DISTRIBUTIONAL ESTIMATES FOR MULTILINEAR OPERATORS

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by
DMYTRO BILYK

Dr. Loukas Grafakos, Dissertation Supervisor

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The undersigned, appointed by the Dean of the Graduate School, have examined the (thesis or dissertation) entitled

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presented by Dmitriy Bilyk

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and hereby certify that in their opinion it is worthy of acceptance.

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Dmytro Bilyk

Dr. Loukas Grafakos, Dissertation Supervisor

ABSTRACT

We prove that if a multilinear operator and all its adjoints map $L^1 \times \cdots \times L^1$ to $L^{1/m,\infty}$, then the distribution function of the operator applied to characteristic functions of sets of finite measure has exponential decay at infinity. These estimates are based only on the boundedness properties and not the specific structure of the operator. The result applies to multilinear Calderón-Zygmund operators and several maximal operators.

We have also obtained similar distributional estimates for the bilinear Hilbert transform:

$$|\{|H_\alpha(\chi_{F_1}, \chi_{F_2})| > \lambda\}| \leq C \left(|F_1|^{\frac{1}{p_1}} |F_2|^{\frac{1}{p_2}}\right)^{\frac{p_1 p_2}{p_1+p_2}} \begin{cases} \lambda^{-\frac{p_1 p_2}{p_1+p_2}} (1 + \log \frac{1}{\lambda})^{\frac{2p_1 p_2}{p_1+p_2}} & \text{when } \lambda < 1 \\ e^{-c\sqrt{\lambda}} & \text{when } \lambda \geq 1 \end{cases}$$

for the cases $p_1 = 1$, $p_2 \geq 2$ (or vice versa) and $p_1, p_2 > 1$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{3}{2}$. These estimates reflect the exponential decay of the distribution function at infinity and also, up to a logarithmic factor, cover the endpoint cases of the region treated by Lacey and Thiele. Distributional estimates of this type also imply the boundedness of the operator on other rearrangement invariant spaces, in particular, the local exponential integrability.

Introduction

In recent years the theory of multilinear singular integral operators has enjoyed an outburst of activity, starting with the pioneering work of Lacey and Thiele ([22], [23]) on the bilinear Hilbert transform. This is an operator defined as

$$H_\alpha(f_1, f_2)(x) = \lim_{\varepsilon\to 0} \int_{|t|\geq\varepsilon} f_1(x-t) f_2(x+\alpha t) \frac{dt}{t} \ ,$$

for functions f_1, f_2 on the line. These operators serve as building blocks for bilinear singular integrals in a way analagous to that in which (linear) directional Hilbert transforms build (linear) singular integrals via the Calderón-Zygmund method of rotations (cf. [3]). Writing a bilinear singular operator as an average of bilinear Hilbert transforms is an idea of A. Calderón who first observed that this can be done for the first order commutator. Indeed, Calderón wrote the first order commutator $\mathcal{C}_1$ as

$$\mathcal{C}_1(f; A)(x) = \text{p.v.} \int_{\mathbf{R}} \frac{A(x) - A(y)}{(x-y)^2} f(y)\, dy = \int_{-1}^{0} H_\alpha(f, A')(x)\, d\alpha \,,$$

and posed a question of L^p boundedness of these operators.

Properties of these operators remained elusive until the appearance of the fundamental work of Lacey and Thiele [22], [23] in the late nineties who established their boundedness on certain products of L^p spaces.

Their work was based on a remarkable set of techniques called time-frequency analysis and revealed a fundamental and deep connection with almost everywhere convergence of Fourier series and in particular, the boundedness of the Carleson-Hunt operator; on the latter the work of Fefferman [8] was influential. The Carleson-Hunt operator is defined as

$$\mathcal{C}(f)(x) = \sup_{N>0} \left| \int_{-N}^{+N} \widehat{f}(\xi) e^{2\pi i x \xi} d\xi \right|$$

where $\widehat{f}(\xi) = \int_{\mathbf{R}} f(x) e^{-2\pi i x \xi} dx$ is the Fourier transform of the function f on the line, or, respectively,

$$\mathcal{C}(f)(x) = \sup_{N>0} \left| \sum_{n=-N}^{+N} \widehat{f}(n) e^{2\pi i n x} \right|$$

for functions on the torus.

The boundedness of this operator on L^2 was proved by Carleson [4]. It implies the almost everywhere convergence of the Fourier series of functions in L^2 of the torus, thus answering a long-standing conjecture posed by Lusin. Very soon this result was extended to L^p by Hunt [18]. Hunt obtained the L^p boundedness of the Carleson operator for $1 < p < \infty$ as a consequence of the following powerful distributional estimate

$$\left|\left\{\mathcal{C}(\chi_F) > \lambda\right\}\right| \leq C\,|F| \begin{cases} \frac{1}{\lambda}(1+\log\frac{1}{\lambda}) & \text{when } \lambda \leq 1 \\ e^{-c\lambda} & \text{when } \lambda > 1. \end{cases} \tag{0.0.1}$$

This, in turn, followed from the estimate

$$\lambda |\{x \in \mathbf{T} : \mathcal{C}(\chi_F)(x) > \lambda\}|^{\frac{1}{p}} \leq C \frac{p^2}{p-1} |F|^{\frac{1}{p}}.$$

These distributional inequalities together with extrapolation arguments similar to Yano's interpolation theorem (see [30]) lead to the norm estimate

$$\|\mathcal{C}(f)\|_{L^1(\mathbf{T})} \leq C \int_{\mathbf{T}} |f(x)|(\log^+ |f(x)|)^2 dx + C,$$

which implies the almost everywhere convergence of functions in $L(\log L)^2(\mathbf{T})$. Later, P. Sjölin [27] used these distributional estimates in a more efficient way to obtain the almost everywhere convergence for functions in $L \log L \log\log L(\mathbf{T})$. Finally, N.Yu. Antonov ([1], [2]) has been able to extract even more information from these distributional inequalities and showed that one can obtain from them the almost everywhere convergence of functions in $L \log L \log\log\log L(\mathbf{T})$ (see also Sjölin and Soria [28]).

This shows that the estimates for the distribution function of an operator applied to the characteristics functions of sets can carry a lot of valuable information. In particular, such estimates can imply the boundedness of the operator on Lebesgue spaces L^p to other L^p spaces or to *weak*–L^p spaces $L^{p,\infty}$ (these arise naturally as they are defined via distribution function estimates). Besides, through the equality

$$\int \phi(|f(x)|)dx = \int_0^\infty \phi'(\lambda) \left|\{x : |f(x)| > \lambda\}\right| d\lambda\,,$$

for a function ϕ with $\phi(0) = 0$, estimates similar to (0.0.1) imply that the operator maps to other rearrangement invariant (Orlicz-type) spaces. In particular, as we shall see, it may imply the exponential integrability.

The fact that we demand that the estimates of type (0.0.1) hold just for characteristic functions of sets of finite measure rather than general functions simplifies matters significantly, but, at the same time, it is not a very restrictive condition, as such restricted boundedness assumptions suffice for a wide range of interpolation results (cf. [19], [15], [12] etc.).

In this dissertation we will be mainly concerned with proving distributional estimates similar to (0.0.1) for multilinear operators.

In Chapter 1 we outline a general approach to distributional estimates, showing that they are, in fact, equivalent to estimates of the integral of the operator over a large subset of a given sets. Using this approach, we prove, via an inductive procedure, distributional estimates for a wide range of multilinear operators, which together with their adjoints satisfy certain boudedness properties. In particular, we show that if an m-linear operator and all its adjoints map $L^1 \times \cdots \times L^1$ to $L^{\frac{1}{m},\infty}$, then the distribution function of this operator applied to characteristic functions of sets of finite measure has exponential decay at infinity. We also prove a maximal version of such estimates. These results apply to the multilinear Calderón-Zygmund operators and their maximal analogs, thus generalizing a well-known fact about the Hilbert transform. These distributional estimates also imply local exponential integrability of the operator.

In Chapter 2 we prove similar distributional estimates for the bilinear Hilbert transform, which is a more singular operator than the standard multilinear Calderón-Zygmund operators and does not satisfy the assumptions required for the results of Chapter 1. However, elaborating on the time-frequency decomposition technique of Lacey and Thiele ([22], [23]), the simplified argument of Lacey ([21]), and an enhanced "energy" estimate (cf. Grafakos, Tao, Terwilleger [14]; Muscalu, Tao, Thiele [25]), we are enabled to prove that the bilinear Hilbert transform satisfies similar restricted weak-type estimates up to some logarithmic factors. This estimate already allows us to obtain the boundedness of the bilinear Hilbert transform in the range of spaces treated by Lacey and Thiele. An inductive procedure, similar to the one utilized in Chapter 1, but more complicated technically, leads us to the distribution function estimates, which yield a blowup of the order $\lambda^{-\frac{2}{3}}$ modulo logarithmic factors near zero and decay of the order $e^{-c\sqrt{\lambda}}$ at infinity. As a consequence, we also obtain that the square root of the bilinear Hilbert transform is locally exponentially integrable.

In Chapter 3 we discuss the relations between the L^p boundedness properties of the bilinear Hilbert transform and the L^p convergence of bilinear Fourier series over parallelograms. We also discuss the pointwise (almost everywhere) convergence of such series and its relation to the boundedness of the *bi-Carleson* operator (see [24]). We note, via Kolmogorov's counterexample ([20]), that this operator is not bounded on $L^1 \times L^p$.

In the sequel, we use the notation $|F|$ for the Lebesgue measure of the set F in $\mathbf{R}^n$, or for a general measure when this notation doesn't cause confusion. We will occasionally utilize the notation $A \lesssim B$ to say that there exists a constant $C > 0$ such that $A \leq CB$, and $A \approx B$ shall mean that A and B are equivalent, i.e. $A \lesssim B$ and $B \lesssim A$.

Chapter 1

Distributional estimates

1.1 A different approach to distribution function estimates

In this section we will show a method allowing to derive distributional estimates for operators. Of course, the most natural distributional estimate arising in various applications is the so-called weak type estimate.

We will be working with a multilinear operator T defined on the m-fold product of spaces of measurable functions on measure spaces (X_j, μ_j) that contain the simple functions. We assume that T takes values in the set of measurable functions on another measure space (X, μ). We will sometimes use the convention $\mu_0 = \mu$.

Let $1 \le p_j \le \infty$, $q > 0$. The operator is said to be of *weak type* $(p_1, \ldots, p_m, q)$ if there exists a constant $C > 0$ such that the following inequality is satisfied for all functions $f_j \in L_p(X_j, \mu_j)$:

$$\mu\{x \in X : |T(f_1, \ldots, f_m)(x)| > \lambda\} \le \frac{C^q}{\lambda^q} \big(\prod_{j=1}^{m} \|f_j\|_{L^{p_j}(X_j,\mu_j)} \big)^q , \qquad (1.1.1)$$

i.e. T maps the product of Lebesgue spaces $L^{p_j}(X_j, \mu_j)$ boundedly into $L^{q,\infty}(X, \mu)$:

$$T : L^{p_1}(X_1, \mu_1) \times \ldots L^{p_m}(X_m, \mu_m) \longrightarrow L^{q,\infty}(X, \mu).$$

We shall often be working with operators which have the *restricted* weak type property, which means that the estimate above only holds in the case when all the functions f_j are characteristic functions of sets of finite measure. More precisely, we say that a multilinear operator T is of restricted weak type $(p_1, \dots, p_m, q)$ if there is a positive constant C such that for all measurable sets $F_j \in X_j$ $(j = 1, \dots, m)$ of finite measure we have

$$\|T(\chi_{F_1}, \dots, \chi_{F_m})\|_{L^{q,\infty}} \leq A\, \mu_1(F_1)^{\frac{1}{p_1}} \dots \mu_m(F_m)^{\frac{1}{p_m}} . \tag{1.1.2}$$

Here the smallest constant C such that (1.1.2) is satisfied for all sets $F_1, \dots, F_m$ is called the restricted weak type $(p_1, \dots, p_m, q)$ constant of T. We shall also use the notation

$$T :\, L^{p_1}(X_1, \mu_1) \times \dots L^{p_m}(X_m, \mu_m) \xrightarrow{\ r\ } L^{q,\infty}(X, \mu).$$

The restricted weak type property is obviously much easier to work with than the weak type property. But even being a weaker condition, it suffices for most practical purposes such as interpolation (e.g. the classical result of Stein and Weiss [29] in the linear case; for the multilinear version see [12]).

Next, we present a condition equivalent to the definition of the restricted weak type.

Lemma 1.1.1. *The following two conditions are equivalent:*

(i) $T :\, L^{p_1}(X_1, \mu_1) \times \dots L^{p_m}(X_m, \mu_m) \xrightarrow{\ r\ } L^{q,\infty}(X, \mu)$.

(ii) There is a constant C_1 *such that for all measurable sets of finite measure* $E, F_1, \dots, F_m$ *there is a subset* S *of* E *such that* $|S| \geq \frac{1}{2}|E|$ *and*

$$\left| \int_S T(\chi_{F_1}, \dots, \chi_{F_m})\, d\mu \right| \leq C_1\, |E|^{1-\frac{1}{q}} |F_1|^{\frac{1}{p_1}} \dots |F_m|^{\frac{1}{p_m}} . \tag{1.1.3}$$

Proof. $(i) \Rightarrow (ii)$. Define

$$\Omega = \left\{ x : |T(\chi_{F_1}, \ldots, \chi_{F_m})(x)| > 2^{\frac{1}{q}} C \frac{|F_1|^{\frac{1}{p_1}} \ldots |F_m|^{\frac{1}{p_m}}}{|E|^{\frac{1}{q}}} \right\},$$

where C is the restricted weak type $(p_1, \ldots, p_m, q)$ constant of T. Then, since T has restricted weak type, we obtain

$$|\Omega| \leq C^q \left(2^{\frac{1}{q}} C \frac{|F_1|^{\frac{1}{p_1}} \ldots |F_m|^{\frac{1}{p_m}}}{|E|^{\frac{1}{q}}} \right)^{-q} \left(|F_1|^{\frac{1}{p_1}} \ldots |F_m|^{\frac{1}{p_m}} \right)^q = \frac{1}{2} |E|.$$

Now we set $S = E \setminus \Omega$. We have $|S| \geq \frac{1}{2}|E|$ and also using a pointwise estimate for the integrand we get

$$\left| \int_S T(\chi_{F_1}, \ldots, \chi_{F_m}) \, d\mu \right| \leq 2^{\frac{1}{q}} C \frac{|F_1|^{\frac{1}{p_1}} \ldots |F_m|^{\frac{1}{p_m}}}{|E|^{\frac{1}{q}}} |E| = C_1 \, |E|^{1-\frac{1}{q}} |F_1|^{\frac{1}{p_1}} \ldots |F_m|^{\frac{1}{p_m}} .$$

$(ii) \Rightarrow (i)$.For a given $\lambda > 0$, we set

$$\begin{aligned} E_\lambda^1 &= \{\mathrm{Re}\, T(\chi_{F_1}, \ldots, \chi_{F_m})(x) > \lambda\}, \\ E_\lambda^2 &= \{\mathrm{Re}\, T(\chi_{F_1}, \ldots, \chi_{F_m})(x) < -\lambda\}, \\ E_\lambda^3 &= \{\mathrm{Im}\, T(\chi_{F_1}, \ldots, \chi_{F_m})(x) > \lambda\}, \\ E_\lambda^4 &= \{\mathrm{Im}\, T(\chi_{F_1}, \ldots, \chi_{F_m})(x) < -\lambda\}. \end{aligned}$$

We will only show the required estimate for E_λ^1, the other cases being absolutely identical. Apply condition (ii) with $E = E_\lambda^1$. Then there exists a subset S of E such that $|S| \geq \frac{1}{2}|E|$ for which we have

$$\lambda \frac{|E|}{2} \leq \left| \int_S T(\chi_{F_1}, \ldots, \chi_{F_m}) \, d\mu \right| \leq C_1 \, |E|^{1-\frac{1}{q}} |F_1|^{\frac{1}{p_1}} \ldots |F_m|^{\frac{1}{p_m}} .$$

Thus,

$$|E| \leq \frac{(2C_1)^q}{\lambda^q} \left(|F_1|^{\frac{1}{p_1}} \ldots |F_m|^{\frac{1}{p_m}} \right)^q,$$

which proves the restricted weak type $(p_1, \ldots, p_m, q)$ of T. □

Remark 1.1.1. Note that the exponents in the equation (1.1.3) are the same that we should have obtained, via Hölder's inequality, had the operator been of strong type. Due to the fact that we deal with restricted *weak* type, we have to pay the price of throwing away a portion of the set. This lemma is an analog of duality for the restricted weak type.

Besides, in the multilinear case, in many natural examples (such as the multilinear Calderón-Zygmund operators or the bilinear Hilbert transform), the homogeneity conditions imply that the exponents satisfy

$$\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{q}.$$

Thus, in many cases, the target space $L^{q,\infty}$ has index $q < 1$, for which regular duality fails. Lemma 1.1.1 gives a way to use duality type estimates for $q < 1$.

The idea of Lemma 1.1.1 may be taken one step further to obtain the following equivalence condition for more general distributional estimates. Suppose $\mathcal{F} : \mathbf{R}_+ \to \mathbf{R}_+$ is a decreasing function. We have the following generalization of Lemma 1.1.1:

Lemma 1.1.2. *The following two conditions are equivalent:*

(i) There are positive constants C, c such that for all measurable sets of finite measure $F_1, \ldots, F_m$

$$\mu\{x \in X : |T(\chi_{F_1}, \ldots, \chi_{F_2})(x)| > \lambda\} \le C\mathcal{F}(c\lambda)|F_1|^{\alpha_1} \ldots |F_m|^{\alpha_m}. \tag{1.1.4}$$

(ii) There are constants C_1, $c_1 > 0$ such that for all measurable sets of finite measure $E, F_1, \ldots, F_m$ there is a subset S of E such that $|S| \ge \frac{1}{2}|E|$ and

$$\left| \int_S T(\chi_{F_1}, \ldots, \chi_{F_m})\, d\mu \right| \le C_1\, |E| \mathcal{F}^{-1}\left(c_1 \frac{|E|}{|F_1|^{\alpha_1} \ldots |F_m|^{\alpha_m}} \right). \tag{1.1.5}$$

The proof of this statement is verbatim the same as the proof of Lemma 1.1.1. This lemma outlines the approach that we are going to take in some of the results below. Namely, to prove a distributional estimate, we shall extract a suitable subset of a given set and estimate the integral of the operator on this subset.

1.2 The main result

It is a classical result that the Hilbert transform H and its maximal counterpart H_* defined for functions f on the line by the identities

$$H(f)(x) = \lim_{\varepsilon\to 0} \frac{1}{\pi} \int_{|t|\geq\varepsilon} f(x-t)\,\frac{dt}{t}\,, \qquad H_*(f)(x) = \sup_{\varepsilon>0} \frac{1}{\pi} \left| \int_{|t|\geq\varepsilon} f(x-t)\,\frac{dt}{t} \right|,$$

satisfy, for all measurable sets F of finite measure, the distributional estimates

$$|\{|H(\chi_F)| > \lambda\}| \leq C\,|F| \begin{cases} \lambda^{-1} & \text{when } \lambda < 1 \\ e^{-c\lambda} & \text{when } \lambda \geq 1, \end{cases} \tag{1.2.1}$$

$$|\{|H^*(\chi_F)| > \lambda\}| \leq C^*\,|F| \begin{cases} \lambda^{-1} & \text{when } \lambda < 1 \\ e^{-c^*\lambda} & \text{when } \lambda \geq 1 \end{cases} \tag{1.2.2}$$

for some constants C, c, C^*, c^*. For a proof of this result we refer to Garsia [9] in which explicit properties of the kernel $1/t$ of H are exploited.

In this chapter we will show that distributional estimates of the type (1.2.1) hold for a variety of linear (and sublinear) operators that may not have the rich structure of the Hilbert transform. In fact, we prove that any linear operator of restricted weak type $(1,1)$, whose adjoint is also of restricted weak type $(1,1)$, must satisfy the distributional estimate (1.2.1), provided it has a bounded kernel or can be written as a pointwise limit of linear operators with bounded kernels. Our results also apply to m-linear operators that are of restricted weak type $(1, \dots, 1, 1/m)$ and whose adjoints have the same property. Extensions of this result to certain maximal operators are also obtained.

It is illustrative to demonstrate the main idea of this chapter on an easy example, e.g. the Hilbert transform mentioned above. We will thus start off by proving the estimate (1.2.1). We note that in this proof we use the weak type $(1,1)$ of H and $H^* = -H$ which is "essentially" the same operator, and also the L^2-boundedness of H, but not the exact structure of the operator and its kernel.

Proof of estimate (1.2.1):

It is obviously enough to prove the second line of the estimate, as the first line is merely the restricted weak type $(1,1)$. To prove the second part, we will use the following inductive procedure:

Suppose $|E| \leq |F|$. Let us denote $F_0 = F$. We now proceed inductively. At the j^{th} step we use weak type $(1,1)$ of the operator H^* and Lemma 1.1.1 with the roles of E and F_j swapped to extract a subset $S_j \subset F_j$ such that $|S_j| \geq \frac{1}{2}|F_j|$ and

$$\left| \int_{S_j} H^*(\chi_E)(x)\, dx \right| \leq C\, |E|.$$

We now define $F_{j+1} = F_j \setminus S_j$. The size of F_j decreases by at least a half with each step since $|S_j| \geq \frac{1}{2}|F_j|$. We continue like this until we come to the point when $|E| \geq |F_n|$. Obviously, the number of steps n is at most $C(1 + \log \frac{|F|}{|E|})$.

We can thus estimate

$$\begin{aligned}
\left|\int_E H(\chi_F)(x)\,dx\right| &= \left|\int_E H(\chi_{S_0}+\chi_{F_1})\,dx\right| \\
&\leq \left|\int_{S_0} H^*(\chi_E)(x),dx\right| + \left|\int_E H(\chi_{F_1})(x)\,dx\right| \\
&\leq C\,|E| + \left|\int_E H(\chi_{F_1})(x)\,dx\right|.
\end{aligned}$$

Proceeding in the same manner until the "stopping time" we obtain that the previous expression is bounded by

$$\begin{aligned}
& n\,C\,|E| + \left|\int_E H(\chi_{F_n})(x)\,dx\right| \\
\leq\ & C\,|E|\Big(1+\log\frac{|F|}{|E|}\Big) + \|H\|_{L^2\to L^2}\,|F_n|^{\frac{1}{2}}\,|E|^{\frac{1}{2}} \\
\leq\ & C_2\,|E|\Big(1+\log\frac{|F|}{|E|}\Big).
\end{aligned}$$

We thus arrive to the following estimate:

$$\left|\int_E H(\chi_F)(x)\,dx\right| \leq C\,|E|\Big(1+\log\frac{|F|}{|E|}\Big). \tag{1.2.3}$$

We now set $E=\{x : H(\chi_F)(x)>\lambda\}$ (the case $E=\{x : H(\chi_F)(x)<-\lambda\}$ works out analogously). By the above estimates we obtain:

$$\lambda\,|E| < \left|\int_E H(\chi_F)(x)\,dx\right| \leq C\,|E|\Big(1+\log\frac{|F|}{|E|}\Big). \tag{1.2.4}$$

Thus $\lambda < C\Big(1+\log\frac{|F|}{|E|}\Big)$, i.e. $|E| \leq C\,e^{-c\lambda}|F|$.

Remark. In terms of Lemma 1.1.2 we have proved that estimate (1.1.5) holds with $\mathcal{F}^{-1}(\alpha)=1+\log(\alpha^{-1})$ for $\alpha<1$. Thus for $\lambda>1$ we have $\mathcal{F}(\lambda)=e^{-c\lambda}$. $\square$

To put in other words, we can say that we are actually looking at the bilinear form $\langle H(\chi_F),\chi_E\rangle$ and inductively use the restricted weak type estimate with respect to the set which has greater size. This will be the core idea for similar results regarding multilinear operators.

1.3 Multilinear version

We denote by T^{*j} the adjoint of the operator T with respect to the jth variable, where $j \in \{1, 2, \ldots, m\}$. The operator T^{*j} is defined by the equality

$$\int g\, T(f_1, \ldots, f_m)\, d\mu = \int f_j\, T^{*j}(\ldots, f_{j-1}, g, f_{j+1}, \ldots)\, d\mu_j \tag{1.3.1}$$

for all functions $f_1, \ldots, f_m, g$ in the corresponding domains; (an implicit assumption is that all integrals converge absolutely.) We also set $T^{*0} = T$ for convenience purposes.

For the purposes of our discussion we have to introduce the following property:

Definition 1.3.1. We say that an m-linear operator T satisfies the *restricted weak continuity property* if for all sequences of sets $\{F_j^{(n)}\}_{n\geq 1}$, $j = 0, \ldots, m$ with the property that $\mu_j(F_j^{(n)}) \to 0$ as $n \to \infty$ we have

$$\langle T(\chi_{F_1^{(n)}}, \ldots, \chi_{F_m^{(n)}}), \chi_{F_0^{(n)}} \rangle \longrightarrow 0 \quad \text{as} \quad \mathrm{n} \to \infty. \tag{1.3.2}$$

Remark 1.3.1. We may note that this property is readily satisfied in many natural cases. For example, if T is an integral operator with a bounded kernel, or if T maps continuously a product of Lebesgue spaces $L^{p_1} \times \cdots \times L^{p_m}$ into some other space L^q, in which case this is a consequence of Hölder's inequality.

As we have seen in the case of the Hilbert transform, weak type of the operator and its adjoints yields exponential decay of the distribution function at infinity. A similar result is true for multilinear operators. We have the following theorem:

Theorem 1.3.1. *Suppose that for some $p_k \geq 1$ $(k = 1, \ldots, m)$ T is of restricted weak type $(p_1, \ldots, p_m, q)$, where q satisfies $\frac{1}{q} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$.*

Suppose that for all $j = 1, \ldots, m$, T^{*j} is of restricted weak type $(p_{1,j}, \ldots, p_{m,j}, q_j)$, where $\frac{1}{q_j} = \frac{1}{p_{1,j}} + \cdots + \frac{1}{p_{m,j}}$, $p_{k,j} \geq 1$, and $p_{j,j} = 1$.

Suppose also that T satisfies the restricted weak continuity property.

Then there are constants C, c (depending on the previous indices, T, and m) such that for all measurable sets $F_1, \ldots, F_m$ of finite measure we have

$$\mu(\{|T(\chi_{F_1}, \ldots, \chi_{F_m})| > \lambda\}) \leq C \left(\mu_1(F_1)^{\frac{1}{p_1}} \ldots \mu_m(F_m)^{\frac{1}{p_m}}\right)^q \begin{cases} \lambda^{-q} & \text{when } \lambda < 1 \\ e^{-c\lambda} & \text{when } \lambda \geq 1. \end{cases} \tag{1.3.3}$$

The formulation of theorem 1.3.1 in such generality is motivated by properties of the bilinear Hilbert transform which satisfies similar restricted weak type assumptions modulo some logarithmic factors. A similar conclusion for this operator is valid as well (cf. the the following chapter).

However, for many practical purposes, a partial case of the above result would be useful. Setting all exponents p_k, $p_{k,j}$ equal to 1, we obtain the following important corollary:

Corollary 1.3.2. *Suppose that for $j = 0, 1, \ldots m$, T^{*j} is of restricted weak type $(1, \ldots, 1, 1/m)$. Suppose also that T satisfies the restricted weak continuity property. Then there are constants C, c such that for all measurable sets $F_1, \ldots, F_m$ of finite measure we have*

$$\mu(\{|T(\chi_{F_1}, \ldots, \chi_{F_m})| > \lambda\}) \leq C \left(\mu_1(F_1) \ldots \mu_m(F_m)\right)^{1/m} \begin{cases} \lambda^{-1/m} & \text{when } \lambda < 1 \\ e^{-c\lambda} & \text{when } \lambda \geq 1. \end{cases} \tag{1.3.4}$$

This corollary, in particular, applies to multilinear Calderón-Zygmund operators (cf. the subsequent section). In the linear case, it yields estimate (1.2.1) for the Hilbert transform and general Calderón-Zygmund operators.

The distributional estimates (1.3.4) imply that T is bounded from $L^{pm} \times \cdots \times L^{pm}$ to L^p for all indices $1/m < p < \infty$. It follows by duality and standard multilinear interpolation [19], [12], that T is bounded from $L^{p_1} \times \cdots \times L^{p_m}$ to L^p whenever $1 < p_1, \ldots, p_m < \infty$ and $p^{-1} = p_1^{-1} + \cdots + p_m^{-1}$. Therefore Corollary 1.3.2 recovers and strengthens the result in [15] in this case.

In particular, in the linear case (i.e., $m = 1$), Corollary 1.3.2 implies that estimate (1.3.4) holds for the Hilbert transform and other self adjoint (or skew adjoint) singular integrals that are of weak type $(1, 1)$. This corollary also applies to multilinear Calderón-Zygmund operators (cf. the subsequent section).

In Section 3 we extend Corollary 1.3.2 for certain maximal singular integral operators.

1.4 The proof of Theorem 1.3.1

For simplicity we denote the measure of any set S that appears in the sequel by $|S|$ with the understanding that this may be either $\mu_j(S)$ or $\mu(S)$ depending on the context. Let T be as in the statement of Theorem 1.3.1. We first prove the following lemma.

Lemma 1.4.1. *There is a constant C_2 such that for all measurable sets of finite measure $E, F_1, \ldots, F_m$ that satisfy $|E|^{\frac{1}{q}} \le |F_1|^{\frac{1}{p_1}} \ldots |F_m|^{\frac{1}{p_m}}$ we have*

$$\left| \int_E T(\chi_{F_1}, \ldots, \chi_{F_m})\, d\mu \right| \le C_2\, |E| \left(1 + \log \frac{|F_1|^{\frac{1}{p_1}} \ldots |F_m|^{\frac{1}{p_m}}}{|E|^{\frac{1}{q}}} \right).$$

Proof. Let us denote $F_i^{(0)} = F_i$ for $i = 1, \ldots, m$. We now proceed inductively. At the j^{th} step we choose the index k_j such that $|F_{k_j}^{(j)}| = \max(|F_1^{(j)}|, \ldots, |F_m^{(j)}|)$. By Lemma 1.1.1 applied to T^{*k_j} for exponents $p_{1,k_j}, \ldots, , p_{m,k_j}, q_{k_j}$ with the roles of E and F_{k_j} interchanged, we can choose $S_{k_j}^{(j)} \subset F_{k_j}^{(j)}$ such that $|S_{k_j}^{(j)}| \geq \frac{1}{2}|F_{k_j}^{(j)}|$ and

$$\left| \int_{S_{k_j}^{(j)}} T^{*k_j}(\chi_{F_1}, \ldots, \chi_E, \ldots \chi_{F_m})\, d\mu_{k_j} \right| \leq C\, \frac{|E| \prod_{i \neq k_j} |F_i^{(j)}|^{\frac{1}{p_{i,k_j}}}}{|F_{k_j}^{(j)}|^{\frac{1}{q_{k_j}}-1}} \leq C\,|E|.$$

We now define $F_i^{(j+1)} = F_i^{(j)} \setminus S_i^{(j)}$ for all $i = 1, \ldots, m$, where we set $S_i^{(j)} = \emptyset$ for $i \neq k_j$. We proceed by induction and we stop at the first integer n such that $|E|^{\frac{1}{q}} \geq |F_1^{(n)}|^{\frac{1}{p_1}} \ldots |F_m^{(n)}|^{\frac{1}{p_m}}$. (Such an integer always exists since the quantity $|F_1^{(j)}|^{\frac{1}{p_1}} \ldots |F_m^{(j)}|^{\frac{1}{p_m}}$ gets smaller by at least a factor of $(\frac{1}{2})^{\frac{1}{\max p_i}}$ when j is replaced by $j+1$.) Obviously, the number of steps n is at most $C(1 + \log \frac{|F_1|^{\frac{1}{p_1}} \ldots |F_m|^{\frac{1}{p_m}}}{|E|^{\frac{1}{q}}})$.

We now have the sequence of estimates

$$\begin{aligned}
\left| \int_E T(\chi_{F_1}, \ldots, \chi_{F_m})\, d\mu \right| &= \left| \int_E T(\ldots, \chi_{S_{k_0}^{(0)}} + \chi_{F_{k_0}^{(1)}}, \ldots)\, d\mu \right| \\
&\leq \left| \int_{S_{k_0}^{(0)}} T^{*k_0}(\ldots, \chi_E, \ldots)\, d\mu_{k_0} \right| + \\
&\quad + \left| \int_E T(\chi_{F_1^{(1)}}, \ldots, \chi_{F_m^{(1)}})\, d\mu \right| \\
&\leq C\,|E| + \left| \int_E T(\chi_{F_1^{(1)}}, \ldots, \chi_{F_m^{(1)}})\, d\mu \right|.
\end{aligned}$$

Writing $\chi_{F_{k_1}^{(1)}}$ as $\chi_{S_{k_1}^{(1)}} + \chi_{F_{k_1}^{(2)}}$ and applying this argument $n-1$ more times we obtain that the previous expression is controlled by

$$\begin{aligned}
& n\, C\,|E| + \left| \int_E T(\chi_{F_1^{(n)}}, \ldots, \chi_{F_m^{(n)}})\, d\mu \right| \\
\leq\ & C\,|E| \left(1 + \log \frac{|F_1|^{\frac{1}{p_1}} \ldots |F_m|^{\frac{1}{p_m}}}{|E|^{\frac{1}{q}}}\right),
\end{aligned}$$

where we have assumed that the remainder term $\left|\int_E T(\chi_{F_1^{(1)}}, \dots, \chi_{F_m^{(1)}})\, d\mu\right|$ is estimated by $C|E|$. We prove this bound using a similar iteration procedure:

First of all, we note that in Lemma 1.1.1 the size of the set S can be taken to be actually equal to $\frac{1}{2}|E|$ (since in the proof we put the absolute values inside and use a pointwise estimate, we may just as well integrate over a smaller set). In the iteration that follows we will use this more careful selection. By this remark we may assume (setting $E^{(n)} = E$) that

$$\big(\frac{1}{2}|E^{(n)}|\big)^{\frac{1}{q}} \leq |F_1^{(n)}|^{\frac{1}{p_1}} \dots |F_m^{(n)}|^{\frac{1}{p_m}} \leq |E^{(n)}|^{\frac{1}{q}}. \tag{1.4.1}$$

We make use of Lemma 1.1.1 for T to choose $S_1 \subset E^{(n)}$ with size half of size of $E^{(n)}$ such that

$$\left|\int_{S_1} T(\chi_{F_1}, \dots, \chi_{F_m})\, d\mu\right| \leq C_1\, |E^{(n)}| \frac{|F_1|^{\frac{1}{p_1}} \dots |F_m|^{\frac{1}{p_m}}}{|E|^{\frac{1}{q}}} \leq C|E^{(n)}|.$$

Next we apply the same procedure to $E \setminus S_1$ and extract S_2 and so on. We stop as soon as $|F_1^{(n)}|^{\frac{1}{p_1}} \dots |F_m^{(n)}|^{\frac{1}{p_m}} \geq |E \setminus (S_1 \cup \dots \cup S_N)|^{\frac{1}{q}}$. Because of the inequality (1.4.1), the number of steps N is bounded by an absolute constant depending on the exponents p_i and q. We set $E^{(n+1)} = E^{(n)} \setminus (S_1 \cup \dots \cup S_N)$. We have $|E^{(n+1)}| \leq \big(\frac{1}{2}\big)^N |E^{(n)}|$.

We now run the cycle of the same type as in the beginning of the proof of this theorem and stop when

$$\big(\frac{1}{2}|E^{(n+1)}|\big)^{\frac{1}{q}} \leq |F_1^{(n+M)}|^{\frac{1}{p_1}} \dots |F_m^{(n+M)}|^{\frac{1}{p_m}} \leq |E^{(n+1)}|^{\frac{1}{q}}.$$

Notice that due to (1.4.1), the number of steps in this part M is also bounded by a constant. We set $E^{(n+M)} = E^{(n+1)}$.

Thus we obtain,

$$\left|\int_{E^{(n)}} T(\chi_{F_1^{(1)}},\dots,\chi_{F_m^{(1)}})\,d\mu\right| \le C(N+M)|E^{(n)}|+\left|\int_{E^{(n+M)}} T(\chi_{F_1^{(1)}},\dots,\chi_{F_m^{(1)}})\,d\mu\right|, \tag{1.4.2}$$

where $C(N+M) \le C''$.

We now repeat the procedure starting from the inequality (1.4.1) with $n := n+M$, and thus, iterating (1.4.2) infinitely many times, we can estimate

$$\left|\int_{E} T(\chi_{F_1^{(1)}},\dots,\chi_{F_m^{(1)}})\,d\mu\right| \le C''(|E|+\frac{1}{2}|E|+\frac{1}{4}|E|+\dots) \le 2C''|E|\,, \tag{1.4.3}$$

where passing to the limit is justified by the weak continuity property. Thus the lemma is proved.

□

Remark 1.4.1. We should note that if the operator T maps $L^{\alpha p_1}\times\dots\times L^{\alpha p_m}$ into $L^{\alpha q}$ (and this is indeed the case by the multilinear interpolation with adjoints theorem of [15], if it is an integral operator with a bounded kernel or may be written as a limit of such operators), then the second part of the iteration procedure, may be replaced by Hölder's inequality:

$$\left|\int_{E} T(\chi_{F_1^{(n)}},\dots,\chi_{F_m^{(n)}})\,d\mu\right| \le C\,|F_1^{(n)}|^{\frac{1}{\alpha p_1}}\dots|F_m^{(n)}|^{\frac{1}{\alpha p_m}}\,|E|^{1-\frac{1}{\alpha q}} \le C\,|E|,$$

where $C = \|T\|_{(\alpha p_1,\dots,\alpha p_m,\alpha q)}$. Theorem 1.3.1 actually strengthens the mentioned interpolation result.

Combining Lemmata 1.1.1 and 1.4.1 we obtain the following:

Corollary 1.4.2. *There is a constant C_3 such that for all $E, F_1, \dots, F_m$ measurable sets of finite measure there is a subset $S = S_{E,F_1,\dots,F_m}$ of E with $|S| \ge \frac{1}{2}|E|$ such that*

$$\begin{aligned} & \left| \int_S T(\chi_{F_1}, \dots, \chi_{F_m})\, d\mu \right| \le \\ \le \quad & C_3\, |E| \min\left(1\,,\ \frac{|F_1|^{\frac{1}{p_1}} \dots |F_m|^{\frac{1}{p_m}}}{|E|^{\frac{1}{q}}}\right) \left(1 + \log^+ \frac{|F_1|^{\frac{1}{p_1}} \dots |F_m|^{\frac{1}{p_m}}}{|E|^{\frac{1}{q}}}\right). \end{aligned}$$

We are now ready to prove the distributional estimate (1.3.3).

For a given $\lambda > 0$, we set

$$\begin{aligned} E_\lambda^1 &= \{\mathrm{Re}\ T(\chi_{F_1}, \dots, \chi_{F_m}) > \lambda\}\,, \\ E_\lambda^2 &= \{\mathrm{Re}\ T(\chi_{F_1}, \dots, \chi_{F_m}) < -\lambda\}\,, \\ E_\lambda^3 &= \{\mathrm{Im}\ T(\chi_{F_1}, \dots, \chi_{F_m}) > \lambda\}\,, \\ E_\lambda^4 &= \{\mathrm{Im}\ T(\chi_{F_1}, \dots, \chi_{F_m}) < -\lambda\}\,. \end{aligned}$$

We shall prove the required estimate for a fixed E_λ^j. Suppose that

$$|E_\lambda^j|^{\frac{1}{q}} \ge |F_1|^{\frac{1}{p_1}} \dots |F_m|^{\frac{1}{p_m}}.$$

Then by Corollary 1.4.2 there is a subset S_λ^j of E_λ^j of at least half its measure so that

$$\frac{\lambda}{2}|E_\lambda^j| \le \lambda |S_\lambda^j| \le \left| \int_{S_\lambda^j} T(\chi_{F_1}, \dots, \chi_{F_m})\, d\mu \right| \le C_3\, \frac{|F_1|^{\frac{1}{p_1}} \dots |F_m|^{\frac{1}{p_m}}}{|E_\lambda^j|^{\frac{1}{q}-1}},$$

which implies

$$|E_\lambda^j| \le (2C_3)^q \left(|F_1|^{\frac{1}{p_1}} \dots |F_m|^{\frac{1}{p_m}}\right)^q \lambda^{-q}\,.$$

But this in turn implies that if $\lambda > 2C_3$, we must have $|E_\lambda^j|^{\frac{1}{q}} \le |F_1|^{\frac{1}{p_1}} \dots |F_m|^{\frac{1}{p_m}}$.

In this case, Corollary 1.4.2 gives

$$\frac{\lambda}{2}|E_\lambda^j| \le C_3\,|E_\lambda^j|\left(1+\log\frac{|F_1|^{\frac{1}{p_1}}\ldots|F_m|^{\frac{1}{p_m}}}{|E_\lambda^j|^{\frac{1}{q}}}\right),$$

from which one easily deduces that $|E_\lambda^j| \le C\,e^{-c\lambda}\big(|F_1|^{\frac{1}{p_1}}\ldots|F_m|^{\frac{1}{p_m}}\big)^q$. Summing over $j = 1,2,3,4$ we deduce the required conclusion with a constant four times as large.

Remark. The estimate in Corollary 1.4.2 can be interpreted in the notation of Lemma 1.1.2 as $\mathcal{F}^{-1}(\alpha) = \min\{1,\alpha^{-\frac{1}{q}}\}(1+\log^+\alpha^{-\frac{1}{q}})$, inverting this we obtain $\mathcal{F}(\lambda) = \min\{\frac{1}{\lambda^q}, e^{-c\lambda}\}$.

1.5 Extensions to (multi)sublinear operators

Next we prove the following extension of Corollary 1.3.2 for operators that may be sublinear in each variable. Naturally, the applications for such a result are maximal operators. In particular, the the Hardy-Littlewood maximal function of an operator satisfying the estimate (1.3.4) will also satisfy a distributional estimate with exponential decay at infinity. This will in its turn, via a Cotlar-type inequality, yield an estimate of the type (1.3.4) for the maximally truncated singular integrals, as we will see later.

Our setting here will be $\mathbf{R}^n$ (endowed with Lebesgue measure) and M will denote the Hardy-Littlewood maximal operator.

Theorem 1.5.1. *Suppose a positive sublinear operator T_* satisfies the following Cotlar-type inequality*

$$T_*(f_1,\ldots,f_m) \le A\Big[M(T(f_1,\ldots,f_m)) + \prod_{j=1}^m M(f_j)\Big]$$

for some operator T that satisfies estimate (1.3.3). *Then there exist constants $C_*, c_* > 0$ such that for $\lambda > 1$*

$$|\{T_*(\chi_{F_1}, \ldots, \chi_{F_m}) > \lambda\}| \leq C_* \, e^{-c_*\lambda}\big(|F_1|^{\frac{1}{p_1}} \ldots |F_m|^{\frac{1}{p_m}}\big)^q.$$

Proof. Obviously, it is enough to show that $M(T(\chi_{F_1}, \ldots, \chi_{F_m}))$ satisfies the required distributional estimate, since $M(\chi_{F_j}) \leq 1$. We denote

$$\Omega_\lambda = \{M(T(\chi_{F_1}, \ldots, \chi_{F_m})) > \lambda\}$$

and $f = T(\chi_{F_1}, \ldots, \chi_{F_m})$ and we set $E_j = \{x : 2^j\lambda < |f(x)| \leq 2^{j+1}\lambda\}$ for $j \geq 0$. By our assumption we have

$$|E_j| \leq C \, e^{-c2^j\lambda}\big(|F_1|^{\frac{1}{p_1}} \ldots |F_m|^{\frac{1}{p_m}}\big)^q.$$

We claim that there exists a constant $B > 0$, such that for all $x \in \Omega_\lambda$ there is an integer $k \geq 0$ and a ball I containing x with the property that

$$|I \cap E_k| \geq B 2^{-2k}|I|. \tag{1.5.1}$$

Indeed, if it were not the case, then for any $B > 0$ there would exist an $x \in \Omega_\lambda$ such that for any ball I containing x we would have

$$\begin{aligned}
\frac{1}{|I|}\int_I |f(z)|dz &= \frac{1}{|I|}\int_{I\cap\{|f|\leq\frac{\lambda}{2}\}} |f(z)|dz + \sum_{j=0}^{\infty}\frac{1}{|I|}\int_{I\cap E_j}|f(z)|dz \\
&\leq \frac{\lambda}{2} + B\sum_{j=0}^{\infty} 2^{-2j}2^{j+1}\lambda \\
&= \lambda(\frac{1}{2} + 4B) < \frac{3}{4}\lambda
\end{aligned}$$

for $B < \frac{1}{16}$. But this would imply that $M(f)(x) < \lambda$ and that $x \notin \Omega_\lambda$, a contradiction.

For each $x \in \Omega_\lambda$ we denote by k_x the smallest k for which (1.5.1) holds and we set $\Omega_\lambda^k = \{x \in \Omega_\lambda :\ k_x = k\}$. It is easy to see that

$$\Omega_\lambda^k \subset \{M(\chi_{E_k}) \geq B2^{-2k}\}.$$

Thus, using weak type (1,1) property of M, we obtain

$$|\Omega_\lambda^k| \leq B' 2^{2k} |E_k| \leq B'' 2^{2k - c' 2^k \lambda} \big(|F_1|^{\frac{1}{p_1}} \dots |F_m|^{\frac{1}{p_m}}\big)^q.$$

Now the required estimate for $\lambda \geq 1$ is obtained by summing the series

$$|\Omega_\lambda| = \sum_{j=0}^{\infty} |\Omega_\lambda^j| \,.$$

□

1.6 Applications to m-linear Calderón-Zygmund operators

Bounded m-linear operators from $L^{p_1}(\mathbf{R}^n) \times \cdots \times L^{p_m}(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ (for some exponents $1 < p_1, \dots, p_m < \infty$ with $p^{-1} = p_1^{-1} + \cdots + p_m^{-1}$) are called multilinear Calderón-Zygmund if they have the form

$$T(f_1, \dots, f_m)(x) = \int_{(\mathbf{R}^n)^m} K(x, y_1, \dots, y_m) f_1(y_1) \dots f_m(y_m)\, dy_1 \dots dy_m, \quad (1.6.1)$$

for some distributional kernel $K(x, y_1, \dots, y_m)$ that coincides with a function defined away from the diagonal $x = y_1 = y_2 = \cdots = y_m$ that satisfies the size estimate

$$|K(y_0, y_1, \dots, y_m)| \leq \frac{A}{\big(\sum_{k,\,l=0}^{m} |y_k - y_l|\big)^{mn}}, \quad (1.6.2)$$

and, for some $\epsilon > 0$, the regularity condition

$$|K(y_0,\dots,y_j,\dots,y_m)-K(y_0,\dots,y_j',\dots,y_m)| \le \frac{A}{\left(\sum_{k,\,l=0}^{m}|y_k-y_l|\right)^{mn+\epsilon}}, \quad (1.6.3)$$

whenever $0 \le j \le m$ and $|y_j - y_j'| \le \frac{1}{2}\max_{0\le k\le m}|y_j - y_k|$. In view of the result in [16], an m-linear Calderón-Zygmund operator T must be bounded from the product $L^1 \times \cdots \times L^1$ to $L^{1/m,\infty}$. As the properties of the kernel K are symmetric in all variables, it follows that for any j between 1 and m we also have that $T^{*j} : L^1 \times \cdots \times L^1 \to L^{1/m,\infty}$. Thus multilinear Calderón-Zygmund operators satisfy the hypotheses of Corollary 1.3.2. It follows that they must also satisfy the distributional estimates (1.3.4).

Next, we show that the maximal multilinear Calderón-Zygmund operators also satisfy the distributional estimates (1.3.4). We define the maximal truncated operator as

$$T_*(f_1,\dots,f_m) = \sup_{\delta>0}|T_\delta(f_1,\dots,f_m)|,$$

where we set

$$T_\delta(f_1,\dots,f_m)(x) = \int_{|x-y_1|^2+\cdots+|x-y_m|^2<\delta^2} K(x,y_1,\dots,y_m)f_1(y_1)\dots f_m(y_m)\,dy_1\dots dy_m.$$

It was proved in [17] that T_* satisfies the pointwise estimate

$$T_*(f_1,\dots,f_m) \le C_\eta\Big[(M(|T(f_1,\dots,f_m)|^\eta))^{1/\eta} + \prod_{j=1}^{m} M(f_j)\Big]. \quad (1.6.4)$$

for some $C_\eta > 0$ whenever $0 < \eta < \infty$, which is a multilinear version of Cotlar's inequality ([6]).

Using Theorem 1.5.1 we therefore deduce the following conclusion.

Proposition 1.6.1. *If T is a multilinear Calderón-Zygmund operator, then T_* satisfies the distributional estimate* (1.3.4).

Proof. The estimate for $\lambda < 1$ follows from the weak type $(1, \ldots, 1, \frac{1}{m})$ property of T_* (cf. [17]); the estimate for $\lambda > 1$ follows from Theorem 1.5.1 and (1.6.4) with $\eta = 1$. $\square$

The next corollary is an immediate consequence of the results just obtained. Naturally the same conclusion applies to any operator that satisfies the hypotheses of Theorem 1.3.1 or Theorem 1.5.1 accordingly.

Corollary 1.6.2. *There is a constant $c_1 > 0$ so that for any multilinear Calderón-Zygmund operator T, for any ball B, and for any tuple of measurable sets $F_1, \ldots, F_m$ of finite measure we have*

$$\int_B e^{c_1 T(\chi_{F_1}, \ldots, \chi_{F_m})} \, dx + \int_B e^{c_1 T_*(\chi_{F_1}, \ldots, \chi_{F_m})} \, dx < \infty \, .$$

Proof. Denote $f = T(\chi_{F_1}, \ldots, \chi_{F_m})$. Then

$$\begin{aligned} \int_B e^{c_1 f(x)} \, dx &= \int_B \int_{-\infty}^{f(x)} c_1 e^{c_1 \lambda} d\lambda \, dx = c_1 \int_{-\infty}^{\infty} e^{c_1 \lambda} \int_{B \cap \{f(x) > \lambda\}} dx \, d\lambda \\ &\leq c_1 |B| + c_1 \int_0^{\infty} e^{c_1 \lambda} e^{-c\lambda} |F_1|^{\frac{1}{p_1}} \ldots |F_m|^{\frac{1}{p_m}} \, d\lambda < \infty \end{aligned}$$

for some $0 < c_1 < c$. The case of T_* is analogous.

$\square$

1.7 Concluding remarks

One may wonder if the conclusion of Theorem 1.3.1 is still valid if it is assumed that T and its adjoints are bounded on some product of Lebesgue spaces $L^{p_1} \times \cdots \times L^{p_m}$

with all $p_j > 1$. We show that this is not the case even when $m = 1$.

Consider a linear operator T that maps L^q into $L^{q,\infty}$ for some $1 < q < 2$ and suppose that T^* has the same property. Then both T and T^* are L^p bounded for all $p \in (q, q')$. Suppose, furthermore, that T does not map L^r into itself for any $r \notin [q, q']$. Following the same procedure discussed in the proof of Theorem 1.3.1 we deduce that there is a constant C such that for all sets F of finite measure we have

$$|\{|T(\chi_F)| > \lambda\}| \leq C\,|F| \begin{cases} \lambda^{-q} & \text{when } \lambda < 1 \\ \lambda^{-q'}(1+\log\lambda)^{q'} & \text{when } \lambda \geq 1\ , \end{cases} \tag{1.7.1}$$

where $q' = q/(q-1)$. It is clear that the term $\lambda^{-q'}(1+\log\lambda)^{q'}$ cannot be replaced by a term of the form $e^{-c\lambda}$ as this would imply that T is bounded on L^p for all $p > q$ which we assume is not the case.

Chapter 2

Distributional estimates for the bilinear Hilbert transform

2.1 Introduction

Certain bilinear singular integral operators can be expressed as averages of bilinear Hilbert transforms in a way analogous to that which linear singular integrals can be written as averages of linear directional Hilbert transforms. The bilinear Hilbert transforms were introduced in the early sixties to play exactly this role by A. Calderón in his study of the first commutator. The L^p boundedness of these operators was explored in the ground-breaking work of Lacey and Thiele, who proved the following:

Theorem. ([22], [23]) For all $\alpha \in \mathbf{R} \setminus \{-1, 0\}$ the bilinear Hilbert transform H_α maps $L^{p_1} \times L^{p_2}$ to L^q for $1 < p_1, p_2 \leq \infty$, $\frac{2}{3} < q < \infty$, $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}$.

It is interesting that the target space may have exponents less than 1, whereas the (linear) Hilbert transform doesn't map into L^q for $q \leq 1$.

This work also showed a connection with almost everywhere convergence of Fourier series and in particular, the boundedness of the Carleson-Hunt operator;

on the latter the work of Fefferman [8] was influential. (The Carleson-Hunt theorem was proved using the time-frequency analysis methods in [8], [24], [14].)

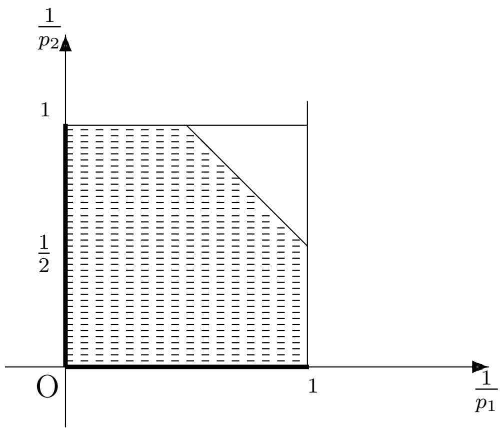

Figure 1. The range of exponents for which the bilinear Hilbert transform is known to be bounded on $L^{p_1} \times L^{p_2}$.

The main purpose of this chapter is to prove a distribution function estimate analogous to that in (0.0.1) for the bilinear Hilbert transform H_α. This operator is defined for a parameter $\alpha \in \mathbf{R}$ by

$$H_\alpha(f,g)(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|t| \geq \varepsilon} f(x-t)g(x+\alpha t)\frac{dt}{t}, \qquad x \in \mathbf{R}$$

for functions f, g on the line. For the cases $\alpha = -1$, 0, or ∞, we obtain that H_α is, respectively, $H(f \cdot g)$, $H(f) \cdot g$, or $f \cdot H(g)$, where H is the (linear) Hilbert transform.

Our approach uses the model sum reduction of Lacey and Thiele [22],[23], a tree analysis based on a selection inspired by Lacey [21], and relies on an improved "energy estimate" that appeared in the proof of estimate (0.0.1) by Grafakos, Tao, and Terwilleger [14]. A variant of this energy estimate also appeared in the related work of Muscalu, Thiele, and Tao [25].

The main result of this chapter is the following:

Theorem 2.1.1. *Let* $2 \le p_2 < \infty$ *and* $\alpha \in \mathbf{R} \setminus \{0, -1\}$. *Then there exist constants* $C = C(\alpha, p_2), c = c(\alpha, p_2)$ *such that for all measurable sets* F_1, F_2 *of finite measure we have*

$$|\{|H_\alpha(\chi_{F_1}, \chi_{F_2})| > \lambda\}| \le C \left(|F_1|\,|F_2|^{\frac{1}{p_2}}\right)^{\frac{p_2}{p_2+1}} \begin{cases} \lambda^{-\frac{p_2}{p_2+1}}(1+\log\frac{1}{\lambda})^{\frac{2p_2}{p_2+1}} & \text{when } \lambda < 1 \\ e^{-c\sqrt{\lambda}} & \text{when } \lambda \ge 1. \end{cases} \tag{2.1.1}$$

Analogously, the following estimate is valid for $2 \le p_1 < \infty$:

$$|\{|H_\alpha(\chi_{F_1}, \chi_{F_2})| > \lambda\}| \le C \left(|F_1|^{\frac{1}{p_1}}\,|F_2|\right)^{\frac{p_1}{p_1+1}} \begin{cases} \lambda^{-\frac{p_1}{p_1+1}}(1+\log\frac{1}{\lambda})^{\frac{2p_1}{p_1+1}} & \text{when } \lambda < 1 \\ e^{-c\sqrt{\lambda}} & \text{when } \lambda \ge 1. \end{cases} \tag{2.1.2}$$

These estimates correspond to the line segments $\{(\frac{1}{p_1}, \frac{1}{p_2}) : \ p_1 = 1,\ 2 \le p_2 < \infty\}$ and $\{(\frac{1}{p_1}, \frac{1}{p_2}) : \ 2 \le p_1 < \infty,\ p_2 = 1\}$. As a corollary we obtain the following distributional estimate corresponding to the line segment $\{(\frac{1}{p_1}, \frac{1}{p_2}) : \ 1 \le p_1, p_2 \le 2,\ \frac{1}{p_1} + \frac{1}{p_2} = \frac{3}{2}\}$.

Corollary 2.1.2. *For any* $\alpha \in \mathbf{R} \setminus \{0, -1\}$ *there exist constants* $C = C(\alpha), c = c(\alpha)$ *such that for all measurable sets* F_1, F_2 *of finite measure we have*

$$|\{|H_\alpha(\chi_{F_1}, \chi_{F_2})| > \lambda\}| \le C \left(|F_1|^{\frac{1}{2}}|F_2|^{\frac{1}{2}} \min|F_i|^{\frac{1}{2}}\right)^{\frac{2}{3}} \begin{cases} \lambda^{-\frac{2}{3}}(1+\log\frac{1}{\lambda})^{\frac{4}{3}} & \text{when } \lambda < 1 \\ e^{-c\sqrt{\lambda}} & \text{when } \lambda \ge 1. \end{cases}$$

(2.1.3)

Remark. In the distributional estimate (2.1.3), the expression $|F_1|^{\frac{1}{2}}|F_2|^{\frac{1}{2}}\min|F_j|^{\frac{1}{2}}$ can be dominated by $|F_1|^{\frac{1}{p_1}}|F_2|^{\frac{1}{p_2}}$, where $1 \le p_j \le 2$ and $\frac{1}{p_1}+\frac{1}{p_2}=\frac{3}{2}$. Thus this estimate (up to a logarithmic term) is similar to restricted weak type estimate for such exponents.

It should be noted that the estimates of Theorem 2.1.1 and Corollary 2.1.2 are, up to a logarithmic factor, the restricted weak type estimates, on the boundary of the range of p's for which the boundedness has been proved by Lacey and Thiele. Notice that the exponential decay at infinity for H_α is not as strong as in the case of the Carleson-Hunt operator and at present we don't know if it is sharp. Estimates of this sort capture the boundedness of H_α on products of Lebesgue spaces but also provide crucial quantitative information that yield other important information such as local exponential integrability and boundedness on other rearrangement invariant spaces. Refinements of this estimate also concern endpoint cases that could not have been treated using previously known techniques.

We also have the corollary concerning the exponential integrability of H_α.

Corollary 2.1.3. *Let $\alpha \in \mathbf{R}\setminus\{0,-1\}$ and $c=c(\alpha)$ be as in Corollary 2.1.2. Then there is a constant $C'=C'(\alpha)$ such athat for any bounded measurable set K and for all measurable sets F_1, F_2 of finite measure the following holds:*

$$\int_K e^{c'|H_\alpha(\chi_{F_1},\chi_{F_2})(x)|^{\frac{1}{2}}}\,dx \le C'\left(|K| + \left(|F_1|^{\frac{1}{2}}|F_2|^{\frac{1}{2}}\min(|F_1|,|F_2|)^{\frac{1}{2}}\right)^{\frac{2}{3}}\right)$$

for any $0<c'<c$.

The proof of this corollary is verbatim the same as that of Corollary 1.6.2.

2.2 Decomposition of the bilinear Hilbert transforms

In this section we will decompose the bilinear Hilbert transform into "model sums" using the so-called wave packets. Later on, this will allow us to use combinatorial methods to treat the operator. This decomposition is essentially the one presented in [22], [23].

In the sequel we will drop the dependence of H_α on α and simply denote it by H. We will use the notation $|A|$ for the Lebesgue measure of a set A and $\langle f, g\rangle$ for the complex inner product $\int f(x)\overline{g(x)}dx$. For a number $a > 0$ and an interval I we denote aI an interval of length $a|I|$ concentric with I and by $a \otimes I$ the interval $[ap, aq]$ if $I = [p, q]$.

Our goal will be to study the trilinear form

$$(f_1, f_2, f_3) \to \int H(f_1, f_2)(x) f_3(x) dx$$

for three functions f_1, f_2, f_3 which for our purposes will be characteristic functions of sets of finite measure, i.e. $f_1 = \chi_{F_1}$, $f_2 = \chi_{F_2}$, and $f_3 = \chi_{E'}$, similar to the ideas of the previous chapter.

We fix L to be the smallest integer greater than $2^{10} \max\{|\alpha|, \frac{1}{|\alpha|}, \frac{1}{|1+\alpha|}\}^3$. The dependence of the bounds on α will enter the proof through polynomial dependence on L.

We begin by noting that the distribution $p.v.\frac{1}{t}$ that appears in the definition of H can be written as $c_1\delta_0 + c_2\gamma$ for some constants c_1, c_2, where δ_0 is the Dirac mass at the origin and γ is another distribution that satisfies $\widehat{\gamma} = \chi_{(0,\infty)}$. Since all the estimates that we are going to be proving in this chapter are trivial for δ_0, we may restrict our attention to γ. Let θ be a smooth function which is equal to 1 on $(-\infty, 2L)$ and 0 on $(3L, \infty)$. Define

$$\widehat{\psi}(\xi) = \theta(\xi) - \theta(2\xi).$$

Observe that $\widehat{\psi}$ is nonzero and is supported in $[L, 3L]$. For each integer k we define

$$\psi_k(x) = 2^{-\frac{k}{2}} \psi(2^{-k}x).$$

Then we have

$$\gamma = \sum_{k \in \mathbf{Z}} 2^{-\frac{k}{2}} \psi_k.$$

Indeed, if we look at the Fourier transform of the righthand side of the identity above, we get a telescopic sum:

$$\Big(\sum_{k \in \mathbf{Z}} 2^{-\frac{k}{2}} \psi_k \Big)^{\wedge}(\xi) = \lim_{N \to \infty} \sum_{-N}^{N} \left[\theta(2^k \xi) - \theta(2^{k+1}\xi) \right] = \lim_{N \to \infty} \left[\theta(2^{-N}\xi) - \theta(2^{N+1}\xi) \right] = \widehat{\gamma}.$$

It clearly suffices to study the trilinear form

$$\Lambda(f_1, f_2, f_3) := \sum_{k \in \mathbf{Z}} 2^{-\frac{k}{2}} \int \int f_1(x-t) f_2(x + \alpha t) f_3(x) \psi_k(t) \, dt \, dx. \tag{2.2.1}$$

We can further break the function ψ into a sum of at most $2L$ functions $\psi^{(M)}$ such that $\widehat{\psi^{(M)}}$ is supported in the interval $[M - \frac{1}{2}, M + \frac{1}{2}]$ for $L \leq M \leq 2L$. It would suffice to study each piece separately. For notational convenience, we will omit the dependence on M and will just write ψ.

For further decomposition we fix a Schwartz function ϕ of L^2 norm 1, with Fourier transform supported in $[-\frac{1}{2}, \frac{1}{2}]$, which also has the property that for all

$\xi \in \mathbf{R}$ we have

$$\sum_{l\in\mathbf{Z}} \left|\widehat{\phi}(\xi - l/2)\right|^2 \equiv C_0$$

for some constant $C_0 > 0$. To produce such a function, one may start with an arbitrary Schwartz function and then divide it by the expression in the left hand side of the above equation.

Let $u = I_u \times \omega_u$ be a rectangle in $\mathbf{R}^2$ and define the adaptation of ϕ to the recangle u as follows:

$$\phi_u(x) = |I_u|^{-\frac{1}{2}}\phi\Big(\frac{x - c(I_u)}{|I_u|}\Big)e^{2\pi i c(\omega_u)x},$$

where $c(J)$ denotes the center of the interval J.

For each $k \in \mathbf{Z}$ we consider the set of dyadic rectangles of scale k:

$$\mathbf{S}_k = \{(2^k n, 2^k(n+1)) \times (2^{-k}m/2, 2^{-k}(m/2+1)) \,|\, m, n \in \mathbf{Z}\}.$$

Then $\mathbf{S} = \bigcup_k \mathbf{S}_k$ is the set of all dyadic rectangles of area 1 in $\mathbf{R}^2$.

It is an easy calculation to verify that for all $f \in L^2$

$$f = \frac{1}{C_0}\sum_{u\in\mathbf{S}_k}\langle f, \phi_u\rangle\phi_u$$

where the convergence is in L^2 (cf. [7]). Moreover, the series also converges a.e. for all $f \in L^p$, $1 < p < \infty$, see [13]. Using this decomposition of the identity in the k^{th} term of (2.2.1), as in [23], we obtain

$$\Lambda(f_1, f_2, f_3) := \sum_{k\in\mathbf{Z}}\sum_{u_1,u_2,u_3\in\mathbf{S}_k} C_{k,u_1,u_2,u_3}\Lambda_{k,u_1,u_2,u_3}(f_1, f_2, f_3), \tag{2.2.2}$$

where

$$C_{k,u_1,u_2,u_3} = C_0^{-3}\int_{\mathbf{R}}\int_{\mathbf{R}} \phi_{u_1}(x-t)\phi_{u_2}(x+\alpha t)\phi_{u_3}(x)\psi_k(t)\,dt\,dx$$

and

$$\Lambda_{k,u_1,u_2,u_3}(f_1,f_2,f_3) = 2^{-\frac{k}{2}}\langle f_1,\phi_{u_1}\rangle\langle f_2,\phi_{u_2}\rangle\langle f_3,\phi_{u_3}\rangle.$$

We now take a closer look at the coefficients C_{k,u_1,u_2,u_3} in two different ways. First,

$$\begin{aligned}
& |C_{k,u_1,u_2,u_3}| \\
\leq\ & C\int\int\left|\phi\Big(\frac{x-t-c(I_{u_1})}{|I_{u_1}|}\Big)\phi\Big(\frac{x+\alpha t-c(I_{u_2})}{|I_{u_2}|}\Big)\phi\Big(\frac{x-c(I_{u_3})}{|I_{u_3}|}\Big)\psi(2^{-k}t)\right|2^{-2k}\,dt\,dx \\
=\ & C\int\int\Big|\phi\big(x-t-A_1\big)\phi\big(x+\alpha t-A_2\big)\phi\big(x-A_3\big)\psi(t)\Big|\,dt\,dx,
\end{aligned}$$

where $A_i = \frac{c(I_{u_i})}{|I_{u_i}|}$ for $i = 1, 2, 3$ (these numbers are half-integers). Observe that

$$\begin{aligned}
A_2 - A_1 &= (x-t-A_1) - (x+\alpha t - A_2) + (1+\alpha)t, \\
A_3 - A_1 &= (x-t-A_1) - (x-A_3) + t, \\
A_3 - A_2 &= (x+\alpha t - A_2) - (x-A_3) - \alpha t.
\end{aligned}$$

This implies that at least one of the arguments in the last displayed double integral has to have size at least $\frac{1}{4L}\mathrm{diam}\{A_i\}$. Since ϕ and ψ are Schwartz functions, it follows that, for any positive integer m, there exists a constant C_m such that

$$|C_{k,u_1,u_2,u_3}| \leq C_m\left(1+\frac{\mathrm{diam}\{A_i\}}{4L}\right)^{-m} = C_m\left(1+\frac{\max_{i,j}|c(I_{u_i})-c(I_{u_j})|}{2^k\,4L}\right)^{-m}. \tag{2.2.3}$$

Secondly, we set $F_1(x,t) = \phi_{u_1}(x-t)\phi_{u_2}(x+\alpha t)$, $F_2(x,t) = \phi_{u_3}(x)\psi_k(t)$. These are Schwartz functions of two variables. We have

$$\begin{aligned}
\widehat{F_1}(\xi,\tau) &= \widehat{\phi_{u_1}}\left(\frac{\alpha\xi-\tau}{1+\alpha}\right)\widehat{\phi_{u_2}}\left(\frac{\xi+\tau}{1+\alpha}\right), \\
\widehat{F_2}(\xi,\tau) &= \widehat{\phi_{u_3}}(\xi)\widehat{\psi_k}(\tau).
\end{aligned}$$

Thus, applying the two-dimensional Plancherel formula, we obtain

$$|C_{k,u_1,u_2,u_3}| \le \frac{C}{|1+\alpha|} \int\int \left|\phi\left(\frac{\alpha\xi-\tau}{1+\alpha}-B_1\right)\phi\left(\frac{\xi+\tau}{1+\alpha}-B_2\right)\phi(\xi-B_3)\psi(\tau)\right| d\xi d\tau, \tag{2.2.4}$$

where $B_i = \frac{c(\omega_{u_i})}{|\omega_{u_i}|} = 2^k c(\omega_{u_i})$ (notice that this is an integer or a half-integer).

Assume that the integral above is not zero. Then we must have

$$\frac{\alpha\xi-\tau}{1+\alpha} - B_1 \in \left[-\frac{1}{2}, \frac{1}{2}\right], \ \frac{\xi+\tau}{1+\alpha} - B_2 \in \left[-\frac{1}{2}, \frac{1}{2}\right],$$

$$\xi - B_3 \in \left[-\frac{1}{2}, \frac{1}{2}\right], \ \tau \in \left[M-\frac{1}{2}, M+\frac{1}{2}\right],$$

which imply

$$B_1 \in \left[\frac{\alpha}{1+\alpha}B_3 - \frac{1}{1+\alpha}M - \frac{1+|\alpha|+|1+\alpha|}{2|1+\alpha|}, \frac{\alpha}{1+\alpha}B_3 - \frac{1}{1+\alpha}M + \frac{1+|\alpha|+|1+\alpha|}{2|1+\alpha|}\right], \tag{2.2.5}$$

and

$$B_2 \in \left[\frac{1}{1+\alpha}B_3 + \frac{1}{1+\alpha}M - \frac{2+|1+\alpha|}{2|1+\alpha|}, \frac{1}{1+\alpha}B_3 + \frac{1}{1+\alpha}M + \frac{2+|1+\alpha|}{2|1+\alpha|}\right]. \tag{2.2.6}$$

This means that the triple of parameters B_1, B_2, B_3 really depends only on the parameter B_3 as for each value of B_3, the quantities B_1 and B_2 can take only a finite number of values depending on α. Also, (2.2.5) and (2.2.6) show that $B_1 + B_2 = B_3$ up to an error that can only take a finite number of integer values (depending on α.)

We introduce parameters $\nu_1, \nu_2, \mu_1, \mu_2$ by setting

$$A_1 = A_3 + \nu_1, \qquad A_2 = A_3 + \nu_2, \qquad B_1 = \frac{\alpha}{\alpha+1}B_3 + \mu_1, \qquad B_2 = \frac{1}{\alpha+1}B_3 + \mu_2.$$

We also set $\nu = \max|\nu_i|$. We aim to reduce the sum over $u_1, u_2, u_3 \in \mathbf{S}_k$ as the rapidly converging sum over $\nu_1, \nu_2, \mu_1, \mu_2$ of the sum over the tiles u_3.

For N sufficiently large we have

$$|\Lambda(f_1, f_2, f_3)| \leq \tag{2.2.7}$$
$$\sum_{\nu=0}^{\infty} C_N(1+\tfrac{\nu}{4L})^{-N} \sum_{\substack{(\nu_1,\nu_2):\\ \max|\nu_i|=\nu}} \sum_{\mu_1} \sum_{\mu_2} \Big| \sum_{k\in\mathbf{Z}} \sum_{u_3\in\mathbf{S}_k} \varepsilon_{\nu_1,\nu_2,\mu_1,\mu_2,u_3} \Lambda_{k,u_1,u_2,u_3}(f_1, f_2, f_3) \Big|$$

where $u_1 = u_1(u_3)$ and $u_2 = u_2(u_3)$ are uniquely determined by u_3 in terms of $\nu_1, \nu_2, \mu_1, \mu_2$, $\varepsilon_{\nu_1,\nu_2,\mu_1,\mu_2,u_3}$ is a constant of modulus at most 1, and $\mu_1 \in \frac{1}{2}\mathbf{Z} - \frac{\alpha}{1+\alpha}\frac{1}{2}\mathbf{Z}$ and $\mu_2 \in \frac{1}{2}\mathbf{Z} - \frac{1}{1+\alpha}\frac{1}{2}\mathbf{Z}$ range in the intervals

$$\mu_1 \in \Big[-\frac{1}{1+\alpha}M - \frac{1+|\alpha|+|1+\alpha|}{2|1+\alpha|}, -\frac{1}{1+\alpha}M + \frac{1+|\alpha|+|1+\alpha|}{2|1+\alpha|} \Big],$$
$$\mu_2 \in \Big[\frac{1}{1+\alpha}M - \frac{2+|1+\alpha|}{2|1+\alpha|}, \frac{1}{1+\alpha}M - \frac{2+|1+\alpha|}{2|1+\alpha|} \Big].$$

Thus μ_1 and μ_2 take only a finite number of values depending on α. (Note that $\varepsilon_{\nu_1,\nu_2,\mu_1,\mu_2,s_3}$ is the ratio of C_{k,u_1,u_2,u_3} by $C_N(1+\frac{\max|\nu_i|}{4L})^{-N}$.)

It will clearly suffice to study the boundedness of the expression inside the absolute values in (2.2.7) and to obtain bounds independent of μ_i and polynomial in ν, since for each ν , there are of the order of ν pairs (ν_1, ν_2) with $\max|\nu_i| = \nu$.

Next, we further separate the triples in such a way that for two triples (u_1, u_2, u_3) and (u_1', u_2', u_3') from the same group the following conditions hold:

the separation of scales:

$$\text{if } k \neq k', \text{ then } |k-k'| > L^{10}, \tag{2.2.8}$$

the separation of space:

$$\text{if } A_3 \neq A_3', \text{ then } |A_3 - A_3'| > \nu L^{10}, \tag{2.2.9}$$

the separation of frequencies:

$$\text{if } B_3 \neq B_3', \text{ then } |B_3 - B_3'| > L^{10}. \tag{2.2.10}$$

Obviously, the number of such groups is polynomial in L and ν.

To facilitate the study of the sums above, we introduce *tri-tiles*. A tri-tile is a rectangle $s = I_s \times \omega_s$ and three subrectangles s_1, s_2, s_3 built in the following way:

Let (u_1, u_2, u_3) be a triple of rectangles participating in the sum in (2.2.7). Define $I_s = I_{s_i} = I_{u_3}$. Defining the frequency projections requires a little bit more work, we cannot just use the dyadic grid. We want these projections to satisfy the following properties:

$$\mathcal{J} = \bigcup_{s \in S} \left(\omega_s \cup \omega_{s_1} \cup \omega_{s_2} \cup \omega_{s_3}\right) \text{ is a grid.} \tag{2.2.11}$$

$$\text{If } \omega_{s_i} \subsetneq J \text{ for some } J \in \mathcal{J}, \text{ then } \omega_{s_j} \subsetneq J \text{ for some } J \in \mathcal{J} \text{ for all } j = 1, 2, 3. \tag{2.2.12}$$

$$\omega_{s_i} \neq \omega_{s_j} \text{ for } i \neq j \tag{2.2.13}$$

We build these intervals by induction on the cardinality of the set U of triples of rectangles. If this set is nonempty, we pick the triple (u_1, u_2, u_3), such that k, where $|\omega_{u_3}| = 2^k$, is maximal. Let $U' = U \setminus (u_1, u_2, u_3)$. By induction we find the intervals ω_s, ω_{s_i} $(i = 1, 2, 3)$, corresponding to the elements of U'. If there is an element $u' \in U'$ such that $\omega_{u_3'} = \omega_{u_3}$, then we define ω_s, ω_{s_i} $(i = 1, 2, 3)$ to be the same as the corresponding intervals for u'. Otherwise we define (for $i = 1, 2$, or 3) ω_{s_i} to be the convex hull of the interval $C_i \omega_{u_i}$ ($C_1 = \frac{1+\alpha}{\alpha}$, $C_2 = 1 + \alpha$, $C_3 = 1$) and all

all sets $\omega_{s'}$ that intersect it. (Note that, because of the separation of scales (2.2.8) and of frequencies (2.2.10), what we get is only slightly smaller than the interval itself, since the frequency projections of other triples are either much smaller or are located far away.) Next, we define ω_s as follows: take $[a,b]$ to be the convex hull of ω_{s_i} $(i=1,2,3)$, then set ω_s to be the convex hull of $[a,b]$ and all intervals $\omega_{s'}$ that intersect $[a,b]$. Properties (2.2.11) and (2.2.12) are obvious in view of (2.2.10) and (2.2.8). Also $|\omega_s|$ and $|\omega_{s_i}|$ are comparable to 2^k with a factor depending on L. Property (2.2.13) follows from (2.2.5), (2.2.6), and the separation of scales.

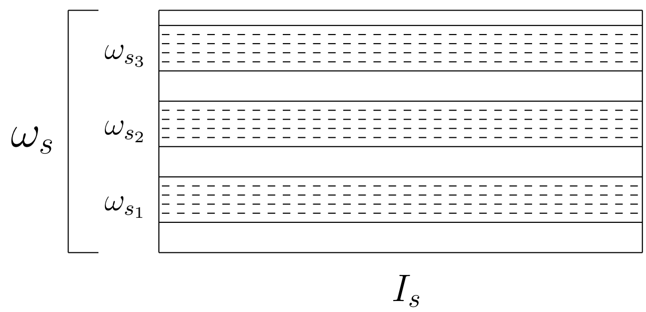

Figure 2. A tri-tile

We define the functions adapted to the tri-tile s with parameters $\nu_1, \nu_2, \mu_1, \mu_2$ as follows:

$$\begin{aligned}
\varphi_{s_1}^{\nu_1,\mu_1,\alpha}(x) &= |I_s|^{-\frac{1}{2}}\phi\left(\frac{x-c(I_s)}{|I_s|}-\nu_1\right)e^{2\pi i\left(\frac{\alpha}{\alpha+1}c(\omega_{s_1})+\theta_{s_1}|\omega_{s_1}|\right)x} = \phi_{u_1}(x),\\
\varphi_{s_2}^{\nu_2,\mu_2,\alpha}(x) &= |I_s|^{-\frac{1}{2}}\phi\left(\frac{x-c(I_s)}{|I_s|}-\nu_2\right)e^{2\pi i\left(\frac{1}{\alpha+1}c(\omega_{s_2})+\theta_{s_2}|\omega_{s_2}|\right)x} = \phi_{u_2}(x),\\
\varphi_{s_3}^{\alpha}(x) &= |I_s|^{-\frac{1}{2}}\phi\left(\frac{x-c(I_s)}{|I_s|}\right)e^{2\pi i\left(c(\omega_{s_3})+\theta_{s_3}|\omega_{s_3}|\right)x} = \phi_{u_3}(x),
\end{aligned}$$

where the error terms θ_{s_i} in the modulations are chosen so that $\frac{\alpha}{\alpha+1}c(\omega_{s_1}) + \theta_{s_1}|\omega_{s_1}| = c(\omega_{u_1})$, $\frac{1}{\alpha+1}c(\omega_{s_2}) + \theta_{s_2}|\omega_{s_2}| = c(\omega_{u_2})$, and $c(\omega_{s_2}) + \theta_{s_3}|\omega_{s_3}| = c(\omega_{u_3})$. Obviously, $|\theta_{s_i}| \le CL$.

Then the expression inside the absolute values in (2.2.7) becomes exactly

$$\sum_{s_3 \in \bigcup\limits_{k\in\mathbf{Z}} \mathbf{S}_k} |I_s|^{-\frac{1}{2}} \varepsilon_{\nu_1,\nu_2,\mu_1,\mu_2,s} \langle f_1, \varphi_{s_1}^{\nu_1,\mu_1,\alpha}\rangle \langle f_2, \varphi_{s_2}^{\nu_2,\mu_2,\alpha}\rangle \langle f_3, \varphi_{s_3}^{\alpha}\rangle.$$

This expression needs to be controlled with bounds that grow polynomially in the parameters ν_1, ν_2, and are independent of μ_1, μ_2. We will work with sums over finite sets of tri-tiles and get bounds independent of the choice of the finite set, which is clearly sufficient by a limiting argument.

Note that if ω_{u_i} and $\omega_{u_i'}$ were not disjoint, then neither are ω_{s_i} and $\omega_{s_i'}$. Thus we have an orthogonality relation

$$\text{if } \omega_{s_i} \cap \omega_{s_i'} = \emptyset, \text{ then } \langle \varphi_{s_i}, \varphi_{s_i'}\rangle = 0.$$

For notational convenience, in the sequel we will suppress the dependence of the functions φ_{s_j} on the parameters $\nu_1, \nu_2, \mu_1, \mu_2$. Notice that

$$|\varphi_{s_k}(x)| \le C\left(1 + \left|\frac{x - c(I_s)}{|I_s|} - \nu_k\right|\right)^{-10} \le C\left(1 + \left|\frac{x - c(I_s)}{|I_s|}\right|\right)^{-10} (1+\nu)^{10}.$$

2.3 Estimates for model sums. The case $I_s \subseteq \Omega$.

Let S be a finite set of tri-tiles with fixed data ν_1, ν_2, μ_1, and μ_2. Then we define the "model sum" associated with S as follows:

$$H_S(f_1, f_2)(x) = \sum_{s\in S} |I_s|^{-\frac{1}{2}} \varepsilon_s \langle f_1, \varphi_{s_1}\rangle \langle f_2, \varphi_{s_2}\rangle \varphi_{s_3}(x).$$

We set

$$\Omega = \left\{x :\ M(\chi_{F_1})(x) > 8\,\frac{|F_1|}{|E|}\right\} \bigcup \left\{x :\ M(\chi_{F_2})(x) > 8\,\frac{|F_2|}{|E|}\right\},$$

where M is the Hardy-Littlewood maximal function. Since M if of weak type $(1,1)$ with constant at most 2, it is easy to see that $|\Omega| < \frac{1}{2}|E|$. We now set $E' = E \setminus \Omega$. Obviously, then $|E'| \geq \frac{1}{2}|E|$.

The main idea of this chapter, resembling the one in the previous chapter, is to obtain a good estimate for the expression

$$\int_{E'} H_S(\chi_{F_1}, \chi_{F_2})(x)dx = \langle H_S(\chi_{F_1}, \chi_{F_2}), \chi_{E'} \rangle.$$

To do so we will break the model sum into two parts: the sum over those $s \in S$ for which $I_s \subseteq \Omega$ (easier case) and the sum over tiles with $I_s \nsubseteq \Omega$.

We begin with the easier case. For a dyadic interval J we set

$$\omega(x) = \left(1 + \Big(\frac{|x - c(J)|}{|J|}\Big)^2\right)^5$$

and

$$S_J = \{s \in S : I_s = J\}.$$

We have the following inequalities (for $i = 1, 2, 3$):

$$\|(\langle f, \varphi_{s_i} \rangle)\|_{\ell^\infty(S_J)} \lesssim (1+\nu)^{10} |J|^{-\frac{1}{2}} \|f\|_{L^1(\omega^{-1})} \tag{2.3.1}$$

$$\|\sum_{s \in S_J} \alpha_s \varphi_{s_i}\|_{L^2(\omega)} \lesssim (1+\nu)^{10} \|(\alpha_s)\|_{\ell^2(S_J)} \tag{2.3.2}$$

$$\|(\langle f, \varphi_{s_i} \rangle)\|_{\ell^2(S_J)} \lesssim (1+\nu)^{10} \|f\|_{L^2(\omega^{-1})}. \tag{2.3.3}$$

Indeed, to prove (2.3.1), for any $s \in S_J$ we have

$$\begin{aligned} |\langle f, \varphi_{s_i} \rangle| &= \left| \int_{\mathbb{R}} f(x) |J|^{-\frac{1}{2}} \varphi\Big(\frac{x - c(J)}{|J|} - \nu_i\Big) e^{2\pi i x (C_i c(\omega_{s_i}) + \theta |\omega_{s_i}|)} dx \right| \\ &\leq C\,(1+\nu)^{10} |J|^{-\frac{1}{2}} \|f\|_{L^1(\omega^{-1})}. \end{aligned}$$

Next, we prove (2.3.2), which is an analogue of Bessel's inequality. Although the functions φ_{s_i} are no longer orthogonal in the weighted space $L^2(\omega)$, we will see that they are still "almost" orthogonal in this space. It is straightforward to check that

$$\begin{aligned}|\langle\varphi_{s_i},\varphi_{s_i'}\rangle_\omega| &= \Big|\big(|\varphi(y-\nu_i)|^2(1+y^2)^5\big)\widehat{\ }\big(\big(K_i(c(\omega_{s_i})-c(\omega_{s_i'}))+(\theta_{s_i}-\theta_{s_i'})|\omega_{s_i}|\big)|J|\big)\Big| \\ &\le C(1+\nu)^{10}\big(1+K_i|c(\omega_{s_i})-c(\omega_{s_i'})|\,|J|\big)^{-10},\end{aligned}$$

since $|\varphi(y-\nu_i)|^2(1+y^2)^5$ (and its Fourier transform) is a Schwartz function (here $K_1=\frac{\alpha}{\alpha+1}$, $K_2=\frac{1}{\alpha+1}$, $K_3=1$). Now we have

$$\begin{aligned}\Big\|\sum_{s\in S_J}\alpha_s\varphi_{s_i}\Big\|^2_{L^2(\omega)} &\le \sum_{s,s'\in S_J}|\alpha_s|\,|\alpha_{s'}|\,|\langle\varphi_{s_i},\varphi_{s_i'}\rangle_\omega| \\ &\le C(1+\nu)^{10}\sum_{k,m}|\alpha_k|\,|\alpha_m|(1+|k-m|)^{-10} \\ &\le 2C(1+\nu)^{10}\sum_{k\in\mathbb{Z}}\Big(|\alpha_k|^2\sum_{m\in\mathbb{Z}}(1+|k-m|)^{-10}\Big) \\ &\le C'(1+\nu)^{10}\|(\alpha_k)\|^2_{\ell^2}.\end{aligned}$$

Note that (2.3.3) is the dual statement of (2.3.2).

Next, we prove the following estimate:

Lemma 2.3.1. *For $A>1$ we have*

$$\|H_{S_J}(\chi_{F_1},\chi_{F_2})\|_{L^1((AJ)^c)}\le(1+\nu)^{20}C_MA^{-M}\,|J|\,\inf_{x\in J}M(\chi_{F_1})(x)\,\inf_{x\in J}M_2(\chi_{F_2})(x).$$

Proof. If we write $H_{S_J}(\chi_{F_1},\chi_{F_2})=(H_{S_J}(\chi_{F_1},\chi_{F_2})\omega^{\frac12})(\omega^{-\frac12})$ and use Hölder's inequality, we obtain:

$$
\begin{aligned}
\|H_{S_J}(\chi_{F_1},\chi_{F_2})\|_{L^1((AJ)^c)} &\leq \|\omega^{-\frac{1}{2}}\|_{L^2((AJ)^c)}\,\|H_{S_J}(\chi_{F_1},\chi_{F_2})\|_{L^2(\omega)}\\
&\leq C\,A^{-M}|J|^{\frac{1}{2}}\Big\|\sum_{s\in S_J}|J|^{-\frac{1}{2}}\langle\chi_{F_1},\varphi_{s_1}\rangle\langle\chi_{F_2},\varphi_{s_2}\rangle\varphi_{s_3}\Big\|_{L^2(\omega)}\\
&\leq C\,A^{-M}\|(\langle\chi_{F_1},\varphi_{s_1}\rangle\langle\chi_{F_2},\varphi_{s_2}\rangle)\|_{\ell^2(S_J)}\\
&\leq C\,A^{-M}\|(\langle\chi_{F_1},\varphi_{s_1}\rangle)\|_{\ell^\infty(S_J)}\|(\langle\chi_{F_2},\varphi_{s_2}\rangle)\|_{\ell^2(S_J)}\\
&\leq C\,(1+\nu)^{20}A^{-M}|J|^{-\frac{1}{2}}\|\chi_{F_1}\|_{L^1(\omega^{-1})}\|\chi_{F_2}\|_{L^2(\omega^{-1})}\\
&\leq C\,(1+\nu)^{20}A^{-M}|J|^{-\frac{1}{2}}|J|\inf_x M(\chi_{F_1})\,|J|^{\frac{1}{2}}\inf_J M_2(\chi_{F_2}),
\end{aligned}
$$

where M is the Hardy-Littlewood maximal function, and $M_2(f) = M^{\frac{1}{2}}(f^2)$. In the last estimate we have used the fact that

$$
1+\Big(\frac{|x-c(J)|}{|J|}\Big)^2 \geq 1+\Big(\frac{|x-\theta|}{|J|}-\frac{1}{2}\Big)^2
$$

for all $\theta \in J$. □

The main conclusion is the following:

Lemma 2.3.2. *For all F_1, F_2 we have the estimate*

$$
\Big|\int_{E'}\sum_{s:I_s\subseteq\Omega}|I_s|^{-\frac{1}{2}}\langle\chi_{F_1},\phi_{s_1}\rangle\langle\chi_{F_2},\phi_{s_2}\rangle\phi_{s_3}(x)\,dx\Big| \leq C_\nu\min_{i=1,2}|F_i|^{\frac{1}{2}}|F_1|^{\frac{1}{2}}|F_2|^{\frac{1}{2}}|E|^{-\frac{1}{2}}, \tag{2.3.4}
$$

where $C_\nu \leq C\,(1+\nu)^{20}$.

Proof. Since the roles of F_1 and F_2 are symmetric, it will suffice to prove that (2.3.4) holds with the expression $C\,|F_1|\,|F_2|^{\frac{1}{2}}|E|^{-\frac{1}{2}}$ on the right hand side of the inequality.

We organize all dyadic intervals $J \subseteq \Omega$ into sets $\mathcal{F}_k$ $(k \geq 0)$ in the following way:

$$\mathcal{F}_k = \{J : 2^k J \subseteq \Omega,\, 2^{k+1} J \nsubseteq \Omega\}.$$

We note that

$$\sum_{J \in \mathcal{F}_k} |J| \leq 4|\Omega| \leq 2|E|.$$

Indeed, assume J_{max} is a maximal element of $\mathcal{F}_k$ with respect to inclusion. If $J \subseteq J_{max}$ and $|J| < |J_{max}|$, then J must have a common endpoint with J_{max} (otherwise, we would have $2^{k+1}J = 2^k(2J) \subseteq 2^k J_{max} \subseteq \Omega$, thus $J \notin \mathcal{F}_k$). Thus, for each particular scale, J_{max} may contain at most 2 intervals belonging to $\mathcal{F}_k$. Therefore

$$\sum_{J \in \mathcal{F}_k, J \subseteq J_{max}} |J| \leq \sum_{k=0}^{\infty} 2^{-k+1} |J_{max}| \leq 4|J_{max}|.$$

Since the maximal elements of $\mathcal{F}_k$ are disjoint, summing over them we obtain the required conclusion.

Also, for any $J \in \mathcal{F}_k$ we have $E' \subseteq (\Omega)^c \subseteq (2^k J)^c$. Thus we have:

$$\begin{aligned}
& \left| \int_{E'} H_{\{I_s \subseteq \Omega\}}(\chi_{F_1}, \chi_{F_2}) dx \right| \\
\leq \; & \sum_{J \subseteq \Omega} \left| \int_{E'} H_{S_J}(\chi_{F_1}, \chi_{F_2}) dx \right| \\
= \; & \sum_{k=0}^{\infty} \sum_{J \in \mathcal{F}_k} \left| \int_{E'} H_{S_J}(\chi_{F_1}, \chi_{F_2}) dx \right| \\
\leq \; & \sum_{k=0}^{\infty} \sum_{J \in \mathcal{F}_k} \| H_{S_J}(\chi_{F_1}, \chi_{F_2}) \|_{L^1((2^k J)^c)} \\
\leq \; & C_M (1+\nu)^{20} \sum_{k=0}^{\infty} \sum_{J \in \mathcal{F}_k} |J| 2^{-kM} \inf_{x \in J} M(\chi_{F_1}) \inf_{x \in J} M_2(\chi_{F_2})
\end{aligned}$$

$$
\begin{aligned}
&\le C_M(1+\nu)^{20}\sum_{k=0}^{\infty}2^{-kM}C_0^{2k+2}\sum_{J\in\mathcal{F}_k}|J|\inf_{2^{k+1}J}M(\chi_{F_1})\inf_{2^{k+1}J}M_2(\chi_{F_2})\\
&\le C'(1+\nu)^{20}\sum_{k=0}^{\infty}2^{-kM}C_0^{2k+2}\sum_{J\in\mathcal{F}_k}|J|\frac{|F_1|}{|E|}\left(\frac{|F_2|}{|E|}\right)^{\frac{1}{2}}\\
&\le C(1+\nu)^{20}|F_1|\,|F_2|^{\frac{1}{2}}|E|^{-\frac{1}{2}}.
\end{aligned}
$$

□

Remark. We notice that what Lemma 2.3.2 really says is that this part of the model sum actually satisfies the restricted weak type $(p_1, p_2, \frac{2}{3})$ estimates for $1 \le p_i \le 2$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{3}{2}$ (see Lemma 1.1.1).

2.4 Estimates for model sums. The case $I_s \not\subseteq \Omega$.

We will now deal with the harder case $I_s \not\subseteq \Omega$. This part of the proof is based on an adaptation of the $L^2 \times L^2 \to L^{1,\infty}$ estimate in [21].

We denote by P the set of all tri-tiles $s \in S$, for which $I_s \not\subseteq \Omega$. Tri-tiles admit a partial order. We say that $s < s'$ if $I_s \subseteq I_{s'}$ and $\omega_{s'} \subseteq \omega_s$. We note that s and s' intersect as rectangles if and only if they are comparable under " $<$ ".

The separation of scales (2.2.8) allows to say that if $s < s'$, then $\omega_{s'} \subseteq \omega_{s_i}$ for some $i = 1, 2, 3$ or it is disjoint with all ω_{s_i}'s.

We say that a collection of tri-tiles T is a tree with top t if for all $s \in T$, $s < t$. Every finite collection of tri-tiles S is a union of trees. Indeed, if we denote by S^* the set of all elements in S which are maximal under " $<$ ", and, for each $t \in S^*$, T_t is the maximal tree in S with top t, then $S = \cup_{t\in S^*} T_t$. We refine the notion of the tree by saying that T is a *j-tree* ($j = 1, 2, 3$) if T is a tree with top T and for every $s \in T$, $\omega_{s_j} \cap \omega_t = \emptyset$.

For a tree T, $s \in T$, $s \neq t$, at most one of the intervals ω_{s_i} can intersect ω_t. Thus if we denote $T_k = \{s \in T : \omega_{s_k} \cap \omega_t \neq \emptyset\}$, $k = 1, 2, 3$, then T_k is a j-tree for $j \neq k$ (there are also elements such that $\omega_{s_i} \cap \omega_t = \emptyset$ for all $i = 1, 2, 3$, but those may be added to any of the T_k's). Then $T = \cup_{k=1}^{3} T_k$, i.e. any tree is a union of at most three subtrees which are j-trees for at least two choices of j.

For a k-tree T we set

$$\Delta(T,k) = \frac{1}{\|f_k\|_2}\left(|I_t|^{-1}\sum_{s\in T}|\langle f_k, \varphi_{s_k}\rangle|^2\right)^{\frac{1}{2}}.$$

and we define the k-energy of a finite set of tiles S by

$$\mathcal{E}_k(S) = \sup \Delta(T,k), \tag{2.4.1}$$

where the supremum is taken over all k-trees $T \subseteq S$. Note that a singleton $\{s\}$ is a k-tree for all k, so for all $s \in S$,

$$|I_s|^{-\frac{1}{2}}|\langle f_k, \varphi_{s_k}\rangle| \leq \mathcal{E}_k(S)\,\|f_k\|_2.$$

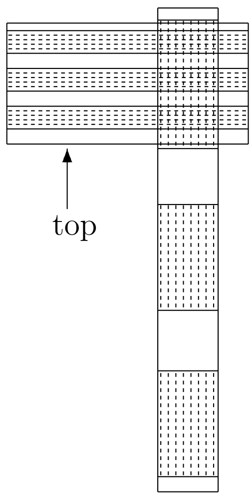

Figure 3. A tree, which is a 2-tree,a 3-tree, but not a 1-tree.

Now fix some $j = 1, 2, 3$ and let T be a k-tree for $k \neq j$. Applying the above estimate and the Cauchy-Schwarz inequality, we deduce

$$\begin{aligned} |\langle H_T(f_1, f_2), f_3\rangle| &\leq \sum_{s\in T} \frac{|\langle f_j, \varphi_{s_j}\rangle|}{|I_s|^{\frac{1}{2}}} \prod_{k\neq j} |\langle f_k, \varphi_{s_k}\rangle| \\ &\leq \mathcal{E}_j(S)\, \|f_j\|_2 \sum_{s\in T} \prod_{k\neq j} |\langle f_k, \varphi_{s_k}\rangle| \\ &\leq \mathcal{E}_j(S)\, \|f_j\|_2\, |I_t| \prod_{k\neq j} \Delta(T, k) \|f_k\|_2 \\ &\leq |I_t| \prod_{j=1}^{3} \mathcal{E}_j(S)\, \|f_j\|_2. \end{aligned} \tag{2.4.2}$$

This is a crucial estimate on a single tree that will be used in conjunction with the idea that any tree can be written as a union of three trees of the above type.

Next, we state the main lemma which will allow us to obtain the estimates for the model sums (cf. [21]).

Lemma 2.4.1. *Let S be a finite set of tri-tiles. Then S can be written as a union of two sets $S = S_1 \cup S_2$, which have the following properties. Let S_1^* be the set of elements which are maximal in S_1 under " $<$ " (i.e. S_1 is a union of trees with tops in S_1^*). We then have*

$$\sum_{t\in S_1^*} |I_t| \leq C_1(1+\nu)^{20} \mathcal{E}_k(S)^{-2}, \tag{2.4.3}$$

$$\mathcal{E}_k(S_2) \leq \frac{1}{2}\, \mathcal{E}_k(S). \tag{2.4.4}$$

This lemma only yields weak-type estimates from $L^2 \times L^2$ into $L^{1,\infty}$. But the fact that we are now working with the set of tiles $P = \{s \in S : I_s \nsubseteq \Omega\}$ and all functions are characteristic of some sets gives us an advantage quantified by the following energy estimate which appeared in [14], [10], and is essentially contained in [25]:

Lemma 2.4.2. *For $k = 1, 2$ and $f_k = \chi_{F_k}$, there exists a constant $C > 0$, such that the following estimate is valid:*

$$\mathcal{E}_k(P) \leq C|E|^{-\frac{1}{2}} \min \left[\left(\frac{|F_k|}{|E|} \right)^{\frac{1}{2}}, \left(\frac{|F_k|}{|E|} \right)^{-\frac{1}{2}} \right] \tag{2.4.5}$$

With these two lemmata at hand we can derive an estimate of the model sum for the case $I_s \not\subseteq \Omega$ in the following way. We construct inductively the sequence of pairwise disjoint sets P_j such that

$$P = \bigcup_{j=-\infty}^{n_0} P_j$$

and the following properties are satisfied:

(1) $\mathcal{E}_k(P_j) \leq 2^{j+1}$ for $k = 1, 2, 3$.

(2) P_j is a union of trees T_{jk} such that $\sum_k |I_{top(T_{jk})}| \leq C_0(1+\nu)^{20} 2^{-2j}$ for all $j \leq n_0$.

(3) $\mathcal{E}_k(P \setminus (P_{n_0} \cup \cdots \cup P_j)) \leq 2^j$ for $k = 1, 2, 3$.

This sequence is constructed in the following way: We start the induction at the number $j = n_0$ such that $\mathcal{E}_k \leq 2^{n_0}$ for $k = 1, 2, 3$. We set $P_{n_0} = \emptyset$. Then properties (1), (2), and (3) are clearly satisfied. Assuming that we have already constructed the set P_n, we construct P_{n-1} as follows. Let $S = P \setminus (P_{n_0} \cup \cdots \cup P_n)$. First, if $\mathcal{E}_1(S) > 2^{n-1}$, then apply Lemma 2.4.1 to S with $k = 1$, thus obtaining the sets $S_1^{(1)}$ with a control of the sum of the tops and $S_2^{(1)}$ with small 1-energy, otherwise just skip this step (i.e. $S_1^{(1)} = \emptyset$) . Then, in the same fashion, if $\mathcal{E}_2(S_2^{(1)}) > 2^{n-1}$, we apply this lemma to $S_2^{(1)}$ obtaining the set $S_1^{(2)}$ and $S_2^{(2)}$ (otherwise again skipping this step, $S_1^{(2)} = \emptyset$). And, finally, we apply Lemma 2.4.1 for the third time with $k = 3$ to the set $S_2^{(2)}$ to obtain $S_1^{(3)}$ and $S_2^{(3)}$ (we also skip this step, if $\mathcal{E}_1(S_1^{(2)}) \leq$

2^{n-1}). We set $P_{n-1} = S_1^{(1)} \cup S_1^{(2)} \cup S_1^{(3)}$. Observe that if all three steps were skipped, then $P_{n-1} = \emptyset$. We have to verify that properties (1)-(3) indeed hold.

First, for $k = 1, 2, 3$:

$$\mathcal{E}_k(P \setminus (P_{n_0} \cup \cdots \cup P_{n-1})) \leq \frac{1}{2}\mathcal{E}_k(P \setminus (P_{n_0} \cup \cdots \cup P_n)) \leq 2^{n-1}$$

by Lemma 2.4.1 (and the fact that we just skipped the corresponding step if this was already so for some k) , thus verifying (3). Then,

$$\mathcal{E}_k(P_{n-1}) \leq \mathcal{E}_k(P \setminus (P_{n_0} \cup \cdots \cup P_n)) \leq 2^n = 2^{(n-1)+1},$$

which proves (1). And finally, using Lemma 2.4.1, we have (with the convention that $S_1^{(0)} = S$):

$$\sum_k |I_{top(T_{jk})}| \leq C(1+\nu)^{20} \sum_{k=1}^{3} \mathcal{E}_k(S_1^{(k-1)})^{-2} \leq 3C(1+\nu)^{20} 2^{-2(n-1)},$$

since the sum actually ranges over those values of k for which $\mathcal{E}_k(S_1^{(k-1)}) > 2^{n-1}$, otherwise the corresponding part of P_{n-1} is empty.

Taking into account the above families P_j, we obtain the following:

$$\begin{aligned}
& |\langle H_P(\chi_{F_1}, \chi_{F_2}), \chi_{E'}\rangle| \qquad (2.4.6)\\
\leq\ & \sum_{j=-\infty}^{\infty} \sum_k |\langle H_{T_{jk}}(\chi_{F_1}, \chi_{F_2}), \chi_{E'}\rangle| \\
\leq\ & C \sum_{j=-\infty}^{\infty} \big(\sum_k |I_{top_{T_{jk}}}|\big) \mathcal{E}_1(\chi_{F_1}, S_j)\mathcal{E}_2(\chi_{F_2}, S_j)\mathcal{E}_3(\chi_{E'}, S_j)|F_1|^{\frac{1}{2}}|F_2|^{\frac{1}{2}}|E|^{\frac{1}{2}} \\
\leq\ & C' \sum_{j=-\infty}^{\infty} 2^{-2j} \min\big(|F_1|^{\frac{1}{2}}, \frac{|F_1|^{\frac{1}{2}}}{|E|}, 2^j\big) \min\big(|F_2|^{\frac{1}{2}}, \frac{|F_2|^{\frac{1}{2}}}{|E|}, 2^j\big) 2^j\, |F_1|^{\frac{1}{2}}|F_2|^{\frac{1}{2}}|E|^{\frac{1}{2}} \\
=\ & C' \sum_{j=-\infty}^{\infty} 2^{-j} \min\big(|F_1|^{-\frac{1}{2}}, \frac{|F_1|^{\frac{1}{2}}}{|E|}, 2^j\big) \min\big(|F_2|^{-\frac{1}{2}}, \frac{|F_2|^{\frac{1}{2}}}{|E|}, 2^j\big)\, |F_1|^{\frac{1}{2}}|F_2|^{\frac{1}{2}}|E|^{\frac{1}{2}},
\end{aligned}$$

where we used the estimate on a single tree (2.4.2) and the improved energy estimate (2.4.5).

We control (2.4.6) in different cases:

A) Suppose $|E| \geq |F_2| \geq |F_1|$. Then (2.4.6) is bounded by

$$\Bigg(\sum_{j=-\infty}^{\log\frac{|F_1|^{\frac12}}{|E|}} 2^j + \sum_{j=\log\frac{|F_1|^{\frac12}}{|E|}}^{\log\frac{|F_2|^{\frac12}}{|E|}} |F_1|^{\frac12}|E|^{-1} + \sum_{j=\log\frac{|F_2|^{\frac12}}{|E|}}^{\infty} 2^{-j}|F_1|^{\frac12}|F_1|^{\frac12}|E|^{-2}\Bigg)|F_1|^{\frac12}|F_2|^{\frac12}|E|^{\frac12}$$

$$\lesssim |F_1|\,|F_2|^{\frac12}|E|^{-\frac12}\left(1+\log\frac{|F_2|}{|F_1|}\right).$$

So, by symmetry, in the case $|E| \geq |F_1|, |F_2|$ the expression (2.4.6) can be controlled by

$$\min(|F_1|^{\frac12}, |F_2|^{\frac12})|F_1|^{\frac12}\,|F_2|^{\frac12}|E|^{-\frac12}\left(1+\left|\log\frac{|F_2|}{|F_1|}\right|\right). \tag{2.4.7}$$

We may also note that in this case $|\log\frac{|F_2|}{|F_1|}| \leq \log\frac{|E|^2}{|F_1|\,|F_2|}$.

B) Suppose that $|F_1| \leq |E| \leq |F_2|$ and $|E|^2 \geq |F_1|\,|F_2|$. In this case we can bound (2.4.6) by

$$\Bigg(\sum_{j=-\infty}^{\log\frac{|F_1|^{\frac12}}{|E|}} 2^j + \sum_{j=\log\frac{|F_1|^{\frac12}}{|E|}}^{\log|F_2|^{-\frac12}} |F_1|^{\frac12}|E|^{-1} + \sum_{j=\log|F_2|^{-\frac12}}^{\infty} 2^{-j}|F_1|^{\frac12}|F_1|^{-\frac12}|E|^{-1}\Bigg)|F_1|^{\frac12}|F_2|^{\frac12}|E|^{\frac12}$$

$$\lesssim |F_1|\,|F_2|^{\frac12}|E|^{-\frac12}\left(1+\log\frac{|E|^2}{|F_1|\,|F_2|}\right).$$

Thus, by symmetry, in the case when $|E|$ is between $|F_1|$ and $|F_2|$ and $|E|^2 \geq |F_1|\,|F_2|$ we obtain that (2.4.6) is bounded by

$$\min(|F_1|^{\frac12}, |F_2|^{\frac12})|F_1|^{\frac12}\,|F_2|^{\frac12}|E|^{-\frac12}\left(1+\log\frac{|E|^2}{|F_1|\,|F_2|}\right). \tag{2.4.8}$$

The other cases work in a similar way:

C) If $|E|$ is between $|F_1|$ and $|F_2|$, but $|E|^2 \leq |F_1|\,|F_2|$, the bound is

$$\min(|F_1|^{\frac{1}{2}}, |F_2|^{\frac{1}{2}})|E|^{\frac{1}{2}}\left(1 + \log\frac{|F_1|\,|F_2|}{|E|^2}\right). \tag{2.4.9}$$

D) For $|E| \leq |F_1|$, $|F_2|$, we obtain the bound

$$\min(|F_1|^{\frac{1}{2}}, |F_2|^{\frac{1}{2}})|E|^{\frac{1}{2}}\left(1 + \left|\log\frac{|F_1|}{|F_2|}\right|\right). \tag{2.4.10}$$

Combining the four cases A), B), C), and D) we obtain the following inequality for the case when the tiles s satisfy $I_s \nsubseteq \Omega$:

$$\begin{aligned}&\left|\int_{E'} H_{\{s:\, I_s \nsubseteq \Omega\}}(\chi_{F_1}, \chi_{F_2})dx\right| \\ &\leq C_1 \min(|F_1|^{\frac{1}{2}}, |F_2|^{\frac{1}{2}}) \min\left(\frac{|F_1|^{\frac{1}{2}}\,|F_2|^{\frac{1}{2}}}{|E|^{\frac{1}{2}}}, |E|^{\frac{1}{2}}\right)\left(1 + \left|\left|\log\frac{|F_1|}{|E|}\right| - \left|\log\frac{|F_2|}{|E|}\right|\right|\right).\end{aligned} \tag{2.4.11}$$

As a consequence of the results so far we deduce the following:

Proposition 2.4.3. *There exists a constant C_1 such that, for any sets E,F_1,F_2 with the property that $|E|^2 \geq |F_1|\,|F_2|$ there exists a set $E' \subseteq E$ with $|E'| \geq \frac{1}{2}|E|$ such that for any set of tri-tiles S we have the following estimate:*

$$\left|\int_{E'} H_S(\chi_{F_1}, \chi_{F_2})(x)dx\right| \leq C_1 \min(|F_1|, |F_2|)^{\frac{1}{2}}|F_1|^{\frac{1}{2}}\,|F_2|^{\frac{1}{2}}|E|^{-\frac{1}{2}}\left(1+\log\frac{|E|^2}{|F_1|\,|F_2|}\right). \tag{2.4.12}$$

This estimate is also valid for the bilinear Hilbert transform H.

Proof. The result for H_S follows from the estimates (2.3.4) and (2.4.11). Note that the construction of E' did not depend on the choice of the set of tri-tiles, so E' is the same for any S, and by an averaging argument this estimate is also valid for H. □

It is clear that, since both adjoints of H_S, are "essentially" the same operators, the same estimate (with different constants) also holds for them.

2.5 Proof of the Energy Lemma 2.4.1

To prove the lemma we have to further refine the notion of a tree. We say that a k-tree T is a *left tree* if for each $s \in T$, $s \neq t$, ω_{s_k} lies to the left of (below) ω_t. It is worth noting that for s, s' in a left tree T, $s < s'$ implies that ω_{s_k} lies to the left of $\omega_{s'_k}$. The right trees are defined correspondingly. Obviously, any k-tree is a union of a left and a right tree, so we can deal with left and right trees separately. First, we deal with the left trees.

We construct the set S_{1L} as a union of trees. First let us choose a left tree T_1, $T_1 \subseteq S$, which has top $t(1)$ and satisfies the following conditions: $\Delta_k(T_1) \geq \frac{1}{4}\mathcal{E}_k(S)$. (Either left or right trees with this property exist by pigeonhole principle.) T_1 is maximal with respect to inclusion; $t(1)$ is a maximal element of S with respect to "$<$". Finally, $\omega_{t(1)}$ is leftmost, i.e. $\min\{\xi : \xi \in \omega_{t(1)}\}$ is minimal among all such trees.

After we have selected T_1, we let $\widetilde{T_1}$ to be the maximal tree in S with top $t(1)$. We now remove $\widetilde{T_1}$ from S and repeat the procedure inductively until there are no such left trees. We set $S_{1L} = \cup \widetilde{T}_l$. Obviously, if $T \subseteq S \setminus S_{1L}$ is a left tree, then $\Delta_k(T) < \frac{1}{4}\mathcal{E}_k(S)$. Also, if $s \in T_l$, $s' \in T_{l'}$, and $\omega_{s_k} \subsetneqq \omega_{s'_k}$, then

$$I_{t(l)} \cap I_{s'} = \emptyset. \tag{2.5.1}$$

Indeed, we have $\omega_{t(j)} \subseteq \omega_s \subseteq \omega_{s'_k}$ (the last inclusion, which is not necessarily strict, is true due to $\omega_{s_k} \subsetneqq \omega_{s'_k}$ and the separation of scales). Thus, $\omega_{t(l)}$, as $\omega_{s'_k}$, lies to the left of $\omega_{t(l')}$, which implies that T_l was selected first. So, s' is not in $\widetilde{T}_l$, i.e. $t(l)$ and s' are not comparable under " $<$ ". But their frequency projections intersect forcing the time projections to be disjoint.

Also, we have the following combinatorial property: Let $s' \in T_{l'}$ and $s'' \in T_{l''}$ be to different tri-tiles. If both $\omega_{s_k} \subsetneqq \omega_{s'_k}$ and $\omega_{s_k} \subsetneqq \omega_{s''_k}$, then

$$I_{s'} \cap I_{s''} = \emptyset. \tag{2.5.2}$$

To prove it, observe that, since $\omega_{s'_k}$ and $\omega_{s''_k}$ intersect, three things may happen: (a) $\omega_{s''_k} \subseteq \omega_{s'_k} \subsetneqq \omega_{s'_k}$, in which case $I_{s''}$ is disjoint from $I_{t(l')}$ by (2.5.1)and thus also from $I_{s'}$; (b)$\omega_{s'_k} \subseteq \omega_{s''_k}$, which is similar; (c) $\omega_{s''_k} = \omega_{s'_k}$, but then since s' and s'' are different, $I_{s'}$ and $I_{s''}$ are disjoint.

We do the same procedure with the right trees in $S \setminus S_{1L}$ to construct S_{1R}. Finally, we put $S_1 = S_{1L} \cup S_{1R}$, $S_2 = S \setminus S_1$. Property (2.4.4) is now obvious, since we have excluded all the trees with large energy.

It remains to verify (2.4.3). Let us define $S'_1 = \cup T_l$, where T_l's are the left and right trees chosen in the inductive procedures above. We have $\Delta_k(T_l) \geq \frac{1}{4}\,\mathcal{E}_k(S)$, which implies

$$|I_{t(l)}| \leq \frac{16}{\mathcal{E}_k(S)^2 \|f_k\|_2^2} \sum_{s \in T_l} |\langle f_k, \varphi_{s_k} \rangle|^2.$$

Let us set $B := \| \sum_{s\in S_1'} \langle f_k, \varphi_{s_k}\rangle \varphi_{s_k}\|_2$. It will suffice to show that $B \le K\|f_k\|_2$ as the following calculation shows:

$$\begin{aligned}
\sum_l |I_{t(l)}| &\le \frac{16}{\mathcal{E}_k(S)^2\|f_k\|_2^2}\sum_{s\in S_1'} |\langle f_k, \varphi_{s_k}\rangle|^2 \qquad (2.5.3)\\
&= \frac{16}{\mathcal{E}_k(S)^2\|f_k\|_2^2}\langle f_k, \sum_{s\in S_1'}\langle f_k, \varphi_{s_k}\rangle\varphi_{s_k}\rangle\\
&\le \frac{16}{\mathcal{E}_k(S)^2\|f_k\|_2^2}\|f_k\|_2\, B\\
&\le \frac{16K(1+\nu)^{20}}{\mathcal{E}_k(S)^2}.
\end{aligned}$$

Now, we expand B^2 to get a diagonal term $\mathcal{D}$ and off-diagonal term $\mathcal{O}$. The diagonal term is

$$\mathcal{D} = \sum_{\substack{s,s'\in S_1'\\ \omega_s=\omega_{s'}}} |\langle f_k, \varphi_{s_k}\rangle|\,|\langle f_k, \varphi_{s_k'}\rangle|\,|\langle \varphi_{s_k}, \varphi_{s_k'}\rangle|.$$

The off-diagonal term is

$$\mathcal{O} := 2\sum_{s\in S_1'} |\langle f_k, \varphi_k\rangle| \mathcal{O}(s),$$

where

$$\mathcal{O}(s) := \sum_{s'\in S(s)} |\langle \varphi_{s_k}, \varphi_{s_k'}\rangle|\,|\langle f_k, \varphi_{s_k'}\rangle|$$

and $S(s) = \{s' \in S_1' : \omega_{s_k} \subsetneq \omega_{s_k'}\}$.

Before we estimate these terms, let us examine the expressions of the form $|\langle \varphi_{s_k}, \varphi_{s_k'}\rangle|$.

Lemma 2.5.1. *For any $k = 1,2,3$ and any tri-tiles s, s' we have*

$$|\langle \varphi_{s_k}, \varphi_{s_k'}\rangle| \le C(1+\nu)^{20}\frac{\min_{\pm}\left(\frac{|I_s|}{|I_{s'}|}\right)^{\pm\frac{1}{2}}}{\left(1+\frac{|c(I_s)-c(I_{s'})|}{\max(|I_s|,|I_{s'}|)}\right)^{10}}$$

In particular, for the case $\omega_s = \omega_{s'}$, we get

$$|\langle \varphi_{s_k}, \varphi_{s'_k}\rangle| \le C(1+\nu)^{20}(1+\frac{|c(I_s)-c(I_{s'})|}{|I_s|})^{-10}.$$

If $|I_s| \ge |I_{s'}|$, we have

$$|\langle \varphi_{s_k}, \varphi_{s'_k}\rangle| \le C\Big(\frac{|I_s|}{|I_{s'}|}\Big)^{\frac{1}{2}} \int_{I_{s'}} \frac{|I_s|^{-1}dx}{\Big(1+\frac{|x-c(I_s)|}{|I_s|}\Big)}$$

First we estimate the diagonal term $\mathcal{D}$. Using the inequality $2ab \le a^2+b^2$, we estimate it by

$$\begin{aligned}
&\sum_{s\in S'_1} |\langle f_k, \varphi_{s_k}\rangle|^2 \sum_{\substack{s'\in S'_1\\ \omega_{s'}=\omega_s}} |\langle \varphi_{s_k}, \varphi_{s'_k}\rangle| \qquad (2.5.4)\\
\le\ & \sum_{s\in S'_1} |\langle f_k, \varphi_{s_k}\rangle|^2 \sum_{\substack{s'\in S'_1\\ \omega_{s'}=\omega_s}} C(1+\nu)^{20}\big(1+\frac{|c(I_s)-c(I_{s'})|}{|I_s|}\big)^{-10}\\
=\ & C(1+\nu)^{20}\sum_{s\in S'_1} |\langle f_k, \varphi_{s_k}\rangle|^2 \sum_{m\in\mathbf{Z}}(1+|m|)^{-10}\\
\le\ & C'(1+\nu)^{20}\sum_{s\in S'_1} |\langle f_k, \varphi_{s_k}\rangle|^2\\
\le\ & C''(1+\nu)^{20}\|f_k\|_2 B.
\end{aligned}$$

The final inequality has been shown in (2.5.3). Next, we take care of the off-diagonal case. First of all, notice that by Cauchy-Schwarz

$$\mathcal{O} \le 2\big(\sum_{s\in S'_1} |\langle f_k, \varphi_{s_k}\rangle|^2\big)^{\frac{1}{2}}\big(\sum_{s\in S'_1}\mathcal{O}(s)^2\big)^{\frac{1}{2}} \le 2(B\|f_k\|_2)^{\frac{1}{2}}\big(\sum_{s\in S'_1}\mathcal{O}(s)^2\big)^{\frac{1}{2}},$$

where we have once again used the calculation in (2.5.3). Next, we are going to show that for each tree T_l

$$\sum_{s\in T_l}\mathcal{O}(s)^2 \le C(1+\nu)^{40}|I_{t(l)}|\mathcal{E}_k(S)^2\|f_k\|_2^2. \qquad (2.5.5)$$

This will finish the proof of the lemma, since then we would, based on (2.5.3), (2.5.4), and (2.5.5), have

$$\begin{aligned}
B^2 &\leq \mathcal{D}+\mathcal{O} &(2.5.6)\\
&\leq C_1(1+\nu)^{20}\|f_k\|_2 B + C_2(B\|f_k\|_2)^{\frac{1}{2}}(1+\nu)^{20}\mathcal{E}_k(S)\|f_k\|_2\big(\sum_l |I_{t(l)}|\big)^{\frac{1}{2}}\\
&\leq C_1(1+\nu)^{20}\|f_k\|_2 B + C_2' B^{\frac{1}{2}}(1+\nu)^{20}\left(\frac{B}{\mathcal{E}_k(S)^2\|f_k\|_2}\right)^{\frac{1}{2}}\\
&= K(1+\nu)^{20}B\|f_k\|_2,
\end{aligned}$$

which implies $B \leq K\|f_k\|_2$.

Now we prove (2.5.5). Fix a tree T_l with top $t(l)$ and $s \in T_l$. If $s' \in S(s)$, then obviously s' cannot be in the same k-tree as s, since ω_{s_k} intersects $\omega_{s'_k}$, but then, by (2.5.1), $I_{t(l)} \cap I_{s'} = \emptyset$. It follows from (2.5.2) that for each point $x \in I_{t(l)}$, there are at most two intervals $I_{s'}$ with $s' \in S(s)$ that cover it (one may come from a left tree, another from a right tree). Also we must have $|I_s| \geq |I_{s'}|$, thus using lemma 2.5.1 and the fact that any singleton is a tree we get

$$\begin{aligned}
\mathcal{O}(s) &= \sum_{s'\in S(s)} |\langle \varphi_{s_k}, \varphi_{s'_k}\rangle|\,|\langle f_k, \varphi_{s'_k}\rangle| &(2.5.7)\\
&\leq C\|f_k\|_2|I_{s'}|^{\frac{1}{2}}\mathcal{E}_k(S)(1+\nu)^{20}\sum_{s'\in S(s)}\left(\frac{|I_s|}{|I_{s'}|}\right)^{\frac{1}{2}}\int_{I_{s'}}\frac{|I_s|^{-1}dx}{\left(1+\frac{|x-c(I_s)|}{|I_s|}\right)^{10}}\\
&\leq C\|f_k\|_2|I_s|^{\frac{1}{2}}\mathcal{E}_k(S)(1+\nu)^{20}|I_s|^{\frac{1}{2}}2\int_{I^c_{t(l)}}\frac{|I_s|^{-1}dx}{\left(1+\frac{|x-c(I_s)|}{|I_s|}\right)^{10}}\\
&\leq C'\|f_k\|_2|I_s|^{\frac{1}{2}}\mathcal{E}_k(S)(1+\nu)^{20}|I_s|^{\frac{1}{2}}\left(1+\frac{\text{dist}\,(I_s, I^c_{t(l)})}{|I_s|}\right)^{-9}.
\end{aligned}$$

Now, summing this expression over $s \in T_l$ we obtain (keeping in mind that I_s's are dyadic intervals contained in $I_{t(l)}$):

$$\begin{aligned}\sum_{s\in T_l}\mathcal{O}(s)^2 &\le C'\|f_k\|_2^2\mathcal{E}_k(S)^2(1+\nu)^{40}\sum_{j=0}^{\infty}|I_s|\sum_{m=1}^{2^{j-1}}2(1+m)^{-9}\\ &\le C''\|f_k\|_2^2\mathcal{E}_k(S)^2(1+\nu)^{40}\sum_{j=0}^{\infty}|I_{t(l)}|2^{-j}\int_{\mathbf{R}}(1+|x|)^{-9}dx\\ &\le C''\|f_k\|_2^2\mathcal{E}_k(S)^2(1+\nu)^{40}|I_{t(l)}|,\end{aligned}$$

which proves (2.5.5), and thus the whole lemma.

2.6 Proof of the Improved Energy Estimate, Lemma 2.4.2

In the proof of the estimate (2.4.5) we closely follow [10]. Fix a j-tree ($j=1$ or 2) $\mathbf{T}$ contained in $\mathbf{P}$ and let $t=\mathrm{top}(\mathbf{T})$ denote its top. We will show that

$$\frac{1}{|I_t|}\sum_{s\in\mathbf{T}}\left|\left\langle\chi_{F_j},\varphi_{s_j}\right\rangle\right|^2\le C\,\min\left\{1,\left(\frac{|F_j|}{|E|}\right)^2\right\} \tag{2.6.1}$$

for some constant C independent of F and $\mathbf{T}$. Then (2.4.5) will follow from (2.6.1) by taking the supremum over all 2-trees $\mathbf{T}$ contained in $\mathbf{P}$. For convenience we set $F=F_j$. Also, we consider left and right trees separately. We only prove the estimate for a left tree, the other case being symmetric. We also shall not care about the factors $(1+\nu)^m$.

We write the function χ_F as $\chi_{F\cap 3I_t}+\chi_{F\cap(3I_t)^c}$. We begin by observing that for s in $\mathbf{P}$ we have

$$\left|\left\langle\chi_{F\cap(3I_t)^c},\varphi_s\right\rangle\right| \le \frac{C_M|I_s|^{\frac12}\inf_{I_s}M(\chi_F)}{\left(1+\dfrac{\mathrm{dist}((3I_t)^c,c(I_s))}{|I_s|}\right)^M}\le C_M|I_s|^{\frac12}\min\left(1,\frac{|F|}{|E|}\right)\left(\frac{|I_s|}{|I_t|}\right)^M$$

(2.6.2)

since I_s meets the complement of Ω for every $s \in \mathbf{P}$. Square this inequality and sum over all s in $\mathbf{T}$ to obtain

$$\sum_{s\in\mathbf{T}} |\langle \chi_{F\cap(3I_t)^c}, \varphi_s\rangle|^2 \leq C\,|I_t| \,\min\left(1, \frac{|F|}{|E|}\right)^2 .$$

We now turn to the corresponding estimate for the function $\chi_{F\cap 3I_t}$. At this point it will be convenient to distinguish the simple case $|F| > |E|$ from the difficult case $|F| \leq |E|$. In the first case we have

$$\sum_{s\in\mathbf{T}} \big|\big\langle \chi_{F\cap 3I_t}, \varphi_s\big\rangle\big|^2 \leq C\,\big\|\chi_{F\cap 3I_t}\big\|_{L^2}^2 \leq C\,|I_t| \,\leq C\,|I_t|\,\min\left(1, \frac{|F|}{|E|}\right)^2 ,$$

since $|F| > |E|$.

We can therefore concentrate on the case $|F| \leq |E|$. In proving (2.4.5) we may assume that there exists a point $x_0 \in I_t$ such that

$$M(\chi_F)(x_0) \leq 8\,\frac{|F|}{|E|}\,;$$

otherwise there is nothing to prove.

We write the set $\Omega_F = \{M(\chi_F) > 8\,\frac{|F|}{|E|}\}$ (since $I_s \not\subseteq \Omega$, we have $I_s \not\subseteq \Omega_F$) as a disjoint union of dyadic intervals J'_ℓ such that the dyadic parent $\widetilde{J}'_\ell$ of J'_ℓ is not contained in Ω_F and therefore

$$|F \cap J'_\ell| \leq |F \cap \widetilde{J}'_\ell| \leq 16\,\frac{|F|}{|E|}\,|J'_\ell|\,.$$

Now some of these dyadic intervals may have size larger than or equal to $|I_t|$. Let J'_ℓ be such an interval. Then we split J'_ℓ in $\frac{|J'_\ell|}{|I_t|}$ intervals $J'_{\ell,m}$ each of size exactly

$|I_t|$. Since there is an $x_0 \in I_t$ with $M(\chi_F)(x_0) \le 8\,\frac{|F|}{|E|}$, if K is the smallest interval that contains x_0 and $J'_{\ell,m}$, then

$$\frac{1}{|K|}\int_K \chi_F\,dx \le 8\left(\frac{|F|}{|E|}\right) \implies |F \cap J'_{\ell,m}| \le |F \cap K| \le 8\left(\frac{|F|}{|E|}\right)|I_t|\,\frac{|K|}{|I_t|}.$$

We conclude that

$$|F \cap J'_{\ell,m}| \le\; 8\,\frac{|F|}{|E|}\,|I_t|\left(1+\frac{\operatorname{dist}(I_t, J'_{\ell,m})}{|I_t|}\right). \tag{2.6.3}$$

We now have a new collection of dyadic intervals $\{J_k\}_k$ contained in Ω_F consisting of all the previous J'_ℓ when $|J'_\ell| < |I_t|$ and the $J'_{\ell,m}$'s when $|J'_{\ell,m}| \ge |I_t|$. In view of the construction we have

$$|F \cap J_k| \le \begin{cases} 16\,\frac{|F|}{|E|}\,|J_k| & \text{when } |J_k| < |I_t| \\ 16\,\frac{|F|}{|E|}\,|J_k|\left(1+\dfrac{\operatorname{dist}(I_t, J_k)}{|I_t|}\right) & \text{when } |J_k| = |I_t| \end{cases} \tag{2.6.4}$$

for all k. We now define the "bad functions"

$$b_k(x) = \left(e^{-2\pi i K_j c(\omega_t)x}\chi_{F\cap 3I_t}(x) - \frac{1}{|J_k|}\int_{J_k} e^{-2\pi i K_j c(\omega_t)y}\chi_{F\cap 3I_t}(y)\,dy\right)\chi_{J_k}(x)$$

which are supported in J_k, have mean value zero, and satisfy

$$\big\|b_k\big\|_{L^1} \le 16\,\frac{|F|}{|E|}\,|J_k|\left(1+\frac{\operatorname{dist}(I_t, J_k)}{|I_t|}\right).$$

Remember that $K_1 = \frac{\alpha}{\alpha+1}$ and $K_2 = \frac{1}{\alpha+1}$. We also set

$$g(x) = e^{-2\pi i K_j c(\omega_t)x}\chi_{F\cap 3I_t}(x) - \sum_k b_k(x)$$

the "good function" of the Calderón-Zygmund decomposition. We have therefore decomposed the function $\chi_{F\cap 3I_t}$ as follows:

$$\chi_{F\cap 3I_t}(x) = g(x)e^{2\pi i K_j c(\omega_t)x} + \sum_k b_k(x)e^{2\pi i K_j c(\omega_t)x}. \tag{2.6.5}$$

We check that $\|g\|_{L^\infty} \le C\frac{|F|}{|E|}$. Indeed, for x in J_k we have

$$g(x) = \frac{1}{|J_k|}\int_{J_k} e^{-2\pi i K_j c(\omega_t)y}\chi_{F\cap 3I_t}(y)\,dy\,,$$

which implies

$$|g(x)| \le \frac{|F\cap 3I_t \cap J_k|}{|J_k|} \le \begin{cases} \dfrac{|F\cap J_k|}{|J_k|} & \text{when } |J_k| < |I_t| \\[2ex] \dfrac{|F\cap 3I_t|}{|I_t|} & \text{when } |J_k| = |I_t| \end{cases}$$

and both of the preceding are at most a multiple of $\frac{|F|}{|E|}$; the latter is because there is an $x_0 \in I_t$ with $M(\chi_F)(x_0) \le 8\,\frac{|F|}{|E|}$. Also, for $x \in (\cup_k J_k)^c = (\Omega_F)^c$ we have

$$|g(x)| = \chi_{F\cap 3I_t}(x) \le M(\chi_F)(x) \le 8\,\frac{|F|}{|E|}\,.$$

We conclude that $\|g\|_{L^\infty} \le C\,\frac{|F|}{|E|}$. Moreover,

$$\|g\|_{L^1} \le \sum_k \int_{J_k} \frac{|F\cap 3I_t\cap J_k|}{|J_k|}\,dx + \|\chi_{F\cap 3I_t}\|_{L^1} \le C\,|F\cap 3I_t| \le C\,\frac{|F|}{|E|}\,|I_t|$$

since the J_k are disjoint. It follows that

$$\|g\|_{L^2} \le C\left(\frac{|F|}{|E|}\right)^{\frac12}\left(\frac{|F|}{|E|}\right)^{\frac12}|I_t|^{\frac12} = C\frac{|F|}{|E|}\,|I_t|^{\frac12}\,.$$

We have

$$\sum_{s\in\mathbf{T}} \left|\left\langle g e^{2\pi i c(\omega_t)(\cdot)}, \varphi_{s_j}\right\rangle\right|^2 \le C\,\|g\|_{L^2}^2\,,$$

from which we obtain the required conclusion for the first function in the decomposition (2.6.5).

Next we turn to the corresponding estimate for the second function

$$\sum_k b_k e^{2\pi i K_j c(\omega_t)(\cdot)}$$

in the decomposition (2.6.5), which requires some further analysis. We have the following two estimates for all s and k:

$$\big|\big\langle b_k e^{2\pi i K_j c(\omega_t)(\cdot)}, \varphi_{s_j}\big\rangle\big| \leq \frac{C_M\,|F|\,|J_k|^2\,|I_s|^{-\frac{3}{2}}}{(1+\frac{\operatorname{dist}(J_k,I_s)}{|I_s|})^M}\,, \tag{2.6.6}$$

$$\big|\big\langle b_k e^{2\pi i K_j c(\omega_t)(\cdot)}, \varphi_{s}\big\rangle\big| \leq \frac{C_M\,|F|\,|I_s|^{\frac{1}{2}}}{(1+\frac{\operatorname{dist}(J_k,I_s)}{|I_s|})^M} \tag{2.6.7}$$

for all $M > 0$, where C_M depends only on M.

To prove (2.6.6) we use the mean value theorem together with the fact that b_k has vanishing integral to write for some ξ_y

$$\begin{aligned}
&\big|\big\langle b_k e^{2\pi i K_j c(\omega_t)(\cdot)}, \varphi_{s_j}\big\rangle\big| \\
&= \Big|\int_{J_k} b_k(y) e^{2\pi i K_j c(\omega_t) y}\overline{\varphi_{s_j}(y)}\,dy\Big| \\
&= \Big|\int_{J_k} b_k(y)\big(e^{2\pi i K_j c(\omega_t) y}\overline{\varphi_{s_j}(y)} - e^{2\pi i K_j c(\omega_t) c(J_k)}\overline{\varphi_{s_j}(c(J_k))}\,dy\Big| \\
&\leq |J_k|\int_{J_k}|b_k(y)|\Bigg[2\pi\frac{|K_j c(\omega_t)-(K_j c(\omega_{s_j})+\theta_{s_j}|\omega_{s_j}|)|\,\big|\varphi\big(\frac{\xi_y-c(I_s)}{|I_s|}-\nu_j\big)\big|}{|I_s|^{\frac{1}{2}}}+ \\
&\quad+\frac{\big|\varphi'\big(\frac{\xi_y-c(I_s)}{|I_s|}-\nu_j\big)\big|}{|I_s|^{\frac{3}{2}}}\Bigg]\,dy \\
&\leq \big\|b_k\big\|_{L^1}|J_k|\sup_{\xi\in J_k}\frac{C_M|I_s|^{-\frac{3}{2}}}{(1+\frac{|\xi-c(I_s)|}{|I_s|})^{M+1}} \\
&\leq C_M\,\frac{|F|}{|E|}\,|J_k|\Big(1+\frac{\operatorname{dist}(J_k,I_t)}{|I_t|}\Big)\,\frac{|J_k|\,|I_s|^{-\frac{3}{2}}}{(1+\frac{\operatorname{dist}(J_k,I_s)}{|I_s|})^{M+1}} \\
&\leq \frac{C_M\,\frac{|F|}{|E|}\,|J_k|^2\,|I_s|^{-\frac{3}{2}}}{(1+\frac{\operatorname{dist}(J_k,I_s)}{|I_s|})^M}\,,
\end{aligned}$$

where we used the fact that $1+\frac{\operatorname{dist}(J_k,I_t)}{|I_t|}\leq 1+\frac{\operatorname{dist}(J_k,I_s)}{|I_s|}$ and that

$$|K_j c(\omega_t)-(K_j c(\omega_{s_j})+\theta_{s_j}|\omega_{s_j}|)| \leq C_L|\omega_s| \leq C_L'|I_s|^{-1}$$

since $\omega_t \subseteq \omega_s$. To prove estimate (2.6.7) we note that

$$\left|\left\langle b_k e^{2\pi i K_j c(\omega_t)(\cdot)}, \varphi_{s_j}\right\rangle\right| \le \frac{C_M\,|I_s|^{\frac{1}{2}}\,\inf\limits_{I_s} M(b_k)}{(1+\frac{\operatorname{dist}(J_k,I_s)}{|I_s|})^M}$$

and that

$$M(b_k) \le M(\chi_F) + \frac{|F \cap 3I_t \cap J_k|}{|J_k|} M(\chi_{J_k}),$$

and since $I_s \nsubseteq \Omega_F$ we have $\inf_{I_s} M(\chi_F) \le 8\,\frac{|F|}{|E|}$ while the second term in the sum was observed earlier to be at most $C\,\frac{|F|}{|E|}$.

Finally, we have the estimate

$$\left|\left\langle b_k e^{2\pi i K_j c(\omega_t)(\cdot)}, \varphi_{s_j}\right\rangle\right| \le \frac{C_M\,\frac{|F|}{|E|}\,|J_k|\,|I_s|^{-\frac{1}{2}}}{(1+\frac{\operatorname{dist}(J_k,I_s)}{|I_s|})^M}, \tag{2.6.8}$$

which follows by taking the geometric mean of (2.6.6) and (2.6.7).

Now for a fixed $s \in \mathbf{P}_F$ we may have either $J_k \subseteq I_s$ or $J_k \cap I_s = \emptyset$ (since I_s is not contained in Ω_F). Therefore, for fixed $s \in \mathbf{P}_F$ there are only three possibilities for J_k:

(a) $J_k \subseteq 3I_s$

(b) $J_k \cap 3I_s = \emptyset$

(c) $J_k \cap I_s = \emptyset$, $J_k \cap 3I_s \neq \emptyset$, and $J_k \nsubseteq 3I_s$

Observe that case (c) is equivalent to the following statement:

(c) $J_k \cap I_s = \emptyset$, $\operatorname{dist}(J_k, I_s) = 0$, and $|J_k| \ge 2|I_s|$

Note that in case (c) for each I_s there exists exactly one $J_k = J_{k(s)}$ with the previous properties; but for a given J_k there may be a sequence of I_s's that lie on the left of J_k such that $|J_k| \geq 2|I_s|$ and $\text{dist}\,(J_k, I_s) = 0$ and another sequence with similar properties on the right of J_k. The I_s's that lie on either side of J_k must be nested and their lengths add up to $|I^L_{s_k}| + |I^R_{s_k}|$ where $I^L_{s_k}$ is the largest one among them on the left of J_k and $I^R_{s_k}$ is the largest one among them on the right of J_k. Using (2.6.7), we obtain

$$\sum_{s\in\mathbf{T}} \Big| \sum_{\substack{k:\ J_k\cap I_s=\emptyset \\ \text{dist}\,(J_k,I_s)=0 \\ |J_k|\geq 2|I_s|}} \left\langle b_k e^{2\pi i K_j c(\omega_t)(\cdot)}, \varphi_{s_j} \right\rangle \Big|^2 = \sum_{s\in\mathbf{T}} \Big| \left\langle b_{k(s)} e^{2\pi i K_j c(\omega_t)(\cdot)}, \varphi_{s_j} \right\rangle \Big|^2$$

$$\leq C \left(\frac{|F|}{|E|}\right)^2 \sum_{\substack{s\in\mathbf{T}:\ J_k\cap I_s=\emptyset \\ \text{dist}\,(J_k,I_s)=0 \\ |J_k|\geq 2|I_s|}} |I_s| \leq C \left(\frac{|F|}{|E|}\right)^2 \sum_k |I^L_{s_k}| + |I^R_{s_k}|\,.$$

But note that $I^L_{s_k} \subseteq 2J_k$ and since $I^L_{s_k} \cap J_k = \emptyset$ we must have $I^L_{s_k} \subseteq 2J_k \setminus J_k$ (and likewise for $I^R_{s_k}$). We define $I^{L+}_{s_k} = I^L_{s_k} + \frac{1}{2}|J_k|$ and $I^{R-}_{s_k} = I^R_{s_k} - \frac{1}{2}|J_k|$. We have $I^{L+}_{s_k} \cup I^{R-}_{s_k} \subseteq J_k$ and hence the sets $I^{L+}_{s_k}$ are pairwise disjoint for different k and the same is true for the $I^{R-}_{s_k}$. Moreover, since $\frac{1}{2}|J_k| \leq \frac{1}{2}|I_t|$ for all k, all the shifted sets $I^{L+}_{s_k}$, $I^{R-}_{s_k}$ are contained in $3I_t$. We conclude that

$$\sum_k |I^L_{s_k}| + \sum_k |I^R_{s_k}| = \sum_k |I^{L+}_{s_k}| + |I^{R-}_{s_k}| \leq \Big| \bigcup_k I^{L+}_{s_k} \Big| + \Big| \bigcup_k I^{R-}_{s_k} \Big| \leq 2\,|3I_t|\,,$$

which combined with the previously obtained estimate yields the required result in case (c).

We now consider case (a). Using (2.6.6), we can write

$$\left(\sum_{s\in\mathbf{T}} \Big| \sum_{k: J_k\subseteq 3I_s} \left\langle b_k e^{2\pi i K_j c(\omega_t)(\cdot)}, \varphi_s \right\rangle \Big|^2 \right)^{\frac{1}{2}} \leq C_M \left(\frac{|F|}{|E|}\right)^2 \left(\sum_{s\in T} \Big| \sum_{k:\ J_k\subseteq 3I_s} |J_k|^{\frac{1}{2}} \frac{|J_k|^{\frac{3}{2}}}{|I_s|^{\frac{3}{2}}} \Big|^2 \right)^{\frac{1}{2}} \tag{2.6.9}$$

and we control the second expression by

$$C_M\left(\frac{|F|}{|E|}\right)\Big\{\sum_{s\in\mathbf{T}}\Big(\sum_{k:\ J_k\subseteq 3I_s}|J_k|\Big)\Big(\sum_{k:\ J_k\subseteq 3I_s}\frac{|J_k|^3}{|I_s|^3}\Big)\Big\}^{\frac{1}{2}}$$

$$\le C_M\left(\frac{|F|}{|E|}\right)\Big\{\sum_{k:\ J_k\subseteq 3I_t}|J_k|^3\sum_{\substack{s\in\mathbf{T}\\ J_k\subseteq 3I_s}}\frac{1}{|I_s|^2}\Big\}^{\frac{1}{2}},$$

where we used that the dyadic intervals J_k are disjoint and the Cauchy-Schwarz inequality. We note that the last sum is equal to at most $C|J_k|^{-2}$ since for every dyadic interval J_k there exist at most 3 dyadic intervals of a given length whose triples contain it. The required estimate $C\,\frac{|F|}{|E|}\,|I_t|^{\frac{1}{2}}$ now follows in case (a).

Finally, we deal with case (b) which is the most difficult case. We split the set of k into two subsets, those for which $J_k\subseteq 3I_t$ and those for which $J_k\nsubseteq 3I_t$, (recall $|J_k|\le |I_t|$.) Whenever $J_k\nsubseteq 3I_t$, we have $\operatorname{dist}(J_k,I_s)\approx\operatorname{dist}(J_k,I_t)$. In this case we use Minkowski's inequality and estimate (2.6.8) to deduce

$$\Big(\sum_{s\in\mathbf{T}}\Big|\sum_{k:\ J_k\nsubseteq 3I_t}\big\langle b_k e^{2\pi i K_j c(\omega_t)(\cdot)},\varphi_{s_j}\big\rangle\Big|^2\Big)^{\frac{1}{2}}$$

$$\le\sum_{k:\ J_k\nsubseteq 3I_t}\Big(\sum_{s\in\mathbf{T}}\big|\big\langle b_k e^{2\pi i K_j c(\omega_t)(\cdot)},\varphi_{s_j}\big\rangle\big|^2\Big)^{\frac{1}{2}}$$

$$\le C_M\left(\frac{|F|}{|E|}\right)\sum_{k:\ J_k\nsubseteq 3I_t}|J_k|\Big(\sum_{s\in\mathbf{T}}\frac{|I_s|^{2M-1}}{\operatorname{dist}(J_k,I_s)^{2M}}\Big)^{\frac{1}{2}}$$

$$\le C_M\left(\frac{|F|}{|E|}\right)\sum_{k:\ J_k\nsubseteq 3I_t}\frac{|J_k|}{\operatorname{dist}(J_k,I_t)^M}\Big(\sum_{s\in\mathbf{T}}|I_s|^{2M-1}\Big)^{\frac{1}{2}}$$

$$\le C_M\left(\frac{|F|}{|E|}\right)|I_t|^{M-\frac{1}{2}}\sum_{k:\ J_k\nsubseteq 3I_t}\frac{|J_k|}{\operatorname{dist}(J_k,I_t)^M}$$

$$\le C_M\left(\frac{|F|}{|E|}\right)|I_t|^{M-\frac{1}{2}}\sum_{l=1}^{\infty}\sum_{\substack{k:\\ \operatorname{dist}(J_k,I_t)\approx 2^l|I_t|}}\frac{|J_k|}{(2^l|I_t|)^M}.$$

But note that all the J_k with $\operatorname{dist}(J_k,I_t)\approx 2^l|I_t|$ are contained in $2^{l+2}I_t$ and since they are disjoint we estimate the last sum by $C2^l|I_t|(2^l|I_t|)^{-M}$. The required estimate $C_M\frac{|F|}{|E|}\,|I_t|^{\frac{1}{2}}$ follows.

Next we consider the case $J_k\subseteq 3I_t,\ J_k\cap 3I_s=\emptyset$, and $|J_k|\le|I_s|$ in which we use estimate (2.6.6). We have

$$\Bigg(\sum_{s\in\mathbf{T}}\Big|\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|\le|I_s|}}\langle b_k e^{2\pi iK_jc(\omega_t)(\cdot)},\varphi_{s_j}\rangle\Big|^2\Bigg)^{\frac{1}{2}}$$

$$\le C_M\left(\frac{|F|}{|E|}\right)\Bigg(\sum_{s\in\mathbf{T}}\Big|\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|\le|I_s|}}|J_k|^2|I_s|^{-\frac{3}{2}}\frac{|I_s|^M}{\operatorname{dist}(J_k,I_s)^M}\Big|^2\Bigg)^{\frac{1}{2}}$$

$$\le C_M\left(\frac{|F|}{|E|}\right)\Bigg\{\sum_{s\in\mathbf{T}}\Bigg[\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|\le|I_s|}}\frac{|J_k|^3}{|I_s|^2}\Big(\frac{|I_s|}{\operatorname{dist}(J_k,I_s)}\Big)^M\Bigg]$$

$$\Bigg[\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|\le|I_s|}}\frac{|J_k|}{|I_s|}\Big(\frac{\operatorname{dist}(J_k,I_s)}{|I_s|}\Big)^{-M}\Bigg]\Bigg\}^{\frac{1}{2}}$$

$$\le C_M\left(\frac{|F|}{|E|}\right)\Bigg\{\sum_{s\in\mathbf{T}}\Bigg[\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|\le|I_s|}}\frac{|J_k|^3}{|I_s|^2}\Big(\frac{|I_s|}{\operatorname{dist}(J_k,I_s)}\Big)^M\Bigg]$$

$$\Bigg[\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|\le|I_s|}}\int_{J_k}\Big(\frac{|x-c(I_s)|}{|I_s|}\Big)^{-M}\frac{dx}{|I_s|}\Bigg]\Bigg\}^{\frac{1}{2}}$$

$$\le C_M\left(\frac{|F|}{|E|}\right)\Bigg\{\sum_{s\in\mathbf{T}}\Bigg[\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|\le|I_s|}}\frac{|J_k|^3}{|I_s|^2}\Big(\frac{|I_s|}{\operatorname{dist}(J_k,I_s)}\Big)^M\Bigg]$$

$$\Bigg[\int_{(3I_s)^c}\Big(\frac{|x-c(I_s)|}{|I_s|}\Big)^{-M}\frac{dx}{|I_s|}\Bigg]\Bigg\}^{\frac{1}{2}}$$

$$\le C_M\left(\frac{|F|}{|E|}\right)\Bigg\{\sum_{s\in\mathbf{T}}\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|\le|I_s|}}|J_k|^3|I_s|^{-2}\Big(\frac{|I_s|}{\operatorname{dist}(J_k,I_s)}\Big)^M\Bigg\}^{\frac{1}{2}}.$$

But since the last integral contributes at most a constant factor, we can estimate the last displayed expression by

$$\le C_M\left(\frac{|F|}{|E|}\right)\Bigg\{\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|\le |I_s|}}|J_k|^3\sum_{m\ge\log|J_k|}2^{-2m}\sum_{\substack{s\in\mathbf{T}\\ |I_s|=2^m}}\Big(\frac{\mathrm{dist}\,(J_k,I_s)}{2^m}\Big)^{-M}\Bigg\}^{\frac{1}{2}}$$

$$\le C_M\left(\frac{|F|}{|E|}\right)\Bigg\{\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|\le |I_s|}}|J_k|^3\sum_{m\ge\log|J_k|}2^{-2m}\Bigg\}^{\frac{1}{2}}$$

$$\le C_M\left(\frac{|F|}{|E|}\right)\Bigg\{\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|\le |I_s|}}|J_k|^3|J_k|^{-2}\Bigg\}^{\frac{1}{2}}$$

$$\le C_M\left(\frac{|F|}{|E|}\right)|I_t|^{\frac{1}{2}}\,.$$

There is also the subcase of case (b) in which $|J_k|>|I_s|$. Here we have the two special subcases: $I_s\cap 3J_k=\emptyset$ and $I_s\subseteq 3J_k=\emptyset$. We begin with the first of these special subcases in which we use estimate (2.6.7). We have

$$\Bigg(\sum_{s\in\mathbf{T}}\Big|\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|>|I_s|\\ I_s\cap 3J_k=\emptyset}}\langle b_k e^{2\pi iK_jc(\omega_t)(\cdot)},\varphi_{s_j}\rangle\Big|^2\Bigg)^{\frac{1}{2}}$$

$$\le C_M\left(\frac{|F|}{|E|}\right)\Bigg(\sum_{s\in T}\Big|\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|>|I_s|\\ I_s\cap 3J_k=\emptyset}}|I_s|^{\frac{1}{2}}\frac{|I_s|^M}{\mathrm{dist}\,(J_k,I_s)^M}\Big|^2\Bigg)^{\frac{1}{2}}$$

$$\le C_M\left(\frac{|F|}{|E|}\right)\Bigg\{\sum_{s\in\mathbf{T}}\Bigg[\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|>|I_s|\\ I_s\cap 3J_k=\emptyset}}\frac{|I_s|^2}{|J_k|}\frac{|I_s|^M}{\mathrm{dist}\,(J_k,I_s)^M}\Bigg]\Bigg[\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|>|I_s|\\ I_s\cap 3J_k=\emptyset}}\frac{|J_k|}{|I_s|}\frac{|I_s|^M}{\mathrm{dist}\,(J_k,I_s)^M}\Bigg]\Bigg\}^{\frac{1}{2}}.$$

Since $I_s\cap 3J_k=\emptyset$ we have that $\mathrm{dist}\,(J_k,I_s)\approx|x-c(I_s)|$ for every $x\in J_k$ and therefore the second term inside square brackets satisfies

$$\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|>|I_s|\\ I_s\cap 3J_k=\emptyset}} \frac{|J_k|}{|I_s|}\frac{|I_s|^M}{\text{dist}\,(J_k,I_s)^M} \le \sum_k \int_{J_k} \Big(\frac{|x-c(I_s)|}{|I_s|}\Big)^{-M}\frac{dx}{|I_s|} \le C_M\,.$$

Using this estimate, we obtain

$$C_M\left(\frac{|F|}{|E|}\right)\Bigg\{\sum_{s\in\mathbf{T}}\Bigg[\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|>|I_s|\\ I_s\cap 3J_k=\emptyset}}\frac{|I_s|^2}{|J_k|}\frac{|I_s|^M}{\text{dist}\,(J_k,I_s)^M}\Bigg]\Bigg[\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|>|I_s|\\ I_s\cap 3J_k=\emptyset}}\frac{|J_k|}{|I_s|}\frac{|I_s|^M}{\text{dist}\,(J_k,I_s)^M}\Bigg]\Bigg\}^{\frac12}$$

$$\le C_M\left(\frac{|F|}{|E|}\right)\Bigg\{\sum_{s\in\mathbf{T}}\Bigg[\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|>|I_s|\\ I_s\cap 3J_k=\emptyset}}\frac{|I_s|^2}{|J_k|}\frac{|I_s|^M}{\text{dist}\,(J_k,I_s)^M}\Bigg]\Bigg\}^{\frac12}$$

$$= C_M\left(\frac{|F|}{|E|}\right)\Bigg\{\sum_{k:\ J_k\subseteq 3I_t}\frac{1}{|J_k|}\sum_{\substack{s\in\mathbf{T}\\ J_k\cap 3I_s=\emptyset\\ |J_k|>|I_s|\\ I_s\cap 3J_k=\emptyset}}|I_s|^2\frac{|I_s|^M}{\text{dist}\,(J_k,I_s)^M}\Bigg\}^{\frac12}$$

$$\le C_M\left(\frac{|F|}{|E|}\right)\Bigg\{\sum_{k:\ J_k\subseteq 3I_t}\frac{1}{|J_k|}\sum_{m=-\infty}^{\log_2|J_k|}2^{2m}\sum_{\substack{s\in\mathbf{T}:\ |I_s|=2^m\\ J_k\cap 3I_s=\emptyset\\ |J_k|>|I_s|\\ I_s\cap 3J_k=\emptyset}}\frac{|I_s|^M}{\text{dist}\,(J_k,I_s)^M}\Bigg\}^{\frac12}$$

$$\le C_M\left(\frac{|F|}{|E|}\right)\Bigg\{\sum_{k:\ J_k\subseteq 3I_t}\frac{1}{|J_k|}\sum_{m=-\infty}^{\log_2|J_k|}2^{2m}\Bigg\}^{\frac12}$$

$$\le C_M\left(\frac{|F|}{|E|}\right)\Bigg\{\sum_{k:\ J_k\subseteq 3I_t}\frac{1}{|J_k|}|J_k|^2\Bigg\}^{\frac12}$$

$$\le C_M\left(\frac{|F|}{|E|}\right)|I_t|^{\frac12}\,.$$

Finally there is the subcase of case (b) in which $|J_k|\ge|I_s|$ and $I_s\subseteq 3J_k=\emptyset$. Here again we use estimate (2.6.7). We have

$$\left\{ \sum_{s\in\mathbf{T}} \Big| \sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|>|I_s|\\ I_s\subseteq 3J_k}} \langle b_k e^{2\pi i K_j c(\omega_t)(\cdot)}, \varphi_{s_j}\rangle \Big|^2 \right\}^{\frac{1}{2}}$$

$$\le\ C_M \left(\frac{|F|}{|E|}\right) \left\{ \sum_{s\in\mathbf{T}} |I_s| \Big| \sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|>|I_s|\\ I_s\subseteq 3J_k}} \frac{|I_s|^M}{\mathrm{dist}\,(J_k, I_s)^M} \Big|^2 \right\}^{\frac{1}{2}} . \tag{2.6.10}$$

Let us make some observations. For a fixed s there exist at most finitely many J_k's contained in $3I_t$ with size at least $|I_s|$. Let $J_L^1(s)$ be the interval that lies to the left of I_s and is closest to I_s among all J_k that satisfy the conditions in the preceding sum. Then $|J_L^1(s)| > |I_s|$ and

$$\mathrm{dist}(J_L^1(s), I_s) \ge |I_s| .$$

Let $J_L^2(s)$ be the interval to the left of $J_L^1(s)$ that is closest to $J_L^1(s)$ and that satisfies the conditions of the sum. Since $3J_L^2(s)$ contains I_s, it follows that $|J_L^2(s)| > 2|I_s|$ and

$$\mathrm{dist}(J_L^2(s), I_s) \ge 2|I_s| .$$

Continuing in this way, we can find a finite number of intervals $J_L^r(s)$ that lie to the left of I_s and inside $3I_t$, satisfy $|J_L^r(s)| > 2^r|I_s|$ and $\mathrm{dist}(J_L^r(s), I_s) \ge 2^r|I_s|$, and whose triples contain I_s. Likewise we find a finite collection of intervals $J_R^1(s), J_R^2(s)\dots$ that lie to the right of I_s and satisfy similar conditions. Then, using the Cauchy-Schwarz inequality, we obtain

$$\Big| \sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|>|I_s|\\ I_s\subseteq 3J_k}} \frac{|I_s|^M}{\mathrm{dist}\,(J_k, I_s)^M} \Big|^2$$

$$\leq 2\left|\sum_{r=1}^{\infty}\frac{|I_s|^{\frac{M}{2}}}{\mathrm{dist}(J_L^r(s),I_s)^{\frac{M}{2}}}\frac{1}{2^{\frac{rM}{2}}}\right|^2+2\left|\sum_{r=1}^{\infty}\frac{|I_s|^{\frac{M}{2}}}{\mathrm{dist}(J_R^r(s),I_s)^{\frac{M}{2}}}\frac{1}{2^{\frac{rM}{2}}}\right|^2$$

$$\leq C_M\sum_{r=1}^{\infty}\frac{|I_s|^M}{\mathrm{dist}(J_L^r(s),I_s)^M}+C_M\sum_{r=1}^{\infty}\frac{|I_s|^M}{\mathrm{dist}(J_R^r(s),I_s)^M}$$

$$\leq C_M\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|>|I_s|\\ I_s\subseteq 3J_k}}\frac{|I_s|^M}{\mathrm{dist}\,(J_k,I_s)^M}\,.$$

We use this estimate to control the expression on the left in (2.6.10) by

$$C_M\left(\frac{|F|}{|E|}\right)\left\{\sum_{s\in\mathbf{T}}|I_s|\sum_{\substack{k:\ J_k\subseteq 3I_t\\ J_k\cap 3I_s=\emptyset\\ |J_k|>|I_s|\\ I_s\subseteq 3J_k}}\frac{|I_s|^M}{\mathrm{dist}\,(J_k,I_s)^M}\right\}^{\frac{1}{2}}$$

$$\leq\ C_M\left(\frac{|F|}{|E|}\right)\left\{\sum_{k:J_k\subseteq 3I_t}|J_k|\sum_{m=0}^{\infty}2^{-m}\sum_{\substack{s:\,I_s\subseteq 3J_k\\ J_k\cap 3I_s=\emptyset\\ |I_s|=2^{-m}|J_k|}}\frac{|I_s|^M}{\mathrm{dist}\,(J_k,I_s)^M}\right\}^{\frac{1}{2}}.$$

Since the last sum is at most a constant, it follows that the term on the left in (2.6.10) also satisfies the estimate $C_M\,|F|\,|I_t|^{\frac{1}{2}}$. This concludes the proof of Lemma 2.4.2.

2.7 $L^{r_1} \times L^{r_2} \to L^r$ boundedness of the model sums

In this section we will show that estimates (2.3.4) and (2.4.11) imply boundedness of the model sum operator H_S from $L^{r_1} \times L^{r_2}$ to L^r for $\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r}$, $r_1, r_2 > 1$, $r > \frac{2}{3}$. We include this section for the sake of completeness (as we will use this result in the sequel), but we point out that the reader may wish to skip it and cite the results of Lacey and Thiele [22],[23].

Take some p_1, p_2 such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{3}{2}$ and $p_1, p_2 > 1$. We will show that H_S is of restricted weak type (r_1, r_2, r) where $\frac{1}{r_1} = \frac{1}{p_1} - \varepsilon$, $\frac{1}{r_2} = \frac{1}{p_2} - \varepsilon$ and $\frac{1}{r} = \frac{3}{2} - 2\varepsilon$. Then the strong boundedness for the claimed range of exponents follows by the interpolation theorem of Grafakos and Tao [15] as the operator H_S has bounded kernel whenever S is a finite set.

We recall that a bilinear operator T is of restricted weak type (r_1, r_2, r) if and only if the following is valid (see Lemma 1.1.1): for any sets E, F_1, F_2 of finite measure there exists a set $E' \subset E$ with $|E'| \geq \frac{1}{2}|E|$, such that

$$\left| \int_{E'} T(\chi_{F_1}, \chi_{F_2})(x) dx \right| \lesssim \frac{|F_1|^{\frac{1}{r_1}} |F_2|^{\frac{1}{r_2}}}{|E|^{\frac{1}{r}-1}}. \tag{2.7.1}$$

Take arbitrary sets E, F_1, F_2 of finite positive measure. It follows from (2.3.4) and (2.4.11) that

$$\left| \int_{E'} H_S(\chi_{F_1}, \chi_{F_2})\, dx \right| \lesssim \frac{|F_1|^{\frac{1}{p_1}} |F_2|^{\frac{1}{p_2}}}{|E|^{\frac{1}{2}}} \left(1 + \left| \log \frac{|F_1|}{|E|} \right| \right) \left(1 + \left| \log \frac{|F_2|}{|E|} \right| \right). \tag{2.7.2}$$

We will use the fact that $1 + \log x \lesssim x^{\varepsilon}$ for $x \geq 1$. In the case when $|E| \geq \max(|F_1|, |F_2|)$ we can estimate the righthand side of (2.7.2) by the expression

$$\frac{|F_1|^{\frac{1}{p_1}} |F_2|^{\frac{1}{p_2}}}{|E|^{\frac{1}{2}}} \left(1 + \log \frac{|E|}{|F_1|}\right) \left(1 + \log \frac{|E|}{|F_2|}\right) \lesssim \frac{|F_1|^{\frac{1}{p_1}-\varepsilon} |F_2|^{\frac{1}{p_2}-\varepsilon}}{|E|^{\frac{1}{2}-2\varepsilon}} = \frac{|F_1|^{\frac{1}{r_1}} |F_2|^{\frac{1}{r_2}}}{|E|^{\frac{1}{r}-1}}.$$

Now consider the case $|F_1| \leq |E| \leq |F_2|$ (as the case $|F_2| \leq |E| \leq |F_1|$ is symmetric). Fix some $\varepsilon_1 > 2\varepsilon$. Put $\alpha = \frac{1}{p_1} - \varepsilon + \varepsilon_1$ (ε and ε_1 have to be chosen small enough, so that $\alpha \leq 1$) and $\beta = \frac{1}{p_2} - \varepsilon_1 + \varepsilon$ (thus $\beta \leq 1$ also). We have $\alpha + \beta = \frac{3}{2}$. Thus, similarly to (2.7.2), we obtain:

$$\begin{aligned}\left|\int_{E'} H_S(\chi_{F_1}, \chi_{F_2})\, dx\right| &\lesssim \frac{|F_1|^\alpha\, |F_2|^\beta}{|E|^{\frac{1}{2}}}\left(1 + \log\frac{|E|}{|F_1|}\right)\left(1 + \log\frac{|F_2|}{|E|}\right) \\ &\lesssim \frac{|F_1|^\alpha\, |F_2|^\beta}{|E|^{\frac{1}{2}}}\left(\frac{|E|}{|F_1|}\right)^{\varepsilon_1}\left(\frac{|F_2|}{|E|}\right)^{\varepsilon_1 - 2\varepsilon} \\ &= \frac{|F_1|^{\frac{1}{p_1}-\varepsilon}\, |F_2|^{\frac{1}{p_2}-\varepsilon}}{|E|^{\frac{1}{2}-2\varepsilon}} = \frac{|F_1|^{\frac{1}{r_1}}\, |F_2|^{\frac{1}{r_2}}}{|E|^{\frac{1}{r}-1}}.\end{aligned}$$

The remaining case is $|E| \leq \min(|F_1|, |F_2|)$. We observe that in this case the set Ω is empty, since $M(\chi_{F_i}) \leq 1$. We therefore only need to use (2.4.6) which for $|E|$ small yields:

$$\begin{aligned}\left|\int_{E'} H_S(\chi_{F_1}, \chi_{F_2})\, dx\right| &\lesssim \min(|F_1|, |F_2|)^{\frac{1}{2}}\, |E|^{\frac{1}{2}}\left(1 + \log\frac{|F_1|}{|E|}\right)\left(1 + \log\frac{|F_2|}{|E|}\right) \\ &\lesssim |F_1|^{\frac{1}{p_1}-\frac{1}{2}}\, |F_2|^{\frac{1}{p_2}-\frac{1}{2}} |E|^{\frac{1}{2}}\left(\frac{|F_1|}{|E|}\right)^{\frac{1}{2}-\varepsilon}\left(\frac{|F_2|}{|E|}\right)^{\frac{1}{2}-\varepsilon} \\ &= \frac{|F_1|^{\frac{1}{p_1}-\varepsilon}\, |F_2|^{\frac{1}{p_2}-\varepsilon}}{|E|^{\frac{1}{2}-2\varepsilon}} = \frac{|F_1|^{\frac{1}{r_1}}\, |F_2|^{\frac{1}{r_2}}}{|E|^{\frac{1}{r}-1}}.\end{aligned}$$

Thus, for any measurable sets E and F_1, F_2, H_S satisfies (2.7.1) and this implies that it is of restricted weak type (r_1, r_2, r). The strong type estimates for the same range of exponents can now be obtained by varying r_1 and r_2 and using the result on interpolation between adjoint operators (cf. [15]).

2.8 Estimates corresponding to the case $p_1 = 1$, $2 \le p_2 < \infty$.

Fix $2 \le p_2 < \infty$. There are some minor differences in the treatment of the cases $p_2 = 2$ and $p_2 > 2$. In the case $p_2 = 2$ for the moment we shall assume that $|F_1| \le |F_2|$.

CASE: $p_2 = 2$, $|E|^{\frac{3}{2}} \ge |F_1|\,|F_2|^{\frac{1}{2}}$, $|F_1| \le |F_2| \le |E|$.

Since $|E|^{\frac{3}{2}} \ge |F_1|\,|F_2|^{\frac{1}{2}}$ and $|F_2| \ge |F_1|$, we have $|E|^2 \ge |F_1|\,|F_2|$. Using estimate (2.4.12) we obtain

$$\left|\int_{E'} H(\chi_{F_1}, \chi_{F_2})dx\right| \le C_1 \frac{|F_1|\,|F_2|^{\frac{1}{2}}}{|E|^{\frac{1}{2}}}\left(1 + \log \frac{|E|^{\frac{3}{2}}}{|F_1|\,|F_2|^{\frac{1}{2}}}\right). \tag{2.8.1}$$

We note that this estimate is also valid if $|E| \ge \max|F_i|$, even when $|F_1| \ge |F_2|$. We will use this estimate in the inductive procedures below.

CASE: $p_2 > 2$, $|E|^{1+\frac{1}{p_2}} \ge |F_1|\,|F_2|^{\frac{1}{p_2}}$, $|E| \ge |F_2|$.

Let $\alpha = \frac{1}{2} - \frac{1}{p_2} > 0$, $\beta = 1 - \frac{1}{p_2} > 0$. Since $|E| \ge |F_2|$ we must have $|E|^2 \ge |F_1|\,|F_2|$. Using (2.4.12) we obtain

$$\begin{aligned}\left|\int_{E'} H(\chi_{F_1}, \chi_{F_2})(x)\,dx\right| &\le C_1 \min(|F_1|, |F_2|)^{\frac{1}{2}}|F_1|^{\frac{1}{2}}\,|F_2|^{\frac{1}{2}}|E|^{-\frac{1}{2}}\left(1 + \log\frac{|E|^2}{|F_1|\,|F_2|}\right)\\ &\le C_1 \frac{|F_1|\,|F_2|^{\frac{1}{p_2}}}{|E|^{\frac{1}{p_2}}}\left(\frac{|F_2|}{|E|}\right)^{\alpha}\left(1 + \log\frac{|E|^{1+\frac{1}{p_2}}}{|F_1|\,|F_2|^{\frac{1}{p_2}}} + \log\frac{|E|^{\beta}}{|F_2|^{\beta}}\right)\\ &\lesssim \frac{|F_1|\,|F_2|^{\frac{1}{p_2}}}{|E|^{\frac{1}{p_2}}}\left(1 + \log\frac{|E|^{1+\frac{1}{p_2}}}{|F_1|\,|F_2|^{\frac{1}{p_2}}}\right),\end{aligned} \tag{2.8.2}$$

since the function $f(x) = x^{\alpha}(1 + \log\frac{1}{x^{\beta}})$ is bounded on $[0, 1]$ when $\alpha > 0$ (here

$x = \frac{|F_2|}{|E|}$).

CASE: $p_2 \geq 2$, $|E|^{1+\frac{1}{p_2}} \geq |F_1|\,|F_2|^{\frac{1}{p_2}}$, $|E| \leq |F_2|$ (which implies $|E| \geq |F_1|$).

In this case we will obtain an estimate via an iterative procedure. The iteration procedure will consist of two parts. At first, we set $F_2^0 = F_2$. We will continue this part of the iteration until the first integer n such that $|F_2^n| \leq |E|$. At the j^{th} step, according to the estimates above, we choose a subset S^j of F_2^j with $|S^j| \geq \frac{1}{2}|F_2^j|$, such that:

$$\Big|\int_{S^j} H^{*2}(\chi_{F_1}, \chi_E)(x)dx\Big| \lesssim \frac{|F_1|\,|E|^{\frac{1}{p_2}}}{|F_2^j|^{\frac{1}{p_2}}}\left(1+\log\frac{|F_2^j|^{1+\frac{1}{p_2}}}{|F_1|\,|E|^{\frac{1}{p_2}}}\right) \leq |F_1|\left(1+\log\frac{|F_2|^{1+\frac{1}{p_2}}}{|F_1|\,|E|^{\frac{1}{p_2}}}\right).$$

Then we set $F_2^{j+1} = F_2^j \setminus S^j$. Obviously, for the number of steps n we have $n \lesssim 1 + \log\frac{|F_2|}{|E|}$. Thus, we have

$$\begin{aligned}
\left|\int_E H(\chi_{F_1}, \chi_{F_2})\,dx\right| &\lesssim |F_1|\left(1+\log\frac{|F_2|^{1+\frac{1}{p_2}}}{|F_1|\,|E|^{\frac{1}{p_2}}}\right)\left(1+\log\frac{|F_2|}{|E|}\right) + \\
&\quad + \left|\int_E H(\chi_{F_1}, \chi_{F_2^n})\,dx\right| \\
&\lesssim \frac{|F_1|\,|F_2|^{\frac{1}{p_2}}}{|E|^{\frac{1}{p_2}}}\left(1+\log\frac{|E|^{1+\frac{1}{p_2}}}{|F_1|\,|F_2|^{\frac{1}{p_2}}}\right)^2 + \left|\int_E H(\chi_{F_1}, \chi_{F_2^n})\,dx\right|.
\end{aligned}$$

In the last line we have used the following simple inequality (with $a = \frac{|E|}{|F_1|}$, $b = \frac{|F_2|}{|E|}$):

For $a \geq 1$, $b \geq 1$, such that $ab^{-\frac{1}{p_2}} \geq 1$ we have

$$\left(1+\log\big(ab^{1+\frac{1}{p_2}}\big)\right)\left(1+\log b\right) \lesssim \left(1+\log\frac{a}{b^{\frac{1}{p_2}}}\right)^2 b^{\frac{1}{p_2}}. \tag{2.8.3}$$

To prove (2.8.3) we note that if $b^{\frac{1}{p_2}} \leq \sqrt{a}$, then $\log\frac{a}{b^{\frac{1}{p_2}}} \geq \log\sqrt{a} = \frac{1}{2}\log a$ and we have

$$\left(1+\log\big(ab^{1+\frac{1}{p_2}}\big)\right)\left(1+\log b\right) \lesssim \left(1+\log a\right)^2 \lesssim \left(1+\log\frac{a}{b^{\frac{1}{p_2}}}\right)^2 b^{\frac{1}{p_2}},$$

while when $\sqrt{a} \le b^{\frac{1}{p_2}} \le a$, then

$$\Big(1+\log\big(ab^{1+\frac{1}{p_2}}\big)\Big)\Big(1+\log b\Big) \lesssim \Big(1+\log b\Big)^2 \lesssim b^{\frac{1}{p_2}}.$$

It remains to estimate the term

$$\Big|\int_E H(\chi_{F_1},\chi_{F_2^n})\,dx\Big|.$$

In the second part of the iteration process we proceed in a similar manner, only now we will be splitting either F_2 or E, depending on which one is larger in size. We set $E^n = E$. At the j^{th} step, if $|E^j| \ge |F_2^j|$, we choose $S^j \subset E^j$ such that $|S^j| \ge \frac{1}{2}|E^j|$ and

$$\begin{aligned}\Big|\int_{S^j} H(\chi_{F_1},\chi_{F_2^j})\,dx\Big| &\lesssim \frac{|F_1|\,|F_2^j|^{\frac{1}{p_2}}}{|E^j|^{\frac{1}{p_2}}}\left(1+\log\frac{|E^j|^{1+\frac{1}{p_2}}}{|F_1|\,|F_2^j|^{\frac{1}{p_2}}}\right)\\ &\le |F_1|\frac{|F_2^j|^{\frac{1}{p_2}}}{|E^j|^{\frac{1}{p_2}}}\left(1+\log\frac{|E^j|^{\frac{1}{p_2}}}{|F_2^j|^{\frac{1}{p_2}}}+\log\frac{|E|}{|F_1|}\right)\\ &\lesssim |F_1|\left(1+\log\frac{|E|}{|F_1|}\right),\end{aligned}$$

where we have once again made use of the fact that $f(x) = x\cdot\log\frac{1}{x}$ is bounded on $[0,1]$ ($x = \frac{|F_2^j|^{\frac{1}{p}}}{|E^j|^{\frac{1}{p}}} \le 1$).

In the other case, when $|F_2^j| \ge |E^j|$, we choose $S^j \subset F_2^j$ with $|F_2^j| \ge |E^j|$ such that

$$\Big|\int_{S^j} H^{*2}(\chi_{F_1},\chi_{E^j})\,dx\Big| \lesssim \frac{|F_1|\,|E^j|^{\frac{1}{p_2}}}{|F_2^j|^{\frac{1}{p_2}}}\left(1+\log\frac{|F_2^j|^{1+\frac{1}{p_2}}}{|F_1|\,|E^j|^{\frac{1}{p_2}}}\right).$$

An identical calculation and the fact that $|F_2^j| \le |E|$ show that this can also be dominated by $|F_1|\big(1+\log\frac{|E|}{|F_1|}\big)$.

In the first case we set $F_2^{j+1} = F_2^j$, $E^{j+1} = E^j \setminus S^j$. In the second case we set $F_2^{j+1} = F_2^j \setminus S^j$, $E^{j+1} = E^j$. We proceed until the first integer m such that both $|E^m|, |F_2^m| \leq |F_1|$. Obviously, the number of steps in the second part $m \lesssim \big(1 + \log \frac{|E|}{|F_1|}\big)$. We now have

$$
\begin{aligned}
\Big|\int_E H(\chi_{F_1}, \chi_{F_2^n})\,dx\Big| &= \Big|\int_{E^{n+1}\cup S^n} H(\chi_{F_1}, \chi_{F_2^j})\,dx\Big| \\
&\leq \Big|\int_{S^n} H(\chi_{F_1}, \chi_{F_2^j})(x)\,dx\Big| + \Big|\int_{E^{j+1}} H(\chi_{F_1}, \chi_{F_2^{j+1}})\,dx\Big| \\
&\lesssim |F_1|\Big(1 + \log \frac{|E|}{|F_1|}\Big) + \Big|\int_{E^{j+1}} H(\chi_{F_1}, \chi_{F_2^{j+1}})\,dx\Big| \\
&\lesssim \ldots \\
&\lesssim m\,|F_1|\Big(1 + \log \frac{|E|}{|F_1|}\Big) + \Big|\int_{E^m} H(\chi_{F_1}, \chi_{F_2^m})\,dx\Big| \\
&\lesssim |F_1|\Big(1 + \log \frac{|E|}{|F_1|}\Big)^2 + |E^m|^{\frac{1}{2}}|F_1|^{\frac{1}{4}}|F_2^m|^{\frac{1}{4}} \\
&\lesssim |F_1|\frac{|F_2|^{\frac{1}{p_2}}}{|E|^{\frac{1}{p_2}}}\Big(1 + \log \frac{|E|^{1+\frac{1}{p_2}}}{|F_1|\,|F_2|^{\frac{1}{p_2}}}\Big)^2,
\end{aligned}
$$

where we made use of the boundedness of H on $L^4 \times L^4 \to L^2$ and the following inequality: For any $a \geq 1$, $b \geq 1$, such that $ba^{-\frac{1}{p_2}} \geq 1$ we have

$$
(1 + \log b)^2 \lesssim (1 + \log(ba^{-\frac{1}{p_2}}))^2 a^{\frac{1}{p_2}},
$$

with $a = \frac{|F_2|}{|E|}$, $b = \frac{|E|}{|F_1|}$. The proof of this inequality is similar to that in (2.8.3) and is omitted.

CASE: $p_2 \geq 2$, $|E|^{1+\frac{1}{p_2}} \leq |F_1|\,|F_2|^{\frac{1}{p_2}}$. (We are still assuming that $|F_1| \leq |F_2|$ when $p_2 = 2$.)

Here we will need the following lemma.

Lemma 2.8.1. *Let $2 \leq p_2 < \infty$. For all measurable sets E, F_1, F_2 of finite measure satisfying $|E|^{1+\frac{1}{p_2}} \leq |F_1|\,|F_2|^{\frac{1}{p_2}}$ (and also $|F_1| \leq |F_2|$ when $p_2 = 2$) we* *have*

$$\left|\int_E H(\chi_{F_1},\chi_{F_2})(x)dx\right| \lesssim |E|\left(1+\log\frac{|F_1|\,|F_2|^{\frac{1}{p_2}}}{|E|^{1+\frac{1}{p_2}}}\right)^2.$$

Proof. Let us denote $F_i^0 = F_i$ for $i = 1,2$. We shall now employ an inductive procedure similar to the one described above. At the j^{th} step among the sets F_1^j and F_2^j we choose the one which has greater size and denote it by F_{max}^j and the other one by F_{min}^j. By H^{*max} we shall denote the expression $H^{*1}(\chi_E, \chi_{F_2^{(j)}})$ in the case when $F_{max}^j = F_1^j$ and $H^{*2}(\chi_{F_1^{(j)}}, \chi_E)$ in the other case. By (2.8.2) or (2.8.1) applied to the respective adjoint of H with the roles of E and F_{max}^j interchanged, we can choose $S^j \subset F_{max}^j$ such that $|S^j| \geq \frac{1}{2}|F_{max}^j|$ and

$$\left|\int_{S^j} H^{*max}(x), dx\right| \lesssim \frac{|E|\,|F_{min}^j|^{\frac{1}{p_2}}}{|F_{max}^j|^{\frac{1}{p_2}}}\left(1+\log\frac{|F_{max}^j|^{1+\frac{1}{p_2}}}{|E|\,|F_{min}^j|^{\frac{1}{p_2}}}\right). \tag{2.8.4}$$

We define $F_i^{j+1} = F_i^j \setminus S_i^j$ for all $i = 1,2$, where we set $S_i^j = S^j$ if $F_{max}^j = F_i^j$ and $S_i^j = \emptyset$ otherwise. Let us examine the righthand side of the inequality above. If $|E| \leq |F_{min}^j|$, it is easy to check that

$$\frac{|F_{max}^j|^{1+\frac{1}{p_2}}}{|F_{min}^j|^{\frac{1}{p_2}}|E|} \leq \left(\frac{|F_{min}^j|\,|F_{max}^j|^{\frac{1}{p_2}}}{|E|^{1+\frac{1}{p_2}}}\right)^{p_2+1} \leq \left(\frac{|F_{min}|\,|F_{max}|^{\frac{1}{p_2}}}{|E|^{1+\frac{1}{p_2}}}\right)^{p_2+1} \leq \left(\frac{|F_1|\,|F_2|^{\frac{1}{p_2}}}{|E|^{1+\frac{1}{p_2}}}\right)^{p_2+1}.$$

Thus, in this case we can estimate the righthand side by $(p_2+1)|E|\left(1+\log\frac{|F_1|\,|F_2|^{\frac{1}{p_2}}}{|E|^{1+\frac{1}{p_2}}}\right)$.

In the case when $|E| \geq |F_{min}^j|$, we have

$$\frac{|F_{max}^{1+\frac{1}{p_2}}|^2}{|F_{min}^j|^{\frac{1}{p}}|E|} \leq \frac{|F_{max}|^{1+\frac{1}{p_2}}}{|F_{min}|^{1+\frac{1}{p_2}}}.$$

So, in this case the righthand side of the inequality can be estimated by

$$|E|\left(\frac{|F_{min}^j|}{|F_{max}^j|}\right)^{\frac{1}{p_2}}\left(1+\log\frac{|F_{max}^j|}{|F_{min}^j|}\right) \lesssim |E|,$$

since the function $f(x) = x^{\frac{1}{p_2}}(1 + log\frac{1}{x})$ is bounded for $x \in [0, 1]$.

Thus, in each case we get

$$\Big|\int_{S^j} H^{*max}(x)dx\Big| \leq C''\,|E|\bigg(1 + \log\frac{|F_1|\,|F_2|^{\frac{1}{p}}}{|E|^{1+\frac{1}{p}}}\bigg).$$

We proceed by induction and we stop at the first integer n such that

$$|E|^{1+\frac{1}{p}} \geq |F_1^n|\,|F_2^n|^{\frac{1}{p_2}}.$$

(Such an integer always exists since the quantity $|F_1^{(n)}|\,|F_2^{(n)}|^{\frac{1}{p_2}}$ gets smaller by at least a factor of $\frac{1}{2^{\frac{1}{p_2}}}$ when j is replaced by $j + 1$.) Obviously, the number of steps $n \lesssim 1 + \log\frac{|F_1|\,|F_2|^{\frac{1}{p}}}{|E|^{1+\frac{1}{p_2}}}$.

We can now estimate

$$\begin{aligned}
\Big|\int_E H(\chi_{F_1}, \chi_{F_2})\,dx\Big| &= \Big|\int_E H(\ldots, \chi_{S^0} + \chi_{F^1_{max}}, \ldots)\,dx\Big| \\
&\leq \Big|\int_{S^0} H^{*max}(x)\,dx\Big| + \Big|\int_E H(\chi_{F_1^1}, \chi_{F_2^1})\,dx\Big| \\
&\lesssim |E|\bigg(1 + \log\frac{|F_1|\,|F_2|^{\frac{1}{p_2}}}{|E|^{1+\frac{1}{p_2}}}\bigg) + \Big|\int_E H(..., \chi_{S^1} + \chi_{F^2_{max}}, ...)\,dx\Big| \\
&\leq \ldots \\
&\lesssim n\,|E|\bigg(1 + \log\frac{|F_1|\,|F_2|^{\frac{1}{p_2}}}{|E|^{1+\frac{1}{p_2}}}\bigg) + \Big|\int_E H(\chi_{F_1^n}, \chi_{F_2^n})\,dx\Big| \\
&\lesssim |E|\,\Big(1 + \log\frac{|F_1|\,|F_2|^{\frac{1}{p_2}}}{|E|^2}\Big)^2 + |E|^{1-\frac{p_2+1}{\theta p_2}}|F_1^n|^{\frac{1}{\theta}}|F_2^n|^{\frac{1}{\theta p_2}} \\
&\leq C_2\,|E|\,\Big(1 + \log\frac{|F_1|\,|F_2|^{\frac{1}{p_2}}}{|E|^{1+\frac{1}{p_2}}}\Big)^2,
\end{aligned}$$

where in the second line from the bottom we have used the Hölder inequality and the fact that H is of strong type $(\theta, \theta p_2, \theta\frac{p_2}{p_2+1})$ for some large θ. $\square$

In the case $p_2 > 2$ we obtain the following estimate : For any sets F_1, F_2, and E of finite measure we can find $E' \subset E$ with $|E'| \geq \frac{1}{2}|E|$ such that

$$\left| \int_{E'} H(\chi_{F_1}, \chi_{F_2})\, dx \right| \lesssim |E| \min\left[1, \frac{|F_1|\,|F_2|^{\frac{1}{p_2}}}{|E|^{1+\frac{1}{p_2}}}\right]\left[1 + \left|\log \frac{|E|^{1+\frac{1}{p_2}}}{|F_1|\,|F_2|^{\frac{1}{p_2}}}\right|\right]^2. \qquad (2.8.5)$$

We now remove the assumption that $|F_1| \leq |F_2|$ when $p_2 = 2$. For $p_2 = 2$, we can consider the (symmetric) case when $|F_1| \geq |F_2|$, proceed as above with the roles of F_1 and F_2 interchanged and putting together the two estimates we obtain: For any sets F_1, F_2, and E of finite measure we can find a set $E' \subset E$ with $|E'| \geq \frac{1}{2}|E|$ such that

$$\left| \int_{E'} H(\chi_{F_1}, \chi_{F_2})\, dx \right| \lesssim |E| \min\left[1, \frac{\min(|F_i|)^{\frac{1}{2}}|F_1|^{\frac{1}{2}}\,|F_2|^{\frac{1}{2}}}{|E|^{\frac{3}{2}}}\right]\left[1 + \left|\log \frac{|E|^{\frac{3}{2}}}{\min(|F_i|)^{\frac{1}{2}}|F_1|^{\frac{1}{2}}\,|F_2|^{\frac{1}{2}}}\right|\right]^2. \qquad (2.8.6)$$

2.9 Distributional estimates for the bilinear Hilbert transform

We can now prove Theorem 2.1.1.

Proof. For a given $\lambda > 0$, we set

$$E_\lambda^+ = \{H(\chi_{F_1}, \chi_{F_2}) > \lambda\},$$

$$E_\lambda^- = \{H(\chi_{F_1}, \chi_{F_2}) < -\lambda\}.$$

We shall prove the required estimate for E_λ^+, the other case is identical. Suppose that $|E_\lambda^+|^{1+\frac{1}{p_2}} \geq |F_1|\,|F_2|^{\frac{1}{p_2}}$. Then by (2.8.5) there is a subset S_λ^+ of E_λ^+ of at least half its measure so that

$$\frac{\lambda}{2}|E_\lambda^+| \leq \left| \int_{S_\lambda^+} H(\chi_{F_1}, \chi_{F_2})\, dx \right| \lesssim \frac{|F_1|\,|F_2|^{\frac{1}{p_2}}}{|E_\lambda^+|^{\frac{1}{p_2}}} \left(1 + \log \frac{|E_\lambda^+|^{1+\frac{1}{p_2}}}{|F_1|\,|F_2|^{\frac{1}{p_2}}} \right)^2,$$

which implies

$$|E_\lambda^+| \leq C_4 \left(|F_1|\,|F_2|^{\frac{1}{p_2}} \right)^{\frac{p_2}{p_2+1}} \cdot \lambda^{-\frac{p_2}{p_2+1}} \left(1 + \log \frac{1}{\lambda} \right)^{\frac{2p_2}{p_2+1}}.$$

But this in turn implies that there is a $\lambda_0 > 0$ such that for $\lambda > \lambda_0$ we must have $|E_\lambda^+|^{1+\frac{1}{p_2}} \leq |F_1|\,|F_2|^{\frac{1}{p_2}}$. In this case, estimate (2.8.5) gives

$$\lambda |E_\lambda^+| \leq C_3\, |E_\lambda^+| \left(1 + \log \frac{|F_1|\,|F_2|^{\frac{1}{p_2}}}{|E_\lambda^+|^{1+\frac{1}{p_2}}} \right)^2,$$

from which one easily deduces that $|E_\lambda^+| \leq \frac{1}{2} C\, e^{-c\sqrt{\lambda}} \big(|F_1|\,|F_2|^{\frac{1}{p_2}} \big)^{\frac{p_2}{p_2+1}}$. The same argument applies for the set E_λ^-.

For $p_2 = 2$ we run the same argument for estimate (2.8.6) and in the end dominate the expression $\min(|F_1|^{\frac{1}{2}}, |F_2|^{\frac{1}{2}}) |F_1|^{\frac{1}{2}}\, |F_2|^{\frac{1}{2}}$ by $|F_1|\,|F_2|^{\frac{1}{2}}$.

Replacing the constants C, c by different ones we may take $\lambda_0 = 1$ and thus estimate (2.1.1) is now proved. Estimate (2.1.2) is proved likewise. Finally, Corollaries 2.1.2 and 2.1.3 are easy consequences of these estimates. $\square$

Chapter 3

Bilinear Fourier series related to the bilinear Hilbert transform.

The bilinear Hilbert transform can be represented as a bilinear Fourier multiplier operator

$$H(f,g)(x) = p.v.\frac{1}{\pi}\int_{\mathbf{R}} f(x-t)g(x+t)\frac{dt}{t} = \int_{\mathbf{R}}\int_{\mathbf{R}} \widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i(\xi+\eta)x}\mathrm{sgn}\,(\xi-\eta)d\xi\, d\eta. \tag{3.0.1}$$

Obviously, its properties are equivalent to the properties of the following bilinear multiplier operator:

$$T(f,g)(x) = \int_{\mathbf{R}}\int_{\mathbf{R}} \widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i(\xi+\eta)x}\chi_{\{\xi>\eta\}}(\xi,\eta)d\xi\, d\eta, \tag{3.0.2}$$

as $H(f,g) = -f\cdot g + 2T(f,g)$. From now on we will be working with T instead of H.

By the transference theorems for multilinear operators, cf. [11], we can instead work with the operator defined for the functions on the torus:

$$\mathcal{H}(f,g)(x) = \sum_{(m,n):\, m\geq n} \widehat{f}(m)\widehat{g}(n)e^{2\pi i(m+n)x}, \tag{3.0.3}$$

where $\widehat{f}(m) = \int_{\mathbf{T}} f(x)e^{-2\pi imx}dx$ denotes the m^{th} Fourier coefficient of f.

First of all, we would like to show that (just as it is the case with its linear counterpart) the fact that $\mathcal{H}$ maps $L^{p_1} \times L^{p_2}$ to L^q is equivalent to the $L^q(\mathbf{T})$ convergence of the bilinear Fourier series, where the partial sums are taken over parallelograms $A_N = \{(m, n) \in \mathbf{Z}^2 : |n| \leq N, |m - n| \leq N\}$:

$$S_N(f, g)(x) = \sum_{(m,n) \in A_N} \widehat{f}(m)\widehat{g}(n)e^{2\pi i(m+n)x}. \tag{3.0.4}$$

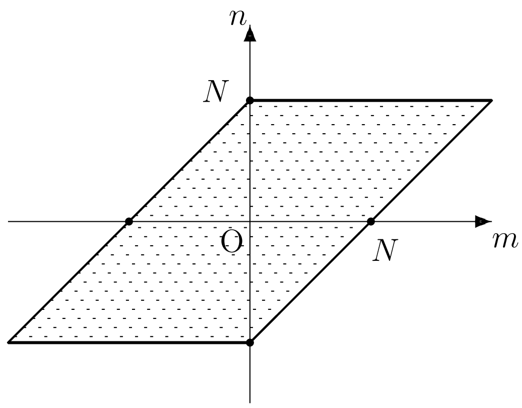

Figure 4. The parallelogram A_N.

Theorem 3.0.1. *Let* $1 \leq p_1 < \infty$, $1 < p_2 < \infty$ *and* $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}$. *Then the following statements are equivalent:*

1. $\mathcal{H} : L^{p_1} \times L^{p_2} \longrightarrow L^q$.
2. *For all* $f \in L^{p_1}(\mathbf{T})$ *and* $g \in L^{p_2}(\mathbf{T})$, *we have* $\lim_{N\to\infty} S_N(f, g) = f \cdot g$ *in* L^q.
3. $sup_N \|S_N\|_{L^{p_1} \times L^{p_2} \to L^q} < \infty$.

Proof. The proof of the equivalence of (2) and (3) is standard.

Suppose (1) holds. We denote by $P_+ f(x) = \sum_{m \geq 0} \widehat{f}(m)e^{2\pi imx}$ the projection of f onto positive frequencies (this operator maps $L^{p_2}(\mathbf{T})$ to itself for $p_2 > 1$), and by $M^n f(x) = e^{2\pi inx} f(x)$ the modulation of f. Let us notice that we can write the partial sum operator S_N in the following form:

$$S_N(f,g)(x) = \big(M^{-N}\mathcal{H}(f, M^N g_N) - M^{N+1}\mathcal{H}(f, M^{-(N+1)} g_N)\big)(x), \tag{3.0.5}$$

where

$$g_N(x) = M^{-N}P_+M^N g(x) - M^{N+1}P_+M^{-(N+1)}g(x).$$

Indeed, it is easy to see that $M^{-s}P_+M^s g(x) = \sum_{n\geq -s}\widehat{g}(n)e^{2\pi inx}$, and thus $g_N(x) = \sum_{n=-N}^{N}\widehat{g}(n)e^{2\pi inx}$.

Also, we check that

$$M^{-s}\mathcal{H}(f, M^s g)(x) = \sum_{m\geq n}\widehat{f}(m)\widehat{g}(n-s)e^{2\pi i(m+n-s)x} = \sum_{m-n\geq s}\widehat{f}(m)\widehat{g}(n)e^{2\pi i(m+n)x}. \tag{3.0.6}$$

Thus, $M^{-N}\mathcal{H}(\cdot, M^N\cdot) - M^{N+1}\mathcal{H}(\cdot, M^{-(N+1)}\cdot)$ is a bilinear multiplier operator with the symbol equal to the characteristic function of the set $\{(m,n)\in\mathbf{Z}^2 : |m-n|\leq N\}$, which proves the equation (3.0.5).

From (3.0.5), it is easy to see that if $\mathcal{H}$ is bounded from $L^{p_1}\times L^{p_2}$ to L^q, then so is S_N with constant independent of N, i.e. condition (3) holds.

Now assume that condition (3) holds. Define the operators

$$\begin{aligned} A_N(f,g)(x) &= \sum_{|n|\leq N}\sum_{0\leq m-n\leq 2N}\widehat{f}(m)\widehat{g}(n)e^{2\pi i(m+n)x} \\ &= \sum_{|n|\leq N}\sum_{|m'-n|\leq N}\widehat{f}(m'+N)\widehat{g}(n)e^{2\pi i(m'+N+n)x} \\ &= M^N S_N(M^{-N}f, g)(x). \end{aligned}$$

Thus we see that $\sup_n \|A_N\|_{L^{p_1}\times L^{p_2}\to L^q} < \infty$. For $f,g\in C^\infty(\mathbf{T})$ obviously, we have $\lim_{N\to\infty} A_N(f,g) = \mathcal{H}(f,g)$ pointwise, and thus by Fatou's Lemma,

$$\|\mathcal{H}(f,g)\|_{L^q} = \|\lim_{N\to\infty} A_N(f,g)\|_{L^q} \leq \liminf_{N\to\infty}\|A_N(f,g)\|_{L^q} \leq C\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}.$$

Thus $\mathcal{H}$ extends as a bounded operator on $L^{p_1}\times L^{p_2}$, i.e. condition (1) holds. □

Due to the results of Lacey and Thiele (cf. [22], [23]) we can see that the bilinear Fourier series (3.0.4) converges in $L^q(\mathbf{T})$ for functions $f \in L^{p_1}(\mathbf{T})$, $g \in L^{p_2}(\mathbf{T})$, where $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}$, $q > \frac{2}{3}$, $1 < p_1, p_2 \leq \infty$.

It is now easy to see that this bilinear Fourier series does not converge in $L^{\frac{p}{p+1}}(\mathbf{T})$ for functions $f \in L^p(\mathbf{T})$, $g \in L^1(\mathbf{T})$, $p > 1$. To show this we again use transference and work with the bilinear Hilbert transform of functions on the line. We prove the following Lemma:

Lemma 3.0.2. *The bilinear Hilbert transform does not map $L^p \times L^1$ into $L^{\frac{p}{p+1}}$.*

Proof. Define

$$f_N(x) = \frac{1}{x^{\frac{1}{p}}} \cdot \chi_{[1,N]}(x), \quad g_N(x) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(x).$$

Then

$$H(f_N, g_N)(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \frac{1}{(x+t)^{\frac{1}{p}}} \frac{dt}{t} \approx \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} t^{-\frac{p+1}{p}} \approx x^{-\frac{p+1}{p}}$$

for $x \in [\frac{1}{2}, N + \frac{1}{2}]$, and it is 0 otherwise. Thus we have

$$\|H(f_N, g_N)\|_{L^{\frac{p+1}{p}}} \approx \big(\log N\big)^{\frac{p+1}{p}},$$

$$\|f_N\|_{L^p} \approx \big(\log N\big)^{\frac{1}{p}}, \quad \|g_N\|_{L^1} = 1.$$

And so,

$$\frac{\|H(f_N, g_N)\|_{L^{\frac{p+1}{p}}}}{\|f_N\|_{L^p}\|g_N\|_{L^1}} \approx \log N \longrightarrow \infty,$$

which concludes the proof. □

Applying multilinear transference ([11]) and Theorem 3.0.1 we conclude that the bilinear Fourier series (3.0.4) does not converge in $L^{\frac{p+1}{p}}$ for $f \in L^p$, $g \in L^1$.

We now turn to the almost everywhere convergence of the series (3.0.4). It is a standard argument that this convergence would follow from the weak-type estimates for the following maximal operator

$$\mathcal{C}(f,g)(x) = \sup_{N>0} \left| \sum_{(m,n)\in A_N} \widehat{f}(m)\widehat{g}(n)e^{2\pi i(m+n)x} \right|. \tag{3.0.7}$$

This is a bilinear analog of the *Carleson* operator for functions on the torus. Using equation (3.0.5), one can easily see that weak-type bounds for this operator would follow from estimates for the operator

$$T(f,g)(x) = \sup_{N>0} \left| \sum_{m\le n\le N} \widehat{f}(m)\widehat{g}(n)e^{2\pi i(m+n)x} \right|, \tag{3.0.8}$$

which, by the multilinear transference, is equivalent to the estimates for the *bi-Carleson* operator acting on functions on the line, defined by

$$B(f,g)(x) = \sup_{N>0} \left| \int\int_{\xi_1\le\xi_2\le N} \widehat{f}(\xi_1)\widehat{g}(\xi_2)e^{2\pi i(\xi_1+\xi_2)x} \right|. \tag{3.0.9}$$

This operator was considered by Muscalu, Tao, Thiele ([26]). They proved the L^p boundedness of this operator for the same range of exponents that was obtained by Lacey and Thiele for the bilinear Hilbert transform:

Theorem 3.0.3. *[26] The bi-Carleson operator maps $L^{p_1}\times L^{p_2}$ to L^q for $1 < p_1, p_2 \le \infty$, $\frac{2}{3} < q < \infty$, $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}$.*

This theorem actually implies the result of Lacey and Thiele([22], [23]) and the Carleson-Hunt theorem ([4]), [18]).

We notice that the if $g(x) \equiv 1$ for all $x \in \mathbf{T}$, then the partial sums of the bilinear series (3.0.4) become the partial sum of the (linear) Fourier series of the function f:

$$S_N(f,g)(x) = \sum_{(m,n)\in A_N} \widehat{f}(m)\widehat{g}(n)e^{2\pi i(m+n)x} = \sum_{m=-N}^{N} \widehat{f}(m)e^{2\pi imx}.$$

Now, taking $f \in L^1(\mathbf{T})$ to be the Kolmogorov's counterexample (see [20]) of an L^1 function whose Fourier series diverges everywhere, we obtain that

$$\limsup_{N\to\infty} |S_N(f,g)(x)| = \limsup_{N\to\infty} |\sum_{m=-N}^{N} \widehat{f}(m) e^{2\pi i m x}| = \infty$$

for all $x \in \mathbf{T}$. Thus, for these functions the bilinear Fourier series diverges everywhere. Since $f \in L^1(\mathbf{T})$ and $g \equiv 1 \in L^p(\mathbf{T})$ for all $p \geq 1$, we arrive to the following result

Theorem 3.0.4. *The bi-Carleson operator doesn't map* $L^1 \times L^p$ *to* $L^{\frac{p}{p+1},\infty}$.

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VITA

Dmytro (Dmitriy) Bilyk was born March 12, 1979, in Kharkiv, Ukraine (USSR). After attending the Lyceum of Physics and Mathematics in Kharkiv, he received the M.Sc. degree from Kharkiv National University (2001). He is presently a graduate student at the Mathematics Department at University of Missouri - Columbia.