TOPICS IN FUNCTIONAL ANALYSIS AND CONVEX GEOMETRY

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The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

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ABSTRACT

In this thesis we study different problems in Convex Geometry with the aid of the Fourier Transform and tools of Functional Analysis.

In the first chapter we collect all necessary definitions and preliminary results.

In the second chapter we construct an example of a non-intersection body all of whose central sections are intersection bodies.

The third chapter is devoted to the study of the geometry of L_0. We introduce the definition of embedding of a normed space in L_0, give a characterization of subspaces of L_0 and confirm the place of L_0 in the scale of L_p spaces.

In the fourth chapter we modify the assumptions of the original Busemann-Petty problem in order to obtain the positive answer in all dimensions.

Chapter five is focused on L_p-centroid bodies and generalization of some results of Lutwak and Grinberg, Zhang to $-1 < p < 1$.

Contents

Chapter 1

Preliminaries

A compact set K in $\mathbb{R}^n$ is called an origin-symmetric star body if every straight line passing through the origin crosses the boundary of K at exactly two points, the boundary is continuous, and the origin is an interior point of K.

Let K be a convex origin-symmetric body in $\mathbb{R}^n$. Our definition of a convex origin-symmetric body assumes that the origin is an interior point of K . The *radial function* of K is given by

$$\rho_K(x) = \max\{a > 0 : ax \in K\}, \quad x \in \mathbb{R}^n \setminus \{0\}$$

If $x \in S^{n-1}$, $\rho_K(x)$ is the distance from the origin to the boundary of K in the direction of x.

The Minkowski *norm* of K is defined by

$$\|x\|_K = \min\{a \geq 0 : x \in aK\}.$$

Clearly $\rho_K(x) = \|x\|_K^{-1}$.

We denote by $(\mathbb{R}^n, \|\cdot\|_K)$ the Euclidean space equipped with the Minkowski functional of the body K. Clearly, $(\mathbb{R}^n, \|\cdot\|_K)$ is a normed space if and only if the body K is convex.

Writing the volume of K in polar coordinates, one can express the volume in terms of the Minkowski norm:

$$\mathrm{vol}_n(K) = \frac{1}{n} \int_{S^{n-1}} \|\theta\|_K^{-n} d\theta. \tag{1.1}$$

We say that a body K is infinitely smooth if its radial function ρ_K restricted to the unit sphere S^{n-1} belongs to the space $C^\infty(S^{n-1})$ of infinitely differentiable functions on the unit sphere.

The *radial metric* on the set of all origin symmetric star bodies is defined by

$$\rho(K, L) = \max_{x \in S^{n-1}} |\rho_K(x) - \rho_L(x)|.$$

As usual, we denote by $\mathcal{S}(\mathbb{R}^n)$ the space of infinitely differentiable rapidly decreasing functions on $\mathbb{R}^n$ (test functions), and by $\mathcal{S}'(\mathbb{R}^n)$ the space of distributions over $\mathcal{S}(\mathbb{R}^n)$.

The Fourier transform of a distribution f is defined by $\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle$ for every test function ϕ from the Schwartz space $\mathcal{S}$ of rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$. For any even distribution f, we have $(\hat{f})^\wedge = (2\pi)^n f$.

A distribution is *positive definite* if its Fourier transform is a positive distribution in the sense that $\langle \hat{f}, \phi \rangle \geq 0$ for every non-negative test function ϕ; see, for example, [GV, p.152].

We say that a distribution is positive (negative) outside of the origin in $\mathbb{R}^n$ if it assumes non-negative (non-positive) values on non-negative Schwartz's test functions with compact support outside of the origin.

Let ϕ be an integrable function on $\mathbb{R}^n$ that is also integrable on hyperplanes, let $\xi \in S^{n-1}$, and let $t \in \mathbb{R}^n$. Then

$$\mathcal{R}\phi(\xi;t) = \int_{(x,\xi)=t} \phi(x)dx$$

is the *Radon transform of ϕ in the direction ξ at the point t*. A simple connection between the Fourier and Radon transforms is that for every fixed $\xi \in \mathbb{R}^n \setminus \{0\}$

$$\hat{\phi}(s\xi) = (\mathcal{R}\phi(\xi;t))^\wedge(s), \quad \forall s \in \mathbb{R} \tag{1.2}$$

where in the right hand side we have the Fourier transform of the function $t \to \mathcal{R}\phi(\xi;t)$, see e.g. [Ko11, Lemma 2.11].

Let $\xi \in S^{n-1}$ and $(x,\xi) = t$ be the hyperplane orthogonal to ξ at the distance t from the origin. Define the *parallel section function* of a star body K in the direction of ξ by

$$A_{K,\xi}(t) = \mathrm{vol}_{n-1}(K \cap \{(x,\xi) = t\}), \quad t \in \mathbb{R}.$$

Let f be an integrable continuous function on $\mathbb{R}$, m-times continuously differentiable in some neighborhood of zero, $m \in \mathbb{N}$. For a number $q \in (m-1, m)$ the *fractional derivative* of the order q of the function f at zero is defined as follows [Ko11, Section 2.5]:

$$f^{(q)}(0) = \frac{1}{\Gamma(-q)} \int_0^\infty t^{-1-q}\Big(f(t) - f(0) - tf'(0) - \cdots - \\ - \frac{t^{m-1}}{(m-1)!}f^{(m-1)}(0)\Big)dt.$$

Note that without dividing by $\Gamma(-q)$ the expression for the fractional derivative represents an analytic function in the domain $\{q \in \mathbb{C}, -1 < \mathrm{Re}\ q < m\}$ not including integers and has simple poles at non-negative integers. The function

$\Gamma(-q)$ is analytic in the same domain and also has simple poles at non-negative integers. Therefore, after division we get an analytic function in the whole domain $\{q \in \mathbb{C}, -1 < \text{Re } q < m\}$, which also defines fractional derivatives of integer orders. Moreover, computing the limit as $q \to k$, where k is a non-negative integer and $k < m$, we see that the fractional derivatives of integer orders coincide with usual derivatives up to a sign:

$$f^{(k)}(0) = (-1)^k \frac{d^k}{dt^k} f(t)|_{t=0}.$$

More details on fractional derivatives may be found in [Ko11, Section 2.6].

For $\xi \in S^{n-1}$, consider a function $A_{K,\xi,p}$ on $\mathbb{R}$

$$A_{K,\xi,p}(t) = \int_{K \cap \langle x,\xi \rangle = t} |x|_2^{-p} dx,$$

where $p < n - 1$.

Chapter 2

Non-intersection bodies all of whose central sections are intersection bodies.

2.1 Introduction

The concept of an intersection body was introduced by Lutwak [Lu1] in 1988. Let K and L be origin symmetric star bodies in $\mathbb{R}^n$. Following [Lu1], we say that K is the *intersection body of* L if the radius of K in every direction is equal to the volume of the central hyperplane section of L perpendicular to this direction, i.e. for every $\xi \in S^{n-1}$,

$$\|\xi\|_K^{-1} = \mathrm{vol}_{n-1}(L \cap \xi^\perp).$$

The closure in the radial metric of the class of intersection bodies of star bodies gives the class of *intersection bodies*.

Intersection bodies played an important role in the solution of the Busemann-Petty problem (see [Ko11, Chapter 5] for the solution and [Ko11, Chapter 1] for the historical details). Posed in 1956, [BP], the Busemann-Petty problem asks the following. Let K and L be two origin-symmetric convex bodies in $\mathbb{R}^n$ so that the $(n-1)$-dimensional volume of every central hyperplane section of K is smaller that the same for L. Does it follow that the n-dimensional volume of K is smaller than the n-dimensional volume of L? The answer turns out to be affirmative for dimensions $n \leq 4$ and negative for $n \geq 5$.

The connection between intersection bodies and the Busemann-Petty problem was found by Lutwak [Lu1]. First, the answer to the problem is affirmative if K is an intersection body and L is any origin-symmetric star body. On the other hand, if L is an origin-symmetric convex body that is not an intersection body, one can perturb L to construct a counterexample to the Busemann-Petty problem. Hence, a solution to the problem in $\mathbb{R}^n$ is affirmative if and only if every infinitely smooth origin-symmetric convex body in $\mathbb{R}^n$ is an intersection body, which is the case for dimensions $n \leq 4$. Examples of non-intersection bodies in dimensions 5 and higher were constructed in [Ga2], [Zh1], [GKS], [Ko6].

In this chapter we present a result from [Ya]. We are interested in the following problem. Does there exist a convex body K that is not an intersection body, but every its section by a central hyperplane is an intersection body? We construct an example of such a body for dimensions $n \geq 5$. Our result can also be considered as a new way of constructing non-intersection bodies.

This question was motivated by results of W.Weil [W] and A.Neyman [N]. In 1982 W.Weil, [W], showed that it is not possible to characterize zonoids by means of their projections. He constructed a convex body in $\mathbb{R}^n$ $(n \geq 3)$ that is not a zonoid but all its projections onto hyperplanes are zonoids. A.Neyman in [N] showed that there are n-dimensional normed spaces that do not embed in L_p, but all their $(n-1)$-dimensional subspaces embed in L_p for $p > 0$. He used this to prove that for $p > 0$, $p \neq 2$, L_p is not characterized by a finite number of equations. Let us note that M.Burger in [Bu] used another approach to show that, for $n \geq 3$, zonoids cannot be characterized by a finite number of piecewise inequalities. A.Koldobsky in [Ko7] introduced the concept of embedding of a normed spaces in L_p, $p < 0$, and proved that intersection bodies are the unit balls of spaces that embed in L_{-1}. Therefore, our result can be considered as an extension of Neyman's example to negative p.

2.2 Main Results

The main result of this chapter is the following

Theorem 2.2.1. *There exists a convex body K in $\mathbb{R}^n$, $n \geq 5$, that is not an intersection body, but for every $(n-1)$-dimensional subspace V of $\mathbb{R}^n$, $K \cap V$ is an intersection body.*

We will use Lemma 3.16 from [Ko11, p.58]. It states the following:

Lemma 2.2.2. *Let $k \in \mathbb{N} \cup \{0\}$ and $f \in C^{2k}(S^{n-1})$, f is even, $q \leq 2k$, q is not an odd integer. Let $x = (r, \theta)$ be polar coordinates in $\mathbb{R}^n$, so that $f_\varepsilon(\theta) r^p = f_\varepsilon\left(\frac{x}{|x|_2}\right)|x|_2^{-p}$. Then:*

1. *The Fourirer transform of the distribution $f(\theta)r^{-n+q+1}$ is a homogeneous of degree $-1-q$ continuous on $\mathbb{R}^n \setminus \{0\}$ function. If $q < 2k$ then for every*

$x \in \mathbb{R}^n$,

$$|x|_2^{2k}\left(f(\theta)r^{-n+q+1}\right)^\wedge(x) = \frac{(-1)^k\pi}{-2\Gamma(2k-q)\sin(\pi(2k-q-1)/2)}$$

$$\times \int_{S^{n-1}} |(x,\xi)|^{2k-q-1}\Delta^k\left(f(\theta)r^{-n+q+1}\right)(\xi)d\xi.$$

If $q = 2k$ then

$$\left(f(\theta)r^{-n+q+1}\right)^\wedge(x) = (-1)^k\pi|x|_2^{-1-2k}$$

$$\times \int_{S^{n-1}\cap(x/|x|_2)^\perp} \Delta^k\left(f(\theta)r^{-n+q+1}\right)(\xi)d\xi,$$

where Δ is the Laplace operator in $\mathbb{R}^n$.

2. *If $f \in C^\infty(S^{n-1})$ then there exist an even function $g \in C^\infty(S^{n-1})$ so that for every $x = t\xi \in \mathbb{R}^n$, $t \neq 0$, $\xi \in S^{n-1}$,*

$$\left(f(\theta)r^{-n+q+1}\right)^\wedge(x) = g(\xi)t^{-1-q},$$

so the Fourier transform of $f(\theta)r^{-n+q+1}$ is an infinitely smooth function on $\mathbb{R}^n \setminus \{0\}$.

Fix a point x_0 on the unit sphere S^{n-1}. Define a function f_ε as follows:

$$f_\varepsilon(x) = \begin{cases} 2e^{-\frac{|x-x_0|^2}{\varepsilon^2-|x-x_0|^2}} & \text{if } |x-x_0| < \varepsilon \\ 2e^{-\frac{|x+x_0|^2}{\varepsilon^2-|x+x_0|^2}} & \text{if } |x+x_0| < \varepsilon \\ 0 & \text{otherwise.} \end{cases}$$

Clearly f_ε is an infinitely differentiable function. Define a body K by

$$\|x\|_K^{-1} = \left((1-f_\varepsilon(\theta))r^{-n+1}\right)^\wedge(x), \quad x \in \mathbb{R}^n \setminus \{0\}. \tag{2.1}$$

The function $\|x\|_K^{-1}$ is infinitely smooth on $\mathbb{R}^n \setminus \{0\}$ by Lemma 2.2.2. It will be shown in Lemma 2.2.4 that $\left(f_\varepsilon(\theta)r^{-n+1}\right)^\wedge(x)$ is of the order ε^{n-2} uniformly with respect to $x \in S^{n-1}$ and the Fourier transform of r^{-n+1} is strictly greater then zero, therefore $\|x\|_K$ is positive for a small ε.

Lemma 2.2.3. *For any $\varepsilon > 0$, K is not an intersection body.*

Proof. Since f_ε is an even function, we have

$$\left(\|x\|_K^{-1}\right)^\wedge = (2\pi)^n (1 - f_\varepsilon(\theta)) r^{-n+1},$$

which is negative for θ in some neighborhood of x_0. Therefore, by [Ko5, Theorem 1], K is not an intersection body.

□

Lemma 2.2.4. *There exist numbers D_1, D_2, D_3 that does not depend on ε so that for every $x \in S^{n-1}$*

$$\left| \left(f_\varepsilon(\theta) r^{-n+1}\right)^\wedge (x) \right| \leq D_1 \varepsilon^{n-2}$$

$$\left| \frac{\partial}{\partial x_i} \left(f_\varepsilon(\theta) r^{-n+1}\right)^\wedge (x) \right| \leq D_2 |\ln \varepsilon| \cdot \varepsilon^{n-3}$$

and

$$\left| \frac{\partial^2}{\partial x_i \partial x_j} \left(f_\varepsilon(\theta) r^{-n+1}\right)^\wedge (x) \right| \leq D_3 \varepsilon^{n-4}.$$

Proof. First we show the estimate for the second derivative. Using the connection between the Fourier transform and differentiation and Lemma 2.2.2 with $q = 2$ and $k = 1$, we get

$$\frac{\partial^2}{\partial x_i x_j} \left(f_\varepsilon(\theta) r^{-n+1}\right)^\wedge (x) = - \left(f_\varepsilon \left(\frac{y}{|y|} \right) y_i y_j |y|^{-n+1} \right)^\wedge (x)$$

$$= -\left(f_\varepsilon\left(\frac{y}{|y|}\right)\frac{y_i y_j}{|y|^2}|y|^{-n+3}\right)^\wedge(x) = -\left(g_\varepsilon(\theta)|y|^{-n+3}\right)^\wedge(x)$$

$$= \pi|x|_2^{-3}\int_{S^{n-1}\cap(x/|x|_2)^\perp} \triangle\left(g_\varepsilon(\theta)|y|^{-n+3}\right)(\xi)d\xi, \tag{2.2}$$

where

$$g_\varepsilon(\theta) = f_\varepsilon\left(\frac{y}{|y|}\right)\frac{y_i y_j}{|y|^2}, \quad \theta = \frac{y}{|y|} \in S^{n-1}.$$

Note that the function $g_\varepsilon(\theta)$ is supported in $B_\epsilon(x_0)$ and $B_\epsilon(-x_0)$, where

$$B_\epsilon(x_0) = \{x \in S^{n-1} : |x - x_0| < \varepsilon\}.$$

The volume of these balls is of the order ε^{n-1}.

Now we want to show that $\frac{\partial^2}{\partial x_i \partial x_j}\left(f_\varepsilon(\theta)r^{-n+1}\right)^\wedge(x)$ can be made as small as desirable if ε is small. First let us show that $\frac{\partial^2}{\partial y_i^2}\left(g_\varepsilon\left(\frac{y}{|y|}\right)\right)$ is of the order ε^{-2}.

To prove this, consider the function

$$f_1(x) = \begin{cases} 2e^{-\frac{|x-x_0|^2}{1-|x-x_0|^2}} & \text{if } |x - x_0| < 1 \\ 2e^{-\frac{|x+x_0|^2}{1-|x+x_0|^2}} & \text{if } |x + x_0| < 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $x \in S^{n-1}$.

The function f_1 is infinitely differentiable, so all its derivatives are bounded. Also $f_1\left(\frac{x/|x| - x_0}{\varepsilon} + x_0\right) = f_\varepsilon(x/|x|)$. Therefore,

$$\left|\frac{\partial^k}{\partial x_i^k} f_\varepsilon(x/|x|)\right| = \left|\frac{\partial^k}{\partial x_i^k} f_1\left(\frac{x/|x| - x_0}{\varepsilon} + x_0\right)\right| \le C_k\varepsilon^{-k},$$

where C_k depends on k but not on x.

The same is true for the derivatives of $g_\varepsilon(\theta) = f_\varepsilon\left(\frac{y}{|y|}\right)\frac{y_i^2}{|y|^2}$, i.e.

$$\left|\frac{\partial^k}{\partial y_i^k}\, g_\varepsilon\left(\frac{y}{|y|}\right)\right| \le \tilde{C}_k\varepsilon^{-k}.$$

So $\triangle\left(g_\varepsilon(\theta)|y|^{-n+3}\right)(\xi)$ is also of the order ε^{-2} since the main term when ε is small is $\dfrac{\partial^2}{\partial y_i^2}\left(g_\varepsilon\left(\dfrac{y}{|y|}\right)\right)|y|^{-n+3}$.

Therefore,

$$\left|\int_{S^{n-1}\cap(x/|x|_2)^\perp}\triangle\left(g_\varepsilon(\theta)|y|^{-n+3}\right)(\xi)d\xi\right|\le$$

$$\le\tilde{C}_2\sup\left|\triangle\left(g_\varepsilon(\theta)|y|^{-n+3}\right)\right|\cdot\int_{(B_\varepsilon(x_0)\cup B_\varepsilon(-x_0))\cap(x/|x|_2)^\perp}d\xi$$

$$=O(\varepsilon^{-2}\varepsilon^{n-2})=O(\varepsilon^{n-4}),$$

since the volume of the balls $B_\varepsilon(x_0)$ and $B_\varepsilon(-x_0)$ is of the order ε^{n-1} and the volume of their intersection with a hyperplane is of the order ε^{n-2}.

So, using equality (2.2) we get $\dfrac{\partial^2}{\partial x_i\partial x_j}\left(f_\varepsilon(\theta)r^{-n+1}\right)^\wedge(x)=O(\varepsilon^{n-4})$ and for $n>4$ it can be made as small as desirable uniformly with respect to $x\in S^{n-1}$.

Using the same argument, we prove that $\left(f_\varepsilon(\theta)r^{-n+1}\right)^\wedge(x)=O(\varepsilon^{n-2})$.

To get the estimate for $\dfrac{\partial}{\partial x_i}\left(f_\varepsilon(\theta)r^{-n+1}\right)^\wedge(x)$ we again use Lemma 2.2.2 and the connection between the Fourier transform and differentiation. Take α close to 1. Then, if $g_\varepsilon(\theta)=f_\varepsilon\left(\dfrac{y}{|y|}\right)\dfrac{y_i}{|y|},\quad\theta=\dfrac{y}{|y|}\in S^{n-1}$,

$$\frac{\partial}{\partial x_i}\left(f_\varepsilon(\theta)r^{-n+\alpha+1}\right)^\wedge(x)=-\left(g_\varepsilon(\theta)|y|^{-n+\alpha+2}\right)^\wedge(x)$$

$$=\frac{-\pi|x|_2^{-2}\int_{S^{n-1}}|(x,\xi)|^{1-\alpha}\Delta(g_\varepsilon(\theta)|y|^{-n+\alpha+1})(\xi)d\xi}{2\Gamma(2-\alpha)\sin\frac{\pi(1-\alpha)}{2}}.\tag{2.3}$$

When α approaches 1, the numerator and denominator in the right hand side approach zero. Indeed, let us show that the limit of the numerator is zero:

$$\lim_{\alpha\to1}\int_{S^{n-1}}|(x,\xi)|^{1-\alpha}\Delta(g_\varepsilon(\theta)|y|^{-n+\alpha+1})(\xi)d\xi=\int_{S^{n-1}}\Delta(g_\varepsilon(\theta)|y|^{-n+2})(\xi)d\xi.$$

Recall the relation between the spherical Laplacian Δ_S and Euclidean Laplacian Δ (see, for example, [Gr, p.7]): if f is a homogeneous function of degree m, then on the sphere

$$\Delta_S f_1 = \Delta f_1 - m(m+n-2)f_1.$$

Since $g_\varepsilon(\theta)|y|^{-n+2}$ has degree of homogeneity $-n+2$, the previous formula implies $\Delta(g_\varepsilon(\theta)|y|^{-n+2})(\xi) = \Delta_S(g_\varepsilon(\theta)|y|^{-n+2})(\xi)$. Due to the fact that Δ_S is a self-adjoint operator, [Gr, p.7], we have

$$\int_{S^{n-1}} \Delta_S(g_\varepsilon(\theta)|y|^{-n+2})(\xi)d\xi = 0.$$

Now to compute the limit of (2.3) as $\alpha \to 0$, apply l'Hopital's rule:

$$\frac{\partial}{\partial x_i}\left(f_\varepsilon(\theta)r^{-n+2}\right)^\wedge(x) \quad = \quad |x|_2^{-2}\int_{S^{n-1}} \ln|(x,\xi)|\Delta(g_\varepsilon(\theta)|y|^{-n+2})(\xi)d\xi.$$

Recall that the function $g_\varepsilon(\theta)$ is supported in the balls $B_\varepsilon(x_0)$ and $B_\varepsilon(-x_0)$. Then

$$\left|\int_{S^{n-1}} \ln|(x,\xi)|\Delta(g_\varepsilon(\theta)|y|^{-n+2})(\xi)d\xi\right| \le \sup\left|\Delta(g_\varepsilon(\theta)|y|^{-n+2})(\xi)\right| \times$$

$$\times \int_{B_\varepsilon(x_0)\cup B_\varepsilon(-x_0)} |\ln|(x,\xi)||\,d\xi.$$

Now we want to estimate the latter integral. Note that it is enough to estimate just $\int_{B_\varepsilon(x_0)} |\ln|(x,\xi)||\,d\xi$. Consider two cases. First, suppose that x is not perpendicular to any $y \in B_{2\varepsilon}(x_0)$. In this case one can check that $|(x,\xi)| > \varepsilon/2$ and therefore

$$\int_{B_\varepsilon(x_0)} |\ln|(x,\xi)||\,d\xi \le |\ln(\varepsilon/2)|\cdot \mathrm{vol}(B_\varepsilon(x_0)) = O(|\ln\varepsilon|\cdot\varepsilon^{n-1})$$

since the volume of the ball $B_\varepsilon(x_0)$ is of the order ε^{n-1}.

In the second case there exists $y \in B_{2\varepsilon}(x_0)$ such that $x \perp y$. Consider the ball $B_{4\varepsilon}(y)$. Clearly, $B_{2\varepsilon}(x_0) \subset B_{4\varepsilon}(y)$, therefore

$$\int_{B_\varepsilon(x_0)} |\ln |(x, \xi)|| \, d\xi \leq \int_{B_{4\varepsilon}(y)} |\ln |(x, \xi)|| \, d\xi.$$

Let us make a change of coordinates from $\xi \in S^{n-1}$ to $\zeta \in S^{n-2}$ and $t \in [-\pi/2, \pi/2]$ such that $\xi = y\sqrt{1-t^2} + \zeta t$. The Jacobian is equal to $\dfrac{t^{n-2}}{\sqrt{1-t^2}}$. If $\xi \in B_{4\varepsilon}(y)$ then $t \in [0, 4\varepsilon\sqrt{1-4\varepsilon^2}] \subset [0, 4\varepsilon]$. Using the fact that x is perpendicular to y, we get

$$\begin{aligned}
\int_{B_{4\varepsilon}(y)} |\ln |(x, \xi)|| \, d\xi &\leq \int_{S^{n-2}} \int_0^{4\epsilon} |\ln |(x, \zeta) t|| \, \frac{t^{n-2}}{\sqrt{1-t^2}} dt d\zeta \\
= \int_{S^{n-2}} &\left(\int_0^{4\epsilon} |\ln |(x, \zeta)|| \, \frac{t^{n-2}}{\sqrt{1-t^2}} dt + \int_0^{4\epsilon} |\ln t| \, \frac{t^{n-2}}{\sqrt{1-t^2}} dt \right) d\xi \\
&\leq 2 \int_{S^{n-2}} \left(\int_0^{4\epsilon} |\ln |(x, \zeta)|| \, t^{n-2} dt + \int_0^{4\epsilon} |\ln t| \, t^{n-2} dt \right) d\xi
\end{aligned}$$

for a small ε. The first integral can be estimated in the following way:

$$\int_{S^{n-2}} |\ln |(x, \zeta)|| \, d\xi \int_0^{4\epsilon} t^{n-2} dt = O(\varepsilon^{n-1}).$$

Using integration by parts in the second integral, we get that

$$\begin{aligned}
\int_0^{4\epsilon} |\ln t| \, t^{n-2} dt &= \frac{1}{n-1} \ln t \cdot t^{n-1} \Big|_0^{4\varepsilon} - \frac{1}{n-1} \int_0^{4\epsilon} t^{n-2} dt \\
&= O(|\ln \varepsilon| \cdot \varepsilon^{n-1}).
\end{aligned}$$

Therefore,

$$\int_{B_\varepsilon(x_0) \cup B_\varepsilon(-x_0)} |\ln |(x, \xi)|| \, d\xi = O(|\ln \varepsilon| \cdot \varepsilon^{n-1})$$

and

$$\frac{\partial}{\partial x_i}\left(f_\varepsilon(\theta)r^{-n+\alpha+1}\right)^\wedge(x) = O(|\ln\varepsilon|\cdot\varepsilon^{n-3}).$$

□

Lemma 2.2.5. *If $n \geq 5$, the body K is convex for small enough ε.*

Proof. Geometrically the proof is based on the idea of adding in radial metric of an arbitrary small function to a radial function of the body which is uniformly convex, i.e. the curvature is a constant bounded from zero. In our case this body is the Euclidean ball. So, due to this property of a uniformly convex body, we can always choose a function in such a way that the body we get is still convex.

Let $|\cdot|_2$ be the Euclidean norm. By [GS, p.363] the Fourier transform of $|x|_2^q$, $q \in (-n, 0)$ equals

$$(|x|_2^q)^\wedge(t) = 2^{q+n}\pi^{n/2}\frac{\Gamma(\frac{q+n}{2})}{\Gamma(\frac{-q}{2})}|t|_2^{-n-q}.$$

Using this formula and the definition of the body K

$$\begin{aligned}\|x\|_K^{-1} &= \left((1-f_\varepsilon(\theta))r^{-n+1}\right)^\wedge(x)\\ &= C_n|x|_2^{-1} - \left(f_\varepsilon(\theta)r^{-n+1}\right)^\wedge(x), \qquad (2.4)\end{aligned}$$

where $C_n = \dfrac{2\pi^{(n+1)/2}}{\Gamma(\frac{n-1}{2})}$.

Let K_W be the section of K by a 2-dimensional central plane W with an orthonormal basis ξ_1, ξ_2. So, if $x \in W \cap S^{n-1}$, then $x = \xi_1\cos\phi + \xi_2\sin\phi$, $\phi \in [0, 2\pi]$. To show that K is convex, it is enough to show that K_W is convex for any W.

Consider a function

$$\rho_W(\phi) = \left(f_\varepsilon(\theta)r^{-n+1}\right)^\wedge(\xi_1\cos\phi + \xi_2\sin\phi).$$

By the definition of K, the radial function of K_W is given by

$$\rho(\phi) = C_n - \rho_W(\phi).$$

To prove that K_W is convex, we need to show that for small ε

$$J(W, \varepsilon, \phi) = 2(\rho')^2 - \rho''\rho + \rho^2 > 0$$

for every W and ϕ, see [Ga3, p.25].

Computing the derivatives,

$$\begin{aligned}
\rho'(\phi) &= -\frac{d}{d\phi}\left(\rho_W(\phi)\right) \\
\rho''(\phi) &= -\frac{d^2}{d^2\phi}\left(\rho_W(\phi)\right).
\end{aligned}$$

To estimate ρ' and ρ'', consider

$$\begin{aligned}
\left|\frac{d}{d\phi}\left(\rho_W(\phi)\right)\right| &= \left|\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(f_\varepsilon(\theta) r^{-n+1}\right)^\wedge(x)\right| \left|\frac{dx_i}{d\phi}\right| \\
&\leq 2\left|\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(f_\varepsilon(\theta) r^{-n+1}\right)^\wedge(x)\right|,
\end{aligned}$$

since $x = \xi_1 \cos\phi + \xi_2 \sin\phi$ and

$$\left|\frac{dx_i}{d\phi}\right| = |-\xi_{1,i} \sin\phi + \xi_{2,i}\cos\phi| \leq 2.$$

Similarly,

$$\left|\frac{d^2}{d\phi^2}\left(\rho_W(\phi)\right)\right| \leq 4\left|\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\left(f_\varepsilon(\theta) r^{-n+1}\right)^\wedge(x)\right|.$$

By Lemma 2.2.4 we have $\rho' = O(|\ln\varepsilon|\varepsilon^{n-3})$ and $\rho'' = O(\varepsilon^{n-4})$. Since these estimates are uniform with respect to ϕ and W, it follows that, for small enough ε, $J(W, \varepsilon, \phi) > 0$ for every ϕ and W. □

In [Ko4, Lemma 1] the following was proved. Let f be an even continuous homogeneous function of degree $-n+1$ on $\mathbb{R}^n \setminus \{0\}$. Then $\hat{f}$ is a continuous function outside of the origin and for every $\xi \in S^{n-1}$,

$$\hat{f}(\xi) = \pi \int_{S^{n-1}\cap\{(\theta,\xi)=0\}} f(\theta)d\theta. \tag{2.5}$$

Using this formula,

$$\left((1-f_\varepsilon(\theta))r^{-n+1}\right)^\wedge(x) = \pi \int_{S^{n-1}\cap\{(x,\theta)=0\}} (1-f_\varepsilon(\theta))d\theta. \tag{2.6}$$

Lemma 2.2.6. *For every $(n-1)$-dimensional subspace V of $\mathbb{R}^n$, the body $K \cap V$ is an intersection body.*

Proof. Fix an $(n-1)$-dimensional subspace V of $\mathbb{R}^n$. By the definition (2.1) of the body K and formula (2.6)

$$\frac{1}{\pi}\|x\|_K^{-1} = \int_{S^{n-1}\cap\{(x,\theta)=0\}} (1-f_\varepsilon(\theta))d\theta.$$

In particular, for $x \in V$,

$$\frac{1}{\pi}\|x\|_{K\cap V}^{-1} = \int_{S^{n-1}\cap\{(x,\theta)=0, x\in V\}} (1-f_\varepsilon(\theta))d\theta,$$

where $\{(x,\theta)=0, x \in V\}$ is the hyperplane of all θ that are perpendicular to x for a fixed $x \in V$.

Let us change coordinates from $\theta \in S^{n-2} = S^{n-1} \cap \{(x,\theta)=0, x \in V\}$ to $\phi \in [0,\pi]$ and $\xi \in S^{n-3} = S^{n-2} \cap \{(x,\theta)=0, x \in V\}$. The Jacobian is equal to $(\sin\phi)^{n-3}$, cf. [M, p.1]. So

$$\frac{1}{\pi}\|x\|_{K\cap V}^{-1} = \int_{S^{n-2}\cap\{(x,\theta)=0, x\in V\}} \left(\int_0^\pi (1-f_\varepsilon(\xi,\phi))(\sin\phi)^{n-3}d\phi\right) d\xi$$

$$= \int_{S^{n-2}\cap\{(x,\theta)=0, x\in V\}} \left(\int_0^\pi (\sin\phi)^{n-3} d\phi - \int_0^\pi f_\varepsilon(\xi,\phi)(\sin\phi)^{n-3} d\phi \right) d\xi.$$

Taking the Fourier transform of both sides, as functions of the variable $x \in V$, and using (2.5)

$$\left(\|x\|_{K\cap V}^{-1}\right)^\wedge(\theta) = \pi \int_0^\pi (\sin\phi)^{n-3} d\phi - \pi \int_0^\pi f_\varepsilon(\theta)(\sin\phi)^{n-3} d\phi$$

for $\theta \in S^{n-2}$.

By the definition, f_ε is non-zero only in an ε-neighborhood of x_0, so there exists a set $R_\varepsilon \subset [0,\pi]$ such that $f_\varepsilon(\xi,\phi) = 0$ for $\phi \in [0,\pi] \setminus R_\varepsilon$. Also, $|f_\varepsilon| \le 2$ and $|R_\varepsilon|$, the length of the one-dimensional set R_ε, is of the order ε. Therefore,

$$\begin{aligned} \int_0^\pi f_\varepsilon(\xi,\phi)(\sin\phi)^{n-3} d\phi &= \int_{R_\varepsilon} f_\varepsilon(\xi,\phi)(\sin\phi)^{n-3} d\phi \\ &\le 2|R_\varepsilon| = C\varepsilon, \end{aligned}$$

where C does not depend of the choice of V.

Since $\int_0^\pi (\sin\phi)^{n-3} d\phi$ is equal to some positive constant depending on n only,

$$\int_0^\pi (\sin\phi)^{n-3} d\phi - \int_0^\pi f_\varepsilon(\theta)(\sin\phi)^{n-3} d\phi > 0$$

for a sufficiently small ε, which means we can find ε small enough that $\left(\|x\|_{K\cap V}^{-1}\right)^\wedge(\theta) > 0$ for all θ. Therefore by [Ko5, Thm.1], for small enough ε, for every $(n-1)$-dimensional subspace V of $\mathbb{R}^n$, the body $K \cap V$ is an intersection body. $\square$

Now Theorem 2.2.1 follows from Lemmas 2.2.3, 2.2.5 and 2.2.6.

Remark 1. By a result of Neyman [N], for any finite system of equations and inequalities involving the norms, there exists a subspace of L_p, $0 < p < 2$ that does not satisfy this system. Since, by [Ko5], the unit ball of every finite dimensional subspace of L_p, $0 < p < 2$ is an intersection body, we conclude that the class of intersection bodies cannot be characterized by a finite number of equations or inequalities.

Remark 2. For $n = 5$ the fact that there exists a non-intersection body whose central sections are intersection bodies follows from the solution of the Busemann-Petty problem since every four-dimensional symmetric convex body is an intersection body, see [GKS], [Zh2].

Remark 3. It is known that the polar of zonoid is an intersection body (see [Ko11, p.127]), but the converse is not true and the polar of an intersection body may not be a zonoid, for example a cube in $\mathbb{R}^4$. Therefore, the result in this paper does not follow from the result of W.Weil on zonoids by applying duality argument and a new example should be constructed.

Chapter 3

The geometry of L_0.

3.1 Introduction

This chapter is based on the results from [KKYY]. Suppose that we have the unit Euclidean ball in $\mathbb{R}^n$ and are allowed to construct new bodies using three operations - linear tranformations, multiplicative summation and closure in the radial metric. The *multiplicative sum* $K +_0 L$ of star bodies K and L is defined by

$$\|x\|_{K+_0L} = \sqrt{\|x\|_K \|x\|_L}, \tag{3.1}$$

where $\|x\|_K = \min\{a \geq 0 : \ x \in aK\}$ is the Minkowski functional of a star body K. What class of bodies do we get from the unit ball by means of these three operations?

We are going to prove that in dimension $n = 3$ we get all origin-symmetric convex bodies, while in dimension 4 and higher this is no longer the case. However, the class of bodies that we get in arbitrary dimension also has a clear interpretation. We introduce the concept of embedding in L_0 and show that the bodies that we get by means of these three operations are exactly the unit balls of spaces that embed in L_0.

The idea of this interpretation comes from a similar result for L_p-spaces with $p \in [-1,1],\ p \neq 0$. Namely, if we replace the multiplicative summation by p-summation

$$\|x\|_{K+_pL} = (\|x\|_K^p + \|x\|_L^p)^{1/p} \tag{3.2}$$

then we get the unit balls of all spaces that embed in L_p. The case $p = 1$ is well-known (see [Ga3, Corollary 4.1.12]) and the unit balls of subspaces of L_1 have a clear geometric meaning - these are the polar projection bodies (see [Bo]). On the other hand, it was proved by Goodey and Weil [GW] that if $p = -1$ (this case corresponds to the radial summation) then we get the class of intersection bodies in $\mathbb{R}^n$. As shown in [Ko8], intersection bodies are the unit balls of spaces that embed in L_{-1}. The concept of embedding in $L_p,\ p < 0$ was introduced in [Ko7] as an analytic extension of the same property for $p > 0$, see [KK2] for related results. The result of Goodey and Weil can easily be extended to $p \in (-1, 1),\ p \neq 0$. Note that this construction provides a continuous (except for $p = 0$) path from polar projection bodies to intersection bodies, which is important for understanding the duality between projections and sections of convex bodies. One of the goals of this article is to fill the gap in this scheme at $p = 0$ and better understand the geometry of this intermediate case.

Another interesting similarity of our result with other values of p is that for $p = 1$ the procedure defined above gives all origin-symmetric convex bodies only in dimension 2. This follows from a result of Schneider [Sc] that every origin-symmetric convex body is a polar projection body only in dimension 2. When

$p = -1$ we get all origin-symmetric convex bodies only in dimensions 4 and lower, because, by results from [Ga1], [Zh2], [GKS], only in these dimensions every origin-symmetric convex body is an intersection body. The transition between the dimensions 2 and 3 in the case $p = 1$ and the transition between the dimensions 4 and 5 in the case $p = -1$ directly correspond to the transition between the affirmative and negative answers in the Shephard and Busemann-Petty problems, respectively. It would be interesting to find a similar geometric result corresponding to the transition between dimensions 3 and 4 in the case $p = 0$. We refer the reader to the book [Ko11, Chapter 6] for more details and history of the connection between convex geometry and the theory of L_p-spaces.

3.2 The definition of embedding in L_0.

A well-known result of P.Lévy, see [BL, p. 189] or [Ko11, Section 6.1], is that a space $(\mathbb{R}^n, \|\cdot\|)$ embeds into L_p, $p > 0$ if and only if there exists a finite Borel measure μ on the unit sphere so that, for every $x \in \mathbb{R}^n$,

$$\|x\|^p = \int_{S^{n-1}} |(x, \xi)|^p d\mu(\xi). \tag{3.3}$$

On the other hand, the definition of embedding in L_p with $p < 0$ from [Ko7] implies that a space $(\mathbb{R}^n, \|\cdot\|)$ embeds into L_p, $p \in (-n, 0)$ if and only if there exists a finite symmetric measure μ on the sphere S^{n-1} so that for every test function ϕ,

$$\int_{\mathbb{R}^n} \|x\|^p \phi(x) dx = \int_{S^{n-1}} d\mu(\xi) \int_{\mathbb{R}} |t|^{-p-1} \hat{\phi}(t\xi) dt. \tag{3.4}$$

Both representations (5.6) and (3.4) are invariant with respect to p-summation. This gives an idea of defining embedding in L_0 by means of a representation that is invariant with respect to multiplicative summation. Note that the multiplicative summation is the limiting case of p-summation as $p \to 0$.

Definition 3.2.1. *We say that a space* $(\mathbb{R}^n, \|\cdot\|)$ *embeds in* L_0 *if there exist a finite Borel measure* μ *on the sphere* S^{n-1} *and a constant* $C \in \mathbb{R}$ *so that, for every* $x \in \mathbb{R}^n$,

$$\ln \|x\| = \int_{S^{n-1}} \ln |(x, \xi)| d\mu(\xi) + C. \tag{3.5}$$

While being similar to (5.6) and (3.4), this definition has its unique features. First, the measure μ must be a probability measure on S^{n-1}. In fact, put $x = ky$, $k > 0$ in (5.19). Then

$$\ln k + \ln \|y\| = \int_{S^{n-1}} \ln k \; d\mu(\xi) + \int_{S^{n-1}} \ln |(y, \xi)| d\mu(\xi) + C$$

and, again by (5.19) with $x = y$, we get $\ln k = \int_{S^{n-1}} \ln k \; d\mu(\xi)$, so $\int_{S^{n-1}} d\mu(\xi) = 1$.

Secondly, the constant C depends on the norm and can be computed precisely. In order to compute this constant, integrate the equality (5.19) over the uniform measure on the unit sphere. We get

$$\begin{aligned} C \cdot |S^{n-1}| &= \int_{S^{n-1}} \ln \|x\| dx - \int_{S^{n-1}} \int_{S^{n-1}} \ln |(x, \theta)| \, d\mu(\theta) dx \\ &= \int_{S^{n-1}} \ln \|x\| dx - \int_{S^{n-1}} \int_{S^{n-1}} \ln |(x, \theta)| \, dx \, d\mu(\theta) \\ &= \int_{S^{n-1}} \ln \|x\| dx - \int_{S^{n-1}} \ln |(x, \theta)| \, dx, \end{aligned}$$

since $\int_{S^{n-1}} \ln |(x, \theta)| \, dx$ is rotationally invariant and, therefore, is a constant for $\theta \in S^{n-1}$, and μ is a probability measure.

To compute the latter integral, we are going to use the well-known formula (see

[Ko11, Section 6.4])

$$\int_{S^{n-1}} |(x,\theta)|^p \, dx = \frac{2\pi^{(n-1)/2}\Gamma((p+1)/2)}{\Gamma((n+p)/2)}.$$

Differentiating with respect to p and letting $p = 0$ we get

$$\int_{S^{n-1}} \ln|(x,\theta)| \, dx = \pi^{(n-1)/2} \left[\frac{\Gamma'(1/2)}{\Gamma(n/2)} - \sqrt{\pi}\frac{\Gamma'(n/2)}{\Gamma^2(n/2)} \right]$$

Note that

$$|S^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)},$$

so

$$C = \frac{1}{|S^{n-1}|}\int_{S^{n-1}} \ln\|x\| dx - \frac{1}{2\sqrt{\pi}}\Gamma'(1/2) + \frac{1}{2}\frac{\Gamma'(n/2)}{\Gamma(n/2)}.$$

Let us remark that Definition 3.2.1 is equivalent to the following. A finite-dimensional normed space $X = (\mathbb{R}^n, \|\cdot\|)$ embeds into L_0 if and only if there is a probability space (Ω, μ) and a linear map $T : X \to \mathcal{M}(\Omega, \mu)$ (where $\mathcal{M}(\Omega, \mu)$ denotes the space of $\mu-$measurable functions on Ω) such that

$$\int_\Omega \ln|Tx(\omega)| d\mu(\omega) < \infty, \qquad x \in X$$

and

$$\ln\|x\| = \int_\Omega \ln|Tx(\omega)| d\mu(\omega), \qquad x \in X.$$

Indeed if such an operator T exists we can write it in the form

$$Tx(\omega) = h(\omega)(x, \xi(\omega)), \qquad x \in X$$

where $h : \Omega \to \mathbb{R}^+$ and $\xi : \Omega \to S^{n-1}$ are measurable. Then for each $\omega \in \Omega$

$$\int_{S^{n-1}} \ln|(x, \xi(\omega))| dx > -\infty$$

so that it follows for some $x \in S^{n-1}$, $\omega \to \ln |(x, \xi(\omega)|$ is $\mu-$integrable. Hence so is $\ln h$ and further

$$\ln \|x\| = \int \ln h(\omega)\, d\mu(\omega) + \int \ln |(x, \xi(\omega))| d\mu(\omega).$$

Now we can induce a probability measure μ' on S^{n-1} by $\mu'(B) = \mu\{\omega : \xi(\omega) \in B\}$ and we have the same situation as Definition 2.1.

On the other hand, if X satisfies Definition 3.2.1, we may take $\Omega = S^{n-1}$ and μ is a probability measure. If we define $Tx(\xi) = e^C(x, \xi)$ then $T : X \to \mathcal{M}(S^{n-1}, \mu)$ satisfies our conditions.

One advantage of this viewpoint is that we can make sense of the statement that an infinite-dimensional Banach space embeds into L_0.

3.3 A Fourier analytic characterization of subspaces of L_0

The fact that the Fourier transform is useful in the study of subspaces of L_p has been known for a long time. A well-known result of P.Levy is that a finite dimensional normed space $(\mathbb{R}^n, \|\cdot\|)$ embeds isometrically in $L_p,\ 0 < p \leq 2$ if and only if $\exp(-\|\cdot\|^p)$ is a positive definite function on $\mathbb{R}^n$. It was proved in [Ko2] that a space $(\mathbb{R}^n, \|\cdot\|)$ embeds isometrically in $L_p,\ p > 0, p \notin 2\mathbb{N}$ if and only if the Fourier transform of the function $\Gamma(-p/2)\|x\|^p$ (in the sense of distributions) is a positive distribution outside of the origin. If $-n < p < 0$ a similar fact was proved in [Ko7]: a space $(\mathbb{R}^n, \|\cdot\|)$ embeds in L_p if and only if the Fourier transform of $\|\cdot\|^p$ is a positive distribution in the whole $\mathbb{R}^n$. These characterizations have proved to be useful in the study of subspaces of L_p and intersection bodies, see [Ko11, Chapter 6]. In this section we prove a similar characterization of spaces that embed in L_0.

Theorem 3.3.1. *Let K be an origin symmetric star body in $\mathbb{R}^n$. The space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0 if and only if the Fourier transform of $\ln\|x\|_K$ is a negative distribution outside of the origin in $\mathbb{R}^n$.*

Proof. First, assume that $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0. Let ϕ be a non-negative even test function with compact support outside of the origin. By the definition of embedding in L_0, formula (1.2)(note that $\hat{\hat{\phi}} = (2\pi)^n\phi$ for even ϕ) and the Fubini theorem,

$$\begin{aligned}
\langle (\ln\|x\|)^\wedge, \phi\rangle &= \langle \ln\|x\|, \hat{\phi}(x)\rangle \\
&= \int_{S^{n-1}}\int_{\mathbb{R}^n} \ln|(x,\xi)|\hat{\phi}(x)\; dx\; d\mu(\xi) + C\int_{\mathbb{R}^n}\hat{\phi}(x)dx \\
&= \int_{S^{n-1}} \left\langle \ln|t|, \int_{(x,\xi)=t}\hat{\phi}(x)\; dx\right\rangle d\mu(\xi) \\
&= (2\pi)^{-1}\int_{S^{n-1}} \left\langle (\ln|t|)^\wedge(z), \left(\int_{(x,\xi)=t}\hat{\phi}(x)\; dx\right)^\wedge(z)\right\rangle d\mu(\xi) \\
&= (2\pi)^{n-1}\int_{S^{n-1}}\int_{\mathbb{R}} (\ln|t|)^\wedge(z)\phi(z\xi)\; dz\; d\mu(\xi) \qquad (3.6)
\end{aligned}$$

since $\int_{\mathbb{R}^n}\hat{\phi}(x)dx = (2\pi)^n\phi(0) = 0$. Now, the formula for the Fourier transform of $\ln|t|$ from [GS, p.362] implies that

$$(\ln|t|)^\wedge(z) = -\pi|z|^{-1} < 0 \qquad (3.7)$$

outside of the origin, so (3.6) is negative (recall that ϕ is non-negative with support outside of the origin). This means that $(\ln\|x\|)^\wedge$ is a negative distribution.

To prove the other direction, note that, by [Ko11, Section 2.6], a distribution that is positive outside of the origin coincides with a finite Borel measure on every set of the form

$$A \times (a,b) = \{x \in \mathbb{R}^n : x = t\theta, t \in (a,b), \theta \in A\},$$

where A is an open subset of S^{n-1} and $0 < a < b < \infty$.

Denote by $\mu = -(\ln \|x\|)^\wedge$. This distribution coincides with a finite Borel measure on each set $A \times (a,b)$, as above, so for any test function ϕ supported outside of the origin

$$\begin{aligned} \langle -(\ln \|x\|)^\wedge, \phi \rangle &= \langle \mu, \phi \rangle \\ &= \int_{\mathbb{R}^n} \phi(x) d\mu(x). \end{aligned} \tag{3.8}$$

Now for every test function ϕ with support outside of the origin and $t > 0$, we have $(\phi(x/t))^\wedge(z) = t^n \hat{\phi}(tz)$, so

$$\begin{aligned} \langle \mu(x), \phi(x/t) \rangle &= -\langle (\ln \|x\|)^\wedge(x), \phi(x/t) \rangle \\ &= -\int_{\mathbb{R}^n} \ln \|z\| \hat{\phi}(tz) t^n dz \\ &= -\int_{\mathbb{R}^n} \hat{\phi}(\tilde{x}) \ln \|\frac{1}{t}\tilde{x}\| d\tilde{x} \\ &= -\int_{\mathbb{R}^n} \hat{\phi}(\tilde{x}) \ln \|\tilde{x}\| d\tilde{x} + \ln |t| \int_{\mathbb{R}^n} \hat{\phi}(\tilde{x}) d\tilde{x} \\ &= -\int_{\mathbb{R}^n} \hat{\phi}(\tilde{x}) \ln \|\tilde{x}\| d\tilde{x} \\ &= \langle \mu(x), \phi(x) \rangle. \end{aligned} \tag{3.9}$$

Let $\chi_{A\times(a,b)}$ be the indicator of the set $A \times (a,b)$. Approximating $\chi_{A\times(a,b)}$ by test functions and using (3.9), we get for any $(a,b) \subset (0,\infty)$ and $A \subset S^{n-1}$

$$\begin{aligned} \mu(A \times (a,b)) &= \int_{\mathbb{R}^n} \chi_{A\times(a,b)}(x) d\mu(x) = \int_{\mathbb{R}^n} \chi_{A\times(1,b/a)}(x/a) d\mu(x) \\ &= \int_{\mathbb{R}^n} \chi_{A\times(1,b/a)}(x) d\mu(x) = \mu(A \times (1, b/a)). \end{aligned}$$

Applying this formula n times,

$$\mu\left(A \times (1, a^n)\right) = n\mu\left(A \times (1, a)\right) \tag{3.10}$$

for $n \in \mathbb{N}$. Moreover, we can extend formula (3.10) to $n \in \mathbb{R}$. So, for any $a \in (0, \infty)$, $A \subset S^{n-1}$

$$\mu\left(A \times [1, a]\right) = \mu\left(A \times [1, e^{\ln a}]\right) = \ln a \cdot \mu\left(A \times [1, e]\right)$$

Now for every $(a, b) \subset (0, \infty)$ and $A \subset S^{n-1}$ we have

$$\begin{aligned} \mu\left(A \times (a, b)\right) &= \mu\left(A \times (1, b/a)\right) \\ &= \ln\left(\frac{b}{a}\right)\mu\left(A \times (1, e)\right) \\ &= \left(\ln(b) - \ln(a)\right)\mu(A \times (1, e)). \end{aligned}$$

Define a measure μ_0 on S^{n-1} by

$$\mu_0(A) = \frac{\mu\left(A \times (a, b)\right)}{\left(\ln(b) - \ln(a)\right)} = \mu(A \times (1, e))$$

for every Borel set $A \subset S^{n-1}$. We have

$$\begin{aligned} \int_{S^{n-1}} d\mu_0(\theta) \int_0^\infty |t|^{-1} \chi_{A\times(a,b)}(t\theta)dt &= \left(\ln(b) - \ln(a)\right)\mu_0(A) \\ &= \mu(A \times (a, b)) \\ &= \int_{\mathbb{R}^n} \chi_{A\times(a,b)}(x)d\mu(x) \end{aligned} \tag{3.11}$$

Therefore, for an arbitrary even test function ϕ supported outside of the origin,

$$\begin{aligned} \langle \mu, \phi \rangle &= \int_{S^{n-1}} d\mu_0(\theta) \int_0^\infty |t|^{-1}\phi(t\theta)dt \\ &= \frac{1}{2}\int_{S^{n-1}} d\mu_0(\theta) \int_{\mathbb{R}} |t|^{-1}\phi(t\theta)dt \end{aligned} \tag{3.12}$$

since A, a, b are arbitrary in (3.11).

Using $\mu = -\left(\ln \|x\|\right)^\wedge$, we get

$$\langle \left(\ln \|x\|\right)^\wedge (\xi), \phi \rangle = -\frac{1}{2} \int_{S^{n-1}} d\mu_0(\theta) \int_{\mathbb{R}} |t|^{-1} \phi(t\theta) dt.$$

Define a new measure $\tilde{\mu}_0 = (2\pi)^n \mu_0$. By (5.23), (3.12) and the connection between the Fourier and Radon transforms

$$\begin{aligned}
\langle \ln \|x\|, \hat{\phi}(x) \rangle &= -\frac{1}{2(2\pi)^n} \int_{S^{n-1}} d\tilde{\mu}_0(\theta) \int_{\mathbb{R}} |t|^{-1} \phi(t\theta) dt \\
&= \int_{S^{n-1}} \langle \ln |z|, \mathcal{R}\hat{\phi}(\theta; z) \rangle d\tilde{\mu}_0(\theta) \\
&= \int_{S^{n-1}} d\tilde{\mu}_0(\theta) \int_{\mathbb{R}} \ln |z| \left(\int_{(x,\theta)=z} \hat{\phi}(x) dx \right) dz \\
&= \int_{S^{n-1}} d\tilde{\mu}_0(\theta) \int_{\mathbb{R}^n} \ln |(x,\theta)| \, \hat{\phi}(x) dx
\end{aligned}$$

Thus, we have proved that for any even test function ϕ supported outside of the origin

$$\langle (\ln \|x\|)^\wedge, \phi \rangle = \Big\langle \left(\int_{S^{n-1}} \ln |(x,\theta)| \, d\tilde{\mu}_0(\theta) \right)^\wedge, \phi \Big\rangle.$$

Therefore the distributions $\ln \|x\|$ and $\int_{S^{n-1}} \ln |(x,\theta)| \, d\tilde{\mu}_0(\theta)$ can differ only by a polynomial. Clearly, this polynomial cannot contain terms homogeneous of degree different from zero, so it is a constant.

□

Remark 3.3.2. *Let K be an infinitely smooth body. From the proof of the previous theorem it follows that the measure μ from Definition 3.2.1 is equal to restriction of the Fourier transform of $\ln \|x\|_K$ to the sphere. In the next section we are going to prove that this is a function, therefore*

$$d\mu(\xi) = -\frac{1}{(2\pi)^n} \left(\ln \|x\|_K\right)^\wedge (\xi) d\xi.$$

In particular, since μ is a probability measure, for any infinitely smooth body K we get

$$-\frac{1}{(2\pi)^n}\int_{S^{n-1}} (\ln\|x\|_K)^\wedge(\xi)d\xi = 1.$$

3.4 A geometric characterization of subspaces of L_0.

If K has an infinitely smooth boundary then the function $A_{K,\xi}(t)$ is an infinitely differentiable function of t in some neighborhood of zero and as was shown in [GKS] the fractional derivatives of $A_{K,\xi}(t)$ can be computed in terms of the Fourier transform of the Minkowski functional raised to certain powers. Namely, for $q \in \mathbb{C}$, $q \neq n-1$,

$$A^{(q)}_{K,\xi}(0) = \frac{\cos\frac{q\pi}{2}}{\pi(n-q-1)}\left(\|x\|_K^{-n+q+1}\right)^\wedge(\xi), \tag{3.13}$$

and, in particular, $\left(\|x\|_K^{-n+q+1}\right)^\wedge$ is a continuous function on $\mathbb{R}^n \setminus \{0\}$. Here we extend $A^{(q)}_{K,\xi}(0)$ from the sphere to the whole $\mathbb{R}^n$ as a homogeneous function of the variable ξ of degree $-q-1$. Note that $\langle A^{(q)}_{K,\xi}(0), \phi\rangle$ is an analytic function of q for any fixed test function ϕ.

Since the right-hand side of formula (3.13) is not defined for $q = n-1$, in our next Theorem we use a limiting argument to extend this formula to the case $q = n-1$.

Let $\mathcal{D}$ be an open set in $\mathbb{R}^n$, f, g two distributions. We say that $f = g$ on $\mathcal{D}$ if $\langle f, \phi\rangle = \langle g, \phi\rangle$ for any test function ϕ with compact support in $\mathcal{D}$.

Theorem 3.4.1. *Let K be an infinitely smooth origin symmetric star body in $\mathbb{R}^n$. Extend $A^{(n-1)}_{K,\xi}(0)$ to a homogeneous function of degree $-n$ of the variable*

$\xi \in \mathbb{R}^n \setminus \{0\}$. Then $(\ln \|\cdot\|_K)^\wedge$ is a continuous function on $\mathbb{R}^n \setminus \{0\}$ and

$$A_{K,\xi}^{(n-1)}(0) = -\frac{\cos(\pi(n-1)/2)}{\pi} (\ln \|\cdot\|_K)^\wedge (\xi), \tag{3.14}$$

as distributions (of the variable ξ) acting on test functions with compact support outside of the origin. In particular,

i) if n is odd

$$(\ln \|x\|_K)^\wedge (\xi) = (-1)^{(n+1)/2} \pi A_{K,\xi}^{(n-1)}(0), \quad \xi \in \mathbb{R}^n \setminus \{0\}$$

ii) if n is even, then for $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$(\ln \|x\|_K)^\wedge (\xi) = a_n \int_0^\infty \frac{A_\xi(z) - A_\xi(0) - A_\xi''(0)\frac{z^2}{2} - \dots - A_\xi^{n-2}(z)\frac{z^{n-2}}{(n-2)!}}{z^n} dz,$$

where $a_n = 2(-1)^{n/2+1}(n-1)!$

Proof. Let us start with the case where n is odd. Let ϕ be a test function supported outside of the origin.

Using formula (3.13) for q close to $n-1$, we have

$$\begin{aligned}
\langle A_{K,\xi}^{(q)}(0), \phi(\xi)\rangle &= \frac{\cos(\pi q/2)}{\pi(n-q-1)} \langle \left(\|x\|^{-n+q+1}\right)^\wedge (\xi), \phi(\xi)\rangle \\
&= \frac{\cos(\pi q/2)}{\pi(n-q-1)} \langle \|x\|^{-n+q+1}, \hat{\phi}(x)\rangle \\
&= \frac{\cos(\pi q/2)}{\pi(n-q-1)} \int_{\mathbb{R}^n} \|x\|^{-n+q+1} \hat{\phi}(x) dx \\
&= \frac{\cos(\pi q/2)}{\pi(n-q-1)} \int_{\mathbb{R}^n} \left(\|x\|^{-n+q+1} - 1\right) \hat{\phi}(x) dx \\
&\quad + \frac{\cos(\pi q/2)}{\pi(n-q-1)} \int_{\mathbb{R}^n} \hat{\phi}(x) dx \\
&= \frac{\cos(\pi q/2)}{\pi} \int_{\mathbb{R}^n} \frac{\|x\|^{-n+q+1} - 1}{n-q-1} \hat{\phi}(x) dx,
\end{aligned}$$

since $\int_{\mathbb{R}^n} \hat{\phi}(x)dx = (2\pi)^n \phi(0) = 0$. Taking the limit of both sides as $q \to n-1$, we get

$$\langle A_{K,\xi}^{(n-1)}(0), \phi(\xi)\rangle = \Big\langle -\frac{\cos(\pi(n-1)/2)}{\pi} \left(\ln \|x\|\right)^\wedge(\xi), \phi(\xi)\Big\rangle$$

since

$$\begin{aligned}
\lim_{q\to n-1} \int_{\mathbb{R}^n} \frac{\|x\|^{-n+q+1}-1}{n-q-1}\hat{\phi}(x)dx &= -\int_{\mathbb{R}^n} \ln\|x\|\hat{\phi}(x)dx \\
&= \langle -\left(\ln\|x\|\right)^\wedge(\xi), \phi(\xi)\rangle.
\end{aligned}$$

When n is odd the formula of i) follows immediately.

When n is even, both sides of (3.14) are equal to zero, and we repeat the reasoning from Theorem 1 in [GKS]. Divide both sides of (3.13) by $\cos(\frac{\pi q}{2})$

$$\Big\langle \frac{\left(\|x\|_K^{-n+q+1}\right)^\wedge(\xi)}{(n-q-1)}, \phi(\xi)\Big\rangle = \pi\Big\langle \frac{A_{K,\xi}^{(q)}(0)}{\cos\frac{\pi q}{2}}, \phi(\xi)\Big\rangle$$

and take the limit of both sides when $q \to n-1$.

We have already proved that

$$\lim_{q\to n-1} \Big\langle \frac{\left(\|x\|_K^{-n+q+1}\right)^\wedge(\xi)}{(n-q-1)}, \phi(\xi)\Big\rangle = \langle -\left(\ln\|x\|\right)^\wedge(\xi), \phi(\xi)\rangle$$

for any test function ϕ supported outside of the origin.

To compute the limit of $\dfrac{A_{K,\xi}^{(q)}(0)}{\cos\frac{q\pi}{2}}$ we use the definition of fractional derivatives in exactly the same way as it was done in [GKS, Theorem 1].

$$\lim_{q\to n-1} \Gamma(-q)A_{K,\xi}^{(q)}(0) = \int_0^\infty \frac{A_\xi(z) - A_\xi(0) - A_\xi''(0)\frac{z^2}{2} - \ldots - A_\xi^{n-2}(z)\frac{z^{n-2}}{(n-2)!}}{z^n}dz$$

and

$$\lim_{q\to n-1} \Gamma(-q)\sin\frac{(q+1)\pi}{2} = \frac{\pi}{2}(-1)^{n/2}\frac{1}{(n-1)!}.$$

Combining these two formulas we get the formula in the statement ii) of the Theorem. $\square$

An immediate application of Theorem 5.3.2 is

Corollary 3.4.2. *Let K be an infinitely smooth body in $\mathbb{R}^n$. Then*

i) if n is odd, $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0 if and only if

$$(-1)^{(n-1)/2} A_{K,\xi}^{(n-1)}(0) \geq 0, \quad \forall \xi \in S^{n-1};$$

ii) if n is even, $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0 if and only if, for every $\xi \in S^{n-1}$,

$$(-1)^{(n+2)/2} \int_0^\infty \frac{A_\xi(z) - A_\xi(0) - A''_\xi(0)\frac{z^2}{2} - \ldots - A_\xi^{n-2}(z)\frac{z^{n-2}}{(n-2)!}}{z^n} dz \geq 0.$$

Corollary 3.4.3. *Every 3-dimensional normed space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0.*

Proof. The unit ball K of a normed space is an origin-symmetric convex body. First assume that K is infinitely smooth. By Brunn's theorem the central section of a convex body has maximal volume among all sections perpendicular to a given direction. Therefore, for any ξ the function $A_{K,\xi}(t)$ attains its maximum at $t = 0$, hence $A''_{K,\xi}(0) \leq 0$. So, by Theorem 5.3.2, for smooth convex bodies in $\mathbb{R}^3$ the distribution $-(\ln \|x\|)^\wedge$ is positive outside of the origin, and our result follows from Theorem 3.3.1. For general convex bodies the result follows from the facts that any convex body can be approximated by smooth convex bodies and that positive definiteness is preserved under limits. In fact, let $\{K_i\}$ be a sequence of infinitely smooth convex bodies that approach K in the radial metric. Then for any non-negative test function ϕ supported outside of the origin we have

$$-\int_{\mathbb{R}^n} \ln \|x\|_{K_i} \hat{\phi}(x) dx = \langle -\ln \|x\|_{K_i}, \hat{\phi}(x) \rangle = \langle -(\ln \|x\|_{K_i})^\wedge(\xi), \phi(\xi) \rangle \geq 0$$

Since K_i approximate K there is a constant $C > 0$, such that

$$|\ln \|x\|_{K_i}| \leq C + |\ln |x|_2| \,,$$

therefore the functions $|\ln\|x\|_{K_i}\hat{\phi}(x)|$ are majorated by an integrable function $(C + |\ln|x|_2|)|\hat{\phi}(x)|$ and by the Lebesgue Dominated Convergence Theorem we get

$$\begin{aligned} -\lim_{i\to\infty}\int_{\mathbb{R}^n}\ln\|x\|_{K_i}\hat{\phi}(x)dx &= -\int_{\mathbb{R}^n}\ln\|x\|_K\hat{\phi}(x)dx \\ &= \langle -(\ln\|x\|_K)^\wedge(\xi), \phi(\xi)\rangle \geq 0 \end{aligned}$$

□

Our next result shows that that the previous statement is no longer true in $\mathbb{R}^n$, $n \geq 4$.

Theorem 3.4.4. *There exists an origin-symmetric convex body K in $\mathbb{R}^n$, $n \geq 4$ so that the space $(\mathbb{R}^n, \|\cdot\|_K)$ does not embed in L_0.*

Proof. It is enough to construct a convex body for which the distribution $-(\ln\|x\|)^\wedge$ is not positive. The construction will be similar to that from [GKS].

Define $f_N(x) = (1 - x^2 - Nx^4)^{1/3}$, let $a_N > 0$ be such that $f_N(a_N) = 0$ and $f_N(x) > 0$ on the interval $(0, a_N)$. Define a body K in $\mathbb{R}^4$ by

$$K = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \; x_4 \in [-a_N, a_N] \text{ and } \sqrt{x_1^2 + x_2^2 + x_3^2} \leq f_N(x_4)\}.$$

The body K is strictly convex and infinitely smooth. By Theorem 5.3.2,

$$-\left(\ln\|x\|_K\right)^\wedge(\xi) = 12\int_0^\infty \frac{A_\xi(z) - A_\xi(0) - A''_\xi(0)\frac{z^2}{2}}{z^4}dz.$$

The function $A_{K,\xi}$ can easily be computed:

$$A_{K,\xi}(x) = \frac{4\pi}{3}(1 - x^2 - Nx^4).$$

We have

$$\int_0^\infty \frac{A_\xi(z) - A_\xi(0) - A''_\xi(0)\frac{z^2}{2}}{z^4}dz = \frac{4\pi}{3}(-Na_N + \frac{1}{a_N} - \frac{1}{3a_N^3}).$$

The latter is negative for N large enough, because $N^{1/4} \cdot a_N \to 1$ as $N \to \infty$.

$\square$

3.5 Addition in L_0

It is clear from the definition that the class of bodies K for which $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0 is closed with respect to multiplicative summation, i.e. if two spaces $(\mathbb{R}^n, \|\cdot\|_{K_1})$ and $(\mathbb{R}^n, \|\cdot\|_{K_2})$ embed in L_0 and $K = K_1 +_0 K_2$, then $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0. In this section we are going to prove that the unit ball of every space $(\mathbb{R}^n, \|\cdot\|_K)$ that embeds in L_0 can be obtained from the Euclidean ball by means of multiplicative summation, linear transformations and closure in the radial metric, i.e. it can be approximated in the radial metric by multiplicative sums of ellipsoids.

Consider the set of bodies K for which $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0. As mentioned above, this set is closed with respect to multiplicative summation, also from the proof of Corollary 3.4.3 it follows that this set is closed with respect to limits in the radial metric. Let us show that it is closed with respect to linear transformations. Suppose that $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0. By Theorem 3.3.1 $(\ln \|x\|_K)^\wedge$ is a negative distribution outside of the origin. Let T be an invertible linear transformation in $\mathbb{R}^n$, then for any non-negative test function ϕ with support outside of the origin, we have

$$
\begin{aligned}
\langle (\ln \|Tx\|_K)^\wedge, \phi\rangle &= \langle \ln \|Tx\|_K, \hat{\phi}(x)\rangle \\
&= \int_{\mathbb{R}^n} \ln \|Tx\|_K \hat{\phi}(x) dx \\
&= |\det T|^{-1} \int_{\mathbb{R}^n} \ln \|x\|_K \hat{\phi}(T^{-1}x) dx \\
&= \int_{\mathbb{R}^n} \ln \|x\|_K \left(\phi(T^* y)\right)^\wedge (x) dx \\
&= \langle \ln \|x\|_K, \left(\phi(T^* y)\right)^\wedge (x)\rangle, \\
&= \langle (\ln \|x\|_K)^\wedge (y), \phi(T^* y)\rangle \le 0.
\end{aligned}
$$

So $(\ln \|Tx\|_K)^\wedge$ is a negative distribution outside of the origin. By Theorem 3.3.1, $(\mathbb{R}^n, \|\cdot\|_{TK})$ embeds in L_0.

Moreover, if $(\ln \|x\|)^\wedge$ is a function, then

$$
(\ln \|Tx\|)^\wedge (y) = |\det T|^{-1} (\ln \|x\|)^\wedge ((T^*)^{-1} y). \tag{3.15}
$$

To prove the main result of this section we need a few lemmas. For a fixed $x \in S^{n-1}$, let $E_{a,b}(x)$ be an ellipsoid with the norm

$$
\|\theta\|_{E_{a,b}(x)} = \left(\frac{(x,\theta)^2}{a^2} + \frac{1-(x,\theta)^2}{b^2} \right)^{1/2}, \quad \text{for } \theta \in S^{n-1}.
$$

Lemma 3.5.1. *For all $\theta \in S^{n-1}$,*

$$
\left(\ln \|\xi\|_{E_{a,b}(x)}\right)^\wedge_\xi (\theta) = -\frac{2^{n-1}\pi^{n/2}\Gamma(n/2)}{a^{n-1}b} \|\theta\|^{-n}_{E_{b,a}(x)}.
$$

Proof. For $-n < \lambda < 0$ the following formula holds (see [GS, p.192]):

$$
\left(|x|_2^\lambda\right)^\wedge (\xi) = 2^{\lambda+n}\pi^{n/2} \frac{\Gamma((\lambda+n)/2)}{\Gamma(-\lambda/2)} |\xi|_2^{-\lambda-n}.
$$

Dividing both sides by λ, using the formula $x\Gamma(x) = \Gamma(1+x)$ and sending $\lambda \to 0$ we get

$$(\ln |x|_2)^\wedge (\xi) = -2^{n-1}\pi^{n/2}\Gamma(n/2)|\xi|_2^{-n},$$

as distributions outside of the origin. Note that, by rotation, it is enough to prove Lemma for the ellipsoids $E_{a,b}(x)$ with $x = (0, 0, \dots, 0, 1)$.

$$\|\xi\|_{E_{a,b}(x)} = \left(\frac{\xi_n^2}{a^2} + \frac{\xi_1^2 + \cdots + \xi_{n-1}^2}{b^2}\right)^{1/2}.$$

Since this norm can be obtained from the Euclidean norm by an obvious linear transformation, one can use formula (3.15) to get

$$\begin{aligned}\left(\ln \|\xi\|_{E_{a,b}(x)}\right)_\xi^\wedge (\theta) &= -2^{n-1}\pi^{n/2}\Gamma(n/2)ab^{n-1}\|\theta\|_{E_{1/a,1/b}(x)}^{-n} \\ &= -\frac{2^{n-1}\pi^{n/2}\Gamma(n/2)}{a^{n-1}b}\|\theta\|_{E_{b,a}(x)}^{-n}.\end{aligned}$$

□

Lemma 3.5.2. *Let K be a star body, then $\ln \|x\|_K$ can be approximated in the space $C(S^{n-1})$ by the functions of the form*

$$f_{a,b}(x) = \frac{1}{|S^{n-1}|a^{n-1}b}\int_{S^{n-1}} \ln \|\theta\|_K \|\theta\|_{E_{b,a}(x)}^{-n} d\theta, \tag{3.16}$$

as $a \to 0$ and b is fixed.

Proof. The proof is similar to that of [GW, Lemma 2]. First, note that the space $\mathbb{R}^n$ with the Euclidean norm embeds in L_0, so $(\mathbb{R}^n, \|\cdot\|_E)$ embeds in L_0 for any ellipsoid E with center at the origin. Therefore, by Remark 3.3.2 and Lemma 3.5.1 we get

$$\int_{S^{n-1}} \frac{1}{|S^{n-1}|a^{n-1}b}\|\theta\|_{E_{b,a}(x)}^{-n} d\theta = 1,$$

for all values of a and b. From now on b will be fixed.

We have

$$\begin{aligned}
&\left|\ln\|x\|_K - \frac{1}{|S^{n-1}|a^{n-1}b}\int_{S^{n-1}} \ln\|\theta\|_K \|\theta\|^{-n}_{E_{b,a}(x)} d\theta\right| \\
&\le \frac{1}{|S^{n-1}|a^{n-1}b}\int_{S^{n-1}} \left|\ln\|x\|_K - \ln\|\theta\|_K\right| \|\theta\|^{-n}_{E_{b,a}(x)} d\theta \\
&= \frac{1}{|S^{n-1}|a^{n-1}b}\int_{|(x,\theta)|\ge\delta} \left|\ln\|x\|_K - \ln\|\theta\|_K\right| \|\theta\|^{-n}_{E_{b,a}(x)} d\theta \\
&+ \frac{1}{|S^{n-1}|a^{n-1}b}\int_{|(x,\theta)|<\delta} \left|\ln\|x\|_K - \ln\|\theta\|_K\right| \|\theta\|^{-n}_{E_{b,a}(x)} d\theta \\
&= I_1 + I_2.
\end{aligned}$$

For the first integral I_1 use the uniform continuity of $\ln\|x\|_K$ on the sphere. For any given $\epsilon > 0$ there exists $\delta \in (0,1)$, δ close to 1, so that $|(x,\theta)| \ge \delta$ implies $\left|\ln\|x\|_K - \ln\|\theta\|_K\right| < \epsilon/2$. Therefore

$$\begin{aligned}
I_1 &= \frac{1}{|S^{n-1}|a^{n-1}b}\int_{|(x,\theta)|\ge\delta} \left|\ln\|x\|_K - \ln\|\theta\|_K\right| \|\theta\|^{-n}_{E_{a,b}(x)} d\theta \\
&\le \frac{\epsilon}{2}\left[\frac{1}{|S^{n-1}|a^{n-1}b}\int_{|(x,\theta)|\ge\delta} \|\theta\|^{-n}_{E_{a,b}(x)} d\theta\right] \le \frac{\epsilon}{2}.
\end{aligned}$$

Now fix δ chosen above and estimate the integral I_2 as follows

$$\begin{aligned}
I_2 &= \frac{1}{|S^{n-1}|a^{n-1}b}\int_{|(x,\theta)|<\delta} \left|\ln\|x\|_K - \ln\|\theta\|_K\right| \|\theta\|^{-n}_{E_{b,a}(x)} d\theta \\
&\le \frac{C(n,b,K)}{a^{n-1}}\int_{|(x,\theta)|<\delta} \|\theta\|^{-n}_{E_{b,a}(x)} d\theta,
\end{aligned}$$

where

$$C(n,b,K) = \frac{2\max_{S^{n-1}} |\ln\|x\|_K|}{|S^{n-1}|b}.$$

For the latter integral we use an elementary formula (see e.g. [Ko11, Section 6.4])

$$\int_{|(x,\theta)|<\delta} f((x,\theta))d\theta = |S^{n-2}|\int_{-\delta}^{\delta}(1-t^2)^{(n-3)/2} f(t)dt, \quad \text{for } x \in S^{n-1}.$$

Now,

$$\begin{aligned}
I_2 &\leq \frac{C(n,b,K)|S^{n-2}|}{a^{n-1}} \int_{-\delta}^{\delta} (1-t^2)^{(n-3)/2} \left(\frac{t^2}{b^2} + \frac{1-t^2}{a^2}\right)^{-n/2} dt \\
&\leq \frac{C(n,b,K)|S^{n-2}|}{a^{n-1}} \int_{-\delta}^{\delta} (1-t^2)^{(n-3)/2} \left(\frac{1-t^2}{a^2}\right)^{-n/2} dt \\
&= a \cdot C(n,b,K)|S^{n-2}| \int_{-\delta}^{\delta} (1-t^2)^{-3/2} dt \\
&\leq a \cdot C(n,b,K)|S^{n-2}| \frac{2\delta}{(1-\delta^2)^{3/2}}.
\end{aligned}$$

Now we can choose a so small that $I_2 \leq \epsilon/2$. □

Lemma 3.5.3. *If μ is a probability measure on S^{n-1} and $a, b > 0$, then the function*

$$f(x) = \int_{S^{n-1}} \ln \|\xi\|_{E_{a,b}(x)} d\mu(\xi)$$

can be approximated in $C(S^{n-1})$ by the sums of the form

$$\sum_{i=1}^{m} \frac{1}{p_i} \ln \|x\|_{E_i},$$

where E_1,...,E_m are ellipsoids and $1/p_1 + \cdots + 1/p_m = 1$.

Proof. Let $\sigma > 0$ be a small number and choose a finite covering of the sphere by spherical σ-balls $B_\sigma(\eta_i) = \{\eta \in S^{n-1} : |\eta - \eta_i| < \sigma\}$, $\eta_i \in S^{n-1}$, $i = 1, \ldots, m = m(\delta)$. Define

$$\widetilde{B}_\sigma(\xi_1) = B_\sigma(\xi_1)$$

and

$$\widetilde{B}_\sigma(\xi_i) = B_\sigma(\xi_i) \setminus \bigcup_{j=1}^{i-1} B_\sigma(\xi_j), \qquad \text{for } i = 2, ..., m.$$

Let $1/p_i = \mu(\widetilde{B}_\sigma(\xi_i))$. Clearly, $1/p_1 + \cdots + 1/p_m = 1$.

Let $\rho(E_{a,b}(\xi), x)$ be the radial function of the ellipsoid $E_{a,b}(\xi)$, that is

$$\rho(E_{a,b}(\xi), x) = \|x\|_{E_{a,b}(\xi)}^{-1}.$$

Note that $\rho(E_{a,b}(\xi),x)=\rho(E_{a,b}(x),\xi)$, therefore

$$|\rho(E_{a,b}(\xi),x)-\rho(E_{a,b}(\theta),x)|\leq C_{a,b}|\xi-\theta|,$$

with a constant $C_{a,b}$ that depends on a and b. Also note that, since we consider a close to zero and b fixed, we may assume

$$a\leq\rho(E_{a,b}(\xi),x)\leq b,\qquad x\in S^{n-1}.$$

Then,

$$\begin{aligned}
&\left|\int_{S^{n-1}}\ln\rho(E_{a,b}(\xi),x)d\mu(\xi)-\sum_{i=1}^{m}\frac{1}{p_i}\ln\rho(E_{a,b}(\xi_i),x)\right|=\\
&=\left|\sum_{i=1}^{m}\left(\int_{\widetilde{B}_\sigma(\xi_i)}\ln\rho(E_{a,b}(\xi),x)d\mu(\xi)-\int_{\widetilde{B}_\sigma(\xi_i)}\ln\rho(E_{a,b}(\xi_i),x)d\mu(\xi)\right)\right|\leq\\
&\leq\sum_{i=1}^{m}\int_{\widetilde{B}_\sigma(\xi_i)}\left|\ln\frac{\rho(E_{a,b}(\xi),x)}{\rho(E_{a,b}(\xi_i),x)}\right|d\mu(\xi)\leq\\
&\leq\sum_{i=1}^{m}\int_{\widetilde{B}_\sigma(\xi_i)}\left|\ln\frac{\rho(E_{a,b}(\xi_i),x)+[\rho(E_{a,b}(\xi),x)-\rho(E_{a,b}(\xi_i),x)]}{\rho(E_{a,b}(\xi_i),x)}\right|d\mu(\xi)\leq\\
&\leq\sum_{i=1}^{m}\int_{\widetilde{B}_\sigma(\xi_i)}\left|\ln(1\pm C'_{a,b}|\xi-\xi_i|)\right|d\mu(\xi)\leq\\
&\leq\left|\ln(1\pm C'_{a,b}\sigma)\right|,
\end{aligned}$$

and the result follows since σ is arbitrarily small.

□

Now we are ready to prove the following

Theorem 3.5.4. *Let K be an origin symmetric star body in $\mathbb{R}^n$. The space $(\mathbb{R}^n,\|\cdot\|_K)$ embeds in L_0 if and only if $\|x\|_K$ is the limit (in the radial metric) of finite products $\|x\|_{E_1}^{1/p_1}\cdots\|x\|_{E_m}^{1/p_m}$, where $E_1,\dots,E_m$ are ellipsoids and $1/p_1+\cdots+1/p_m=1$.*

Proof. The "if" part is a consequence of the fact that L_0 is closed with respect to the three operations as discussed above.

The proof of "only if" part easily follows the Lemmas we have proved.

Suppose that $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0 with the corresponding probability measure μ on S^{n-1} and constant C. By Remark 3.3.2, $(\mathbb{R}^n, \|\cdot\|_{E_{a,b}(x)})$ embeds in L_0 with the measure $-\frac{1}{(2\pi)^n}\left(\ln\|x\|_E\right)^\wedge(\theta)d\theta$ and some constant $C_{E_{a,b}}$. Note, this constant does not depend on x. We have

$$\begin{aligned}
&\int_{S^{n-1}} \ln\|\xi\|_{E_{a,b}(x)} d\mu(\xi) \\
= &\int_{S^{n-1}}\int_{S^{n-1}} \ln|(\xi,\theta)| \left(-\frac{1}{(2\pi)^n}\right)\left(\ln\|x\|_{E_{a,b}(x)}\right)^\wedge(\theta)d\theta d\mu(\xi) + C_{E_{a,b}} \\
= &\int_{S^{n-1}}\left[\int_{S^{n-1}} \ln|(\xi,\theta)|d\mu(\xi) + C_K\right]\left(-\frac{1}{(2\pi)^n}\right)\left(\ln\|x\|_{E_{a,b}(x)}\right)^\wedge(\theta)d\theta \\
&+C_{E_{a,b}} - C_K \\
= &\int_{S^{n-1}} \ln\|\theta\|_K \left(-\frac{1}{(2\pi)^n}\right)\left(\ln\|x\|_{E_{a,b}(x)}\right)^\wedge(\theta)d\theta + C_{E_{a,b}} - C_K \\
= &\int_{S^{n-1}} \ln\|\theta\|_K \left(-\frac{1}{(2\pi)^n}\right)\left(\ln\|x\|_{E_{a,b}(x)}\right)^\wedge(\theta)d\theta + C_{E_{a,b}} - C_K \\
= &\frac{1}{|S^{n-1}|a^{n-1}b}\int_{S^{n-1}} \ln\|\theta\|_K\|\theta\|^{-n}_{E_{b,a}(x)}d\theta + C_{E_{a,b}} - C_K
\end{aligned}$$

In Lemma 3.5.2 we proved that $\ln\|x\|_K$ can be uniformly approximated by the integrals of the form

$$\frac{1}{|S^{n-1}|a^{n-1}b}\int_{S^{n-1}} \ln\|\theta\|_K\|\theta\|^{-n}_{E_{b,a}(x)}d\theta,$$

as $a \to 0$. Therefore, using the previous calculations, one can see that $\ln\|x\|_K$ can be uniformly approximated by

$$\int_{S^{n-1}} \ln\|\xi\|_{E_{a,b}(x)} d\mu(\xi) + C'.$$

Hence, by Lemma 3.5.3, $\ln \|x\|_K$ can be uniformly approximated by the sums

$$\sum_{i=1}^{m} \frac{1}{p_i} \ln \|x\|_{E_i} + C'.$$

Replacing E_1 by another ellipsoid E_1' given by $\|x\|_{E_1'}^{1/p_1} = e^{C'} \|x\|_{E_1}^{1/p_1}$, we get the statement of the Theorem.

□

Corollary 3.5.5. *Any convex body in* $\mathbb{R}^3$ *can be obtained from the Euclidean unit ball by means of three operations: linear transformations, multiplicative addition and closure in the radial metric.*

Proof. As was proved in Theorem 3.5.4, any convex body can be approximated by the finite products of the type $\|x\|_{E_1}^{1/p_1} \cdots \|x\|_{E_m}^{1/p_m}$. Since any number $1/p$ can be approximated by the sums

$$\frac{1}{2^{i_1}} + \frac{1}{2^{i_2}} + \cdots + \frac{1}{2^{i_k}},$$

the result follows.

□

A proof similar to that of Theorem 3.5.4 can be used to show that the previous theorem holds for p-summation with $-1 < p < 1$, $p \neq 0$, in place of the multiplicative summation.

Theorem 3.5.6. *Let* K *be an origin symmetric star body in* $\mathbb{R}^n$. *The space* $(\mathbb{R}^n, \|\cdot\|_K)$ *embeds in* L_p, $-1 < p < 1$, $p \neq 0$ *if and only if* $\|x\|_K^p$ *is the limit (in the radial topology) of finite sums* $\|x\|_{E_1}^p + \cdots + \|x\|_{E_m}^p$, *where* $E_1, \ldots, E_m$ *are ellipsoids.*

3.6 Confirming the place of L_0 in the scale of L_p-spaces.

In this section we establish the relations between embedding in L_0 and in L_p with $p \neq 0$, which confirm the place of L_0 between L_p with $p > 0$ and $p < 0$. We are going to use the following result from [Ko7, Theorem 1]:

Theorem 3.6.1. *An n-dimensional homogeneous space* $(\mathbb{R}^n, \|\cdot\|_K)$ *embeds in* L_{-p}, $p \in (0, n)$ *if and only if* $\|x\|_K^{-p}$ *is a positive definite distribution.*

We also use a well-known result of P.Levy (see [BL, p.189], also [BDK] for the infinite dimensional case):

Theorem 3.6.2. *A space* $(\mathbb{R}^n, \|\cdot\|_K)$ *embeds in* L_p, $p \in (0, 2]$ *if and only if the function* $\exp(-\|x\|_K^p)$ *is positive definite.*

Now we are ready to prove

Theorem 3.6.3. *Let K be an origin symmetric star body in* $\mathbb{R}^n$. *If the space* $(\mathbb{R}^n, \|\cdot\|_K)$ *embeds in* L_0 *then it also embeds in* L_{-p}, $0 < p < n$.

Proof. By Theorem 3.5.4, $\|x\|_K$ is the limit of finite products $\|x\|_{E_1}^{1/p_1} \cdots \|x\|_{E_m}^{1/p_m}$. Consider $\|x\|_K^{-p}$ for $0 < p < n$. It is the limit of the products of the form $\|x\|_{E_1}^{-p/p_1} \cdots \|x\|_{E_m}^{-p/p_m}$. Using the formula

$$\|x\|^{-p} = \frac{2}{\Gamma(p/2)} \int_0^\infty t^{p-1} \exp(-t^2\|x\|^2) dt,$$

we get

$$\|x\|_{E_1}^{-p/p_1}\cdots\|x\|_{E_m}^{-p/p_m} = C\int_0^\infty\cdots\int_0^\infty t_1^{p/p_1-1}\cdots t_m^{p/p_m-1}\times$$
$$\times\exp(-t_1^2\|x\|_{E_1}^2-\cdots-t_m^2\|x\|_{E_m}^2)dt_1\cdots dt_m,$$

where

$$C=\frac{2^m}{\Gamma(p/2p_1)\cdots\Gamma(p/2p_m)}.$$

Therefore, for any non-negative test function ϕ we have

$$\langle(\|x\|_{E_1}^{-p/p_1}\cdots\|x\|_{E_m}^{-p/p_m})^\wedge(\xi),\phi(\xi)\rangle=\langle\|x\|_{E_1}^{-p/p_1}\cdots\|x\|_{E_m}^{-p/p_m},\hat{\phi}(x)\rangle=$$
$$=C\int_0^\infty\cdots\int_0^\infty t_1^{p/p_1-1}\cdots t_m^{p/p_m-1}\times$$
$$\times\langle\exp(-t_1^2\|x\|_{E_1}^2-\cdots-t_m^2\|x\|_{E_m}^2),\hat{\phi}(x)\rangle dt_1\cdots dt_m=$$
$$=C\int_0^\infty\cdots\int_0^\infty t_1^{p/p_1-1}\cdots t_m^{p/p_m-1}\times$$
$$\times\langle(\exp(-t_1^2\|x\|_{E_1}^2-\cdots-t_m^2\|x\|_{E_m}^2))^\wedge(\xi),\phi(\xi)\rangle dt_1\cdots dt_m.$$

We claim that the latter expression is non-negative. Indeed, $(\mathbb{R}^n,\|x\|_E)$ embeds in L_2 for any ellipsoid, therefore the 2-sum of ellipsoids $t_1^2\|x\|_{E_1}^2+\cdots+t_m^2\|x\|_{E_m}^2$ embeds in L_2, and hence by Theorem 3.6.2, the function $\exp(-t_1^2\|x\|_{E_1}^2-\cdots-t_m^2\|x\|_{E_m}^2)$ is positive definite. Now the fact that $\langle(\|x\|_K^{-p})^\wedge,\phi\rangle\ge 0$ follows by an approximation argument, as in Corollary 3.4.3.

□

Theorem 3.6.4. *Let K be an origin symmetric star body in $\mathbb{R}^n$. If the space $(\mathbb{R}^n,\|\cdot\|_K)$ embeds in L_{-p} for every $p\in(0,\epsilon)$, then it also embeds in L_0.*

Proof. The space $(\mathbb{R}^n,\|\cdot\|_K)$ embeds in L_{-p}, so by Theorem 3.6.1 the distribution $\|x\|^{-p}$ is positive definite. Then for every non-negative test function ϕ supported outside of the origin,

$$-\int_{\mathbb{R}^n} \ln \|x\| \hat{\phi}(x) dx = \lim_{p\to 0} \frac{1}{p} \int_{\mathbb{R}^n} (\|x\|^{-p} - 1)\hat{\phi}(x) dx$$
$$= \lim_{p\to 0} \frac{1}{p} \int_{\mathbb{R}^n} \|x\|^{-p} \hat{\phi}(x) dx \geq 0.$$

The result follows from Theorem 3.3.1.

□

Theorem 3.6.5. *There are normed spaces that embed in L_0, but do not embed in L_p for $p > 0$.*

Proof. As proved above, every 3-dimensional normed space embeds in L_0, hence l_q^3 with $q > 2$ does. On the other hand, l_q^3, $q > 2$ does not embed in L_p for $0 < p \leq 2$ (see [Ko1]).

□

Let us also mention that one can use the approach of [KK1] to produce examples in the same spirit. It follows from [KK1], Proposition 3.5 that $\mathbb{R} \oplus_2 \ell_1$ does not embed isometrically into L_p for $p > 0$; hence neither does $\mathbb{R} \oplus_2 \ell_1^n$ for large enough n.

Proposition 3.6.6. *For any $n \in \mathbb{N}$ the space $\mathbb{R} \oplus_2 \ell_1^n$ embeds in L_0.*

Proof. Let $(f_n)_{n=1}^\infty$ be a sequence of functions on some probability space which are independent and 1-stable symmetric, so that $\mathbb{E}(e^{itf_j}) = e^{-|t|}$ (i.e. the f_j have the Cauchy distribution). Then it is clear that

$$\mathbb{E} \ln |\sum_{j=1}^n a_j f_j| = \ln \sum_{j=1}^n |a_j|.$$

Indeed this follows from the fact that

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\ln|x|}{1+x^2}dx = 0.$$

On the other hand if $f = \sum_{j=1}^n a_j f_j$ where $\sum_{j=1}^n |a_j| = 1$ then f has the Cauchy distribution and so has the same distribution as g_1/g_2 where g_1, g_2 are independent normalized Gaussians. Hence

$$\begin{aligned}\mathbb{E}\ln|a+bf| &= \mathbb{E}(\ln|ag_2+bg_1| - \ln|g_2|)\\ &= \ln(a^2+b^2)^{\frac{1}{2}}.\end{aligned}$$

Now for any $a_0, a_1, \dots, a_n \in \mathbb{R}$ we have

$$\mathbb{E}|a_0 + \sum_{j=1}^n a_j f_j| = \ln\left(|a_0|^2 + (\sum_{j=1}^n |a_j|)^2\right)^{\frac{1}{2}}.$$

This shows (using the remarks at the end of §3.2) that $\mathbb{R}\oplus_2 \ell_1^n$ embeds into L_0 for every n. □

Theorem 3.6.7. *Let K be an origin symmetric star body in $\mathbb{R}^n$. If the space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_{p_0} , $0 < p_0 \le 2$, then it also embeds in L_0.*

Proof. Since $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_{p_0}, $0 < p_0 \le 2$, by [Ko7, Theorem 2] it also embeds in L_{-p} for any $p \in (0, n)$ and hence, by Theorem 3.6.4, it embeds in L_0.

□

Chapter 4

Modified Busemann-Petty problem on sections of convex bodies

4.1 Introduction

The classical Minkowski's uniqueness theorem states that an origin-symmetric star body in $\mathbb{R}^n$ is uniquely determined by the volumes of its central hyperplane sections in all directions, see for example [Ko11, Corollary 3.9]. This result provides a strong intuition towards an affirmative answer in the following Busemann-Petty problem [BP]: given two convex origin-symmetric bodies K and L in $\mathbb{R}^n$ such that

$$\text{vol}_{n-1}(K \cap H) \leq \text{vol}_{n-1}(L \cap H)$$

for every central hyperplane H in $\mathbb{R}^n$, does it follow that

$$\text{vol}_n(K) \leq \text{vol}_n(L)?$$

The solution has been completed a few years ago and appeared as the result of work of many mathematicians (see [GKS], [Zh2] or [Ko11, Chapter 5] for the solution and historical details). Surprisingly, the answer is affirmative only if the dimension $n \leq 4$, and it is negative if $n \geq 5$. In view of this answer, it is natural to ask what information about the volumes of central hyperplane sections of two bodies does allow to compare the volumes of these bodies in all dimensions. Here we present results from [KYY].

For an origin-symmetric convex body K in $\mathbb{R}^n$, consider the section function

$$S_K(\xi) = \mathrm{vol}_{n-1}(K \cap \xi^\perp), \qquad \xi \in S^{n-1},$$

where $\xi^\perp$ is the central hyperplane in $\mathbb{R}^n$ orthogonal to ξ. We extend S_K from the sphere to the whole $\mathbb{R}^n$ as a homogeneous function of degree -1. Our goal is to find a condition in terms of the section functions of two bodies only that allows to compare the n-dimensional volumes of these bodies. We prove in this paper that, for two origin-symmetric smooth bodies K, L in $\mathbb{R}^n$ and $\alpha \in \mathbb{R}$, $\alpha \geq n-4$, the inequalities

$$(-\Delta)^{\alpha/2} S_K(\xi) \leq (-\Delta)^{\alpha/2} S_L(\xi), \qquad \forall \xi \in S^{n-1} \tag{4.1}$$

imply that $\mathrm{vol}_n(K) \leq \mathrm{vol}_n(L)$, while for $\alpha < n-4$ this is not necessarily true. Here Δ is the Laplace operator on $\mathbb{R}^n$, and the fractional powers of the Laplacian are defined by

$$(-\Delta)^{\alpha/2} f = \frac{1}{(2\pi)^n} (|x|_2^\alpha \hat{f}(x))^\wedge,$$

where the Fourier transform is considered in the sense of distributions, and $|x|_2$ stands for the Euclidean norm in $\mathbb{R}^n$. Of course, if α is an even integer and f is an even distribution we get the Laplacian applied $\alpha/2$ times. The fact that both sides of (4.1) represent continuous functions of the variable ξ follows from [Ko11, Lemma 3.16].

This result means that one has to differentiate the section functions at least $n-4$ times in order to compare the n-dimensional volumes. The case $\alpha = 0$ corresponds to the original Busemann-Petty problem, so our result can also be considered as a "continuous" generalization of the problem. Other generalizations of the Busemann-Petty problem and related open questions can be found in [BZ], [Ko4], [Ko8], [Ko10], [MiP], [RZ], [Y], [Zv].

Let us briefly outline the idea of the proof. As shown in [Ko3], the section function can be expressed in terms of the Fourier transform, as follows:

$$S_K(\xi) = \frac{1}{\pi(n-1)}(\|x\|_K^{-n+1})^\wedge(\xi), \tag{4.2}$$

so the condition (4.1) can be written as

$$(|x|_2^\alpha \|x\|_K^{-n+1})^\wedge \le (|x|_2^\alpha \|x\|_K^{-n+1})^\wedge. \tag{4.3}$$

Now let us write the volume in polar coordinates and use a spherical version of Parseval's formula from [Ko4], which allows to remove the Fourier transforms of homogeneous functions in the integrals over the sphere under the condition that the degrees of homogeneity of these functions add up to $-n$:

$$n \text{ vol}_n(K) = \int_{S^{n-1}} \|x\|_K^{-n}\, dx = \int_{S^{n-1}} |x|_2^{-\alpha}\|x\|_K^{-1}|x|_2^{\alpha}\|x\|_K^{-n+1}\, dx$$
$$= \frac{1}{(2\pi)^n}\int_{S^{n-1}} \left(|x|_2^{-\alpha}\|x\|_K^{-1}\right)^\wedge(\xi)\left(|x|_2^{\alpha}\|x\|_K^{-n+1}\right)^\wedge(\xi)\, d\xi.$$

Suppose that the distribution $|x|_2^{-\alpha}\|x\|_K^{-1}$ is positive definite, so its Fourier transform is non-negative. Then the latter equality combined with (4.3) implies that

$$n \text{ vol}_n(K) \le \int_{S^{n-1}} \|x\|_K^{-1}\|x\|_L^{-n+1}\, dx,$$

and applying Hölder's inequality to the right-hand side we get that $\mathrm{vol}_n(K) \le \mathrm{vol}_n(L)$. On the other hand, if $|x|_2^{-\alpha}\|x\|_K^{-1}$ is not positive definite one can construct a counterexample using a more or less standard perturbation procedure.

Thus, the problem is essentially reduced to the question, for which α is the distribution $|x|_2^{-\alpha}\|x\|_K^{-1}$ positive definite, for every origin-symmetric convex body K in $\mathbb{R}^n$. Note that for $\alpha = 0$ this happens only if the dimension $n \le 4$, as proved in [GKS]. We prove that this function is positive definite for $\alpha \ge n-4$ and any symmetric convex body K in $\mathbb{R}^n$ by an argument modifying the proof from [GKS]. If $\alpha < n-4$ we construct examples of bodies for which this distribution is not positive definite. The latter requires a substantial technical effort.

4.2 Distributions of the form $|x|_2^{-r}\|x\|_K^{-s}$

Let us note that a simple approximation argument reduces the original Busemann-Petty problem (as well as all generalizations mentioned in the introduction) to the case where the bodies K and L are infinitely smooth.

In this section we establish some regularity properties of the function $A_{K,\xi,p}$ and express its fractional derivatives in terms of the Fourier transform. We assume that K is an infinitely smooth body.

For a real number q define the ceiling function $\lceil q \rceil$, which gives the smallest integer greater than or equal to q.

Lemma 4.2.1. *Let $\xi \in S^{n-1}$, $k \in \mathbb{N}$, $0 \le p < n-k-1$. Then the function $A_{K,\xi,p}$ is k-times continuously differentiable (uniformly with respect to ξ) in some neighborhood of zero.*

For fixed $q \in \mathbb{C}$, the fractional derivative $A^{(q)}_{K,\xi,p}(0)$ is a continuous function of the variable $\xi \in S^{n-1}$, and, for fixed $\xi \in S^{n-1}$, it is an analytic function of q in the domain $\{q \in \mathbb{C}\colon -1 < \lceil Re\ q \rceil < n-p-1\}$, with convergence in the derivatives by q being uniform with respect to ξ.

The proof is similar to that of [Ko11, Lemma 2.4]. The only difference is that in our case the function is differentiable only up to a certain order. To explain this, write the function in the form

$$A_{K,\xi,p}(t) = \int_{S_t^{n-2}} \left(\int_0^{\rho_{K\cap H_t}(\theta)} r^{n-2}(r^2+t^2)^{-p/2} dr \right) d\theta,$$

where $\rho_{K\cap H_t}(\theta)$ is the radial function of the body $K \cap H_t$ and S_t^{n-2} is the unit sphere in $H_t = \{x \in \mathbb{R}^n \colon \langle x, \xi \rangle = t\}$. If we differentiate by t too many times the integral stops being convergent when $t = 0$, which is why we have restrictions on k and q.

The following Lemma is a generalization of Theorem 2 from [GKS].

Lemma 4.2.2. *Let K be an infinitely smooth origin-symmetric convex body in $\mathbb{R}^n$, $q > -1$, $q \neq n-p-1$ and $0 \le p < n - \lceil q \rceil - 1$. Then for every $\xi \in S^{n-1}$,*

$$A^{(q)}_{K,\xi,p}(0) = \frac{\cos\frac{\pi q}{2}}{\pi(n-p-q-1)} (\|x\|_K^{-n+p+q+1} \cdot |x|_2^{-p})^\wedge(\xi).$$

Proof. We simply write $\|\cdot\|$ for $\|\cdot\|_K$. By [Ko11, Lemma 3.16], $(\|x\|^{-n+p+q+1} \cdot |x|_2^{-p})^\wedge$ is a continuous function on $\mathbb{R}^n \setminus \{0\}$.

Suppose first that $-1 < q < 0$. The function

$$A_{K,\xi,p}(t) = \int_{K\cap\langle x,\xi\rangle = t} |x|_2^{-p} dx = \int_{\langle x,\xi\rangle = t} \chi(\|x\|)|x|_2^{-p} dx$$

is even. Applying Fubini's theorem and passing to spherical coordinates, we get

$$\begin{aligned}
A^{(q)}_{K,\xi,p}(0) &= \frac{1}{\Gamma(-q)}\int_0^\infty t^{-q-1}A_{K,\xi,p}(t)dt \\
&= \frac{1}{2\Gamma(-q)}\int_{-\infty}^\infty |t|^{-q-1}A_{K,\xi,p}(t)dt \\
&= \frac{1}{2\Gamma(-q)}\int_{-\infty}^\infty |t|^{-q-1}\int_{\langle x,\xi\rangle=t}\chi(\|x\|)|x|_2^{-p}dxdt \\
&= \frac{1}{2\Gamma(-q)}\int_{\mathbb{R}^n}|\langle x,\xi\rangle|^{-q-1}\chi(\|x\|)|x|_2^{-p}dx \\
&= \frac{1}{2\Gamma(-q)}\int_{S^{n-1}}|\langle \theta,\xi\rangle|^{-q-1}\int_0^\infty r^{-q-1}\chi(r\|\theta\|)r^{-p}r^{n-1}drd\theta \\
&= \frac{1}{2\Gamma(-q)}\int_{S^{n-1}}|\langle \theta,\xi\rangle|^{-q-1}\int_0^{\frac{1}{\|\theta\|}} r^{n-p-q-2}drd\theta \\
&= \frac{1}{2\Gamma(-q)(n-p-q-1)}\int_{S^{n-1}}|\langle \theta,\xi\rangle|^{-q-1}\|\theta\|^{-n+p+q+1}d\theta.
\end{aligned}$$

Now we extend $A^{(q)}_{K,\xi,p}(0)$ to $\mathbb{R}^n$ as a homogeneous function of ξ of degree $-1-q$. Then for every even test function $\phi\in\mathcal{S}$,

$$\begin{aligned}
\langle A^{(q)}_{K,\xi,p}(0),\phi(\xi)\rangle &= \frac{1}{2\Gamma(-q)(n-p-q-1)}\times \\
&\times\int_{S^{n-1}}\|\theta\|^{-n+p+q+1}\int_{\mathbb{R}^n}|\langle\theta,\xi\rangle|^{-q-1}\phi(\xi)d\xi d\theta.
\end{aligned}$$

Using Lemma 5 from [GKS]

$$\begin{aligned}
&= \frac{-1}{4\Gamma(-q)\Gamma(1+q)(n-p-q-1)\sin\frac{q\pi}{2}}\times \\
&\qquad\times\int_{S^{n-1}}\|\theta\|^{-n+p+q+1}\int_{-\infty}^\infty |t|^q\hat{\phi}(t\theta)dtd\theta \\
&= \frac{-\sin(-\pi q)}{2\pi(n-p-q-1)\sin\frac{q\pi}{2}}\langle(\|x\|^{-n+p+q+1}\cdot|x|_2^{-p})^\wedge(\xi),\phi(\xi)\rangle.
\end{aligned}$$

The latter follows from the fact that $\Gamma(-q)\Gamma(q+1)=-\pi/\sin(q\pi)$ and the calculation

$$\begin{aligned}\langle (\|x\|^{-n+p+q+1}\cdot |x|_2^{-p})^\wedge(\xi),\phi(\xi)\rangle &= \int_{R^n}\|x\|^{-n+p+q+1}\cdot |x|_2^{-p}\hat{\phi}(x)dx\\ &= \int_{S^{n-1}}\|\theta\|^{-n+p+q+1}\int_0^\infty t^{-n+p+q+1}t^{-p}t^{n-1}\hat{\phi}(t\theta)dtd\theta\\ &= \int_{S^{n-1}}\|\theta\|^{-n+p+q+1}\int_0^\infty t^q\hat{\phi}(t\theta)dtd\theta.\end{aligned}$$

We have proved that

$$\langle A^{(q)}_{K,\xi,p}(0),\phi(\xi)\rangle = \frac{\cos\frac{\pi q}{2}}{\pi(n+p-q-1)}\langle (\|x\|^{-n+p+q+1}\cdot |x|_2^{-p})^\wedge(\xi),\phi(\xi)\rangle$$

for $-1<q<0$. Since both $A^{(q)}_{K,\xi,p}(0)$ and $(\|x\|^{-n+p+q+1}\cdot |x|_2^{-p})^\wedge(\xi)$ are continuous functions of $\xi\in\mathbb{R}^n\setminus\{0\}$, we get the statement of the Lemma for $-1<q<0$.

To prove the Lemma for other values of q we use the fact that for every even test function ϕ the functions

$$q\mapsto \langle A^{(q)}_{K,\xi,p}(0),\phi(\xi)\rangle$$

and

$$q\mapsto \frac{\cos\frac{\pi q}{2}}{\pi(n-p-q-1)}\langle (\|x\|^{-n+p+q+1}\cdot |x|_2^{-p})^\wedge(\xi),\phi(\xi)\rangle$$

are analytic in the domain $\{q\in\mathbb{C}:\ -1<\lceil \mathrm{Re}\, q\rceil<n-p-1\}$. (The fact, that $(\|x\|^{-n+p+q+1}\cdot |x|_2^{-p})^\wedge(\xi)$ is analytic with respect to q, can be seen from the argument of [Ko11, Lemma 2.22]). The result of the Lemma follows, since these analytic functions coincide for $q\in(-1,0)$, ϕ is arbitrary and, by Lemma 4.2.1, the fractional derivative is a continuous function of ξ outside of the origin.

□

Lemma 4.2.3. *Let K be an origin-symmetric convex body in $\mathbb{R}^n$. Assume that $q \in (-1, 2]$ and $0 \le p < n - \lceil q \rceil - 1$, then $\|x\|_K^{-n+p+q+1} \cdot |x|_2^{-p}$ is a positive definite distribution on $\mathbb{R}^n$.*

Proof. First we prove that

$$A_{K,\xi,p}(t) \le A_{K,\xi,p}(0), \qquad \text{for all } t \ge 0 \tag{4.4}$$

If $p = 0$, it follows from Brunn's theorem (see [Ko11, Theorem 2.3]) that the central hyperplane section of an origin-symmetric convex body has maximal volume among all hyperplane sections orthogonal to a given direction. If $p > 0$ one can see that

$$|x|_2^{-p} = p \int_0^\infty \chi(z|x|_2) z^{p-1} dz,$$

therefore

$$\begin{aligned} A_{K,\xi,p}(t) &= \int_{K \cap \langle x,\xi \rangle = t} |x|_2^{-p} dx \\ &= p \int_{K \cap \langle x,\xi \rangle = t} \int_0^\infty \chi(z|x|_2) z^{p-1} dz dx \\ &= p \int_0^\infty z^{p-1} \int_{K \cap \langle x,\xi \rangle = t} \chi(z|x|_2) dx dz \\ &= p \int_0^\infty z^{p-1} \int_{B_{1/z} \cap K \cap \langle x,\xi \rangle = t} dx dz \\ &\le p \int_0^\infty z^{p-1} \int_{B_{1/z} \cap K \cap \langle x,\xi \rangle = 0} dx dz = A_{K,\xi,p}(0) \end{aligned}$$

by Brunn's theorem applied to the convex origin-symmetric body $B_{1/z} \cap K$, where $B_{1/z}$ is a ball of radius $1/z$.

Now consider $q \in (1, 2)$. Here $\cos \dfrac{q\pi}{2}$ is negative, therefore we need to prove that $A^{(q)}_{K,\xi,p}(0) \le 0$. Using inequality (4.4), the formula for fractional derivatives for

$q \in (1,2)$ and the fact that $A'(0) = 0$ we get

$$\begin{aligned} A^{(q)}_{K,\xi,p}(0) &= \frac{1}{\Gamma(-q)} \int_0^\infty t^{-q-1}(A(t) - A(0) - tA'(0))dt \\ &= \frac{1}{\Gamma(-q)} \int_0^\infty t^{-q-1}(A(t) - A(0))dt \leq 0 \end{aligned}$$

since $\Gamma(-q)$ is positive.

If $q \in (0,1)$ then $\cos\frac{q\pi}{2}$ is positive and

$$A^{(q)}_{K,\xi,p}(0) = \frac{1}{\Gamma(-q)} \int_0^\infty t^{-q-1}(A(t) - A(0))dt \geq 0$$

since $\Gamma(-q) < 0$ for these values of q.

Finally if $q \in (-1,0)$ then $\cos\frac{q\pi}{2}$ is positive, $\Gamma(-q)$ is also positive and

$$A^{(q)}_{K,\xi,p}(0) = \frac{1}{\Gamma(-q)} \int_0^\infty t^{-q-1}A(t)dt \geq 0$$

We still have to prove the Lemma for $q = 0, 1, 2$.

When $q = 0$, $\cos\frac{\pi q}{2} = 1$ and

$$A^{(0)}_{K,\xi,p}(0) = (-1)^0 A_{K,\xi,p}(0) \geq 0.$$

When $q = 2$, $\cos\frac{\pi q}{2} = -1$ and

$$A^{(2)}_{K,\xi,p}(0) = (-1)^2 A''_{K,\xi,p}(0) \leq 0,$$

since $A_{K,\xi,p}(t)$ has maximum at 0.

When $q = 1$, take small $\varepsilon > 0$. By what we just proved for non-integer q, for any non-negative test function ϕ,

$$\langle (|x|_2^{-p}\|x\|_K^{-n+p+2+\varepsilon})^\wedge, \phi\rangle \geq 0.$$

Since $\|x\|_K \leq C|x|_2$ for some C, it follows that

$$\|x\|_K^{-n+p+2+\varepsilon}|x|_2^{-p} \leq \tilde{C}|x|_2^{-n+2+\varepsilon} \leq \tilde{C}|x|_2^{-n+1},$$

the latter being a locally-integrable function on $\mathbb{R}^n$.

Set $g(x) = \tilde{C}|x|_2^{-n+1}|\hat{\phi}(x)|$ for $|x|_2 < 1$ and $g(x) = \tilde{C}|\hat{\phi}(x)|$ for $|x|_2 > 1$. The function $g(x)$ is integrable on $\mathbb{R}^n$ and for small ε we have that $\|x\|_K^{-n-p+2+\varepsilon}|x|_2^p\hat{\phi}(x) \leq g(x)$. Therefore by the Lebesgue dominated convergence theorem,

$$\langle(\|x\|_K^{-n+p+2}|x|_2^{-p})^\wedge, \phi\rangle = \int_{\mathbb{R}^n} \|x\|_K^{-n+p+2}|x|_2^{-p}\hat{\phi}(x)dx =$$

$$= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} \|x\|_K^{-n+p+2+\varepsilon}|x|_2^{-p}\hat{\phi}(x)dx = \lim_{\varepsilon\to 0}\langle(\|x\|_K^{-n+p+2+\varepsilon}|x|_2^{-p})^\wedge, \phi\rangle \geq 0$$

$\square$

4.3 The proof of the main result

Theorem 4.3.1. *Let $\alpha \in [n-4, n-1)$, K and L be origin-symmetric infinitely smooth convex bodies in $\mathbb{R}^n$, $n \geq 4$, so that for every $\xi \in S^{n-1}$*

$$(-\Delta)^{\alpha/2}S_K(\xi) \leq (-\Delta)^{\alpha/2}S_L(\xi). \tag{4.5}$$

Then

$$\text{vol}_n(K) \leq \text{vol}_n(L).$$

On the other hand, for any $\alpha \in [n-5, n-4)$ there are convex origin-symmetric bodies $K, L \in \mathbb{R}^n$, $n \geq 5$ that satisfy (4.5) *for every $\xi \in S^{n-1}$ but* $\text{vol}_n(L) < \text{vol}_n(K)$.

Proof of the affirmative part. Let $S_K(\xi) = \text{vol}_{n-1}(K \cap \xi^\perp)$, $\xi \in S^{n-1}$, the central section function defined in the Introduction. Then, as proved in [Ko3]

$$S_K(\xi) = \frac{1}{\pi(n-1)} (\|x\|_K^{-n+1})^\wedge(\xi). \tag{4.6}$$

Extending $S_K(\xi)$ to $\mathbb{R}^n$ as a homogeneous function of degree -1 and using the definition of fractional powers of the Laplacian we get

$$(-\Delta)^{\alpha/2} S_L(\theta) = \frac{1}{\pi(n-1)} (|x|_2^\alpha \|x\|_L^{-n+1})^\wedge(\theta),$$

therefore

$$\begin{aligned}
(2\pi)^n & \int_{S^{n-1}} \|x\|_K^{-1} \|x\|_L^{-n+1} dx = \\
= \quad & (2\pi)^n \int_{S^{n-1}} (|x|_2^{-\alpha} \|x\|_K^{-1})(|x|_2^\alpha \|x\|_L^{-n+1}) dx \\
= \quad & \int_{S^{n-1}} (|x|_2^{-\alpha} \|x\|_K^{-1})^\wedge(\theta)(|x|_2^\alpha \|x\|_L^{-n+1})^\wedge(\theta) d\theta \\
= \quad & \pi(n-1) \int_{S^{n-1}} (|x|_2^{-\alpha} \|x\|_K^{-1})^\wedge(\theta)(-\Delta)^{\alpha/2} S_L(\theta) d\theta
\end{aligned}$$

Here we used Parseval's formula on the sphere (see [Ko4, Lemma 3]) and (4.6).

By Lemma 4.2.3 with $p = \alpha$ and $q = n - \alpha - 2$, $(|x|_2^{-\alpha} \|x\|_K^{-1})^\wedge$ is a non-negative function on S^{n-1}, therefore using the condition of the theorem and repeating the above calculation in the opposite order, we get

$$\int_{S^{n-1}} \|x\|_K^{-1} \|x\|_K^{-n+1} dx \le \int_{S^{n-1}} \|x\|_K^{-1} \|x\|_L^{-n+1} dx.$$

Then by Hölder's inequality and the polar formula for the volume (1.1),

$$n \ \mathrm{vol}_n(K) \le \left(\int_{S^{n-1}} \|\theta\|_K^{-n} d\theta \right)^{1/n} \left(\int_{S^{n-1}} \|\theta\|_L^{-n} d\theta \right)^{(n-1)/n} =$$

$$n(\mathrm{vol}_n(K))^{1/n} (\mathrm{vol}_n(L))^{(n-1)/n},$$

which yields the statement of the positive part of the theorem.

Proof of the negative part. Let $\alpha \in [n-5, n-4)$. We need to construct two convex origin-symmetric bodies $K, L \in \mathbb{R}^n$, $n \geq 5$ such that for every $\xi \in S^{n-1}$

$$(-\Delta)^{\alpha/2} S_K(\xi) \leq (-\Delta)^{\alpha/2} S_L(\xi),$$

but

$$\mathrm{vol}_n(L) < \mathrm{vol}_n(K).$$

First let us prove the following Lemma.

Lemma 4.3.2. *Let $\alpha \in [n-5, n-4)$. There exists an infinitely smooth origin-symmetric convex body L with positive curvature, so that*

$$\|x\|_L^{-1} \cdot |x|_2^{-\alpha}$$

is not a positive definite distribution.

Proof. First assume that $\alpha \in (n-5, n-4)$. Put $q = n - \alpha - 2$, so $q \in (2, 3)$. Our goal is to construct a body L so that there is a $\xi \in S^{n-1}$ satisfying

$$\int_0^\infty t^{-q-1} \left(A_{L,\xi,\alpha}(t) - A_{L,\xi,\alpha}(0) - A''_{L,\xi,\alpha}(0) \frac{t^2}{2} \right) dt < 0. \tag{4.7}$$

If we construct such a body L the result of this lemma immediately follows from Lemma 4.2.2 and the definition of fractional derivatives.

Consider the function

$$f(t) = \left(1 - t^2 - N t^4\right)^{\frac{1}{n-\alpha-1}}$$

Let a_N be the positive real root of the equation $f(t) = 0$. Define the body $L \in \mathbb{R}^n$ as follows.

$$L = \left\{ (x_1, ..., x_n) \in \mathbb{R}^n \colon x_n \in [-a_N, a_N] \text{ and } \left(\sum_{i=1}^{n-1} x_i^2 \right)^{1/2} \le f(x_n) \right\},$$

which is a strictly convex infinitely differentiable body.

Take ξ to be the unit vector in the direction of the x_n-axis. Then for $t \in [0, a_N]$,

$$\begin{aligned} A_{L,\xi,\alpha}(t) &= \int_{S^{n-1}} \int_0^{f(t)} (t^2 + r^2)^{-\alpha/2} r^{n-2} dr \, d\theta \\ &= C_n \int_0^{f(t)} (t^2 + r^2)^{-\alpha/2} r^{n-2} dr \end{aligned}$$

where $C_n = |S^{n-1}|$, and for $t > a_N$ we have $A_{L,\xi,\alpha}(t) = 0$.

One can compute:

$$A_{L,\xi,p}(0) = \frac{C_n}{n - \alpha - 1},$$

and

$$A''_{L,\xi,p}(0) = -C_n \left[\frac{\alpha}{n - \alpha - 3} + \frac{2}{n - \alpha - 1} \right].$$

In order to estimate the integral (4.7), we split it into three parts: over $[0, b_N]$, $[b_N, a_N]$ and $[a_N, \infty)$, where b_N is the positive real root of the equation $1 - t^2 - Nt^4 = t^{q+1}$. Recall that a_N was defined as the positive real root of the equation $1 - t^2 - Nt^4 = 0$. It is easy to check that $a_N \simeq b_N \simeq N^{-1/4}$ for large N. Also note that on $[0, a_N]$ we have $f(t) \ge 0$, and $f(t) \ge t$ if and only if $t \in [0, b_N]$.

First consider the interval $[0, b_N]$. For all t from this interval we have $t \le f(t)$. Then we can break the integral:

$$\int_0^{f(t)} (t^2 + r^2)^{-\alpha/2} r^{n-2} dr = I_1 + I_2$$

into two parts, where the first one can be estimated as follows

$$I_1 = \int_0^t (t^2+r^2)^{-\alpha/2} r^{n-2} dr \leq \int_0^t (r^2)^{-\alpha/2} r^{n-2} dr = \frac{t^{n-\alpha-1}}{n-\alpha-1}$$

and for the second one we will use the inequality:

$$(1+x)^{-\gamma} \leq 1-\gamma x + \frac{\gamma(\gamma+1)}{2}x^2, \quad \text{for } \gamma > 0 \text{ and } 0 < x < 1.$$

Then

$$\begin{aligned}
I_2 &= \int_t^{f(t)} (t^2+r^2)^{-\alpha/2} r^{n-2} dr \\
&= \int_t^{f(t)} \left(1+\frac{t^2}{r^2}\right)^{-\alpha/2} r^{n-\alpha-2} dr \leq \\
&\leq \int_t^{f(t)} \left(1-\frac{\alpha}{2}\frac{t^2}{r^2}+\frac{\frac{\alpha}{2}\left(\frac{\alpha}{2}+1\right)}{2}\frac{t^4}{r^4}\right) r^{n-\alpha-2} dr \\
&= \left[\frac{r^{n-\alpha-1}}{n-\alpha-1}-\frac{\alpha}{2}\frac{t^2 r^{n-\alpha-3}}{n-\alpha-3}+\frac{\frac{\alpha}{2}\left(\frac{\alpha}{2}+1\right)}{2}\frac{t^4 r^{n-\alpha-5}}{n-\alpha-5}\right]_t^{f(t)} \\
&= \frac{f^{n-\alpha-1}(t)}{n-\alpha-1}-\frac{\alpha}{2}\frac{t^2}{n-\alpha-3}f^{n-\alpha-3}(t)+ \\
&\quad +\frac{\frac{\alpha}{2}\left(\frac{\alpha}{2}+1\right)}{2}\frac{t^4}{n-\alpha-5}f^{n-\alpha-5}(t)+Ct^{n-\alpha-1} \\
&\leq \frac{f^{n-\alpha-1}(t)}{n-\alpha-1}-\frac{\alpha}{2}\frac{t^2}{n-\alpha-3}f^{n-\alpha-3}(t)+Ct^{n-\alpha-1} \\
&= \frac{1-t^2-Nt^4}{n-\alpha-1}-\frac{\alpha}{2}\frac{t^2}{n-\alpha-3}(1-t^2-Nt^4)^{\frac{n-\alpha-3}{n-\alpha-1}}+Ct^{n-\alpha-1}
\end{aligned}$$

for some constant C. The last inequality follows from $f(t) \geq 0$ on $[0, b_N]$ and $\alpha \in (n-5, n-4)$.

Using the inequality:

$$(1-x)^{\gamma} \geq 1-\gamma x(1-x)^{\gamma-1}, \quad \text{for } 0 < \gamma < 1 \text{ and } 0 < x < 1,$$

applied to the second term in the previous expression, we get

$$\begin{aligned}
I_2 &\leq \frac{1-t^2-Nt^4}{n-\alpha-1} - \frac{\alpha}{2}\frac{t^2}{n-\alpha-3} \times \\
&\times \left(1 - \frac{n-\alpha-3}{n-\alpha-1}(1-t^2-Nt^4)^{\frac{n-\alpha-3}{n-\alpha-1}-1}(t^2+Nt^4)\right) + Ct^{n-\alpha-1} \\
&= \frac{1-t^2-Nt^4}{n-\alpha-1} - \frac{\alpha}{2}\frac{t^2}{n-\alpha-3} + \\
&\qquad + C_1\frac{t^4+Nt^6}{(1-t^2-Nt^4)^{\frac{2}{n-\alpha-1}}} + Ct^{n-\alpha-1}
\end{aligned}$$

Now using the estimates for I_1 and I_2 we get

$$\int_0^{b_N} t^{-q-1}\left(A_{L,\xi,\alpha}(t) - A_{L,\xi,\alpha}(0) - A''_{L,\xi,\alpha}(0)\frac{t^2}{2}\right) dt \leq$$

$$\begin{aligned}
&\leq C_n \int_0^{b_N} t^{-q-1}\Big(\frac{1-t^2-Nt^4}{n-\alpha-1} - \frac{\alpha}{2}\frac{t^2}{n-\alpha-3} + C_1\frac{t^4+Nt^6}{(1-t^2-Nt^4)^{\frac{2}{n-\alpha-1}}} \\
&\qquad + Ct^{n-\alpha-1} - \frac{1}{n-\alpha-1} + \left[\frac{\alpha}{n-\alpha-3} + \frac{2}{n-\alpha-1}\right]\frac{t^2}{2}\Big)dt \\
&= C_n \int_0^{b_N} t^{-q-1}\left(\frac{-Nt^4}{n-\alpha-1} + C_1\frac{t^4+Nt^6}{(1-t^2-Nt^4)^{\frac{2}{n-\alpha-1}}} + Ct^{n-\alpha-1}\right) dt
\end{aligned}$$

Now one can estimate each term of the last integral separately. Since $b_N \simeq N^{-1/4}$, we get that

$$\int_0^{b_N} t^{-q-1}\frac{-Nt^4}{n-\alpha-1}dt \simeq -C_2 N^{q/4}$$

for a positive constant C_2.

For the second term, we change the variable of integration: $u = N^{1/4}t$. Then

$$\begin{aligned}
&\int_0^{b_N} t^{-q-1}\frac{t^4+Nt^6}{(1-t^2-Nt^4)^{\frac{2}{n-\alpha-1}}}dt \\
&\qquad = N^{q/4}\int_0^{b_N N^{1/4}} u^{-q-1}\frac{u^4N^{-1}+u^6N^{-1/2}}{(1-N^{-1/2}u^2-u^4)^{\frac{2}{n-\alpha-1}}}du \\
&\qquad \leq N^{(q-2)/4}\int_0^{b_N N^{1/4}} u^{-q-1}\frac{u^4+u^6}{(1-N^{-1/2}u^2-u^4)^{\frac{2}{n-\alpha-1}}}du \\
&\qquad \leq C_3 N^{(q-2)/4},
\end{aligned}$$

since $b_N N^{1/4} \to 1$ as $N \to \infty$, and the integral

$$\int_0^1 u^{-q-1} \frac{u^4 + u^6}{(1-u^4)^{\frac{2}{n-\alpha-1}}} du$$

converges both at 0 and 1.

And finally the integral of the last term is small for large values of N, since $n - \alpha - 1 = q + 1$. From what we have obtained one can see that the integral over $[0, b_N]$ will be negative for large values of N since the leading term is $-C_2 N^{q/4}$.

Now consider the integral over $[b_N, a_N]$. The expression $A_{L,\xi,\alpha}(t) - A_{L,\xi,\alpha}(0) - A''_{L,\xi,\alpha}(0)t^2/2$ can be estimated from above by a constant C, not depending on N. Indeed, $A_{L,\xi,\alpha}(t) \leq A_{L,\xi,\alpha}(0)$, $A''_{L,\xi,\alpha}(0)$ is a constant independent of N, and $t \leq a_N \simeq N^{-1/4} \leq 1$ for N large enough. Therefore

$$\int_{b_N}^{a_N} t^{-q-1} \left(A_{L,\xi,\alpha}(t) - A_{L,\xi,\alpha}(0) - A''_{L,\xi,\alpha}(0)\frac{t^2}{2} \right) dt \leq$$
$$\leq C \int_{b_N}^{a_N} t^{-q-1} dt \leq C \int_{b_N}^{a_N} (b_N)^{-q-1} dt = C \frac{a_N - b_N}{(b_N)^{q+1}}$$

Recalling that a_N and b_N satisfy the equations

$$1 - a_N^2 - N a_N^4 = 0 \quad \text{and} \quad 1 - b_N^2 - N b_N^4 = b_N^{q+1}$$

we conclude that

$$b_N^{q+1} = (a_N^2 - b_N^2)(1 + N(a_N^2 + b_N^2)).$$

Therefore

$$C \int_{b_N}^{a_N} t^{-q-1} dt \leq \frac{C}{(a_N + b_N)(1 + N(a_N^2 + b_N^2))} \simeq C N^{-1/4}.$$

Finally, the integral over $[a_N, \infty)$ can be computed as follows

$$\int_{a_N}^{\infty} t^{-q-1} \left(-A_{L,\xi,\alpha}(0) - A''_{L,\xi,\alpha}(0)\frac{t^2}{2} \right) dt \quad \simeq \quad -D_1 N^{q/4} + D_2 N^{(q-2)/4}$$

where $D_1 > 0$. Therefore, this integral is negative for N large enough.

Combining all the integrals one can see that for N large enough the desired integral (4.7) is negative. This means that for some direction $\xi \in S^{n-1}$ the function $(\|x\|_L^{-1} \cdot |x|_2^{-\alpha})^\wedge(\xi)$ is negative, if $\alpha \in (n-5, n-4)$.

If $\alpha = n-5$, both sides of the equality in the statement of Lemma 4.2.2 vanish, therefore we need to apply the argument from [GKS] (see the proof of Theorem 1). Then

$$(\|x\|_L^{-1} \cdot |x|_2^{-n+5})^\wedge(\xi) =$$
$$= C \int_0^\infty t^{-4} \left(A_{L,\xi,\alpha}(t) - A_{L,\xi,\alpha}(0) - A''_{L,\xi,\alpha}(0)\frac{t^2}{2} \right) dt$$

for a positive constant C. Considering the same body as before, we get that $(\|x\|_L^{-1} \cdot |x|_2^{-n+5})^\wedge(\xi)$ is also negative at some point ξ.

□

Now we are ready to finish the proof of the negative part. Apply Lemma 4.3.2 to construct an infinitely smooth origin-symmetric body L with positive curvature for which $(\|x\|_L^{-1} \cdot |x|_2^{-\alpha})^\wedge(\xi) < 0$ for some direction ξ. By Lemma 4.2.2, the function $(\|x\|_L^{-1} \cdot |x|_2^{-\alpha})^\wedge$ is continuous on the sphere S^{n-1}, hence there is a neighborhood of ξ where it is negative.

Let

$$\Omega = \{\theta \in S^{n-1} \colon (\|x\|_L^{-1} \cdot |x|_2^{-\alpha})^\wedge(\theta) < 0\}.$$

Choose a non-positive infinitely differentiable even function v supported on Ω. Extend v to a homogeneous function $r^{-\alpha-1}v(\theta)$ of degree $-\alpha-1$ on $\mathbb{R}^n$. By [Ko11, Lemma 3.16], the Fourier transform of $|x|_2^{-\alpha-1}v(x/|x|_2)$ is equal to $|x|_2^{-n+\alpha+1}g(x/|x|_2)$

for some infinitely differentiable function g on S^{n-1}.

Define a body K by

$$\|x\|_K^{-n+1} = \|x\|_L^{-n+1} + \varepsilon |x|_2^{-n+1} g(x/|x|_2)$$

for some small ε so that the body K is convex (see for example [Ko11, Theorem 5.3] for this standard perturbation argument). Multiply both sides by $\frac{1}{\pi(n-1)}|x|_2^{\alpha}$ and apply the Fourier transform:

$$(-\Delta)^{\alpha/2} S_K = (-\Delta)^{\alpha/2} S_L + \frac{\varepsilon (2\pi)^n}{\pi(n-1)} |x|_2^{-\alpha-1} v(x/|x|_2) \leq (-\Delta)^{\alpha/2} S_L,$$

since v is non-positive.

On the other hand,

$$\begin{aligned}
&\int_{S^{n-1}} (\|x\|_L^{-1} \cdot |x|_2^{-\alpha})^\wedge(\theta)(-\Delta)^{\alpha/2} S_K d\theta = \\
&= \int_{S^{n-1}} (\|x\|_L^{-1} \cdot |x|_2^{-\alpha})^\wedge(\theta)(-\Delta)^{\alpha/2} S_L d\theta \\
&\qquad + \varepsilon \frac{(2\pi)^n}{\pi(n-1)} \int_{S^{n-1}} (\|x\|_L^{-1} \cdot |x|_2^{-\alpha})^\wedge(\theta) v(\theta) d\theta \\
&> \int_{S^{n-1}} (\|x\|_L^{-1} \cdot |x|_2^{-\alpha})^\wedge(\theta)(-\Delta)^{\alpha/2} S_L d\theta.
\end{aligned}$$

Repeating the argument from the proof of the affirmative part we get:

$$\mathrm{vol}_n(L) < \mathrm{vol}_n(K).$$

□

Remarks. (i) The negative part is formulated only for $q \in [n-5, n-4)$, because we wanted this to work for $n = 5$. In fact, for bigger n one can take $q \in [0, n-4)$. Also the condition (1) can be written in terms of the Fourier transforms so that no smoothness of the bodies is required.

(ii) In the case where $q = n - 4$ and n is an even integer, the result of Theorem 4.3.1 was proved in [Ko10] using an induction argument. The proof from [Ko10] can not be extended to other values of q and n and does not produce any results in the negative direction.

(iii) Shephard's problem (see, for example, [Ko11, Section 8.4]) asks whether convex origin-symmetric bodies with smaller projections necessarily have smaller n-dimensional volume. As proved independently by Petty [Pe] and Schneider [Sc], the answer to this problem is affirmative only in dimension $n = 2$, so one may try to modify Shephard's problem to guarantee the affirmative answer in all dimensions. However, attempts to repeat the proof of Theorem 4.3.1 for Shephard's problem fail, since the section function $A_{K,\xi,p}$ may not be sufficiently differentiable.

Chapter 5

Centroid bodies and comparison of volumes.

5.1 Introduction

Here we state results from [YYa]. Let K be a star body in $\mathbb{R}^n$, then the centroid body of K is a convex body ΓK defined by its support function:

$$h_{\Gamma K}(\xi) = \frac{1}{\mathrm{vol}(K)} \int_K |(x,\xi)| dx, \quad \xi \in \mathbb{R}^n.$$

Let K and L be two origin-symmetric star bodies in $\mathbb{R}^n$ such that $\Gamma K \subset \Gamma L$, what can be said about the volumes of K and L? Lutwak [Lu2] proved that, if L is a polar projection body then $\mathrm{vol}(K) \leq \mathrm{vol}(L)$. On the other hand, if K is not a polar projection body, then there is a body L, so that $\Gamma K \subset \Gamma L$, but $\mathrm{vol}(K) > \mathrm{vol}(L)$. Since in $\mathbb{R}^2$ every convex body is a polar projection body [Sc], the results of Lutwak imply the following:

Suppose that K and L are two origin-symmetric convex bodies in $\mathbb{R}^n$ such that $\Gamma K \subset \Gamma L$. If $n = 2$, then we necessarily have $\mathrm{vol}(K) \leq \mathrm{vol}(L)$, while this is no longer true if $n \geq 3$.

Let K be a star body in $\mathbb{R}^n$ and $p \geq 1$, then the p-centroid body of K is the body $\Gamma_p K$ defined by:

$$h_{\Gamma_p K}(\xi) = \left(\frac{1}{\text{vol}(K)} \int_K |(x, \xi)|^p dx \right)^{1/p}, \quad \xi \in \mathbb{R}^n. \tag{5.1}$$

Clearly, $h_{\Gamma_p K}$ is a homogeneous function of degree 1, and if $p \geq 1$, then this function is convex, and, therefore, $\Gamma_p K$ is well-defined. The polar of $\Gamma_p K$ is called the polar p-centroid body of K and denoted by $\Gamma_p^* K$. Since the support function of a body is the norm of its polar, $h = \|\cdot\|_*$, the polar p-centroid body of K is given by

$$\|\xi\|_{\Gamma_p^* K} = \left(\frac{1}{\text{vol}(K)} \int_K |(x, \xi)|^p dx \right)^{1/p}, \quad \xi \in \mathbb{R}^n. \tag{5.2}$$

The p-centroid bodies and their polars have recently been studied by different authors, see e.g. [CG], [GZ], [Lu2], [LYZ], [LZ]. In [GZ] Grinberg and Zhang generalized the results of Lutwak discussed in the beginning of this section. Namely, let K and L be two origin-symmetric star bodies in $\mathbb{R}^n$ such that for $p \geq 1$

$$\Gamma_p K \subset \Gamma_p L.$$

They prove that if the space $(\mathbb{R}^n, \|\cdot\|_L)$ embeds in L_p, then we necessarily have

$$\text{vol}(K) \leq \text{vol}(L).$$

On the other hand, if $(\mathbb{R}^n, \|\cdot\|_K)$ does not embed in L_p, then there is a body L so that $\Gamma_p K \subset \Gamma_p L$, but $\text{vol}(K) \leq \text{vol}(L)$.

Note, that if $p = 1$ the positive answer holds for all convex bodies in $\mathbb{R}^2$, while if $p > 1$ there is no dimension where this is always true. The preceding remark suggests considering $p < 1$ in order to make the answer affirmative in higher dimensions.

If $p < 1$, then the function $h_{\Gamma_p K}(\xi)$ in (5.1) is not necessarily convex, therefore it is not a support function, but the definition of the polar p-centroid body still makes sense, even though these bodies may be non-convex. So for all $p > -1$, $p \neq 0$ we define the polar p-centroid body of a star body K by the formula:

$$\|\xi\|_{\Gamma_p^* K} = \left(\frac{1}{\mathrm{vol}(K)} \int_K |(x,\xi)|^p dx \right)^{1/p}, \quad \xi \in \mathbb{R}^n. \tag{5.3}$$

For $p = 0$, this definition looks as follows (if we send $p \to 0$):

$$\|\xi\|_{\Gamma_0^* K} = \exp\left(\frac{1}{\mathrm{vol}(K)} \int_K \ln |(x,\xi)| dx \right), \quad \xi \in \mathbb{R}^n. \tag{5.4}$$

Now we can ask the question discussed above for all $p > -1$. Namely, suppose that

$$\Gamma_p^* L \subset \Gamma_p^* K, \tag{5.5}$$

for origin-symmetric star bodies K and L. Does it follow that we have an inequality for the volumes of K and L? In this paper we show that if $(\mathbb{R}^n, \|\cdot\|_L)$ embeds in L_p, $p > -1$, then we have $\mathrm{vol}(K) \leq \mathrm{vol}(L)$. However if $(\mathbb{R}^n, \|\cdot\|_K)$ does not embed in L_p, we construct counterexamples to the latter result.

These results can also be reformulated as follows:

(i) If $0 < p < 1$, then in $\mathbb{R}^2$ the condition (5.5) implies that $\mathrm{vol}(K) \leq \mathrm{vol}(L)$, while this is no longer true in dimensions $n \geq 3$.

(ii) If $-1 < p \leq 0$, (5.5) implies that $\mathrm{vol}(K) \leq \mathrm{vol}(L)$ if and only if $n \leq 3$.

Clearly the integral in (5.3) diverges if $p \leq -1$, but still we can make sense of this integral considering fractional derivatives. Indeed, if $-1 < p < 0$

$$\begin{aligned}\frac{1}{\mathrm{vol}(K)}\int_K |(x,\xi)|^p dx &= \frac{1}{\mathrm{vol}(K)}\int_{-\infty}^{\infty}|z|^p\int_{(x,\xi)=z}\chi(\|x\|_K)dx\ dz\\ &= \frac{1}{\mathrm{vol}(K)}\int_{-\infty}^{\infty}|z|^p A_{K,\xi}(z)dz\\ &= \frac{2\Gamma(p+1)}{\mathrm{vol}(K)}A_{K,\xi}^{(-p-1)}(0),\end{aligned}$$

where $A_{K,\xi}(z)$ is the parallel section function of K, and $A_{K,\xi}^{(-p-1)}(0)$ is its fractional derivative at zero. (For details on fractional derivatives, see e.g. [Ko11, Section 2.6]). So, in such terms our problem can be written as follows:

Suppose K and L are two origin-symmetric star bodies, so that for all $\xi \in S^{n-1}$:

$$\frac{A_{K,\xi}^{(-p-1)}(0)}{\mathrm{vol}(K)} \le \frac{A_{L,\xi}^{(-p-1)}(0)}{\mathrm{vol}(L)}.$$

Do we necessarily have an inequality for the volumes of K and L?

Note that Koldobsky already considered such inequalities (see e.g. [Ko9]) without dividing by volumes. So, for $-1 < p < 0$ the positive part of our results can also be obtained from the results of Koldobsky, but we give our own proof. The case $p = -1$ leads to the following modification of the Busemann-Petty problem. Let K and L be two convex origin-symmetric bodies in $\mathbb{R}^n$ such that

$$\frac{\mathrm{vol}_{n-1}(K\cap\xi^\perp)}{\mathrm{vol}(K)} \le \frac{\mathrm{vol}_{n-1}(L\cap\xi^\perp)}{\mathrm{vol}(L)}.$$

Does this imply an inequality for the volumes of K and L?

It is easy to show that in dimensions $n \le 4$ we have $\mathrm{vol}(L) \le \mathrm{vol}(K)$. The proof is almost identical to that of the original solution of the Busemann-Petty problem from [GKS]. The counterexamples in dimensions $n \ge 5$ from [GKS] also work in this situation.

In view of all these remarks one can consider our results as a certain bridge between the results of Lutwak-Grinberg-Zhang about p-centroid bodies and the results of Busemann-Petty type obtained by Koldobsky.

5.2 Centroid inequalities for $-1 < p < 1$, $p \neq 0$.

The *support function* of a convex body K in $\mathbb{R}^n$ is defined by

$$h_K(x) = \max_{\xi \in K}(x, \xi), \quad x \in \mathbb{R}^n.$$

If K is origin-symmetric, then h_K is the Minkowski norm of the polar body K^*.

A well-known result going back to P.Lévy, (see [BL, p. 189] or [Ko11, Section 6.1]), is that a space $(\mathbb{R}^n, \|\cdot\|)$ embeds into L_p, $p > 0$ if and only if there exists a finite Borel measure μ on the unit sphere so that, for every $x \in \mathbb{R}^n$,

$$\|x\|^p = \int_{S^{n-1}} |(x, \xi)|^p d\mu(\xi). \tag{5.6}$$

On the other hand, this can be considered as the definition of embedding in L_p, $-1 < p < 0$ (cf. [Ko7]).

It was proved in [Ko2] that a space $(\mathbb{R}^n, \|\cdot\|)$ embeds isometrically in L_p, $p > 0$, $p \notin 2\mathbb{N}$ if and only if the Fourier transform of the function $\Gamma(-p/2)\|x\|^p$ (in the sense of distributions) is a positive distribution outside of the origin. If $-n < p < 0$ a similar fact was proved in [Ko7]: a space $(\mathbb{R}^n, \|\cdot\|)$ embeds in L_p if and only if the Fourier transform of $\|\cdot\|^p$ is a positive distribution in the whole $\mathbb{R}^n$.

Now we are ready to prove our first result.

Theorem 5.2.1. *Let $-1 < p < 1$, $p \neq 0$. Let K and L be origin-symmetric convex* *bodies in* $\mathbb{R}^n$*, so that* $(\mathbb{R}^n, \|\cdot\|_K)$ *embeds in* L_p *and*

$$\Gamma_p^* K \subset \Gamma_p^* L. \tag{5.7}$$

Then $\mathrm{vol}(L) \le \mathrm{vol}(K)$.

Proof. First let us prove the case $0 < p < 1$. Since $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_p, there exists a measure μ_K on the unit sphere S^{n-1} such that

$$\|x\|_K^p = \int_{S^{n-1}} |(x,\xi)|^p d\mu_K(\xi).$$

Note that (5.7) can be written as

$$\frac{1}{\mathrm{vol}(L)} \int_L |(x,\xi)|^p dx \le \frac{1}{\mathrm{vol}(K)} \int_K |(x,\xi)|^p dx. \tag{5.8}$$

Integrating both sides of the last inequality over S^{n-1} with the measure μ_K, we get

$$\frac{1}{\mathrm{vol}(L)} \int_{S^{n-1}} \int_L |(x,\xi)|^p dx \; d\mu_K(\xi) \le \frac{1}{\mathrm{vol}(K)} \int_{S^{n-1}} \int_K |(x,\xi)|^p dx \; d\mu_K(\xi).$$

Applying Fubini's Theorem,

$$\frac{1}{\mathrm{vol}(L)} \int_L \|x\|_K^p dx \le \frac{1}{\mathrm{vol}(K)} \int_K \|x\|_K^p dx. \tag{5.9}$$

Note that

$$\begin{aligned} \int_K \|x\|_K^p dx &= \int_{S^{n-1}} \left(\int_0^{\|\theta\|_K^{-1}} \|r\theta\|_K^p \; r^{n-1} dr \right) d\theta \\ &= \frac{1}{n+p} \int_{S^{n-1}} \|\theta\|_K^{-n} d\theta = \frac{n}{n+p} \mathrm{vol}(K). \end{aligned}$$

Therefore, (5.9) can be rewritten as

$$\frac{1}{\mathrm{vol}(L)} \int_L \|x\|_K^p dx \le \frac{n}{n+p}.$$

Using the inequality

$$\frac{1}{\mathrm{vol}(L)}\int_L \|x\|_K^p dx \ge \frac{n}{n+p}\left(\frac{\mathrm{vol}(L)}{\mathrm{vol}(K)}\right)^{p/n} \tag{5.10}$$

from [MiP, Section 2.2], we get

$$\frac{n}{n+p} \quad\ge\quad \frac{1}{\mathrm{vol}(L)}\int_L \|x\|_K^p dx \ge \frac{n}{n+p}\left(\frac{\mathrm{vol}(L)}{\mathrm{vol}(K)}\right)^{p/n},$$

therefore $\mathrm{vol}(L) \le \mathrm{vol}(K)$, which proves the theorem for $0 < p < 1$.

Now consider $-1 < p < 0$. In this case (5.7) is equivalent to

$$\frac{1}{\mathrm{vol}(L)}\int_L |(x,\xi)|^p dx \ge \frac{1}{\mathrm{vol}(K)}\int_K |(x,\xi)|^p dx. \tag{5.11}$$

Since $(\mathbb{R}^n, \|\cdot\|_K)$ embeds into L_p, $p > -1$, there exists a measure μ_K on the unit sphere such that

$$\|x\|_K^p = \int_{S^{n-1}} |(x,\xi)|^p d\mu_K(\xi).$$

Integrating both sides of (5.11) over S^{n-1} with the measure μ_K and using the same argument as in the first part of the proof, we get

$$\frac{1}{\mathrm{vol}(L)}\int_L \|x\|_K^p dx \ge \frac{n}{n+p}. \tag{5.12}$$

Passing to spherical coordinates and applying Hölder's inequality

$$\begin{aligned}\int_L \|x\|_K^p dx &= \int_{S^{n-1}}\left(\int_0^{\|\theta\|_L^{-1}} r^{n+p-1}\|\theta\|_K^p dr\right)d\theta \\ &= \frac{1}{n+p}\int_{S^{n-1}} \|\theta\|_L^{-n-p}\|\theta\|_K^p d\theta \\ &\le \frac{1}{n+p}\left(\int_{S^{n-1}} \|\theta\|_L^{-n} d\theta\right)^{(n+p)/n}\left(\int_{S^{n-1}} \|\theta\|_K^{-n} d\theta\right)^{-p/n} \\ &= \frac{n}{n+p}\left(\mathrm{vol}(L)\right)^{(n+p)/n}\left(\mathrm{vol}(K)\right)^{-p/n}.\end{aligned}$$

So (5.12) can be written as

$$\begin{aligned} 1 &\le \frac{1}{\mathrm{vol}(L)} (\mathrm{vol}(L))^{(n+p)/n} (\mathrm{vol}(K))^{-p/n} \\ &= (\mathrm{vol}(L))^{p/n} (\mathrm{vol}(K))^{-p/n} . \end{aligned}$$

Therefore, using the fact that $p < 0$, we get $\mathrm{vol}(L) \le \mathrm{vol}(K)$.

□

Since all 2-dimensional spaces embed in L_1, and therefore in L_p with $-2 < p < 1$ (see e.g. [Ko11, Chapter 6]), and all 3-dimensional spaces embed in L_0, and therefore in L_p with $-3 < p < 0$ (see [KKYY]), we have the following

Corollary 5.2.2. *Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$, so that $\Gamma_p^* K \subset \Gamma_p^* L$. Then*

i) if $0 < p < 1$, we necessarily have $\mathrm{vol}(L) \le \mathrm{vol}(K)$ in dimension $n = 2$,

ii) if $-1 < p < 0$, we necessarily have $\mathrm{vol}(L) \le \mathrm{vol}(K)$ in dimensions $n = 2$ and 3.

In order to show a negative counterpart of Theorem 5.2.1, we need some lemmas. The following Lemma is [Ko11, Corollary 3.15] with $k = 0$ and $p = -q - 1$.

Lemma 5.2.3. *Let $-1 < p < 1$, $p \neq 0$. For an origin-symmetric convex body K in $\mathbb{R}^n$ we have*

$$\left(\|x\|_K^{-n-p}\right)^\wedge (\xi) = -\frac{\pi}{2\Gamma(p+1)\sin(\pi p/2)} \int_{S^{n-1}} |(\theta, \xi)|^p \; \|\theta\|_K^{-n-p} d\theta.$$

We will use this formula in the following form:

$$\left(\|x\|_K^{-n-p}\right)^\wedge (\xi) = -\frac{\pi(n+p)}{2\Gamma(p+1)\sin(\pi p/2)} \int_K |(x, \xi)|^p dx.$$

Also we can write this formula in terms of fractional derivatives of the parallel section function of K. Recall that the parallel section function of a an origin-symmetric star body K is defined by

$$A_{K,\xi}(z) = \int_{(x,\xi)=z} \chi(\|x\|_K)dx.$$

For $-1 < q < 0$ the fractional derivative of this function at zero is defined by

$$A^{(q)}_{K,\xi}(0) = \frac{1}{2\Gamma(-q)} \int_{-\infty}^{\infty} |z|^{-1-q} A_{K,\xi}(z)dz = \frac{1}{2\Gamma(-q)} \int_K |(x,\xi)|^{-1-q} dx.$$

In fact one can see that this is analytically extendable to $q < -1$. Therefore Lemma 5.2.3 can be reformulated as follows. Let $-1 < p < 1$, $p \neq 0$, then

$$\left(\|x\|_K^{-n-p}\right)^\wedge(\xi) = -\frac{\pi(n+p)}{\sin(\pi p/2)} A^{(-p-1)}_{K,\xi}(0).$$

Note, that for $-1 < p < 0$ this formula was proved in [GKS].

Now recall a version of Parseval's formula on the sphere proved by Koldobsky [Ko4].

Lemma 5.2.4. *If K and L are origin-symmetric infinitely smooth bodies in $\mathbb{R}^n$ and $0 < p < n$, then $(\|x\|_K^{-p})^\wedge$ and $(\|x\|_L^{-n+p})^\wedge$ are continuous functions on S^{n-1} and*

$$\int_{S^{n-1}} \left(\|x\|_K^{-p}\right)^\wedge(\xi) \left(\|x\|_L^{-n+p}\right)^\wedge(\xi)d\xi = (2\pi)^n \int_{S^{n-1}} \|x\|_K^{-p}\|x\|_L^{-n+p}dx.$$

Remark 5.2.5. A proof of this formula via spherical harmonics was given in [Ko9]. Repeating this proof word by word and using the above definition of the fractional derivative of order $q < -1$, one can easily extend this result to $-1 < p < 0$.

Now we prove a negative counterpart of Theorem 5.2.1.

Theorem 5.2.6. *Let L be an infinitely smooth origin-symmetric strictly convex body in $\mathbb{R}^n$, for which $(\mathbb{R}^n, \|\cdot\|_L)$ does not embed in L_p, $-1 < p < 1$, $p \neq 0$. Then there exists an origin-symmetric convex body K in $\mathbb{R}^n$ such that*

$$\Gamma_p^* K \subset \Gamma_p^* L.$$

but

$$\mathrm{vol}(L) > \mathrm{vol}(K).$$

Proof. First consider $0 < p < 1$. Since $(\mathbb{R}^n, \|\cdot\|_L)$ does not embed in L_p, there exists a $\xi \in S^{n-1}$ such that $(\|x\|_L^p)^\wedge(\xi)$ is positive; for more details see [Ko2]. Because $(\|x\|_L^p)^\wedge(\theta)$ is a continuous function on S^{n-1}, there exists a neighborhood of ξ where it is positive. Define

$$\Omega = \{\theta \in S^{n-1} : (\|x\|_L^p)^\wedge(\theta) > 0\}.$$

Choose a non-positive infinitely-smooth even function v supported in Ω. Extend v to a homogeneous function $|x|_2^{-n-p} v(x/|x|_2)$ of degree $-n-p$ on $\mathbb{R}^n$. By [Ko11, Lemma 3.16], the Fourier transform of $|x|_2^{-n-p} v(x/|x|_2)$ is equal to $|x|_2^p\, g(x/|x|_2)$ for some infinitely smooth function g on S^{n-1}.

Define a body K by

$$\|x\|_K^{-n-p} = \|x\|_L^{-n-p} + \epsilon |x|_2^{-n-p} g(x/|x|_2)$$

for some small ϵ so that the body K is convex (see e.g. the perturbation argument from [Ko11, p.96]). Applying the Fourier transform to both sides we get

$$\left(\|x\|_K^{-n-p}\right)^\wedge(\xi) = \left(\|x\|_L^{-n-p}\right)^\wedge(\xi) + \epsilon (2\pi)^n |\xi|_2^p v(\xi/|\xi|_2).$$

So using the formula from Lemma 5.2.3

$$\left(\|x\|_K^{-n-p}\right)^\wedge(\xi) = \Gamma(-p)\sin\left(\frac{\pi(p+1)}{2}\right)\int_K |(x,\xi)|^p dx$$

we have

$$\int_L |(x,\xi)|^p dx < \int_K |(x,\xi)|^p dx. \tag{5.13}$$

Consider the integral

$$\begin{aligned}
&\int_{S^{n-1}} \left(\|x\|_L^p\right)^\wedge(\xi)\left(\|x\|_K^{-n-p}\right)^\wedge(\xi)d\xi \\
&= \int_{S^{n-1}} \left(\|x\|_L^p\right)^\wedge(\xi)\left(\|x\|_L^{-n-p}\right)^\wedge(\xi)d\xi + \epsilon(2\pi)^n \int_{S^{n-1}} \left(\|x\|_L^p\right)^\wedge(\xi)v(\xi)d\xi \\
&< \int_{S^{n-1}} \left(\|x\|_L^p\right)^\wedge(\xi)\left(\|x\|_L^{-n-p}\right)^\wedge(\xi)d\xi \\
&= (2\pi)^n \int_{S^{n-1}} \|x\|_L^p\|x\|_L^{-n-p}dx = (2\pi)^n n\mathrm{vol}(L).
\end{aligned} \tag{5.14}$$

Here we used a version of Parseval's formula (Lemma 5.2.4 and Remark 5.2.5) and the fact that v is negative on Ω.

On the other hand, again using Parseval's formula and (5.10)

$$\begin{aligned}
&\int_{S^{n-1}} \left(\|x\|_L^p\right)^\wedge(\xi)\left(\|x\|_K^{-n-p}\right)^\wedge(\xi)d\xi = (2\pi)^n\int_{S^{n-1}} \|x\|_L^p\|x\|_K^{-n-p}dx \\
&= (2\pi)^n(n+p)\int_K \|x\|_L^p dx \geq (2\pi)^n n\mathrm{vol}(K)\left(\frac{\mathrm{vol}(L)}{\mathrm{vol}(L)}\right)^{p/n}.
\end{aligned} \tag{5.15}$$

Combining (5.14) and (5.15) we get

$$\mathrm{vol}(K) < \mathrm{vol}(L). \tag{5.16}$$

Now from (5.16) and (5.13) it follows that

$$\frac{1}{\mathrm{vol}(L)}\int_L |(x,\xi)|^p dx \leq \frac{1}{\mathrm{vol}(K)}\int_K |(x,\xi)|^p dx,$$

which is equivalent to

$$\Gamma_p^* K \subset \Gamma_p^* L.$$

Now consider the case $-1 < p < 0$. Since $(\mathbb{R}^n, \|\cdot\|_L)$ does not embed in L_p, there exists a $\xi \in S^{n-1}$ such that $(\|x\|_L^p)^\wedge(\xi)$ is negative, see [Ko7, Theorem 1]. Define

$$\Omega = \{\theta \in S^{n-1} : (\|x\|_L^p)^\wedge(\theta) < 0\}$$

and choose $v(\theta)$ the same way as in the first part.

Define a body K by

$$\frac{\|x\|_K^{-n-p}}{\mathrm{vol}(K)} = \frac{\|x\|_L^{-n-p}}{\mathrm{vol}(L)} + \epsilon |x|_2^{-n-p} g(x/|x|_2)$$

for some small ϵ so that the body K is convex. Applying Fourier transform to both sides we get

$$\frac{1}{\mathrm{vol}(K)} \left(\|x\|_K^{-n-p}\right)^\wedge(\xi) = \frac{1}{\mathrm{vol}(L)} \left(\|x\|_L^{-n-p}\right)^\wedge(\xi) + \epsilon(2\pi)^n |\xi|_2^p v(\xi/|\xi|_2).$$

Again using the formula from Lemma 5.2.3 and the fact that $v(\theta)$ is non-positive, we have

$$\frac{1}{\mathrm{vol}(K)} \int_K |(x,\xi)|^p dx < \frac{1}{\mathrm{vol}(L)} \int_L |(x,\xi)|^p dx,$$

which is the same as

$$\Gamma_p^* K \subset \Gamma_p^* L,$$

since $-1 < p < 0$.

Consider the integral

$$\frac{1}{\mathrm{vol}(K)} \int_{S^{n-1}} (\|x\|_L^p)^\wedge(\xi) \left(\|x\|_K^{-n-p}\right)^\wedge(\xi) d\xi$$

$$= \frac{1}{\mathrm{vol}(L)} \int_{S^{n-1}} \left(\|x\|_L^p\right)^\wedge(\xi) \left(\|x\|_L^{-n-p}\right)^\wedge(\xi) d\xi + \epsilon(2\pi)^n \int_{S^{n-1}} \left(\|x\|_L^p\right)^\wedge(\xi) v(\xi) d\xi$$

$$> \frac{1}{\mathrm{vol}(L)} \int_{S^{n-1}} \left(\|x\|_L^p\right)^\wedge(\xi) \left(\|x\|_L^{-n-p}\right)^\wedge(\xi) d\xi = (2\pi)^n n. \tag{5.17}$$

Here we used Parseval's formula and the fact that v is negative on Ω.

On the other hand, again using Parseval's formula and Hölder's inequality

$$\int_{S^{n-1}} \left(\|x\|_L^p\right)^\wedge(\xi) \left(\|x\|_K^{-n-p}\right)^\wedge(\xi) d\xi = (2\pi)^n \int_{S^{n-1}} \|x\|_L^p \|x\|_K^{-n-p} dx$$

$$\leq (2\pi)^n \left(\int_{S^{n-1}} \|x\|_L^{-n} dx\right)^{-p/n} \left(\int_{S^{n-1}} \|x\|_K^{-n} dx\right)^{(n+p)/n}$$

$$= (2\pi)^n n \left(\mathrm{vol}(L)\right)^{-p/n} \left(\mathrm{vol}(K)\right)^{(n+p)/n}. \tag{5.18}$$

So combining (5.17) and (5.18) we get $\mathrm{vol}(L) > \mathrm{vol}(K)$.

□

The result of Theorem 5.2.6 can be formulated as follows:

Corollary 5.2.7. *i) Let $-1 < p < 0$. There exist origin-symmetric convex bodies K and L in $\mathbb{R}^4$, so that $\Gamma_p^* K \subset \Gamma_p^* L$, but* $\mathrm{vol}(L) > \mathrm{vol}(K)$.

ii) Let $0 < p < 1$. There exist origin-symmetric convex bodies K and L in $\mathbb{R}^3$, so that $\Gamma_p^* K \subset \Gamma_p^* L$, but $\mathrm{vol}(L) > \mathrm{vol}(K)$.

Proof. Consider only the case $-1 < p < 0$, the other case is similar. In view of the previous theorem it is enough to construct an origin-symmetric infinitely smooth convex body $L \in \mathbb{R}^4$ for which the distribution $(\|x\|_L^p)^\wedge$ is not positive. The construction will be similar to that from [GKS].

Define $f_N(x) = (1 - x^2 - Nx^4)^{1/3}$; let $a_N > 0$ be such that $f_N(a_N) = 0$ and $f_N(x) > 0$ on the interval $(0, a_N)$. Define a body L in $\mathbb{R}^4$ by

$$L = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \, x_4 \in [-a_N, a_N] \text{ and } \sqrt{x_1^2 + x_2^2 + x_3^2} \leq f_N(x_4)\}.$$

The body L is strictly convex and infinitely smooth.

By the formula

$$A^{(q)}_{L,\xi}(0) = \frac{\cos\frac{\pi q}{2}}{\pi(n-q-1)} \left(\|x\|_L^{-n+q+1}\right)^\wedge(\xi)$$

from [GKS] and the definition of fractional derivatives, we get

$$\begin{aligned}\left(\|x\|_L^p\right)^\wedge(\xi) &= \frac{\pi p}{\cos\frac{\pi(3+p)}{2}} A^{(3+p)}_{L,\xi}(0) \\ &= \frac{\pi p}{\Gamma(-3-p)\cos\frac{\pi(3+p)}{2}} \int_0^\infty \frac{A_{L,\xi}(z) - A_{L,\xi}(0) - A''_{L,\xi}(0)\frac{z^2}{2}}{z^{4+p}} dz.\end{aligned}$$

Note that the coefficient in the latter formula is positive, therefore it is enough to show that the integral is negative.

The function $A_{L,\xi}$ can easily be computed:

$$A_{L,\xi}(x) = \frac{4\pi}{3}(1 - x^2 - Nx^4).$$

We have

$$\begin{aligned}&\int_0^\infty \frac{A_\xi(z) - A_\xi(0) - A''_\xi(0)\frac{z^2}{2}}{z^{4+p}} dz = \\ &= \frac{4\pi}{3}\left(-\frac{1}{1+p} N a_N^{1+p} + \frac{1}{(1+p)a_N^{(1+p)}} - \frac{1}{(3+p)a_N^{3+p}}\right).\end{aligned}$$

The latter is negative for N large enough, because $N^{1/4} \cdot a_N \to 1$ as $N \to \infty$.

□

5.3 Centroid inequalities for $p = 0$.

In this section we extend the results of the previous section to $p = 0$. First we need some preliminary results. The concept of embedding in L_0 was introduced in [KKYY]:

Definition 5.3.1. *We say that a space* $(\mathbb{R}^n, \|\cdot\|)$ *embeds in* L_0 *if there exist a finite Borel measure* μ *on the sphere* S^{n-1} *and a constant* $C \in \mathbb{R}$ *so that, for every* $x \in \mathbb{R}^n$,

$$\ln \|x\| = \int_{S^{n-1}} \ln |(x,\xi)| d\mu(\xi) + C. \tag{5.19}$$

It follows directly from the definition that μ is a probability measure, and the constant C equals

$$C = \frac{1}{|S^{n-1}|}\int_{S^{n-1}} \ln \|x\| dx - \frac{1}{2\sqrt{\pi}}\Gamma'(1/2) + \frac{1}{2}\frac{\Gamma'(n/2)}{\Gamma(n/2)}. \tag{5.20}$$

Also it was proved that if K is an infinitely smooth body then $(\ln \|x\|_K)^\wedge(\xi)$ is a homogeneous of degree $-n$ function on $\mathbb{R}^n \setminus \{0\}$, as seen from the following result.

Theorem 5.3.2. *[KKYY, Theorem 4.1] Let* K *be an infinitely smooth origin-symmetric star body in* $\mathbb{R}^n$. *Extend* $A^{(n-1)}_{K,\xi}(0)$ *to a homogeneous function of degree* $-n$ *of the variable* $\xi \in \mathbb{R}^n \setminus \{0\}$. *Then*

i) if n *is odd*

$$(\ln \|x\|_K)^\wedge(\xi) = (-1)^{(n+1)/2}\pi A^{(n-1)}_{K,\xi}(0), \quad \xi \in \mathbb{R}^n \setminus \{0\}$$

ii) if n *is even, then for* $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$(\ln \|x\|_K)^\wedge(\xi) = a_n \int_0^\infty \frac{A_\xi(z) - A_\xi(0) - A''_\xi(0)\frac{z^2}{2} - \dots - A^{n-2}_\xi(z)\frac{z^{n-2}}{(n-2)!}}{z^n} dz,$$

where $a_n = 2(-1)^{n/2+1}(n-1)!$

In particular, for an infinitely smooth origin-symmetric star body K, $(\ln \|x\|_K)^\wedge(\xi)$ is a continuous function on S^{n-1}, and moreover the measure in Definition 5.3.1 equals

$$d\mu(\xi) = -\frac{1}{(2\pi)^n} \left(\ln \|x\|_K\right)^\wedge(\xi) d\xi.$$

Since μ is a probability measure, one can see that

$$\int_{S^{n-1}} (\ln \|x\|_K)^\wedge(\theta) d\theta = -(2\pi)^n \tag{5.21}$$

for any infinitely smooth origin-symmetric star body K (see [KKYY, Remark 3.2]).

In our next Lemma we prove that a representation similar to (5.19) holds for all infinitely smooth bodies, with μ being a signed measure.

Lemma 5.3.3. *Let K be an infinitely smooth origin-symmetric star body in $\mathbb{R}^n$. Then*

$$\ln \|x\|_K = -\frac{1}{(2\pi)^n} \int_{S^{n-1}} \ln |(x,\xi)| \left(\ln \|x\|_K\right)^\wedge(\xi) d\xi + C_K, \tag{5.22}$$

where C_K is the constant from (5.20).

Proof. Since the body K is infinitely smooth, by Theorem 5.3.2, $\left(\ln \|x\|_K\right)^\wedge(\xi)$ is a continuous homogeneous function of degree $-n$ on $\mathbb{R}^n \setminus \{0\}$.

Let ϕ be an even test function supported outside of the origin, then

$$\begin{aligned}
&\left\langle \left(\int_{S^{n-1}} \ln |(x,\xi)| \left(\ln \|x\|_K\right)^\wedge(\xi) d\xi \right)^\wedge, \phi \right\rangle \\
&= \left\langle \int_{S^{n-1}} \ln |(x,\xi)| \left(\ln \|x\|_K\right)^\wedge(\xi) d\xi, \hat{\phi}(x) \right\rangle \\
&= \int_{\mathbb{R}^n} \left[\int_{S^{n-1}} \ln |(x,\xi)| \left(\ln \|x\|_K\right)^\wedge(\xi) d\xi \right] \hat{\phi}(x) dx \\
&= \int_{S^{n-1}} \left[\int_{\mathbb{R}^n} \ln |(x,\xi)| \hat{\phi}(x) dx \right] \left(\ln \|x\|_K\right)^\wedge(\xi) d\xi
\end{aligned}$$

Now compute the inner integral using Fubini's theorem and the connection between the Radon and Fourier transforms (see e.g. [Ko11, Lemma 2.11]):

$$\int_{\mathbb{R}^n} \ln|(x,\xi)|\hat{\phi}(x)dx = \int_{\mathbb{R}} \ln|t| \int_{(x,\xi)=t} \hat{\phi}(x)dxdt$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} (\ln|t|)^\wedge(z) \left(\int_{(x,\xi)=t} \hat{\phi}(x)dx\right)^\wedge (z)dz = -\frac{1}{2}\int_{\mathbb{R}} |z|^{-1}\hat{\hat{\phi}}(z\xi)dz$$

$$= -2^{n-1}\pi^n \int_{\mathbb{R}} |z|^{-1}\phi(z\xi)dz = -(2\pi)^n \int_0^\infty z^{-1}\phi(z\xi)dz$$

Here we used the formula for the Fourier transform of $\ln|t|$ (see [GS, p.362])

$$(\ln|z|)^\wedge(t) = -\pi|t|^{-1} \tag{5.23}$$

outside of the origin. Therefore, passing from polar to Euclidean coordinates and recalling from Theorem 5.3.2, that $(\ln\|x\|_K)^\wedge$ is a homogeneous function of degree $-n$ on $\mathbb{R}^n \setminus \{0\}$, we get

$$\langle\left(\int_{S^{n-1}} \ln|(x,\xi)|\,(\ln\|x\|_K)^\wedge(\xi)d\xi\right)^\wedge, \phi\rangle$$

$$= -(2\pi)^n \int_{S^{n-1}} \left[\int_0^\infty z^{-1}\phi(z\xi)dz\right] (\ln\|x\|_K)^\wedge(\xi)d\xi$$

$$= -(2\pi)^n \int_{\mathbb{R}^n} \phi(y)\,(\ln\|x\|_K)^\wedge(y)dy = -(2\pi)^n\langle(\ln\|x\|_K)^\wedge, \phi\rangle.$$

It follows that

$$\left(\int_{S^{n-1}} \ln|(x,\xi)|\,(\ln\|x\|_K)^\wedge(\xi)d\xi\right)^\wedge = -(2\pi)^n\,(\ln\|x\|_K)^\wedge$$

as distributions outside of the origin. Hence, the functions $-(2\pi)^n\ln\|x\|_K$ and $\int_{S^{n-1}} \ln|(x,\xi)|\,(\ln\|x\|_K)^\wedge(\xi)d\xi$ may differ only by a polynomial. But

$$\frac{1}{(2\pi)^n}\int_{S^{n-1}} \ln|(x,\xi)|\,(\ln\|x\|_K)^\wedge(\xi)d\xi + \ln\|x\|_K$$

is a homogeneous function of degree zero, therefore this polynomial is some constant C, which is exactly the constant from Definition 5.3.1, as computed in [KKYY].

□

Now we need a version of Parseval's formula for L_0. How does the formula of Lemma 5.2.4 look if we pass to the limit as $p \to 0$? The answer to this question is given in our next Lemma. Even though in the proof we use an argument based on Lemma 5.3.3, one can obtain the following Lemma by taking the limit in Parseval's formula.

Lemma 5.3.4. *Let K and L be infinitely smooth origin-symmetric star bodies in $\mathbb{R}^n$. Then*

$$-\frac{1}{(2\pi)^n}\int_{S^{n-1}}\left[\int_L \ln|(x,\xi)|dx\right](\ln\|x\|_K)^\wedge(\xi)d\xi = \int_L(\ln\|x\|_K - C_K)dx.$$

Proof. By Lemma 5.3.3 we have

$$-\frac{1}{(2\pi)^n}\int_{S^{n-1}}\ln|(x,\xi)|\,(\ln\|x\|_K)^\wedge(\xi)d\xi = \ln\|x\|_K - C_K.$$

Integrating this equality over the body L we get the statement of the Lemma.

□

Now we prove the main result of this section.

Theorem 5.3.5. *Let K and L be two origin-symmetric star bodies in $\mathbb{R}^n$ such that $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0 and*

$$\Gamma_0^* K \subset \Gamma_0^* L \tag{5.24}$$

for every $\xi \in S^{n-1}$. Then

$$\mathrm{vol}(L) \le \mathrm{vol}(K).$$

Proof. Since $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0, there exist a probability measure μ_K on S^{n-1} (which is the restriction of the Fourier transform of $\ln\|x\|_K$ to the unit sphere) and a constant C_K from Definition 5.3.1.

Rewrite inequality (5.24) as follows:

$$\frac{\int_L \ln|(x,\xi)|dx}{\mathrm{vol}(L)} \le \frac{\int_K \ln|(x,\xi)|dx}{\mathrm{vol}(K)},$$

and integrate it over S^{n-1} with respect to μ_K to get

$$\int_{S^{n-1}} \frac{\int_L \ln|(x,\xi)|dx}{\mathrm{vol}(L)} d\mu_K(\xi) \le \int_{S^{n-1}} \frac{\int_K \ln|(x,\xi)|dx}{\mathrm{vol}(K)} d\mu_K(\xi).$$

Using the Fubini theorem and the definition of embedding in L_0, we get

$$\frac{1}{\mathrm{vol}(L)} \int_L (\ln\|x\|_K - C_K)dx \le \frac{1}{\mathrm{vol}(K)} \int_K (\ln\|x\|_K - C_K)dx.$$

Therefore

$$\frac{1}{\mathrm{vol}(L)} \int_L \ln\|x\|_K dx \le \frac{1}{\mathrm{vol}(K)} \int_K \ln\|x\|_K dx = -\frac{1}{n},$$

where the latter equality follows from the formula

$$\frac{1}{\mathrm{vol}(K)} \int_K \|x\|_K^p dx = \frac{n}{n+p},$$

that we had earlier, after differentiating and letting $p = 0$.

Now use the following inequality from Milman and Pajor [MiP, Section 2.2]:

$$\frac{1}{\mathrm{vol}(L)} \int_L \ln\|x\|_K dx \ge -\frac{1}{n} + \frac{1}{n}[\ln(\mathrm{vol}(L)) - \ln(\mathrm{vol}(K))]. \tag{5.25}$$

Therefore

$$\mathrm{vol}(L) \le \mathrm{vol}(K).$$

□

Remark 5.3.6. *Since every three dimensional normed space embeds in L_0 (see [KKYY, Corollary 4.3]), the previous theorem holds for all convex bodies in $\mathbb{R}^3$.*

To prove our next Theorem we need the following Lemma.

Lemma 5.3.7. *Let K be an origin-symmetric star body in $\mathbb{R}^n$, then the Fourier transform of $\|x\|_K^{-n}$ is a continuous function on $\mathbb{R}^n \setminus \{0\}$ and equals*

$$\begin{aligned}(\|x\|_K^{-n})^\wedge(\xi) = & - n\int_K \ln|(x,\xi)|dx + \\ & + (n\Gamma'(1)-1)\mathrm{vol}(K) - \int_{S^{n-1}} \|\theta\|_K^{-n}\ln\|\theta\|_K d\theta.\end{aligned}$$

Proof. Let ϕ be an even test function. Using the definition of the action of a homogeneous function of degree $-n$ (see [GS, p.303]) we get

$$\begin{aligned}&\langle(\|x\|_K^{-n})^\wedge,\phi\rangle = \langle\|x\|_K^{-n},\hat\phi(x)\rangle \\ &= \int_{B_1(0)}\|x\|_K^{-n}(\hat\phi(x)-\hat\phi(0))dx + \int_{\mathbb{R}^n\setminus B_1(0)}\|x\|_K^{-n}\hat\phi(x)dx \\ &= \int_{S^{n-1}}\int_0^1 r^{-1}\|\theta\|_K^{-n}(\hat\phi(r\theta)-\hat\phi(0))drd\theta + \int_{S^{n-1}}\int_1^\infty r^{-1}\|\theta\|_K^{-n}\hat\phi(r\theta)drd\theta \\ &= \int_{S^{n-1}}\|\theta\|_K^{-n}\left(\int_0^1 r^{-1}(\hat\phi(r\theta)-\hat\phi(0))dr + \int_1^\infty r^{-1}\hat\phi(r\theta)dr\right)d\theta \\ &= \frac{1}{2}\int_{S^{n-1}}\|\theta\|_K^{-n}\langle|r|^{-1},\hat\phi(r\theta)\rangle d\theta \\ &= \frac{1}{2}\int_{S^{n-1}}\|\theta\|_K^{-n}\langle 2\Gamma'(1)-2\ln|t|,\int_{(\theta,\xi)=t}\phi(\xi)d\xi\rangle d\theta \\ &= \langle\int_{S^{n-1}}\|\theta\|_K^{-n}\left(\Gamma'(1)-\ln|(\theta,\xi)|\right)d\theta,\phi(\xi)\rangle.\end{aligned}$$

Here we used the formula for the Fourier transform of $|r|^{-1}$ from [GS, p.361]:

$$(|r|^{-1})^\wedge(t) = 2\Gamma'(1) - 2\ln|t|.$$

Thus we have proved that

$$(\|x\|_K^{-n})^\wedge(\xi) = \int_{S^{n-1}}\|\theta\|_K^{-n}\Big(\Gamma'(1)-\ln|(\theta,\xi)|\Big)d\theta. \tag{5.26}$$

Next, let us compute the following:

$$\int_K \ln|(x,\xi)|dx = \int_{S^{n-1}} \int_0^{\|\theta\|_K^{-1}} r^{n-1} \ln|(r\theta,\xi)|drd\theta$$

$$= \int_{S^{n-1}} \int_0^{\|\theta\|_K^{-1}} r^{n-1} \ln r dr d\theta + \int_{S^{n-1}} \ln|(\theta,\xi)| \int_0^{\|\theta\|_K^{-1}} r^{n-1} dr d\theta$$

$$= -\frac{1}{n} \int_{S^{n-1}} \left(\|\theta\|_K^{-n} \ln \|\theta\|_K + \frac{1}{n} \|\theta\|_K^{-n} \right) d\theta + \frac{1}{n} \int_{S^{n-1}} \|\theta\|_K^{-n} \ln|(\theta,\xi)|d\theta.$$

Therefore

$$\int_{S^{n-1}} \|\theta\|_K^{-n} \ln|(\theta,\xi)|d\theta =$$

$$= n \int_K \ln|(x,\xi)|dx + \int_{S^{n-1}} \left(\|\theta\|_K^{-n} \ln \|\theta\|_K + \frac{1}{n} \|\theta\|_K^{-n} \right) d\theta.$$

Combining this formula with the formula (5.26), we get

$$(\|x\|_K^{-n})^\wedge(\xi) = - n \int_K \ln|(x,\xi)|dx +$$
$$+ (n\Gamma'(1) - 1)\mathrm{vol}(K) - \int_{S^{n-1}} \|\theta\|_K^{-n} \ln \|\theta\|_K d\theta.$$

□

Theorem 5.3.8. *There are convex bodies K and L in $\mathbb{R}^n$, $n \geq 4$ such that*

$$\Gamma_0^* K \subset \Gamma_0^* L$$

for every $\xi \in S^{n-1}$, but

$$\mathrm{vol}(K) < \mathrm{vol}(L).$$

Proof. Let L be a strictly convex infinitely smooth body in $\mathbb{R}^n$, $n \geq 4$, for which $-(\ln \|x\|_L)^\wedge$ is not positive everywhere. (See [KKYY, Theorem 4.4] for an explicit construction of such a body.)

Let $\xi \in S^{n-1}$ be such that $-(\ln \|x\|_L)^\wedge(\xi) < 0$. By continuity of the function $(\ln \|x\|_L)^\wedge(\theta)$ on the sphere there is a neighborhood of ξ where this function is negative. Let

$$\Omega = \{\theta \in S^{n-1} : -(\ln \|x\|_L)^\wedge(\theta) < 0\}.$$

Choose an infinitely smooth body D whose Minkowski norm $\|x\|_D$ is equal to 1 outside of Ω and $\|x\|_D < 1$ for $x \in \Omega$. Let v be a homogeneous function of degree 0 on $\mathbb{R}^n \setminus \{0\}$, defined as follows:

$$v(x) = \ln \|x\|_D - \ln |x|_2.$$

Clearly $v(x) < 0$ if $x \in \Omega$ and $v(x) = 0$ if $x \in S^{n-1} \setminus \Omega$.

In view of Theorem 5.3.2, the Fourier transforms of $\ln \|x\|_D$ and $\ln |x|_2$ outside of the origin are some homogeneous functions of degree $-n$, therefore the Fourier transform of $v(x)$ outside of the origin is equal to $|x|_2^{-n} g(x/|x|_2)$ for some infinitely smooth function g on S^{n-1}. Since by (5.21)

$$\int_{S^{n-1}} (\ln \|x\|_D)^\wedge(\theta) d\theta = \int_{S^{n-1}} (\ln |x|_2)^\wedge(\theta) d\theta = -(2\pi)^n,$$

we have

$$\int_{S^{n-1}} g(\theta) d\theta = 0. \tag{5.27}$$

Define a body K by the formula:

$$\frac{\|x\|_K^{-n}}{\mathrm{vol}(K)} = \frac{\|x\|_L^{-n}}{\mathrm{vol}(L)} + n(2\pi)^{-n} \epsilon |x|_2^{-n} g(x/|x|_2). \tag{5.28}$$

Note that formula (5.27) validates this definition, since integrating the last equality over the unit sphere we get the same quantity in both sides. Also, since L is strictly convex, there is an ϵ small enough, so that K is also convex (see e.g. the perturbation argument from [Ko11, p.96]). From now on we fix such an ϵ.

Now we will show that K together with L constructed above satisfy the assumptions of the theorem. Apply the Fourier transform to both sides of (5.28). Note, that the Fourier transform of $|x|_2^{-n} g(x/|x|_2)$ is equal to $(2\pi)^n v$ on test functions, whose Fourier transform is supported outside of the origin. Such distributions can differ only by a polynomial, which must be a constant in this case, since both functions cannot grow faster than a logarithm (see Lemma 5.3.7). So

$$\left(|x|_2^{-n} g(x/|x|_2)\right)^{\wedge} = (2\pi)^n (v + \alpha),$$

for some constant α whose value has no significance for us. Hence, by Lemma 5.3.7, the Fourier transform of (5.28) looks as follows:

$$-\frac{n \int_K \ln |(x,\xi)| dx}{\mathrm{vol}(K)} = -\frac{n \int_L \ln |(x,\xi)| dx}{\mathrm{vol}(L)} + n\epsilon \cdot v(\xi) + C, \tag{5.29}$$

where the constant C equals

$$C = \frac{\int_K \|\theta\|_K^{-n} \ln \|\theta\|_K d\theta dx}{\mathrm{vol}(K)} - \frac{\int_L \|\theta\|_L^{-n} \ln \|\theta\|_L d\theta dx}{\mathrm{vol}(L)} + n\epsilon \cdot \alpha.$$

Since the bodies L and D are fixed, dilating the body K we can make this constant equal to zero. Indeed, multiply the Minkowski functional of K by a positive constant λ, then

$$\begin{aligned}
C &= \frac{\int_K (\lambda \|\theta\|_K)^{-n} \ln \lambda \|\theta\|_K d\theta dx}{\lambda^{-n} \mathrm{vol}(K)} - \frac{\int_L \|\theta\|_L^{-n} \ln \|\theta\|_L d\theta dx}{\mathrm{vol}(L)} + n\epsilon \cdot \alpha \\
&= \frac{\int_K \|\theta\|_K^{-n} \left[\ln \lambda + \ln \|\theta\|_K\right] d\theta dx}{\mathrm{vol}(K)} - \frac{\int_L \|\theta\|_L^{-n} \ln \|\theta\|_L d\theta dx}{\mathrm{vol}(L)} + n\epsilon \cdot \alpha \\
&= n \ln \lambda + \frac{\int_K \|\theta\|_K^{-n} \ln \|\theta\|_K d\theta dx}{\mathrm{vol}(K)} - \frac{\int_L \|\theta\|_L^{-n} \ln \|\theta\|_L d\theta dx}{\mathrm{vol}(L)} + n\epsilon \cdot \alpha.
\end{aligned}$$

One can choose a $\lambda > 0$ so that $C = 0$. Therefore from (5.29) we get

$$\frac{\int_K \ln |\langle x,\xi\rangle| dx}{\mathrm{vol}(K)} = \frac{\int_L \ln |\langle x,\xi\rangle| dx}{\mathrm{vol}(L)} - \epsilon\, v(\xi) \geq \frac{\int_L \ln |\langle x,\xi\rangle| dx}{\mathrm{vol}(L)}, \tag{5.30}$$

since v is non-positive. Therefore

$$\Gamma_0^* K \subset \Gamma_0^* L.$$

Now using Parseval's formula and inequality (5.30) we get

$$\begin{aligned}
&\frac{1}{\mathrm{vol}(K)}\int_K (\ln\|x\|_L - C_L)dx = \\
&= -\frac{1}{(2\pi)^n}\frac{1}{\mathrm{vol}(K)}\int_{S^{n-1}}\left[\int_K \ln|\langle x,\xi\rangle|dx\right](\ln\|x\|_L)^\wedge(\xi)d\xi \\
&= -\frac{1}{(2\pi)^n}\int_{S^{n-1}}\left[\frac{1}{\mathrm{vol}(L)}\int_L \ln|\langle x,\xi\rangle|dx - \epsilon v(\xi)\right](\ln\|x\|_L)^\wedge(\xi)d\xi \\
&= -\frac{1}{(2\pi)^n}\frac{1}{\mathrm{vol}(L)}\int_{S^{n-1}}\left[\int_L \ln|\langle x,\xi\rangle|dx\right](\ln\|x\|_L)^\wedge(\xi)d\xi \\
&\quad +\frac{1}{(2\pi)^n}\frac{1}{\mathrm{vol}(L)}\int_{S^{n-1}}\epsilon v(\xi)(\ln\|x\|_L)^\wedge(\xi)d\xi \\
&< -\frac{1}{(2\pi)^n}\frac{1}{\mathrm{vol}(L)}\int_{S^{n-1}}\left[\int_L \ln|\langle x,\xi\rangle|dx\right](\ln\|x\|_L)^\wedge(\xi)d\xi \\
&= \frac{1}{\mathrm{vol}(L)}\int_L (\ln\|x\|_L - C_L)dx,
\end{aligned}$$

where the inequality follows from the fact that v is non-positive and supported on the set where $-(\ln\|x\|_L)^\wedge(\xi) < 0$.

Recalling the inequality (5.25)

$$-\frac{1}{n} \geq \frac{1}{\mathrm{vol}(K)}\int_K \ln\|x\|_L dx \geq -\frac{1}{n} + \frac{1}{n}[\ln(\mathrm{vol}(K)) - \ln(\mathrm{vol}(L))],$$

we get

$$\mathrm{vol}(K) < \mathrm{vol}(L).$$

□

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