frame $\Psi = \{S^{-1/2}\varphi_i\}_{i=1}^M$ satisfies*

$$d(\Phi, \Psi) \leq N(2 - \epsilon - 2\sqrt{1-\epsilon}) \leq \frac{N\epsilon^2}{4}.$$

It is also nearly equal norm with the bounds:

$$\frac{(1-\epsilon)^2}{1+\epsilon}\frac{N}{M} \leq \|\psi_i\|^2 \leq \frac{(1+\epsilon)^2}{1-\epsilon}\frac{N}{M}.$$

It is known [9, 37, 55] that the canonical Parseval frame is the closest Parseval frame (with the distance function above) to a given frame. It is also known that this constant is best possible in general. So we are not giving up anything by working with a simpler variation of the Paulsen Problem:

Problem 16 (Parseval Paulsen Problem)**.** *Find the function $f(\epsilon, N, M)$ so that whenever $\Phi = \{\varphi_i\}_{i=1}^M$ is an ϵ-nearly equal norm Parseval frame, then there is an equal norm Parseval frame Ψ so that*

$$d(\Phi, \Psi) \leq f(\epsilon, N, M).$$

2.3 Preliminary Results

Let us first outline the proof of the equivalence of the Paulsen Problem and the Projection Problem. This will explain the results we develop in this section.

First we will assume that the Parseval Paulsen Problem function $f(\epsilon, N, M)$ is given and let P be a rank N projection on $\mathcal{H}_M$ with ϵ-nearly constant diagonal. We need to find a constant diagonal projection whose distance to P is on the order of $f(\epsilon, N, M)$. To do this, we consider $\Phi = \{Pe_i\}_{i=1}^M$ a nearly equal norm Parseval frame for $\mathcal{H}_N$. It follows that there is an equal norm Parseval frame $\Psi = \{\psi_i\}_{i=1}^M$ for $\mathcal{H}_N$ with

$$d(\Phi, \Psi) \leq f(\epsilon, N, M).$$

Now let Q be the projection onto $\mathrm{im}(\Psi^*)$ so that

$$\Psi^* \psi_i = Qe_i, \text{ for all } i = 1, 2, \ldots, M.$$

It is the problem of finding $d(P, Q)$ we will address in this section.

Conversely, if we assume the Projection Problem function $g(\epsilon, N, M)$ is given, we choose a nearly equal norm Parseval frame $\Phi = \{\varphi_i\}_{i=1}^M$ with analysis operator $\Phi^* : \mathcal{H}_N \to \mathcal{H}_M$ an isometry and let P be the orthogonal projection onto $\mathrm{im}(\Phi^*)$. We need to find an equal norm Parseval frame which is close to Φ. By our assumption, P is a projection with nearly constant diagonal. By the Projection Problem, there is a projection Q on $\mathcal{H}_M$ with $d(P, Q) \leq g(\epsilon, N, M)$. It follows that $\{Qe_i\}_{i=1}^M$ is a equal norm Parseval frame. We will be done if we can find an equal norm Parseval frame $\Psi = \{\psi_i\}_{i=1}^M$ for $\mathcal{H}_N$ with analysis operator Ψ^* satisfying:

$$\Psi^* \psi_i = Qe_i, \text{ and } d(\Phi, \Psi) \approx g(\epsilon, N, M). \tag{2.1}$$

So it is the problem of finding Ψ we address in this section. This problem is made more difficult by the fact that there are many frames Ψ satisfying the first part of

2.1 and most of them are not close to Φ. In particular, if $\Psi = \{\psi_i\}_{i=1}^M$ satisfies the first part of 2.1, and U is any unitary operator on $\mathcal{H}_N$, then $U(\Psi) = \{U\psi_i\}_{i=1}^N$ also satisfies the first part of 2.1. To address this problem, we will introduce the *chordal distance* between subspaces of a Hilbert space and give a computation of this distance in terms of our distance function. Using this, we will be able to construct the required frame Ψ.

We need a result from [14] and for completeness include its proof.

Theorem 17. *Let $\Phi = \{\varphi_i\}_{i=1}^M, \Psi = \{\psi_i\}_{i=1}^M$ be Parseval frames for $\mathcal{H}_N$ with analysis operators Φ^*, Ψ^* respectively. If*

$$d(\Phi, \Psi) = \sum_{i=1}^M \|\varphi_i - \psi_i\|^2 < \epsilon,$$

then

$$\sum_{i=1}^M \|\Phi^*\varphi_i - \Psi^*\psi_i\|^2 < 4\epsilon.$$

Proof. Note that for all $j \in \{1, ..., M\}$,

$$\Phi^*\varphi_j = \sum_{i=1}^M \langle\varphi_j, \varphi_i\rangle e_i, \text{ and } \Psi^*\psi_j = \sum_{i=1}^M \langle\psi_j, \psi_i\rangle e_i.$$

Hence,

$$\begin{aligned}
\|\Phi^*\varphi_j - \Psi^*\psi_j\|^2 &= \sum_{i=1}^M |\langle\varphi_j, \varphi_i\rangle - \langle\psi_j, \psi_i\rangle|^2 \\
&= \sum_{i=1}^M |\langle\varphi_j, \varphi_i - \psi_i\rangle + \langle\varphi_j - \psi_j, \psi_i\rangle|^2 \\
&\leq 2\sum_{i=1}^M |\langle\varphi_j, \varphi_i - \psi_i\rangle|^2 + 2\sum_{i=1}^M |\langle\varphi_j - \psi_j, \psi_i\rangle|^2.
\end{aligned}$$

Summing over j and using the fact that our frames Φ and Ψ are Parseval gives

$$\begin{aligned}
\sum_{j=1}^{M} \|\Phi^* \varphi_j - \Psi^* \psi_j\|^2 &\leq 2\sum_{j=1}^{M}\sum_{i=1}^{M} |\langle \varphi_j, \varphi_i - \psi_i \rangle|^2 + 2\sum_{j=1}^{M}\sum_{i=1}^{M} |\langle \varphi_j - \psi_j, \psi_i \rangle|^2 \\
&= 2\sum_{i=1}^{M}\sum_{j=1}^{M} |\langle \varphi_j, \varphi_i - \psi_i \rangle|^2 + 2\sum_{j=1}^{M} \|\varphi_j - \psi_j\|^2 \\
&= 2\sum_{i=1}^{M} \|\varphi_i - \psi_i\|^2 + 2\sum_{j=1}^{M} \|\varphi_j - \psi_j\|^2 \\
&= 4\sum_{j=1}^{M} \|\varphi_j - \psi_j\|^2.
\end{aligned}$$

□

Now recall the definition of principal angles from the introduction, (Definition 6). The *chordal distance* between two N-dimensional subspaces W_1, W_2 is given by

$$d_c^2(W_1, W_2) = \sum_{j=1}^{N} sin^2\theta_j.$$

So by the definition, there exists orthonormal bases $\{a_j\}_{j=1}^N$, $\{b_j\}_{j=1}^N$ for W_1, W_2 respectively satisfying

$$\|a_j - b_j\| = 2sin\left(\frac{\theta_j}{2}\right), \text{ for all } j = 1, 2, \ldots, N.$$

It follows that

$$sin^2\theta_j \leq 4sin^2\left(\frac{\theta_j}{2}\right) = \|a_j - b_j\|^2 \leq 4sin^2\theta_j, \text{ for all } j = 1, 2, \ldots, N.$$

Hence,

$$d_c^2(W_1, W_2) \le \sum_{j=1}^{N} \|a_j - b_j\|^2 \le 4d_c^2(W_1, W_2). \tag{2.2}$$

The following well known formula gives an easy way to compute the chordal distance between two subspaces:

Lemma 18. *If $\mathcal{H}_M$ is an M-dimensional Hilbert space and P, Q are rank N orthogonal projections onto subspaces W_1, W_2 respectively, then the chordal distance $d_c(W_1, W_2)$ between the subspaces satisfies*

$$d_c^2(W_1, W_2) = M - Tr\ PQ.$$

Next we give the precise connection between chordal distance for subspaces and the distance between the projections onto these subspaces. This result can be found in [44] in the language of Hilbert-Schmidt norms. We give our own proof for the sake of completeness.

Proposition 19. *Let $\mathcal{H}_M$ be an M-dimensional Hilbert space with orthonormal basis $\{e_i\}_{i=1}^M$. Let P, Q be the orthogonal projections of $\mathcal{H}_M$ onto N-dimensional subspaces W_1, W_2 respectively. Then the chordal distance between W_1, W_2 satisfies*

$$d_c^2(W_1, W_2) = \frac{1}{2} \sum_{i=1}^{M} \|Pe_i - Qe_i\|^2.$$

In particular, there are orthonormal bases $\{a_j\}_{j=1}^N$ for W_1 and $\{b_j\}_{j=1}^N$ for W_2 satisfying

$$\frac{1}{2} \sum_{i=1}^{M} \|Pe_i - Qe_i\|^2 \le \sum_{j=1}^{N} \|a_j - b_j\|^2 \le 2 \sum_{i=1}^{M} \|Pe_i - Qe_i\|^2.$$

Proof. We compute:

$$\begin{aligned}
\sum_{i=1}^{M} \|Pe_i - Qe_i\|^2 &= \sum_{i=1}^{M} \langle Pe_i - Qe_i, Pe_i - Qe_i \rangle \\
&= \sum_{i=1}^{M} \|Pe_i\|^2 + \sum_{i=1}^{M} \|Qe_i\|^2 - 2\sum_{i=1}^{M} \langle Pe_i, Qe_i \rangle \\
&= 2N - 2\sum_{i=1}^{M} \langle PQe_i, e_i \rangle \\
&= 2N - 2Tr\ PQ \\
&= 2N - 2[N - d_c^2(W_1, W_2)] \\
&= 2d_c^2(W_1, W_2).
\end{aligned}$$

Where we have used Lemma 18 for the last equality. This combined with Equation 2.2 completes the proof. $\square$

Now we are ready to answer the second problem we need to address in this section.

Theorem 20. *Let P and Q be projections of rank N on $\mathcal{H}_M$ and let $\{e_i\}_{i=1}^M$ be an orthonormal basis of $\mathcal{H}_M$. Further assume that $\Phi = \{\varphi_i\}_{i=1}^M$ is a Parseval frame from $\mathcal{H}_N$ such that* $\mathrm{im}(\Phi^*) = \mathrm{im}(P)$ *If*

$$\sum_{i=1}^{M} \|Pe_i - Qe_i\|^2 < \epsilon,$$

then there is a Parseval frame $\Psi = \{\psi_i\}_{i=1}^M$ satisfying

$$\Psi^*\psi_i = Qe_i, \text{ for all } i = 1, 2, \ldots, M,$$

and

$$\sum_{i=1}^{M} \|\varphi_i - \psi_i\|^2 < 2\epsilon.$$

Moreover, if $\{Qe_i\}_{i=1}^M$ *is equal norm, then* $\{\psi_i\}_{i=1}^M$ *is also equal norm.*

Proof. By Proposition 19, there are orthonormal bases $\{a_j\}_{j=1}^N$ and $\{b_j\}_{j=1}^N$ for W_1, W_2 respectively satisfying

$$\sum_{j=1}^{N} \|a_j - b_j\|^2 < 2\epsilon.$$

Let A and B be the $M \times N$ matrices whose j^{th} rows are a_j and b_j respectively, and let a_{ij} and b_{ij} be the (i,j) entry of A, B respectively. Finally, let $\{\varphi_i^{'}\}_{i=1}^M$ and $\{\psi_i^{'}\}_{i=1}^M$ be the i^{th} columns of A and B respectively. Then we have

$$\begin{aligned}
\sum_{i=1}^{M} \|\varphi_i^{'} - \psi_i^{'}\|^2 &= \sum_{i=1}^{M}\sum_{j=1}^{N} |a_{ij} - b_{ij}|^2 \\
&= \sum_{j=1}^{N}\sum_{i=1}^{M} |a_{ij} - b_{ij}|^2 \\
&= \sum_{i=1}^{N} \|a_j - b_j\|^2 \\
&\leq 2\epsilon.
\end{aligned}$$

Since the rows of A form an orthonormal basis for W_1, we know that $\{\varphi_i^{'}\}_{i=1}^M$ is a Parseval frame which is isomorphic to $\{\varphi_i\}_{i=1}^M$. Thus there is a unitary operator $U : \mathcal{H}_N \to \mathcal{H}_N$ with $U\varphi_i^{'} = \varphi_i$. Now let $\{\psi_i\}_{i=1}^M = \{U\psi_i^{'}\}_{i=1}^M$. Then

$$\sum_{i=1}^{M} \|\varphi_i - \psi_i\|^2 = \sum_{i=1}^{M} \|U(\varphi_i^{'}) - U(\psi_i^{'})\|^2 = \sum_{i=1}^{M} \|\varphi_i^{'} - \psi_i^{'}\|^2 \leq 2\epsilon.$$

Finally, if Ψ^* is the analysis operator for the Parseval frame $\{\psi_i\}_{i=1}^M$, then Ψ^* is an isometry and since $\{\Psi^*(\psi_i)\}_{i=1}^M = \{Qe_i\}_{i=1}^M$, for all $i = 1, 2, \ldots, M$, if Qe_i is equal norm, so is $\Psi^*(\psi_i)$ and hence so is $\{\psi_i\}_{i=1}^M$. $\square$

2.4 The Equivalence of our Problems

Now we can show that the Parseval Paulsen Problem and the Projection Problem are equivalent in the sense that their functions $f(\epsilon, N, M)$ and $g(\epsilon, N, M)$, respectively, are equal up to a factor of 4.

Theorem 21. *If $f(\epsilon, N, M)$ is the function for the Paulsen Problem and $g(\epsilon, N, M)$ is the function for the Projection Problem, then*

$$g(\epsilon, N, M) \leq 4f(\epsilon, N, M) \leq 8g(\epsilon, N, M).$$

Proof. First, assume that Problem 14 holds with function $g(\epsilon, N, M)$. Let $\{\varphi_i\}_{i=1}^M$ be a Parseval frame for $\mathcal{H}_N$ satisfying

$$(1-\epsilon)\frac{N}{M} \leq \|\varphi_i\|^2 \leq (1+\epsilon)\frac{N}{M}.$$

Let Φ^* be the analysis operator of $\{\varphi_i\}_{i=1}^M$ and let P be the projection of $\mathcal{H}_M$ onto $\operatorname{im}(\Phi^*)$, so that $\Phi^*\varphi_i = Pe_i$, for all $i = 1, 2, \ldots, M$. By our assumption that Problem 14 holds, there is a projection Q on $\mathcal{H}_M$ with constant diagonal so that

$$\sum_{i=1}^{M} \|Pe_i - Qe_i\|^2 \leq g(\epsilon, N, M).$$

By Theorem 20, there is a a Parseval frame $\{\psi_i\}_{i=1}^M$ for $\mathcal{H}_N$ with analysis operator

Ψ^* so that $\Psi^*\psi_i = Qe_i$ and

$$\sum_{i=1}^{M} \|\varphi_i - \psi_i\|^2 \leq 2g(\epsilon, N, M).$$

Since Ψ is an isometry and $\{\Psi^*\psi_i\}_{i=1}^M$ is equal norm, it follows that $\{\psi_i\}_{i=1}^M$ is an equal norm Parseval frame satisfying the Paulsen problem.

Conversely, assume the Parseval Paulsen problem has a positive solution with function $f(\epsilon, N, M)$. Let P be an orthogonal projection on $\mathcal{H}_M$ satisfying

$$(1-\epsilon)\frac{N}{M} \leq \|Pe_i\|^2 \leq (1+\epsilon)\frac{N}{M}.$$

Then $\{Pe_i\}_{i=1}^M$ is a Parseval frame for $\mathcal{H}_N$ and by the Parseval Paulsen problem, there is an equal norm Parseval frame $\{\psi_i\}_{i=1}^M$ so that

$$\sum_{i=1}^{M} \|\varphi_i - \psi_i\|^2 < f(\epsilon, N, M).$$

Let Ψ^* be the analysis operator of $\{\psi_i\}_{i=1}^M$. Letting Q be the projection onto the $\mathrm{im}(\Psi^*)$, we have that $Qe_i = \Psi^*\psi_i$, for all $i = 1, 2, \ldots, M$. By Theorem 17, we have that

$$\sum_{i=1}^{M} \|Pe_i - \Psi^*\psi_i\|^2 = \sum_{i=1}^{M} \|Pe_i - Qe_i\|^2 \leq 4f(\epsilon, N, M).$$

Since Ψ^* is an isometry and $\{\psi_i\}_{i=1}^M$ is equal norm, it follows that Q is a constant diagonal projection. $\square$

2.5 The Paulsen Problem and Naimark Complements

In this section we will use Naimark complements to show that we only need to solve the Paulsen problem for $M \leq 2N$. Recall that if $\Phi = \{\varphi_i\}_{i=1}^M$ is a Parseval frame for $\mathcal{H}_N$, then its Naimark complement is a Parseval frame for $\mathcal{H}_{M-N}$. Now we will compare the Paulsen function for a Parseval frame to the Paulsen function for its Naimark complement.

Theorem 22. *If $f(\epsilon, N, M)$ is the Paulsen constant then*

$$f(\epsilon, N, M) \leq 8f(\epsilon\frac{N}{M-N}, M-N, M).$$

Proof. Assume that $\Phi = \{\varphi_i\}_{i=1}^M$ is a ϵ-nearly equal norm Parseval frame for $\mathcal{H}_N$ with analysis operator Φ which is an isometry. Then there is a projection P on $\mathcal{H}_M$ so that $Pe_i = \Phi^*\varphi_i$, for all $i = 1, 2, \ldots, M$. It follows that $\{(I-P)e_i\}_{i=1}^M$ is a Parseval frame and

$$\begin{aligned}
\|(I-P)e_i\|^2 &= 1 - \|Pe_i\|^2 \\
&\leq 1 - (1-\epsilon)\frac{N}{M} \\
&= \left(1 + \epsilon\frac{N}{M-N}\right)\left(1 - \frac{N}{M}\right).
\end{aligned}$$

Similarly,

$$\|(I-P)e_i\|^2 \geq \left(1 - \epsilon\frac{N}{M-N}\right)\left(1 - \frac{N}{M}\right).$$

Choose a Parseval frame $\{\psi_i\}_{i=1}^M$ for $\mathcal{H}_{M-N}$ with analysis operator Ψ^* satisfying $\Psi^* g_i = (I-P)e_i$. Since Ψ^* is an isometry, it follows that $\Psi = \{\psi_i\}_{i=1}^M$ is a $\epsilon\frac{N}{M-N}$-

nearly equal norm Parseval frame. Hence, there is an equal norm Parseval frame $\Gamma = \{\gamma_i\}_{i=1}^M$ for $\mathcal{H}_{M-N}$ with

$$d(\Psi, \Gamma) \leq f(\epsilon \frac{N}{M-N}, M-N, M),$$

where f is the Paulsen function for N vectors in $\mathcal{H}_{M-N}$. Let Γ^* be the analysis operator for Γ. Applying Theorem 17, we have that

$$d(\Gamma^*(\{\gamma_i\}_{i=1}^M), \Psi^*(\{\psi_i\}_{i=1}^M)) \leq 4f(\epsilon \frac{N}{M-N}, M-N, M).$$

Let $I - Q$ be the orthogonal projection onto $\Gamma^*(\mathcal{H}_{M-N})$. Now we check

$$\begin{aligned} d(\{Pe_i\}_{i=1}^M, \{Qe_i\}_{i=1}^M) &= \sum_{i=1}^M \|Pe_i - Qe_i\|^2 \\ &= \sum_{i=1}^M \|(I-P)e_i - (I-Q)e_i\|^2 \\ &\leq 4f(\epsilon \frac{N}{M-N}, M-N, M). \end{aligned}$$

By Theorem 20, we can choose a equal norm Parseval frame $\Delta = \{\delta_i\}_{i=1}^M$ for $\mathcal{H}_{M-N}$ with analysis operator Δ^* satisfying $\Delta^*\delta_i = Qe_i$, for all $i = 1, 2, \ldots, M$ and

$$d(\Phi, \Delta) \leq 8f(\epsilon \frac{N}{M-N}, M-N, M).$$

□

Given $M \geq N$, then either $M \leq 2N$ or $M \leq 2(M-N)$. So we have:

Corollary 23. *To solve the Paulsen problem, it suffices to solve it for Parseval frames*

$\{\varphi_i\}_{i=1}^M$ *for* $\mathcal{H}_N$ *with* $M \leq 2N$.

Chapter 3

Nonorthogonal fusion frames

3.1 Introduction

Fusion frames were introduced in [36] and further developed in [39]. Recently there has been much activity around the idea of fusion frames, see [38] and references therein. In [25] we introduced the idea of nonorthogonal fusion frames in order to achieve the sparsity of the fusion frame operator.

Definition 24. *Let $\{P_i\}_{i=1}^M$ be a collection of projections on $\mathcal{H}_N$ and $\{v_i\}_{i=1}^M$ a collection of positive real numbers. Then we say $\{(P_i, v_i)\}_{i=1}^M$ is a nonorthogonal fusion frame for $\mathcal{H}_N$ if there exist constants $0 < A \leq B < \infty$ such that*

$$A\|f\|^2 \leq \sum_{i=1}^{M} v_i^2 \|P_i f\|^2 \leq B\|f\|^2$$

for every $f \in \mathcal{H}_N$. We say it is tight if $A = B$.

Definition 25. *Given a nonorthogonal fusion frame* $\{(v_i, P_i)\}_{i=1}^M$ *we define the nonorthogonal fusion frame operator* $S : \mathcal{H}_N \to \mathcal{H}_N$ *by*

$$Sf = \sum_{i=1}^{M} v_i^2 P_i^* P_i f.$$

Throughout this chapter we will always use the notation of Proposition 4; *i.e.*, P will always stand for a projection, W will always be the image of P, and W^* will always be the image of P^*. Furthermore, we will always use the symbol π_W to denote the *orthogonal* projection onto the subspace $W \subseteq \mathcal{H}_N$.

This chapter is organized as follows: In the next section for a fixed self-adjoint operator T we will classify the projections P for which $T = P^*P$. In the next section we apply these results to get some new results on the existence of tight nonorthogonal fusion frames. In particular, in subsection 3.3.1 we give a complete classification of tight nonorthogonal fusion frames with 2 projections. The work in this chapter originally appeared in [22].

3.2 Classification of self adjoint operators via projections

Let $T : \mathcal{H}_N \to \mathcal{H}_N$ be a positive, self adjoint, linear operator. The main point of this section is to classify the set

$$\Omega(T) = \{P : P^2 = P, P^*P = T\}.$$

The spectral theorem tells us that $T = \sum_{j=1}^{N} \lambda_j \pi_j$ where the λ_j's are the eigenvalues of T and π_j is the orthogonal projection onto the one dimensional span of the jth eigenvector of T. Therefore $P \in \Omega(T)$ if and only if P^*P has the same eigenvalues and eigenvectors as T. Also note that if $P \in \Omega(T)$ then $\ker(P) = \operatorname{im}(T)^\perp$, and since a projection is uniquely determined by its kernel and its image we have a natural bijection between $\Omega(T)$ and the set

$$\tilde{\Omega}(T) := \{W \subseteq \mathbb{R}^n : \operatorname{im}(P) = W \text{ for some } P \in \Omega(T)\}.$$

given by

$$\Omega(T) \ni P \mapsto \operatorname{im}(P) \in \tilde{\Omega}(T).$$

We start with two elementary lemmas.

Lemma 26. *Let P be a projection and let $\{e_j\}_{j=1}^k$ be an orthonormal basis of W^* consisting of eigenvectors of P^*P with corresponding nonzero eigenvalues $\{\lambda_j\}$. Then $\{Pe_j\}_{j=1}^k$ is an orthogonal basis for W and $\|Pe_j\| = \sqrt{\lambda_j}$.*

Proof. Just observe that $\langle Pe_j, Pe_\ell\rangle = \langle P^*Pe_j, e_\ell\rangle = \lambda_j\langle e_j, e_\ell\rangle$. □

Lemma 27. *Let P be a projection and suppose λ is an eigenvalue of P^*P, $\lambda \neq 0$. Then $\lambda \geq 1$. Moreover, $\lambda = 1$ if and only if the corresponding eigenvector is in $W \cap W^*$.*

Proof. Note that $W^* = \operatorname{im} P^*P$, so all eigenvectors of P^*P corresponding to nonzero eigenvalues are in W^*. Let $x \in W^*$ and write $Px = x + (P - I)x$. Since $x \perp (I - P)x$,

$$\|Px\|^2 = \|x\|^2 + \|(P - I)x\|^2 \geq \|x\|^2. \tag{3.1}$$

By the same argument on P^* we get $\|P^*Px\| \geq \|Px\| \geq \|x\|$ for all $x \in W^*$. Therefore, if $P^*Px = \lambda x$ we have that $\lambda \geq 1$.

Finally, by equation (3.1), $\lambda = 1$ if and only if $(I - P)x = 0$, or $x = Px \in W$. Hence $x \in W \cap W^*$. $\square$

The next proposition allows us reduce our problem to the case when $\text{rank}(T) \leq N/2$

Proposition 28. *Let P be a projection, then we can write*

$$P = P' + \pi_{W \cap W^*}$$

where $\pi_{W \cap W^*}$ is the orthogonal projection onto $W \cap W^*$, and P' is a projection such that all nonzero eigenvalues of P'^*P' are strictly greater than 1.

Proof. First note that Lemma 27 says that $W \cap W^* = \{x : P^*Px = x\}$. Now let W' be the orthogonal complement of $W \cap W^*$ in W and let P' be the projection onto W' along $\ker(P) + W \cap W^*$. Then $P'\pi_{W\cap W^*} = \pi_{W\cap W^*}P' = 0$, so $(P' + \pi_{W\cap W^*})^2 = P'^2 + \pi^2_{W\cap W^*} = P' + \pi_{W\cap W^*}$. It is clear that $\text{im}(P' + \pi_{W\cap W^*}) = W$. Since $\ker P = W^{*\perp} \subseteq (W \cap W^*)^\perp$ it follows that $\ker(P) \subseteq \ker(P' + \pi_{W\cap W^*})$ so we must have $\ker(P) = \ker(P' + \pi_{W\cap W^*})$. Therefore $P = P' + \pi_{W\cap W^*}$, and the nonzero eigenvalues of P'^*P' are precisely the nonzero eigenvalues of P^*P which are greater than 1. $\square$

We can now state the main theorem of this section:

Theorem 29. *Let $T : \mathcal{H}_N \to \mathcal{H}_N$ be a positive, self-adjoint operator of rank $k \leq \frac{N}{2}$. Let $\{\lambda_j\}_{j=1}^k$ be the nonzero eigenvalues of T and suppose $\lambda_j \geq 1$ for $i = 1, ..., k$ and* *let* $\{e_j\}_{j=1}^k$ *be an orthonormal basis of* $im(T)$ *consisting of eigenvectors of* T*. Then*

$$\tilde{\Omega}(T) = \{span\{\frac{1}{\sqrt{\lambda_j}}e_j + \sqrt{\frac{\lambda_j - 1}{\lambda_j}}e_{j+k}\} : \{e_j\}_{j=1}^{2k} \text{ is orthonormal}\}.$$

Proof. First suppose $W \in \tilde{\Omega}(T)$ and let P be the projection onto W along $\mathrm{im}(T)^\perp$. Be Lemma 26 we know that $\{\frac{Pe_j}{\|Pe_j\|}\}_{j=1}^k$ is an orthonormal basis for W. We also know that $\|Pe_j\| = \sqrt{\lambda_j}$ so

$$\lambda_j = \|e_j\|^2 + \|(P-I)e_j\|^2 = 1 + \|(P-I)e_j\|^2$$

which means

$$\|(P-I)e_j\| = \sqrt{\lambda_j - 1}$$

so if we set

$$e_{j+k} = \frac{(P-I)e_j}{\sqrt{1-\lambda_j}},$$

then $\{e_j\}_{j=1}^{2k}$ is an orthonormal set and

$$\frac{Pe_j}{\|Pe_j\|} = \frac{1}{\sqrt{\lambda_j}}e_j + \sqrt{\frac{\lambda_j - 1}{\lambda_j}}e_{j+k}.$$

Conversely suppose $W = \mathrm{span}\{\frac{1}{\sqrt{\lambda_j}}e_j + \sqrt{\frac{\lambda_j-1}{\lambda_j}}e_{j+k}\}$ with $\{e_j\}_{j=1}^{2k}$ orthonormal. Let P be the projection onto W along $\mathrm{im}(T)^\perp$. Notice that $e_j = e_j + \sqrt{\lambda_j - 1}e_{j+k} - \sqrt{\lambda_j - 1}e_{j+k}$ with $e_j + \sqrt{\lambda_j - 1}e_{j+k} \in W$ and $-\sqrt{\lambda_j - 1}e_{j+k} \in \mathrm{im}(T)^\perp$, so $Pe_j = e_j + \sqrt{\lambda_j - 1}e_{j+k}$ for $j = 1, ..., k$. Similarly $e_j + \sqrt{\lambda_j - 1}e_{j+k} = \lambda_j e_j + (1-\lambda_j)e_j + \sqrt{\lambda_j - 1}e_{j+k}$ with $\lambda_j e_j \in W^* = \mathrm{im}P^*$ and $(1-\lambda_j)e_j + \sqrt{\lambda_j - 1}e_{j+k} \in W^\perp = \ker(P^*)$, so $P^*Pe_j = \lambda_j e_j$ for $j = 1, ..., k$. Therefore, P^*P has the same eigenvectors and corresponding eigenvalues as T, so $P^*P = T$, and $W \in \tilde{\Omega}(T)$. $\square$

Before proceeding we remark that Theorem 29 is independent of our choice of eigenbasis for T. To see this let $\{e'_j\}_{j=1}^k$ be any other eigenbasis for T and let $W = \text{span}\{\frac{1}{\sqrt{\lambda_j}}e'_j + \sqrt{\frac{\lambda_j - 1}{\lambda_j}}e_{j+k}\}$ with $\{e'_j\}_{j=1}^{2k}$ orthonormal. By the second part of the proof of Theorem 29 we have that $W \in \tilde{\Omega}(T)$, and so by the first part of the proof we have that in fact $W = \text{span}\{\frac{1}{\sqrt{\lambda_j}}e_j + \sqrt{\frac{\lambda_j - 1}{\lambda_j}}e_{j+k}\}$ with $\{e_j\}_{j=1}^{2k}$ orthonormal.

We now state several consequences of Theorem 29.

Corollary 30. *If T is a positive self-adjoint operator of rank $\leq \frac{N}{2}$ with all nonzero eigenvalues ≥ 1, then there is a projection P so that $T = P^*P$.*

Corollary 31. *If T is a positive self-adjoint operator of rank $\leq \frac{N}{2}$, then there is a projection P and a weight $v > 0$ so that $T = v^2P^*P$.*

Proof. Let λ_k be the smallest non-zero eigenvalue of T. So all nonzero eigenvalues of $\frac{1}{\lambda_k}T$ are greater than or equal to 1 and by Corollary 30 there is a projection P so that $P^*P = \frac{1}{\lambda_k}T$. Let $v = \sqrt{\lambda_k}$ to finish the proof. $\square$

In the rest of this section we will analyze the case where $\text{rank}(T) > N/2$.

Proposition 32. *Let T be a positive self-adjoint operator of rank $k > \frac{N}{2}$ with eigenvectors $\{e_j\}_{j=1}^N$ and respective eigenvalues $\{\lambda_j\}_{j=1}^N$. The following are equivalent:*

(1) There is a projection P so that $T = P^*P$.

(2) The nonzero eigenvalues of T are greater than or equal to 1 and we have

$$|\{j : \lambda_j > 1\}| \leq |\{j : \lambda_j = 0\}|.$$

In particular,

$$|\{j : \lambda_j = 1\}| \geq k - \left\lfloor \tfrac{N}{2} \right\rfloor .$$

Proof. Let $A_1 = \{j : \lambda_j > 1\}$, $A_2 = \{j : \lambda_j = 0\}$, and $A_3 = \{j : \lambda_j = 1\}$, and let π_i be the orthogonal projection onto $\text{span}\{e_j : j \in A_i\}$ for $i = 1, 2, 3$.

(1) $\Rightarrow$ (2): By Proposition 28, we can write

$$P = P' + \pi_{W \cap W^*},$$

where $\pi_{W \cap W^*}$ is the orthogonal projection onto $W \cap W^*$, and P' is the projection onto the orthogonal complement W' of $W \cap W^*$ in W along *ker* $P + W \cap W^*$. Define $W'^* \equiv \text{im}\, P'^*$. Then P' is an invertible operator from W'^* onto W', $W'^* \perp W \cap W^*$ and $W' \perp W \cap W^*$, and $W' \cap W'^* = \{0\}$. Hence,

$$\begin{aligned} 2 \dim W'^* &= \dim W' + \dim W'^* \\ &= \dim(W' + W'^*) \\ &\leq \dim W'^* + \dim \text{span}\{e_j : j \in A_2\}. \end{aligned}$$

Since $W'^* = \text{span}\ \{e_j : j \in A_1\}$, it follows that $|A_1| \leq |A_2|$.

(2) $\Rightarrow$ (1): Let $T_1 = T(\pi_1 + \pi_2)$, so $T = T_1 + \pi_3$. By our assumption

$$\text{rank } T_1 \leq \frac{N}{2},$$

and all non-zero eigenvalues of T_1 are strictly greater than 1. By Theorem 29 there is a projection P' so that $P'^* P' = T_1$. Let $P = P' + \pi_3$. Then $P'\pi_3 = \pi_3 P' = 0$. Hence,

$P = P^2$ is a projection and

$$P^*P = P'^*P' + \pi_3 = T_1 + \pi_3 = T.$$

$\square$

Corollary 33. *If rank$(T) = k > \frac{N}{2}$ and T does not have 1 as an eigenvalue with multiplicity at least $k - \lfloor \frac{N}{2} \rfloor$, then $\Omega(T) = \emptyset$.*

Remark 34. *Similar to the proof of Corollary 31, if T is a positive self-adjoint operator of rank $> \frac{N}{2}$ with eigenvalues $\{\lambda_1 \geq \cdots \geq \lambda_k > 0 = \lambda_{k+1} = \cdots \lambda_N\}$, then there is a projection P and a weight $v = \sqrt{\lambda_k}$ so that $T = v^2 P^* P$ if and only if*

$$|\{j : \lambda_j > \lambda_k\}| \leq |\{j : \lambda_j = 0\}|.$$

Proposition 35. *Let $T : \mathcal{H}_N \to \mathcal{H}_N$ be a positive, self adjoint operator of rank $k > \frac{N}{2}$ whose nonzero eigenvalues are all greater than or equal to 1. If either*

(1) N is even, or

(2) N is odd and T has at least one eigenvalue in the set $\{0, 1, 2\}$

then there are two projections P_1 and P_2 such that $T = P_1^* P_1 + P_2^* P_2$.

Proof. Let $\{e_j\}_{j=1}^N$ be an orthonormal basis of $\mathcal{H}_N$ consisting of eigenvectors of T with respective eigenvalues $\{\lambda_j\}_{j=1}^N$, in decreasing order.

Case 1: N is even.

Let $V = \text{span}\{e_j\}_{j \in I}$, $|I| = \frac{N}{2}$. Note that $T = T\pi_V + T\pi_{V^\perp}$. Also, since T, π_V, and $\pi_{V^\perp}$ are all diagonal with respect to $\{e_j\}_{j=1}^N$ it follows that T commutes with both π_V and $\pi_{V^\perp}$. Therefore $(T\pi_V)^* = \pi_V^* T^* = \pi_V T = T\pi_V$, so by Theorem 29 there is a projection P_1 such that $T\pi_V = P_1^* P_1$. Similarly we can find a projection P_2 such that $T\pi_{V^\perp} = P_2^* P_2$.

Case 2: N is odd and T has an eigenvalue in the set $\{0, 1, 2\}$.

We will look at the case for each eigenvalue separately.

Subcase 1: $\lambda_N = 0$.

Let $\mathcal{H}_1 = \text{span}\{e_j : 1 \leq j \leq N-1\}$. Then $\dim(\mathcal{H}_1)$ is even so we can apply the same argument as above to $\mathcal{H}_1$.

Subcase 2: $\lambda_N = 1$.

Define T_1, T_2 by

$$T_1 e_j = \begin{cases} Te_j & \text{if } \ j = 1, 2, \ldots, \frac{N-1}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$T_2 e_j = \begin{cases} Te_j & \text{if } \ j = \frac{N-1}{2} + 1, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases}$$

Then $\text{rank}(T_1) = \text{rank}(T_2) = \frac{N-1}{2} < \frac{N}{2}$ so by Corollary 30, we can write

$$T_i = P_i^* P_i, \quad i = 1, 2.$$

Let π be the orthogonal projection of $\mathcal{H}_N$ onto $\text{span}\{e_N\}$ and let

$$Q = P_2 + \pi,$$

which is clearly a projection. Then we have

$$T = P_1^* P_1 + Q^* Q.$$

Subcase 3: $\lambda_j = 2$ for some j.

Without loss of generality, re-index $\{\lambda_j\}$ so that $\lambda_N = 2$. Define T_1, T_2, and π as above. As in the previous case, define two projections $\{P_i\}_{i=1}^2$ so that

$$T_i = P_i^* P_i.$$

Now let $Q_i = P_i + \pi$, $i = 1, 2$. Then

$$T = Q_1^* Q_1 + Q_2^* Q_2.$$

□

Corollary 36. *Let $T : \mathcal{H}_N \to \mathbb{H}_N$ be a positive, self adjoint operator of rank $k > \frac{N}{2}$. There is a weight v and projections $\{P_i\}_{i=1}^2$ so that*

$$T = v^2 P_1^* P_1 + v^2 P_2^* P_2.$$

Proof. Let T have eigenvectors $\{e_j\}_{j=1}^N$ with respective eigenvalues $\{\lambda_1 \geq \lambda_2 \geq \lambda_k > 0 = \lambda_{k+1} = \ldots = \lambda_N\}$. If N is even, we are done by Proposition 35. If N is odd, let $T_1 = \frac{1}{\lambda_k} T$. Then the smallest eigenvalue of T_1 equals 1. By Proposition 35, we can find projections $\{P_i\}_{i=1}^2$ so that

$$\frac{1}{\lambda_k}T = T_1 = P_1^*P_1 + P_2^*P_2.$$

Letting $v = \sqrt{\lambda_k}$ finishes the proof. □

It is important to note that, without weighting, we can always write every positive self-adoint T as the sum of $P_i^*P_i$ with three projections.

Corollary 37. *If $T : \mathcal{H}_N \to \mathbb{H}_N$ is a positive, self adjoint operator of rank $k > \frac{N}{2}$ whose nonzero eigenvalues are all greater than or equal to 1, then there are projections $\{P_i\}_{i=1}^3$ so that*

$$T = P_1^*P_1 + P_2^*P_2 + P_3^*P_3.$$

Proof. If N is even, we can write T as the sum of two projections. Suppose N is odd and let $\{e_j\}_{j=1}^N$ be an eigenbasis of T. Suppose $J_1 \cup J_2 \cup J_3 = \{1, ..., N\}$ with $|J_i| < \frac{N}{2}$ and let π_i be the orthogonal projection onto span$\{e_j : j \in J_i\}$ for $i = 1, 2, 3$. Then $T = T(\pi_1 + \pi_2 + \pi_3)$ and $T\pi_i$ satisfies Corollary 30 for each i. □

We note before leaving this section that the classification results may also be expanded a bit more to a set of self-adjoint operators T that are not necessarily positive.

Corollary 38. *Suppose $T = T_1 - T_2$ where T_1, T_2 are positive, self-adjoint operators. Then there are projections $\{P_i\}_{i=1}^4$ and weights $\{v_i\}_{i=1}^2$ so that*

$$T = v_1^2(P_1^*P_1 + P_2^*P_2) - v_2^2(P_3^*P_3 + P_4^*P_4).$$

3.3 Tight nonorthogonal fusion frames

In this section we address some issues regarding tight nonorthogonal fusion frames. The first theorem addresses the issue of which sets of dimensions allow the existence of a tight nonorthogonal fusion frame. The corresponding problem for fusion frames has received considerable attention and proven to be quite difficult, see [64], [33], and [18].

Theorem 39. *Suppose $n_1 + \cdots + n_M \geq N$, $n_i \leq \frac{N}{2}$. Then there exists a tight nonorthogonal fusion frame $\{P_i\}_{i=1}^M$ ($v_i = 1$ for every i) for $\mathcal{H}_N$ such that rank$(P_i) = n_i$ for $i = 1, ..., M$.*

Proof. Choose an orthonormal basis $\{e_j\}_{j=1}^N$ for $\mathcal{H}_N$ and choose a collection of subspaces $\{W_i\}_{i=1}^M$ such that:

1) $W_i = \text{span}\{e_j\}_{j\in J_i}$ with $|J_i| = n_i$ for each $i = 1, ..., M$, and

2) $W_1 + \cdots + W_M = \mathcal{H}_N$.

Let π_i be the orthogonal projection onto W_i and let $S = \sum_{i=1}^M \pi_i$. Observe that $I_N = S^{-1}S = \sum_{i=1}^M S^{-1}\pi_i$. Since each π_i is diagonal with respect to $\{e_j\}_{j=1}^N$ it follows that S^{-1} commutes with π_i, so $S^{-1}\pi_i$ is positive and self adjoint for every $i = 1, ..., M$. Let γ be the smallest nonzero eigenvalue of any $S^{-1}\pi_i$, then $\frac{1}{\gamma}S^{-1}\pi_i$ satisfies the hypotheses of Corollary 30 so there is a projection P_i so that $P_i^*P_i = \frac{1}{\gamma}S^{-1}\pi_i$, and we have

$$\sum_{i=1}^M P_i^*P_i = \frac{1}{\gamma}I_N.$$

□

Theorem 39 should be compared with Theorem 3.2.2 in [64]. Also note that the proof of Theorem 39 is constructive, cf [33]. The next theorem deals with adding projections to a given nonorthogonal fusion frame it order to get a tight nonorthogonal fusion frame. Somewhat surprisingly, this can always be achieved with only two projections.

Theorem 40. *Let $\{P_i\}_{i=1}^M$ be projections on $\mathcal{H}_N$, $N \geq 2$. Then there are two projections $\{P_i\}_{i=M+1}^{M+2}$ and a λ so that*

$$\sum_{i=1}^{M+2} P_i^* P_i = \lambda I_N.$$

Proof. Let

$$S = \sum_{i=1}^{M} P_i^* P_i,$$

and let $\lambda = \lambda_1 + 1$ where λ_1 is the biggest eigenvalue of S. Set

$$T = \lambda I_N - S.$$

Then T is a positive self-adjoint operator with all of its eigenvalues greater than or equal to 1 and at least one eigenvalue equal to one. By Proposition 35, we can find projections $\{P_i\}_{i=M+1}^{M+2}$ so that

$$T = P_{M+1}^* P_{M+1} + P_{M+2}^* P_{M+2}.$$

Thus,

$$\lambda I_N = S + T = \sum_{i=1}^{M+2} P_i^* P_i.$$

□

No such theorem exists for frames or regular (orthogonal) fusion frames. In general we need to add $n-1$ vectors to a frame in $\mathcal{H}_N$ in order to get a tight frame (see Proposition 2.1 in [42]). However, in this context Theorem 40 may be misleading, as the ranks of the projections we need to add could be quite large. The next result tells us how to deal with the case where we want small rank projections.

Proposition 41. *If $\{P_i\}_{i=1}^M$ are projections on $\mathcal{H}_N$ and $k \leq \frac{N}{2}$, there are projections $\{Q_i\}_{i=1}^L$ with $L = \lceil \frac{N}{k} \rceil$ and rank$(Q_i) \leq k$, and a λ so that*

$$\sum_{i=1}^M P_i^* P_i + \sum_{j=1}^L Q_i^* Q_i = \lambda I_N.$$

Proof. Let $S = \sum_{i=1}^M P_i^* P_i$ and assume S has eigenvectors $\{e_j\}_{j=1}^N$ with respective eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$. Partition the set $\{1, ..., N\}$ into sets $J_1, ..., J_L$ with $|J_\ell| \leq k$ for every $\ell = 1, ..., L$. Let π_ℓ denote the orthogonal projection onto span$\{e_j\}_{j \in J_\ell}$. Set $\lambda = \lambda_1 + 1$ and let $T_\ell = (\lambda I - S)\pi_\ell$. Then each T_ℓ satisfies the hypotheses of Corollary 30 so choose any projection $Q_\ell \in \Omega(T_\ell)$. Now we have that

$$\begin{aligned} \sum_{i=1}^M P_i^* P_i + \sum_{\ell=1}^L Q_\ell^* Q_\ell &= S + \sum_{\ell=1}^L T_\ell \\ &= S + \lambda I_N - S = \lambda I_N. \end{aligned}$$

$\square$

3.3.1 2 projections

As an application of the results of the previous section we will give a complete description of when there are two projections $P_i : \mathcal{H}_N \to \mathcal{H}_N$, $i = 1, 2$ such that

$$P_1^* P_1 + P_2^* P_2 = \lambda I_N. \tag{3.2}$$

Let $W_1 = \text{im}(P_1), W_1^* = \text{im}(P_1^*), W_2 = \text{im}(P_2), W_2^* = \text{im}(P_2^*)$. We will examine this in several cases but first we make some general remarks. Note that if $x \in W_1^*$ such that $P_1^* P_1 x = \alpha x$ (for $\alpha \in \mathbb{R}$) then $P_2^* P_2 x = (\lambda - \alpha)x$, so there is an orthonormal bases $\{e_j\}_{j=1}^n$ consisting of eigenvectors of both $P_1^* P_1$ and $P_2^* P_2$. Furthermore, if $P_1^* P_1 x = 0$ then $P_2^* P_2 x = \lambda x$, so $\ker P_1 = W_1^{*\perp} \subseteq W_2^*$, and similarly $W_2^{*\perp} \subseteq W_1^*$.

It follows from (3.2) that $\text{rank}(P_1) + \text{rank}(P_2) \geq N$. We will examine the cases of equality and strict inequality separately.

Proposition 42. *Suppose P_1 and P_2 are projections on $\mathcal{H}_N$ such that $P_1^* P_1 + P_2^* P_2 = \lambda I_N$ and that rank(P_1) + rank$(P_2) = N$. Then either rank$(P_1) \neq$ rank(P_2) and $\lambda = 1$ or rank$(P_1) =$ rank$(P_2) = \frac{N}{2}$ and $\lambda \geq 1$.*

Proof. First suppose without loss of generality that $\text{rank}(P_1) = k > \text{rank}(P_2)$. In this case we have that $k > \frac{N}{2}$, so $\dim(W_1 \cap W_1^*) \geq 2k - N > 0$. Then by Proposition 28 we have that $P_1 = P_1' + \pi_{W_1 \cap W_1^*}$ and $P_1^* P_1 + P_2^* P_2 = P_1'^* P_1' + \pi_{W_1 \cap W_1^*} + P_2^* P_2$. If $x \in W_1 \cap W_1^*$, then $P'x = 0$, and since $x \notin W_2^*$ it follows that $P_2 x = 0$. Therefore $(P_1^* P_1 + P_2^* P_2)x = x$ which means $\lambda = 1$, both P_1 and P_2 are orthogonal projections, and $W_j^* = W_j$ $j = 1, 2$.

Now suppose that N is even, and $\dim(W_1) = \dim(W_2) = \frac{N}{2}$. In this case we have that $W_1^* = W_2^{*\perp}$, so it follows immediately that $P_1^* P_1 = \lambda \pi_{W_1^*}$ and $P_2^* P_2 = \lambda \pi_{W_2^*}$. $\square$

Proposition 43. *Suppose P_1 and P_2 are projections on $\mathcal{H}_N$ such that $P_1^*P_1 + P_2^*P_2 = \lambda I_N$ and that rank(P_1) + rank$(P_2) > N$. Then rank(P_1) = rank(P_2), $\lambda = 2$, and $W_1^* \cap W_1 = W_2^* \cap W_2$.*

Proof. First suppose $\dim(W_1) = k > \ell = \dim(W_2)$. Note that $k > \frac{N}{2}$. By the remarks above we know that 0 must be an eigenvalue of $P_1^*P_1$ with multiplicity $N-k$, λ must be an eigenvalue of $P_1^*P_1$ with multiplicity $N-\ell$, and 1 must be an eigenvalue of $P_1^*P_1$ with multiplicity $\dim(W_1^* \cap W_2^*) \geq 2k - N$. Adding up these multiplicities we get $(N-k) + (N-\ell) + (2k-N) = N + k - \ell = N$ which contradicts the fact that $k > \ell$. Therefore, we may assume that $\dim(W_1) = \dim(W_2)$.

By the remarks above we can choose an orthonormal basis $\{e_j\}_{j=1}^N$ of $\mathcal{H}_N$ so that

$$\begin{aligned} P_1^*P_1e_j &= \lambda e_j \text{ and } P_2^*P_2e_j = 0 \text{ for } j = 1, ..., N-k, \\ P_1^*P_1e_j &= 0 \text{ and } P_2^*P_2e_j = \lambda e_i \text{ for } j = k+1, ..., N. \end{aligned}$$

Since $\dim(W_1 \cap W_1^*), \dim *(W_2 \cap W_2^*) \geq 2k - N$ it follows that

$$P_1^*P_1e_j = e_j = P_2^*P_2e_j \text{ for } j = N-k+1, ..., k.$$

Therefore $\lambda = 2$ and $W_1 \cap W_1^* = W_2 \cap W_2^*$. □

Note that in this case it is possible to have $W_1 = W_2$, *i.e.*, there are tight nonorthogonal fusion frames consisting of two different projections onto the same subspace.

Ideally we would like analogous theorems for any number of projections, but this seems to be quite a difficult problem.

Chapter 4

Dual frames and projections

4.1 Introduction

Given two frames Φ and Ψ we define their *cross gramian* $G_{\Phi,\Psi} = \Phi^*\Psi = [\langle\varphi_i, \psi_j\rangle]$. Note that $G_{\Phi,\Psi}$ is a projection if and only if Φ and Ψ are dual frames. Indeed, $\Phi^*\Psi = \Phi^*\Psi\Phi^*\Psi$ if and only i $\Psi\Phi^* = I_N$. Also note that in this case $G_{\Phi,\Psi} = G_\Phi G_\Psi$. Now observe that $\mathrm{im}(G_{\Phi,\Psi}) = \mathrm{im}(\Phi^*)$ and $\ker(G_{\Phi,\Psi}) = \ker(\Psi)$, so $G_{\Phi,\Psi}$ is an orthogonal projection if and only if $\mathrm{im}(\Phi^*) = \ker(\Psi)^\perp = \mathrm{im}(\Psi^*)$, *i.e.*, if and only if Φ and Ψ are isomorphic, which is further equivalent to the fact that Φ and Ψ are canonic duals.

In this chapter we will explore this relationship between projections and pairs of dual frames. This chapter is organized as follows: In section 4.2 we make this relationship precise by establishing a bijection between projections and isomorphism classes of dual frame pairs. In section 4.3 we give a classification of when two tight frames can be dual to each other. In section 4.4 we define a notion of Naimark complement for dual pairs and establish some basic properties. Finally, in section 4.5 we establish how the properties of dual pairs change under small perturbations of the corresponding projection. The work in this chapter is still a work in progress, [24].

4.2 Subspace characterization of the set of all duals

Theorem 44. *Let $\{\varphi_i\}_{i=1}^M$ be a frame. Let $X = \mathrm{im}(\Phi^*) \subset \mathcal{H}_M$, and let $Y \subset \mathcal{H}_M$ be any subspace such that $\dim Y = N$ and $Y \cap X^\perp = \{0\}$. Then there is a (unique) dual frame $\{\psi_i\}_{i=1}^M$ such that $\mathrm{im}(\Psi^*) = Y$.*

Proof. Let $\Gamma \in M_{N\times M}$ be any matrix whose rows y_j form a basis for Y, and call the ith column γ_i. Define $T : \mathbb{K}^N \to \mathbb{K}^N$ by $T = \Gamma\Phi^*$. Since $Y \cap X^\perp = \{0\}$ it follows that $\Gamma|_X$ is invertible, and since the columns of Φ^* are linearly independent, it follows that the columns of T are linearly independent and hence T is invertible. Therefore, we have that $T^{-1}\Gamma\Phi^* = I_N$ which means that $\{T^{-1}\gamma_i\}_{i=1}^M$ is a dual frame to $\{\varphi_i\}_{i=1}^M$. Since $\mathrm{im}(\Gamma^* T^{*-1}) = \mathrm{span}\,\{y_j\}_{j=1}^N$, the proof is complete. $\square$

Remark 45. *Since the Grassmanian $\mathcal{G}_{N,M}$ of N-dimensional subspaces in $\mathcal{H}_M$ is $(M-N)N$-dimensional, we see (again) that the set of duals is $(M-N)N$-dimensional.*

If (Φ, Ψ) is a dual frame pair, then so is $(T\Phi, (T^*)^{-1}\Psi)$ for all $T \in GL_N$ and the range of the analysis operators are the same, i.e., the associated projections are the same. Hence, if we mod out the action of GL_N on dual pairs of frames (Φ, Ψ) associated with the two subspaces $X = \mathrm{im}(\Phi^*)$ and $Y = \mathrm{im}(\Psi^*)$, then there is a one-to-one correspondence between isomorphism classes of dual frame pairs and projections.

4.3 Projections associated with two tight frames

Recall that the non-zero singular values of a projection P satisfies $\sigma_j \geq 1$ for $j = 1, \ldots, \operatorname{rank} P$, and the projection P is orthogonal if and if $\sigma_j = 1$ for $i = 1, \ldots, \operatorname{rank} P$. The following theorems classifies when two tight frames can form a dual pair.

Theorem 46. *Let $\sigma \geq 1$. Suppose that Φ and Ψ are dual frames of M vectors for $\mathcal{H}_N$ and that Φ is Parseval. Let $P = \Phi^*\Psi$ be the associated projection of rank N. Then the nonzero singular values of the $P = \Phi^*\Psi$ are constant and equal to σ if and only if Ψ is σ^2-tight and. In this case it is possible to have $\sigma > 1$ only when $M \geq 2N$. If $\sigma > 1$, then Φ and Ψ are non-canonical tight duals.*

Proof. Suppose all nonzero singular values of $P = \Phi^*\Psi$ are equal to σ. Let $P = U\Sigma V^*$ be a singular value decomposition with $\Sigma = [\sigma_{i,j}]$ and $\sigma_{i,i} = \sigma$ for $i = 1, \ldots, N$ and $\sigma_{i,j} = 0$ otherwise. Let u_j and v_i be the jth and ith column of the complex conjugated unitaries $\overline{U}$ and $\overline{V}$, respectively. Then

$$P = U\Sigma V^* = \sigma \begin{bmatrix} | & & | \\ \overline{u_1} & \cdots & \overline{u_N} \\ | & & | \end{bmatrix} \begin{bmatrix} — & v_1 & — \\ & \vdots & \\ — & v_N & — \end{bmatrix}.$$

Now define

$$\Phi = \begin{bmatrix} — & u_1 & — \\ & \vdots & \\ — & u_N & — \end{bmatrix} \quad \text{and} \quad \Psi = \sigma \begin{bmatrix} — & v_1 & — \\ & \vdots & \\ — & v_N & — \end{bmatrix}.$$

Since $\{v_1, \ldots v_N\}$ are orthonormal vectors, we have that $\Psi\Psi^* = \sigma^2 I_N$ hence Ψ is a σ^2-tight frame. They are dual frames since $P = \Phi^*\Psi$ is a projection of rank N.

On the other hand, if Φ and Ψ satisfy our hypotheses then we have:

$$\Phi = \begin{bmatrix} — & u_1 & — \\ & \vdots & \\ — & u_N & — \end{bmatrix} \quad \text{and} \quad \Psi = \sigma \begin{bmatrix} — & v_1 & — \\ & \vdots & \\ — & v_N & — \end{bmatrix},$$

where $\{u_1, \dots u_N\}$ and $\{v_1, \dots v_N\}$ are orthonormal vectors. Since $P = \Phi^*\Psi = U\Sigma V^*$ it follows by uniqueness of the singular values that the diagonal terms of Σ are $\sigma_{i,i} = \sigma$ for $i = 1, \dots, N$. $\square$

In terms of principal angles between $\operatorname{im}(P)$ and $\operatorname{im}(P^*)$, Theorem 46 says that dual pairs of tight frames correspond to isoclinic subspaces.

Note that Theorem 46 shows that a Parseval frame cannot have a non-canonical Parseval dual. Also note that even when the singular values of the projection $P = \Phi^*\Psi$ are not constant we can always make one of the dual frames Parseval. This follows from the factorization:

$$P = U\Sigma V^* = \begin{bmatrix} | & & | \\ \overline{u_1} & \cdots & \overline{u_N} \\ | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \cdots & & & 0 \\ 0 & \ddots & 0 & \cdots & & \vdots \\ \vdots & & \sigma_N & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} — & v_1 & — \\ — & v_2 & — \\ & \vdots & \\ — & v_M & — \end{bmatrix}$$
$$= \begin{bmatrix} | & & | \\ \sigma_1\overline{u_1} & \cdots & \sigma_n\overline{u_N} \\ | & & | \end{bmatrix} \begin{bmatrix} — & v_1 & — \\ & \vdots & \\ — & v_N & — \end{bmatrix} =: \Phi^*\Psi$$

4.4 Naimark complements

Recall that given a Parseval frame Φ_1 for $\mathcal{H}_N$ a Naimark complement is a Parseval frame Φ_2 for $\mathcal{H}_{M-N}$ such that the matrix

$$\Phi = \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix}$$

becomes a unitary $M \times M$ matrix. Using Naimark complements we can parametrize the set of all duals of a Parseval frame in a particularly nice way.

Proposition 47. *Suppose that Φ_1 and Φ_2 are a pair of Naimark complement Parseval frames for $\mathcal{H}_N$ and $\mathcal{H}_{M-N}$ respectively. Then Ψ_1 is a dual frame for Φ_1 if and only if there exists some $N \times (M-N)$ matrix T such that $\Psi_1 = \Phi_1 + T\Phi_2$.*

Proof. First suppose $\Psi_1 = \Phi_1 + T\Phi_2$. We need to show that $\Phi_1^*\Psi_1 = \Phi_1^*\Phi_1 + \Phi_1^*T\Phi_2$ is a projection. To do this just compute

$$(\Phi_1^*\Phi_1 + \Phi_1^*T\Phi_2)^2 = \Phi_1^*\Phi_1\Phi_1^*\Phi_1 + \Phi_1^*\Phi_1\Phi_1^*T\Phi_2 + \Phi_1^*T\Phi_2\Phi_1^*\Phi_1 + \Phi_1^*T\Phi_2\Phi_1^*T\Phi_2.$$

Now apply the facts that $\Phi_1^*\Phi_1$ is a projection, $\Phi_1\Phi_1^* = I_N$, and $\Phi_2\Phi_1^* = 0$.

For the other direction suppose that Ψ_1 is a dual for Φ_1, and let $\Phi = \left[\begin{smallmatrix} \Phi_1 \\ \Phi_2 \end{smallmatrix}\right]$. Note that $\Psi_1 = \Psi_1\Phi^*\Phi$ since $\Phi^*\Phi = I_M$. Therefore

$$\begin{aligned} I_N &= \Phi_1\Psi_1^* \\ &= \Phi_1(\Psi_1\Phi^*\Phi)^* \\ &= (\Phi_1\Phi^*)(\Phi\Psi_1^*) \\ &= [I_N\ 0_{N\times(M-N)}]\Phi\Psi_1^*. \end{aligned}$$

This implies that $\Psi_1\Phi^* = [I_N\ T]$ for some $N \times (M-N)$ matrix T which further implies (be multiplying on the left by Φ) that $\Psi_1 = \Phi_1 + T\Phi_2$. □

Note that in the above proof we uncovered the following fact which we have no use for at the moment, but we will formally state because we find it interesting:

Proposition 48. *If P be a projection, then we can write $P = \pi + Q$ where π is the orthogonal projection onto $\mathrm{im}(P)$ and $Q^2 = 0$.*

By anology with the case of Parseval frames, we make the following definition of Naimark complements for dual pairs:

Definition 49. *Suppose (Φ_1, Ψ_1) is a pair of dual frames for $\mathcal{H}_N$ and (Φ_2, Ψ_2) is a pair of dual frames for $\mathcal{H}_{M-N}$. We say these two pairs are Naimark complements if $G_{\Phi_1,\Psi_1} = I_M - G_{\Psi_2,\Phi_2}$.*

The definition is stated the way that it is in order to make the following proposition hold:

Proposition 50. *Suppose (Φ_1, Ψ_1) is a pair of dual frames for $\mathcal{H}_N$ and (Φ_2, Ψ_2) is a pair of dual frames for $\mathcal{H}_{M-N}$ and that Φ_1 and Φ_2 are Parseval frames. Then*

(Φ_1, Ψ_1) and (Φ_2, Ψ_2) are Naimark complement dual pairs if and only if Φ_1 and Φ_2 are Naimark complement Parseval frames.

We have the following classification of Naimark complement dual pairs in terms of Naimark complemenet Parseval frames:

Proposition 51. *Suppose Φ_1 and Φ_2 are Naimark complement Parseval frames and that $\Psi_1 = \Phi_1 + T\Phi_2$ is a dual for Φ_1 and $\Psi_2 = \Phi_2 + R\Phi_1$ is a dual for Φ_2. Then (Φ_1, Ψ_1) and (Φ_2, Ψ_2) are Naimark complement dual pairs if and only if $R = -T^*$.*

Proof. We have

$$\begin{aligned} G_{\Phi_1,\Psi_1} + G_{\Psi_2,\Phi_2} &= \Phi_1^*\Phi_1 + \Phi_1^* T\Phi_2 + \Phi_2^*\Phi_2 + \Phi_1^* R^*\Phi_2 \\ &= I_M + \Phi_1^*(T + R^*)\Phi_2. \end{aligned}$$

□

4.4.1 Linear independence and spanning

Lemma 52. *Fix an orthonormal basis $\{e_i\}_{i=1}^M$ for $\mathcal{H}_M$ and for any $I \subseteq \{1, ..., M\}$ let π_I denote the orthogonal projection from $\mathcal{H}_M$ onto span$\{e_i\}_{i\in I}$, and let T be any $M \times M$ matrix. Then $\operatorname{rank}(\pi_I T\pi_I) = |I|$ if and only if $\ker(T) \cap \operatorname{im}(\pi_I) = \{0\}$ and $\operatorname{im}(T) \cap \ker(\pi_I) = \{0\}$.*

Proof. First note that $\operatorname{rank}(\pi_I T\pi_I) = |I|$ if and only if $\pi_I T\pi_I$ acts as an invertible operator on $\operatorname{im}(\pi_I)$. Therefore, if there is a $0 \neq x \in \ker(T) \cap \operatorname{im}(\pi_I)$ then $x \in \ker(\pi_I T\pi_I)$ so $\operatorname{rank}(\pi_I T\pi_I) < |I|$.

Now suppose $\operatorname{rank}(\pi_I T \pi_I) < |I|$. That is, there exists a nonzero $x \in \operatorname{im}(\pi_I)$ such that $\pi_I T \pi_I x = 0$. Since $\pi_I x = x$ we have that $\pi_I T x = 0$. This means that either $Tx = 0$ in which case $x \in \ker(T) \cap \operatorname{im}(\pi_I)$, or $Tx \in \ker(\pi_I)$ but by definition $Tx \in \operatorname{im}(T)$. □

Theorem 53. *Let P be an $M \times M$ projection of rank N and suppose $I \subseteq \{1, ..., M\}$ and $|I| = N$. Then*

$$\operatorname{rank}(\pi_I P \pi_I) = N \Leftrightarrow \operatorname{rank}(\pi_{I^c}(I_M - P^*)\pi_{I^c}) = M - N.$$

Proof. Suppose $\operatorname{rank}(\pi_I P \pi_I) = N$. Then by Lemma 52

$$\begin{aligned} \ker(P) \cap \operatorname{im}(\pi_I) = \{0\} \quad &\Leftrightarrow \quad \ker(P)^{\perp} \cap \operatorname{im}(\pi_I)^{\perp} = \{0\} \\ &\Leftrightarrow \quad \operatorname{im}(I_M - P^*) \cap \ker(\pi_{I^c}) = \{0\}, \end{aligned}$$

where we have used that $\ker(P)$ and $\operatorname{im}(\pi_I)$ are complementary subspaces and Proposition 4.

Similarly,

$$\begin{aligned} \operatorname{im}(P) \cap \ker(\pi_I) = \{0\} \quad &\Leftrightarrow \quad \operatorname{im}(P)^{\perp} \cap \ker(\pi_I)^{\perp} = \{0\} \\ &\Leftrightarrow \quad \ker(I_M - P^*) \cap \operatorname{im}(\pi_{I^c}) = \{0\}. \end{aligned}$$

Now since $|I^c| = M - N$ we can apply Lemma 52 to see that $\operatorname{rank}(\pi_{I^c}(I_M - P^*)\pi_{I^c}) = M - N$. The other direction follows by symmetry. □

Corollary 54. *Suppose $\Phi_1 = \{\varphi_i^1\}_{i=1}^M \subseteq \mathcal{H}_N$ and $\Phi_2 = \{\varphi_i^2\}_{i=1}^M \subseteq \mathcal{H}_{(M-N)}$ are Naimark complement Parseval frames, and let $I \subseteq \{1, ..., M\}$. Then $\{\varphi_i^1\}_{i\in I}$ is lin-*

early independent if and only if $\mathrm{span}\{\varphi_i^2\}_{i\in I^c} = \mathcal{H}_{M-N}$.

Theorem 53 should say something about Naimark complement dual pairs which is analogous to what Corollary 54 says for Naimark complement Parseval frames, but we still have to decipher exactly what it means.

4.5 Perturbation results

Definition 55. *A projection P is said to be ϵ-nearly orthogonal if*

$$\|P-\pi\|_F^2 < \epsilon,$$

where π is the orthogonal projection onto $\mathrm{im}(P)$.

Proposition 56. *Let P be projection of rank N. P is ϵ-nearly orthogonal if and only if*

$$\|P\|_F^2 < N + \epsilon.$$

Proof. First note that

$$\begin{aligned}\langle P-\pi, \pi\rangle &= \mathrm{tr}\,(P-\pi)\pi \\ &= \mathrm{tr}\,(P\pi - \pi^2) \\ &= \mathrm{tr}\,(\pi - \pi) \\ &= 0.\end{aligned}$$

Therefore we have

$$\|P\|_F^2 = \|P - \pi\|_F^2 + \|\pi\|_F^2 = \|P - \pi\|_F^2 + N.$$

$\square$

Proposition 57. *Let P be a projection of rank N and let π be the orthogonal projection onto* $\operatorname{im}(P)$. *Let $\{\theta_i\}_{i=1}^N$ be the principal angles between* $\ker(P)$ *and* $\ker(\pi)$. *Then*

$$\|P - \pi\|_F^2 = \sum_{i=1}^{N} \tan^2 \theta_i.$$

Proof. As in the proof of Proposition 56 we have

$$\begin{aligned}
\|P - \pi\|_F^2 &= \|P\|_F^2 - N \\
&= \left(\sum_{i=1}^{N} \frac{1}{\cos^2 \theta_i} \right) - N \\
&= \sum_{i=1}^{N} \frac{1 - \cos^2 \theta_i}{\cos^2 \theta_i} \\
&= \sum_{i=1}^{N} \tan^2 \theta_i.
\end{aligned}$$

$\square$

Definition 58. *A frame Φ is said to be ϵ-nearly Parseval if*

$$\|I_N - \Phi\Phi^*\|_F < \epsilon.$$

Theorem 59. *Suppose Φ and Ψ are dual frames in $\mathcal{H}_N$ and that Φ is Parseval. Then $\Phi^*\Psi$ is ϵ-nearly orthogonal if and only if Ψ is ϵ-nearly Parseval.*

Proof. Write $\Psi = \Phi + \Gamma$ with $\Phi\Gamma^* = \Gamma\Phi^* = 0_N$, where 0_N is the $N \times N$ zero matrix. Then

$$\Phi^*\Psi = \Phi^*(\Phi + \Gamma) = \Phi^*\Phi + \Phi^*\Gamma \tag{4.1}$$

and

$$\Psi\Psi^* = (\Phi + \Gamma)(\Phi + \Gamma)^* = E + \Gamma\Gamma^*, \tag{4.2}$$

and so, using (4.1),

$$\|\Phi^*\Psi - \Phi^*\Phi\|_F^2 = \|\Phi^*\Gamma\|_F^2 = \operatorname{tr}(\Gamma^*\Phi\Phi^*\Gamma) = \operatorname{tr}(\Gamma^*\Gamma) = \operatorname{tr}(\Gamma\Gamma^*) \tag{4.3}$$

Assume now that $\Phi^*\Psi$ is ϵ-nearly orthogonal. Note that $\pi = \Phi^*\Phi$ since $\operatorname{im}(\pi) = \operatorname{im}(\Phi^*)$. From our assumption and (4.3), we see that $\operatorname{tr}(\Gamma\Gamma^*) < \varepsilon$. By (4.2) it follows that

$$\|I_N - \Psi\Psi^*\|_F^2 = \|\Gamma\Gamma^*\|_F^2 = \operatorname{tr}(\Gamma\Gamma^*)^2 < \epsilon^2.$$

which shows that Ψ is ϵ-nearly Parseval.

Reversing the order of the arguments yields the other direction. $\square$

Note that we actually see that $I_N \leq \Psi\Psi^*$ whenever Ψ is dual to a Parseval frame with equality only when Ψ is the canonical dual.

Chapter 5

Other operators and frames

5.1 Introduction

Letting $\{\varphi_i\}_{i=1}^M$ be a frame for $\mathcal{H}_N$ with frame operator S, and letting T be an invertible operator on $\mathcal{H}_N$, a fundamental, but poorly understood, question in frame theory is the following: How do the frame properties of $\{T\varphi_i\}_{i=1}^M$ relate to the frame properties of $\{\varphi_i\}_{i=1}^M$? Deserving particular attention are the following subproblems due to their relevance for theoretical as well as practical applicability of the generated frames.

- *Frame operator.* Often a particular frame operator is desired, which yields the question: Is it possible to classify the operators T under which the frame operator is invariant? More generally, can we classify those invertible operators such that the generated frame has a prescribed frame operator?

- *Norm.* It is essential to have a means to control the norms of the generated frame elements due to, for instance, numerical stability issues. This raises the following question: Can we classify those invertible operators which map a frame to another frame which is equal norm? Another variant is the classification problem of all invertible operators T for which $\|\varphi_i\| = c\|T\varphi_i\|$ for all $1 \leq i \leq M$.

- *Parseval frame.* Parseval frames are crucial for applications, since from a numerical standpoint, they are optimally stable. Also for theoretical purposes those are the most useful frames to utilize, for instance, for the decomposition of mathematical objects. Often, however it is not possible to construct an 'exact' Parseval frame, which leads to the problem of deriving a deep understanding of operators which map a frame to a nearly Parseval frame.

In this chapter, we will provide comprehensive answers to these questions, which are in fact long-standing open problems or are close to such, giving proof to both their significance and their difficulty.

An answer to the first question will presumably – as we will indeed see – provide a way to index frames which possess the same frame operator. This in fact solves a problem which has been debated in frame theory for several years, see, for instance, [15].

The second group of problems has been discussed at meetings for many years but has not formally been stated in the literature. These are fundamental questions which often arise when one is trying to improve frame properties by applying an invertible operator to the frame.

Relating to the third problem, despite the significance of equal norm Parseval frames for applications, their class is one of the least understood classes of frames. The reason is that for any frame $\{\varphi_i\}_{i=1}^M$ with frame operator S, the closest Parseval frame is the canonical Parseval frame $\{S^{-1/2}\varphi_i\}_{i=1}^M$, which is rarely equal norm. Because of the difficulty of finding equal norm Parseval frames, the famous Paulsen Problem is still open to date. We refer the reader to [14, 21, 31] for some recent results on this problem. The question we deal with in this paper is closely related to the Paulsen Problem, and we anticipate our answer to provide a new direction of attack.

This chapter is organized as follows. In Section 5.2, we first classify those operators which leave the frame operator invariant. A characterization of operators mapping frames to frames with comparable norms of the frame vectors is provided in Section 5.3. Section 5.4 is then devoted to the study of operators which generate nearly Parseval frames. The work in this chapter can be found in [23].

5.2 Prescribed Frame Operators

5.2.1 Invariance of the Frame Operator

The first question in this section to tackle is the characterization of invertible operators which leave the frame operator invariant. More precisely, given a frame $\{\varphi_i\}_{i=1}^M$ for a Hilbert space $\mathcal{H}_N$ with frame operator S, we aim to classify the invertible operators T on $\mathcal{H}_N$ for which the frame operator for $\{T\varphi_i\}_{i=1}^M$ equals S. We remark that not even unitary operators possess this property, the reason being that although a unitary operator applied to a frame will maintain the eigenvalues of the frame operator, it will however in general not maintain the eigenvectors.

The mathematical exact formulation of this question is the following:

Question 1. *Given a frame $\{\varphi_i\}_{i=1}^M$ with frame operator S, can we classify the invertible operators T so that the frame operator of the generated frame $\{T\varphi_i\}_{i=1}^M$ equals S?*

We start with a well known result identifying the frame operator of the frame $\{T\varphi_i\}_{i=1}^M$. Since the proof is just one line, we include it for completeness.

Theorem 60. *If $\{\varphi_i\}_{i=1}^M$ is a frame for $\mathcal{H}_N$ with frame operator S, and T is an operator on $\mathcal{H}_N$, then the frame operator for $\{T\varphi_i\}_{i=1}^M$ equals TST^*. If T is invertible, then $\{T\varphi_i\}_{i=1}^M$ also constitutes a frame for $\mathcal{H}_N$.*

Proof. The claim follows from the fact that the frame operator for $\{T\varphi_i\}_{i=1}^M$ is given by

$$\sum_{i=1}^M \langle f, T\varphi_i\rangle T\varphi_i \;=\; T\left(\sum_{i=1}^M \langle T^* f, \varphi_i\rangle \varphi_i\right) = TST^* f. \quad \square$$

This leads to the following reformulation of Question 1:

Question 2. *Given a positive, self-adjoint, invertible operator S, can we classify the invertible operators T for which $TST^* = S$?*

For the case of Parseval frames, the answer is well-known [35]. Since known proofs were highly non-trivial, we provide a trivial proof, which seems to have been overlooked in previous publications.

Corollary 61. *If $\{\varphi_i\}_{i=1}^M$ and $\{\psi_i\}_{i=1}^M$ are isomorphic Parseval frames, then they are unitarily equivalent.*

In particular, in the case of Parseval frames, the desired set of operators in Question 1 are the unitary operators.

Proof. Since $\{\varphi_i\}_{i=1}^M$ and $\{\psi_i\}_{i=1}^M$ are isomorhic, there exists an invertible operator T so that $\varphi_i = T\psi_i$ for every $i = 1, 2, ..., M$. By Theorem 60, the fact that both frames constitute Parseval frames implies that $T\, Id\, T^* = Id$. $\square$

5.2.2 General Characterization Result

Instead of directly providing an answer to Question 1 for any frame, we will now first state the generalization of this question whose answer will then include the solution to this problem. For this, we will start with the following consequence of Theorem 60.

Corollary 62. *Let $\{\varphi_i\}_{i=1}^M$ and $\{\psi_i\}_{i=1}^M$ be frames for $\mathcal{H}_N$ with frame operators S_1 and S_2, respectively. Then there exists an invertible operator T on $\mathcal{H}_N$ such that S_1 is the frame operator of $\{T\psi_i\}_{i=1}^M$.*

Proof. Letting $T = S_1^{1/2} S_2^{-1/2}$, by Theorem 60, we obtain

$$TS_2T^* = (S_1^{1/2} S_2^{-1/2}) S_2 (S_1^{1/2} S_2^{-1/2})^* = S_1. \qquad \square$$

Hence asking for the generation of frames with a prescribed frame operator can be formulated in terms of operator theory as the following generalization of Question 2 shows.

Question 3. *Given two positive, invertible, self-adjoint operators S_1 and S_2 on a Hilbert space $\mathcal{H}$, can we classify the invertible operators T on $\mathcal{H}$ for which $S_1 = TS_2T^*$?*

A first classification result answering Question 3 is the following. In this context, we mention that Condition (ii) shall also be compared with the choice of T in the proof of Corollary 62.

Theorem 63. *Let S_1 and S_2 be positive, self-adjoint, invertible operators on a Hilbert space $\mathcal{H}$, and let T be an invertible operator on $\mathcal{H}$. Then the following conditions are equivalent.*

(i) $S_2 = TS_1T^*$.

(ii) There exists a unitary operator U on $\mathcal{H}$ such that $T = S_2^{1/2}US_1^{-1/2}$.

Proof. (i) $\Rightarrow$ (ii). We set $U = S_2^{-1/2}TS_1^{1/2}$, which is a unitary operator, since

$$(S_2^{-1/2}TS_1^{1/2})(S_2^{-1/2}TS_1^{1/2})^* = S_2^{-1/2}TS_1T^*S_2^{-1/2} = S_2^{-1/2}S_2S_2^{-1/2} = I_N.$$

Moreover, we have

$$S_2^{1/2}US_1^{-1/2} = S_2^{1/2}(S_2^{-1/2}TS_1^{1/2})S_1^{-1/2} = T.$$

(ii) $\Rightarrow$ (i). Since $T = S_2^{1/2}US_1^{-1/2}$, we obtain

$$TS_1T^* = S_2^{1/2}US_1^{-1/2}S_1S_1^{-1/2}U^*S_2^{1/2} = S_2^{1/2}UU^*S_2^{1/2} = S_2^{1/2}\,I_N\,S_2^{1/2} = S_2,$$

which is (i). □

This result is however not entirely satisfactory, since one might prefer to have an explicit construction of all invertible operators T satisfying $S_1 = TS_2T^*$.

5.2.3 Constructive Classification

We start with some preparatory lemmas, the first being an easy criterion for identifying the eigenvectors of a positive, self-adjoint operator.

Lemma 64. *Let $T : \mathcal{H}_N \to \mathcal{H}_N$ be an invertible operator on $\mathcal{H}_N$, and let $\{e_j\}_{j=1}^N$ be an orthonormal basis for $\mathcal{H}_N$. Then the following conditions are equivalent.*

(i) $\{Te_j\}_{j=1}^N$ is an orthogonal set.

(ii) $\{e_j\}_{j=1}^N$ is an eigenbasis for T^*T with respective eigenvalues $\|Te_j\|^2$.

In particular, T must map some orthonormal basis to an orthogonal set.

Proof. For any $1 \leq j, k \leq N$, we have

$$\langle (T^*T)e_j, e_k\rangle = \langle Te_j, Te_k\rangle.$$

Hence, $\{Te_j\}_{j=1}^N$ is an orthogonal set if and only if $\langle (T^*T)e_j, e_k\rangle = 0$ for all $1 \leq j \neq k \leq N$. This in turn is equivalent to the condition that $T^*Te_j = \lambda_j e_j$ for all $j = 1, 2, \ldots, N$ with

$$\lambda_j = \langle (T^*T)e_j, e_j\rangle = \langle Te_j, Te_j\rangle = \|Te_j\|^2.$$

This is (ii), and the lemma is proved. □

In the language of frames, the next lemma will describe the impact of invertible operators on the eigenvalues and eigenvectors of a frame.

Lemma 65. *Let S_1, S_2 be positive, self-adjoint, invertible operators on $\mathcal{H}_N$, and let $\{e_j\}_{j=1}^N$ be an eigenbasis for S_1 with corresponding eigenvalues $\{\lambda_j\}_{j=1}^N$. Further, let T*

be an invertible operator on $\mathcal{H}_N$ satisfying $S_1 = T^* S_2 T$. Then the following conditions hold.

(i) $\{S_2^{1/2} T^* e_j\}_{j=1}^N$ *is an orthogonal set.*

(ii) $\|S_2^{1/2} T^* e_j\|^2 = \lambda_j$ *for all* $j = 1, \ldots, N$.

Proof. We have

$$\langle S_2^{1/2} T^* e_j, S_2^{1/2} T^* e_k\rangle = \langle T S_2 T^* e_j, e_k\rangle = \langle S_1 e_j, e_k\rangle = \begin{cases} 0 & \text{if } \; j \neq k \\ \lambda_j & \text{if } \; j = k. \end{cases}$$

The claims follow immediately. □

The property which is a crucial ingredient of Lemma 65 will be fundamental for the following characterization results. Hence we manifest a formal notation of it.

Definition 66. *Let $\mathcal{E} = \{e_j\}_{j=1}^N$ and $\mathcal{G} = \{g_j\}_{j=1}^N$ be orthonormal bases for $\mathcal{H}_N$, and let $\Lambda = \{\lambda_j\}_{j=1}^N$ and $\Gamma = \{\gamma_j\}_{j=1}^N$ be sequences of positive constants. An operator $T : \mathcal{H}_N \to \mathcal{H}_N$ is called* admissible for $(\mathcal{E}, \mathcal{G}, \Lambda, \Gamma)$, *if there exists some orthonormal basis $\{h_j\}_{j=1}^N$ for $\mathcal{H}_N$ satisfying*

$$T^* e_j = \sum_{k=1}^N \sqrt{\frac{\lambda_j}{\gamma_k}} \langle h_j, g_k\rangle g_k, \quad \textit{for all } j = 1, \ldots, N.$$

This notion now allows us to formulate the main classification theorem answering Question 3 in a constructive manner.

Theorem 67. *Let S_1 and S_2 be positive, self-adjoint, invertible operators on a Hilbert space $\mathcal{H}_N$, and let $\mathcal{E} = \{e_j\}_{j=1}^N$ and $\mathcal{G} = \{g_j\}_{j=1}^N$ be eigenvectors with eigenvalues*

$\Lambda = \{\lambda_j\}_{j=1}^N$ and $\Gamma = \{\gamma_j\}_{j=1}^N$ for S_1 and S_2, respectively. Further, let T be an invertible operator on $\mathcal{H}_N$. Then the following conditions are equivalent.

(i) $S_1 = TS_2T^*$.

(ii) T *is an admissible operator for* $(\mathcal{E}, \mathcal{G}, \Lambda, \Gamma)$.

Proof. (i) $\Rightarrow$ (ii). By Lemma 65, $\{S_2^{1/2}T^*e_j\}_{j=1}^N$ is an orthogonal set satisfying $\|S_2^{1/2}T^*e_j\|^2 = \lambda_j$ for all $j = 1, 2, \ldots, N$. Hence,

$$\{h_j\}_{j=1}^N := \left\{ \frac{1}{\sqrt{\lambda_j}} S_2^{1/2}T^*e_j \right\}_{j=1}^N$$

is an orthonormal set. Thus, for each $j = 1, 2, \ldots, N$,

$$\frac{1}{\sqrt{\lambda_j}} S_2^{1/2}T^*e_j = \sum_{k=1}^N \langle h_j, g_k\rangle g_k.$$

This implies

$$T^*e_j = \sqrt{\lambda_j} S_2^{-1/2} \sum_{j=1}^N \langle h_j, g_k\rangle g_k = \sum_{k=1}^N \sqrt{\frac{\lambda_j}{\gamma_k}} \langle h_j, g_k\rangle g_k.$$

Hence, we proved that T is admissible for $(\mathcal{E}, \mathcal{G}, \Lambda, \Gamma)$.

(ii) $\Rightarrow$ (i). Now assume that T is admissible for $(\mathcal{E}, \mathcal{G}, \Lambda, \Gamma)$, which implies that

$$S_2^{1/2}T^*e_j = S_2^{1/2} \sum_{k=1}^N \sqrt{\frac{\lambda_j}{\gamma_k}} \langle h_j, g_k\rangle g_j = \sum_{k=1}^N \sqrt{\gamma_k} \sqrt{\frac{\lambda_j}{\gamma_k}} \langle h_j, g_k\rangle g_k = \sqrt{\lambda_j} h_j.$$

Hence, $\{S_2^{1/2}T^*e_j\}_{j=1}^N$ is an orthogonal set, and, for all $j = 1, \ldots, N$, we have

$$\|S_2^{1/2}T^*e_j\|^2 = \lambda_j.$$

By Lemma 64, we obtain

$$(S_2^{1/2}T^*)^*(S_2^{1/2}T^*)e_j = \lambda_j e_j \quad \text{for all } j = 1, 2 \ldots, N,$$

which yields

$$S_1 = (S_2^{1/2}T^*)^*(S_2^{1/2}T^*) = (TS_2^{1/2})S_2^{1/2}T^* = TS_2T^*,$$

i.e., condition (i). □

We remark that this theorem provides a way to index frames which possess the same frame operator, solving a problem which has been debated in frame theory for several years, see, for instance, [15].

5.3 Prescribed Norms

We now focus on the second question, namely to derive a classification of all invertible operators which map frames to frames such that the norms of its frame elements are a fixed multiple of the norms of the original frame vectors. Formalizing, we face the following problem:

Question 4. *Given a constant $c > 0$ and a frame $\{\varphi_i\}_{i=1}^M$ for $\mathcal{H}_N$, can we classify the invertible operators $T : \mathcal{H}_N \to \mathcal{H}_N$ which satisfy $\|T\varphi_i\| = c\|\varphi_i\|$ for all $i =$*

$1, 2, \ldots, M$?

5.3.1 Main Classification Result

We first observe that without loss of generality we can assume that each frame vector is non-zero, since for a zero vector φ_i, say, the condition $\|T\varphi_i\| = c\|\varphi_i\|$ is trivially fulfilled. Furthermore, note that a solution to Question 4 for a particular $c > 0$ immediately implies a solution for any $c > 0$ just by multiplying the operators by an appropriate constant.

We start with a very simple lemma.

Lemma 68. *Let T be an invertible operator on $\mathcal{H}_N$, and let $f \in \mathcal{H}_N$. Then the following conditions are equivalent:*

(i) $\|Tf\|^2 = c^2\|f\|^2$.

(ii) $\langle (T^*T - c^2 I_N)f, f\rangle = 0$.

Proof. We have

$$\langle (T^*T - c^2 I_N)f, f\rangle = \langle T^*Tf, f\rangle - \langle c^2 f, f\rangle = \|Tf\|^2 - \|cf\|^2 = \|Tf\|^2 - c^2\|f\|^2.$$

The result is immediate from here. □

The following definition is required for our main result. It will give rise to a special class of vectors associated to a frame and an orthonormal basis.

Definition 69. *Let $\Phi = \{\varphi_i\}_{i=1}^M$ be a frame for $\mathcal{H}_N$, and let $\mathcal{E} = \{e_j\}_{j=1}^N$ be an* *orthonormal basis for $\mathcal{H}_N$. Then we define*

$$\mathcal{H}(\Phi, \mathcal{E}) = \operatorname{span}\{(|\langle \varphi_i, e_j \rangle|^2)_{j=1}^N\}_{i=1}^M \subset \mathcal{H}_N.$$

We can now state the main result of this section, which will subsequently solve Question 4 completely.

Theorem 70. *Let $\{\varphi_i\}_{i=1}^M$ be a frame for $\mathcal{H}_N$, and let $\{c_i\}_{i=1}^M$ be positive scalars. Further, let T be an invertible operator on $\mathcal{H}_N$, and let $\{e_j\}_{j=1}^N$ be the eigenvectors for T^*T with respective eigenvalues $\{\lambda_j\}_{j=1}^N$. Then the following conditions are equivalent.*

(i) We have

$$c_i^2 \|\varphi_i\|^2 = \|T\varphi_i\|^2 \quad \text{for all } i = 1, 2, \ldots, M.$$

(ii) We have

$$\left\langle \sum_{j=1}^N (\lambda_j - c_i^2) e_j, \sum_{j=1}^N |\langle \varphi_i, e_j \rangle|^2 e_j \right\rangle = 0 \quad \text{for all } i = 1, 2, \ldots, N.$$

Proof. By Lemma 68, (i) is equivalent to

$$\langle (T^*T - c_i^2 I_N) f_i, f_i) \rangle = 0 \quad \text{for all } i = 1, 2, \ldots, N.$$

But, for all $i = 1, 2, \ldots, N$,

$$\begin{aligned}
0 &= \langle (T^*T - c_i^2 I_N)\varphi_i, \varphi_i \rangle \\
&= \left\langle \sum_{j=1}^{N} (\lambda_j - c_i^2)\langle \varphi_i, e_j \rangle e_j, \sum_{j=1}^{N} \langle \varphi_i, e_j \rangle e_j \right\rangle \\
&= \sum_{j=1}^{N} (\lambda_j - c_i^2) |\langle \varphi_i e_j \rangle|^2 \\
&= \left\langle \sum_{j=1}^{N} (\lambda_j - c_i^2) e_j, \sum_{j=1}^{N} |\langle \varphi_i, e_j \rangle|^2 e_j \right\rangle .
\end{aligned}$$

The result is immediate from here. □

Now we can answer Question 4. The result follows directly from Theorem 70. We want to caution the reader that T^*T is *not* the frame operator for the frame $\{T\varphi_i\}_{i=1}^M$. In fact, as discussed before, the frame operator for this frame is TST^*, where S is the frame operator for $\{\varphi_i\}_{i=1}^M$.

Theorem 71. *Let $\Phi = \{\varphi_i\}_{i=1}^M$ be a frame for $\mathcal{H}_N$, and let $c > 0$. Further, let T be an invertible operator on $\mathcal{H}_N$, and let T^*T have the orthonormal basis $\mathcal{E} = \{e_j\}_{j=1}^N$ as eigenvectors with respective eigenvalues $\{\lambda_j\}_{j=1}^N$. Then the following conditions are equivalent:*

(i) $\|T\varphi_i\| = c\|\varphi_i\|$, *for all* $i = 1, 2, \ldots, M$.

(ii) $(\lambda_1 - c^2, \lambda_2 - c^2, \ldots, \lambda_N - c^2) \perp \mathcal{H}(\Phi, \mathcal{E})$.

In particular, if $\mathcal{H}(\Phi, \mathcal{E}) = \mathcal{H}_N$, then $\lambda_i = c^2$ for all $i = 1, 2, \ldots, N$, and hence T is a multiple of a unitary operator.

Given a frame $\{\varphi_i\}_{i=1}^M$, Theorem 71 now provides us with a unique method for constructing all operators T so that $\|T\varphi_i\| = c\|\varphi_i\|$ for all $i = 1, 2, \ldots, M$, detailed in the following remark.

Remark 72. *Let $\Phi = \{\varphi_i\}_{i=1}^M$ be a frame for $\mathcal{H}_N$, and let $c > 0$. First, we choose any orthonormal basis $\mathcal{E} = \{e_j\}_{j=1}^N$ for $\mathcal{H}_N$, and consider $\mathcal{H}(\Phi, \mathcal{E})$. We distinguish two cases:*

Case $\mathcal{H}(\Phi, \mathcal{E}) = \mathcal{H}_N$. In this case only unitary operators T can map $\{\varphi_i\}_{i=1}^M$ to an equal norm frame and satisfy that the operator T^*T has $\mathcal{E}$ as its eigenvectors.

Case $\mathcal{H}(\Phi, \mathcal{E}) \neq \mathcal{H}_N$. In this case, choose a vector

$$\sum_{j=1}^N a_j e_j = (a_1, a_2, \ldots, a_N) \in \mathcal{H}(\Phi, \mathcal{E})^\perp,$$

which satisfies $c^2 + a_j > 0$ for all $j = 1, 2, \ldots, N$. Set $\lambda_j := c^2 + a_j$ for all $j = 1, 2, \ldots, N$. Then choose any operator T on $\mathcal{H}_N$ such that $\{Te_j\}_{j=1}^N$ forms an orthogonal set and satisfies

$$\|Te_j\|^2 = \lambda_j \quad \text{for all } j = 1, 2, ..., N.$$

By Lemma 64,

$$T^*Te_j = (c^2 + a_j)e_j = \lambda_j e_j \quad \text{for all } j = 1, 2, \ldots, N.$$

Moreover, by our choice of $\{\lambda_j\}_{j=1}^N$,

$$(\lambda_1 - c^2, \lambda_2 - c^2, \ldots, \lambda_N - c^2) \perp \mathcal{H}(\Phi, \mathcal{E}).$$

Since

$$\begin{aligned}
\|T\varphi_i\|^2 - c^2\|f_i\|^2 &= \langle (T^*T - c^2 I_N)\varphi_i, \varphi_i\rangle \\
&= \sum_{j=1}^{N} (\lambda_j - c^2)|\langle \varphi_i, e_j\rangle|^2 \\
&= \left\langle \sum_{j=1}^{N} (\lambda_j - c^2)e_j, \sum_{j=1}^{N} |\langle \varphi_i, e_j\rangle|^2 e_j \right\rangle \\
&= \left\langle \sum_{j=1}^{N} a_j e_j, \sum_{j=1}^{N} |\langle \varphi_i, e_j\rangle|^2 e_j \right\rangle \\
&= 0,
\end{aligned}$$

it follows that $\|T\varphi_i\|^2 = c^2\|\varphi_i\|^2$ *for all* $i = 1, 2, \ldots, M$.

5.3.2 Generating Equal Norm Frames

Now we regard Theorem 71 from a different standpoint. In fact, for a given invertible operator T, Theorem 71 identifies all unit norm frames which T maps to equal norm frames. Namely, this is the family of frames for which there exists some $c > 0$ such that

$$(\lambda_1 - c^2, \lambda_2 - c^2, \ldots, \lambda_N - c^2) \perp \mathcal{H}(\Phi, \mathcal{E}).$$

Based on this observation, we derive several rather surprising corollaries from Theorem 71.

Corollary 73. *Let* $\Phi = \{\varphi_i\}_{i=1}^N$ *be a unit norm frame for* $\mathcal{H}_N$ *(i.e. a unit norm linearly independent set). Then for every non-unitary operator* T *on* $\mathcal{H}_N$ *which maps*

$\{\varphi_i\}_{i=1}^N$ to an equal norm spanning set, we have

$$\mathcal{H}(\Phi, \mathcal{E}) \neq \mathcal{H}_N.$$

Proof. Towards a contradiction, assume that $\mathcal{H}(\Phi, \mathcal{E}) = \mathcal{H}_N$. Then, by Corollary **??**(ii), $\{\varphi_i\}_{i=1}^M$ is not equivalent to an equal norm Parseval frame, a contradiction. □

Corollary 74. *Every invertible operator T on a Hilbert space $\mathcal{H}_N$ maps some equal norm Parseval frame to an equal norm frame.*

Proof. Let T be an invertible operator on $\mathcal{H}_N$, and let $\{e_j\}_{j=1}^N$ be an eigenbasis for T^*T with respective eigenvalues $\{\lambda_j\}_{j=1}^N$. Set

$$c^2 = \frac{1}{N}\sum_{j=1}^N \lambda_j \quad \text{and} \quad f = \sum_{i=1}^N e_j.$$

Then

$$\langle T^*T - c^2 I_N\, f, f\rangle = \sum_{j=1}^N (\lambda_j - c^2) \cdot 1 = 0,$$

which means

$$(1, 1, \ldots, 1) \perp (\lambda_1 - c^2, \lambda_2 - c^2, \ldots, \lambda_N - c^2).$$

Next, consider the frame

$$\{\varphi_i\}_{i=1}^{2^N} = \left\{\sum_{j=1}^N \varepsilon_j e_j\right\}_{\{\varepsilon_j\}\in\{1,-1\}^N}.$$

For every $g = \sum_{j=1}^{N} a_j e_j$, we obtain

$$\sum_{i=1}^{2^N} |\langle g, \varphi_i \rangle|^2 = \sum_{i=1}^{2^N} |\sum_{j=1}^{N} \varepsilon_j a_j|^2 = 2^N \sum_{j=1}^{N} |a_j|^2 = 2^N \|g\|^2.$$

Thus $\left\{ \frac{1}{\sqrt{2^N}} \varphi_i \right\}_{i=1}^{2^N}$ forms an equal norm Parseval frame, and we have $\mathcal{H}(\Phi, \mathcal{E}) = \text{span}\{(1, 1, \ldots, 1)\}$. By Theorem 71, this implies that $\{T\varphi_i\}_{i=1}^{2^N}$ is an equal norm frame with $\|T\varphi_i\|^2 = c^2$ for all $i = 1, 2, \ldots, 2^N$. $\square$

We now provide an example of an equal norm Parseval frame and a non-unitary operator T, which maps it to a unit norm frame.

Example 1. *Let $\varphi_1, \ldots, \varphi_4$ be the vectors in $\mathbb{R}^3$ defined by*

$$\varphi_1 = \frac{1}{2}(1, 1, 1), \ \varphi_2 = \frac{1}{2}(-1, 1, 1), \ \varphi_3 = \frac{1}{2}(1, -1, 1), \ \varphi_4 = \frac{1}{2}(1, 1, -1).$$

$\{\varphi_i\}_{i=1}^{4}$ is an equal norm Parseval frame for $\mathbb{R}^3$. If we let $\{e_j\}_{j=1}^{3}$ denote the standard unit vector basis, then

$$\mathcal{H}(\Phi, \mathcal{E}) = span\{(1, 1, 1)\}.$$

We next choose a vector g such that $g \perp \mathcal{H}(\Phi, \mathcal{E})$ by $g = (1, -1, 0)$. Let now, for instance, $c = 2$, and set

$$\lambda_1 = 1 + c^2 = 5, \ \lambda_2 = -1 + c^2 = 3, \ \lambda_3 = 0 + c^2 = 4.$$

Then define an operator T such that $\{Te_j\}_{j=1}^{3}$ is an orthogonal set and $\|Te_j\| = \lambda_j$.

One example of such an operator is defined by

$$T(1,0,0) = (5,0,0),\ T(0,1,0) = \left(0, \sqrt{\frac{3}{2}},\ \sqrt{\frac{3}{2}}\right),\ T(0,0,1) = (0,\sqrt{2},-\sqrt{2}).$$

Thus

$$\begin{aligned}
T\varphi_1 &= \frac{1}{2}(\sqrt{5}, \sqrt{\frac{3}{2}} + \sqrt{2}, \sqrt{\frac{3}{2}} - \sqrt{2}), \\
T\varphi_2 &= \frac{1}{2}(-\sqrt{5}, \sqrt{\frac{3}{2}} + \sqrt{2}, \sqrt{\frac{3}{2}} - \sqrt{2}), \\
T\varphi_3 &= \frac{1}{2}(\sqrt{5}, -\sqrt{\frac{3}{2}} + \sqrt{2}, \sqrt{\frac{3}{2}} - \sqrt{2}), \\
T\varphi_4 &= \frac{1}{2}(\sqrt{5}, \sqrt{\frac{3}{2}} - \sqrt{2}, \sqrt{\frac{3}{2}} + \sqrt{2}),
\end{aligned}$$

and we indeed obtain

$$\|T\varphi_i\|^2 = \frac{1}{4}(5+3+4) = 3$$

as desired.

5.4 Generating Nearly Parseval Frames

We finally tackle the question of deriving a deep understanding of operators which map a frame to a nearly Parseval frame.

5.4.1 Parseval Frames and Determinants

As a first step, we will draw an interesting connection between Parseval frames and determinants. We remark that this result is closely related to results by Cahill [20], see also [27], who pioneered the utilization of the Plücker embedding from Algebraic Geometry for characterization results in frame theory.

We start by pointing out some important, but simple consequences of the well known arithmetic-geometric mean inequality, which we first state for reference.

Theorem 75 (Arithmetic/Geometric Mean Inequality)**.** *Let* $\{x_j\}_{j=1}^N$ *be a sequence of positive real numbers, then*

$$\left(\prod_{j=1}^N x_j\right)^{1/n} \leq \frac{1}{N}\sum_{j=1}^N x_j$$

with equality if and only if $x_j = x_k$ *for every* $j, k = 1, \ldots, N$.

In terms of positive self-adjoint matrices, this results reads as follows.

Corollary 76. *Let* S *be a positive, self-adjoint* $N \times N$ *matrix such that* $Tr(S) = N$ *and* $\det(S) = 1$*, then* S *is the identity matrix.*

We now use this result to draw a connection between Parseval frames and determinants, namely of the matrix representation of the associated frame operator.

Theorem 77. *Let* $\{\varphi_i\}_{i=1}^M$ *be a frame for* $\mathcal{H}_N$ *with frame operator* S*. If* $\det(S) \geq 1$ *and* $\sum_{i=1}^M \|f_i\|^2 = N$*, then* $\{\varphi_i\}_{i=1}^M$ *constitutes a Parseval frame.*

Proof. Set

$$\{\psi_i\}_{i=1}^M = \left\{\frac{\varphi_i}{\det(S)}\right\}_{i=1}^M.$$

Now, let $\{\lambda_j\}_{j=1}^N$ denote the eigenvalues of S, and let the eigenvalues of the frame operator for $\{\psi_i\}_{i=1}^M$ be denoted by $\{\lambda_j'\}_{j=1}^N$. Then we obtain

$$\sum_{j=1}^N \lambda_j' = \frac{\sum_{j=1}^N \lambda_j}{\det(S)^2} = \frac{N}{\det(S)^2},$$

which implies

$$\frac{\sum_{j=1}^{N} \lambda_j'}{N} = \frac{1}{\det(S)^2} \leq 1 = \prod_{j=1}^{N} \lambda_j'.$$

However, this contradicts the arithmetic-geometric mean inequality unless $\lambda_j' = 1$ for all $j = 1, 2, \ldots N$, *i.e.*, unless $\{\varphi_i\}_{i=1}^{M}$ constitutes a Parseval frame. $\square$

5.4.2 Characterization of Unitary Operators

We will next provide a classification of unitary operators as those operators of determinant one which map some Parseval frame to a set of vectors with the same norms of the frame vectors.

Theorem 78. *Let T be an operator on $\mathcal{H}_N$. Then the following conditions are equivalent.*

(i) *T is unitary.*

(ii) *$|\det(T)| = 1$, and there exists some Parseval frame $\{\varphi_i\}_{i=1}^{M}$ for $\mathcal{H}_N$ such that $\|\varphi_i\| = \|T\varphi_i\|$ for all $i = 1, 2, \ldots, M$.*

(iii) *$|\det(T)| = 1$, and there exists some Parseval frame $\{\varphi_i\}_{i=1}^{M}$ for $\mathcal{H}_N$ such that $\sum_{i=1}^{M} \|T\varphi_i\|^2 = N$.*

(iv) *$|\det(T)| = 1$, and there exists some frame $\{\varphi_i\}_{i=1}^{M}$ for $\mathcal{H}_N$ such that $\sum_{i=1}^{M} \|\varphi_i\|^2 = N$ and $\{T\varphi_i\}_{i=1}^{M}$ is a Parseval frame for $\mathcal{H}_N$.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are clear.

(iii) $\Rightarrow$ (i). By Theorem 60, the frame operator of $\{T\varphi_i\}_{i=1}^{M}$ equals TT^*, since $\{\varphi_i\}$ is a Parseval frame. Now, let $\lambda_1, ..., \lambda_N$ denote the eigenvalues of TT^*. Then, it follows that

$$\sum_{j=1}^{N} \lambda_j = \sum_{i=1}^{M} \|T\varphi_i\|^2 = \sum_{i=1}^{M} \|\varphi_i\|^2 = N,$$

where the last equality follows from the fact that $\{\varphi_i\}_{i=1}^M$ is a Parseval frame. On the other hand, we have

$$\prod_{j=1}^{N} \lambda_j = \det(TT^*) = |\det(T)|^2 = 1.$$

Hence, we have equality in the arithmetic-geometric mean inequality. Thus, all λ_j's coincide, which is only possible provided that $\lambda_j = 1$ for every $j = 1, ..., N$. This implies $TT^* = I_N$.

(iv) $\Leftrightarrow$ (i). This follows by applying the previous argument to T^{-1}. $\square$

One might wonder whether the assumption in Theorem 78 that either $\{\varphi_i\}_{i=1}^M$ or $\{T\varphi_i\}_{i=1}^M$ is a Parseval frame is indeed necessary. This is in fact the case, even for a linearly independent set. In the following we will construct such an illuminating example in $\mathbb{R}^4$ by using the results of the previous section.

Example 2. *In $\mathbb{R}^4$, let $\{e_j\}_{j=1}^4$ denote the standard unit orthonormal basis, and define*

$$\varphi_1 = (1, 1, 2, 2),\ \varphi_2 = (1, -1, 2, 2),\ \varphi_3 = (1, 1, -2, 2),\ \varphi_4 = (1, 1, 2, -2).$$

An easy computation shows that

$$\mathcal{H}(\Phi, \mathcal{E}) = \mathit{span}\{(1, 1, 2, 2)\}.$$

Now, let

$$a_1 = 1,\ a_2 = 1,\ a_3 = -\frac{1}{2},\ a_4 = -\frac{1}{2},$$

and

$$\lambda_1 = 1 + a_1 = 2,\ \lambda_2 = 1 + a_2 = 2,\ \lambda_3 = 1 + a_3 = -\frac{1}{2},\ \lambda_4 = 1 + a_4 = -\frac{1}{2}.$$

This choice ensures that, for every $i = 1, 2, 3, 4$,

$$\left\langle \sum_{j=1}^{4} (\lambda_j - 1) e_j, \sum_{j=1}^{4} |\langle \varphi_i, e_j \rangle|^2 e_j \right\rangle = 0.$$

By Remark 72, if we choose any – in particular, a non-unitary – operator T on $\mathbb{R}^4$ such that $\{Te_j\}_{j=1}^4$ an orthogonal set with $\|Te_j\|^2 = \lambda_i$ for $j = 1, 2, 3, 4$, then $\|T\varphi_i\| = \|\varphi_i\|$ for all $i = 1, 2, \ldots, 4$ and $\det T = \prod_{j=1}^{4} \lambda_j = 1$.

5.4.3 Extension of the Arithmetic-Geometric Mean Inequality

We now proceed to analyze when an invertible operator can map an equal norm frame to a nearly Parseval frame. For the next proposition, we first need the following special case of a result from [60].

Theorem 79 ([60])**.** *Let $N \geq 2$, and let $x_j \geq 0$ for all $j = 1, 2, \ldots, N$. Then*

$$\frac{1}{N(N-1)} \sum_{1 \leq j < k \leq N} (x_j^{1/2} - x_k^{1/2})^2 \leq \frac{\sum_{j=1}^{N} x_j}{N} - \left(\prod_{j=1}^{N} x_j \right)^{1/N}.$$

The following quantitative version of the arithmetic-geometric mean inequality will be crucial for our main result in this section.

Theorem 80. *Let $N \geq 2$, and let $0 \leq x_j \leq N$ for all $j = 1, 2, \dots, N$. If*

$$\frac{\sum_{j=1}^{N} x_j}{N} - \left(\prod_{j=1}^{N} x_j \right)^{1/N} < \varepsilon,$$

then there exists a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ with

$$|x_j - x_k| \leq f(\varepsilon) \quad \text{for all } j, k = 1, 2, \dots, N$$

and

$$1 - f(\varepsilon) \leq x_j \leq 1 + f(\varepsilon) \quad \text{for all } j = 1, 2, \dots, N.$$

Moreover, f is bounded by

$$f(\varepsilon) \leq 2\varepsilon^{1/2} N^{3/2}.$$

Proof. By Theorem 79,

$$\frac{1}{N(N-1)} \sum_{1 \leq j < k \leq N} (x_j^{1/2} - x_k^{1/2})^2 \leq \frac{\sum_{j=1}^{N} x_j}{N} - \left(\prod_{j=1}^{N} x_j \right)^{1/N} < \varepsilon.$$

Therefore, for all $1 \leq j < k \leq N$,

$$|x_j^{1/2} - x_k^{1/2}|^2 \leq \sum_{1 \leq \tilde{j} < \tilde{k} \leq N} (x_{\tilde{j}}^{1/2} - x_{\tilde{k}}^{1/2})^2 \leq N(N-1)\varepsilon.$$

Since $x_j \leq N$, it follows that $x_j^{1/2} \leq N^{1/2}$ for all $j = 1, 2, \dots, N$. Thus,

$$|x_j - x_k|^2 = |x_j^{1/2} - x_k^{1/2}|^2 \, |x_j^{1/2} + x_k^{1/2}|^2 \leq N(N-1)\varepsilon^2 4N \leq 4N^3\varepsilon,$$

which implies

$$|x_j - x_k| \leq 2N^{3/2}\varepsilon^{1/2}.$$

Further, for any $1 \leq j \leq N$, we obtain

$$\begin{aligned}
x_j &= \frac{\sum_{k=1}^N x_k + x_j - x_k}{N} \\
&\leq \frac{\sum_{k=1}^N x_k}{N} + \frac{\sum_{k=1}^N |x_j - x_k|}{N} \\
&\leq 1 + \frac{N 2N^{3/2}\varepsilon^{1/2}}{N} \\
&= 1 + 2N^{3/2}\varepsilon^{1/2}.
\end{aligned}$$

The inequality $x_j \geq 1 - 2N^{3/2}\varepsilon^{1/2}$ can be similarly proved. □

The bound on f in Theorem 80 is certainly not optimal. We believe that the *optimal bound* is of the order of εN; our intuition is supported by the following example.

Example 3. *Fix K, N and let*

$$x_1 = K,\ x_2 = \frac{1}{K},\ \text{and}\ x_j = 1 \quad \text{for all}\ j = 3, 4, \ldots, N.$$

Then $\prod_{j=1}^N x_j = 1$. Also, since $KN > N$, we can conclude that

$$\frac{\sum_{j=1}^N x_j}{N} = \frac{K}{N} + \frac{1}{KN} + 1 - \frac{2}{N} \leq 1 + \frac{K}{N} - \frac{1}{N} = 1 + \varepsilon,$$

where $\varepsilon = (K-1)/N$. For N large, ε is arbitrarily small. Moreover, $\lambda_1 = K \approx \varepsilon N$, which implies

$$f(\varepsilon) \geq \varepsilon N.$$

5.4.4 Main Results

We now head towards our main results. The next proposition gives a first estimate of how close an equal norm frame is to being Parseval in terms of the determinant of the frame operator.

Proposition 81. *Let $\{\varphi_i\}_{i=1}^M$ be a frame for $\mathcal{H}_N$ with frame operator S satisfying $\|\varphi_i\|^2 = \frac{N}{M}$ for all $i = 1, \ldots, M$ and $(1-\varepsilon)^N \le |\det(S)|$. Then*

$$(1 - f(\varepsilon))I_N \le S \le (1 + f(\varepsilon))I_N,$$

where

$$f(\varepsilon) \le 2\varepsilon^{1/2} N^{3/2}.$$

Proof. Let $\{\lambda_j\}_{j=1}^N$ denote the eigenvalues of S. By hypothesis,

$$(1-\varepsilon)^N \le |\det(S)| = \prod_{j=1}^N \lambda_j,$$

which implies

$$1 - \varepsilon \le \left(\prod_{j=1}^N \lambda_j\right)^{1/N}. \tag{5.1}$$

On the other hand,

$$\frac{\sum_{j=1}^N \lambda_j}{N} = \frac{\sum_{i=1}^M \|\varphi_i\|^2}{N} = \frac{\sum_{i=1}^M \frac{N}{M}}{N} = 1. \tag{5.2}$$

This implies $\lambda_j \le N$ for all $j = 1, 2, \dots, N$, and combining (5.1) and (5.2), we obtain

$$\frac{\sum_{j=1}^{N} \lambda_j}{N} - \left(\prod_{j=1}^{N} \lambda_j \right)^{1/N} < \varepsilon.$$

The result now follows from Theorem 80. □

Our first main theorem, which we will state in the sequel, now provides sufficient conditions for when there exists a mapping of an arbitrary frame to an equal norm nearly Parseval frame. It also in a certain sense weakens the assumption of Proposition 81.

Theorem 82. *Let $\{\varphi_i\}_{i=1}^{M}$ be a frame for $\mathcal{H}_N$ with frame operator S satisfying $(1-\varepsilon)^N \le |\det(S)|$. Further, let T be an operator with $|\det(T)| \ge 1$ and $\|T\varphi_i\|^2 = \frac{N}{M}$ for all $i = 1, 2, \dots, M$. Also, let S_1 be the frame operator of $\{T\varphi_i\}_{i=1}^{M}$ and denote its eigenvalues by $\{\mu_j\}_{j=1}^{N}$. Then*

$$(1-\varepsilon)\frac{\sum_{j=1}^{N} \mu_j}{N} \le \left(\prod_{j=1}^{N} \mu_j \right)^{1/N} \le \frac{\sum_{j=1}^{N} \mu_j}{N}.$$

Moreover,

$$(1 - f(\varepsilon))I_N \le S_1 \le (1 + f(\varepsilon))I_N.$$

Proof. By hypothesis,

$$|\det(TST^*)| = |\det(T)|^2 |\det(S)| \ge |\det(S)| \ge (1-\varepsilon)^N.$$

Thus

$$\prod_{j=1}^{N} \mu_j \geq (1-\varepsilon)^N, \quad \text{hence } 1-\varepsilon \leq \left(\prod_{j=1}^{N} \mu_j\right)^{1/N}. \tag{5.3}$$

Since $\|T\varphi_i\|^2 = \frac{N}{M}$ for all $i = 1, 2, \ldots, M$,

$$\sum_{j=1}^{N} \mu_j = N, \quad \text{hence } \frac{\sum_{j=1}^{N} \mu_j}{N} = 1. \tag{5.4}$$

By (5.3) and (5.4), it follows that

$$(1-\varepsilon)\frac{\sum_{j=1}^{N} \mu_j}{N} = 1-\varepsilon \leq \left(\prod_{j=1}^{N} \mu_j\right)^{1/N} \leq \frac{\sum_{j=1}^{N} \mu_j}{N} = 1.$$

The *moreover* part is immediate by Theorem 80. □

Our final main results provides a generalization of Theorem 82, in the sense that it gives a bound on TT^* in the situation that T maps an equal norm frame to an equal norm frame.

Theorem 83. *Let* $\{\varphi_i\}_{i=1}^M$ *be a frame for* $\mathcal{H}_N$, *and let* T *be an invertible operator on* $\mathcal{H}_N$ *with* $|\det(T)| \geq 1$ *and satisfying*

(1) $\|\varphi_i\|^2 = \|T\varphi_i\|^2 = \frac{N}{M}$, *and*

(2) $(1-\varepsilon)^N \leq |\det(S)|$.

Then

$$\frac{1-f(\varepsilon)}{1+f(\varepsilon)} I_N \leq TT^* \leq \frac{1+f(\varepsilon)}{1-f(\varepsilon)} I_N.$$

Proof. Since $(1-\varepsilon)^N \leq |\det(S)| \leq |\det(TST^*)|$, applying Proposition 81 to both

$\{\varphi_i\}_{i=1}^M$ and $\{T\varphi_i\}_{i=1}^M$ yields

$$(1 - f(\varepsilon))I_N \leq S \leq (1 + f(\varepsilon))I_N \tag{5.5}$$

and

$$(1 - f(\varepsilon))I_N \leq TST^* \leq (1 + f(\varepsilon))I_N. \tag{5.6}$$

By applying T from the left and T^* from the right to (5.5), we obtain

$$(1 - f(\varepsilon))TT^* \leq T(1 - f(\varepsilon))T^* \leq TST^*.$$

Combining with (5.6), we can conclude that

$$(1 - f(\varepsilon))TT^* \leq (1 + f(\varepsilon))I_N,$$

which implies

$$TT^* \leq \frac{1 + f(\varepsilon)}{1 - f(\varepsilon)} I_N.$$

Similar arguments lead to the other inequality. □

Chapter 6

Frames as projections

6.1 Introduction

In this chapter we will take a slightly different point of view on frames. Rather than thinking about a frame as a collection of vectors, we instead think of a frame as a collection of rank 1 operators. In particular, we associate to each vector the rank 1 projection onto the one dimensional span of that vector scaled by the norm squared of the vector. We will see that certain problems that seem quite difficult when thinking of frames as sets of vectors, become quite simple from this point of view. We remark that in this chapter we must make a distinction between frames for a complex Hilbert space versus frames for a real Hilbert space. Therefore we will refer to either $\mathbb{R}^N$ or $\mathbb{C}^N$ rather than $\mathcal{H}_N$.

We will work in the space $\mathbb{H}_{N\times N}$ of all $N \times N$ Hermitian matrices. Note that this is a **real** vector space of dimension N^2 (it is not a space over the complex numbers since a Hermitian matrix multiplied by a complex scalar is no longer Hermitian). The inner product on this space is given by $\langle S, T\rangle = \text{Trace}(ST)$ and the norm induced by this inner product is the Froebenius norm, *i.e.*, $\langle S, S\rangle = \|S\|_F^2$.

Consider the mapping from $\mathbb{C}^N$ to $\mathbb{H}_{N\times N}$ given by

$$x \mapsto xx^*.$$

Note that xx^* is the rank one projection onto $\text{span}\{x\}$ scaled by $\|x\|^2$. xx^* is called the *outer product* of x with itself. Also note that if $x = \lambda y$ for $\lambda \in \mathbb{C}$ then $xx^* = (\lambda y)(\lambda y)^* = |\lambda|^2 yy^*$.

Before stating our first theorem we need one more definition. A subset $Q \subseteq \mathbb{R}^n$ is called *generic* if there exists a nonzero polynomial $p(x_1, ..., x_n)$ such that $Q^c \subseteq \{(x_1, ..., x_n) \in \mathbb{R}^n : p(x_1, ..., x_n) = 0\}$. It is a standard fact that generic sets are open, dense, and full measure. When we talk about a generic set in $\mathbb{C}^n$ we mean that it is generic when we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$.

Theorem 84. *For a generic choice of vectors $\{\varphi_i\}_{i=1}^{N^2} \subseteq \mathbb{C}^N$ we have that*

$$\text{span}\{\varphi_i\varphi_i^*\}_{i=1}^{N^2} = \mathbb{H}_{N\times N}.$$

Proof. First let $\{T_i\}_{i=1}^{N^2}$ be any basis for $\mathbb{H}_{N\times N}$. Since each T_i is Hermitian we can use the spectral theorem to get a decomposition $T_i = \sum_{j=1}^N \lambda_{ij} P_{ij}$ where each P_{ij} is rank 1. So it follows that $\text{span}\{P_{ij}\} = \mathbb{H}_{N\times N}$ and therefore this set contains a basis of $\mathbb{H}_{N\times N}$. Thus, we have constructed a basis of $\mathbb{H}_{N\times N}$ consisting only of rank 1 matrices.

Now observe that for a given choice of vectors $\{\varphi_i\}_{i=1}^{N^2}$ we have that $\text{span}\{\varphi_i\varphi_i^*\} = \mathbb{H}_{N\times N}$ if and only if the determinant of the frame operator is nonzero (note that we are refering to the frame operator of $\{\varphi_i\varphi_i^*\}_{i=1}^{N^2}$ as an operator on $\mathbb{H}_{N\times N}$, not the frame operator of $\{\varphi_i\}_{i=1}^{N^2}$ as an operator on $\mathbb{C}^N$). But the determinant of the frame operator is a polynomial in the (real and imaginary parts) of the entries of the φ_i's, and by the first paragraph we know that there is at least one choice for which this does not vanish, so we can conclude that for a generic choice it does not vanish. $\square$

Corollary 85. *If $M \leq N^2$ then for a generic choice of vectors $\{\varphi_i\}_{i=1}^M \subseteq \mathbb{C}^N$ we have that $\{\varphi_i\varphi_i^*\}_{i=1}^M$ is linearly independent.*

This chapter is split into two sections, each dealing with a different problem. The next section deals with what are known as scalable frames, and the work in this section is from [26]. The last section is about a problem known as phase retrieval which is a very popular topic at the time of this writing. The work in that section is from [13].

6.2 Scalable frames

A frame $\{\varphi_i\}_{i=1}^M$ is said to be *scalable* if there exists a collection of scalars $\{v_i\}_{i=1}^M \subseteq \mathbb{C}$ so that $\{v_i\varphi_i\}_{i=1}^M$ is a Parseval frame. In this case, we call the vector $(|v_1|^2, ..., |v_M|^2) \in \mathbb{R}_+^M$ a *scaling* of $\{\varphi_i\}_{i=1}^M$. Scalable frames have been studied previously in [63].

In what follows we will always consider frames in the complex space $\mathbb{C}^N$, however all of our results hold in the real space $\mathbb{R}^N$ as well. The only difference is in this case we must replace the space $\mathbb{H}_{N\times N}$ with its subspace $\mathbb{S}_{N\times N}$ consisting of all $N\times N$ real symmetric matrices, which is a real vector space of dimension $N(N+1)/2$. Thus, if one replaces $\mathbb{H}_{N\times N}$ with $\mathbb{S}_{N\times N}$ and N^2 with $N(N+1)/2$ all of our results will hold for frames $\{\varphi_i\}_{i=1}^M \subseteq \mathbb{R}^N$ and the same proofs will work.

6.2.1 Scaling generic frames

Given a frame $\{\varphi_i\}_{i=1}^M$, in this setting we have that the frame operator is given by

$$S = \sum_{i=1}^{M} \varphi_i \varphi_i^*,$$

so $\{\varphi_i\}_{i=1}^M$ is scalable if and only if there exists a collection of **positive** scalars $\{w_i\}_{i=1}^M$ so that

$$\sum_{i=1}^{M} w_i \varphi_i \varphi_i^* = I_N,$$

in this case $\{\sqrt{w_i}\varphi_i\}_{i=1}^M$ is a Parseval frame, and the vector $(w_1, ..., w_M) \in \mathbb{R}_+^M$ is the scaling.

Given a frame $\{\varphi_i\}_{i=1}^M \subseteq \mathbb{C}^N$ define the operator $\mathcal{A} : \mathbb{R}^M \to \mathbb{H}_{N\times N}$ by

$$\mathcal{A}w = \sum_{i=1}^{M} w_i \varphi_i \varphi_i^*$$

where $w = (w_1, ..., w_M)^T$. To determine whether $\{\varphi_i\}_{i=1}^M$ is scalable boils down to finding a nonnegative solution to

$$\mathcal{A}w = I_N.$$

In the generic case when $\{\varphi_i \varphi_i^*\}_{i=1}^M$ is linearly independent, this system is guaranteed to have either no solution, or one unique solution. So if it either has a solution with a negative entry or has no solution we can conclude that this frame is not scalable, and if it has a nonnegative solution then it is scalable and this solution tells us the unique scalars to use. We summarize this in the following corollary:

Corollary 86. *Given frame* $\{\varphi_i\}_{i=1}^M \subseteq \mathbb{C}^N$ *such that* $\{\varphi_i\varphi_i^*\}_{i=1}^M$ *is linearly independent in* $\mathbb{H}_{N\times N}$*, we can determine its scalability by solving the linear system*

$$\mathcal{A}w = I_N. \tag{6.1}$$

Furthermore, in this case if it is scalable then it is scalable in a unique way.

In particular, if $M \leq N^2$ *then with probability 1, determining the scalability of* $\{\varphi_i\}_{i=1}^M$ *is equivalent to solving the linear system given in* (6.1).

6.2.2 Linearly dependent outer products

In this section we will address the situation when $\{\varphi_i\varphi_i^*\}_{i=1}^M$ is linearly dependent. The main problem here is that the system $\mathcal{A}w = I_N$ may have many solutions, and possibly none of them are nonnegative. In this section we will find it convenient to assume that $\|\varphi_i\| = 1$ for every $i = 1, ..., M$, note that we lose no generality by making this assumption.

Given a collection of vectors $\{x_i\}_{i=1}^M \subseteq \mathbb{R}^N$ we define their *affine span* as

$$\mathrm{aff}\{x_i\}_{i=1}^M := \{\sum_{i=1}^M c_i x_i : \sum_{i=1}^M c_i = 1\}$$

and we say that $\{x_i\}_{i=1}^M$ is *affinely independent* if

$$x_j \notin \mathrm{aff}\{x_i\}_{i\neq j}$$

for every $j = 1, ..., M$. We also define their *convex hull* as

$$\text{conv}\{x_i\}_{i=1}^M := \{\sum_{i=1}^M c_i x_i : c_i \geq 0, \sum_{i=1}^M c_i = 1\}.$$

We say a set $\mathcal{P} \subseteq \mathbb{R}^N$ is called a *polytope* if it is the convex hull of finitely many points.

Proposition 87. *Given a collection of unit norm vectors $\{\varphi_i\}_{i=1}^M \subseteq \mathbb{C}^N$ we have that $\{\varphi_i \varphi_i^*\}_{i=1}^M$ is linearly independent if and only if it is affinely independent.*

Proof. Clearly linear independence always implies affine independence. So suppose that $\{\varphi_i \varphi_i^*\}_{i=1}^M$ is not linearly independent. Then we have an equation of the form

$$\varphi_j \varphi_j^* = \sum_{i \neq j} c_i \varphi_i \varphi_i^*$$

for some j. Also note that since $\|\varphi_i\| = 1$ it follows that $\langle \varphi_i \varphi_i^*, I_N \rangle = 1$ for every $i = 1, ..., M$. Therefore, we have

$$\begin{aligned} 1 &= \langle \varphi_j \varphi_j^*, I_N \rangle = \langle \sum_{i \neq j} c_i \varphi_i \varphi_i^*, I_N \rangle \\ &= \sum_{i \neq j} c_i \langle \varphi_i \varphi_i^*, I_N \rangle = \sum_{i \neq j} c_i. \end{aligned}$$

Therefore $\{\varphi_i \varphi_i^*\}_{i=1}^M$ is not affinity independent. $\square$

Proposition 88. *A unit norm frame $\{\varphi_i\}_{i=1}^M \subseteq \mathbb{C}^N$ is scalable if and only if $\frac{1}{N} I_N \in \text{conv}\{\varphi_i \varphi_i^*\}_{i=1}^M$. Furthermore, if $\lambda I_N \in \text{conv}\{\varphi_i \varphi_i^*\}_{i=1}^M$ then $\lambda = \frac{1}{N}$ and if $\sum_{i=1}^M w_i \varphi_i \varphi_i^* = \frac{1}{N} I_N$ then $\sum_{i=1}^M w_i = 1$.*

Proof. Suppose we have a scaling w so that

$$I_N = \sum_{i=1}^{M} w_i \varphi_i \varphi_i^*.$$

Then

$$\begin{aligned} N &= \langle I_N, I_N \rangle = \langle \sum_{i=1}^{M} w_i \varphi_i \varphi_i^*, I_N \rangle \\ &= \sum_{i=1}^{M} w_i \langle \varphi_i \varphi_i^*, I_N \rangle = \sum_{i=1}^{M} w_i. \end{aligned}$$

Thus, $\sum_{i=1}^{M} \frac{w_i}{N} = 1$ and since $w_i \geq 0$ for every $i = 1, ..., M$ it follows that $\frac{1}{N} I_N = \sum_{i=1}^{M} \frac{w_i}{N} \varphi_i \varphi_i^* \in \operatorname{conv}\{\varphi_i \varphi_i^*\}_{i=1}^{M}$. The converse is obvious.

The furthermore part follows from a similar argument. Suppose $\lambda I_N = \sum_{i=1}^{M} w_i \varphi_i \varphi_i^*$ with $\sum_{i=1}^{M} w_i = 1$. Then

$$N\lambda = \langle \lambda I_N, I_N \rangle = \sum_{i=1}^{M} w_i = 1.$$

Now suppose $\frac{1}{N} I_N = \sum_{i=1}^{M} w_i \varphi_i \varphi_i^*$. Then

$$1 = \langle \sum_{i=1}^{M} w_i \varphi_i \varphi_i^*, I_N \rangle = \sum_{i=1}^{M} w_i.$$

□

The following theorem is known as Carathéodory's theorem:

Theorem 89. *Given a set of points* $\{x_i\}_{i=1}^{M} \subseteq \mathbb{R}^N$ *suppose* $y \in \operatorname{conv}\{x_i\}_{i=1}^{M}$. *Then there exists a subset* $I \subseteq \{1, ..., M\}$ *such that* $y \in \operatorname{conv}\{x_i\}_{i \in I}$ *and* $\{x_i\}_{i \in I}$ *is affinely* *independent.*

Corollary 90. *Suppose $\{\varphi_i\}_{i=1}^M \subseteq \mathbb{C}^N$ is a scalable frame. Then there is a subset $\{\varphi_i\}_{i\in I}$ which is also scalable and $\{\varphi_i\varphi_i^*\}_{i\in I}$ is linearly independent.*

Given a unit norm frame $\{\varphi_i\}_{i=1}^M \subseteq \mathbb{C}^N$ we define the set

$$\mathcal{P}(\{\varphi_i\}_{i=1}^M) := \{(w_1, ..., w_M) : w_i \geq 0, \sum_{i=1}^M w_i\varphi_i\varphi_i^* = \frac{1}{N} I_N\}.$$

Proposition 88 tells us two things about this set: first we have that $w \in \mathcal{P}(\{\varphi_i\}_{i=1}^M)$ if and only if $N \cdot w$ is a scaling of $\{\varphi_i\}_{i=1}^M$, and second, that $\mathcal{P}(\{\varphi_i\}_{i=1}^M)$ is a (possibly empty) polytope (see, for example, Theorem 1.1 in [74]).

Suppose $\{\varphi_i\}_{i=1}^M \subseteq \mathbb{C}^N$ is a scalable frame, and we are given a scaling $w = (w_1, ..., w_M)$. We say the scaling is *minimal* if $\{\varphi_i : w_i > 0\}$ has no proper subset which is scalable.

Theorem 91. *Suppose $\{\varphi_i\}_{i=1}^M \subseteq \mathbb{C}^N$ is a scalable, unit norm frame. If $w = (w_1, ..., w_M)$ is a minmal scaling then $\{\varphi_i\varphi_i^*\}_{i=1}^M$ is linearly independent. Furthermore, $\mathcal{P}(\{\varphi_i\}_{i=1}^M)$ is the convex hull of the minimal scalings, i.e., every scaling is a convex combination of minimal scalings.*

Proof. The first statement follows directly from Corollary 90.

We now show that every vertex of $\mathcal{P}(\{\varphi_i\}_{i=1}^M)$ is indeed a minimal scaling. Let $u \in \mathcal{P}(\{\varphi_i\}_{i=1}^M)$ be a vertex and assume to the contrary that u is not minimal, then there exists a $v \in P$ such that $\mathrm{supp}(v) \subsetneq \mathrm{supp}(u)$. Let $w(t) = v + t(u - v)$, and $t_0 = \min\{\frac{v_i}{v_i - u_i} : v_i > u_i\}$. We observe that $t_0 > 1$ and $w(t_0)_i \geq 0$ since supp(v) $\subsetneq$ supp(u). This means $w(t_0) \in P$, and u lies on the line segment connecting v and $w(t_0)$ which contradicts the fact that u is a vertex.

Finally we show that every minimal scaling is a vertex of $\mathcal{P}(\{\varphi_i\}_{i=1}^M)$. Suppose we are given a minimal scaling w which is not a vertex of $\mathcal{P}(\{\varphi_i\}_{i=1}^M)$. Then we can write w as a convex combination of vertices, say $w = \sum t_i v_i$, where we know at least two t_i's are nonzero, without loss of generality say t_1 and t_2. Since both t_1 and t_2 are positive and all the entries of v_1 and v_2 are nonnegative, it follows that $\mathrm{supp}(v_1) \cup \mathrm{supp}(v_2) \subseteq \mathrm{supp}(w)$, which contradicts the fact the w is a minimal scaling. $\square$

Theorem 91 reduces the problem of understanding the scalings of the frame $\{\varphi_i\}_{i=1}^M$ to that of finding the vertices of the polytope $\mathcal{P}(\{\varphi_i\}_{i=1}^M)$. Relatvely fast algorithms for doing this are known, see [8].

6.2.3 When are outer products linearly independent?

Since most of the results in this section deal with linear independence of the outer products of subsets of our frame vectors we will address this issue in this section. It would be nice if there were conditions on a frame $\{\varphi_i\}_{i=1}^M$ which could guarantee that the set of outer products $\{\varphi_i\varphi_i^*\}_{i=1}^M$ is linearly independent, or conversely if knowing that $\{\varphi_i\varphi_i^*\}_{i=1}^M$ is linearly independent tells anything about the frame $\{\varphi_i\}_{i=1}^M$. One obvious condition is that in order for $\{\varphi_i\varphi_i^*\}_{i=1}^M$ to be linearly independent we must have $M \leq N^2$, and when this is satisfied Theorem 84 tells us that this will usually be the case.

Another condition which is easy to prove is that if $\{\varphi_i\}_{i=1}^M$ is linearly independent then so is $\{\varphi_i\varphi_i^*\}_{i=1}^M$. The converse of this is certainly not true, and since we are usually interested in frames for which $M > N$ this condition is not very useful. The main idea here is that while the frame vectors live in a N-dimensional space the outer products live in a N^2-dimensional space, so there is much more "room" for them to be linearly independent.

Given a frame $\{\varphi_i\}_{i=1}^M$ we define its *spark* to be the size of its smallest linearly dependent subset, more precisely

$$\text{spark}(\{\varphi_i\}_{i=1}^M) := \min\{|I| : \{\varphi_i\}_{i\in I} \text{ is linearly dependent}\}.$$

Clearly for a frame $\{\varphi_i\}_{i=1}^M \subseteq \mathbb{C}^N$ we must have that $\text{spark}(\{\varphi_i\}_{i=1}^M) \leq N+1$, if its spark is equal to $N+1$ we say it is *full spark*. For more background on full spark frames see [2].

Proposition 92. *Suppose $\{\varphi_i\}_{i=1}^M \subseteq \mathbb{C}^N$ is a frame with $M \leq 2N-1$. If $\{\varphi_i\}_{i=1}^M$ is full spark then $\{\varphi_i\varphi_i^*\}_{i=1}^M$ is linearly independent.*

Proof. Suppose by way of contradiction that $\{\varphi_i\}_{i=1}^M$ is full spark but $\{\varphi_i\varphi_i^*\}_{i=1}^M$ is linearly dependent. Then we can write an equation of the form

$$\sum_{i\in I} a_i\varphi_i\varphi_i^* = \sum_{j\in J} b_j\varphi_j\varphi_j^*$$

with $a_i > 0$ for every $i \in I$, $b_j > 0$ for every $j \in J$, and $I \cap J = \emptyset$. This implies that

$$\begin{aligned} \text{span}(\{\varphi_i\}_{i\in I}) &= \text{Im}(\sum_{i\in I} a_i\varphi_i\varphi_i^*) \\ &= \text{Im}(\sum_{j\in J} b_j\varphi_j\varphi_j^*) = \text{span}(\{\varphi_j\}_{j\in J}). \end{aligned}$$

But since $M \leq 2N-1$ we have either $|I| \leq N-1$ or $|J| \leq N-1$, so this contradicts the fact the $\{\varphi_i\}_{i=1}^M$ is full spark. $\square$

We first remark that the converse of Proposition 92 is not true:

Example 4. *Let* $\{e_1, e_2, e_3\}$ *be an orthonormal basis for* $\mathbb{C}^3$ *and consider the frame* $\{e_1, e_2, e_3, e_1{+}e_2, e_2{+}e_3\}$*. Clearly this frame is not full spark and yet it is easy to verify that* $\{e_1e_1^*, e_2e_2^*, e_3e_3^*, (e_1+e_2)(e_1+e_2)^*, (e_2+e_3)(e_2+e_3)^*\}$ *is linearly independent.*

Next we remark that the assumption $M \leq 2N-1$ is necessary:

Example 5. *Let* $\{e_1, e_2\}$ *be an orthonormal basis for* $\mathbb{C}^2$ *and consider the frame* $\{e_1, e_2, e_1+e_2, e_1-e_2\}$*. Clearly this frame is full spark but*

$$e_1e_1^* + e_2e_2^* = I_2 = \frac{1}{2}((e_1+e_2)(e_1+e_2)^* + (e_1-e_2)(e_1-e_2)^*).$$

Finally we remark that with only slight modifications the proof of Propostion 92 can be used to prove the following more general result:

Proposition 93. *If* $\operatorname{spark}(\{\varphi_i\}_{i=1}^M) \geq s$ *then* $\operatorname{spark}(\{\varphi_i\varphi_i^*\}_{i=1}^M) \geq 2s-2$.

Unfortunately, the converse of Proposition 93 is still not true. The main problem here is that given any three vectors such that no one of them is a scalar multiple of another, the corresponding outer products will be linearly independent (we leave the proof of this as an exercise). Therefore it is easy to make examples (such as Example 4 above) of frames that have tiny spark, but the corresponding outer products are linearly independent.

We conclude our discussion of spark by remarking that in [2] it is shown that computing the spark of a general frame is NP-hard. Thus, the small amount of insight we gain from Proposition 93 is of little practical use.

Another property worth mentioning in this section is known as the *complement property*. A frame $\{\varphi_i\}_{i=1}^M \subseteq \mathbb{C}^N$ has the complement property if for every $I \subseteq$

$\{1, ..., M\}$ we have either $\text{span}(\{\varphi_i\}_{i\in I}) = \mathbb{C}^N$ or $\text{span}(\{\varphi_i\}_{i\in I^c}) = \mathbb{C}^N$. We will take a deeper look at the complement property in the next section, but for now we make some simple observations.

If a frame $\{\varphi_i\}_{i=1}^M \subseteq \mathbb{C}^N$ has the complement property then clearly we must have $M \geq 2N - 1$ (if not we could partition the frame into two sets each of size at most $N - 1$) and that in this case full spark implies the complement property. If $M = 2N - 1$ then the complement property is equivalent to full spark, but for $M > 2N - 1$ the complement property is (slightly) weaker. One might ask if the complement property tells us anything about the linear independence of the outer products, or vice versa. Example 4 above is an example of a frame which does not have the complement property but the outer products are linearly independent, and Example 5 is an example of a frame that does have the complement property but the outer products are linearly dependent. So it seems like the complement property has nothing to do with the linear independence of the outer products.

Given a frame with the complement property we can add any set of vectors to it without losing the complement property. Thus it seems natural to ask whether every frame with the complement property has a subset of size $2N - 1$ which is full spark. This also turns out to be not true as the following example shows:

Example 6. *Consider the frame in Example 4 with the vector $e_1 + e_3$ added to it. It is not difficult to verify that this frame does have the complement property, but no subset of size 5 is full spark.*

We conclude by noting that as in the proof of Proposition 92, a set of outer products $\{\varphi_i\varphi_i^*\}_{i=1}^M$ is linearly dependent if and only if we have an equation of the form

$$\sum_{i\in I} a_i\varphi_i\varphi_i^* = \sum_{j\in J} b_j\varphi_j\varphi_j^*$$

with $a_i > 0$ for every $i \in I$, $b_j > 0$ for every $j \in J$, and $I \cap J = \emptyset$. This is equivalent to $\{\varphi_i\}_{i=1}^M$ having two disjoint subsets, namely $\{\varphi_i\}_{i\in I}$ and $\{\varphi_j\}_{j\in J}$, which can be scaled to have the same frame operator. Thus, determining whether $\{\varphi_i\varphi_i^*\}_{i=1}^M$ is linearly independent is equivalent to solving a more difficult scaling problem than the one presented in this paper.

6.3 Phase retrieval

Signals are often passed through linear systems, and in some applications, only the pointwise absolute value of the output is available for analysis. For example, in high-power coherent diffractive imaging, this loss of phase information is eminent, as one only has access to the power spectrum of the desired signal [19]. *Phase retrieval* is the problem of recovering a signal from absolute values (squared) of linear measurements, called *intensity measurements*. Note that phase retrieval is often impossible—intensity measurements with the identity basis effectively discard the phase information of the signal's entries, and so this measurement process is not at all injective; the power spectrum similarly discards the phases of Fourier coefficients. This fact has led many researchers to invoke a priori knowledge of the desired signal, since intensity measurements might be injective when restricted to a smaller signal class. Unfortunately, this route has yet to produce practical phase retrieval guarantees, and practitioners currently resort to various ad hoc methods that often fail to work.

Thankfully, there is an alternative approach to phase retrieval, as introduced in 2006 by Balan, Casazza and Edidin [12]: Seek injectivity, not by finding a smaller signal class, but rather by designing a larger ensemble of intensity measurements. In [12], Balan et al. characterized injectivity in the real case and further leveraged algebraic geometry to show that $4N-2$ intensity measurements suffice for injectivity over N-dimensional complex signals. This realization that so few measurements can lend injectivity has since prompted a flurry of research in search of practical phase retrieval guarantees [1, 10, 11, 28, 29, 30, 45, 48, 73]. Notably, Candès, Strohmer and Voroninski [30] viewed intensity measurements as Hilbert-Schmidt inner products between rank-1 operators, and they applied certain intuition from compressed sensing to stably reconstruct the desired N-dimensional signal with semidefinite programming using only $\mathcal{O}(N\log N)$ random measurements; similar alternatives and refinements have since been identified [28, 29, 45, 73]. Another alternative exploits the polarization identity to discern relative phases between certain intensity measurements; this method uses $\mathcal{O}(N\log N)$ random measurements in concert an expander graph, and comes with a similar stability guarantee [1].

Despite these recent strides in phase retrieval algorithms, there remains a fundamental lack of understanding about what it takes for intensity measurements to be injective, let alone whether measurements lend stability (a more numerical notion of injectivity). For example, until very recently, it was believed that $3N-2$ intensity measurements sufficed for injectivity (see for example [28]); this was disproved by Heinosaari, Mazzarella and Wolf [52], who used embedding theorems from differential geometry to establish the necessity of $(4+o(1))N$ measurements. As far as stability is concerned, the most noteworthy achievement to date is due to Eldar and Mendelson [48], who proved that $\mathcal{O}(N)$ Gaussian random measurements separate distant M-dimensional real signals with high probability. Still, the following problem remains wide open:

Problem 94. *What are the necessary and sufficient conditions for measurement vectors to lend injective and stable intensity measurements?*

In this section we address this problem in a number of ways. First we focus on injectivity, and start by providing the first known characterization of injectivity in the complex case (Theorem 97). Next, we make a rather surprising identification: that intensity measurements are injective in the complex case precisely when the corresponding phase-only measurements are injective in some sense (Theorem 98). We then use this identification to prove the necessity of the complement property for injectivity (Theorem 100). Later, we conjecture that $4N-4$ intensity measurements are necessary and sufficient for injectivity in the complex case, and we prove this conjecture in the cases where $N=2,3$ (Theorems 103 and 105). Our proof for the $N=3$ case leverages a new test for injectivity, which we then use to verify the injectivity of a certain quantum-mechanics-inspired measurement ensemble.

Given a collection of measurement vectors $\Phi=\{\varphi_i\}_{i=1}^{M}$ in $V=\mathbb{R}^N$ or $\mathbb{C}^N$, consider the intensity measurement process defined by

$$(\mathcal{A}(x))(i):=|\langle x,\varphi_i\rangle|^2.$$

Note that $\mathcal{A}(x)=\mathcal{A}(y)$ whenever $y=cx$ for some scalar c of unit modulus. As such, the mapping $\mathcal{A}\colon V\rightarrow\mathbb{R}^M$ is necessarily not injective. To resolve this (technical) issue, throughout this section, we consider sets of the form V/S, where V is a vector space and S is a multiplicative subgroup of the field of scalars. By this notation, we mean to identify vectors $x, y \in V$ for which there exists a scalar $c \in S$ such that $y = cx$; we write $y \equiv x \bmod S$ to convey this identification. Most (but not all) of the time, V/S is either $\mathbb{R}^N/\{\pm 1\}$ or $\mathbb{C}^N/\mathbb{T}$ (here, $\mathbb{T}$ is the complex unit circle), and we view the intensity measurement process as a mapping $\mathcal{A}\colon V/S \to \mathbb{R}^M$; it is in this way that we will consider the measurement process to be injective or stable.

6.3.1 Injectivity and the complement property

Phase retrieval is impossible without injective intensity measurements. In their seminal work on phase retrieval [12], Balan, Casazza and Edidin introduce the following property to analyze injectivity:

Definition 95. *We say $\Phi = \{\varphi_i\}_{i=1}^M$ in $\mathbb{R}^N$ ($\mathbb{C}^N$) satisfies the complement property (CP) if for every $I \subseteq \{1, \ldots, M\}$, either $\{\varphi_i\}_{i \in I}$ or $\{\varphi_i\}_{i \in I^c}$ spans $\mathbb{R}^N$ ($\mathbb{C}^N$).*

In the real case, the complement property is characteristic of injectivity, as demonstrated in [12]. We provide the proof of this result below; it contains several key insights which we will apply throughout this paper.

Theorem 96. *Consider $\Phi = \{\varphi_i\}_{i=1}^M \subseteq \mathbb{R}^N$ and the mapping $\mathcal{A}\colon \mathbb{R}^N/\{\pm 1\} \to \mathbb{R}^M$ defined by $(\mathcal{A}(x))(i) := |\langle x, \varphi_i \rangle|^2$. Then $\mathcal{A}$ is injective if and only if Φ satisfies the complement property.*

Proof. We will prove both directions by obtaining the contrapositives.

($\Rightarrow$) Assume that Φ is not CP. Then there exists $I \subseteq \{1, \ldots, M\}$ such that neither $\{\varphi_i\}_{i \in I}$ nor $\{\varphi_i\}_{i \in I^c}$ spans $\mathbb{R}^N$. This implies that there are nonzero vectors $u, v \in \mathbb{R}^N$

such that $\langle u, \varphi_i \rangle = 0$ for all $i \in I$ and $\langle v, \varphi_i \rangle = 0$ for all $i \in I^c$. For each i, we then have

$$|\langle u \pm v, \varphi_i \rangle|^2 = |\langle u, \varphi_i \rangle|^2 \pm 2\langle u, \varphi_i \rangle \langle v, \varphi_i \rangle + |\langle v, \varphi_i \rangle|^2 = |\langle u, \varphi_i \rangle|^2 + |\langle v, \varphi_i \rangle|^2.$$

Since $|\langle u+v, \varphi_i \rangle|^2 = |\langle u-v, \varphi_i \rangle|^2$ for every i, we have $\mathcal{A}(u+v) = \mathcal{A}(u-v)$. Moreover, u and v are nonzero by assumption, and so $u + v \neq \pm(u - v)$.

($\Leftarrow$) Assume that $\mathcal{A}$ is not injective. Then there exist vectors $x, y \in \mathbb{R}^N$ such that $x \neq \pm y$ and $\mathcal{A}(x) = \mathcal{A}(y)$. Taking $I := \{i : \langle x, \varphi_i \rangle = -\langle y, \varphi_i \rangle\}$, we have $\langle x + y, \varphi_i \rangle = 0$ for every $i \in I$. Otherwise when $i \in I^c$, we have $\langle x, \varphi_i \rangle = \langle y, \varphi_i \rangle$ and so $\langle x - y, \varphi_i \rangle = 0$. Furthermore, both $x + y$ and $x - y$ are nontrivial since $x \neq \pm y$, and so neither $\{\varphi_i\}_{i \in S}$ nor $\{\varphi_i\}_{i \in S^c}$ spans $\mathbb{R}^N$. $\square$

Note that [12] erroneously stated that the first part of the above proof also gives that CP is necessary for injectivity in the complex case. Although it demonstrates that $u + v \neq \pm(u - v)$, it fails to establish that $u + v \not\equiv u - v \bmod \mathbb{T}$; for instance, it could very well be the case that $u + v = \mathrm{i}(u - v)$, and so injectivity would not be violated *in the complex case.*

Theorem 97. *Consider* $\Phi = \{\varphi_i\}_{i=1}^M \subseteq \mathbb{C}^N$ *and the mapping* $\mathcal{A}\colon \mathbb{C}^N/\mathbb{T} \to \mathbb{R}^M$ *defined by* $(\mathcal{A}(x))(i) := |\langle x, \varphi_i \rangle|^2$. *Viewing* $\{\varphi_i \varphi_i^* u\}_{i=1}^M$ *as vectors in* $\mathbb{R}^{2N}$, *denote* $S(u) := \operatorname{span}_{\mathbb{R}}\{\varphi_i \varphi_i^* u\}_{i=1}^M$. *Then the following are equivalent:*

(a) $\mathcal{A}$ *is injective.*

(b) $\dim S(u) \geq 2N - 1$ *for every* $u \in \mathbb{C}^N \setminus \{0\}$.

(c) $S(u) = \operatorname{span}_{\mathbb{R}}\{\mathrm{i}u\}^\perp$ *for every* $u \in \mathbb{C}^N \setminus \{0\}$.

Before proving this theorem, note that unlike the characterization in the real case, it is not clear whether this characterization can be tested in finite time; instead of being a statement about all (finitely many) partitions of $\{1,\ldots,M\}$, this is a statement about all $u\in\mathbb{C}^N\setminus\{0\}$. However, we can view this characterization as an analog to the real case in some sense: In the real case, the complement property is equivalent to having $\operatorname{span}\{\varphi_i\varphi_i^*u\}_{i=1}^M=\mathbb{R}^N$ for all $u\in\mathbb{R}^N\setminus\{0\}$. As the following proof makes precise, the fact that $\{\varphi_i\varphi_i^*u\}_{i=1}^N$ fails to span all of $\mathbb{R}^{2N}$ is rooted in the fact that more information is lost with phase in the complex case.

Proof of Theorem 97. (a) $\Rightarrow$ (c): Suppose $\mathcal{A}$ is injective. We need to show that $\{\varphi_i\varphi_i^*u\}_{i=1}^M$ spans the set of vectors orthogonal to $\mathrm{i}u$. Here, orthogonality is with respect to the real inner product, which can be expressed as $\langle a,b\rangle_{\mathbb{R}}=\operatorname{Re}\langle a,b\rangle$. Note that

$$|\langle u\pm v,\varphi_i\rangle|^2=|\langle u,\varphi_i\rangle|^2\pm 2\operatorname{Re}\langle u,\varphi_i\rangle\langle\varphi_i,v\rangle+|\langle v,\varphi_i\rangle|^2,$$

and so subtraction gives

$$|\langle u+v,\varphi_i\rangle|^2-|\langle u-v,\varphi_i\rangle|^2=4\operatorname{Re}\langle u,\varphi_i\rangle\langle\varphi_i,v\rangle=4\langle\varphi_i\varphi_i^*u,v\rangle_{\mathbb{R}}. \tag{6.2}$$

In particular, if the right-hand side of (6.2) is zero, then injectivity implies that there exists some ω of unit modulus such that $u+v=\omega(u-v)$. Since $u\neq 0$, we know $\omega\neq -1$, and so rearranging gives

$$v=-\frac{1-\omega}{1+\omega}u=-\frac{(1-\omega)(1+\overline{\omega})}{|1+\omega|^2}u=-\frac{2\operatorname{Im}\omega}{|1+\omega|^2}\,\mathrm{i}u.$$

This means $S(u)^\perp\subseteq\operatorname{span}_{\mathbb{R}}\{\mathrm{i}u\}$. To prove $\operatorname{span}_{\mathbb{R}}\{\mathrm{i}u\}\subseteq S(u)^\perp$, take $v=\alpha\mathrm{i}u$ for some

$\alpha \in \mathbb{R}$ and define $\omega := \frac{1+\alpha\mathrm{i}}{1-\alpha\mathrm{i}}$, which necessarily has unit modulus. Then

$$u + v = u + \alpha\mathrm{i}u = (1+\alpha\mathrm{i})u = \frac{1+\alpha\mathrm{i}}{1-\alpha\mathrm{i}}(u - \alpha\mathrm{i}u) = \omega(u - v).$$

Thus, the left-hand side of (6.2) is zero, meaning $v \in S(u)^\perp$.

(b) $\Leftrightarrow$ (c): First, (b) immediately follows from (c). For the other direction, note that $\mathrm{i}u$ is necessarily orthogonal to every $\varphi_i\varphi_i^*u$:

$$\langle \varphi_i\varphi_i^*u, \mathrm{i}u\rangle_{\mathbb{R}} = \operatorname{Re}\langle \varphi_i\varphi_i^*u, \mathrm{i}u\rangle = \operatorname{Re}\langle u, \varphi_i\rangle\langle \varphi_i, \mathrm{i}u\rangle = -\operatorname{Re}\mathrm{i}|\langle u, \varphi_i\rangle|^2 = 0.$$

Thus, $\operatorname{span}_{\mathbb{R}}\{\mathrm{i}u\} \subseteq S(u)^\perp$, and by (b), $\dim S(u)^\perp \leq 1$, both of which gives (c).

(c) $\Rightarrow$ (a): This portion of the proof is inspired by Mukherjee's analysis in [67]. Suppose $\mathcal{A}(x) = \mathcal{A}(y)$. If $x = y$, we are done. Otherwise, $x - y \neq 0$, and so we may apply (c) to $u = x - y$. First, note that

$$\langle \varphi_i\varphi_i^*(x-y), x+y\rangle_{\mathbb{R}} = \operatorname{Re}\langle \varphi_i\varphi_i^*(x-y), x+y\rangle = \operatorname{Re}(x+y)^*\varphi_i\varphi_i^*(x-y),$$

and so expanding gives

$$\begin{aligned}\langle \varphi_i\varphi_i^*(x-y), x+y\rangle_{\mathbb{R}} &= \operatorname{Re}\Big(|\varphi_i^*x|^2 - x^*\varphi_i\varphi_i^*y + y^*\varphi_i\varphi_i^*x - |\varphi_i^*y|^2\Big)\\ &= \operatorname{Re}\Big(-x^*\varphi_i\varphi_i^*y + \overline{x^*\varphi_i\varphi_i^*y}\Big)\\ &= 0.\end{aligned}$$

Since $x + y \in S(x-y)^\perp = \operatorname{span}_{\mathbb{R}}\{\mathrm{i}(x-y)\}$, there exists $\alpha \in \mathbb{R}$ such that $x + y = \alpha\mathrm{i}(x-y)$, and so rearranging gives $y = \frac{1-\alpha\mathrm{i}}{1+\alpha\mathrm{i}}x$, meaning $y \equiv x \bmod \mathbb{T}$. $\square$

The above theorem leaves a lot to be desired; it is still unclear what it takes for a complex ensemble to yield injective intensity measurements. While in pursuit of a more clear understanding, we established the following bizarre characterization: A complex ensemble lends injective intensity measurements precisely when it lends injective *phase-only* measurements (in some sense). This is made more precise in the following theorem statement:

Theorem 98. *Consider* $\Phi = \{\varphi_i\}_{i=1}^M \subseteq \mathbb{C}^N$ *and the mapping* $\mathcal{A}\colon \mathbb{C}^N/\mathbb{T} \to \mathbb{R}^M$ *defined by* $(\mathcal{A}(x))(i) := |\langle x, \varphi_i\rangle|^2$*. Then* $\mathcal{A}$ *is injective if and only if the following statement holds: If for every* $i = 1, \ldots, M$*, either* $\arg(\langle x, \varphi_i\rangle^2) = \arg(\langle y, \varphi_i\rangle^2)$ *or one of the sides is not well-defined, then* $x = 0$*,* $y = 0$*, or* $y \equiv x \bmod \mathbb{R}\setminus\{0\}$*.*

Proof. By Theorem 97, $\mathcal{A}$ is injective if and only if

$$\forall x \in \mathbb{C}^N \setminus \{0\}, \qquad \operatorname{span}_{\mathbb{R}}\{\varphi_i\varphi_i^* x\}_{i=1}^M = \operatorname{span}_{\mathbb{R}}\{\mathrm{i}x\}^\perp \quad \forall i = 1, \ldots, M. \tag{6.3}$$

Taking orthogonal complements of both sides, note that regardless of $x \in \mathbb{C}^N \setminus \{0\}$, we know $\operatorname{span}_{\mathbb{R}}\{\mathrm{i}x\}$ is necessarily a subset of $(\operatorname{span}_{\mathbb{R}}\{\varphi_i\varphi_i^* x\}_{i=1}^M)^\perp$, and so (6.3) is equivalent to

$$\forall x \in \mathbb{C}^N \setminus \{0\}, \quad \operatorname{Re}\langle \varphi_i\varphi_i^* x, \mathrm{i}y\rangle = 0 \quad \forall i = 1, \ldots, M \Longrightarrow y = 0 \text{ or } y \equiv x \bmod \mathbb{R}\setminus\{0\}.$$

Thus, we need to determine when $\operatorname{Im}\langle x, \varphi_i\rangle\overline{\langle y, \varphi_i\rangle} = \operatorname{Re}\langle \varphi_i\varphi_i^* x, \mathrm{i}y\rangle = 0$. We claim that this is true if and only if $\arg(\langle x, \varphi_i\rangle^2) = \arg(\langle y, \varphi_i\rangle^2)$ or one of the sides is not well-defined. To see this, we substitute $a := \langle x, \varphi_i\rangle$ and $b := \langle y, \varphi_i\rangle$. Then to complete the proof, it suffices to show that $\operatorname{Im} a\bar{b} = 0$ if and only if $\arg(a^2) = \arg(b^2)$, $a = 0$, or $b = 0$.

($\Leftarrow$) If either a or b is zero, the result is immediate. Otherwise, if $2\arg(a) = \arg(a^2) = \arg(b^2) = 2\arg(b)$, then 2π divides $2(\arg(a) - \arg(b))$, and so $\arg(a\bar{b}) = \arg(a) - \arg(b)$ is a multiple of π. This implies that $a\bar{b} \in \mathbb{R}$, and so $\operatorname{Im} a\bar{b} = 0$.

($\Rightarrow$) Suppose $\operatorname{Im} a\bar{b} = 0$. Taking the polar decompositions $a = re^{\mathrm{i}\theta}$ and $b = se^{\mathrm{i}\phi}$, we equivalently have that $rs \sin(\theta - \phi) = 0$. Certainly, this can occur whenever r or s is zero, i.e., $a = 0$ or $b = 0$. Otherwise, a difference formula then gives $\sin\theta\cos\phi = \cos\theta\sin\phi$. From this, we know that if θ is an integer multiple of $\pi/2$, then ϕ is as well, and vice versa, in which case $\arg(a^2) = 2\arg(a) = \pi = 2\arg(b) = \arg(b^2)$. Else, we can divide both sides by $\cos\theta\cos\phi$ to obtain $\tan\theta = \tan\phi$, from which it is evident that $\theta \equiv \phi \bmod \pi$, and so $\arg(a^2) = 2\arg(a) = 2\arg(b) = \arg(b^2)$. $\square$

We will use this result to (correctly) prove the necessity of CP for injectivity. First, we need the following lemma, which is interesting in its own right:

Lemma 99. *Consider $\Phi = \{\varphi_i\}_{i=1}^M \subseteq \mathbb{C}^N$ and the mapping $\mathcal{A} \colon \mathbb{C}^N/\mathbb{T} \to \mathbb{R}^M$ defined by $(\mathcal{A}(x))(i) := |\langle x, \varphi_i\rangle|^2$. If $\mathcal{A}$ is injective, then the mapping $\mathcal{B} \colon \mathbb{C}^N/\{\pm 1\} \to \mathbb{C}^M$ defined by $(\mathcal{B}(x))(i) := \langle x, \varphi_i\rangle^2$ is also injective.*

Proof. Suppose $\mathcal{A}$ is injective. Then we have the following facts (one by definition, and the other by Theorem 98):

(i) If $\forall i = 1, \ldots, M$, $|\langle x, \varphi_i\rangle|^2 = |\langle y, \varphi_i\rangle|^2$, then $y \equiv x \bmod \mathbb{T}$.

(ii) If $\forall i = 1, \ldots, M$, either $\arg(\langle x, \varphi_i\rangle^2) = \arg(\langle y, \varphi_i\rangle^2)$ or one of the sides is not well-defined, then $x = 0$, $y = 0$, or $y \equiv x \bmod \mathbb{R} \setminus \{0\}$.

Now suppose we have $\langle x, \varphi_i\rangle^2 = \langle y, \varphi_i\rangle^2$ for all $i = 1, \ldots, M$. Then their moduli and arguments are also equal, and so (i) and (ii) both apply. Of course, $y \equiv x \bmod \mathbb{T}$

implies $x = 0$ if and only if $y = 0$. Otherwise both are nonzero, in which case there exists $\omega \in \mathbb{T} \cap \mathbb{R} \setminus \{0\} = \{\pm 1\}$ such that $y = \omega x$. In either case, $y \equiv x \bmod \{\pm 1\}$, so $\mathcal{B}$ is injective. $\square$

Theorem 100. *Consider $\Phi = \{\varphi_i\}_{i=1}^M \subseteq \mathbb{C}^N$ and the mapping $\mathcal{A}\colon \mathbb{C}^N/\mathbb{T} \to \mathbb{R}^M$ defined by $(\mathcal{A}(x))(i) := |\langle x, \varphi_i \rangle|^2$. If $\mathcal{A}$ is injective, then Φ satisfies the complement property.*

We are now ready to prove the necessity of the complement property in the complex case.

Proof of Theorem 100. Recall that if $\mathcal{A}$ is injective, then so is the mapping $\mathcal{B}$ of Lemma 99. Therefore, it suffices to show that Φ is CP if $\mathcal{B}$ is injective. To complete the proof, we will obtain the contrapositive (note the similarity to the proof of Theorem 96). Suppose Φ is not CP. Then there exists $I \subseteq \{1, \ldots, M\}$ such that neither $\{\varphi_i\}_{i \in I}$ nor $\{\varphi_i\}_{i \in I^c}$ spans $\mathbb{C}^N$. This implies that there are nonzero vectors $u, v \in \mathbb{C}^N$ such that $\langle u, \varphi_i \rangle = 0$ for all $i \in I$ and $\langle v, \varphi_i \rangle = 0$ for all $i \in I^c$. For each i, we then have

$$\langle u \pm v, \varphi_i \rangle^2 = \langle u, \varphi_i \rangle^2 \pm 2\langle u, \varphi_i \rangle \langle v, \varphi_i \rangle + \langle v, \varphi_i \rangle^2 = \langle u, \varphi_i \rangle^2 + \langle v, \varphi_i \rangle^2.$$

Since $\langle u + v, \varphi_i \rangle^2 = \langle u - v, \varphi_i \rangle^2$ for every i, we have $\mathcal{B}(u + v) = \mathcal{B}(u - v)$. Moreover, u and v are nonzero by assumption, and so $u + v \neq \pm(u - v)$. $\square$

Note that the complement property is necessary but not sufficient for injectivity. To see this, consider measurement vectors $(1, 0)$, $(0, 1)$ and $(1, 1)$. These certainly satisfy the complement property, but $\mathcal{A}((1, \mathrm{i})) = (1, 1, 2) = \mathcal{A}((1, -\mathrm{i}))$, despite the fact that $(1,\mathrm{i}) \not\equiv (1,-\mathrm{i}) \bmod \mathbb{T}$; in general, real measurement vectors fail to lend injective intensity measurements in the complex setting since they do not distinguish complex conjugates.

6.3.2 The $4N-4$ Conjecture

In this section we address the following problem:

Problem 101. *For any dimension N, what is the smallest number $M^*(N)$ of injective intensity measurements, and how do we design such measurement vectors?*

To be clear, this problem was completely solved in the real case by Balan, Casazza and Edidin [12]. Indeed, Theorem 96 immediately implies that $2N-2$ intensity measurements are necessarily not injective, and furthermore that $2N-1$ measurements are injective if and only if the measurement vectors are full spark. As such, we will focus our attention to the complex case.

In the complex case, Problem 101 has some history in the quantum mechanics literature. For example, [72] presents *Wright's conjecture* that three observables suffice to uniquely determine any pure state. In phase retrieval parlance, the conjecture states that there exist unitary matrices U_1, U_2 and U_3 such that $\Phi = [U_1\ U_2\ U_3]$ lends injective intensity measurements (here, the measurement vectors are the columns of Φ). Note that Wright's conjecture actually implies that $M^*(N) \leq 3N-2$; indeed, U_1 determines the norm (squared) of the signal, rendering the last column of both U_2 and U_3 unnecessary. Finkelstein [49] later proved that $M^*(N) \geq 3N-2$; combined with Wright's conjecture, this led many to believe that $M^*(N) = 3N-2$ (for example, see [28]). However, both this and Wright's conjecture were recently disproved in [52], in which Heinosaari, Mazzarella and Wolf invoked embedding theorems from differential geometry to prove that

$$M^*(N) \geq 4N - 2\alpha(N-1) - 3 \quad \text{for all } N \tag{6.4}$$

where $\alpha(N-1) \leq \log_2(N)$ is the number of 1's in the binary representation of $N-1$. By comparison, Balan, Casazza and Edidin [12] proved that $M^*(N) \leq 4N-2$, and so we at least have the asymptotic expression $M^*(N) = (4+o(1))N$.

At this point, we should clarify some intuition for $M^*(N)$ by explaining the nature of these best known lower and upper bounds. First, the lower bound (6.4) follows from an older result that complex projective space $\mathbb{C}\mathbf{P}^n$ does not smoothly embed into $\mathbb{R}^{4n-2\alpha(n)}$ (and other slight refinements which depend on n); this is due to Mayer [65], but we highly recommend James's survey on the topic [54]. To prove (6.4) from this, suppose $\mathcal{A}\colon \mathbb{C}^N/\mathbb{T} \rightarrow \mathbb{R}^M$ were injective. Then $\mathcal{E}$ defined by $\mathcal{E}(x) := \mathcal{A}(x)/\|x\|^2$ embeds $\mathbb{C}\mathbf{P}^{N-1}$ into $\mathbb{R}^M$, and as Heinosaari et al. show, the embedding is necessarily smooth; considering $\mathcal{A}(x)$ is made up of rather simple polynomials, the fact that $\mathcal{E}$ is smooth should not come as a surprise. As such, the nonembedding result produces the best known lower bound. To evaluate this bound, first note that Milgram [66] constructs an embedding of $\mathbb{C}\mathbf{P}^n$ into $\mathbb{R}^{4n-\alpha(n)+1}$, establishing the importance of the $\alpha(n)$ term, but the constructed embedding does not correspond to an intensity measurement process. In order to relate these embedding results to our problem, consider the real case: It is known that for odd $n \geq 7$, real projective space $\mathbb{R}\mathbf{P}^n$ smoothly embeds into $\mathbb{R}^{2n-\alpha(n)+1}$ [69], which means the analogous lower bound for the real case would necessarily be smaller than $2(N-1)-\alpha(N-1)+1 = 2N-\alpha(N-1)-1 < 2N-1$. This indicates that the $\alpha(N-1)$ term in (6.4) might be an artifact of the proof technique, rather than of $M^*(N)$.

There is also some intuition to be gained from the upper bound $M^*(N) \leq 4N-2$, which Balan et al. proved by applying certain techniques from algebraic geometry (some of which we will apply later in this section). In fact, their result actually gives that $4N-2$ or more measurement vectors, if chosen generically, will lend injective intensity measurements. This leads us to think that $M^*(N)$ generic measurement vectors might also lend injectivity.

The lemma that follows will help to refine our intuition for $M^*(N)$, and it will also play a key role in the main theorems of this section (a similar result appears in [52]). Given an ensemble of measurement vectors $\{\varphi_i\}_{i=1}^M \subseteq \mathbb{C}^N$, define the *super analysis operator* $\mathbf{A}\colon \mathbb{H}_{N\times N} \rightarrow \mathbb{R}^M$ by $(\mathbf{A}H)(i) = \langle H, \varphi_i\varphi_i^*\rangle$. Note that $\mathbf{A}$ is a linear operator, and yet

$$\begin{aligned}
(\mathbf{A}xx^*)(i) &= \langle xx^*, \varphi_i\varphi_i^*\rangle \\
&= \mathrm{Tr}[\varphi_i\varphi_i^* xx^*] \\
&= \mathrm{Tr}[\varphi_i^* xx^* \varphi_i] \\
&= \varphi_i^* xx^* \varphi_i \\
&= |\langle x, \varphi_i\rangle|^2 \\
&= (\mathcal{A}(x))(i).
\end{aligned}$$

In words, $x \bmod \mathbb{T}$ can be "lifted" to xx^*, thereby linearizing the intensity measurement process at the price of squaring the dimension of the vector space of interest; this identification has been exploited by some of the most noteworthy strides in modern phase retrieval [11, 30]. As the following lemma shows, this identification can also be used to characterize injectivity:

Lemma 102. *$\mathcal{A}$ is not injective if and only if there exists a matrix of rank 1 or 2 in the null space of $\mathbf{A}$.*

Proof. ($\Rightarrow$) If $\mathcal{A}$ is not injective, then there exist $x, y \in \mathbb{C}^N/\mathbb{T}$ such that $\mathcal{A}(x) = \mathcal{A}(y)$. That is, $\mathbf{A}xx^* = \mathbf{A}yy^*$, and so $xx^* - yy^*$ is in the null space of $\mathbf{A}$.

($\Leftarrow$) First, suppose there is a rank-1 matrix H in the null space of $\mathbf{A}$. Then there exists $x \in \mathbb{C}^N$ such that $H = xx^*$ and $(\mathcal{A}(x))(i) = (\mathbf{A}xx^*)(i) = 0 = (\mathcal{A}(0))(i)$. But $x \not\equiv 0 \bmod \mathbb{T}$, and so $\mathcal{A}$ is not injective. Now suppose there is a rank-2 matrix H in the null space of $\mathbf{A}$. Then by the spectral theorem, there are orthonormal $u_1, u_2 \in \mathbb{C}^N$ and nonzero $\lambda_1 \geq \lambda_2$ such that $H = \lambda_1 u_1 u_1^* + \lambda_2 u_2 u_2^*$. Since H is in the null space of $\mathbf{A}$, the following holds for every i:

$$0 = \langle H, \varphi_i\varphi_i^* \rangle = \langle \lambda_1 u_1 u_1^* + \lambda_2 u_2 u_2^*, \varphi_i\varphi_i^* \rangle = \lambda_1 |\langle u_1, \varphi_i \rangle|^2 + \lambda_2 |\langle u_2, \varphi_i \rangle|^2. \tag{6.5}$$

Taking $x := |\lambda_1|^{1/2} u_1$ and $y := |\lambda_2|^{1/2} u_2$, note that $y \not\equiv x \bmod \mathbb{T}$ since they are nonzero and orthogonal. We claim that $\mathcal{A}(x) = \mathcal{A}(y)$, which would complete the proof. If λ_1 and λ_2 have the same sign, then by (6.5), $|\langle x, \varphi_i \rangle|^2 + |\langle y, \varphi_i \rangle|^2 = 0$ for every i, meaning $|\langle x, \varphi_i \rangle|^2 = 0 = |\langle y, \varphi_i \rangle|^2$. Otherwise, $\lambda_1 > 0 > \lambda_2$, and so $xx^* - yy^* = \lambda_1 u_1 u_1^* + \lambda_2 u_2 u_2^* = A$ is in the null space of $\mathbf{A}$, meaning $\mathcal{A}(x) = \mathbf{A}xx^* = \mathbf{A}yy^* = \mathcal{A}(y)$. $\square$

Lemma 102 indicates that we want the null space of $\mathbf{A}$ to avoid nonzero matrices of rank ≤ 2. Intuitively, this is easier when the "dimension" of this set of matrices is small. To get some idea of this dimension, let's count real degrees of freedom. By the spectral theorem, almost every matrix in $\mathbb{H}_{N\times N}$ of rank ≤ 2 can be uniquely expressed as $\lambda_1 u_1 u_1^* + \lambda_2 u_2 u_2^*$ with $\lambda_1 \leq \lambda_2$. Here, (λ_1, λ_2) has two degrees of freedom. Next, u_1 can be any vector in $\mathbb{C}^N$, except its norm must be 1. Also, since u_1 is only unique up to global phase, we take its first entry to be nonnegative without loss of generality. Given the norm and phase constraints, u_1 has a total of $2N-2$ real degrees of freedom. Finally, u_2 has the same norm and phase constraints, but it must also be orthogonal to u_1, that is, $\mathrm{Re}\langle u_2, u_1\rangle = \mathrm{Im}\langle u_2, u_1\rangle = 0$. As such, u_2 has $2N-4$ real degrees of freedom. All together, we can expect the set of matrices in question to have $2 + (2N-2) + (2N-4) = 4N-4$ real dimensions.

If the set R_2 of matrices of rank at most 2 formed a subspace of $\mathbb{H}_{N\times N}$ (it doesn't), then we could expect the null space of $\mathbf{A}$ to intersect that subspace nontrivially whenever $\dim \mathrm{null}(\mathbf{A}) + (4N-4) > \dim(\mathbb{H}_{N\times N}) = N^2$. By the rank-nullity theorem, this would indicate that injectivity requires

$$M \geq \mathrm{rank}(\mathbf{A}) = N^2 - \dim \mathrm{null}(\mathbf{A}) \geq 4N-4. \tag{6.6}$$

Of course, this logic is not technically valid since R_2 is not a subspace. It is, however, a special kind of set: a real projective variety. The definition of projective variety is not important here, we just remark that if r_2 were a projective variety over an algebraically closed field (it's not), then the projective dimension theorem (Theorem 7.2 of [51]) says that S intersects $\mathrm{null}(\mathbf{A})$ nontrivially whenever the dimensions are large enough: $\dim \mathrm{null}(\mathbf{A}) + \dim R_2 > \dim \mathbb{H}_{N\times N}$, thereby implying that injectivity requires (6.6). Unfortunately, this theorem is not valid when the field is $\mathbb{R}$; for example, the cone defined by $x^2 + y^2 - z^2 = 0$ in $\mathbb{R}^3$ is a projective variety of dimension 2, but its intersection with the 2-dimensional xy-plane is trivial, despite the fact that $2+2>3$.

In the absence of a proof, we pose the natural conjecture:

The $4N-4$ Conjecture. *Consider* $\Phi = \{\varphi_i\}_{i=1}^M \subseteq \mathbb{C}^N$ *and the mapping* $\mathcal{A}\colon \mathbb{C}^N/\mathbb{T} \to \mathbb{R}^M$ *defined by* $(\mathcal{A}(x))(i) := |\langle x, \varphi_i\rangle|^2$*. If* $N \geq 2$*, then the following statements hold:*

(a) If $M < 4N-4$*, then* $\mathcal{A}$ *is not injective.*

(b) If $M \geq 4N-4$*, then* $\mathcal{A}$ *is injective for generic* Φ*.*

Note that the work of Balan, Casazza and Edidin [12] already proves this for $M \geq 4N-2$. Also note that the analogous statement of (b) holds in the real case: Full spark measurement vectors are generic, and they satisfy the complement property whenever $M \geq 2N-1$. Also, the dimension of the set of real symmetric matrices of rank at most 2 is $2N-1$.

At this point, it is fitting to mention that after we initially formulated this conjecture, Bodmann presented a Vandermonde construction of $4N-4$ injective intensity measurements at a phase retrieval workshop at the Erwin Schrödinger International Institute for Mathematical Physics. The result has since been documented in [16], and it establishes one consequence of the $4N-4$ conjecture: $M^*(N) \leq 4N-4$.

As incremental progress toward solving the $4N-4$ conjecture, we offer the following result:

Theorem 103. *The* $4N-4$ *Conjecture is true when* $N = 2$*.*

Proof. (a) Since $\mathbf{A}$ is a linear map from 4-dimensional real space to M-dimensional real space, the null space of $\mathbf{A}$ is necessarily nontrivial by the rank-nullity theorem. Furthermore, every nonzero member of this null space has rank 1 or 2, and so Lemma 102 gives that $\mathcal{A}$ is not injective.

(b) For a generic choice of 4 vectors $\{\varphi_i\}_{i=1}^4 \subseteq \mathbb{C}^2$, Theorem 84 says that

$$\operatorname{span}\{\varphi_i\varphi_i^*\}_{i=1}^4 = \mathbb{H}_{2\times 2},$$

so $\ker(\mathbf{A}) = \{0\}$. $\square$

We also have a proof for the $N = 3$ case, but we first introduce Algorithm 6.3.2, namely the *HMW test* for injectivity; we name it after Heinosaari, Mazarella and Wolf, who implicitly introduce this algorithm in their paper [52].

Algorithm 1 The HMW test for injectivity when $N = 3$

Input: Measurement vectors $\{\varphi_i\}_{i=1}^M \subseteq \mathbb{C}^3$
Output: Whether $\mathcal{A}$ is injective

Define $\mathbf{A}\colon \mathbb{H}_{3\times 3} \to \mathbb{R}^M$ such that $\mathbf{A}H = \{\langle H, \varphi_i\varphi_i^*\rangle_{\mathrm{HS}}\}_{i=1}^M$
if $\dim \operatorname{null}(\mathbf{A}) = 0$ **then**
  "INJECTIVE" {if $\mathbf{A}$ is injective, then $\mathcal{A}$ is injective}
else
  Pick $H \in \operatorname{null}(\mathbf{A})$, $H \neq 0$
  if $\dim \operatorname{null}(\mathbf{A}) = 1$ and $\det(H) \neq 0$ **then**
    "INJECTIVE" {if $\mathbf{A}$ only maps invertible matrices to zero, then $\mathcal{A}$ is injective}
  else
    "NOT INJECTIVE"
  end if
end if

Theorem 104 (cf. Proposition 6 in [52]). *When $N = 3$, the HMW test correctly determines whether $\mathcal{A}$ is injective.*

Proof. First, if $\mathbf{A}$ is injective, then $\mathcal{A}(x) = \mathbf{A}xx^* = \mathbf{A}yy^* = \mathcal{A}(y)$ if and only if $xx^* = yy^*$, i.e., $y \equiv x \bmod \mathbb{T}$. Next, suppose $\mathbf{A}$ has a 1-dimensional null space. Then Lemma 102 gives that $\mathcal{A}$ is injective if and only if the null space of $\mathbf{A}$ is spanned by a matrix of full rank. Finally, if the dimension of the null space is 2 or more, then there exist linearly independent (nonzero) matrices A and B in this null space. If $\det(A) = 0$, then it must have rank 1 or 2, and so Lemma 102 gives that $\mathcal{A}$ is not injective. Otherwise, consider the map

$$f\colon t \mapsto \det(A\cos t + B\sin t) \qquad \forall t \in [0, \pi].$$

Since $f(0) = \det(A)$ and $f(\pi) = \det(-A) = (-1)^3 \det(A) = -\det(A)$, the intermediate value theorem gives that there exists $t_0 \in [0, \pi]$ such that $f(t_0) = 0$, i.e., the matrix $A\cos t_0 + B\sin t_0$ is singular. Moreover, this matrix is nonzero since A and B are linearly independent, and so its rank is either 1 or 2. Lemma 102 then gives that $\mathcal{A}$ is not injective. $\square$

As an example, we may run the HMW test on the columns of the following matrix:

$$\Phi = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 & 1 & \mathrm{i} \\ -1 & 0 & 0 & 1 & 1 & -1 & -2 & 2 \\ 0 & 1 & -1 & 1 & -1 & 2\mathrm{i} & \mathrm{i} & -1 \end{bmatrix}. \tag{6.7}$$

In this case, the null space of $\mathbf{A}$ is 1-dimensional and spanned by a nonsingular matrix. As such, $\mathcal{A}$ is injective. We will see that the HMW test has a few important applications. First, we use it to prove the $4N-4$ Conjecture in the $N = 3$ case:

Theorem 105. *The $4N-4$ Conjecture is true when $N = 3$.*

Proof. (a) Suppose $M < 4N - 4 = 8$. Then by the rank-nullity theorem, the super analysis operator $\mathbf{A}\colon \mathbb{H}_{3\times 3} \to \mathbb{R}^M$ has a null space of at least 2 dimensions, and so by the HMW test, $\mathcal{A}$ is not injective.

(b) Consider a 3×8 matrix of 48 real variables $\Phi(x)$ (the real and imaginary parts of each entry are variables). Then $\mathcal{A}$ is injective whenever $x \in \mathbb{R}^{48}$ produces an ensemble $\{\varphi_i(x)\}_{i=1}^{8} \subseteq \mathbb{C}^3$ that passes the HMW test. To pass, the rank-nullity theorem says that the null space of the super analysis operator had better be 1-dimensional and spanned by a nonsingular matrix. Let's use an orthonormal basis for $\mathbb{H}_{3\times 3}$ to find an 8×9 matrix representation of the super analysis operator $\mathbf{A}(x)$; it is easy to check that the entries of this matrix (call it $\mathbf{A}(x)$) are polynomial functions of x. Consider the matrix

$$B(x,y) = \begin{bmatrix} y^{\mathrm{T}} \\ \mathbf{A}(x) \end{bmatrix},$$

and let $u(x)$ denote the vector of $(1,j)$th cofactors of $B(x,y)$. Then $\langle y, u(x)\rangle = \det(B(x,y))$. This implies that $u(x)$ is in the null space of $\mathbf{A}(x)$, since each row of $\mathbf{A}(x)$ is necessarily orthogonal to $u(x)$.

We claim that $u(x) = 0$ if and only if the dimension of the null space of $\mathbf{A}(x)$ is 2 or more, that is, the rows of $\mathbf{A}(x)$ are linearly dependent. First, $(\Leftarrow)$ is true since the entries of $u(x)$ are signed determinants of 8×8 submatrices of $\mathbf{A}(x)$, which are necessarily zero by the linear dependence of the rows. For $(\Rightarrow)$, we have that $0 = \langle y, 0\rangle = \langle y, u(x)\rangle = \det(B(x,y))$ for all $y \in \mathbb{R}^9$. That is, even if y is nonzero and orthogonal to the rows of $\mathbf{A}(x)$, the rows of $B(x,y)$ are linearly dependent, and so the rows of $\mathbf{A}(x)$ must be linearly dependent. This proves our intermediate claim.

We now use the claim to prove the result. The entries of $u(x)$ are coordinates of a matrix $U(x) \in \mathbb{H}^{3\times 3}$ in the same basis as before. Note that the entries of $U(x)$ are polynomials of x. Furthermore, $\mathcal{A}$ is injective if and only if $\det U(x) \neq 0$. To see this, observe three cases:

Case I: $U(x) = 0$, i.e., $u(x) = 0$, or equivalently, $\dim \operatorname{null}(\mathbf{A}(x)) \geq 2$. By the HMW test, $\mathcal{A}$ is not injective.

Case II: The null space is spanned by $U(x) \neq 0$, but $\det U(x) = 0$. By the HMW test, $\mathcal{A}$ is not injective.

Case III: The null space is spanned by $U(x) \neq 0$, and $\det U(x) \neq 0$. By the HMW test, $\mathcal{A}$ is injective.

Since $U(x)$ is a nonzero polynomial the proof is complete. $\square$

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VITA

Jameson Cahill was born in Anchorage, Alaska in 1981. He moved to the outskirts of Sacramento, California in 1997. After graduating high school in 1999 he attended community college for several years and eventually transferred to San Francisco State University in 2003. He completed his undergraduate work in 2006, and stayed at San Francisco State University, completing a MA in Mathematics in 2009. He then transferred to University of Missouri in 2009 to start work on his PhD. Upon completing his PhD. in 2013 he will start a post-doctoral position at Duke University.