A GENERALIZED APPROACH FOR CALCULATION OF THE EIGENVECTOR SENSITIVITY FOR VARIOUS EIGENVECTOR NORMALIZATIONS

A Thesis
presented to
the Faculty of the Graduate School
University of Missouri - Columbia

In Partial Fulfillment
of the Requirements for the Degree

Master of Science

---

by
VIJENDRA SIDDHI

Dr. Douglas E. Smith, Thesis Supervisor

DECEMBER 2005

The undersigned, appointed by the Dean of the Graduate School, have examined the thesis entitled

A GENERALIZED APPROACH FOR CALCULATION
OF THE EIGENVECTOR SENSITIVITY
FOR VARIOUS EIGENVECTOR NORMALIZATIONS

presented by Vijendra Siddhi

a candidate for the degree of Master of Science

and hereby certify that in their opinion it is worthy of acceptance.

Advisor: Dr. Douglas E. Smith

Readers: Dr. P. Frank Pai

Dr. V.S. Gopalaratnam

Dr. Noah Manring

To my parents, who are always with me even though they are thousands of miles away.

ACKNOWLEDGEMENTS

First I would like to thank my advisor Dr. Smith for all the support, guidance, help and everything else he has given me. It was my privilege to work for him and nothing would have been possible without him. Without the backing of my parents I would not have pursued my masters' degree. I would like to thank my research mates David Jack, Qi Wang and Vijay Rajendran for giving me some valuable feedback. Finally I would like to thank all the faculty in MAE department and all of my friends who helped me get to this point of my life.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

A GENERALIZED APPROACH FOR CALCULATION OF THE EIGENVECTOR SENSITIVITY FOR VARIOUS EIGENVECTOR NORMALIZATIONS

Vijendra Siddhi

Dr. Douglas E. Smith, Thesis Supervisor

ABSTRACT

Sensitivity analysis is an important step in any gradient based optimization problem. Eigenvalue and Eigenvector Sensitivity Analysis has been a major area for more than three decades in structural optimization. An efficient and generalized method is required to do the sensitivity analysis as it can reduce computational time for large industrial problems. Previous methods focus mainly on calculating the eigenvector sensitivity for mass normalized eigenvectors only. A new generalized method is presented to calculate the first and second order eigenvector sensitivities for eigenvectors with any normalization condition. This new generalized method incorporates the use of normalization condition in the eigenvector sensitivity calculation in a manner similar to the calculation of the eigenvectors themselves. This generalized method also reduces to the well known Nelson's method, which is generally accepted as the most efficient and exact method for eigenvector sensitivity analysis for mass normalized eigenvectors. Equations to compute eigenvector sensitivities when the normalization condition is changed are also derived. The effect of the eigenvector normalization condition on the eigenvector sensitivity is discussed. Examples are provided to illustrate the generalized method for the calculation of first-order and second-order eigenvector sensitivities and the use of rescaling equations.

CHAPTER 1

INTRODUCTION

As modern computing capabilities increase, considerable effort is devoted to the development of computational tools for the analysis, design, control, and optimization of complex physical systems. A critical, and sometimes expensive, first step in the process of computer-aided analysis and design is the formulation of a detailed and accurate model of the system. The physical performance of the design can often be modeled with a system of mathematical equations, usually ordinary or partial differential equations. For many engineering applications, analysis of the design requires efficient construction and manipulation of complex geometries along with fast and accurate numerical methods for solving partial differential equations. Generally, there are a number of physical parameters (or design variables) that designers can adjust (either manually or within a computer simulation) to improve the design. In applications such as aerodynamic design, growth and control of thin films, or design of smart materials, some design variables may influence the shape (geometry) of the design. Consequently, designers then become interested in how sensitive the state variables are to small changes in the design variables. For example, when analyzing a composite material, one may be interested in the sensitivity of the heat flow through the material to small changes in orientation of the suspended fibers. Sensitivity analysis is a mathematical tool that provides a methodology for investigating such questions.

Sensitivity Analysis has evolved over the past four decades and has been found to be useful in many engineering applications. In the early stages, sensitivity analysis was used to calculate the effect of changes in design variables of analytical models. The early development of Sensitivity theory was discussed in texts of Tomovic [1], Brayton and Spence [2], Frank [3], and Radanovic [4]. Later, it was used in various areas of optimization, such as finding search directions required to compute optimal solutions in automated structural optimization [5] and optimal control [6]. In addition, researchers have developed and applied sensitivity analysis for approximated analysis, analytical model improvement and assessment of design trends. Other areas of research such as physiology [7], thermodynamics [8], physical chemistry [9], and aerodynamics [10–12], have also started using sensitivity analysis to quantify the effect of varying parameters in the models.

The study of eigenvalue and eigenvector response is an area of high importance because in most vibration problems the response of the structure to dynamic excitation is primarily a function of its eigenvalues (fundamental frequencies) and eigenvectors (mode shapes). Structural optimization is one of the main areas in which eigenvalue and eigenvector sensitivity analysis is used. Knowing the responses of the eigenvectors with respect to physical variables can help an engineer optimize a structure's design or minimize its sensitivity to the variables. In structural control systems, for example, these eigen sensitivities have direct application in system identification and robust performance tests. With the knowledge of the derivatives, an engineer can construct a parameterized evaluation model containing the structure's natural modes. Experimental data can then be used to identify best-fit values for the parameters. Alternatively, the closed-loop control system can be tested to determine if they will perform satisfactorily for all parameter values in a given set. For structural design, these derivatives can be used to optimize the mode frequencies and mode shapes of a structure by varying its design parameters. Mode shape sensitivities can also be used to do damage assessment as shown by Parloo [13]. Due to the singularity of the characteristic matrix associated with eigenvalues, some technical complexities arise in the calculation of the sensitivity of eigenvalues and eigenvectors which makes the sensitivity calculations more complicated.

Several review papers have appeared on sensitivity analysis. For example, a comprehensive survey by Adelman and Haftka [14] on sensitivity analysis of discrete structural systems gives a list of the references related to various methods used for the sensitivity analysis. Murthy and Haftka [15] authored another on eigen sensitivity analysis of a general complex matrix and a review by Grandhi [16] describes the use of eigen sensitivity analysis in structural optimization, providing extensive references by area of application. In addition, Tortorelli and Michaleris [17] provided an overview of design sensitivity analysis for linear elliptic systems which included eigen sensitivities.

One of the earliest researchers to study eigenvalue sensitivity was Jacobi in 1846 who derived equations for the sensitivity of unique eigenvalues. In the early 1960s Lancaster [18] provided equations to calculate the eigenvalue sensitivities for the repeated eigenvalue problem. Fox and Kapoor [19] were the first to do work on eigenvector sensitivities. They developed an approximate approach to calculate eigenvector derivatives for symmetric matrices, which showed that eigenvector derivatives can be written as a linear combination of the eigenvectors themselves. Juang et al. [20], Bernard and Bronowicki [21] and Akgin [22] extended this approach to systems with repeated eigenvalues. While Akgin's method [22] showed improved convergence, Juang et al. [20] considered the repeated eigenvalue derivative case. Lim et al. [23] used singular value decomposition to calculate eigenvector derivatives for repeated eigenvalues. Hou and Kenny [24] also presented an approximate analysis when repeated eigenvalues exist. In addition, Rogers [25], Plaut and Husseyin [26], Rudisill [27], Rudisill and Chu [28], and Doughty [29] studied the eigenvector derivative problem for nonsymmetric matrices.

The most significant development in the study of eigenvector derivatives appeared in 1976, through the pivotal work of Nelson [30] which has since seen wide range of applications throughout engineering analysis and design. The significant advantage of Nelson's approach for calculating eigenvector sensitivities is that it only requires those eigenvectors that are to be differentiated. His algorithm also preserves the bandedness and symmetry of the matrices which reduces the computing time and allows the use of efficient storage and solution techniques. Unfortunately, this method could not be used with repeated eigenvalues and it was valid only for the eigenvectors evaluated using mass normalization. A decade later in 1986, Ojalvo [31, 32] extended Nelson's approach to incorporate repeated eigenvalues that have unique eigenvalue sensitivities. Ojalvo's method was further modified by Mills-Curran [33] and Dailey [34], addressing some of the issues in references [31,32]. While Mills-Curran's method was easy to implement on a computer, it was not robust whereas Dailey's method was more rigorous but difficult to incorporate into programs. But even these methods were not perfect as they assumed unique second-order eigenvalue sensitivities, which is not always the case. Friswell [35] addressed this issue in 1996 with a method that evaluates the eigenvector sensitivities when there are repeated eigenvalues and eigenvalue sensitivities. Zhang and Wang [36] addressed the same issue with a different approach. More recently in 1997, Lee and Jung [37] presented an algebraic method to calculate the sensitivities for both distinct and repeated eigenvalues.

Although most of the work in the eigen sensitivity involves first-order derivatives, interest in the second and higher order eigen sensitivity has also received attention. This is due in part to the need for more accurate approximations when large design parameter changes exist. Rudisill and Chu [28] were the first to give a direct method to calculate the second and higher order eigenvalue and eigenvector sensitivities. Brandon [38, 39] provided an explanation for the significance of second-order modal design sensitivities. He derived the second-order sensitivities with a series expression of the eigenvalues and eigenvectors which has some inherent drawbacks. Chen [40, 41] calculated the second-order sensitivities using a perturbation method and used them for reanalysis of modified structures and to compute the eigenvalue bounds in structural vibration systems. Tan [42] developed an iterative method to calculate the second-order sensitivities of eigenvalues and eigenvectors. Jankovic [43] gave an exact method to calculate n-th order sensitivity of eigenvalues and eigenvectors. All of the above methods were computationally expensive as they needed large amount of data to be calculated and stored. In addition, Chen [44] developed a second-order shape design sensitivity analysis method for three-dimensional elastic solids using a continuum approach. Friswell [45] published a technical note on calculating the higher order sensitivities for eigenvalues and eigenvectors which extended Nelson's method [30] for first-order sensitivity and thus solved many of the computational efficiency issues found in the earlier methods.

Nelson's method [30] has served as a basis for much of the eigenvector sensitivity analysis since its introduction nearly three decades ago. Therefore, most of subsequent derivations and applications [31–37] consider eigenvectors with mass normalization only, as Nelson did. As a result, little attention has been given to the role of the normalization condition in the computation of eigenvector sensitivities. This thesis provides a generalized approach for calculating the eigenvector sensitivities which can accommodate any normalization condition. Nelson's method is explained in a new perspective and it is shown that it is indeed a specific cases of this new approach. A series of equations are also derived such that the design sensitivities for any normalization can be obtained from any other normalization. The new method is extended to evaluate second-order sensitivities as well, and is compared to Friswell's method. Examples are provided to illustrate the new approach where eigenvalue and eigenvector design sensitivities are computed for a simple spring system and a rectangular plate.

CHAPTER 2

THE EIGENVALUES AND EIGENVECTORS

Design Sensitivity Analysis is the study of the relationship between a design variable (or design parameter) and a design performance measure (or design response). The design variables can define geometric variables such as width, thickness etc. or material constants which can be varied such as stiffness, mass etc. Design performance measures can be the displacement, stress, natural frequency etc. As mentioned in Chapter 1, it is very helpful for an engineer to accurately and efficiently calculate the design sensitivity. In this thesis, the eigenvalues and eigenvectors are considered as the design performance measures.

A discussion on the eigenvalue problem is provided before going into the details of the sensitivities of eigenvalues and eigenvectors. The eigenvalue problem can be discussed in terms of two different problems. First as a basic problem in the analysis of free undamped vibration and structural stability and then for a classic problem in linear algebra. The eigenproblem derived for the vibration and structural systems may be written in terms of the eigenvalue λ and eigenvector $\mathbf{\Phi}$ as

$$[\mathbf{K} - \lambda \mathrm{M}]\, \mathbf{\Phi} = \mathbf{0} \tag{2.1}$$

where $\mathbf{K}$ is the stiffness matrix and $\mathbf{M}$ is the mass matrix in a vibration analysis. Equation 2.1 may be used to evaluate the structures buckling load when $\mathbf{M}$ is replaced by the geometric stiffness matrix. Note that all of these matrices are of dimension $n \times n$ where n is the number of free degrees-of-freedom in the discretized system of equations. In the vibration problem, the eigenvalue $\lambda = \omega^2$ is the square of the natural frequency ω and in a buckling analysis, λ is the load factor. The eigenvector $\mathbf{\Phi}$ represents the mode shape.

The eigenproblem may also be written as the classical problem from linear algebra where an eigenvalue is computed to satisfy the system

$$[\mathbf{A} - \lambda\mathbf{I}]\,\mathbf{\Phi} = \mathbf{0} \tag{2.2}$$

The nonzero quantity λ here is the eigenvalue or characteristic value of $\mathbf{A}$. The vector $\mathbf{\Phi}$ is the eigenvector or characteristic vector belonging to λ. The set of eigenvalues of $\mathbf{A}$ is called the spectrum of $\mathbf{A}$ where the largest of the absolute values of the eigenvalues of $\mathbf{A}$ is called the spectral radius of $\mathbf{A}$. To calculate λ and $\mathbf{\Phi}$ the characteristic equation is formed, which is

$$det(\mathbf{K} - \lambda\mathbf{M}) = \mathbf{0} \tag{2.3}$$

for a structural system or

$$det(\mathbf{A} - \lambda\mathbf{I}) = \mathbf{0} \tag{2.4}$$

for a linear algebra problem, where $det()$ indicates the determinant.

Simplifying the above equations gives a polynomial of n-th degree in λ, the roots of this polynomial are the eigenvalues. As the polynomial is of n-th degree, exactly n eigenvalues are obtained. Each of these n eigenvalues will have a corresponding eigenvector $\mathbf{\Phi}$. As it is known from basic linear algebra, for every polynomial the roots can be of three different types: real distinct, real repeated and imaginary.

Similarly eigenvalues can also be of three different types: real distinct eigenvalues, real repeated eigenvalues, imaginary eigenvalues. In most of the physical cases the eigenvector are real, and imaginary eigenvalues are rare.

In the sensitivity analysis presented in this thesis only the structural eigenproblem problem is considered with both $\mathbf{K}$ and $\mathbf{M}$ symmetric and $\mathbf{K}$ as positive definite, which yields real positive λ. One can always convert a structural problem in equation 2.1 into linear algebra problem in equation 3.1 by using the relation $\mathbf{A} = \mathbf{M}^{-1}\mathbf{K}$.

Equation 2.1 has a nontrivial solution only when the characteristic equation 2.3 is valid, the solutions of which will provide eigenvalues λ_i, $i = 1, 2, \ldots, n$. When each distinct λ_i is substituted into equation 2.1, the matrix $[\mathbf{K} - \lambda_i \mathbf{M}]$ is rank deficient by order 1, rendering each nontrivial $\mathbf{\Phi}_i$ being determined within an arbitrary scaling factor. Therefore, normalization condition is required to uniquely define each eigenvector. Common normalization conditions for computing eigenvectors are:

1) Mass normalization:

$$\mathbf{\Phi}_i \cdot \mathbf{M}\mathbf{\Phi}_i = 1 \qquad (2.5)$$

2) Defining a component of $\mathbf{\Phi}_i$:

$$\lfloor 0 \ \ldots \ 0 \ \overbrace{1}^{p\text{-th}} \ 0 \ \ldots \ 0 \rfloor \mathbf{\Phi}_i = \alpha_i \qquad (2.6)$$

3) Defining the magnitude of $\mathbf{\Phi}_i$:

$$\sqrt{\mathbf{\Phi}_i \cdot \mathbf{\Phi}_i} = \beta_i \qquad (2.7)$$

where the p-th component is prescribed a value of α_i in equation 2.6 and the magnitude of $\mathbf{\Phi}_i$ is β_i in equation 2.7. Note that there may be other normalizations which could be used. Any one of equations 2.5 - 2.7 may replace any one of the equations in equation 2.1 to remove the singularity of the matrix $[\mathbf{K} - \lambda_i \mathbf{M}]$. This is achieved by replacing one of the equations in the system of equations in 2.1 with one of the equations 2.5-2.7. For example, when equation 2.6 is used as normalization condition, the modified $[\mathbf{K} - \lambda_i \mathbf{M}]$ becomes full rank and its bandwidth remains same as the original $[\mathbf{K} - \lambda_i \mathbf{M}]$ matrix which proves to be very useful while inverting the matrix. Note that while other normalizations are common (see e.g. equations 2.5 or 2.7), the modified system may become non-linear when these normalizations are used.

A unique eigenvector can be calculated for each eigenvalue with an assumed or given normalization condition, when the eigenvalues are real and distinct. However, when the eigenvalues are real and repeated, the matrix $[\mathbf{K} - \lambda_i \mathbf{M}]$ becomes rank deficient by N, when there are N repeated eigenvalues. There are many research papers available on sensitivities of eigenvectors for repeated eigenvalue case (see for e.g. [31], [33] and [35]).

CHAPTER 3

FIRST-ORDER DESIGN SENSITIVITY ANALYSIS

As mentioned in chapter 1, eigenvalue and eigenvector DSA has received considerable attention during the past four decades. The most efficient and widely used method for the non-repeated eigenvalue case was provided by Nelson [30]. Nelson presented his method considering an algebraic eigensystem and his approach applied to the structural problem appears in Appendix A. Nelson's method is often been considered the standard for eigenvector sensitivity calculations. Nelson's approach has been extended to the repeated eigenvalue case by many researchers, but very few have noticed that his method was only valid for mass normalized eigenvectors. Therefore, even after thirty years there is not much literature on calculation of sensitivities for eigenvectors with different normalization conditions besides mass normalization.

A novel generalized method is presented here, which has some advantages over Nelson's method. Moreover, Nelson's method is shown to be a subset of the new approach present here since Nelson temporarily changes the normalization of the eigenvector sensitivity and scales it back to the original mass normalization. In this new approach, the normalization which is used to calculate the eigenvectors is also employed to calculate the eigenvector sensitivity.

Design sensitivities quantify the relationship between design variables b_j, $j = 1, 2, \ldots, N$, which form the N-dimensional vector $\mathbf{b}$, and computed eigenpairs $(\lambda_i, \mathbf{\Phi}_i)$, $i = 1, 2, \ldots, n$. To better illustrate the sensitivity analysis, the dependence of the eigenproblem and normalization conditions on the design variable vector $\mathbf{b}$ is emphasized, respectively, as

$$[\mathbf{K}(\mathbf{b}) - \lambda_i(\mathbf{b})\mathbf{M}(\mathbf{b})]\, \mathbf{\Phi}_i(\mathbf{b}) = \mathbf{0} \tag{3.1}$$

and

$$G(\mathbf{\Phi}_i(\mathbf{b}), \mathbf{b}) = 0 \tag{3.2}$$

which defines the scalar function $G = 0$, in general, having an explicit dependence on $\mathbf{b}$ in addition to being implicitly defined in terms of $\mathbf{b}$ through the eigenvector $\mathbf{\Phi}_i$. The various normalization conditions defined as equations 2.5 - 2.7 can be written as scalar functions in the form of equation 3.2 as

1) Mass normalization:

$$G_1(\mathbf{\Phi}_i) = \mathbf{\Phi}_i \cdot \mathbf{M}\mathbf{\Phi}_i - 1 = 0 \tag{3.3}$$

2) Defining a component of $\mathbf{\Phi}_i$:

$$G_2(\mathbf{\Phi}_i) = \lfloor 0 \ \ldots \ 0 \ \overbrace{\ 1\ }^{p\text{-th}} \ 0 \ \ldots \ 0 \rfloor \mathbf{\Phi}_i - \alpha_i = 0 \tag{3.4}$$

3) Defining the magnitude of $\mathbf{\Phi}_i$:

$$G_3(\mathbf{\Phi}_i) = \sqrt{\mathbf{\Phi}_i \cdot \mathbf{\Phi}_i} - \beta_i = 0 \tag{3.5}$$

It is possible to define any other normalization conditions that can be used to calculate the eigenvectors, in the form of a scalar function $G = 0$.

3.1 First-Order Eigenvalue Sensitivity

The calculation of Eigenvalue sensitivities is known to be a simple and straightforward computation and there are many papers which discuss this in detail. It was explained first by Fox and Kapoor [19] in 1968 and is summarized here. Differentiating the eigen problem equation 3.1 with respect to the design variable b_j yields

$$\left[\frac{\partial \mathbf{K}}{\partial b_j} - \frac{d\lambda_i}{db_j}\mathbf{M} - \lambda_i \frac{\partial \mathbf{M}}{\partial b_j}\right] \mathbf{\Phi}_i + [\mathbf{K} - \lambda_i \mathbf{M}] \frac{d\mathbf{\Phi}_i}{db_j} = \mathbf{0} \tag{3.6}$$

Premultiplying equation 3.6 with $\mathbf{\Phi}_i$ and rearranging the terms yields the first-order eigenvalue design derivative as

$$\frac{d\lambda_i}{db_j} = \frac{1}{C_i}\mathbf{\Phi}_i \cdot \left[\frac{\partial \mathbf{K}}{\partial b_j} - \lambda_i \frac{\partial \mathbf{M}}{\partial b_j}\right] \mathbf{\Phi}_i \tag{3.7}$$

where $C_i = \mathbf{\Phi}_i \cdot \mathbf{M}\mathbf{\Phi}_i$ is the i-th generalized mass which is equated to unity when mass normalization in equation 3.3 is employed. Both Fox [19] and Nelson [30] assume mass normalization for the eigenvectors, and therefore the eigenvalue sensitivity for that case simplifies to

$$\frac{d\lambda_i}{db_j} = \mathbf{\Phi}_i \cdot \left[\frac{\partial \mathbf{K}}{\partial b_j} - \lambda_i \frac{\partial \mathbf{M}}{\partial b_j}\right] \mathbf{\Phi}_i \tag{3.8}$$

Most of the previous work for eigenvalue sensitivity only consider mass normalization and therefore use equation 3.8 instead of equation 3.7. However, when other normalizations are considered, C_i plays a very important role in the eigenvalue sensitivity as seen in equation 3.7. The numerical value of the eigenvalue sensitivity is exactly the same for all normalization conditions since C_i appears in the equation 3.7, which is therefore the generalized equation to calculate the eigenvalue sensitivity for any normalization condition.

Finally it is emphasized that equation 3.7 is the required expression to get the explicit sensitivity for the i-th eigenvalue. This computation only requires the matrices $\partial\mathbf{M}/\partial b_j$ and $\partial\mathbf{K}/\partial b_j$ followed by relatively simple matrix calculations which avoid additional computationally expensive eigenproblem solutions. Furthermore, the calculation of the eigenvalue sensitivity $d\lambda_i/db_j$ requires only the i-th eigenpair $(\lambda_i, \mathbf{\Phi}_i)$ in its calculation. This result may now be used in the vibration problem to calculate the design sensitivity of the natural frequency $\omega_i = \sqrt{\lambda_i}\ cycles/sec$ with respect to the design variable b_j as

$$\frac{d\omega_i}{db_j} = \frac{1}{2\omega_i}\frac{d\lambda_i}{db_i}$$

3.2 First-Order Eigenvector Design Sensitivity Analysis

The first-order design sensitivities of the eigenvectors with respect to the design b_j, represented by $d\mathbf{\Phi}_i/db_j$ for $i = 1, 2, ..., n$ and $j = 1, 2, ..., N$ are calculated here. Equation 3.6 is rearranged as

$$[\mathbf{K} - \lambda_i \mathbf{M}] \frac{d\mathbf{\Phi}_i}{db_j} = - \left[\frac{\partial \mathbf{K}}{\partial b_j} - \frac{d\lambda_i}{db_j} \mathbf{M} - \lambda_i \frac{\partial \mathbf{M}}{\partial b_j} \right] \mathbf{\Phi}_i \tag{3.9}$$

where all of the terms on the right-hand-side are known once $d\lambda_i/db_j$ is computed from equation 3.7.

Unfortunately, equation 3.9 cannot be solved for $d\mathbf{\Phi}_i/db_j$ since the original eigenvalue solution computed each λ_i to make $[\mathbf{K} - \lambda_i \mathbf{M}]$ singular. In this regard, problems when computing eigenvector sensitivities from equation 3.9 are the same as those faced when solving the original eigenvector problem of equation 2.1. In the design sensitivity analysis problem in equation 3.9, it is assumed that a normalization condition represented by equation 3.2 has been selected from equations 3.3 - 3.5 or similar to calculate the eigenvector $\mathbf{\Phi}_i$. This normalization condition can be differentiated to obtain

$$\frac{dG}{db_j} = \frac{\partial G}{\partial \mathbf{\Phi}_i} \cdot \frac{d\mathbf{\Phi}_i}{db_j} + \frac{\partial G}{\partial b_j} = 0 \tag{3.10}$$

which can be rearranged as

$$\frac{\partial G}{\partial \mathbf{\Phi}_i} \cdot \frac{d\mathbf{\Phi}_i}{db_j} = - \frac{\partial G}{\partial b_j} \tag{3.11}$$

In the above equation, the term $\partial G/\partial \mathbf{\Phi}_i$ represents a row matrix and the term

$\partial G/\partial b_j$ represents a scalar quantity. When equation 3.11 is replaces one of the equations in equation 3.9 (in a manner similar to that described above for the evaluation of eigenvectors), the system matrix on the left-hand-side of equation 3.9 is modified such that it becomes non-singular, making the solution of $d\mathbf{\Phi}_i/db_j$ possible.

To employ equation 3.11 when computing eigenvector sensitivities with equation 3.9, $\partial G/\partial \mathbf{\Phi}_i$ and $\partial G/\partial \mathbf{\Phi}_j$ for the selected eigenvector normalization are required. The calculation of these terms for some common normalization conditions is shown below:

1) **Mass normalization:** Let us consider the case when the eigenvectors are normalized with respect to the mass matrix as used in Nelson's method. The mass normalization equation 3.3 can be written in the form of the general normalization equation 3.2 to give

$$G_1(\mathbf{\Phi}_i(\mathbf{b}), \mathbf{b}) = \mathbf{\Phi}_i(\mathbf{b}) \cdot \mathbf{M}(\mathbf{b})\mathbf{\Phi}_i(\mathbf{b}) - 1 = 0 \tag{3.12}$$

differentiating the above equation with respect to the eigenvector $\mathbf{\Phi}_i$ and the design variable b_i separately yields the following

$$\begin{aligned} \left\lfloor \frac{\partial G_1}{\partial \mathbf{\Phi}_i} \right\rfloor &= 2\mathbf{\Phi}_i \cdot \mathbf{M} \\ \frac{\partial G_1}{\partial b_j} &= \mathbf{\Phi}_i \cdot \frac{\partial \mathbf{M}}{\partial b_j} \mathbf{\Phi}_i \end{aligned} \tag{3.13}$$

where the symmetry of $\mathbf{M}$ is used to simplify equation 3.13. Equations 3.13 may now be substituted into equation 3.10 and the result used to eliminate the singularity issue in equation 3.9 when computing $d\mathbf{\Phi}_i/db_j$. Note that the eigenvector $\mathbf{\Phi}_i$ on the right hand side is mass normalized.

2) Defining a component of $\mathbf{\Phi}_i$: As a second example, consider the case where the eigenvector is normalized such that its p-th component is prescribed a value of α_i. This condition can be written in the form of equation 3.2 as

$$G_2(\mathbf{\Phi}_i(\mathbf{b}), \mathbf{b}) = \lfloor 0 \;\ldots\;\ldots\; 0 \;\overbrace{1}^{p\text{-th}}\; 0 \;\ldots\; 0 \rfloor \cdot \mathbf{\Phi}_i(\mathbf{b}) - \alpha_i = 0 \qquad (3.14)$$

where p identifies the prescribed component of $\mathbf{\Phi}_i$, *i.e.*, $(\mathbf{\Phi}_i)_p = \alpha_i$. Differentiation of equation 3.14 yields

$$\left\lfloor \frac{\partial G_2}{\partial \mathbf{\Phi}_i} \right\rfloor = \lfloor 0 \;\ldots\;\ldots\; 0 \;\overbrace{1}^{p\text{-th}}\; 0 \;\ldots\; 0 \rfloor \qquad (3.15)$$

$$\frac{\partial G_2}{\partial b_j} = 0 \qquad (3.16)$$

which simply states that $(d\mathbf{\Phi}_i/db_j)_p = 0$. When $\alpha_i = 1$ and p is assigned to the eigenvector component having the maximum absolute value, it is similar to the assumption Nelson makes to calculate $\boldsymbol{\nu}_{ij}$ in the equation A.1. The similarity to Nelson's assumption is considered in detail below. Note that when G_2 is used to calculate the design sensitivities the modified matrix on the left hand side of the sensitivity equation 3.9 is same as the modified matrix on the left hand side of the equation used to calculate the eigenvector itself (see equation 2.1). This property can be utilized very effectively to significantly reduce time and cost for the calculation.

3) Defining the magnitude of $\mathbf{\Phi}_i$: As a third example, consider the case where the eigenvector is normalized such that its magnitude is prescribed a value of β_i which is often equated to unity. This condition can be written in the form of equation 3.2 as

$$G_3(\mathbf{\Phi}_i(\mathbf{b}), \mathbf{b}) = |\mathbf{\Phi}_i(\mathbf{b})| - \beta_i = 0 \tag{3.17}$$

where

$$|\mathbf{\Phi}_i(\mathbf{b})| = \sqrt{\mathbf{\Phi}_i \cdot \mathbf{\Phi}_i} \tag{3.18}$$

is the magnitude of the eigenvector $\mathbf{\Phi}_i$. Differentiation of equation 3.17 yields

$$\begin{aligned} \left\lfloor \frac{\partial G_3}{\partial \mathbf{\Phi}_i} \right\rfloor &= \frac{1}{\beta_i}\{\mathbf{\Phi}_i\}^T \\ \frac{\partial G_3}{\partial b_j} &= 0 \end{aligned} \tag{3.19}$$

As explained above using this approach, the terms $\partial G/\partial \mathbf{\Phi}_i$ and $\partial G/\partial b_j$ can be calculated for any selected normalization condition, therefore this method is considered as a generalized method. It is known that the eigenvectors with any normalization only differ in their magnitude, i.e the direction remains the same. In other words eigenvectors with one normalization can be scaled to another normalization by multiplying with a constant. However this property does not hold for eigenvector design sensitivity. The sensitivities of eigenvectors with one normalization are different in both magnitude and direction to the sensitivities of eigenvectors with a different normalization. The sensitivities of eigenvectors with one normalization can be obtained from sensitivities of eigenvectors with another normalization as illustrated in the following section.

It should be clear from equation 3.7 that the eigenvector normalization does not effect the eigenvalue sensitivity due to the presence of the generalized mass term

C_i. However, it is shown here that the normalization approach directly defines the eigenvector sensitivity, as might be expected. Further proof in this regard is provided by the numerical examples provided in the following chapters.

The calculation of sensitivities for eigenvectors with different normalization conditions have different computational advantages with this approach. When G_2 is used as the normalization condition, the band width of and symmetry of the system matrix $[\mathbf{K} - \lambda\mathbf{M}]$ is preserved, even after the removal of singularity. If G_1 and G_3 are used as normalization conditions, the sensitivity problem will always be linear even though the original eigen problem could be non-linear. These properties could be effectively used to reduce computational time and cost.

3.3 Effect of Normalization Condition

In the previous section, a generalized method to calculate the design sensitivity of an eigenvector with any normalization is presented. In some cases it might be desired to rescale the eigenvector and then calculate its sensitivity with the modified normalization condition without re-evaluating equations 3.9 to 3.11. A set of equations are derived in this chapter using the relationship between the eigenvectors with different normalization conditions with which the sensitivities can be easily converted from one normalization condition to another. These equations can be very helpful in eigenvector matching problems for structural systems and they also show the difference in the sensitivities of eigenvectors with different normalization conditions.

Once computed, an eigenvector $\mathbf{\Phi}_i$ may be re-scaled by a constant c to give

$$\hat{\mathbf{\Phi}}_i(\mathbf{b}) = c(\mathbf{b})\mathbf{\Phi}_i(\mathbf{b}) \tag{3.20}$$

and still satisfy equation 3.1. In other words, assuming $\mathbf{\Phi}_i$ as an eigenvector vector with a given normalization condition, it can be transformed or scaled into another eigenvector $\hat{\mathbf{\Phi}}_i$ with a new normalization condition using the equation 3.20. Here we note that the scaling parameter $c(\mathbf{b})$ is a function of the design variable $\mathbf{b}$. The constant c can be obtained by simple calculations shown below and it depends only on the normalization condition of $\hat{\mathbf{\Phi}}_i$.

To compute the sensitivity of the scaled eigenvector $\hat{\mathbf{\Phi}}_i$, equation 3.20 may be differentiated as

$$\frac{d\hat{\mathbf{\Phi}}_i}{db_j} = \frac{\partial c}{\partial b_j}\mathbf{\Phi}_i + c\frac{d\mathbf{\Phi}_i}{db_j} \tag{3.21}$$

where the eigenvector sensitivity with the modified normalization condition $d\hat{\mathbf{\Phi}}_i/db_j$ can be calculated from the eigenvector sensitivity evaluated with the original normalizaion condition $d\mathbf{\Phi}_i/db_j$.

For example, an eigenvector $\mathbf{\Phi}_i(\mathbf{b})$ can be scaled such that it is normalized with respect to the mass matrix using equation 3.20 with

$$c(\mathbf{b}) = 1/\sqrt{C_i(\mathbf{b})} \tag{3.22}$$

where

$$C_i(\mathbf{b}) = \mathbf{\Phi}_i(\mathbf{b}) \cdot \mathbf{M}(\mathbf{b})\mathbf{\Phi}_i(\mathbf{b}) \tag{3.23}$$

is the generalized mass. Differentiating the equation 3.22 and substituting the result into equation 3.21 yields the design sensitivity of the scaled eigenvector as

$$\frac{d\hat{\mathbf{\Phi}}_i}{db_j} = -\frac{1}{2\sqrt{C_i}^3}\frac{dC_i}{db_j}\mathbf{\Phi}_i + \frac{1}{\sqrt{C_i}}\frac{d\mathbf{\Phi}_i}{db_j} \tag{3.24}$$

where equation 3.23 is differentiated to obtain

$$\frac{dC_i}{db_j} = 2\mathbf{\Phi}_i \cdot \mathbf{M}\frac{d\mathbf{\Phi}_i}{db_j} + \mathbf{\Phi}_i \cdot \frac{\partial \mathbf{M}}{\partial b_j}\mathbf{\Phi}_i \tag{3.25}$$

In the above equation, the symmetry of $\mathbf{M}$ is used to simplify the expression. Note that when a component of $\mathbf{\Phi}_i$ is arbitrarily set to a prescribed value, the generalized mass becomes a function of design. Alternatively, had $\mathbf{\Phi}_i$ been normalized with respect to the mass matrix, then $C_i = 1$ for all design variables $\mathbf{b}$ and $dC_i/db_j = 0$ which results in $d\hat{\mathbf{\Phi}}_i/db_j = d\mathbf{\Phi}_i/db_j$, as expected.

Note that c and $\partial c/\partial b_j$ depend upon the normalization to which the sensitivities need to be converted and they are not arbitrary as suggested by elsewhere (see e.g. [15]).

It follows that if an eigenvector is re-scaled to satisfy G_2 then c and $\partial c/\partial b_j$ in equation 3.20 and 3.21 are, respectively,

$$c(\mathbf{b}) = \frac{\alpha_i}{(\mathbf{\Phi}_i)_p} \tag{3.26}$$

and

$$\frac{\partial c}{\partial b_j} = -c^2 \left(\frac{\partial \mathbf{\Phi}_i}{\partial b_j}\right)_p \tag{3.27}$$

where p is the location of the maximum component of $\mathbf{\Phi}_i$. Similarly, to rescale the eigenvector to satisfy G_3 the following equations may be employed

$$c(\mathbf{b}) = \beta_i |\mathbf{\Phi}_i| \tag{3.28}$$

and

$$\frac{\partial c}{\partial b_j} = -c^3 \ \mathbf{\Phi}_i \cdot \frac{\partial \mathbf{\Phi}_i}{\partial b_j} \tag{3.29}$$

3.4 Interpretation of Nelson's Method

As mentioned in chapter 1, Nelson [30] developed a very powerful tool to calculated the eigenvector sensitivities for mass normalized eigenvectors. In this section, the rescaling equations from previous section are used to show that the generalized approach presented in this thesis reduces to the Nelson's method [30] upon making the appropriate assumptions. Let $d\hat{\mathbf{\Phi}}_i/db_j$ represent a mass normalized eigenvector sensitivity and $d\mathbf{\Phi}_i/db_j$ represent a non-mass normalized eigenvector sensitivity. Nelson method [30] presented in detail in appendix A, computes the eignevector sensitivity as

$$\frac{d\hat{\mathbf{\Phi}}_i}{db_j} = \boldsymbol{\nu}_{ij} + c_{ij}\hat{\mathbf{\Phi}}_i \tag{3.30}$$

where

$$c_{ij} = -\hat{\mathbf{\Phi}}_i \cdot \mathbf{M}\boldsymbol{\nu}_{ij} - \frac{1}{2}\hat{\mathbf{\Phi}}_i \cdot \frac{\partial \mathbf{M}}{\partial b_j}\hat{\mathbf{\Phi}}_i \tag{3.31}$$

and $\boldsymbol{\nu}_{ij}$ is a vector associated with the i-th mode and j-th design variable computed from a linear combination of the eigenvectors $\mathbf{\Phi}_m$, $m = 1, 2, \ldots n, m \neq i$. The equation 3.30 used by Nelson is similar to the re-scaling equation 3.21 derived in the previous section. Therefore, let us rewrite all the terms on the right hand side of equation 3.21 in terms of $\hat{\mathbf{\Phi}}_i$. Using the re-scaling equation 3.20 for eigenvectors, equation 3.21 can be written as

$$\frac{d\hat{\mathbf{\Phi}}_i}{db_j} = \mathbf{A}_{ij} + \frac{1}{c}\frac{\partial c}{\partial b_j}\hat{\mathbf{\Phi}}_i \tag{3.32}$$

where the vector A_{ij} for mode i and design variable j is evaluated from

$$\mathbf{A}_{ij} = c\frac{d\mathbf{\Phi}_i}{db_j} \tag{3.33}$$

Nelson calculates $\boldsymbol{\nu}_{ij}$ by assuming the rows and columns related to component of maximum value to zero represent in the equation A.3, which is identical to calculating the sensitivity of the eigenvector with maximum value equal to one normalization condition G_2 represented by equation 3.14 with α equal to one. Hence, it can be said that $\mathbf{A}_{ij}$ is equal to $\boldsymbol{\nu}_{ij}$ when $\mathbf{\Phi}_i$ is assumed to be normalized with maximum value equal to one normalization condition G_2.

As $\hat{\mathbf{\Phi}}_i$ is normalized with respect to mass, equation 3.22 is used to evaluate c. Using the values of c and $\partial c/\partial b_j$ from the equations 3.22 and 3.25 respectively, the term $\frac{1}{c}\frac{\partial c}{\partial b_j}$ can be manipulated as

$$\begin{aligned}
\frac{1}{c}\frac{\partial c}{\partial b_j} &= -\frac{1}{2c\sqrt{C_i}^3}\frac{dC_i}{db_j} \\
&= -\frac{c^2}{2}\frac{dC_i}{db_j} \\
&= -\frac{c^2}{2}\left\{2\mathbf{\Phi}_i \cdot \mathbf{M}\frac{d\mathbf{\Phi}_i}{db_j} + \mathbf{\Phi}_i \cdot \frac{\partial \mathbf{M}}{\partial b_j}\mathbf{\Phi}_i\right\} \\
&= -c\mathbf{\Phi}_i \cdot \mathbf{M}c\frac{d\mathbf{\Phi}_i}{db_j} - \frac{1}{2}c\mathbf{\Phi}_i \cdot \frac{\partial \mathbf{M}}{\partial b_j}c\mathbf{\Phi}_i \\
&= -\hat{\mathbf{\Phi}}_i \cdot \mathbf{M}\mathbf{A}_{ij} - \frac{1}{2}\hat{\mathbf{\Phi}}_i \cdot \frac{\partial \mathbf{M}}{\partial b_j}\hat{\mathbf{\Phi}}_i \\
&= -\hat{\mathbf{\Phi}}_i \cdot \mathbf{M}\boldsymbol{\nu}_{ij} - \frac{1}{2}\hat{\mathbf{\Phi}}_i \cdot \frac{\partial \mathbf{M}}{\partial b_j}\hat{\mathbf{\Phi}}_i \\
&= c_{ij}
\end{aligned} \tag{3.34}$$

where c_{ij} is used in Nelson's method as shown in appendix A.

Therefore, when the terms $\mathbf{A}_{ij}$ and $\frac{1}{c}\frac{\partial c}{\partial b_j}$ are replaced by $\boldsymbol{\nu}_{ij}$ and c_{ij} respectively, on the right hand side of equation 3.32 it becomes equation 3.30. Therefore, the re-scaling equation 3.21 reduces to 3.30 when $\mathbf{\Phi}_i$ satisfies equation 3.4 with α_i =1. This shows that Nelson's method [30] can be derived from our generalized approach, by using the re-scaling equation. In other words, Nelson first calculates the eigenvector sensitivity for maximum value equal one normalized eigenvector in the form of $\boldsymbol{\nu}_{ij}$ and then re-scales it to match the normalization of the original eigenvector which is mass normalization with the equation 3.30. Note that Nelson's method preserves the band width and symmetry of the matrices. This technique of preserving the band width while calculate the sensitivities for can be applied not only for mass normalized eigenvectors but also for any other normalized eigenvectors with the help of the re-scaling equations derived in the previous section.

CHAPTER 4

SECOND-ORDER DESIGN SENSITIVITY ANALYSIS

Interest in second- and higher-order sensitivities is increasing, particularly to estimate the eigensystems of modified structures. For large design parameter changes, the linear approximation inherent in the use of first-order sensitivities may not be sufficient. When the roots are non-repeated, Nelson's method can be extended to derive second-order sensitivities. This was first proposed by Friswell [35] and a short description of the method is given in Appendix B. Just as Friswell extended Nelson's method to second-order eigenvector sensitivity analysis, the generalized method in this thesis can also be extended to the second-order design derivatives. Calculating the second-order design derivatives is very similar to that of the first-order, which are assumed to be evaluated as described above.

4.1 Second-Order Eigenvalue Sensitivity

To generalize the calculation of the second-order eigenvalue sensitivity, the design variable b_k is also considered in addition to b_j from Chapter 3. The variable b_k may be the same or different than the variable b_j. Differentiating equation 3.6 for first-order eigen sensitivity with respect to the design variable b_k yields

$$\begin{aligned}
&\left[\frac{\partial^2\mathbf{K}}{\partial b_j\partial b_k}-\frac{d^2\lambda_i}{db_jdb_k}\mathbf{M}-\frac{d\lambda_i}{db_k}\frac{\partial\mathbf{M}}{\partial b_j}-\frac{d\lambda_i}{db_j}\frac{\partial\mathbf{M}}{\partial b_k}-\lambda_i\frac{\partial^2\mathbf{M}}{\partial b_j\partial b_k}\right]\mathbf{\Phi}_i\\
&+\left[\frac{\partial\mathbf{K}}{\partial b_j}-\frac{d\lambda_i}{db_j}\mathbf{M}-\lambda_i\frac{\partial\mathbf{M}}{\partial b_j}\right]\frac{d\mathbf{\Phi}_i}{db_k}+\left[\frac{\partial\mathbf{K}}{\partial b_k}-\frac{d\lambda_i}{db_k}\mathbf{M}-\lambda_i\frac{\partial\mathbf{M}}{\partial b_k}\right]\frac{d\mathbf{\Phi}_i}{db_j}\\
&+\left[\mathbf{K}-\lambda_i\mathbf{M}\right]\frac{d^2\mathbf{\Phi}_i}{db_jdb_k} = \mathbf{0} \qquad (4.1)
\end{aligned}$$

where it is assumed that $\partial^2\mathbf{M}/\partial b_j\partial b_k$ and $\partial^2\mathbf{K}/\partial b_j\partial b_k$ on the right hand side are known. Premultiplying equation 4.1 by $\mathbf{\Phi}_i$ and rearranging yields the second-order eigenvalue design derivative for the eigenvalue λ_i as

$$\begin{aligned}\frac{d^2\lambda_i}{db_jdb_k} = \frac{1}{C_i} \quad \mathbf{\Phi}_i \quad &\cdot \left[\frac{\partial^2\mathbf{K}}{\partial b_j\partial b_k} - \frac{d\lambda_i}{db_k}\frac{\partial\mathbf{M}}{\partial b_j} - \frac{d\lambda_i}{db_j}\frac{\partial\mathbf{M}}{\partial b_k} - \lambda_i\frac{\partial^2\mathbf{M}}{\partial b_j\partial b_k}\right]\mathbf{\Phi}_i \\ &+ \quad \mathbf{\Phi}_i \cdot \left[\frac{\partial\mathbf{K}}{\partial b_j} - \frac{d\lambda_i}{db_j}\mathbf{M} - \lambda_i\frac{\partial\mathbf{M}}{\partial b_j}\right]\frac{d\mathbf{\Phi}_i}{db_k} \\ &+ \quad \mathbf{\Phi}_i \cdot \left[\frac{\partial\mathbf{K}}{\partial b_k} - \frac{d\lambda_i}{db_k}\mathbf{M} - \lambda_i\frac{\partial\mathbf{M}}{\partial b_k}\right]\frac{d\mathbf{\Phi}_i}{db_j}\end{aligned} \tag{4.2}$$

where, as it was in the first-order eigenvalue sensitivity, $C_i = \mathbf{\Phi}_i \cdot \mathbf{M}\mathbf{\Phi}_i$ is the i-th generalized mass. Again the importance of C_i in the calculation of eigenvalue sensitivity is exposed such that the second-order eigenvalue sensitivity is invariant to the assumed normalization conditions. Therefore equation 4.2 is the explicit equation to get the second-order sensitivity for the i-th eigenvalue.

4.2 Second-Order Eigenvector Sensitivity Analysis

The calculation of the second-order eigenvector sensitivities may be obtained by rearranging equation 4.1 as

$$\begin{aligned}
[\mathbf{K}-\lambda_i\mathbf{M}]\frac{d^2\mathbf{\Phi}_i}{db_j db_k} &= -\left[\frac{\partial^2\mathbf{K}}{\partial b_j\partial b_k}-\frac{d^2\lambda_i}{db_j db_k}\mathbf{M}-\frac{d\lambda_i}{db_k}\frac{\partial\mathbf{M}}{\partial b_j}-\frac{d\lambda_i}{db_j}\frac{\partial\mathbf{M}}{\partial b_k}-\lambda_i\frac{\partial^2\mathbf{M}}{\partial b_j\partial b_k}\right]\mathbf{\Phi}_i \\
&- \left[\frac{\partial\mathbf{K}}{\partial b_j}-\frac{d\lambda_i}{db_j}\mathbf{M}-\lambda_i\frac{\partial\mathbf{M}}{\partial b_j}\right]\frac{d\mathbf{\Phi}_i}{db_k}-\left[\frac{\partial\mathbf{K}}{\partial b_k}-\frac{d\lambda_i}{db_k}\mathbf{M}-\lambda_i\frac{\partial\mathbf{M}}{\partial b_k}\right]\frac{d\mathbf{\Phi}_i}{db_j}
\end{aligned} \tag{4.3}$$

where all of the terms on the right-hand-side are known once $d\lambda_i/db_j$ and $d\lambda_i/db_k$ is calculated from equation 4.2 and $d\mathbf{\Phi}_i/db_j$, $d\mathbf{\Phi}_i/db_j$ are evaluated from equations 3.9 and 3.10, and $\frac{d^2\lambda_i}{db_j db_k}$ are obtained from equation 4.2.

Similar to the first-order sensitivity evaluation, equation 4.3 cannot be solved as the matrix $[\mathbf{K}-\lambda_i\mathbf{M}]$ is singular. Therefore the same technique used in the first-order differentiation is also employed to evaluate second-order derivatives as well. It follows that the selected normalization condition may be differentiated twice to obtain

$$\begin{aligned}
\frac{d^2G}{db_j db_k} = \frac{\partial^2 G}{\partial\mathbf{\Phi}_i\partial b_k}\cdot\frac{d\mathbf{\Phi}_i}{db_j}+\frac{d\mathbf{\Phi}_i}{db_k}\cdot\frac{\partial^2 G}{\partial^2\mathbf{\Phi}_i}\frac{d\mathbf{\Phi}_i}{db_j}+\frac{\partial G}{\partial\mathbf{\Phi}_i}\cdot\frac{d^2\mathbf{\Phi}_i}{db_j db_j} \\
+\frac{\partial^2 G}{\partial b_j\partial b_k}+\frac{\partial^2 G}{\partial b_j\partial\mathbf{\Phi}_i}\cdot\frac{d\mathbf{\Phi}_i}{db_k} \quad = \quad 0
\end{aligned} \tag{4.4}$$

which is rearranged as

$$\frac{\partial G}{\partial\mathbf{\Phi}_i}\cdot\frac{d^2\mathbf{\Phi}_i}{db_j db_k} = -\left[\frac{\partial^2 G}{\partial\mathbf{\Phi}_i\partial b_k}\cdot\frac{d\mathbf{\Phi}_i}{db_j}+\frac{d\mathbf{\Phi}_i}{db_k}\cdot\frac{\partial^2 G}{\partial\mathbf{\Phi}_i^2}\frac{d\mathbf{\Phi}_i}{db_j}+\frac{\partial^2 G}{\partial b_j\partial b_k}+\frac{\partial^2 G}{\partial b_j\partial\mathbf{\Phi}_i}\cdot\frac{d\mathbf{\Phi}_i}{db_k}\right] \tag{4.5}$$

One of the equations in the system of equations 4.3 may be replaced with equation 4.5 to make the system matrix on left hand side non-singular as described above.

For each normalization condition considered in Chapters 2 and 3 the derivatives in equation 4.5 are calculated as follows:

1) Mass normalization: As defined above in equation 3.12 the mass normalization equation is written in terms of the design variable $\mathbf{b}$ as

$$G_1(\mathbf{\Phi}_i(\mathbf{b}), \mathbf{b}) = \mathbf{\Phi}_i(\mathbf{b}) \cdot \mathbf{M}(\mathbf{b})\mathbf{\Phi}_i(\mathbf{b}) - 1 = 0$$

By performing partial differentiation on G_1 we obtain

$$\begin{aligned} \left\lfloor \frac{\partial G_1}{\partial \mathbf{\Phi}_i} \right\rfloor &= 2\mathbf{\Phi}_i \cdot \mathbf{M} \\ \left\lfloor \frac{\partial^2 G_1}{\partial \mathbf{\Phi}_i^2} \right\rfloor &= 2\mathbf{M} \\ \frac{\partial^2 G_1}{\partial b_j \partial b_k} &= \mathbf{\Phi}_i \cdot \frac{\partial^2 \mathbf{M}}{\partial b_j \partial b_k}\mathbf{\Phi}_i \\ \frac{\partial^2 G_1}{\partial \mathbf{\Phi}_i \partial b_j} = \frac{\partial^2 G_1}{\partial b_j \partial \mathbf{\Phi}_i} &= 2\mathbf{\Phi}_i \cdot \frac{\partial \mathbf{M}}{\partial b_j} \end{aligned} \tag{4.6}$$

Substituting these terms in equation 4.5 and then replacing one of the equations from the system of equation 4.3 with the result makes it possible to obtain the second-order eigenvector sensitivity $d^2\mathbf{\Phi}_i/db_jdb_k$. As mentioned in the first-order calculations, all the eigenvectors and the first-order sensitivities on the right hand side are obtained using mass normalization. It can also be seen that modified $[\mathbf{K} - \lambda\mathbf{M}]$ is the same for both first-order and second-order sensitivity analysis. Therefore, once the first-order design derivatives are computed, second-order sensitivities are readily evaluated by forming an additional right-hand side vector and performing an additional back substitution into the inverted or decomposed modified system matrix.

2) Defining a component of $\mathbf{\Phi}_i$: Similarly, when the eigenvector is normalized by defining one of its components, it can be represented by equation 3.14 written in terms of $\mathbf{b}$ as

$$G_2(\mathbf{\Phi}_i(\mathbf{b}),\mathbf{b}) = \lfloor 0\ \ldots\ \ldots\ 0\ \overbrace{1}^{p\text{-th}}\ 0\ \ldots\ 0 \rfloor \cdot \mathbf{\Phi}_i(\mathbf{b}) - \alpha_i = 0$$

which may be differentiated to obtain

$$\begin{aligned}
\left\lfloor \frac{\partial G_2}{\partial \mathbf{\Phi}_i} \right\rfloor &= \lfloor 0\ \ldots\ \ldots\ 0\ \overbrace{1}^{p\text{-th}}\ 0\ \ldots\ 0 \rfloor \\
\left\lfloor \frac{\partial^2 G_2}{\partial^2 \mathbf{\Phi}_i} \right\rfloor &= \mathbf{0} \\
\frac{\partial^2 G_2}{\partial b_j \partial b_k} &= 0 \\
\frac{\partial^2 G_2}{\partial \mathbf{\Phi}_i \partial b_j} = \frac{\partial^2 G_2}{\partial b_j \partial \mathbf{\Phi}_i} &= \lfloor 0\ \ldots\ \ldots\ 0\ 0\ 0\ \ldots\ 0 \rfloor
\end{aligned} \tag{4.7}$$

As explained above these equations are substituted in the system of equations to make the matrix $[\mathbf{K}-\lambda\mathbf{M}]$ non-singular. In these calculations the eigenvector $\mathbf{\Phi}_i$ is normalized by equating a component to α_i.

3) Defining the magnitude of $\mathbf{\Phi}_i$: This normalization condition can be represented in terms of $\mathbf{b}$ as

$$G_3(\mathbf{\Phi}_i(\mathbf{b}),\mathbf{b}) = |\mathbf{\Phi}_i(\mathbf{b})| - \beta_i = 0$$

The partial derivatives required for the equation 4.5 are calculated below

$$\begin{aligned}
\left\lfloor \frac{\partial G_3}{\partial \mathbf{\Phi}_i} \right\rfloor &= \frac{1}{\beta_i}\{\mathbf{\Phi}_i\}^T \\
\left\lfloor \frac{\partial^2 G_3}{\partial^2 \mathbf{\Phi}_i} \right\rfloor &= \frac{1}{\beta_i}\mathbf{I} \\
\frac{\partial^2 G_3}{\partial b_j \partial b_k} &= 0 \\
\frac{\partial^2 G_3}{\partial \mathbf{\Phi}_i \partial b_j} = \frac{\partial^2 G_3}{\partial b_j \partial \mathbf{\Phi}_i} &= \mathbf{0}
\end{aligned} \tag{4.8}$$

Where $\mathbf{0}$ and $\mathbf{I}$ are null matrix and identity matrix respectively. For this normalization condition, the modified $[\mathbf{K} - \lambda\mathbf{M}]$ is the same as that used for the calculation of first-order eigenvector sensitivity.

The calculation of the second-order sensitivities for eigenvectors offers computational advantages with the approach presented here. For all the above normalization conditions, the modified $[\mathbf{K} - \lambda\mathbf{M}]$ matrices are same for both first-order and second-order calculations, therefore once it is formed and inverted in the first-order sensitivity analysis it can be used again for the second-order sensitivity. Note that for the G_2 normalization, the modified $[\mathbf{K} - \lambda_i\mathbf{M}]$ is the same as that used to calculate the eigenvector itself so it can be used for all three calculations. Also when G_2 is used as the normalization condition, the band width of and symmetry of the system matrix $[\mathbf{K} - \lambda\mathbf{M}]$ is preserved even after the removal of the singularity in all of the sensitivity calculations. If G_1 and G_3 are used as normalization conditions, the sensitivity problem will always be linear even though the original eigen problem could be non-linear. These properties could be effectively used to reduce computational time.

4.3 Effect of Normalization Condition

A generalized method to calculate the second-order eigenvector sensitivity of an eigenvector with any normalization was presented in the previous section of this chapter. In some cases it might be required to calculate the sensitivity of an eigenvector with a different normalization after the calculation with the first normalization is performed. Instead of again solving equation 4.3 and 4.5 for the new scaling, a set of equations are derived below that can be used to convert the eigenvector sensitivities from one normalization condition to another. These equations also expose the difference in the second-order sensitivities of eigenvectors having different normalization conditions in a manner that is similar to that considered in Chapter 3 for first-order eigenvector sensitivity.

In this derivation, the eigenvector $\mathbf{\Phi}_i$ may be re-scaled to $\hat{\mathbf{\Phi}}_i$ with equation 3.20 shown below

$$\hat{\mathbf{\Phi}}_i(\mathbf{b}) = c(\mathbf{b})\mathbf{\Phi}_i(\mathbf{b})$$

where the dependence on the design variable $\mathbf{b}$ is identified. To compute the second-order sensitivity of the scaled eigenvector $\hat{\mathbf{\Phi}}_i$, differentiate equation 3.20 with respect to the design variables b_j and b_k to obtain

$$\frac{d^2\hat{\mathbf{\Phi}}_i}{db_j db_k} = \frac{\partial^2 c}{\partial b_j \partial b_k}\mathbf{\Phi}_i + \frac{\partial c}{\partial b_j}\frac{d\mathbf{\Phi}_i}{db_k} + \frac{\partial c}{\partial b_k}\frac{d\mathbf{\Phi}_i}{db_j} + c\frac{d^2\mathbf{\Phi}_i}{db_j db_k} \tag{4.9}$$

where the first-order derivatives of c for various normalization conditions are computed in Chapter 3 using equations 3.22 to 3.29. Note that all of the terms on right-hand side of equation 4.9 are known except the second-order term $\partial^2 c/\partial b_j \partial b_k$ which is obtained by differentiating the first-order partial derivative $\partial c/\partial b_j$ with respect to the design variable b_k. For example, when $\hat{\mathbf{\Phi}}_i$ is mass normalized, the first-order derivative $\partial c/\partial b_j$ follows from equation 3.24 as

$$\frac{\partial c}{\partial b_j} = -\frac{1}{2\sqrt{C_i}^3}\frac{dC_i}{db_j} \tag{4.10}$$

which may be differentiated with respect to b_k to yield

$$\frac{\partial^2 c}{\partial b_j \partial b_k} = \frac{3}{4\sqrt{C_i}^5}\frac{dC_i}{db_j}\frac{dC_i}{db_k} - \frac{1}{2\sqrt{C_i}^3}\frac{d^2 C_i}{db_j db_k} \tag{4.11}$$

where d^2C_i/db_jdb_k is obtained by differentiating equation 3.25. Now substituting equations 4.10 and 4.11 into equation 4.9 yields the second-order eigenvector sensitivity for a mass normalized eigenvector.

To convert second-order sensitivity of eigenvectors with any normalization to eigenvectors normalized by defining the maximum value, equations 3.26 and 3.27 are employed from the first-order sensitivity repeated here as

$$c(\mathbf{b}) = \frac{\alpha_i}{(\mathbf{\Phi}_i)_p}$$

$$\frac{\partial c}{\partial b_j} = -c^2 \left(\frac{\partial \mathbf{\Phi}_i}{\partial b_j}\right)_p$$

to obtain

$$\frac{\partial^2 c}{\partial b_j \partial b_k} = 2c^3 \left(\frac{\partial \mathbf{\Phi}_i}{\partial b_j}\right)_p \left(\frac{\partial \mathbf{\Phi}_i}{\partial b_k}\right)_p - c^2 \left(\frac{\partial^2 \mathbf{\Phi}_i}{\partial b_j \partial b_k}\right)_p \tag{4.12}$$

Similarly, to rescale second-order sensitivities computed for $\mathbf{\Phi}_i$ with any normalization to $\hat{\mathbf{\Phi}}_i$ which is normalized by defining its magnitude, substitute equations 3.28 and 3.29 repeated here

$$c(\mathbf{b}) = \frac{\beta_i}{|\mathbf{\Phi}_i|}$$

$$\frac{\partial c}{\partial b_j} = -c^3 \;\; \mathbf{\Phi}_i \cdot \frac{\partial \mathbf{\Phi}_i}{\partial b_j}$$

into equation 4.9 along with the second-order derivative of the scaling factor c given as

$$\frac{\partial^2 c}{\partial b_j \partial b_k} = 3c^5 \;\; \left(\mathbf{\Phi}_i \cdot \frac{\partial \mathbf{\Phi}_i}{\partial b_j}\right)\left(\mathbf{\Phi}_i \cdot \frac{\partial \mathbf{\Phi}_i}{\partial b_k}\right) - c^3 \;\; \left(\frac{\partial \mathbf{\Phi}_i}{\partial b_k} \cdot \frac{\partial \mathbf{\Phi}_i}{\partial b_j} + \mathbf{\Phi}_i \cdot \frac{\partial^2 \mathbf{\Phi}_i}{\partial b_j \partial b_k}\right) \quad (4.13)$$

This concept can also be extended to higher-order sensitivities of eigenvectors.

4.4 Interpretation of Friswell's Method

Friswell [35] in his technical note extended Nelson's approach [30] to calculate the second- and higher-order eigenvector sensitivities while retaining all the advantages associated with it. In his extension, only the eigenvalue and eigenvector of interest and their lower-order derivatives are required to calculate the second- or higher-order derivatives. A brief summary of Friswell's technical note is provided in Appendix B. Following Nelson, Friswell assumes the second-order sensitivity as equation B.3, which is

$$\frac{\partial^2 \mathbf{\Phi}_i}{\partial b_j \partial b_k} = \boldsymbol{\nu}_{ijk} + c_{ijk} \mathbf{\Phi}_i$$

where

$$
\begin{aligned}
c_{ijk} = -\frac{1}{2}\boldsymbol{\Phi}_i \cdot \frac{\partial^2 \mathbf{M}}{\partial b_j \partial b_k}\boldsymbol{\Phi}_i \quad &- \quad \boldsymbol{\Phi}_i \cdot \frac{\partial \mathbf{M}}{\partial b_j}\frac{d\boldsymbol{\Phi}_i}{db_k} - \boldsymbol{\Phi}_i \cdot \frac{\partial \mathbf{M}}{\partial b_k}\frac{d\boldsymbol{\Phi}_i}{db_j} \\
&- \quad \frac{d\boldsymbol{\Phi}_i}{db_j} \cdot \mathbf{M}\frac{d\boldsymbol{\Phi}_i}{db_k} - \boldsymbol{\Phi}_i \cdot \mathbf{M}\boldsymbol{\nu}_{ijk}
\end{aligned}
$$

$\boldsymbol{\nu}_{ijk}$ is calculated by replacing the row and column corresponding to the maximum value to zero in the second-order eigen problem equation similar to Nelson's method for first-order sensitivity. Friswell's method can be interpreted similar to the interpretation of the Nelson's method in Chapter 3. For example, consider equation 4.9 rewritten as

$$
\frac{d^2\hat{\boldsymbol{\Phi}}_i}{db_j db_k} = \frac{\partial^2 c}{\partial b_j \partial b_k}\boldsymbol{\Phi}_i + \frac{\partial c}{\partial b_j}\frac{d\boldsymbol{\Phi}_i}{db_k} + \frac{\partial c}{\partial b_k}\frac{d\boldsymbol{\Phi}_i}{db_j} + c\frac{d^2\boldsymbol{\Phi}_i}{db_j db_k}
$$

The last term on the right hand side $c(d^2\boldsymbol{\Phi}_i/db_j db_k)$ reduces to $c_{ijk}\boldsymbol{\Phi}_i$ after some manipulations and the remaining terms on the right-hand-side are equivalent to $\boldsymbol{\nu}_{ijk}$. This again shows that first the second-order sensitivity of maximum value equals one normalized eigenvector is calculated in $\boldsymbol{\nu}_{ijk}$ and then it is scaled back using the rescaling equation to get the second-order sensitivity of mass normalized eigenvectors.

CHAPTER 5

NUMERICAL EXAMPLES

5.1 Simple Spring-Mass System

Consider an example problem where the design sensitivity of the eigenvalues and eigenvectors is computed for the simple spring-mass system shown in figure 5.1. The number of degrees-of-freedom in this example is $n = 2$ and the stiffness and mass matrix for the system are, respectively,

$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \qquad \text{and} \qquad \mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

First the eigenproblem of equation 2.1 is solved for $k_1 = 4$, $k_2 = 5$, $k_3 = 3$, $m_1 = 2$ and $m_2 = 3$ to yield the eigenvalues and the mass-normalized eigenvectors

$$\lambda_1 = 1.34571 \qquad \mathbf{\Phi}_1 = \begin{Bmatrix} 0.38417 \\ 0.48471 \end{Bmatrix}$$

$$\lambda_2 = 5.82095 \qquad \mathbf{\Phi}_2 = \begin{Bmatrix} -0.59365 \\ 0.31367 \end{Bmatrix}$$

The design problem for this example considers $N = 2$ design variables $\mathbf{b} = \lfloor k_1 \;\; m_1 \rfloor^T$ where $b_1 = k_1 = 4$ and $b_2 = m_1 = 2$. To evaluate the design sensitivities for the eigenproblem, first the design derivatives of the system matrices are evaluated as

$$\left[\frac{\partial \mathbf{K}}{\partial b_1}\right] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \left[\frac{\partial \mathbf{K}}{\partial b_2}\right] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \left[\frac{\partial^2 \mathbf{K}}{\partial b_1^2}\right] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \left[\frac{\partial^2 \mathbf{K}}{\partial b_2^2}\right] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\left[\frac{\partial \mathbf{M}}{\partial b_1}\right] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \left[\frac{\partial \mathbf{M}}{\partial b_2}\right] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \left[\frac{\partial^2 \mathbf{M}}{\partial b_1^2}\right] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \left[\frac{\partial^2 \mathbf{M}}{\partial b_2^2}\right] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

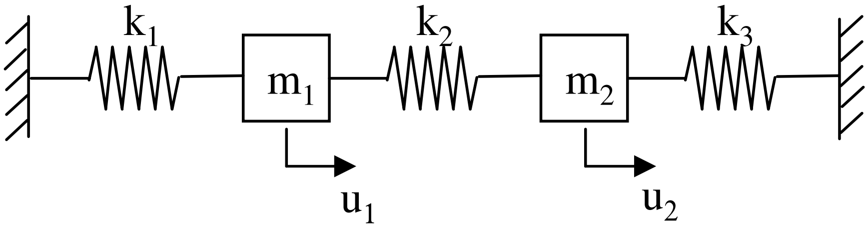


Figure 5.1: Spring Mass Model for Eigenproblem Design Sensitivity Analysis Example.

5.1.1 First-Order Eigenvalue Design Sensitivity Analysis

The first-order eigenvalue sensitivities are computed from equation 3.7 for mode 1 as

$$\begin{aligned}\frac{d\lambda_1}{db_1} &= 0.14759 \\ \frac{d\lambda_1}{db_2} &= -0.19861\end{aligned}$$

and similarly for mode 2

$$\begin{aligned}\frac{d\lambda_2}{db_1} &= 0.35242 \\ \frac{d\lambda_2}{db_2} &= -2.05139\end{aligned}$$

It has to be noted here that $C_1 = C_2 = 1$ from equation 3.7 since the eigenvectors here are mass normalized and that no additional eigenproblem solutions or matrix inverse computations are required to obtain the final results. It can be seen that increasing the stiffness k_1 increases the natural frequencies and increasing the mass m_1 decreases them, as expected.

5.1.2 First-Order Eigenvector Design Sensitivity Analysis

The first-order sensitivity of the eigenvectors with respect to the design variable $\mathbf{b}$ are evaluated here for the spring-mass system. First consider mode 1 where the system matrix on the left-hand-side of equation 3.9 is

$$\begin{aligned}\mathbf{K} - \lambda_1 \mathbf{M} &= \begin{bmatrix} 9 & -5 \\ -5 & 8 \end{bmatrix} - (1.34571) \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 6.30857 & -5 \\ -5 & 3.96286 \end{bmatrix}\end{aligned}$$

which is singular, as expected. For the design variable $b_1 = k_1$, the right-hand-side of equation 3.9 becomes

$$\begin{aligned}& - \left[\frac{\partial \mathbf{K}}{\partial b_1} - \frac{d\lambda_1}{db_1} \mathbf{M} - \lambda_1 \frac{\partial \mathbf{M}}{\partial b_1} \right] \mathbf{\Phi}_1 \\ &= - \left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - 0.14759 \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} - 1.34571 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right] \begin{Bmatrix} 0.38417 \\ 0.48471 \end{Bmatrix} \\ &= \begin{Bmatrix} -0.27077 \\ 0.21461 \end{Bmatrix}\end{aligned}$$

Substituting these results into equation 3.9 gives the system of equations

$$\begin{bmatrix} 6.30857 & -5 \\ -5 & 3.96286 \end{bmatrix} \frac{d\mathbf{\Phi}_1}{db_1} = \begin{Bmatrix} -0.27077 \\ 0.21461 \end{Bmatrix}$$

which contains only one linearly independent equation. Since the eigenvectors have been normalized with respect to the mass matrix, equation 3.13 with equation 3.10 is used to obtain

$$2\mathbf{\Phi}_1 \cdot \mathbf{M} \frac{d\mathbf{\Phi}_1}{db_1} = -\mathbf{\Phi}_1 \cdot \frac{\partial \mathbf{M}}{\partial b_1} \mathbf{\Phi}_1$$

which for this example becomes

$$2 \begin{Bmatrix} 0.38417 \\ 0.48471 \end{Bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \frac{d\mathbf{\Phi}_1}{db_1} = - \begin{Bmatrix} 0.38417 \\ 0.48471 \end{Bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} 0.38417 \\ 0.48471 \end{Bmatrix}$$

or

$$\lfloor -1.53667 \quad -2.90826 \rfloor \frac{d\mathbf{\Phi}_1}{db_1} = 0$$

Now this result is combined with the first equation from the previous singular system to obtain the following modified non-singular system of equations

$$\begin{bmatrix} 6.30857 & -5 \\ -1.53667 & -2.90826 \end{bmatrix} \frac{d\mathbf{\Phi}_1}{db_1} = \begin{Bmatrix} -0.27077 \\ 0 \end{Bmatrix}$$

where it is noted that the second equation could have been chosen to be retained from the singular system, instead. Finally, the eigenvector sensitivity becomes

$$\frac{d\mathbf{\Phi}_1}{db_1} = \begin{Bmatrix} -0.03025 \\ 0.01599 \end{Bmatrix}$$

Likewise, $d\mathbf{\Phi}_1/db_2$ where $b_2 = m_1$ may be computed and equation 3.9 becomes

$$\begin{bmatrix} 6.30857 & -5 \\ -5 & 3.96286 \end{bmatrix} \frac{d\mathbf{\Phi}_1}{db_2} = \begin{Bmatrix} 0.36438 \\ -0.28880 \end{Bmatrix}$$

Note that the system matrix on the left-hand-side is the same as that computed above for b_1 which is singular rendering the calculation of $d\mathbf{\Phi}_1/db_2$ impossible. However, as before, the normalization condition is introduced from equation 3.13 for design variable b_2 to yield

$$\lfloor -1.53667 \quad -2.90826 \rfloor \frac{d\mathbf{\Phi}_1}{db_2} = -0.14758$$

The modified non-singular system of equations is again formed by replacing the second equation in the singular system above with this result to give

$$\begin{bmatrix} 6.30857 & -5 \\ -1.53667 & -2.90826 \end{bmatrix} \frac{d\mathbf{\Phi}_1}{db_2} = \begin{Bmatrix} 0.36438 \\ -0.14758 \end{Bmatrix}$$

which has the same system matrix on the left-hand-side as that used to compute $d\mathbf{\Phi}_1/db_1$ above. Therefore, without any further matrix inverse computations, the eigenvector sensitivity is obtained as

$$\frac{d\mathbf{\Phi}_1}{db_2} = \left\{ \begin{array}{r} 0.01236 \\ -0.05728 \end{array} \right\}$$

In a similar manner, the design sensitivities of the eigenvector associated with the second mode are evaluated

$$\frac{d\mathbf{\Phi}_2}{db_1} = \left\{ \begin{array}{r} -0.01958 \\ -0.02470 \end{array} \right\} \quad \text{and} \quad \frac{d\mathbf{\Phi}_2}{db_2} = \left\{ \begin{array}{r} 0.21856 \\ 0.08851 \end{array} \right\}$$

5.1.3 Second-Order Eigenvalue Design Sensitivity Analysis

Equation 4.2 is used to calculate the second-order sensitivities of the first eigenvalue with respect to the design variables b_1 and b_2 to get

$$\frac{d^2\lambda_1}{db_1^2} = -0.02324$$

$$\frac{d^2\lambda_1}{db_2^2} = 0.01653$$

Similar calculations for the second eigenvalue are performed to obtain the sensitivities and are presented in table 5.1

Table 5.1: Eigenvalue and eigenvalue design sensitivity for spring-mass example with respect to design variables b_1 and b_2.

mode number	λ $(1/s^2)$	design variable	$d\lambda/db$ $(1/in \cdot s^2)$	$d^2\lambda/db^2$ $(1/in^2 \cdot s^2)$
1	1.3457	b_1	0.1475	-0.0232
		b_2	-0.1986	0.0165
2	5.8209	b_1	0.3524	0.0232
		b_2	-2.0514	2.2334

5.1.4 Second-Order Eigenvector Design Sensitivity Analysis

To calculate the second-order eigenvector sensitivity the right-hand-side of the equation 4.3 is calculated to obtain

$$
\begin{aligned}
& - \left[\frac{\partial^2 \mathbf{K}}{\partial b_1^2} - \frac{d^2\lambda_1}{db_1^2}\mathbf{M} - 2\frac{d\lambda_1}{db_1}\frac{\partial \mathbf{M}}{\partial b_1} - \lambda_1 \frac{\partial^2 \mathbf{M}}{\partial^2 b_1} \right] \mathbf{\Phi}_1 \\
& \qquad -2 \left[\frac{\partial \mathbf{K}}{\partial b_1} - \frac{d\lambda_1}{db_1}\mathbf{M} - \lambda_1 \frac{\partial \mathbf{M}}{\partial b_1} \right] \frac{d\mathbf{\Phi}_1}{db_1} \\
= & \left\{ \begin{array}{r} 0.0247 \\ -0.0195 \end{array} \right\}
\end{aligned}
$$

Substituting the above results and the value of $[\mathbf{K} - \lambda_1 \mathbf{M}]$ from the first-order sensitivity into equation 4.3 gives the system of equations

$$
\begin{bmatrix} 6.30857 & -5 \\ -5 & 3.96286 \end{bmatrix} \frac{d^2\mathbf{\Phi}_1}{db_1^2} = \left\{ \begin{array}{r} 0.02477 \\ -0.01959 \end{array} \right\}
$$

which has only one linearly independent equation. Since the eigenvectors are mass normalized, equation 4.5 is used to obtain

$$
2\mathbf{\Phi}_1 \cdot \mathbf{M}\frac{d^2\mathbf{\Phi}_1}{db_1^2} = - \left[4\mathbf{\Phi}_1 \cdot \frac{\partial \mathbf{M}}{\partial b_1}\frac{d\mathbf{\Phi}_1}{db_1} + 2\frac{d\mathbf{\Phi}_1}{db_1} \cdot \mathbf{M}\frac{d\mathbf{\Phi}_1}{db_1} + \mathbf{\Phi}_1 \cdot \frac{\partial^2 \mathbf{M}}{\partial b_1^2}\mathbf{\Phi}_1 \right]
$$

which for this example becomes

$$
\lfloor 1.53667 \;\; 2.90826 \rfloor \frac{d^2\mathbf{\Phi}_1}{db_1^2} = -0.00517
$$

this result is combined with the first equation from the previous singular system to obtain the following modified non-singular system of equations

$$\begin{bmatrix} 6.30857 & -5 \\ 1.53667 & 2.90826 \end{bmatrix} \frac{d^2\mathbf{\Phi}_1}{db_1^2} = \begin{Bmatrix} 0.02477 \\ -0.00517 \end{Bmatrix}$$

here it is noted first, that the second equation could have been retained from the singular system, instead and second, that the modified $[\mathbf{K} - \lambda\mathbf{M}]$ is same as the one obtained for the first-order sensitivity and therefore it is not required to be inverted again. Finally, the second-order eigenvector sensitivity becomes

$$\frac{d^2\mathbf{\Phi}_1}{db_1^2} = \begin{Bmatrix} 0.00177 \\ -0.00272 \end{Bmatrix}$$

Likewise, $d^2\mathbf{\Phi}_1/db_2^2$ can be computed where $b_2 = m_1$ and equation 3.9 becomes

$$\begin{bmatrix} 6.30857 & -5 \\ -5 & 3.96286 \end{bmatrix} \frac{d^2\mathbf{\Phi}_1}{db_2^2} = \begin{Bmatrix} -0.1164 \\ 0.0923 \end{Bmatrix}$$

Note that the system matrix on the left-hand-side is the same as that computed above for b_1 which is singular making the calculation of $d^2\mathbf{\Phi}_1/db_2^2$ impossible. However, as before, the normalization condition is introduced from equation 4.5 for design variable b_2 to yield

$$\lfloor 1.53667 \;\; 2.90826 \rfloor \frac{d^2\mathbf{\Phi}_1}{db_2^2} = -0.0393$$

The modified non-singular system of equations is again formed by replacing the second equation in the singular system above with this result to give

$$\begin{bmatrix} 6.30857 & -5 \\ 1.53667 & 2.90826 \end{bmatrix} \frac{d^2\mathbf{\Phi}_1}{db_2^2} = \begin{Bmatrix} -0.1164 \\ -0.0393 \end{Bmatrix}$$

which again has the same system matrix on the left-hand-side as that used to compute $d^2\mathbf{\Phi}_1/db_1^2$ above. Therefore, without any further matrix inverse computations, the eigenvector sensitivity is obtained as

$$\frac{d^2\mathbf{\Phi}_1}{db_2^2} = \left\{ \begin{array}{c} -0.02055 \\ -0.00264 \end{array} \right\}$$

In a similar manner, the design sensitivities of the eigenvector associated with mode 2 are evaluated to be

$$\frac{d^2\mathbf{\Phi}_2}{db_1^2} = \left\{ \begin{array}{c} -0.00333 \\ -0.00144 \end{array} \right\} \quad \text{and} \quad \frac{d^2\mathbf{\Phi}_2}{db_2^2} = \left\{ \begin{array}{c} 0.143140 \\ 0.031342 \end{array} \right\}$$

All the sensitivity results obtained here have been verified with forward finite difference calculations with an increment of 10^{-10}. Numerical results associated with the calculation of the sensitivities for this example have been tabulated and presented systematically in the next few pages. Table 5.2 gives the eigenvectors and their sensitivities for various normalization conditions. It can be seen from the table that the normalization condition changes both the magnitude and the direction of the eigenvector sensitivity. For the $G_2 = 0$ normalization condition which is maximum value equals one, the first- and second-order sensitivities are zeros for the component of $\boldsymbol{\phi}$ with the maximum value, which is expected as the maximum value is fixed to unity. Table 5.3 shows the re-scaling factors and their derivatives for various normalization conditions. To re-scale the eigenvectors and their sensitivities with $G_2 = 0$ normalization to $G_1 = 0$ normalization, the values for $c_1, \partial c_1/\partial b_1, \partial^2 c_1/\partial b_1^2$ from the G_2 column are used in the re-scaling equations for eigenvectors (cf. equation 3.20) and their sensitivities (cf. equations 3.21 and 4.9). Consider, for example, computing the eigenvector design sensitivities when the first eigenvector is re-scaled with equation 3.20 from having a unity magnitude (i.e., $G_2 = 0$ and $\alpha_i = 1$ in equation 2.6) to a mass normalized eigenvector where $G_1 = 0$ in equation 2.5. From the G_2 normalization column under mode 1 in Table 5.3 $c_1 = 0.4847$, $dc_1/db_1 =$ -0.0159 are obtained and from Table 5.2 the eigenvector Φ_1 with $G_2 = 0$ normalization and its sensitivity $\mathbf{\Phi}_{11} = -0.7925$, $d\mathbf{\Phi}_{11}/db_1 = -0.0885$ are obtained, so that the value of the eigenvector derivative with $G_1 = 0$ normalization becomes $d\mathbf{\Phi}_{11}/db_1 = (-0.0159)(-0.7925) + (-0.0885)(0.4847) = -0.03025$ from equation 3.21. It has to be noted here that the eigenvectors can be multiplied by a negative sign and still remain the same. In a similar manner, values in the G_1 normalization column under mode 1 of Table 5.3 may also be used to calculate values of the second-order design derivative when eigenvector is re-scaled for $G_2 = 0$. In this case, equation 4.9 yields $d^2\mathbf{\Phi}_{11}/db_1 = (0.0160)(0.3841) + 2(-0.068)(-0.0302) + (2.0631)(0.00177) = 0.01394$ as seen in Table 5.2. Similar calculations may be performed for mode 2. The diagonal terms in this table are ones and zeros as the re-scaling factors are directly related to their corresponding normalization conditions.

Table 5.2: Eigenvectors and eigenvector design sensitivities with respect to design variable b_1 for spring-mass example when $\mathbf{\Phi}_I$ is normalized by G_1 - G_3 from equations 2.5 - 2.7, respectively, for $\alpha = 1$ and $\beta = 1$ (p indicates vector component).

Normalization		G_1	G_2	G_3
$\mathbf{\Phi}_1$	$p = 1$	0.38417	0.79257	0.62114
	$p = 2$	0.48471	1.00000	0.78370
$\mathbf{\Phi}_2$	$p = 1$	0.59365	1.00000	0.88416
	$p = 2$	-0.31367	-0.52838	-0.46718
$\frac{d\mathbf{\Phi}_1}{db_1}$	$p = 1$	-0.030252	-0.088551	-0.042623
	$p = 2$	0.015985	0.000000	0.033782
$\frac{d\mathbf{\Phi}_2}{db_1}$	$p = 1$	0.019577	0.000000	0.021560
	$p = 2$	0.024701	0.059034	0.040804
$\frac{d^2\mathbf{\Phi}_1}{db_1^2}$	$p = 1$	0.001772	0.013946	0.001201
	$p = 2$	-0.002722	0.000000	-0.004726
$\frac{d^2\mathbf{\Phi}_2}{db_1^2}$	$p = 1$	-0.003334	0.000000	-0.004227
	$p = 2$	-0.001447	-0.009298	-0.003441

Table 5.3: Eigenvector re-scaling factors and their design sensitivities with respect to design variable b_1 for spring-mass example when $\mathbf{\Phi}_I$ is normalized by G_1 - G_3 (from equations 2.5 - 2.7, respectively) for $\alpha = 1$ and $\beta = 1$.

Mode	$\mathbf{\Phi}_1$			$\mathbf{\Phi}_2$		
Normalization	G_1	G_2	G_3	G_1	G_2	G_3
c_1	1.0000	0.4847	0.6184	1.0000	0.5936	0.6714
c_2	2.0631	1.0000	1.2759	1.6845	1.0000	1.1310
c_3	1.6168	0.7837	1.0000	1.4894	0.8841	1.0000
dc_1/db_1	0.0000	-0.0159	0.0062	0.0000	-0.0195	-0.0057
dc_2/db_1	-0.0680	0.0000	-0.0550	-0.0555	0.0000	-0.0275
dc_3/db_1	0.0163	0.0337	0.0000	-0.0128	0.0215	0.0000
d^2c_1/db_1^2	0.0000	-0.0009	0.0024	0.0000	-0.0011	0.0009
d^2c_2/db_1^2	0.0160	0.0000	0.0124	0.0131	0.0000	0.0067
d^2c_3/db_1^2	-0.0017	-0.0047	0.0000	0.0020	-0.0042	0.0000

Table 5.4: Eigenvectors and eigenvector design sensitivities with respect to design variable b_2 for spring-mass example when $\mathbf{\Phi}_I$ is normalized by G_1 - G_3 from equations 2.5 - 2.7, respectively, for $\alpha = 1$ and $\beta = 1$ (p indicates vector component).

Normalization		G_1	G_2	G_3
$\mathbf{\Phi}_1$	$p = 1$	0.38417	0.79257	0.62114
	$p = 2$	0.48471	1.00000	0.78370
$\mathbf{\Phi}_2$	$p = 1$	0.59365	1.00000	0.88416
	$p = 2$	-0.31367	-0.52838	-0.46718
$\frac{d\mathbf{\Phi}_1}{db_2}$	$p = 1$	0.01236	0.11916	0.05735
	$p = 2$	-0.05727	0.000000	-0.04546
$\frac{d\mathbf{\Phi}_2}{db_2}$	$p = 1$	-0.21856	0.000000	-0.04546
	$p = 2$	-0.08851	-0.34363	-0.23751
$\frac{d^2\mathbf{\Phi}_1}{db_2^2}$	$p = 1$	-0.02055	-0.00991	-0.01475
	$p = 2$	-0.002649	0.000000	0.00486
$\frac{d^2\mathbf{\Phi}_2}{db_2^2}$	$p = 1$	0.14314	0.000000	-0.05477
	$p = 2$	0.03134	-0.07283	0.05080

Table 5.5: Eigenvector re-scaling factors and their design sensitivities with respect to design variable b_2 for spring-mass example when $\mathbf{\Phi}_I$ is normalized by G_1 - G_3 (from equations 2.5 - 2.7, respectively) for $\alpha = 1$ and $\beta = 1$.

Mode	$\mathbf{\Phi}_1$			$\mathbf{\Phi}_2$		
Normalization	G_1	G_2	G_3	G_1	G_2	G_3
c_1	1.0000	0.4847	0.6184	1.0000	0.5936	0.6714
c_2	2.0631	1.0000	1.2759	1.6845	1.0000	1.1310
c_3	1.6168	0.7837	1.0000	1.4894	0.8841	1.0000
dc_1/db_2	0.0000	0.0572	0.0372	0.0000	0.2185	0.1518
dc_2/db_2	0.2437	0.0000	0.0740	0.6201	0.0000	0.1605
dc_3/db_2	0.0972	-0.0454	0.0000	0.3369	-0.1254	0.0000
d^2c_1/db_2^2	0.0000	-0.0010	-0.0092	0.0000	0.2172	0.2163
d^2c_2/db_2^2	0.0688	0.0000	0.00067	0.0505	0.0000	0.1156
d^2c_3/db_2^2	0.0418	0.0048	0.0000	-0.2032	-0.0547	0.0000

5.2 Finite Element Calculation Procedure

The eigenvector sensitivities are shown for various eigenvector normalizations since the choice of normalization condition does directly influence the design derivative of the eigenvector. First the steps involved in calculating the sensitivities for the isoparametric plate example problem are explained here.

- **Build the model in ABAQUS CAE**: First a plate model with the given dimensions and physical properties is built in the finite element software ABAQUS CAE [46]. All of the dimensions, boundary conditions and the mesh are obtained accurately using CAE which is used to generate the input (.inp) file which has all the plate model information in it. The eigenvalues and eigenvectors are calculated using ABAQUS to validate the MATLAB [47] results.

- **Develop the MATLAB-based finite element code to calculate the sensitivities**: A MATLAB based finite element code is developed which calculates the first- and second- order sensitivities of the eigenvectors with respect to a given design variable. This code reads in an .txt input file which has all the information about the problem. The **K** and **M** are formed for the structure.

- **Generate the input file for the finite element code**: The input file for the MATLAB code has to be created manually. To get the mesh of the plate, the input file from the first step is taken and manually re-formatted to fit the requirements. Input files for small problems like the first spring problem can be typed in easily, but the mesh of the plate problem has 240 elements, the input file for which is very long therefore the input file from ABAQUS CAE is used. The input file for the spring problem is given in Appendix C.

- **Obtain the results from the code in NASTRAN punch format**: The MATLAB code is written such that it writes the results in NASTRAN [48] punch format. These files are recognized by Hypermesh [49] and are translated into Hypermesh result files. To view these files in Hypermesh, a model which contains all the nodes and elements corresponding to the result files is needed.

- **Import the plate model from ABAQUS to Hypermesh to view the results**: The ABAQUS input file created earlier is now imported to Hypermesh and saved as a Hypermesh model. All of the nodes, elements, element properties and boundary conditions are imported into the Hypermesh model. This model is now used to view the results obtained from the MATLAB code which were converted into Hypermesh result files. The results at each node are mapped on to the corresponding node on the model and a contour plot is obtained. This contour plot can also be viewed in animation. The complete procedure is shown as a flowchart for better understanding in figure 5.2

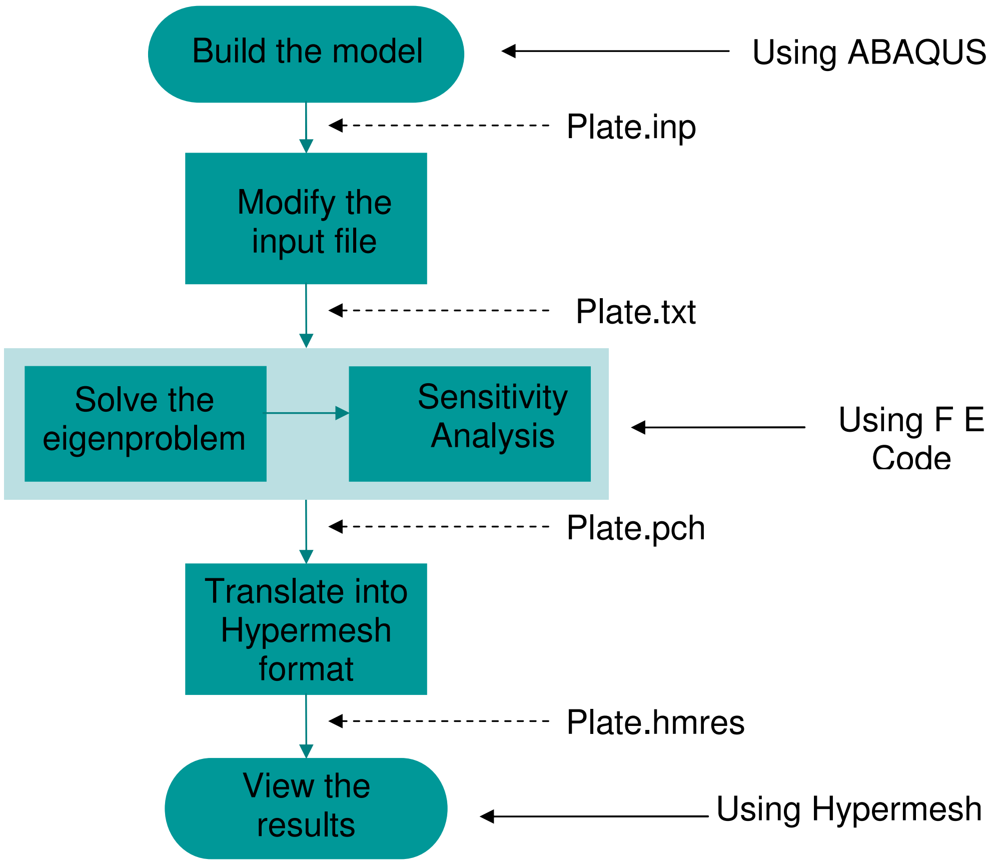


Figure 5.2: Flow chart for Eigenvector Design Sensitivity Analysis.

5.3 2-D Plate Problem

To further illustrate the eigenvalue and eigenvector design sensitivity computations described in the previous chapter, consider the rectangular plate appearing in figure 5.3 where the thickness is 1 inch in the regions marked A and 1.2 inch in the region marked B. Note that if a square or rectangular plate of uniform thickness was chosen, the eigenvalues or eigenvalue first-order sensitivities, respectively, would have been repeated, in which case the method presented above would no longer be valid. In this analysis, the plate is clamped around its perimeter. The plate is assumed to be made of a homogeneous isotropic material with a modulus of elasticity $E = 2.07E11$, a Poisson's ratio $\nu = 0.3$, and a density $\rho = 7750.3$. Only the vibration problem is considered here and the buckling analysis is ignored.

Eigenvalues, eigenvectors, and the design sensitivities of interest are evaluated using the finite element method, where the stiffness and mass matrices are, respectively, evaluated from the expressions

$$\mathbf{K} = \sum_e \int_{A_e} \mathbf{B}^T \mathbf{D} \mathbf{B}\, t^3 \, dA \tag{5.1}$$

and

$$\mathbf{M} = \sum_e \int_{A_e} \mathbf{N}^T \mathbf{N}\, \rho t \, dA \tag{5.2}$$

where $\mathbf{N}$ is the element shape function matrix, $\mathbf{D}$ is the linear elastic modulus matrix, $\mathbf{B}$ is the shape function derivative matrix, t is the plate thickness, and A, A_e are the total area and elemental area of the plate respectively, in the xy-plane shown in figure 5.3. These equations were obtained from references [50] and [51]. The summation is performed over all elements. The plate in figure 5.4 is modeled with 20 equally spaced elements in the x-direction, and 12 in the y-direction. A design variable b_1 is defined which describes the thickness t in the A regions of figure 5.3.

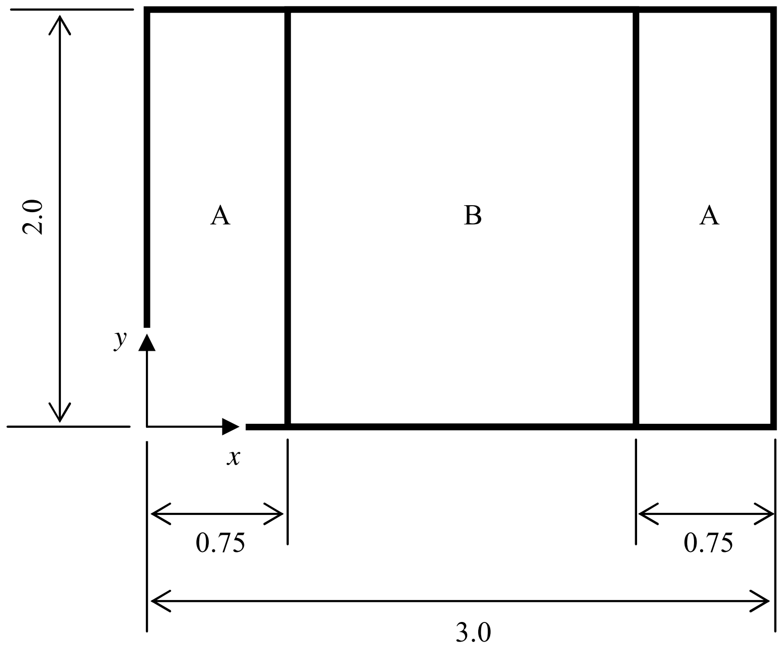


Figure 5.3: 2-D Plate Example.

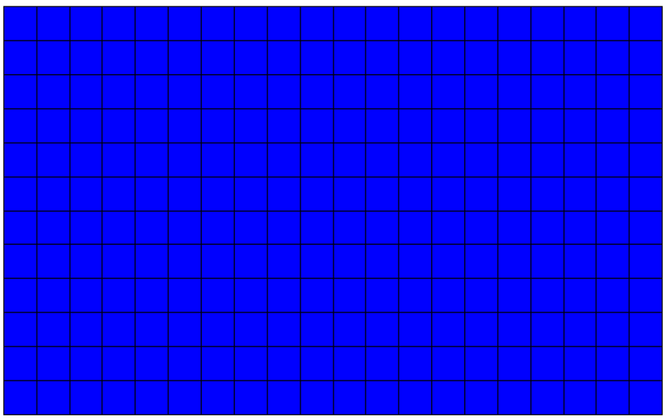


Figure 5.4: Finite Element Mesh of 2-D Plate.

Partial derivatives used in all of the sensitivity computations described above are evaluated by differentiating equations 5.1 and 5.2 with respect to $b = t$ for the elements in regions A. The expressions for $\partial \mathbf{M}/\partial b$ and $\partial \mathbf{K}/\partial b$ are obtained as

$$\frac{\partial \mathbf{K}}{\partial b} = 3 \sum_{\hat{e}} \int_{A_{\hat{e}}} \mathbf{B}^T \mathbf{D} \mathbf{B}\, t^2 \, dA \tag{5.3}$$

and

$$\frac{\partial \mathbf{M}}{\partial b} = \sum_{\hat{e}} \int_{A_{\hat{e}}} \mathbf{N}^T \mathbf{N}\, \rho \, dA \tag{5.4}$$

where the integrations are limited to region 'A' elements $\hat{e}$ only (cf. figures 5.2 and 5.3). Differentiating these expressions again with respect to the design variable $b = t$ yields the second-order partial differentials for mass and stiffness matrices. For this plate problem the second-order partial derivative of the mass matrix is the zero matrix as the mass is linearly dependent on t. The expression for $\partial^2 \mathbf{K}/\partial b^2$ is

$$\frac{\partial^2 \mathbf{K}}{\partial b^2} = 6 \sum_{\hat{e}} \int_{A_{\hat{e}}} \mathbf{B}^T \mathbf{D} \mathbf{B}\, t \, dA \tag{5.5}$$

These expressions are incorporated into a MATLAB based finite element code which generates $\mathbf{K}$, $\mathbf{M}$ and their partial derivatives, and evaluates various equations to calculates the eigenvalues, eigenvectors and their sensitivities. The eigen equation 2.1 is solved first to get the eigenvalues and their corresponding eigenvectors. Figures 5.5 - 5.9 here represent the eigenvectors for the above plate example with maximum value equated to unity which is represented as $G_2 = 0$ in the equation 3.4. These figures, obtained using the method described in the last section, are contours of out-of-plane displacement magnitude only, the rotational components of the eigenvectors are not shown.

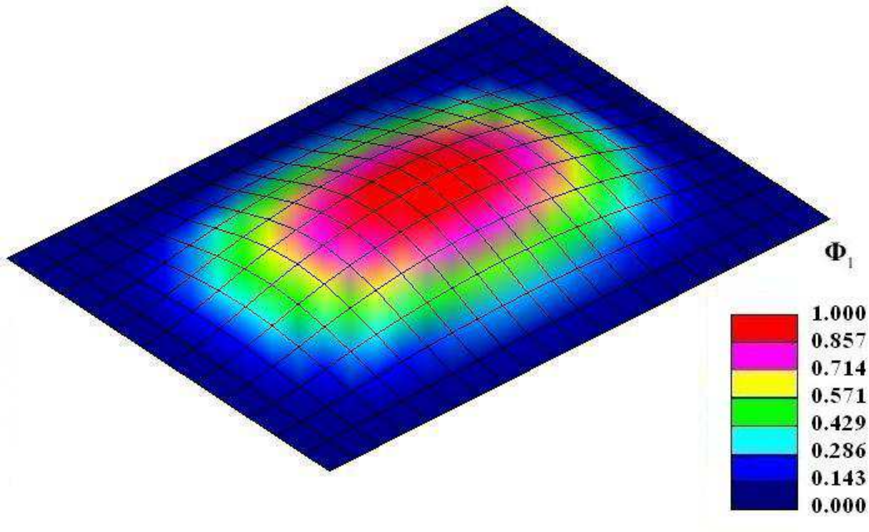


Figure 5.5: Plate Model First Mode Shape.

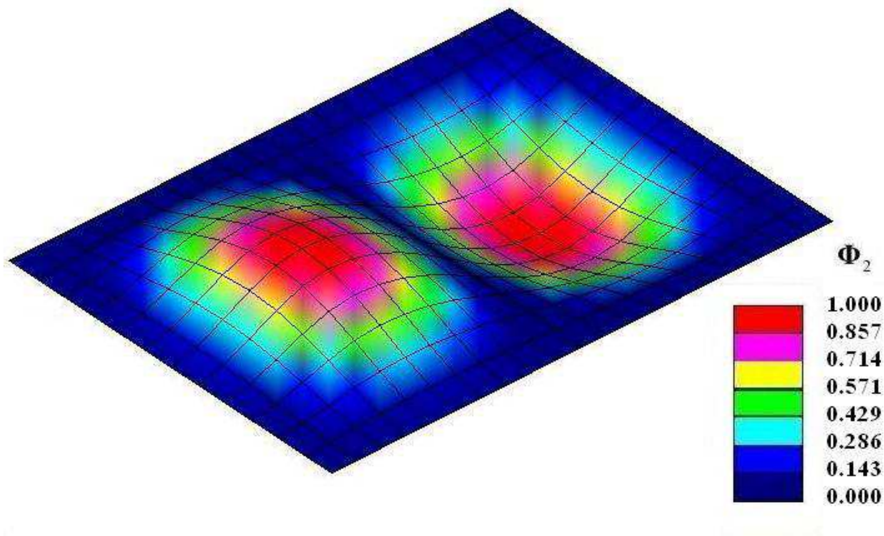


Figure 5.6: Plate Model Second Mode Shape.

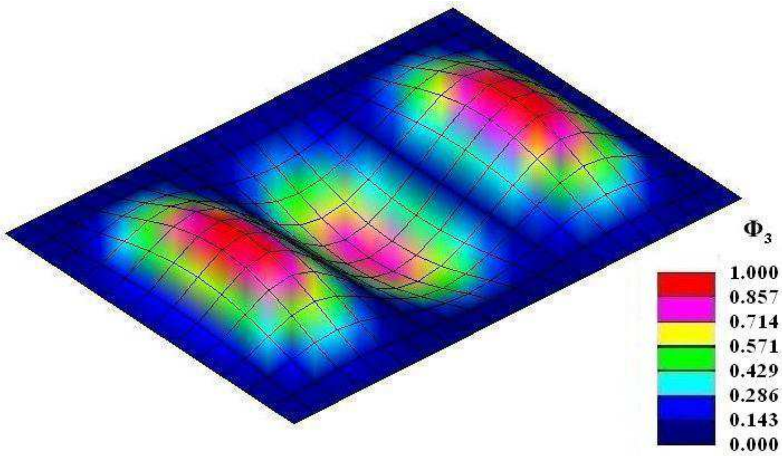


Figure 5.7: Plate Model Third Mode Shape.

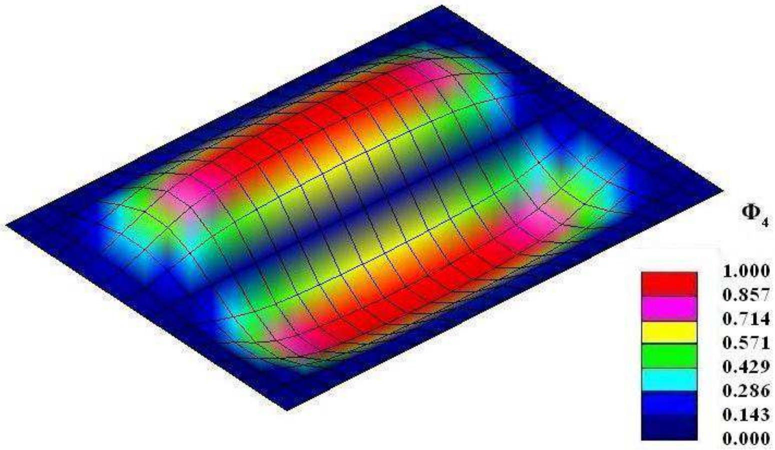


Figure 5.8: Plate Model Fourth Mode Shape.

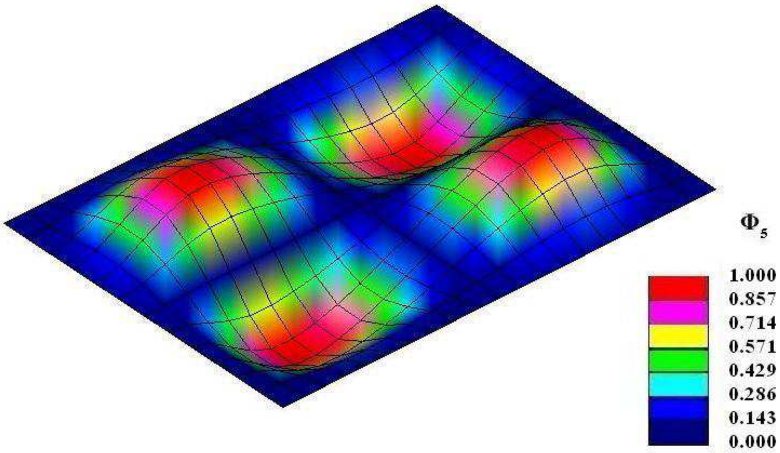


Figure 5.9: Plate Model Fifth Mode Shape .

The eigenvalues obtained from the MATLAB code which uses the above expressions are compared to the eigenvalues obtained from ABAQUS for the same Plate model and boundary conditions. Because of the different approximations used to calculate the mass and stiffness matrices there is a slight difference in these two results. This difference increase for higher eigenvalues as seen in the table 5.6.

Table 5.6: Eigenvalues from the MATLAB code and ABAQUS

Mode number	F E Code λ $1/sec^2$	ABAQUS λ $1/sec^2$
1	2.207E+04	2.25E+04
2	4.747E+04	4.802E+04
3	1.181E+05	1.305E+05
4	1.361E+05	1.353E+05
5	1.766E+05	2.063E+05

Recall that the mode shape itself does not change when other normalizations are considered, but are instead simply scaled by equation 3.20. The shape of the mode remains the same whereas the magnitude is different. Scaling factors appear as c_1, c_2 and c_3 in Table 5.7. Looking at the mode shapes closely it is seen that the step in the middle of the plate does not have much effect on the mode shape itself, but its effect is seen further in the first and second order sensitivities.

Table 5.7: Eigenvector re-scaling factors (i.e c values) and their design sensitivities for simply supported plate example when $\mathbf{\Phi}_I$ is normalized by G_1 - G_3 (from equations 2.5 - 2.7, respectively) for modes 1 and 2 only with $\alpha = 1$ and $\beta = 1$.

mode	$\mathbf{\Phi}_1$			$\mathbf{\Phi}_2$		
normalization	G_1	G_2	G_3	G_1	G_2	G_3
c_1	1.000	128.532	1417.36	1.000	171.415	1830.82
c_2	0.0078	1.000	11.027	0.0058	1.000	10.680
c_3	0.0007	0.0907	1.000	0.0005	0.0936	1.000
dc_1/db	0.000	-1500.20	1.72E+4	0.000	1.10E+03	5.31E+4
dc_2/db	-0.0908	0.000	-262.5848	0.3743	0.000	375.368
dc_3/db	0.0085	2.1593	0.000	0.0158	-3.2905	0.000
d^2c_1/db^2	0.000	-1.12E+6	-2.18E+7	0.000	-1.17E+6	-2.15E+7
d^2c_2/db^2	565.322	0.000	8.41E+05	1472.273	0.000	2.69E+6
d^2c_3/db^2	-2.1786	-6818.91	0.000	-0.9780	-2.33E+4	0.000

5.3.1 First- and Second-Order Eigenvalue Design Sensitivity Analysis

Table 5.8 contains the lowest 5 eigenvalues λ that solve equation 2.1 for the plate example. Also shown are the eigenvalue design sensitivities computed from equation 3.7. Note that the computed eigenvalue sensitivities are not a function of the eigenvector normalization, as described in the previous section.

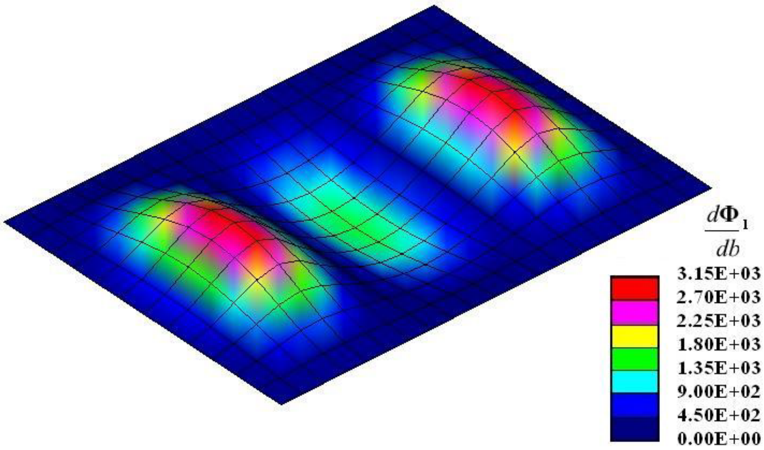
Table 5.8: Eigenvalue and eigenvalue design sensitivity for fixed plate example.

Mode number	λ	$d\lambda/db$	$d^2\lambda/db^2$
1	2.207E+04	9.092E+05	-1.792E+08
2	4.747E+04	3.955E+06	-9.867E+08
3	1.181E+05	1.032E+07	-2.294E+09
4	1.361E+05	4.852E+06	-1.529E+09
5	1.766E+05	1.259E+07	-3.168E+09

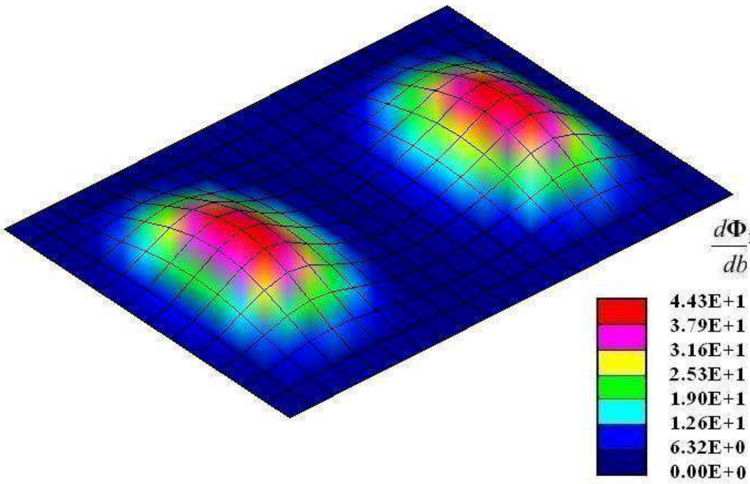
5.3.2 First-Order Eigenvector Design Sensitivity Analysis

Figures 5.10 through 5.14 show the first-order eigenvector design sensitivities of the first 5 modes for three normalization conditions. Note that in all of the following eigenvector sensitivity plots, the first figure of each page is the eigenvector sensitivity for the eigenvector normalized with respect to the mass matrix as given by $G_1 = 0$ in equation 2.5. The second and the third plots in each figure represent the sensitivities of eigenvectors normalization with maximum value equals unity and magnitude equals unity normalizations, respectively. In all these cases, as it can be seen from the plots, the eigenvector sensitivity is not simply scaled when the normalization condition is changed. Instead, the shape of the eigenvector sensitivity itself is changed as expected from equation 3.21.

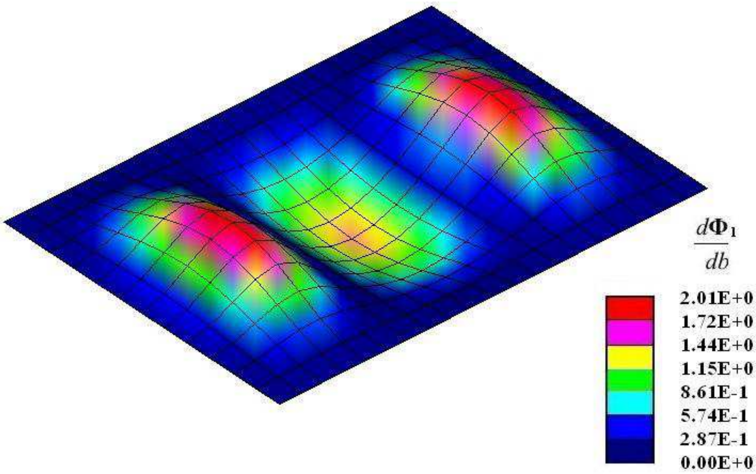
The derivatives of the scaling factors c_i appear in Table 5.7 which may be used with equation 3.21 to compute new eigenvector design derivatives when the normalization condition on $\mathbf{\Phi}$ is changed. It is clearly seen from the eigenvector plots that the eigenvector sensitivities vary with a change in the normalization condition. The effect of change in normalization condition is more pronounced in the first few modes and the effect reduces for higher modes. In figure 5.10, sensitivities for the first mode for three normalizations are shown. Note that for mass normalized eigenvector, the sensitivity follows the design of the plate as a step in the sensitivity is clearly visible. For the sensitivity of the eigenvector normalized with maximum displacement equal to unity, the center of the plate is zero as the maximum out-of-plane displacement is fixed to unity at that point. Also, since the magnitude is fixed to unity for $G_3 = 0$ normalization, the amplitude for its sensitivity very low compared to the other normalization conditions. Table 5.7 shows the re-scaling factors and their derivatives for various normalization conditions. For example, to re-scale the eigenvectors and their sensitivities with $G_3 = 0$ normalization to $G_1 = 0$ normalization, the values for $c_1, \partial c_1/\partial b, \partial^2 c_1/\partial b^2$ from the G_3 column are used in the re-scaling equations for eigenvectors and their sensitivities, as explained for the spring-mass example. Consider, for example, computing the eigenvector design sensitivities when the mode 1 eigenvector appearing in figure 5.5 is re-scaled with equation 3.20 from having a unity maximum displacement (i.e., $G_2 = 0$ and $\alpha_i = 1$ in equation 2.6, where i represents the component with maximum displacement) to a mass normalized eigenvector where $G_1 = 0$ in equation 2.5 (cf. figure 5.14b for the shape of the sensitivity of mode 1 with unity magnitude). From the G_2 normalization column under mode 1 in Table 5.7 $c_1 = 128.53$ and $dc_1/d_b = -1.5E + 03$ are obtained so that the value of the eigenvector derivative at the plates center in figure 5.14a is $d\mathbf{\Phi}_1/db = (-1.5E + 03)(1) + (128.53)(0) = -1.5E + 03$ from equation 3.21. In a similar manner, values in the G_2 normalization column under mode 1 of Table 5.7 may also be used to calculate values of the second-order design derivative when $\mathbf{\Phi}_1$ is re-scaled for $G_1 = 0$ (cf. equation 2.5). In this case, equation 4.9 yields $d^2\mathbf{\Phi}_1/db^2 = (-1.12E + 06)(1) + 2(-1.5E + 03)(0) + (128.53)(0) = -1.12E + 06$ as seen in figure 5.15a. Similar calculations may be performed for mode 2. The diagonal terms in this table are ones and zeros as the re-scaling factors are directly related to their corresponding normalization conditions.



a: Mass normalization ($G_1 = 0$)

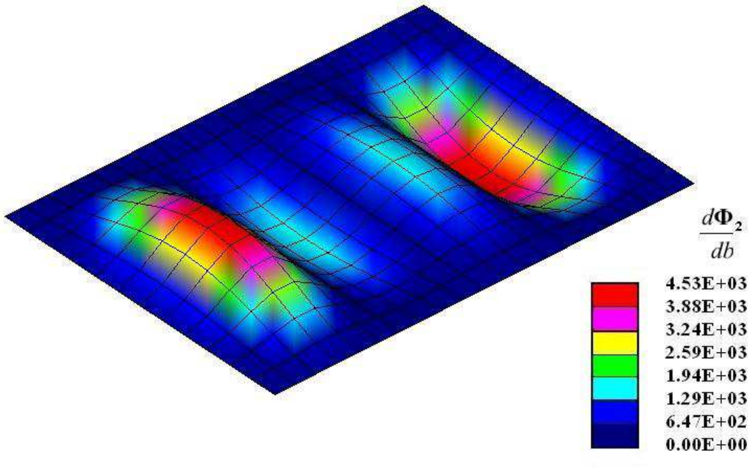


b: Maximum displacement equals unity ($G_2 = 0$) normalization

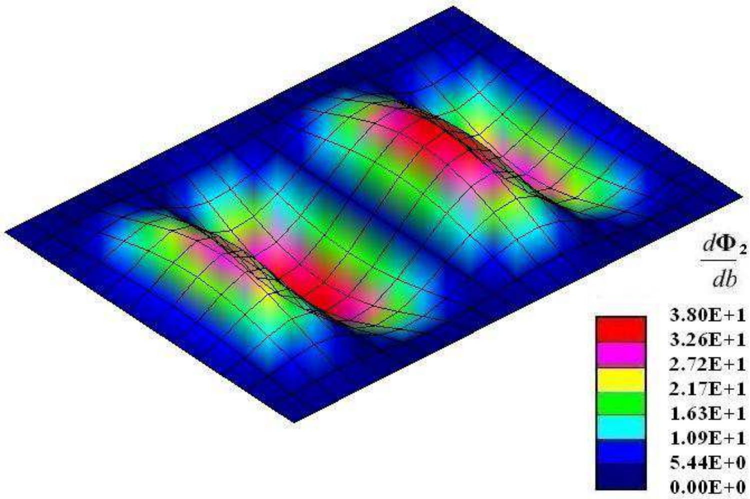


c: Magnitude equals unity ($G_3 = 0$) normalization

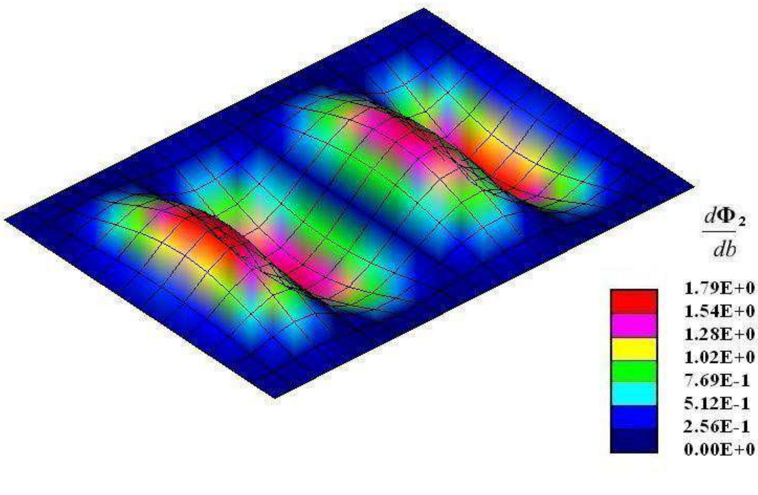
Figure 5.10: First-order sensitivity for the first mode



a: Mass normalization ($G_1 = 0$)

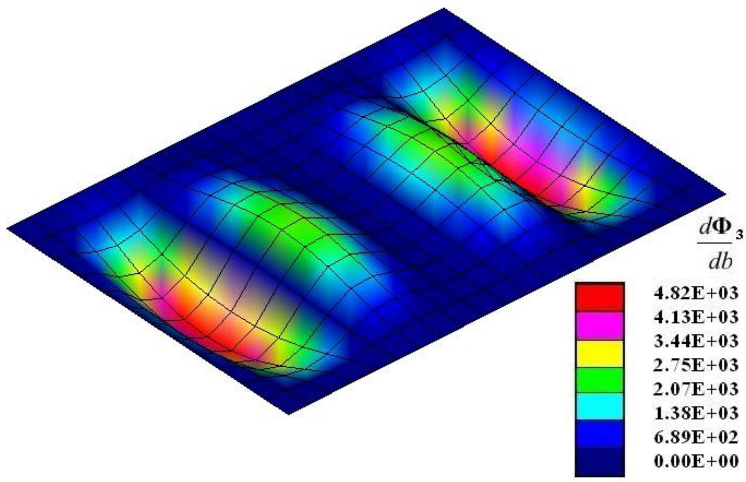


b: Maximum displacement equals unity ($G_2 = 0$) normalization

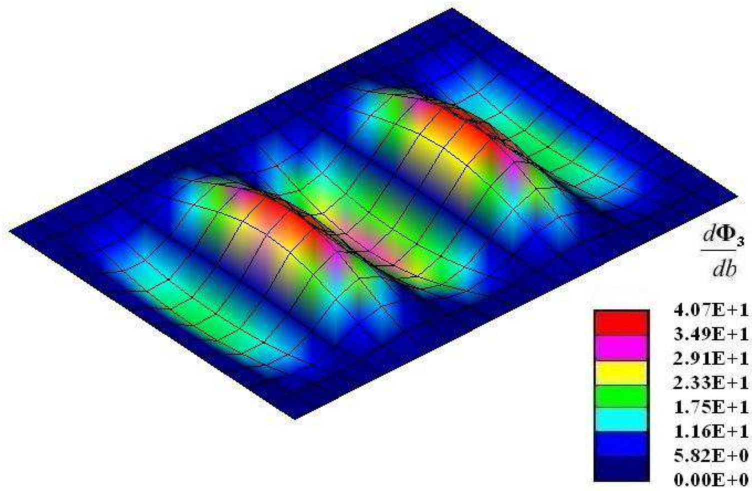


c: Magnitude equals unity ($G_3 = 0$) normalization

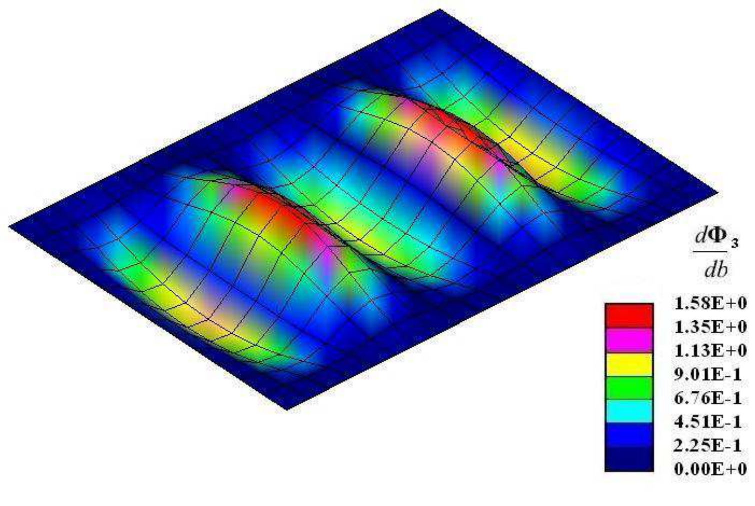
Figure 5.11: First-order sensitivity for the second mode



a: Mass normalization ($G_1 = 0$)

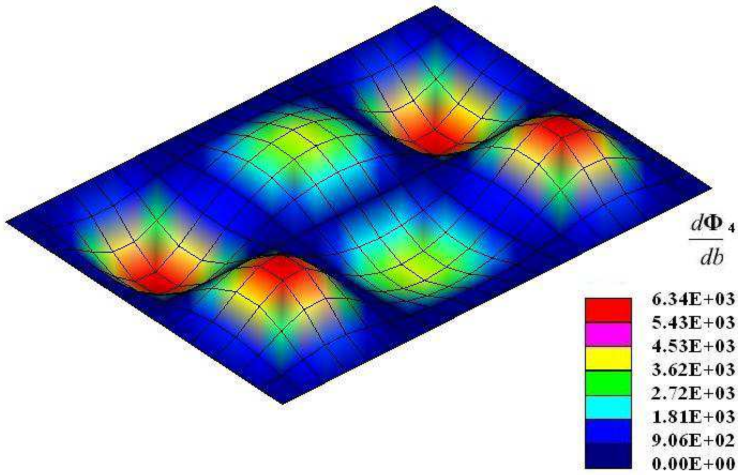


b: Maximum displacement equals unity ($G_2 = 0$) normalization

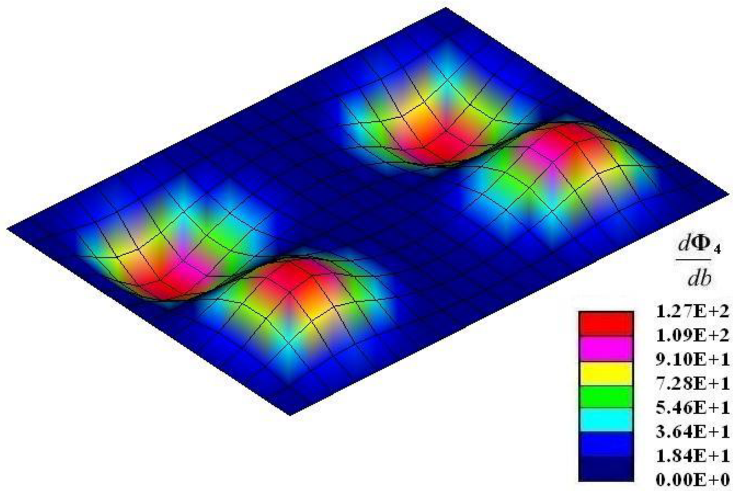


c: Magnitude equals unity ($G_3 = 0$) normalization

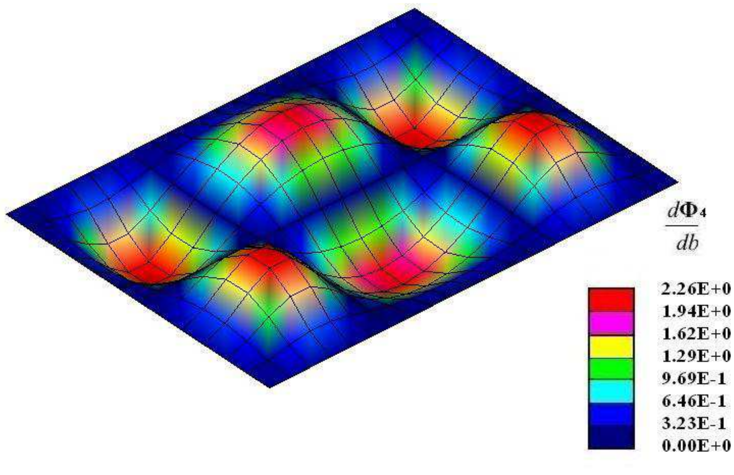
Figure 5.12: First-order sensitivity for the third mode



a: Mass normalization ($G_1 = 0$)

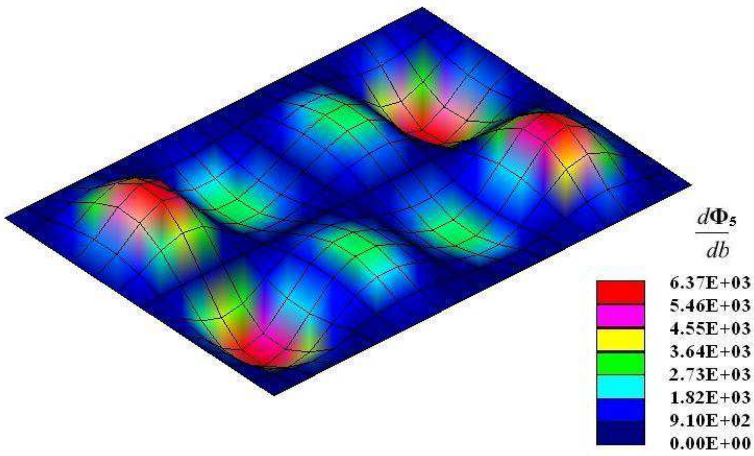


b: Maximum displacement equals unity ($G_2 = 0$) normalization

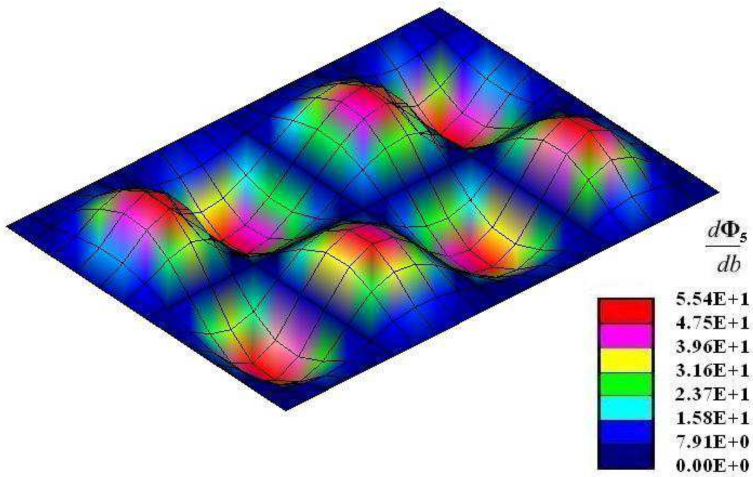


c: Magnitude equals unity ($G_3 = 0$) normalization

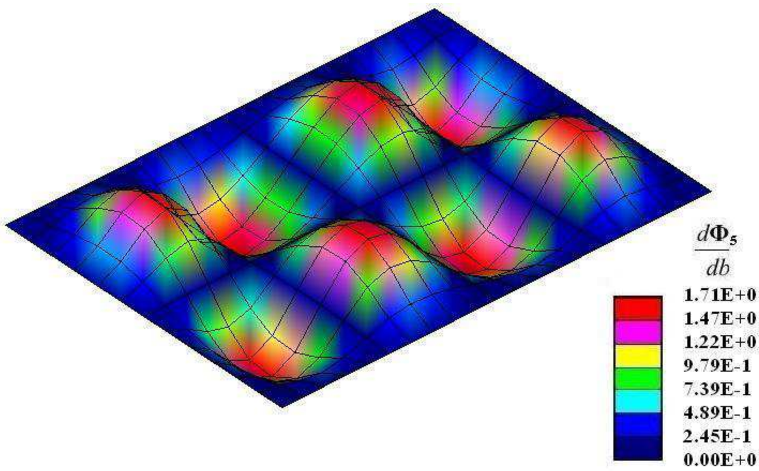
Figure 5.13: First-order sensitivity for the fourth mode



a: Mass normalization ($G_1 = 0$)



b: Maximum displacement equals unity ($G_2 = 0$) normalization

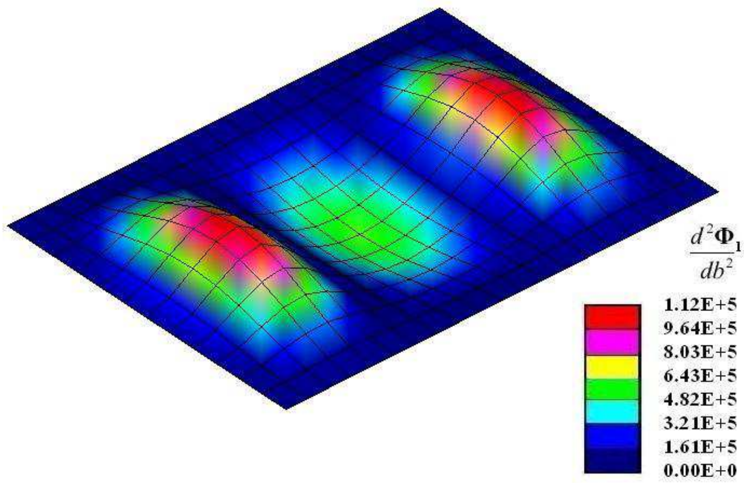


c: Magnitude equals unity ($G_3 = 0$) normalization

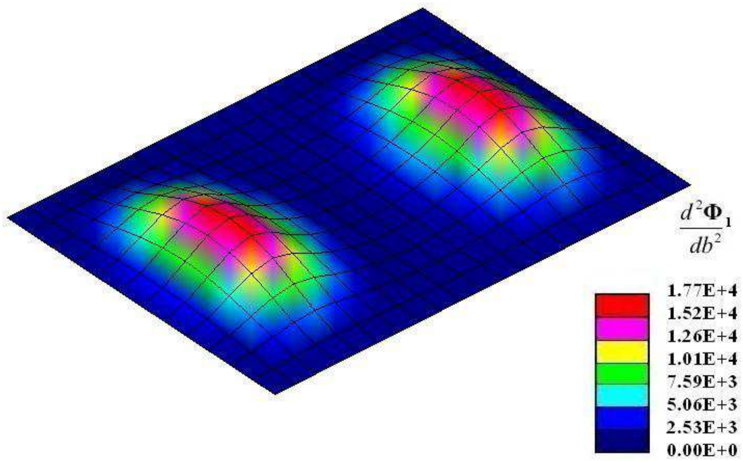
Figure 5.14: First-order sensitivity for the fifth mode

5.3.3 Second-Order Eigenvector Design Sensitivity Analysis

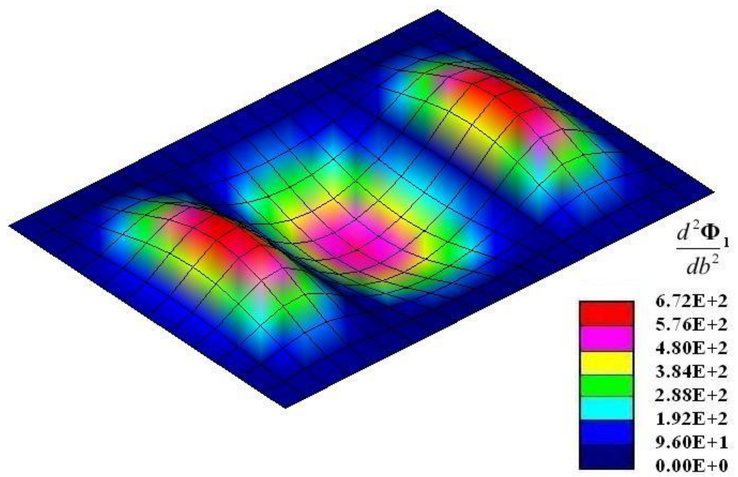
Figures 5.15 through 5.19 present the second-order eigenvector design sensitivities of the first 5 modes for three normalization conditions. Again it is seen from these plots that the normalization condition does have an effect on the second-order sensitivities of the mode shapes. The effect of normalization is not the same in all the modes, it has greater effect on the first and fourth modes whereas for the eigenvector sensitivity shapes of other modes does not appear to be as dependent on the normalization. Even though the plate problem presented here is a very simple symmetric structure the effect of normalization condition is very clearly seen. For more complicated problems like a 3-D dashboard or an airplane wing problems the effects may be more pronounced and therefore should be accounted for in the design and analysis of structures.



a: Mass normalization ($G_1 = 0$)

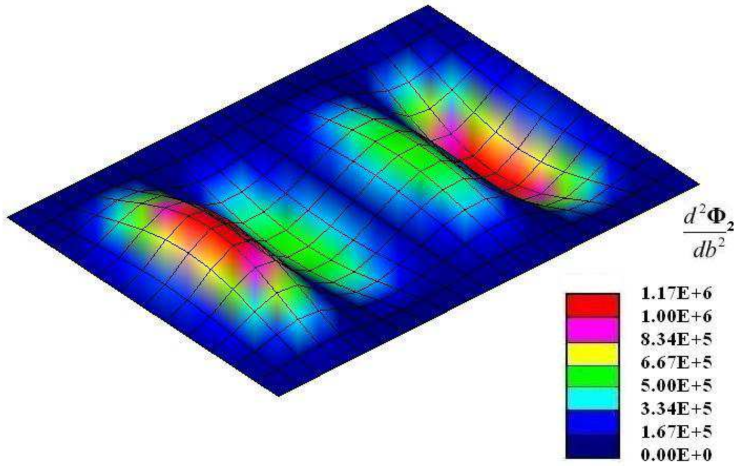


b: Maximum displacement equals unity ($G_2 = 0$) normalization

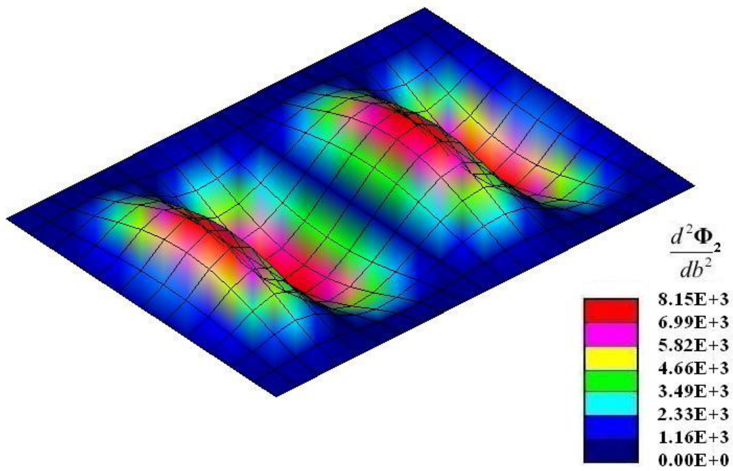


c: Magnitude equals unity ($G_3 = 0$) normalization

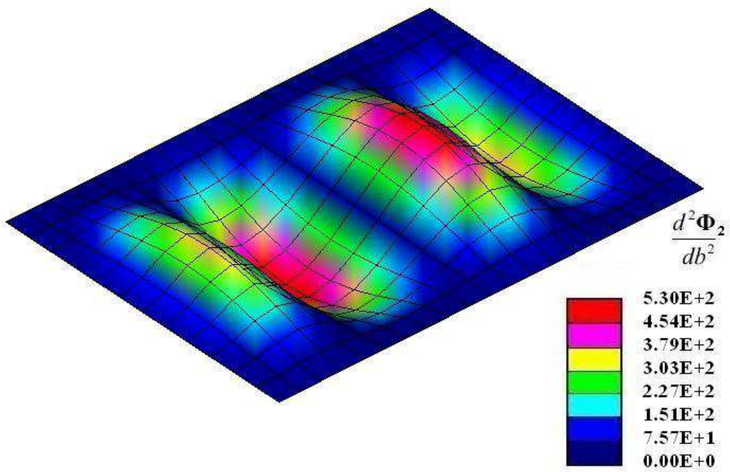
Figure 5.15: Second-order sensitivity for the first mode



a: Mass normalization ($G_1 = 0$)

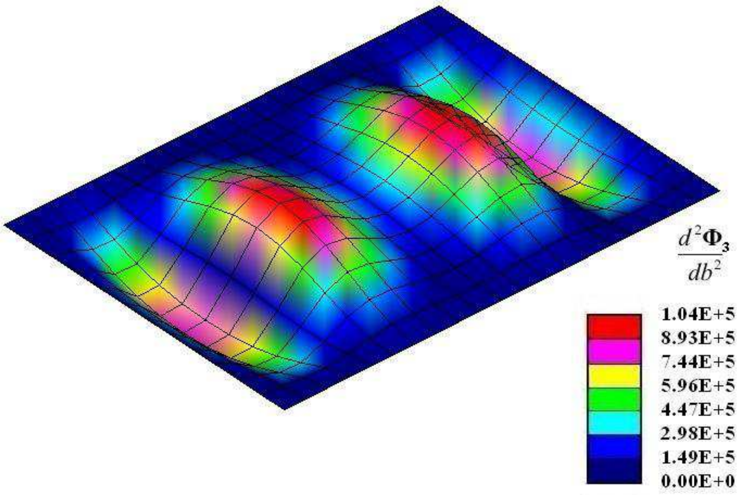


b: Maximum displacement equals unity ($G_2 = 0$) normalization

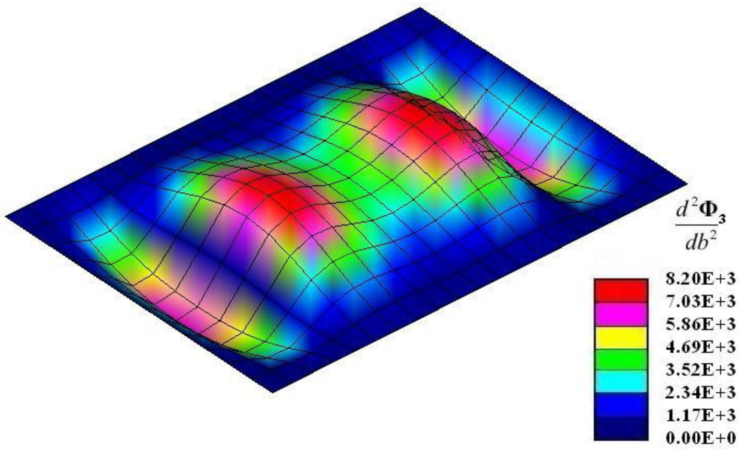


c: Magnitude equals unity ($G_3 = 0$) normalization

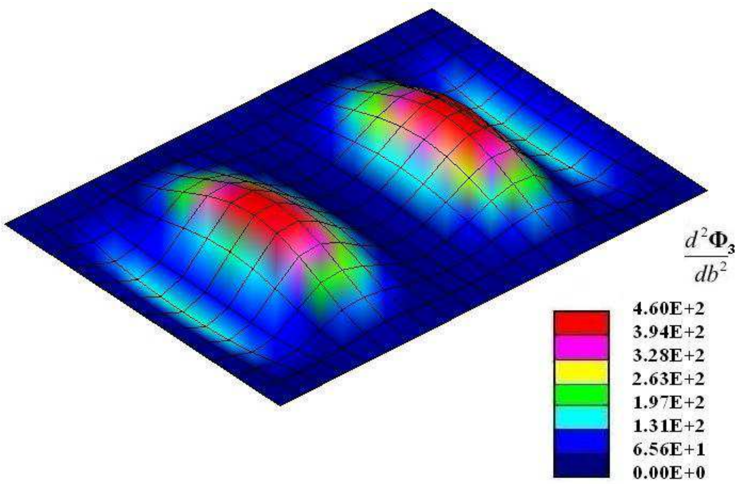
Figure 5.16: Second-order sensitivity for the second mode



a: Mass normalization ($G_1 = 0$)

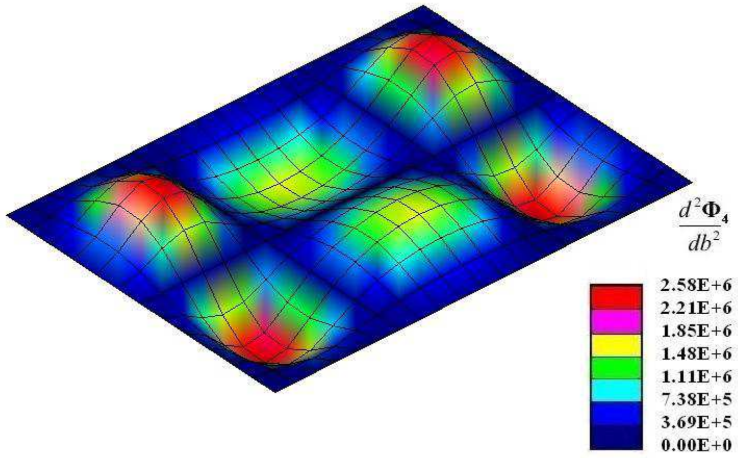


b: Maximum displacement equals unity ($G_2 = 0$) normalization

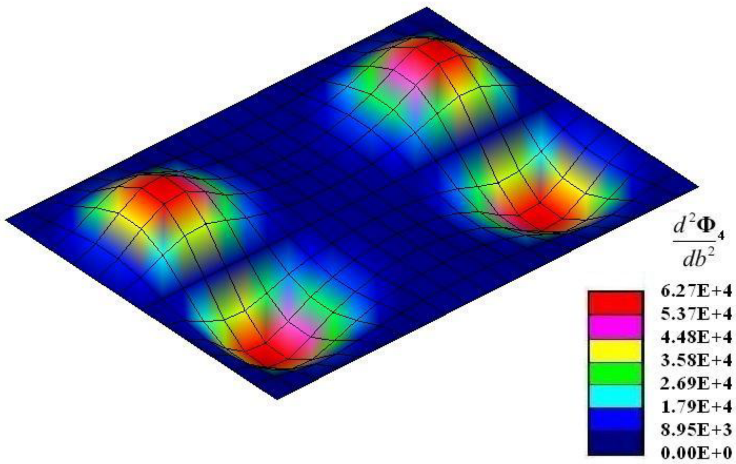


c: Magnitude equals unity ($G_3 = 0$) normalization

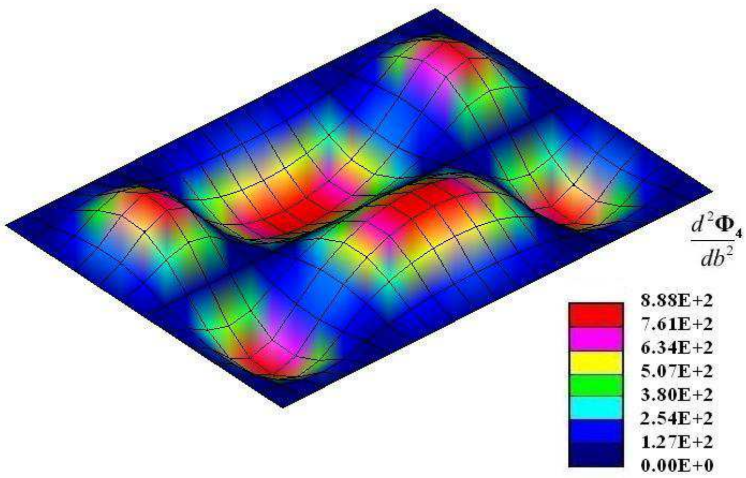
Figure 5.17: Second-order sensitivity for the third mode



a: Mass normalization ($G_1 = 0$)

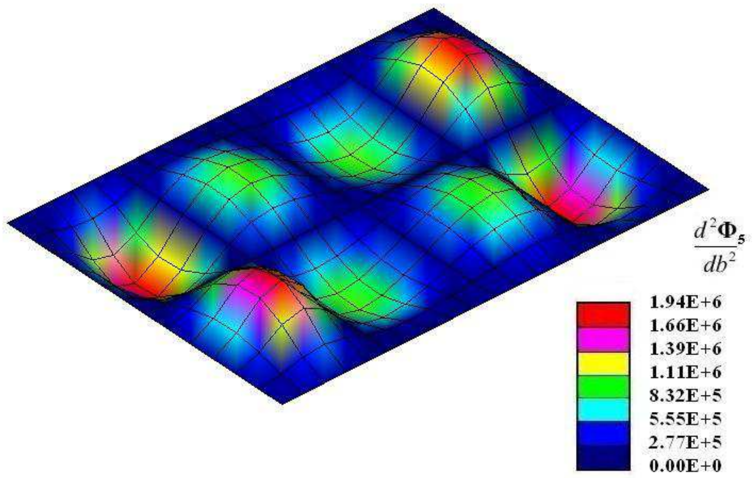


b: Maximum displacement equals unity ($G_2 = 0$) normalization

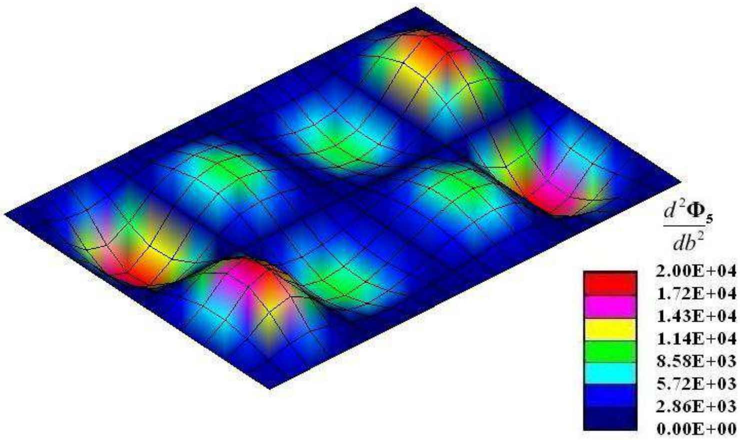


c: Magnitude equals unity ($G_3 = 0$) normalization

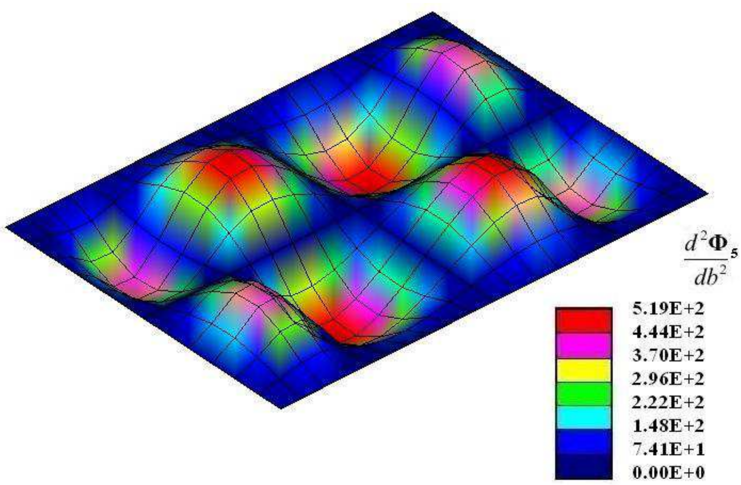
Figure 5.18: Second-order sensitivity for the fourth mode



a: Mass normalization ($G_1 = 0$)



b: Maximum displacement equals unity ($G_2 = 0$) normalization



c: Magnitude equals unity ($G_3 = 0$) normalization

Figure 5.19: Second-order sensitivity for the fifth mode

CHAPTER 6

CONCLUSIONS

Eigenvector design sensitivity is important in many structural and optimization problems. Nelson's algorithm has been considered a benchmark for calculating eigenvector sensitivity for over three decades. His method is applicable to mass normalized eigenvectors only, and is, therefore, not valid for eigenvector with other normalization conditions. A generalized method to calculate the eigenvector sensitivities for any normalization condition is presented in this thesis.

In this new method the the modified $[\mathbf{K} - \lambda\mathbf{M}]$ matrix on the left hand side of the design sensitivity equation is the same for both the first- and second-order sensitivities for all normalizations, which can be used very effectively to reduce the calculation time and cost. For the normalization that specifies the maximum value equals unity normalization condition, which is the most commonly used normalization, the modified $[\mathbf{K} - \lambda\mathbf{M}]$ matrix on the left hand side for first- and second-order sensitivity is same as that used to compute the eigenvector itself. Therefore this method is most effective for maximum value equals unity normalization.

Following the evaluation of eigenvector sensitivities for a given normalization condition, the need may arise for the eigenvector sensitivities for another normalization condition. Instead of repeating the entire process, the scaling properties of the eigenvectors are used to calculate new eigenvector sensitivities for the rescaled eigenvectors. These equations to calculate the scaling factors and re-scale the eigenvector sensitivities were derived. A change in normalization condition for an eigenvector only effects its magnitude, the shape of the eigenvector remains the same. But that is not the case for eigenvector sensitivity. Changing the normalization condition is shown to effect both the shape and magnitude of the eigenvector sensitivity. This was shown using the theoretical equations and the examples provided above.

The equations presented here can be easily incorporated into a finite element program to calculate the sensitivities for various normalization condition. This was done by writing a MATLAB-based finite element code to calculate the sensitivities.

Two examples are provided to further illustrate the methodology involved in the calculation of the eigenvalues, eigenvectors and their sensitivities. A spring-mass example is provided to show a step-by-step explanation of how eigenvector sensitivities are calculated with the proposed general procedure. In addition, a plate example showed that this method can be applied to more complicated problems and also further illustrates the usefullness of this research. The calculation for these examples was performed in the MATLAB code discussed earlier, although the spring example can be done manually.

This approach can be extended to more practical problems like composite structures, dashboard problems and aircraft wings etc. The finite element code can be made more efficient by utilizing the memory and time saving properties of this method. This approach can also be incorporated into commercial analysis packages such as ABAQUS, ANSYS [52] etc.

BIBLIOGRAPHY

[1] Tomovic, R. *Sensitivity Analysis of Dynamic Systems.* McGraw-Hill Book Co., New York, 1963.

[2] Brayton, R.K. and R. Spence. *Sensitivity and Optimization.* Elsevier, New York, 1980.

[3] Frank, P.M. *Introduction to Sensitivity Theory.* Academic Press, Orlando, FL, 1978.

[4] Radanovic, L. *Sensitivity Methods in Control Theory.* Pergamon Press, Oxford, England, 1966.

[5] Kelly, H.J. Methods of Gradients. In *Optimization Techniques with Applications to Aerospace Systems.* Academic Press, Orlando, FL., 1962.

[6] Schmit, L.A. Jr. Structural Synthesis - Its Genisis and Developement. *AIAA Jn.*, **19**:1249–1263, 1981.

[7] Leonard, J.I. The Application of Sensitivity Analysis to Models of Large Scale Physiological Systems. Technical Report CR-160228, NASA, 1974.

[8] Irwin, C.L. and T.J. O'Brien. Sensitivity of Thermodynamic Calculations. Technical Report DOE/METC/82-53, U.S. Dept. of Energy, 1982.

[9] Hwang, J.T., E.P. Dougherty, S. Rabitz, and H. Rabitz. The Green's Function Method of Sensitivity Analysis in Chemical Kinetics. *Jn. of Chemical Physics*, **69**:5180–5191, 1978.

[10] Dwyer, H.A. and T. Peterson. A Study of Turbulent Flow with Sensitivity Analysis. *AIAA Jn.*, **19**:1249–1263, 1981.

[11] Dwyer, H.A., T. Peterson, and J. Brewer, editors. *Sensitivity Analysis Applied to Boundary Layer Flow.* Proceedings of the 5th International Conference on Numerical Methods in Fluid Dynamics, Springer-Verlag,N.Y, 1976.

[12] Bristow, D.R. and J.D. Hawk. Subsonic Panel Method for Designing Wing Surfaces from Pressure Deistributions. Technical Report CR-3713, NASA, 1983.

[13] Parloo, E., P. Guillaume, and M. Van Overmeire. Damage Assesment using mode shape sensitivities. *Mechanical Systems and Signal Processing*, **17**(3):499–518, 2001.

[14] Adelman, H.M. and R.T. Haftka. Sensitivity Analysis of Discrete Structural System. *AIAA Jn.*, **24**(5):823–832, 1986.

[15] Murthy, D.V. and R.T. Haftka. Derivatives of Eigenvalues and Eigenvectors of a General Complex Matrix. *Int. Jn. Numerical Methods Engineering*, **26**:293–311, 1988.

[16] Grandhi, R. Structural Optimization with Frequency Constraints – A Review. *AIAA Jn.*, **31**(12):2296–2303, 1993.

[17] Tortorelli, D. A. and P. Michaleris. Design Sensitivity Analysis: Overview and Review. *Inverse Problems in Engineering*, **1**(1):71–105, 1993.

[18] Lancaster, P. On Eigenvalues of Matrices Dependent on a Parameter. *Numerische Mathematik*, **6**:377–387, 1964.

[19] Fox, R.L. and M.P. Kapoor. Rate of Change of Eigenvalues and Eigenvectors. *AIAA Jn.*, **6**(12):2426–2429, 1968.

[20] Juang, J.N., P. Ghaemmaghami, and K. B. Lim. Eigenvalue and Eigenvector Derivatives of a Nondefective Matrix. *Jn. Guidance, Control and Dynamics*, **12**(4):480–486, 1989.

[21] Bernard,M.L. and A.J. Bronowicki. Modal Expansion method for Eigensensitivity with Repeated Roots. *AIAA Jn.*, **32**(7):1500–1506, 1994.

[22] Akgin, M.A. New Family of Modal Methods for Calculating Eigenvector Derivatives. *AIAA Jn.*, **32**(2):379–386, 1994.

[23] Lim, K.B., P. Ghaemmaghami, and J.N. Juang. Eigenvector Derivatives of Repeated Eigenvalues using Singular Value Decomposition. *Jn. Guidance, Control and Dynamics*, **12**(2):282–283, 1989.

[24] Hou, G.J.W. and S.P. Kenny. Eigenvalue and Eigenvector Approximate Analysis for Repeated Eigenvalue Problems. *AIAA Jn.*, **30**(9):2317–2323, 1992.

[25] Rogers, L.C. Derivatives of Eigenvalues and Eigenvectors. *AIAA Jn.*, **8**(5):943–944, 1970.

[26] Plaut, R.H. and K. Husseyian. Derivatives of Eigenvalues and Eigenvector in Non-Self-Adjoint Systems. *AIAA Jn.*, **11**(2):250–251, 1973.

[27] Rudisill, C.S. Derivatives of Eigenvalues and Eigenvectors of a General Matrix. *AIAA Jn.*, **12**(5):721–722, 1974.

[28] Rudisill, C.S. and Y. Chu. Numerical Methods for Evaluating the Derivatives of Eigenvalues and Eigenvectors. *AIAA Jn.*, **13**(6):834–837, 1975.

[29] Doughty, S. Eigenvalue Derivatives of Damped Torsional Vibrations. *ASME Jn. Mechanical Design*, **104**(4):463–465, 1982.

[30] Nelson, R.B. Simplified Calculation of Eigenvector Derivatives. *AIAA Jn.*, 14(9):1201–1205, 1976.

[31] Ojalvo, I.U. Gradients for Large Structural Models with Repeated Frequencies. *SPE Technical Paper Series*, (861789), 1986.

[32] Ojalvo, I.U. Efficient Computation of Mode-Shape Derivatives for Large Dynamic Systems. *AIAA Jn.*, **25**(10):1386–1390, 1987.

[33] Mills-Curran, W.C. Calculation of Eigenvector Derivatives for Structures with Repeated Eigenvalues. *AIAA Jn.*, **26**(7):867–871, 1988.

[34] Dailey, R.L. Eigenvector Derivatives with Repeated Eigenvalues. *AIAA Jn.*, **27**(4):486–491, 1989.

[35] Friswell, M.I. The Derivatives of Repeated Eigenvalues and Their Associated Eigenvectors. *ASME Jn. Vibration and Acoustics*, **118**(7):390–397, 1996.

[36] Zhang, Y.Q. and W.L. Wang. Eigenvector Derivatives of Generalized Nondefective Eigenproblems With Repeated Eigenvalues. *Jn. Eng. Gas Turbines and Power*, **117**(1):207–212, 1995.

[37] Lee, I.W. and G.H Jung. An Efficient Algebraic Method for the Computation of Natural Frequency and Mode Shape Sensitivities- Part I. Distinct Natural Frequencies. *Computers and Structures*, 62(3):429–435, 1997.

[38] Brandon, J.A. Derivation and Significance of Second-Order Modal Design Sensitivities. *AIAA Jn.*, **22**(5):723–724, 1984.

[39] Brandon, J.A. Second-Order Design Sensitivities to Asses the Applicability of Sensitivity Analysis. *AIAA Jn.*, **29**(1):135–139, 1991.

[40] Chen, S.H., D.T. Song, and A.J. Ma. Eigensolution Reanalaysis of Modified Structures Using Perturbations and Rayleigh Quotients. *Communications in Numerical Methods in Engineering*, **10**(2):111–119, 1994.

[41] Chen, S.H., Z. Qiu, and Z. Liu. Perturbation Method for Computing Eigenvalue Bounds in Structural Vibration Systems with Interval Parameters. *Communications in Numerical Methods in Engineering*, **10**(2):121–134, 1994.

[42] Tan, R.C.E., A.L. Andrew, and F.M.L. Hong. Iterative Computation of Second-Order Derivatives of Eigenvalues and Eigenvectors. *Communications in Numerical Methods in Engineering*, **10**(1):1–9, 1994.

[43] Jankovic, M.S. Exact nth Derivatives of Eigenvalues and Eigenvectors. *Jn. of Guidance, Control, and Dynamics*, **17**(1):136–144, 1994.

[44] Chen, C. J. and K. K. Choi. Continuum Approach for Second-Order Shape Design Sensitivity of Three-Dimensional Elastic Solids. *AIAA Jn.*, **32**(10):2099–2108, 1994.

[45] Friswell, M.I. Calculation of Second and Higher Order Eigenvectors Derivatives. *Jn. of Guidance: Engineering Notes*, **18**(4):919–921, 1995.

[46] ABAQUS CAE. *ABAQUS, Inc.* www.abaqus.com.

[47] MATLAB. *The Mathworks, Inc.* www.mathworks.com.

[48] MSC.NASTRAN. *MSC.Software Corporation.* www.mscsoftware.com.

[49] Hyperworks Hypermesh. *Altair Engineering, Inc.* www.altair.com.

[50] Shames, I.H. and C.L. Dym. *Energy and Finite Element Methods in Structural Mechanics*, volume **1**. Hemisphere Publishing Corportation, Washington, 1985.

[51] Pai, P.F. Notes of Introduction to Finite Element Methods Class (MAE 386). 2002 Fall.

[52] ANSYS. *ANSYS, Inc.* www.ansys.com.

APPENDIX A

NELSON'S METHOD

Though eigenvalue sensitivities are the most commonly required sensitivities, eigenvector sensitivities are also desired in applications. Nelson [30] presented a powerful algorithm for computing the eigenvalue and eigenvector derivatives of general real matrices with non-repeated eigenvalues. His method is a significant advance as it requires knowledge of only those eigenvectors that are to be differentiated. All of the previous methods required calculation of all or most of the eigenvectors, which may require significant calculations. Nelson's method also preserved the band structure and symmetry of the matrices. Nelson gives his method with the an algebraic eigensystem of the form $[\mathbf{A} - \lambda_i \mathbf{I}]\mathbf{\Phi}_i = \mathbf{0}$ which can be interpreted as a structural problem of the form $[\mathbf{K} - \lambda_i \mathbf{M}]\mathbf{\Phi}_i = \mathbf{0}$ where $\mathbf{\Phi}_i$ is the eigenvector, λ_i is the eigenvalue and $\mathbf{K}$ and $\mathbf{M}$ are real symmetric matrices. He also assumes that the eigenvectors are mass normalized as given in equation 2.5.

Solution for Eigenvector Derivatives

The derivatives of the eigenvector with respect to the design parameter b_j, is obtained by differentiating equation 3.1 with respect to b_j which gives equation 3.6 shown below

$$\left[\frac{\partial \mathbf{K}}{\partial b_j} - \frac{d\lambda_i}{db_j}\mathbf{M} - \lambda_i \frac{\partial \mathbf{M}}{\partial b_j}\right] \mathbf{\Phi}_i + [\mathbf{K} - \lambda_i \mathbf{M}] \frac{\partial \mathbf{\Phi}_i}{\partial b_i} = \mathbf{0}$$

The eigenvector sensitivity cannot be calculated directly from equation 3.6 as $[\mathbf{K} -$

$\lambda_i \mathbf{M}]$ is singular as its rank is $n-1$ for distinct λ. Therefore Nelson [30] proposed that the eigenvector derivative be written as

$$\frac{\partial \mathbf{\Phi}_i}{\partial b_j} = \boldsymbol{\nu}_{ij} + c_{ij}\mathbf{\Phi}_i \tag{A.1}$$

for some vector $\boldsymbol{\nu}_{ij}$ and a scalar constant c_{ij} which can be calculated. Substituting equation A.1 in equation 3.6 and simplifying to yield

$$[\mathbf{K} - \lambda_i \mathbf{M}]\, \boldsymbol{\nu}_{ij} = \mathbf{F}_{ij} \tag{A.2}$$

the vector on the right-hand-side of equation A.2 is computed from

$$\mathbf{F}_{ij} = -\left[\frac{\partial \mathbf{K}}{\partial b_j} - \frac{d\lambda_i}{db_j}\mathbf{M} - \lambda_i \frac{\partial \mathbf{M}}{\partial b_j}\right]\mathbf{\Phi}_i$$

$[\mathbf{K} - \lambda \mathbf{M}]$ on the left hand side of the equations is still singular, therefore Nelson proposed that the p^{th} component of $\boldsymbol{\nu}_{ij}$ be set to zero, where p is the location at which the eigenvector $\mathbf{\Phi}$ has the maximum absolute value. He achieves this by replacing the p^{th} row and column of $[\mathbf{K} - \lambda \mathbf{M}]$ matrix with zeros except for the diagonal term, which is set to one and setting the k^{th} component of the vector $\mathbf{F}_{ij}$ to zero. The resulting partitioned form can be represented in the equation form as

$$\begin{bmatrix} [\mathbf{K} - \lambda_i \mathbf{M}]_{11} & \mathbf{0} & [\mathbf{K} - \lambda_i \mathbf{M}]_{13} \\ \mathbf{0} & 1 & \mathbf{0} \\ [\mathbf{K} - \lambda_i \mathbf{M}]_{31} & \mathbf{0} & [\mathbf{K} - \lambda_i \mathbf{M}]_{33} \end{bmatrix} \boldsymbol{\nu}_{ij} = \begin{Bmatrix} \{\mathbf{F}_{ij}\}_1 \\ 0 \\ \{\mathbf{F}_{ij}\}_3 \end{Bmatrix} \tag{A.3}$$

The new matrix formed is now non-singular and it is possible to calculate $\boldsymbol{\nu}_{ij}$ using any numerical methods available. Once $\boldsymbol{\nu}_{ij}$ is calculated, the value of the scalar constant c_{ij} is determined by differentiating the mass normalization condition in equation 2.5 with respect to b_j as

$$\mathbf{\Phi}_i \cdot \frac{\partial \mathbf{M}}{\partial b_j}\mathbf{\Phi}_i + 2\mathbf{\Phi}_i \cdot \mathbf{M}\frac{\partial \mathbf{\Phi}_i}{\partial b_j} = 0 \tag{A.4}$$

Substituting the expression for eigenvector sensitivity from equation A.1 in equation A.4 and using the mass normalization condition itself gives the solution for c_{ij} as

$$c_{ij} = -\frac{1}{2}\mathbf{\Phi}_i \cdot \frac{\partial \mathbf{M}}{\partial b_j}\mathbf{\Phi}_i - \mathbf{\Phi}_i \cdot \mathbf{M}\boldsymbol{\nu}_{ij} \tag{A.5}$$

Once $\boldsymbol{\nu}_{ij}$, c_{ij} and $\mathbf{\Phi}_i$ are known, they may be substitute back into equation A.1 to get the the eigenvector sensitivity $\partial\mathbf{\Phi}_i/\partial b_j$. This method works with mass normalized eigenvectors having non-repeated eigenvalues.

APPENDIX B

FRISWELL'S METHOD

Friswell [45] developed a procedure similar to that used to calculate the first-order eigenvector sensitivities, developed by Nelson [30], for the second-order eigenvector sensitivity. Consider the second-order sensitivity equation 4.1 which may be written as

$$[\mathbf{K} - \lambda_i \mathbf{M}] \frac{d^2 \mathbf{\Phi}_i}{db_j db_k} = \mathbf{F}_{ijk} \tag{B.1}$$

where

$$\begin{aligned} \mathbf{F}_{ijk} = &\left[\frac{\partial^2 \mathbf{K}}{\partial b_j \partial b_k} - \frac{d^2 \lambda_i}{db_j db_k}\mathbf{M} - \frac{d\lambda_i}{db_k}\frac{\partial \mathbf{M}}{\partial b_j} - \frac{d\lambda_i}{db_j}\frac{\partial \mathbf{M}}{\partial b_k} - \lambda_i \frac{\partial^2 \mathbf{M}}{\partial b_j \partial b_k}\right] \mathbf{\Phi}_i \\ &+ \left[\frac{\partial \mathbf{K}}{\partial b_j} - \frac{d\lambda_i}{db_j}\mathbf{M} - \lambda_i \frac{\partial \mathbf{M}}{\partial b_j}\right] \frac{d\mathbf{\Phi}_i}{db_k} + \left[\frac{\partial \mathbf{K}}{\partial b_k} - \frac{d\lambda_i}{db_k}\mathbf{M} - \lambda_i \frac{\partial \mathbf{M}}{\partial b_k}\right] \frac{d\mathbf{\Phi}_i}{db_j} \end{aligned} \tag{B.2}$$

It is known that $[\mathbf{K} - \lambda_i \mathbf{M}]$ is singular and the sensitivity cannot be directly calculated from equation B.1. Therefore, as Nelson did in the first-order derivative, Friswell's evaluations suggest that the eigenvector sensitivity can be of the form

$$\frac{\partial^2 \mathbf{\Phi}_i}{\partial b_j \partial b_k} = \boldsymbol{\nu}_{ijk} + c_{ijk}\mathbf{\Phi}_i \tag{B.3}$$

where $\boldsymbol{\nu}_{ijk}$ is a non-unique vector and c_{ijk} is a constant. Equation B.3 when substituted in equation B.1 will reduce to

$$[\mathbf{K} - \lambda_i \mathbf{M}]\, \boldsymbol{\nu}_{ijk} = \mathbf{F}_{ijk} \tag{B.4}$$

The component of $\boldsymbol{\nu}_{ijk}$ corresponding to that having the largest absolute value in the eigenvector is set to zero. Now it is possible to calculate a unique $\boldsymbol{\nu}_{ijk}$ vector which satisfies the above equation.

To calculate the value of the scalar constant c_{ijk}, the mass normalization equation 2.5 is differentiated twice to yield

$$\begin{aligned} 2\left[\boldsymbol{\Phi}_i \cdot \frac{\partial \mathbf{M}}{\partial b_j}\frac{d\boldsymbol{\Phi}_i}{db_k} + \boldsymbol{\Phi}_i \cdot \frac{\partial \mathbf{M}}{\partial b_k}\frac{d\boldsymbol{\Phi}_i}{db_j} + \frac{d\boldsymbol{\Phi}_i}{db_j} \cdot \mathbf{M}\frac{d\boldsymbol{\Phi}_i}{db_k} + \boldsymbol{\Phi}_i \cdot \mathbf{M}\frac{d^2\boldsymbol{\Phi}_i}{db_j db_k}\right] \\ +\boldsymbol{\Phi}_i \cdot \frac{\partial^2 \mathbf{M}}{\partial b_j \partial b_k}\boldsymbol{\Phi}_i = \mathbf{0} \end{aligned} \tag{B.5}$$

Substituting the equation B.3 in equation B.5 and using mass normalization condition will give the expression for c_{ijk} as

$$\begin{aligned} c_{ijk} = -.5\boldsymbol{\Phi}_i \cdot \frac{\partial^2 \mathbf{M}}{\partial b_j \partial b_k}\boldsymbol{\Phi}_i \; &- \; \boldsymbol{\Phi}_i \cdot \frac{\partial \mathbf{M}}{\partial b_j}\frac{d\boldsymbol{\Phi}_i}{db_k} - \boldsymbol{\Phi}_i \cdot \frac{\partial \mathbf{M}}{\partial b_k}\frac{d\boldsymbol{\Phi}_i}{db_j} \\ &- \; \frac{d\boldsymbol{\Phi}_i}{db_j} \cdot \mathbf{M}\frac{d\boldsymbol{\Phi}_i}{db_k} - \boldsymbol{\Phi}_i \cdot \mathbf{M}\boldsymbol{\nu}_{ijk} \end{aligned} \tag{B.6}$$

All the terms required to calculate the second-order eigenvector sensitivity using the equation B.3 are known. Substituting the values of c_{ijk}, $\boldsymbol{\nu}_{ijk}$ and $\boldsymbol{\Phi}_i$ in the equation gives the required second-order sensitivity. Note that this is very similar to what was done by Nelson for the first-order sensitivity and therefore the interpretation of Nelson's method given in the previous section can also be extended to Friswell's second-order calculations. Friswell [45] also mentions that higher order eigenvalue and eigenvector sensitivities for non-repeated eigenvalues can be calculated by extending his method and provides general equations for m-th order sensitivities.