EXAMINING SECONDARY STUDENTS' ALGEBRAIC REASONING:

FLEXIBILITY AND STRATEGY USE

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by

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The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled.

EXAMINING SECONDARY STUDENTS' ALGEBRAIC REASONING:

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A candidate for the degree of Doctor of Philosophy

And hereby certify that in their opinion it is worthy of acceptance.

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ABSTRACT

Recent curricular recommendations (NCTM, 2000; RAND, 2003) call for the development of student flexibility in relation to algebraic reasoning. In response to these recommendations, this study focused on the algebraic strategies employed by the participants and their flexibility in understanding various generalization strategies when generalizing numeric situations. Algebraic flexibility consisted of two components: (a) Within-task flexibility (recognizing appropriate generalization strategies that could be used for a particular task) and cross-task flexibility (recognizing when a generalization strategy could be applied to various tasks).

Eleven tenth-grade students from two rural schools participated in active interviews (Holstein & Gubrium, 1995) centered on developing generalizations for

contextualized algebraic tasks. Following the development of a generalization for a particular task, participants were provided alternative student strategies to examine.

The results demonstrated that secondary students employ the same generalization strategies as elementary and middle level students: explicit, whole-object, recursive, and chunking. Participants used recursive (92.3%) and chunking (90%) strategies with the greatest success, while the explicit strategy was the least effective (correctly used 60% of the time).

Participants classified as exhibiting a high level of flexibility did not necessarily demonstrate that ability in initially generalizing tasks. The participants fell along a range for both within-task and cross-task flexibility. Participants classified as exhibiting a high level of flexibility were able to determine the applicability of a strategy and develop contextually-justified rules. Students with low flexibility were unable to determine the applicability of a strategy or justify their rules.

CHAPTER I

THE PROBLEM AND ITS BACKGROUND

In an age where mathematical sophistication and utility are defined in terms of ability to solve complex problems, the current goals of mathematics education have changed in an attempt to meet societal demands. Active engagement and flexible problem solving represent factors critical to future mathematical and professional success (National Council of Teachers of Mathematics, 2000, pp. 20-21). With the development of efficient solutions for non-routine problems representing a key workforce skill, a student's ability to work flexibly in real-world problem situations has become paramount. As the skills required in today's world change, schools must adjust to ensure that tomorrow's societal members are adequately prepared to contribute. In terms of mathematics, the search for improvement has led to greater expectations vis-à-vis a student's ability to "explore, conjecture, and reason logically, as well as the ability to use a variety of mathematical methods effectively to solve non-routine problems" (NCTM, 1989, p. 5).

While the aforementioned abilities represent expectations of students working in any mathematical content strand, this study examined them in light of algebraic reasoning. Specifically, this study documented the generalization strategies employed by secondary algebra students and measured their flexibility in using these strategies.

In this chapter, I offer a rationale for looking more closely at the current state of secondary student flexibility in terms of algebraic generalization. The chapter begins with a look at current classroom recommendations, followed by a discussion outlining the merits of studying student flexibility, leading to the statement of the research questions. Sections follow containing the theory guiding the study and pertinent definitions of terms. The chapter concludes with an argument for the significance of the study.

NCTM's Vision of the Classroom

The National Council of Teachers of Mathematics (NCTM) (1989, 1991, 1995, 2000) set forth a vision of the mathematics classroom where interactions within the social milieu are paramount; mathematical situations are explored, conjectures are made and refined, ideas are shared, and mathematical truth is reached through negotiation, based on the validity of the arguments. As Simon and Blume (1996) noted, the mathematics classroom should involve students who "actively participate, taking on a role that is analogous to the role of mathematician, creating mathematics, evaluating mathematics that has been created by members of the classroom mathematics community, and negotiating shared approaches to and standards for these activities" (p. 3). Such activities remain at the forefront of Standards-based reform efforts in mathematics education.

In order for students to gainfully participate in activities that support this vision, the manner in which they "do mathematics" must be dramatically different from what students experience in a typical American classroom, where "the teacher and textbook serve as the source of mathematics and the evaluators of mathematical validity" (Simon & Blume, 1996, p. 3). In the vision laid out by the *Principles and Standards for School Mathematics* (NCTM, 2000), the responsibility for mathematical authority lies more in the hands of the student learners, who serve as “flexible and resourceful problem solvers” that “value mathematics and engage actively in learning it” (p. 3). A key to achieving this vision is the cultivation of flexible problem-solvers, whose charge is to create and substantiate the mathematics of the classroom.

Since publication of the Standards documents, the idea of mathematical “flexibility” has become more and more a focus in discussions surrounding K-12 mathematical proficiency (RAND, 2003). As mathematics teachers and researchers work to implement the core tenets of NCTM’s vision, classroom events and the accompanying research have begun to exhibit an eye towards identifying and promoting flexible student reasoning.

The Importance of Algebra in the Curriculum

The meaningful learning of algebra is an important goal for every student. As noted by Romberg and Spence (1995), “All students, as members of tomorrow’s work force, need to see algebra as important and useful, and to regard it as making sense” (p. 177). This is due to the role of algebra in the workplace, where algebra “pervades computing and business modeling, from everyday spreadsheets to sophisticated scheduling systems and financial planning strategies” (Hoachlander, 1997, p. 135). Policy position documents have taken a similar stance with “algebra for all” representing the end goal. NCTM (2000) championed this sentiment in stating that, “Algebraic competence is important in adult life, both on the job and as preparation for postsecondary education. All students should learn algebra” (p. 37).

Although algebra is important due to its real world applications, the learning of algebra is critical to the study and usage of all mathematics. The ability of students to develop strategies to construct generalizations for algebraic tasks represents a key element of mathematical flexibility given the importance placed on algebra in the mathematics curriculum. The vaunted status of algebra embodies its utility and necessity in mathematical and societal contexts. Algebra serves as the fundamental backbone of most mathematical endeavors by providing the language and tools necessary for representing and analyzing quantitative relationships, modeling situations, solving problems, and stating generalizations (RAND, 2003, p. 44). Romberg and Spence (1995) concurred with this view, noting that, “algebra is a tool for making sense of the world” (p. 186). Without a firm foundation in algebraic thinking, students do not have the means necessary to succeed in other areas of mathematics or many aspects of the real world. Algebraic notation, thinking, and concepts remain critical assets in a number of workplace contexts and in the interpretation of information by Americans on a daily basis (NCTM, 2000; RAND, 2003).

In addition, algebra serves as a gatekeeper in K-12 education (RAND, 2003), with high-stakes decisions often resting on the relative success or failure of a student’s performance on algebraic tasks. For many secondary schools, states require the demonstration of algebraic competency prior to graduation (Education Commission of the States, 2002; Kelderman, 2004). At the postsecondary level, algebra has been used in a similar gatekeeper role, with qualifying and placement exams for certain programs containing a strong algebraic component (U.S. Department of Education, 1997). The importance placed upon algebraic proficiency at most levels of formal education indicates that the teaching and learning of algebra will remain of primary importance, as is the need for teachers and researchers to understand the development of algebraic reasoning.

Flexibility in Algebraic Reasoning

While the teaching and learning of algebra has taken different forms over the years (NCTM, 1991; Romberg & Spence, 1995; Wheeler, 1989), recent recommendations call for greater student flexibility in terms of algebraic reasoning. In describing expectations for algebraic proficiency, the RAND (2003) panel noted the importance of students being able to work "flexibly and meaningfully with formulas or algebraic relations--to use them to represent situations, to manipulate them, and to solve the equations they represent" (pg. 44). While each of these algebraic goals is important, this study focuses primarily on students' abilities to represent situations algebraically.

Due to the value placed on a particular procedure or strategy, students' algebraic experiences often provide access to only a small portion of the strategies available to them for solving algebraic tasks (Kaput, 1995). In many cases, these methods involve establishing explicit rules that are to be manipulated to produce correct answers.

However, research has shown that several factors (including the students' prior knowledge, the mathematical characteristics of the particular task, and the social milieu) influence how students approach and eventually model algebraic situations (Lannin, Townsend, & Barker, under review-a), often resulting in the use of several different strategies (Healey & Hoyles, 1999; Lannin, 2001; Stacey, 1989; Swafford & Langrall, 2000). Depending on the input values given, the mathematical structure of the task, the social interactions, and their prior knowledge (including previous tasks and strategies), students may use any number of strategies to solve a particular task, or across tasks, depending on the perceived utility of the strategy. For linear situations, these strategies include recursive, chunking, whole-object, and explicit reasoning (Lannin, Barker, & Townsend, under review-a). Each of these strategies is described in detail later in this chapter. Given the complexity of real-world problems and the fact that relatively few situations can actually be modeled with an explicit relation (Kaput, 1995), flexibility in algebraic strategy selection becomes paramount.

Purpose of the Study

The purpose of this study is to examine secondary students' algebraic strategy use and flexibility. The primary medium for exploring strategy use and flexibility was contextualized algebraic tasks.

Specifically, this study seeks to:

1. define and document student algebraic flexibility. While research and policy documents agree that flexibility in solving algebraic problems is an important skill to cultivate and possess (NCTM, 2000; RAND, 2003), little agreement exists on what is meant by mathematical flexibility in algebra and few studies have documented this ability. This study builds upon the current literature by offering a definition of algebraic flexibility that is based on prior research and by providing a framework for the documentation of algebraic flexibility.
2. document algebraic strategies employed by secondary students who have experienced an algebra curriculum. Myriad studies have examined student learning, especially struggles and misunderstandings with respect to K-12 algebra (Booth, 1984; Buswell & Judd, 1925; Radatz, 1979; Stefanich & Rokusek, 1992), but few studies have looked at student generalization strategies and their

classification. While a few studies have documented student strategy use at the elementary and middle levels (Healy and Hoyles (1999); Lannin, 2001; Stacey, 1989; Swafford & Langrall, 2000), none of the research has focused on the algebraic strategies employed by secondary school students. This study seeks to add to the literature by providing this perspective.

3. provide a useful framework for identifying and developing flexible strategy use in the classroom. Given that students enter algebra classrooms with considerable knowledge of mathematics (NCTM, 2000) and a particular understanding of various algebraic concepts, it ultimately remains the teacher's charge to identify and support the expansion of student conceptions and approaches to different algebraic situations. Without a clear method for strategy identification and an understanding of how to encourage diversification of strategy use, it will remain difficult for algebra teachers to ensure that their students develop algebraic flexibility. While frameworks have been developed and continue to be modified for characterizing elementary and middle school algebraic thinking, no such framework exists for secondary algebra students.

Research Questions

The following research questions for this study are based on the field of mathematics education's needs for a deeper understanding of the algebraic strategies employed by secondary students and their flexibility of use.

1. *What strategies do secondary students use when generalizing numeric situations and how do they use these strategies?*

2. *To what extent do students exhibit within-task strategic flexibility when generalizing algebraic tasks?* In other words, to what extent are secondary students who have completed a formal algebra curriculum able to produce mathematically correct strategies given various constraints within a single problem situation? To what extent do these students understand how various strategies apply to the same problem situation? How general do the students view the rules that they produce? What justifications do students offer for the rules they produce?

3. *To what extent do students exhibit cross-task strategic flexibility when generalizing algebraic situations?* Specifically, to what extent are secondary students who have completed a formal algebra curriculum able to apply, modify, and/or develop algebraic generalization strategies for different mathematical situations?

In order to document and measure the attributes noted in the research questions, a conceptual framework was developed to guide the collection and analysis of the data.

Conceptual Framework

Guiding my understanding of students' perspectives on algebraic reasoning, and therefore the design of this study, is a multi-tiered framework built upon an active view of student learning. The first level of the framework is illustrated in Figure 1.

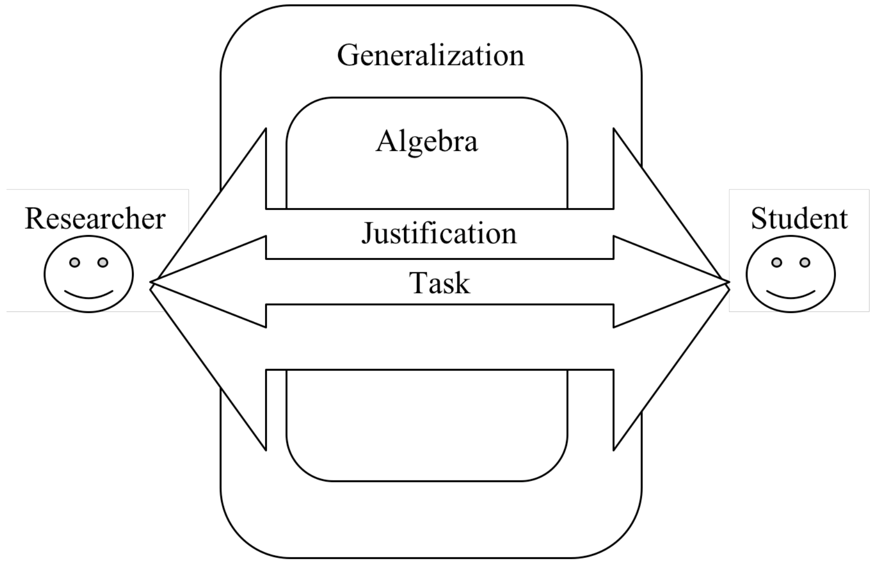


Figure 1. Overall Conceptual Framework

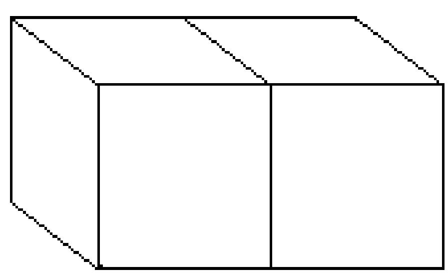
Generalization represents the outer layer of this model due to its centrality to all mathematical processes, including algebra (Mason, 1996). As Gattegno (1990) noted, "Something is mathematical, only if it is shot through with infinity." In other words, generality is not only an important part of mathematics, but it is, or should be, the goal of every mathematical endeavor. In terms of this study, algebraic generalization represents the focus of the research questions. The development of generalizations and the strategies employed by the students and the flexibility they demonstrate in doing so are at the heart of this study.

Within generalization lies the content strand, algebra, on which this study is based. Algebra, as defined by Sfard (1995), is "any kind of mathematical endeavor concerned with generalized computational processes" (p. 18). This study is primarily concerned with developing a deeper understanding of these processes. A more detailed rationale for choosing algebra as the content strand for this study is provided in Chapter II.

Instructional tasks provide the means for eliciting algebraic generalizations. As illustrated in Figure 1, the task plays an important role in that it provides a link to the student's thoughts on algebraic generalization. For example, in the Cube sticker problem (Figure 2), generalization is encouraged through the input values asked. From the particular values, a student could develop a general recursive rule (computing values term to term) that she believes will always work due to a pattern she noticed in the context or in the values she found. A larger value, like 49, provides the impetus for students to consider more efficient generalization strategies. The task serves as the medium upon which the researcher and student can build illuminating dialogue. A sample task is provided in Figure 2 below.

Cube Sticker Problem

A company makes colored rods by joining cubes in a row and using a sticker machine to place stickers on the rods. The machine places exactly one sticker on each exposed face of each cube. Every exposed face of each cube has to have a sticker, so this length two rod would need 10 stickers.



How many stickers would you need for rods of length 7? Length 10? Length 20? Length 49? Explain how you determined these values.

Explain how you could find the number of stickers needed for a rod of any length. Write a rule that you could use to determine this.

Figure 2. Cube Sticker Problem

Although a student's work with a task can afford the researcher or teacher a glimpse at the student's algebraic generalization strategies, it remains challenging to ascertain how students develop their generalizations (i.e., what role did the context and the values in the problem play in determining their rule), and how they see the generalizability of the rule in the context of the problem (for which values of the implied domain would their rule work). Much can be learned about the student's thinking regarding these issues through student/researcher interactions that center on student justification.

Justification provides a window into student thinking (Lins, 2001) that the researcher/teacher otherwise would not have. For example, in generalizing a task such as the Cube Sticker problem (see Figure 2), a student might generate this rule: the number of stickers = 4 • (the number of sides) + 2. While this equation provides the researcher with important information (i.e., the student seems to be thinking explicitly and the rule correctly models the problem situation), much more could be known through engaging in dialogue that focuses on justification. Questions aimed at ascertaining the students' views of the generality of their rules can provide insight into the general nature of the student's reasoning. Through discourse focused on the student's reasoning and understanding of the situation, a deeper understanding of the student's strategies, as well as their ability to apply these strategies, can be gained. Both are critical in terms of answering the research questions.

Single Task View

Expanding further on the previous framework, student decisions regarding generalization strategies when solving algebraic tasks are represented at the second level of the conceptual framework. The "task" noted in Figure 3 represents a task similar to the task in Figure 1 that was discussed in the previous section. Figure 3 depicts the strategies generated, and their accompanying justifications, as a result of student work with a particular task.

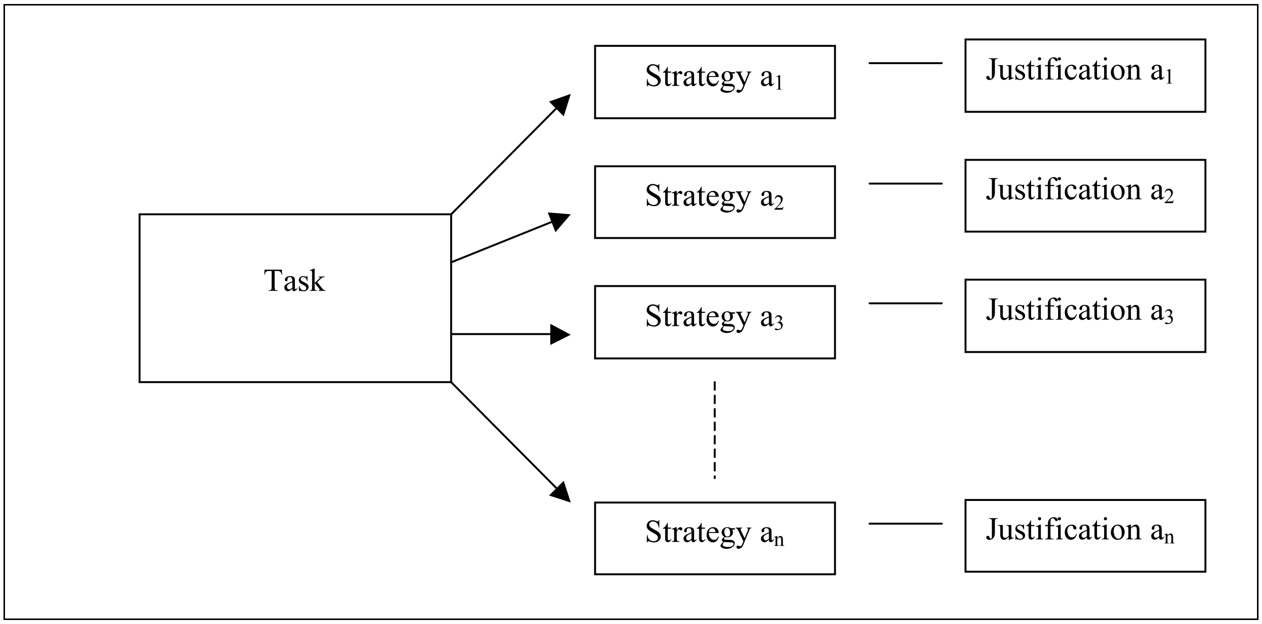


Figure 3. Within-Task Flexibility Framework

As indicated in the above model, students can select from among a number of strategies, viable or otherwise, when generalizing algebraic tasks. Lannin, Barker, and Townsend (under review-a) noted how student strategy selection is based upon several determining and contributing factors, including social, task (input values, mathematical structure), and cognitive (mental image, prior experiences including strategies) influences. A flexible problem solver recognizes the viability of various strategies as well as the strengths and limitations of these strategies. As noted in the research questions (pp. 7-8), both the documentation of the student generalization strategies and the assessment of their respective *within-task flexibility* were primary goals of this study. To that end, this framework played a crucial role.

Cross-Task View

While the single task perspective allowed for examination of the strategies employed for a particular task, the assessment of *cross-task flexibility* required a different look at strategy use, as illustrated in Figure 4.

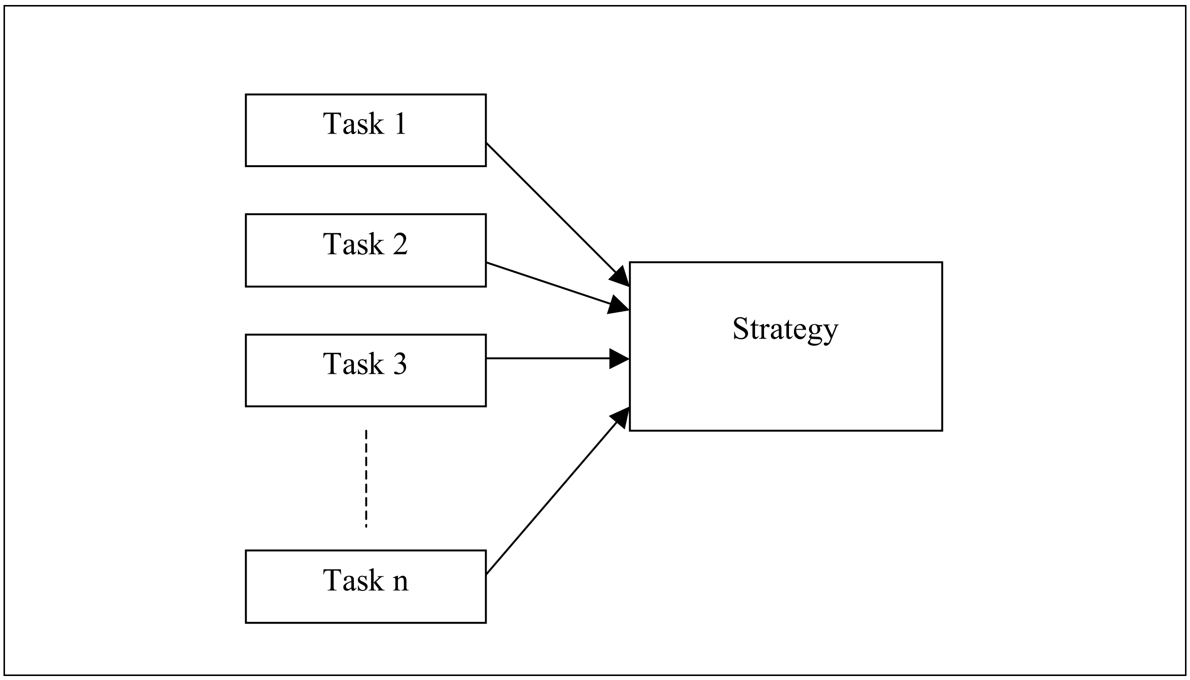


Figure 4. Cross-task Flexibility Framework

Cross-task flexibility is the ability of a student to deduce the applicability of a particular strategy to various problem situations. As illustrated in Figure 4, a particular strategy, such as explicit reasoning, can be utilized in various situations. However, contexts exist where the strategy does not apply. A student's ability to discern the contexts for which a strategy does and does not work is their cross-task flexibility.

For example, consider the student who generalizes three algebraic tasks. In doing so, she utilizes several strategies for each task, documenting each strategy that she considers. She correctly employs explicit rules for two of the tasks, and correctly notes that an explicit rule is not possible for the third. In terms of the explicit strategy, this student would be considered relatively flexible. Depending on her understanding of the other strategies and her subsequent flexibility ratings for each, her overall level of strategic algebraic flexibility, as determined by these tasks, could vary. Further discussion of *within-task* and *cross-task flexibility* can be found in Chapter 2.

Student Algebraic Generalization Strategies

The final component of the conceptual framework utilized current knowledge of how elementary and middle level children reason in algebraic situations. This study was informed by prior research on generalization strategies that contributed to development of the framework. Healy and Hoyles (1999), Stacey (1989), and Swafford and Langrall (2000) described similar strategies employed by middles grades students when generalizing contextualized tasks. Lannin (2001) focused on a subset of these strategies, which he termed explicit and recursive, in his study of sixth grade students. Building upon the work of these researchers, Lannin, Townsend, and Barker (under review-a) demonstrated that fifth grade students utilized similar strategies that could be classified into four categories: recursive, chunking, whole-object, and explicit (see Figure 5 for an adapted framework of these strategies).

In the columns following each algebraic generalization strategy an example is provided for how students could develop this strategy. For each strategy, a student could develop a rule through a relation discovered within the context of the problem situation (contextual) or through a numerical pattern in the values derived from the task (numeric).

Strategy	Contextual	Numeric
Explicit	An explicit rule is constructed based on the information provided in the situation by connecting to a counting technique [e.g., Cube Sticker: There are four stickers for each cube, so I took four times the length of the rod, then I added two stickers for the ends of the rod.]	The student recognizes that the rate of change in the results is the same and multiplies by the rate factor to establish an explicit rule. Then an adjustment is made by adding or subtracting a constant to reach a particular value previously determined for the situation. [Cube Sticker: The number of stickers increases by 4 each time so I multiplied by 4. Then I added two, because a rod of length 1 requires 6 stickers and 4*1+2 = 6.]
Whole-Object (also referred to as Unitizing)	The student uses a portion as a unit to construct a larger unit using multiples of the unit. The student adjusts for over (or under) counting due to the overlap that occurs when units are connected. [Beam problem: A rod of length 10 has 42 stickers, so a rod of length 20 would have 42*2 – 2 because the stickers between the two length 10 sections would need to be removed.]	The student uses a portion as unit to construct a larger unit using multiples of the unit. The student fails to adjust for any over or undercounting, where applicable [Beam Problem: A rod of length 10 has 42 stickers, so a rod of length 20 would have 42*2 or 84 stickers (incorrect)].
Chunking	A recursive rule is established based on a relationship established in the context, adding a unit onto known values of the desired attribute. [Beam problem: For a rod of length 10 there are 42 stickers, so for a rod of length 15, I would take 42 + 5(4) because each cube adds four stickers.]	The student builds on a recursive pattern, devoid of relation to the context (perhaps referring a table of values), by building a unit onto known values of the desired attribute. [Beam problem: For a rod of length 10 there are 42 stickers, so for a rod of length 5, I would take 42 + 5(4) because the number of stickers increases by 4 each time.]
Recursive	The student describes a relationship that occurs in the situation between consecutive values of the independent variable. [Cube Sticker: Each additional cube adds 5 stickers, and one sticker must be removed when the new cube is added to the rod, making a total of 4 stickers added for each cube.]	The student notices a pattern in the results for consecutive values of the independent variable. [Cube Sticker: The number of stickers goes up 4 each time when the length of rod is increased by 1. The number of stickers goes 6, 10, 14, 18, etc.]

Figure 5. Generalization Strategy Framework

To determine how students developed their rules, as well as how they view their generalizations, an understanding of a student's justification for his generalization is paramount. Student explanations of the various components of their rules provide information as to how their rules were developed and how generally applicable they view their rules to be (Lannin, 2005). For example, in working the Cube Sticker problem (Figure 2), a student produces the rule $S = 4n + 2$, where S is the number of stickers required for a rod of length n. In describing how he developed his rule, the student notices that four rows with n stickers comprise the length of the rod ($4n$) and two more stickers are needed for the ends. The student's justification implies that his rule is based in the context of the situation and that his visual image of the problem situation led to the development of his explicit rule. The general nature of his statement suggests that he could apply this rule to any length of rod. Subsequent questions could provide more information regarding the development and perceived generality of the rule.

A student's view of the domain of a representation impacts the student's ability to apply the rule within the task, as well as to other tasks. Therefore, ascertaining student understanding of the generality of their representations is key to gauging algebraic flexibility.

In gauging strategy use, the Generalization Strategy Framework (Figure 5) plays a critical role by providing a means for classifying the strategies that students employ, as well as providing a key perspective in classifying the strategies as contextual or numeric. This framework informed both the study design and data analysis of this study.

Theoretical Perspective

Given that this study centers on individual student cognition (the strategies employed by students and their flexibility in employing these strategies), a Piagetian perspective on learning was utilized in the design of this study and guided the subsequent data collection and analytical decisions. In such a view, the acquisition of knowledge is seen as a process of perpetual self-construction, where perturbations lead to accommodations that either maintain or re-establish equilibrium (von Glasersfeld, 1995, p. 68). In his studies of human development, Piaget (1985) hypothesized about the organization of sets of skills for performing particular tasks and denoted these sets as "schema." These cognitive structures became evident to Piaget as students engaged in various tasks. Reflection on certain tasks by a child appeared to lead to a reorganization (accommodation) of their thinking or to an integration of the information into the child's current schema (assimilation). Piaget's work served as the theoretical foundation of this study; the students participating in this study harboured myriad schema that came into play while working to solve the algebraic tasks. These included general mathematical, problem-solving, algebra, and strategy schema, among others. How the students were able to utilize and modify these schemes played an integral role in determining the strategies that they used and, ultimately, their algebraic flexibility.

While this study focuses on the individual mental constructs of the student participants, social factors were present during the data collection. These students were active participants in an interview with the researcher, necessitating social interaction, and subsequently, a social view of learning. Particular to this study, as much as possible, the focus of the social interactions was on gaining insight to the student's individual cognitions. While the interactions inevitably affected the outcomes to some extent, the questions that comprised the bulk of the interactions focused on clarification, not suggestion or instruction. Therefore, the guiding theoretical framework focused on the Piagetian cognitive perspective.

Definition of Terms

As is often the case with fairly specialized areas of research, several terms used in this study require precise definition.

Mathematical Power, as defined by NCTM (1989) is the ability to "explore, conjecture, and reason logically, as well as to use a variety of mathematical methods effectively to solve nonroutine problems" (p. 5). In terms of this study, students who demonstrated mathematical flexibility were, by definition, able to use a variety of methods effectively, thus exhibiting *mathematical power.*

Within-task flexibility, denoted as *strategic flexibility* by NRC (2001), is the ability to develop and understand viable strategies given the context and constraints of a particular problem. The term *within-task* was chosen to highlight the fact that this type of flexibility occurs within the confines of a particular task. While *strategic flexibility* has been coined to describe this phenomenon, such nomenclature could be confusing in this study, given the focus on strategy use within and across tasks.

Cross-task flexibility is defined as the ability to recognize commonalities and differences within a class of problems and to choose among a set of strategies gleaned from prior experience with the problems. As students work through problem situations, they access and modify their processes, strategies, and techniques for solving such tasks.

As noted in the research questions, this study seeks to examine students' abilities to notice and utilize this prior knowledge.

Algebraic flexibility is the term used to represent the possession and the ability to utilize both within-task flexibility and cross-task flexibility synergistically in solving an algebraic problem situation.

A *recursive strategy* is a rule that generates terms in a sequence through the preceding term or terms (NCTM, 2000). Recursive strategies are usually employed when a student knows a particular term and needs to find the next term or a value fairly close to it (Lannin, Townsend, Barker, under review-a).

Use of a *whole-object strategy* includes multiples of a previously derived amount being used to construct the solution for a different input value.

An *explicit strategy* uses index-to-term reasoning that relates the independent variable to the dependent variable(s), allowing for the immediate calculation of any output value.

A *chunking strategy* is best described as a recursive strategy where multiple "chunks" of the recursive value are added on to a known value. Examples of each of these strategy types can be found in Figure 4.

Contextual justification is an explanation of a generalization that ties each part of the rule to the context of the problem situation. For example, a student providing a contextual justification might say, "there are four sides showing for each cube and one side on each end, so the total number of stickers needed to cover the sides would be $4n+2$ for n cubes."

Numeric justification involves a description of a numerical pattern gleaned from a list of numbers that were derived from the problem situation. For example, a student offering an empirical justification could note, “I noticed that there were 14 stickers for a rod of length three and 18 stickers for a rod of length four. After guessing and checking and a few modifications, I came up with $4n+2$ for my rule.”

Significance of Study

Researchers (Lannin, 2001; Stacey, 1989; Swafford & Langrall, 2000) have documented the generalization strategies of students at the upper elementary and lower middle school levels when working in algebraic situations. To date, studies designed to document and categorize the algebraic strategies employed by secondary students have not been completed. Likewise, the research that has been conducted on algebraic flexibility has focused primarily on representational flexibility or flexibility in algebraic manipulation. This study seeks to expand the literature base by: (a) documenting the types of strategies that secondary students use when generalizing algebraic situations, and (b) measuring their flexibility in completing these tasks.

Given that the mathematical preparation of today’s students requires that they become efficient solvers of complex, real-world problems, the curricula that these students experience will necessarily be focused toward working effectively and efficiently in various contexts and situations. Within the realm of algebra, the real-life situations facing these students will often require them to draw upon their prior experiences with similar problems to model the situation correctly. Implicit to this ability to solve these types of problems is a student’s ability to generalize the situation effectively and efficiently given the constraints of the task. Knowledge of flexibility can be an asset to teachers in nurturing their students to become better problem solvers. This study seeks to assist in this endeavor by capturing and providing descriptions of the various strategies that these secondary students of algebra employ and by documenting the flexibility these students exhibit in this venture.

Summary

Mathematical flexibility is critical to students as they prepare to enter the workforce. Current reform efforts support an increased emphasis on the study of algebra and algebraic flexibility. This study seeks to document secondary students' algebraic generalizations and the flexibility of these students in using these generalizations. Student work on algebraic generalization tasks and student justification provide a medium for determining strategy use and flexibility. Four generalization strategies gleaned from prior research frame the determination of student strategy use. Flexibility, as defined for this study, is made up of two components: within-task flexibility and cross-task flexibility. Both facets of flexibility impact overall student algebraic flexibility. A Piagetian perspective on student learning underpins the theoretical framework utilized for this research.

The remaining chapters further delineate the details of this study. Chapter II provides a review of the current literature on algebraic reasoning, generalization, problem solving, and mathematical flexibility. Chapter III includes a description of the participants, algebraic generalization tasks, data sources, data gathering procedures, and methods of analysis (including the Alternative Generalization Strategies). Chapter IV offers an interpretation of the data, focusing specifically on algebraic strategy use and flexibility. Chapter V provides a discussion of the results, limitations of the study, and implications for curriculum and instruction.

CHAPTER II

REVIEW OF RELATED LITERATURE

This chapter begins with a look at algebra and generalization, including a review of the current state of algebraic learning in the classroom and recommendations from research. This is followed by a discussion of the algebraic strategies that students employ when solving generalization tasks and the role of justification in interpreting student generalizations. The chapter concludes with a discussion of flexibility, including different types of algebraic flexibility and the derivation of flexibility as used in this study, and a look at problem solving literature pertinent to gauging flexibility.

Student View of and Difficulties with Algebra

Many students, through their school experiences, develop a narrow perspective of what it means to do mathematics, particularly algebra (Kieran, 1992; Sfard & Linchevski, 1994). The view of algebra as a static field, made up of isolated topics and particular skills--such as symbol manipulation and equation identification and application--to be mastered through memorization has been fostered through the organization and teaching of algebra in schools (Lampert, 1990; Romberg & Spence, 1995; Schoenfeld, 1992). Students experience algebra lessons where the teacher and the textbook serve as the mathematical authorities for the correctness of answers (Weller, 1991), subject matter is packaged into single “one-rule-per-section” chunks (Wenger, 1987), and student variation from the prescribed processes results in immediate correction from the teacher (Weller, 1991). Such algebraic experiences provide the impression that a straightforward, single rule will always suffice (Wenger, 1987). These experiences often lead to student misconceptions about the nature of algebra. In this section, I provide an overview of research on student difficulties in algebra, focusing on issues relevant to this study.

The transition from arithmetic to algebra has provided an obstacle to student understanding of algebraic conventions. Collis (1974) noted that beginning algebra students view algebraic expressions as statements that are somehow incomplete. This causes cognitive dissonance for the students, due to the fact that younger children cannot hold unevaluated operations in suspension. For example, younger children require that numbers connected by an operation be physically replaced by the result of that operation. This causes a problem in algebra as expressions such as $n - 1$ are not always able of being replaced. Booth (1984) also wrote about student misunderstandings related to the simplification of algebraic expressions. She found that students view algebraic symbols as unknowns (standing for specific numbers), rather than as varying quantities.

Similarly, Kuchemann (1981) reported that high school age students had difficulties representing word problems with equations. In his study, students struggled to correctly produce a cost equation when given two colors of pencils cost different amounts. Many of the students viewed the variables as representing labels for the different sets of pencils (i.e. “r” for red and “b” for blue), rather than as variable quantities. Similarly, Carpenter et al. (1981) reported that many students who had completed at least one year of algebra had trouble solving the following NAEP problem:

> Carol earned *D* dollars during the week. She spent *C* dollars for clothes and *F* dollars for food. Write an expression using *D*, *C* and *F* that shows the number of dollars she has left.

The translation of the words into an algebraic expression caused considerable difficulty for these students, demonstrating the misunderstandings that exist in the algebraic backgrounds of many students.

Other studies have demonstrated that college students' attempts to represent situations are influenced by an impoverished understanding of variables and their use in equations. Research by Clement (1982) with the student-professor problem (i.e., There are six students for each professor. Write an equation to represent this relationship.) suggested two sources of common error with such a task: syntactic and semantic translations of the problem. In a syntactic translation, the student assumes that the sequence of words maps directly into a corresponding sequence of symbols (in the case of the student-professor problem, $6s = p$.) In a semantic translation, students link the equation to the meaning of the problem. The equation generated by the student to represent the situation is not viewed as an indication of equivalence, but as an illustration of relative size (more students than professors, so $6s = p$.)

Later research on the student-professor problem showed similar results with respect to the reversal of terms, even when the context explicitly provided the order for the equation (Stacey & MacGregor, 1993). Philipp's (1992) work with the student-professor problem reiterated the findings of Kuchemann, demonstrating that students often misunderstood the meaning of the symbols in algebraic situations, leading to the use of symbols as labels rather than as varying quantities.

Prior research demonstrates that classroom experiences have often promoted the view of algebra as involving only symbol manipulation. Such a view has contributed to difficulties by students of all ages in learning algebra. Research has documented that these difficulties lie in representing algebraic situations symbolically and understanding what these symbols actually mean. This study seeks to take a different view in examining the learning of algebra. While I will be investigating similar aspects in the learning of algebra (the understanding and representation of an algebraic situation), my study focuses on the strategy use of the students, instead of students' views of symbolic notation.

Current Recommendations for the Teaching and Learning of Algebra

Mathematics education literature presents an alternative view for learning algebra. As noted by NCTM (2000):

> Algebra is more than moving symbols around. Students need to understand the concepts of algebra, the structures and principles that govern the manipulation of the symbols, and how the symbols themselves can be used for recording ideas and gaining insights into situations (p. 36).

Pegg and Redden (1990) noted that algebra should emerge through student activity instead of being imparted as fact. They recommend that students experience tasks that facilitate the recognition of number patterns, create rules that describe those patterns, and then write the rule(s) in an abbreviated form.

Mathematics is an evolving discipline with new applications arising as business, technology, and other real-world entities demand solutions to emerging problems. Due to the inherent ties between mathematics and the real world, students should experience and learn mathematics through real-world settings and applications. More specific to this study, *algebra* is best learned through real-world contexts (Usiskin, 1995; Nathan & Koedinger, 2000). NCTM is also a strong advocate for the learning of algebra through contextual means. The 1989 *Curriculum and Evaluation Standards* calls for students to use algebra in real-world situations (p. 151). NCTM (2000) notes that, "working in real-world contexts may help students make sense of the underlying mathematical concepts and may foster an appreciation of those concepts" (p. 296).

Generalization is the Core of Mathematics

Generalization represents a key element of mathematics and a guiding goal in the mathematics classroom. As Gattegno (1990) asserted, something should only be considered mathematical when fully generalized or "shot through with infinity." Mason (1996) brought this idea to the classroom in noting that if students fail to express their own generalizations, "mathematical thinking is not taking place" (p. 65). The process of mathematical generalization involves students looking across particular cases for meaningful commonalities, such as patterns and structures, and identifying and exposing these relationships (Kaput, 1999; Mason 1996). For example, Mason (1996) described a five-year-old student who worked through a succession of paired arithmetic questions where the order was reversed between the first and second question (i.e. 4 + 3 = __, 3 + 4 = __.) After answering a few of the questions, the student suddenly noted, "Something plus something is…the same as something plus…something" (p. 70). The student was able to at least hypothesize the generalization that the addition of two numbers is commutative. This movement led Mason (1996) to describe generality as "seeing a generality through a particular and seeing the particular in the general" (p. 65).

Harel and Tall (1991) provide a framework for classifying the types of generalization that individuals employ. The three generalization categories are: (a) *expansive generalization* in which the learner extends the applicable range of an existing schema without reconstructing it, (b) *reconstructive generalization* which occurs when the learner reconstructs an existing schema in order to widen the range of applicability, and (c) *disjunctive generalization* where the learner constructs a new schema in order to deal with new information. Expansive and reconstructive generalization require the learner to extend or modify previous schema while disjunctive generalization requires a Piagettian accommodation for the generalization. For example, students learning to solve linear equations might encounter the following two problems: $3x = 21$ and $(2/3)x = 21$. If a student understands these equations to be mathematically similar and sees the processes of solving the equations to be essentially the same, then the student has exhibited either expansive or reconstructive generalization. The learner has likely either extended or reconstructed to widen the applicability of his schema for solving one of the equations to include the solving of the other. If a student views the two equations and their methods of solution as unrelated, then this student has likely experienced disjunctive generalization for the solving of these two equations.

The authors suggest that expansive and reconstructive generalizations are more appropriate for cognitive development. To provoke generalization in students, Harel and Tall suggest the use of a generic example to help students generalize a particular idea. A generic example represents a successful generalization when a student views "one or more specific examples as typical of a wider range of examples embodying an abstract concept" (p. 6). This idea of a generic example relates to Mason's view of seeing the generic in the particular.

Mathematical generalization represents a major focus of this study, as can be seen in the conceptual framework (Figure 1). The tasks used for this study harbored the capacity to serve as generic examples. When generalizing these tasks, students were asked to produce descriptions and/or formulae that serve as generalizations for each problem situation. Specific questions aimed at ascertaining the generality with which the student was viewing their rules were asked (see Interview Protocol in Appendix A).

Algebra is the Language of Generalization

While generalization is at the heart of all mathematical activity (Kaput, 1999; Mason, 1996), it is also considered to be at the heart of algebra. Students must view algebraic rules as generalizations--mathematical statements that model situations for any value in the domain of the variable(s)--to demonstrate an understanding of algebra (Dienes, 1961). In this study, generalization is viewed as a key component of algebra. Such a position is represented in the conceptual framework that guides this study and is supported in the literature. For example, Mason (1996) noted that "generalization is central to all mathematical processes" (p. 74), while speaking specifically about algebra (the combining of arithmetic operations). Lee (1996) supported this position, noting that, "algebra and indeed all of mathematics is about generalizing patterns" (p. 103). Clearly, generalization and algebra are deeply intertwined with generalization representing a critical element of algebra. For example, the student described by Mason (1996) in the previous paragraph developed a generalization about the addition of two numbers. Algebraic notation provided the student with a means for representing his generalization symbolically. In terms of this study (as illustrated in the conceptual framework), algebra represented the medium through which generalization was viewed in the form of patterning situations.

Linking Generalization and Justification

While the focus of this study is on student generalizations and the flexibility the students exhibit in providing and utilizing these generalizations, little could be ascertained about how and why they create and use particular strategies without detailed explanations concerning their thoughts and choices. As noted by Lannin (2005), "generalization cannot be separated from justification" (p. 235).

Justification plays two key roles related to understanding student thinking: (a) It allows the researcher insight into why a student used a particular strategy, and (b) it provides a window for ascertaining the degree to which students view the generality of their rules (Lins, 2001). For example, when attempting to generalize the Cube Sticker problem (see Figure 2), a student produces the rule $S = 4n + 2$, where S is the number of stickers required for a rod of length n. In describing how he developed his rule, the student noticed that four rows of stickers comprise the length of the rod ($4n$) and two more stickers are needed for the ends. The student's justification implies that his rule is based in the context of the situation and suggests that his visual image of the problem situation led to the development of his explicit rule. The general nature of his statement suggests that he could apply this rule to any length rod. Subsequent questions such as "For which values would your rule work?" could provide more information regarding the development and perceived generality of the rule.

Without justification, the degree of generality implied by a student for a particular rule would be nearly impossible to ascertain. Justification is also important in determining a student's level of flexibility, which is discussed in the next section.

Strategy Use

Relatively few studies have focused on student strategies for generalizing algebraic tasks. Of the extant literature on generalization strategies, most research has been conducted with students in late elementary through middle school grades. These studies, however, provide a strong foundation for investigating secondary student strategy use. Stacey (1989) described several strategies used by 9-11 year old students in solving contextualized linear generalization tasks. She found that students employed a counting strategy, a difference strategy, a whole-object strategy, and a linear strategy. For the counting strategy, the students counted the number of items from a drawing. The difference method involved multiplying by the common difference (similar to both the chunking strategy and explicit strategy used in this study). The whole-object strategy used a multiple of a previous value as a new value, implicitly assuming direct variation. Stacey's final strategy category, linear, described the development of a linear (explicit) model to find the solutions.

These strategies were similar to those found by Healy and Hoyles (1999) who engaged students aged 12-13 in specific contextualized linear tasks and Swafford and Langrall (2000) who worked with 6th grade students in generalizing algebraic tasks prior to formal algebra instruction. Lannin, Townsend, and Barker (under review-a) studied the strategies of 5th grade students using similar, contextualized tasks. Their categorizations of recursive and chunking, whole-object, and explicit strategies represent adaptations of

Stacey's difference, whole-object, and linear strategies. They also meld well with Healy & Hoyles' differences between terms, multiplicative, and dependent and independent variables strategies, respectively. These categories serve as the basis for classification of strategies in this study. A description of these strategies and the literature surrounding them is provided below.

Explicit Strategies

Explicit reasoning has long been the focus of the algebra curricula (Kaput, 1999). Explicit, or closed-form, generalizations allow for the immediate calculation of any value for the particular situation by relating the independent variable to the dependent variable. For example, in the Cube Sticker Problem (see Figure 2), the explicit rule $S = 4(n\text{-}2) + 10$ provides a method for calculating the number of stickers, S, needed for a rod of length n.

Until recently, developing explicit rules has been the primary focus for algebra textbooks. Even with other strategies considered as viable alternatives (Kaput, 1995; Sandefur, 1992), explicit reasoning remains the primary end goal of many lessons. Such focus is not without merit. The National Council of Teachers of Mathematics has supported the use of explicit thinking throughout iterations of standards. For example, the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) and the *Principles and Standards for School Mathematics* (NCTM, 2000) both provided examples that support the goal of using explicit reasoning at various grade levels. One particular example occurs in the Grades 6-8 Algebra Standard (NCTM, 2000). The "Super Chocolates" problem provides a context for developing explicit reasoning (p. 226). The Super Chocolates problem is similar to the one presented in Figure 6 below.

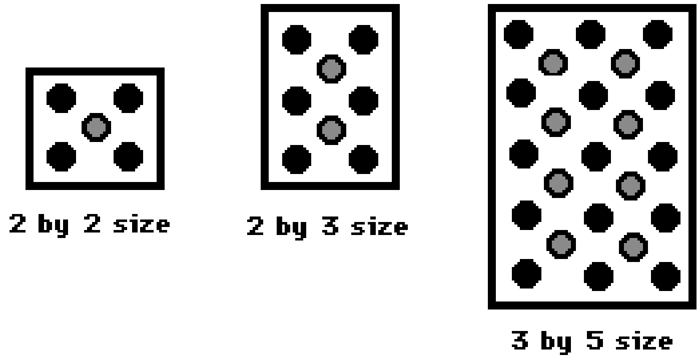


Figure 6. Chocolates problem.

In this problem, arrays of caramel candies are surrounded by arrays of chocolates. Students must determine the number of caramels in any box if the dimensions of the box (the number of chocolates in each direction of the array) are known. NCTM (2000) reiterates the importance of explicit reasoning in the 9-12 Algebra Standard, where one student expectation involves “generalizing patterns using explicitly defined” functions (p. 296).

While many examples can be found in current classrooms and textbooks that refer to the use explicit strategies, other strategies are often treated as relatively unimportant or are not valued at all. However, NCTM (2000) recommended the use of non-explicit strategies, such as recursion, in the 9-12 grade level expectations. NCTM’s call for alternative strategy use seems warranted given the pragmatic utility of such generalizations. For example, Kaput (1993) noted that most mathematical situations cannot be modeled using a closed-form (explicit) representation. As noted in Chapter I, if the mathematics curriculum is expected to prepare students for the realities of the outside world, students should be expected to solve problems that require alternative generalizations. Given that current technology allows for the efficient use of non-explicit strategies (Kaput, 1995), it seems prudent to consider alternative strategies.

Recursive Strategies

Recursive reasoning can often provide a viable option for solving problems of algebraic nature. Recursive reasoning involves recognizing and applying the change from term-to-term in the dependent variable. Swafford and Langrall (2000) noted that students naturally employ general recursive reasoning to problem situations. Maurer (1995) described the importance of recursive reasoning in noting "the essential ingredient in many problems is relating the arbitrary case to earlier cases" (p. 95). Booth (1989) wrote that research evidence on the kinds of procedures that students use naturally points to a greater use of recursive techniques. NCTM (2000) agreed, stating that "students should study sequences best defined by recursion," (p. 37) specifically pointing to the many that appear naturally in various contexts.

The National Council of Teachers of Mathematics considered recursive reasoning important enough to recommend the study of recursive relationships at both the high school level (NCTM, 1989, p. 178, 296) and in the middle grades (p. 263). In the *Principles and Standards for School Mathematics,* NCTM (2000) called for secondary students to engage in increasingly more and diverse experiences with potentially recursively-modeled situations such as interest-rate and models of growth problems (p. 305). Thus, students should have opportunities to generalize tasks that can be represented through a variety of strategies so that connections among strategies can emerge.

It should be noted that there is not complete agreement regarding the study of recursive relationships. In fact, Stacey and MacGregor (2001) have recommended that particular tasks not be used when the emphasis is on encouraging explicit reasoning. Their argument stems from possible student confusion with the recursive notation and their belief that explicit reasoning might be overshadowed by the students' work with recursion.

Whole-Object Strategies

The Whole-Object strategy (Stacey, 1989), also referred to as unitizing (Lamon, 1993) is quite different from the other three strategies. Stacey defined whole-object reasoning as using a multiple of the output for a smaller input value to find the output for a larger input value. The use of this method, without a correction, implies that the task represents a direct variation situation. For example, a rod of length 10 has 42 stickers, so a student might reason that a rod of length 20 would have 42*2 or 84 stickers. In this case, the student failed to adjust for over-counting that occurred due to the ends of the rod also being doubled. A student who has a good visual image of the problem situation may adjust for the over-counting in this case by subtracting two for the ends of the rods that were joined together in the doubling (i.e. a rod of length 10 has 42 stickers, so a rod of length 20 would have 42*2 – 2 because the stickers between the two length-10 sections would need to be removed.)

Chunking Strategies

Given the relative infancy of the definition of and research on the chunking strategy, it is not surprising that much less literature has been devoted to understanding student use of this strategy than has focused on recursive, explicit, and whole-object reasoning. However, Stacey (1989) dealt with chunking in her difference method (repeated addition implies multiplication), which also included recursive reasoning. Healy and Hoyles (1999) described chunking strategies in terms of student operations on differences between terms. In this study, chunking played an important role in algebraic generalization in that it represents a potential link between recursive and explicit reasoning. When using a chunking strategy, a student builds on a recursive pattern by adding a unit onto known values of the desired attribute. For example, on a rod of length 10 there are 42 stickers. For a rod of length 15, one would take 42 + 5(4) because the number of stickers increases by 4 each time. The quantity 5•4 represents a "chunk" of five iterations of recursion that is added on to the previous total. As previously noted, the research on chunking is brief.

Research has provided a strong base for looking at student strategy use. Looking across the studies for commonalities reveals that elementary and middle grades students employ four main algebraic generalization strategies: explicit, whole-object, chunking, and recursive strategies. However, the current research base does not extend beyond the middle school level in terms of the strategies employed by students.

Algebraic Flexibility

Algebraic flexibility represents a key component of this study, as research questions two and three focus on this topic. For this research, I focused on a particular form of strategic flexibility for solving algebraic generalization problems. In this section, I discuss the extant literature surrounding the meaning and importance of algebraic flexibility. Following this review, I detail the definition of algebraic flexibility that was used to guide the data analysis for this study.

Mathematics education research has demonstrated the importance of understanding students' algebraic flexibility. Krutetskii (1976) provided an early look at algebraic flexibility in his work documenting student ability. Lewis (1981) and Starr (2001) also investigated student flexibility in working with particular algebraic situations. Although in every case algebraic flexibility is referred to as a positive student attribute, a consensus does not exist regarding what algebraic, or even mathematical, "flexibility" entails. It is often difficult to ascertain exactly how "algebraic flexibility" is being defined when no formal definition is given or the context under which the term was used leaves a lot to be deduced.

In looking at the various contexts under which algebraic flexibility has been studied, it seems that algebraic flexibility has myriad meanings. The definitions for flexibility include the ability to:

1. Move between interpreting notation as a process to do something (procedural) and as an object to think with and about (conceptual), depending upon the context (Gray & Tall, 1994).

2. Move among representations and understand how each says the same thing (PSSM, 2000; Davis & McGowen, 2002).

3. Use more than one strategy to solve the same problem or equation (Hollands, 1972; Lewis, 1981; NRC, 2001).

4. Apply various sequences of steps to solve several similar problems (Star, 2001).

5. Switch from one solution method to another (Krutetskii, 1976).

The definitions used by Gray and Tall, NCTM, and Davis and McGowen describe flexibility in ways that were not useful for this study. They do, however, provide alternative perspectives on how the term "flexibility" is being used within the same mathematical domain. I discuss each of these definitions below.

Gray and Tall (1994) described flexibility in terms of an ability to view notation as a process to do something (procedural) and as an object to think with and about (conceptual), depending upon the context. For example, the expression $x + 4$ represents both the process "add four" and for the product of that process, the quantity "$x + 4$." This process of viewing an expression in two different ways simultaneously was not included in the definition of flexibility that guided this study.

The National Council of Teachers of Mathematics (1989, 2000) calls for students to be mathematically flexible vis-à-vis representation by the middle grades. "Students should be able to understand the relationships among tables, graphs, and symbols and to judge the advantages and disadvantages of each way of representing relationships for particular purposes (NCTM, 2000, p. 37). The *Curriculum and Evaluation Standards* (NCTM, 1989) recommends that students in grades 5-8 should understand the relationship between data in tables, algebraic generalizations, and graphical representations. Similarly, Davis and McGowen (2002) describe flexibility in terms of connections between various representations of functions, including tables, graphs, and algebraic syntax. In these examples, *flexibility* is described as the ability to use and understand various representations of mathematical data. For example, a student trying to find the solution to the system of equations containing $y = 3x + 4$ and $x + y = 7$ might be inclined to construct a graph, generate a table, or work with the formal symbols. The student's ability to understand and move between the different representations is at the heart of this description of flexibility. Once again, this view of algebraic flexibility was not included in the definition of flexibility employed for this study.

The remaining three views of flexibility are related to this study, with each adopting a slightly different perspective in terms of what it means to algebraically flexible. These three descriptions of algebraic flexibility were influential in the development of the definition used for this study.

Lewis (1981) defined flexibility as "the lack of consistent use of a single process for solving a given equation." Under this definition, students who routinely provided more than one solution for the same equation were considered to have flexibility. For example, a student who could usually produce more than one explicit rule, such as $S = 4n+2$ and $S = 4\ (n-1) + 6$ for the Cube Sticker problem, would be classified as flexible. Hollands (1972) provided a similar description of flexibility (flexibility is demonstrated through a suggestion of a variety of methods) in his work on creativity. One could imagine that changing the word "equation" to "problem" or "task" in Lewis's description would provide a broader, more inclusive definition that would work for contextualized situations.

The National Research Council (2001) introduced a similar definition for algebraic flexibility when discussing strategic competence. Under this definition, a student exhibits flexibility if he can devise several different approaches to a nonroutine problem and "chose flexibly among reasoning, guess-and-check, algebraic, or other methods" (p. 127) in solving the particular problem. Lewis' definition, along with the one suggested by the NRC, for flexibility centers on providing different solutions for the same situation (*within-task*), thus neglecting the potential for students who solve mathematically similar tasks (e.g., problems that could be modeled linearly) in different manners or tasks exhibiting different constraints to be considered flexible.

Star (2001) defined flexibility as "the ability to vary the sequence of steps that one uses to solve a group of similar problems" (p. 89). In his study, Star looked at the number of different ways that sixth grade students, who had not experienced instruction on symbolic equation solving, could solve linear equations. Star's view of flexibility effectively expands upon the single-problem constraint inherent in Lewis' definition and provides a useful definition for the context of students using different steps to solve similar equations. However, unless "sequence of steps" is construed broadly, this definition does not seem to include problem-solving situations that are not procedural in nature.

In his work on the identification of mathematically capable seventh grade students, Krutetskii (1976) discussed the idea of flexibility as the ability to switch from one mental operation to another (p. 222), "from one method of approach to another, from one method of solution to another" (p. 188). Differentiating his work from the previous references, Krutetskii did not focus on whether or not the students found several methods of solution. He was more interested in their abilities to move from one method of solution to another. In his study, students were asked to ascertain the maximum number of ways to solve each problem in a particular set. The problems used in this investigation included traditional algebraic tasks such as sailing with and against the current. To distinguish between students who found the same number of solution strategies, Krutetskii documented the amount of time that it took for a student to find each solution for each problem. He found that it was more difficult for "average" pupils to switch to a new method of solving a problem once they already solved it. The less capable students had even more difficulty, "as if the solution that had been found cut off any possibility of their switching to a new method of operations" (p. 278). Ultimately, Krutetskii demonstrated that the students who were able to grasp the structure of the problem demonstrated the greatest algebraic/problem solving flexibility.

Although Krutetskii spoke mainly of flexibility in terms the ability to switch between different ways of solving the same problem, he also implied that there are actually two aspects of flexibility: flexibility working within a particular task (*within-task flexibility*) and flexibility that applies across tasks (*cross-task flexibility*). Although Krutetskii did not explicitly use these terms, both within-task and cross-task flexibility were alluded to in his work. Within-task flexibility involved the ability to move between multiple methods for solving a particular task. Likewise, cross-task flexibility deals with the effect, or lack thereof, that working a particular problem had on a subsequent one; the set of problems for Krutetskii's study included variations of the same problem that required different methods of solution. Since the amount of time it took to produce solutions for each problem was documented, he used a ratio of the time it took to solve the original problem and its variation to measure how "bound" a student was to his previous method of solution. In other words, student experience with what was considered to be a "similar" problem could impact them negatively in terms of flexibility.

Certainly, flexibility can and does mean different things to different people working within different contexts and topics. However, it seems that a well defined, yet broad description of flexibility is needed to allow researchers and practitioners to more clearly understand and communicate thoughts and findings related to this important facet of student learning. With that goal, my experience, and other's writings in mind, this study sought to provide such a definition.

In establishing the definition for flexibility used to guide this study, it was important for flexibility to represent a valuable characteristic of mathematical thinking that ultimately enhances a student's mathematical power in terms of problem solving. To ensure that such criteria are met, mathematical flexibility needs to account for thinking within and across tasks. As noted in the review of definitions of flexibility in mathematics education research, flexibility has been defined in particular cases to refer to flexibility within a task (Lewis, 1981), flexibility across tasks (Star, 2001), and a combination of both (Krutetskii, 1976). By defining flexibility as such, the expectation for achieving mathematically flexible status is markedly higher, with a deeper understanding of the structure of the mathematics behind the problem required. Within-task and cross-case flexibility remain critical aspects of the definition. The achievement of each type of flexibility is certainly no less important, but it is the attainment of both elements concurrently that leads to the characterization of a more comprehensive understanding of the underlying mathematics and is seemingly, therefore, a crucial ingredient in consideration for becoming "algebraically flexible."

Given the afore-mentioned preferential aspects of algebraic flexibility, I draw upon the work of Krutetskii (1976) and Star (2001) in fashioning this definition. Krutetskii alluded to the need for flexibility to include both within-task flexibility and cross-task flexibility. However, Krutetskii was more concerned with the amount of time it took a student to change strategies than in her ability to understand multiple strategies.

Star valued the correct use of multiple strategies within a set of similar situations, but did not focus on the ability of students to apply strategies across varied tasks. A combination of the two perspectives provides a definition in which the broader aspects of flexibility noted by Krutetskii (within and cross-task flexibility) meld with the importance of multiple strategies presented by Star.

For the purpose of this study, flexibility was defined as the ability to utilize and demonstrate both *within-task flexibility* and *cross-task flexibility* in solving a mathematical task. *Within-task flexibility* [referred to as strategic flexibility in NRC (2001)] represents the ability to recognize the viability of various strategies given the context and constraints of a particular problem. Cross-task flexibility is defined as the ability to recognize to the applicability of a particular strategy to various situations. This definition of flexibility requires students to be able to solve a variety of problems within different contexts, under myriad conditions.

Flexibility and Problem Solving

While Krutetskii discussed student algebraic flexibility, his work also intertwined flexibility with problem solving. Taking his lead, I explored the problem-solving literature to gain further insight into algebraic flexibility. In this section, I discuss the relationship between problem solving and algebraic flexibility, factors that contribute to problem solving ability, and how these factors relate to the determination of algebraic flexibility.

"The central point of education is to teach people to think, to use their rational powers, to become better problem solvers" (Gagne, 1980, p. 85). Educational researchers have considered problem solving a critical skill due to its everyday utility. As each of the students in this study worked to solve the tasks, they engaged in problem solving. To that end, problem solving was employed in this study. However, the literature on problem solving also plays a key role in informing this study due to the focus on algebraic flexibility. In this study, algebraic flexibility involves the ability of a student to understand and utilize various generalization strategies. Similarly, problem solving involves the ability of a student to understand and utilize various problem solving approaches or heuristics (Stewin & Anderson, 1974). Since little literature exists on how algebraic flexibility can be determined, I looked to the research on problem solving, as I hypothesized that the qualities that impact problem solving ability would have implications for determining algebraic flexibility. Several factors have been shown to contribute to problem solving ability. These factors are described below.

Stewin and Anderson (1974) noted that students who consider more alternative approaches are better problem solvers. A key component of algebraic flexibility, as defined in this study, is the ability to use and understand multiple strategies when solving an algebraic task.

As previously noted, Krutetskii found that students who were able to determine the deeper structure of a problem demonstrated the greatest flexibility in terms of solving the algebraic situations. Similarly, research (Bassok, 1997; Blessing & Ross, 1996; Schoenfeld & Herrmann, 1982; Silver, 1981) has shown that novices tend to focus on surface level features of tasks rather than on the deeper relational properties that experts use to solve the problems. In focusing on the surface level features (words or objects presented in the problem), novices are far less successful in solving problems than the experts who understand the structure of the problem and recognize common features that allow them to be successful problem solvers. For example, expert chess players perceive board positions differently than do their novice counterparts. The experts see the board in terms of patterns or broad arrangements, whereas novices do not (Chase & Simon, 1973). In physics, experts and novices were asked to describe their approach to solving particular physics problems. Experts usually mentioned the principles or laws that were applicable, along with their reasoning as to how and why the laws could be applied. Novices, however, mentioned which equations they would use and how they could be manipulated (Chi, Feltovich, & Glaser, 1981). With respect to this study, the students who were able to see and utilize the mathematical similarities and differences between problems certainly had an advantage in producing a correct generalization. They were also likely to be more algebraically flexible in that they were able to generalize the functionality of a strategy across problems.

Summary

Students often view algebra as a static field in which algebraic problem solving involves the application of a single rule to solve a problem. Consequently, many students have difficulties understanding and utilizing algebraic symbols. Research has suggested that the learning of algebra emerges through student activity, employing the use of contextualized algebraic tasks.

The determination of student generalization represented a main goal of this study and guided the selection of algebraic tasks. Justification of a generalization was also discussed as a critical component in ascertaining student understanding, as it provides a window to student perception of generality. Literature surrounding the development and classification of algebraic strategies was provided, which included a description of each of the four algebraic generalization strategies (explicit, whole-object, chunking, and recursive) that formed the foundation for strategy classification in this study.

Various definitions of algebraic flexibility presented in literature were examined and used to construct the definition for flexibility that guided the analysis of algebraic flexibility for this study. Algebraic flexibility is comprised of within-task and cross-task flexibility. Within-task flexibility refers to a student's ability to recognize the viability of various strategies given the context and constraints of a particular algebraic task. Cross-task flexibility deals with a student's ability to recognize to the applicability of a particular strategy across various situations. The chapter concludes with a discussion of the problem solving literature focused on descriptions of the various factors that impact problem solving ability and how these studies inform the determination of flexibility. The next chapter details the specific methods used for this study, guided by the literature noted in this chapter.

Chapter III

RESEARCH DESIGN AND METHODOLOGY

For this study, I analyzed secondary student strategy use and characterized the algebraic flexibility of these students. In particular, I examined the strategies that secondary students use, recognize, and value when solving particular algebraic tasks. Also, I ascertained the extent to which the secondary students in this study exhibited within-task strategic flexibility and cross-task strategic flexibility when working in the afore-mentioned contexts. This chapter provides a detailed description of the methods I employed to select the participants for my study and the methodology and tools used in the analysis of their algebraic reasoning and flexibility. In the following sections, I first describe the methods that were used to select the students for participation in the study. Next, I present the design of the interviews, followed by the instrumentation, including tasks that were used in data gathering and analysis of student algebraic strategy use. This is followed by a description of the data sources that were collected for analysis and an illustrative example of the coding scheme. The chapter concludes with details regarding the data analysis.

Skemp (1987) defined a methodology as "a collection of methods for constructing (building and testing) theories, together with a rationale that decides whether or not a method is sound. This includes both constructing a new theory *ab initio*, and improving an existing theory by extending its domain or increasing its accuracy and completeness"

(p. 130). This research project contributes to theory by defining and analyzing student flexibility and extends existing theory by adding to the literature base concerning student strategy use.

Participants

For this study, 11 tenth-grade students participated from two rural Midwestern schools, hereafter designated School A and School B. Tenth grade students were chosen due to the students' recent completion of an introductory algebra course (i.e., Algebra I). Participant completion of an introductory algebra course was important due to the lack of current literature about the strategy use of students at this level. The two schools were selected due to their use of non-NSF funded mathematics curricula throughout K-12. Given that most students in the United States currently do not experience NSF-funded curricula, schools employing only non-NSF funded curricula were chosen in an effort to enhance the potential generalizability of the findings.

School A is located in a growing town that is situated midway between two larger cities. Many of the inhabitants of this community commute to the cities for work. Seventeen percent of students qualify for free and reduced lunches. For School A, initial contact with school personnel was made through the principal. After the principal provided consent to conduct the study, I distributed a call for potential participants through 10^{th} grade teachers. These teachers allowed for contact with the entire 10^{th} grade student population at School A. Once the pool of potential participants was identified, 8 student volunteers were contacted to ascertain their level of interest in taking part in the study. All 8 students agreed to participate in the study. Seven of the 8 students participated in three 30-45 minute interviews, with one student withdrawing from the study after the first full interview. All students selected in the study represent a convenience sample, as the sample selection process was not random.

Additional students from another high school, School B, were selected to broaden the sample. School B is located in a small agricultural town where 48% of the students qualify for free or reduced lunches. Initial contact was made through the high school mathematics teacher at School B. Due to the small size of this rural school, few students were available for participation. Once the principal consented to involve students in his school, the selection process began. Three students from School B were each able to participate in two interview sessions. The teacher identified these students as individuals that would provide the richest data (i.e., the students most likely to verbalize their reasoning). These three students were selected and each participated in two one-hour interviews.

Generalization Tasks

In determining the tasks in which the participants were to be engaged, efforts were made to identify questions that elicit algebraic generalizations. These tasks are often embedded in a contextual situation that requires the calculation of particular values. The ultimate goal of these tasks is the generation of a rule or rules that could be used to determine other particular instances of the pattern. Research (Kenney, Zawojewski, & Silver, 1998; Stacey, 1989; Swafford & Langrall, 2000) has shown that such activities lead students to the construction of a variety of generalizations.

Given that the goals of this study center on the ability of students to generate, understand, and efficiently utilize different generalizations, problems of the variety listed in the previous paragraph were employed extensively. However, given the reality that the students in this study have mainly experienced classrooms more representative of the "one-rule-per-section" teaching philosophy, the format of the tasks that were employed for this study likely did not represent the norm for the participants. While all students attended class periods specifically designed for preparation for the Missouri Assessment Program exams (which were described as at least somewhat driven by the discussion of contextualized problem situations), this study incorporated only contextually situated algebraic tasks. The selection of these tasks follows current recommendations for mathematics teaching and learning. A more detailed breakdown of the tasks that were used is provided below.

Two different problem types were utilized in this study: (a) contextualized patterning situations that included visual models and (b) contextualized patterning problems that did not include models were presented to students during the interview.

Contextualized problems encourage students to draw upon their prior experiences (Kaput, 1989), promoting reflection on their mathematical representations and thinking (Filloy & Sutherland, 1996), and increasing students' understanding of the topic (Van Reeuwijk, 1995). Van Reeuwijk notes that given the opportunity to choose their own solution strategy for these contextualized tasks, students fair even better. Furthermore, algebra students are more successful in solving contextual-type algebra problems than similar problems presented outside of a context (Nathan & Kodinger, 2000). Contextualized tasks provide opportunities for the students to reason more flexibly, drawing upon their verbal reasoning skills to produce inventive non-standard strategies. Problems presented through traditional algebraic notation provide little opportunity for students to employ such alternative strategies, instead often prompting the use of potentially maligned manipulation skills.

Other contextualized problems without visual representations (diagrams) were also provided. Such tasks require students to create their own models of the phenomena through various means, including "drawing on their knowledge of many classes of functions—to decide, for instance, whether a situation would best be modeled with a linear function or a quadratic function—and be able to draw conclusions about the situation by analyzing the model" (NCTM, 2000, p. 39).

Specific Tasks

The selection of the tasks used in the active interviews (Holstein & Gubrium, 1995) was based on a few factors inherent to the tasks: (a) they were algebraic in nature, and (b) they offered the potential to elicit various strategies. Given the wide range of potential tasks, the first criterion required further definition. After consideration of problems from various studies and texts, tasks that included linear, quadratic, and exponential functions were chosen. Since this study looked at students who had recently completed an initial algebra course, the topics represented in the tasks were chosen to match those covered in an Algebra I course. Given the focus of this study on student algebraic flexibility, task selection was performed with an eye toward tasks that could contribute to determining algebraic flexibility. Previously published tasks were afforded preferred status, given that they had undergone field-testing and piloting during prior research.

From this set of tasks, the problems that elicited the widest range of strategies while also providing appropriate levels of difficulty during the pilot study comprised the corpus of tasks. Three student volunteers from a private high school participated in the pilot study in fall of 2004. Each participant completed three interviews in which they solved seven algebraic generalization tasks. Based on student responses to the tasks during the pilot study, the tasks were modified. Changes in the tasks included clarification of wording and changes in input values.

After the final selection and modifications were completed, six tasks represented the elements of the interview battery. The tasks used in this study were as follows: the Theater Seats problem, the Calling Tree problem, the Cube Sticker problem, the Streets and Lampposts problem, the Carwash problem, and the Brick problem. The first three tasks were given in the order listed with the remaining tasks given as needed. Three of these problems represented linear increasing situations (i.e. ax + b), with one each of exponential increasing and decreasing (ka^x), and quadratic increasing (ax^2 + bx + c). The tasks and their mathematical structures are listed below in Table 1.

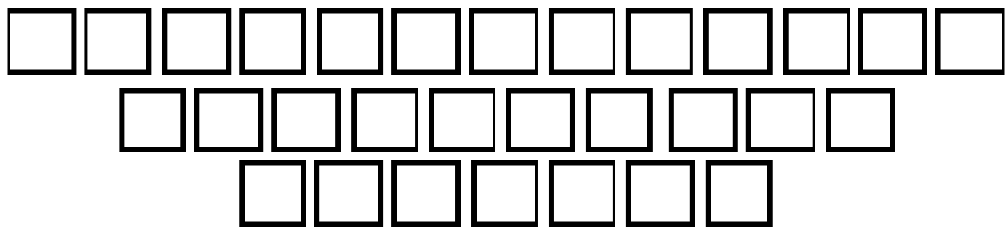
Task	**Mathematical Structure**
Theater Seats	Linear Increasing
Calling Tree	Exponential Increasing
Cube Sticker	Linear Increasing
Streets & Lampposts	Quadratic Increasing
Carwash	Exponential Decreasing
Brick	Linear Increasing

Table 1. Tasks and their mathematical structures

Theater Seats problem. The Theater Seats problem was chosen due to its ability to promote multiple strategies. In prior research (Lannin, Barker, & Townsend, under review-a) and the pilot study, this task proved capable of eliciting four generalization strategies. The Theater Seats problem is provided in Figure 7.

Theater Seats Problem

In a theater there are seven seats in the first row. The increase in the number of seats is the same from row to row. Below is a diagram of the first three rows in the theater.



1. How many seats are there in the 4th row of the theater? In the 5th row? In the 10th row? In the 20th row? In the 38th row?

2. How many seats are in the 138th row of the theater? Explain how you determined this. In the 139th row?

3. Write a rule that would allow you to calculate the number of seats in any row. Explain your rule.

Figure 7. Theater Seats problem

The Theater Seats problem is a linear increasing situation that establishes a recursive relationship for the number of seats in consecutive rows. The chunking strategy represents a useful bridge between the recursive and explicit strategies for this task. For example, a student who noticed the recursive relation “add” 3 between successive rows of the theater could “chunk” groups of 3 together to find larger values; building on the fact that the third row of the theater contains 13 seats, a student could add a “chunk” of three 3s, or 9, to find that there are 21 seats in the sixth row. From there, she might add 2 more 3s on to the value of 21 to find the number of seats for row eight. After using the chunking strategy to calculate the number of seats for a few rows, a student might notice that she can “chunk” together any number of rows and add that value to a particular row that she had already found. At this point, the strategy moves from chunking on to the previous total to using an explicit strategy to find the number of seats for any row in the theater.

Whole-object reasoning for the Theater Seats problem is possible and can be completed correctly through an adjustment. However, it would be difficult for a student to use the whole-object strategy correctly for this task. For example, after calculating 19 seats for the fifth row, a student decides that he can double this number to find the number of seats for the tenth row. After investigating the effects of doubling between the second and fourth rows, he quickly reasons that a doubling of the output does not work in this case. For the doubling to provide the correct solution, the student would have to realize that the initial value of seven provided in the first row cannot be included in the doubling since that seven seats includes the first set of “3” seats being added; only the multiples of three added to each row as the seats numbers increase should be doubled. To find the correct total, a student would have to first subtract the four “extra” seats from the row she is doubling, then double the amount, and add the four seats back on to produce the correct total.

Development of an explicit formula for this task could also be challenging due to the lack of direct correspondence between the row number and the number of seats. For example, after discovering the recursive ”add 3” strategy that exists between each row and noticing that the first row has seven seats, a student might hypothesize that $S = 3n + 7$, where S is the number of seats in a row and n is the row number, represents an explicit rule for modeling this situation. However, this would produce an incorrect result as the number of times three seats are added to each row is one less than the row number.

Calling Tree problem. The Calling Tree problem represents an exponentially increasing situation where, as in the Theater Seats problem, the recursive relationship is described within the context of the problem. The Calling Tree problem is presented in Figure 8.

Calling Tree

Suzy, Maria, and Dominique need to call everyone in the school about a last-minute change in the schedule for the upcoming dance. Fortunately they have designed a calling tree for just such an emergency. The calling tree is designed so that each person calls two other people. Suzy, Maria, and Dominique each call their two contact students in about one minute. Assume that it takes about one minute for each person to contact the two students that they are assigned to contact.

1. How many students were contacted during the 5th minute? During the 7th minute? During the 10th minute? During the 20th minute?

2. How many students were contacted during the 37th minute?

3. Explain how you could determine how many students were contacted during any given minute. Write a rule that would allow you to calculate the number of students contacted during any particular minute. Explain your rule.

Figure 8. Calling Tree Problem

The chunking strategy provides a useful and available extension to those students who use a recursive strategy. As with the Theater Seats problem, an explicit rule is viewed as challenging for the student to develop. However, for this problem, the explicit rule is hypothesized as more difficult to develop due to its exponential nature. If the student does not have a good understanding of exponential rules, it is likely that she will continue to use recursive or chunking strategies to solve the problem. A whole-object strategy is not useful for this situation, as a doubling of output values does not result in a correct value for a doubling of input values and a linear adjustment (such as subtracting to correct for an overlap) does not consistently produce a correct value.

Cube Sticker problem. For the Cube Sticker problem (Figure 2, p. 10), a linear increasing situation, prior studies have demonstrated that this task elicits all four generalization strategies noted in the Generalization Strategy Framework (Figure 5, p. 15). The recursive, explicit, and chunking strategies are accessible, with a correct whole-object strategy requiring a small adjustment. The recursive strategy is discernable through the addition of another block. An explicit rule could be constructed that does not require adding to the length of the rod before multiplying. For example, the explicit expression $4n+2$ could represent the number of stickers on a rod of length n, with the digit four accounting for the four sides of the cubes and the two representing the stickers on the end of the rod. A chunking rule could be developed through iterations of the recursive "add four" rule. For example, after finding that 18 stickers are needed for a rod of length 4, a student could chunk together three 4's to find the number of stickers for a rod of length 7. The whole-object strategy could be correctly applied if the student saw that that a doubling of rods leads to an overlap of two stickers. For example, a student who found that a rod of length 5 required 22 stickers might assume that a rod of length 10 would require twice as many (44 stickers). This amount would be incorrect due to an over-count of two stickers produced when the ends of the two length-5 rods were put together to form the rod of length 10. A correction could be made by subtracting the two stickers that are no longer on the outside of the rod.

Carwash problem. The Carwash problem represents a more challenging situation in that the recursive relationship is not immediately discernable. The change from term to term is difficult to discern due to the fact that the change is not the same and calculation of successive terms is not possible without having already constructed an effective explicit or whole-object strategy. The Carwash problem is presented in Figure 9.

Carwash Problem

A team of 40 car washers can clean 100 cars in only 2 hours. Assuming that each car washer continues to clean at the same rate, answer the following questions:

1. How long will it take a team of 20 car washers to wash 100 cars? How long if there are 30 car washers? How long if there are 10 car washers?

2. How long will it take a team of 60 car washers to clean 100 cars?

3. Write a rule or formula for how long it will take a team with any number of car washers to clean 100 cars. Explain your rule or formula.

Figure 9. Carwash problem

This inverse variation situation is most readily modeled through a whole-object strategy. A student could reason that if 40 washers require 2 hours to wash the cars, half as many washers (20) would take twice as long (4 hours) to complete the job. The same reasoning could easily be used to find the number of hours needed for 10, 5, or even 80 washers. However, problems could occur if a student attempted to use whole-object reasoning for an input value that is not a power of two. For example, a student who uses a whole-object strategy to find the number of hours needed when there are 30 washers would have to have a more sophisticated understanding of proportional reasoning. To answer the problem correctly, a student could reason that 30 washers is ¾ of the number that washed the cars in 2 hours. Even this reasoning could prove problematic if the student did not understand how to complete the computation. Given that recursion is problematic, a chunking strategy would prove just as difficult, as the chunking of recursive steps requires prior recursion.

Streets and Lampposts problem. The Streets and Lampposts problem represents a situation in which an explicit rule is more difficult to obtain, due to its quadratic nature. It was hypothesized that such a task would challenge student beliefs that an explicit rule is always attainable and beneficial to develop. The Streets and Lampposts problem is presented in Figure 10.

Streets and Lampposts

In Crazytown every street intersects every other street exactly once. The streets are not necessarily straight. Only two streets meet at any one intersection. The city wishes to erect a lamppost at every intersection.

1. How many lampposts would be needed if there was 1 street in Crazytown? 2 streets? 3 streets? 5 streets? 10 streets? 20 streets? Explain how you determined these.

2. How many lampposts would be needed if there were 42 streets in Crazytown? 138 streets? 139 streets? Explain how you determined these values.

3. Write a rule or formula that would allow you to find the number of lampposts for any number of streets. Explain your rule or formula.

Figure 10. Streets and Lampposts problem

The construction of an explicit formula is likely more difficult due to the variable rate of change that exists as the number of streets increases. For example, the number of posts needed for two, three, and four streets is 1, 3, and 6, respectively. Without a strong visualization of how the streets intersect each other or knowledge of developing quadratic formulas through given values, a student would likely not model this situation explicitly, as the recursive relationship does not provide for the determination of linear slope. As previously noted, the rate of change between successive streets is not the same, making the recursive relationship less apparent than in linear situations. However, the recursive strategy remains accessible to most students. For example, a student who understands why the number of new posts needed for four streets (3) is one more than was needed for three streets (2) could generalize that the number of new posts needed for each successive street is one more than the previous number required. The student might also realize that the number of new posts needed for each new street is one less than the street number. While the recursively added term changes for each iteration, the chunking strategy remains a viable strategy for students working this task. Once a student discovers the recursive pattern, it is likely that she would be able to "chunk" several values together to find the value for a larger input. For example, a student who understands the recursive relationship could take the number of posts required for four streets (6) and add a chunk of 15 (4+5+6) to find the number of posts needed for seven streets. The whole-object strategy is essentially unusable for this task. For example, a student who reasons that eight streets (22 posts) should require twice as many posts as four streets (6 posts) would not produce a correct solution. Any sort of correction would require an advanced understanding of quadratic relationships.

Brick Problem. The Brick problem is a task in which all four strategies are fairly easy to develop. This task is a linear increasing, direct variation situation. The Brick problem is provided in Figure 11.

Brick Problem

Black	Black	Black	Black
Gold	Gold	Gold	Gold
Black	Black	Black	Black

1. a) How many total bricks are needed to make Tiger Path of length 7? 10? 20? 25?

2. How many total bricks are needed for a Tiger Path of length 187? Explain how you found this.

3. Write a rule or a formula to find the total number bricks needed to make a Tiger Path of any length.

Figure 11. Brick problem

The recursive strategy is easily found as the rate of change is constant between successive terms (3 bricks) and the recursive relationship is illustrated in each column of the diagram. Chunking is similarly accessible, as the chunks of 3 follow directly from the recursive relationship. A whole-object strategy will also work as no adjustment is needed for any iteration of values. For example, the number of bricks required for a path of length six (18) is twice the number needed for a path of length three (9). The explicit strategy is also easily deduced as the length of the path multiplied by 3 produces the correct number of bricks; there is no "initial" value to be considered when developing an explicit rule for this situation. Given that all four strategies are accessible to most students, I hypothesized that most students would solve this problem explicitly, due to the utility of an explicit strategy is finding the outputs for large values. The impetus for the inclusion of this task was to draw a contrast between the other linear increasing situations and this one so that students could grapple with the proper use of the whole-object strategy.

Core Task Selection

At the beginning of the study, I created a list of core tasks that would be completed by each student. Given that every student agreed to three 30-minute sessions or two 45-minute sessions, I felt comfortable that each student could complete at least three tasks. As noted earlier in this chapter, the six tasks selected as possibilities for this study were chosen due to their abilities to provoke various strategies and their mathematical structure. The same criteria were used in the selection of the three tasks that comprised the core.

The Theater Seats problem (Figure 7) and the Cube Sticker (Figure 2, p. 10) problem were chosen as core problems due to their propensity to elicit multiple strategies and for their similar mathematical structure (each represented a linear increasing situation that did not vary directly). Prior research (Lannin, Barker, & Townsend, under review-a) that utilized these two problems demonstrated that each was capable of eliciting all four previously documented strategies (explicit, whole-object, chunking, and recursive). Given that one of the foci of this study centered on the strategies that students chose to employ, problems that have been shown to elicit all four strategies were critical to ensuring the accuracy of the results. As previously noted, the Theater Seats and Cube Sticker problems share the trait of being linear increasing situations, a facet that played an important role in determining cross-task flexibility.

The Calling Tree problem was selected due to its different mathematical structure, along with its ability to elicit multiple strategies. Whereas the Theater Seats problem and the Cube Sticker problem represent linear increasing situations, the Calling Tree problem is an exponential increasing situation. By situating this problem between the two linear situations, the students were unable to directly apply their same strategy from one problem to the next. Once again, this helped to preserve the integrity of the measure of cross-task flexibility by providing a mathematical context that required a different approach. The Calling Tree Problem also played an important role in determining strategy use. Student responses in the pilot study demonstrated that this situation elicited multiple strategies.

After the core tasks were administered, the remaining tasks were selected for each student on a case-by-case basis. Depending on the mathematical issues that had either been provoked or resolved, different tasks were used. For example, students who exhibited difficulty explaining why the whole-object strategy did not work while solving the Theater Seats problem or the Cube Sticker problem (i.e., they did not appear to see the impact of the initial value in the problem) were given the Brick problem in an attempt to get the students to grapple with the intricacies of the strategy when used in similar, yet different, mathematical contexts. For example, if a student had difficulty understanding why the whole-object strategy was not applicable to these situations (without a correction), the Brick problem was given in an attempt to offer the students another perspective on the phenomenon. The Brick problem is a linear increasing situation that varies directly, which allows for the whole-object strategy to be applied without adjustment. Responses during the pilot study suggested that this problem could help students grapple with the whole-object strategy.

The final two tasks that were used were the Carwash problem and the Streets and Lampposts problem. To complete either of these tasks required a significant amount of time, generally meaning that students would likely not attempt both tasks. Ultimately, the Carwash problem was given preference due to its proclivity in eliciting the whole-object strategy, as per the pilot study.

Due to the wide range of participant performance levels, variation occurred in the number of tasks completed by each participant. Every participant in the study, except one, completed the three "core" tasks that were used to determine overall strategy use. Once the participants had completed the three core tasks, most were given additional tasks to solve. Decisions surrounding the particular tasks that each participant was given to complete were based on the amount of remaining time available and the particular participant's responses to prior tasks and questions. For example, Dave moved through the core tasks fairly quickly and was able to complete all six tasks within the three sessions. Chrissy, on the other hand, worked meticulously through the problems and was only able to complete the three core tasks and the Carwash problem. Since Gavin seemed to have a good understanding of whole-object strategy and its use, he was not provided the Brick problem. The Brick problem was provided to participants who did not have a good understanding of the whole-object strategy in an effort to stimulate disequilibrium about the viability of the whole-object strategy in different linear increasing situations; the Brick problem does not require an adjustment for doubling as it has no "initial value." A list of tasks completed by each participant in presented in Table 2.

Participant	Core Tasks			Other Tasks		
Adam	Theater	Call	Cube	S & L	Carwash	Brick
Bridgette	Theater	Call	Cube		Carwash	
Chrissy	Theater	Call	Cube		Carwash	
Dave	Theater	Call	Cube	S & L	Carwash	Brick
Elizabeth	Theater	Call	Cube	S & L		Brick
Fran	Theater	Call				
Gavin	Theater	Call	Cube	S & L	Carwash	
Hailey	Theater	Call	Cube			Brick
Iza	Theater	Call	Cube			Brick
John	Theater	Call	Cube			Brick
Karen	Theater	Call	Cube		Carwash	Brick

Theater = Theater Seats problem
Call = Calling Tree problem
Cube = Cube Sticker problem
S & L = Streets and Lampposts problem

Table 2. Tasks completed by participants

Seven of the eight participants at School A attended three interview sessions, with the exception of Fran. Fran declined further participation in the study after time constraints made subsequent participation difficult. As noted in Table 2, 2 of the 7 participants from School A completed all six tasks in the allotted three sessions.

The three participants from School B each completed two interviews. Due to scheduling agreements made with officials at School B, two interviews were scheduled and completed with each participant instead of three. Given the fewer meetings, these participants had less total time to complete the tasks. However, each of the participants from School B completed at least four tasks (Theater Seats, Calling Tree, Cube Sticker, and Brick problems), with one participant also engaging in the Carwash problem.

Alternative Strategies

Alternative strategies were developed for the Theater Seats problem (see Figure 7). The Theater Seats problem was the first task administered to each participant and the alternative strategies for the Theater Seats problem were shown to the students immediately after their work with the task. As previously noted, one alternative student strategy was provided for each of the four generalization strategies. The alternative strategies are provided in Figure 12 in the order presented to the participants: explicit, whole-object, recursive, and chunking.

<u>Abby</u>
It goes up 3 each time and there are 7 seats in the first row, so my rule is 3*N + 7.

<u>Bobby</u>
To find the number of seats in the 40^{th} row, just double the number in the 20^{th} row. There are 64 seats in the 20^{th} row so there are 2 x 64 or 128 seats in the 40^{th} row.

<u>Claire</u>
To find the number of seats in the next row, just add 3 each time. I wrote my rule: NOW + 3 = NEXT.

<u>Danny</u>
For the 13^{th} row, I know there are 34 seats in the 10^{th} row, so I added a total of 9 seats for the next 3 rows.

Figure 12. Alternative Strategies

Each alternative strategy was written from the perspective of another student who had completed the Theater Seats problem. The rationale for introducing these strategies as student generated rules was to minimize the potential for assumed correctness; it was hypothesized that students would be more apt to believe that the rules were correct if they originated from the researcher. By situating them in the context of a potential peer, I hypothesized that the students would be more likely to consider the rule in light of the context before rendering judgment. Given that the students in the pilot study and this dissertation study evaluated some rules as viable and others as not viable, depending on their understanding of the context, the presentation of the rules as student responses seemed to be justified.

While the alternative strategies illustrated in Figure 12 pertain explicitly to the Theater Seats problem, this set of alternative strategies was presented to the students for consideration for each of the six problem situations. The decision was made to use the same set of strategies for a few reasons. First, it was hypothesized that maintaining the same set of strategies would allow students to focus on an individual strategy without having to consider a different presentation of each strategy for each problem. I also thought that using the same set of strategies would force the students to re-conceptualize each strategy for the new situation. This required the students to focus on the strategy itself, instead of the correctness of the answers that the numbers would provide if the strategies were specific to each problem.

Utilizing elements of an active interview, each student was asked to discuss whether or not they believed each fictitious student strategy would work to model a particular situation and justified his or her claim. If a student considered a strategy as viable, he or she was asked to discuss the potential advantages and limitations of using the strategy. If a student did not view a strategy as viable, he or she was queried as to what changes, if any, could be made to the strategy so that it would apply to the particular situation or whether he or she could describe a situation where the strategy would be valid. For example, a student considering Abby's strategy (Figure 12) says that it is not valid. However, after being asked if it could be modified to produce a correct rule, the student notes that it would be correct if the + 7 was changed to + 4. Students were also asked how the strategy compared to other strategies in terms of preference of use and likeness to their own. The questions regarding the viability and utility of strategies provided data to answer research questions two and three by shedding further light into students' views of the strengths and limitations of various strategies. The interview protocol can be found in Appendix A.

Procedure

Each student participated in two or three 45-minute active interviews (Holstein & Gubrium, 1995) conducted by the author. Interviews were chosen as the data collection method due to their ability to "incite the production of meanings that address issues relating to particular research concerns" (Holstein & Gubrium, 1995, p. 17). Active interviews were specifically selected because they "eschew the image of the vessel waiting to be tapped in favor of the notion that the subject's interpretive capabilities must be activated, stimulated, and cultivated" (p. 17). Active interviews focus on conversations with the participants instead of coaxing them into preferred answers.

During each interview session, the participants were asked to generalize a variety of tasks, one at a time. The tasks were purposefully ordered in an attempt to best capture cross-task flexibility. Linear increasing situations were not presented consecutively so that the same strategies would not necessarily apply to successive tasks. The Theater Seats problem, a linear increasing situation, was the first task given to each student. This task was followed by the Calling Tree problem, an exponentially increasing situation and another linear increasing situation, the Cube Sticker problem, to complete the core tasks that every student was to solve. As each participant attempted to generalize various parts of a particular task, the researcher prompted the student for justifications of the various rules and solutions that the student constructed. The data produced from the student solutions for the tasks was utilized to answer research question one concerning student algebraic strategy use.

After participant completion of each task and the subsequent discussion of his or her strategies, I then shared sample student strategies that represented each of the four algebraic generalization strategies (recursive, explicit, whole-object, and chunking). These alternative student strategies (Figure 12, p. 65) were used in an attempt to gauge participant understanding of the generalization strategies, and their accompanying justifications, including those they might not have considered or those they chose not to articulate. Information gleaned from participant work with the alternative student strategies aided the researcher in answering research questions two and three that deal with within-task and cross-task flexibility by providing data on the participants' abilities to understand various strategies for a particular problem situation. Participants were asked questions to determine the following about their understanding of each alternative strategy: (a) whether or not the participant found the strategy to be valid (this included information concerning if and how the student could change the strategy to make it valid for the particular situation), (b) whether or not the participant could develop a rule to model the current task using the particular strategy, (c) whether the participant provided an empirical or contextual justification for their response, (d) advantages and disadvantages of the particular strategy (if they found it to be valid), and (e) if they would use the strategy (if valid). This information was used, along with the strategies used by the students when solving the tasks, to determine student flexibility, as noted in research questions two and three.

Data Sources

Given the nature of the active interview and the potential richness of the available data, several data sources were gathered for this study. Following the recommendations of Rochelle (2000), technological data gathering sources were utilized along with more traditional methods.

Each student interview was captured on both digital and analog audio media so that no verbal cues would be missed from the interactions. The digital audio recorder served as the primary recording device due to the higher quality in recording and preserving the data. Written student work completed during the sessions, along with my observation notes, were collected to provide another layer of accuracy to the data. The observation notes served to document nonverbal cues that might not have been captured on the audio recordings and offered a means for documenting memos for analysis and subsequent interview sessions.

Analysis

Each interview session was transcribed verbatim, with student artifacts and observation notes used for triangulation. A data reduction approach (Miles & Huberman, 1994) was employed to capture the salient mathematical events of each session. This involved the coding of the transcripts with an eye towards generalization strategy (see Figure 5, p. 15). When a particular strategy was noted, the input values, calculations used by the student, and output values were documented. This documentation of student strategies was used to answer research question one, which deals with the generalization strategies employed by student when solving algebraic tasks.

Strategy codes

A chronological mapping of the strategy codes was created for each task. These strategy tables (i.e. within-case displays, Miles & Huberman, 1994) helped to provide a visual representation of the strategies used, and the order in which they occurred, while working through a particular task. An example of a strategy schematic for the Cube Sticker Problem is provided in Table 3.

Cube Sticker			
Strategy Classification	Input Value	Calculations	Output Values
Whole-Object	4	10-2(for the ends)=8 8*2+2=18	18
Recursive	5	18+4=22	22
Whole-Object	10	22-2=20 20*2+2	42
Whole-Object	20	Same	82
Whole-Object	38	40*3+16*2+2	154
Whole-Object	120	80*6+2	482
Recursive	121	482+4	486
Explicit	R	4*N+2	

Table 3: Example strategy schematic for analysis

This particular schematic illustrates the path taken by a student in solving the Cube Sticker problem (Figure 2, p. 10). The student employed a whole-object strategy for a rod of length four by subtracting 2 stickers for the ends of the given rod, doubling the remaining stickers, and adding the ends back on to bring his total to 18 stickers for a rod of length four. After using a recursive strategy of adding 4 to find the value for a rod of length five, the student returned to his whole-object reasoning for values ten, 20, 38, and 120. For the case of a rod of length 121, the student returned to using a recursive strategy. While this schematic lists only the whole-object, recursive, and explicit strategies, the use of the chunking strategy would have been similarly represented in the schematic as well. After the student developed a generalization for the task, the fictitious student strategies that were discussed were documented in a separate schematic. A sample alternative student strategy schematic is provided below in Table 4.

Brick						
Strategy	Valid	Rule	Justify	Explanation	Advantage/Disadvantage	Domain
Explicit	Y	Y	Contex tual	Change +7 to +4	A-Everything D-None	None Stated
Whole-Object	Y	Y	Contex tual	Because 2 rows would be 6 and 4 rows would be 12	D-To find twice of something, you'd have to know what 1/3 or 1/4 of something is.	A lot of the time, if we need to double.
Recursive	Y	Y	Contex tual	It's basically his rule. +3	A- Use if you needed the next row D-is harder because of not multiplying. Would take longer.	Every time
Chunking	Y	N	Contex tual	Exactly like his, except in words (incorrect).		Every time

Table 4. Sample alternative student strategy schematic

The type of strategy, the student's view of the viability of the strategy, whether or not the participant was able to develop a mathematically correct rule representing the particular strategy, the type of justification used to explain the rule (conceptual or empirical), the participant's explanation of the strategy or changes she would make to the strategy (including any rules developed), the advantages and disadvantages noted for each strategy, and the values or times when the participant would use the particular strategy are contained in these tables.

Check coding for two of the eleven students was performed by a mathematics education faculty member to ensure that the coding accurately represented the happenings of particular sessions. No discrepancies were found between my codes and the check-coding.

After all sessions were coded for strategy use and alternative strategy use, within-task and cross-task flexibility were considered. Individual within-task flexibility was determined using only those codes used within a particular task. For example, flexibility for a particular problem was determined by a student's use and understanding of explicit, whole-object, chunking, and recursive strategies for that task. Cross-task flexibility was determined by the strategies utilized by students across the various tasks. For example, consider a student who completed the Theater Seats problem and the Cube Sticker problem.

If the student developed a mathematically correct explicit rule for the Theater Seats Problem, could provide a contextual justification for her rule, and could do the same for the Cube Sticker problem, the student was considered to exhibit a high level of cross-task flexibility for the explicit strategy. The discourse and student justification of how they developed the generalization were important for determining the degree of cross-task flexibility demonstrated by the student. Assisting in this analysis were the diagrams of participant strategy use. When considered alongside the diagrams of alternative strategy understanding, a more accurate and complete description of cross-task flexibility emerged.

Alternative Strategy Criteria

With a goal of determining student flexibility in strategy use, criteria were developed to categorize student strategy understanding. While much of the information used to construct the criteria emerged from the data, the determination of the levels of strategy understanding relied heavily on previous research (Lannin, Barker, & Townsend, under review-a) to establish core tenets of what it means to have high, moderate, and low flexibility for each task. While some of the requirements for a particular rating were dependent upon the task, a set of core elements for high, medium, and low ratings for each strategy were developed to provide consistency across tasks. For example, students who were able to produce contextual justifications for their explicit rules appeared to have better understanding of the intricacies of their rules and where a strategy could be applied. Therefore, contextual justifications represented one of the key factors for a classification of high flexibility, regardless of the task. An outline of the core tenets for determination of participant understanding of the explicit strategy is provided in Figure 13.

High: The student states that an explicit rule exists, provides a correct general rule, and offers a valid contextual justification.

Medium: The student states that an explicit rule exists, provides a correct general rule, and offers an empirical justification.

Low: The student states that an explicit rule exists, but does not provide a correct general rule or does not offer a valid contextual or empirical justification, OR the student states that an explicit rule does not exist.

Figure 13. Explicit strategy criteria

Each strategy for each problem situation was classified by two to three levels in the rubric, representing either high, medium, and low or high and low levels of understanding of the particular strategy within the problem context. The rubrics underwent several iterations to produce consistent descriptions of levels of flexibility by task. The rubrics of criteria for each problem are located in Appendix B.

For some tasks, the participants exhibited more difficulty generating a general rule than for other tasks. For example, finding a general explicit rule to model the Streets and Lampposts or Carwash problem was more difficult than developing one for the Brick problem. This was primarily due to the mathematical structure of this task; the Streets and Lampposts task is a quadratic increasing situation and the Carwash problem is an inverse variation situation, while the Brick problem represents a linear increasing direct variation situation.

Once the criteria were developed, each participant response regarding a particular strategy for a particular task was categorized as high, medium, or low based on the rubric. This initial categorization served as a key component in determining algebraic flexibility.

However, participant strategy use also played a key role in ultimately determining algebraic flexibility, as well as in checking the accuracy of the alternative strategy rubric. In considering algebraic flexibility, the alternative strategies provided a look at participant understanding of different strategies, regardless of the strategies that were used when initially working through the problem situation. However, flexibility of strategy use for a particular situation inherently includes the strategies that the participants employed when initially generalizing the problem situation. In short, both the understanding of alternative strategies and the strategies used in solving a particular problem represent demonstrations of algebraic flexibility.

As previously noted, the categorizations of the participant interpretations of the alternative strategies comprised the initial flexibility framework. The participant-developed strategies were then categorized using the alternative strategy rubrics and checked against the categorizations of the alternative strategies. In the case of a disagreement between the two categorizations, the assessment of the participant's actual work was taken as the more representative example of the participant's understanding of the particular strategy within a particular problem context. For example, consider a participant that received a low classification for the explicit strategy when considering alternative strategies for the Cube Sticker problem. If the same participant correctly employed an explicit strategy when generalizing a task, the participant produced explicit strategy was considered in place of the alternative strategy and was scored using the alternative strategy rubric. The reason behind this decision centered on the participant having a better grasp of the rule that they developed than the rule that they interpreted and changed to represent a particular strategy. The cases representing disagreement were few in number and represented only 6% of the total classifications. For example, Adam received a medium rating for his understanding of the whole-object strategy in terms of the Theater Seats problem under alternative strategy scoring, due to his empirical justification that a whole-object strategy would not work for the task. Given that Adam employed a modified whole-object strategy for when generalizing the task initially, his use of the strategy was judged and compared to his alternative strategy score. In this case, as in most of the cases, the scores were the same. Most of the disagreements were due to either a participant not understanding how the given rule applied to the particular situation or, in a few cases, the participant providing brief responses due to time constraints.

Summary

Eleven tenth-grade students from two rural schools participated in interviews centered on the completion of contextualized algebraic generalization tasks. After the student completed a task and answered questions about how his rules were developed, alternative student strategies were provided to the student for consideration. All interviews were digitally audio recorded and transcribed. The transcripts, researcher observation notes and student work comprised the data analyzed. Open coding, followed by axial coding was performed so that salient categories could emerge. Student strategies were examined and classified for mathematical correctness. Student reactions to alternative strategies for each problem were documented as well. Both the student strategies and the students' understandings of the alternative strategies contributed to their respective levels of flexibility for each problem.

In the next chapter, I provide my interpretations of the collected data, focusing on algebraic strategy use and flexibility.

CHAPTER IV

DATA INTERPRETATIONS

This study documents the generalization strategies used by secondary students as they generalized algebraic problems as well as documents the flexibility of these students in understanding and using algebraic generalization strategies. Algebraic flexibility has two components: (a) within-task flexibility (determination of the viability of various strategies for a particular task), and (b) cross-task flexibility (determination of the applicability of a strategy across tasks). In this chapter I provide an analysis of active interview sessions for which eleven participants generalized algebraic tasks and considered alternative student strategies.

This chapter begins with a look at the algebraic generalization strategies that the participants employed as they were solving the generalization problems, as outlined in research question one: What strategies do secondary students use when generalizing numeric situations and how do they use these strategies? Participant strategy use data were analyzed and described from several perspectives. Participant strategy use by task is described first, followed by a discussion of overall participant strategy use. This section of the findings also includes a description of how the strategies documented in this study align with those noted in the Algebraic Generalization Strategy Framework.

The algebraic generalization strategy section of this chapter concludes with a look at the effectiveness of the participants in using the strategies and strategy use by participant.

To address research question two (To what extent do students exhibit within-task strategic flexibility when generalizing algebraic tasks?) and research question three (To what extent do students exhibit cross-task strategic flexibility when generalizing algebraic situations?), the data sources were analyzed with an eye towards determining participant algebraic flexibility. Algebraic flexibility was analyzed separately by its components, within-task flexibility and cross-task flexibility. The section of this chapter on flexibility begins with a look at within-task flexibility. Individual participant within-task flexibility ratings are discussed for each of the three core tasks, followed by a discussion of overall within-task flexibility for each of the eleven participants. The section concludes with a look at cross-task flexibility. As with within-task flexibility, individual cross-task flexibility ratings are provided for each of the core tasks, with overall cross-task flexibility discussed for each participant.

Participant Strategy Use

Participant work on tasks, transcribed audio recordings of the interview sessions, and researcher notes were used to determine each participant's use of algebraic generalization strategies for a particular task and across the corpus of tasks. More specifically, the participant work on the tasks provided documentation of the particular strategies used for each task. The audio taped transcriptions allowed for the documentation of strategies that might not have been scribed by a participant during the completion of a task, provided information regarding the generality of the strategies provided by the participants, and served as triangulation for the data gleaned from the participant work. The researcher notes offered insight into specific instances where participant work, remarks, or actions were of particular importance. In this section of the analysis, these data provided a look at the initial strategies employed by the participants in generalizing the algebraic tasks.

Overall Participant Strategy Use

When I selected tasks to be used for this study, algebraic generalization situations that harbored the capacity to elicit multiple strategies were preferred. Ultimately, tasks that had been shown to elicit multiple strategies during previous research studies were chosen for the study.

After I coded each participant's interview transcripts for strategy use (utilizing the Generalization Strategy Framework and elements of grounded theory so that new strategies could emerge), participant-by-task tables were created that depicted the strategies that were used, the input values for the particular strategies, information on how the strategies were applied, and the output values resulting from the computations. These tables will be discussed later in this chapter under the section "Individual participant strategy use." Information (i.e., the task, the participant produced strategy, strategy classification, and contextual/numerical determination) from these individual tables were combined in tables that depicted the strategies used, by problem, for this study.

With the exception of the Theater Seats problem, the strategies that the participants produced for the various problem situations were developed after the participants had seen the fictitious student strategies for at least the Theater Seats problem. For example, a participant working through the Cube Sticker problem was previously exposed to the set of alternative strategies twice: once after completing the Theater Seats problem and then again after working the Calling Tree problem. While sharing fictitious student strategies with participants may have impacted the participants' knowledge of the alternative strategies, the data supply little evidence that the impact was substantial. For example, Bridgette employed only a recursive strategy for the Theater Seats problem. After viewing the alternative student strategies, Bridgette continued to use only a recursive strategy for the subsequent problems, except for the Car Wash problem (which is considered as a special case later in this chapter). Dave similarly used two strategies for the Theater Seats problem and continued to use only those strategies throughout his work on the tasks. In other cases where a new strategy did emerge, the factor that seemed to impact the participant's strategy use was inherent to the task. For example, Hailey did not use an explicit strategy for either of the first two tasks. However, when she began work on the Cube Sticker problem, Hailey developed an explicit rule to model the situation before she employed any other strategy. She explained that there were four sides for each cube and there were two sides on the ends. Her ability to visualize the context of the problem seemed to lead her to the development of an explicit strategy for the task.

The following table illustrates how the various strategies were employed for the problems in this study.

Task	Strategy	Total	Percent Used
Theater Seats	Explicit	27	30.3
	Whole-Object	1	1.1
	Chunking	13	14.6
	Recursive	46	51.7
Calling Tree	Explicit	23	20.7
	Whole-Object	0	0.0
	Chunking	13	11.7
	Recursive	75	67.6
Cube Sticker	Explicit	42	45.2
	Whole-Object	12	12.9
	Chunking	4	4.3
	Recursive	35	37.6
Brick	Explicit	38	92.7
	Whole-Object	0	0.0
	Chunking	0	0.0
	Recursive	3	7.3
Streets & Lampposts	Explicit	2	15.4
	Whole-Object	0	0.0
	Chunking	0	0.0
	Recursive	11	84.6
Carwash	Explicit	5	20.0
	Whole-Object	18	72.0
	Chunking	0	0.0
	Recursive	2	8.0

Table 5. Overall participant strategy use

The “Total” column represents the total number of times the strategy was used by the participants in the study while initially generalizing the problem situations. The next column, percent used, denotes the percentage of the total number of strategies used of the particular strategy. For example, an explicit strategy was used around 30% of the time by the participants in this study when solving the Theater Seats problem.

Four Algebraic Generalization Strategies

Although the analysis for documenting participant strategy use was done with an eye for emergent categorical classifications, only the four strategies listed in the Generalization Strategy Framework (Figure 5) are depicted in the table above. All strategies used by the participants in this study could be classified as explicit, whole-object, chunking, or recursive. Thus, the strategies that are used by elementary and middle grades students are the same strategies used by secondary students who have completed a course in algebra.

A few specific cases of potential new categories were noted during the analysis. However, each of the instances was considered individually and found to be representative of one of the four documented strategies, specifically whole-object or chunking. For example, when working to solve the Cube Sticker problem (Figure 2, p. 10), Chrissy began to use what was later realized to be a proportional reasoning strategy to approximate what she thought to be the correct solution. To find the number of stickers needed for a rod of length 38, Chrissy divided the number of stickers for a rod of length 20 (82) by the ratio (20/38). Initially, this strategy was coded as a potential "new" strategy. However, after further analysis, the strategy was marked as "whole-object," after the realization that Chrissy's strategy involved proportional reasoning.

As can be seen in Table 5, the Theater Seats problem, Calling Tree problem, and Cube Sticker problem produced the greatest diversity of strategy use by the students. For each of these problems, the participants used at least three of the four strategies a nontrivial number of times (more than five). For the Theater Seats problem, the students primarily employed recursive, explicit, and chunking reasoning, with only one occurrence of whole-object reasoning. Likewise, for the Calling Tree problem, recursive, explicit, and chunking strategies were used. Regarding the Cube Sticker problem, students mostly used recursive, explicit and whole-object strategies, with the chunking strategy employed only four times by a single student.

While each of these tasks allowed students to use multiple strategies, particular problem situations appeared to encourage the use of particular strategies. For example, students used the recursive strategy three times more than any of the other strategies to generalize the Calling Tree problem. The Calling Tree problem is provided in Figure 8, p. 55. Likewise, the Cube Sticker problem (Figure 2) had the highest percentage of explicit strategies of the three core problems.

The Brick problem, Streets and Lampposts problem, and the Carwash problem all produced one main generalization strategy, with one or two other strategies used minimally. For example, 93% of the strategies used to generalize the Brick problem were explicit, with recursion representing the only other strategy used. While all four of the strategies would have been fairly easy to visualize for the Brick problem, the fact that the participants overwhelmingly employed an explicit strategy would seemingly indicate that it is explicitly preferred, at least at the secondary level. For the Streets and Lampposts problem, participants used a recursive strategy nearly 85% of the time.

Seventy-two percent of the strategies employed when working the Carwash problem (Figure 9) were of the whole-object variety. As the participants attempted to determine the amount of time required for 20 car washers to wash the cars given that 40 washers took 2 hours, they used proportional reasoning to solve the problem. In fact, the participants who succeeded in developing an explicit rule for the situation were only able to do so by using their knowledge of division and guessing and checking with the values that they obtained using the whole-object strategy. Dave explained his reasoning for the explicit rule that he developed. He said, "I knew that you had to somehow divide 30 by 40 or 40 by 30 to get a number and then multiply or divide it by two to get the time. So I just guessed and checked to figure out which one would work right…Dividing is going to make this number bigger."

Effective Strategy Use

The three core problems were considered to determine how effective participants were in using the various strategies. The Theater Seats, Calling Tree, and Cube Sticker problems were used in this analysis due to the fact that they elicited multiple strategies. Given that these tasks provided the most opportunities for the participants to use the available strategies, these problems provided the basis for the effectiveness analysis.

By combining the number of instances in which a participant used a strategy correctly and incorrectly when initially attempting to generalize each of the three core problems, the following table was produced. This table represents correct and incorrect strategy use by strategy type, as well as the percent of the strategies used correctly (% correct).

Strategy	Correct	Incorrect	% Correct
Explicit	55	37	59.8
W/O	9	4	69.2
Chunking	27	3	90.0
Recursive	144	12	92.3

Table 6. Effective Generalization Strategy Use

As can be seen Table 6, not only was the recursive strategy used the most number of times for the three core problem situations, but it was also used most successfully. In looking at the individual participant strategy tables, the recursive strategy was often the first strategy used and was usually used correctly with the smaller values associated with the beginning of the tasks. Participants often did not move to other strategies until they were asked to determine the output values for larger input values and the recursive strategy proved inefficient. For example, when discussing the advantages and disadvantages of the recursive strategy in terms of the Cube Sticker problem, John noted that he knew the recursive strategy would produce the correct answer, due to his extraction of the recursive relation from the context. However, he added the caveat, "but it is going to take forever if you want to find the length for 121."

While exceptions existed to the use of the recursive strategy prior to other strategies (e.g., Adam employed the whole-object strategy for the Cube Sticker problem prior to using a recursive strategy), most participants followed this pattern of strategy use. Participants consistently used recursion when asked to find the output value for a large input value after finding the input value that immediately preceded it. For example, if a participant was asked to find the number of stickers for a rod of length 121, after finding the number of stickers for a length-120 rod, participants would often employ recursive reasoning. Although there were cases where a participant would continue to use her explicit rule, for example, to find the next value, most of the time the participants used the recursive relationship to find this value. For example, in working to find the number of stickers for a rod of length 121 for the Cube Sticker problem, Gavin noted, "I could use the (explicit) rule, but I just figured, well, I am just going to add four [stickers] more again. Because [the length of the rod] is just one ahead of 120." As the preceding statement suggests, participants were likely to arrive at a correct answer when employing a recursive strategy for this reason as well.

The chunking strategy was the strategy that the participants used with the second greatest level of success. While the participants did not use the chunking strategy as often as recursive or explicit strategies, the participants in this study were generally effective at using the strategy when they did employ it. Along with recursion, chunking was the only other strategy used successfully at least 90% of the time.

The explicit strategy represented the least effective strategy used by the participants in this study. As can be seen in Table 6, explicit reasoning was used correctly 60% of the time.

The whole-object strategy was used less frequently than the other strategies for the three core problems. The fact that it was used few times made it difficult to compare its success to the other strategies. However, whole-object reasoning, like explicit reasoning, was not used as successfully as recursive or chunking reasoning.

Individual Participant Strategy Use

The eleven participants of this study fell into four main groups in relation to their strategy use. Each group was characterized by the generalization strategies the participants used to generalize the three core problems. Ultimately, consideration of 5 of the 6 tasks, excluding the Carwash problem, would produce the same results; the strategies that a particular participant used for the three core tasks (Theater Seats problem, Calling Tree problem, and Cube Sticker problem) were not augmented by the strategies that the same participant used for the Streets and Lampposts and the Brick problems. The groups are named as follows: All group; Recursive, Explicit, Chunking (REC) group; Recursive/Explicit group; and Recursive group.

Strategy Use Group	**Participant**
All	Adam, Chrissy
REC	Elizabeth, Fran, Gavin, Hailey, Karen
Recursive/Explicit	Dave, Iza, John
Recursive	Bridgette

Table 7. Participant strategy groups

All Group Strategies

The All group represents participants who employed all four algebraic generalization strategies when generalizing the three core tasks. Adam and Chrissy comprised the All group. Adam used all four strategies in working to solve the Theater Seats Problem. He began by determining the recursive relationship that existed between each row of seats. This information was used to calculate the number of seats for the fifth row from the number of seats in the third row. When asked to find the number of seats for the tenth row, Adam tried to find the answer two different ways. First, he used the recursive relationship that the number of seats increased by three for each row, and multiplied the increase in the number of rows from the fifth row to the tenth row (five) by the number of seats added from one row to the next (three). He then added this total to the number of seats that he found for the fifth row (19), arriving at 34 seats in row 10. Noticing that the movement from the fifth row to the tenth row was a double in terms of rows, Adam thought he might be able to simply double the number of seats in the fifth row (19) to get the number of seats for the tenth row. However, he questioned the validity of this whole-object reasoning when he found that this result (38) conflicted with the answer found when using his previous chunking strategy (34). Although he seemed to believe the chunking strategy provided the correct answer, he returned to the +3 recursive strategy that he used to find the initial values of the problem for assurance that his chunking strategy produced a correct solution. This check seemed to reinforce his belief that his chunking rule provided correct results for the number of seats as evidenced by his subsequent use of this strategy for row 20. However, after using a chunking strategy for row 20, he again returned to his recursive strategy to check the validity of his answer, beginning with seven and adding three for each row until he reached 20. At this point, Adam continued to use a form of this chunking rule to find the all of the values for the rest of the problem. Had he continued to build upon subsequent values, these strategies would have been classified as chunking strategies. Instead, Adam continued to base his values off of the number of seats for the fifth row (19). Given that this point remained fixed for the remainder of his attempts for the Cube Sticker problem, his strategy was considered to be an explicit strategy. Adam's strategy schematics for the core tasks are provided below in Table 8.

Theater Strategy	**In**	**Calcs**	**Out**
Recursive	5	3+3+13	19
Chunking	10	(10-5) *3+19	34
W/O(a)	10	5*2=10, so 19*2=38	38
Recursive	10	+3 to check	34
Chunking	20	(20-5) *3+19	64
Recursive	20	+3 to check	64
Explicit	38	(38-5) *3+19	108
Explicit	138	(138-5) *3+19	418
Explicit	139	(139-5) *3+20	421
Explicit	R	3(x-5) +19	

Call Strategy	**In**	**Calcs**	**Out**
Recursive	1-5	6*2=12, 12*2=24 …	5=96
Chunking	7	96*4= 384	384
Chunking	10	384*3	3072
Chunking	20	(20-10)=10 2^10* value for 10	3.14 m
Chunking	37	37-20 =17 2^17* value for 20	
Explicit	R	2^(x-5) *96	

Cube			
W/O	4	10-2 (ends) =8 8*2+2=18	18
	5		22
W/O	10	22-2=20 20*2+2	42
W/O	20	Same	82
W/O	38	40*3+16 *2+2	154
W/O	120	80*6+2	482
Recursive	121	482+4	486
Explicit	R	4*N+2	

Table 8. Adam's strategy use schematic

The first column of numbers in each schematic represents the input value that the participant was considering. The second column depicts the calculations that the participant used when finding a solution for the particular input value. The last column illustrates the output values that resulted from the calculations.

As can be seen in Table 8, Adam followed a similar pattern when generalizing the Calling Tree problem. After noticing the contextual recursive pattern, Adam successfully employed a recursive strategy to find the solutions to input values one through five. However, when asked to find the number of students contacted during the seventh minute, Adam chunked the differences together to find the answer. As the numbers (and the resulting differences between the output values) became larger, Adam searched for a more efficient means for performing his chunking strategy. For the 20th and 37th minutes, Adam used a chunking strategy that built off of the previous value by multiplying that value by two to the power of the difference between the number he found and the previous number. His desire to not have to multiply his output repeatedly by two caused him to seek out the more efficient chunking strategy, which he ultimately converted into an explicit rule by continually building off of the value for the fifth minute ($2^{x-5} \bullet 96$).

Adam used the whole-object, recursive, and explicit strategies to solve the Cube Sticker problem. Whereas Adam abandoned the whole-object strategy when working the Theater Seats problem due to its role in the production of an incorrect solution, he was able to successfully adjust the whole-object strategy when working to solve the Cube Sticker problem. When he began working the problem, Adam seemed to quickly develop an accurate visual image (Lannin, Barker, Townsend, under review-a) of what was going on in the problem situation. Given this visual image, Adam was able to employ the whole-object strategy to achieve correct results. He explained,

> "I think that every time it doubles in length, it doubles in stickers and so if the length of five equals 22 stickers, you have to take off two for the faces, and then since the length of five doubles to equal the length of ten, the length of ten would have to be, um, take off the two front and back, the length of five which would be twenty, multiply by two for the length of twenty which would be forty, add the two faces on to it, which would be 42…"

Adam's visual image allowed him to effectively use the whole-object strategy to find the solutions for various input values for the Cube Sticker problem. For example, to determine the number of stickers for a length-38 rod, he broke 38 into three groups of 10 and two groups of 4, applying the whole-object strategy to these four segments. He then added the five pieces together along with the two ends. Adam offered the following description of his actions:

> I think that if you can multiply halves of a number to find a whole number, then you can multiply thirds to get a whole number, and so to find the 38th [row, I] divided [38] by ten [which] is three and four-fifths… The number of [stickers for a rod of length] ten, subtracted by two and multiplied by three (gives the number of stickers for 30 cubes) and then the eight you divide into fours and subtract two, multiplied by two…and then I added the…ends on.

Adam used similar reasoning to find the number of stickers for a rod of length 120. He then used recursion to find the value for length 121. Adam's explanation of his thinking is provided below.

> Okay, so 120 divided by 20 is 6 and if…we know what the size of 20 is and subtract the ends, 20 would equal 80. So 80 times 6 is 480 plus the two ends would be 482. [For] 121, since you only have to add 4 more sides on to each cube, you add it on, and then 120 is 482 plus 4 is 486.

Adam broke the length-120 rod into six rods of length 20. He then mentally removed the ends of the length-20 rods and multiplied 20 by 4, producing the number of stickers for the sides of each length-20 rod (80). Since he had six length-20 rods, he multiplied the number of stickers on a length-20 rod (80) by 6, and finished his calculation by adding the two end stickers back onto the total. Adam's visual image of the problem situation led him to recognize that ends of the smaller rods had to be removed before iterations of the rods could be put together. This understanding allowed Adam to correctly apply whole-object reasoning to various values in the task.

Chrissy was also included in the All group. Chrissy did not use all four strategies during any one problem, but all four strategies were used over the course of the problems. Three of the four strategies (explicit, chunking, and explicit) were used in both the Theater Problem and the Calling Tree Problem. It should be noted that Chrissy exhibited considerable difficulty using any of the four strategies effectively. In the case of the Theater Seats problem, most of her incorrect answers seemed to be a result of focusing on numeric relationships instead of the context of the situation. For example, after correctly determining the number of seats for rows four, five, ten, and 20 using a recursive relationship that Chrissy developed from the context, she applied a chunking strategy to find the number of seats for row 138. While her initial thinking seemed to be rooted in the context (she took 138-20 to give her the number of rows that needed to be added to the 20^{th} row and multiplied that by three because there was an increase of three seats for every row), she soon seemed to leave, or at least misunderstand, the context in that she failed to add her resulting number of seats to her prior total. The resulting answer was, therefore, off by the number of seats in the 20^{th} row. After realizing that her rule didn't produce the correct answer, Chrissy began trying to find a pattern in the numbers that would produce the correct answer. Eventually, she did find a rule that would work ($(n+1)*3+1$, where n is the row number), but seemed to forget the intricacies of her rule (adding 1 to the row number before multiplying) later when asked to find the number of seats in the 300^{th} row. Once again, this certainly could have been due to her rule not being anchored in the context of the problem.

Similar instances occurred where attempted Chrissy to find patterns in the numbers, remained disconnected from the context, and produced rules that ultimately did not generalize within a problem situation. Chrissy's strategy schematics are provided in Table 9.

Theater Strategy	In put	Calculations	Out put
Recursive	4	+3	16
Recursive	5	+3	19
Recursive	10	+3	34
Recursive	20	+3	63
Chunking	138	138-20=118 118*3=354	354
Explicit	138	138*3+3	417
Explicit	137	138*3+1	
Explicit	138	139*3+1	418
Explicit	139	140*3+1	421
Explicit	R	(n+1)*3+1	All
Explicit	300	n*3+1	901

Call Strategy	In put	Calculations	Out put
Explicit	5	5*6=30	30
Recursive	2	6*2	12
Recursive	3	12*2	24
Recursive	3	12+6	18
Explicit	3	6*3	18
Recursive	4	18+6	24
Recursive	5	24+6	30
Recursive	6- 10	+6	
Recursive	3	12*2	24
Recursive	4	24*2	48
Recursive	5	48*2	96
Recursive	6- 10	*2	
Recursive	R	Now*2=Next	All
Chunking	20	(20-10)*10 (a)	

Cube			
Recursive , W/O	4	add a block, add 5 Since 10 for length 2, double it.	
Recursive	5	adding 5 each time	25
	3	counts	14
Recursive	4	+5	19
Recursive	4	+5-1	18
Recursive	5	-1+5	22
Recursive	10	-1+5 (rep)	42
W/O	8	value for 4 *2-2	34
W/O	10	value of 5*2- 2	42
W/O	20	value of 10*2-2	82
W/O	38	(20/38)=.526 (82/.52)= 157.69	158
W/O	120	Same	510
W/O	2	Same	22.5

Table 9. Chrissy's strategy schematics

Recursive, Explicit, Chunking Group Strategies

The Recursive, Explicit, Chunking (REC) group denotes participants who used recursive, chunking, and explicit strategies to generalize the tasks provided in this study. This group was the largest group with Karen, Elizabeth, Fran, Gavin, and Hailey categorized as members. The two participants in this group who worked the Carwash Problem employed whole-object reasoning for that task. However, they did not use the whole-object strategy for any of the other tasks.

Elizabeth typified the strategies used by the REC group. Elizabeth, much like Adam in the Calling Tree problem, used a recursive strategy initially for the Theater Seats problem when provided small input values. She developed a chunking strategy from her recursive rule that ultimately resulted in an explicit strategy [$(n - 1) * 3 + 7$, where n is the row number] based on the value for the first row of the theater. Elizabeth attempted to replicate this line of thinking in the Calling Tree problem, but was unable to develop a chunking rule to make finding the values for larger numbers more efficient. Although she knew that she needed to multiply her answer for the seventh minute by 2 thirty times to find the number of students contacted in the 37th minute, she was unable to transfer that knowledge into an efficient chunking rule, and ultimately, an explicit rule. This appeared to have been due to her lack of understanding of exponents.

Theater Strategy	In put	Calculations	Out put
Recursive	4	+3	16
Recursive	5	+3	19
Recursive	10	+3	34
Recursive	20	+3	61
Chunking	38	38-20=18, 61+(3*18) = 115	115
Chunking	138	138-38=100, (100*3)+115 = 415	415
Chunking	Rule	(row-row you know)*3+ # of seats in that row	
Explicit	Rule	(row-1)*3+7	

Call Strategy	In put	Calculations	Out put
	1		6
Recursive	2	(after discussion) 6*2	12
Recursive	3	12*2	24
Recursive	4	24*2	48
Recursive	5	48*2	96
Recursive	6	96*2	
Recursive	7	192*2	384
Recursive	37	37-7=30 - number of times she has to multiply by 2	

Table 10. Elizabeth's Theater Seats and Calling Tree strategy schematics

Gavin also provided an example of the REC group. Similar to Elizabeth, Gavin began the Theater Seats problem with a recursive strategy rooted in the context, followed by a chunking rule for the 20th row as he sought a more efficient strategy. His chunking strategy quickly gave way to an explicit rule [(n - 1) *3 + 7, where n is the row number] that centered on the number of seats in the first row, once again similar to Elizabeth. While Gavin noted that he would use his explicit rule for most values, he returned to his recursive rule to find the number of seats for the 139th row, as it was only one row away from a previously calculated row (138).

Theater Strategy	Input	Calculations	Output
Recursive	4	+3	16
Recursive	5-7	+3	
Recursive	10	+3	34
Chunking	20	(20-10)*3 to see how many seats to add. +34	64
Explicit	138	(138-1)*3 +7	418
Recursive	139	+3	421
Explicit	Rule	(#-1)*3+7	All

Table 11. Gavin's Theater Seats strategy schematic

Recursive/Explicit Group Strategies

The Recursive/Explicit group included participants who used only recursive and explicit strategies, outside of the Carwash problem. Three participants, Iza, John, and Dave, comprised this group. The difference between the Recursive/Explicit group and the REC group centered on the Recursive/Explicit participants' lack of use of the chunking strategy. Not using a chunking strategy, however, did not appear to benefit or hinder the participants of the Recursive/Explicit group. In fact, John and Dave performed well without the use of a chunking strategy, while Iza struggled with moving from successful recursive reasoning to correct explicit reasoning. For example in the Theater Seats problem, Dave was able to make the jump from finding the number of seats for the first few rows using a recursive strategy to employing an explicit rule for the rest of the input values. His ability to develop an explicit rule seemed to be aided by his connection to the context. Dave noted, "So then I made the equation, because you added three to every row, but the first row. Then you multiply…So then you have to subtract one…"

Dave also developed an explicit rule ($4c + 2 = s$, where c is the number of cubes in the rod and s is the number of stickers needed) for the Cube Sticker problem, due to his accurate visual image of the problem situation. Describing how he constructed his rule, he said, “Cause there is four sides exposed on every one and then there is two on each end that you are going to have to add.” This visual image also allowed him to utilize the recursive relationship to find the value for a rod of length 121 when he knew the value for length 120.

Cube Strategy	Input	Calculations	Output
Explicit	Rule	4c+2=s	All
Explicit	120	4(120)+2	482
Recursive	121	+4	486

Table 12. Dave’s Cube Sticker strategy schematic

Iza, on the other hand, drew little connection between the recursive relationship and her quest for an explicit rule. For example, after using a recursive relationship to find the values for the second and third minutes in the Calling Tree problem, Iza mistakenly believed that she could multiply the input value by six to attain the number of people contacted during that minute. This assertion was not supported by the context and the relation only provided an accurate empirical value (6 people contacted) for the first minute. Although she later recognized this as an error and returned to her recursive strategy, she repeated this behavior when working on the Cube Sticker problem. Presented as a given in the context of the problem, a length-two rod has ten stickers. Seeing this as a direct variation situation involving a multiplication by 5, Iza mistakenly applied this rule to the entire domain. It was not until she was asked to physically draw and count length-three and length-four rods that she realized the recursive relationship existed. Ultimately, she used the recursive strategy to calculate the number of stickers needed for rods of length 1 through 10.

Cube Strategy	**Input**	**Calculations**	**Out put**
Explicit	Rule(a)	Length*5	All
Explicit	4	4*5	20
Explicit	5	*5	25
Explicit	10	*5	50
Explicit	20	*5	100
Explicit	38	*5	190
Explicit	3	*5	15
Explicit	3	counted &*5-1	14
	4	Counted	18
	1		6
Recursive	5	+4	22
Recursive	Rule	+4	
Recursive	1-10	+4	

Table 13. Iza's Cube Sticker strategy schematic

Recursive Preferred Group

The final "group," the Recursive preferred group, might be considered an outlier as there is only one participant, Bridgette, in this group. Regardless of the problem situation, Bridgette used only a recursive strategy to determine output values for each input value in the problem situation. Larger input values did not prompt her to use another strategy. In each problem that she attempted, outside of the Carwash problem, Bridgette recognized the recursive relationship that existed between consecutive values and used it extensively. For example, when pressed as to how she would find the 139th row in the Theater Seats problem, Bridgette reasoned, "Um, you divide them, but I don't know by what. If this was a test, I would probably divide them by three just so I would have an answer, because I don't know." It appeared that outside of recursion, arbitrary number operations represented her only alternative strategy.

Theater Strategy	**In put**	**Calcula tions**	**Out put**
Recursive	4	+3	15/16
Recursive	5	+3	18
Recursive	10	+3	34
Recursive	20	+3	

Call Strategy	**In put**	**Calcula tions**	**Out put**
	1		6
Recursive	2	6*2	12
Recursive	5	12*2=24, 24*2=48 48*2=96	96
Recursive	7	96*2*2	384
Recursive	10	384*2*2*2	307 2
Recursive	37	Same	

Cube			
Recursive	4	+5	20
	1	Counted	6
	3	Counted	14
	4		18
Recursive	5	+4	22
Recursive	10	+4	42
Recursive	20	+4	82
Recursive	38	+4	152
Recursive	Rule	Add 4	

Table 14. Bridgette's strategy schematics

Algebraic Flexibility

Another important element of this study centers on determining the algebraic flexibility of secondary students. As noted in Chapter I, the mathematics classrooms advocated by NCTM (2000) center on mathematical discourse from multiple perspectives. A critical element of a student's algebraic flexibility, and ultimately their respective success, is that student's ability to consider, make sense of, and determine the usefulness of other strategies. Given these reasons, it was important for the participants in this study to consider and grapple with other potential strategies to determine their ability to understand, use, and value them. The participants' understandings of these strategies, along with their respective strategy use, for each task determined their algebraic flexibility.

Algebraic flexibility is discussed by its parts: within-task flexibility and cross-task flexibility. Within-task flexibility refers to a student's understanding of the applicability of the four algebraic generalization strategies for a particular problem. Cross-task flexibility entails a student's understanding of the applicability of a particular generalization strategy to various algebraic tasks. This section details the within-task flexibility and cross-task flexibility of the participants in this study.

Determination of Flexibility

Transcribed audio recordings of the interview sessions, researcher notes, and participant work on tasks were used to ascertain each participant's understanding of the various algebraic generalization strategies for a particular task and across the corpus of tasks. More specifically, the audio taped transcriptions allowed for the documentation of participant responses to questions regarding the viability, utility, and nature and generality of alternative student strategies. My notes provided additional insight into specific instances where participant work, remarks, or actions were of particular importance. The participant work served to document the participant-developed strategies for the particular tasks, which provided triangulation for the participant work with alternative participant strategies.

The theoretical framework for within-task flexibility (Figure 3, p. 12**)** illustrates the importance of ascertaining the strategies that a student believes accurately model a particular situation, as well as those that the student thinks do not provide a correct generalization for the task. Alternative student strategies prompted the participants of this study to grapple with four algebraic generalization strategies, even when they had not considered each strategy when initially solving the problems.

Alternative Student Strategies

As mentioned in Chapter III, after each participant finished working through each of the problem situations, they were given a set of alternative student strategies (Figure 12, p. 65) representing each of the four strategy types to consider, based on the Theater Seats problem. Each participant was asked whether or not they thought something like the "student's rule" provided as an alternative strategy would work for the particular situation that they had just completed working. The participants were also asked if they could develop a particular rule for each strategy, the advantages and disadvantages to each of the strategies that they thought would work for a particular situation, and were always asked to explain their reasoning.

Within-Task Flexibility

As previously noted, each participant's understanding of each strategy in the context of each task was assessed and categorized using the student strategy rubrics. The following represents Karen's strategy classifications for the core tasks that she completed.

Task	Explicit	W/O	Recursive	Chunking	Within
Cube Sticker	M	M	M	H	M
Calling Tree	L	H	H	M	M
Theater	H	L	H	H	H

Table 15. Karen's strategy ratings

To demonstrate how these scores were determined, the following section provides examples of high, medium, and low classifications for Karen, illustrated through excerpts from the interview transcripts. Each example references the strategy rubric qualifications for determining each instance. The strategy criteria can be found in Appendix B.

For the Cube Sticker problem, Karen received a medium score for her understanding of the explicit strategy due to her ability to develop a correct explicit rule and her numeric justification. Her development of a rule was fairly easy to gauge. However, the justification of the elements of her rule proved much more difficult. To illustrate the process involved in determining Karen's understanding of the explicit strategy for the Cube Sticker problem, several vignettes are needed. The first excerpt is from the discussion of the alternative explicit strategy regarding the Cube Sticker problem. This excerpt demonstrates her ability to produce a mathematically correct rule. For more information regarding the explanation she is considering, see Figure 12 (p. 65) for the Alternative Strategies.

Brian: So do you think that there is a rule like Abby's [explicit rule] that applies to this situation?

Karen: Yeah, cause it is kind of the same thing [as her rule], it is only written backwards.

Brian: Okay.

Karen: Cause like 3 is where they would put 4, *n* is where the number of boxes added [to length one] would be…and instead of 7 you would put 6 and it would basically be the same thing.

Brian: Basically be the same thing?

Karen: Yeah

Brian: Okay, so Abby's is basically like your last rule there?

Karen: Yes, my last rule.

Karen was able to see the alternative explicit rule as similar to the chunking-turned-explicit rule [6 + 4 (number of boxes added)] that she used to calculate several values. However, this vignette does not provide information about the nature of her understanding of the rule. A return to her development of the rule is needed to determine whether she provided a contextual or numeric justification. A vignette focused on her development of this rule is provided below.

Karen: Yes, so you have 6 the first time and then you add 1 box, 4 times 1, would give you 6 times 4 and that would equal 10.

Brian: Oh, so this is even a different rule.

Karen: Well, it is, but it is kind of what I did the first time, only I started with 2 instead of 1, cause I knew 2.

Brian: So to find the length, the number of stickers for length 10, what would you plug in there?

Karen: 6 plus 4 times 9

Brian: Okay, and is that always going to work?

Karen: Yeah, it will.

This exchange provides context for her understanding of the alternative explicit rule. However, Karen did not explain why she added multiples of 4 to her initial value. In order to ascertain her justification for that element of her rule, I looked to a discussion on her understanding of the recursive "plus 4" relationship. A dialogue centering on her development of a recursive rule is provided to demonstrate her view of the 4 in her rule.

Karen: You add on 4.

Brian: How does that work out, where are the 4 coming from?

Karen: The 4 are coming from the 4 sides that you added.

Brian: The 4, which 4 are they? So like if I add on, if I just added on this block to the first one, where are the 4 sides that I added?

Karen: I don't know now. Because like I know that you add them together, and so like you have that shape and that shape and you put them together, you take away those 2 sides and that is how we got 10.

Brian: Okay,

Karen: But I don't know like which ones.

Brian: Okay, but do you think that it adds 4 each time?

Karen: Yes.

Although Karen could see that 4 stickers were added for each additional length, she was unable to pinpoint these stickers within the problem context. In terms of generalization, this is problematic in that this lack of knowledge prohibits her from understanding why these four stickers should continue to be added.

In line with the strategy rubric, her ability to generate a correct explicit rule and to justify the rule numerically results in a medium score for flexibility in terms of the Cube Sticker Problem.

Determination of Karen's understanding of the whole-object strategy in terms of the Cube Sticker problem was more straight-forward. For the whole-object strategy, Karen received a medium rating due to her numeric justification that the whole-object strategy does not work for this situation.

Brian: Would a rule like Bobby's work here?

Karen: No, because you don't double it. You just add 4 each time, nothing doubles.

Brian: Okay, so like from the 10^{th} to the 20^{th} length, it doesn't double?

Karen: Well, no it doesn't.

Brian: And why don't you think it works to double?

Karen: Because you are adding 4 each time, it is not, like doubling. [It] is going up the same amount, but each time it has to increase.

Karen could see that the whole-object strategy did not work for the problem situation. However, her reasoning seemed to center on the empirical evidence that doubling did not produce the correct answer. Her justification that adding four is not the same as doubling seemed to confirm that she did not have a good understanding of why the situation did not allow for a whole-object doubling.

For the Calling Tree problem, Karen received a low rating for her understanding of the explicit strategy. While she initially thought that an explicit rule existed for the situation, she never developed an explicit rule and ultimately claimed that such a rule did not exist for this problem. The following transcript illustrates her thinking.

Brian: So do you think that there is a rule like Abby's that applies to the calling tree situation?

Karen: Yes.

Brian Okay.

Karen: Maybe.

Brian: And why do you think that?

Karen: Because it has got to increase each time, but I don't think it would work exactly because it doesn't increase by the same amount each time.

Brian: I see.

Karen: Like it doubles, but you can't …multiply the same thing by a number because it changes each time.

Brian: Okay, so you think that something like Abby's would work or something like Abby's won't work?

Karen: It won't work, I don't think.

Brian: Okay. Because that goes up steadily each time?

Karen: Yeah, and this one doesn't.

Karen did not think that an explicit rule could model the change that occurs in this exponential situation. This belief kept her from producing an explicit rule and resulted in her low rating for the explicit strategy in the Calling Tree problem.

A high rating was determined for Karen's understanding of the recursive strategy for the Theater Seats problem. She produced a correct general rule to model the situation and provide a valid contextual justification for the rule. As illustrated in the transcription below, Karen succeeded in doing both.

Brian: Okay, how about Claire's [recursive] rule?

Karen: Um, like it would but the first one, like you have 7 seats plus 3, but then like it really couldn't tell you, I mean, you would have to go up by each increment to find it. You couldn't like skip around.

Brian: Okay, so you think that it would work?

Karen: Yeah, but it would just take longer.

Brian: Take longer.

Karen: Cause you could put like 7 plus 3 and then you know it would equal 10, but then you couldn't skip from like the 1st to like the 13th row, cause you wouldn't know what you had for like the 12th row.

Brian: But for what numbers do you think that it would work, Claire's rule?

Karen: Um

Brian: Will it work for anything? I mean,

Karen: It would, probably it would only work on like the 1st one and then, I don't know.

Brian: What if I wanted to find the 20^{th} row, could I use Claire's rule?

Karen: You could, but it would take you a long time.

Brian: Okay, so it would work for the 20^{th} row.

Karen: Yeah, it is just not the fastest.

While Karen seemed somewhat ambivalent about the use of a recursive rule for this situation, her responses seem to indicate that she was concerned about the amount of time it would take to use the rule for larger numbers, and not the ability of the rule to generate correct values. However, this vignette did not provide information regarding her understanding of the nature of this rule. To determine this facet, I looked at her use of the recursive strategy during her initial solving of the problem. In this vignette, Karen refers to the incremental increase in the context of the problem.

Brian: Okay for the 4^{th} row, you have 16. Fifth row you have 19. How did you find those?

Karen: I just added 3 to 13.

Brian: Okay, and why did you do that?

Karen: Because it says there that it increases by the same amount for every row and I counted that it increases by 3 each row.

While the recursive relationship was easily deduced, due to its reference in the problem context, Karen effectively used this information to ground her understanding of the recursive relationship. This understanding allowed her to achieve a high rating in terms of the recursive strategy for the Theater Seats problem.

Overall Within-Task Flexibility by Task

As demonstrated in the previous section, each participant received an individual score for his or her understanding of each strategy in terms of each task. While each of these ratings provide important information for teachers trying to improve their students' understanding of algebra, overall individual student flexibility for a particular task provides a measure of how well a student can algebraically model the problem situation. This measure of a student's understanding of the applicability of the four generalization strategies to a particular task is within-task flexibility (see Figure 3, p. 12). To determine overall within-task flexibility for a particular task**,** all strategies considered for a particular task were considered together. The following classification guide (Table 16) was developed to classify within-task flexibility for each task.

Four Independent Strategy Classifications				**Result**
H	H	H	H	H
H	H	H	M	H
H	H	H	L	H
H	H	M	M	M
H	H	M	L	M
M	M	M	M	M
H	M	M	M	M
H	M	M	L	M
H	H	L	L	L
H	M	L	L	L
M	M	L	L	L
H	L	L	L	L
M	L	L	L	L
L	L	L	L	L

Table 16. Classification Guide for within-task flexibility by task

Discussion of Within-Task Flexibility Ratings

Although each individual strategy rating for each problem provides information about how well a student understands each strategy in terms of the particular problem context, research question number two necessitated that an overall rating be determined so that task-specific flexibility could be discussed and comparisons could be made across the students in terms of within-task flexibility.

To achieve such a rating, task-specific flexibility was determined for each participant for each task. This rating represents a composite of the four ratings given for each of the strategies for a particular task. If a participant received three or four of the same rating for each strategy, then their overall flexibility for that particular task was classified with the rating that included the three or four individual ratings (e.g., medium, medium, high, medium was classified as medium). However, cases where a participant had only two ratings the same also had to be considered. For example, a participant who received two low scores, one medium, and one high score represents a less clear picture of task-specific flexibility. To determine the within-task flexibility rating in these cases, the core tenets of what it means to be flexible were considered. Ultimately, I decided that a participant needed at least three of a particular rating to be considered in that category. For example, the participant described above would have been considered to exhibit "low" task-specific flexibility for the particular task. A participant that received a low, medium, and two high ratings would be considered "moderately" flexible for a particular problem, given that the participant received three scores of *at least* medium.

Using Table 16, each participant was classified as exhibiting high, medium, or low flexibility for each of the problem situations. Table 17 depicts the ratings of a few sample participants by strategy and by task and includes the within-task flexibility classification.

Task	Participant	Explicit	W/O	Recursive	Chunking	Within
Cube Sticker	Dave	H	H	H	H	H
Calling Tree	Dave	M	H	H	H	H
Theater	Dave	H	H	H	H	H
Cube Sticker	Hailey	H	H	L	L	L
Calling Tree	Hailey	L	H	H	L	L
Theater	Hailey	H	H	H	H	H
Cube Sticker	Iza	L	M	H	M	M
Calling Tree	Iza	L	L	H	M	L
Theater	Iza	L	L	H	M	L

Table 17. Participant flexibility ratings by strategy and within-task flexibility by task

Table 17 provides an example of the ratings of within-task flexibility for each strategy that were developed for each participant. When these within-task participant strategy ratings were combined by task, a picture of the within-task flexibility for each task emerged. Table 18 below represents within-task flexibility by task.

Theater Seats Flexibility	
High	Adam, Dave, Elizabeth, Gavin, Hailey, Karen
Medium	Chrissy, Fran, John
Low	Bridgette, Iza

Cube Sticker Flexibility	
High	Adam, Dave, Gavin
Medium	Bridgette, Elizabeth, Hailey, Iza, John, Karen
Low	Chrissy

Calling Tree Flexibility	
High	Adam, Dave, Fran, Gavin
Medium	Chrissy, Karen
Low	Elizabeth, Hailey, Iza, John

Brick Flexibility	
High	Adam, Dave, Elizabeth, Hailey, Iza, John, Karen
Medium	
Low	

Carwash Flexibility	
High	Adam, Dave
Medium	
Low	Karen

Streets & Lampposts Flexibility	
High	Dave, Gavin
Medium	Elizabeth
Low	

Table 18. Within-Task flexibility

As can be seen in Table 18, particular participants scored consistently high, regardless of the task. In fact, Adam, Dave, and Gavin were classified as highly flexible for every task that they attempted. These participants are characterized by their abilities to correctly identify the applicability of the generalization strategies, to generate correct rules to model the situations for each applicable strategy, and to provide contextual justifications for their rules.

The ability to determine the applicability of each of the generalization strategies for each of the problems provided these participants with a strong foundation for attacking algebraic problems in a classroom environment. These participants had the capacity to consider several strategies and pick out those that provided the best solutions for the particular instance.

The ability to generate a correct rule for each of the applicable strategies allowed these participants to move from the consideration of potential strategies to adapting these strategies to accurately model a new situation. The participants demonstrated an ability to modify a strategy considered in a different context to fit the constraints of a new problem. This knowledge should provide these participants with the ability to apply their understanding of the various strategies to various contexts presented in the classroom or outside world.

The ability of Adam, Dave, and Gavin to consistently provide contextual justifications for their algebraic generalizations implies that these participants have a strong understanding of the generality of their rules. In describing how various facets of their rules related to the context of the problem, these participants appeared to see not only the domain for their particular rules, but also the reason that their rules would apply to that domain. These participants understood how their rules modeled the facets of the situation that changed and those that stayed the same, and they understood why this pattern would continue. For example, Adam provided the following justification for his development of an explicit rule for the Cube Sticker problem.

Adam: Alright…. I think I have it. Alright. You would take 4 times the number of lengths that you have and add 2.

Brian: Now how did you figure that out? How did you figure out 4 times the length times 2?

Adam: Well, I used, I just took off the ends completely, and I am thinking that if you can have a length of 2 equals 8 sides not including the ends, and 1 equals 4, and the other equals 8, and the 3rd one equals 12, without the ends, then it would be like multiplying 4 by 4 by 4 and so in essence it would only be multiplying 4 by the number of lengths you were trying to figure out in order to find the stickers.

Brian: Okay, so the 4 is represented by…

Adam: The lengths of sides you have for one cube, without the ends.

Brian: Without the ends. And the 2?

Adam: It would be with the [number of sides on the] ends.

Adam's understanding of his $4n + 2$ rule stemmed directly from his view of the context. He could see that each additional cube added to the rod added four new sides that required stickers. As he noted, this was represented in his rule by the multiplication of 4. Using this rule and his understanding of the context, Adam appeared to have generalized the task, in terms of the explicit strategy, at a high level. Dave and Gavin exhibited similar capabilities and understandings with respect to the Cube Sticker problem. In working to answer research question two, this finding represents a good start towards determining the group of participants who could be considered highly flexible.

None of the other participants displayed consistent understandings of the strategies across the tasks. Even if only the three core tasks are considered, none of the remaining participants were scored consistently as high, medium or low in terms of their overall use and understanding of the four generalization strategies for the tasks. Specifically, Hailey appeared highly, moderately, and lowly flexible, depending on the particular situation. Hailey is discussed later as a special case.

As was the case with initial strategy use, the Brick problem was accessible to all participants in terms of flexibility. Every participant in the study was able to use and understand all four generalization strategies at a high level for this task. Given the low level of difficulty presented by this task, it was not surprising to find that all of the participants could use and understand all four strategies for this problem.

Overall Within-Task Flexibility

To develop a system to meaningfully group these participants in terms of flexibility, I considered the variation of within-task flexibility along with other factors, including the reality that not every participant had the opportunity to work every problem. I detail the development of the overall within-task flexibility below.

As described in Chapter III, a total of six problems were used with the participants in this study. However, given that each participant worked through the problems at a different pace, not every participant had the opportunity to complete each of the six tasks in the three allotted research sessions. Therefore, it would be difficult to accurately compare and group participants in terms of overall within-task flexibility, and, later, cross-task flexibility, when participants completed different numbers of tasks or different tasks. To alleviate this difficulty, I considered those tasks that were completed by most of the participants. The Streets and Lampposts problem and the Carwash problem were eliminated, leaving the Cube Sticker problem, the Calling Tree problem, the Theater Seats problem, and the Brick problem for consideration.

The Brick problem represented another difficulty in that it ultimately did not prove overall problematic to any of the participants in the study. Whether it was the nature of the problem (it was the only direct variation problem in the set) or the fact that the alternative strategies were rather easily deduced given the numerical similarities shared by the Brick problem and the Theater Seats problem, on which the alternative strategies were based. Consequently, every participant was able to understand each of the strategies at a high level within the context of the Brick problem. Therefore, this problem provided little discrimination among levels of flexibility. Ultimately, the Brick problem was removed from consideration in determining overall within-task flexibility. However, it was analyzed to determine within-task flexibility for that particular problem. This analysis yielded little information outside of the fact that every participant displayed a high level of within-task flexibility for the Brick problem.

After the removal of the Brick problem for consideration of overall within-task flexibility, three tasks remained for use in making the determination: the Cube Sticker problem, the Calling Tree problem, and the Theater Seats problem. As previously noted in this chapter, these problems comprised the “core tasks” that every participant completed. These tasks represented the problems with the greatest capacity to elicit all four generalization strategies. The Cube Sticker problem and the Theater Seats problem are examples of linear increasing situations that do not vary directly. The Calling Tree problem is exponential in nature. The overall rankings of each participant’s use of the various strategies for these three problems were used to determine overall within-task flexibility for each participant.

Whereas the scheme used to determine the overall within-task flexibility for each task considered four separate classifications, the determination of overall within-task flexibility for the participant involved only three classifications (one overall task classification for each of the three tasks). In a sense, having to consider only three classifications made decisions regarding the determination of overall within-task flexibility somewhat easier; two or three high, medium, or low rankings resulted in the same high, medium, or low overall participant classification. If a participant received at least two ratings that were the same for the three problems, it made sense to provide the participant with an overall score consistent with this majority. A participant scoring one each of high, medium, and low was considered to exhibit a medium level of overall within-task flexibility. With no consistency between the ratings, the average rating was given. These classifications covered each of the participants who had completed the three tasks under consideration. However, two cases existed where the particular participant had only completed two of the tasks. In each of the cases, the participant's scores were only one classification apart; one had a low/medium combination while the other had been scored medium/high. For both cases, I returned to the transcripts to see if a holistic, overall within-task flexibility could be ascertained. In each case, I determined that the participant should be classified as the lower of the two rankings, given their overall understandings of the strategies for the two problems.

The table below illustrates the results of overall within-task flexibility classification.

Overall Within-Task Flexibility	
Classification	**Participant**
High	Adam, Dave, Gavin
Medium	Chrissy, Elizabeth, Fran, John, Karen
Low	Bridgette, Hailey, Iza

Table 19. Overall within-task flexibility

Overall high within-task flexibility. Adam, Dave, and Gavin were classified as exhibiting a high level of within-task flexibility for the three core tasks. A high within-task flexibility rating meant that these four participants demonstrated the ability to employ at least three out of the four algebraic generalization strategies at a high level for at least 2 of the 3 core problem situations. In fact, Adam, Dave, and Gavin were judged to have exhibited high levels of within-task flexibility for the three core tasks.

Adam, Dave, and Gavin are characterized by their abilities to determine the applicability of and to develop correct contextually-justified rules for the explicit, recursive, and chunking strategies of the three core problems. These participants understood how and when these strategies could be applied to these situations and developed rules representing these strategies for the tasks. Overall, these participants were able to tie their rules to the context of each situation, demonstrating the general nature of their rules in modeling each task. For example, Gavin's understanding of the Cube Sticker problem allowed him to generalize the chunking strategy at a high level. A part of Gavin's discussion of the chunking strategy for the Cube Sticker problem is provided below.

Brian: Okay. And how about Danny's rule?

Gavin: Depends on how it is stated, like, he, right now he has it set for rows, and he is taking the difference in the rows and what they would be, and like adding them, so he knows like if between sticker 4 and 5 is a 1 cube difference and if he knew that for a 1 cube difference you are adding 4 sides, you could figure it like that and be right all the way through.

Brian: I see. So if he wrote his from say the 10th to the 20th, which we have over there, what would he do?

Gavin: Then he would figure that there is a 10-cube difference, which multiply that by 4 would give us a 40 sticker difference, and that works adding them together.

Brian: So Danny's rule works?

Gavin: Yeah

Brian: Will it always work in this situation?

Gavin: I think so. Except for when 0 is in there.

This vignette provides a look at Gavin's understanding of the whole-object strategy for the Cube Sticker problem and offers evidence regarding his ability to contextually justify his rule. Specifically, he mentioned that there were 4 sides added for each cube, which could then be multiplied by the number of additional lengths. For corroboration for this view, Gavin's work in solving the Cube Sticker problem provides useful data. Gavin had developed the explicit rule $4n + 2$ and was describing how he had developed the rule.

> Gavin: I determined that for each cube that around it there would be 4 sides and then there would be the top and the bottom of the rod which would count for another two sides, and so I made a formula that said that 2 plus 4 multiplied by whatever the number of cubes is would give me the total number of stickers. Two being counting for the top and the bottom of the rod, and the four being the number of sides multiplied by *x* which was cubes, to determine how many number of sides would be added for each of the cubes.
>
> Brian: Okay, and you did that before you even answered the first question?
>
> Gavin: Yeah
>
> Brian: So how were you able to do that? Were you just looking at the picture?
>
> Gavin: Looking at the picture and by that I mean, just looking at the picture like cause I know the picture has 2 cubes and it has the top and the bottom ends plus the 4 to go around it. And that was 12.

As evidenced in these vignettes, Gavin exhibited the ability to develop an explicit rule for the Cube Sticker problem, even though he had not employed such a rule when initially solving this task. Adam and Dave also displayed the ability to develop rules and contextual justifications for most of the strategies for most of the problem situations.

Outside of their abilities to correctly develop rules for strategies and provide contextual justifications, these participants, in general, seemed to better understand exponential growth, and the vocabulary surrounding it, as presented in the Calling Tree problem. Adam, Dave, Fran, and Gavin were the only participants to be classified as highly flexibly in terms of the Calling Tree Task. Dave described the Calling Tree problem as representing “exponential” growth, while Adam used the word “power.” Adam explained his chunking strategy in the following manner. “Whatever the minute is and I multiply that by 2 to the power of the difference between the minute given, okay, so the difference between the minute given and the number that I am wanting to find.”

While these three participants were able to understand and describe how the four strategies could be used in the various contexts of the core tasks, Adam, Dave, and Gavin represented different groups in terms of strategy use when generalizing problem situations. Adam used all four strategies in solving the core tasks. Gavin used recursion, chunking, and explicit thinking to solve the core tasks. Dave employed only recursive and explicit reasoning in solving the three core tasks.

Overall Medium Within-Task Flexibility. Chrissy, Elizabeth, Fran, John, and Karen were categorized as moderately flexible in terms of overall within-task flexibility. As previously noted, a classification of “moderately flexible” was slightly different in terms of the various students with this label. For example, Elizabeth was determined to be moderately flexible due to her receiving one each of low, medium, and high classifications for the three tasks. John was rated as moderately flexible for receiving two medium rankings and one low ranking for the core problems. Due to a different number of problems worked, Fran received her classification of medium due to one high rating and one medium rating. On the other hand, Bridgette was considered medium due to one low and one medium ranking. While these students certainly differed in their overall understandings of the strategies in terms of the different tasks, they shared the common trait of inconsistency regarding their abilities to understand and use these strategies for different tasks.

The group of participants classified as exhibiting overall medium flexibility for the three core tasks are characterized by several factors. First, these participants were often capable of determining when a strategy could or could not be used for a particular situation, and they were generally able to develop rules for the applicable strategies for each of the problem situations. A main difference between the overall algebraic understanding of these participants and the highly flexible participants lies in their justification of their rules. The participants classified as exhibiting medium flexibility commonly did not provide contextual justifications for their rules. In many of these cases, the justification centered on a numeric pattern that the participant noticed between particular values. For example, Fran provided the following assessment of the chunking strategy for the Theater Seats problem.

Fran: For Danny's [chunking strategy], is it asking like if you are at the 13^{th} row, you add 9?

Brian: Right, let's see, he added 9, a total of 9, for the next three rows to get to the 3^{rd} row.

Fran: I think that would work, because it is a lot easier than 3 and 3 and 3. It would be 9 difference for the 3 rows.

During this discussion, Fran did not provide explicit evidence of her view of the domain for this strategy. Later in the interview, she offered some evidence of her view. She said, "If you are trying to find like every third row, that [the chunking rule] would be easier than someone else's [rule].

Participants classified as medium were characterized by their overall inconsistency in understanding the strategies for the core tasks. Elizabeth, for example, was classified as exhibiting medium within-task flexibility for the Cube Sticker problem, high within-task flexibility for the Theater Seats problem, and low within-task flexibility for the Calling Tree problem. Within that inconsistency existed more inconsistency, as Elizabeth's understanding of the explicit strategy was categorized as low for the Calling Tree problem and high for the Cube Sticker and Theater Seats problems. Her understanding of the chunking strategy was equally variant, as she scored low for the Cube Sticker problem, medium for the Calling Tree problem, and high for the Theater Seats problem.

Overall low within-task flexibility. Bridgette, Hailey, and Iza were categorized as having low flexibility in terms of their understandings of the four algebraic strategies for the three core problems. A classification of low in terms of overall within-task flexibility means that these participants demonstrated little ability to understand and use half of the four strategies for at least two of the three tasks.

Bridgette, Hailey, and Iza were characterized by their inability to correctly determine the applicability of the four algebraic generalization strategies to the core task situations, their inability to develop a mathematically correct generalization for the particular strategy, and/or their inability to provide a correct justification for the rules they developed. While these participants may have had good understandings of some of the strategies for some of the problems, their overall understandings were lacking power. A low rating for within-task flexibility for a particular problem means that a participant was unable to determine the applicability, develop a rule, and/or correctly justify their rule for two out of the four generalization strategies. Bridgette, for example, received low scores for her understanding of half of the strategies, with the other half scored as medium. Iza provided a different perspective, struggling with the explicit and whole-object strategies for the three tasks. Hailey was unique in that she received two low scores and two high scores for the Cube Sticker and the Calling tree problems. Iza was an example of a participant who was unable to produce a rule for a strategy while also not providing a correct justification. Her is an excerpt from Iza's examination of the explicit strategy for the Cube Sticker problem.

Brian: Do you think that there is a rule like Abby's [explicit rule] that will apply in this situation?

Iza: Yeah.

Brian: Okay, why do you think that?

Iza: There is going to be a formula.

Brian: Okay, there is going to be a formula. Why do you think that there is going to be a formula?

Iza: Cause it just does. Everything does.

Iza had similar difficulty in developing an explicit rule and justifying her reasoning when working the Theater Seats problem. The following is a vignette taken from her consideration of the explicit strategy for the Theater Seats problem.

Brian: It is interesting cause you look at the first person, Abby, it goes up 3 each time and there are 7 seats in the row, so my rule is 3n plus 7. Which is exactly what you had. So do you think that a rule like that is going to work?

Iza: No.

Brian: Something like that is going to work?

Iza: Yeah, I couldn't find it, but yeah.

Brian: But not 3n plus 7 exactly.

Iza: No.

Brian: But something like that is going to work. Why do you think that something like that is going to work in this situation?

Iza: Cause you just plug in the numbers.

Brian: Okay. And so you think that if we have a situation like this there is going to be a rule where I can just plug in a number and find out the answer?

Iza: Yes

In neither instance was Iza able to produce an explicit rule. This is somewhat surprising given the fact that the explicit alternative strategy for the Theater Seats problem was an adjustment of the constant term away from being correct. Iza provided the same reasoning for an explicit rule existing in both instances; an explicit rule always exists. Such justification appeared to be informed by a connection to the context of the situation, but instead on an underlying belief that every mathematical situation can be modeled through the use of an explicit rule.

Similar to the group classified as highly flexible, each member of this group represented a different strategy group in terms of the strategies that were used in actually solving the problems. Hailey, like Gavin in the previous example, used the three strategies of recursive, chunking, and explicit to solve the problems. Iza used recursive and explicit thinking exclusively, just as Dave did above. Bridgette, however, represented the opposite end of the spectrum of Adam in terms of within-task flexibility and strategy use; she was characterized as having low flexibility and used only the recursive strategy throughout the core tasks.

Cross-task Flexibility

As with within-task flexibility, cross-task flexibility was determined by a participant's use and understanding of the generalization strategies; the same ratings (see Table 16) that were used to determine a participant's task-specific flexibility for a particular problem were used to determine that participant's cross-task flexibility for each strategy. However, instead of looking across the ratings for the various strategies for a particular problem, a participant's use of a particular strategy across several problems was examined. As can be seen in Figure 4, a participant's ability to use and understand a particular strategy in different problem contexts defines that participant's cross-task flexibility.

Similar to within-task flexibility, each participant received a rating for each strategy for each problem based upon the strategy criteria (Appendix B). As with the case of overall within-task flexibility, only the core tasks were considered in the determination of cross-task flexibility for the reasons cited in the overall within-task flexibility section; the Carwash, Streets and Lampposts, and Brick problems were eliminated from consideration when determining cross-task flexibility. This left, once again, the Cube Sticker, Calling Tree, and Theater Seats problems for determining cross-task flexibility.

The coding scheme used to classify overall within-task flexibility for each participant was once again employed to determine cross-task flexibility for each strategy, due to similar factors being considered. Table 20 depicts the ratings of a few sample participants and includes the cross-task flexibility classification of each participant by strategy.

Task	Participant	Explicit	W/O	Recursive	Chunking
Cube Sticker	Dave	H	H	H	H
Calling Tree	Dave	M	H	H	H
Theater	Dave	H	H	H	H
Overall	Dave	H	H	H	H
Cube Sticker	Hailey	H	H	L	L
Calling Tree	Hailey	L	H	H	L
Theater	Hailey	H	H	H	H
Overall	Hailey	H	H	H	L
Cube Sticker	Iza	L	M	H	M
Calling Tree	Iza	L	L	H	M
Theater	Iza	L	L	H	M
Overall	Iza	L	L	H	M

Table 20. Example of cross-task flexibility ratings

As was the case with within-task flexibility, when the cross-task participant strategy ratings are combined by strategy, a picture of the overall cross-task flexibility for each strategy emerges. Below are tables depicting cross-task flexibility by strategy.

Explicit	
High	Adam, Dave, Elizabeth, Gavin, Hailey
Medium	Fran, John, Karen
Low	Bridgette, Chrissy, Iza

Whole-Object	
High	Dave, Hailey
Medium	Adam, Chrissy, Fran, Gavin, John, Karen
Low	Bridgette, Elizabeth, Iza

Recursive	
High	Adam, Chrissy, Dave, Elizabeth, Fran, Gavin, Hailey, Iza, John, Karen
Medium	
Low	Bridgette

Chunking	
High	Adam, Dave, Gavin, Karen
Medium	Elizabeth, Fran, Iza, John
Low	Bridgette, Chrissy, Hailey

Table 21. Cross-task flexibility by generalization strategy

Table 21 illustrates the participants that exhibited high, medium, and low cross-task flexibility for each generalization strategy.

Examples of Cross-Task Flexibility (Explicit Strategy)

While cross-task flexibility was determined for each of the four generalization strategies, cross-task flexibility for the explicit strategy is used to illustrate the characteristics of the participants in each category.

High cross-task flexibility. As can be seen in Table 21, Adam, Dave, Elizabeth, Gavin, and Hailey were all classified as exhibiting a high level of cross-task flexibility for the explicit strategy for the three core tasks. A high cross-task flexibility rating means that, most of the time, these participants were able to determine that an explicit rule could be used to model the situation, develop an explicit rule, and justify that rule within the context of the problem situation. Gavin, for example, modified the alternative explicit strategy for the Theater Seats problem to mimic his own. An excerpt from his discussion follows.

Gavin: The first one (explicit alternative strategy) doesn't [work]. They made the mistake that I first made. They are not subtracting 1 from the number of rows.

Brian: So you think that is like what you did. Can you fix it to make it work?

Gavin: Yeah, just by doing what I did, by putting the *n* in parentheses and then subtracting 1 from it.

Brian: Okay, so now you think it will work?

Gavin: Yeah

In this vignette, Gavin stated that an explicit rule would model the situation and described his explicit rule. To examine his justification for determination of his view of generality, an excerpt from a discussion about his development of an explicit rule is provided.

Brian: To find row 138, you started off, you said 7 were in the first row.

Gavin: Yeah.

Brian: So you needed to add 7

Gavin: So, cause that is the start of the pattern and so I had to subtract the first, cause I have already done the first row by putting 7 in there. I had to subtract one from the 138, so I got 137, and multiply by 3, because by my pattern, the first one increased by 3 each time.

Brian: I see.

Gavin: And then I added those two together to figure out what it would be.

Brian: Okay, now I am going to ask you some questions about that. So you started with 7, because the first row has 7.

Gavin: Yeah.

Brian: And now you said that you multiplied by 3 because it increases by 3 each time. How does multiplying by 3 relate to the problem?

Gavin: Well, not multiplying 7 by 3, but by the number of other rows by 3.

Brian: How does multiplying the number of rows by 3 do the same thing as adding 3?

Gavin: Well, it is the same thing as adding 3 since what I am doing is multiple times, more than once and by doing it this way, I don't have to sit here and go 3 plus 3 plus 3 plus 3 plus 3 over and over again.

Brian: Okay, and so multiplying by 3 does that for you?

Gavin: Yeah.

Brian: Okay, and now you said that you have to subtract one row first. Why is that? I know you said you started with 7.

Gavin: It is because otherwise I would be counting row 1 twice. I would be adding an additional 3 that is not required.

Brian: Interesting. Very interesting. So the first row already has its 3 being accounted for in the 7?

Gavin: Well, the first row is where the pattern started so it doesn't get 3 added to it, because that is the base where I started getting the pattern where I got row 2 and 3.So I have to get rid of it from the 138 otherwise I would be counting it twice and adding an additional 3, instead of having 7 as the starting number, I would have 10.

In this discussion, Gavin provides a strong justification for how each element of his explicit rule relates to the context of the problem. His ability to determine that an explicit rule is applicable, develop an explicit rule to model the problem situation, and contextually justify his rule led to his high rating for this problem, as well as the other two tasks. The other participants classified as exhibiting a high level of within-task flexibility demonstrated similar abilities with respect to the core-tasks.

Medium cross-task flexibility. Fran, John, and Karen were classified as exhibiting medium within-task flexibility for the explicit strategy. Each participant in this category earned his/her rating due to an inconsistent depth of understanding of this strategy. John and Karen each received a high, medium, and low rating that resulted in their medium status. Likewise, Fran, who completed only the Calling Tree and Theater Seats problems, was rated as exhibiting low and high levels of understanding for these tasks respectively. In each case, the participants "averaged" to their medium rating. This means that the participants in this group were able to understand and use the explicit strategy at a high level (noted the strategy was applicable and developed a correct, contextually-justified explicit rule to model the situation) in the context of one particular task. These participants also were unable to understand the applicability of the explicit strategy and/or were unable to develop a contextually-justified explicit rule for one of the three problem situations. For example, Karen could not see how an explicit rule could be developed for the Calling Tree problem. Below is an excerpt in which she contemplates the explicit alternative student strategy for the Calling Tree problem.

Brian: So do you think that there is a rule like Abby's that applies to the calling tree situation?

Karen: Yes.

Brian: Okay.

Karen: Maybe.

Brian: And why do you think that?

Karen: Because it has got to increase each time, but I don't think it would work exactly because it doesn't increase by the same amount each time.

Brian: I see.

Karen: Like it doubles, but you can't do like, I don't think that you can take like, you can't multiply the same thing by a number because it changes each time.

Brian: Okay, so you think that something like Abby's would work or something like Abby's won't work?

Karen: It won't work, I don't think.

Karen did not seem to think that an explicit strategy could be used to model the exponential situation presented in the Calling Tree problem. This perception kept her attempting to construct an explicit rule, and resulted in her low rating for her understanding of the explicit strategy for the Calling Tree problem.

Low cross-task flexibility. Bridgette, Chrissy, and Iza were classified as exhibiting a low level of cross-task flexibility for the explicit strategy. These participants were characterized by their belief that an explicit strategy does exist for each of tasks but were unable to develop such a rule, for thought that an explicit strategy did not exist for particular tasks.

As previously noted, participants exhibited varying levels of understanding of the whole-object, recursive, and chunking strategies as well. The respective cross-task flexibility categorizations for these strategies were similar to those discussed regarding the explicit strategy. Likewise, the characteristics of the participants in these groups were also similar.

While each of the four strategies had at least two participants who were considered highly flexible in terms of cross-task flexibility, only one participant was consistently categorized as highly flexible: Dave. In almost every instance, Dave was able to understand how the strategy applied to the particular situation, develop a rule representing the strategy to model the situation, and provide a contextual justification of his rule.

As was the case with the highly flexible category, only one participant was consistently rated as having low flexibility for all four strategies: Bridgette. Although it seems somewhat likely that Bridgette would have been classified as having low flexibility in terms of the explicit, whole-object, and chunking strategies, I was surprised to see Bridgette rated as lowly flexible in using the recursive strategy. It is even more surprising that Bridgette was the only participant to be categorized as lowly flexible for the recursive strategy. As previously noted in the strategy use section of this chapter, Bridgette employed the recursive strategy exclusively for each of the core tasks.

Overall Cross-task Flexibility

Overall cross-task flexibility was determined by looking at the four cross-task flexibility ratings given for each strategy. As with within-task flexibility ratings by task, these four strategy ratings were combined into one overall cross-task flexibility rating so that comparisons among groups could be made. To this end, the classification guide (see Table 16, p. 109) used to determine within-task flexibility by task was utilized to determine overall cross-task flexibility, due to the consideration of similar factors.

Overall Cross-Task Flexibility	
Classification	**Participant**
High	Adam, Dave, Gavin, Hailey
Medium	Elizabeth, Fran John, Karen
Low	Bridgette, Chrissy, Iza

Table 22. Overall cross-task flexibility

High overall cross-task flexibility. As can be seen in the above table, Adam, Dave, Gavin, and Hailey exhibited a high level of cross-task flexibility. In other words, these participants demonstrated the ability to use a particular strategy at a high level for at least two of the three core problem situations and received this high score for at least three of the four generalization strategies.

While Adam, Dave, Gavin, and Hailey were able to develop rules and contextual justifications for the explicit, recursive, and chunking strategies for most of the tasks, only Dave was able to consistently develop rules and provide contextual justifications for all four strategies. Adam and Gavin appeared to have good understandings of the explicit, recursive, and chunking strategies for the core tasks. However, both of these participants struggled in their understanding of the whole-object strategy. For example, Adam scored low in terms of his understanding of the whole-object strategy in terms of the Calling Tree problem. After initially believing that a whole-object strategy would work, he quickly decided that it would not. The following is Adam's explanation as to why a whole-object strategy would not apply to the Calling Tree problem.

Adam: Yes, that's correct. No actually that is not correct, because um, it wouldn't.

Brian: It wouldn't? Why not?

Adam: Because the 20th minute multiplied by 2 would be …It would not work because you would be finding the square root of everything…. That 40 subtracted by 20 would be 20 and since that would be, I don't think that you could just square that number to find the 40th. If you could square the number in the 20th row than you could find the number in the 40th row. That did not work.

Adam did not seem to understand the whole-object strategy in terms of the Calling Tree problem. Therefore, he was not able to provide a valid justification for why the strategy would not work. While Gavin did not struggle to this extreme with the whole-object strategy, his understanding of this strategy was not on par with his understanding of the other three. Similarly, Hailey seemed to have a good overall understanding of the explicit, whole-object, and recursive strategies for the core tasks. However, Hailey struggled with the chunking strategy, receiving a low score for her understanding of chunking for the Cube Sticker and Calling Tree tasks.

Due to Dave's overall understanding of each of the strategies for each of the core tasks, Dave was considered the most flexible participant in this study, with respect to cross-task flexibility. Regardless of the situation, Dave was generally able to understand each of the strategies and how they applied to the situation. He was also able to consistently develop a rule, when applicable, and describe the rule within the context of the situation. The only exception to Dave's outstanding performance was his lone medium score for his understanding of the explicit strategy for the Calling Tree problem.

Hailey was unique in that she struggled somewhat consistently only with the chunking strategy. Hailey received an overall low rating for her understanding of the chunking strategy. This stood in contrast to her apparent understanding of the other three strategies. For example, Hailey received a low rating for her understanding of the chunking strategy for the Calling Tree problem. Below is a part of Hailey's conversation about the chunking rule for this problem.

Brian: Okay. How about Danny's [chunking] rule? Is a rule like Danny's going to work here?

Hailey: It could work.

Brian: How is it going to work?

Hailey: You could take, I don't know if this would apply better to Danny or Abby, but you could take like maybe, oh that wouldn't work either. No I guess it wouldn't work. Because what I was thinking of was taking something like the third minute and trying to find the fifth minute, so there are 2 different minutes between the third and fifth.

Brian: Right.

Hailey: Multiply the 2 minutes times the 24 people for the third minute, but that would give you the number in between which would be 48.

Brian: So is there anything that we can multiply to make that jump?

Hailey: Maybe from the third to the seventh, you could multiply between the third and the seventh minute, there is 4 minutes in between, 4 times 24 would be 96, which would give me the fifth minute.

Brian: So you decided to multiply by 4 because of the 4 minutes in between?

Hailey: Yes.

Brian: Okay. Is that going to work? Do you think that is going to give you something?

Hailey: If you do the subtracting of minutes and then multiplying the remaining minutes by the number of people, I suppose that would work, but that would be more Abby, not Danny. So I suppose that Danny wouldn't work.

Hailey seemed to have difficulty with the chunking strategy due to her inability to adjust her linear chunking reasoning to match an exponential situation. Her difficulty in understanding the explicit strategy for this problem seemed to support this idea. However, Hailey also struggled to develop a chunking rule for the Cube Sticker problem, which is a linear situation. For that problem, Hailey was able to develop a contextually-justified explicit rule. However, she did not believe that a chunking rule would work and provided little information regarding her thinking, even after several prompts.

Medium overall cross-task flexibility. Elizabeth, Fran, John, and Karen comprised the medium overall cross-task flexibility group. This group was characterized by their strength in understanding the recursive strategy and their abilities, for the most part, to develop rules to model the other three strategies for each of the situations. However, these participants were not able to tie their rules to the context of the situation. This left ambiguity as to why their rule would continue to work for various input values in the situation. Without this evidence of the generality of their rules, they were not categorized as having a high level of understanding.

One participant in this group, Elizabeth, received a low rating for his understanding of the whole-object strategy. While this rating was low, her understanding of the explicit and recursive strategies was rated as high, which, along with his medium rating for the chunking strategy, resulted in his overall medium cross-task flexibility rating.

Low overall cross-task flexibility. Bridgette, Chrissy, and Iza comprised the low overall cross-task flexibility group. This group was characterized by an inability to understand the applicability of at least two of the algebraic generalization strategies for the three core tasks. Chrissy seemed to have a fairly good understanding of the recursive strategy, but her work with the explicit and chunking strategies was representative of a low level of understanding. Iza was similar, in that she also seemed to understand the recursive strategy well, but struggled with recursive and whole-object strategies.

Bridgette's overall understanding of each of the generalization strategies was low. While she seemed to understand the whole-object, recursive, and chunking strategies at a medium level with regard to the Cube Sticker problem, she exhibited a low level of understanding for each of these strategies when working the Theater Seats problem. At no time was she able to produce a contextual justification for the rules that she developed. In fact, most of the time, she seemed to see rules as working for particular cases, but not in general. Bridgette's work on the Theater Seats problem provides an illustration of her disjunctive thinking with respect to the recursive strategy.

Brian:	Okay so what do you have here? You have 18 for the 5^{th} row, 34 for the 10^{th} row, and so what are you doing to find each of these rows?
Bridgette:	Just adding 3
Brian:	Do you think that will work all the time?
Bridgette:	No.
Brian:	When do you think that wouldn't work?
Bridgette:	Uh, Like, I don't know

After she had completed her work on the Theater Seats problem, Bridgette once again encountered recursive reasoning in her consideration of the alternative strategies. After experiencing some initial difficulty in understanding the terminology (now, next), Bridgette seemed surprised that a "plus 3" rule would work to model the situation. She remarked, "Whoa, that did work." This indicated that not only had she not generalized the recursive strategy that she had developed when initially solving the problem, but she did not connect her initial understanding of the recursive strategy for this situation to the very same recursive strategy presented as an alternative strategy.

While these last measures of overall within-task and cross-task flexibility allow for the comparison of the participants in this study across the three core tasks, the more important measures, from a teacher's perspective, are likely to be the individual within-task flexibility determinations for each task and the cross-task ratings for each generalization strategy. These pieces of information describe a student's understanding of the group of strategies in terms of a particular task and how that student's ability to effectively apply a particular generalization strategy to various tasks. These measures represent the heart of student algebraic flexibility and the assessment of these qualities can provide a foundation for student growth in algebra.

Summary of the Results

The participants employed four algebraic generalization strategies when generalizing the tasks for this study. These strategies were explicit, whole-object, recursive, and chunking.

The participants used a recursive strategy to model their algebraic thinking more than they used the other strategy types combined. This was often due to the participants' beliefs that a recursive strategy would provide a correct answer, due to their understanding of the connection of the recursive strategy to the context of the problem. After recursion, the explicit strategy was used the most often. This was often due to their belief that an explicit rule exists for every problem situation. The participants employed correct recursive reasoning 92% of the time, which was the highest of all the generalization strategies. The chunking strategy was used correctly 90% of the time, but was used far less frequently than recursion. Whole-object and explicit reasoning were used far less successfully than the other two strategies, with whole-object strategies correctly employed 69% of the time and explicit strategies used correctly 60% of the time.

Four participant strategy-use groups emerged from the data: the All Group (participants used all four generalization strategies), the REC group (participants employed recursive, explicit, and chunking strategies), the Recursive/Explicit Group (participants reasoned recursively and explicitly), and the Recursive Group (only recursion was used to generalize the tasks). Most of the participants fell into one of two groups, the REC group or the Recursive/Explicit group.

Participant flexibility was categorized in two parts: within-task and cross-task. In terms of within-task flexibility, every participant was classified as exhibiting at least medium flexibility for at least one of the three core tasks. Thus, every participant was able to determine the applicability of and develop a rule for most of the generalization strategies for at least one of the core tasks. When overall within-task flexibility was considered, three of the participants were classified as exhibiting high flexibility for the core tasks, while five participants were classified as medium, and three as low.

Concerning cross-task flexibility, most of the participants exhibited at least a medium rating for their ability to understand the applicability of the generalization strategies across tasks. Only one participant was classified as consistently exhibiting cross-task flexibility. Cross-task flexibility was highest for the recursive strategy and lowest for whole-object reasoning. In terms of overall cross-task flexibility, a relatively equal number of participants were classified as exhibiting high, medium, and low cross-task flexibility.

CHAPTER V

SUMMARY, DISCUSSION, AND CONCLUSIONS

This study investigated secondary students' use of algebraic generalization strategies and their respective within-task and cross-task flexibility. Specifically, this study documented secondary student strategy use when generalizing contextual algebraic tasks. In addition, I constructed a definition for algebraic flexibility consisting of two parts: within-task flexibility and cross-task flexibility. Conceptual frameworks for these aspects of algebraic flexibility were developed so that within and cross-task flexibility could be assessed. This chapter provides a summary of the study and a discussion of the findings in relation to the current research base. The limitations of this study, the implications for curriculum and instruction, and the recommendations for future research are also addressed.

Summary of the Study and its Findings

Research (Healy & Hoyles, 1999; Lannin, 2001; Stacey, 1989; Swafford & Langrall, 2000) has shown that elementary and middle grade students naturally employ various generalization strategies when working to solve algebraic situations. To date, no studies have employed a similar perspective to document the algebraic generalization strategies employed by students at the secondary level.

Standards documents (NCTM, 2000; NRC, 2000) have called for students to become flexible problem solvers. In terms of algebra, research (Davis & McGowen, 2002; Lewis, 1981; Star, 2001) on flexibility has focused mainly on representational flexibility or flexibility in algebraic manipulation. This study expands the literature concerning these issues by documenting the algebraic strategies of secondary students when generalizing algebraic situations and measuring their flexibility.

Purpose of the Study

This study investigated the secondary students' algebraic reasoning and flexibility through the use of active interviews (Holstein & Gubrium, 1995) in which participants generalized various algebraic situations. Alternative "fictitious student" strategies were used to assist in determining the degree to which the participants understood four generalization strategies (explicit, whole-object, chunking, and recursive) in various situations.

Specifically, this study investigated the following questions:

1. *What strategies do secondary students use when generalizing numeric situations and how do they use these strategies?*
2. *To what extent do students exhibit within-task strategic flexibility when generalizing algebraic tasks?*
3. *To what extent do students exhibit cross-task strategic flexibility when generalizing algebraic situations?*

Methodology

Eleven tenth grade students participated in two to three interviews where they generalized contextualized algebraic situations. Following the development of an initial generalization, the participants were prompted to examine fictitious student strategies for each task and to determine the viability, usefulness, and domain of these strategies. The fictitious student strategies were developed to mimic the algebraic generalization strategies noted in the extant literature. Data sources (transcripts, participant work, and field notes) were analyzed as to allow the natural strategies of the participants to emerge. Had a new strategy been employed, it would have been represented by a new fictitious student strategy. However, no new strategies emerged during the initial or retrospective analyses.

The interview data were analyzed with an eye for the generalization strategies employed by the participants and for the participants' understanding of the various strategies in terms of the different tasks. Criteria were developed to determine participant understanding of strategies for each task. Participant classification according to these criteria provided the basis for determination of within-task and cross-task flexibility. Most of the study employed qualitative analysis, while measures of efficient strategy use required rudimentary quantitative means.

Results of the Study

The participants employed four algebraic generalization strategies when generalizing the tasks for this study. These strategies were explicit, whole-object, recursive, and chunking.

The participants used a recursive strategy to model their algebraic thinking more often than they used the other strategy types combined. This seemed to be because of their belief that a recursive strategy would provide a correct answer, due to their understanding of the connection of the recursive strategy to the context of the problem.

Other than recursion, the explicit strategy was used the most often. This often appeared to be due to the participants' beliefs that an explicit rule exists for every problem situation.

Participants employed correct recursive reasoning 92% of the time, which was the highest of all the generalization strategies. The chunking strategy was used correctly 90% of the time, but was used far less frequently than recursion. Whole-object and explicit reasoning were used less successfully than the other two strategies, with whole-object strategies correctly employed 69% of the time and explicit strategies used correctly 60% of the time.

Four participant strategy-use groups emerged from the data: the All Group (used all four generalization strategies), the REC group (employed recursive, explicit, and chunking strategies), the Recursive/Explicit Group (reasoned recursively and explicitly), and the Recursive Group (used only recursion). Most of the participants fell into one of two groups, the REC group or the Recursive/Explicit group.

Participant flexibility was categorized in two parts: within-task and cross-task. In terms of within-task flexibility, every participant was classified as exhibiting at least medium flexibility for at least one of the three core tasks. Thus, every participant was able to determine the applicability of and develop a rule for most of the generalization strategies for at least one of the core tasks. When overall within-task flexibility was considered, three participants were classified as exhibiting high flexibility for the core tasks, five participants were classified as medium, and three were determined to have low flexibility.

Concerning cross-task flexibility, most participants exhibited at least a medium rating for their ability to understand the applicability of the generalization strategies across tasks. Only one participant was classified as consistently exhibiting cross-task flexibility. Cross-task flexibility was highest for the recursive strategy and lowest for whole-object reasoning. In terms of overall cross-task flexibility, participants were evenly distributed among the categories of high, medium, and low cross-task flexibility.

Discussion of the Findings

This study examined secondary student use of algebraic generalization strategies when solving contextualized generalization tasks and determined the algebraic flexibility of these students in terms of generalization strategies for these tasks. In this section, I discuss the strategies used by the participants and their algebraic flexibility in light of current literature.

Generalization Strategy Use

All strategies employed by the secondary students in this study could be classified as explicit, whole-object, chunking, or recursive. Research (Healy & Hoyles, 1999; Lannin, 2001; Stacey, 1989; Swafford & Langrall, 2000) has demonstrated that upper elementary and middle school children use these generalization strategies when solving algebraic generalization situations. Since prior research had not examined strategy use in this manner above the middle grades, this study adds to the research base by extending this classification of algebraic generalization strategies to the secondary level.

Effective Strategy Use. Similar to findings by Lannin (2005) and Swafford and Langrall (2000), the participants had difficulty developing explicit rules. The explicit strategy was used correctly less than 60% of the time, which was far less than the recursive and chunking strategies (participants used both correctly over 90% of the time) and was around 10% less than the whole-object strategy.

The ineffective use of the explicit strategy appeared to occur for several reasons. One factor that led to the participants' misuse of the explicit strategy involved the participants appearing to notice an explicit relation for a particular value and incorrectly apply it to other cases. For example, Iza noticed that taking the length of the rod in the Cube Sticker problem, times five, provided the correct number of stickers for a rod of length two. Although her rule generated the correct value for a rod length two, it did not apply to other instances of the problem situation. Her success in finding the value for a length-two rod seemingly reinforced by her mistaken mental image of the problem context. When asked how she came up with her rule, Iza explained, "five, because each time you add a cube you have five faces to add." She then incorrectly applied this rule to several other instances. Iza eventually realized her mistake, after sketching additional cubes onto the two that were provided and counting the resulting faces. This mistaken explicit strategy rarely proved successful.

The whole-object strategy was used less frequently than the other strategies for the three core problems. The fact that it was used few times made it difficult to compare its success to the other strategies. However, whole-object reasoning, like explicit reasoning, was not used as successfully as recursive or chunking reasoning. Research (Lannin, Barker, & Townsend, under review-a) has shown that for the whole-object strategy to be used correctly, the participant has to have an accurate visual image of the problem situation. For example, when Adam correctly used the whole-object strategy throughout the Cube Sticker problem, it was due to his seemingly strong mental image of the problem situation. A detailed description of how Adam's visual image helped him to use the whole-object strategy to solve the Cube Sticker problem is provided in the "Individual Participant Strategy Use" section (p. 86).

In contrast, mistakes made when using the whole-object strategy appeared to be due to the participant not having an accurate visual image of the problem situation. For example, when Chrissy attempted to find the number of stickers for a rod of length four in the Cube Sticker problem, she incorrectly doubled the number of stickers for a rod of length two. In performing this operation, Chrissy failed to recognize how doubling the number of the stickers for one rod to get the number of stickers for a rod of twice its length would effect the number of stickers required for this situation.

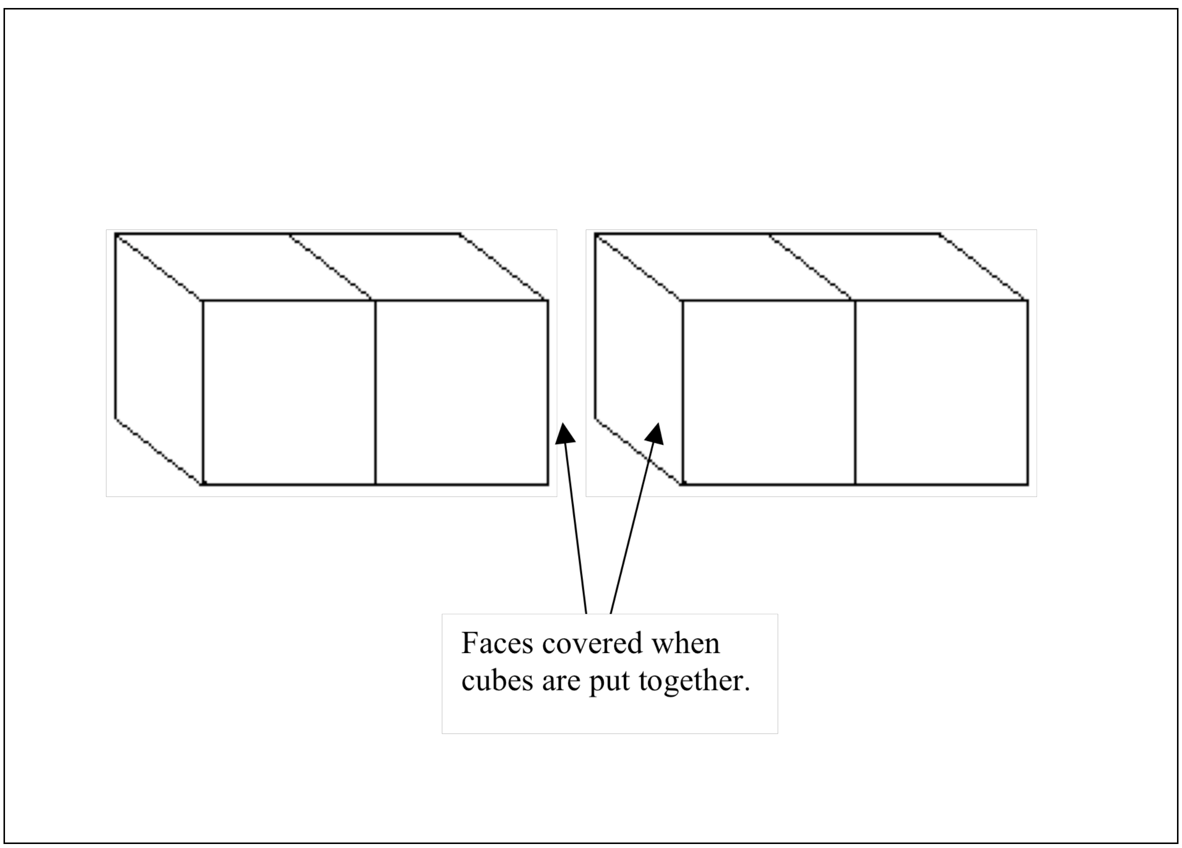


Figure 14. Illustration of whole-object doubling

As illustrated in Figure 14, a simple doubling would lead to one of the end stickers from each rod being covered when the two sections are put together.

Adam seemed to account for the overlap produced by this action through his understanding of mental image of the situation. He subtracted two stickers (representing the ends of the rod), and then employed whole-object reasoning and doubled the amount. He finished the computation by adding two stickers back on to the total, to cover the ends of the rods. This adjustment to the whole-object doubling allowed the participant to use the strategy to compute the mathematically correct value. While a participant's mental image of the problem situation always played a role in that participant's success in using a particular strategy, it appeared critical to the successful use of the whole-object strategy.

Recursion and its use. The recursive strategy often served as a starting point for the participants when working to generalize the problems in this study. This is similar to the findings of Lannin, Barker, and Townsend (under review-a). Every participant in the study employed a recursive strategy for each of the three core tasks, with one exception (Gavin did not use a recursive strategy for the Cube sticker problem). Regardless of their ability, their visualization of the task, and their previous success with other strategies, the participants that generalized the three core tasks found utility in employing a recursive strategy at some point during their solution of the problems. This phenomenon occurred only with recursion. It should be noted, however, that the participants' use of recursion was likely impacted by the generalization tasks.

Not only was recursion used by virtually every participant for each of the three core problems, but recursion was, in general, used correctly by almost every participant for each of the three core tasks. As previously noted, recursion was often the first strategy employed when generalizing the problem situations. Consequently, recursive strategies were used for smaller values in the domain, which likely contributed to the high level of correct usage. Nevertheless, it is important to consider the participants' seemingly natural tendencies to employ recursive reasoning and to exploit this fact in working to expand their abilities in algebraic reasoning.

Recursion as a gateway to other strategies. While others (e.g., Stacey & MacGregor, 2001) have noted that a focus on recursive reasoning can inhibit younger students' abilities to advance to "more efficient" strategies, such as explicit reasoning, Lannin, Barker, and Townsend (under review-b) argued that exploring the ties between recursive and explicit reasoning can lead to a stronger understanding of how both can serve to effectively model a particular problem situation. The authors discussed the importance of facilitating the connection of students' recursive and explicit strategies, to help ensure that the contextual understanding that students exhibited in their recursive rules could carry over to their explicit rules. While the teacher plays an important role in helping students to consider other strategies and the relation between them, the participants of this study demonstrated that students could make the move from recursive reasoning to explicit reasoning essentially on their own, using the chunking strategy as a bridge from their recursive reasoning to explicit thinking. For example, Adam used a chunking rule to move from his recursive rule in the Calling Tree problem to an explicit one. He knew that he needed to multiply by 2 for each additional minute. Adam reasoned that he could "chunk" these multiplications of 2 together and multiply the current output by this chunk. After using a recursive strategy to find the number of people contacted for the first five minutes, Adam used a chunking strategy to find the number of people contacted for minutes 7, 10, 20, and 37. In each case, the new total built upon the total for the previously computed minute. In the end, Adam was able to use this chunking strategy as a springboard into explicit reasoning as evidenced by his development of an explicit rule that built off of the number of students contacted during the fifth minute. Adam noted that his rule, $2^{x-5} * 96$ would work for any value in the domain. In this example, Adams recursive rule led to the development of a chunking strategy, which in turn helped him to develop his explicit strategy.

Iza represented another participant whose knowledge of recursion, and chunking, could help her to better understand other strategies. Iza's Flexibility Schematic is provided below.

Task	Explicit	Whole Object	Recursive	Chunking	Overall
Cube Sticker	L	M	H	M	M
Calling Tree	L	L	H	M	L
Theater	L	L	H	M	L

Table 23. Iza's flexibility schematic

Iza scored consistently low in terms of her understanding of the explicit strategy and the whole-object strategy for the three core tasks. However, she seemed to have a better understanding of the chunking strategy for these tasks, as she received a ranking of medium for all three. Iza also demonstrated strong understanding of when the recursive strategy could be used for the three tasks, as she received a high rating for each. Iza's consistently correct understanding of the recursive strategy would seem to indicate that this strategy represents an area of strength for her. Her recursive reasoning should be encouraged as it provides a foundation for her understanding of other generalization strategies. Iza's consistently medium scorings for the chunking strategy seem to imply that this is an area that, through appropriate task selection and guidance in building on her knowledge of recursion, could become an area of strength for her. Her consistently low classifications for her understanding of the explicit strategy indicate she has been unable to connect her reasoning with recursion and chunking to developing explicit; interventions that focus on the connections between recursive and explicit reasoning and chunking and explicit strategies would likely improve Iza's ability to reason explicitly.

Chunking as a window toward explicit reasoning. The chunking strategy provided an effective tool for developing correct explicit strategies. As previously mentioned in this chapter, not every problem situation can be modeled explicitly. However, for those tasks that can be modeled explicitly, participants struggled to produce correct explicit generalizations. When working through a problem situation, the participants in this study often found the use of a recursive strategy to be too time-consuming and tedious when the input values increased. At these times, the participants developed what they considered to be more efficient strategies; most of the time, this dissatisfaction led these participants to search for an explicit rule. In the move from a recursive rule to an explicit generalization, the participants experienced mixed success. However, those participants who moved from a recursive strategy to an explicit rule via a chunking strategy often proved successful in the transition. For example, Adam employed a chunking strategy for input values 7, 10, 20, and 37 in the Calling Tree problem. Through this experience, Adam realized that he could build a "chunk" to calculate any input value (greater than 5) and could determine the correct output by connecting the chunk to his calculation for the number of people contacted during the fifth minute. By expanding the constraints of his chunking strategy, Adam was able to develop an explicit rule based on the context of the situation that provided him with correct solutions to the task.

Strategy-Preferred Tasks. While each of generalization tasks used in this study allowed students to employ multiple strategies, particular problem situations appeared to encourage the use of particular strategies. As noted in Chapter IV, students used the recursive strategy three times more than any of the other strategies to generalize the Calling Tree problem (Figure 8, p. 55.) This was likely due to how the incremental mathematical relationship was presented within the context and the mathematical structure of the task. The recursive relationship between consecutive terms is clearly stated within the problem situation (each person calls two other people). In terms of the mathematical structure, the exponential nature of the task appeared to make the development of an explicit rule more difficult to construct than for the linear situations. In addition, student depth of understanding of exponents may not be as strong as their understanding of the operations of addition, subtraction, multiplication, and division.

Likewise, the Cube Sticker problem (Figure 2, p. 10) had the highest percentage of explicit strategies of the three core problems. This was likely due to students being more readily able to visualize an explicit relationship that exists within the context of the problem, due to the visual image provided in the situation and the mathematical structure of the task.

The visual image provided for the Cube Sticker problem allows the student to simultaneously view two consecutive iterations of the problem situation. While this would seemingly provide the students with information for a recursive rule, this presentation can also serve as an illustration of the variant and invariant conditions that exist in the problem. In this case, the number of stickers on the sides changes with the length, while the two stickers on the end stay the same. Considering the mathematical structure of the problem, explicit forms could be easily constructed that did not require adding to the length of the rod before multiplying. For example, the explicit rule $4L+2$ could be used to represent the number of stickers on a rod of length L. This particular, and most often used for this study, rule does not require an addition, or subtraction, from the length before the multiplication.

The Brick problem, Streets and Lampposts problem, and the Carwash problem all produced one primary generalization strategy, with one or two other strategies used minimally. For example, the 93% of the strategies used to generalize the Brick problem were explicit in nature, with recursion representing the only other strategy used. Use of the explicit strategy was likely due to the mathematical structure of the task and visual representation provided in the problem. The participants often recognized the mathematical structure of the task as direct variation and quickly developed an explicit rule to model it. As noted by Adam, “Because each row increases by 3 and multiplying is just increasing…by the same number every time.” Hailey said, “There are four lengths and in each length there is three bricks…so I just multiplied the length by three.” The pictorial representation provided for the task also helped to highlight the direct variation structure that underpinned the task; the bricks were illustrated by a pictorial array of three bricks by four bricks.

For the Streets and Lampposts problem, participants used a recursive strategy nearly 85% of the time. The use of a recursive strategy for this problem was likely precipitated by the mathematical structure of the problem. As previously noted, the Streets and Lampposts problem represented the only quadratic task in the set. Given that the recursive relationship is much more accessible than the quadratic rule (due to the variable rate of change that exists as the number of streets increases), it seems appropriate that the participants often employed recursive reasoning. The only instances of non-recursive strategy use for this problem occurred when a participant was able to develop a less apparent explicit rule for the problem situation. Dave initially thought that his rule, $(n/2 - 0.5)*n$ where n represents the number of streets, only applied to odd cases in the domain, but later realized that he could use it for any value greater than one. Similar to the Calling Problem, which was the only exponential problem in the set of six tasks, the Streets and Lampposts problem was a recursively-preferred problem for these participants.

Finally, 72% of the strategies employed when working the Carwash problem (Figure 9, p. 57) were of the whole-object variety. Once again, the use of the whole-object strategy was likely due to the mathematical structure of the problem situation. Given that the Car Wash problem represents an inverse-variation situation, it would be difficult for participants to "chunk" together recursive pieces to arrive at a reasonable solution. A similar argument could be made as to why the recursive strategy was rarely used for this task, and never used mathematically appropriately.

As the participants attempted to determine the amount of time required for 20 car washers to wash the cars given that 40 washers took 2 hours, they used proportional reasoning to solve the problem. In fact, the participants who succeeded in developing an explicit rule for the situation were only able to do so by using their knowledge of division and guessing and checking with the values that they obtained using the whole-object strategy. Dave explained his reasoning for the explicit rule that he developed. He said, "I knew that you had to somehow divide 30 by 40 or 40 by 30 to get a number and then multiply or divide it by two to get the time. So I just guessed and checked to figure out which one would work right…Dividing is going to make this number bigger."

Within and Cross-Task Flexibility

The number of strategies that a participant used to solve a particular problem was not necessarily an indication of the strategies that the participant understood or could use to solve a problem situation. For example, Dave used only recursive and explicit strategies in initially solving the three core tasks. However, Dave was able to understand the applicability of each of the four generalization strategies for these tasks and developed a contextually justified rule for almost every one. Had Dave's strategic flexibility been judged by only his work in solving the tasks, a complete picture of his understanding would not have been realized.

Most of the participants were fairly flexible. While only three participants in this study (Adam, Dave and Gavin) were found to be highly flexible in terms of both within and cross-task flexibility, only two participants were rated as consistently low in terms of flexibility. In fact, 9 out of the 11 participants in this study demonstrated a relatively solid understanding of when the various generalization strategies are applicable and have the ability to develop a rule that models the situation using each strategy. Even though the participants of this study had experienced algebra classes that had not focused on multiple strategies, the participants could still use and could understand different algebraic rules for generalization situations.

The participants of this study recognized that generalization strategies can be modified and applied across tasks and recognized how various strategies can be used to solve particular tasks. Krutetskii's (1976) work on algebraic flexibility provided a look at the ability of students to develop different strategies for solving various algebraic tasks, including contextualized problems and symbol manipulations. It is unclear, however, how the students in Krutetskii's study viewed the generality of the generalizations that they created. Krutetskii's research was focused on the participants' speed in moving from one correct method of solution to another and not on the participants' views of their generalizations. The within-task flexibility framework that guided this study looked beyond the strategies that a student produced to the student's understanding of the strategies and their applicability to a particular situation. This view of within-task flexibility provides a new perspective of a student' ability to model and generalize mathematical situations, which is key to developing mathematical power.

Krutetskii's research included a measure of how contextually similar, yet mathematically different tasks impacted a students ability to produce solutions. Participants were judged on how "bounded" they were to the previous problem situation by how long it took them to produce a new solution. Unlike Krutetskii's view of cross-task "flexibility," I did not consider the amount of time that a student required to generalize a task. The cross-task flexibility framework in this study provided a tool for looking at how students could adjust and apply a particular generalization strategy to new problem situations. This ability to apply prior knowledge to new situations is what others

(Holyoak, 1984; Novick and Holyoak, 1991) have termed *transfer*. "Students develop flexible understanding of when, where, why, and how to use their knowledge to solve new problems if they learn how to extract underlying themes and principles from their learning exercises" (NRC, 2000, p. 236). The cross-task flexibility framework was designed to model this process.

While the participants of this study were, in general, able to demonstrate solid understanding of the generalization strategies, they occasionally over-generalized the use of particular strategies, seemingly lacking a precise demarcation for when and when not to use particular strategies. For example, Karen believed that, most of the time, the whole-object strategy would not work. In the Brick problem, where whole-object reasoning did produce correct results, she felt that it was a "coincidence" and would generalize to other instances in the task. Likewise, many of the participants over-generalized the applicability of the explicit strategy, noting that an explicit strategy exists for every problem situation, even if they were not able to develop or find it. In many cases, this could be due to their experiences in algebra classes where, as Gavin remarked, "There is always a general rule…There has usually been a pattern to follow in almost every problem that I have ever done." Adam echoed this sentiment when asked if there would be a rule like Abby's (explicit) for the Theater Seats problem. He said, "Yeah, I think so, because, like I said that math works in patterns so there will always be a pattern, like $3n + 7$ would be something like that. And the same for all of these [tasks]."

Limitations

The limitations of this study fall into three categories: voluntary sample, task issues, and the nature of alternative strategies. As is the case for many educational research studies, the students chosen to participate in the study did not represent a random sample. The nature of the extensive interviews used in this study limited the population of potential participants.

Along with the voluntary nature of the students selected for participation, the fact that all of the participants in this study resided in two geographically similar locales could be cause for concern. However, in line with prior research in this area, the fact that the population is not geographically diverse should not adversely affect the results; this study sought to document the algebraic strategies and flexibility of algebra students at the secondary level and the results provide a foundation for future research in different demographics.

While every student who participated in the study had a chance to solve the Calling Tree problem, only three participants had the opportunity to work the Carwash problem and/or the Streets and Lampposts problem. While it would have been preferable to have each participant in the study work these tasks, the limited number of sessions did not allow all participants to complete these tasks. Ultimately, the inclusion of these tasks might have encouraged the participants to reconsider the utility of other strategies. However, the limited length of this study is considered a limitation and offers an area for future investigation.

The particular tasks used in this study to elicit generalization strategies likely impacted strategy use. Each of the tasks encouraged generalization through patterning.

In most of the cases, smaller values in the domain were either illustrated or provided with accompanying outputs to provide reference points for patterns. The often smaller and consecutive input values seemed to make recursive reasoning an attractive option, due to its ease of implementation and utility in producing correct solutions, from the students' perspectives.

The alternative strategies that the participants considered following their work with each of the tasks were presented in different contexts. The explicit and recursive rules were presented as generalities while the whole-object and chunking strategies were given as particular instances. For example, the recursive strategy was presented using the following example: "To find the number of seats in the next row, just add 3 each time. I wrote my rule: NOW + 3 = NEXT." For this strategy, no particular row number is provided for when this rule should work; it is provided as a general rule. On the other hand, the chunking strategy was illustrated in the following manner: "For the 13th row, I know there are 34 seats in the 10th row, so I added a total of 9 seats for the next 3 rows." For this strategy, a particular instance is used to demonstrate the strategy. For a student to generalize this rule, they would have to see the general in the particular (Mason & Pimm, 1984). While the participants were asked whether they could apply their rule to other cases in an effort to minimize the differences, the fact that there were differences in the implied generality of the fictitious student strategies could be a limitation.

Implications for Curriculum and Instruction

The results of this study can serve as a guide for curriculum and instructional decisions regarding the development of algebraic thinking in the secondary school. The curriculum recommendations describe how generalizing algebraic situations can be used as a means for facilitating the development of algebraic flexibility in terms of strategy use. The instructional implications emphasize the importance of facilitating this flexibility through sociomathematical norms (Cobb, 2000) that include student interactions that focus on acceptable justification.

Implications for Curriculum

This study demonstrates the range of strategies that students naturally employ when solving contextualized generalization tasks. Curriculum developers seeking to integrate NCTM's (2000) recommendations regarding algebra should consider such tasks for introducing and developing algebraic concepts. Given NCTM's position on the value of multiple strategies and conceptual understanding, contextualized generalization tasks can allow students to use and nurture their understanding of the strategies they develop.

The tasks that I selected for the study impacted the strategies used. Lannin, Barker, and Townsend (under review-a) discussed how certain task factors, such as the mathematical structure and input values, can impact the generalization strategies used by students. They also noted that cognitive factors play a key role in determining student strategy use. These findings were supported by the data in this study. For example, the participants in this study believed that an explicit rule always existed, regardless of the context, even if they could not develop or find it. In many cases, this was due to their prior experiences with algebra where an explicit rule could always be developed to model the situation. It makes sense, then, that for most of the problems, this "prior knowledge" cognitive factor was evident, as the participants made efforts to develop explicit rules to model the algebraic situations.

While it may be the case for many of the problems that these participants have experienced, including this study, it is certainly not the case, in general, that every problem can be represented with an explicit generalization. As noted by Kaput (1995), few situations can actually be modeled explicitly. This is where the understanding of task factors can prove beneficial. As students work to solve various problem situations, it would be to the students' advantage for them to experience mathematical situations where an explicit rule cannot be found to model the situation. While such tasks serve the purpose of expanding the students' horizons in terms of mathematical sophistication and knowledge of mathematical modeling, these tasks would also highlight the importance of other generalization strategies that the students might not have considered or might have deemed inappropriate or inefficient for most mathematical situations.

The inclusion of problems that do not allow for an explicit rule, or at least do not allow for an easily-developed explicit rule, was important to this study. To that end, three tasks were employed during the sessions that represented algebraic situations in which the explicit generalization was difficult to deduce. The three tasks were the Calling Tree problem, the Carwash problem, and the Streets and Lampposts problem. The Calling Tree problem was considered difficult in terms of producing a mathematically correct explicit equation due to its exponential nature. In fact, this hypothesis proved correct with 10 of the 11 participants in this study employing an explicit strategy for this task a total of only four times, with only two of those four representing correct explicit strategies. The other participant used an explicit strategy a total of 19 times, but only one of those 19 represented a mathematically correct explicit strategy. In other words, the Calling Tree problem was difficult to solve explicitly, meaning that the participants had to employ other methods of solution.

The Streets and Lampposts and the Carwash problems provided similar difficulty vis-à-vis developing a mathematically correct explicit generalization. Once again, this was likely due to their mathematical structure. The Carwash problem represented an inverse-variation situation while the Streets and Lampposts task was quadratic. The participants that worked these problems experienced difficulty in producing a mathematically correct explicit generalization.

Using tasks such as the Calling Tree problem, the Streets and Lampposts problem, and the Carwash problem could allow students to discover that the explicit strategy does not always provide the most efficient and accurate method of solution, when the trials of establishing such rules are factored into the equation. Similar tasks that actually cannot be modeled explicitly would help to bring more advanced students who might have developed explicit rules for these three tasks to the same realization.

Implications for Instruction

Student-generated/alternative strategies. As was hypothesized initially in this study, students do not always use all of the strategies that they consider or that they understand. To accurately gauge a student's understanding and ability to use different strategies, the student must be presented with opportunities to consider viable strategies that they may not have articulated. Whether this is done through the normal course of classroom discourse or through the use of fictitious student strategies, providing students opportunities to grapple with other strategies for a particular task should increase their within-task flexibility. As students generalize algebraic tasks, the teacher should have students who have employed different strategies share their ideas. As they share, the teacher should encourage the students to explain their reasoning, so that class can better understand their ideas and see how they developed. These explanations should connect their generalizations to general relations that exist in the problem context (Lannin, 2005). The teacher could have the students grapple with differences and similarities between the strategies, as well as advantages and disadvantages. Such discourse should allow the students to build on their knowledge of strategies and improve in terms of within-task flexibility.

As more tasks are explored, the teacher should ask students to consider how the strategies used to solve the new task compare to the strategies used to solve previous tasks. By focusing on how each strategy is alike or different from previous strategies should improve the students' cross-task flexibility.

Individual versus overall classifications. While the measures of overall within-task and cross-task flexibility allow for the comparison of the participants in this study across the three core tasks, potentially more important for classroom teachers is the determination of within-task flexibility by task and cross-task flexibility by strategy. The overall categorizations provide a global assessment of a particular student's abilities to understand if and how different strategies can be used for a range of tasks or how well the student understands the four algebraic generalization strategies in general. However, the individual classifications of within-task flexibility and cross-task flexibility might prove even more useful. This information describes a student's understanding of the group of strategies in terms of a particular task and how that student's ability to effectively apply a particular generalization strategy to various tasks. These measures represent the heart of student algebraic flexibility and the assessment of these qualities can provide a foundation for student growth in algebra.

Recommendations for Future Research

Future research needs to examine student flexibility at various age levels and track flexibility over time. While prior studies have shown that students utilize different strategies when solving contextualized algebra tasks at the elementary and middle levels, little research has been done to document younger students' abilities to reflect on and adapt others' strategies. Given that the framework for determining algebraic flexibility is presented in this study, no research has documented algebraic flexibility at the elementary, middle, or post-secondary levels. Studies focusing on these areas would provide a more complete picture of student algebraic flexibility. Research that documents how the flexibility of a group of students develops over time would provide further insight into student algebraic flexibility and how it develops at different grade levels.

Additionally, the use of a more diverse sample of participants would allow a more descriptive picture of flexibility to emerge. As previously noted, the participants for this study were drawn from two, similar Midwestern high schools that did not use NSF-funded curricular materials at any level. While any student population might contain participants classified as high, medium, and low in terms of flexibility, future research conducted in different locales could provide alternative perspectives that would allow for comparisons and the determination of influencing factors. For example, studies focused on the algebraic flexibility of participants who have completed an integrated high school mathematics curriculum could provide a comparison for that particular treatment versus a more traditional approach.

Finally, this study analyzed student flexibility when solving algebraic generalization situations, specifically linear and exponential tasks. Future research that focuses on other algebraic topics, such as quadratically- and cubically-modeled tasks, would provide a look at the applicability of the flexibility frameworks of this study to other algebraic topics. Such work would also extend the knowledge base by providing a more complete picture of flexibility within the domain of algebra.

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APPENDIX A

INTERVIEW PROTOCOL

Interview Protocol

Give the task to the participant. Have the participant work through a few instances and see if he or she can generalize the task. Have the participant write a general rule (in words or symbols) and consider the possible values that could exist for N.

Questions:

- How did you develop your rule?
- How does your rule relate to the problem situation?
- Will your rule always work?
- How do you know that your rule will work for (a large input value?)

Have the participant examine the alternative strategies for this situation. Have the participant discuss the viability of each strategy.

If the strategy is viable, ask the participant if she can develop a rule to model the situation using the particular strategy. Then ask her to describe the strengths and weaknesses of the strategy for the situation.

Additional Questions:

- How does the strategy relate to the problem situation?
- Would you use this strategy? If so, for what values?

If the student does not view the strategy as viable, ask the student if changes can be made to the strategy so that it is viable. If yes, then ask about strengths and weaknesses etc.

Have the student discuss which of the rules are preferable for the particular situation.

APPENDIX B

ALTERNATIVE STRATEGY CRITERIA

Theater Seats

Explicit:

- H The participant states that an explicit rule exists and provides a correct general rule for all values in the domain with contextual justification.
- M The participant states that an explicit rule exists and provides a correct general rule for all values in the domain with empirical justification.
- L The participant states that an explicit rule exists, but does not provide an explicit rule for the problem situation OR
  The participant states that an explicit rule does not exist for the problem situation.

W/O:

- H The participant states that a whole-object rule does not exist and provides a valid contextual justification.
- M The participant states that a whole-object rule does not exist and provides an empirical justification.
- L The participant states that a whole-object rule does not exist and does not provide a valid justification OR
  The participant states that a whole-object rule exists.

Recursive:

- H The participant states that a recursive rule exists and provides a valid justification.
- L The participant states that a recursive rule exists, but does not provide a valid justification OR
  The participant states that a recursive rule does not exist.

Chunking:

- H The participant states that a chunking rule exists and describes how it would work in general.
- M The participant states that a chunking rule exists and describes how it would work for particular cases.
- L The participant states that a chunking rule exists and notes that it works for the 10-13 case OR
  The participant states that a chunking rule exists, but does not provide a valid justification OR
  The participant states that a chunking rule does not exist.

Calling Tree

Explicit:

- H The participant states that an explicit rule exists, provides a correct general rule for all values in the domain, and offers a valid contextual justification.
- M The participant states that an explicit rule exists and provides a description of the type of rule that would be used for the situation.
- L The participant states that an explicit rule exists, but changes rule to represent a different strategy OR
  The participant states that an explicit rule exists due to most problems having an explicit rule OR
  The participant states that an explicit rule does not exist for this situation.

W/O

- H The participant states that a whole-object rule exists, changes the supplied rule to a recursive doubling, and provides a valid justification OR
  The participant states that a whole-object rule does not exist and provides a valid general justification.
- M The participant states that a whole-object rule exists and provides a description of the type of rule that would be used for all values in the domain OR
  The participant states that a whole-object rule does not exist and provides an empirical justification.
- L The participant states that a whole-object rule does or does not exist and does not offer a valid justification.

Recursive:

- H The participant states that a recursive rule exists, provides a correct general rule for all values in the domain, and offers a valid justification.
- L The participant states that a recursive rule exists, but does not provide a correct general rule for all values in the domain and a valid justification OR
  The participant states that a recursive rule does not exist.

Chunking:

- H The participant states that a chunking rule exists, provides a correct rule for the situation, and offers a valid justification.
- M The participant states that a chunking rule exists and provides a rule for finding a particular instance (i.e. determining minute thirteen from minute ten) and offers a valid justification.
- L The participant states that a chunking rule exists, but changes rule to represent a different strategy OR
  The participant states that a chunking rule exists, but does not provide a correct general rule for all values in the domain and a valid justification OR
  The participant states that a chunking rule does not exist for this situation.

Cube Sticker

Explicit:

H. The participant states that an explicit rule exists, provides a correct general rule for all values in the domain, and offers a valid contextual justification.
M. The participant states that an explicit rule exists, provides a correct general rule for all values in the domain, and offers an empirical justification.
L. The participant states that an explicit rule exists, but does not provide a correct general rule for all values in the domain and a valid justification OR
The participant states that an explicit rule does not exist for the problem situation.

W/O:

H The participant states that a whole-object rule exists and provides a correct rule with contextual justification of the rule OR
The participant states that a whole-object rule does not exist for the situation and provides correct contextual justification as to why the rule does not work.
M The participant states that a whole-object rule exists and provides a correct rule with empirical justification of the rule. OR
The participant states that a whole-object rule does not exist for the situation and provides empirical justification as to why the rule does not work.
L The participant states that a whole-object rule exists and does not provide a correct justification as to why the rule would work. OR
The participant states that a whole-object rule does not exist and does not provide a correct justification as to why the rule would not work.

Recursive:

H The participant states that a recursive rule exists, offers a valid contextual justification, and provides a correct general rule for the situation.
M The participant states that a recursive rule exists, offers an empirical justification, and provides a correct general rule for the situation.
L The participant states that a recursive rule exists, but does not provide a valid justification and/or rule OR
The participant states that a recursive rule does not exist.

Chunking:

H The participant states that a chunking rule exists, offers a valid justification, and describes the rule as working for all values in the domain.
M The participant states that a chunking rule exists, offers a valid justification, and describes the rule as working for particular cases.
L The participant states that a chunking rule does not exist, or states that a chunking rule exists and does not offer a valid justification.

Streets & Lampposts

Explicit:

H The participant states that an explicit rule exists, provides a correct general rule for all values in the domain, and offers a valid contextual justification.

M The participant states that an explicit rule exists and provides a description of the type of rule that would be used for all values in the domain.

L The participant states that an explicit rule exists, but does not provide a correct general rule for all values in the domain and a valid justification OR
The participant states that an explicit rule does not exist for this situation.

W/O:

H The participant states that a whole-object rule does not exist for this situation and provides a valid justification.

L The participant states that a whole-object rule exists for this situation OR
The participant states that a whole-object rule does not exist for this situation and does not provide a valid justification.

Recursive:

H The participant states that a recursive rule exists for this situation, provides a correct rule, and offers a valid contextual justification.

M The participant states that a recursive rule exists for this situation and provides a correct rule, and offers an empirical justification.

L The participant states that a recursive rule does not exist for this situation OR
The participant states that a recursive rule exists for this situation, but does not provide a correct rule and/or a valid justification.

Chunking:

H The participant states that a chunking rule exists, provides a correct rule for the situation, and offers a valid contextual justification.

M The participant states that a chunking rule exists and provides a correct rule for the situation, and offers an empirical justification.

L The participant states that a chunking rule exists, but does not provide a correct rule for the situation and/or a valid justification OR
The participant states that a chunking rule does not exist for this situation.

Carwash

Explicit:

- H The participant states that an explicit rule exists, provides a correct general rule for all values in the domain, and offers a valid contextual justification.
- M The participant states that an explicit rule exists, provides a correct general rule for all values in the domain, and offers an empirical justification.
- L The participant states that an explicit rule exists, but does not provide a correct general rule for all values in the domain and a valid justification OR
  The participant states that an explicit rule does not exist for this situation.

W/O:

- H The participant states that a whole-object rule exists for this situation, provides a valid rule, and offers a valid justification.
- L The participant states that a whole-object rule does not exist for this situation OR
  The participant states that a whole-object rule exists for this situation, but does not provide a valid rule and a valid justification.

Recursive

- H The participant states that a recursive rule does not exist for this situation and provides a valid justification.
- L The participant states that a recursive rule does not exist for this situation, but does not provide a valid justification OR
  The participant states that a recursive rule does exist for this situation.

Chunking

- H The participant states that a chunking rule does not exist for this situation and provides a valid justification.
- L The participant states that a chunking rule does not exist for this situation and does not provide a valid justification.

Brick

Explicit:

- H The participant states that an explicit rule exists for this situation, provides a correct general rule, and offers a valid contextual justification.
- M The participant states that an explicit rule exists for this situation, provides a correct general rule, and offers an empirical justification.
- L The participant states that an explicit rule exists for this situation, but does not provide a correct general rule and a valid justification OR
  The participant states that an explicit rule does not exist for this situation.

W/O:

- H The participant states that a whole-object rule exists for this situation and provides a valid contextual justification.
- M The participant states that a whole-object rule exists for this situation and provides an empirical justification.
- L The participant states that a whole-object rule exists for this situation, but does not provide a valid justification OR
  The participant states that a whole-object rule does not exist for this situation.

Recursive:

- H The participant states that a recursive rule exists for this situation and provides a valid contextual justification.
- L The participant states that a recursive rule exists for this situation, but does not provide a valid justification OR
  The participant states that a recursive rule does not exist for this situation.

Chunking:

- H The participant states that a chunking rule exists for this situation and provides a valid contextual justification.
- L The participant states that a chunking rule exists for this situation, but does not provide a valid contextual justification OR
  The participant states that a chunking rule does not exist for this situation.

VITA

Brian Townsend was born February 25, 1976, in Girard, Kansas. After attending public schools in Girard, he received the following degrees: B.S. in Education from Pittsburg State University in Pittsburg, Kansas (1998); M.S. in Mathematics from Pittsburg State University (2001); Ph.D. in Mathematics Education from the University of Missouri-Columbia (2005). He is married to the former Karis Davis of Pittsburg, Kansas, and presently has one son, Reese. Brian is currently a member of the Mathematics Department at the University of Northern Iowa, Cedar Falls, Iowa.