SHARP ESTIMATES OF THE TRANSMISSION BOUNDARY VALUE PROBLEM FOR DIRAC OPERATORS ON NON-SMOOTH DOMAINS

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The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

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A candidate for the degree of Doctor of Philosophy

And hereby certify that in their opinion it is worthy of acceptance.

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ABSTRACT

This thesis derives the sharp estimates for the transmission boundary value problems (TBVP) for Dirac operators in Lipschitz domains in the three dimensional setting.

Most of the transmission problems considered in the literature fall under several categories, depending on the nature of the domain and solution. First, there is the class of problems in domains with sufficiently smooth boundaries. Second, there is the class of problems in domains with isolated singularities. Weak (variational) solutions for transmission problems in Lipschitz domains and strong solutions in Dahlberg's sense for transmission problems in Lipschitz domains were discussed in various literatures. Compared to previous work on transmission problems, our results are the first to establish well-posedness and *optimal* estimates in arbitrary Lipschitz domains. Applications to the transmission boundary value problems of the system of Maxwells equations are also presented in the last chapter of this thesis.

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Chapter 1

Introduction

The goal of this thesis is to derive sharp estimates for the transmission boundary value problems (TBVP) for Dirac operators in Lipschitz domains in the three dimensional setting. The underlying domain Ω is assumed to have a boundary which is locally given by graphs of Lipschitz functions considered in some suitable systems of coordinates (in the sequel, such a domain is simply referred to as being *Lipschitz*) and the boundary data are from appropriate Lebesgue and Sobolev spaces. In the case of the Laplace operator, such problems have been relatively recently solved in [EM] by relying on the Serrin-Weinberger asymptotic theory (or, De Giorgi-Nash-Moser theory at infinity). Subsequently, a new approach has been developed in our paper [MMS] based on the Hölder regularity of the Neumann function associated with the transmission problem. Here we further extend the scope of these works by considering *systems* of equations.

This analysis is particular relevant in the study of electromagnetic scattering by domains with a rough boundary. Recall that the propagation of an electromagnetic wave (E, H) in $\mathbb{R}^3$ is governed by the three dimensional Maxwell system

$$\text{curl } E - ikH = 0 \text{ in } \Omega_-, \qquad \text{curl } H + ikE = 0 \text{ in } \Omega_-, \tag{1.1.1}$$

where $k \in \mathbb{C}$ is the so-called wave number and, given a bounded domain Ω, in general, we set $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^3 \setminus \bar{\Omega}$. In this regard, let us note that it has long been understood that there are basic connections between the Maxwell system on the one hand and the Hodge-Dirac operator $\mathbb{D} = d + \delta$ (with d, δ, denoting the exterior derivative and its adjoint, respectively), on the other hand. A classical observation which underscores this point is that Maxwell's system (1.1.1) can be written in the compact form

$$\mathbb{D}_k u = 0, \tag{1.1.2}$$

where

$$\mathbb{D}_k := \mathbb{D} + k\, e_4, \text{ and } u := H - ie_4 E. \tag{1.1.3}$$

Above, the vector fields $E = (E_1, E_2, E_3)$ and $H = (H_1, H_2, H_3)$ are regarded as *Clifford algebra-valued functions*, via the identification

$$E = E_1 e_1 + E_2 e_2 + E_3 e_3, \qquad H = H_1 e_2 e_3 + H_2 e_3 e_1 + H_3 e_1 e_2, \tag{1.1.4}$$

and e_1, e_2, e_3, e_4 are the four anti-commuting imaginary units generating the Clifford algebra $\mathcal{A}_4$. Our goal is to further exploit these connections and present a coherent, unified approach to transmission problems which relies on the Clifford algebra formalism. For more background material and further general references on Clifford algebras and related matters, the interested reader is referred to the monographs [BDS], [HQW], [GW] and [Mi3]; see also the article [McM] for harmonic and Fourier analysis methods in the context of Clifford algebras. An excellent survey of progress in the area of harmonic analysis techniques for nonsmooth elliptic problems until early 1990's, can be found in the monograph [Ke1].

The main result of our work is summarized in the theorem below. To state it, recall that for a (possibly algebra-valued) function u defined in Ω, the *non-tangential maximal function* Nu is given by

$$Nu(x) := \sup\{|u(y)| : y \in \Omega,\ |x-y| \leq \kappa\, \mathrm{dist}\,(y, \partial\Omega)\}, \quad x \in \partial\Omega, \tag{1.1.5}$$

where $\kappa > 1$ is some fixed, large constant. We let 'wedge' and 'backward wedge' denote, respectively, the exterior and interior products (cf. Chapter 3 for a more detailed exposition), and set u_{tan}, u_{nor} for the tangential and normal components of an $\mathcal{A}_4$-valued function u defined on $\partial\Omega$. Also, ν will denote the outward unit normal to $\partial\Omega$.

Theorem 1.1.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$, $1 < p < \infty$, and let $k \in \mathbb{C}$ be a non-zero complex number. Assume that $0 < \mu < 1$ is fixed and consider the following transmission boundary value problem for the perturbed Dirac operator*

$\mathbb{D}_k$:

$$
\begin{cases}
u^{\pm} \in C^1(\Omega_{\pm}, \mathcal{A}_4), \\
\mathbb{D}_k u^{\pm} = 0 \text{ in } \Omega_{\pm}, \\
u^{+}_{nor} - u^{-}_{nor} = f \in L^p(\partial\Omega, \mathcal{A}_4), \\
u^{+}_{tan} - \mu\, u^{-}_{tan} = g \in L^p(\partial\Omega, \mathcal{A}_4), \\
\lim_{|x|\to\infty}(|x| - ie_4 x)u^{-}(x) = 0, \\
N(u^{\pm}),\ N(du^{\pm}) \textit{ and } N(\delta u^{\pm}) \in L^p(\partial\Omega).
\end{cases} \tag{1.1.6}
$$

Above, all boundary traces are taken in the pointwise non-tangential limit sense.

Then there exist $\varepsilon > 0$ and a sequence of real numbers $\{k_j\}_j$ which depend exclusively on the boundary $\partial\Omega$ and the transmission parameter μ, and which have the following significance. For every $p \in (1, 2+\varepsilon)$ and every $k \in \mathbb{C} \setminus \{k_j\}_j$, the transmission problem (1.1.6) has a solution if and only if

$$
f \in \nu \wedge L^{p,\delta}_{tan}(\partial\Omega, \mathcal{A}_4) \quad \textit{and} \quad g \in \nu \vee L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4). \tag{1.1.7}
$$

Furthermore, the solution (u^+, u^-) is unique, satisfies the estimate

$$
\begin{aligned}
&\|N(u^{\pm})\|_{L^p(\partial\Omega)} + \|N(du^{\pm})\|_{L^p(\partial\Omega)} + \|N(\delta u^{\pm})\|_{L^p(\partial\Omega)} \\
&\leq C\Big(\|\nu \vee f\|_{L^{p,\delta}_{tan}(\partial\Omega,\mathcal{A}_4)} + \|\nu \wedge g\|_{L^{p,d}_{nor}(\partial\Omega,\mathcal{A}_4)}\Big),
\end{aligned} \tag{1.1.8}
$$

where $C > 0$ depends only on $\partial\Omega$, μ, k, p, and it can be represented in terms of integral operators acting on the boundary data.

The above well-posedness result is sharp in the class of Lipschitz domains, but extends to $1 < p < \infty$ if the unit normal to $\partial\Omega$ has vanishing mean oscillations. In particular, this is the case when $\partial\Omega \in C^1$.

The spaces appearing in (1.1.7) are defined in Chapter 5, where a detailed analysis of their properties is carried out. Here we only want to point out that an equivalent reformulation of (1.1.8) reads as follows: For any Clifford algebra-valued functions $u^{\pm}$ defined in $\Omega_{\pm}$, satisfying $\mathbb{D}_k u^{\pm} = 0$ in $\Omega_{\pm}$, $N(u^{\pm})$, $N(du^{\pm})$, $N(\delta u^{\pm}) \in L^p(\partial\Omega)$, and for which u^- decays at infinity, there holds,

$$
\begin{aligned}
&\|N(u^{\pm})\|_{L^p(\partial\Omega)} + \|N(du^{\pm})\|_{L^p(\partial\Omega)} + \|N(\delta u^{\pm})\|_{L^p(\partial\Omega)} \\
&\quad \leq C\|\nu \vee u^+ - \nu \vee u^-\|_{L^p(\partial\Omega)} + C\|\nu \vee \delta u^+ - \nu \vee \delta u^-\|_{L^p(\partial\Omega)} \\
&\qquad + C\|\nu \wedge u^+ - \mu\, \nu \wedge u^-\|_{L^p(\partial\Omega)} + C\|\nu \wedge du^+ - \mu\, \nu \wedge du^-\|_{L^p(\partial\Omega)}
\end{aligned}
$$

whenever $1 < p < 2 + \varepsilon$, where $C = C(\partial\Omega, k, p) > 0$ is independent of $u^{\pm}$.

Part of the interest in the transmission boundary-value problem for the Dirac operator $\mathbb{D}_k$ in the statement of Theorem 1.1.1 stems from the fact that this is intimately connected with the transmission boundary-value problem for the Helmholtz equation

$$
(1.1.9) \qquad \begin{cases}
(\Delta + k^2)u^{\pm} = 0 \text{ in } \Omega_{\pm}, \\
N(\nabla u^{\pm}),\ N(u^{\pm}) \in L^p(\partial\Omega), \\
u^+\Big|_{\partial\Omega} - u^-\Big|_{\partial\Omega} = f \in L^p_1(\partial\Omega), \\
\partial_\nu u^+ - \mu\, \partial_\nu u^- = g \in L^p(\partial\Omega), \\
\lim_{|x|\to\infty}\Big(x \cdot \nabla u^- - iku^-\Big) = 0,
\end{cases}
$$

as well as the transmission boundary-value problem for the Maxwell system

$$(1.1.10) \qquad \begin{cases} \operatorname{curl} E_i - ikH_i = 0 \text{ in } \Omega_+, \\ \operatorname{curl} H_i + ikE_i = 0 \text{ in } \Omega_+, \\ \operatorname{curl} E_e - ikH_e = 0 \text{ in } \Omega_-, \\ \operatorname{curl} H_e + ikE_e = 0 \text{ in } \Omega_-, \\ N(E_i),\, N(H_i),\, N(E_e),\, N(H_e) \in L^p(\partial\Omega), \\ \nu \times E_e\Big|_{\partial\Omega} - \nu \times E_i\Big|_{\partial\Omega} = f \in L^{p,Div}_{tan}(\partial\Omega), \\ \nu \times H_e\Big|_{\partial\Omega} - \mu\, \nu \times H_i\Big|_{\partial\Omega} = g \in L^{p,Div}_{tan}(\partial\Omega), \\ \frac{x}{|x|} \times H_e + E_e = o\big(\frac{1}{|x|}\big) \text{ as } |x| \to \infty. \end{cases}$$

Here $L^{p,Div}_{tan}(\partial\Omega)$ is a suitable Sobolev-like space of vector fields on $\partial\Omega$ (consisting of L^p tangential fields whose surface divergence is also in L^p); see Chapter 5 for detailed definitions.

The above problem (1.1.9) models the scattering of acoustic time-harmonic waves by a penetrable bounded obstacle Ω. In this case, k stands for the wave number. See, e.g., [CK1], [DL], [GK]. In order to explain the genesis of this problem, assume for a moment that $\mathbb{R}^3 \setminus \Omega$ is connected. The incident plane wave $u_{\rm in}(x) = e^{ik\langle x,\omega\rangle}$, $x \in \mathbb{R}^3$, with $\omega \in S^2$ the propagation direction, will produce a (radiating) scattered wave u^- in the exterior of Ω and a transmitted wave u^+ in Ω. The waves $u^\pm$ are annihilated by the Helmholtz operator $\Delta + k^2$ and verify the so-called conductive boundary conditions

$$(1.1.11) \qquad u^+ = u_{\rm in} + u^- \quad \text{and} \quad \partial_\nu(u_{\rm in} + u^-) = \mu\, \partial_\nu u^+ \quad \text{on} \quad \partial\Omega.$$

In particular, the scalar transmission problem (1.1.9) corresponds precisely to

(1.1.11) for the choice of boundary data

(1.1.12) $$f := u_{\rm in}\Big|_{\partial\Omega}, \quad g := -\partial_\nu u_{\rm in}.$$

In passing, let us also note from (1.1.12) that the smoothness of $\partial\Omega$ affects, via ν, the smoothness of the boundary data f, g.

Likewise, (1.1.10) models the scattering of electro-magnetic waves by a penetrable bounded obstacle Ω. In this case, k is related to the frequency of the electromagnetic wave and the physical characteristics of the medium. See, e.g., [Mu], [MO], [AK]. Much as before, for each $\omega \in S^2$, the propagation direction, and $p \in \mathbb{R}^3$, the polarization, the incident plane electric wave

$$E^{\rm in}(x;\omega,p) := \frac{i}{k}{\rm curl}\,{\rm curl}\,[p\,e^{ik\langle x,\omega\rangle}] = ik(\omega\times p)\times\omega\,e^{ik\langle x,\omega\rangle}, \quad x\in\mathbb{R}^3,$$

and the incident plane magnetic wave

$$H^{\rm in}(x;\omega,p) := k\,{\rm curl}\,[p\,e^{ik\langle x,\omega\rangle}] = ik^2\omega\times p\,e^{ik\langle x,\omega\rangle}, \quad x\in\mathbb{R}^3,$$

will produce (radiating) scattered fields E_e, H_e in the exterior of Ω and transmitted fields E_i, H_i inside Ω. The vector fields E_i, H_i on one hand and E_e, H_e on the other hand, verify Maxwell's equations and the transmission boundary conditions

$$\nu\times(E^{\rm in}+E_e) = \nu\times E_i \;\;\text{and}\;\; \nu\times(H^{\rm in}+H_e) = \mu\,\nu\times H_i \text{ on } \partial\Omega_\infty.$$

The boundary conditions in (1.1.10) are obtained with

$$f := -\nu\times E^{\rm in} \quad\text{and}\quad g := -\mu\,\nu\times H^{\rm in} \quad\text{on}\quad \partial\Omega.$$

Once again, in the context of scatterers with Lipschitz boundaries, f and g above are, generally speaking, discontinuous vector fields.

In Chapter 9 we find necessary and sufficient conditions for the boundary data which guarantee that problem (1.1.6) decouples into four scalar transmission problems (9.1.57), (9.1.58), (9.1.59), and (9.1.60) of the type (1.1.9) and two vector transmission problems of the type (1.1.10). Hence, from this point of view, Theorem 1.1.1 can essentially be regarded as an 'elliptization' method for the original Maxwell system. In broad terms, the Maxwell system is 'embedded' into a more general, elliptic system via a procedure which also identifies the (more specialized) type of boundary data for which the two systems are actually equivalent. For a more detailed discussion in this regard, see Chapter 9. Here we only want to point out that extending the L^p-theory of transmission problems from single equations to systems of equations presents a whole new set of challenges, as many of the basic ingredients used in the scalar case (most notably, the local Hölder regularity of weak solutions) cease to function in this context.

As is implicit in the statement of the above theorem, we shall employ singular integral operators of Cauchy type, which are defined and systematically treated in Chapter 6. In this regard, an incisive result pertaining to the proof of Theorem 6.5.1 is the following.

Theorem 1.1.2. *Let $\Omega \subset \mathbb{R}^3$ be an arbitrary Lipschitz domain with compact bound-*

ary. Then for each $\lambda \in \mathbb{R}$ with $|\lambda| > \frac{1}{2}$, there exists a sequence of real numbers $\{k_j\}_j$ such that for each $1 < p < 2 + \varepsilon$ and $k \in \mathbb{C} \backslash \{k_j\}_j$, the operator $\lambda I + \nu \wedge C_k$ is an isomorphism of $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4)$.

Since the difficulties of working with boundary integral operators in the non-smooth context are well documented (cf., e.g., the discussion in § 1 of [MMP]), this is a delicate result. We are able to prove it by relying on certain distinguished algebraic identities relating the Cauchy operators in the Clifford algebra setting to the scalar layer potential operators associated with the Helmholtz operator $\Delta + k^2$ and the vector layer potentials which are relevant in the study of the Maxwell system. See Chapter 6 for details.

Most of the transmission problems considered in the literature fall under several categories, depending on the nature of the domain and solution. First, there is the class of problems in domains with sufficiently smooth boundaries (so that they can be flattened and/or pseudo-differential operator techniques –with a limited amount of smoothness– can be used). See, e.g., [LRU], [KP1], [KM1], for scalar equations, and [Wi], [Rei], [BD], [AK], [MO], [CK1], [Mu], for Maxwell's equations. Second, there is the class of problems in domains with isolated singularities (in which scenario, Mellin transforms are applicable); cf. [Re2], [Re1], [NS]. Weak (variational) solutions for transmission problems in Lipschitz domains are discussed in [Sa], [Ag]. Finally, strong solutions in Dahlberg's sense ([Da]) for transmission problems in Lipschitz domains are treated in [EFV], [ES], [MM1], [Seo], for single equations, and [ES], [MM1], for systems (such as Lamé and Maxwell). Compared to previous work on transmission problems, our results are the first to establish well-posedness and *optimal* estimates in arbitrary Lipschitz domains.

Several significant extensions of this body of work which we plan to address in the near future are as follows.

• Given the flexible nature of our approcah to the problem at hand, it is natural to suggest that all our main results continue to hold in the context of variable coefficient Hodge-Dirac operators and, more generally, on Lipschitz subdomains of three dimensional Riemannian manifolds. We plan to address this by further refining the techniques developed here and by relying on the results in [MMS].

• When $2-\varepsilon < p < 2+\varepsilon$, our results should be valid in *all* space dimensions. For related developments, see [MMT], [Mi4].

• It is of interest to investigate the case when the underlying domain Ω is not simply connected. One concrete situation when this case is most relevant is that of a ray of light going trough a layer of glass that has several air bubbles in it.

• While this thesis is concerned with the study of the direct problem, a particularly important aspect of the theory is the corresponding *inverse* problem, aimed at determining the shape of the obstacle from the far-field patterns of the scattered waves. Problems as such have enjoyed a lot of attention in the literature and results in this direction are contained in, e.g., [LRU], [MT1], [Is1], [Is2], [He], [GK]. A basic limitation of most of the literature dealing with this subject, however, is the rather strong smoothness assumption on the boundary of the scatterer, namely that it belongs to the class C^2. One natural goal is to use the advances made here in order to be able to treat the case when the boundary of the scatterer has irregularities. Let us note that Lipschitz interfaces have been also considered in [Is1] but the general framework is different inasmuch as the transmission problem is understood there in an L^2-based variational sense and the concept of solution is weak.

To formulate a concrete conjecture, recall first that the scattered wave u^- has the asymptotic behavior

$$u^s(x) = \frac{e^{ik|x|}}{|x|}\left[u_\infty\Big(\frac{x}{|x|}\Big) + \mathcal{O}\Big(\frac{1}{|x|}\Big)\right], \quad \text{as} \quad |x| \to \infty, \tag{1.1.13}$$

where u_∞ is the so-called *far-field pattern* of u; see, e.g., [CK2]. Also, for any bounded domain $\Omega \subset \mathbb{R}^3$ set $[\Omega] := \mathbb{R}^3 \setminus \Omega_\infty$, where Ω_∞ stands for the unbounded complement of $\mathbb{R}^3 \setminus \Omega$. Clearly, if Ω has a connected complement then $\Omega_\infty = \mathbb{R}^3 \setminus \Omega$ and, hence, $[\Omega] = \Omega$ in this case.

Conjecture. *Suppose that two conductive scatterers occupy the interiors of two bounded Lipschitz domains Ω_1, Ω_2 in $\mathbb{R}^3$. We assume that $\mathbb{R}^3 \setminus \Omega_j$, $j = 1, 2$, have the same wave number k. Suppose that the two far-field patterns for Ω_1 and Ω_2*

corresponding to all incident plane wave coincide. Then $[\Omega_1] = [\Omega_2]$.

Chapter 2

Definitions and Review of Some Basic Results

2.1 The Geometry of Lipschitz Domains

Let U be an open subset of $\mathbb{R}^m$. A function $f : U \to R$ is called *Lipschitz* provided that there exists a constant $M > 0$ such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in U$. The best constant in the above inequality is called the *Lipschitz constant* of f.

The following result of Rademacher (cf. [Wh] p.272) is basic for our entire work.

Lemma 2.1.1. *Let f be a real-valued, Lipschitz function defined in an open set U of $\mathbb{R}^m$. Then for each $1 \leq j \leq m$, $\frac{\partial f}{\partial x_j}$ exists at almost every point in U and $\frac{\partial f}{\partial x_j} \in L^\infty(U, R)$. In fact, $\|\nabla f\|_{L^\infty}$ is the Lipschitz constant of f.*

An open set $\Omega \subset \mathbb{R}^n$ is called a *graph Lipschitz domain* if there exists a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\Omega = \{(x', \varphi(x') + t) : \, x' \in \mathbb{R}^{n-1}, \, t > 0\}, \tag{2.1.1}$$

i.e. Ω is the portion of $\mathbb{R}^n$ lying above the graph of the real-valued Lipschitz function φ. Fix $\kappa = \kappa(\Omega) > 1$ and, at each boundary point $x \in \partial\Omega$, define the (cone-like) non-tangential approach region

$$\Gamma(x) := \{y \in \Omega : \, |x - y| < \kappa \, \mathrm{dist}\,(y, \partial\Omega)\}, \qquad x \in \partial\Omega, \tag{2.1.2}$$

and define the *non-tangential maximal operator* N acting on a measurable function $u : \Omega \to \mathbb{C}$ by

$$(Nu)(x) := \|u\|_{L^\infty(\Gamma(x))}, \qquad x \in \partial\Omega. \tag{2.1.3}$$

If we wish to emphasize the dependence of Γ and N on κ, we shall simply write Γ_κ and N_κ instead. It is well-known that for each $\kappa_1, \kappa_2 > 1$ and $p \in (0, \infty)$ there exist $C_1, C_2 > 0$ such that

$$\|N_{\kappa_1} u\|_{L^p(\partial\Omega)} \leq C_1 \|N_{\kappa_2} u\|_{L^p(\partial\Omega)} \leq C_2 \|N_{\kappa_1} u\|_{L^p(\partial\Omega)} \tag{2.1.4}$$

for any measurable function u in Ω. See, e.g., [Ke2].

Call an open set $\Omega \subset \mathbb{R}^n$ a *bounded Lipschitz domain* if there exists a finite open covering $\{\mathcal{O}_j\}_{1 \leq j \leq N}$ of $\partial\Omega$ with the property that, for every $j \in \{1, ..., N\}$, $\mathcal{O}_j \cap \Omega$ coincides with the portion of $\mathcal{O}_j$ lying above $R_j(\mathrm{graph}\, \varphi_j)$ where $\varphi_j : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function and R_j is a rigid motion of the Euclidean space $\mathbb{R}^n$.

Call $Z \subset \mathbb{R}^n$ a *coordinate cylinder* if Z is an open, right circular doubly truncated cylinder with center at $x_Z \in \partial\Omega$ and which, in addition, has the following properties:

i) If $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ is a rectangular coordinate system such that x_Z corresponds to the origin and the axis of Z is in the direction of $e_n = (0, ..., 0, 1) \in \mathbb{R}^n$, then there exists a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $x_Z = (0', \varphi(0'))$ and

$$\Omega \cap Z = \{x = (x', x_n) : \varphi(x') < x_n\} \cap Z, \\ \partial\Omega \cap Z = \{x = (x', x_n) : \varphi_i(x') = x_n\} \cap Z. \tag{2.1.5}$$

ii) If h and R are the height and the radius of Z, then $h/R > 5\sqrt{1 + \|\nabla\varphi\|^2_{L^\infty}}$.

iii) If tZ denotes the concentric dilation of Z of factor $t > 0$, then (2.1.5) also holds with tZ in place of Z for each $1 < t < 10\sqrt{1 + \|\nabla\varphi\|^2_{L^\infty}}$.

In the sequel, we shall write occasionally $Z = Z(x, h, R, \varphi)$ to indicate that the coordinate cylinder Z is centered at x, has height h, radius R, and that the Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfies (2.1.5).

Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, it is then possible to cover its boundary $\partial\Omega$ with a finite number of coordinate cylinders $\Big\{Z_i(x_{Z_i}, h_i, R_i, \varphi_i)\Big\}_{1 \le i \le N}$. Call this family an atlas for $\partial\Omega$, and say that a quantity depends on the Lipschitz character of Ω if its size is controlled in terms of N and the numbers R_i, h_i, $\|\nabla\varphi_i\|_{L^\infty(\mathbb{R}^{n-1})}$, for $1 \le i \le N$.

In the context of a bounded Lipschitz domain Ω, we shall retain the definitions (2.1.2)-(2.1.3) of the non-tangential approach regions $\Gamma(x)$ and of the non-tangential maximal operator N. In particular, (2.1.4) holds in this case as well.

It is well-known that, given a Lipschitz domain Ω there exists a canonical surface measure $d\sigma$ on $\partial\Omega$, with respect to which ν, the outward unit normal to Ω, is defined almost everywhere on $\partial\Omega$. We shall denote by $\sigma(E)$ the surface measure of a measurable set $E \subset \partial\Omega$. Also, throughout the thesis, we shall let $L^p(\partial\Omega)$, $1 \leq p \leq \infty$, stand for the Lebesgue space of complex-valued, measurable functions which are p-th power integrable with respect to $d\sigma$ on $\partial\Omega$.

2.2 Functional Analysis Elements

Let E and F be normed spaces. We denote by $\mathcal{L}(E,F)$ the space of all continuous linear operators $T : E \to F$ equipped with the operator norm and we also set $\mathcal{L}(E) := \mathcal{L}(E,E)$. We shall denote the set of compact linear operators from E into F by $\mathcal{K}(E,F) \subseteq \mathcal{L}(E,F)$, and abbreviate $\mathcal{K}(E) := \mathcal{K}(E,E)$.

If E is a normed space and $F = \mathbb{R}$ or $\mathbb{C}$ is the field of scalars of E, the space $\mathcal{L}(E,F)$ is called the *dual* of E and is denoted by E^*.

We record two theorems which handle some important properties of compact operators as below. See [La] for the proof.

Theorem 2.2.1. *Let E, F, G, H be normed vector spaces and let*

$$f : E \to F, \qquad u : F \to G, \qquad g : G \to H$$

be continuous linear maps. If u is compact then $u \circ f$ and $g \circ u$ are compact. In particular, $\mathcal{K}(E)$ is a two-sided ideal of $\mathcal{L}(E)$.

Theorem 2.2.2. *Let E, F be Banach spaces and $f : E \to F$ be a compact linear map. Then $f^* : F^* \to E^*$ is compact, where f^* is the adjoint of f.*

For any operator $T \in \mathcal{L}(E, F)$ we denote by $Im\,T$, $Ker\,T$ and $Coker\,T$ the image, the kernel and the cokernel of T correspondingly:

$$Im\,T := \{y \in F : y = Tx, \text{ for some } x \in E\} \subseteq F,$$

$$Ker\,T := \{x \in E : Tx = 0\} \subseteq E,$$

$$Coker\,T := \{f \in F^* : f(Tx) = 0, \text{ for any } x \in E\} \subseteq F^*.$$

One class of operators that we are going to deal with is the one of Fredholm operators, which are named for Erik Ivar Fredholm. An operator $T \in \mathcal{L}(E, F)$ is said to be *Fredholm* if the following three conditions are satisfied:

1) $dim\,(Ker\,T) < \infty$;

2) $Im\,T$ is closed in F;

3) $dim\,(Coker\,T) < \infty$.

In this case, $ind\,T := dim\,(Ker\,T) - dim\,(Coker\,T)$ is called the *index* of the operator T. We denote by $\mathrm{Fred}(E,F)$ the set of Fredholm operates from E into F.

Let us now briefly describe some of the basic properties of Fredholm operators.

Theorem 2.2.3. *Let E, F be Banach spaces. Then* $\mathrm{Fred}(E,F)$ *is open in* $\mathcal{L}(E,F)$*, and the function $T \to ind\,T$ is continuous on* $\mathrm{Fred}(E,F)$*, hence constant on connected components.*

Theorem 2.2.4. *The composite of Fredholm operators is Fredholm. If T is Fredholm and R is compact, then $T+R$ is Fredholm.*

Corollary 2.2.5. *If T is Fredholm and R is compact, then*

$$ind\,(T+R) = ind\,T.$$

Corollary 2.2.6. *If $T \in \mathcal{L}(E,F)$ is an invertible operator, and $R \in \mathcal{K}(E,F)$, then $T+R$ is a Fredholm operator with index zero.*

Theorem 2.2.7. *Let E, F, G be Banach spaces, and let*

$$S : E \to F \quad \text{and} \quad T : F \to G$$

be Fredholm. Then

$$ind\,TS = ind\,T + ind\,S.$$

Next, let us turn our attention to the exact sequence, which will be used in Chapter 6. An *exact sequence* is a sequence of maps

$$T_i : E_i \to E_{i+1} \tag{2.2.6}$$

between a sequence of spaces E_i, which satisfies

$$Im(T_i) = Ker(T_{i+1}). \tag{2.2.7}$$

An exact sequence may be of either finite or infinite length. The special case of length five,

$$0 \to E_1 \to E_2 \to E_3 \to 0 \tag{2.2.8}$$

beginning and ending with zero, meaning the zero space $\{0\}$, is called a *short exact sequence.*

We are now going to present Analytic Fredholm Alternative Theory, which will be invoked in Chapter 6 of this thesis.

Theorem 2.2.8. *Let X be a Banach space and O be an open subset of $\mathbb{C}$. Let the operator*

$$A : O \to \mathcal{L}(X)$$

be analytic, i.e. $\frac{\partial}{\partial z}A(z) = 0$ for any z in O. Also, suppose that $A(0) = 0$ and $A(z) \in \mathcal{K}(X)$. Then there exists a subset E of O, which has no accumulation points, such that the operator

$$I - A(z) : X \to X$$

is invertible for all z in the set $O \backslash E$.

2.3 Calderón-Zygmund Theory

Handling boundary sigular integral operators at the level of generality assumed in this work, i.e. Lipschitz boundaries and L^p-based function spaces, requires the use of the rather sophisticated machinery known as Calderón-Zygmund theory. We now discuss some aspects of the Calderón-Zygmund theory which are most relevant for our work. The first result below models the behavior of sigular integral operators best suited for this work.

Theorem 2.3.1. *Let $A : \mathbb{R}^m \to \mathbb{R}^n$ be a Lipschitz function and $F : \mathbb{R}^n \to \mathbb{R}$ be a smooth and odd function. For any $x, y \in \mathbb{R}^m$ with $x \neq y$ we set the kernel $K(x,y) := \frac{1}{|x-y|^m} F\left(\frac{A(x)-A(y)}{|x-y|}\right)$, and for $\varepsilon > 0$, $f \in Lip_{comp}(\mathbb{R}^m)$, we define the truncated operator $T_\varepsilon f(x) := \int_{|x-y|>\varepsilon} K(x,y) f(y) dy$. Then for each $1 < p < \infty$, the following assertions hold:*

(1) *The maximal operator $T_* f(x) := \sup\limits_{\varepsilon>0} |T_\varepsilon f(x)|$ is bounded on $L^p(\mathbb{R}^m)$;*

(2) *If there exists a dense subspace V in $L^p(\mathbb{R}^m)$ such that for any $f \in V$ the limit $\lim_{\varepsilon \to 0} T_\varepsilon f(x)$ exists for almost every $x \in \mathbb{R}^m$, then this limit exists for any $f \in L^p(\mathbb{R}^m)$ at almost any $x \in \mathbb{R}^m$ and the operator $Tf(x) := \lim_{\varepsilon \to 0} T_\varepsilon f(x)$ is bounded on $L^p(\mathbb{R}^m)$.*

Our next result models the behavior of layer potential-like operatos mapping function defined on a Lipschitz surface to functions defined in its complement.

Theorem 2.3.2. *Let $A : \mathbb{R}^m \to \mathbb{R}^n$, $B : \mathbb{R}^m \to \mathbb{R}$ be two Lipschitz functions and let $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a smooth and odd function which satisfies the decay condition*

$$|F(a,b)| \leq C(1+|b|)^{-m} \tag{2.3.9}$$

uniformly for a in compact subsets of $\mathbb{R}^n$ and arbitrary $b \in \mathbb{R}$. For any $x, y \in \mathbb{R}^m$ with $x \neq y$ and $t > 0$ we set

$$K^t(x,y) := \frac{1}{|x-y|^m} F\left(\frac{A(x)-A(y)}{|x-y|}, \frac{B(x)-B(y)+t}{|x-y|}\right).$$

Also, for each $t > 0$, we introduce the operators $T^t f(x) := \int_{\mathbb{R}^m} K^t(x,y) f(y) dy$, for $f \in Lip_{comp}(\mathbb{R}^m)$, and $T_{**}f(x) := \sup_{|x-z|<\lambda t} |T^t f(z)|$, for some fixed positive λ. Then, for each $1 < p < \infty$, the following assertions are valid:

(1) *The non-tangential maximal operator T_{**} is bounded on $L^p(\mathbb{R}^m)$;*

(2) *If there exists a dense subspace V in $L^p(\mathbb{R}^m)$ such that for any $f \in V$ the (non-tangential) limit*

$$\mathcal{T} f(x) := \lim_{\substack{z \to x, t \to 0 \\ |x-z| < \lambda t}} T^t f(z)$$

exists for almost every $x \in \mathbb{R}^m$, then this limit exists for any $f \in L^p(\mathbb{R}^m)$ at almost any $x \in \mathbb{R}^m$ and the operator $\mathcal{T}$ is bounded on $L^p(\mathbb{R}^m)$.

Chapter 3

The Clifford Algebra Structure

3.1 Construction of a Clifford Algebra

Fix $m \in \mathbb{N}$ and let $\{e_i\}_{0 \leq i \leq m}$ be a collection of objects for which we assume that there exists an associative multiplication such that the following axioms are true:

1. $e_0 = 1$;

2. $e_i^2 = -1$ for $1 \leq i \leq m$;

3. $e_i e_j = -e_j e_i$ for $1 \leq i \neq j \leq m$.

Define *Clifford conjugation* of e_i, $i = 1, 2, \ldots, m$, as follows:

1. $\overline{e}_0 = e_0$.

2. $\overline{e}_i = -e_i$ for $1 \leq i \leq m$.

Remark 3.1.1. *To distinguish between complex conjugation and Clifford conjugation, we will use $(\cdot)^c$ to denote the complex conjugation in this work.*

We denote by $\mathcal{A}_m$ the algebra generated by $\{e_i\}_{0\leq i\leq m}$. That is, $\mathcal{A}_m$ consists of all elements u, which can be represented in the form

$$u := \sum_{l=0}^{m} \sum_{|I|=l}{}' u_I e_I, \tag{3.1.1}$$

where e_I stands for the product $e_{i_1}e_{i_2}...e_{i_l}$ if $I = (i_1, i_2, ..., i_l)$. For each multi-index I, we call l the length of I and denote it by $|I| = i_1 + i_2 + ... + i_l$. And $\sum'$ indicates that the sum is performed only over strictly increasing multi-indices, i.e. $I = (i_1, i_2, ..., i_l)$ with $1 \leq i_1 < i_2 < ... < i_l \leq m$.

In the sequel, we shall refer to $\mathcal{A}_m$ as being *Clifford algebra* generated by m imaginary units.

Remark 3.1.2. *For each $m \geq 1$, the algebra $\mathcal{A}_m$ exists.*

Proof.

Let us prove this remark via a constructive approach.

One can represent the algebra of complex numbers $\mathbb{C}$ as a subset of $M_{2\times 2}(\mathbb{R})$, the set of all 2×2 matrices with coefficients in $\mathbb{R}$. Indeed, if we let I denote the 2×2 identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and we denote the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ by i. Then

$$i^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I.$$

That is, informally, we have $i = \sqrt{-I}$. With this in mind, we can think of $\mathbb{C}$ as a subset of $M_{2\times 2}(\mathbb{R})$. Specifically, $\mathbb{C} = \text{span}\{I, i\}$, where the span is taken over $\mathbb{R}$.

There are a few points worth taking notice of at this stage. First of all, we only need these two matrices to generate $\mathbb{C}$. Also, there are only 2 matrices (i and $-i$) that have a square of $-I$. We can think of $\mathbb{C}$ as an associative algebra, with the usual matrix addition and the usual matrix multiplication as the $+, \cdot$ operations in this algebra. In this case, the algebra $\mathbb{C}$ is commutative. The subsequent algebras that we generate following a similar pattern will not be commutative, though they will all be associative.

In order to construct algebra involving more imaginary units, we need to increase the size of the matrices. For the next level, we will consider matrices in $M_{4\times 4}(\mathbb{R})$. It can be shown that the size of the matrices needs to be *even* for the construction to be feasible. However, in order to construct this new algebra, we also need to produce more imaginary units.

The standard notation to designate these algebras, which we will use in this paper, is $\mathcal{A}_m$. Notice, we have that $\mathcal{A}_0 = \mathbb{R}$ and $\mathcal{A}_1 = \mathbb{C}$. In general, $\mathcal{A}_m$ has m "imaginary units." Our current goal is to construct $\mathcal{A}_2$. To generate $\mathcal{A}_m$, we will need a simple adaptation of the following recursive algorithm.

As previously stated, we will define $\mathcal{A}_2$ as a subset of $M_{2^2\times 2^2}(\mathbb{R})$. First, we need to define some matrices. Let

$$e_0 := I_{2^2\times 2^2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now, using the matrix i from the 2×2 case above, we define

$$e_1 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Similarly, using the matrix I from the 2×2 case above, we define

$$e_2 := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

A simple calculation shows that both e_1 and e_2 are imaginary units, in the sense that $e_1^2 = e_2^2 = -I_{2^2 \times 2^2} = -e_0$.

If we now consider

$$\text{span}_{\mathbb{R}}\{e_0, e_1, e_2, e_1e_2\} = \{a_0e_0 + a_1e_1 + a_2e_2 + a_3e_1e_2 : a_i \in \mathbb{R}\},$$

then this set will be *closed* under multiplication. We can verify this fact with the following multiplication chart. In the chart, the column represents the first factor in the product, and the row represents the second factor.

$\cdot$	e_0	e_1	e_2	e_1e_2
e_0	e_0	e_1	e_2	e_1e_2
e_1	e_1	$-e_0$	e_1e_2	$-e_2$
e_2	e_2	e_2e_1	$-e_0$	e_1
e_1e_2	e_1e_2	e_2	$-e_1$	$-e_0$

Note that the term e_2e_1 appears in the above chart, so it may appear that the span is *not* stable under multiplication. However, one can show that

$$\begin{aligned} e_2 \cdot e_1 &= \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \cdot \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = -e_1 \cdot e_2. \end{aligned}$$

Having clarified this issue, it is easy see that the span is indeed closed under multiplication. In particular, $\text{span}_{\mathbb{R}}\{e_0, e_1, e_2, e_1e_2\}$ is a subalgebra of $M_{2^2 \times 2^2}(\mathbb{R})$, which is not commutative. This is in stark contrast with the case of $\mathcal{A}_0$ and $\mathcal{A}_1$. (This non-commutative nature will also appear in the subsequent algebras). With this in mind, we define

$$\mathcal{A}_2 := \text{span}_{\mathbb{R}}\{e_0, e_1, e_2, e_1e_2\}.$$

Continuing much in the same way, we can construct other subalgebras by simply considering more imaginary units. The next subalgebra will be called $\mathcal{A}_3$ and will be a subset of $M_{2^3 \times 2^3}(\mathbb{R})$. In $\mathcal{A}_3$, we define $e_0 := I_{2^3 \times 2^3}$ and, for $j = 1$ and 2, define $e_j := \begin{pmatrix} e_j^{\text{old}} & 0 \\ 0 & -e_j^{\text{old}} \end{pmatrix}$ where e_j^{old} is the "old" $e_j \in \mathcal{A}_2$ and 0 is the $0 \in \mathcal{A}_2$ (i.e. the $2^2 \times 2^2$ zero matrix). Then, in much the same way as before, we will have that $e_j^2 = -e_0$ for $j = 1$ and 2.

Now, we need to define a "new" imaginary unit. With our previous construction in mind, we define $e_3 := \begin{pmatrix} 0 & -e_0^{\text{old}} \\ e_0^{\text{old}} & 0 \end{pmatrix}$ where e_0^{old} is the "old" $e_0 \in \mathcal{A}_2$ and 0 is the $2^2 \times 2^2$ zero matrix.

We can quickly check that

$$\begin{aligned} e_3^2 &= \begin{pmatrix} 0 & -e_0^{\text{old}} \\ e_0^{\text{old}} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -e_0^{\text{old}} \\ e_0^{\text{old}} & 0 \end{pmatrix} \\ &= \begin{pmatrix} -(e_0^{\text{old}})^2 & 0 \\ 0 & -(e_0^{\text{old}})^2 \end{pmatrix} = \begin{pmatrix} -e_0^{\text{old}} & 0 \\ 0 & -e_0^{\text{old}} \end{pmatrix} = -e_0. \end{aligned}$$

Much as before, we will have that e_1, e_2, and e_3 anti-commute with each other. Also, we need to make the span stable under multiplication. To do this, we need to include all possible products of e_1, e_2, and e_3 with each other. Hence, we define

$$\mathcal{A}_3 := \operatorname{span}_{\mathbb{R}}\{e_0, e_1, e_2, e_3, e_1e_2, e_1e_3, e_2e_3, e_1e_2e_3\}.$$

Since this span is stable under multiplication, this is, as before, a non-commutative yet associative algebra.

In particular, one can repeat this procedure for as long as desired. However, in this work, we shall only need to go up to $\mathcal{A}_4 \subseteq M_{2^4 \times 2^4}(\mathbb{R})$. Once again, this is obtained by creating a new $e_0 = I_{2^4 \times 2^4}$, recycling the basic imaginary units of $\mathcal{A}_3$ to create some new imaginary units e_1, e_2, and e_3, and then constructing a new imaginary unit e_4 from the $e_0 \in \mathcal{A}_3$ (using the $\sqrt{-1}$ recipe). Much as before, when constructing the span, we need to include all possible products of the e_i in order to ensure that the span is stable under multiplication. Thus, we define

$$\begin{aligned} \mathcal{A}_4 := \operatorname{span}_{\mathbb{R}}\{&e_0, e_1, e_2, e_3, e_4, e_1e_2, e_1e_3, e_1e_4, e_2e_3, e_2e_4, e_3e_4, \\ &e_1e_2e_3, e_1e_2e_4, e_1e_3e_4, e_2e_3e_4, e_1e_2e_3e_4\}. \end{aligned} \tag{3.1.2}$$

It is clear that these elements form a basis of $\mathcal{A}_4$, and that the dimension of $\mathcal{A}_4$ as a linear space is $2^4 = 16$. Again, $\mathcal{A}_4$ is a non-commutative, associative algebra. $\square$

Remark 3.1.3. *According to Theorem 1.2 in [Mi3], the representation of u in $\mathcal{A}_m$ is unique.*

3.2 General Properties of $\mathcal{A}_m$

The algebra $\mathcal{A}_m$ has many properties which we will use throughout this thesis. First of all, we define the *Clifford conjugation* of u in $\mathcal{A}_m$ by setting

$$\overline{u} := \sum_{l=0}^{m} {\sum_{|I|=l}}' u_I \overline{e}_I, \tag{3.2.3}$$

where

$$\overline{e}_I := \overline{e_{i_1} e_{i_2} \ldots e_{i_l}} = \overline{e}_{i_l} \ldots \overline{e}_{i_2} \overline{e}_{i_1}. \tag{3.2.4}$$

provided $I = (i_1, i_2, \ldots, i_m)$ is a multi-index. Going further, define the *norm* of u as

$$|u| := \sqrt{\sum_{l=0}^{m} {\sum_{|I|=l}}' |u_I|^2}, \tag{3.2.5}$$

so that $|u|^2 = \langle u, u \rangle$ where, for each $u, v \in \mathcal{A}_m$, we define the *inner product*

$$\langle u, v \rangle := \sum_{l=0}^{m} {\sum_{|I|=l}}' u_I v_I. \tag{3.2.6}$$

For each $l \in \{0, 1, \ldots, m\}$ consider the projection map Π_l onto the l-homogeneous part of u, i.e. by

$$\Pi_l u := {\sum_{|I|=l}}' u_I e_I, \tag{3.2.7}$$

and denote by Λ^l the range of $\Pi_l : \mathcal{A}_m \to \mathcal{A}_m$. It follows that

$$\mathcal{A}_m = \Lambda^0 \oplus \Lambda^1 \oplus \cdots \oplus \Lambda^m. \tag{3.2.8}$$

We shall refer to the elements in Λ^0 as *scalars*, to the elements in Λ^1 as *vectors*, to the elements in Λ^2 as *bi-vectors*, to the elements in Λ^3 as *tri-vectors*, etc.

If $a \in \Lambda^1$ and $u \in \Lambda^j$, then

$$a \cdot u \in \Lambda^{j-1} \oplus \Lambda^{j+1}.$$

This is because multiplication of a homogeneous Clifford element by a vector will either increase or decrease the degree of homogeneity by 1. We then define the *wedge product* $\wedge$ and the *backward wedge product* $\vee$ by

$$a \wedge u := \Pi_{j+1}(a \cdot u) \qquad \text{and} \qquad a \vee u := -\Pi_{j-1}(a \cdot u). \tag{3.2.9}$$

As a result, we have

$$a \cdot u = a \wedge u - a \vee u, \text{ for any } u \in \mathcal{A}_m. \tag{3.2.10}$$

One obvious observation is that both $\wedge$ and $\vee$ are linear maps. This is due to the fact that Π_l is itself a linear map, for each $l \in \{0, 1, 2, \ldots, m\}$.

The *Hodge star operator* can be defined as the unique linear mapping

$$* : \Lambda^l \to \Lambda^{m-l}$$

such that

$$e_I(*e_I) = e_1 e_2 ... e_m, \tag{3.2.11}$$

for every multi-index $I = (i_1, i_2, \dots, i_m)$.

Remark 3.2.1. *Let a be a vector. $a \wedge u = ua$ if u is a scalar, and $a \wedge u = *(a \times u)$ if u is a vector. Moreover, $a \vee u = 0$ if u is a scalar, and $a \vee u = \langle a, u \rangle$ if u is a vector.*

Some properties which are intrinsic to the Clifford structure that are going to be most relevant for our work in the sequel. Hence we collect some of the most important properties in the form a lemma.

Lemma 3.2.2. *Suppose that $a, b \in \Lambda^1$ and $u, v \in \Lambda^l$, for some $l \in \{0, 1, \dots, m\}$. Then the following hold:*

1. $a \wedge (a \wedge u) = 0$ *and* $a \vee (a \vee u) = 0$;

2. $* * u = (-1)^{l(m-l)} u$;

3. $\langle u, *v \rangle = (-1)^{l(m-l)} \langle *u, v \rangle$;

4. $*(a \wedge u) = (-1)^l a \vee (*u)$;

5. $*(a \vee u) = (-1)^{l-1} a \wedge (*u)$;

6. $a \wedge (b \vee u) + b \vee (a \wedge u) = \langle a, b \rangle u$;

7. $\langle a \wedge u, v \rangle = \langle u, a \vee v \rangle$.

Corollary 3.2.3. *For each $a \in \Lambda^1$ with $|a| = 1$, and each $u \in \mathcal{A}_m$,*

$$u = a \wedge (a \vee u) + a \vee (a \wedge u). \tag{3.2.12}$$

Proof.

This follows readily from (6) above. $\square$

In what follows, if Ω is a Lipschitz domain with unit normal ν and $u : \bar{\Omega} \to \mathcal{A}_m$, we set

$$u_{nor} := \nu \wedge (\nu \vee u) \tag{3.2.13}$$

and call it the *normal component* of u, and set

$$u_{tan} := \nu \vee (\nu \wedge u) \tag{3.2.14}$$

and call it the *tangential component* of u.

We record a lemma which will be important for further subsequent development.

Lemma 3.2.4. *Assume that $a \in \Lambda^1$, and that $u \in \mathcal{A}_4$. Then*

$$a \wedge (e_4 u) = -e_4(a \wedge u) \text{ and } a \vee (e_4 u) = -e_4(a \vee u). \tag{3.2.15}$$

Proof.

By linearity, it suffices to treat the cases: $u \in \Lambda^j$, $0 \leq j \leq 3$, and $u = e_4 v$, where $v \in \Lambda^j$, $0 \leq j \leq 3$. Since, in general, $\Pi_{j+1}(e_4 w) = e_4 \cdot \Pi_j(w)$ if $w \in \mathcal{A}_j$, $0 \leq j \leq 3$, the desired conclusion follows readily from definitions. $\square$

Important Convention. *For the remainder of this work, we shall denote by* $\Lambda^0, \Lambda^1, \Lambda^2, \Lambda^3$, *the scalars, vectors, bi-vectors, tri-vectors of* $\mathcal{A}_3$, *which, in turn, is viewed as a subalgebra of* $\mathcal{A}_4$. *In particular,*

$$\mathcal{A}_4 = \mathcal{A}_3 \oplus (e_4 \mathcal{A}_3) \tag{3.2.16}$$

and, hence,

$$\mathcal{A}_4 \quad = \quad \Lambda^0 \oplus (\Lambda^1 \oplus e_4\Lambda^0) \oplus (\Lambda^2 \oplus e_4\Lambda^1) \oplus (\Lambda^3 + e_4\Lambda^2) \oplus e_4\Lambda^3. \tag{3.2.17}$$

Chapter 4

Clifford Analysis

4.1 Dirac Operators

Let Ω be an open subset in $\mathbb{R}^m$. We shall work with $\mathcal{A}_m$-valued functions defined in Ω, i.e. functions $f : \Omega \to \mathcal{A}_m$. If f is $\mathcal{A}_m$-valued, we can decompose f much as we have decomposed Clifford elements. Hence, we can write

$$f(x) := \sum_{l=0}^{m} {\sum_{|I|=l}}' f_I(x) e_I, \tag{4.1.1}$$

where each coefficient $f_I : \Omega \to \mathbb{C}$. Note that algebraic combinations of $\mathcal{A}_m$-valued functions, such as λf, $f \pm g$ and $f \cdot g$, for $\lambda \in \mathbb{C}$ and f, g $\mathcal{A}_m$-valued, are defined in a natural fashion.

Let $\mathbb{N}$ be the set of natural numbers, i.e. $\mathbb{N} = \{1, 2, 3, \dots\}$, and, in addition, set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}_0^m$ be a multi-index. The length of α is defined as

$$|\alpha| := \sum_{j=1}^{m} \alpha_j.$$

Define

$$C^k(\Omega) := \{f : \Omega \to \mathbb{C} : \partial^\alpha f \text{ is continuous for any } |\alpha| \leq k\}.$$

We say that $f : \Omega \to \mathcal{A}_m$ is of class C^k if each $f_I \in C^k(\Omega)$, and denote the set of functions of class C^k by

$$C^k(\Omega, \mathcal{A}_m) := \{f : \Omega \to \mathcal{A}_m : f_I \in C^k(\Omega) \text{ for all } |I| \leq m\}.$$

We now introduce several Dirac-like operators. The classical *Dirac operator* (named after P. Dirac) is the first-order, differential operator given by

$$\mathbb{D} := \sum_{j=1}^{m} e_j \partial_j. \tag{4.1.2}$$

If $f(x) = \sum'_{|I|} f_I(x) e_I$ is of class C^1, then $\mathbb{D}f$ is defined naturally by

$$\mathbb{D}f := \sum_{|I|=0}^{m}{}' \sum_{j=1}^{m} (\partial_j f_I(x)) e_j e_I. \tag{4.1.3}$$

We can also define $f\mathbb{D}$ in a similar way. For $k \in \mathbb{C}$, the *perturbed Dirac operator* $\mathbb{D}_k$ is defined as

$$\mathbb{D}_k = \mathbb{D} + k e_{m+1}, \tag{4.1.4}$$

i.e., if f is an $\mathcal{A}_{m+1}$-valued function, then

$$\mathbb{D}_k f := \sum_{j=1}^{m} e_j \partial_j f + k e_{m+1} f. \tag{4.1.5}$$

We now make some definitions concerning these Dirac operators.

Let $f \in C^1(\Omega, \mathcal{A}_m)$. If $\mathbb{D}f = 0$ in Ω, then f is called *left-monogenic*. If $f\mathbb{D} = 0$ in Ω, then f is called *right-monogenic*. Similarly, if $\mathbb{D}_k f = 0$ (or $f\mathbb{D}_k = 0$) in Ω, then f is called *left-k-monogenic* (or *right-k-monogenic*). Sometimes, the term Clifford analytic or holomorphic is used in place of monogenic.

Now, we discuss connections that between the operators $\mathbb{D}$ and $\mathbb{D}_k$ on the one hand, and Laplace operator

$$\Delta := \sum_{j=1}^{m} \partial_j^2.$$

and the Helmholtz operator $\Delta + k^2$, on the other hand. We have the following standard lemma.

Lemma 4.1.1. *There hold*

$$\mathbb{D}^2 = -\Delta \tag{4.1.6}$$

and, if $k \in \mathbb{C}$ is arbitrary,

$$\mathbb{D}_k^2 = -(\Delta + k^2). \tag{4.1.7}$$

4.2 The Exterior Derivative Operator

In this section, we attempt to define the *exterior derivative operator d* and its (formal) adjoint δ as follows:

$$du := \Pi_{l+1}(\mathbb{D}u) \tag{4.2.8}$$

$$\delta u := \Pi_{l-1}(\mathbb{D}u) \tag{4.2.9}$$

whenever u is a Λ^l-valued function, $0 \leq l \leq m$.

Remark 4.2.1. *The exterior derivative operator d maps any Λ^0-valued function into* 0*, and its adjoint, δ, maps any Λ^m-valued function into* 0*.*

Remark 4.2.2. *If φ is vector-valued, then $d\varphi = \nabla\varphi$.*

We now comment on the connection between the Dirac operator $\mathbb{D}$ and the operators d and δ.

Lemma 4.2.3. *For the exterior derivative operator d and its adjoint δ, the following hold:*

$$\mathbb{D} = d + \delta, \;\; d^2 = 0, \;\; \delta^2 = 0, \;\; \text{and } d\delta + \delta d = -\Delta. \tag{4.2.10}$$

There are also a number of useful properties of the Hodge star operator and the operators d and δ which we would like to summarize here.

Lemma 4.2.4. *Suppose that u is a Λ^l-valued function, $0 \leq l \leq m$. Then the following hold:*

1. $*\delta u = (-1)^l d(*u)$;
2. $\delta(*u) = (-1)^{l+1} * (du)$;
3. $\delta u = (-1)^{m(l+1)+1} * (d(*u))$.

When we restrict our attention to the physically most relevant case, i.e. $m = 3$, and work with $\mathcal{A}_3 \hookrightarrow \mathcal{A}_4$, employing standard three-dimensional notation, we have the following lemma.

Lemma 4.2.5. *For any Λ^1-valued functions u, v, we have*

$$\operatorname{div} u = -\delta u, \;\; \langle u, v\rangle = u \vee v, \;\; \operatorname{curl} u = * \, du, \;\; \text{and} \;\; u \times v = * \, (u \wedge v). \tag{4.2.11}$$

The proof is elementary, hence omitted.

Let us also introduce the following lemma which allows us to interchange the operators d, δ and the imaginary unit e_4.

Lemma 4.2.6. *For the exterior derivative opearator d and its adjoint δ, the following hold:*

$$de_4 = -e_4 d, \quad \text{and} \quad \delta e_4 = -e_4 \delta. \tag{4.2.12}$$

Chapter 5

Smoothness Spaces on the Boundary of a Lipschitz Domain

5.1 Tangential and Normal Spaces

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^m$, and denote by $d\sigma$ and ν the surface measure and outward unit normal to $\partial\Omega$, respectively.

For $1 < p < \infty$, $L^p(\partial\Omega)$ stands for the usual Lebesgue space of functions defined on $\partial\Omega$ which are measurable, and p-th power integrable with respect to the surface measure $d\sigma$. In other words, we have

$$(5.1.1)\quad L^p(\partial\Omega) := \Big\{ f : \partial\Omega \to \mathbb{C} \text{ measurable} : \| f \|_{L^p(\partial\Omega)} := \Big[\int_{\partial\Omega} |f|^p d\sigma \Big]^{\frac{1}{p}} < \infty \Big\}.$$

Next, define the first-order Sobolev space

$$(5.1.2)\qquad L_1^p(\partial\Omega) := \{ f \in L^p(\partial\Omega) : \nabla_{tan} f \in L^p(\partial\Omega) \},$$

where

$$(5.1.3)\qquad \nabla_{tan} f := \nabla f - (\partial_\nu f)\nu = \nabla f - (\nu \cdot \nabla f)\nu$$

is the tangential gradient of f on $\partial\Omega$. When $m = 3$, we also have

$$\nabla_{tan} f = -\nu \times (\nu \times \nabla f). \tag{5.1.4}$$

The norm in $L_1^p(\partial\Omega)$ is defined by

$$\| f \|_{L_1^p(\partial\Omega)} := \| f \|_{L^p(\partial\Omega)} + \| \nabla_{tan} f \|_{L^p(\partial\Omega)} . \tag{5.1.5}$$

Next, we discuss the surface divergence operator. First, for $1 < p < \infty$, we set

$$L_{tan}^p(\partial\Omega) := \{ f \in L^p(\partial\Omega, \mathbb{R}^3) : \langle \nu, f \rangle = 0 \text{ a.e. on } \partial\Omega \} \tag{5.1.6}$$

and introduce the *surface divergence operator*

$$\mathrm{Div} : L_{tan}^p(\partial\Omega) \to L_{-1}^p(\partial\Omega), \quad \int_{\partial\Omega} g \mathrm{Div} f \, d\sigma = -\int_{\partial\Omega} \langle f, \nabla_{tan} g \rangle \, d\sigma \tag{5.1.7}$$

for each $f \in L_{tan}^p(\partial\Omega)$, and $g \in L_1^{p'}(\partial\Omega) = (L_{-1}^p(\partial\Omega))^*$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

For $1 < p < \infty$, another space that is going to be important for us in the sequel is

$$L_{tan}^{p,Div}(\partial\Omega) := \{ f \in L_{tan}^p(\partial\Omega) : \mathrm{Div} f \in L^p(\partial\Omega) \}, \tag{5.1.8}$$

equipped with the norm

$$\| f \|_{L_{tan}^{p,Div}(\partial\Omega)} := \| f \|_{L^p(\partial\Omega, \mathbb{R}^3)} + \| \mathrm{Div} f \|_{L^p(\partial\Omega)} . \tag{5.1.9}$$

For $\mathcal{A}_m$-valued functions, we define

$$L^p(\partial\Omega, \mathcal{A}_m) := \Big\{ f(x) = \sum_{l=0}^{m} {\sum_{|I|=l}}' f_I(x) e_I : \ f_I \in L^p(\partial\Omega) \Big\}. \tag{5.1.10}$$

Going further, let us restrict our attention to the physicall most relevant case, i.e. $m = 3$, and we introduce a new operator, d_∂, by the requirement that

$$\int_{\partial\Omega} \langle d_\partial f, \varphi\rangle \, d\sigma := \int_{\partial\Omega} \langle f, \delta\varphi\rangle \, d\sigma. \tag{5.1.11}$$

for any test function $\varphi \in C^1(\mathbb{R}^3, \mathcal{A}_4)$.

Moreover, we call $f \in L^p(\partial\Omega, \mathcal{A}_4)$ *normal* if $\nu \wedge f = 0$ a.e. on $\partial\Omega$ and set

$$L^p_{nor}(\partial\Omega, \mathcal{A}_4) := \{f \in L^p(\partial\Omega, \mathcal{A}_4) : \nu \wedge f = 0 \textit{ a.e. on } \partial\Omega\}, \tag{5.1.12}$$

then define

$$L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4) := \{f \in L^p_{nor}(\partial\Omega, \mathcal{A}_4) : d_\partial f \in L^p(\partial\Omega, \mathcal{A}_4)\}, \tag{5.1.13}$$

equipped with the norm

$$\| f \|_{L^{p,d}_{nor}(\partial\Omega,\mathcal{A}_4)} := \| f \|_{L^p(\partial\Omega,\mathcal{A}_4)} + \| d_\partial f \|_{L^p(\partial\Omega,\mathcal{A}_4)} . \tag{5.1.14}$$

Similarly, we define another operator, δ_∂, by demanding that

$$\int_{\partial\Omega} \langle \delta_\partial f, \varphi\rangle \, d\sigma := \int_{\partial\Omega} \langle f, d\varphi\rangle \, d\sigma \tag{5.1.15}$$

for any $\varphi \in C^1(\mathbb{R}^3, \mathcal{A}_4)$.

Call $f \in L^p(\partial\Omega, \mathcal{A}_4)$ *tangential* if $\nu \vee f = 0$ a.e. on $\partial\Omega$. Thus, after defining the related space

$$L^p_{tan}(\partial\Omega, \mathcal{A}_4) := \{f \in L^p(\partial\Omega, \mathcal{A}_4) : \nu \vee f = 0 \textit{ a.e. on } \partial\Omega\}, \tag{5.1.16}$$

we can introduce

$$L^{p,\delta}_{tan}(\partial\Omega, \mathcal{A}_4) := \{f \in L^p_{tan}(\partial\Omega, \mathcal{A}_4) : \delta_\partial f \in L^p(\partial\Omega, \mathcal{A}_4)\} \tag{5.1.17}$$

and equip it with the norm

$$\| f \|_{L^{p,\delta}_{tan}(\partial\Omega,\mathcal{A}_4)} := \| f \|_{L^p(\partial\Omega,\mathcal{A}_4)} + \| \delta_\partial f \|_{L^p(\partial\Omega,\mathcal{A}_4)} . \tag{5.1.18}$$

At this stage, We introduce some important properties of the operators d_∂ and δ_∂, which are first proved in the reference [MiD].

Lemma 5.1.1. *If $F \in L^1_{tan}(\partial\Omega, \mathcal{A}_4)$ is such that $\delta_\partial F \in L^1_{tan}(\partial\Omega, \mathcal{A}_4)$, then*

$$\delta_\partial(\delta_\partial F) = 0. \tag{5.1.19}$$

Similarly, if $G \in L^1_{nor}(\partial\Omega, \mathcal{A}_4)$ is such that $d_\partial G \in L^1_{nor}(\partial\Omega, \mathcal{A}_4)$, then

$$d_\partial(d_\partial G) = 0. \tag{5.1.20}$$

Lemma 5.1.2. *Let ν be the outward unit normal to $\partial\Omega$. Let $F \in C^\infty(\Omega, \mathcal{A}_4)$ be such that F and dF have non-tangential boundary traces at almost any point on $\partial\Omega$ and $N(F), N(dF) \in L^1(\partial\Omega)$. Then $d_\partial(\nu \wedge F\big|_{\partial\Omega})$ exists in L^1 and in fact*

$$d_\partial(\nu \wedge F\big|_{\partial\Omega}) = -\nu \wedge (dF)\big|_{\partial\Omega}. \tag{5.1.21}$$

Similarly, if F and δF have non-tangential boundary traces at almost any point on $\partial\Omega$ and $N(F), N(\delta F) \in L^1(\partial\Omega)$. Then $\delta_\partial(\nu \vee F\big|_{\partial\Omega})$ exists in L^1 and

$$\delta_\partial(\nu \vee F\big|_{\partial\Omega}) = -\nu \vee (\delta F)\big|_{\partial\Omega}. \tag{5.1.22}$$

5.2 Decomposition of $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_3)$

Our next result is a decomposition theorem for the space $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_3)$.

Theorem 5.2.1. *For each $1 < p < \infty$, we have*

$$L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_3) = \nu L^p_1(\partial\Omega) \oplus *L^{p,Div}_{tan}(\partial\Omega) \oplus *L^p(\partial\Omega).$$

Proof.

Take $h \in L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_3)$ and write $h = h_0 + h_1 + h_2 + h_3$, where h_j is Λ^j-valued for $j = 0, 1, 2, 3$. By the definition of $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_3)$, we have that $h_j \in L^p(\partial\Omega, \mathcal{A}_3)$, $\nu \wedge h = 0$, and $d_\partial h \in L^p(\partial\Omega, \mathcal{A}_3)$. It is obvious that $\nu \wedge h = 0$ implies that $\nu \wedge h_j = 0$ for $j = 0, 1, 2, 3$.

By corollary 3.2.3, we know that $h_0 = \nu\wedge(\nu\vee h_0)+\nu\vee(\nu\wedge h_0)$. Since $\nu \vee h_0 = 0$ and $\nu \wedge h_0 = 0$, then $h_0 = 0$.

Next, $\nu \wedge h_1 = 0$ implies $*(\nu \wedge h_1) = 0$. Invoking the identity $*(\nu \wedge h_1) = \nu \times h_1$, we obtain $\nu \times h_1 = 0$. Therefore, $h_1 = f\nu$, where f is a scalar-valued function. Since $h_1 \in L^p(\partial\Omega, \mathcal{A}_3)$, then $f \in L^p(\partial\Omega)$.

For any function $\varphi \in C^\infty_0(\partial\Omega, \mathcal{A}_3)$, we can write

$$\varphi = \varphi_0 + \varphi_1 + *\varphi_0' + *\varphi_1',$$

where φ_0 and φ_0' are Λ^0-valued functions, and φ_1 and φ_1' are Λ^1-valued functions. Applying the operator δ to φ, we then get

$$\delta\varphi = \delta\varphi_1 + *(d\varphi_1') - *(d\varphi_0').$$

In particular, choose φ such that $\varphi_0 = \varphi_1 = \varphi_0^{'} = 0$, and $\varphi_1^{'}$ is arbitrary. In this scenario, the identity

$$\int_{\partial\Omega} \langle \delta\varphi, h\rangle \, d\sigma = \int_{\partial\Omega} \langle \varphi, d_{\partial} h\rangle \, d\sigma$$

becomes

$$\int_{\partial\Omega} \langle *(d\varphi_1^{'}), f\nu\rangle \, d\sigma = \int_{\partial\Omega} \langle \varphi_1^{'}, d_{\partial} h\rangle \, d\sigma.$$

Redenoting $\varphi_1^{'}$ by g, we are led to the conclusion that

$$\int_{\partial\Omega} \langle *(dg), \nu\rangle f \, d\sigma = \int_{\partial\Omega} \langle g, d_{\partial} h\rangle \, d\sigma.$$

On the other hand,

$$\int_{\partial\Omega} \langle *(dg), \nu\rangle f \, d\sigma = \int_{\partial\Omega} \langle \nu, \mathrm{curl} g\rangle f \, d\sigma = -\int_{\partial\Omega} \langle \nu \times \nabla f, g\rangle \, d\sigma.$$

Hence, ultimately we have $-\nu \times \nabla f = d_{\partial} h \in L^p(\partial\Omega, \mathcal{A}_3)$. Consequently,

$$\nabla_{tan} f = -\nu \times (\nu \times \nabla f) \in L^p(\partial\Omega, \mathcal{A}_3).$$

This proves that $f \in L_1^p(\partial\Omega)$ and, thus, $h_1 \in \nu L_1^p(\partial\Omega)$.

Next, let us choose φ such that $\varphi_0 = \varphi_1 = \varphi_1^{'} = 0$, and $\varphi_0^{'}$ is arbitrary. In this case, the left-hand side of the identity

$$\int_{\partial\Omega} \langle \delta\varphi, h\rangle \, d\sigma = \int_{\partial\Omega} \langle \varphi, d_{\partial} h\rangle \, d\sigma$$

reduces to

$$\begin{aligned}\int_{\partial\Omega} \langle \delta\varphi, h\rangle \, d\sigma &= -\int_{\partial\Omega} \langle *(d\varphi_0^{'}), h_2\rangle \, d\sigma \\ &= -\int_{\partial\Omega} \langle d\varphi_0^{'}, *h_2\rangle \, d\sigma \\ &= -\int_{\partial\Omega} \langle \nabla\varphi_0^{'}, *h_2\rangle \, d\sigma.\end{aligned}$$

Since $\nu \vee (*h_2) = *(\nu \wedge h_2) = 0$, then $*h_2$ is tangential. Consequnetly,

$$\int_{\partial\Omega} \langle \nabla\varphi_0^{'}, *h_2\rangle \, d\sigma = \int_{\partial\Omega} \langle \nabla_{tan}\varphi_0^{'}, *h_2\rangle \, d\sigma.$$

On the other hand, the right-hand side of the identity

$$\int_{\partial\Omega} \langle \delta\varphi, h\rangle \, d\sigma = \int_{\partial\Omega} \langle \varphi, d_\partial h\rangle \, d\sigma$$

can be written as

$$\int_{\partial\Omega} \langle \varphi, d_\partial h\rangle \, d\sigma = \int_{\partial\Omega} \langle *\varphi_0^{'}, d_\partial h\rangle \, d\sigma = \int_{\partial\Omega} \langle \varphi_0^{'}, *(d_\partial h)\rangle \, d\sigma.$$

The bottom line is that

$$\int_{\partial\Omega} \langle \nabla_{tan}\varphi_0^{'}, *h_2\rangle \, d\sigma = -\int_{\partial\Omega} \langle \varphi_0^{'}, *(d_\partial h)\rangle \, d\sigma, \quad \forall \varphi_0^{'} \in C_0^\infty(\partial\Omega, \mathcal{A}_3),$$

Since $\varphi_0^{'}$ is Λ^0-valued, we can replace $*(d_\partial h)$ by $\Pi_0(*(d_\partial h))$, in the equation above. Then, the equation is now

$$\int_{\partial\Omega} \langle \nabla_{tan}\varphi_0^{'}, *h_2\rangle \, d\sigma = -\int_{\partial\Omega} \langle \varphi_0^{'}, \Pi_0(*(d_\partial h))\rangle \, d\sigma.$$

for any nice $\varphi_0^{'} \in C_0^\infty(\partial\Omega, \mathcal{A}_3)$. By the definition of $L^{p,Div}_{tan}(\partial\Omega, \mathcal{A}_3)$, we may therefore conclude that

$$\mathrm{Div}(*h) = \Pi_0(*(d_\partial h)) \in L^p(\partial\Omega), \text{ hence } *h_2 \in L^{p,Div}_{tan}(\partial\Omega).$$

Finally, since h_3 is a Λ^3-valued function, the condition $\nu \wedge h_3 = 0$ is satisfied automatically, and we can simply view h_3 as an element in $*L^p(\partial\Omega)$. This finishes the proof of the decomposition theorem. $\square$

Chapter 6

The Cauchy Integral and Related Operators

6.1 Definitions and Basic Properties

For each $k \in \mathbb{C}$, let Φ_k stand for the standard radial fundamental solution for the Helmholtz operator $\Delta + k^2$ in $\mathbb{R}^m$, that is

$$\Phi_k(x) := \frac{1}{4i}\Big(\frac{k}{2\pi}\Big)^{\frac{m-2}{2}} \frac{1}{|x|^{\frac{m-2}{2}}}\ H^{(1)}_{\frac{m-2}{2}}(k|x|), \tag{6.1.1}$$

where $x \in \mathbb{R}^m\backslash\{0\}$ and H^1_α denotes the Hankel function of the first kind and order α. In particular, in $\mathbb{R}^3$,

$$\Phi_k(x) := -\frac{e^{ik|x|}}{4\pi|x|}, \tag{6.1.2}$$

where $x \in \mathbb{R}^3\backslash\{0\}$.

Let Ω be a Lipschitz domain of $\mathbb{R}^3$. The associated *single-layer potential operator* is defined by

$$(\mathcal{S}_k f)(x) := \int_{\partial\Omega} \Phi_k(x-y) f(y)\, d\sigma_y, \quad x \notin \partial\Omega, \tag{6.1.3}$$

and its boundary version is given by

$$(6.1.4) \qquad \begin{aligned}(S_k f)(x) &:= \int_{\partial\Omega} \Phi_k(x-y) f(y)\, d\sigma_y, \\ &= -\frac{1}{4\pi}\int_{\partial\Omega} \frac{e^{ik|x-y|}}{|x-y|} f(y) d\sigma_y, \quad x \in \partial\Omega.\end{aligned}$$

We define the *Cauchy operator* $\mathcal{C}_k$ by

$$(6.1.5) \qquad \mathcal{C}_k := \mathbb{D}_k \mathcal{S}_k = d\mathcal{S}_k + \delta\mathcal{S}_k + k\, e_4 \mathcal{S}_k,$$

or, more explicitly,

$$(6.1.6) \qquad (\mathcal{C}_k f)(x) = \int_{\partial\Omega} \mathbb{D}_k \Phi_k(x-y) f(y)\, d\sigma_y, \quad x \notin \partial\Omega.$$

Its boundary version is given by

$$(6.1.7) \qquad (C_k f)(x) = p.v. \int_{\partial\Omega} \mathbb{D}_k \Phi_k(x-y) f(y)\, d\sigma_y, \quad x \in \partial\Omega.$$

We also find it useful to work with the *double-layer potential operator*

$$(6.1.8) \qquad K_k f(x) := p.v. \int_{\partial\Omega} \partial_{\nu_y} \Phi_k(x-y) f(y)\, d\sigma_y, \quad x \in \partial\Omega.$$

and its formal transpose K_k^t. Above, *p.v.* stands for "*principle value*", i.e. the integral is consider over $\{y \in \partial\Omega : |x-y| > \varepsilon\}$ and then we pass the limit as $\varepsilon \to 0^+$.

Remark 6.1.1. *The following identities hold:*

$$(6.1.9) \qquad K_k f = -\mathrm{div} S_k(\nu f) \text{ and } K_k^t f = \nu \cdot \nabla S_k f.$$

Finally, the principle value magnetic dipole operator is given by

$$M_k f(x) := \nu(x) \times \left(p.v. \int_{\partial\Omega} \mathrm{curl}_x \{\Phi_k(x-y) f(y)\} \, d\sigma_y \right), \quad x \in \partial\Omega. \tag{6.1.10}$$

At the end of this section, we introduce some important properties of the operators d_∂ and δ_∂, which are first proved in the reference [MiD].

Lemma 6.1.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^m$, and $1 < p < \infty$. Then for any $F \in L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_m)$ we have*

$$d\mathcal{S}_k F = \mathcal{S}_k(d_\partial F). \tag{6.1.11}$$

Similarly, for any $G \in L^{p,\delta}_{tan}(\partial\Omega, \mathcal{A}_m)$ we have

$$\delta\mathcal{S}_k G = \mathcal{S}_k(\delta_\partial G). \tag{6.1.12}$$

Next we present a version of Lemma 6.1.2 which deals with the case when all integral operators are considered on the boundary.

Corollary 6.1.3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^m$. For $1 < p < \infty$, we have*

$$dS_k F = S_k(d_\partial F) \quad \text{for } F \in L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_m), \tag{6.1.13}$$

and

$$\delta S_k G = S_k(\delta_\partial G) \quad \text{for } G \in L^{p,\delta}_{tan}(\partial\Omega, \mathcal{A}_m). \tag{6.1.14}$$

6.2 Non-tangential Maximal Function Estimates

In this section we shall prove an important result to the effect that functions expressed in terms of the Cauchy operator when acting on suitable spaces satisfy natural non-tangential maximal function estimates. Concretely we have the following theorem.

Theorem 6.2.1. *For each $k \in \mathbb{C}$ and $1 < p < \infty$, there exists $C = C(k, \partial\Omega, p) > 0$ such that if $u := \mathcal{C}_k f$ in Ω, for $f \in L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4)$, then*

$$\|N(u)\|_{L^p(\partial\Omega)} + \|N(du)\|_{L^p(\partial\Omega)} + \|N(\delta u)\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^{p,d}_{nor}(\partial\Omega,\mathcal{A}_4)}. \tag{6.2.15}$$

Proof.

Based on (6.1.5), write

$$u = d\mathcal{S}_k f + \delta\mathcal{S}_k f + ke_4\mathcal{S}_k f. \tag{6.2.16}$$

From Theorem 2.3.1,

$$\|N(u)\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^{p,d}_{nor}(\partial\Omega,\mathcal{A}_4)}. \tag{6.2.17}$$

Next, by Lemma 4.2.3 and Lemma 4.2.6, we have

$$\begin{aligned} du &= d^2\mathcal{S}_k f + d\delta\mathcal{S}_k f + kde_4\mathcal{S}_k f & (6.2.18)\\ &= d\delta\mathcal{S}_k f - ke_4 d\mathcal{S}_k f. & (6.2.19)\end{aligned}$$

Using $d\delta = -\delta d - (\Delta + k^2) + k^2$ and the fact that the Helmholtz operator annihilates the single-layer $\mathcal{S}_k$, we obtain that

$$du = -\delta d \mathcal{S}_k f + k^2 \mathcal{S}_k f - k e_4 d \mathcal{S}_k f. \tag{6.2.20}$$

Recall that $d\mathcal{S}_k f = \mathcal{S}_k(d_\partial f)$ (see Lemma 6.1.2), we finally arrive at the representation

$$du = -\delta \mathcal{S}_k(d_\partial f) + k^2 \mathcal{S}_k f - k e_4 d \mathcal{S}_k f. \tag{6.2.21}$$

To this end, the Calderón-Zygmund theory 2.3.1 applies. Since, $d_\partial f \in L^p(\partial\Omega, \mathcal{A}_4)$ given that $f \in L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4)$, we thus have,

$$\|N(du)\|_{L^p(\partial\Omega)} \leq C \|f\|_{L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4)}. \tag{6.2.22}$$

The case of $N(\delta u)$ is similar, even simpler. Concretely,

$$\delta u = \delta \mathcal{S}_k(d_\partial f) - k e_4 d \mathcal{S}_k f, \tag{6.2.23}$$

and the same analysis applies. □

6.3 Jump Formulas

We next discuss the jump formulas for the exterior derivative operator and its adjoint acting on $\mathcal{S}_k$. These formulas are of basic importance for our work in the sequel.

Theorem 6.3.1. *For any $h \in L^p(\partial\Omega, \mathcal{A}_4)$, and a.e. $x \in \partial\Omega$, we have*

$$d\mathcal{S}_k h\Big|_{\partial\Omega\pm}(x) = \mp\tfrac{1}{2}(\nu \wedge h)(x) + dS_k h(x), \tag{6.3.24}$$

where

$$dS_k h(x) := p.v. \int_{\partial\Omega} (\nabla\Phi_k)(x-y) \wedge h(y)\, d\sigma_y. \tag{6.3.25}$$

and

$$\delta\mathcal{S}_k h\Big|_{\partial\Omega\pm}(x) = \pm\tfrac{1}{2}(\nu \vee h)(x) + \delta S_k h(x), \tag{6.3.26}$$

where

$$\delta S_k h(x) := -p.v. \int_{\partial\Omega} (\nabla\Phi_k)(x-y) \vee h(y)\, d\sigma_y. \tag{6.3.27}$$

These formulas are proved in [MiD] p.82.

Base on these, we can now prove the jump formulas for the Cauchy operator $\mathcal{C}_k$.

Theorem 6.3.2. *For any $h \in L^p(\partial\Omega, \mathcal{A}_4)$, and a.e. $x \in \partial\Omega$, we have*

$$\mathcal{C}_k h\Big|_{\partial\Omega\pm} = (\mp\tfrac{1}{2}\nu\, \cdot + C_k)h. \tag{6.3.28}$$

Proof.

By the definition of $\mathcal{C}_k$, we may write

$$\mathcal{C}_k h\Big|_{\partial\Omega\pm} = d\mathcal{S}_k h\Big|_{\partial\Omega\pm} + \delta\mathcal{S}_k h\Big|_{\partial\Omega\pm} + k\, e_4 \mathcal{S}_k h\Big|_{\partial\Omega\pm}.$$

By (6.3.24), we have that

$$d\mathcal{S}_k h\Big|_{\partial\Omega\pm} = \mp\tfrac{1}{2}(\nu \wedge h)(x) + dS_k h(x).$$

Also the equality (6.3.26) gives us

$$\delta\mathcal{S}_k h\Big|_{\partial\Omega\pm} \;=\; \pm\tfrac{1}{2}(\nu \vee h) + \delta S_k h.$$

Since the single layer-operator $\mathcal{S}_k$ has less singularity, then

$$k\, e_4 \mathcal{S}_k h\Big|_{\partial\Omega\pm} = k\, e_4 S_k h.$$

Therefore,

$$\begin{aligned}
\mathcal{C}_k h\Big|_{\partial\Omega\pm} &= \mp\tfrac{1}{2}(\nu \wedge h) + dS_k h + \pm\tfrac{1}{2}(\nu \vee h) + \delta S_k h + k\, e_4 S_k h \\
&= \mp\tfrac{1}{2}[\nu \wedge h - \nu \vee h] + C_k h \\
&= \mp\tfrac{1}{2}\nu \cdot h + C_k h.
\end{aligned}$$

This finishes the proof. □

6.4 Decay at Infinity

We next discuss the decay at infinity in the form of the following lemma.

Lemma 6.4.1. *If $f \in L^1(\partial\Omega, \mathcal{A}_4)$, where Ω is a bounded Lipschitz domain, and u is defined by*

$$u(x) := \int_{\partial\Omega} \mathbb{D}_k \Phi_k(x-y) f(y)\, d\sigma_y,\;\; x \in \Omega_-,$$

then u satisfies the following decay condition

$$\lim_{|x|\to\infty}(|x| - ie_4 x)u(x) = 0. \tag{6.4.29}$$

Proof.

As mentioned in (6.1.2), $\Phi_k(x)$ is the fundament solution of $\Delta + k^2$ in $\mathbb{R}^3$. That is,

$$\Phi_k(x) := -\frac{1}{4\pi|x|}\ e^{ik|x|}.$$

Computing the partial derivative $\partial_1\Phi_k(x)$, we have

$$\begin{aligned}
\partial_1\Phi_k(x) &= \partial_1\left(-\frac{1}{4\pi|x|}\ e^{ik|x|}\right) \\
&= -\frac{1}{4\pi}\ \partial_1\left(\frac{1}{|x|}\ e^{ik|x|}\right) \\
&= -\frac{1}{4\pi}\ \left[\partial_1((x_1^2+x_2^2+x_3^2)^{-\frac{1}{2}})\ e^{ik|x|} + \frac{1}{|x|}\ \partial_1(e^{ik(x_1^2+x_2^2+x_3^2)^{\frac{1}{2}}})\right] \\
&= -\frac{1}{4\pi}\left[-\frac{x_1}{|x|^3}\ e^{ik|x|} + \frac{ikx_1}{|x|^2}\ e^{ik|x|}\right].
\end{aligned}$$

Similarly,

$$\partial_2\Phi_k(x) = -\frac{1}{4\pi}\left[-\frac{x_2}{|x|^3}\ e^{ik|x|} + \frac{ikx_2}{|x|^2}\ e^{ik|x|}\right],$$

and

$$\partial_3\Phi_k(x) = -\frac{1}{4\pi}\left[-\frac{x_3}{|x|^3}\ e^{ik|x|} + \frac{ikx_3}{|x|^2}\ e^{ik|x|}\right].$$

Considering all these partial derivatives together, we obtain

$$\begin{aligned}
\mathbb{D}_k\Phi_k(x-y) &= \mathbb{D}\Phi_k(x-y) + k\,e_4\Phi_k(x-y) \\
&= \sum_{j=1}^{3}(\partial_j\Phi_k(x-y))e_j + k\,e_4\Phi_k(x-y) \\
&= -\frac{1}{4\pi}\left[-\frac{x-y}{|x-y|^3}\ e^{ik|x-y|} + \frac{ik(x-y)}{|x-y|^2}\ e^{ik|x-y|}\right] \\
&\quad + k\,e_4\left(-\frac{1}{4\pi|x-y|}\ e^{ik|x-y|}\right) \\
&= -\frac{1}{4\pi}\left[-\frac{x-y}{|x-y|^3} + \frac{ik(x-y)}{|x-y|^2} + \frac{k\,e_4}{|x-y|}\right]\ e^{ik|x-y|}.
\end{aligned}$$

Once $\mathbb{D}_k\Phi_k(x-y)$ has been calculated out, we have

$$\begin{aligned}
&(|x| - ie_4x)u(x) \\
&= \int_{\partial\Omega}\mathbb{D}_k\Phi_k(x-y)f(y)\,d\sigma_y \\
&= \int_{\partial\Omega}\frac{1}{4\pi}(|x|-ie_4x)\frac{x-y}{|x-y|^3}\ e^{ik|x-y|}f(y)d\sigma_y \\
&\quad + \int_{\partial\Omega} -\frac{1}{4\pi}(|x|-ie_4x)\left[\frac{ik(x-y)}{|x-y|^2} + \frac{k\,e_4}{|x-y|}\right]\ e^{ik|x-y|}f(y)d\sigma_y.
\end{aligned}$$

Estimating the first integral above gives

$$\begin{aligned}
&\left|\int_{\partial\Omega}\frac{1}{4\pi}(|x|-ie_4x)\frac{x-y}{|x-y|^3}\ e^{ik|x-y|}f(y)d\sigma_y\right| \\
&\qquad \le c\int_{\partial\Omega}|x|\frac{1}{|x-y|^2}|f(y)|d\sigma_y.
\end{aligned}$$

Since $y \in \partial\Omega$ and $|x| \to \infty$ then, as x is large enough, it is not hard to see that

$$\frac{1}{|x-y|^2} \le \frac{c}{|x|^2}.$$

Therefore,

$$
\begin{aligned}
&\left| \int_{\partial\Omega} \frac{1}{4\pi}(|x| - ie_4 x)\frac{x-y}{|x-y|^3}\ e^{ik|x-y|} f(y) d\sigma_y \right| \\
&\quad \le c \int_{\partial\Omega} |x| \frac{1}{|x|^2} |f(y)| d\sigma_y \\
&\quad \le c \frac{1}{|x|} \int_{\partial\Omega} |f(y)| d\sigma_y \\
&\quad \le c \frac{1}{|x|} \parallel f \parallel_{L^1(\partial\Omega)} \rightarrow 0 \ as \ |x| \rightarrow \infty.
\end{aligned}
$$

In order to estimate the second integral, we introduce the function

$$
G(z) := \frac{ik}{|z|^2}[z - ie_4|z|] = ik\frac{z}{|z|^2} + \frac{k\, e_4}{|z|}.
$$

A direct computation of the partial derivatives of $G(z)$ yields the following:

$$
\begin{aligned}
\partial_1 G(z) &= \partial_1 \left(ik\frac{z}{|z|^2} \right) + \partial_1 \left(\frac{k\, e_4}{|z|} \right) \\
&= ik\partial_1 \left(\frac{z}{|z|^2} \right) + k\, e_4 \partial_1 \left(\frac{1}{|z|} \right) \\
&= ik \left[(\partial_1 z) \frac{1}{|z|^2} + z\partial_1 \frac{1}{|z|^2} \right] + k\, e_4 \partial_1 \left(\frac{1}{|z|} \right) \\
&= ik \left[\frac{e_1}{|z|^2} - \frac{z_1 z}{|z|^4} \right] + k\, e_4 \left(-\frac{z_1}{|z|^4} \right).
\end{aligned}
$$

Similarly,

$$
\partial_2 G(z) = ik \left[\frac{e_2}{|z|^2} - \frac{z_2 z}{|z|^4} \right] + k\, e_4 \left(-\frac{z_2}{|z|^4} \right),
$$

and

$$
\partial_3 G(z) = ik \left[\frac{e_3}{|z|^2} - \frac{z_3 z}{|z|^4} \right] + k\, e_4 \left(-\frac{z_3}{|z|^4} \right).
$$

Therefore,

$$|\nabla G(z)| \leq c\frac{1}{|z|^2}.$$

Then we can rewrite

$$\begin{aligned}
&\int_{\partial\Omega} -\frac{1}{4\pi}(|x| - ie_4 x) \left[\frac{ik(x-y)}{|x-y|^2} + \frac{k\,e_4}{|x-y|}\right] \; e^{ik|x-y|} f(y) d\sigma_y \\
&= \int_{\partial\Omega} -\frac{1}{4\pi}(|x| - ie_4 x) \; G(x-y) \; e^{ik|x-y|} f(y) d\sigma_y \\
&= \int_{\partial\Omega} -\frac{1}{4\pi}(|x| - ie_4 x) \; [G(x-y) - G(x)] \; e^{ik|x-y|} f(y) d\sigma_y \\
&\quad + \int_{\partial\Omega} -\frac{1}{4\pi}(|x| - ie_4 x) \; G(x) \; e^{ik|x-y|} f(y) d\sigma_y.
\end{aligned}$$

Let us point out that since $x \cdot x = -|x|^2$, we have

$$\begin{aligned}
&(|x| - ie_4 x) \; G(x) \\
&= \frac{ik}{|x|}(|x| - ie_4 x)(x - ie_4|x|) \\
&= \frac{ik}{|x|} \left[|x|x - ie_4|x|^2 - ie_4 x \cdot x - |x|x\right] \\
&= \frac{ik}{|x|} \left[|x|x - ie_4|x|^2 + ie_4|x|^2 - |x|x\right] \\
&= 0.
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
&\int_{\partial\Omega} -\frac{1}{4\pi}(|x| - ie_4 x) \left[\frac{ik(x-y)}{|x-y|^2} + \frac{k\,e_4}{|x-y|}\right] \; e^{ik|x-y|} f(y) d\sigma_y \\
&= \int_{\partial\Omega} -\frac{1}{4\pi}(|x| - ie_4 x) \; [G(x-y) - G(x)] \; e^{ik|x-y|} f(y) d\sigma_y.
\end{aligned}$$

With the above equality in hands, we estimate

$$\begin{aligned}
&\left|\int_{\partial\Omega} -\frac{1}{4\pi}(|x| - ie_4 x)\left[\frac{ik(x-y)}{|x-y|^2} + \frac{k\, e_4}{|x-y|}\right]\ e^{ik|x-y|} f(y) d\sigma_y\right| \\
&\qquad \le \left|\int_{\partial\Omega} -\frac{1}{4\pi}(|x| - ie_4 x)\ [G(x-y) - G(x)]\ e^{ik|x-y|} f(y) d\sigma_y\right| \\
&\qquad \le c \int_{\partial\Omega} \Big||x| - ie_4 x\Big|\ \ |G(x-y) - G(x)|\ \ \left|e^{ik|x-y|}\right| |f(y)|\, d\sigma_y.
\end{aligned}$$

At this stage, apply the Mean Value Theorem,

$$G(x-y) - G(x) = (\nabla G)(x - \xi y)\cdot(-y),\ \text{for some } \xi \in (0,1),$$

and further estimate

$$\begin{aligned}
&\left|\int_{\partial\Omega} -\frac{1}{4\pi}(|x| - ie_4 x)\left[\frac{ik(x-y)}{|x-y|^2} + \frac{k\, e_4}{|x-y|}\right]\ e^{ik|x-y|} f(y) d\sigma_y\right| \\
&\qquad \le c \int_{\partial\Omega} \Big||x| - ie_4 x\Big|\ \ |\nabla G(x - \xi y)|\ \ |y|\ \ \left|e^{ik|x-y|}\right| |f(y)|\, d\sigma_y \\
&\qquad \le c \int_{\partial\Omega} |x|\ \frac{1}{|x - \xi y|^2}\ \ |f(y)|\, d\sigma_y \\
&\qquad \le c\ \frac{|x|}{|x|^2}\ \int_{\partial\Omega} |f(y)|\, d\sigma_y \\
&\qquad \le c \frac{1}{|x|}\ \|\ f\ \|_{L^1(\partial\Omega)} \to 0\ as\ |x| \to \infty.
\end{aligned}$$

This concludes the proof. $\square$

6.5 Invertibility of the Cauchy-type Operator $\lambda I + \nu \wedge C_k$

We are now ready to show the main result of this chapter.

Theorem 6.5.1. *Let $\Omega \subset \mathbb{R}^3$ be an arbitrary Lipschitz domain with compact boundary. Then for every $\lambda \in \mathbb{R}$ with $|\lambda| \geq \frac{1}{2}$, there exists a sequence of real numbers $\{k_j\}_j$ such that for each $1 < p < 2+\varepsilon$ and $k \in \mathbb{C}\backslash\{k_j\}_j$, the operator $\lambda I + \nu \wedge C_k$ is an isomorphism on $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4)$*

Proof.

For any $f \in L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4)$, it can be rewritten as $f = F + e_4\widetilde{F}$, where both F and $\widetilde{F}$ are in $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_3)$. Recall that

$$\begin{aligned} \mathcal{C}_k F &= \mathbb{D}_k \mathcal{S}_k F \\ &= d\mathcal{S}_k F + \delta\mathcal{S}_k F + k\, e_4 \mathcal{S}_k F \\ &= \mathcal{S}_k(d_\partial F) + \delta\mathcal{S}_k F + k\, e_4\mathcal{S}_k F. \end{aligned} \tag{6.5.30}$$

Restrict $\mathcal{C}_k$ to the boundary, apply the jump formulas for $\mathcal{C}_k$ and $\delta\mathcal{S}_k$, and then apply $\nu\wedge$ to both sides of the equality (6.5.30), we arrive at

$$\nu \wedge (-\tfrac{1}{2}\nu \cdot F + C_k F) = \nu \wedge S_k(d_\partial F) + \nu \wedge (\tfrac{1}{2}\nu \vee F + \delta S_k F) + k\, e_4 \nu \wedge S_k F. \tag{6.5.31}$$

Moreover, considering that F is in $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_3)$, we can further simplify the equality (6.5.31) and write

$$(\nu \wedge C_k)F = \nu \wedge S_k(d_\partial F) + \nu \wedge \delta S_k F + k\, e_4 \nu \wedge S_k F. \tag{6.5.32}$$

By Theorem 5.2.1, we can express F as

$$F = \nu f_0 + *f_1 + *f_0', \tag{6.5.33}$$

where f_0 is in $L^p_1(\partial\Omega)$, f_1 is in $L^{p,Div}_{tan}(\partial\Omega)$, and $f_0^{'}$ is in $L^p(\partial\Omega)$. So

$$\nu \wedge \delta S_k F = \nu \wedge \delta S_k(\nu f_0) + \nu \wedge \delta S_k(*f_1) + \nu \wedge \delta S_k(*f_0^{'}). \tag{6.5.34}$$

For the first term $\nu \wedge \delta S_k(\nu f_0)$ in the equality (6.5.34), we have

$$\begin{aligned}
\nu \wedge \delta S_k(\nu f_0) &= -\nu \wedge p.v. \int_{\partial\Omega} \langle (\nabla \Phi_k)(\cdot - y), \nu(y) \rangle f_0(y)\, d\sigma_y \\
&= \nu K_k f_0. \qquad (6.5.35)
\end{aligned}$$

For the second term, $\nu \wedge \delta S_k(*f_1)$, we write

$$\begin{aligned}
\nu \wedge \delta S_k(*f_1) &= \nu \wedge (*dS_k f_1) \\
&= -*(\nu \vee dS_k f_1) \\
&= -*\left(\nu \vee p.v. \int_{\partial\Omega} d\Phi_k(\cdot - y) \wedge f_1(y)\, d\sigma_y\right) \\
&= -*\left(\nu \vee *p.v. \int_{\partial\Omega} \nabla\Phi_k(\cdot - y) \times f_1(y)\, d\sigma_y\right) \\
&= \nu \wedge p.v. \int_{\partial\Omega} \operatorname{curl}(\Phi_k(\cdot - y) f_1(y))\, d\sigma_y \\
&= *\left(\nu \times p.v. \int_{\partial\Omega} \operatorname{curl}(\Phi_k(\cdot - y) f_1(y))\, d\sigma_y\right) \\
&= *M_k f_1. \qquad (6.5.36)
\end{aligned}$$

Finally, we can rewrite the third term, $\nu \wedge \delta S_k(*f_0^{'})$, as

$$\begin{aligned}
\nu \wedge \delta S_k(*f_0^{'}) &= -*(\nu \vee dS_k f_0^{'}) \\
&= -*\langle \nu, \nabla S_k f_0^{'} \rangle \\
&= -*\partial_\nu S_k f_0^{'} \\
&= -*K_k^t f_0^{'}. \qquad (6.5.37)
\end{aligned}$$

In summary, from the equalities (6.5.35), (6.5.36) and (6.5.37), the equality (6.5.32) becomes

$$\begin{aligned}(\nu \wedge C_k)F &= \nu K_k f_0 + *M_k f_1 - *K_k^t f_0' \\ &\quad +\nu \wedge S_k(d_\partial F) + k\, e_4 \nu \wedge S_k F.\end{aligned}$$

Let I denote the identity operator. For every $\lambda \in \mathbb{R}$, we may write

$$\begin{aligned}&(\lambda I + \nu \wedge C_k)F \\ &= \nu[\lambda I + K_k]f_0 + *[\lambda I + M_k]f_1 + *[\lambda I - K_k^t]f_0' \\ &\quad + \nu \wedge S_k(d_\partial F) + k\, e_4 \nu \wedge S_k F,\end{aligned} \tag{6.5.38}$$

where $F \in L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_3)$. Similarly, for every $\lambda \in \mathbb{R}$, we have

$$\begin{aligned}&(-\lambda I + \nu \wedge C_{-k})F \\ &= \nu[-\lambda I + K_{-k}]f_0 + *[-\lambda I + M_{-k}]f_1 + *[-\lambda I - K_{-k}^t]f_0' \\ &\quad + \nu \wedge S_{-k}(d_\partial F) - k\, e_4 \nu \wedge S_{-k} F.\end{aligned} \tag{6.5.39}$$

Let us compute $(\lambda I + \nu \wedge C_k)f$. We start with writing

$$\begin{aligned}&(\lambda I + \nu \wedge C_k)f \\ &= (\lambda I + \nu \wedge C_k)(F + e_4 \widetilde{F}) \\ &= (\lambda I + \nu \wedge C_k)F + (\lambda I + \nu \wedge C_k)(e_4 \widetilde{F}).\end{aligned}$$

Since $(\nu \wedge C_k)\, e_4 = -e_4\, (\nu \wedge C_{-k})$, we may continue writing

$$
\begin{aligned}
&(\lambda I + \nu \wedge C_k) f \\
&\quad = \quad (\lambda I + \nu \wedge C_k) F - e_4(-\lambda I + \nu \wedge C_{-k})\widetilde{F}.
\end{aligned}
$$

By the equalities (6.5.38) and (6.5.39), we may further express $(\lambda I + \nu \wedge C_k) f$ as

$$
\begin{aligned}
&(\lambda I + \nu \wedge C_k) f \\
&= \quad \nu[\lambda I + K_k] f_0 + *[\lambda I + M_k] f_1 + *[\lambda I - K_k^t] f_0' \\
&\qquad + \nu \wedge S_k(d_\partial F) + k\, e_4 \nu \wedge S_k F \\
&\qquad - e_4 \Big\{ \nu[-\lambda I + K_{-k}] f_0 + *[-\lambda I + M_{-k}] f_1 + *[-\lambda I - K_{-k}^t] f_0' \\
&\qquad + \nu \wedge S_{-k}(d_\partial F) - k\, e_4 \nu \wedge S_{-k} F \Big\}.
\end{aligned} \tag{6.5.40}
$$

Our next goal is to prove that $\lambda I + \nu \wedge C_k$ is a Fredholm operator with index zero on $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4)$.

In order to continue with the proof, let us define a new operator T_1, in the following fashion:

$$
T_1 : L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4) \to L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_3) \oplus L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_3) \tag{6.5.41}
$$

is such that for any $f = F + e_4 \widetilde{F}$,

$$
T_1(f) := (F, \widetilde{F}). \tag{6.5.42}
$$

It is trivial that T_1 in (6.5.42) is an isomorphism.

We shall also need an operator T_2

$$T_2 : L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_3) \oplus L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_3) \quad \to \quad L^p_1(\partial\Omega) \oplus L^{p,Div}_{tan}(\partial\Omega) \oplus L^p(\partial\Omega)$$
$$\oplus\, L^p_1(\partial\Omega) \oplus L^{p,Div}_{tan}(\partial\Omega) \oplus L^p(\partial\Omega)$$

defined such that for any $(F, \widetilde{F})$ in $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_3) \oplus L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_3)$, where

$$F = \nu f_0 + *f_1 + *f_0^{'} \tag{6.5.43}$$

and

$$\widetilde{F} = \nu \widetilde{f_0} + *\widetilde{f_1} + *\widetilde{f_0^{'}}, \tag{6.5.44}$$

we have

$$T_2(F, \widetilde{F}) := (f_0, f_1, f_0^{'}, \widetilde{f_0}, \widetilde{f_1}, \widetilde{f_0^{'}}). \tag{6.5.45}$$

Once again, the operator T_2 is an isomorphism.

Finally, let us consider an operator $\widetilde{Q}$ written as a 6×6 matrix

$$\widetilde{Q} := \begin{bmatrix} \lambda I + K_k & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda I + M_k & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda I - K_k^t & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda I + K_{-k} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda I + M_{-k} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\lambda I - K_{-k}^t \end{bmatrix}.$$

Our next goal is to show that, when considered between appropriate spaces, the operator $\widetilde{Q}$ is Fredholm with index zero. As a preamble, we record some results in the paper [MMP].

Lemma 6.5.2. *Let X, Y, Z be Banach spaces and consider the commutative diagram*

$$\begin{array}{ccccccccc} 0 & \to & X & \to & Y & \to & Z & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & X & \to & Y & \to & Z & \to & 0 \end{array}$$

where all arrows are linear and bounded and the horizontal sequences are exact. Then the following hold:

a. *If two vertical arrows are isomorphsims, then so is the third one.*

b. *If two vertical arrows are Fredholm operators, then so is the third. Moreover, the index of the middle vertical arrow is the sum of the indices of the other two vertical arrows.*

Lemma 6.5.3. *Let X_0, Y_0, X_1, Y_1 be Banach spaces and assume that the following diagram is commutative:*

$$\begin{array}{ccc} X_0 & \xrightarrow{h_0} & Y_0 \\ {\scriptstyle v_0}\downarrow & & \downarrow{\scriptstyle v_1} \\ X_1 & \xrightarrow{h_1} & Y_1 \end{array}$$

Then if three of the four arrows are Fredholm operators, all arrows are Fredholm operators. Furthermore, the difference between the indices of the horizontal arrows is equal to the difference between the indices of the vertical arrows. In other words,

$$h_0 - h_1 = v_0 - v_1, \tag{6.5.46}$$

where h_0 and h_1 are the indices of the first and the second horizontal arrows respectively, and v_0 and v_1 are the indices of the first and the second vertical arrows correspondingly.

We are now in a position to state our claim about the operator $\widetilde{Q}$ in a more concrete fashion.

Lemma 6.5.4. *The operator $\widetilde{Q}$ is Fredholm with index zero when acting on*

$$\begin{aligned} & L^p_1(\partial\Omega) \oplus L^{p,Div}_{tan}(\partial\Omega) \oplus L^p(\partial\Omega) \\ & \oplus L^p_1(\partial\Omega) \oplus L^{p,Div}_{tan}(\partial\Omega) \oplus L^p(\partial\Omega). \end{aligned} \tag{6.5.47}$$

Proof.

We shall proceed in a series of steps, starting with:

Step 1. *For each $k \in \mathbb{C}$, and $\lambda \in \mathbb{R}$ with $|\lambda| > \frac{1}{2}$, $\lambda I + K_0^t$ is an isomorphism of $L^p(\partial\Omega)$ for any $1 < p < 2 + \varepsilon$.*

Recall that this result was proved in the reference [EM].

Step 2. *For each $k \in \mathbb{C}$, and $\lambda \in \mathbb{R}$ with $|\lambda| > \frac{1}{2}$, $\lambda I + K_k^t$ is a Fredholm operator with index zero on $L^p(\partial\Omega)$ whenever $1 < p < 2 + \varepsilon$.*

Indeed, we write

$$\begin{aligned} \lambda I + K_k^t &= \lambda I + K_0^t + (K_k^t - K_0^t) \\ &= (\lambda I + K_0^t)[I - (\lambda I + K_0^t)^{-1} \circ (-K_k^t + K_0^t)]. \end{aligned}$$

The key observation is that $-K_k^t + K_0^t$ is compact on $L^p(\partial\Omega)$. Thus,

$$I - (\lambda I + K_0^t)^{-1} \circ (-K_k^t + K_0^t)$$

is Fredholm with index zero. By Step 1, $\lambda I + K_0^t$ is an isomorphism and, therefore, $\lambda I + K_k^t$ is Fredholm with index zero on $L^p(\partial\Omega)$. (See the disussion in section 2.2).

Step 3. *For each $k \in \mathbb{C}$, and $\lambda \in \mathbb{R}$ with $|\lambda| > \frac{1}{2}$, $\lambda I + K_0$ is an isomorphism of $L_1^p(\partial\Omega)$ whenever $1 < p < 2 + \varepsilon$.*

Recall that this step was proved in the reference [EM].

Step 4. *For each $k \in \mathbb{C}$, and $\lambda \in \mathbb{R}$ with $|\lambda| > \frac{1}{2}$, $\lambda I + K_k$ is a Fredholm operator with index zero on $L_1^p(\partial\Omega)$ whenever $1 < p < 2 + \varepsilon$.*

To see this, we write

$$\begin{aligned} \lambda I + K_k &= \lambda I + K_0 + (K_k - K_0) \\ &= (\lambda I + K_0)[I - (\lambda I + K_0)^{-1} \circ (-K_k + K_0)]. \end{aligned}$$

Once again, one can prove that $-K_k + K_0$ is compact on $L_1^p(\partial\Omega)$ for each $1 < p < \infty$. Consequently,

$$I - (\lambda I + K_0)^{-1} \circ (-K_k + K_0)$$

is Fredholm with index zero on $L_1^p(\partial\Omega)$. By Step 3, $\lambda I + K_0$ is an isomorphism and, hence, $\lambda I + K_k$ is Fredholm with index zero on $L_1^p(\partial\Omega)$.

Step 5. *For each $k \in \mathbb{C}$, and $\lambda \in \mathbb{R}$ with $|\lambda| > \frac{1}{2}$, $\lambda I + M_0$ is Fredholm with index zero on $L_{tan}^{p,0}(\partial\Omega)$ whenever $1 < p < 2 + \varepsilon$.*

The object of this step is to recall some results from [MMP] and [Mi2]. Specifically, as proved in the equation (5.12) of [MMP], we have

$$(\nu \times \nabla)K_0 = M_0(\nu \times \nabla),$$

and $\nu \times \nabla$ is a Fredholm operator with index zero from $L^p_1(\partial\Omega)$ to $L^{p,0}_{tan}(\partial\Omega)$ by Theorem 5.1-(v) in [MMP]. As a consequence, we have the following commutative diagram:

$$\begin{tikzcd}
L^p_1(\partial\Omega) \arrow[r, "\lambda I + K_0"] \arrow[d, "\nu \times \nabla"'] & L^p_1(\partial\Omega) \arrow[d, "\nu \times \nabla"] \\
L^{p,0}_{tan}(\partial\Omega) \arrow[r, "\lambda I + M_0"] & L^{p,0}_{tan}(\partial\Omega)
\end{tikzcd}$$

where the two vertical arrows are Fredholm with index zero. Since, by Step 3, the operator $\lambda I + K_0$ is an isomorphism, it follows from Lemma 6.5.3 that $\lambda I + M_0$ is Fredholm with index zero on $L^{p,0}_{tan}(\partial\Omega)$ for $\lambda \in \mathbb{R}$ with $|\lambda| > \frac{1}{2}$, and $1 < p < 2+\varepsilon$.

Step 6. *For each $k \in \mathbb{C}$, and $\lambda \in \mathbb{R}$ with $|\lambda| > \frac{1}{2}$, $\lambda I + M_0$ is a Fredholm operator with index zero on $L^{p,Div}_{tan}(\partial\Omega)/L^{p,0}_{tan}(\partial\Omega)$ whenever $1 < p < 2 + \varepsilon$.*

From Theorem 5.1-(viii) of [MMP], it follows that Div is a Fredholm operator from the quotient space $L^{p,Div}_{tan}(\partial\Omega)/L^{p,0}_{tan}(\partial\Omega)$ to $L^p(\partial\Omega)$. Making $k = 0$ in Lemma 4.4 of [MMP] gives the following commutative diagram:

$$\begin{array}{ccc}
L^{p,Div}_{tan}(\partial\Omega)/L^{p,0}_{tan}(\partial\Omega) & \xrightarrow{\lambda I + M_0} & L^{p,Div}_{tan}(\partial\Omega)/L^{p,0}_{tan}(\partial\Omega) \\
\Big\downarrow \mathrm{Div} & & \Big\downarrow \mathrm{Div} \\
L^p(\partial\Omega) & \xrightarrow{\lambda I - K_0^t} & L^p(\partial\Omega)
\end{array}$$

Since the two vertical arrows are Fredholm, and since the operator $\lambda I - K_0^t$ is Fredholm with index zero, it follows from Lemma 6.5.3 that the operator $\lambda I + M_0$ is Fredholm with index zero when acting from the quotient space $L^{p,Div}_{tan}(\partial\Omega)/L^{p,0}_{tan}(\partial\Omega)$ into itself for $\lambda \in \mathbb{R}$ with $|\lambda| > \frac{1}{2}$, and $1 < p < 2 + \varepsilon$.

Step 7. *For each $k \in \mathbb{C}$, and $\lambda \in \mathbb{R}$ with $|\lambda| > \frac{1}{2}$, $\lambda I + M_0$ is a Fredholm operator with index zero on $L^{p,Div}_{tan}(\partial\Omega)$ whenever $1 < p < 2 + \varepsilon$.*

Denote the natural inclusion operator from $L^{p,0}_{tan}(\partial\Omega)$ into $L^{p,Div}_{tan}(\partial\Omega)$ by

$$\iota(f) = f \quad \text{for any } f \in L^{p,0}_{tan}(\partial\Omega). \tag{6.5.48}$$

and denote the projection operator π acting on $L^{p,Div}_{tan}(\partial\Omega)$ into the quotient space $L^{p,Div}_{tan}(\partial\Omega)/L^{p,0}_{tan}(\partial\Omega)$ by

$$\pi(f) := \text{the class of } f, \text{ modulo } L^{p,0}_{tan}(\partial\Omega), \quad \text{where } f \in L^{p,Div}_{tan}(\partial\Omega). \tag{6.5.49}$$

Going further, define the spaces X, Y and Z by

$$X := L^{p,0}_{tan}(\partial\Omega), \quad Y := L^{p,Div}_{tan}(\partial\Omega), \quad \text{and} \quad Z := L^{p,Div}_{tan}(\partial\Omega)/L^{p,0}_{tan}(\partial\Omega). \tag{6.5.50}$$

and consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X \arrow[r] & Y \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$

where, in each exact sequence, the second horizontal arrow is the inclusion operator ι and the third horizontal arrow is the projection operator π. We can easily check that the two horizontal sequence in the commutative diagram are exact. Also by Step 5 and Step 6, the first and the third vertical arrows in the above diagram are both Fredholm operators with index zero. Consequently, by Lemma 6.5.2, the second vertical arrow is also Fredholm with index zero, which proves that $\lambda I + M_0$ is Fredholm with index zero on $L^{p,Div}_{tan}(\partial\Omega)$ for $\lambda \in \mathbb{R}$ with $|\lambda| > \frac{1}{2}$, and $1 < p < 2 + \varepsilon$.

Step 8. *For each $k \in \mathbb{C}$ and $\lambda \in \mathbb{R}$ with $|\lambda| > \frac{1}{2}$, $\lambda I + M_k$ is a Fredholm operator with index zero on $L^{p,Div}_{tan}(\partial\Omega)$, whenever $1 < p < 2 + \varepsilon$.*

To see this, we rewrite the operator $\lambda I + M_k$ in the form

$$\lambda I + M_k = \lambda I + M_0 + (M_k - M_0).$$

Due to the weak singularity in the kernel, it is not difficult to prove that the difference $M_k - M_0$ is a compact operator on $L^{p,Div}_{tan}(\partial\Omega)$. By invoking Step 7, this further implies that $\lambda I + M_0$ is Fredholm with index zero on $L^{p,Div}_{tan}(\partial\Omega)$. Then $\lambda I + M_k$ is a Fredholm operator with index zero on $L^{p,Div}_{tan}(\partial\Omega)$ for $\lambda \in \mathbb{R}$ with $|\lambda| > \frac{1}{2}$, and $1 < p < 2 + \varepsilon$.

Step 9. *For each $k \in \mathbb{C}$ and $\lambda \in \mathbb{R}$ with $|\lambda| > \frac{1}{2}$, the operator $\widetilde{Q}$, which is defined right before Lemma 6.5.2, is Fredholm with index zero when acting on the space defined in (6.5.47), whenever $1 < p < 2 + \varepsilon$.*

The proof of this step follows readily from Step 1 to Step 8.

With the Fredholmness of $\widetilde{Q}$ in hands, we define a new operator Q as the composition

$$Q := T_1^{-1} \circ T_2^{-1} \circ \widetilde{Q} \circ T_2 \circ T_1, \tag{6.5.51}$$

where T_1 and T_2 have been introduced in (6.5.42) and (6.5.45). Since the operators T_1 and T_2 are isomorphisms and $\widetilde{Q}$ is Fredholm with index zero, then by Theorem 2.2.7, Q is a Fredholm operator with index zero on $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4)$, whenever $k \in \mathbb{C}$, $\lambda \in \mathbb{R}$ with $|\lambda| > \frac{1}{2}$, and $1 < p < 2 + \varepsilon$.

In order to continue the proof, we consider two projection operators, namely

$$\pi_1 : L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4) \to L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_3) \tag{6.5.52}$$

and

$$\pi_2 : L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4) \to L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_3) \tag{6.5.53}$$

define by

$$\pi_1(f) := F \text{ and } \pi_2(f) := \widetilde{F}, \tag{6.5.54}$$

where $f = F + e_4\widetilde{F}$ with $F, \widetilde{F}$ are $\mathcal{A}_3$-valued. Moreover, for any F in $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_3)$, introduce

$$
(6.5.55)\qquad
\begin{aligned}
R_1(F) &:= \nu \wedge S_k(d_\partial F);\\
R_2(F) &:= k\,\nu \wedge S_k(F);\\
R_3(F) &:= -\nu \wedge S_{-k}(d_\partial F);\\
R_4(F) &:= -k\,\nu \wedge S_{-k}(F).
\end{aligned}
$$

Remark 6.5.5. *The operators R_j, for $j = 1, 2, 3, 4$, are compact from the space $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_3)$ into itself, for $1 < p < \infty$.*

In order to prove this remark, we single out a technical result in the following lemma.

Lemma 6.5.6. *Suppose X is a Banach space, T is a linear bounded operator from X into $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_3)$. Then*

$$T : X \to L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_3) \text{ is compact}$$

if and only if

$$T : X \to L^{p}(\partial\Omega, \mathcal{A}_3) \text{ is compact,}$$

and

$$d_\partial T : X \to L^{p}(\partial\Omega, \mathcal{A}_3) \text{ is compact.}$$

Proof.

Let $\{x_i\}_j$ be a bounded sequence in X. Since $T : X \to L^p(\partial\Omega, \mathcal{A}_3)$ is compact, then there exists a convergent subsequence $\{T(x_{i_j})\}_j$ in $L^p(\partial\Omega, \mathcal{A}_3)$. Let us re-denote this subsequence by $\{T(x_i)\}_i$. Since the operator $d_\partial T : X \to L^p(\partial\Omega, \mathcal{A}_3)$ is compact, there exists a convergent subsequence $\{d_\partial T(x_{i_j})\}_j$ in $L^p(\partial\Omega, \mathcal{A}_3)$.

By its construction, the sequence $\{x_{i_j}\}_j$ is bounded in X, and both $\{T(x_{i_j})\}_j$ and $\{d_\partial T(x_{i_j})\}_j$ are convergent in $L^p(\partial\Omega, \mathcal{A}_3)$. This proves that the operator T is compact from X into $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_3)$, we concluding the proof of the lemma. $\square$

We are now ready to prove Remark 6.5.5.

Proof.

Since the opeator $\nu \wedge S_k$ is compact from $L^p(\partial\Omega, \mathcal{A}_3)$ into itself and since the operator d_∂ is bounded from $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_3)$ into $L^p(\partial\Omega, \mathcal{A}_3)$, the operator R_1 is compact from $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_3)$ into $L^p(\partial\Omega, \mathcal{A}_3)$. (See Theorem 2.2.1).

Moreover, based on Lemma 5.1.1, Lemma 5.1.2 and Corollary 6.1.3, we may write

$$\begin{aligned} d_\partial(R_1 F) &= d_\partial(\nu \wedge S_k(d_\partial F)) \\ &= -\nu \wedge d(S_k(d_\partial F)) \\ &= \nu \wedge S_k(d_\partial^2 F) \\ &= 0. \end{aligned}$$

Hence, obviously, $d_\partial R_1$ is a compact operator from $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_3)$ into $L^p(\partial\Omega, \mathcal{A}_3)$. Then by Lemma 6.5.6, we may conclude that R_1 is compact from $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_3)$ into itself.

We next consider the operator R_2. First, by the compactness of S_k on $L^p(\partial\Omega, \mathcal{A}_3)$, the definition of R_2 and Theorem 2.2.1, we conclude that R_2 is a compact operator from $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_3)$ into $L^p(\partial\Omega, \mathcal{A}_3)$. On the other hand, we have

$$\begin{aligned} d_\partial(R_2F) &= d_\partial(k\,\nu \wedge S_k(F)) \\ &= -k\,\nu \wedge d(S_k(F)) \\ &= k\,\nu \wedge S_k(d_\partial F), \end{aligned}$$

thanks to Lemma 5.1.2 and Corollary 6.1.3. Since $\nu \wedge S_k$ is compact from $L^p(\partial\Omega, \mathcal{A}_3)$ into itself and since the operator d_∂ is bounded from $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_3)$ into $L^p(\partial\Omega, \mathcal{A}_3)$, we infer that $d_\partial R_2$ is a compact operator from $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_3)$ into $L^p(\partial\Omega, \mathcal{A}_3)$. Once again, by applying Lemma 6.5.6, we may conclude that R_2 is a compact operator from $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_3)$ into itself.

Similarly, we can also prove that the other two operators R_3 and R_4 are compact, and this concludes the proof of the remark. $\square$

Recall the operators R_j, for $j = 1, 2, 3, 4$, in (6.5.55). We are now in a position to define the operator

$$R := R_1 \circ \pi_1 + e_4 R_2 \circ \pi_1 + e_4 R_3 \circ \pi_2 + R_4 \circ \pi_2, \tag{6.5.56}$$

where the projection operators π_1 and π_2 have been introduced in (6.5.54). Since π_1 and π_2 are bounded and since R_j, for $j = 1, 2, 3, 4$, are compact, it follows that R is a compact operator itself.

From that definition of the operator Q in (6.5.51), the definition of operator R in (6.5.56) and the equality (6.5.40), one can easily check that

$$\lambda I + \nu \wedge C_k = Q + R.$$

Since Q is a Fredholm operator with index zero on $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4)$ and R is compact, then by Theorem 2.2.4, the operator $\lambda I + \nu \wedge C_k$ is Fredholm with index zero on $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4)$ for $k \in \mathbb{C}$, $\lambda \in \mathbb{R}$ with $|\lambda| > \frac{1}{2}$ and $1 < p < 2 + \varepsilon$.

Our next goal is to prove the following important result.

Theorem 6.5.7. *For each $k \in \mathbb{C}\backslash\mathbb{R}$, and $\lambda \in \mathbb{R}$ with $|\lambda| > \frac{1}{2}$, $\lambda I + \nu \wedge C_k$ is an isomorphism of $L^{2,d}_{nor}(\partial\Omega, \mathcal{A}_4)$.*

Proof.

Let $f \in L^{2,d}_{nor}(\partial\Omega, \mathcal{A}_4)$ be such that

$$\lambda f + \nu \wedge C_k f = 0. \tag{6.5.57}$$

Our first objective is to eventually show that $f = 0$.

Fix $\lambda > \frac{1}{2}$ and set $u^{\pm} := \mathcal{C}_k f$ in $\Omega_{\pm}$, so that, in particular,

$$u^{\pm}\Big|_{\partial\Omega} = \mp\tfrac{1}{2}\nu \cdot f + C_k f = \pm\tfrac{1}{2}\nu \vee f + C_k f.$$

Therefore,

$$\nu \vee u^{+} - \nu \vee u^{-} = 0 \quad \text{on } \partial\Omega, \tag{6.5.58}$$

and

$$
\begin{aligned}
\nu \wedge u^{+} - \mu\, \nu \wedge u^{-} &= \tfrac{1}{2}(\mu+1)f + (1-\mu)\nu \wedge C_k f \\
&= (1-\mu)\left[\frac{1}{2}\frac{1+\mu}{1-\mu} f + \nu \wedge C_k f\right]. \qquad (6.5.59)
\end{aligned}
$$

Choose $\mu \in (0,1)$ such that $\frac{1}{2}\frac{1+\mu}{1-\mu} = \lambda$, where $\lambda > \frac{1}{2}$, then $\mu := \frac{2\lambda-1}{2\lambda+1}$. For this choice, it follows from (6.5.57) that

$$
\nu \wedge u^{+} - \mu\, \nu \wedge u^{-} = 0 \quad \text{on } \partial\Omega. \qquad (6.5.60)
$$

The key observation now is that $u^{\pm}$ solve the homogeneous problem

$$
\begin{cases}
\mathbb{D}_k u^{\pm} = 0 \quad \text{in } \Omega_{\pm}, \\
\nu \vee u^{+}\Big|_{\partial\Omega} - \nu \vee u^{-}\Big|_{\partial\Omega} = 0, \\
\nu \wedge u^{+}\Big|_{\partial\Omega} - \mu\, \nu \wedge u^{-}\Big|_{\partial\Omega} = 0, \\
N(u^{\pm}), N(du^{\pm}), N(\delta u^{\pm}) \in L^2(\partial\Omega).
\end{cases} \qquad (6.5.61)
$$

Our long-term goal is to show that

$$
u^{\pm} = 0 \quad \text{in } \Omega_{\pm}. \qquad (6.5.62)
$$

For now, we recall that $\mathbb{D}_k = d + \delta + k\, e_4$, and note that $\mathbb{D}_k u^{\pm} = 0$ implies that

$$
d\, u^{\pm} + \delta u^{\pm} + k\, e_4\, u^{\pm} = 0 \quad \text{in } \Omega_{\pm}. \qquad (6.5.63)
$$

Applying the opearator δ_∂ to both sides of the equality (6.5.58) gives

$$
\nu \vee \delta u^{+} = -\nu \vee \delta u^{-} \quad \text{on } \partial\Omega. \qquad (6.5.64)
$$

Similarly, by applying the operator to both sides of the equality (6.5.60), we get

$$\nu \wedge du^{+} = -\nu \wedge du^{-} \quad \text{on } \partial\Omega. \tag{6.5.65}$$

Recall that ν stands for the outward unit normal to the boundary of $\Omega = \Omega_+$ and define

$$\nu_+ := \nu \quad \text{and} \quad \nu_- := -\nu.$$

The following integration by parts formulas (cf. [MiD]) are going to be useful for us.

Lemma 6.5.8. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^m$, $0 \le l \le m$, and u and $w \in C^1(\Omega, \Lambda^l)$, which behave well near $\partial\Omega$. Then the following formulas hold:*

$$\int_{\Omega_+} \langle du, w\rangle \, dx = \int_{\Omega_+} \langle u, \delta w\rangle \, dx + \int_{\partial\Omega} \langle \nu_+ \wedge u, w\rangle \, d\sigma; \tag{6.5.66}$$

$$\int_{\Omega_-} \langle du, w\rangle \, dx = \int_{\Omega_-} \langle u, \delta w\rangle \, dx + \int_{\partial\Omega} \langle \nu_- \wedge u, w\rangle \, d\sigma; \tag{6.5.67}$$

$$\int_{\Omega_+} \langle \delta u, w\rangle \, dx = \int_{\Omega_+} \langle u, dw\rangle \, dx - \int_{\partial\Omega} \langle \nu_+ \vee u, w\rangle \, d\sigma; \tag{6.5.68}$$

$$\int_{\Omega_-} \langle \delta u, w\rangle \, dx = \int_{\Omega_-} \langle u, dw\rangle \, dx - \int_{\partial\Omega} \langle \nu_- \vee u, w\rangle \, d\sigma. \tag{6.5.69}$$

Returning to the mainstream discussion, let us now calculate $\int_{\Omega_+} |du^+|^2 \, dx$ and $\int_{\Omega_+} |\delta u^+|^2 \, dx$. First, by definition of the inner product, we have

$$\int_{\Omega_+} |du^+|^2 \, dx \;=\; \int_{\Omega_+} \langle du^+, (du^+)^c\rangle \, dx.$$

Since $du^+ = -\delta u^+ - k\, e_4 u^+$, we have

$$\begin{aligned}\int_{\Omega_+} |du^+|^2\, dx &= \int_{\Omega_+} \langle -\delta u^+ - k\, e_4 u^+, (du^+)^c \rangle\, dx \\ &= -\int_{\Omega_+} \langle \delta u^+, (du^+)^c \rangle\, dx - \int_{\Omega_+} \langle k\, e_4 u^+, (du^+)^c \rangle\, dx.\end{aligned}$$

An integration by parts, (cf. Lemma 6.5.8), gives

$$\int_{\Omega_+} \langle \delta u^+, (du^+)^c \rangle\, dx = \int_{\Omega_+} \langle u^+, d(du^+)^c \rangle\, dx - \int_{\partial\Omega} \langle \nu_+ \vee u^+, (du^+)^c \rangle\, d\sigma.$$

Notice that $d^2 = 0$, we can easily justify

$$\int_{\Omega_+} \langle \delta u^+, (du^+)^c \rangle\, dx = -\int_{\partial\Omega} \langle \nu_+ \vee u^+, (du^+)^c \rangle.$$

Then

$$\int_{\Omega_+} |du^+|^2\, dx = \int_{\partial\Omega} \langle \nu_+ \vee u^+, (du^+)^c \rangle\, d\sigma - \int_{\Omega_+} \langle k\, e_4 u^+, (du^+)^c \rangle\, dx.$$

Using the property $\langle a \wedge u, v \rangle = \langle u, a \vee v \rangle$, we may then write

$$\begin{aligned}&\int_{\Omega_+} |du^+|^2\, dx \\ &= \int_{\partial\Omega} \langle u^+, \nu_+ \wedge (du^+)^c \rangle\, d\sigma - \int_{\Omega_+} \langle k\, e_4 u^+, (du^+)^c \rangle\, dx \\ &= \int_{\partial\Omega} \langle u^+, \mu\, \nu_+ \wedge (du^-)^c \rangle\, d\sigma - \int_{\Omega_+} \langle k\, e_4 u^+, (du^+)^c \rangle\, dx \\ &= \mu \int_{\partial\Omega} \langle \nu_+ \vee u^+, (du^-)^c \rangle\, d\sigma - \int_{\Omega_+} \langle k\, e_4 u^+, (du^+)^c \rangle\, dx \\ &= \mu \int_{\partial\Omega} \langle \nu_+ \vee u^-, (du^-)^c \rangle\, d\sigma - \int_{\Omega_+} \langle k\, e_4 u^+, (du^+)^c \rangle\, dx. \qquad (6.5.70)\end{aligned}$$

This completes our treatment of $\int_{\Omega_+} |du^+|^2\,dx$ and we now turn our attention to calculating $\int_{\Omega_+} |\delta u^+|^2\,dx$. To begin with, the equality (6.5.63) gives

$$\begin{aligned}
\int_{\Omega_+} |\delta u^+|^2\,dx &= \int_{\Omega_+} \langle \delta u^+, (\delta u^+)^c\rangle\,dx \\
&= \int_{\Omega_+} \langle -du^+ - k\,e_4 u^+, (\delta u^+)^c\rangle\,dx \\
&= -\int_{\Omega_+} \langle du^+, (\delta u^+)^c\rangle\,dx - \int_{\Omega_+} \langle k\,e_4 u^+, (\delta u^+)^c\rangle\,dx.
\end{aligned}$$

By repeatedly integrating by parts in the first term above, we obtain

$$\int_{\Omega_+} \langle du^+, (\delta u^+)^c\rangle\,dx = \int_{\Omega_+} \langle u^+, \delta(\delta u^+)^c\rangle\,dx + \int_{\partial\Omega} \langle \nu_+ \wedge u^+, (\delta u^+)^c\rangle\,d\sigma. \tag{6.5.71}$$

Since $\delta^2 = 0$, the first term above is then zero. Consequently, we can wrtie

$$\begin{aligned}
&\int_{\Omega_+} |\delta u^+|^2\,dx \\
&\quad= -\int_{\partial\Omega} \langle \nu_+ \wedge u^+, (\delta u^+)^c\rangle\,d\sigma - \int_{\Omega_+} \langle k\,e_4 u^+, (\delta u^+)^c\rangle\,dx \\
&\quad= -\int_{\partial\Omega} \langle u^+, \nu_+ \vee (\delta u^+)^c\rangle\,d\sigma - \int_{\Omega_+} \langle k\,e_4 u^+, (\delta u^+)^c\rangle\,dx \\
&\quad= -\int_{\partial\Omega} \langle u^+, \nu_+ \vee (\delta u^-)^c\rangle\,d\sigma - \int_{\Omega_+} \langle k\,e_4 u^+, (\delta u^+)^c\rangle\,dx \\
&\quad= -\int_{\partial\Omega} \langle \nu_+ \wedge u^+, (\delta u^-)^c\rangle\,d\sigma - \int_{\Omega_+} \langle k\,e_4 u^+, (\delta u^+)^c\rangle\,dx \\
&\quad= -\mu\int_{\partial\Omega} \langle \nu_+ \wedge u^-, (du^-)^c\rangle\,d\sigma - \int_{\Omega_+} \langle k\,e_4 u^+, (\delta u^+)^c\rangle\,dx. \qquad (6.5.72)
\end{aligned}$$

Similarly, we wish to compute $\mu\int_{\Omega_-} |du^-|^2\,dx$ and $\mu\int_{\Omega_-} |\delta u^-|^2\,dx$. First we deal with $\int_{\Omega_-} |du^-|^2\,dx$, for which following a familiar pattern, based on Lemma 6.5.8, we write

$$\begin{aligned}\int_{\Omega_-} |du^-|^2\,dx &= \int_{\Omega_-} \langle du^-, (du^-)^c\rangle\,dx \\ &= \int_{\Omega_-} \langle -\delta u^- - k\,e_4 u^-, (du^-)^c\rangle\,dx \\ &= -\int_{\Omega_-} \langle \delta u^-, (du^-)^c\rangle\,dx - \int_{\Omega_-} \langle k\,e_4 u^-, (du^-)^c\rangle\,dx.\end{aligned}$$

Once again, based on Lemma 6.5.8, we may further write

$$\begin{aligned}\int_{\Omega_-} |du^-|^2\,dx &= -\left[\int_{\Omega_-} \langle u^-, d(du^-)^c\rangle\,dx - \int_{\partial\Omega} \langle \nu_- \vee u^-, (du^-)^c\rangle\,d\sigma\right] \\ &\quad - \int_{\Omega_-} \langle k\,e_4 u^-, (du^-)^c\rangle\,dx \\ &= \int_{\partial\Omega} \langle \nu_- \vee u^-, (du^-)^c\rangle\,d\sigma - \int_{\Omega_-} \langle k\,e_4 u^-, (du^-)^c\rangle\,dx \\ &= -\int_{\partial\Omega} \langle \nu_+ \vee u^-, (du^-)^c\rangle\,d\sigma - \int_{\Omega_-} \langle k\,e_4 u^-, (du^-)^c\rangle\,dx.\end{aligned}$$

We multiply both sides of the above equality by ν and obtain

$$\begin{aligned}&\mu \int_{\Omega_-} |du^-|^2\,dx \\ &\quad = -\mu \int_{\partial\Omega} \langle \nu_+ \vee u^-, (du^-)^c\rangle\,d\sigma - \mu \int_{\Omega_-} \langle k\,e_4 u^-, (du^-)^c\rangle\,dx.\end{aligned} \tag{6.5.73}$$

There remains to handle $\int_{\Omega_-} |\delta u^-|^2\,dx$. Much as before, we write

$$\begin{aligned}\int_{\Omega_-} |\delta u^-|^2\,dx &= \int_{\Omega_-} \langle \delta u^-, (\delta u^-)^c\rangle\,dx \\ &= \int_{\Omega_-} \langle -du^- - k\,e_4 u^-, (\delta u^-)^c\rangle\,dx \\ &= -\int_{\Omega_-} \langle du^-, (\delta u^-)^c\rangle\,dx - \int_{\Omega_-} \langle k\,e_4 u^-, (\delta u^-)^c\rangle\,dx\end{aligned}$$

An integration by parts, (cf. Lemma 6.5.8), gives

$$
\begin{aligned}
\int_{\Omega_-} |\delta u^-|^2 \, dx &= -\left[\int_{\Omega_-} \langle u^-, \delta(\delta u^-)^c\rangle \, dx + \int_{\partial\Omega} \langle \nu_- \wedge u^-, (\delta u^-)^c\rangle \, d\sigma\right] \\
&\quad - \int_{\Omega_-} \langle k\, e_4 u^-, (\delta u^-)^c\rangle \, dx \\
&= -\int_{\partial\Omega} \langle \nu_- \wedge u^-, (\delta u^-)^c\rangle \, d\sigma - \int_{\Omega_-} \langle k\, e_4 u^-, (\delta u^-)^c\rangle \, dx \\
&= \int_{\partial\Omega} \langle \nu_+ \wedge u^-, (\delta u^-)^c\rangle \, d\sigma - \int_{\Omega_-} \langle k\, e_4 u^-, (\delta u^-)^c\rangle \, dx.
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
&\mu \int_{\Omega_-} |\delta u^-|^2 \, dx \\
&\quad = \mu \int_{\partial\Omega} \langle \nu_+ \wedge u^-, (\delta u^-)^c\rangle \, d\sigma - \mu \int_{\Omega_-} \langle k\, e_4 u^-, (\delta u^-)^c\rangle \, dx.
\end{aligned}
\tag{6.5.74}
$$

Summing up the formulas (6.5.70), (6.5.72), (6.5.73) and (6.5.74), we finally obtain

$$
\begin{aligned}
&\int_{\Omega_+} \left(|du^+|^2 + |\delta u^+|^2\right) \, dx \; + \; \mu \int_{\Omega_-} \left(|du^-|^2 + |\delta u^-|^2\right) \, dx \\
&= -\int_{\Omega_+} \langle k\, e_4 u^+, (du^+ + \delta u^+)^c\rangle \, dx - \mu \int_{\Omega_-} \langle k\, e_4 u^-, (du^- + \delta u^-)^c\rangle \, dx \\
&= -\int_{\Omega_+} \langle k\, e_4 u^+, -k^c e_4 (u^+)^c\rangle \, dx - \mu \int_{\Omega_-} \langle k\, e_4 u^-, -k^c e_4 (u^+)^c\rangle \, dx \\
&= \int_{\Omega_+} |k|^2 |u^+|^2 \, dx \; + \; \mu \int_{\Omega_-} |k|^2 |u^-|^2 \, dx.
\end{aligned}
$$

In summary, so far we have proved that

$$
\begin{aligned}
&\int_{\Omega_+} \left(|du^+|^2 + |\delta u^+|^2\right) \, dx \; + \; \mu \int_{\Omega_-} \left(|du^-|^2 + |\delta u^-|^2\right) \, dx \\
&\quad = \int_{\Omega_+} |k|^2 |u^+|^2 \, dx \; + \; \mu \int_{\Omega_-} |k|^2 |u^-|^2 \, dx.
\end{aligned}
\tag{6.5.75}
$$

On the other hand, Lemma 6.5.8 yields

$$\begin{aligned}\int_{\Omega_+} |du^+|^2\,dx &= \int_{\Omega_+} \langle du^+, (du^+)^c\rangle\,dx \\ &= \int_{\Omega_+} \langle u^+, (\delta du^+)^c\rangle\,dx + \int_{\partial\Omega} \langle \nu_+ \wedge u^+, (du^+)^c\rangle\,d\sigma,\end{aligned}$$

and also

$$\begin{aligned}\int_{\Omega_+} |\delta u^+|^2\,dx &= \int_{\Omega_+} \langle \delta u^+, (\delta u^+)^c\rangle\,dx \\ &= \int_{\Omega_+} \langle u^+, (d\delta u^+)^c\rangle\,dx - \int_{\partial\Omega} \langle \nu_+ \vee u^+, (\delta u^+)^c\rangle\,d\sigma.\end{aligned}$$

We then observe an identity in Ω_+ to the effect that

$$\begin{aligned}&\int_{\Omega_+} |du^+|^2\,dx + \int_{\Omega_+} |\delta u^+|^2\,dx - \int_{\Omega_+} k^2|u^+|^2\,dx \\ &= \int_{\Omega_+} \langle u^+, (\delta d + d\delta - k^2)(u^+)^c\rangle\,dx + \int_{\partial\Omega} \langle \nu_+ \wedge u^+, (du^+)^c\rangle\,d\sigma \\ &\quad - \int_{\partial\Omega} \langle \nu_+ \vee u^+, (\delta u^+)^c\rangle\,d\sigma.\end{aligned}$$

Going further, since $\delta d + d\delta - k^2 = -\Delta - k^2$ and $(-\Delta - k^2)(u^+) = 0$, this further yeilds

$$\begin{aligned}&\int_{\Omega_+} |du^+|^2\,dx + \int_{\Omega_+} |\delta u^+|^2\,dx - \int_{\Omega_+} k^2|u^+|^2\,dx \\ &= \int_{\partial\Omega} \langle \nu_+ \wedge u^+, (du^+)^c\rangle\,d\sigma - \int_{\partial\Omega} \langle \nu_+ \vee u^+, (\delta u^+)^c\rangle\,d\sigma.\end{aligned} \tag{6.5.76}$$

Since $du^+ = -\delta u^+ - k\, e_4 u^+$, the first term in (6.5.76) turns to be

$$
\begin{aligned}
&\int_{\partial\Omega} \langle \nu_+ \wedge u^+, (du^+)^c \rangle \, d\sigma \\
&= \; -\int_{\partial\Omega} \langle \nu_+ \wedge u^+, (\delta u^+)^c \rangle \, d\sigma - \int_{\partial\Omega} \langle \nu_+ \wedge u^+, (k\, e_4 u^+)^c \rangle \, d\sigma \\
&= \; -\int_{\partial\Omega} \langle u_+, \nu_+ \vee (\delta u^+)^c \rangle \, d\sigma - \int_{\partial\Omega} \langle u_+, \nu_+ \vee (k\, e_4 u^+)^c \rangle \, d\sigma \\
&= \; -\mu \int_{\partial\Omega} \langle \nu_+ \wedge u^-, (\delta u^-)^c \rangle \, d\sigma - \mu \int_{\partial\Omega} \langle \nu_+ \wedge u^-, (k\, e_4 u^-)^c \rangle \, d\sigma \\
&= \; \mu \int_{\partial\Omega} \langle \nu_- \wedge u^-, (\delta u^-)^c \rangle \, d\sigma + \mu \int_{\partial\Omega} \langle \nu_- \wedge u^-, (k\, e_4 u^-)^c \rangle \, d\sigma.
\end{aligned}
$$

Similarly, the second term in (6.5.76) can be written as

$$
\begin{aligned}
&\int_{\partial\Omega} \langle \nu_+ \vee u^+, (\delta u^+)^c \rangle \, d\sigma \\
&= \; -\int_{\partial\Omega} \langle \nu_+ \vee u^+, (du^+)^c \rangle \, d\sigma - \int_{\partial\Omega} \langle \nu_+ \vee u^+, (k\, e_4 u^+)^c \rangle \, d\sigma \\
&= \; -\int_{\partial\Omega} \langle u_+, \nu_+ \wedge (du^+)^c \rangle \, d\sigma - \int_{\partial\Omega} \langle u_+, \nu_+ \wedge (k\, e_4 u^+)^c \rangle \, d\sigma \\
&= \; -\mu \int_{\partial\Omega} \langle \nu_+ \vee u^-, (du^-)^c \rangle \, d\sigma - \mu \int_{\partial\Omega} \langle \nu_+ \vee u^-, (k\, e_4 u^-)^c \rangle \, d\sigma \\
&= \; \mu \int_{\partial\Omega} \langle \nu_- \vee u^-, (du^-)^c \rangle \, d\sigma + \mu \int_{\partial\Omega} \langle \nu_- \vee u^-, (k\, e_4 u^-)^c \rangle \, d\sigma.
\end{aligned}
$$

Hence, all together, we have

$$
\begin{aligned}
&\int_{\Omega_+} |du^+|^2 \, dx + \int_{\Omega_+} |\delta u^+|^2 \, dx - \int_{\Omega_+} k^2 |u^+|^2 \, dx \\
&= \; \mu \int_{\partial\Omega} \langle \nu_- \wedge u^-, (\delta u^-)^c \rangle \, d\sigma + \mu \int_{\partial\Omega} \langle \nu_- \wedge u^-, (k\, e_4 u^-)^c \rangle \, d\sigma \\
&\quad -\mu \int_{\partial\Omega} \langle \nu_- \vee u^-, (du^-)^c \rangle \, d\sigma - \mu \int_{\partial\Omega} \langle \nu_- \vee u^-, (k\, e_4 u^-)^c \rangle \, d\sigma. \qquad (6.5.77)
\end{aligned}
$$

A similar analysis can be carried out in Ω_-. More specifically, we have

$$\begin{aligned} &\int_{\Omega_-} |du^-|^2\, dx + \int_{\Omega_-} |\delta u^-|^2\, dx - \int_{\Omega_-} k^2 |u^-|^2\, dx \\ &\quad= \int_{\partial\Omega} \langle \nu_- \wedge u^-, (du^-)^c\rangle\, d\sigma - \int_{\partial\Omega} \langle \nu_- \vee u^-, (\delta u^-)^c\rangle\, d\sigma \\ &\quad= -\int_{\partial\Omega} \langle \nu_- \wedge u^-, (\delta u^-)^c\rangle\, d\sigma - \int_{\partial\Omega} \langle \nu_- \wedge u^-, (k\, e_4 u^-)^c\rangle\, d\sigma \\ &\qquad+ \int_{\partial\Omega} \langle \nu_- \vee u^-, (du^-)^c\rangle\, d\sigma + \int_{\partial\Omega} \langle \nu_- \vee u^-, (k\, e_4 u^-)^c\rangle\, d\sigma. \end{aligned} \tag{6.5.78}$$

By multiplying both sides of (6.5.78) by μ, we get

$$\begin{aligned} &\mu\int_{\Omega_-} |du^-|^2\, dx + \mu\int_{\Omega_-} |\delta u^-|^2\, dx - \mu\int_{\Omega_-} k^2 |u^-|^2\, dx \\ &\quad= -\mu\int_{\partial\Omega} \langle \nu_- \wedge u^-, (\delta u^-)^c\rangle\, d\sigma - \mu\int_{\partial\Omega} \langle \nu_- \wedge u^-, (k\, e_4 u^-)^c\rangle\, d\sigma \\ &\qquad+\mu\int_{\partial\Omega} \langle \nu_- \vee u^-, (du^-)^c\rangle\, d\sigma + \mu\int_{\partial\Omega} \langle \nu_- \vee u^-, (k\, e_4 u^-)^c\rangle\, d\sigma. \end{aligned} \tag{6.5.79}$$

We now desire to combine (6.5.77) and (6.5.79). If we add (6.5.79) to (6.5.77), after a number of cancellation, we obtain

$$\begin{aligned} &\int_{\Omega_+} \left(|du^+|^2 + |\delta u^+|^2\right)\, dx \;+\; \mu\int_{\Omega_-} \left(|du^-|^2 + |\delta u^-|^2\right)\, dx \\ &\quad= \int_{\Omega_+} k^2 |u^+|^2\, dx \;+\; \mu\int_{\Omega_-} k^2 |u^-|^2\, dx. \end{aligned} \tag{6.5.80}$$

All in all, comparing (6.5.75) and (6.5.80) gives

$$(k^2 - |k|^2)\left(\int_{\Omega_+} |u^+|^2\, dx + \mu\int_{\Omega_-} |u^-|^2\, dx\right) = 0.$$

Therefore, for any $k \in \mathbb{C}\backslash\mathbb{R}$, we may conclude that $u^+ = 0$ and $u^- = 0$, as desired.

In order to continue, we need a simply result.

Lemma 6.5.9. *Let ν be the outward unit normal to the boundary of Ω, then*

$$\nu \cdot \nu = -1.$$

Proof.

If ν be the unit normal, then

$$\nu = (\nu_1, \ldots, \nu_m) = \sum_j \nu_j e_j,$$

where $\sum_j {\nu_j}^2 = 1$. So we have

$$\begin{aligned} \nu \cdot \nu &= \sum_i \nu_i e_i \sum_j \nu_j e_j \\ &= \sum_{i,j} \nu_i \nu_j e_i e_j \\ &= \sum_{i<j} \nu_i \nu_j e_i e_j + \sum_{i>j} \nu_i \nu_j e_i e_j + \sum_{i=j} \nu_i \nu_j e_i e_j. \end{aligned}$$

Since $e_i e_j = -e_j e_i$, the first term and the second term above will cancel each other. Also, since $e_i e_i = -1$, then

$$\nu \cdot \nu = -\sum_{i=j} \nu_i \nu_j = -1.$$

This finishes the proof of the lemma. □

Recall that $u^\pm := \mathcal{C}_k f$ in $\Omega_\pm$, and then by the jump formulas for $\mathcal{C}_k$, we have

$$u^+\Big|_{\partial\Omega} = -\tfrac{1}{2}\nu \cdot f + C_k f,$$

and

$$u^-\Big|_{\partial\Omega} = \tfrac{1}{2}\nu \cdot f + C_k f.$$

In particular,

$$u^+ - u^- = -\nu \cdot f \quad \text{on } \partial\Omega.$$

Multiplying both sides of the above equality by ν and then applying Lemma 6.5.9, we may conclude that $f = 0$. Therefore, $\lambda I + \nu \wedge C_k$ is one-to-one from $L^{2,d}_{nor}(\partial\Omega, \mathcal{A}_4)$ into itself for each $\lambda > \frac{1}{2}$ and each $k \in \mathbb{C}\backslash\mathbb{R}$.

Our next step is to prove a similar conclusion for $\lambda < -\frac{1}{2}$. More concretely, fix such a λ and consider $f \in L^{2,d}_{nor}(\partial\Omega, \mathcal{A}_4)$ such that

$$(\lambda I + \nu \wedge C_k) f = 0.$$

Our goal is to show that $f = 0$.

We now choose $\mu \in (0,1)$ such that $\frac{1}{2}\frac{\mu+1}{\mu-1} = \lambda$, (i.e. $\mu := \frac{2\lambda+1}{2\lambda-1}$), and much as before, let $u^{\pm} := \mathcal{C}_k f$ in $\Omega_{\pm}$. Similarly to the equality (6.5.59), we have

$$\begin{aligned}
&\mu\, \nu \wedge u^+ - \nu \wedge u^- \\
&\quad = \tfrac{1}{2}(\mu+1) f + (\mu - 1)\nu \wedge C_k f \\
&\quad = (\mu - 1)\left[\frac{1}{2}\frac{\mu+1}{\mu-1} f + \nu \wedge C_k f\right] \\
&\quad = (\mu - 1)(\lambda I + \nu \wedge C_k) f \\
&\quad = 0.
\end{aligned}$$

Also, recall the equality (6.5.58), which gives

$$\nu \vee u^{+} - \nu \vee u^{-} = 0 \quad \text{on } \partial\Omega, \tag{6.5.81}$$

Thus, for this choice of μ we now make the important observation that $u^{\pm}$ solve the following homogeneous problem

$$\begin{cases} \mathbb{D}_k u^{\pm} = 0 \quad \text{in } \Omega_{\pm}, \\ \nu \vee u^{+}\Big|_{\partial\Omega} - \nu \vee u^{-}\Big|_{\partial\Omega} = 0, \\ \mu\, \nu \wedge u^{+}\Big|_{\partial\Omega} - \nu \wedge u^{-}\Big|_{\partial\Omega} = 0, \\ N(u^{\pm}), N(du^{\pm}), N(\delta u^{\pm}) \in L^2(\partial\Omega). \end{cases} \tag{6.5.82}$$

Repeat the reasoning in the case $\lambda > \frac{1}{2}$ but change μ to $\frac{1}{\mu}$, we arrvie at a similar identity

$$(k^2 - |k|^2)\left(\int_{\Omega_+} |u^+|^2\, dx + \frac{1}{\mu}\int_{\Omega_-} |u^-|^2\, dx\right) = 0.$$

As before, this implies $f = 0$. Therefore, $\lambda I + \nu \wedge C_k$ is one-to-one from $L^{2,d}_{nor}(\partial\Omega, \mathcal{A}_4)$ into itself for each $\lambda < -\frac{1}{2}$ and each $k \in \mathbb{C}\backslash\mathbb{R}$.

As a consequence of the fact that the operator $\lambda I + \nu \wedge C_k$ is one-to-one from $L^{2,d}_{nor}(\partial\Omega, \mathcal{A}_4)$ into itself for $|\lambda| > \frac{1}{2}$ and $k \in \mathbb{R}\backslash\mathbb{C}$, its kernel is the zero space. If we now recall that $\lambda I + \nu \wedge C_k$ is a Fredholm operator with index zero, we arrive at the conclusion that the operator $\lambda I + \nu \wedge C_k$ is onto as well. All together, we may conclude that $\lambda I + \nu \wedge C_k$ is an isomorphism of $L^{2,d}_{nor}(\partial\Omega, \mathcal{A}_4)$ for $k \in \mathbb{C}\backslash\mathbb{R}$, $\lambda \in \mathbb{R}$ with $|\lambda| > \frac{1}{2}$ and $1 < p < 2 + \varepsilon$. $\square$

With the above piece of information in hands, we are finally able to begin the proof of Theorem 6.5.1 in the earnest.

Proof.

Fix $k_0 \in \mathbb{C}\backslash\mathbb{R}$ and write

$$\begin{aligned} \lambda I + \nu \wedge C_k &= \lambda I + \nu \wedge C_{k_0} + \nu \wedge (C_k - C_{k_0}) \\ &= (\lambda I + \nu \wedge C_{k_0})\, T_k \end{aligned} \tag{6.5.83}$$

where, for each $k \in \mathbb{C}$, we set

$$T_k := I + (\lambda I + \nu \wedge C_{k_0})^{-1} \circ \nu \wedge (C_k - C_{k_0}). \tag{6.5.84}$$

Thus, Theorem 2.2.8 applies and gives that there exists a subset D of $\mathbb{C}$, which has no accumulation points, such that T_k is invertible on $L^{2,d}_{nor}(\partial\Omega, \mathcal{A}_4)$ for any k in $\mathbb{C}\backslash D$. From the equality (6.5.83), we know that $\lambda I + \nu \wedge C_k$ is invertible if and only if T_k is invertible. Therefore, the operator $\lambda I + \nu \wedge C_k$ is invertible on $L^{2,d}_{nor}(\partial\Omega, \mathcal{A}_4)$ for any k in $\mathbb{C}\backslash D$. This implies that D is a subset of $\mathbb{R}$. Since it has no accumulation points, D is a countable set. Hence, we can arrange D in the form of a sequence, say

$$D = \{k_j\}_{j\in\mathbb{N}},$$

where k_j are real numbers.

In summary, so far, for any λ real with $|\lambda| > \frac{1}{2}$, there exists a real sequence $\{k_j\}_j$ such that the operator $\lambda I + \nu \wedge C_k$ is an isomorphism of $L^{2,d}_{nor}(\partial\Omega, \mathcal{A}_4)$ for any k in $\mathbb{C}\backslash\{k_j\}_j$.

Claim: *For any $\lambda \in \mathbb{R}$ with $|\lambda| > \frac{1}{2}$, there exists a real sequence $\{k_j\}_j$ such that $\lambda I + \nu \wedge C_k$ is an isomorphism of $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4)$ for any $p \in (1, 2+\varepsilon)$ and any $k \in \mathbb{C}\backslash\{k_j\}_j$.*

Proof.

Recall that the operator $\lambda I + \nu \wedge C_k$ is Fredholm with index zero on $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4)$ for $k \in \mathbb{C}$, $\lambda \in \mathbb{R}$ with $|\lambda| > \frac{1}{2}$ and $1 < p < 2+\varepsilon$. Note that in order to justify the desired conclusion, we need $\lambda I + \nu \wedge C_k$ is either one-to-one or onto. We will consider two cases.

Case I. *For $2 \le p < 2+\varepsilon$, the operator $\lambda I + \nu \wedge C_k$ is one-to-one on the space $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4)$ for every $\lambda \in \mathbb{R}$ with $|\lambda| > \frac{1}{2}$.*

First, by Hölder's inequality and our assumption on p, we have that

$$L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4) \subset L^{2,d}_{nor}(\partial\Omega, \mathcal{A}_4). \tag{6.5.85}$$

Suppose that f is in $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4)$ and that, for some $\lambda \in \mathbb{R}$ with $|\lambda| > \frac{1}{2}$,

$$(\lambda I + \nu \wedge C_k)f = 0.$$

In particular, f can be viewed as a function in $L^{2,d}_{nor}(\partial\Omega, \mathcal{A}_4)$ due to (6.5.85). By the invertibility of $\lambda I + \nu \wedge C_k$ on $L^{2,d}_{nor}(\partial\Omega, \mathcal{A}_4)$ we have that $f = 0$. This proves that the operator $\lambda I + \nu \wedge C_k$ is one-to-one on $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4)$. Hence, from the fact that this operator is Fredholm with index zero on the space $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4)$ for $2 \le p < 2+\varepsilon$, it then follows that the operator $\lambda I + \nu \wedge C_k$ is an isomorphism of $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4)$ for every $2 \le p < 2+\varepsilon$.

Case II. *For $1 \le p < 2$, the operator $\lambda I + \nu \wedge C_k$ is onto on the space $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4)$ for every $\lambda \in \mathbb{R}$ with $|\lambda| > \frac{1}{2}$.*

We first note that the operator $\lambda I + \nu \wedge C_k$ has closed range on $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4)$ since it is Fredholm on that space. Thus, it suffices to show that the range of $\lambda I + \nu \wedge C_k$ is dense. Since the space $L^{2,d}_{nor}(\partial\Omega, \mathcal{A}_4)$ is densely included in $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4)$, the range of the operator $\lambda I + \nu \wedge C_k$ on $L^{2,d}_{nor}(\partial\Omega, \mathcal{A}_4)$ is densely included in the range of $\lambda I + \nu \wedge C_k$ on the space $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4)$. However, since the operator $\lambda I + \nu \wedge C_k$ is an isomorphism of $L^{2,d}_{nor}(\partial\Omega, \mathcal{A}_4)$, the range of $\lambda I + \nu \wedge C_k$ on $L^{2,d}_{nor}(\partial\Omega, \mathcal{A}_4)$ is precisely the space $L^{2,d}_{nor}(\partial\Omega, \mathcal{A}_4)$ itself. All together, we may conclude that the range of $\lambda I + \nu \wedge C_k$ on $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4)$ is densely included in $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4)$. This, together with the closedness of the range, implies that $\lambda I + \nu \wedge C_k$ is onto on $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4)$. Thanks to the fact that this operator is Fredholm with index zero, the operator $\lambda I + \nu \wedge C_k$ is an isomorphism of $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4)$.

This concludes the proof of Theorem 6.5.1. $\square$

We now discuss a theorem which deals with the invertibility of $\lambda I + \nu \vee C_k$. The proof is very similar to the one for the operator $\lambda I + \nu \wedge C_k$ and, hence, it is left to the interested reader.

Theorem 6.5.10. *Let $\Omega \subset \mathbb{R}^3$ be an arbitrary Lipschitz domain with compact boundary. Then for every $\lambda \in \mathbb{R}$ with $|\lambda| \ge \frac{1}{2}$ there exists a sequence of real numbers $\{k_j\}_j$ such that for each $1 < p < 2 + \varepsilon$ and $k \in \mathbb{C} \setminus \{k_j\}_j$ the operator*

$\lambda I + \nu \vee C_k$ is an isomorphism of $L^{p,\delta}_{tan}(\partial\Omega, \mathcal{A}_4)$.

In the last part of this section, we discuss the classical approach for the invertibility of the operator $\lambda I + \nu \wedge C_k$, which, nonetheless, yields a weaker result (more precisely, the set of exceptional values for the wave number k is larger).

First, we state the following lemma.

Lemma 6.5.11. *Let $\Omega \subset \mathbb{R}^3$ be an arbitrary Lipschitz domain with compact boundary. Suppose that $k \in \mathbb{C}$ satisfies $|\mathrm{Im}k| > |\mathrm{Re}k|$ and $\lambda \in \mathbb{R}$ is such that $|\lambda| > \frac{1}{2}$. Then the operator $\lambda I + \nu \wedge C_k$ is one-to-one on the space $L^{2,d}_{nor}(\partial\Omega, \mathcal{A}_4)$.*

Proof.

Fix k and λ as in the statement of the lemma and assume that $f \in L^{2,d}_{nor}(\partial\Omega, \mathcal{A}_4)$, $f \neq 0$ is such that

$$\lambda f + \nu \wedge C_k f = 0. \tag{6.5.86}$$

Set $u^{\pm} := \mathcal{C}_k f$ in $\Omega_{\pm}$ so that, in particular,

$$u^{\pm}\Big|_{\partial\Omega} = \mp\tfrac{1}{2}\nu \cdot f + C_k f = \pm\tfrac{1}{2}\nu \vee f + C_k f. \tag{6.5.87}$$

Recall that

$$\mathbb{D}_k = d + \delta + k\, e_4.$$

Consequently, $\mathbb{D}_k u^{\pm} = 0$, which further implies that

$$du^{\pm} + \delta u^{\pm} + k\, e_4 u^{\pm} = 0 \quad \text{in} \quad \Omega_{\pm}. \tag{6.5.88}$$

Recall that ν stands for the outward unit normal to the boundary of $\Omega_+ := \Omega$ and define

$$\nu_+ := \nu \quad \text{and} \quad \nu_- := -\nu. \tag{6.5.89}$$

By Lemma 6.5.8, we have

$$\begin{aligned}
\int_{\Omega_+} |du^+|^2 \, dx &= \int_{\Omega_+} \langle du^+, (du^+)^c \rangle \, dx \\
&= \int_{\Omega_+} \langle u^+, (\delta du^+)^c \rangle \, dx + \int_{\partial\Omega} \langle \nu_+ \wedge u^+, (du^+)^c \rangle \, d\sigma
\end{aligned} \tag{6.5.90}$$

and

$$\begin{aligned}
\int_{\Omega_+} |\delta u^+|^2 \, dx &= \int_{\Omega_+} \langle \delta u^+, (\delta u^+)^c \rangle \, dx \\
&= \int_{\Omega_+} \langle u^+, (d\delta u^+)^c \rangle \, dx - \int_{\partial\Omega} \langle \nu_+ \vee u^+, (\delta u^+)^c \rangle \, d\sigma.
\end{aligned} \tag{6.5.91}$$

By adding (6.5.90), (6.5.91) and subtracting the term $\int_{\Omega_+} k^2 |u^+|^2 \, dx$, we obtain the following equality in Ω_+ :

$$\begin{aligned}
&\int_{\Omega_+} |du^+|^2 \, dx + \int_{\Omega_+} |\delta u^+|^2 \, dx - \int_{\Omega_+} k^2 |u^+|^2 \, dx \\
&= \int_{\Omega_+} \langle u^+, (\delta d + d\delta - k^2)(u^+)^c \rangle \, dx + \int_{\partial\Omega} \langle \nu_+ \wedge u^+, (du^+)^c \rangle \, d\sigma \\
&\quad - \int_{\partial\Omega} \langle \nu_+ \vee u^+, (\delta u^+)^c \rangle \, d\sigma \\
&= \int_{\Omega_+} \langle u^+, (-\Delta - k^2)(u^+)^c \rangle \, dx + \int_{\partial\Omega} \langle \nu_+ \wedge u^+, (du^+)^c \rangle \, d\sigma \\
&\quad - \int_{\partial\Omega} \langle \nu_+ \vee u^+, (\delta u^+)^c \rangle \, d\sigma \\
&= \int_{\partial\Omega} \langle \nu_+ \wedge u^+, (du^+)^c \rangle \, d\sigma - \int_{\partial\Omega} \langle \nu_+ \vee u^+, (\delta u^+)^c \rangle \, d\sigma.
\end{aligned} \tag{6.5.92}$$

In order to continue, let us now observe that

$$u^+\Big|_{\partial\Omega} = -\tfrac{1}{2}\nu \cdot f + C_k f = \tfrac{1}{2}\nu \vee f + C_k f. \tag{6.5.93}$$

This implies

$$\nu_+ \vee u^+ = \nu_+ \vee C_k f. \tag{6.5.94}$$

Also, from the equality (6.5.93), we have

$$\begin{aligned} \nu_+ \wedge u^+\Big|_{\partial\Omega} &= \nu_+ \wedge \left(\tfrac{1}{2}\nu \vee f\right) + \nu_+ \wedge C_k f \\ &= \tfrac{1}{2} f + \nu_+ \wedge C_k f \\ &= \tfrac{1}{2} f - \lambda f \\ &= \left(\tfrac{1}{2} - \lambda\right) f. \end{aligned} \tag{6.5.95}$$

Now we are ready to compute the second term in the rightmost of (6.5.92).

$$\begin{aligned} &\int_{\partial\Omega} \langle \nu_+ \vee u^+, (\delta u^+)^c \rangle \, d\sigma \\ &= \int_{\partial\Omega} \langle u^+, \nu_+ \wedge (\delta u^+)^c \rangle \, d\sigma \\ &= \int_{\partial\Omega} \langle \nu_+ \wedge (\nu_+ \vee u^+), \nu_+ \wedge (\delta u^+)^c \rangle \, d\sigma \\ &= \int_{\partial\Omega} \langle \nu_+ \wedge (\nu_+ \vee C_k f), -\nu_+ \wedge (du^+ + k\, e_4 u^+)^c \rangle \, d\sigma \\ &= -\int_{\partial\Omega} \langle C_k f, \nu_+ \wedge (du^+ + k\, e_4 u^+)^c \rangle \, d\sigma \\ &= -\int_{\partial\Omega} \langle C_k f, (\nu_+ \wedge du^+ + \nu_+ \wedge k\, e_4 u^+)^c \rangle \, d\sigma. \end{aligned} \tag{6.5.96}$$

Based on Lemma 3.2.4, Lemma 5.1.2 and the equality (6.5.95), we may write

$$
\begin{aligned}
\nu_+ \wedge du^+ &= -d_\partial(\nu_+ \wedge u^+) \\
&= -d_\partial\left(\left(\tfrac{1}{2}-\lambda\right) f\right) \\
&= -\left(\tfrac{1}{2}-\lambda\right) d_\partial f
\end{aligned}
\tag{6.5.97}
$$

and

$$
\begin{aligned}
\nu_+ \wedge k\, e_4 u^+ &= -k\, e_4 \nu_+ \wedge u^+ \\
&= -k\, e_4 \left(\tfrac{1}{2}-\lambda\right) f \\
&= -\left(\tfrac{1}{2}-\lambda\right) k\, e_4 f.
\end{aligned}
\tag{6.5.98}
$$

Therefore, the equality (6.5.96) further implies that

$$
\begin{aligned}
&\int_{\partial\Omega} \langle \nu_+ \vee u^+, (\delta u^+)^c\rangle \, d\sigma \\
&\quad = -\int_{\partial\Omega} \langle C_k f, (\nu_+ \wedge du^+ + \nu_+ \wedge k\, e_4 u^+)^c\rangle \, d\sigma \\
&\quad = \left(\tfrac{1}{2}-\lambda\right) \int_{\partial\Omega} \langle C_k f, (d_\partial f + k\, e_4 f)^c\rangle \, d\sigma.
\end{aligned}
\tag{6.5.99}
$$

Turning our attention to the first term in the rightmost of the equality (6.5.92),

$$
\begin{aligned}
&\int_{\partial\Omega} \langle \nu_+ \wedge u^+, (du^+)^c\rangle \, d\sigma \\
&\quad = \int_{\partial\Omega} \langle u^+, \nu_+ \vee (du^+)^c\rangle \, d\sigma \\
&\quad = -\int_{\partial\Omega} \langle \nu_+ \vee (\nu_+ \wedge u^+), (\nu_+ \vee \delta u^+ + \nu_+ \vee k\, e_4 u^+)^c\rangle \, d\sigma.
\end{aligned}
\tag{6.5.100}
$$

Using again the equality (6.5.95), we obtain the following:

$$\nu_+ \vee (\nu_+ \wedge u^+) = \nu_+ \vee \left(\tfrac{1}{2} - \lambda\right) f = \left(\tfrac{1}{2} - \lambda\right) \nu_+ \vee f, \tag{6.5.101}$$

$$\nu_+ \vee \delta u^+ = -\delta_\partial(\nu_+ \vee u^+) = -\delta_\partial(\nu_+ \vee C_k f), \tag{6.5.102}$$

$$\nu_+ \vee k\, e_4 u^+ = -k\, e_4 \nu_+ \vee C_k f. \tag{6.5.103}$$

From the equalities (6.5.100)-(6.5.103), it readily follows that

$$\begin{aligned} &\int_{\partial\Omega} \langle \nu_+ \wedge u^+, (du^+)^c \rangle \, d\sigma \\ = \;& \left(\tfrac{1}{2} - \lambda\right) \int_{\partial\Omega} \langle \nu_+ \vee f, (\delta_\partial(\nu_+ \vee C_k f) + k\, e_4 \nu_+ \vee C_k f)^c \rangle \, d\sigma. \end{aligned} \tag{6.5.104}$$

On account of (6.5.92), (6.5.99) and (6.5.104), we have that

$$\begin{aligned} &\int_{\Omega_+} |du^+|^2 \, dx + \int_{\Omega_+} |\delta u^+|^2 \, dx - \int_{\Omega_+} k^2 |u^+|^2 \, dx \\ = \;& \left(\tfrac{1}{2} - \lambda\right) \int_{\partial\Omega} \langle \nu_+ \vee f, (\delta_\partial(\nu_+ \vee C_k f) + k\, e_4 \nu_+ \vee C_k f)^c \rangle \, d\sigma \\ & - \left(\tfrac{1}{2} - \lambda\right) \int_{\partial\Omega} \langle C_k f, (d_\partial f + k\, e_4 f)^c \rangle \, d\sigma. \end{aligned} \tag{6.5.105}$$

At this stage, we define

$$A_+ \;:=\; \int_{\Omega_+} |du^+|^2 \, dx + \int_{\Omega_+} |\delta u^+|^2 \, dx - \int_{\Omega_+} k^2 |u^+|^2 \, dx, \tag{6.5.106}$$

and

$$\begin{aligned} B_+ \;:=\; & \int_{\partial\Omega} \langle \nu_+ \vee f, (\delta_\partial(\nu_+ \vee C_k f) + k\, e_4 \nu_+ \vee C_k f)^c \rangle \, d\sigma \\ & - \int_{\partial\Omega} \langle C_k f, (d_\partial f + k\, e_4 f)^c \rangle \, d\sigma. \end{aligned} \tag{6.5.107}$$

Then the equality (6.5.105) can be expressed as

$$A_+ = (\tfrac{1}{2} - \lambda)B_+. \tag{6.5.108}$$

Similarly, we can show that

$$A_- = (\tfrac{1}{2} + \lambda)B_-, \tag{6.5.109}$$

where

$$A_- := \int_{\Omega_-} |du^-|^2\, dx + \int_{\Omega_-} |\delta u^-|^2\, dx - \int_{\Omega_-} k^2|u^-|^2\, dx, \tag{6.5.110}$$

and

$$\begin{aligned} B_- := & \int_{\partial\Omega} \langle \nu_+ \vee f, (\delta_\partial(\nu_+ \vee C_k f) + k\, e_4\nu_+ \vee C_k f)^c\rangle\, d\sigma \\ & - \int_{\partial\Omega} \langle C_k f, (d_\partial f + k\, e_4 f)^c\rangle\, d\sigma. \end{aligned} \tag{6.5.111}$$

It is obvious that $B_+ = B_-$.

For any $k \in \mathbb{C}$ with $k = a + bi$ we have $k^2 = (a^2 - b^2) + 2abi$. By the assumption $|\mathrm{Im}k| > |\mathrm{Re}k|$, we observe $|b| > |a|$. We now introduce some notation in order to simplify the expressions of $A_\pm$ and $B_\pm$. Let

$$\alpha_+ := \int_{\Omega_+} |du^+|^2\, dx + \int_{\Omega_+} |\delta u^+|^2\, dx$$

and

$$\beta_+ := \int_{\Omega_+} |u^+|^2\, dx.$$

Similarly, set

$$\alpha_- := \int_{\Omega_-} |du^-|^2\,dx + \int_{\Omega_-} |\delta u^-|^2\,dx$$

and

$$\beta_- := \int_{\Omega_-} |u^-|^2\,dx.$$

Then

$$A_+ = \alpha_+ - k^2\beta_+ = [\alpha_+ + \beta_+(b^2 - a^2)] - 2ab\beta_+ \tag{6.5.112}$$

and

$$A_- = \alpha_- - k^2\beta_- = [\alpha_- + \beta_-(b^2 - a^2)] - 2ab\beta_-. \tag{6.5.113}$$

Assuming $B_+ = B_- = 0$, we have $A_+ = A_- = 0$ by (6.5.108) and (6.5.109). Therefore, $\mathrm{Re}A_+ = \alpha_+ + \beta_+(b^2 - a^2) = 0$. Since $\alpha_+ \geq 0$, $\beta_+ \geq 0$, and $b^2 - a^2 > 0$, then $\mathrm{Re}A_+ = 0$ implies that $\alpha_+ = \beta_+ = 0$. Furthermore, by the definition of β_+, we have $u^+ = 0$. Similarly, we have $u^- = 0$ as well. Consequently, $f = 0$, which contradicts the assumption that $f \neq 0$. Thus, we may conclude that B_+ and B_- cannot be zero.

Now using the equalities (6.5.108) and (6.5.109), we can express λ as

$$\lambda = \frac{A_+ - A_-}{-2(A_+ + A_-)}.$$

Taking absolute values on both sides, we have

$$|\lambda| = \frac{1}{2}\left|\frac{A_+ - A_-}{A_+ + A_-}\right|.$$

In order to analyze the range of $|\lambda|$, we involve the following lemma whose proof is elementary and, hence, omitted.

Lemma 6.5.12. *Suppose $z_1, z_2 \in \mathbb{C}$, and z_1, z_2 are in the same quadrant, then*

$$|z_1 - z_2| < |z_1 + z_2|.$$

By (6.5.112) and (6.5.113) we know that both A_+ and A_- are either in the first quadrant or in the fourth quadrant, depending on the signs of the parameters a and b.

Applying Lemma (6.5.12) with $z_1 = A_+$ and $z_2 = A_-$, we obtain the inequality $|A_+ - A_-| < |A_+ + A_-|$. Hence

$$|\lambda| = \frac{1}{2}\left|\frac{A_+ - A_-}{A_+ + A_-}\right| < \frac{1}{2},$$

which contradicts the assumption that $|\lambda| > \frac{1}{2}$. Therefore, $f = 0$, and hence the operator $\lambda I + \nu \wedge C_k$ is one-to-one on the space $L^{2,d}_{nor}(\partial\Omega, \mathcal{A}_4)$. This completes the proof of Lemma 6.5.11. $\square$

Chapter 7

Half-Dirichlet Problems for Dirac Operators

7.1 Well-posedness Results

In this section, we consider the Half-Dirichlet Problem for Dirac Operators. First we introduce the Hardy space

$$H_k^p(\Omega) := \{u \in C^1(\Omega, \mathcal{A}_4) : \mathbb{D}_k u = 0 \text{ in } \Omega,\ N(u), N(du), N(\delta u) \in L^p(\partial\Omega)\}$$

and the exterior Hardy space

$$H_k^p(\Omega_-) := \{u \in C^1(\Omega_-) : \mathbb{D}_k u = 0 \text{ in } \Omega_-,\ N(u), N(du), N(\delta u) \in L^p(\partial\Omega)$$
$$\text{and } \lim_{|x|\to\infty} (|x| - ie_4 x)u(x) = 0\}.$$

Consider the following boundary value problems:

$$\text{(7.1.1)} \qquad \begin{cases} u \in H_k^p(\Omega), \\ \nu \vee u\Big|_{\partial\Omega} = f \ \in L^{p,\delta}_{tan}(\partial\Omega, \mathcal{A}_4), \end{cases}$$

$$(7.1.2) \qquad \begin{cases} u \in H^p_k(\Omega), \\ \nu \wedge u\Big|_{\partial\Omega} = f \ \in L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4), \end{cases}$$

$$(7.1.3) \qquad \begin{cases} u \in H^p_k(\Omega_-), \\ \nu \vee u\Big|_{\partial\Omega} = f \ \in L^{p,\delta}_{tan}(\partial\Omega, \mathcal{A}_4), \end{cases}$$

$$(7.1.4) \qquad \begin{cases} u \in H^p_k(\Omega_-), \\ \nu \wedge u\Big|_{\partial\Omega} = f \ \in L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4). \end{cases}$$

Theorem 7.1.1. *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain with compact boundary. Then there exists a sequence of real numbers $\{k_j\}_j$ such that for each $1 < p < 2 + \varepsilon$ and $k \in \mathbb{C}\backslash\{k_j\}_j$, the boundary value problems (7.1.1), (7.1.2), (7.1.3) and (7.1.4) are all well-posed.*

Proof.

It follows from Theorem 6.5.1, Theorem 6.5.10 and Lemma 6.4.1 that

$$(7.1.5) \qquad u := \mathcal{C}_k \left[(\tfrac{1}{2} I + \nu \vee C_k)^{-1} f \right] \ \text{ in } \Omega$$

is the unique solution to the boundary value problem (7.1.1). Similarly, we also have

$$(7.1.6) \qquad u := \mathcal{C}_k \left[(\tfrac{1}{2} I + \nu \wedge C_k)^{-1} f \right] \ \text{ in } \Omega$$

is the unique solution to the boundary value problem (7.1.2).

Considering the boundary value problems in Ω_-, by a similar reasoning we conclude that

$$u := \mathcal{C}_k\left[(\tfrac{1}{2}I + \nu \vee C_k)^{-1} f\right] \text{ in } \Omega_- \tag{7.1.7}$$

and

$$u := \mathcal{C}_k\left[(\tfrac{1}{2}I + \nu \wedge C_k)^{-1} f\right] \text{ in } \Omega_- \tag{7.1.8}$$

are the unique solutions to the boundary value problems (7.1.3) and (7.1.4), respectively. □

For any $f \in L^{p,\delta}_{tan}(\partial\Omega, \mathcal{A}_4)$, by Theorem 7.1.1, there exists a unique u solving the problem (7.1.1). In fact, we have pointed out that

$$u = \mathcal{C}_k\left[(\tfrac{1}{2}I + \nu \vee C_k)^{-1} f\right].$$

Moreover, $\nu \wedge u\Big|_{\partial\Omega} \in L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4)$. The similar conclusion holds for the other boundary values problems (7.1.2), (7.1.3) and (7.1.4). By the spirit of Theorem 7.1.1 and the above statement, we obtain the following corollary.

Corollary 7.1.2. *If* $u \in H^p_k(\Omega)$, *then* $\nu \vee u\Big|_{\partial\Omega} \in L^{p,\delta}_{tan}(\partial\Omega, \mathcal{A}_4)$ *and*

$$u := \mathcal{C}_k\left[(\tfrac{1}{2}I + \nu \vee C_k)^{-1}(\nu \vee u\Big|_{\partial\Omega})\right]; \tag{7.1.9}$$

or $\nu \wedge u\Big|_{\partial\Omega} \in L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4)$ *and*

$$u := \mathcal{C}_k\left[(\tfrac{1}{2}I + \nu \wedge C_k)^{-1}(\nu \wedge u\Big|_{\partial\Omega})\right]. \tag{7.1.10}$$

Similarly, if $u \in H_k^p(\Omega_-)$*, then* $\nu \vee u\Big|_{\partial\Omega} \in L^{p,\delta}_{tan}(\partial\Omega, \mathcal{A}_4)$ *and*

$$u := \mathcal{C}_k\left[(-\tfrac{1}{2}I + \nu \vee C_k)^{-1}(\nu \vee u\Big|_{\partial\Omega})\right]; \tag{7.1.11}$$

or $\nu \wedge u\Big|_{\partial\Omega} \in L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4)$ *and*

$$u := \mathcal{C}_k\left[(-\tfrac{1}{2}I + \nu \wedge C_k)^{-1}(\nu \wedge u\Big|_{\partial\Omega})\right]. \tag{7.1.12}$$

7.2 Invertibility of Cauchy-type Operators $\nu \wedge C_k$ and $\nu \vee C_k$

In this section we would like to consider the Invertibility of Cauchy-type operators $\nu \wedge C_k$ and $\nu \vee C_k$. We begin with introducing the tangential-normal operator TN which maps any f in $L^{p,\delta}_{tan}(\partial\Omega, \mathcal{A}_4)$ into $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4)$ in the following sense:

$$TN(f) := \nu \wedge \mathcal{C}_k\left[(\tfrac{1}{2}I + \nu \vee C_k)^{-1}f\right], \tag{7.2.13}$$

and also define the normal-tangential operator NT which maps any g in $L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4)$ into $L^{p,\delta}_{tan}(\partial\Omega, \mathcal{A}_4)$ by

$$NT(g) := \nu \vee \mathcal{C}_k\left[(\tfrac{1}{2}I + \nu \wedge C_k)^{-1}g\right]. \tag{7.2.14}$$

We intend to develop other expressions of TN and NT in the following lemma.

Lemma 7.2.1. *Assume that TN is the tangential-normal operator defined in (7.2.13) and NT is the normal-tangential operator defined in (7.2.14), then the following hold:*

$$TN = (\nu \wedge C_k)(\tfrac{1}{2}I + \nu \vee C_k)^{-1}, \tag{7.2.15}$$

$$NT = (\nu \vee C_k)(\tfrac{1}{2}I + \nu \wedge C_k)^{-1}. \tag{7.2.16}$$

Proof.

For any $f \in L^{p,\delta}_{tan}(\partial\Omega, \mathcal{A}_4)$, from (7.2.13), we have

$$TN(f) = \nu \wedge \mathcal{C}_k \left[(\tfrac{1}{2}I + \nu \vee C_k)^{-1} f \right].$$

Denoting $(\frac{1}{2}I + \nu \vee C_k)^{-1} f$ by h, we have $h \in L^{p,\delta}_{tan}(\partial\Omega, \mathcal{A}_4)$ and hence $v \vee h = 0$. Next we compute

$$\begin{aligned}
TN(f) &= \nu \wedge \mathcal{C}_k h\Big|_{\partial\Omega} \\
&= \nu \wedge (-\tfrac{1}{2}\nu \cdot h + C_k h) \\
&= \nu \wedge \left[-\tfrac{1}{2}(\nu \wedge h - \nu \vee h) + C_k h \right] \\
&= -\tfrac{1}{2}\nu \wedge (\nu \wedge h) + \nu \wedge C_k h \\
&= \nu \wedge C_k h \\
&= (\nu \wedge C_k)(\tfrac{1}{2}I + \nu \vee C_k)^{-1} f.
\end{aligned}$$

Therefore,

$$TN = (\nu \wedge C_k)(\tfrac{1}{2}I + \nu \vee C_k)^{-1}.$$

The second equality in the lemma is proved in a similar way. □

Theorem 7.2.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. Then there exists a sequence of real numbers $\{k_j\}_j$ such that for each $1 < p < 2 + \varepsilon$ and $k \in \mathbb{C}\backslash\{k_j\}_j$, the operators TN and NT are linear bounded operators and*

$$TN \circ NT = I \quad \text{on} \quad L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4), \tag{7.2.17}$$

$$NT \circ TN = I \quad \text{on} \quad L^{p,\delta}_{tan}(\partial\Omega, \mathcal{A}_4), \tag{7.2.18}$$

where I stands for the identity operator.

In particular, the operators

$$TN : L^{p,\delta}_{tan}(\partial\Omega, \mathcal{A}_4) \xrightarrow{\sim} L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4)$$

and

$$NT : L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4) \xrightarrow{\sim} L^{p,\delta}_{tan}(\partial\Omega, \mathcal{A}_4)$$

are isomorphisms.

Proof.

For each $g \in L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4)$,

$$NT(g) = \nu \vee v\Big|_{\partial\Omega}, \tag{7.2.19}$$

where v is such that

$$\begin{cases} v \in H^p_k(\Omega), \\ \nu \wedge v\Big|_{\partial\Omega} = g. \end{cases} \tag{7.2.20}$$

Then, by (7.2.19),

$$TN \circ NT(g) = TN(\nu \vee v\Big|_{\partial\Omega}). \tag{7.2.21}$$

The definition of TN yields

$$TN(\nu \vee v\Big|_{\partial\Omega}) = \nu \wedge u\Big|_{\partial\Omega}, \tag{7.2.22}$$

where

$$
\begin{cases} u \in H^p_k(\Omega), \\ \nu \vee u\Big|_{\partial\Omega} = \nu \vee v\Big|_{\partial\Omega}. \end{cases} \tag{7.2.23}
$$

Thus, $u = v$ and therefore

$$
TN(\nu \vee v\Big|_{\partial\Omega}) = \nu \wedge v\Big|_{\partial\Omega} = g. \tag{7.2.24}
$$

In conclusion, we have

$$
TN \circ NT = I.
$$

Similarly, we can also prove that $NT \circ TN = I$, which concludes the proof of the theorem. $\square$

Corollary 7.2.3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. Then there exists a sequence of real numbers $\{k_j\}_j$ such that for each $1 < p < 2 + \varepsilon$ and $k \in \mathbb{C}\backslash\{k_j\}_j$, the operators*

$$
\nu \wedge C_k : L^{p,\delta}_{tan}(\partial\Omega, \mathcal{A}_4) \xrightarrow{\sim} L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4) \tag{7.2.25}
$$

and

$$
\nu \vee C_k : L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4) \xrightarrow{\sim} L^{p,\delta}_{tan}(\partial\Omega, \mathcal{A}_4) \tag{7.2.26}
$$

are isomorphisms.

Proof.

On account of the equality (7.2.15), we may write

$$
\nu \wedge C_k = TN \circ (\tfrac{1}{2}I + \nu \vee C_k).
$$

Since the operators TN and $\frac{1}{2}I + \nu \vee C_k$ are both isomorphisms, $\nu \wedge C_k$ is also an isomorphism. Similarly, by using the equality (7.2.16), one can prove that $\nu \vee C_k$ is an isomorphism as well. $\square$

Chapter 8

Formulation of the Main Boundary Value Problem

8.1 Proof of Existence

The current section is going to deal with the existence of the transmission boundary value problem. To begin with, let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. Assume that $0 < \mu < 1$, and $1 < p < \infty$. Also, assume that the functions u^+ and u^- are of class C^1. Consider the following transmission boundary value problem:

$$\begin{cases} \mathbb{D}_k u^{\pm} = 0 \text{ in } \Omega_{\pm}, \\ \nu \vee u^+\Big|_{\partial\Omega} - \nu \vee u^-\Big|_{\partial\Omega} = f \ \in L^{p,\delta}_{tan}(\partial\Omega, \mathcal{A}_4), \\ \nu \wedge u^+\Big|_{\partial\Omega} - \mu\, \nu \wedge u^-\Big|_{\partial\Omega} = g \ \in L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4), \\ \lim_{|x|\to\infty}(|x| - ie_4 x)u^-(x) = 0, \\ N(u^{\pm}), N(du^{\pm}) \text{ and } N(\delta u^{\pm}) \ \in L^p(\partial\Omega). \end{cases} \tag{8.1.1}$$

Theorem 8.1.1. *The transmission boundary value problem (8.1.1) is well-posed in the sense that there exist $\varepsilon > 0$ and a sequence of real numbers $\{k_j\}_j$ which depend* *exclusively on the boundary $\partial\Omega$ and the transmission parameter μ, and which have the following significance. For every $p \in (1, 2+\varepsilon)$ and every $k \in \mathbb{C} \setminus \{k_j\}_j$, the transmission problem (8.1.1) has a solution if and only if*

$$f \in L^{p,\delta}_{tan}(\partial\Omega, \mathcal{A}_4) \quad \textit{and} \quad g \in L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4). \tag{8.1.2}$$

Furthermore, the solution (u^+, u^-) is unique and satisfies the estimate

$$\begin{aligned}
&\|N(u^\pm)\|_{L^p(\partial\Omega)} + \|N(du^\pm)\|_{L^p(\partial\Omega)} + \|N(\delta u^\pm)\|_{L^p(\partial\Omega)} \\
&\quad \leq C\|\nu \vee u^+ - \nu \vee u^-\|_{L^p(\partial\Omega)} + C\|\nu \vee \delta u^+ - \nu \vee \delta u^-\|_{L^p(\partial\Omega)} \\
&\qquad + C\|\nu \wedge u^+ - \mu\, \nu \wedge u^-\|_{L^p(\partial\Omega)} + C\|\nu \wedge du^+ - \mu\, \nu \wedge du^-\|_{L^p(\partial\Omega)}
\end{aligned}$$

whenever $1 < p < 2+\varepsilon$, where $C = C(\partial\Omega, k, p) > 0$ is independent of $u^\pm$.

Proof.

In order to solve the above transmission boundary value problem, we consider the following two auxiliary problems.

First we consider the boundary value problem

$$\begin{cases} \mathbb{D}_k u = 0 \text{ in } \Omega, \\ \nu \vee u\Big|_{\partial\Omega} = f \ \in L^{p,\delta}_{tan}(\partial\Omega, \mathcal{A}_4), \\ N(u), N(du) \text{ and } N(\delta u) \ \in L^p(\partial\Omega, \mathcal{A}_4). \end{cases} \tag{8.1.3}$$

By Theorem 6.3 of [Mi5], this problem is well-posed for $1 < p < 2+\varepsilon$.

We next consider the reduced transmission boundary value problem

$$
(8.1.4) \qquad \begin{cases} \mathbb{D}_k v^\pm = 0 \text{ in } \Omega_\pm, \\ \nu \vee v^+\Big|_{\partial\Omega} - \nu \vee v^-\Big|_{\partial\Omega} = 0, \\ \nu \wedge v^+\Big|_{\partial\Omega} - \mu\, \nu \wedge v^-\Big|_{\partial\Omega} = \widetilde{g}, \\ \lim_{|x|\to\infty}(|x| - ie_4 x)v^-(x) = 0, \\ N(v^\pm), N(dv^\pm), N(\delta v^\pm) \in L^p(\partial\Omega, \mathcal{A}_4), \end{cases}
$$

where $\widetilde{g} = g - \nu \wedge u\big|_{\partial\Omega}$. In order to proceed, we let

$$
(8.1.5) \qquad \begin{cases} u^+ = u + v^+, \\ u^- = v^-. \end{cases}
$$

One can observe that if the problem (8.1.4) can be solved, then the transmission boundary value problem (8.1.1) is also solvable since

$$
\begin{aligned}
\nu \vee u^+ - \nu \vee u^- &= \nu \vee u + \nu \vee v^+ - \nu \vee v^- \\
&= \nu \vee u \\
&= f
\end{aligned}
$$

and

$$
\begin{aligned}
\nu \wedge u^+ - \mu\, \nu \wedge u^- &= \nu \wedge u + \nu \wedge v^+ - \mu\, \nu \wedge v^- \\
&= \nu \wedge u + \widetilde{g} \\
&= g.
\end{aligned}
$$

Now, in order to solve (8.1.4), we take $v^\pm := \mathcal{C}_k h$ in $\Omega_\pm$, where $h \in L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4)$. It has been poved that $\mathbb{D}_k v^\pm = 0$ for this choice of $v^\pm$. Moreover, Lemma 6.4.1

guarantees that v^- decays at infinity. Checking the second boundary condition of (8.1.1), we have

$$\begin{aligned}
\widetilde{g} &= \nu \wedge v^+\Big|_{\partial\Omega} - \mu\,\nu \wedge v^-\Big|_{\partial\Omega} \\
&= \nu \wedge (-\tfrac{1}{2}\nu \cdot h + C_k h) - \mu\,\nu \wedge (\tfrac{1}{2}\nu \cdot h + C_k h) \\
&= \nu \wedge \left[-\tfrac{1}{2}(\nu \wedge h - \nu \vee h) + C_k h\right] \\
&\quad -\mu\,\nu \wedge \left[\tfrac{1}{2}(\nu \wedge h - \nu \vee h) + C_k h\right] \\
&= \tfrac{1}{2}\nu \wedge (\nu \vee h) + \nu \wedge C_k h + \tfrac{1}{2}\mu\,\nu \wedge (\nu \vee h) - \mu\nu \wedge C_k h \\
&= \tfrac{1}{2}\, h_{nor} + \nu \wedge C_k h + \tfrac{1}{2}\,\mu h_{nor} - \mu\nu \wedge C_k h \\
&= \tfrac{1}{2}h + \tfrac{1}{2}\mu h + \nu \wedge C_k h - \mu\nu \wedge C_k h \\
&= \tfrac{1}{2}(\mu + 1)h + (1 - \mu)\nu \wedge C_k h.
\end{aligned}$$

Dividing both sides of the above equality by $1 - \mu$, we obtain

$$\frac{1}{2}\frac{1+\mu}{1-\mu}\, h + \nu \wedge C_k h = \frac{1}{1-\mu}\,\widetilde{g} \;\in L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4). \tag{8.1.6}$$

Let $\lambda := \frac{1}{2}\frac{1+\mu}{1-\mu}$. Then $\frac{1}{2} < \lambda < \infty$, and the equality (8.1.6) can be expressed as

$$(\lambda I + \nu \wedge C_k)\, h = \frac{1}{1-\mu}\,\widetilde{g} \;\in L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4). \tag{8.1.7}$$

By Theorem 6.5.1, there exists an h such that the equality (8.1.7) holds. This concludes the proof of the existence of the solution of the problem (8.1.1).

□

8.2 Proof of Uniqueness

We next turn our attention to proving the uniqueness of the solution the problem (8.1.1). We start with considering the following homogeneous problem

$$\text{(8.2.8)} \qquad \begin{cases} u^+ \in H^p_k(\Omega) \text{ and } u^- \in H^p_k(\Omega_-), \\ \nu \vee u^+\Big|_{\partial\Omega} = \nu \vee u^-\Big|_{\partial\Omega}, \\ \nu \wedge u^+\Big|_{\partial\Omega} = \mu\, \nu \wedge u^-\Big|_{\partial\Omega}. \end{cases}$$

Our goal is to show that $u^+ = u^- = 0$.

Let us denote $\nu \vee u^+\Big|_{\partial\Omega}$ by h, which also happens to coincide with $\nu \vee u^-\Big|_{\partial\Omega}$. Then by Corollary 7.1.2, we have

$$\text{(8.2.9)} \qquad u^+ = \mathcal{C}_k \left[(\tfrac{1}{2}I + \nu \vee C_k)^{-1} h\Big|_{\partial\Omega} \right]$$

and

$$\text{(8.2.10)} \qquad u^- = \mathcal{C}_k \left[(-\tfrac{1}{2}I + \nu \vee C_k)^{-1} h\Big|_{\partial\Omega} \right].$$

Applying Theorem 6.3.2, namely the Jump formulas, to (8.2.9) and (8.2.10), the second boundary condition $\nu \wedge u^+\Big|_{\partial\Omega} = \mu\, \nu \wedge u^-\Big|_{\partial\Omega}$ in (6.5.61) becomes

$$\begin{aligned} & \nu \wedge \left[-\tfrac{1}{2}\nu \cdot (\tfrac{1}{2}I + \nu \vee C_k)^{-1} h\Big|_{\partial\Omega} + C_k(\tfrac{1}{2}I + \nu \vee C_k)^{-1} h\Big|_{\partial\Omega} \right] \\ \text{(8.2.11)} \qquad = \;& \mu\, \nu \wedge \left[\tfrac{1}{2}\nu \cdot (-\tfrac{1}{2}I + \nu \vee C_k)^{-1} h\Big|_{\partial\Omega} + C_k(-\tfrac{1}{2}I + \nu \vee C_k)^{-1} h\Big|_{\partial\Omega} \right]. \end{aligned}$$

Simplifying the left-hand side of the equality (8.2.11), we get

$$
\begin{aligned}
&\nu \wedge \left[-\tfrac{1}{2}\nu \cdot (\tfrac{1}{2}I + \nu \vee C_k)^{-1}h\Big|_{\partial\Omega} + C_k(\tfrac{1}{2}I + \nu \vee C_k)^{-1}h\Big|_{\partial\Omega} \right] \\
&\quad = \quad -\tfrac{1}{2}\nu \wedge \left[\nu \wedge (\tfrac{1}{2}I + \nu \vee C_k)^{-1}h\Big|_{\partial\Omega} - \nu \vee (\tfrac{1}{2}I + \nu \vee C_k)^{-1}h\Big|_{\partial\Omega} \right. \\
&\qquad\qquad \left. + \, C_k(\tfrac{1}{2}I + \nu \vee C_k)^{-1}h\Big|_{\partial\Omega} \right].
\end{aligned} \tag{8.2.12}
$$

Since $(\frac{1}{2}I + \nu \vee C_k)^{-1}h\Big|_{\partial\Omega} \in L^{p,\delta}_{tan}(\partial\Omega, \mathcal{A}_4)$, then $\nu \vee (\frac{1}{2}I + \nu \vee C_k)^{-1}h\Big|_{\partial\Omega} = 0$. Also, we note that $\nu \wedge \left[\nu \wedge (\frac{1}{2}I + \nu \vee C_k)^{-1}h\Big|_{\partial\Omega} \right] = 0$. Therefore, the equality (8.2.12) is reduced to

$$
\begin{aligned}
&\nu \wedge \left[-\tfrac{1}{2}\nu \cdot (\tfrac{1}{2}I + \nu \vee C_k)^{-1}h\Big|_{\partial\Omega} + C_k(\tfrac{1}{2}I + \nu \vee C_k)^{-1}h\Big|_{\partial\Omega} \right] \\
&= \nu \wedge C_k(\tfrac{1}{2}I + \nu \vee C_k)^{-1}h\Big|_{\partial\Omega}.
\end{aligned} \tag{8.2.13}
$$

Similarly, the right-hand side of the equality (8.2.11) reduces to

$$
\begin{aligned}
&\mu\, \nu \wedge \left[\tfrac{1}{2}\nu \cdot (-\tfrac{1}{2}I + \nu \vee C_k)^{-1}h\Big|_{\partial\Omega} + C_k(-\tfrac{1}{2}I + \nu \vee C_k)^{-1}h\Big|_{\partial\Omega} \right] \\
&= \mu\, \nu \wedge C_k(-\tfrac{1}{2}I + \nu \vee C_k)^{-1}h\Big|_{\partial\Omega}.
\end{aligned} \tag{8.2.14}
$$

Consequently, from the equalities (8.2.13) and (8.2.14), the second boundary condition in (6.5.61) becomes

$$
\nu \wedge C_k(\tfrac{1}{2}I + \nu \vee C_k)^{-1}h\Big|_{\partial\Omega} = \mu\, \nu \wedge C_k(-\tfrac{1}{2}I + \nu \vee C_k)^{-1}h\Big|_{\partial\Omega}. \tag{8.2.15}
$$

By Corollary 7.2.3, we may further write

$$
(\tfrac{1}{2}I + \nu \vee C_k)^{-1}h\Big|_{\partial\Omega} = \mu(-\tfrac{1}{2}I + \nu \vee C_k)^{-1}h\Big|_{\partial\Omega}. \tag{8.2.16}
$$

Moving the right-hand side of (8.2.16) to left and factoring out two inverse operators, we get

$$(\tfrac{1}{2}I + \nu \vee C_k)^{-1} \left[(-\tfrac{1}{2}I + \nu \vee C_k) - \mu(\tfrac{1}{2}I + \nu \vee C_k)\right] (-\tfrac{1}{2}I + \nu \vee C_k)^{-1} h\Big|_{\partial\Omega} = 0.$$

After some simple algebraic computation, we can rewrite the above equation as

$$(1-\mu)(\tfrac{1}{2}I + \nu \vee C_k)^{-1}(\lambda I + \nu \vee C_k)(-\tfrac{1}{2}I + \nu \vee C_k)^{-1} h\Big|_{\partial\Omega} = 0, \tag{8.2.17}$$

where $\lambda := -\frac{1}{2}\frac{1+\mu}{1-\mu} \in (-\infty, -\frac{1}{2})$. By Theorem 6.5.10, the operators $(\frac{1}{2}I + \nu \vee C_k)^{-1}$, $\lambda I + \nu \vee C_k$, and $(-\frac{1}{2}I + \nu \vee C_k)^{-1}$ are all isomorphisms. Hence, $h = 0$ and furthermore $u^+ = u^- = 0$.

Finally, this finishes the proof of the uniqueness of the solution of the transmission boundary value problem (8.1.1). $\square$

Chapter 9

Applications to TBVP for Maxwell's Equations

In this chapter we are going to connect the transmission boundary value problem for Dirac operators with Maxwell's equations. In section 1 we will decompose the transmission boundary value problem for Dirac operators into the transmission boundary value problems for Maxwell's equations and Helmholtz operator. In section 2 we give a sufficent and necessary condition which guarantees that the transmission boundary value problem for Dirac operators is equivalent to two Maxwell's systems.

9.1 Another Point of View on TBVP for Dirac Operators

We start with recalling the transmission boundary value problem (8.1.1):

$$(9.1.1)\qquad \begin{cases} \mathbb{D}_k u^\pm = 0 \text{ in } \Omega_\pm, \\ \nu \vee u^+\Big|_{\partial\Omega} - \nu \vee u^-\Big|_{\partial\Omega} = f \ \in L^{p,\delta}_{tan}(\partial\Omega, \mathcal{A}_4), \\ \nu \wedge u^+\Big|_{\partial\Omega} - \mu\, \nu \wedge u^-\Big|_{\partial\Omega} = g \ \in L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4), \\ \lim_{|x|\to\infty}(|x| - ie_4 x)u^-(x) = 0, \\ N(u^\pm), N(du^\pm) \text{ and } N(\delta u^\pm) \ \in L^p(\partial\Omega, \mathcal{A}_4). \end{cases}$$

Assume

$$(9.1.2)\qquad u^\pm = U^\pm - ie_4 \widetilde{U}^\pm,$$

where $U^\pm$ and $\widetilde{U}^\pm$ are $\mathcal{A}_3$-valued functions. Moreover, decompose $U^\pm$ as the following:

$$(9.1.3)\qquad U^\pm = U_0^\pm + *U_0'^\pm + U_1^\pm + *U_1'^\pm,$$

where $U_0^\pm, U_0'^\pm$ are Λ^0-valued functions and $U_1^\pm, U_1'^\pm$ are Λ^1-valued functions. Similarly, we have

$$(9.1.4)\qquad \widetilde{U}^\pm = \widetilde{U}_0^\pm + *\widetilde{U}_0'^\pm + \widetilde{U}_1^\pm + *\widetilde{U}_1'^\pm,$$

where each function carrying the subscript j is Λ^j-valued.

We next define the vector

$$(9.1.5)\qquad U := (U_0^\pm, U_1^\pm, U_0'^\pm, U_1'^\pm, \widetilde{U}_0^\pm, \widetilde{U}_1^\pm, \widetilde{U}_0'^\pm, \widetilde{U}_1'^\pm).$$

Then

$$\mathbb{D}_k u^{\pm} = 0 \Leftrightarrow \mathcal{P}_k U = 0 \tag{9.1.6}$$

where

$$\mathcal{P}_k := \begin{bmatrix} 0 & -\operatorname{div} & 0 & 0 & ik & 0 & 0 & 0 \\ \nabla & 0 & 0 & \operatorname{curl} & 0 & ik & 0 & 0 \\ 0 & \operatorname{curl} & -\nabla & 0 & 0 & 0 & 0 & ik \\ 0 & 0 & \operatorname{div} & 0 & 0 & 0 & ik & 0 \\ -ik & 0 & 0 & 0 & 0 & -\operatorname{div} & 0 & 0 \\ 0 & -ik & 0 & 0 & \nabla & 0 & 0 & \operatorname{curl} \\ 0 & 0 & 0 & -ik & 0 & \operatorname{curl} & -\nabla & 0 \\ 0 & 0 & -ik & 0 & 0 & 0 & 0 & \operatorname{div} \end{bmatrix}. \tag{9.1.7}$$

The equation $\mathcal{P}_k U = 0$ implies the following eight equations:

$$-\operatorname{div} U_1^{\pm} + ik\widetilde{U}_0^{\pm} = 0; \tag{9.1.8}$$

$$\nabla U_0^{\pm} + \operatorname{curl} U_1'^{\pm} + ik\widetilde{U}_1^{\pm} = 0; \tag{9.1.9}$$

$$\operatorname{curl} U_1^{\pm} - \nabla U_0'^{\pm} + ik\widetilde{U}_1'^{\pm} = 0; \tag{9.1.10}$$

$$\operatorname{div} U_0'^{\pm} + ik\widetilde{U}_0'^{\pm} = 0; \tag{9.1.11}$$

(9.1.12)

$$-ikU_0^{\pm} - \operatorname{div} \widetilde{U}_1^{\pm} = 0; \tag{9.1.13}$$

$$-ikU_1^{\pm} + \nabla \widetilde{U}_0^{\pm} + \operatorname{curl} \widetilde{U}_1'^{\pm} = 0; \tag{9.1.14}$$

$$-ikU_1'^{\pm} + \operatorname{curl} \widetilde{U}_1^{\pm} + \nabla \widetilde{U}_0'^{\pm} = 0; \tag{9.1.15}$$

$$-ikU_0'^{\pm} + \operatorname{div} \widetilde{U}_1'^{\pm} = 0. \tag{9.1.16}$$

Equation (9.1.8) implies

$$\widetilde{U}_0^{\pm} = \frac{1}{ik} \operatorname{div} U_1^{\pm}. \tag{9.1.17}$$

By equation (9.1.14), we have

$$U_1^{\pm} = \frac{1}{ik}\,\nabla\,\widetilde{U}_0^{\pm} + \frac{1}{ik}\,\mathrm{curl}\,\widetilde{U}_1'^{\pm}. \tag{9.1.18}$$

Substituting $U_1^{\pm}$ in the equality (9.1.17) by (9.1.18), we have

$$\widetilde{U}_0^{\pm} = \frac{1}{(ik)^2}\,\Delta\,\widetilde{U}_0^{\pm}, \tag{9.1.19}$$

which further yields

$$(\Delta + k^2)\widetilde{U}_0^{\pm} = 0 \text{ in } \Omega_{\pm}. \tag{9.1.20}$$

In a very similar fashion, from equalities (9.1.9) and (9.1.13), we obtain that

$$(\Delta + k^2)U_0^{\pm} = 0 \text{ in } \Omega_{\pm}. \tag{9.1.21}$$

Equalities (9.1.10) and (9.1.16) imply

$$(\Delta + k^2)U_0'^{\pm} = 0 \text{ in } \Omega_{\pm}. \tag{9.1.22}$$

On account of equalities (9.1.11) and (9.1.15), we have

$$(\Delta + k^2)\widetilde{U}_0'^{\pm} = 0 \text{ in } \Omega_{\pm}. \tag{9.1.23}$$

Our next goal is to collect the information from the boundary conditions. Much as before, we define

$$f := F + ie_4\widetilde{F}, \tag{9.1.24}$$

where F and $\widetilde{F}$ are $\mathcal{A}_3$-valued functions. We next decompose F and $\widetilde{F}$ as follows:

$$F = F_0 + *F_0^{'} + F_1 + *F_1^{'}, \tag{9.1.25}$$

and

$$\widetilde{F} = \widetilde{F}_0 + *\widetilde{F}_0^{'} + \widetilde{F}_1 + *\widetilde{F}_1^{'}, \tag{9.1.26}$$

where each function carrying the subscript j is Λ^j-valued.

Similarly, write the function g as

$$g = G + ie_4\widetilde{G}, \tag{9.1.27}$$

where

$$G = G_0 + *G_0^{'} + G_1 + *G_1^{'}, \tag{9.1.28}$$

and

$$\widetilde{G} = \widetilde{G}_0 + *\widetilde{G}_0^{'} + \widetilde{G}_1 + *\widetilde{G}_1^{'}. \tag{9.1.29}$$

In order to make use of the first boundary condition, we need to find the components of $\nu \vee u^{\pm}$. We first focus on $\nu \vee u^{+}$ and note that

$$\begin{aligned} \nu \vee u^{+} &= \nu \vee (U^{+} - ie_4\widetilde{U}^{+}) \\ &= \nu \vee U^{+} - i\nu \vee e_4(\widetilde{U}^{+}) \\ &= \nu \vee U^{+} + ie_4(\nu \vee \widetilde{U}^{+}). \end{aligned}$$

In the last step above, we use Lemma 3.2.4. Now rewrite $\nu \vee U^+$ as

$$\begin{aligned} \nu \vee U^+ &= \nu \vee (U_0^+ + *U_0'^+ + U_1^+ + *U_1'^+) \\ &= \nu \vee U_0^+ + \nu \vee *U_0'^+ + \nu \vee U_1^+ + \nu \vee *U_1'^+. \end{aligned} \tag{9.1.30}$$

The first term $\nu \vee U_0^+$, in (9.1.30), is zero since U_0^+ is Λ^0-valued. Moreover by Remark 3.2.1 and Property 3.2.2-(4), we have

$$\nu \vee *U_0'^+ = *(\nu \wedge U_0'^+) = *(\nu U_0'^+), \tag{9.1.31}$$

and

$$\nu \vee *U_1'^+ = -*(\nu \wedge U_1'^+) = -*(*(\nu \times U_1'^+)) = -\nu \times U_1'^+. \tag{9.1.32}$$

Using Remark 3.2.1 again, we obtain

$$\nu \vee U_1^+ = \langle \nu, U_1^+ \rangle. \tag{9.1.33}$$

By the equalities (9.1.31), (9.1.32) and (9.1.33), we observe that the equality (9.1.30) becomes

$$\nu \vee U^+ = *(\nu U_0'^+) + \langle \nu, U_1^+ \rangle - \nu \times U_1'^+. \tag{9.1.34}$$

Then it is easy to see that

$$\begin{aligned} \nu \vee u^+ &= \nu \vee U^+ + ie_4(\nu \vee \widetilde{U}^+) \\ &= *(\nu U_0'^+) + \langle \nu, U_1^+ \rangle - \nu \times U_1'^+ \\ &\quad + ie_4[*(\nu \widetilde{U}_0'^+) + \langle \nu, \widetilde{U}_1^+ \rangle - \nu \times \widetilde{U}_1'^+]. \end{aligned} \tag{9.1.35}$$

A similar argument shows that

$$\begin{aligned}\nu \vee u^- &= *(\nu U_0^{\prime -}) + \langle \nu, U_1^- \rangle - \nu \times U_1^{\prime -}\\ &\quad + ie_4[*(\nu \widetilde{U}_0^{\prime -}) + \langle \nu, \widetilde{U}_1^- \rangle - \nu \times \widetilde{U}_1^{\prime -}].\end{aligned} \tag{9.1.36}$$

Comparing the difference between (9.1.35) and (9.1.36) with f componentwisely, we conclude that the first boundary condition implies the following six equations on the boundary:

$$U_0^{\prime +}\Big|_{\partial\Omega} - U_0^{\prime -}\Big|_{\partial\Omega} = F_0^{\prime}; \tag{9.1.37}$$

$$\langle \nu, U_1^+ \rangle\Big|_{\partial\Omega} - \langle \nu, U_1^- \rangle\Big|_{\partial\Omega} = F_0; \tag{9.1.38}$$

$$-\nu \times U_1^{\prime +}\Big|_{\partial\Omega} + \nu \times U_1^{\prime -}\Big|_{\partial\Omega} = F_1; \tag{9.1.39}$$

$$\widetilde{U}_0^{\prime +}\Big|_{\partial\Omega} - \widetilde{U}_0^{\prime -}\Big|_{\partial\Omega} = \widetilde{F}_0^{\prime}; \tag{9.1.40}$$

$$\langle \nu, \widetilde{U}_1^+ \rangle\Big|_{\partial\Omega} - \langle \nu, \widetilde{U}_1^- \rangle\Big|_{\partial\Omega} = \widetilde{F}_0; \tag{9.1.41}$$

$$-\nu \times \widetilde{U}_1^{\prime +}\Big|_{\partial\Omega} + \nu \times \widetilde{U}_1^{\prime -}\Big|_{\partial\Omega} = \widetilde{F}_1. \tag{9.1.42}$$

Similarly, it is not hard to check that

$$\begin{aligned}\nu \wedge u^{\pm} &= \nu U_0^{\pm} + *(\nu \times U_1^{\pm}) + *\langle \nu, U_1^{\prime \pm} \rangle\\ &\quad + ie_4[\nu \widetilde{U}_0^{\pm} + *(\nu \times \widetilde{U}_1^{\pm}) + *\langle \nu, \widetilde{U}_1^{\prime \pm} \rangle].\end{aligned} \tag{9.1.43}$$

Then the second boundary condition implies the other six equations on the boundary:

$$U_0^+\Big|_{\partial\Omega} - \mu U_0^-\Big|_{\partial\Omega} = G_0; \tag{9.1.44}$$

$$\langle \nu, U_1'^+\rangle\Big|_{\partial\Omega} - \mu\langle \nu, U_1'^-\rangle\Big|_{\partial\Omega} = G_0'; \tag{9.1.45}$$

$$\nu \times U_1^+\Big|_{\partial\Omega} - \mu\,\nu \times U_1^-\Big|_{\partial\Omega} = G_1; \tag{9.1.46}$$

$$\widetilde{U}_0^+\Big|_{\partial\Omega} - \mu \widetilde{U}_0^-\Big|_{\partial\Omega} = \widetilde{G}_0; \tag{9.1.47}$$

$$\langle \nu, \widetilde{U}_1'^+\rangle\Big|_{\partial\Omega} - \mu\langle \nu, \widetilde{U}_1'^-\rangle\Big|_{\partial\Omega} = \widetilde{G}_0'; \tag{9.1.48}$$

$$\nu \times \widetilde{U}_1^+\Big|_{\partial\Omega} - \mu\,\nu \times \widetilde{U}_1^-\Big|_{\partial\Omega} = \widetilde{G}_1. \tag{9.1.49}$$

The equations (9.1.9) and (9.1.15) with the boundary conditions (9.1.39) and (9.1.49) give us the first Maxwell's equations

$$\begin{cases} \operatorname{curl} \widetilde{U}_1^\pm - ikU_1'^\pm = -\nabla \widetilde{U}_0'^\pm, \\ \operatorname{curl} U_1'^\pm + ik\widetilde{U}_1^\pm = -\nabla U_0^\pm, \\ \nu \times \widetilde{U}_1^+\Big|_{\partial\Omega} - \mu\,\nu \times \widetilde{U}_1^-\Big|_{\partial\Omega} = \widetilde{G}_1, \\ \nu \times U_1'^+\Big|_{\partial\Omega} - \nu \times U_1'^-\Big|_{\partial\Omega} = -F_1. \end{cases} \tag{9.1.50}$$

On the other hand, the equations (9.1.10) and (9.1.14) with the boundary conditions (9.1.42) and (9.1.46) form the second Maxwell's equations

$$\begin{cases} \operatorname{curl} \widetilde{U}_1'^\pm - i\,k\,U_1^\pm = -\nabla \widetilde{U}_0^\pm, \\ \operatorname{curl} U_1^\pm + i\,k\,\widetilde{U}_1'^\pm = \nabla U_0'^\pm, \\ \nu \times U_1^+\Big|_{\partial\Omega} - \mu\,\nu \times U_1^-\Big|_{\partial\Omega} = G_1, \\ \nu \times \widetilde{U}_1'^+\Big|_{\partial\Omega} - \nu \times \widetilde{U}_1'^-\Big|_{\partial\Omega} = -\widetilde{F}_1. \end{cases} \tag{9.1.51}$$

We now turn our attention to finding the transmission boundary value problems for the Helmholtz operator that are implicitly implied by (9.1.1). We first focus on solving a boundary condition

$$\partial_\nu U_0^+\Big|_{\partial\Omega} - \partial_\nu U_0^-\Big|_{\partial\Omega} = \langle \nu, \nabla U_0^+\rangle\Big|_{\partial\Omega} - \langle \nu, \nabla U_0^-\rangle\Big|_{\partial\Omega}. \tag{9.1.52}$$

By (9.1.9) we have

$$\nabla\, U_0^+ = -ik\widetilde{U}_1^+ - \operatorname{curl} U_1'^+. \tag{9.1.53}$$

Therefore,

$$\begin{aligned} \langle \nu, \nabla U_0^+\rangle\Big|_{\partial\Omega} &= -ik\langle \nu, \widetilde{U}_1^+\rangle\Big|_{\partial\Omega} - \langle \nu, \operatorname{curl} U_1'^+\rangle\Big|_{\partial\Omega} \\ &= -ik\langle \nu, \widetilde{U}_1^+\rangle\Big|_{\partial\Omega} - \operatorname{Div}(\nu \times U_1'^+)\Big|_{\partial\Omega}. \end{aligned} \tag{9.1.54}$$

Similarly,

$$\langle \nu, \nabla U_0^-\rangle\Big|_{\partial\Omega} = -ik\langle \nu, \widetilde{U}_1^-\rangle\Big|_{\partial\Omega} - \operatorname{Div}(\nu \times U_1'^-)\Big|_{\partial\Omega}. \tag{9.1.55}$$

Applying the equalities (9.1.54) and (9.1.55) to (9.1.52), we obtain that

$$\begin{aligned} &\partial_\nu U_0^+\Big|_{\partial\Omega} - \partial_\nu U_0^-\Big|_{\partial\Omega} \\ &= -ik\langle \nu, \widetilde{U}_1^+\rangle\Big|_{\partial\Omega} - ik\langle \nu, \widetilde{U}_1^-\rangle\Big|_{\partial\Omega} \\ &\quad -\operatorname{Div}(\nu \times U_1'^+)\Big|_{\partial\Omega} - \operatorname{Div}(\nu \times U_1'^-)\Big|_{\partial\Omega} \\ &= -ik\langle \nu, \widetilde{U}_1^+ - \widetilde{U}_1^-\rangle\Big|_{\partial\Omega} - \operatorname{Div}(\nu \times U_1'^+ - \nu \times U_1'^-)\Big|_{\partial\Omega}. \end{aligned}$$

By the boundary conditions (9.1.39) and (9.1.41), we have

$$\partial_\nu U_0^+\Big|_{\partial\Omega} - \partial_\nu U_0^-\Big|_{\partial\Omega} = -ik\widetilde{F}_0 - \mathrm{DivF}_1. \tag{9.1.56}$$

According to the equation (9.1.20) and the boundary conditions (9.1.44) and (9.1.56), we have the first transmission boundary value problem for the the Helmholtz operator:

$$\begin{cases} (\Delta + k^2)U_0^\pm = 0 \text{ in } \Omega_\pm, \\ U_0^+\Big|_{\partial\Omega} - \mu U_0^-\Big|_{\partial\Omega} = G_0, \\ \partial_\nu U_0^+\Big|_{\partial\Omega} - \partial_\nu U_0^-\Big|_{\partial\Omega} = -ik\widetilde{F}_0 - \mathrm{DivF}_1. \end{cases} \tag{9.1.57}$$

Similarly, we have the other three transmission boundary value problems for the the Helmholtz operator as follows:

$$\begin{cases} (\Delta + k^2)\widetilde{U}_0^\pm = 0 \text{ in } \Omega_\pm, \\ \widetilde{U}_0^+\Big|_{\partial\Omega} - \mu \widetilde{U}_0^-\Big|_{\partial\Omega} = \widetilde{G}_0, \\ \partial_\nu \widetilde{U}_0^+\Big|_{\partial\Omega} - \partial_\nu \widetilde{U}_0^-\Big|_{\partial\Omega} = ikF_0 - \mathrm{Div}\widetilde{\mathrm{F}}_1, \end{cases} \tag{9.1.58}$$

$$\begin{cases} (\Delta + k^2)U_0'^\pm = 0 \text{ in } \Omega_\pm, \\ U_0'^+\Big|_{\partial\Omega} - U_0'^-\Big|_{\partial\Omega} = F_0', \\ \partial_\nu U_0'^+\Big|_{\partial\Omega} - \mu\partial_\nu U_0'^-\Big|_{\partial\Omega} = ik\widetilde{G}_0' - \mathrm{DivG}_1, \end{cases} \tag{9.1.59}$$

$$
(9.1.60) \qquad \begin{cases} (\Delta + k^2)\widetilde{U}_0'^{\pm} = 0 \text{ in } \Omega_{\pm}, \\ \widetilde{U}_0'^{+}\Big|_{\partial\Omega} - \widetilde{U}_0'^{-}\Big|_{\partial\Omega} = \widetilde{F}_0', \\ \partial_\nu \widetilde{U}_0'^{+}\Big|_{\partial\Omega} - \mu\partial_\nu \widetilde{U}_0'^{-}\Big|_{\partial\Omega} = ikG_0' + \mathrm{Div}\widetilde{\mathrm{G}}_1. \end{cases}
$$

All in all, the problem (9.1.1) implicitly implies four transmission boundary value problems for the Helmholtz operator and two transmission boundary value problems for Maxwell's equations.

9.2 Connections with Maxwell's Equations

The main result of this section is stated as the following theorem.

Theorem 9.2.1. *For each Ω, bounded Lipschitz domain in $\mathbb{R}^3$, there exist $\varepsilon > 0$ and a sequence of real numbers $\{k_j\}_j$ such that the following is true:*

For each $1 < p < 2 + \varepsilon$, $k \in \mathbb{C}\backslash\{k_j\}_j$, the boundary problem (8.1.1), with $u^{\pm}$ written in (9.1.2),(9.1.3),(9.1.4), f written in (9.1.24), (9.1.25), (9.1.26) and g written in (9.1.27), (9.1.28), (9.1.29) componentwisely, reduces to two Maxwell's systems (with opposite wave numbers), i.e.

$$
(9.2.61) \qquad \begin{cases} \operatorname{curl} \widetilde{U}_1^{\pm} - ikU_1'^{\pm} = 0, \\ \operatorname{curl} U_1'^{\pm} + ik\widetilde{U}_1^{\pm} = 0, \\ \nu \times \widetilde{U}_1^{+}\Big|_{\partial\Omega} - \mu\,\nu \times \widetilde{U}_1^{-}\Big|_{\partial\Omega} = \widetilde{G}_1, \\ \nu \times U_1'^{+}\Big|_{\partial\Omega} - \nu \times U_1'^{-}\Big|_{\partial\Omega} = -F_1, \end{cases}
$$

and

(9.2.62)
$$\begin{cases} \operatorname{curl} \widetilde{U}_1'^{\pm} - ikU_1^{\pm} = 0, \\ \operatorname{curl} U_1^{\pm} + ik\widetilde{U}_1'^{\pm} = 0, \\ \nu \times U_1^{+}\Big|_{\partial\Omega} - \mu\, \nu \times U_1^{-}\Big|_{\partial\Omega} = G_1, \\ \nu \times \widetilde{U}_1'^{+}\Big|_{\partial\Omega} - \nu \times \widetilde{U}_1'^{-}\Big|_{\partial\Omega} = -\widetilde{F}_1. \end{cases}$$

with boundary data $G_1, \widetilde{G}_1 \in L^{p,d}_{nor}(\partial\Omega, \mathcal{A}_4)$ *and* $F_1, \widetilde{F}_1 \in L^{p,\delta}_{tan}(\partial\Omega, \mathcal{A}_4)$, *if and only if*

$$d_\partial g + k\, e_4 g \text{ and } \delta_\partial f + k\, e_4 f \text{ are } (\Lambda^2 + e_4\Lambda^2) - \text{valued functions.}$$

Proof.

Due to the well-posedness of the BVPs (9.1.57), (9.1.58), (9.1.59) and (9.1.60), $U_0^{\pm}, U_0'^{\pm}, \widetilde{U}_0^{\pm}, \widetilde{U}_0'^{\pm}$ are zero if and only the boundary data of these problems are zero. Hence it suffices to show that the boundary data of the above BVPs are zero if and only if $d_\partial g + k\, e_4 g$ and $\delta_\partial f + k\, e_4 f$ are $(\Lambda^2 + e_4\Lambda^2)$-valued functions.

We begin with computing $d_\partial G$, where G is given by (9.1.28) and hence G is $\mathcal{A}_3$-valued. By Theorem 5.2.1, we can decompose G as follows:

(9.2.63)
$$G = \nu G_0 + *G_1 + *G_0',$$

where νG_0 is Λ^1-valued, $*G_1$ is Λ^2-valued and $*G_0'$ is Λ^3-valued.

Note that for any function $\varphi \in C_0^\infty(\partial\Omega, \mathcal{A}_3)$, we can write

(9.2.64)
$$\varphi = \varphi_0 + \varphi_1 + *\varphi_0' + *\varphi_1',$$

where φ_0 and $\varphi_0^{'}$ are Λ^0-valued, and φ_1 and $\varphi_1^{'}$ are Λ^1-valued. Applying the operator δ to both sides of the equality (9.2.64), we get

$$\delta\varphi = \delta\varphi_1 + *(d\varphi_1^{'}) - *(d\varphi_0^{'}). \tag{9.2.65}$$

Then, using the equality (9.2.63), we have

$$\int_{\partial\Omega} \langle G, \delta\varphi\rangle \, d\sigma = \int_{\partial\Omega} \langle \nu G_0, *(d\varphi_1^{'})\rangle \, d\sigma - \int_{\partial\Omega} \langle *G_1, *(d\varphi_0^{'})\rangle \, d\sigma. \tag{9.2.66}$$

By Lemma 4.2.5, we have $*(d\varphi_1^{'}) = \operatorname{curl}\varphi_1^{'}$, which further implies that

$$\begin{aligned}
\int_{\partial\Omega} \langle \nu G_0, *(d\varphi_1^{'})\rangle \, d\sigma &= \int_{\partial\Omega} \langle \nu, \operatorname{curl}\varphi_1^{'}\rangle G_0 \, d\sigma \\
&= -\int_{\partial\Omega} \operatorname{Div}(\nu \times \varphi_1^{'}) G_0 \, d\sigma \\
&= \int_{\partial\Omega} \langle \nu \times \varphi_1^{'}, \nabla G_0\rangle \, d\sigma \\
&= \int_{\partial\Omega} \langle *\varphi_1^{'}, *(\nu \times \nabla G_0)\rangle \, d\sigma.
\end{aligned} \tag{9.2.67}$$

On the other hand, we note that

$$\begin{aligned}
\int_{\partial\Omega} \langle *G_1, *(d\varphi_0^{'})\rangle \, d\sigma &= \int_{\partial\Omega} \langle G_1, d\varphi_0^{'}\rangle \, d\sigma \\
&= \int_{\partial\Omega} \langle G_1, \nabla\varphi_0^{'}\rangle \, d\sigma \\
&= -\int_{\partial\Omega} \langle \operatorname{Div} G_1, \varphi_0^{'}\rangle \, d\sigma \\
&= -\int_{\partial\Omega} \langle *(\operatorname{Div} G_1), *\varphi_0^{'}\rangle \, d\sigma.
\end{aligned} \tag{9.2.68}$$

Having the equalities (9.2.66), (9.2.67) and (9.2.68) together, we obtain

$$\begin{aligned}
&\int_{\partial\Omega} \langle G, \delta\varphi\rangle \, d\sigma \\
&\quad = \int_{\partial\Omega} \langle *\varphi_1', *(\nu \times \nabla G_0)\rangle \, d\sigma + \int_{\partial\Omega} \langle *\varphi_0', *(\operatorname{Div} G_1)\rangle \, d\sigma \\
&\quad = \int_{\partial\Omega} \langle \varphi, *(\nu \times \nabla G_0) + *(\operatorname{Div} G_1)\rangle \, d\sigma. \qquad (9.2.69)
\end{aligned}$$

Recall the definition of d_∂ in (5.1.11), we have

$$\int_{\partial\Omega} \langle d_\partial G, \varphi\rangle \, d\sigma = \int_{\partial\Omega} \langle G, \delta\varphi\rangle \, d\sigma. \qquad (9.2.70)$$

Comparing (9.2.69) with (9.2.70), we conclude that

$$d_\partial G = *(\nu \times \nabla G_0) + *(\operatorname{Div} G_1), \qquad (9.2.71)$$

where $*(\nu \times \nabla G_0)$ is Λ^2-valued and $*(\operatorname{Div} G_1)$ is Λ^3-valued.

Similarly, we have the decomposition of $\widetilde{G}$.

$$\widetilde{G} = \nu \widetilde{G}_0 + *\widetilde{G}_1 + *\widetilde{G}_0', \qquad (9.2.72)$$

where νG_0 is Λ^1-valued, $*G_1$ is Λ^2-valued and $*G_0'$ is Λ^3-valued. Then, by a similar fashion of proof, we have

$$d_\partial \widetilde{G} = *(\nu \times \nabla \widetilde{G}_0) + *(\operatorname{Div} \widetilde{G}_1), \qquad (9.2.73)$$

where $*(\nu \times \nabla \widetilde{G}_0)$ is Λ^2-valued and $*(\operatorname{Div} \widetilde{G}_1)$ is Λ^3-valued. Now we are ready to compute $d_\partial g + k\, e_4 g$. First, we have

$$d_\partial g = d_\partial (G + ie_4 \widetilde{G}) = d_\partial G - ie_4 d_\partial \widetilde{G}. \qquad (9.2.74)$$

Next, we observe

$$k\, e_4 g = k\, e_4(G + ie_4\widetilde{G}) = k\, e_4 G - ik\widetilde{G}. \tag{9.2.75}$$

On account of the equalities (9.2.74) and (9.2.75), we have

$$d_\partial g + k\, e_4 g = (d_\partial G - ik\widetilde{G}) - ie_4(d_\partial \widetilde{G} + ikG), \tag{9.2.76}$$

where $d_\partial G - ik\widetilde{G}$ and $d_\partial \widetilde{G} + ikG$ are $\mathcal{A}_3$-valued.

Now by the decompositions (9.2.71) and (9.2.72), we conclude that the Λ^0-component of $d_\partial G - ik\widetilde{G}$ is zero.

Also the Λ^1-component of $d_\partial G - ik\widetilde{G}$ is $-ik\,\nu\widetilde{G}_0$. As a consequence, the Λ^1-component of $d_\partial G - ik\widetilde{G} = 0$ if and only if $\widetilde{G}_0 = 0$, where $\widetilde{G}_0$ is the first boundary data in problem (9.1.58).

Moreover, the Λ^3-component of $d_\partial G - ik\widetilde{G}$ is $*\mathrm{Div}\, G_1 - ik * \widetilde{G}_0'$. Hence, the Λ^3-component of $d_\partial G - ik\widetilde{G} = 0$ if and only if $ik\widetilde{G}_0' - \mathrm{Div}\, G_1 = 0$. Note here $ik\widetilde{G}_0' - \mathrm{Div}\, G_1$ is the second boundary condition in (9.1.59).

Now from the observation of the decompositions (9.2.63) and (9.2.73), we have that the Λ^0-component of $d_\partial \widetilde{G} + ikG$ is zero. The Λ^1-component of $d_\partial \widetilde{G} + ikG$ is $ik\,\nu G_0$. This observation further implies that the Λ^1-component of $d_\partial \widetilde{G} + ikG$ is zero if and only if $G_0 = 0$, where G_0 is the first boundary condition in (9.1.57). At last, the Λ^3-component of $d_\partial \widetilde{G} + ikG$ is $*(\mathrm{Div}\, \widetilde{G}_1) + ik * G_0'$. Then $*(\mathrm{Div}\, \widetilde{G}_1) + ik * G_0'$ is zero if and only if $ikG_0' + \mathrm{Div}\, \widetilde{G}_1 = 0$, which is the second boundary condition in (9.1.60).

In summary, $d_\partial g + k\, e_4 g$ is $(\Lambda^2 + e_4\Lambda^2)$-valued if and only if the four boundary conditions that were mentioned above, i.e. the first boundary conditions of problems (9.1.57) and (9.1.58), the second boundary conditions of problems (9.1.59) and (9.1.60), are all zero. In a similar way, by decomposing the boundary data f in $L^{p,\delta}_{tan}(\partial\Omega, \mathcal{A}_4)$, one can check that $\delta_\partial f + k\, e_4 f$ is $(\Lambda^2 + e_4\Lambda^2)$-valued if and only if the other four boundary conditions of TBPVs (9.1.57)-(9.1.60) are zero.

Once all the boundary conditions in TBVPs (9.1.57), (9.1.58), (9.1.59) and (9.1.60) are zero, by the well-posedness of these problems, we can conclude that all the solutions of these four TBVPs are zero, which is going to give us the reduced Maxwell's equations (9.2.61) and (9.2.62).

This concludes the proof of Theorem 9.2.1. $\square$

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