Stability estimates for semigroups and partly parabolic reaction diffusion equations

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The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

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ABSTRACT

The purpose of this thesis is to apply methods of the abstract theory of strongly continuous operator semigroups and of evolution semigroups to study the spectral properties of a class of differential operators on the line that appears when one linearizes partial differential equations about such special solutions as steady states or traveling waves.

We begin by discussing the stability of the traveling wave solutions of a reaction-diffusion system with a degenerate diffusion matrix. We demonstrate that under some reasonable assumptions on the system, its spectral stability directly implies the linear stability. In particular, we study asymptotic spectral properties of certain first order matrix differential operators, thus generalizing some results known for the evolution semigroups.

We then turn to abstract strongly continuous operator semigroup on Banach spaces, revisit a quantitative version of Datko's Stability Theorem and obtain the estimates for the constant M satisfying the inequality $\|T(t)\| \leqslant M e^{\omega t}$, for all $t \geqslant 0$, in terms of the norm of the convolution, L^p-Fourier multipliers, and other operators involved in Datko's Stability Theorem. This generalizes recent results for the Hilbert spaces on estimating M in terms of the norm of the resolvent of the generator of the semigroup in the right half-plane.

Chapter 1

Introduction

The literature on infinitely dimensional dynamical systems is well-known to be abundant with insights and concepts, both old and new. One of its most enduring and consistently popular subjects is undoubtedly the problem of stability of traveling waves (see [KP], [S2]). In essence, a traveling wave is a time independent solution of a partial differential equation written in a co-moving coordinate frame. To understand how this concept came into being, let us consider an example: a reaction-diffusion equation in one space dimension,

$$Y_t = \mathcal{D}Y_{xx} + \mathcal{R}(Y),\ x \in \mathbb{R},\ t \geqslant 0,\ \mathcal{R} : \mathbb{R}^N \to \mathbb{R}^N, \tag{1.1}$$

where $\mathcal{D}$ is an $N \times N$ diagonal matrix and the nonlinear map $\mathcal{R}$ is smooth. Passing to the coordinates $\xi = x - ct$, we arrive at the equation

$$Y_t = \mathcal{D}Y_{\xi\xi} + cY_\xi + \mathcal{R}(Y),\ \xi \in \mathbb{R},\ t \geqslant 0. \tag{1.2}$$

A t-independent solution Y_* of (1.2) is called the *traveling wave* for (1.1). A crucial characteristic of traveling waves is that they move with a constant velocity c, but their shape remains unchanged. Although the traveling waves are not the simplest solutions possible (such as spatially constant solutions), they are by far the simplest among the non-trivial ones. Traveling waves arise in many applied problems in diverse areas of optical communications, combustion theory, biomathematics, chemistry and many others. In these applications, the special interest is allotted to the traveling waves that connect two spatially constant solutions, that is, such that

$$\lim_{\xi \to \pm\infty} Y_*(\xi) = Y_\pm.$$

In the literature it is customary to call Y_* a *pulse* if $Y_+ = Y_-$. If $Y_+ \neq Y_-$, then the traveling wave Y_* is called a *front*.

Now, having the three solutions Y_*, $Y_\pm$, we can linearize (1.2) about them. By doing so, we will obtain the linear partial differential equations

$$Y_t = LY := (\mathcal{D}\partial_{\xi\xi} + c\partial_\xi + D\mathcal{R}(Y_*))Y, \tag{1.3}$$

$$Y_t = L_\pm Y := (\mathcal{D}\partial_{\xi\xi} + c\partial_\xi + D\mathcal{R}(Y_\pm))Y, \tag{1.4}$$

where the differential expressions L, $L_\pm$ generate differential operators $\mathcal{L}$, $\mathcal{L}_\pm$ with appropriately chosen domains acting on a suitable function space $\mathcal{E}$. The space can be chosen in a variety of ways, depending on the problem; the common choices being $\mathcal{E} = L^2(\mathbb{R}; \mathbb{C}^N)$; $\mathcal{E} = H^1(\mathbb{R}; \mathbb{C}^N)$, the Sobolev space; and $\mathcal{E} = BUC(\mathbb{R}, \mathbb{C}^N)$, the Banach space of bounded uniformly continuous functions.

The spectral theory of differential operators appearing in (1.3), (1.4) and their generalizations, and the operator semigroups generated by the operators of their type have been studied by many authors, see, e.g. [He], [S2], [KP], [GLSS], [GLS2] and the vast literature cited therein. These studies have in turn paved a way for many interesting concepts in analysis of abstract semigroups of linear operators on Banach and Hilbert spaces ([BJ], [GJLS]).

In this thesis we address two related issues from this circle of ideas. First, we look at the operator $\mathcal{L}$ that generalizes the operator that arise after the linearization of so-called partly parabolic systems, and establish the relationship between its essential spectrum and the essential spectrum of the operator semigroup that $\mathcal{L}$ generates. In fact, we use exponential dichotomies to arrive at resolvent estimates for the differential operator $\mathcal{L}$ with the diffusion matrix $\mathcal{D}$, under the assumption that the matrix $\mathcal{D}$ might have some zero entries. The well known abstract Gearhart-Prüss spectral mapping theorem (see [EN]) then allows us to make appropriate conclusions regarding stability of the operator semigroup generated by $\mathcal{L}$.

Second, working with abstract strongly continuous operator semigroups, we prove a Banach-space generalization of a recent important theorem in [HS] that allows one to gain explicit formulas for the stability estimates for the semigroup as soon as estimates on the resolvent of its generator are known. As a byproduct of our approach, we also add several missing pieces in the so-called quantitative Datko's Theorem, an important and well-studied result in the abstract stability theory for strongly continuous operator semigroups ([ABHN], [CL], [EN], [V]).

To motivate our interest in the first problem, we recall that (1.1) is called partly parabolic (see [SY], [RM2], [KV1], [GLSR] and the literature therein) if the entries d_j of the diagonal matrix $\mathcal{D} = \mathrm{diag}\{d_1, ..., d_N\}$ satisfy $d_j > 0$ for $j = 1, ..N_1$ for some $N_1 < N$ and $d_j = 0$ for $j = +1, ..., N$; it is called parabolic if $N_1 = N$. Thus, (1.1) has diffusion in some equations (those for which $d_i > 0$) and no diffusion in others. Examples include equations modeling nerve impulses, such as Hodgkin-Huxley and FitzHugh-Nagumo; combustion and chemical reaction equations in which some of the reactants are solid (and hence do not diffuse); intracellular calcium dynamics in the presence of immobile buffers ([KV2], [KV3], [KV4], [T2], [TS]); population interaction models in which some populations diffuse and others do not ([CM], [HL], [HSc]); and models for malignant tumor growth [LT].

One direction of study of partly parabolic systems has been existence and properties of attractors of the associated semiflows; see references in [SY]. In addition, examples of traveling waves in partly parabolic equations have been studied for a long time. However, interest in stability of traveling waves for partly parabolic equations as a class appears to be recent. For commentary on this class of equations, see Tsai [T2], Rottmann-Matthes [RM2] and a recent review [GLSR].

The following definitions are widely accepted in the literature on stability of traveling waves:

(i) The traveling wave Y_* (or the respective linear operator $\mathcal{L}$) is called *spectrally stable* in a space $\mathcal{E}$ if the spectrum of the operator $\mathcal{L}$ in $\mathcal{E}$ is contained in the half plane $\{z \in \mathbb{C} : \operatorname{Re} z \leqslant -\gamma\}$ for some $\gamma > 0$, except for a simple eigenvalue at zero.

The presence of the zero eigenvalue in the spectrum is due to the fact that the derivative of the wave solution, Y_*', clearly satisfies the equation $LY_*' = 0$. Assume that 0 is indeed a simple eigenvalue of $\mathcal{L}$ on the space $\mathcal{E}$, and let $\mathcal{Y}$ denote the null space of the spectral projection of the operator $\mathcal{L}$ on the span of Y_*'.

(ii) The traveling wave (or the respective operator $\mathcal{L}$) is called *linearly stable* if $\mathcal{L}$ generates a strongly continuous semigroup $\mathrm{e}^{t\mathcal{L}}$ that, when restricted to $\mathcal{Y}$, is uniformly exponentially stable, that is, satisfies the estimate $\|\mathrm{e}^{t\mathcal{L}}|_{\mathcal{Y}}\| \leqslant M\mathrm{e}^{\omega t}$ for some $M > 0$ and $\omega < 0$.

(iii) A traveling wave is called *nonlinearly stable* (or *orbitally stable*) in $\mathcal{E}$ if a solution to (1.2) that starts near $Y_* + \mathcal{E}$ converges to a particular shift $Y_*(\cdot + \zeta_0)$ of

Y_* as $t \to +\infty$.

The ultimate aim of the stability analysis of a traveling wave is to derive the viable information on nonlinear stability (instability) from the spectral information about the linearization of the wave. An impressive example of this type can be glimpsed from the results of [He]. According to it, if the reaction-diffusion equation is *parabolic* then the spectral stability implies nonlinear stability, that is, (i) implies (ii) and (ii) implies (iii). The latter implication requires some nonlinear arguments based on the Lipschitsh properties of the nonlinearity and the variation of constant formula. The linear part of the argument – that (i) implies (ii) – for the *parabolic* case holds because the linearized operator is sectorial (and therefore generates an analytic semigroup), and the following general abstract fact applies (cf. [EN], [P]): if A is a sectorial operator generating an analytic semigroup $\{T(t)\}_{t\geqslant 0}$ then for the operator spectrum $\sigma(\cdot)$ the following *spectral mapping theorem* holds:

$$\sigma(T(t))\backslash\{0\} = \mathrm{e}^{t\sigma(A)},\ t \geqslant 0. \tag{1.5}$$

We recall that the spectral inclusion "$\supseteq$" in (1.5) as well as the spectral mapping theorem (1.5) for the point spectrum, holds for *any*, not necessarily analytic, but merely strongly continuous semigroup. However, for the essential (Fredholm) spectrum and nonsectorial generators the spectral mapping theorem for non-analytic semigroups frequently fails. This is a well studied phenomenon, and we refer to [ABHN, CL, EN, V] for a detailed discussion of the subject.

Thus, the whole point of the spectral mapping theorem for sectorial generators as it applies in stability theory is that the Fredholm spectral bound

$$s_F(A) := \sup\{\mathrm{Re} z :\ \text{the operator } A - zI \text{ is not Fredholm}\}$$

is equal to the essential growth bound

$$\omega_{ess}(A) := \lim_{t\to\infty} t^{-1} \log \|T(t)\|_{\mathcal{C}},$$

where $\|T\|_{\mathcal{C}} = \inf\{\|T + \mathcal{C}\| : \mathcal{C} \text{ is compact}\}$ so that $\omega_{ess}(A)$ is t^{-1} times the logarithm of the radius of the essential spectrum of $T(t)$, defined as the set of all points in the spectrum except for the set of isolated eigenvalues of finite algebraic multiplicity. Thus, the spectral stability (i) implies the linear stability (ii) for parabolic equations (1.3) because

$$\omega_{ess}(\mathcal{L}) = s_F(\mathcal{L}) \tag{1.6}$$

due to the fact that $\mathcal{L}$ is sectorial. Unfortunately, the equality in the spectral mapping theorem (1.5) does not hold for more general classes of equations and thus the inequality $s_F(A) \leqslant \omega_{ess}(A)$ can actually be strict for many practically important differential operators ([EN], [CL], [V]).

In case of *partly parabolic* systems one can still show that the linear stability (ii) implies the nonlinear stability (iii) (see, e.g., [BJ], [GLSS], [GLSR]), and thus the whole point of stability analysis in this case is therefore to prove that (i) implies (ii). Such a proof for the pulse solutions has been demonstrated in [BJ]. The proof in [BJ] is based on a compact perturbation argument tailored for the particular operator considered therein. Much later, in [GLSS], [GLS2], [RM2] there was a proof that (i) implies (ii) for the front solutions. In particular, [GLS2] contains the proof of the semigroup equality (1.6) for a class of partly parabolic system, while [RM2] shows that (i) implies (ii) for a class of differential operators using the Laplace transform instead of semigroup methods.

In the first part of this thesis we establish equality (1.6) for the matrix differential operator $\mathcal{L}$ on $L^2(\mathbb{R};\mathbb{C}^N)$ of the following type:

$$\mathcal{L} = \begin{bmatrix} \mathcal{A} & R_{12} \\ R_{21} & \mathcal{G} \end{bmatrix}, \qquad \mathcal{A} = d\partial_{\xi\xi} + a\partial_\xi + R, \qquad \mathcal{G} = B\partial_\xi + D, \tag{1.7}$$

where $N = N_1 + N_2$, $d = \mathrm{diag}\{d_1, ..., d_{N_1}\}$ is a constant diagonal matrix with positive entries, $a = a(\xi)$, $R = R(\xi)$ are bounded continuous $(N_1 \times N_1)$ matrix valued functions on $\mathbb{R}$, $B = B(\xi)$, $D = D(\xi)$ are bounded continuous $(N_2 \times N_2)$ matrix valued functions on $\mathbb{R}$, and $R_{12} = R_{12}(\xi)$, $R_{21} = R_{21}(\xi)$ are bounded[1] continuous $(N_1 \times N_2)$ and $(N_2 \times N_1)$ matrix valued functions on $\mathbb{R}$. We assume the existence of limiting values as $\xi \to \pm\infty$, denoted by $a^\pm = \lim_{\xi\to\pm\infty} a(\xi)$, $R^\pm = \lim_{\xi\to\pm\infty} R(\xi)$, etc., and denote by $\mathcal{L}^\pm$, $\mathcal{A}^\pm$, $\mathcal{G}^\pm$ the asymptotic constant coefficients differential operators obtained by replacing a, R, B, D, R_{12}, R_{21} in (1.7) by $a^\pm$, $R^\pm$, $B^\pm$, $D^\pm$, $R_{12}^\pm$, $R_{21}^\pm$ respectively. Further assumptions are listed in the main body of the thesis. In particular, they cover a more general than (1.3) class of operators with $c\partial_\xi$ replaced by a diagonal term $\mathrm{diag}\{c_1., c_n\}\,\partial_\xi$ with different c_j (possibly, ξ-dependent). Note, that the relationship between the spectral and linear stability of this type of problems has so far remained an open question (see the related discussion in [GLS2]). We prove the following result.

Theorem 1.1. *Under the assumptions listed in the main body of the thesis, equality $\omega_{ess}(\mathcal{L}) = s_F(\mathcal{L})$ holds for the differential operator (1.7) in $L^2(\mathbb{R};\mathbb{C}^N)$. In particular, the spectral stability of $\mathcal{L}$ implies its linear stability.*

The proof of this theorem uses two important ingredients. The first one is the following celebrated abstract spectral mapping theorem due to Gearhart and Prüss

[1] Throughout, we use the sane symbol, say R, to denote a matrix valued function $R = R(\cdot)$ and the operator of multiplication by the matrix valued function acting by the rule $(Ru)(\xi) = R(\xi)u(\xi)$, $\xi \in \mathbb{R}$, on $u \in L^2(\mathbb{R};\mathbb{C}^{N_2})$ or any other suitable function space.

(see, e.g., [ABHN], [EN], [V]).

Theorem 1.2. *If A is the generator of a strongly continuous semigroup $\{T(t)\}_{t\geqslant 0}$ on a Hilbert space $\mathcal{X}$ then $\mathrm{e}^z \notin \sigma(T(t))$ if and only if $z + 2\pi ik \notin \sigma(A)$ for all $k \in \mathbb{Z}$ and $\sup_{k\in\mathbb{Z}} \|(A - (z + 2\pi ik))^{-1}\| < \infty$. In other words, the semigroup satisfies the estimate*

$$\|T(t)\| \leqslant M\mathrm{e}^{\omega t} \text{ for all } t \geqslant 0 \tag{1.8}$$

for some $\omega \in \mathbb{R}$ and $M = M(\omega)$ if and only if $\{z \in \mathbb{C} : \ \mathrm{Re} z > \omega\}$ is a subset of the resolvent set $\rho(A)$ for A and

$$N := \sup_{\mathrm{Re} z \geqslant \omega} \|R(z, A)\| < \infty. \tag{1.9}$$

This result shows that in order to control the spectrum of the semigroup operator $T(t)$ and, eventually, the value of $\omega_{ess}(A)$, one has to control the growth of the resolvent operator for A, in particular, when the absolute value of the imaginary part of the spectral parameter z becomes large. This serves as a motivation for introducing the following useful quantity, $s_0^\infty(A)$, (cf. [ABHN, p.385]).

Definition 1.3. Let A be the generator of a strongly continuous semigroup. Define the number $s_0^\infty(A)$ as the infimum of all $\omega \in \mathbb{R}$ such that for some positive number $r = r(\omega)$ the set

$$\Omega_{\omega,r} = \{z = x + iy \in \mathbb{C} : \ x > \omega, |y| > r\} \tag{1.10}$$

belongs to the resolvent set of A and $\sup_{z\in\Omega_{\omega,r}} \|(z - A)^{-1}\| < \infty$.

We remark that $s_0^\infty(A) = -\infty$ provided A is a sectorial operator (also, see Remark 2.2 for more properties of $s_0^\infty(A)$).

Returning back to the the operator $\mathcal{L}$, it is not difficult to see that the operators $\mathcal{A}$, R_{12},R_{21} in the block-representation (1.7) of $\mathcal{L}$ are all sectorial. Using this, one can prove that $s_0^\infty(\mathcal{L}) = s_0^\infty(\mathcal{G})$. Combining this with Gearhart-Prüss Theorem, one can then prove the inequality $s_0^\infty(\mathcal{G}) \geqslant \omega_{ess}(\mathcal{L})$. On the other hand, a well-known Palmer's Theorem (see [S2], [P1], [P2]) helps to show that $s_F(\mathcal{L}) \geqslant \max s_0^\infty(\mathcal{G}^\pm)$, where $\mathcal{G}^\pm$ are asymptotic to $\mathcal{G}$ constant coefficient operators. This brings us to the second essential ingredient of the proof of Theorem 1.1 which is the spectral analysis of the first order differential operator $\mathcal{G} = B\partial_\xi + D$. Our main result in this direction can be summarized in the following theorem.

Theorem 1.4. *Under the assumptions listed in the main body of the thesis, equality*

$$s_0^\infty(\mathcal{G}) = \max\{s_0^\infty(\mathcal{G}^+), s_0^\infty(\mathcal{G}^-)\} \tag{1.11}$$

holds in $L^2(\mathbb{R};\mathbb{C}^{N_2})$ *for the differential operator* $\mathcal{G} = B\partial_\xi + D$.

As soon as this result is established, the chain of inequalities proved in Section 2.3,

$$\begin{aligned} s_F(\mathcal{L}) &\geqslant \max s_F(\mathcal{L}^\pm) \geqslant \max \omega(\mathcal{L}^\pm) \geqslant \max s_0^\infty(\mathcal{L}^\pm) \\ &= \max s_0^\infty(\mathcal{G}^\pm) = s_0^\infty(\mathcal{G}) = s_0^\infty(\mathcal{L}), \end{aligned}$$

shows that $s_F(\mathcal{L}) \geqslant \omega_{ess}(\mathcal{L})$ and thus completes the proof of Theorem 1.1 (we remark that the inequality $s_F(\mathcal{L}) \leqslant \omega_{ess}(\mathcal{L})$ holds for any strongly continuous semigroup due to the spectral inclusion "$\supseteq$" in (1.5)).

We will now briefly review the literature related to Theorems 1.1 and 1.4. In fact, Theorem 1.1 is a direct generalization of the principal result in [GLS2], see Theorem 3.1 there. The main difference between the setting of [GLS2] and the current work is that in [GLS2] the authors assume that the matrix-function $B(\xi)$ in (1.7) is ξ-independent and scalar, that is, $B(\xi) = cI_{N_2\times N_2}$ for some $c \in \mathbb{R}$, while in the current work we allow for B to be ξ-dependent (and when $B(\xi)$ is a constant diagonal matrix $\mathrm{diag}\{c_1, ..., c_{N_2}\}$ we allow $c_j \neq c_k$ for $j \neq k$). Furthermore, even under the more restrictive assumptions of [GLS2], the proof of Theorem 1.1 provided here is simpler than that given in [GLS2]; this is due to the fact that [GLS2] did not not take into account the significance of the spectral abscissa $s_0^\infty(\mathcal{G})$.

Another very general result relating the spectral and linear stability of the semigroups generated by the operators of type (1.7) has been obtained by J. Rottmann-Matthes in [RM2, RM1]. The methods of [RM2] are very different from the ones employed in the current work (for instance, the theory of operator semigroups such as Theorem 1.2 is not used in [RM2] and replaced by an intensive usage of the Laplace transform). The main assumptions on the coefficients of the operator (1.7) are different from the assumptions made in our work (e.g., $B(\xi)$ is assumed to converge to a diagonal matrix $\mathrm{diag}\{c_1, ..., c_{N_2}\}$ as both $\xi \to -\infty$ and $\xi \to +\infty$, cf. [RM2], Assumption 4.24).

There have been of course many earlier works prior to [GLS2] and [RM2] on connections between the spectral and linear stability for (1.7) since this class of operators includes linearization of the FitzHugh-Nagumo equation and other models for electrical activity in neurons [He]; certain combustion models [GLSS]; population models in which some species diffuse and others do not [HS], [M1] (section 13.8), [M2] (pp. 7-9, sections 13.4, 13.5, 13.9) and "buffered" reaction diffusion systems, in which a diffusing reaction product can be absorbed by stationary buffers [KV2], [T1]. We stress that our approach works for traveling fronts and requires no special structural assumptions on the coefficients in (1.7), quite unlike the earlier contributions to the field in [GLSS], [Ni], [S1], [Y].

Theorem 1.4 can be viewed as a generalization of respective theorems on spectral properties of the generators of evolution semigroups [CL]. Indeed, if $B(\xi)$ in (1.7) is a constant scalar matrix, that is, if $B(\xi) = cI_{N_2 \times N_2}$ for some $c \in \mathbb{R}$, then $\mathcal{G} = c\partial_\xi + D(\xi)$ is, essentially, the generator $\mathcal{E} = -\partial_\xi - c^{-1}D(\xi)$ of an evolution semigroup. These operators are known to have many nice properties (e.g., the norm of the resolvent of $\mathcal{E}$ is constant along vertical lines, etc.) that greatly simplify their studies. Many of these properties have been indispensable for the proof given in [GLS2], which means that passing to the case of a non-scalar ξ-dependent $B(\xi)$ can not be done by a simple generalization of the main result of [GLS2]. Jens Rottman-Matthes [RM2] was the first who studied the spectrum of the operator $\mathcal{G} = B(\xi)\partial_\xi + D(\xi)$, although from a different angle, for the case when $B(\xi)$ is a block-diagonal matrix with scalar ξ-dependent blocks. In fact, we have refined and implemented in our proof some of the technical tools specifically developed in [RM2]. In addition, we would like to mention [L], [N], [LGM] as the very nice works on spectral mapping theorems for semigroups generated by linear hyperbolic ordinary differential equations on finite segments (which is quite different from the situation of the full line considered in the current work).

In the second part of this dissertation we concentrate on an abstract strongly continuous operator semigroup $\{T(t)\}_{t\geqslant 0}$ on a Banach space $\mathcal{X}$, and prove a Banach space version of a recent theorem in [HS] relating the stability constant M in (1.8) to the resolvent bound N from (1.9). In addition, we prove a quantitative version of a theorem known in the literature as Datko's Stability Theorem [ABHN, Sec.5.1]. We will now briefly review the results in this part of the thesis.

In the last two decades a significant progress has been made in asymptotic theory of strongly continuous semigroups (see [ABHN, Chapter 5], [CL, Chapters 1-4], [EN, Chapters IV, V], [P, Chapter 4], [V] and the literature cited therein). One of the major results in this direction is Theorem 1.2 which can be rephrased to say that a strongly continuous semigroup on a Hilbert space is uniformly exponentially stable if and only if the norm of the resolvent of its infinitesimal generator A is uniformly bounded in the right half-plane $\{z \in \mathbb{C} : \mathrm{Re}z \geqslant 0\}$, see, e.g., [ABHN, Theorem 5.2.1] or [EN, Theorem V.1.11]. Various versions of this theorem are due to many authors including L. Gearhart, G. Greiner, I. Herbst, J. Howland, F. Huang and J. Prüss, see, e.g., [ABHN, Sec.5.7] for a historical account and further references, and it is usually called the Gearhart-Prüss Theorem.

We recall that for any strongly continuous semigroup $\{T(t)\}_{t\geqslant 0}$ there exist (possibly large) constants λ and $L = L(\lambda)$ such that the following inequality holds:

$$\|T(t)\| \leqslant L\mathrm{e}^{\lambda t} \quad \text{for all } t \geqslant 0. \tag{1.12}$$

In many problems, it is easy to obtain a rough exponential estimate of this type, but one is interested in decreasing λ as much as possible. The infimum of all λ for which there exists an $L = L(\lambda)$ such that (1.12) holds is called the semigroup growth bound, and is denoted by $\omega(T)$. The semigroup is called uniformly exponentially stable if the inequality $\omega(T) < 0$ holds for its growth bound $\omega(T) = \lim_{t\to\infty} \frac{1}{t} \ln\|T(t)\|$, that is, if the inequality (1.8) holds for some negative ω and $M = M(\omega)$. Another useful quantity is the abscissa of uniform boundedness of the resolvent, $s_0(A)$, defined as the infimum of all real $\omega \in \mathbb{R}$ such that $\{z : \mathrm{Re}z > \omega\} \subset \rho(A)$ and $\sup_{\mathrm{Re}z\geqslant\omega} \|R(z, A)\| <$

∞. Then, the Gearhart-Prüss Theorem 1.2 says that if the semigroup acts on a Hilbert space then $\omega(T) = s_0(A)$.

Naturally, one would like to evaluate the constant M in (1.8) via the uniform bound of the norm of the resolvent $R(z, A) = (z - A)^{-1}$ in the half-plane $\{z \in \mathbb{C} : \mathrm{Re} z \geqslant \omega\}$ provided it is finite, that is, via the quantity N defined in (1.9) and via the known constants λ, L that enter the general inequality (1.12). In particular, the necessity in this evaluation arises in many applied issues related to stability of traveling waves, cf. [GLS1, GLS2, GLSS].

In a recent beautiful paper, B. Helffer and J. Sjöstrand proved the following result relating these constants (see [HS, Proposition 2.1]).

Theorem 1.5 (B. Helffer and J. Sjöstrand). *Let* $\{T(t)\}_{t\geqslant 0}$ *be a strongly continuous semigroup on a Hilbert space. If* $\omega, \lambda \in \mathbb{R}$ *are such that* $\omega < \lambda$ *and* (1.12), (1.9) *hold, then* (1.8) *holds with the constant*

$$M = L\big(1 + 2LN(\lambda - \omega)\big). \tag{1.13}$$

In the current work we examine the situation when the strongly continuous semigroup acts on a *Banach* space X. As it is well known, in the Banach space case the Gearhart-Prüss Theorem does not hold (see [ABHN, Example 5.2.2] for an example of $s_0(A) < \omega(T)$ on $L^p(0, 1)$, $p \neq 2$, and more examples in [V]). For Banach spaces, there are several known replacements of the Gearhart-Prüss Theorem proved by R. Datko, M. Hieber, Y. Latushkin, S. Montgomery-Smith, A. Pazy, F. Räbiger, R. Shvydkoy, J. van Neerven, L. Weis, and others (see again some historical comments in [ABHN, Sec.5.7], [CL, LS], [V] and the bibliography therein). These results are summarized in Section 3.2, and sometimes are collectively called the Datko Theorem. It says that a strongly continuous semigroup on a Banach space X is uniformly exponentially stable if and only if a convolution operator, $\mathcal{K}^+$, is bounded on $L^p(\mathbb{R}_+; X)$. Throughout, we fixed p such that $1 \leqslant p < \infty$; here $\mathcal{K}^+$ is defined by

$$(\mathcal{K}^+ u)(t) = \int_0^t T(t-s)u(s)\,ds, \quad t \geqslant 0. \tag{1.14}$$

In Section 3.2 we prove Theorem 3.7, a quantitative version of this theorem, which relates the norm of the convolution operator and the norms of some other operators whose boundedness is also equivalent to the uniform exponential stability of the semigroup.

One can consider the convolution operator, for an $\omega \in \mathbb{R}$, on the exponentially weighted space $L^p_\omega(\mathbb{R}_+; X)$ of the functions $u : \mathbb{R}_+ \to X$ such that $\mathrm{e}^{-\omega(\cdot)}u(\cdot) \in L^p(\mathbb{R}_+; X)$, and ask how to evaluate the constant M in (1.8) via the norm of $\mathcal{K}^+$ on $L^p_\omega(\mathbb{R}_+; X)$ provided it is finite, that is, via the quantity

$$K := \left\|\mathcal{K}^+\right\|_{\mathcal{B}(L^p_\omega(\mathbb{R}_+;X))} < \infty, \tag{1.15}$$

and via the known constants λ, L in (1.12) (alternatively, by the Datko Theorem the operator $\mathcal{K}^+$ in (1.14) can be replaced by any other operator mentioned in this theorem). Our principal result in Chapter 3 is the following theorem.

Theorem 1.6. *Let $\{T(t)\}_{t\geqslant 0}$ be a strongly continuous semigroup on a Banach space, $p \geqslant 1$, $p^{-1} + q^{-1} = 1$. If $\omega, \lambda \in \mathbb{R}$ are such that $\omega < \lambda$ and* (1.12)*,* (1.15) *hold, then* (1.8) *holds with the constant*

$$M = L\big(1 + 4p^{-1/p}q^{-1/q}LK(\lambda - \omega)\big). \tag{1.16}$$

In Chapter 3 we also show how one can derive Theorem 1.5 from Theorem 1.6 using L^p-Fourier multipliers and Parseval's identity.

We conclude the Introduction by saying that one of the goals of this dissertation is to demonstrate how to apply abstract results such as Theorems 1.5 and 1.6 in the study of concrete operators of the type $\mathcal{L}$ as in (1.7). Indeed, the application of a version of Theorem 1.6, recorded below as Corollary 3.8, is a critical ingredient of the proof of Proposition 2.35, which, in turn, is the main equivalency leading to (1.11).

Chapter 2

A spectral mapping result for partly parabolic systems

In this chapter we study a partly parabolic linear matrix differential operator $\mathcal{L}$ that often appears after linearization of the reaction diffusion systems having several zero diffusion coefficients. "Partly parabolic" means that the operator $\mathcal{L}$ can be represented as a (2×2) block operator such that one of the blocks on the main diagonal, $\mathcal{A}$, is a second order sectorial differential operator, and another operator on the main diagonal, $\mathcal{G}$, is a first order differential operator, while the off-diagonal blocks are operators of multiplication by matrix valued functions. We prove that the essential growth bound of the operator $\mathcal{L}$ coincides with the spectral bound of its Fredholm spectrum. This effectively means that stability of the strongly continuous semigroup generated by $\mathcal{L}$ is controlled by its spectrum. The main technical tool used to prove this result is a relation between the abscissa of uniform boundedness of the resolvent at $\pm i\infty$, denoted s_0^∞, for the operators $\mathcal{L}$ and $\mathcal{G}$. To control $s_0^\infty(\mathcal{G})$ we will need to relate it to the similar quantities, $s_0^\infty(\mathcal{G}^\pm)$, for the constant coefficient differential operators asymptotic to $\mathcal{G}$. This relation is the second major result of this chapter and is of independent interest as it can be viewed as a generalization of well-known theorems on spectral properties of generators of evolution semigroups.

The plan of this chapter is as follows. In Section 2.1 we collect some preliminaries. This includes definitions of various spectral characteristics of generators of strongly continuous semigroups and their basic properties. This information will be also used in Chapter 3. In addition, we briefly summarize a well-known method of conjugation of projections used below to relate $s_0^\infty(\mathcal{G})$ and $s_0^\infty(\mathcal{G}^\pm)$. Also, we prove that $\mathcal{G}$ generates a strongly continuous semigroup. In Section 2.2 we set up the partly parabolic system in question, discuss assumptions and formulate the main results of Chapter 2. In Section 2.3 we show how to use $s_0^\infty(\mathcal{L})$ and $s_0^\infty(\mathcal{G})$ to relate the essential growth bound and the Fredholm spectral bound for $\mathcal{L}$. Finally, in Section 2.4 we prove the relation between $s_0^\infty(\mathcal{G})$ and $s_0^\infty(\mathcal{G}^\pm)$.

2.1 Preliminaries

2.1.1 Semigroups

This subsection contains notations and preliminaries regarding strongly continuous operator semigroups on Banach spaces.

Let $\mathcal{X}$ be a Banach space, and let $A : \mathcal{X} \to \mathcal{X}$ be a closed, densely defined linear operator. Its *resolvent set* $\rho(A)$ is the set of $\lambda \in \mathbb{C}$ such that $A - \lambda I$ has a bounded inverse. The complement of $\rho(A)$ is the *spectrum* $\sigma(A)$. It is the union of the *discrete spectrum* $\sigma_{\mathrm{d}}(A)$, which is the set of isolated eigenvalues of A of finite algebraic multiplicity, and the *essential spectrum* $\sigma_{\mathrm{ess}}(A)$, which is the rest. We also define the *point spectrum* $\sigma_p(A)$, which is the set of all eigenvalues of A.

A is *Fredholm* if its range is closed, its kernel has finite dimension n, and its range has finite codimension m. The *index* of a Fredholm operator A is $n - m$. The *Fredholm resolvent set* $\rho_{\mathrm{F}}(A)$ is the set of all $\lambda \in \mathbb{C}$ such that $A - \lambda I$ is Fredholm of index zero. The set $\rho_{\mathrm{F}}(A)$ is open, and its complement, the *Fredholm spectrum* $\sigma_F(A)$, is contained in $\sigma_{ess}(A)$. The set of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$ is denoted by $\mathcal{B}(\mathcal{X}, \mathcal{Y})$; and we abbreviate $\mathcal{B}(\mathcal{X}) = \mathcal{B}(\mathcal{X}, \mathcal{X})$.

We define:

- The *spectral bound* $s(A) = \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\}$.

- The *essential spectral bound* $s_{\mathrm{ess}}(A)$, the infimum of all real ω such that $\sigma(A) \cap \{\lambda : \mathrm{Re}\lambda > \omega\}$ is a subset of $\sigma_{\mathrm{d}}(A)$ and has only finitely many points.

- The *Fredholm spectral bound* $s_F(A) = \sup\{\mathrm{Re}\lambda : \lambda \in \sigma_F(A)\}$.

For a *bounded* linear operator $\mathcal{T} : \mathcal{X} \to \mathcal{Y}$, we define:

- The *spectral radius* of $\mathcal{T}$, the supremum of $\{|\lambda| : \lambda \in \sigma(\mathcal{T})\}$.

- The *essential spectral radius* of $\mathcal{T}$, the supremum of $\{|\lambda| : \lambda \in \sigma_{ess}(\mathcal{T})\}$.

- The seminorm $\|\mathcal{T}\|_{\mathcal{C}} = \inf_{\mathcal{K}} \|\mathcal{T} + \mathcal{C}\|$, where the infimum is over the set of all compact operators $\mathcal{C} : \mathcal{X} \to \mathcal{X}$.

If A generates a C_0-semigroup $\mathcal{T}(t)$, $t \geqslant 0$, we define:

- The *growth bound* $\omega(A) = \lim_{t\to\infty} t^{-1} \log \|\mathcal{T}(t)\|$ (sometimes denoted by $\omega(T)$).

- The *essential growth bound* $\omega_{\mathrm{ess}}(A) = \lim_{t\to\infty} t^{-1} \log \|\mathcal{T}(t)\|_{\mathcal{C}}$.

The following proposition summarizes various well-known facts about these sets and numbers.

Proposition 2.1. *Suppose $A : \mathcal{X} \to \mathcal{X}$ generates the C_0-semigroup e^{tA}, $t \geqslant 0$. Then*

1. *There exists* $\omega \in \mathbb{C}$ *such that* $\{\lambda : \mathrm{Re}\lambda > \omega\} \subset \rho(A)$.

2. *For each* $t > 0$, $e^{t\sigma(A)} \subset \sigma(\mathrm{e}^{tA})$, $\mathrm{e}^{t\sigma_{\mathrm{ess}}(A)} \subset \sigma_{\mathrm{ess}}(\mathrm{e}^{tA})$, *and* $\mathrm{e}^{t\sigma_p(A)} = \sigma_p(\mathrm{e}^{tA}) \setminus \{0\}$.

3. $s_F(A) \leqslant s_{\mathrm{ess}}(A) \leqslant s(A)$.

4. $s(A) \leqslant \omega(A)$ *and* $s_{\mathrm{ess}}(A) \leqslant \omega_{\mathrm{ess}}(A)$.

5. *For each* $t > 0$, $\mathrm{e}^{t\omega(A)}$ *is the spectral radius of* e^{tA}, *and* $\mathrm{e}^{t\omega_{ess}(A)}$ *is the essential spectral radius of* e^{tA}.

6. *Let* $\omega > \omega_{\mathrm{ess}}(A)$ *be a number such that no element of* $\sigma(A)$ *has real part* ω. *Then there is a finite set* $\{\lambda_1, \dots, \lambda_k\} \subset \mathbb{C}$ *such that*

$$\sigma(A) \cap \{\lambda : \mathrm{Re}\lambda > \omega\} = \sigma_{\mathrm{d}}(A) \cap \{\lambda : \mathrm{Re}\lambda > \omega\} = \{\lambda_1, \dots, \lambda_k\}.$$

Let $E_1, \dots, E_k$ *be the generalized eigenspaces of* $\lambda_1, \dots, \lambda_k$ *respectively; they are finite-dimensional. Then there is a closed subspace* E_0 *of* $\mathcal{X}$ *such that* $\mathcal{X} = E_0 \times E_1 \times \dots \times E_k$ *and* E_0 *is invariant under* A. *Moreover, there is a number* $K > 0$ *such that* $\|\mathrm{e}^{tA}|E_0\| \leqslant K\mathrm{e}^{\omega t}$.

Proof. For the equality $e^{t\sigma_p(A)} = \sigma_p(e^{tA}) \setminus \{0\}$, see [EN, Sec. IV.3b]. The rest of the proposition is discussed in [GLSS] where further references are provided. ∎

In addition, we will use the following spectral bound $s_0^\infty(A)$ for the semigroup generator (cf. [ABHN, p.385]),

$$\begin{aligned} s_0^\infty(A) = \inf\{\omega \in \mathbb{R} : \ & \text{there exists } r > 0 \text{ such that } \Omega_{\omega,r} \subset \rho(A) \text{ and} \\ & \sup_{z \in \Omega_{\omega,r}} \|(A - z)^{-1}\| < \infty\}. \end{aligned} \tag{2.1}$$

Here, and everywhere, we use the following notation:

$$\Omega_{\omega,r} = \{z = x + iy \in \mathbb{C} : x > \omega, |y| > r\}, \quad \omega \in \mathbb{R}, \quad r > 0. \tag{2.2}$$

Remark 2.2. The definition of the abscissa $s_0^\infty(A)$ immediately leads to the following easily checkable facts:

(i) $s(A) \leqslant s_0^\infty(A) \leqslant \omega(T)$, because $(A-z)^{-1}$ is the Laplace transform of the semigroup for $\mathrm{Re}z > \omega(T)$;

(ii) $s_0^\infty(A+\omega) = s_0^\infty(A) + \omega$ for any $\omega \in \mathbb{R}$;

(iii) $s_0^\infty(A) = -\infty$ provided A is a sectorial operator (see fig. 1)

(iv) By (i) above, $\sup_{\mathrm{Re}z > \widehat{\omega}} \|(A - zI)^{-1}\| < \infty$ for any $\widehat{\omega} > \omega(T)$. This shows that $\Omega_{\omega,r}$ in definition (2.1) can be replaced by the set $\Omega_{\omega,r} \cap \{z \in \mathbb{C} : \mathrm{Re}z < \widehat{\omega}\}$. Thus, the generator A is allowed to have spectrum in the bounded rectangle $\{z \in \mathbb{C} : \omega \leqslant \mathrm{Re}z \leqslant \widehat{\omega}, |\mathrm{Im}z| \leqslant r\}$. What is not allowed is that for any $\omega > s_0^\infty(A)$ the spectrum of A escapes to infinity along vertical lines $\{\mathrm{Re}z = \omega\}$; moreover, the norm of the resolvent operator $(A - zI)^{-1}$ should be bounded as $|\mathrm{Im}z| \to \infty$ with $\mathrm{Re}z = \omega$. This property differs $s_0^\infty(A)$ from the more common abscissa of uniform boundedness of the resolvent, $s_0(A)$, defined in say [ABHN, p.342] by $s_0(A) = \inf\{\omega \in \mathbb{R} : \Omega_\omega \subset \rho(A)$ and $\sup_{z\in\Omega_\omega} \|(A-z)^{-1}\| < \infty\}$, where $\Omega_\omega = \{z = x + iy \in \mathbb{C} : x > \omega\}$.

In the proof of Theorem 1.4 we will use the following lemma (see [GLS2], Lemma 6.2 and Remark 6.3).

Lemma 2.3. *Let A be the generator of a strongly continuous semigroup $\{T(t))\}_{t\geqslant 0}$ on a Hilbert space $\mathcal{X}$. Assume:*

(i) 1 *is an isolated point of* $\sigma(T(2\pi))$.

(ii) There is a vertical strip Π *around the imaginary axis and an integer* $K > 0$ *such that* $\sigma(A) \cap \Pi \subset \{in : |n| \leqslant K\}$ *and such that* $\sigma(A) \cap \Pi \subset \sigma_d(A)$.

(iii) $\sup_{|n|\geqslant K} \|(A - in)^{-1}\| < \infty$.

For the operator $T(2\pi)$ let $\mathcal{P}$ be the Riesz projection corresponding to the element 1 of its spectrum. Then the range of $\mathcal{P}$ is finitely dimensional.

2.1.2 Evolution semigroups and their properties. Exponential dichotomies. Palmer's Theorem.

Let $A = A(\xi)$ be a piece-wise continuous bounded function of $\xi \in \mathbb{R}$ with values in the set of $(N \times N)$ matrices with complex entries. Let $\mathcal{E} = -\partial_\xi + A(\xi)$ be the first order variable coefficient differential operator on $L^2(\mathbb{R};\mathbb{C}^N)$ with the domain $\operatorname{dom}(\mathcal{E}) = H^1(\mathbb{R};\mathbb{C}^N)$, the usual Sobolev space. Let $\Psi_A(\xi,\xi')$, $\xi \geqslant \xi' \in \mathbb{R}$ denote the Cauchy operator (propagator) of the linear ODE $u' = A(\xi)u$ such that $\Psi_A(\xi,\xi) = I_{N\times N}$. The operator $\mathcal{E}$ is the infinitesimal generator of the strongly continuous semigroup $\{T_A(t)\}_{t\geqslant 0}$ on $L^2(\mathbb{R};\mathbb{C}^N)$, called the evolution semigroup, [CL], defined by

$$(T_A(t)u)(\xi) = \Psi_A(\xi,\xi-t)u(\xi-t), \quad t \geqslant 0,\ \xi \in \mathbb{R},\ u \in L^2(\mathbb{R};\mathbb{C}^N). \tag{2.3}$$

The generators of evolution semigroups have nice properties listed in the next proposition; the proofs can be found in [CL].

Proposition 2.4. *Let $\{T(t)\}_{t\geqslant 0}$ be an evolution semigroup on $L^2(\mathbb{R};\mathbb{C}^N)$ with the differential equation $\mathcal{E} = -\partial_\xi + A(\xi)$. Then the following is true:*

(i) $0 \in \rho(\mathcal{E})$ if and only if $1 \in \rho(T_A(1))$ which in turn holds if and only if the differential equation $u' = A(\xi)u$ has an exponential dichotomy on $\mathbb{R}$.

(ii) $\sigma(\mathcal{E})$ is invariant with respect to translations along the imaginary axis and $\sigma(T_A(t))$ is invariant with respect to rotations centered at zero.

(iii) If $z \in \rho(\mathcal{E})$ then $z + iy \in \rho(\mathcal{E})$ and $\|(A - z - iy)^{-1}\| = \|(A - z)^{-1}\|$ for all $y \in \mathbb{R}$.

(iv) Consequently, $\sigma(\mathcal{E})$ is a union of vertical stripes and

$$s(\mathcal{E}) = s_0^\infty(\mathcal{E}) = s_{ess}(\mathcal{E}) = s_F(\mathcal{E}) = \omega(\mathcal{E}) = \omega_{ess}(\mathcal{E}). \tag{2.4}$$

(v) $\|T_A(t)\|_{B(L^2(\mathbb{R};\mathbb{C}^N))} = \sup_{\xi\in\mathbb{R}} \|\Psi_A(\xi, \xi - t)\|_{\mathbb{C}^{N\times N}}$, $t \geqslant 0$.

(vi) $\omega(T_A) = \inf\{\omega \in \mathbb{R}$: there is an $M = M(\omega)$ such that $\|\Psi_A(\xi, \xi')\|_{\mathbb{C}^{N\times N}} \leqslant M\mathrm{e}^{\omega(\xi-\xi')}$ for all $\xi \geqslant \xi' \in \mathbb{R}\}$.

(vii) If $\omega(T_A) < 0$ then $(\mathcal{E}^{-1}u)(\xi) = \int_{-\infty}^{\infty} \Psi_A(\xi, \xi')u(\xi')d\xi'$ for all $\xi \in \mathbb{R}$, $u \in L^2(\mathbb{R};\mathbb{C}^N)$.

Property (iv) will be frequently used below. If $c \in \mathbb{R}$ then the operator $-c\partial_\xi + A(\xi) = c(-\partial_\xi + c^{-1}A(\xi))$ has the same properties as in (iii), (iv). However, (iii) fails if $c\partial_\xi$ is replaced by, say, $\mathrm{diag}\{c_1, ..., c_N\}\partial_\xi$ with different diagonal entries c_j (let alone by the term $B(\xi)\partial_\xi$ with a ξ-dependent matrix B).

We recall that a differential equation $u' = A(\xi)u$ is said to have exponential dichotomy on $\mathbb{R}$, $\mathbb{R}_+$ or $\mathbb{R}_-$ if there is a continuous bounded projection-valued $(N \times N)$ matrix function $P = P(\xi)$ and constants C, $\alpha > 0$, such that

$$\Psi_A(\xi, \xi')P(\xi') = P(\xi)\Psi_A(\xi, \xi'), \quad \xi \geqslant \xi' \in \mathbb{R},$$

$$\|\Psi_A(\xi, \xi')P(\xi')\| \leqslant C\mathrm{e}^{-\alpha(\xi-\xi')}, \quad \xi \geqslant \xi',$$

$$\|\Psi_A(\xi, \xi')(I - P(\xi'))\| \leqslant C\mathrm{e}^{\alpha(\xi-\xi')}, \quad \xi \leqslant \xi'.$$

Exponential dichotomies are robust under small perturbations. "Smallness" of the perturbation may mean that the perturbation decays to zero in matrix norm at infinities, or that its norm is a summable function. In fact, we will use the following well-known roughness result (see, e.g. [C, Sec.4] or [DK, Sec IV.5]).

Theorem 2.5. *Assume that the differential equation $u' = A(\xi)u$ with a piecewise $(N \times N)$ matrix coefficients $A(\cdot)$ has an exponential dichotomy on $\mathbb{R}_+$. Assume that* *a piece-wise continuous $(N \times N)$ matrix perturbation $B(\cdot)$ satisfies one of the two conditions:*

(a) $\lim_{\xi\to+\infty} \|B(\xi)\|_{\mathbb{C}^{N\times N}} = 0$*, or*

(b) $\|B(\cdot)\|_{\mathbb{C}^{N\times N}} \in L^1(\mathbb{R}_+)$.

Then the perturbed equation $u' = (A(\xi) + B(\xi))u$ has an exponential dichotomy on R.

In what follows we will often consider the situation when the coefficient $A(\cdot)$ in the differential equation $u' = A(\xi)u$ is asymptotically constant. This means that there exist constant matrices A^+ and A^- such that $A^\pm = \lim_{\xi\to\pm\infty} A(\xi)$ (alternatively, such that $\|A(\cdot) - A^\pm\|_{\mathbb{C}^{N\times N}} \in L^1(\mathbb{R}_\pm)$). In this case the top exponential growth of the propagator $\Phi_A(\xi, \xi')$ and therefore the spectrum of the generator $\mathcal{E}$ of the respective evolution semigroup is determined by the real part of the eigenvalues of $A^\pm$ as follows:

$$s(\mathcal{E}) = \omega(T_A) = \max\{\sup_{\lambda\in\sigma(A^+)} \mathrm{Re}\lambda, \sup_{\lambda\in\sigma(A^-)} \mathrm{Re}\lambda\}, \tag{2.5}$$

see e.g. [CL, Chapter 2] and also [GLS2, Prop.5.6 (2), Remark 5.9.]. In particular, the following fact holds:

Lemma 2.6. *Assume $A^\pm = \lim_{\xi\to\pm\infty} A(\xi)$, $A(\cdot)$ is continuous and* $\sup_{\xi\in\mathbb{R}}\{\mathrm{Re}\lambda : \lambda \in \sigma(A(\xi))\} < -\varepsilon < 0$. *Then the differential equation $u' = A(\xi)u$ is uniformly exponentially stable, that is, the propagator satisfies the estimate*

$$\|\Psi_A(\xi, \xi')\|_{\mathbb{C}^{N\times N}} \leqslant M\mathrm{e}^{-\varepsilon(\xi-\xi')}, \text{ for all } \xi \geqslant \xi' \in \mathbb{R},$$

and the spectrum of the generator of the respective evolution semigroup satisfies the inequality $\mathrm{Re}\sigma(\mathcal{E}) < -\varepsilon$.

Finally, if $\mathcal{E}^\pm = -\partial_\xi + A^\pm$ is the generator of the evolution semigroup $T_{A^\pm}(t)$ defined by $(T_{A^\pm}(t)u)(\xi) = \mathrm{e}^{tA^\pm}u(\xi - t)$, then (see [GLS2, Prop.5.6]) the following equalities hold:

$$s(\mathcal{E}) = \omega(\mathcal{E}) = s_0^\infty(\mathcal{E}) = \max\{s_0^\infty(\mathcal{E}^+), s_0^\infty(\mathcal{E}^-)\} = \max\{s(\mathcal{E}^+), s(\mathcal{E}^-)\}. \tag{2.6}$$

Exponential dichotomy of the differential equation $u' = A(\xi)u$ on both half lines $\mathbb{R}_+$ and $\mathbb{R}_-$ is equivalent to the fact that the generator $\mathcal{E} = -\partial_\xi + A(\xi)$ of the respective evolution semigroup is Fredholm in $L^2(\mathbb{R};\mathbb{C}^N)$. This is the content of the celebrated Palmer's Theorem [P1, P2, S2], formulated next.

Theorem 2.7. *Let $A(\cdot)$ be a bounded continuous $(N\times N)$ matrix valued function on $\mathbb{R}$. Then the operator $\mathcal{E} = -\partial_\xi + A(\xi)$ is Fredholm in $L^2(\mathbb{R};\mathbb{C}^N)$ if and only if the differential equation $u' = A(\xi)u$ has exponential dichotomies with projection $\mathcal{P}_\pm(\xi)$ on $\mathbb{R}_\pm$. In this case*

$$\dim\ker\mathcal{E} = \dim(\operatorname{ran}\mathcal{P}_+(0)\cap\ker\mathcal{P}_-(0)),$$

$$\operatorname{codim}\operatorname{ran}\mathcal{E} = \operatorname{codim}(\operatorname{ran}\mathcal{P}_+(0) + \operatorname{ran}\mathcal{P}_-(0)),$$

$$\operatorname{index}\mathcal{E} = \dim\operatorname{ran}\mathcal{P}_+(0) - \dim\operatorname{ran}\mathcal{P}_-(0).$$

Spectral properties of higher order differential operators can be recast in terms of the evolution semigroup generators whose coefficients are obtained by re-writing an eigenvalue differential equation as a first order system. We present a result of this type for the operator $\mathcal{L}$ from (1.7); its proof is identical to the proof of Theorem 4.1 in [GLS2] (see Appendix A there) which in turn is a generalization of a similar result in [SS]. Given the coefficients of the operator $\mathcal{L}$ from (1.7), and $\lambda\in\mathbb{C}$, let us define for $\xi\in\mathbb{R}$ the following $(2N_1+N_2)\times(2N_1+N_2)$ matrix valued function

$$\mathbb{A}(\xi,\lambda) = \begin{pmatrix} 0_{N_1\times N_1} & I_{N_1\times N_1} & 0_{N_1\times N_2} \\ -d^{-1}(R(\xi)-\lambda I_{N_1\times N_1}) & -d^{-1}a(\xi) & -d^{-1}R_{12}(\xi) \\ -B^{-1}(\xi)R_{21}(\xi) & 0_{N_2\times N_1} & -B^{-1}(\xi)(D(\xi)-\lambda I_{N_2\times N_2}) \end{pmatrix} \tag{2.7}$$

and the related first order differential operator $\mathcal{E}_\lambda = -\partial_\xi + A(\xi,\lambda)$.

Theorem 2.8. *The operator $\mathcal{L} - \lambda I$ is Fredholm on $L^2(\mathbb{R}; \mathbb{C}^{N_1+N_2})$ if and only if the operator $\mathcal{E}_\lambda$ is Fredholm on $L^2(\mathbb{R}; \mathbb{C}^{2N_1+N_2})$. In this case* $\dim\ker(\mathcal{L}-\lambda I) = \dim\ker \mathcal{E}_\lambda$*,* $\operatorname{codim}\operatorname{ran}(\mathcal{L} - \lambda I) = \operatorname{codim}\operatorname{ran}\mathcal{E}_\lambda$ *and hence* $\operatorname{index}(\mathcal{L} - \lambda I) = \operatorname{index}\mathcal{E}_\lambda$*.*

2.1.3 Similarity of projections and conjugation (transformation) operators

In this subsection we remind the reader of the way to construct the operator that conjugates two families of projections. This construction will be repeatedly used in Subsections 2.4.1 and 2.4.2 below. Our exposition follows [DK, Sec.IV.1, IV.2] which in turn is based on an earlier work [DKr]. It is interesting to note that the brief historical account in [DK, Chapter IV] does contain a reference to the paper [K2] but not to the paper [K1] (the latter material can be found in [K, Sec. II.4]) nor to an important paper [W] which is widely used and cited in contemporary literature (see e.g. [GLZ, p.165]). We stress however that projections in [DK, DKr] depend on a parameter continuously while in [K1, K2, W] the implied dependence is analytic.

Let us consider in a Banach space two decompositions of identity, $\sum_{i=1}^n Q_i = I$ and $\sum_{i=1}^n P_i = I$, that are composed of pairwise disjoint projections, i.e. such that $Q_iQ_j = 0$ and $P_iP_j = 0$, for all $i \neq j$. The conjugation (transformation) operator T is defined by the formula

$$T = \sum_{i=1}^{n} P_i Q_i. \tag{2.8}$$

Clearly, $TQ_i = P_iT$ and T maps $\operatorname{ran} Q_i$ into $\operatorname{ran} P_i$ for all $i = 1, ..., n$. A short calculation shows that (2.8) can be written as follows:

$$T = I + \sum_{i=1}^{n} P_i(Q_i - P_i).$$

Therefore, if

$$\sum_{i=1}^{n} \|P_i\| \|Q_i - P_i\| < 1 \tag{2.9}$$

then the operator T has bounded inverse, the above mentioned inclusions become $T(\operatorname{ran} Q_j) = \operatorname{ran} P_i$ and projections Q_i, P_i become similar:

$$TQ_iT^{-1} = P_i, \quad i = 1, .., n. \tag{2.10}$$

We note in passing, that a frequently used construction of the transformation operator concerns the situation when $P_i = P_i(t)$ are the pairwise disjoint spectral projections of a bounded operator $\mathcal{F}(t)$ that depends on a parameter t (real or complex). Letting $Q_i = P_i(\tau)$ at a different value τ of the parameter, one is interested in finding the invertible transformation operator $T = T(t, \tau)$ such that $T(t, \tau)P_i(\tau) = P_i(t)T(t, \tau)$. If the operator valued function $\mathcal{F}$ is smooth then $T(t, \tau)$ can be chosen as the propagator of the differential equation (see [DK, Theorem IV.1.1]):

$$\frac{d}{dt}T = \left(\sum_{i=1}^{n} P_i'(t)P_i(t) \right) T, \quad t \geqslant 0.$$

In this thesis the transformation operator (2.8) will be used in a slightly different situation. Given a bounded smooth $(N \times N)$ matrix valued function $\xi \mapsto A(\xi)$, $\xi \in \mathbb{R}$, we let $Q_i(\xi)$ denote its spectral projections of $A(\xi)$. In addition, let $\xi \mapsto B(\xi, \alpha)$ be a bounded smooth $(N \times N)$ matrix valued function which depends on an auxiliary parameter $\alpha \in \mathbb{C}$ such that

$$\sup_{\xi \in \mathbb{R}} \|B(\xi, \alpha)\| \to 0 \text{ as } |\alpha| \to \infty. \tag{2.11}$$

We let $P_i(\xi, \alpha)$ denote the spectral projections of the matrix $A(\xi) + B(\xi, \alpha)$. We now define $T(\xi, \alpha)$ by formula (2.8) with $P_i = P_i(\xi, \alpha)$ and $Q_i = Q_i(\xi)$ and observe that (2.9) holds uniformly in ξ for all sufficiently large $|\alpha|$. Thus, (2.10) holds for $T = T(\xi,\alpha)$, $Q_i = Q_i(\xi)$, $P_i = P_i(\xi,\alpha)$. We now introduce yet another matrix valued function,

$$M(\xi,\alpha) = T^{-1}(\xi,\alpha)(A(\xi) + B(\xi,\alpha))T(\xi,\alpha), \tag{2.12}$$

and note that $M(\xi,\alpha)$ in this direct sum decomposition $\mathbb{C}^N = \sum_{i=1}^{n} \operatorname{ran} Q_i(\xi)$ is block diagonal, that is, commutes with $Q_i(\xi)$:

$$\begin{aligned} M(\xi,\alpha)Q_i(\xi) &= T^{-1}(\xi,\alpha)(A(\xi) + B(\xi,\alpha))T(\xi,\alpha)Q_i(\xi) \\ &= T^{-1}(\xi,\alpha)(A(\xi) + B(\xi,\alpha))P_i(\xi)T(\xi,\alpha) \\ &= T^{-1}(\xi,\alpha)P_i(\xi)(A(\xi) + B(\xi,\alpha))T(\xi,\alpha) \\ &= Q_i(\xi)M(\xi,\alpha). \end{aligned}$$

This allows one to construct a block diagonal differential equation $w' = M(\xi,\alpha)w$ which, up to a small perturbation, is kinematically similar to the differential equation $u' = (A(\xi) + B(\xi,\alpha))u$. The latter statement means the following (see, e.g., [DK, Sec.IV.2]): making a change of variable $u(\xi) = T(\xi,\alpha)w(\xi)$ we obtain the equation

$$w' = M(\xi,\alpha)w - T^{-1}(\xi,\alpha)T'(\xi,\alpha)w$$

with the tern $T^{-1}(\xi,\alpha)T'(\xi,\alpha)$ small in norm due to (2.11) for large $|\alpha|$.

The construction just described is inspired by the interesting proof in [RM2], see Lemma A.2 there, which, in fact, implicitly used the transformation operator (2.8) in the particular situation considered therein.

2.1.4 Strongly continuous semigroups generated by first order differential operators

In this subsection we show that the first order differential operator

$$\mathcal{G} = B(\xi)\partial_\xi + D(\xi), \quad B(\cdot), D(\cdot) \in C_b^1(\mathbb{R};\mathbb{C}^{N\times N}), \tag{2.13}$$

in $L^2(\mathbb{R};\mathbb{C}^N)$ with the domain $\operatorname{dom}\mathcal{G} = H^1(\mathbb{R};\mathbb{C}^N)$, the Sobolev space, generates a strongly continuous semigroup. Our basic assumption in (2.13) is that $B(\cdot)$, $D(\cdot)$ belongs to the space of $(N \times N)$ matrix valued bounded functions with continuous bounded derivatives. Specifically, we prove the following result (cf. [L, N, LGM] for results of this type on finite segments).

Theorem 2.9. *Assume that the matrix functions $B(\cdot), D(\cdot) \in C_b^1(\mathbb{R};\mathbb{C}^{N\times N})$ in* (2.13) *satisfy the following assumptions:*

(i) $B(\xi) = B(\xi)^*$ is an invertible self-adjoint matrix for each $\xi \in \mathbb{R}$;

(ii) The differential equation $u' = \alpha B^{-1}(\xi)u$ has exponential dichotomy on $\mathbb{R}$ for all sufficiently large values of an auxiliary parameter $\alpha > 0$.

Then the operator $\mathcal{G}$ defined by $(\mathcal{G}u)(\xi) = B(\xi)u'(\xi) + D(\xi)u(\xi)$ in $L^2(\mathbb{R};\mathbb{C}^N)$ with the domain $\operatorname{dom}\mathcal{G} = H^1(\mathbb{R};\mathbb{C}^N)$ *generates a strongly continuous semigroup.*

Proof. We recall from [ABHN, Sec.3.4] that an operator A on Hilbert space $\mathcal{X}$ is called *dissipative* if the inequality $\operatorname{Re}\langle Au, u\rangle_{\mathcal{X}} \leqslant 0$, holds for all $u \in \operatorname{dom} A$; here $\operatorname{Re}\langle\cdot,\cdot\rangle_{\mathcal{X}}$ is the scalar product in $\mathcal{X}$. The operator A is called *M-dissipative* for some $M \in \mathbb{R}$ if $A - MI$ is dissipative, that is, if the following inequality holds:

$$\operatorname{Re}\langle Au, u\rangle_{\mathcal{X}} \leqslant M\|u\|^2, \quad u \in \operatorname{dom} A. \tag{2.14}$$

The fundamental Lümer-Phillips Theorem (see e.g. [ABHN, Theorem 3.4.5]) says that a densely defined operator A generates a strongly continuous semigroup of contractions on $\mathcal{X}$ if and only if A is dissipative and the operator $A - \lambda I$ is surjective for some (or all) $\lambda > 0$.

We will now apply this to $\mathcal{G}$ in (2.13). Since $\sup_{\xi\in\mathbb{R}} \|D(\xi)\|_{\mathbb{C}^{N\times N}} < \infty$ by assumption, the operator of multiplication $(Du)(\xi) = D(\xi)u(\xi)$ is bounded. Thus, it suffices to show that the operator $A = B\partial_\xi$ generates a strongly continuous semigroup. By the preceding paragraph we need to check the following two assertions:

(a) $A = B\partial_\xi$ is M-dissipative for some $M \in \mathbb{R}$, and

(b) operator $B\partial_\xi - (M+\lambda)$ is invertible for some $\lambda > 0$.

The proof of assertion (a) for $u \in H^1(\mathbb{R};\mathbb{C}^N)$ requires the following chain of integrations by parts:

$$\begin{aligned}
\operatorname{Re}\langle B\partial_\xi u, u\rangle_{L^2(\mathbb{R};\mathbb{C}^N)} &= \frac{1}{2}\int_{-\infty}^{\infty}\Big(\langle B(\xi)u'(\xi), u(\xi)\rangle_{\mathbb{C}^N} + \overline{\langle B(\xi)u'(\xi), u(\xi)\rangle}_{\mathbb{C}^N}\Big)\,d\xi \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\Big(-\langle u(\xi), (Bu)'(\xi)\rangle_{\mathbb{C}^N} + \overline{\langle B(\xi)u'(\xi), u(\xi)\rangle}_{\mathbb{C}^N}\Big)\,d\xi \\
&= -\frac{1}{2}\int_{-\infty}^{\infty}\langle u(\xi), B'(\xi)u(\xi)\rangle_{\mathbb{C}^N}\,d\xi \\
&+ \frac{1}{2}\int_{-\infty}^{\infty}\Big(-\langle u(\xi), B(\xi)u'(\xi)\rangle_{\mathbb{C}^N} + \overline{\langle B(\xi)u'(\xi), u(\xi)\rangle}_{\mathbb{C}^N}\Big)\,d\xi \\
&= -\frac{1}{2}\int_{-\infty}^{\infty}\langle B'(\xi)u(\xi), u(\xi)\rangle_{\mathbb{C}^N}\,d\xi \\
&\leqslant -\frac{1}{2}\int_{-\infty}^{\infty}\min\{\lambda : \lambda \in \sigma(B'(\xi))\}\|u(\xi)\|^2 d\xi \\
&\leqslant M\|u\|^2_{L^2(\mathbb{R};\mathbb{C}^N)},
\end{aligned}$$

where we denote $M = -\frac{1}{2}\inf_{\xi\in\mathbb{R}}\min\{\lambda : \lambda \in \sigma(B'(\xi))\}$, and where we have used a standard inequality $\langle Tx, x\rangle_{\mathbb{C}^N} \geqslant \min\sigma(T)\|x\|^2_{\mathbb{C}^N}$ for self-adjoint matrix $T = B'(\xi)$ and also the fact that $\cup_{\xi\in\mathbb{R}}\sigma(B'(\xi))$ is a bounded set since $\sup_{\xi\in\mathbb{R}}\|B'(\xi)\|_{\mathbb{C}^{N\times N}} < \infty$ by assumption. This finishes the proof of assertion (a).

To show (b), let us use assumption (ii) in Theorem 2.9 and fix $\alpha_0 > 0$ such that for all $\alpha \geqslant \alpha_0$ the differential equation $u' = \alpha B^{-1}(\xi)u$ has exponential dichotomy on $\mathbb{R}$. By Proposition 2.4 (i) we conclude that the operator $\mathcal{E}_\alpha = -\partial + \alpha B^{-1}(\xi)$ is invertible for all $\alpha \geqslant \alpha_0$. But $B\partial_\xi - (M+\lambda)I = -B(\cdot)\mathcal{E}_\alpha$ with $\alpha = M+\lambda$ and thus assertion (b) holds since the operator of multiplication by $B(\cdot)$ is invertible. ∎

2.1.5 Miscellaneous.

In this subsection we collect some known facts used elsewhere in the thesis for references. We begin with the Grönwall inequality. Originally introduced by Thomas Hakon Grönwall in his 1919 article [Gr], the Grönwall inequality is a handy tool that establishes the upper estimate on the solutions of various ordinary differential equations. We will provide only the assertion; for the proof the reader can refer either to [Gr] or [Ha, Theorem 1.1, Corollary 4.4].

Let $u(\cdot)$, $v(\cdot)$ be two nonnegative functions continuous on $[a,b] \subset \mathbb{R}$ and let $C \geqslant 0$ be a constant. Then

$$u(t) \leqslant C + \int_a^t v(s)u(s)ds \quad \text{implies} \quad u(t) \leqslant C \exp \int_a^t v(s)ds \tag{2.15}$$

for all $t \in [a,b]$. In particular,

$$u(t) \leqslant C_1 + C_2 \int_a^t u(s)ds \quad \text{implies} \quad u(t) \leqslant C_1 \mathrm{e}^{C_2(t-a)}. \tag{2.16}$$

We now recall the variation of constants formula and a standard propagator estimate. Assume that $\Phi_A(\xi,\xi')$ is the propagator of the matrix $(N \times N)$ differential equation $u' = A(\xi)u$ and $\Phi_{A+B}(\xi,\xi')$ is the propagator of the differential equation $u' = (A(\xi)+B(\xi))u$, both normalized to be equal to $I_{N\times N}$ at $\xi = \xi'$. Then, see, e.g. [DK, Sec.III.1.5],

$$\Phi_{A+B}(\xi,\xi') = \Phi_A(\xi,\xi') + \int_\xi^{\xi'} \Phi_A(\xi,s)\Phi_{A+B}(s,\xi')ds, \tag{2.17}$$

$$\|\Phi_A(\xi,\xi')\|_{\mathbb{C}^{N\times N}} \leqslant \exp \int_\xi^{\xi'} \|A(s)\|_{\mathbb{C}^{N\times N}} ds, \quad \xi \geqslant \xi'. \tag{2.18}$$

2.2 The Setting and Main Results

This section contains the settings and formulations of the main results of this chapter, Theorems 2.20 and 2.19 (which have been also formulated as Theorems 1.1 and 1.4 in the Introduction). We give the proofs of these theorems in the next section.

We study the linear differential operator $\mathcal{L}$ associated with the right hand side of the following system of partial differential equations:

$$\begin{aligned} \partial_t u(t,\xi) &= d\partial_{\xi\xi}u(t,\xi) + a(\xi)\partial_\xi u(t,\xi) + R(\xi)u(t,\xi) + R_{12}(\xi)v(t,\xi), \\ \partial_t v(t,\xi) &= B(\xi)\partial_\xi v(t,\xi) + D(\xi)v(t,\xi) + R_{21}(\xi)u(t,\xi), \end{aligned} \tag{2.19}$$

with $\xi \in \mathbb{R}$, $t \geqslant 0$ and the vectors $u = u(t,\xi) \in \mathbb{R}^{N_1}$, $v = v(t,\xi) \in \mathbb{R}^{N_2}$ with nonzero $N_1, N_2 \in \mathbb{N}$. The diagonal matrix $d = \operatorname{diag}\{d_1, ..., d_{N_1}\}$ with $d_j > 0$ is assumed to be constant; all other coefficients in (2.19) are ξ-dependent matrix valued functions of appropriate dimensions having limits as $\xi \to \pm\infty$. The assumptions on the coefficients are discussed in Subsection 2.2.1.

System (2.19) is partly parabolic in the sense that some of the equations in this system have diffusion while others do not. As we have mentioned in the Introduction, our main motivation to study (2.19) comes from the theory of stability of traveling waves where linearization along the wave leads to particular cases of (2.19). Respectively, many papers dealt with systems of this type. The closest paper to our setting is [GLS2] where (2.19) was studied assuming that $B(\xi) = cI_{N_2\times N_2}$ is a constant scalar matrix. This case is significantly simpler than the general case of $B = B(\xi)$ as one can directly apply many of the available results on evolution semigroups [CL]. Finally, the most comprehensive study of systems of type (2.19) has been conducted by J. Rottman-Matthes [RM2]. As a very particular example of his setting, he studied the case (mentioned in [GLS2] as an open question) where $B(\xi) = \mathrm{diag}\{c_1, ..., c_{N_2}\}$, where c_j are different constants. More general, a somewhat similar case of the variable coefficient $B(\cdot)$ has been previously studied in [RM1]. However, both the assumptions and the results of this work are different from that in [RM1, RM2]; in particular, we concentrate more on the semigroup properties of the current operator defined by the right-hand side of (2.19). We also mention the work of M. Lichtner [L] (see also [LGM] and [N]) who proved a spectral mapping theorem for the semigroups associated with equations of the (2.19) type but on finite intervals (which makes a significant difference).

2.2.1 Partly Parabolic Operators

We impose the following basic assumptions on the coefficients in (2.19).

Hypothesis 2.10. *Assume that the variable coefficients in* (2.19) *are bounded continuously differentiable matrix valued functions with bounded derivatives,*

$$\begin{aligned} &a(\cdot),\ R(\cdot) \in C_b^1(\mathbb{R}; \mathbb{C}^{N_1\times N_1}), \qquad R_{12}(\cdot) \in C_b^1(\mathbb{R}; \mathbb{C}^{N_1\times N_2}),\\ &B(\cdot),\ D(\cdot) \in C_b^1(\mathbb{R}; \mathbb{C}^{N_2\times N_2}), \qquad R_{21}(\cdot) \in C_b^1(\mathbb{R}; \mathbb{C}^{N_2\times N_1}), \end{aligned} \tag{2.20}$$

while $d = diag\{d_1, .., d_{N_1}\}$ with constants $d_j > 0$ for $j = 1, .., N_1$.

We suspect that the assumptions on a, R, R_{12}, R_{21} and D can be relaxed to replace C_b^1 by L^1 but such a generalization lies out of the scope of this thesis.

Assuming Hypothesis 2.10 we define the block matrix differential operator $\mathcal{L}$ as follows:

$$\mathcal{L} = \begin{bmatrix} A & R_{12} \\ R_{21} & \mathcal{G} \end{bmatrix}, \text{ where } A = d\partial_{\xi\xi} + a(\cdot)\partial_\xi + R(\cdot), \quad \mathcal{G} = B(\cdot)\partial_\xi + D(\cdot). \tag{2.21}$$

Since the off-diagonal terms in (2.21) are bounded, we set $\mathrm{dom}\,\mathcal{L} = \mathrm{dom}\,A \oplus \mathrm{dom}\,\mathcal{G}$, where we define $\mathrm{dom}\,A = H^2(\mathbb{R}; \mathbb{C}^{N_1})$ and $\mathrm{dom}\,\mathcal{G} = H^1(\mathbb{R}; \mathbb{C}^{N_2})$. The second order differential operator A is sectorial, see e.g. [He, pp.136-137] or [P, Sec.3.2, Corollary 2.3], and thus generates an analytic semigroup (see [He, Theorem 1.3.4]). Under further assumptions on $B(\cdot)$ we will see in Proposition 2.18 below that $\mathcal{G}$ generates a strongly continuous semigroup.

Let us now return to the assumptions imposed on the coefficients of (2.19) (and, therefore, (2.21)).

Hypothesis 2.11. *In addition to Hypothesis* 2.10*, we assume that the variable coefficients in* (2.21) *are asymptotically constant as* $\xi \to +\infty$ *and* $\xi \to -\infty$*. This means that there exist constant matrices*

$$a^+, a^-, R^+, R^-, R_{12}^+, R_{12}^-, R_{21}^+, R_{21}^-, B^+, B^-, D^+, D^- \tag{2.22}$$

such that the following hold:

$$\lim_{\xi\to\pm\infty} a(\xi) = a^\pm, \quad \lim_{\xi\to\pm\infty} R(x) = R^\pm, \quad \lim_{\xi\to\pm\infty} R_{12}(\xi) = R_{12}^\pm, \tag{2.23}$$

$$\lim_{\xi\to\pm\infty} B(\xi) = B^\pm, \quad \lim_{\xi\to\pm\infty} D(x) = D^\pm, \quad \lim_{\xi\to\pm\infty} R_{21}(\xi) = R_{21}^\pm. \tag{2.24}$$

Remark 2.12. Note that some of the results obtained in this thesis hold under alternative assumptions that matrices in (2.22) satisfy conditions:

$$\begin{aligned} &\|a(\cdot) - a^\pm\|_{\mathbb{C}^{N_1\times N_1}} \in L^1(\mathbb{R}_\pm), \quad \|R(\cdot) - R^\pm\|_{\mathbb{C}^{N_1\times N_1}} \in L^1(\mathbb{R}_\pm), \\ &\|R_{12}(\cdot) - R_{12}^\pm\|_{\mathbb{C}^{N_1\times N_2}} \in L^1(\mathbb{R}_\pm), \quad \|B(\cdot) - B^\pm\|_{\mathbb{C}^{N_2\times N_2}} \in L^1(\mathbb{R}_\pm), \\ &\|D(\cdot) - D^\pm\|_{\mathbb{C}^{N_2\times N_2}} \in L^1(\mathbb{R}_\pm), \quad \|R_{21}(\cdot) - R_{21}^\pm\|_{\mathbb{C}^{N_2\times N_1}} \in L^1(\mathbb{R}_\pm). \end{aligned} \tag{2.25}$$

The alternative (2.23), (2.24) versus (2.25) is related to the fact that roughness of exponential dichotomies on $\mathbb{R}_\pm$ can be proven either assuming that the perturbation in small at $\pm\infty$ (see condition (a) in Theorem 2.5) or its norm is summable (see condition (b) in Theorem 2.5). Below, we follow [GLS2, RM2] and impose Hypothesis 2.11 rather than conditions (2.25).

For further notational convenience we let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ and will say that a property holds for all $\xi \in \overline{\mathbb{R}}$ if it holds for the matrices $a(\xi), .., R_{21}(\xi)$ in (2.20) for each $\xi \in \mathbb{R}$ and for the matrices in (2.22) as well (for instance, we say that $B(\xi)$ is invertible for $\xi \in \overline{\mathbb{R}}$ if $B(\xi)$ is invertible for each $\xi \in \mathbb{R}$ and the matrices B^+ and B^- are invertible).

Given the constant matrices in (2.22), we now define two constant coefficient differential operators , $\mathcal{L}^+$ and $\mathcal{L}^-$, obtained by replacing the variable coefficients in (2.21) by their limiting values at $+\infty$ and $-\infty$, respectively:

$$\mathcal{L}^{\pm} = \begin{bmatrix} A^{\pm} & R_{12}^{\pm} \\ R_{21}^{\pm} & \mathcal{G}^{\pm} \end{bmatrix}, \text{ where } A^{\pm} = d\partial_{\xi\xi} + a^{\pm}\partial_{\xi} + R^{\pm}, \quad \mathcal{G}^{\pm} = B^{\pm}\partial_{\xi} + D^{\pm}. \tag{2.26}$$

We will now impose additional assumptions on the matrix function $B = B(\xi)$ in (2.21)

Hypothesis 2.13. *In addition to Hypothesis* 2.11 *we assume that* $B(\xi) = B(\xi)^*$ *are invertible selfadjoint matrices for each* $\xi \in \overline{\mathbb{R}}$

These two conditions seem to be quite natural to ensure the well posedness of the partial differential equation $u_t(t,\xi) = B(\xi)u_{\xi}(t,\xi) + D(\xi)u(t,\xi)$.

We now list additional assumptions on $B(\cdot)$ needed in what follows. In fact, we will impose two independent sets of assumptions, for our results will hold if either set is satisfied. Descriptively, the first set of assumptions ensures the existence of a direct sum decomposition of $\mathbb{C}^{N_2}$ with respect to which $B(\xi)$ is (2×2) block diagonal matrix with two diagonal block having negative and positive spectrum respectively. The second set of assumptions ensures the existence of a finer direct sum decomposition such that $B(\xi)$ is block diagonal with the diagonal blocks being ξ-dependent scalar multiples of the respective projection operators (thus $B(\xi)$ for $\xi \in \mathbb{R}$ are operators of the simplest type in the sense of [DK]).

To begin with the first set of assumptions we recall that the function that maps ξ into the spectrum $\sigma(B^{-1}(\xi))$ is upper semicontinuous due to $B(\cdot) \in C_b(\mathbb{R}; \mathbb{C}^{N_2 \times N_2})$, Hypothesis 2.11 and Hypothesis 2.13, and also semicontinuity of the spectrum (cf. e.g. [DK, Theorem I.2.1]). It follows that $\cup_{\xi \in \mathbb{R}} (\sigma(B^{-1}(\xi)) \cap \mathbb{R}_-)$ is a bounded subset of $\mathbb{R}_-$ separated from zero. We fix a contour γ, symmetric with respect to the real line, which encloses this subset and define for each $\xi \in \overline{\mathbb{R}}$ the Riesz orthogonal projection $Q(\xi)$ in $\mathbb{C}^{N_2}$, corresponding to the negative part of the spectrum of the selfadjoint $B^{-1}(\xi)$ by the formula

$$Q(\xi) = \frac{1}{2\pi i} \int_\gamma (\lambda - B^{-1}(\xi))^{-1} d\lambda, \quad \xi \in \overline{\mathbb{R}}. \tag{2.27}$$

Clearly, $Q(\xi)$ commutes with $B^{-1}(\xi)$ (being the Reisz projection) and thus commutes with $B(\xi)$ for each $\xi \in \overline{\mathbb{R}}$. Our first assumption on the projection $Q(\xi)$ is that it commutes with the derivative of $B(\xi)$.

Hypothesis 2.14. *In addition to Hypothesis* 2.13*, we assume that* $Q(\xi)$ *commutes with* $B'(\xi)$ *for each* $\xi \in \overline{\mathbb{R}}$.

Clearly, if Hypothesis 2.14 holds then

$$(B^{-1}(\xi))' Q(\xi) = Q(\xi) (B^{-1}(\xi))', \quad \xi \in \overline{\mathbb{R}}, \tag{2.28}$$

because $(B^{-1}(\xi))' = -B^{-1}(\xi) B'(\xi) B^{-1}(\xi)$ and $Q(\xi)$ commutes with both $B^{-1}(\xi)$ and $B'(\xi)$. The following simple lemma shows that if B satisfies Hypothesis 2.14 then in fact $Q(\xi)$ is ξ-independent and $B(\xi)$ has a desired block diagonal representation.

Lemma 2.15. *Assume Hypothesis* 2.13. *Then Hypothesis* 2.14 *holds if and only if there exists a ξ-independent projection Q_0 in $\mathbb{C}^{N_2}$ such that in the direct sum decomposition $\mathbb{C}^{N_2} = \operatorname{ran} Q_0 \oplus \ker Q_0$ for each $\xi \in \overline{\mathbb{R}}$ the matrix $B(\xi)$ can be written into a block diagonal form $B(\xi) = B_s(\xi) \oplus B_u(\xi)$, such that $\sigma(B_s(\xi)) \subset (-\infty, -\varepsilon)$ and $\sigma(B_u(\xi)) \subset (\varepsilon, +\infty)$ for some $\varepsilon > 0$ and all $\xi \in \overline{\mathbb{R}}$.*

Proof. We note that $B_s(\xi) = Q_0 B(\xi) Q_0$, $B_u(\xi) = (I - Q_0)B(\xi)(I - Q_0)$ and thus the "only if" part of the lemma is obvious. Assuming Hypothesis 2.14, differentiating (2.27) yields

$$Q'(\xi) = \frac{1}{2\pi i}\int_\gamma (\lambda - B^{-1}(\xi))^{-1}(B^{-1}(\xi))'(\lambda - B^{-1}(\xi))^{-1} d\lambda \tag{2.29}$$

and thus $Q'(\xi)Q(\xi) = Q(\xi)Q'(\xi)$ implies (2.28). Since $Q(\xi)$ is a projection, differentiating $Q(\xi) = Q^2(\xi)$ yields $(I - Q(\xi))Q'(\xi) = Q'(\xi)Q(\xi)$. Then

$$\begin{aligned} Q(\xi)Q'(\xi)Q(\xi) = Q(\xi)(I - Q(\xi))Q'(\xi) &= 0, \\ (I - Q(\xi))Q'(\xi)Q(\xi) &= 0, \\ Q(\xi)Q'(\xi)(I - Q(\xi)) &= 0, \\ (I - Q(\xi))Q'(\xi)(I - Q(\xi)) &= 0, \end{aligned}$$

because $Q'(\xi)$ and $Q(\xi)$ commute. So, in the direct sum decomposition $\mathbb{C}^{N_2} = \operatorname{ran} Q(\xi) \oplus \ker Q(\xi)$, the matrix $Q'(\xi)$ is the zero matrix. Thus, the Riesz projection $Q(\xi)$ is in fact ξ-independent yielding the desired result. ∎

We stress that Hypothesis 2.14 does not impose any additional restrictions on the diagonal blocks $B_s(\xi)$, $B_u(\xi)$ in Lemma 2.15. For instance, let us suppose that

$B_{s,u}(\xi) = \sum_{j=1}^{n_\pm} b_j^{s,u}(\xi) Q_j^{s,u}$ where $Q_j^{s,u}$ are given mutually disjoint projections such that $\sum_{j=1}^{n_+} Q_j^s = Q_0$ and $\sum_{j=1}^{n_-} b_j^u(\xi) Q_j^u = (I - Q_0)$ and $b_j^{s,u}(\cdot)$ are scalar real valued functions. Even in this particular case (a generalization of the situation considered in [RM2]) we do not impose any additional assumptions on $b_j^{s,u}(\xi)$ except for the requirement that $b_j^s(\xi) < 0$, $b_j^u(\xi) > 0$ and both are separated from zero for all $\xi \in \overline{\mathbb{R}}$ which actually follows from the Hypothesis 2.13.

We believe that assumptions in Hypothesis 2.14 should be sufficient to obtain all results listed in the next subsection. Indeed, almost all elements of our proof require merely Hypothesis 2.14. However, we are not able to complete the proof at this point without imposing the following strengthened version of the first assertion in equation (2.24).

Hypothesis 2.16. *In addition to Hypothesis* 2.14 *we assume that there exists a constant* $\mathcal{N} > 0$ *such that* $B(\xi) = B^+$ *for* $\xi \geqslant \mathcal{N}$ *and* $B(\xi) = B^-$ *for* $\xi \leqslant -\mathcal{N}$.

The assumption in Hypothesis 2.16 looks technical, and corresponds to the standard step in analysis of differential operators when one reduces the problem to the case of compactly supported coefficients (indeed, one can approximate any $B(\cdot)$ satisfying Hypothesis 2.14 by $B_{\mathcal{N}}(\cdot)$ satisfying Hypothesis 2.16). A problem which one faces next is passing to the limit as $\mathcal{N} \to +\infty$. This is not a trivial issue though because the term $B_{\mathcal{N}}\partial_\xi$ is of leading order in the operator $\mathcal{G} = B_{\mathcal{N}}\partial_\xi + D$.

We will now describe the second set of assumptions on B related to a finer than (2×2) block decomposition. We recall from [DK, Chapter IV] that $B(\xi)$ is called a matrix of simplest type if there exists a family of mutually disjoint projections Q_j, $j = 1, .., m$, such that $\sum_{j=1}^{m} Q_j = I_{\mathbb{C}^{N_2}}$ and scalar valued function $b_j(\cdot)$, $j = 1, .., m$

such that

$$B(\xi) = \sum_{j=1}^{m} b_j(\xi) Q_j(\xi), \quad \xi \in \overline{\mathbb{R}}. \tag{2.30}$$

Differential equations of the type $u' = B(\xi)u(\xi) + R(\xi)u(\xi)$ are called L-diagonal provided $B(\xi)$ is of simplest type, functions $b_j(\cdot)$ in (2.30) satisfy the additional property

$$\inf_{\xi \in \mathbb{R}} |b_j(\xi) - b_k(\xi)| > 0, \text{ for all } k \neq j, \quad k, j = 1, .., m, \tag{2.31}$$

and the matrix valued function $R(\cdot)$ is integrable on $\mathbb{R}$. It is known that under these assumptions the differential equation is kinematically similar (that is, reducible) to the equation $u' = B(\xi)u$, see [DK, Problem IV.7].

Hypothesis 2.17. *In addition to Hypothesis* 2.13*, assume that* (2.30), (2.31) *hold.*

In particular, functions $b_j(\cdot)$ are real valued, boundedly continuously differentiable and separated from zero. Clearly, if $B(\cdot)$ satisfies Hypothesis 2.17 then it satisfies the significantly more general Hypothesis 2.14.

Hypothesis 2.17 can be viewed as a coordinate-free version of the assumptions imposed on $B(\cdot)$ in [RM2], where $B(\xi)$ was assumed to be of a block-diagonal form $\{b_j(\xi) I_{n_j \times n_j}\}_{j=1}^m$, $\sum_{j=1}^m n_j = N_2$. This prompted the use of Gershgorin's Theorem in [RM2]. By using the coordinate independent representation (2.30) and general transformation/conjugation operators techniques described in Subsection 2.1.3, we were able to modify the proof in [RM2] to avoid the usage of Gershgorin Theorem. Note that no assumptions of the type of Hypothesis 2.16 is needed when Hypothesis 2.17 is imposed.

We will now show that $\mathcal{G}$ generates a strongly continuous semigroup provided Hypothesis 2.14 holds (and so does Hypothesis 2.17).

Proposition 2.18. *Assume Hypothesis* 2.14. *Then the operator* $\mathcal{G} = B\partial_\xi + D$ *generates a strongly continuous semigroup on* $L^2(\mathbb{R}; \mathbb{C}^{N_2})$.

Proof. For proof we will use Theorem 2.9. Assumption (i) of this theorem holds by Hypothesis 2.13. We now show that Hypothesis 2.14 implies (ii) in Theorem 2.9.

We need to show that the equation $u' = \alpha B^{-1}(\xi)u$ has exponential dichotomy on $\mathbb{R}$ for all sufficiently large $\alpha > 0$ (see Theorem 2.9 (ii)). Using Lemma 2.15, we know that $B^{-1}(\xi) = B_s^{-1}(\xi) \oplus B_u^{-1}(\xi)$ in the direct sum decomposition $\mathbb{C}^{N_2} = \operatorname{ran} Q_0 \oplus \ker Q_0$. It suffices to show then that both equations $u' = \alpha B_s^{-1}(\xi)u$ and $u' = \alpha B_u^{-1}(\xi)u$ have exponential dichotomies on $\mathbb{R}$. Since $\sigma(B_s(\xi)) \subset (-\infty, -\varepsilon)$ by Lemma 2.15, we have $\sigma(\alpha B_s^{-1}(\xi)) = \alpha\sigma(B_s^{-1}(\xi)) \subset (-\infty, -\alpha/\varepsilon)$ for all $\xi \in \overline{\mathbb{R}}$ and any $\alpha > 0$. By Lemma 2.6 the differential equation $u' = \alpha B_s^{-1}(x)u$ is stable on $\mathbb{R}$, that is, trivially has exponential dichotomy. A similar argument applies to $\alpha B_u^{-1}(x)$, proving the proposition. ∎

2.2.2 Main Theorems

In this subsection we formulate main results of this chapter. Their proofs are given in Section 2.3 below.

We begin with the operator

$$\mathcal{G} = B(\xi)\partial_\xi + D(\xi), \quad \xi \in \mathbb{R} \text{ in } L^2(\mathbb{R}; \mathbb{C}^{N_2}), \tag{2.32}$$

whose domain, $\operatorname{dom} \mathcal{G} = H^1(\mathbb{R}; \mathbb{C}^{N_2})$ is the usual Sobolev space. Using Hypothesis

2.11, we also define the asymptotic constant coefficient operators,

$$\mathcal{G}^{\pm} = B^{\pm}\partial_{\xi} + D^{\pm}, \quad B^{\pm} = \lim_{\xi\to\pm\infty} B(\xi), \quad D^{\pm} = \lim_{\xi\to\pm\infty} D(\xi). \tag{2.33}$$

Assuming Hypothesis 2.14, we know that $\mathcal{G}$, $\mathcal{G}^{\pm}$ generate strongly continuous semigroups (see Proposition 2.18). Thus, we may define $s_0^{\infty}(\mathcal{G}), s_0^{\infty}(\mathcal{G}^{\pm})$ with $A = \mathcal{G}, \mathcal{G}^{\pm}$ in (2.1), that is,

$$\begin{aligned} s_0^{\infty}(\mathcal{G}) = \inf\{\omega \in \mathbb{R} :\ &\text{there exists } r > 0 \text{ such that } \Omega_{\omega,r} \subset \rho(\mathcal{G}) \text{ and} \\ &\sup_{z\in\Omega_{\omega,r}} \{\|(\mathcal{G} - z)^{-1}\| < \infty\}, \end{aligned} \tag{2.34}$$

$$\begin{aligned} s_0^{\infty}(\mathcal{G}^{\pm}) = \inf\{\omega \in \mathbb{R} :\ &\text{there exists } r > 0 \text{ such that } \Omega_{\omega,r} \subset \rho(\mathcal{G}^{\pm}) \text{ and} \\ &\sup_{z\in\Omega_{\omega,r}} \{\|(\mathcal{G}^{\pm} - z)^{-1}\| < \infty\}. \end{aligned} \tag{2.35}$$

We recall our usual notation $\Omega = \Omega_{\omega,r} = \{z = x + iy \in \mathbb{C} : x > \omega, |y| > r\}$, $\omega, r \in \mathbb{R}$ see (2.2).

Our first main result shows that the asymptotic bounds of the resolvents of the operator $\mathcal{G}$ and the asymptotic operators $\mathcal{G}^{\pm}$ are directly correlated.

Theorem 2.19. *Assume either Hypothesis* 2.16 *or Hypothesis* 2.17. *Then*

$$s_0^{\infty}(\mathcal{G}) = \max\{s_0^{\infty}(\mathcal{G}^{+}), s_0^{\infty}(\mathcal{G}^{-})\}. \tag{2.36}$$

Equality (2.36) is derived from Hypothesis 2.16 in Subsection 2.2.1. The main instrument of the proof is construction of a kinematic similarity of the equation $u' = B^{-1}(\xi)u + z^{-1}B^{-1}(\xi)D(\xi)u$ to a (2×2) block diagonal equation $u' = M(\xi, z)u$ which holds up to a term of order $O(|z|^{-1})$ as $|z| \to \infty$.

Equality (2.36) is derived from Hypothesis 2.17 in Subsection 2.2.1. Here, the "almost" kinematically similar equation $u' = M(\xi, z)u$ will have m diagonal blocks; each of them is a multiple of a ξ-independent projection by a scalar valued function.

Equality (2.36) is a generalization of the basic relation (2.6), stemming from the theory of evolution semigroups. As far as we know, this is a new result (e.g., the quantity $s_0^\infty(\mathcal{G})$ was never used in [GLS2]), although the block diagonalization techniques used in its proof is inspired by [RM2] (in fact, the methods of [RM2] are combined with the transformation (conjugation) of projections techniques in Subsection 2.1.3).

Our second main result concerns the (2×2) block operator (2.21), that is,

$$\mathcal{L} = \begin{bmatrix} \mathcal{A} & R_{12} \\ R_{21} & \mathcal{G} \end{bmatrix}, \text{ where } \mathcal{A} = d\partial_{\xi\xi}^2 + a(\cdot)\partial_\xi + R(\cdot), \mathcal{G} = B(\cdot)\partial_\xi + D(\cdot), \quad (2.37)$$

in $L^2(\mathbb{R};\mathbb{C}^{N_1+N_2})$ with the domain $\operatorname{dom}\mathcal{L} = H^2(\mathbb{R};\mathbb{C}^{N_1}) \oplus H^1(\mathbb{R};\mathbb{C}^{N_2})$. Since $\mathcal{A}$ generates an analytic semigroup, and the operators of multiplications by $R_{12}(\cdot)$ and $R_{21}(\cdot)$ are bounded, using Proposition 2.18 and a standard perturbation result from [EN, Sec.III.1], we conclude that $\mathcal{L}$ generates a strongly continuous semigroup as soon as Hypothesis 2.14 holds. Thus, one can introduce all spectral and growth bounds as described in Section 2.1.1, in particular, we will use the bound

$$s_F(\mathcal{L}) = \sup\{\mathrm{Re}\lambda : \mathcal{L} - \lambda \text{ is not a Fredholm index zero operator}\},$$

the Fredholm spectral bound, and the essential growth bound $\omega_{ess}(\mathcal{L})$ equals to times the logarithm of the essential spectral radius of the operator $T_\mathcal{L}(1)$. In principle, $s_F(\mathcal{L})$ can be calculated as $\max\{s_0^\infty(\mathcal{L}^+), s_0^\infty(\mathcal{L}^-)\}$, while calculation of $\omega_{ess}(\mathcal{L})$ requires knowledge of the semigroup $T_\mathcal{L}(t)$ itself. Thus the equality of $s_F(\mathcal{L})$ and $\omega_{ess}(\mathcal{L})$ is a useful fact which also implies that the spectral stability of the operator $\mathcal{L}$ implies the linear stability semigroup generated by $\mathcal{L}$.

Theorem 2.20. *Assume that Hypothesis* 2.11 *holds, together with the conclusion of*

Theorem 2.19. *Then*

$$\omega_{ess}(\mathcal{L}) = s_F(\mathcal{L}) \text{ in } L^2(\mathbb{R};\mathbb{C}^{N_1+N_2}). \tag{2.38}$$

In particular, spectral stability of $\mathcal{L}$ implies its linear stability.

Equality (2.38) is proven in the next subsection. This is a generalization of a similar result in [GLS2] where only the case $B(\xi) = cI_{N_2\times N_2}$ was considered. The proof in [GLS2] did not use the bounds $s_0^\infty(\mathcal{G}), s_0^\infty(\mathcal{G}^\pm)$ and thus was less transparent. An important result relating the linear and spectral stability was proved in [RM2] while using techniques different from the strongly continuous semigroup techniques used in this thesis.

2.3 Proof of theorem 1.1

In this subsection we use the relation $s_0^\infty(\mathcal{G}) = \max\{s_0^\infty(\mathcal{G}^\pm)\}$ and the Gearhart-Prüss spectral mapping Theorem 1.2 to show equality $s_F(\mathcal{L}) = \omega_{ess}(\mathcal{L})$ for the operator $\mathcal{L}$ from (2.21) induced by a partly parabolic system, thus proving Theorem 2.20 (or Theorem 1.1 from the Introduction).

By items (3) and (4) in Proposition 2.1, we know that $s_F(\mathcal{L}) \leqslant \omega_{ess}(\mathcal{L})$ since $\mathcal{L}$ generates a strongly continuous semigroup. Thus, we only need to show that

$$\omega_{ess}(\mathcal{L}) \leqslant s_F(\mathcal{L}). \tag{2.39}$$

Throughout this section, we assume that Hypothesis 2.11 and the conclusions of Theorem 2.19 hold.

The proof naturally splits into the following four steps recorded in Propositions 2.21 – 2.24. Before we begin, we recall again that we can use the relation $s_0^\infty(\mathcal{G}) =$

$\max\{s_0^\infty(\mathcal{G}^+), s_0^\infty(\mathcal{G}^-)\}$, which holds by Theorem 2.19 (the proof of Theorem 2.19 is given in the next section).

Proposition 2.21. *Assume Hypothesis* 2.11*. Then*

$$s_F(\mathcal{L}) = \max\{s_F(\mathcal{L}^+), s_F(\mathcal{L}^-)\}. \tag{2.40}$$

The proof of this fact uses Palmer's Theorem 2.7 and is almost verbatim identical to the proof of Proposition 3.2 in [GLS2]. We will indicate the main steps of the proof in a moment.

Proposition 2.22. *Let A be any of the constant coefficient differential operators $\mathcal{A}^\pm$, $\mathcal{G}^\pm$, $\mathcal{L}^\pm$. Then*

$$s(A) = s_{ess}(A) = s_F(A) = \omega(A) = \omega_{ess}(A). \tag{2.41}$$

Proof. Fourier transform converts A into a multiplication operator for which these properties are known, see[EN, Propositions I.4.2, I.4.10, IV.3.13]. ∎

Proposition 2.23. *Assume Hypothesis* 2.11*. Then*

$$s_0^\infty(\mathcal{G}) = s_0^\infty(\mathcal{L}), \tag{2.42}$$

and, in particular,

$$s_0^\infty(\mathcal{G}^\pm) = s_0^\infty(\mathcal{L}^\pm). \tag{2.43}$$

Proposition 2.24. *Assume Hypothesis* 2.11*. Then*

$$\max\{s_F(\mathcal{L}), s_0^\infty(\mathcal{L})\} \geqslant \omega_{ess}(\mathcal{L}). \tag{2.44}$$

We postpone the proof of the last two propositions and finish the the proof of the statement of Theorem 1.1 first.

Fix $z \in \mathbb{C}$ such that $\mathrm{Re} z > s_F(\mathcal{L})$. We need to show that $\mathrm{Re} z > \omega_{ess}(\mathcal{L})$. This is due to the following chain of implications:

- $\mathrm{Re} z > s_F(\mathcal{L})$ implies $\mathrm{Re} z > \max\{s_F(\mathcal{L}^{\pm})\}$ by Proposition 2.21,
- ...which implies $\mathrm{Re} z > \max\{\omega(\mathcal{L}^{\pm})\}$ by Proposition 2.22 for $A = \mathcal{L}^{\pm}$,
- ...which implies $\mathrm{Re} z > s_0^{\infty}(\mathcal{L}^{\pm})$ by Remark 2.2(i),
- ...implying $\mathrm{Re} z > \max\{s_0^{\infty}(\mathcal{G}^{\pm})\}$ by (2.43) in Proposition 2.23,
- ...implying $\mathrm{Re} z > s_0^{\infty}(\mathcal{G})$ by Theorem 1.4,
- ...implying $\mathrm{Re} z > s_0^{\infty}(\mathcal{L})$ by (2.42) in Proposition 2.23,
- ...which finally implies $\mathrm{Re} z > \omega_{ess}(\mathcal{L})$ by Proposition 2.24.

This completes the proof of Theorem 1.1. We are now ready to present the proofs of Propositions 2.21, 2.23 and 2.24. We begin with the proof of Proposition 2.21.

Proof. We follow the strategy of [GLS2, Proposition 3.2]. Consider the following matrix-valued function (cf. (2.7)),

$$\mathbb{A}_z(\xi) = \begin{pmatrix} 0 & I & 0 \\ -d^{-1}(R(\xi) - zI) & -d^{-1}a(\xi) & -d^{-1}R_{12}(\xi) \\ -B^{-1}(\xi)R_{21}(\xi) & 0 & -B^{-1}(\xi)(D(\xi) - zI) \end{pmatrix} \tag{2.45}$$

and the related first order differential operator $\mathcal{J}_z = \partial_\xi - \mathbb{A}_z(\xi)$.

Lemma 2.25. *The operator $\mathcal{L} - zI$ is Fredholm in $L^2(\mathbb{R}; \mathbb{C}^N)$ if and only if the operator $\mathcal{J}_z$ is Fredholm on $L^2(\mathbb{R}; \mathbb{C}^{N_1} \oplus \mathbb{C}^{N_1} \oplus \mathbb{C}^{N_2})$. In this case* $\dim\ker(\mathcal{L} - zI) = \dim\ker \mathcal{J}_z$, $\operatorname{codim}\operatorname{ran}(\mathcal{L} - zI) = \operatorname{codim}\operatorname{ran} \mathcal{J}_z$.

Proof. This is a slight generalization of Theorem 4.1 in [GLS2], which, in turn, is a generalization of a result in [SS]. The proof is the same as in [GLS2, Appendix A]. ∎

If $z \in \sigma(\mathcal{L}^+)$ then $\mathcal{J}_z^+ = \partial_\xi - \mathbb{A}_z^+$ is not Fredholm by Lemma 2.25. Then $u' = \mathbb{A}_z^+ u$ has no exponential dichotomy on $\mathbb{R}_+$ by Palmer's Theorem 2.7. Since exponential dichotomy persists under perturbations, due to Hypothesis 2.10 and Theorem 2.5, $u' = \mathbb{A}_z(\xi)u$ has no exponential dichotomy on $\mathbb{R}_+$. Thus, $\mathcal{J}_z$ is not Fredholm by Palmer's Theorem 2.7 and $z \in \sigma_F(\mathcal{L})$ by Lemma 2.25. This proves $s_F(\mathcal{L}^+) \leqslant s_F(\mathcal{L})$; the proof of $s_F(\mathcal{L}^-) \leqslant s_F(\mathcal{L})$ is similar, and the inverse inequalities follows as in [GLS2, Proposition 4.9] ∎

Next, we prove Proposition 2.23.

Proof. It suffices to show (2.42) since (2.43) is a particular case of (2.42). The proof is based on the following block structure of the resolvent of the operator $\mathcal{L}$ described in [GLS2, Lemma 8.1]. We introduce the following operator families

$$\mathcal{H}(z) = \mathcal{G} - zI - R_{21}(\mathcal{A} - zI)^{-1}R_{12}, \quad z \in \rho(\mathcal{A}); \tag{2.46}$$

$$\mathcal{F}(z) = (\mathcal{G} - zI)^{-1}R_{21}(\mathcal{A} - zI)^{-1}R_{12}, \quad z \in \rho(\mathcal{A}) \cap \rho(\mathcal{G}); \tag{2.47}$$

$$\mathcal{Q}(z) = \mathcal{A} - zI - R_{12}(\mathcal{G} - zI)^{-1}R_{21}, \quad z \in \rho(\mathcal{G}). \tag{2.48}$$

and cite the following result from [GLS2]:

Lemma 2.26. *If $z \in \rho(\mathcal{A})$, then $z \in \rho(\mathcal{L})$ if and only if $\mathcal{H}(z)$ is invertible. In this case the resolvent of $\mathcal{L}$ is the block operator matrix $(\mathcal{L} - zI)^{-1} = (\mathcal{R}_{ij}(z))_{i,j=1}^2$, where:*

$$\mathcal{R}_{11}(z) = (\mathcal{A} - zI)^{-1} + (\mathcal{A} - zI)^{-1}R_{12}\mathcal{H}(z)^{-1}R_{21}(\mathcal{A} - zI)^{-1}, \tag{2.49}$$

$$\mathcal{R}_{12}(z) = -(\mathcal{A} - zI)^{-1}R_{12}\mathcal{H}(z)^{-1}, \tag{2.50}$$

$$\mathcal{R}_{21}(z) = -\mathcal{H}(z)^{-1}R_{21}(\mathcal{A} - zI)^{-1}, \tag{2.51}$$

$$\mathcal{R}_{22}(z) = \mathcal{H}(z)^{-1}. \tag{2.52}$$

Proof. For $z \in \rho(\mathcal{A})$, we have

$$\begin{aligned} \mathcal{L} - zI &= \begin{pmatrix} \mathcal{A} - zI & 0 \\ R_{21} & I \end{pmatrix} \begin{pmatrix} I & (\mathcal{A} - zI)^{-1} R_{12} \\ 0 & \mathcal{H}(z) \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ R_{21}(\mathcal{A} - zI)^{-1} & \mathcal{H}(z) \end{pmatrix} \begin{pmatrix} \mathcal{A} - zI & R_{12} \\ 0 & I \end{pmatrix}. \end{aligned} \tag{2.53}$$

The lemma follows from the triangular structure of the block operator matrices in (2.53). ∎

Recall the definition of $s_0^\infty(A)$ in (2.1) and $\Omega_{\omega,r}$ in (2.2). We claim that for $\omega \in \mathbb{R}$ the following three assertions, (i), (ii) and (iii), are equivalent:

(i) $\omega > s_0^\infty(\mathcal{L})$;

(ii) there exists $r = r(\omega)$ such that $\Omega_{\omega,r}$ has the following properties:

(ii.1) $\Omega_{\omega,r} \subset \rho(\mathcal{A})$ and thus $\mathcal{H}(z)$ in (2.46) is well defined for each $z \in \Omega_{\omega,r}$;

(ii.2) $\mathcal{H}(z)$ is invertible for each $z \in \Omega_{\omega,r}$;

(ii.3) $\sup_{z\in\Omega_{\omega,r}} \|\mathcal{H}(z)^{-1}\| < \infty$;

(iii) $\omega > s_0^\infty(\mathcal{G})$.

Clearly, $s_0^\infty(\mathcal{L}) = s_0^\infty(\mathcal{G})$ holds as soon as the equivalence of (i) and (iii) in the claim is established. We will now justify the claim.

Proof of (i) ⇒ (ii). Since $\omega > s_0^\infty(\mathcal{L})$, there exists $r = r(\omega)$ such that $\Omega_{\omega,r} \subset \rho(\mathcal{L})$ and

$$\sup_{z\in\Omega_{\omega,r}} \|(\mathcal{L} - z)^{-1}\| < \infty. \tag{2.54}$$

Since $\mathcal{A}$ is sectorial, $\sigma(\mathcal{A})$ is contained in a sector, and so by choosing an appropriately large r we end up with $\Omega_{\omega,r} \subset \rho(\mathcal{A})$ and thus (ii.1) hold. By Lemma 2.26

then (ii.2) holds. Since $\mathcal{R}_{22}(z) = \mathcal{H}^{-1}(\xi)$ is a block of $(\mathcal{L} - zI)^{-1}$, inequality (2.54) yields (ii.3).

Proof of (ii) ⇒ (i). Since $\|(\mathcal{A} - z)^{-1}\| = O(|z|^{-1})$ for all z outside of a sector because of sectoriality of $\mathcal{A}$, the required implication follows from (2.49)-(2.50) in Lemma 2.26.

Proof of (ii) ⇒ (iii). As follows from Lemma 2.26, we can write

$$\mathcal{G} - zI = \mathcal{H}(z)(I - \mathcal{J}_1(z)) \tag{2.55}$$

with $\mathcal{J}_1(z) = -\mathcal{H}^{-1}(z)\mathcal{R}_{21}(\mathcal{A} - zI)^{-1}\mathcal{R}_{12}$.

As before, one should make use of sectoriality of $\mathcal{A}$; by choosing an appropriately large r one will obtain $\|\mathcal{J}_1(z)\| \leqslant \frac{1}{2}$ uniformly in $z \in \Omega_{\omega,r}$, yielding (iii).

Proof of (iii) ⇒ (ii). By sectoriality of $\mathcal{A}$, we have (ii1) for all r large enough. Rewriting the operator $\mathcal{H}(z)$ from Lemma 2.26 as

$$\mathcal{H}(z) = (\mathcal{G} - zI)(I - \mathcal{J}_2(z)) \tag{2.56}$$

with $\mathcal{J}_2(z) = -(\mathcal{G} - zI)^{-1}\mathcal{R}_{21}(A - zI)^{-1}\mathcal{R}_{12}$, and increasing r (if necessary), one can obtain the inequality $\|\mathcal{J}_2(z)\| \leqslant \frac{1}{2}$ uniformly in $z \in \Omega_{\omega,r}$, yielding (ii2) and (ii3). ∎

Finally, we prove Proposition 2.24.

Proof. As we have seen, if $\mathrm{Re}z > s_F(\mathcal{L})$ then $\mathrm{Re}z > s_0^\infty(\mathcal{L})$, that is, $s_F(\mathcal{L}) \geqslant s_0^\infty(\mathcal{L})$. Fix any z with $\mathrm{Re}z > s_F(\mathcal{L}) \geqslant s_0^\infty(\mathcal{L})$. Our objective is to prove that

$$\mathrm{e}^z \in \rho(\mathrm{e}^{\mathcal{L}}) \cup \sigma_d(\mathrm{e}^{\mathcal{L}}). \tag{2.57}$$

Indeed, since $\exp(\omega_{ess}(\mathcal{L}))$ is the radius of $\sigma_{ess}(\mathrm{e}^{\mathcal{L}})$ by Proposition 2.1(5), inclusion (2.57) yields $\mathrm{Re}z > \omega_{ess}(\mathrm{e}^{\mathcal{L}})$, as required. We have the following two possibilities:

(a) $z_k := z + 2\pi i k \in \rho(\mathcal{L})$ for all $k \in \mathbb{Z}$;

(b) $\{z + 2\pi i k : k \in \mathbb{Z}\} \cap \sigma(\mathcal{L}) \neq \emptyset$.

Assume that(a) holds. Since $\mathrm{Re}z > s_0^\infty(\mathcal{L})$, we know that $\sup_{k\in\mathbb{Z}} \|(\mathcal{L} - z_k I)^{-1}\| < \infty$. By Gearhart-Prüss Theorem 1.2 then $\mathrm{e}^z \in \rho(\mathrm{e}^{\mathcal{L}})$ and (2.57) holds.

In order to analyze case (b), we should invoke the following lemma (cf. [GLS2, Corollary 7.3]):

Lemma 2.27. *Assume Hypothesis* 2.11. *Then* $s_{ess}(\mathcal{L}) = s_F(\mathcal{L})$.

Proof. Since $s_F(\mathcal{L}) \leqslant s_{ess}(\mathcal{L})$ by Proposition 2.1(3), we only need to show the inequality $s_{ess}(\mathcal{L}) \leqslant s_F(\mathcal{L})$. Since $\mathcal{L}$ generates a strongly continuous semigroup, by Proposition 2.1(1) there exists $\omega_0 \in \mathbb{R}$ such that $\{\lambda : \mathrm{Re}\lambda > \omega_0\} \subset \rho(\mathcal{L})$. Let $s_F(\mathcal{L}) < \omega < \omega_0$. Since $s_F(\mathcal{L}) \geqslant s_0^\infty(\mathcal{L})$, we have $\omega > s_0^\infty(\mathcal{L})$. In particular, this implies that $\sigma(\mathcal{L}) \cap \{\lambda : \mathrm{Re}\lambda > \omega\}$ is contained in a compact set $\mathcal{K} = \{z \in \mathbb{C} : \omega \leqslant \mathrm{Re}z \leqslant \omega_0$ and $|\mathrm{Im}z| \leqslant r\}$ for some $r > 0$. If $\sigma(\mathcal{L}) \cap \mathcal{K}$ were an infinite set, there would exist a point $\lambda_0 \in \mathcal{K}$ such that every deleted neighborhood of λ_0 contains both points of $\rho(\mathcal{L})$ and points $\lambda \in \sigma(\mathcal{L})$ at which $\mathcal{L} - \lambda I$ is Fredholm of index zero. This would contradict Theorem IV.5.31 of [K]. Therefore $\sigma(\mathcal{L}) \cap \mathcal{K}$ is finite and $\mathcal{L}$ has no essential spectrum in $\{\lambda : \mathrm{Re}\lambda > \omega\}$, proving $s_{ess}(\mathcal{L}) \leqslant s_F(\mathcal{L})$. ∎

Assume that (b) holds. Since $\mathrm{Re}z > s_F(\mathcal{L})$, by Lemma 2.27 we know that the intersection $\{z+2\pi i k : k \in \mathbb{Z}\} \cap \sigma(\mathcal{L})$ consists of points in $\sigma_d(\mathcal{L})$ and contains finitely many points. In addition, there is a thin vertical strip that includes these points but does not include any other point of $\sigma(\mathcal{L})$. Clearly, $\mathrm{e}^z \in \sigma_p(\mathrm{e}^{\mathcal{L}})$ by Proposition 2.1(2). By Gearhart-Prüss Theorem 1.2 we see that e^z is then an isolated point in $\sigma(\mathrm{e}^{\mathcal{L}})$ because the strip is located to the right from $s_0^\infty(\mathcal{L})$. To complete the proof of

$e^z \in \sigma_d(e^{\mathcal{L}})$, it remains to show that the spectral projection of $e^{\mathcal{L}}$ that corresponds to e^z us finite dimensional. This follows from Lemma 6.2 and Remark 6.3 of [GLS2] (see Lemma 2.3) applied to the operator $\mathcal{A} = \frac{1}{2\pi}(\mathcal{L}-zI)$ since $\sup_{|n|\geqslant K}\|(\mathcal{A}-inI)^{-1}\| < \infty$, a condition needed in Lemma 2.3, due to $\mathrm{Re}z > s_0^\infty(\mathcal{L})$, concluding the proof of Proposition 2.24. ∎

2.4 Proof of Theorem 1.4

In this section we prove Theorem 1.4 formulated in the Introduction (or Theorem 2.19), that is, we show that

$$s_0^\infty(\mathcal{G}) = \max\{s_0^\infty(\mathcal{G}^+), s_0^\infty(\mathcal{G}^-)\}, \tag{2.58}$$

for the operator $\mathcal{G} = B(\xi)\partial_\xi + D(\xi), \xi \in \mathbb{R}$. In fact, we give two proofs of equality (2.58): In Section 2.4.1 we prove it assuming Hypothesis 2.16 and in Section 2.4.2 we prove it assuming Hypothesis 2.17. In other words, assuming that $\omega \in \mathbb{R}$ satisfies the inequality $\omega > \max\{s_0^\infty(\mathcal{G}^+), s_0^\infty(\mathcal{G}^-)\}$, we want to show that $\omega > s_0^\infty(\mathcal{G})$ and, conversely, assuming $\omega > s_0^\infty(\mathcal{G})$, we want to show that $\omega > \max\{s_0^\infty(\mathcal{G}^+), s_0^\infty(\mathcal{G}^-)\}$. In each case, without loss of generality we may assume that $\omega > \max\{s_0^\infty(\mathcal{G}^), s_0^\infty(\mathcal{G}^\pm)\} > 0$ and take $z \in \Omega_{\omega,r} = \{\mathrm{Re}z > \omega, |\mathrm{Im}z| > r\}$ with $r > 0$ sufficiently large. It will be convenient to introduce a new differential operator

$$\mathcal{E}(z) = -\partial_\xi + zA(\xi, z), \quad \text{so that} \quad \mathcal{E}(z) = -B^{-1}(\xi)(\mathcal{G} - zI), \tag{2.59}$$

where

$$A(\xi, z) = B^{-1}(\xi) - z^{-1}B^{-1}(\xi)D(\xi), \quad \xi \in \mathbb{R}, \quad z \in \Omega_{\omega,r}. \tag{2.60}$$

Using (2.59), we see that $\omega > s_0^\infty(\mathcal{G})$ if and only if the following two assertions hold: For $r > 0$ large enough,

$$\begin{aligned} &\mathcal{E}(z) \text{ is invertible in } L^2(\mathbb{R};\mathbb{C}^{N_2}) \text{ for each } z \in \Omega_{\omega,r}, \text{ and} \\ &\sup_{z\in\Omega_{\omega,r}} \|\mathcal{E}(z)^{-1}\| < \infty. \end{aligned} \tag{2.61}$$

Using Hypothesis 2.11, we can define the asymptotic operators

$$\mathcal{E}^\pm(z) = -\partial_\xi + zA^\pm(z), \text{ so that } \mathcal{E}^\pm(z) = -(B^\pm)^{-1}(\mathcal{G}^\pm - zI), \tag{2.62}$$

where

$$A^\pm(z) = (B^\pm)^{-1} - z^{-1}(B^\pm)^{-1}D^\pm, \quad z \in \Omega_{\omega,r}. \tag{2.63}$$

Therefore, to show (2.58) we need to prove that assertions (2.61) hold if and only if the following two assertions hold: For $r > 0$ large enough,

$$\begin{aligned} &\mathcal{E}^\pm(z) \text{ is invertible in } L^2(\mathbb{R};\mathbb{C}^{N_2}) \text{ for each } z \in \Omega_{\omega,r}, \text{ and} \\ &\sup_{z\in\Omega_{\omega,r}} \|\mathcal{E}^\pm(z)^{-1}\| < \infty. \end{aligned} \tag{2.64}$$

The advantage of using (2.59) and (2.60) instead of $\mathcal{G}$ is that $\mathcal{E}$ is the generator of an evolution semigroup for which many nice properties are known (see Section 2.1.2). Sometimes we will use a more general family of the generators of evolution semigroups,

$$E(z) = -\partial_\xi + A_0(\xi) + A_1(\xi, z), \quad \xi \in \mathbb{R}, \quad z \in \Omega_{\omega,r}, \tag{2.65}$$

where $A_0(\cdot), A_1(\cdot, z) \in C_b^1(\mathbb{R};\mathbb{C}^{N_2\times N_2})$ are matrix valued functions that have limits $A_0^\pm, A_1^\pm(z)$ as $\xi \to \pm\infty$ uniformly in $z \in \Omega_{\omega,r}$ and the asymptotic operators

$$E^\pm(z) = -\partial_\xi + A_0^\pm + A_1^\pm(z), \quad z \in \Omega_{\omega,r}. \tag{2.66}$$

Remark 2.28. By Hypothesis 2.11, we see that for $A(\xi, z)$ defined in (2.60) there indeed exist limits $A^{\pm}(z) := \lim_{\xi\to\pm\infty} A(\xi, z)$; moreover, these limits exist uniformly for $z \in \Omega_{\omega,r}$ (or, equivalently, uniformly for z satisfying $|z| > r_0$ for each r_0 fixed in advance).

2.4.1 Stable and unstable block diagonalization of B.

Throughout this subsection we assume Hypothesis 2.14. We recall that $0 \in \rho(B(\xi))$ for all $\xi \in \overline{\mathbb{R}}$ and the spectral projection $Q(\xi)$ corresponding to $\sigma(B(\xi)|_{\operatorname{ran} Q(\xi)}) < 0$ satisfies (2.28). Our proof of (2.58) consists of two parts. First, we will block-diagonalize the operator $\mathcal{G}$ and reduce it to the study of the operator family (2.65) where $\sigma(A_0(\xi)) < -\varepsilon < 0$ for all $\xi \in \overline{\mathbb{R}}$ and $z \mapsto \|A(\cdot, z)\|_{L^\infty}$ is bounded on $\Omega_{\omega,r}$. In the second part of the proof, we will use the Hilbert space version of Corollary 3.8 from the last chapter of the thesis to complete the proof of (2.58). We will now proceed with the first part of the proof.

Part 1. Block-diagonalization.

We begin by reminding that $B(\xi)^* = B(\xi)$ by assumption. Let γ be the contour which encloses the set $\cup_{\xi\in\overline{\mathbb{R}}}(\sigma(B^{-1}(\xi)) \cap \{\mathrm{Re}\lambda < 0\})$ and which also symmetric about the real line. The assumptions imposed on the behavior of $B(\xi)$ and $D(\xi)$ in Hypothesis 2.14 imply that there exist such $r_0 > 0$ that for all $|z| \geqslant r_0$ the set

$$\cup_{\xi\in\overline{\mathbb{R}}}(\sigma(B^{-1}(\xi) - z^{-1}B^{-1}(\xi)D(\xi)) \cap \{\mathrm{Re}\lambda < 0\})$$

is also enclosed by the contour γ. This allows us to define for each $\xi \in \overline{\mathbb{R}}$ two spectral projections, $Q(\xi) = Q(\xi)^*$ and $P(\xi, z)$, as follows:

$$Q(\xi) = \frac{1}{2\pi i}\int_\gamma (\lambda - B^{-1}(\xi))^{-1} d\lambda. \tag{2.67}$$

$$
\begin{aligned}
P(\xi, z) &= \frac{1}{2\pi i} \int_\gamma (\lambda - A(\xi, z))^{-1} d\lambda \\
&= \frac{1}{2\pi i} \int_\gamma \left(\lambda - B^{-1}(\xi)\left(I - z^{-1} D(\xi)\right)\right)^{-1} d\lambda.
\end{aligned} \tag{2.68}
$$

By Hypothesis 2.11 and remark 2.28, there exist limits $Q^\pm = \lim_{\xi \to \pm\infty} Q(\xi)$, $P^\pm(z) = \lim_{\xi \to \pm\infty} P(\xi, z)$ which are uniform for $z \in \Omega_{\omega,r}$ (or, equivalently, for z satisfying $|z| > r_0$ for any fixed $r_0 > 0$).

The projections $Q(\xi)$ and $P(\xi, z)$ satisfy a number of important relations. The first of these relations has to do with the fact that for large $|z|$ the projections $Q(\xi)$ and $P(\xi, z)$ differ just by an L^∞ function. To explain the exact relationship between them, it will help to introduce a binary relation "$\approx$" that acts as follows: we say that two functions $F(\xi, z)$ and $G(\xi, z)$ from $C_b^1((\mathbb{R} \times \Omega_{\omega,r}); \mathbb{C}^{n \times n})$ satisfy $F(\xi, z) \approx G(\xi, z)$ if and only if for all $|z|$ large enough

$$
F(\xi, z) - G(\xi, z) = z^{-1} \beta(\xi, z), \tag{2.69}
$$

where β is a generic function from $C_b((\mathbb{R} \times \Omega_{\omega,r}); \mathbb{C}^{n \times n})$, i.e. $\sup_{z \in \Omega} \sup_{\xi \in \overline{\mathbb{R}}} \|\beta(\xi, z)\| < \infty$ such that there exist limits $\lim_{\xi \to \pm\infty} \beta(\xi, z) = \beta^\pm(z)$ uniformly for $|z| \geqslant r_0$. It is easy to see that the relation "$\approx$" is reflexive, symmetric and transitive and is therefore an equivalence relation. Also, for any pair of bounded functions $a(\xi, z)$ and $b(\xi, z)$ having uniform for $|z| \geqslant r_0$ limits as $\xi \to \pm\infty$, the relationship $F(\xi, z) \approx G(\xi, z)$ implies $aFb \approx aGb$.

Returning back to the projections $Q(\xi)$ and $P(\xi, z)$, we claim that the following properties hold.

Claim 2.29. For all $|z| \geqslant r_0$ with r_0 large enough, one has

$$
P(\xi, z) \approx Q(\xi). \tag{2.70}
$$

Proof. Indeed, looking at the difference of $P(\xi, z)$ and $Q(\xi)$ yields:

$$\begin{aligned}
&P(\xi, z) - Q(\xi)\\
&= \frac{1}{2\pi i}\int_{\gamma(\xi)} \left((\lambda - B^{-1}(\xi) + z^{-1}B^{-1}(\xi)D(\xi))^{-1} - (\lambda - B^{-1}(\xi))^{-1}\right) d\lambda\\
&= \frac{-1}{2\pi i z}\int_{\gamma(\xi)} (\lambda - B^{-1}(\xi) + z^{-1}B^{-1}(\xi)D(\xi))^{-1}(B^{-1}(\xi)D(\xi))(\lambda - B^{-1}(\xi))^{-1} d\lambda\\
&= z^{-1}\beta(\xi, z),
\end{aligned} \tag{2.71}$$

where we have used Remark 2.28 to ensure that $\sup_{\xi\in\mathbb{R}} \|B^{-1}(\xi)D(\xi)\|/|z| \to 0$ as $|z| \to \infty$. Formula (2.60) yields:

$$\partial_\xi A(\xi, z) = \partial_\xi B^{-1}(\xi) - z^{-1}\partial_\xi(B^{-1}(\xi)D(\xi)) \approx (B^{-1}(\xi))', \tag{2.72}$$

where we have use the fact that both $B, D \in C_b^1(\mathbb{R}, \mathbb{C}^{N_2\times N_2})$. ∎

Claim 2.30. For all $|z| \geqslant r_0$ with r_0 large enough, one has

$$\partial_\xi(P(\xi, z))Q(\xi) \approx Q(\xi)\partial_\xi(P(\xi, z)). \tag{2.73}$$

Proof. Indeed, from definition (2.68) we obtain:

$$\begin{aligned}
\partial_\xi(P(\xi, z))Q(\xi) &= \frac{1}{2\pi i}\int_\gamma \partial_\xi(\lambda - A(\xi, z))^{-1} d\lambda \cdot Q(\xi)\\
&= \frac{1}{2\pi i}\int_\gamma (\lambda - A(\xi, z))^{-1}\partial_\xi A(\xi, z)(\lambda - A(\xi, z))^{-1}Q(\xi) d\lambda\\
&\approx \frac{1}{2\pi i}\int_\gamma (\lambda - A(\xi, z))^{-1}\partial_\xi A(\xi, z)(\lambda - A(\xi, z))^{-1}P(\xi, z) d\lambda,
\end{aligned} \tag{2.74}$$

where we have used the fact that $\partial_\xi A(\xi, z) \in C_b^0(\mathbb{R}, \mathbb{C}^{N_2\times N_2})$ for all $|z| > 0$, together with property (2.70). Recalling that $P(\xi, z)$ is the Riesz projection and commutes with $A(\xi, z)$, and using Remark 2.28, assertion (2.72) yields:

$$\partial_\xi(P(\xi, z))Q(\xi) \approx \frac{1}{2\pi i}\int_\gamma (\lambda - A(\xi, z))^{-1}(B^{-1}(\xi))'Q(\xi)(\lambda - A(\xi, z))^{-1} d\lambda. \tag{2.75}$$

Now we should go all way back to Hypothesis 2.14; according to (2.28), the projection $Q(\xi)$ commutes with the matrix $(B^{-1}(\xi))'$. Applying this here and then going back to $P(\xi,z)$ and $A(\xi,z)$ yields

$$\begin{aligned}
\partial_\xi(P(\xi,z))Q(\xi) &\approx \frac{1}{2\pi i}\int_\gamma(\lambda - A(\xi,z))^{-1}Q(\xi)(B^{-1}(\xi))'(\lambda - A(\xi,z))^{-1}d\lambda\\
&\approx \frac{1}{2\pi i}\int_\gamma(\lambda - A(\xi,z))^{-1}P(\xi,z)\partial_\xi(A(\xi,z))(\lambda - A(\xi,z))^{-1}d\lambda\\
&= \frac{1}{2\pi i}\int_\gamma P(\xi,z)(\lambda - A(\xi,z))^{-1}\partial_\xi(A(\xi,z))(\lambda - A(\xi,z))^{-1}d\lambda\\
&= P(\xi,z)\partial_\xi(P(\xi,z))\\
&\approx Q(\xi)\partial_\xi(P(\xi,z)),
\end{aligned}$$

which proves the claim. ∎

We recall that $Q'(\xi) = 0$ for all $\xi \in \overline{\mathbb{R}}$ by Lemma 2.15 and thus,

$$Q'(\xi)Q(\xi) = Q(\xi)Q'(\xi) = 0, \quad \xi \in \overline{\mathbb{R}}. \tag{2.76}$$

The projection valued function $Q = Q(\xi)$ defines in $L^2(\mathbb{R};\mathbb{C}^{N_2})$ a projection acting by the formula $(Qu)(\xi) = Q(\xi)u(\xi), u \in L^2(\mathbb{R};\mathbb{C}^{N_2})$. With a slight abuse of notation we will denote this projection also by Q. Since $Q(\cdot)$ is ξ-independent, the projection Q in $L^2(\mathbb{R};\mathbb{C}^{N_2})$ commutes with the operator of differentiation:

$$\partial_\xi Q = Q\partial_\xi \text{ in } L^2((\mathbb{R};\mathbb{C}^{N_2}). \tag{2.77}$$

Now that we are familiar with the properties of projections $P(\xi,z)$ and $Q(\xi)$, we construct the transformation (conjugation) operator described in Subsection 2.1.3:

$$T(\xi,z) = P(\xi,z)Q(\xi) + (I - P(\xi,z))(I - Q(\xi)), \qquad \xi \in \overline{\mathbb{R}},\ z \in \Omega_{\omega,r}. \tag{2.78}$$

This transformation operator acts as an intertwining operator: it is easy to see that for all $\xi \in \overline{\mathbb{R}}$ and $z \in \Omega_{\omega,r}$ we have

$$P(\xi,z)T(\xi,z) = T(\xi,z)Q(\xi), \quad \xi \in \overline{\mathbb{R}}, \quad z \in \Omega_{\omega,r}. \tag{2.79}$$

Since $A(\xi, z)$ and $B^{-1}(\xi)$ are continuously differentiable in ξ, so are the Riesz projections $P(\xi, z)$ and $Q(\xi)$. Denoting by $'$ the derivative with respect to ξ, we end up with the following formula (we omit the arguments (ξ, z) and ξ for brevity):

$$T' = P'Q + PQ' - P'(I - Q) - (I - P)Q' = -P' - Q' + 2P'Q + 2PQ'. \quad (2.80)$$

Second, from definition (2.78) and property (2.70) it follows that for all $|z| > r_0$ one has (again, omitting the arguments):

$$\begin{aligned} T &= PQ + I - P - Q + PQ = I - P - Q + 2PQ \\ &= I - P^2 + 2PQ - Q^2 = I + (P - Q)^2 \approx I. \end{aligned}$$

This means that $T(\xi, z) = I + z^{-1}\beta(\xi, z)$ with a uniformly in z bounded function $\beta(\xi, z)$. Therefore, for $|z| \geqslant r_0$ with r_0 large enough the transformation operator $T(\xi, z)$ is invertible and (2.79) can be rewritten as

$$T(\xi, z)^{-1}P(\xi, z) = Q(\xi)T(\xi, z)^{-1}, \quad \xi \in \overline{\mathbb{R}}, \quad |z| \geqslant r_0. \quad (2.81)$$

Moreover, for sufficiently large r_0 and all $|z| \geqslant r_0$ we have, using (2.78) and Claim 2.29:

$$T^{-1}(\xi, z) \approx I \text{ and } T(\xi, z) \approx I. \quad (2.82)$$

Let us choose r_0 such that (2.82) is satisfied for all $|z| \geqslant r_0$, and let us introduce a function $M(\xi, z)$ as follows (here $A(\xi, z)$ is defined in (2.60)):

$$M(\xi, z) = T^{-1}(\xi, z)A(\xi, z)T(\xi, z), \quad \xi \in \overline{\mathbb{R}}, \quad |z| \geqslant r_0. \quad (2.83)$$

As before, for each $z \in \Omega_{\omega,r}$ the family of matrices $T(\xi, z), \xi \in \overline{\mathbb{R}}$, defines on $L^2(\mathbb{R}; \mathbb{C}^{N_2})$ a bounded operator of multiplication (which will be also denoted by T)

acting by the formula $(Tu)(\xi) = T(\xi, z)u(\xi), \xi \in \overline{\mathbb{R}}, u \in L^2(\mathbb{R}; \mathbb{C}^{N_2})$. The bounded operator of multiplication by the derivative $T'(\xi, z)$ will be denoted by T' so that $\partial_\xi T = T\partial_\xi + T'$ as operators in $L^2(\mathbb{R}; \mathbb{C}^{N_2})$ with natural domains, e.g., $\operatorname{dom} \partial_\xi = H^1(\mathbb{R}; \mathbb{C}^{N_2})$, the Sobolev space. As above, we use notations T^{-1}, M, P, A, Q to denote respective operators of multiplication in $L^2(\mathbb{R}; \mathbb{C}^{N_2})$ by $T^{-1}(\cdot, z), M(\cdot, z), P(\cdot, z), A(\cdot, z), Q(\cdot)$. Finally, if $F(\xi, z)$ and $G(\xi, z)$ are two bounded functions satisfying $F(\xi, z) \approx G(\xi, z)$ then the respective operators of multiplication F and G differ by $O(r_0^{-1})$, that is, $F - G = z^{-1}\beta(\cdot, z)$ where $\beta \in C_b(\mathbb{R} \times \Omega_{\omega,r}; \mathbb{C}^{N_2 \times N_2})$ is a function, uniformly bounded in $\xi \in \overline{\mathbb{R}}$ and $z \in \Omega_{\omega,r}$. This fact will be recorded as $F \approx G$.

Our choice of projectors (2.67), (2.68) and the intertwining property (2.81) imply the following equalities of the operators:

$$MQ = T^{-1}ATQ = T^{-1}APT = T^{-1}PAT = QT^{-1}AT = QM. \tag{2.84}$$

But then, applying the transformation operators T and T^{-1} to the differential operator $\mathcal{E}$ defined in (2.59), we end up with the following operator equalities:

$$\begin{aligned} T^{-1}\mathcal{E}T &= T^{-1}(-\partial_\xi + zA)T \\ &= T^{-1}(-\partial_\xi)T + zT^{-1}AT \\ &= -T^{-1}T\partial_\xi - T^{-1}T' + zM \\ &= -\partial_\xi + zM - T^{-1}T', \end{aligned} \tag{2.85}$$

where we made use of the properties of the operator T.

According to (2.77) and (2.84), the space $\operatorname{ran} Q$ is invariant under the operators ∂_ξ and zM for any $|z| \geqslant r_0$. As for the operator $T^{-1}T'$, according to (2.80) and properties (2.70), (2.73), (2.76) of the Riesz projectors $P(\xi, z)$ and $Q(\xi)$ we have (again, the arguments ξ, z are omitted for brevity):

$$\begin{aligned} T^{-1}T'Q &= T^{-1}(-P' - Q' + 2P'Q + 2PQ')Q \\ &\approx T^{-1}(-P' - Q' + 2P'Q + 2QQ')Q \\ &= T^{-1}(-P'Q - Q'Q + 2P'Q + 2QQ'Q) \\ &\approx T^{-1}(-QP' - QQ' + 2QP' + 2QQ') \\ &= T^{-1}Q(-P' - Q' + 2QP' + 2QQ') \\ &\approx T^{-1}Q(-P' - Q' + 2QP' + 2PQ') \\ &= T^{-1}QT' \end{aligned} \tag{2.86}$$

So, by the intertwining property (2.81) we conclude that

$$T^{-1}T'Q \approx QT^{-1}T', \quad |z| \geqslant r_0, \tag{2.87}$$

as operators in $L^2(\mathbb{R}; C^{N_2})$. So, according to (2.77), (2.84), (2.87) we have the following three identities:

$$\begin{aligned} &\partial_\xi Q = Q\partial_\xi Q, \\ &MQ = QM, \\ &T^{-1}T'Q \approx QT^{-1}T', \end{aligned} \tag{2.88}$$

and it is easy to show that similar identities for $I - Q$ are also true:

$$\begin{aligned} &\partial_\xi(I - Q) = (I - Q)\partial_\xi(I - Q), \\ &M(I - Q) = (I - Q)M, \\ &T^{-1}T'(I - Q) \approx (I - Q)T^{-1}T'. \end{aligned} \tag{2.89}$$

Now recall the definition of $A(\xi, z)$ in (2.60) and consider the differential equation

$$y' = zA(\xi, z)y, \quad \xi \in \mathbb{R}, \quad z \in \Omega_{\omega,r}. \tag{2.90}$$

By Proposition 2.4(i), the existence of exponential dichotomy of (2.90) on $\mathbb{R}$ is equivalent to invertibility of the differential operator $\mathcal{E}(z) = -\partial_\xi + zA(\xi, z)$. Consider a kinematic similarity

$$y = T(\xi, z)u. \tag{2.91}$$

Under (2.91), equation (2.90) transforms into the following equation:

$$T'(\xi, z)u(\xi, z) + T(\xi, z)u'(\xi, z) = zA(\xi, z)T(\xi, z)u(\xi, z). \tag{2.92}$$

According to the discussion above, for all $|z| \geqslant r_0$ the transformation operator $T(\xi, z)$ is invertible, which leads to the following equation on u:

$$-u' + (zM(\xi, z) - T^{-1}(\xi, z)T'(\xi, z))u = 0. \tag{2.93}$$

On the other hand, due to the invariance properties (2.88), (2.89), we have

$$\begin{aligned} &-\partial_\xi + zM - T^{-1}T' \\ &= Q(-\partial_\xi + zM - T^{-1}T')Q \\ &\quad + (I - Q)(-\partial_\xi + zM - T^{-1}T')(I - Q) + z^{-1}\beta(\xi, z), \end{aligned} \tag{2.94}$$

where β is a uniformly bounded function. This implies that we have the following decomposition:

$$\begin{aligned} &T^{-1}\mathcal{E}(z)T \\ &= Q(-\partial_\xi + zM - T^{-1}T')Q \\ &\quad + (I - Q)(-\partial_\xi + zM - T^{-1}T')(I - Q) + z^{-1}\beta(\xi, z). \end{aligned} \tag{2.95}$$

We now introduce the following two differential operators, $\mathcal{E}_s(z)$ and $\mathcal{E}_u(z)$, acting in the subspaces $\operatorname{ran} Q$ and $\operatorname{ran}(I - Q)$, respectively, by the formulas:

$$\begin{aligned} &\mathcal{E}_s(z) = Q(-\partial_\xi + zM - T^{-1}T')Q, \\ &\mathcal{E}_u(z) = (I - Q)(-\partial_\xi + zM - T^{-1}T')(I - Q) + z^{-1}\beta(\xi, z). \end{aligned} \tag{2.96}$$

Therefore, the following decomposition holds:

$$T^{-1}\mathcal{E}(z)T = \mathcal{E}_s(z) + \mathcal{E}_u(z) + z^{-1}\beta(\xi, z),$$

where $\sup_{\xi\in\mathbb{R},z\in\Omega_{\omega,r}}\|\beta(\xi,z)\|<\infty$. By definition of $s_0^\infty(\mathcal{G})$, and using formula (2.59), if $\omega>s_0^\infty(\mathcal{G})$ then $\mathcal{E}(z)$ is invertible in $z\in\Omega_{\omega,r}$ and

$$\sup_{z\in\Omega_{\omega,r}}\|\mathcal{E}(z)^{-1}\|<\infty. \tag{2.97}$$

But, according to (2.95) this also means that

$$\sup_{z\in\Omega_{\omega,r}}\|\mathcal{E}_s(z)^{-1}\|<\infty, \qquad \sup_{z\in\Omega_{\omega,r}}\|\mathcal{E}_u(z)^{-1}\|<\infty. \tag{2.98}$$

Conversely, (2.98) implies (2.97). Thus, it suffices to work with just one of these operators, $\mathcal{E}_s(z)$ or $\mathcal{E}_u(z)$. Without loss of generality, let us consider the operator $\mathcal{E}_s(z)$. It follows from (2.60) and (2.82) that $A\approx B^{-1}$, $T\approx I$ and $T^{-1}\approx I$. Hence, in the subspace $\operatorname{ran}Q$, we have, by the definition of M and boundedness of $T'(\xi,z)$, that

$$\mathcal{E}_s(z)=-\partial_\xi-zB^{-1}(\xi)Q(\xi)+\beta(\xi,z).$$

This means that instead of the operator $\mathcal{E}_s(z)$ one can study the operator

$$E(z)=-\partial_\xi+zA_0(\xi)+A_1(\xi,z), \quad z\in\Omega_{\omega,r}, \tag{2.99}$$

where the matrix $A_0(\xi)=Q(\xi)B^{-1}(\xi)Q(\xi)$ is self-adjoint, there exist such $\varepsilon>0$ that $\sigma(A_0(\xi))<-\varepsilon<0$ and $A_1(\xi,z)$ is a function from $C_b(\mathbb{R}\times\Omega_{\omega,r};\mathbb{C}^{N_2\times N_2})$ having limits $A_1^\pm(z)$ as $\xi\to\pm\infty$ uniformly for $z\in\Omega_{\omega,r}$.

Remark 2.31. We are ready to summarize the results of the first part of the proof of Theorem 1.4 or, which is the same, of the equivalency of the assertions (2.61) and (2.64). Consider the operator $E(z)$ defined in (2.99) and its counterpart

$$E^\pm(z)=-\partial_\xi+zA_0^\pm+A_1^\pm(z), \quad z\in\Omega_{\omega,r}. \tag{2.100}$$

It suffices to show that the following two assertions are equivalent: the assertion

$$\begin{aligned}&E(z) \text{ is invertible for each } z \in \Omega_{\omega,r} \text{ and}\\ &\sup_{z\in\Omega_{\omega,r}} \|E(z)^{-1}\| < \infty;\end{aligned} \tag{2.101}$$

and the assertion

$$\begin{aligned}&E^{\pm}(z) \text{ is invertible for each } z \in \Omega_{\omega,r} \text{ and}\\ &\sup_{z\in\Omega_{\omega,r}} \|E^{\pm}(z)^{-1}\| < \infty.\end{aligned} \tag{2.102}$$

We stress again that the coefficients $A_0(\cdot) \in C_b(\mathbb{R}; \mathbb{C}^{N_2\times N_2})$ and $A_1(\cdot, z) \in C_b(\mathbb{R} \times \Omega_{\omega,r}; \mathbb{C}^{N_2\times N_2})$ in (2.99) satisfy the following conditions (they are implied by Hypothesis 2.16):

Hypothesis 2.32. *(i)* $A_0(\xi) = A_0(\xi)^*, \quad \xi \in \overline{\mathbb{R}}$;

(ii) $\sigma(A_0(\xi)) \leqslant -\varepsilon < 0, \quad \xi \in \overline{\mathbb{R}}$*;*

(iii) there is $\mathcal{N} > 0$ *such that* $A_0(\xi) = A_0^{\pm}$ *for* $\pm\xi \geqslant \mathcal{N}$*;*

(iv) for each $\delta > 0$ *there exists* $\alpha > 0$ *such that:*

$$\|A_1(\xi, z) - A_1^{\pm}(z)\| < \delta, \text{ for all } z \in \Omega_{\omega,r}, \text{ and } \pm\xi \geqslant \alpha.$$

We are ready to begin the second part of the proof of Theorem 1.4.

Part 2. Uniform stability.

Our objective is to prove that assertions (2.101) and (2.102) are equivalent. Our strategy is as follows. We will prove in Proposition 2.35 that assertion (2.101) is equivalent to the fact that the differential equation associated with $E(z)$ is exponentially stable for each $z \in \Omega_{\omega,r}$ and the respective constants in stability estimates are uniformly bounded for $z \in \Omega_{\omega,r}$. The same proposition applies to $E^{\pm}(z)$ and its associated differential equation. We will then complete the proof by showing that the uniform stability of the equation associated with $E(z)$ is equivalent to the uniform stability of the differential equation associated with $E^{\pm}(z)$.

For the proof of Proposition 2.35 we will need the following two lemmas regarding the operator $E(z)$ defined in (2.99). Throughout, we assume Hypothesis 2.32.

Lemma 2.33. *If $E(z)$ is invertible for all $z \in \Omega_{\omega,r}$ and*

$$\sup_{z\in\Omega_{\omega,r}} \|(E(z))^{-1}\| < \infty, \tag{2.103}$$

then there exists $\varepsilon > 0$ such that $\mathrm{Re}\sigma(E(z)) \leqslant -\varepsilon$ and

$$\sup_{\mathrm{Re}\lambda\geqslant-\varepsilon}\ \sup_{z\in\Omega} \|(E(z)-\lambda)^{-1}\| < \infty. \tag{2.104}$$

Proof. Since $E(z)$ is the generator of an evolution semigroup, by Proposition 2.4 (ii), (iii), the spectrum of the operator $E(z)$ is invariant with respect to the vertical translations by $i\nu$ for each $\nu \in \mathbb{R}$. Therefore, if $0 \in \rho(E(z))$, then $i\mathbb{R} \subset \rho(E(z))$. Moreover, if that is the case, then

$$\|(E(z)-i\nu)^{-1}\| = \|(E(z))^{-1}\| \text{ for all } \nu \in \mathbb{R}. \tag{2.105}$$

Consider an identity

$$E(z) - \alpha = E(z)(I - E(z)^{-1}\alpha), \quad \alpha \in \mathbb{R}. \tag{2.106}$$

If we choose $\alpha \in \mathbb{R}$ such that $|\alpha| \leqslant (2\sup_{z\in\Omega}\|E(z)\|)^{-1}$ we immediately conclude that $\|E(z)^{-1}\alpha\| \leqslant 1/2$. But if that is the case, then $0 \notin \sigma(I - E(z)^{-1}\alpha)$, and equation (2.106) tells us that α must lie in the resolvent set of $E(z)$. Using the observation made above about the evolution semigroups, we conclude that for all α such that $|\alpha| \leqslant (2\sup_{z\in\Omega_{\omega,r}}\|E(z)\|)^{-1}$ and for all $\nu \in \mathbb{R}$ it is true that $\lambda = \alpha + i\nu \in \rho(E(z))$.

Or, in other words,

$$|\mathrm{Re}\ \sigma(E(z))| > \frac{1}{2} \inf_{z\in\Omega_{\omega,r}} \left(\|E(z)\|^{-1}\right) =: \varepsilon.$$

and, moreover, that

$$\sup_{-\varepsilon\leqslant \mathrm{Re}\lambda\leqslant\varepsilon}\ \sup_{z\in\Omega_{\omega,r}} \|(E(z)-\lambda)^{-1}\| < \infty. \tag{2.107}$$

We also know by general properties of strongly continuous semigroups that there exist a large enough $\omega_0 \geqslant \omega_0(E(z)) > 0$, such that for all $\mathrm{Re}\lambda \geqslant \omega_0$ the operator $(E(z)-\lambda)$ is invertible and

$$\sup_{\mathrm{Re}\lambda\geqslant\omega_0}\ \sup_{z\in\Omega_{\omega,r}} \|(E(z)-\lambda)^{-1}\| < \infty. \tag{2.108}$$

We will prove in the next Lemma 2.34 that ω_0 is uniform in $z \in \Omega_{\omega,r}$.

The rest of the proof of (2.104) follows by Adamar's Three Lines Theorem for a fixed $z \in \Omega$ (see [RS, Appendix to Sec. XII.4]). Let $\phi_z(\lambda) = (E(z)-\lambda)^{-1}$, Re $\lambda \in [0, \omega_0]$. Owing to the properties of the evolution semigroups in Proposition 2.4(iii), for every $z \in \Omega_{\omega,r}$ we have

$$\sup_{0\leqslant\mathrm{Re}\lambda\leqslant w_0} \|(E(z)-\lambda)^{-1}\| = \sup_{0\leqslant\mathrm{Re}\lambda\leqslant\omega_0} \|(E(z)-\mathrm{Re}\lambda)^{-1}\| \leqslant \infty.$$

According to (2.103) and (2.108), there exist such uniform in z constants M_0, M_1 that

$$\sup_{\mathrm{Re}\lambda=0} \|\phi_z(\lambda)\| \leqslant M_0, \qquad \sup_{\mathrm{Re}\lambda=\omega_0} \|\phi_z(\lambda)\| \leqslant M_1,$$

hence, by Adamar's theorem

$$\sup_{0\leqslant\mathrm{Re}\lambda\leqslant\omega_0} \|\phi_z(\lambda)\| \leqslant M_0 M_1,$$

which, coupled with (2.107) and (2.108) yields the required inequality (2.104). ∎

So, the question that remained unanswered is whether indeed there exist such ω_0 as in the proof of Lemma 2.33 that will be uniform in z. In fact, it does, because the semigroups generated by $E(z)$ are exponentially bounded by the following lemma:

Lemma 2.34. *Assume that the coefficients $A_0 \in C_b^1(\mathbb{R};\mathbb{C}^{N_2\times N_2})$ and $A_1 \in C_b^1(\mathbb{R}\times\Omega_{\omega,r};\mathbb{C}^{N_2\times N_2})$ satisfy assumptions (i) – (iv) listed in Hypothesis* 2.32. *Then the propagator $\Psi(\xi,\xi',z)$ of the differential equation $u' = (zA_0(\xi)+A_1(\xi,z))u$ is exponentially bounded with constants uniform in $z \in \Omega_{\omega,r}$, that is, equivalently, there exist $\omega_0 > 0$ large enough and L independent of z such that:*

$$\|\mathrm{e}^{tE(z)}\| = \sup_{\xi\in\mathbb{R}} \|\Psi(\xi,\xi-t,z)\| \leqslant L\mathrm{e}^{\omega_0 t}, \quad t \geqslant 0, \quad z \in \Omega_{\omega,r}. \tag{2.109}$$

Proof. Let $z = x+iy \in \Omega_{\omega,r}$ with $x,y \in \mathbb{R}$, and let $\Phi_0(\xi,z)$ be the fundamental matrix solution for the differential equation $u' = iyA_0(\xi)u$. We are omitting the argument ξ when possible for the sake of brevity. Then Φ_0 is unitary because $(\Phi_0^*\Phi_0)' = (-iy\Phi_0^*A_0)\Phi_0 + \Phi_0^*(iyA_0\Phi_0^*) = 0$ and $\Phi_0(0,z) = I$. Moreover,

$$-(\Phi_0 u)' = -\Phi_0' u - \Phi_0 u' = -iyA_0\Phi_0 u - \Phi_0 u'.$$

Thus, $E(z)$ is unitary equivalent to the operator

$$E_0(z) = -\partial_\xi + \Phi_0^{-1}(\xi,z)\left(A_1(\xi,z) + xA_0(\xi)\right)\Phi_0(\xi,z),$$

for

$$\begin{aligned}\Phi_0^{-1}E(z)\Phi_0 &= \Phi_0^{-1}(-\partial_\xi + iyA_0(\xi) + xA_0(\xi) + A_1(\xi,z))\Phi_0 \\ &= -\partial_\xi - \Phi_0^{-1}iyA_0\Phi_0 + \Phi_0^{-1}iyA_0\Phi_0 + \Phi_0^{-1}\left(A_1 + xA_0\right)\Phi_0.\end{aligned}$$

Let $\Phi(\xi,\xi',z)$ denote the propagator of the differential equation $u' = A(\xi,z)u$, where

$$A(\xi,z) = \Phi_0^{-1}(\xi,z)\left(A_1(\xi,z) + xA_0(\xi)\right)\Phi_0(\xi,z),$$

and let $\Psi(\xi, \xi', z)$ denote the propagator of the differential equation

$$u' = (A_1(\xi, z) + zA_0(\xi))\, u. \tag{2.110}$$

It is easy to see that

$$\Psi(\xi, \xi', z) = \Phi_0(\xi, z)\Phi(\xi, \xi', z)\Phi_0^{-1}(\xi', z),$$

hence, the evolution semigroup, associated with (2.110) satisfies the following property:

$$\begin{aligned}
\mathrm{e}^{tE(z)}u(\xi) &= \Psi(\xi, \xi - t, z)\ u(\xi - t) \\
&= \Phi_0(\xi, z)\ \Phi(\xi, \xi - t, z)\ \Phi_0^{-1}(\xi - t, z)\ u(\xi - t) \\
&= \Phi_0(\xi, z)\ \mathrm{e}^{tE_0(z)}\left(\Phi_0^{-1}(\cdot, z)u\right)(\xi),
\end{aligned}$$

or, in other words, $\mathrm{e}^{tE(z)} = \Phi_0(\xi, z)\ \mathrm{e}^{tE_0(z)}\Phi_0^{-1}(\xi, z)$. Thus, by Proposition 2.4(v), we infer

$$\begin{aligned}
\|\mathrm{e}^{tE(z)}\| &= \|\Phi_0(\xi, z)\ \mathrm{e}^{tE_0(z)}\Phi_0^{-1}(\xi, z)\| \\
&= \|\mathrm{e}^{tE_0(z)}\| = \max_{\xi \in \mathbb{R}} \|\Phi(\xi, \xi - t)\|.
\end{aligned}$$

Using the standard propagator estimate (2.18), the norm of the semigroup is then bounded from above as follows:

$$\|\mathrm{e}^{tE_0(z)}\| \leqslant \max_{\xi \in \mathbb{R}}\ \exp\left(\int_{\xi - t}^{\xi} \|A(s, z)\| ds\right),$$

which implies that

$$\begin{aligned}
\|\mathrm{e}^{tE(z)}\| &\leqslant \max_{\xi \in \mathbb{R}}\ \exp\left(\int_{\xi - t}^{\xi} \|A_1(s, z) + xA_0(s)\| ds\right) \\
&\leqslant \exp\left(t(\sup_{\substack{s \in \mathbb{R} \\ z \in \Omega_{\omega, r}}} \|A_1(s, z)\| + x \sup_{s \in \mathbb{R}} \|A_0(s)\|)\right).
\end{aligned}$$

For each compact $[0, T] \ni x$ and all $|y| \geqslant r$ we have a uniform estimate as needed in (2.109).

Continuing the proof of Lemma 2.34, our next goal would be to show that $x > 0$ and $\sigma(\Phi_0(\xi, z)A_0(\xi)\Phi_0^{-1}(\xi, z)) \leqslant -\varepsilon < 0$ collectively imply that there exists a sufficiently large T such that

$$\max_{\xi\in\mathbb{R}} \|\Phi(\xi, \xi - t)\| \leqslant C\mathrm{e}^{-\varepsilon_0 t}, \quad t \geqslant 0,$$

with constants C, $\varepsilon_0 > 0$ uniform for $x \geqslant T$ and all $|y| \geqslant r$. To simplify the notation, let us denote

$$\widehat{A}_0(\xi) = \Phi_0(\xi, z)A_0(\xi)\Phi_0^{-1}(\xi, z), \quad \widehat{A}_1(\xi, z) = \Phi_0(\xi, z)A_1(\xi, z)\Phi_0^{-1}(\xi, z).$$

It is easy to see that with this notation $\Phi(\xi, \xi', z)$ is the propagator for the differential equation

$$u' = \Big(\widehat{A}_1(\xi, z) + x\widehat{A}_0(\xi)\Big)\, u. \tag{2.111}$$

We have already noticed earlier that Φ_0 is unitary. This, together with the fact that A_0 is self-adjoint with negative spectrum, implies the following:

$$\begin{aligned} &(\widehat{A}_0(\xi))^* = \widehat{A}_0(\xi), \\ &\sigma(\widehat{A}_0(\xi)) = \sigma(A_0(\xi)) \leqslant -\varepsilon < 0. \end{aligned} \tag{2.112}$$

Let $\Psi_0(\xi, \xi', x)$ be the propagator of the differential equation

$$u' = x\widehat{A}_0(\xi)u, \tag{2.113}$$

and let us introduce the following notation:

$$W(s, s_0, v) = \|\Psi_0(s, s_0, x)v\|^2, \text{ for all } s \geqslant s_0 > 0, \|v\| = 1.$$

Then, keeping in mind that $x \in \mathbb{R}$ and using (2.112), we get:

$$\begin{aligned}
&\frac{d}{ds} W(s, s_0, v) \\
&\quad = \frac{d}{ds} \langle \Psi_0(s, s_0, x)v, \Psi_0(s, s_0, x)v \rangle \\
&\quad = \left\langle x\widehat{A}_0(s)\Psi_0(s, s_0, x)v, \Psi_0(s, s_0, x)v \right\rangle + \left\langle \Psi_0(s, s_0, x)v, x\widehat{A}_0(s)\Psi_0(s, s_0, x)v \right\rangle \\
&\quad = 2x \left\langle \widehat{A}_0(s)\Psi_0(s, s_0, x)v, \Psi_0(s, s_0, x)v \right\rangle \\
&\quad \leqslant -2\varepsilon x \|\Psi_0(s, s_0, x)v\|^2 \\
&\quad = -2\varepsilon x \; W(s, s_0, v).
\end{aligned}$$

Which in turn implies that

$$\int_{s_0}^{s} \frac{W'(\tau, s_0, v)}{W(\tau, s_0, v)} d\tau \leqslant -2\varepsilon x(s - s_0)$$

and, therefore, $\ln(W(s, s_0, v)) \leqslant -2\varepsilon x(s - s_0)$. Returning back to the definition of $W(s, s_0, v)$ we conclude that

$$\|\Psi_0(s, s_0, x)v\| \leqslant \mathrm{e}^{-\varepsilon x(s - s_0)}, \quad s \geqslant s_0. \tag{2.114}$$

The propagator $\Phi(\xi, \xi', z)$ of the differential equation (2.111) is related to $\Psi_0(\xi, \xi', x)$ by the standard variation of constants formula (2.17):

$$\Phi(\xi, \xi - t, z) = \Psi_0(\xi, \xi - t, x) + \int_{\xi - t}^{\xi} \Psi_0(\xi, s, x)\widehat{A}_1(s, z)\Phi(s, \xi - t, z)ds.$$

As follows from (2.114),

$$\|\Phi(\xi, \xi - t, z)\| \leqslant \mathrm{e}^{-\varepsilon x t} + \sup_{\substack{\xi \in \mathbb{R} \\ z \in \Omega_{\omega, r}}} \|\widehat{A}_1(\xi, z)\| \int_{\xi - t}^{\xi} \mathrm{e}^{-\varepsilon x(\xi - s)} \|\Phi(s, \xi - t, z)\| ds.$$

Denoting $\mathcal{K} = \sup_{\xi \in \mathbb{R}} \sup_{z \in \Omega_{\omega, r}} \|\widehat{A}_1(\xi, z)\|$ and $u(s) = \mathrm{e}^{\varepsilon x(s - (\xi - t))} \|\Phi(s, \xi - t, z)\|$ yields

$$u(\xi) \leqslant 1 + \mathcal{K} \int_{\xi - t}^{\xi} u(s) ds,$$

which, by the Grönwall Inequality (2.16) results in $u(\xi) \leqslant \mathrm{e}^{\mathcal{K}t}$ or

$$\|\Phi(\xi, \xi - t)\| \leqslant \mathrm{e}^{(-\varepsilon x + \mathcal{K})t} \leqslant \mathrm{e}^{(-\varepsilon T + \mathcal{K})t} \leqslant \mathrm{e}^{-t}$$

for all $x \geqslant T$ as long as $T \geqslant \varepsilon^{-1}(\mathcal{K} + 1)$, just as required. ∎

We say that a differential equation $u' = A(\xi, z)u$ has a *z-uniform stability* for $z \in \Omega_{\omega,r}$ if its propagator $\Phi(\xi, \xi', z)$ satisfy the following inequality

$$\|\Phi(\xi, \xi', z)\| < K\mathrm{e}^{-\alpha(\xi - \xi')}, \quad \xi \geqslant \xi', \quad z \in \Omega_{\omega,r}, \tag{2.115}$$

with $K, \alpha > 0$ being uniform in $z \in \Omega_{\omega,r}$.

Proposition 2.35. *Let the coefficients of the operator*

$$E(z) = -\partial + zA_0(\xi) + A_1(\xi, z) \tag{2.116}$$

satisfy Hypothesis 2.32*. Then the following assertions are equivalent:*

(i) $E(z)$ *is invertible for all* $z \in \Omega_{\omega,r}$ *and* $\sup_{z \in \Omega_{\omega,r}} \|E(z)^{-1}\| < \infty$.

(ii) The differential equation

$$u' = (zA_0(\xi) + A_1(\xi, z))u, \tag{2.117}$$

is uniformly stable on $\mathbb{R}$ *for all* $z \in \Omega_{\omega,r}$.

Proof. Let $\Phi(\xi, \xi', z)$ denote the propagator of the differential equation (2.117). We recall from Proposition 2.4 (v), (vii) that

$$(\mathrm{e}^{tE(z)}u)(\xi) = \Phi(\xi, \xi - t, z)u(\xi - t), \quad \xi \in \mathbb{R}, \quad t \geqslant 0, \tag{2.118}$$

$$\|\mathrm{e}^{tE(z)}\| = \max_{\xi \in \mathbb{R}} \|\Phi(\xi, \xi - t, z)\|, \quad z \in \Omega_{\omega,r}, \tag{2.119}$$

$$((E(z) - \lambda)^{-1}u)(\xi) = \int_{-\infty}^{\xi} \mathrm{e}^{-\lambda(\xi - \xi')}\Phi(\xi, \xi', z)u(\xi')d\xi', \quad \xi \in \mathbb{R}, \tag{2.120}$$

for all $\mathrm{Re}\lambda > 0$ provided (2.117) is stable on $\mathbb{R}$.

Assume that assertion (i) holds. By Lemma 2.33 we know (2.104). By Lemma 2.34 we know (2.109). Since both estimates are uniform for $z \in \Omega_{\omega,r}$, by the Hilbert space part of Proposition 3.8 we conclude that $\|e^{tE(z)}\| \leqslant Me^{-\varepsilon t}$ for uniform for $z \in \Omega_{\omega,r}$ constants $M, \varepsilon > 0$. Now (2.119) implies assertion (ii).

Assume that assertion (ii) holds. Then $\lambda \in \rho(E(z))$ for all $\mathrm{Re}\lambda > 0$ by Proposition 2.4 and (2.120) combined with (2.115) yield assertion (i). ∎

We are finally ready to finish the proof of Theorem 1.4. As we have shown above, it suffices to prove that (2.101) and (2.102) are equivalent. Moreover, as follows from the Proposition 2.35, the task can be effectively reduced to showing that the differential equation

$$u' = (zA_0(\xi) + A_1(\xi, z))u, \tag{2.121}$$

is uniformly stable on $\mathbb{R}$ if and only if the following two differential equations

$$u' = (zA_0^{\pm} + A_1^{\pm}(z))u \tag{2.122}$$

are uniformly stable on $\mathbb{R}$.

Let us assume that (2.122) are uniformly stable. This means that there exists such $\varepsilon > 0$ that the propagator $\Phi_2(\xi, \xi')$ of (2.122) satisfies

$$\|\Phi_2(\xi, \xi')v\| \leqslant Ce^{-\varepsilon(\xi-\xi')}, \quad \xi \geqslant \xi', \tag{2.123}$$

for some constant $C > 0$. Consider now the differential equation

$$u' = (zA_0^{\pm} + A_1(\xi, z))u. \tag{2.124}$$

Rewriting it as $u' = (zA_0^{\pm} + A_1^{\pm}(z))u + (A_1(\xi,z) - A_1^{\pm}(z))u$, we see that it is nothing but a perturbation of (2.122) and so, the propagator $\Phi(\xi,\xi',z)$ of the differential equation (2.124) satisfies the variation of constant formula (2.17):

$$\Phi(\xi,\xi',z) = \Phi_2(\xi,\xi',z) + \int_{\xi'}^{\xi} \Phi_2(\xi,s)(A_1(s,z) - A_1^{\pm}(z))\Phi(s,\xi',z))ds.$$

As follows from (2.123),

$$\|\Phi(\xi,\xi',z)\| \leqslant C\mathrm{e}^{-\varepsilon(\xi-\xi')} + C\int_{\xi'}^{\xi} \mathrm{e}^{-\varepsilon(\xi-s)}\|A_1(s,z) - A_1^{\pm}(z)\|\|\Phi(s,\xi',z)\|ds, \quad (2.125)$$

which can be rewritten as

$$u(\xi) \leqslant C + C\int_{\xi'}^{\xi} \|A_1(s,z) - A_1^{\pm}(z)\|u(s)ds,$$

where we denote $u(s) = \mathrm{e}^{\varepsilon(s-\xi')}\|\Phi(s,\xi',z)\|$. By Grönwall Inequality (2.16) we infer

$$u(\xi) \leqslant C\mathrm{e}^{\int_{\xi'}^{\xi} C\|A_1(s,z)-A_1^{\pm}(z)\|ds}.$$

By Hypothesis 2.32 it follows that (2.124) possesses the uniform stability:

$$\|\Phi(\xi,\xi',z)\| \leqslant C\mathrm{e}^{-\varepsilon_1(\xi-\xi')}, \quad \xi \geqslant \xi', \quad \varepsilon_1 > 0, \quad z \in \Omega_{\omega,r}. \quad (2.126)$$

We are now ready to compare (2.124) and (2.121). Let $\Phi_1(\xi,\xi')$ denote the propagator of (2.121). Rewriting (2.121) as

$$u' = (zA_0^{\pm} + A_1(\xi,z))u + z(A_0(\xi) - A_0^{\pm})u,$$

variation of constants formula (2.17) yields

$$\Psi(\xi,\xi',z) = \Phi(\xi,\xi',z) + \int_{\xi'}^{\xi} \Phi(\xi,s,z)z(A_0(s) - A_0^{\pm})\Psi(s,\xi',z)ds \quad (2.127)$$

for all $\xi \geqslant \xi'$. We will now use the critical assumption (iii) in Hypothesis 2.32 saying that $A_0(s) - A_0^{\pm} = 0$ provided $\pm s \geqslant \mathcal{N}$. Indeed, assume that $\xi \geqslant \xi' \geqslant \mathcal{N}$. Using (2.127) with the plus sign, and (2.126) we then have

$$\|\Phi(\xi, \xi', z)\| = \|\Psi(\xi, \xi', z)\| \leqslant C\mathrm{e}^{-\varepsilon_1(\xi-\xi')}, \quad \xi \geqslant \xi' \geqslant \mathcal{N}.$$

Similar argument with (2.127) with the minus sign works for $\xi \leqslant \xi' \leqslant -\mathcal{N}$. If $-\mathcal{N} \leqslant \xi' \leqslant \xi \leqslant \mathcal{N}$ then we will use Lemma 2.34 saying that $\Psi(\xi, \xi', z)$ is exponentially bounded with uniform for $z \in \Omega_{\omega,r}$ constants:

$$\|\Phi(\xi, \xi', z)\| \leqslant L\mathrm{e}^{\omega(\xi-\xi')} \leqslant L\mathrm{e}^{\omega 2\mathcal{N}} = L\mathrm{e}^{\omega 2\mathcal{N}}\mathrm{e}^{\varepsilon_1(\xi-\xi')}\mathrm{e}^{-\varepsilon_1(\xi-\xi')} = C_1\mathrm{e}^{-\varepsilon_1(\xi-\xi')},$$

provided it is indeed the case that $-\mathcal{N} \leqslant \xi' \leqslant \xi \leqslant \mathcal{N}$. All other cases are now reduced to the ones that have already been considered; for instance, if $\xi' \leqslant \mathcal{N} \leqslant \xi$ then

$$\|\Psi(\xi, \xi', z)\| = \|\Psi(\xi, \mathcal{N}, z)\Psi(\mathcal{N}, \xi', z)\| \leqslant C\mathrm{e}^{-\varepsilon_1(\xi-\mathcal{N})}\mathrm{e}^{-\varepsilon_1(\mathcal{N}-\xi')} = C_1\mathrm{e}^{-\varepsilon_1(\xi-\xi')}.$$

This means that the existence of uniform stability for (2.122) indeed implies the existence of this property for (2.121).

The proof in the opposite direction follows similarly, thus, finally concluding the proof of the Theorem 1.4.

Remark 2.36. Applying Grönwall inequality (2.15) in (2.127) yields

$$w(\xi) \leqslant C\exp(C|z|\int_{x'}^{\xi} \|A_0(\xi) - A_0^{\pm}\|ds),$$

where $w(\xi) = \mathrm{e}^{\varepsilon_1(\xi-\xi')}\|\Psi(\xi, \xi', z)\|$, provided $\|A_0(\xi) - A_0^{\pm}\| \in L^1(\mathbb{R}_{\pm})$. This fact shows that the differential equation (2.117) has exponential stability (not necessarily uniform for $z \in \Omega_{\omega,r}$) for each $z \in \Omega_{\omega,r}$ if and only if the asymptotic differential equation has it even *without using* assumption (iii) in Hypothesis 2.32. Therefore, we can prove that the operators $\mathcal{E}(z)$ and $\mathcal{E}^{\pm}(z)$ are invertible for each $z \in \Omega_{\omega,r}$ at the same time without imposing assumption (iii). This is "one half" of the equivalency of the assertions (2.61) and (2.64) needed to prove Theorem 1.4 (of course, the implication "$\mathcal{E}(z)$ is invertible implies $\mathcal{E}^{\pm}(z)$ is invertible" follows using Palmer's Theorem, but the reverse implication is nontrivial and requires machinery developed in the current work).

2.4.2 Finer block-diagonalization of B

In this subsection we offer yet another proof of Theorem 1.4 (that is, Theorem 2.19) under a different set of assumptions on the coefficient $B(\xi)$ of the operator $\mathcal{G} = B(\xi)\partial_\xi + D(\xi)$. Specifically, we assume that $B(\xi)$ is an operator of the simplest type [DK], that is, we assume that Hypothesis 2.17 holds. In the proof given in the previous subsection we block-diagonalized $\mathcal{G}$ into a direct sum of two operators of the same type with respective coefficients having positive and negative spectra. This allowed us to apply an abstract result in Proposition 3.8 to show uniform stability estimates for all $z \in \Omega_{\omega,r}$ needed for the proof. In this subsection we use a finer block-diagonalization into several block-operators such that in each block $B(\xi)$ is a scalar multiple of the identity. This allows one to to apply a standard evolution semigroup trick (cf. [CL, Prop.2.36]) to show that certain resolvent estimates do not depend on $y = \mathrm{Im} z$ for $z \in \Omega_{\omega,r}$.

We remark that our assumptions in this subsection are close to the assumptions imposed in the important work [RM2]. Although our proof is very different from the one in [RM2], a part of it could be viewed as a modification of [RM2]. (As we have mentioned above, the problem of equating $s_0^\infty(\mathcal{G})$ and $s_0^\infty(\mathcal{G}^\pm)$ discussed in this section was not considered in [RM2].)

We recall the notation $\mathcal{G}^\pm = B^\pm\partial_\xi + D^\pm$ for the asymptotic operators. It is convenient to split the proof of the equality $s_0^\infty(\mathcal{G}) = \max\{s_0^\infty(\mathcal{G}^+), s_0^\infty(\mathcal{G}^-)\}$ into two parts: Proposition 2.37 for "$\leqslant$" and Proposition 2.39 for "$\geqslant$".

Proposition 2.37. *Assume Hypothesis* 2.17. *Then*

$$\max\{s_0^\infty(\mathcal{G}^+), s_0^\infty(\mathcal{G}^-)\} \geqslant s_0^\infty(\mathcal{G}). \tag{2.128}$$

Proof. Without loss of generality, we may assume that $s_0^\infty(\mathcal{G}^\pm) > 0$, for if necessary one can always replace $D(\xi)$ with $D(\xi) - \omega_0$ where $\omega_0 < \min\{s_0^\infty(\mathcal{G}^+), s_0^\infty(\mathcal{G}^-)\}$. Due to the structure of the operator $B(\xi)$, for every $j = 1, .., m$ there exist non-zero functions $b_j^{-1}(\xi)$ such that

$$B^{-1}(\xi) = \sum_{j=1}^{m} b_j^{-1}(\xi) Q_j. \tag{2.129}$$

Let us recall from (2.59), (2.62) the operators

$$\mathcal{E}(z) = -\partial + zA(\xi, z), \qquad A(\xi, z) = B^{-1}(\xi) - z^{-1}B^{-1}(\xi)D(\xi) \tag{2.130}$$

$$\mathcal{E}^\pm(z) = -\partial + zA^\pm(z), \qquad A^\pm(z) = (B^\pm)^{-1} - z^{-1}(B^\pm)^{-1}D^\pm. \tag{2.131}$$

Then $\mathcal{G} - z = -B(\xi)\mathcal{E}(z)$ and $\mathcal{G}^\pm - z = -B^\pm\mathcal{E}^\pm(z)$ for all $z \in \Omega_{\omega,r}$.

Pick $\omega > s_0^\infty(\mathcal{G}^\pm) > 0$ and $r = r(\omega)$ so that for $\Omega = \{\mathrm{Re} z > \omega, |\mathrm{Im} z| > r\}$ we have (cf. 2.64):

$$\sup_{z\in\Omega} \|(\mathcal{E}^\pm(z))^{-1}\| < \infty. \tag{2.132}$$

Our goal is to show (cf. (2.61)) that there exists $r' \geqslant r$ such that for any $z \in \Omega' = \{\text{Re}z > \omega, |\text{Im}z| > r'\}$ the operator $\mathcal{E}(z)$ is also invertible and

$$\sup_{z\in\Omega'} \|\mathcal{E}(z)^{-1}\| < \infty. \tag{2.133}$$

Part 1. A scalar multiple of identity.

We will begin by studying the case $m = 1$, i.e. when $B(\xi) = b(\xi)I$. Without loss of generality, using Hypothesis 2.17, we may assume that there exist such $\varepsilon > 0$ that $b(\xi) < -\varepsilon < 0$ for all $\xi \in \mathbb{R}$ (indeed, if $b(\xi) > 0$ then we may pass to the operator $-\mathcal{E}(z)^*$). Let $z = x + iy \in \Omega$. Following the idea of the proof of [CL, Prop.2.36], let us denote

$$\Phi(\xi) = \exp(iy \int_0^\xi b^{-1}(s)ds). \tag{2.134}$$

Note that $|\Phi(\xi)| = 1$ for all $\xi \in \mathbb{R}$ and thus multiplication by Φ is a unitary operator in $L^2(\mathbb{R})$. Furthermore, $D(\xi)\Phi(\xi) = \Phi(\xi)D(\xi)$ and, for the respective operators of multiplication in $L^2(\mathbb{R};\mathbb{C}^{N_2})$, we have:

$$-\partial_\xi \Phi = \Phi(-\partial_\xi - iyb^{-1}(\cdot)). \tag{2.135}$$

On the other hand,

$$\mathcal{E}(z) = -\partial_\xi + zB^{-1} - B^{-1}D = -\partial_\xi + iyb^{-1}(\xi)I - b^{-1}(\xi)D(\xi) + xb^{-1}(\xi)I, \tag{2.136}$$

which implies (cf. (2.135)) that

$$\begin{aligned}\Phi^{-1}\mathcal{E}(z)\Phi &= \Phi^{-1}(-\partial_\xi + iyb^{-1}(\xi)I - b^{-1}(\xi)D + xb^{-1}(\xi)I)\Phi \\ &= -\partial_\xi - b^{-1}(\xi)D(\xi) + xb^{-1}(\xi)I.\end{aligned} \tag{2.137}$$

This means that estimates (2.132), (2.133) *do not* depend on the imaginary part of $z = x + iy$, provided that $y \geqslant r$. In other words, the invertibility of the operators

$$\mathcal{E}_0(x) = -\partial - b^{-1}(\xi)D(\xi) + xb^{-1}(\xi)I, \tag{2.138}$$

$$\mathcal{E}_0^{\pm}(x) = -\partial - (b_{\pm})^{-1}D^{\pm} + x(b_{\pm})^{-1}I. \tag{2.139}$$

is completely equivalent to the invertibility of the operators $\mathcal{E}(x+iy)$ and $\mathcal{E}^{\pm}(x+iy)$ for any $|y| > r$ and, furthermore,

$$\|(\mathcal{E}_0(x))^{-1}\| = \|(\mathcal{E}(x+iy))^{-1}\|, \quad |y| > r, \quad x > \omega,$$

$$\|(\mathcal{E}_0^{\pm}(x))^{-1}\| = \|(\mathcal{E}^{\pm}(x+iy))^{-1}\|, \quad |y| > r, \quad x > \omega.$$

Thus, our task is reduced to proving the following statement:

Claim 2.38. If $\mathcal{E}_0^{\pm}(x)$ is invertible for any $x > \omega$ and

$$\sup_{x>\omega} \|\mathcal{E}_0^{\pm}(x)\| < \infty,$$

then $\mathcal{E}_0(x)$ is also invertible for all $x > \omega$ and

$$\sup_{x>\omega} \|\mathcal{E}_0(x)\| < \infty.$$

If $\mathcal{E}_0^{\pm}(x)$ is indeed invertible, then for any $x > \omega$ the spectrum of the matrices $-(b_{\pm})^{-1}D^{\pm} + x(b_{\pm})^{-1}$ does not intersect $i\mathbb{R}$ (see Proposition 2.4). Since, by the negativity assumption on $b(\xi)$, the spectrum of these matrices also lies in the negative half-plane $\{\mathrm{Re}\ z < 0\}$ for all sufficiently large $x > 0$, we conclude that $\mathrm{Re}\sigma(-(b_{\pm})^{-1}D^{\pm} + x(b_{\pm})^{-1}) < 0$ for all $x > \omega$. Thus,

$$\mathrm{e}^{t(-b_{\pm}^{-1}D^{\pm} + xb_{\pm}^{-1})} = \mathrm{e}^{txb_{\pm}^{-1}}\mathrm{e}^{-tb_{\pm}^{-1}D^{\pm}} \tag{2.140}$$

is a stable semigroup for each $x > \omega$. Moreover,

$$\|\mathrm{e}^{t(x-\omega)b_{\pm}^{-1}}\mathrm{e}^{t\omega b_{\pm}^{-1}}\mathrm{e}^{-b_{\pm}^{-1}Dt}\| \leqslant \mathrm{e}^{tb_{\pm}^{-1}(x-\omega)}\|\mathrm{e}^{t(\omega b_{\pm}^{-1} - b_{\pm}^{-1}D)}\|, \tag{2.141}$$

which goes to 0 as $x \to \infty$ due to $b_{\pm}^{-1} < 0$. Therefore, there exist uniform in $x \geqslant \omega$ constants $K, \alpha^{\pm} > 0$ such that

$$\|\mathrm{e}^{t(xb_{\pm}^{-1} - b_{\pm}^{-1}D)}\| \leqslant K\mathrm{e}^{-\alpha^{\pm}t}. \qquad t \geqslant 0. \tag{2.142}$$

We claim that there exist uniform in $x \geqslant \omega$ constants $L, \beta > 0$ such that the propagator $\Psi(\xi, \xi')$ of the differential equation $u' = (-b^{-1}D(\xi) + xb(\xi)^{-1})u$ on $\mathbb{R}$ satisfies:

$$\|\Psi(\xi, \xi')\| \leqslant L\mathrm{e}^{-\beta(\xi-\xi')}, \qquad \xi \geqslant \xi' \in \mathbb{R}. \tag{2.143}$$

Let $\Psi_0(\xi, \xi')$ denote the propagator of the differential equation

$$u' = -b^{-1}(\xi)D(\xi)u, \quad \xi \in \mathbb{R},$$

so that

$$\Psi(\xi, \xi') = \exp\left(x \int_{\xi'}^{\xi} b^{-1}(s)ds\right) \Psi_0(\xi, \xi'), \qquad \xi \geqslant \xi' \in \mathbb{R}. \tag{2.144}$$

By the variation of constants formula 2.17 we infer:

$$\begin{aligned}\Psi_0(\xi, \xi') = {} & \mathrm{e}^{(\xi-\xi')(-b_+^{-1}D^+)} \\ & + \int_{\xi'}^{\xi} \mathrm{e}^{(\xi-\zeta)(-b_+^{-1}+D^+)} \left(-b(\zeta)^{-1}D(\zeta) + b_+^{-1}D^+\right) \Psi_0(\zeta, \xi')d\zeta.\end{aligned} \tag{2.145}$$

By (2.142) we know that

$$\|\mathrm{e}^{t(-b_+^{-1}D^+)}\| = \|\mathrm{e}^{t(-b_+^{-1}D^+ + \omega b_+^{-1})}\mathrm{e}^{-t\omega b_+^{-1}}\| \leqslant K\mathrm{e}^{-\alpha_+ t}\mathrm{e}^{-t\omega b_+^{-1}}. \tag{2.146}$$

Then, introducing $w(\zeta) = \|-b(\zeta)^{-1}D(\zeta) + b_+^{-1}D^+\|$ and using (2.145) we will end up with the estimate

$$\begin{aligned}\|\Psi_0(\xi, \xi')\| \leqslant & K\mathrm{e}^{-\alpha_+(\xi-\xi')}\mathrm{e}^{-(\xi-\xi')\omega b_+^{-1}} \\ & + \int_{\xi'}^{\xi} K\mathrm{e}^{-\alpha_+(\xi-\zeta)}\mathrm{e}^{-(\xi-\zeta)\omega b_+^{-1}} w(\zeta)\|\Psi_0(\zeta, \xi')\|d\zeta.\end{aligned} \tag{2.147}$$

Let $v(\zeta) = \|\Psi_0(\zeta, \xi')\|\mathrm{e}^{\alpha_+(\zeta-\xi')+(\zeta-\xi')\omega b_+^{-1}}$. Then

$$v(\xi) \leqslant K + K\int_{\xi'}^{\xi} w(\zeta)v(\zeta)d\zeta, \tag{2.148}$$

and, by using the Grönwall inequality (2.15), it follows that

$$v(\xi) \leqslant K \exp\left(K \int_{\xi'}^{\xi} w(\zeta)d\zeta\right). \tag{2.149}$$

This, together with (2.144) and the definition of v, yields

$$\begin{aligned}
\|\Psi(\xi,\xi')\| &= \mathrm{e}^{x\int_{\xi'}^{\xi} b(\zeta)^{-1}d\zeta}\|\Psi_0(\xi,\xi')\| \\
&= \exp\left(x\int_{\xi'}^{\xi} b(\zeta)^{-1}d\zeta - \alpha_+(\xi-\xi') - (\xi-\xi')\omega b_+^{-1}\right) v(\xi) \\
&\leqslant K\exp\left(K\int_{\xi'}^{\xi} w(\zeta)d\zeta - \alpha_+(\xi-\xi') + x\int_{\xi'}^{\xi} b(\zeta)^{-1}d\zeta - \omega\int_{\xi'}^{\xi} b_+^{-1}d\zeta\right) \\
&= K\exp\left(K\int_{\xi'}^{\xi} w(\zeta)d\zeta - \alpha_+(\xi-\xi') + (x-\omega)\int_{\xi'}^{\xi} b(\zeta)^{-1}d\zeta\right)\times \\
&\quad\times\exp\left(\omega\int_{\xi'}^{\xi}(b(\zeta)^{-1}-b_+^{-1})d\zeta\right) \\
&\leqslant K\exp\left(K\int_{\xi'}^{\xi} w(\zeta)d\zeta - \alpha_+(\xi-\xi') + \omega\int_{\xi'}^{\xi}|b(\zeta)^{-1}-b_+^{-1}|d\zeta\right),
\end{aligned}$$

where we have used the facts that $b^{-1}(\xi) < 0$, $\xi \geqslant \xi'$, and $x \geqslant \omega > 0$.

Using (2.24) in Hypothesis 2.11, and the definition of $w(\xi)$, for any $\epsilon > 0$ there exist such $\mathcal{N} > 0$ that for all $\xi \geqslant \xi' \geqslant \mathcal{N}$ one infers

$$\|\Psi(\xi,\xi')\| \leqslant K\mathrm{e}^{-\alpha_+(\xi-\xi')}\mathrm{e}^{(K\varepsilon+\omega\varepsilon)(\xi-\xi')}, \tag{2.150}$$

and (2.143) also holds. Similar arguments can be applied for the cases $0 \geqslant \xi \geqslant \xi'$ and for $\xi \geqslant 0 \geqslant \xi'$. This completes the proof of (2.143).

Since the differential equation $u' = (-b(\xi)^{-1}D(\xi) + xb(\xi)^{-1})u$ is stable, Proposition 2.4(vii) yields

$$(\mathcal{E}_0(x)^{-1}u)(\xi) = \int_{-\infty}^{\xi} \Psi(\xi,\xi')u(\xi')d\xi',$$

and, using (2.143), for the operator norm in $L^2(\mathbb{R};\mathbb{C}^{N_2})$ we have

$$\sup_{x\geqslant\omega}\|\mathcal{E}_0(x)^{-1}\| < \infty,$$

because L, β are uniform in $x \geqslant \omega$, thus proving Claim 2.38.

This concludes our analysis for the particular case of $m = 1$. Let us now turn our attention to the case when $m > 1$ in (2.30).

Part 2. Matrices of simplest type.

Let $\beta(\xi, z)$ be a generic matrix such that $\beta \in C^1_b(\mathbb{R} \times \Omega_{\omega,r}; \mathbb{C}^{N_2 \times N_2})$, in particular, $\|\beta\|_\infty = \sup_{z\in\Omega} \sup_{\xi\in\mathbb{R}} |\beta(\xi, z)| < \infty$ (we have used this notation in the previous subsection). In addition, we assume that $\beta_\pm(z) = \lim_{\xi\to\pm\infty} \beta(\xi, z)$ uniformly in $z \in \Omega_{\omega,r}$, that is, for any $\varepsilon > 0$ there exists a corresponding $\mathcal{N} > 0$ such that for all $\pm\xi > \mathcal{N}$ the following inequality holds:

$$\sup_{z\in\Omega} \|\beta(\xi, z) - \beta_\pm(z)\| \leqslant \varepsilon. \tag{2.151}$$

Our objective for the remainder of the proof of Proposition 2.37 is to find a bounded invertible transformation $T(\xi, z)$ such that, for the operator $\mathcal{E}(z)$ defined in (2.130) we have

$$T^{-1}(\xi, z)\mathcal{E}(z)T(\xi, z) = -\partial_\xi + zM(\xi, z) + z^{-1}\beta(\xi, z), \tag{2.152}$$

where the matrix function $M(\xi, z)$ is block-diagonal with respect to the decomposition $\sum_{j=1}^m Q_j = I$, that is, for all $\xi \in \mathbb{R}$, $z \in \Omega_{\omega,r}$ and $j = 1, ..., m$ one has $M(\xi, z)Q_j = Q_jM(\xi, z)$. In addition, we will require that each diagonal block $Q_jM(\xi, z)Q_j$ coincides with $B^{-1}(\xi)Q_j$ up to a term of the type $z^{-1}\beta(\xi, z)$, that is, the following relations hold:

$$Q_jM(\xi, z)Q_j = B^{-1}(\xi)Q_j + z^{-1}Q_j\beta_0(\xi, z)Q_j, \qquad j = 1, .., m. \tag{2.153}$$

We postpone the construction of T until later.

If such T exists then the operator of differentiation can be decomposed as $-\partial = \sum_{j=1}^m Q_j(-\partial)Q_j$, and, denoting by $r_j = \operatorname{rank} Q_j$, we effectively have the representation

$$T^{-1}\mathcal{E}(z)T = \sum_{j=1}^{m} Q_j(-\partial + zb_j^{-1}(\xi)I_{r_j} + \beta_j(\xi,z))Q_j + z^{-1}\beta_*(\xi,z), \qquad \xi \in \overline{R}. \tag{2.154}$$

Here, we have introduced m β-type matrix-valued functions $\beta_j = Q_j\beta_0 Q_j$ of sizes $r_j \times r_j$. Let us denote the corresponding operators by

$$\mathcal{E}_j(z) = -\partial_\xi + zb_j^{-1}(\xi)I + \beta_j(\xi,z), \quad \mathcal{E}_j^{\pm} = -\partial_\xi + (zb_j^{\pm})^{-1}I + \beta_j^{\pm}(z).$$

We remark that each $\mathcal{E}_j(z)$ satisfies the hypothesis imposed earlier in this subsection for $m = 1$.

Let us apply (2.152) at $\xi = \pm\infty$ and recall the condition

$$\varepsilon = \max\left\{(\sup_{z\in\Omega}\|\mathcal{E}^+(z)^{-1}\|)^{-1}, (\sup_{z\in\Omega}\|\mathcal{E}^-(z)^{-1}\|)^{-1}\right\} > 0$$

from (2.132). Clearly,

$$\begin{aligned} T(\sum_{j=1}^{m}\mathcal{E}_j(z))T^{-1} &= T(\sum_{j=1}^{m} Q_j(-\partial + z(b_j^{\pm})^{-1} + \beta_j^{\pm}(z))T^{-1} \\ &= \mathcal{E}^{\pm}(z) + z^{-1}\beta_*^{\pm}(z) = \mathcal{E}^{\pm}(z)\left(I + z^{-1}(\mathcal{E}^{\pm}(z))^{-1}\beta_*^{\pm}(z)\right). \end{aligned} \tag{2.155}$$

Thus, for $|z| > 2\varepsilon^{-1}\|\beta_*\|_\infty$ we have

$$\begin{aligned} \|(z\mathcal{E}^{\pm}(z))^{-1}\beta_*^{\pm}(z)\| &\leqslant \frac{1}{|z|}\|\beta_*\|_\infty\|(\mathcal{E}^{\pm}(z))^{-1}\| \leqslant \frac{1}{|z|}\|\beta_*\|_\infty \sup_{z\in\Omega}\|(\mathcal{E}^{\pm}(z))^{-1}\| \\ &\leqslant \frac{1}{|z|}\|\beta_*\|_\infty \frac{1}{\varepsilon} < \frac{1}{2}, \end{aligned}$$

and therefore, using (2.155), we have that

$$\max_{1\leqslant j\leqslant m}\sup_{z\in\Omega}\|(\mathcal{E}_j^{\pm}(z))^{-1}\| < \infty.$$

This shows that (2.132) is satisfied for each $\mathcal{E}_j^{\pm}(z)$. But for each j we may apply the result obtained in the first part of the proof. This yields (2.133) for each $\mathcal{E}_j(z)$.

Using (2.154) in the same way, we arrive at (2.133), thus completing the proof of the proposition.

Let us now concentrate on constructing T as indicated in (2.152). Choose $\delta > 0$ so small that for any $\xi \in \mathbb{R}$:

$$\{\lambda : |\lambda - b_j^{-1}(\xi)| \leqslant 2\delta\} \cap \{\lambda : |\lambda - b_k^{-1}(\xi)| \leqslant 2\delta\} = \emptyset \tag{2.156}$$

for all $j \neq k$. Let $\gamma_j(\xi)$ denote the circle of radius δ centered at $b_j^{-1}(\xi)$. Since $B(\xi) = \sum_{j=1}^m b_j(\xi) Q_j$ by (2.30), we have (2.129) and $\sum_j Q_j = I_{N_2}$, and infer that

$$\begin{aligned} &\frac{1}{2\pi i}\int_{\gamma_j(\xi)} (\lambda I_{N_2} - B^{-1}(\xi))^{-1} d\lambda \\ &= \frac{1}{2\pi i}\sum_{k=1}^m Q_k \int_{\gamma_j(\xi)} (\lambda - b_k^{-1}(\xi))^{-1} Q_k d\lambda \\ &= \frac{1}{2\pi i}\int_{\gamma_j(\xi)} Q_j(\lambda - b_j^{-1}(\xi))^{-1} Q_j d\lambda = Q_j, \end{aligned} \tag{2.157}$$

where we have used the fact that the values of the functions $b_k^{-1}(\xi)$ are outside of $\gamma_j(\xi)$ and thus

$$\int_{\gamma_j(\xi)} Q_k(\lambda - b_j^{-1}(\xi)) Q_k = 0 \text{ for all } j \neq k.$$

Since $A(\xi, z) = B^{-1}(\xi) - z^{-1}B^{-1}(\xi)D(\xi) \to B^{-1}(\xi)$ as $|z| \to \infty$, and the spectrum of a matrix is semicontinuous from above, we may consider $|z|$ so large that for any $\xi \in \overline{\mathbb{R}}$ one has the following inclusion of the spectra: $\sigma\left(A(\xi, z)\right) \subset \cup_{j=1}^m \{\lambda : |\lambda - b_j^{-1}(\xi)| \leqslant \delta\}$. As we will see in a second, the Riesz projections for the matrices $A(\xi, z)$ are given by the following integrals for all $|z| \geqslant r_0$ where r_0 is large enough:

$$P_j(\xi, z) = \frac{1}{2\pi i}\int_{\gamma_j(\xi)} (\lambda - A(\xi, z))^{-1} d\lambda, \qquad \xi \in \overline{\mathbb{R}}. \tag{2.158}$$

We claim that the following conjugation (transformation) operator (see Subsection

2.1.3),

$$T(t,z)=\sum_{j=1}^{m}P_j(\xi,z)Q_j,\quad \xi\in\overline{\mathbb{R}},\quad |z|\geqslant r_0, \tag{2.159}$$

satisfies requirements (2.152), (2.153). Let us use (2.31) in Hypothesis 2.17 and introduce

$$\delta_0=\inf_{j\neq k}\inf_{\xi\in\mathbb{R}}|b_j^{-1}(\xi)-b_k^{-1}|>0,$$

where, by Hypothesis 2.13, $b_j^{-1}\in C_b^1(\mathbb{R})$ for any $1\leqslant j\leqslant m$. Then (2.156) holds for all $0<\delta<\delta_0/4$. For each such $\delta>0$ there exist $r_0>0$ such that for all $|z|>r_0$ the spectrum of $A(\xi,z)$ is located inside of the union G of the discs bounded by the curves $\gamma_j(\xi)$ because

$$\sup_{\xi\in\mathbb{R}}\|B^{-1}(\xi)D(\xi)\|/|z|\to 0 \text{ as } |z|\to\infty, \tag{2.160}$$

and the spectrum of $B^{-1}(\xi)$ is located inside of the union G. Thus, $P_j(\xi,z)$ is indeed the Riesz projection for $A(\xi,z)$. Recalling from (2.157) that

$$Q_j=\frac{1}{2\pi i}\int_{\gamma_j(\xi)}\big(\lambda I_{N_2}-B^{-1}(\xi)\big)^{-1}\,d\lambda,\quad \xi\in\overline{\mathbb{R}}, \tag{2.161}$$

we conclude that, for all $|z|\geqslant r_0$ with r_0 large enough we have:

$$\begin{aligned}
&P_j(\xi,z)-Q_j(\xi)\\
&\qquad=\frac{1}{2\pi i}\int_{\gamma_j(\xi)}\Big((\lambda I_{N_2}-B^{-1}(\xi)\\
&\qquad\qquad+z^{-1}B^{-1}(\xi)D(\xi))^{-1}-(\lambda I_{N_2}-B^{-1}(\xi))^{-1}\Big)d\lambda\\
&\qquad=\frac{1}{z}(-\frac{1}{2\pi i}\int_{\gamma_j(\xi)}(\lambda I_{N_2}-B^{-1}(\xi)\\
&\qquad\qquad+z^{-1}B^{-1}(\xi)D(\xi))^{-1}B^{-1}(\xi)D(\xi)(\lambda I_{N_2}-B^{-1}(\xi))^{-1}d\lambda)\\
&\qquad=z^{-1}\beta(\xi,z).
\end{aligned} \tag{2.162}$$

Next, let us note that there exist such $\mathcal{N} > 0$ such that for all $|\xi| > \mathcal{N}$ the circles $\gamma_j(\xi)$ are ξ-independent uniformly in $z \in \Omega$ because the circles can be chosen centered at $b_j^{\pm}$ for $\pm\xi > \mathcal{N}$. Since $[-\mathcal{N}, \mathcal{N}]$ is compact, for each $\xi_0 \in [-\mathcal{N}, \mathcal{N}]$ we may choose a small open set $U \ni \xi_0$ such that $\gamma_j(\xi)$ can be replaced by a ξ-independent circle $\gamma_j(\xi_0)$ for each $\xi \in U$. By taking a finite covering, we may assume that $\gamma_j(\xi)$ are in fact ξ-independent.

Taking this into account, we differentiate $P_j(\xi, z)$ from (2.158) as follows:

$$\begin{aligned}\frac{d}{d\xi}P_j(\xi, z) = -\frac{1}{2\pi i}\int_{\gamma_j(\xi)} &(\lambda I_{N_2} - A(\xi, z))^{-1}\Big((B^{-1})'(\xi) \\ &- z^{-1}(B^{-1}(\xi)D^{-1}(\xi))'\Big)(\lambda - A(\xi, z))^{-1}d\lambda \\ = \beta_0(\xi, z) + z^{-1}\beta(\xi, z),&\end{aligned}$$

where, using (2.129) again, we introduce the function

$$\beta_0(\xi, z) = -\frac{1}{2\pi i}\int_{\gamma_j(\xi)}(\lambda - A(\xi, z))^{-1}\sum_{k=1}^{m}\frac{db_k^{-1}(\xi)}{d\xi}Q_k(\lambda - A(\xi, z))^{-1}d\lambda. \quad (2.163)$$

We claim that $T(\xi, z)$ is invertible and $T^{-1}(\xi, z) = I + \beta(\xi, z)$. Indeed, by the definition of T (cf. (2.8)) we derive that

$$T(\xi, z) = I + \sum_{j=1}^{m} P_j(\xi, z)(P_j(\xi, z) - Q_j), \quad \xi \in \overline{\mathbb{R}}.$$

From (2.162) we know that for some $\beta = \beta(\xi, z) \in C_b^1(\mathbb{R} \times \Omega_{\omega,r})$ one has

$$T = I + z^{-1}\beta, \text{ for } |z| \geqslant r_0, \quad (2.164)$$

which implies that T is invertible for all z with $|z| \geqslant r_0$ and r_0 large enough, and also that the inverse can be written as follows:

$$T^{-1} = I - z^{-1}\beta + O(|z|^{-2}) = I - z^{-1}\beta. \quad (2.165)$$

As discussed in Subsection 2.1.3, we also know that (see [DK, IV.1.5] and (2.10))

$$P_j(\xi, z) = T(\xi, z)Q_jT^{-1}(\xi, z). \tag{2.166}$$

Now let us introduce the matrix $M(\xi, z)$, block-diagonal with respect to the decomposition $I_{N_2} = \sum_{j=1}^m Q_j$, by the usual formula, see (2.12),

$$M(\xi, z) = T^{-1}(\xi, z)A(\xi, z)T(\xi, z), \quad \xi \in \overline{\mathbb{R}}, |z| \geqslant r_0.$$

With formulae (2.130), (2.131) in mind, we can write, using $\partial_\xi T = T\partial_\xi + T'$, that

$$T^{-1}\mathcal{E}(z)T = -T^{-1}\partial_\xi T + zT^{-1}AT = -\partial_\xi - T^{-1}T' + zM. \tag{2.167}$$

Using (2.164) and (2.165) we have:

$$\begin{aligned} zM &= zT^{-1}AT = (I - z^{-1}\beta)(zB^{-1} - B^{-1}D)(I + z^{-1}\beta) \\ &= (zB^{-1} + \beta + z^{-1}\beta)(1 + z^{-1}\beta) = zB^{-1} + \beta + z^{-1}\beta. \end{aligned}$$

On the other hand, we know that m is block diagonal, that is,

$$M(\xi, z) = (\sum_{j=1}^m Q_j)M(\xi, z)(\sum_{j=1}^m Q_j) = \sum_{j=1}^m Q_jM(\xi, z)Q_j,$$

and so equation (2.153) satisfies

$$\begin{aligned} zM(\xi, z) &= \sum_{j=1}^m Q_j \left(zB^{-1}(\xi) + \beta + z^{-1}\beta\right) Q_j \\ &= \sum_{j=1}^m Q_j \left(z^{-1}b_j^{-1}(\xi) + \beta(\xi, z)\right) Q_j + z^{-1}\beta(\xi, z). \end{aligned} \tag{2.168}$$

In what follows it will be beneficial to use again the binary relation "$\approx$" introduced in (2.69), that is, we write $A(\xi, z) \approx B(\xi, z)$ if and only if for all z with $|z| > r_0$ and r_0 large enough $A(\xi, z) - B(\xi, z) = z^{-1}\beta(\xi, z)$ for $\xi \in \overline{\mathbb{R}}$. For any $l = 1, ...m$, using definition (2.163) for $\beta_0(\xi,z)$ results in

$$\begin{aligned} P_j'(\xi,z)Q_l &\approx \beta_0(\xi,z)Q_l \\ &= -\sum_{k=1}^m \frac{(b_k^{-1}(\xi))'}{2\pi i}\int_{\gamma_j(\xi)} (\lambda - A)^{-1}(Q_k - P_k + P_k)(\lambda - A)^{-1} d\lambda Q_l \\ &\approx -\sum_{k=1}^m \frac{(b_k^{-1}(\xi))'}{2\pi i}\int_{\gamma_j(\xi)} (\lambda - A)^{-1}P_k(\lambda - A)^{-1} d\lambda Q_l, \end{aligned}$$

where we made use of equation (2.162). Note, that since $\sigma(A(\xi,z)P_k(\xi,z)) \cap \gamma_j(\xi) = \emptyset$ for all $k \neq j$, we have the identity

$$\int_{\gamma_j(\xi)} (A(\xi,z) - \lambda)^{-2} P_k(\xi,z) d\lambda = 0, \qquad \text{for all } j \neq k, \xi \in \overline{\mathbb{R}}, |z| \geqslant r_0.$$

Taking this into account and recalling that $P_k(\xi,z)$ commutes with $A(\xi,z)$ results, for all $j \neq l$, in the following identity:

$$\begin{aligned} P_j'(\xi,z)Q_l &\approx -\frac{(b_j^{-1}(\xi))'}{2\pi i}\int_{\gamma_j(\xi)} (\lambda - A(\xi,z))^{-2} P_j(\xi,z) Q_l d\lambda \\ &= -\frac{(b_j^{-1}(\xi))'}{2\pi i}\int_{\gamma_j(\xi)} (\lambda - A(\xi,z))^{-2} (P_j(\xi,z) - Q_j + Q_j) Q_l d\lambda \\ &\approx -\frac{(b_j^{-1}(\xi))'}{2\pi i}\int_{\gamma_j(\xi)} (\lambda - A(\xi,z))^{-2} Q_j Q_l d\lambda = 0. \end{aligned}$$

Similar calculations yields the following result:

$$\begin{aligned} &P_l(\xi,z)P_j'(\xi,z) \\ &\approx -\frac{(b_j^{-1}(\xi))'}{2\pi i} P_l(\xi,z)\int_{\gamma_j(\xi)} (\lambda - A(\xi,z))^{-2} P_j(\xi,z) d\lambda = 0, \text{ for all } l \neq j. \end{aligned}$$

Armed with this result and using (2.159), (2.166), we conclude that, whenever $j \neq l$,

$$Q_l T^{-1} T' Q_j = Q_l T^{-1} P_l \left(\sum_{k=1}^m P_k' Q_k \right) Q_j = Q_l T^{-1} P_l P_j' Q_j \approx 0,$$

while $Q_j T^{-1}(\xi,z) T'(\xi,z) Q_j = Q_j \beta Q_j$. This implies that, since $\sum_{j=1}^m Q_j = I$, we also have

$$T^{-1}(\xi,z)T'(\xi,z) \approx \sum_{j=1}^m Q_j \beta(\xi,z) Q_j.$$

Combining this with (2.167) and (2.168) we have:

$$T^{-1}\mathcal{E}(z)T = z^{-1}\beta(\xi,z) + \sum_{j=1}^{m} Q_j\left(-\partial_\xi + zb_j^{-1}(\xi) + \beta(\xi,z)\right)Q_j,$$

which exactly corresponds to the formulas (2.152), (2.153), (2.154), and therefore concludes the proof of the theorem. ∎

We can now prove the inequality "$\geqslant$" in Theorem 1.4.

Proposition 2.39. *Assume* 2.17. *Then*

$$s_0^\infty(\mathcal{G}) \geqslant \max\{s_0^\infty(\mathcal{G}^+), s_0^\infty(\mathcal{G}^-)\}. \tag{2.169}$$

Proof. We have to show that for any ω with $\omega > s_0^\infty(\mathcal{G})$ we have $\omega > s_0^\infty(\mathcal{G}^\pm)$. Using the same arguments as in Part 1 of the proof of the proposition and the diagonalization procedure as the one discussed in Part 2 of the proof of Proposition 2.37, the problem can be effectively reduced to the task of proving the following assertion: If the operator $\mathcal{E}_0(x)$ defined in (2.138) is invertible for any $x \geqslant \omega$ and $\sup_{x>\omega}\|\mathcal{E}_0(x)^{-1}\| < \infty$, then the operator $\mathcal{E}_0^\pm(x)$ defined in (2.139) is also invertible for all $x \geqslant \omega$ and $\sup_{x>\omega}\|\mathcal{E}_0^\pm(x)^{-1}\| < \infty$.

Once again, we may assume without loss of generality that there exists such $\varepsilon > 0$ that $b(\xi) < -\varepsilon < 0$ for all $\xi \in \mathbb{R}$. From the invertibility of $\mathcal{E}_0(x)$ and Proposition 2.4(iii), we deduce that $\sigma(\mathcal{E}_0(x)) \cap i\mathbb{R} = \emptyset$ for all $x \geqslant \omega$. On the other hand, the spectrum of the asymptotic operators $\mathcal{E}_0^\pm$ belongs to the boundary of $\sigma(\mathcal{E}_0(x))$ (cf. Proposition 2.4(iv)), and so $\sigma(\mathcal{E}_0^\pm(x)) \cap i\mathbb{R} = \emptyset$, for all $x \geqslant \omega$. Moreover, since $\mathcal{E}_0^\pm = -\partial + A^\pm(x)$ are constant-coefficient differential operators, by Proposition 2.4(i) it is also true that for all $x \geqslant \omega$ one has $\sigma(A^\pm(x)) \cap i\mathbb{R} = \emptyset$, and for all such x we have a stable semigroup $\{e^{tA^{\pm}(x)}\}_{t\geqslant 0}$. Following the calculations as in (2.140) – (2.141), it is easy to show that there exist uniform in $x \geqslant \omega$ constants $K, \alpha^{\pm} > 0$ such that

$$\|e^{tA^{\pm}(x)}\| \leqslant K e^{-\alpha^{\pm} t}, \qquad t \geqslant 0. \tag{2.170}$$

Recalling from Proposition 2.4(vii) that $(\mathcal{E}_0^{\pm})^{-1}$ is the convolution with the semigroup $e^{tA^{\pm}(x)}$, we conclude that for all $x \geqslant \omega$ the operator $\mathcal{E}_0^{\pm}(x)$ is indeed invertible and bounded, and moreover, $\sup_{x>\omega} \|\mathcal{E}_0^{\pm}(x)\| < \infty$. ∎

Chapter 3

Stability estimates for semigroups on Banach spaces

In this chapter we deal with an abstract strongly continuous semigroup $\{T(t)\}_{t\geqslant 0}$ on a Banach space X with the generator A. If X is a Hilbert space then the Gearhart-Prüss Theorem asserts that the stability of the semigroup is controlled by the norm of the resolvent of A. If X is not a Hilbert space, then the Gearhart-Prüss Theorem does not hold. Its replacement, Datko's Theorem, says that now the stability of the semigroup is controlled by certain operators, acting on the space $L_p(\mathbb{R}; X)$, $p \geqslant 1$, of X-valued functions, such as the operator of convolution with the semigroup, L^p-Fourier multiplier induced by the resolvent of A, etc.

Our objectives in this chapter are twofold: we revisit Datko's Theorem, and prove its quantitative version relating the norms of various operators involved in this theorem. We also prove a generalization of a recent result of B. Helffer and J. Sjöstrand, and compute the constants in stability estimates for the semigroup in terms of the norm of these operator.

The plan of this section is as follows. In the next section we setup the stage and formulate the first principal result of this chapter, Theorem 3.2, relating the stability estimates and the norms of the operators from Datko's Theorem. The quantitative version of Datko's Theorem 3.7 is formulated and proved in Section 3.2. Finally, the proof of Theorem 3.2 is given in Section 3.3.

3.1 The Setting and Main Result

We recall that for any strongly continuous semigroup $\{T(t)\}_{t\geqslant 0}$ on a Banach space X there exist (possibly large) constants λ and $L = L(\lambda)$ such that the following inequality holds:

$$\|T(t)\| \leqslant L\mathrm{e}^{\lambda t} \text{ for all } t \geqslant 0. \tag{3.1}$$

In many problems, it is easy to obtain a rough exponential estimate of this type, but one is interested in decreasing λ as much as possible. The infimum of all λ for which there exists an $L = L(\lambda)$ such that (3.1) holds is called the semigroup growth bound, and is denoted by $\omega(T)$ (cf. Subsection 2.1.1). The semigroup is called uniformly exponentially stable if $\omega(T) < 0$, that is, if the inequality

$$\|T(t)\| \leqslant M\mathrm{e}^{\omega t} \text{ for all } t \geqslant 0 \tag{3.2}$$

holds for some negative ω and $M = M(\omega)$. Another useful quantity mentioned in Subsection 2.1.1 is the abscissa of uniform boundedness of the resolvent, $s_0(A)$, defined as the infimum of all real $\omega \in \mathbb{R}$ such that $\{z : \mathrm{Re}z > \omega\} \subset \rho(A)$ and $\sup_{\mathrm{Re}z\geqslant\omega} \|R(z, A)\| < \infty$. Then, the Gearhart-Prüss Theorem 1.2 says that if the semigroup acts on a Hilbert space then $\omega(T) = s_0(A)$.

Our objective in this chapter is to evaluate the constant M in (3.2) via the uniform bound of the norm of the resolvent $R(z, A) = (z - A)^{-1}$ in the half-plane $\{z \in \mathbb{C} :$

$\mathrm{Re}z \geqslant \omega\}$ provided it is finite, that is, via the quantity

$$N := \sup_{\mathrm{Re}z \geqslant \omega} \|R(z,A)\| < \infty, \tag{3.3}$$

and via the known constants λ, L that enter the general inequality (3.1), and to give Banach-space version of this result.

B. Helffer and J. Sjöstrand recently proved the following result relating these constants for Hilbert spaces (see [HS, Proposition 2.1]).

Theorem 3.1 (B. Helffer and J. Sjöstrand)**.** *Let $\{T(t)\}_{t \geqslant 0}$ be a strongly continuous semigroup on a Hilbert space. If $\omega, \lambda \in \mathbb{R}$ are such that $\omega < \lambda$ and* (3.1), (3.3) *hold, then* (3.2) *holds with the constant*

$$M = L\big(1 + 2LN(\lambda - \omega)\big). \tag{3.4}$$

In the current chapter we examine the situation when the strongly continuous semigroup acts on a *Banach* space X. As it is well known, in the Banach space case the Gearhart-Prüss Theorem does not hold (see [ABHN, Example 5.2.2] for an example of $s_0(A) < \omega(T)$ on $L^p(0,1)$, $p \neq 2$, and more examples in [V]). For Banach spaces, there are several known replacements of the Gearhart-Prüss Theorem. These results are summarized in Theorem 3.7 below, and sometimes are collectively called the Datko Theorem (see Section 5.1 in [ABHN] and historical comments therein). This theorem describes several operators whose boundedness is equivalent to uniform exponential stability of a semigroup. In particular, it says that a strongly continuous semigroup on a Banach space X is uniformly exponentially stable if and only if a convolution operator, $\mathcal{K}^+$, is bounded on $L^p(\mathbb{R}_+; X)$. Throughout, we fixed p such that $1 \leqslant p < \infty$; here $\mathcal{K}^+$ is defined by $(\mathcal{K}^+u)(t) = \int_0^t T(t-s)u(s)\,ds$, $t \geqslant 0$. One of our primary results in this direction is a quantitative version of this theorem given below, see Theorem 3.7, which relates the norm of the convolution operator and the norms of some other operators whose boundedness is also equivalent to the uniform exponential stability of the semigroup.

One can consider the convolution operator, for an $\omega \in \mathbb{R}$, on the exponentially weighted space $L^p_\omega(\mathbb{R}_+; X)$ of the functions $u : \mathbb{R}_+ \to X$ such that $\mathrm{e}^{-\omega(\cdot)}u(\cdot) \in L^p(\mathbb{R}_+; X)$, and ask how to evaluate the constant M in (3.2) via the norm of $\mathcal{K}^+$ on $L^p_\omega(\mathbb{R}_+; X)$ provided it is finite, that is, via the quantity

$$K := \left\|\mathcal{K}^+\right\|_{\mathcal{B}(L^p_\omega(\mathbb{R}_+;X))} < \infty, \tag{3.5}$$

and via the known constants λ, L in (3.1) (alternatively, by Datko's Theorem 3.7, the operator $\mathcal{K}^+$ in (3.5) can be replaced by any other operator mentioned in this theorem). Using, essentially, the techniques of [HS], we show the following result (see the proof in Section 3.3).

Theorem 3.2. *Let $\{T(t)\}_{t\geqslant 0}$ be a strongly continuous semigroup on a Banach space, $p \geqslant 1$, $p^{-1} + q^{-1} = 1$. If $\omega, \lambda \in \mathbb{R}$ are such that $\omega < \lambda$ and* (3.1)*,* (3.5) *hold, then* (3.2) *holds with the constant*

$$M = L\big(1 + 4p^{-1/p}q^{-1/q}LK(\lambda - \omega)\big). \tag{3.6}$$

One can derive Theorem 3.1 from Theorem 3.2. For, we note that if $p = 2$ then (3.6) becomes $M = L\big(1 + 2LK(\lambda - \omega)\big)$ and if X is a Hilbert space then $K \leqslant N$. The latter inequality is a consequence of the quantitative Datko Theorem 3.7 and Parseval's identity. Indeed, let us recall that a bounded operator valued function $m \in L^\infty(\mathbb{R}; \mathcal{B}(X))$ is called an L^p-Fourier multiplier if the operator $\mathcal{M}_m$ defined by

$\mathcal{M}_m = \mathcal{F}^{-1}\big(m(\cdot)(\mathcal{F}u)(\cdot)\big)$ is bounded on $L^p(\mathbb{R}; X)$; here $\mathcal{F}$ is the Fourier transform. By Parseval's identity, if X is a Hilbert space then $\mathcal{F}$ is an $L^2(\mathbb{R}; X)$-isomorphism, and thus $\|\mathcal{M}_m\|_{\mathcal{B}(L^2(\mathbb{R};X))} = \sup_{s\in\mathbb{R}} \|m(s)\|_{\mathcal{B}(X)}$. An application of the quantitative Datko Theorem 3.7 shows that K in (3.5) is equal to the norm of $\mathcal{M}_m$ on $L^2(\mathbb{R}; X)$ with $m(s) = R(is, A - \omega)$, $s \in \mathbb{R}$, see (3.41). Thus, if X is a Hilbert space, then

$$K = \|\mathcal{M}_m\|_{\mathcal{B}(L^2(\mathbb{R};X))} = \sup_{s\in\mathbb{R}} \|R(is, A-\omega)\|_{\mathcal{B}(X)} \leqslant \sup_{\mathrm{Re} z\geqslant 0} \|R(z, A-\omega)\|_{\mathcal{B}(X)} = N,$$

as required.

We stress that the results of the current chapter have been used in Chapter 2 to prove a generalization of the spectral mapping theorem from [GLS2] for a strongly continuous (but not analytic) semigroup whose generator is induced by a partly parabolic system of partial differential equations arising in stability analysis of travelling fronts. Specifically, in the proof of Theorem 1.1 (that is, in the proof of Proposition 2.35, to be precise) we needed the following result: Given a family of strongly continuous semigroups on a Hilbert space depending on a complex parameter $\beta \in \Omega \subset \mathbb{C}$, that is, $\{T(t, \beta)\}_{t\geqslant 0}$, $\beta \in \Omega$, let us assume that β-dependent constants $L(\beta)$ and $N(\beta)$ in (3.1), (3.3) are bounded from above uniformly for $\beta \in \Omega$. We then conclude that the constants $M(\beta)$ in (3.2) are bounded from above uniformly for $\beta \in \Omega$. The respective Banach space version of this fact is formulated below in Corollary 3.8.

3.2 The quantitative Datko Theorem

Datko's Theorem asserts that a strongly continuous semigroup $\{T(t)\}_{t\geqslant 0}$ with the infinitesimal generator A acting on a Banach space X is uniformly exponentially stable if and only if certain operators are bounded. The quantitative version of this theorem relates the norms of these operators.

3.2.1 Convolutions

We begin by introducing two convolution operators. The operator $\mathcal{K}^+$ is defined on the space $L^p(\mathbb{R}_+; X)$, with some $p \geqslant 1$, as the convolution with the operator valued function $T(\cdot)$, that is, it is defined by the formula

$$(\mathcal{K}^+ u)(t) = \int_0^t T(t-s)u(s)\,ds,\ t \geqslant 0. \tag{3.7}$$

For any $\omega \in \mathbb{R}$ let $L^p_\omega(\mathbb{R}_+; X)$ denote the space with the exponential weight $\mathrm{e}^{-\omega(\cdot)}$, that is, the space of the functions u such that $\mathrm{e}^{-\omega(\cdot)}u(\cdot) \in L^p(\mathbb{R}_+; X)$ with the norm $\|u\|_{L^p_\omega} = \big(\int_0^\infty ||u(s)\mathrm{e}^{-\omega s}\|^p_X\,ds\big)^{1/p}$. Let J_ω be the isometry acting from $L^p(\mathbb{R}_+; X)$ onto $L^p_\omega(\mathbb{R}_+; X)$ by the rule $(J_\omega u)(s) = \mathrm{e}^{\omega s}u(s)$, $s \in \mathbb{R}$. Let $\mathcal{K}^+_\omega$ denote the convolution operator defined as in (3.7) but with the semigroup $\{T(t)\}_{t\geqslant 0}$ replaced by the rescaled semigroup $\{T_\omega(t)\}_{t\geqslant 0}$, where $T_\omega(t) = \mathrm{e}^{-\omega t}T(t)$. A trivial calculation

$$(\mathcal{K}^+ J_\omega u)(t) = \int_0^t T(t-s)\mathrm{e}^{\omega s}u(s)\,ds = \mathrm{e}^{\omega t}\int_0^t T_\omega(t-s)u(s)\,ds = (J_\omega \mathcal{K}^+_\omega u)(t)$$

shows that the operator $\mathcal{K}^+$ acting on $L^p_\omega(\mathbb{R}_+; X)$ is isometrically isomorphic to the operator $\mathcal{K}^+_\omega$ acting on $L^p(\mathbb{R}_+; X)$.

Remark 3.3. The well-known Datko-van Neerven Theorem says that $\omega(T) < 0$ if and only if the operator $\mathcal{K}^+$ is bounded, that is, $\mathcal{K}^+ \in \mathcal{B}(L^p(\mathbb{R}_+; X))$ (see, e.g., [ABHN, Theorem 5.1.2, (i)⇔(iii)] or [V, Theorem 3.3.1 (i)⇔(ii)]). The easy "only if" part of this equivalence follows from the fact that the norm of the convolution operator satisfies the estimate $\|\mathcal{K}^+\|_{\mathcal{B}(L^p(\mathbb{R}_+;X))} \leqslant \|T(\cdot)\|_{L^1(\mathbb{R}_+)}$. The "if" part is usually proved by contradiction. Alternatively, it can be derived using Theorem 3.2. Indeed, as we will show in Remark 3.4 below, if $\mathcal{K}^+ \in \mathcal{B}(L^p(\mathbb{R}_+;X))$ then $\mathcal{K}^+_\omega \in \mathcal{B}(L^p(\mathbb{R}_+;X))$ for negative ω with small $|\omega|$. Theorem 3.2 gives an estimate of M in (3.2) in terms of $K = \|\mathcal{K}^+_\omega\|_{\mathcal{B}(L^p(\mathbb{R}_+;X))}$ thus implying that $\omega(T) < 0$. $\diamond$

Next, we introduce an operator, $\mathcal{K}$, defined on the space $L^p(\mathbb{R};X)$ as convolution with the operator valued function $T(\cdot)$, that is, by the formula

$$(\mathcal{K}u)(t) = \int_{-\infty}^{t} T(t-s)u(s)\,ds,\ t \in \mathbb{R}. \tag{3.8}$$

Similarly to the semi-line case, for any $\omega \in \mathbb{R}$ one can consider the operator $\mathcal{K}$ on the space $L^p_\omega(\mathbb{R},X)$ with the norm $\|u\|_{L^p_\omega} = \big(\int_{\mathbb{R}} \|u(s)\mathrm{e}^{-\omega s}\|^p\,ds\big)^{1/p}$; this operator is again isometrically isomorphic to the operator $\mathcal{K}_\omega$ on $L^p(\mathbb{R};X)$, where $\mathcal{K}_\omega$ is defined as in (3.8), but with the semigroup $\{T(t)\}_{t\geqslant 0}$ replaced by the rescaled semigroup $\{T_\omega(t)\}_{t\geqslant 0}$.

It is known that $\mathcal{K} \in \mathcal{B}(L^p(\mathbb{R};X))$ if and only if $\omega(T) < 0$ (see, e.g., [CL, Subsection 4.2.1]). Indeed, we recall that the semigroup $\{T(t)\}_{t\geqslant 0}$ is called hyperbolic if $\sigma\big(T(1)\big) \cap \{z \in \mathbb{C} : |z| = 1\} = \varnothing$. Let $\mathcal{P}$ denote the Riesz spectral projection corresponding to the part of the spectrum of $T(1)$ located inside of the unite disc. In particular, $\mathcal{P} = I$ if and only if $\omega(T) < 0$. The main Green's function $\mathcal{G}$ is defined by the formula $\mathcal{G}(t) = T^t\mathcal{P}$ for $t > 0$ and $\mathcal{G}(t) = -T^t(I - \mathcal{P})$ for $t < 0$, see, e.g., [CL, Sect.4.2]. It is known that the semigroup $\{T(t)\}_{t\geqslant 0}$ is hyperbolic if and only if the operator of convolution with $\mathcal{G}$ is bounded on $L^p(\mathbb{R},X)$, see [CL, Theorem 4.22]. The operator of convolution with $\mathcal{G}$ is equal to $\mathcal{K}$ from (3.8) if and only if $\mathcal{P} = I$. This shows that $\mathcal{K} \in \mathcal{B}(L^p(\mathbb{R};X))$ if and only if $\omega(T) < 0$.

Next, we show that if either one of the operators $\mathcal{K}_+$ or $\mathcal{K}$ is bounded (equivalently, the inequality $\omega(T) < 0$ holds) then

$$\|\mathcal{K}^+\|_{\mathcal{B}(L^p(\mathbb{R}_+;X))} = \|\mathcal{K}\|_{\mathcal{B}(L^p(\mathbb{R};X))}. \tag{3.9}$$

To establish the inequality "$\leqslant$" in (3.9), let us use the following notation. If u is a function on $\mathbb{R}$, then $u|_{\mathbb{R}_+}$ will denote its restriction on R_+; if v is a function on R_+, then $[v]_{\mathbb{R}}$ will denote its extension to $\mathbb{R}$ defined by $[v]_{\mathbb{R}}(t) = v(t)$ for $t \geqslant 0$, and $[v]_{\mathbb{R}}(t) = 0$ for $t < 0$. Then, for any $u \in L^p(\mathbb{R}_+;X)$,

$$\begin{aligned}\|\mathcal{K}^+u\|_{L^p(\mathbb{R}_+;X)} &= \|[\mathcal{K}^+u]_{\mathbb{R}}\|_{L^p(\mathbb{R};X)} = \|\mathcal{K}[u]_{\mathbb{R}}\|_{L^p(\mathbb{R};X)} \\ &\leqslant \|\mathcal{K}\|_{\mathcal{B}(L^p(\mathbb{R};X))}\|[u]_{\mathbb{R}}\|_{L^p(\mathbb{R};X)} = \|\mathcal{K}\|_{\mathcal{B}(L^p(\mathbb{R};X))}\|u\|_{L^p(\mathbb{R}_+;X)},\end{aligned} \tag{3.10}$$

yielding "$\leqslant$" in (3.9). To show the inequality "$\geqslant$"in (3.9), we fix $\epsilon > 0$ and $u \in L^p(\mathbb{R};X)$ such that $\|u\|_{L^p(\mathbb{R};X)} = 1$ and $\|\mathcal{K}\|_{\mathcal{B}(L^p(\mathbb{R};X))} \leqslant \|\mathcal{K}u\|_{L^p(\mathbb{R};X)} + \epsilon$. Let $\{S_t\}_{t\geqslant 0}$ denote the standard isometric shift on $L^p(\mathbb{R};X)$, that is, let us denote $\big(S_tu\big)(s) = u(s-t)$, $s \in \mathbb{R}$. By inspection, $S_t\mathcal{K} = \mathcal{K}S_t$ for all $t \geqslant 0$. Choose $u_n \to u$ in $L^p(\mathbb{R};X)$ such that $\operatorname{supp} u_n \subset (-n,\infty)$ and note that $\operatorname{supp} S_nu_n \subset (0,\infty)$. Then

$$\begin{aligned}\|\mathcal{K}u_n\|_{L^p(\mathbb{R};X)} &= \|S_n\mathcal{K}u_n\|_{L^p(\mathbb{R};X)} = \|\mathcal{K}S_nu_n\|_{L^p(\mathbb{R};X)} \\ &= \|\mathcal{K}[(S_nu_n)|_{\mathbb{R}_+}]_{\mathbb{R}}\|_{L^p(\mathbb{R};X)} = \Big\|\big[\mathcal{K}^+((S_nu_n)|_{\mathbb{R}_+})\big]_{\mathbb{R}}\Big\|_{L^p(\mathbb{R};X)} \\ &= \Big\|\mathcal{K}^+\big((S_nu_n)|_{\mathbb{R}_+}\big)\Big\|_{L^p(\mathbb{R}_+;X)} \\ &\leqslant \|\mathcal{K}^+\|_{\mathcal{B}(L^p(\mathbb{R};X))}\|S_nu_n\|_{L^p(\mathbb{R};X)} \\ &= \|\mathcal{K}^+\|_{\mathcal{B}(L^p(\mathbb{R};X))}\|u_n\|_{L^p(\mathbb{R};X)}.\end{aligned} \tag{3.11}$$

Passing to the limits as $n \to \infty$ and $\epsilon \to 0$ yields the desired inequality in (3.9).

3.2.2 Fourier multipliers and evolution semigroup generators

Let $\mathcal{S}(\mathbb{R})$ denote the Schwartz class of rapidly decaying functions. For $\phi \in \mathcal{S}(\mathbb{R})$ and $x \in X$ we denote by $\phi \otimes x$ the X-valued function from $\mathcal{S}(\mathbb{R};X)$ defined by

$(\phi \otimes x)(s) = \phi(s)x$. The linear span of the functions $\phi \otimes x$ is dense in $L^p(\mathbb{R}; X)$. Given an operator-valued function $m \in L^\infty(\mathbb{R}; \mathcal{B}(X))$, we define $\mathcal{M}_m$ on $\phi \otimes x$ by $\mathcal{M}_m(\phi \otimes x) = \mathcal{F}^{-1}\big(m(\cdot)\mathcal{F}(\phi \otimes x)\big)$, where

$$(\mathcal{F}u)(t) = \int_{\mathbb{R}} u(s)\mathrm{e}^{-ist}\,ds, \qquad (\mathcal{F}^{-1}u)(s) = \frac{1}{2\pi}\int_{\mathbb{R}} u(t)\mathrm{e}^{ist}\,dt \tag{3.12}$$

are the Fourier transforms. We say that m is an $L^p(\mathbb{R}; X)$-Fourier multiplier, if the operator $\mathcal{M}_m$ extends to a bounded operator on $L^p(\mathbb{R}; X)$. We refer to [A, H1] for more information regarding operator valued Fourier multipliers.

A strongly continuous semigroup $\{T(t)\}_{t \geqslant 0}$ on a Banach space X is hyperbolic if and only if $\sigma(A) \cap i\mathbb{R} = \varnothing$ and $m(s) = (is - A)^{-1}$ is an $L^p(\mathbb{R}; X)$-Fourier multiplier, see [LS, Theorem 2.7 (1) $\Leftrightarrow$ (2)]. The assertion $m \in L^\infty(\mathbb{R}; \mathcal{B}(X))$ here is a part of the definition of $\mathcal{M}_m$. Moreover, $\omega(T) < 0$ if and only if $\sigma(A) \subset \{z : \mathrm{Re} z < 0\}$, $s_0(A) \leqslant 0$ and m is an $L^p(\mathbb{R}; X)$-Fourier multiplier, cf. [LR, Corollary 3.8] and [H2].

Assume that $\omega(T) < 0$. Then one can apply Fubini's Theorem to check that $\mathcal{F}\mathcal{K}(\phi \otimes x) = m(\cdot)\mathcal{F}(\phi \otimes x)$, for all $\phi \in \mathcal{S}(\mathbb{R})$ and $x \in X$. Thus, $\mathcal{K} = \mathcal{M}_m$ via a density argument, and

$$\|\mathcal{K}\|_{\mathcal{B}(L^p(\mathbb{R};X))} = \|\mathcal{M}_m\|_{\mathcal{B}(L^p(\mathbb{R};X))}, \tag{3.13}$$

where both norms are finite if and only if $\omega(T) < 0$. Here, the condition $\omega(T) < 0$ was used to make sure that Fubini's Theorem applies in the proof of the identity $\mathcal{K}(\phi \otimes x) = \mathcal{M}_m(\phi \otimes x)$. However, a simple analytic continuation argument given in [LR, Lemma 3.5] shows that $\mathcal{K}(\phi \otimes x) = \mathcal{M}_m(\phi \otimes x)$ provided $\sigma(A) \subset \{\mathrm{Re} z < 0\}$ and $s_0(A) \leqslant 0$. Thus, conditions $\mathcal{M}_m \in \mathcal{B}(L^p(\mathbb{R}; X))$, $\sigma(A) \subset \{\mathrm{Re} z < 0\}$ and $s_0(A) \leqslant 0$ imply $\mathcal{K} \in \mathcal{B}(L^p(\mathbb{R}; X))$ and (3.13). On the other hand, if $\mathcal{K} \in \mathcal{B}(L^p(\mathbb{R}; X))$ then

$\mathcal{K}^+ \in \mathcal{B}(L^p(\mathbb{R}_+;X))$ by the proof of the inequality "$\leqslant$" in (3.9). This yields $\omega(T) < 0$ which implies $\sigma(A) \subset \{\mathrm{Re} z < 0\}$ and $s_0(A) \leqslant 0$ thus proving the equalities $\mathcal{K} = \mathcal{M}_m$ and (3.13).

We will now define the evolution semigroups and their generators on the spaces $L^p(\mathbb{R}_+;X)$ and $L^p(\mathbb{R};X)$ and relate the inverses of the generators to the convolution operators introduced in Subsection 3.2.1.

The evolution semigroup $\{E_A^+(t)\}_{t\geqslant 0}$ on $L^p(\mathbb{R}_+;X)$ is defined as

$$\big(E_A^+(t)u\big)(s) = \begin{cases} \mathrm{e}^{tA}u(s-t) & \text{for } s \geqslant t, \\ 0 & \text{for } 0 \leqslant s < t. \end{cases} \tag{3.14}$$

Let ∂_0 be the operator of differentiation on $L^p(\mathbb{R}_+;X)$ with the domain $\mathrm{dom}\,\partial_0 = \{f \in W_1^p(\mathbb{R}_+;X) : f(0) = 0\}$. We keep notation A for the operator on $L^p(\mathbb{R}_+;X)$ defined by $(Au)(s) = Au(s)$, $s \in \mathbb{R}_+$, for $u \in \mathrm{dom}\,A = \{u \in L^p(\mathbb{R}_+;X) : u(s) \in \mathrm{dom}_X A$ a.a. and $Au(\cdot) \in L^p(\mathbb{R}_+;X)\}$. Let $-\partial_0 + A$ denote the sum of the two operators defined on $\mathrm{dom}(\partial_0) \cap \mathrm{dom}_{L^p(\mathbb{R}_+;X)} A$. Let G_A^+ denote the infinitesimal generator of the semigroup $\{E_A^+(t)\}_{t\geqslant 0}$. One can view $E_A^+(t)$ as the product $T(t)S_t^+$, where $S_t^+ = e^{-\partial_0 t}$ is the shift semigroup defined on $L^p(\mathbb{R}_+;X)$ by $(S_t^+u)(s) = u(s-t)$ for $s \geqslant t$ and $(S_t^+u)(s) = 0$ for $0 \leqslant s < t$. Since the semigroups $T(t)$ and S_t^+ commute, G_A^+ is the closure of the operator $-\partial_0 + A$, cf. [CL, Remark 2.35] and [CL, Sect.2.2.3].

In particular, if $\phi = \psi|_{\mathbb{R}_+}$ for some $\psi \in \mathcal{S}(\mathbb{R})$ with $\psi(s) = 0$ for $s \leqslant 0$, and $x \in \mathrm{dom}\,A$ then $\phi \otimes x \in \mathrm{dom}\,G_A^+$ and $G_A^+(\phi \otimes x) = -\phi' \otimes x + \phi \otimes Ax$. Therefore, using integration by parts, for any $\tau \geqslant 0$ one has:

$$\begin{aligned}\big(\mathcal{K}^+G_A^+(\phi \otimes x)\big)(\tau) &= \int_0^\tau \big(-\phi'(s) \otimes T(t-s)x + \phi(s) \otimes AT(t-s)x\big)\, ds \\ &= -(\phi \otimes x)(\tau).\end{aligned} \tag{3.15}$$

Also, $G_A^+\mathcal{K}^+(\phi\otimes x) = -\phi\otimes x$ for $x\in X$ because

$$\big(E_A^+(t)\mathcal{K}^+(\phi\otimes x)\big)(\tau) = \begin{cases} \int_t^\tau \phi(\tau-s)T(s)x\,ds & \text{for any } \tau\geqslant t,\\ 0, & \text{for any } 0\leqslant\tau<t. \end{cases} \tag{3.16}$$

Since the linear span of the functions $\phi\otimes x$ is dense, it follows that

$$\mathcal{K}^+G_A^+u = -u \text{ for all } u\in\operatorname{dom}G_A^+ \text{ and } G_A^+\mathcal{K}^+u = -u \text{ for all } u\in L^p(\mathbb{R}_+;X).$$

Thus, cf. [CL, Sect.2.2.3], $\omega(T)<0$ if and only if $\mathcal{K}^+\in\mathcal{B}(L^p(\mathbb{R}_+;X))$ if and only if $0\in\rho(G_A^+)$; also,

$$\|\mathcal{K}^+\|_{\mathcal{B}(L^p(\mathbb{R}_+;X))} = \|(G_A^+)^{-1}\|_{\mathcal{B}(L^p(\mathbb{R}_+;X))}. \tag{3.17}$$

Remark 3.4. We conclude this fragment by showing how Theorem 3.2 implies the conclusion "$\mathcal{K}^+\in\mathcal{B}\big(L^p(\mathbb{R}_+;X)\big)$ yields $\omega(T)<0$" in the Datko-van Neerven Theorem (see [ABHN, Theorem 5.1.2 (iii) $\Rightarrow$ (i)], [V, Theorem 3.3.1 (ii) $\Rightarrow$ (i)] and (ii) $\Rightarrow$ (i) in Theorem 3.7 below). Indeed, if $\mathcal{K}^+\in\mathcal{B}\big(L^p(\mathbb{R}_+;X)\big)$ then $0\in\rho(G_A^+)$ because $\mathcal{K}^+=(G_A^+)^{-1}$. But then, by a standard perturbation argument, $0\in\rho(G_{A-\omega}^+)$ for a sufficiently close to zero negative ω because $G_{A-\omega}^+=G_A^+-\omega$. Then $\mathcal{K}_\omega^+=(G_{A-\omega}^+)^{-1}\in\mathcal{B}(L^p(\mathbb{R}_+;X))$. Now (3.2) holds by Theorem 3.2, and thus $\omega(T)<0$. $\diamond$

Next, let us introduce the evolution semigroup $\{E_A(t)\}_{t\geqslant0}$, defined on $L^p(\mathbb{R},X)$ by $(E_A(t)u)(s)=\mathrm{e}^{tA}u(s-t)$, $s\in\mathbb{R}$. The generator of this semigroup will be denoted by G_A. As before, G_A is the closure of the operator $-\partial+A$, where ∂ is the operator of differentiation on $L^p(\mathbb{R},X)$ with the domain $W_1^p(\mathbb{R},X)$. If $\phi\in\mathcal{S}(\mathbb{R})$ and $x\in\operatorname{dom}_X A$ then $G_A(\phi\otimes x)=-\phi'\otimes x+\phi\otimes Ax$. Similarly to the semi-line case, we have

$$\mathcal{K}G_Au=-u \text{ for } u\in\operatorname{dom}G_A \text{ and } G_A\mathcal{K}u=-u \text{ for } u\in L^p(\mathbb{R},X).$$

Thus, cf. [CL, Sect.2.2.2, 4.2.1], $\omega(T) < 0$ if and only if $\sigma(G_A) \subset \{\mathrm{Re} z < 0\}$, which it turn is true if and only if $\mathcal{K} \in \mathcal{B}(L^p(\mathbb{R}, X))$; also

$$\|\mathcal{K}\|_{\mathcal{B}(L^p(\mathbb{R};X))} = \|(G_A)^{-1}\|_{\mathcal{B}(L^p(\mathbb{R};X))}. \tag{3.18}$$

Indeed, one of the main properties of the evolution semigroup is that the growth bound of the semigroup $\{T(t)\}_{t\geqslant 0}$ is equal to the spectral bound of the semigroup $\{E_A(t)\}_{t\geqslant}$, that is, $\omega(T) = \sup\{\mathrm{Re} z : z \in \sigma(G_A)\}$, see [CL, Corollary 2.40]. This yields the first of the two equivalent statements just made. Since $\mathcal{K} \in \mathcal{B}(L^p(\mathbb{R}; X))$ if and only if $\mathcal{K}^+ \in \mathcal{B}(L^p(\mathbb{R}_+; X))$ by (3.9), the second equivalence follows from the Datko-van Neerven Theorem (see [ABHN, Theorem 5.1.2 (iii) $\Leftrightarrow$ (i)]).

We will now show that condition $\sigma(G_A) \subset \{\mathrm{Re} z < 0\}$ implies

$$\|(G_A)^{-1}\|_{\mathcal{B}(L^p(\mathbb{R};X))} = \sup_{\mathrm{Re} z \geqslant 0} \|R(G_A, z)\|. \tag{3.19}$$

Indeed, another nice property of the evolution semigroup generator is that $\sigma(G_A)$ is invariant with respect to vertical translations and that

$$\|R(G_A, z)\|_{\mathcal{B}(L^p(\mathbb{R};X))} = \|R(G_A, z + i\zeta)\|_{\mathcal{B}(L^p(\mathbb{R};X))} \text{ for all } z \in \rho(G_A), \zeta \in \mathbb{R}, \tag{3.20}$$

see [CL, Proposition 2.36(b)] and its proof. Since G_A is a semigroup generator, $\|R(G_A, s)\|_{\mathcal{B}(L^p(\mathbb{R};X))} \to 0$ as $s \to \infty$, $s \in \mathbb{R}$. Thus, if $\sigma(G_A) \subset \{\mathrm{Re} z < 0\}$ then

$$\sup_{\mathrm{Re} z \geqslant 0} \|R(G_A, z)\|_{\mathcal{B}(L^p(\mathbb{R};X))} < \infty. \tag{3.21}$$

Now (3.19) holds by applying the maximal principle to $R(G_A, \cdot)$ on a long horizontal rectangle with the left side belonging to $i\mathbb{R}$. A similar argument based on [CL, Proposition 3.21] shows that G_A and $\mathbb{R}$ in (3.19) can be replaced by G_A^+ and $\mathbb{R}_+$.

Finally, we prove that the condition $\sigma(G_A) \subset \{\mathrm{Re} z < 0\}$ is equivalent to the assertions $(G_A)^{-1} \in \mathcal{B}(L^p(\mathbb{R}; X))$, $\sigma(A) \subset \{\mathrm{Re} z < 0\}$, and $s_0(A) \leqslant 0$. Indeed, if $\sigma(G_A) \subset \{\mathrm{Re} z < 0\}$ then $\omega(T) < 0$ as remarked above, which implies the required assertions. On the other hand, if $(G_A)^{-1} \in \mathcal{B}(L^p(\mathbb{R}; X))$ then $\{T(t)\}_{t \geqslant 0}$ is hyperbolic by [LS, Theorem 2.7 (1) $\Rightarrow$ (2)]. Moreover, cf. [LS, Remark 2.11], using elementary properties of Fourier transform, we have

$$\begin{aligned} &\mathcal{M}_m G_A(\phi \otimes x) = -\phi \otimes x, \text{ for } x \in \operatorname{dom} A, \phi \in \mathcal{S}(\mathbb{R}), \\ &G_A \mathcal{M}_m(\phi \otimes x) = -\phi \otimes x, \text{ for } x \in X, \phi \in \mathcal{S}(\mathbb{R}), \end{aligned} \tag{3.22}$$

yielding $\mathcal{M}_m = (G_A)^{-1} \in \mathcal{B}(L^p(\mathbb{R}; X))$. This, combined with the assertions $\sigma(A) \subset \{\mathrm{Re} z < 0\}$ and $s_0(A) \leqslant 0$, yields $\mathcal{K} \in \mathcal{B}(L^p(\mathbb{R}; X))$, as has been noted before, and therefore $\omega(T) < 0$.

3.2.3 Datko's constant

Let us consider an operator, $\mathcal{D}$, acting from the Banach space X into $L^p(\mathbb{R}_+; X)$ by the formula $(\mathcal{D}x)(t) = T(t)x$, $x \in X$, $t \geqslant 0$. This operator is bounded if and only if there is a Datko constant, D, such that the following (Datko-Pazy) inequality is satisfied,

$$\int_0^\infty \|T(t)x\|_X^p \, dt \leqslant D^p \|x\|_X^p \text{ for all } x \in X, \tag{3.23}$$

and $\|\mathcal{D}\|_{\mathcal{B}(X; L^p(\mathbb{R}_+; X))}$ is the infimum of all constants D such that (3.23) holds (cf. [ABHN, eqn. (5.5.4)] or [P, eqn. (4.4.3)]). With a slight abuse of notation we will sometimes denote $\|\mathcal{D}\|_{\mathcal{B}(X; L^p(\mathbb{R}_+; X))}$ by D. By the classical Datko-Pazy Theorem, $D < \infty$ if and only if the semigroup $\{T(t)\}_{t \geqslant 0}$ is uniformly exponentially stable, see [ABHN, Theorem 5.1.2, (i)$\Leftrightarrow$(ii)] or [P, Theorem 4.1]. We will now obtain an estimate from above for D in terms of $\|\mathcal{K}\|_{\mathcal{B}(L^p(\mathbb{R}_+:X))}$, and an estimate from above for $\|\mathcal{K}\|_{\mathcal{B}(L^p(\mathbb{R}_+:X))}$ in terms of D.

Given (3.1), fix any positive $w > \lambda$ and denote $g(t) = e^{-wt}T(t)x$ for an $x \in X$ and all $t \geqslant 0$. Then

$$\begin{aligned} w^{-1}\big(1-\mathrm{e}^{-wt}\big)T(t)x &= \int_0^t \mathrm{e}^{-ws}T(t)x\,ds = \int_0^t T(t-s)\mathrm{e}^{-ws}T(s)x\,ds \\ &= (\mathcal{K}^+g)(t), \qquad t \geqslant 0, \end{aligned} \tag{3.24}$$

and (3.1) imply

$$\begin{aligned} \|T(\cdot)x\|_{L^p(\mathbb{R}_+;X)} &= w\left\|w^{-1}\big(T(\cdot)x - \mathrm{e}^{-w(\cdot)}T(\cdot)x + \mathrm{e}^{-w(\cdot)}T(\cdot)x\big)\right\|_{L^p(\mathbb{R}_+;X)} \\ &\leqslant w\left\|\mathcal{K}^+g\right\|_{L^p(\mathbb{R}_+;X)} + \|g\|_{L^p(\mathbb{R}_+;X)} \\ &\leqslant \left(w\left\|\mathcal{K}^+\right\|_{\mathcal{B}(L^p(\mathbb{R}_+;X))} + 1\right)\left(\int_0^\infty \mathrm{e}^{-wpt}L^p\mathrm{e}^{\lambda pt}\,dt\right)^{\frac{1}{p}}\|x\| \\ &= L\left(w\left\|\mathcal{K}^+\right\|_{\mathcal{B}(L^p(\mathbb{R}_+;X))} + 1\right)\left(p\,(w-\lambda)\right)^{-1/p}\|x\|. \end{aligned} \tag{3.25}$$

Thus $D = \|\mathcal{D}\|_{\mathcal{B}(X;L^p(\mathbb{R}_+;X))}$ is majorated by the pre-factor in (3.25).

To obtain an estimate from above for $\|\mathcal{K}\|_{L^p(\mathbb{R}_+;X)}$ in terms of D satisfying (3.23), we remark that (3.23) implies (3.2) with some $\omega < 0$ by the Datko-Pazy Theorem (see [ABHN, Theorem 5.1.2 (i) $\Leftrightarrow$ (ii)]). Since $\mathcal{K}^+$ is a convolution operator,

$$\|\mathcal{K}^+\|_{\mathcal{B}(L^p(\mathbb{R}_+;X))} \leqslant \|T(\cdot)\|_{L^1(\mathbb{R}_+)} \leqslant M\int_0^\infty \mathrm{e}^{\omega t}\,dt = -M\omega^{-1}. \tag{3.26}$$

An effective estimate for M in terms of D is given in the next proposition whose proof follows the standard argument in the classical Datko-Pazy Theorem, see [P, Theorem 4.1], [EN, Theorem V.1.8].

Proposition 3.5. *Assume* (3.1) *with* $\lambda > 0$ *and* (3.23). *Denote, for brevity,* $C = L(p\lambda D^p + 1)^{1/p}$. *Then* (3.2) *holds for each* ω *satisfying* $-C^{-1}(DC\mathrm{e}^{1/C})^{-p} < \omega < 0$ *and* $M = C\mathrm{e}^{1/C}$.

Proof. **Step 1.** We claim that

$$\|T(t)\|_{\mathcal{B}(X)} \leqslant C := L(p\lambda D^p + 1)^{1/p}, \text{ for all } t \geqslant 0. \tag{3.27}$$

Indeed, for $x \in X$ and all $t \geqslant 0$ we infer

$$\begin{aligned}(p\lambda)^{-1}\big(1 - \mathrm{e}^{-p\lambda t}\big)\|T(t)x\|^p &= \int_0^t \mathrm{e}^{-p\lambda s}\|T(s)T(t-s)x\|^p\,ds \\ \leqslant L^p \int_0^t \|T(t-s)x\|^p\,ds &= L^p \int_0^t \|T(s)x\|^p\,ds \leqslant L^p D^p \|x\|^p.\end{aligned} \tag{3.28}$$

This and (3.1) imply

$$\|T(t)\|_{\mathcal{B}(X)} \leqslant \sup_{t\geqslant 0} \min\Big\{L\mathrm{e}^{\lambda t},\, LD\big((p\lambda)^{-1}(1-\mathrm{e}^{-p\lambda t})\big)^{-1/p}\Big\} = L\mathrm{e}^{\lambda t_0} = C, \tag{3.29}$$

where $t = t_0$ is the unique solution of the equation $e^{\lambda t} = D\big((p\lambda)^{-1}(1-\mathrm{e}^{-p\lambda t})\big)^{-1/p}$. Finding this solution proves claim (3.27).

Step 2. We claim that

$$\lim_{t\to\infty} \|T(t)\|_{\mathcal{B}(X)} = 0. \tag{3.30}$$

Indeed, by (3.23) and (3.27), for any $x \in X$ and $t > 0$,

$$t\,\|T(t)x\|^p = \int_0^t \|T(s)T(t-s)x\|^p\,ds \leqslant C^p \int_0^t \|T(t-s)x\|^p\,ds \leqslant C^p D^p \|x\|, \tag{3.31}$$

yielding $\|T(t)\|_{\mathcal{B}(X)} \leqslant CDt^{-1}$ and (3.30).

Step 3. Denote $\alpha = C^{-1}\mathrm{e}^{-1/C}$. By (3.30), for every $x \in X$ there exists a finite positive number $t(x)$ defined as

$$t(x) = \sup\big\{t > 0 : \|T(s)x\| \geqslant \alpha\|x\|, \text{ for all } 0 \leqslant s \leqslant t\big\}. \tag{3.32}$$

Using (3.23), we have

$$t(x)\alpha^p\,\|x\|^p \leqslant \int_0^{t(x)} \|T(s)x\|^p\,ds \leqslant D^p\|x\|^p, \tag{3.33}$$

and thus $t(x) \leqslant (D/\alpha)^p$. Suppose that $t_1 > (D/\alpha)^p$. Then

$$\|T(t_1)x\| \leqslant \|T(t_1 - t(x))\| \, \|T(t(x))x\| \leqslant C\alpha\|x\| = e^{-1/C}\|x\| \tag{3.34}$$

due to (3.27) and (3.32). Fix any $t_1 > (D/\alpha)^p$ and let $t = nt_1 + s$, $0 \leqslant s < t_1$, $n = 1, 2, \dots$. Then

$$\begin{aligned} \|T(t)\| \leqslant \|T(s)\| \, \|T(t_1)\|^n \leqslant C(e^{-1/C})^n = C(e^{-1/C})^{-s/t_1}(e^{-1/C})^{t/t_1} \\ \leqslant C\mathrm{e}^{1/C}\, e^{-(t_1C)^{-1}t}, \text{ for all } t \geqslant t_1. \end{aligned} \tag{3.35}$$

If $0 \leqslant t < t_1$ then, by (3.27),

$$\|T(t)\| \leqslant C = C\mathrm{e}^{1/C}\mathrm{e}^{-(t_1C)^{-1}t}\mathrm{e}^{-\frac{1}{C}+\frac{t}{t_1C}} \leqslant C\mathrm{e}^{1/C}\mathrm{e}^{-(t_1C)^{-1}t}. \tag{3.36}$$

Thus, we conclude that if $t_1 > (D/\alpha)^p = (DC\mathrm{e}^{1/C})^p$ then

$$\|T(t)\| \leqslant C\mathrm{e}^{1/C}\mathrm{e}^{-(t_1C)^{-1}t}, \text{ for all } t \geqslant 0. \tag{3.37}$$

Step 4. We are finally ready to finish the proof of the proposition. Assume that $0 > \omega > -C^{-1}(DC\mathrm{e}^{1/C})^{-p}$ and choose $t_1 > 0$ such that the following inequalities hold:

$$-C^{-1}(DC\mathrm{e}^{1/C})^{-p} < -(t_1C)^{-1} < \omega. \tag{3.38}$$

Then $t_1 > (DC\mathrm{e}^{1/C})^p$ and (3.37) yields (3.2) with $M = C\mathrm{e}^{1/C}$, as required. ∎

In particular, letting $\omega = -\frac{1}{2}C^{-1}(DC\mathrm{e}^{1/C})^{-p}$ in (3.26) yields:

$$\|\mathcal{K}^+\|_{\mathcal{B}(L^p(\mathbb{R}_+;X))} \leqslant 2D^pC^{2+p}\mathrm{e}^{(1+p)/C}, \tag{3.39}$$

where $C = L(p\lambda D^p + 1)^{1/p}$.

Remark 3.6. Of course, the range for ω in Proposition 3.5 is not optimal: By the quantitative version of Datko's Theorem proved by J. Van Neerven, one has $\omega(T) \leqslant -1/(pD)$, see. [V, Theorem 3.18]. It would have been interesting to obtain an optimal estimate in (3.39). ◇

3.2.4 Summary of the results

We summarize the results obtained earlier in this section as follows. We recall the notation $m(s) = R(is, A)$, $s \in \mathbb{R}$, and emphasize again that the Fourier multiplier $\mathcal{M}_m = \mathcal{F}^{-1}\big(m(\cdot)\mathcal{F}\big)$ is defined provided $\sup_{s\in\mathbb{R}} \|m(s)\| < \infty$. The convolution operators $\mathcal{K}^+$ and $\mathcal{K}$ are defined in Subsection 3.2.1. The operators $G_A = \overline{-\partial + A}$ and $G_A^+ = \overline{-\partial_0 + A}$ are defined in Subsection 3.2.2. The operator $(\mathcal{D}x)(t) = T(t)x$, $t \geqslant 0$, $x \in X$, is defined in Subsection 3.2.3. Also, $s_0(A)$ denotes the abscissa of uniform boundedness of the resolvent and $\omega(T)$ denotes the growth bound.

Theorem 3.7 (Quantitative Datko's Theorem)**.** *For a strongly continuous semigroup $\{T(t)\}_{t\geqslant 0}$ on a Banach space X, the following assertions are equivalent:*

(*i*) $\omega(T) < 0$;

(*ii*) $\mathcal{K}^+ \in \mathcal{B}(L^p(\mathbb{R}_+; X))$;

(*iii*) $\mathcal{K} \in \mathcal{B}(L^p(\mathbb{R}; X))$;

(*iv*) $\mathcal{M}_m \in \mathcal{B}(L^p(\mathbb{R}; X))$ *and* $\sigma(A) \subset \{z : \mathrm{Re} z < 0\}$, $s_0(A) \leqslant 0$;

(*v*) $(G_A^+)^{-1} \in \mathcal{B}(L^p(\mathbb{R}_+; X))$;

(*vi*) $\sigma(G_A) \subset \{z : \mathrm{Re} z < 0\}$;

(*vii*) $\mathcal{D} \in \mathcal{B}(X; L^p(\mathbb{R}_+; X))$.

Moreover, if one of the equivalent assertions holds then $\omega(T) \leqslant -1/(pD)$ *and*

$$\|\mathcal{K}^+\|_{\mathcal{B}(L^p(\mathbb{R}_+;X))} = \|\mathcal{K}\|_{\mathcal{B}(L^p(\mathbb{R};X))} \tag{3.40}$$

$$= \|\mathcal{M}_m\|_{\mathcal{B}(L^p(\mathbb{R};X))} \tag{3.41}$$

$$= \|(G_A^+)^{-1}\|_{\mathcal{B}(L^p(\mathbb{R}_+;X))} = \sup_{\mathrm{Re} z\geqslant 0} \|R(G_A^+; z)\|_{\mathcal{B}(L^p(\mathbb{R}_+;X))} \tag{3.42}$$

$$= \|(G_A)^{-1}\|_{\mathcal{B}(L^p(\mathbb{R};X))} = \sup_{\mathrm{Re} z\geqslant 0} \|R(G_A; z)\|_{\mathcal{B}(L^p(\mathbb{R};X))}. \tag{3.43}$$

Furthermore, if $w > \max\{0, \lambda\}$ *and* (3.1) *holds then we have the estimate*

$$\|\mathcal{D}\|_{\mathcal{B}(X;L^p(\mathbb{R}_+;X))} \leqslant L\big(w\|\mathcal{K}^+\|_{\mathcal{B}(L^p(\mathbb{R}_+;X))} + 1\big)\big(p(w-\lambda)\big)^{-1/p}, \tag{3.44}$$

and, abbreviating $C = L\big(p\lambda\|\mathcal{D}\|^p_{\mathcal{B}(X;L^p(\mathbb{R}_+;X))} + 1\big)^{1/p}$, *we have the estimate*

$$\|\mathcal{K}^+\|_{\mathcal{B}(L^p(\mathbb{R}_+;X))} \leqslant 2\|\mathcal{D}\|_{\mathcal{B}(X;L^p(\mathbb{R}_+;X))} C^{2+p} \mathrm{e}^{(1+p)/C}. \tag{3.45}$$

Finally, if $p = 2$ *and* X *is a Hilbert space then the equal norms of all operators in* (3.40) – (3.43) *are estimated as follows:*

$$\begin{aligned}\|\mathcal{K}^+\|_{\mathcal{B}(L^2(\mathbb{R}_+;X))} &= \cdots = \|(G_A)^{-1}\|_{\mathcal{B}(L^2(\mathbb{R};X))}\\ &= \sup_{s\in\mathbb{R}} \|R(A, is)\|_{\mathcal{B}(X)}\\ &\leqslant \sup_{\mathrm{Re} z\geqslant 0} \|R(A, z)\|_{\mathcal{B}(X)}.\end{aligned} \tag{3.46}$$

Proof. Clearly, (i) implies $(ii), (iii), (vii)$ because (3.2) holds with some negative ω. As indicated in the proof of the inequality "$\leqslant$" in (3.9), assertion (iii) implies (ii). That (ii) implies (i) is the Datko-van Neerven characterization of stability via convolutions, see [ABHN, Theorem 5.1.2 (i) $\Leftrightarrow$ (iii)] or [V, Theorem 3.3.1]. However, the fact that (ii) implies (i) also follows from Theorem 1.2 as described in Remark 3.4. Thus, the first three assertions are equivalent.

The equivalence of (i) and (iv) is proven in [LR, Corollary 3.8]. Alternatively, we indicated the proof of $(iii) \Leftrightarrow (iv)$ in Subsection 3.2.2, see (3.13) and subsequent comments. The equivalence of (i) and (v) is proved in [CL, Proposition 2.43]; it also follows from the fact that $-\mathcal{K}^+ = (G_A^+)^{-1}$, see (3.17). The equivalence of (i) and (vi) is contained in [CL, Corollary 2.40]; also, it follows from the fact that $(G_{A-\omega})^{-1} \in \mathcal{B}(L^p(\mathbb{R}; X))$ if and only if the rescaled semigroup $T_\omega(t)$ is hyperbolic for some $\omega \geqslant 0$, and, in turn, by (iv), using [LS], if and only if $m_\omega(s) = R(A - \omega, is)$ is an $L^p(\mathbb{R}; X)$-Fourier multiplier. We also recall that $(G_A)^{-1} = -\mathcal{K}$. The equivalence of (vi) and

(vii) is discussed at the end of Subsection 3.2.2. The equivalence of (i) and (vii) is the subject of the classical Datko Theorem [ABHN, Theorem 5.1.2 (i) $\Leftrightarrow$ (ii)]; also, inequalities (3.44), (3.45) show that $(vii) \Leftrightarrow (ii)$.

The estimate $\omega(T) \leqslant -1/(pD)$ is the quantitative version of Datko's Theorem, see [V, Theorem 3.1.8]. Equality (3.40) is proved in (3.9); also, see [CL, Theorem 2.4.9 (iii)] for $\|(G_A)^{-1}\|_{\mathcal{B}(L^p(\mathbb{R};X))} = \|(G_A^+)^{-1}\|_{\mathcal{B}(L^p(\mathbb{R}_+;X))}$. Equality (3.41) is (3.13), the first equalities in (3.42), (3.43) are (3.17), (3.18), respectively, while the second equalities are given by (3.19) and discussed in subsequent comments. Estimates (3.44), (3.45) are obtained in (3.25), (3.39). Finally, (3.46) holds because $\mathcal{F}$ is an $L^2(\mathbb{R};X)$-isometry in the Hilbert space case, and thus $\|\mathcal{M}_m\|_{\mathcal{B}(L^2(\mathbb{R};X))} = \sup_{s\in\mathbb{R}} \|R(A,is)\|_{\mathcal{B}(X)}$. ∎

Corollary 3.8. *Let $\{T(t,\beta)\}_{t\geqslant 0}$ be a one parametric family of strongly continuous semigroups $\{T(\cdot,\beta)\}$ on a Banach space X, parameterized by an auxiliary complex parameter $\beta \in \Omega \subseteq \mathbb{C}$ such that $T(t,\cdot) \in L^\infty(\Omega, \mathcal{B}(X))$ for each t and Ω is an open domain in $\mathbb{C}$. Assume that there exists uniform in $\beta \in \Omega$ constants L and λ such that $\|T(t,\beta)\|_{\mathcal{B}(X)} \leqslant L\mathrm{e}^{\lambda t}$ for all $t \geqslant 0$ and $\beta \in \Omega$. Furthermore, assume that $\omega < \lambda$ and that*

$$\sup_{\beta\in\Omega} \|\mathcal{K}^+(\beta)\|_{\mathcal{B}(L^p_\omega(\mathbb{R}_+;X))} < \infty, \tag{3.47}$$

where $\mathcal{K}^+(\beta)$ is the convolution on $L^p(\mathbb{R}_+;X)$ with the semigroup $\{T(\cdot,\beta)\}$. Then, with a uniform in β constant M, one has $\{T(t,\beta)\}_{\mathcal{B}(X)} \leqslant M\mathrm{e}^{\omega t}$ for all $t \geqslant 0$ and $\beta \in \Omega$. Moreover, there is a uniform in $\beta \in \Omega$ constant D such that $\omega\big(T(\cdot,\beta)\big) \leqslant \omega - 1/(pD)$. Finally, if X is a Hilbert space then assumption (3.47) *can be replaced* *by the assumption*

$$\sup_{\beta\in\Omega}\sup_{\mathrm{Re}z\geqslant\omega}\|R\big(A(\beta),z\big)\|<\infty, \tag{3.48}$$

where $A(\beta)$ is the infinitesimal generator of the semigroup $\{T(t,\beta)\}_{t\geqslant 0}$.

Proof. Computing $M(\beta)$ for $T(\cdot,\beta)$ by (3.6) and using the assumptions, we conclude that $M=\sup_{\beta\in\Omega}M(\beta)<\infty$. The quantitative Datko's Theorem 3.7, see [V, Theorem 3.1.8], shows that

$$\omega(T_\omega(\cdot,\beta))=\omega(T(\cdot,\beta))-\omega\leqslant -1/(pD(\beta)) \tag{3.49}$$

for the rescaled semigroup $T_\omega(t,\beta)=e^{-\omega t}T(t,\beta)$ and a constant $D(\beta)$. But then assumption (3.47) and inequality (3.44) imply $D(\beta)\leqslant D$ for $w>\max\{\lambda,0\}$ and

$$D=\sup_{\beta\in\Omega}L(w\|K^+(\beta)\|_{\mathcal{B}(L^p(\mathbb{R}_+;X))}+1)\big(p(w-\lambda)\big)^{-1/p}, \tag{3.50}$$

thus yielding the desired uniform estimate for $\omega(T(\cdot,\beta))$.

Finally, the Hilbert space part of the corollary follows from (3.46) applied to the rescaled semigroup $\{T_\omega(\cdot,\beta)\}$. ∎

3.3 The proof of Theorem 3.2

Proof. Let us fix an $x\in X$, and denote $u(t)=T(t)x$, for $t\geqslant 0$. If ϕ is a differentiable scalar valued function on $\mathbb{R}_+$ with bounded derivative then

$$\begin{aligned}\mathcal{K}^+(\phi' u)(t)&=\int_0^t T(t-s)\phi'(s)T(s)x\,ds\\&=\int_0^t T(t)x\phi'(s)\,ds=u(t)\big(\phi(t)-\phi(0)\big).\end{aligned} \tag{3.51}$$

For each $t>0$, let us choose a function ϕ on $\mathbb{R}_+$ which is monotonically increasing on the interval $(t/2,t)$ and such that $\phi(s)=0$ for $0\leqslant s\leqslant t/2$ and $\phi(s)=1$ for $s\geqslant t$

so that $\mathrm{supp}(\phi') \subset (t/2, t)$. Then (3.51) yields

$$\mathrm{e}^{-\omega t}\|T(t)x\| = \mathrm{e}^{-\omega t}\|u(t)\| = \mathrm{e}^{-\omega t}\|\phi(t)u(t)\| = \mathrm{e}^{-\omega t}\|\mathcal{K}^+(\phi' u)(t)\|, \tag{3.52}$$

and by the definition of the operator $\mathcal{K}^+$,

$$\mathrm{e}^{-\omega t}\|T(t)x\| \leqslant \int_{\mathrm{supp}(\phi')} \mathrm{e}^{-\omega t}\|T(t-s)\phi'(s)u(s)\|\, ds. \tag{3.53}$$

Next, let us introduce another function, ψ, as $\psi(s) = \phi(t-s)$ for $0 \leqslant s \leqslant t$ and $\psi(s) = 0$ for $s > t$. Of course, $\phi(s) = \psi(t-s)$ for $0 \leqslant s \leqslant t$ and $\phi(s) = 1$ for $s \geqslant t$, so that $\mathrm{supp}(\psi') \subset (0, t/2)$. Replacing ϕ by ψ in (3.53) results in

$$\begin{aligned} &\int_{\mathrm{supp}(\phi')} \mathrm{e}^{-\omega t}\|T(t-s)\psi'(t-s)u(s)\|\, ds \\ &\qquad \leqslant \int_{\mathrm{supp}(\phi')} \Big(\mathrm{e}^{-\omega(t-s)}\|T(t-s)\psi'(t-s)\|_{\mathcal{B}(X)}\Big)\big(\mathrm{e}^{-\omega s}\|u(s)\|\big)\, ds. \end{aligned} \tag{3.54}$$

Taking $1 \leqslant p,\ q < \infty$ such that $1/p + 1/q = 1$ and applying Holder's inequality together with (3.1), we infer that (3.54) is smaller than or equal to

$$\begin{aligned} &\Big(\int_{\mathrm{supp}(\phi')} \|\mathrm{e}^{-\omega(t-s)}T(t-s)\psi'(t-s)\|^q_{\mathcal{B}(X)} ds\Big)^{1/q}\Big(\int_{\mathrm{supp}(\phi')} \|\mathrm{e}^{-\omega s}u(s)\|^p ds\Big)^{1/p} \\ &\quad \leqslant \Big(\int_{\mathrm{supp}(\psi')} \mathrm{e}^{-\omega q s}\|T(s)\psi'(s)\|^q_{\mathcal{B}(X)} ds\Big)^{1/q}\Big(\int_{\mathrm{supp}(\phi')} \mathrm{e}^{-\omega p s}\|u(s)\|^p ds\Big)^{1/p} \\ &\quad \leqslant \Big(\int_0^{t/2} \mathrm{e}^{-\omega q s}|L\mathrm{e}^{\lambda s}\psi'(s)|^q\, ds\Big)^{1/q}\Big(\int_{t/2}^{t} \mathrm{e}^{-\omega p s}\|(1-\psi(s))u(s)\|^p ds\Big)^{1/p}, \end{aligned} \tag{3.55}$$

where we used the fact that $\psi(s) = 0$ for all $s \in \mathrm{supp}(\phi')$. Moreover, $\psi(0) = 1$ and (3.51) imply that for any $s > 0$ one has

$$-\mathcal{K}^+\left((\psi)'u\right)(s) = \left(1 - \psi(s)\right)u(s). \tag{3.56}$$

Hence, we can rewrite (3.55) as a product of two norms and, using (3.1) again, estimate the right-hand side of (3.53) by the expression

$$\begin{aligned} &L\|\mathrm{e}^{(\lambda-\omega)(\cdot)}\psi'(\cdot)\|_{L^q(0,t/2)}\,\|\mathcal{K}^+\left(\psi' u\right)\|_{L^p_\omega((t/2,t);X)} \\ &\quad \leqslant L\ \|\mathcal{K}^+\|_{\mathcal{B}(L^p_\omega(\mathbb{R}_+;X))}\,\|\mathrm{e}^{(\lambda-\omega)(\cdot)}\psi'(\cdot)\|_{L^q(0,t)}\,\|\psi' u\|_{L^p_\omega((0,t);X)} \\ &\quad \leqslant L\|\mathcal{K}^+\|_{\mathcal{B}(L^p_\omega(\mathbb{R}_+;X))}\,\|\mathrm{e}^{(\lambda-\omega)(\cdot)}\psi'(\cdot)\|_{L^q(0,t)}\,\|\mathrm{e}^{-\omega(\cdot)}\psi' T(\cdot)x\|_{L^p((0,t);X)} \\ &\quad \leqslant L^2\ \|\mathcal{K}^+\|_{\mathcal{B}(L^p_\omega(\mathbb{R}_+;X))}\,\|\mathrm{e}^{(\lambda-\omega)(\cdot)}\psi'\|_{L^q(0,t)}\,\|\mathrm{e}^{(\lambda-\omega)(\cdot)}\psi'\|_{L^p((0,t);X)}\,\|x\|. \end{aligned} \tag{3.57}$$

Therefore, by (3.53), (3.57) we end up with the following conclusion:

$$\begin{aligned} &\mathrm{e}^{-\omega t}\|T(t)\|_{\mathcal{B}(X)} \\ &\quad \leqslant L^2\ \|\mathcal{K}^+\|_{\mathcal{B}(L^p_\omega(\mathbb{R}_+;X))}\,\|\mathrm{e}^{(\lambda-\omega)(\cdot)}\psi'(\cdot)\|_{L^q(0,t)}\,\|\mathrm{e}^{(\lambda-\omega)(\cdot)}\psi'(\cdot)\|_{L^p(0,t)}. \end{aligned} \tag{3.58}$$

Letting $g(s) = \mathrm{e}^{(\lambda-\omega)s}\psi'(s)$, and combining (3.58) with $\mathrm{e}^{-\omega t}\|T(t)\| \leqslant L\mathrm{e}^{(\lambda-\omega)t}$, $t \geqslant 0$, we obtain the following formula for the constant M in (3.2):

$$M = \sup_{t\geqslant 0}\min\left\{L\mathrm{e}^{(\lambda-\omega)t},\ L^2\|\mathcal{K}^+\|_{\mathcal{B}(L^p_\omega(\mathbb{R}_+;X))}\|g\|_{L^p(0,t)}\|g\|_{L^q(0,t)}\right\}. \tag{3.59}$$

The right-hand side of (3.59) depends on the choice of the function ψ. Let us choose it to allow an easy calculation of the norms in $L^p(0,t)$ and $L^q(0,t)$. Suppose we choose ψ such that

$$\psi(s) = (\mathrm{e}^{2(\omega-\lambda)s} - \mathrm{e}^{(\omega-\lambda)t})/(1-\mathrm{e}^{(\omega-\lambda)t}) \text{ for } 0 \leqslant s \leqslant t/2 \text{ and } \psi(s) = 0 \text{ for } s \geqslant t/2.$$

Then

$$g = 2(\omega-\lambda)\mathrm{e}^{(\omega-\lambda)s}(1-\mathrm{e}^{(\omega-\lambda)t})^{-1} \text{ for } 0 \leqslant s \leqslant t/2 \text{ and } g(s) = 0 \text{ for } s \geqslant t/2.$$

Computing

$$\|g(\cdot)\|_{L^p(0,t)} = 2(\lambda-\omega)^{1-1/p}p^{-1/p}(1-\mathrm{e}^{(\omega-\lambda)pt/2})^{1/p}(1-\mathrm{e}^{(\omega-\lambda)t})^{-1} \tag{3.60}$$

and $\|g\|_{L^q(0,t)}$, and introducing notations $\tau = \mathrm{e}^{(\omega-\lambda)t/2} \in (0,1)$ and

$$R = 4(\lambda - \omega)p^{-1/p}q^{-1/q}L\|\mathcal{K}^+\|_{\mathcal{B}(L^p_\omega(\mathbb{R}_+;X))},$$

and the function $f(\tau) = R(1-\tau^p)^{1/p}(1-\tau^q)^{1/q}(1-\tau^2)^{-2}$, $\tau \in (0,1)$, we infer that equality (3.59) becomes

$$\begin{aligned} M = \sup_{t\geqslant 0} \min\Big\{&L\mathrm{e}^{(\lambda-\omega)t}, \\ &R\big(1-\mathrm{e}^{(\omega-\lambda)pt/2}\big)^{\frac{1}{p}}\big(1-\mathrm{e}^{(\omega-\lambda)qt/2}\big)^{\frac{1}{q}}\big(1-\mathrm{e}^{(\omega-\lambda)t}\big)^{-2}\Big\} \\ = L \sup_{\tau\in(0,1)} &\min\{\tau^{-2}, f(\tau)\}\,. \end{aligned} \tag{3.61}$$

To finish the proof of Theorem 3.2, we will now show the inequality

$$\sup_{\tau\in(0,1)} \min\{\tau^{-2}, f(\tau)\} \leqslant R+1, \tag{3.62}$$

which yields (3.6).

First, we note that

$$(1-\tau^p)^{1/p}(1-\tau^q)^{1/q} \leqslant 1-\tau^2, \qquad \tau\in(0,1). \tag{3.63}$$

Indeed, letting $a_1 = (1-\tau^p)^{1/p}$, $b_1 = (1-\tau^q)^{1/q}$, $a_2 = b_2 = \tau$ and applying Hölder inequality to $\sum_i a_i b_i$ yields (3.63).

Next, we claim that there is a unique $\tau_0 \in (0,1)$ such that $\tau_0^{-2} = f(\tau_0)$.

Assuming the claim, we prove (3.62) as follows. If $\tau_0^{-2} = f(\tau_0)$ then

$$\tau_0^{-2} - 1 = (1-\tau_0^2)\tau_0^{-2} = R(1-\tau_0^p)^{1/p}(1-\tau_0^q)^{1/q}(1-\tau_0^2)^{-1} \leqslant R \tag{3.64}$$

by (3.63). Thus, $\tau_0^{-2} = f(\tau_0) \leqslant R+1$. By inspection, $f(0) = R$ and $\lim_{\tau\to 1} = +\infty$. Using the claim above, it follows that $\min\{\tau^{-2}, f(\tau)\}$ is equal to $f(\tau)$ for $\tau \leqslant \tau_0$, and to τ^{-2} for $\tau \geqslant \tau_0$. Thus, to establish (3.62), it remains to show that $f(\tau) \leqslant R+1$

provided $\tau \leqslant \tau_0$, that is, provided $f(\tau) \leqslant \tau^{-2}$. We will consider two cases: First, if $\tau^2 \geqslant (R+1)^{-1}$ and $f(\tau) \leqslant \tau^{-2}$ then $f(\tau) \leqslant \tau^{-2} \leqslant R+1$, as required. Second, if $\tau^2 < (R+1)^{-1}$ and $f(\tau) \leqslant \tau^{-2}$ then $1 - \tau^2 > 1 - (R+1)^{-1} = R(R+1)^{-1}$. Using this and (3.63), we infer:

$$f(\tau) = R(1-\tau^2)^{-1} \cdot (1-\tau^p)^{1/p}(1-\tau^q)^{1/q}(1-\tau^2)^{-1} \leqslant (R+1), \tag{3.65}$$

which completes the proof of the required inequality $f(\tau) \leqslant R+1$ for all $\tau \leqslant \tau_0$.

It remains to show the claim. Letting $h(\tau) = \tau^2 f(\tau)$, we note that $\tau_0^{-2} = f(\tau_0)$ if and only if $h(\tau_0) = 1$. By inspection, $h(0) = 0$ and $\lim_{\tau \to 1} h(\tau) = +\infty$, and thus it suffices to show that $h'(\tau) > 0$ for $\tau \in (0,1)$. Logarithmic differentiation yields $h'(\tau) = h_0(\tau)((1 - 2\tau^p + \tau^2)(1-\tau^q) + (1 - 2\tau^q + \tau^2)(1-\tau^p))$, where $h_0(\tau) = h(\tau)/(\tau(1-\tau^2)(1-\tau^p)(1-\tau^q))$. Since $1+\tau^2 > 2\tau > 2\max\{\tau^p, \tau^q\}$ for $\tau \in (0,1)$ and $p, q \geqslant 1$, one has $h'(\tau) > 0$ as needed. ∎

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