POTENTIAL THEORY AND HARMONIC ANALYSIS METHODS FOR QUASILINEAR AND HESSIAN EQUATIONS

A Dissertation
presented to
the Faculty of the Graduate School
University of Missouri–Columbia

In Partial Fulfillment
of the Requirements for the Degree

Doctor of Philosophy

---

by
NGUYEN CONG PHUC

Dr. Igor E. Verbitsky, Dissertation Supervisor

MAY 2006

The undersigned, appointed by the Dean of the Graduate School, have examined the thesis entitled

POTENTIAL THEORY AND HARMONIC ANALYSIS METHODS
FOR QUASILINEAR AND HESSIAN EQUATIONS

Presented by Nguyen Cong Phuc

A candidate for the degree of Doctor of Philosophy

And hereby certify that in their opinion it is worthy of acceptance.

Professor Igor E. Verbitsky

Professor Loukas Grafakos

Professor Stephen Montgomery-Smith

Professor Konstantin Makarov

Professor Van Pham

ACKNOWLEDGMENTS

I wish to express my sincere gratitude to my teacher, Professor Igor E. Verbitsky, for introducing me to the subject and for his continuous advice, support and encouragement during this work.

I am also grateful to Professors Loukas Grafakos, Nigel Kalton, Konstantin Makarov and Stephen Montgomery-Smith for their interest and support.

Finally, I am indebted to the Department of Mathematics, University of Missouri at Columbia for its financial support and for facilitating my work.

POTENTIAL THEORY AND HARMONIC ANALYSIS METHODS
FOR QUASILINEAR AND HESSIAN EQUATIONS

Nguyen Cong Phuc

Dr. Igor E. Verbitsky, Dissertation Supervisor

ABSTRACT

The existence problem is solved, and global pointwise estimates of solutions are obtained for quasilinear and Hessian equations of Lane–Emden type, including the following two model problems:

$$-\Delta_p u = u^q + \mu, \qquad F_k[-u] = u^q + \mu, \qquad u \geq 0,$$

on $\mathbb{R}^n$, or on a bounded domain $\Omega \subset \mathbb{R}^n$. Here Δ_p is the p-Laplacian defined by $\Delta_p u = \operatorname{div}(\nabla u |\nabla u|^{p-2})$, and $F_k[u]$ is the k-Hessian defined as the sum of $k \times k$ principal minors of the Hessian matrix $D^2 u$ $(k = 1, 2, \ldots, n)$; μ is a nonnegative measurable function (or measure) on Ω.

The solvability of these classes of equations in the renormalized (entropy) or viscosity sense has been an open problem even for good data $\mu \in L^s(\Omega)$, $s > 1$. Such results are deduced from our existence criteria with the sharp exponents $s = \frac{n(q-p+1)}{pq}$ for the first equation, and $s = \frac{n(q-k)}{2kq}$ for the second one. Furthermore, a complete characterization of removable singularities for each corresponding homogeneous equation is given as a consequence of our solvability results.

TABLE OF CONTENTS

ACKNOWLEDGMENTS ii

ABSTRACT iii

1 Introduction 1

2 Preliminaries 15

2.1 Some notations 15

2.2 $\mathcal{A}$-superharmonic functions 17

2.3 k-convex functions 22

3 Renormalized solutions and global potential estimates 27

3.1 Global estimates for renormalized solutions 27

3.2 Global estimates for k-Hessian equations 36

4 Discrete models of nonlinear equations 41

4.1 Discrete Wolff type inequalities 42

4.2 Criteria for solvability 45

5 Quasilinear equations on $\mathbb{R}^n$ 51

5.1 Continuous Wolff type inequalities 51

5.2 Criteria for solvability 55

6 Quasilinear equations on bounded domains 65

5.1 Sharp integral estimates 65

5.2 Criteria for solvability 71

5.2 Removable singularities of $-\mathrm{div}\mathcal{A}(x, \nabla u) = u^q$ 76

7 Hessian equations 79

5.1 Hessian equations on $\mathbb{R}^n$ 79

5.2 Hessian equations on bounded domains 83

Appendix A Proof of Theorem 2.4 93

Appendix B Proof of Theorem 2.5 99

Bibliography 107

VITA 117

Chapter 1

Introduction

In this work, we study a class of quasilinear and fully nonlinear equations and inequalities with nonlinear source terms, which appear in such diverse areas as quasi-regular mappings, non-Newtonian fluids, elasticity, reaction-diffusion problems, and stochastic control. In particular, the following two model equations are of substantial interest:

$$-\Delta_p u = f(x,u), \qquad F_k[-u] = f(x,u), \tag{1.1}$$

on $\mathbb{R}^n$, or on a bounded domain $\Omega \subset \mathbb{R}^n$, where $f(x,u)$ is a nonnegative function, convex and nondecreasing in u for $u \geq 0$. Here $\Delta_p u = \operatorname{div}(\nabla u\,|\nabla u|^{p-2})$ is the p-Laplacian ($p > 1$), and $F_k[u]$ is the k-Hessian ($k = 1, 2, \ldots, n$) defined by

$$F_k[u] = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}, \tag{1.2}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the Hessian matrix $D^2 u$. In other words, $F_k[u]$ is the sum of the $k \times k$ principal minors of $D^2 u$, which coincides with the

Laplacian $F_1[u] = \Delta u$ if $k = 1$, and the Monge–Ampère operator $F_n[u] = \det(D^2 u)$ if $k = n$.

The form in which we write the second equation in (1.1) is chosen only for the sake of convenience, in order to emphasize the profound analogy between the quasilinear and Hessian equations. Obviously, it may be stated as $(-1)^k F_k[u] = f(x, u)$, $u \geq 0$, or $F_k[u] = f(x, -u)$, $u \leq 0$.

The existence and regularity theory, local and global estimates of sub- and super-solutions, the Wiener criterion, and Harnack's inequalities associated with the p-Laplacian, as well as more general quasilinear operators, can be found in [HKM], [IM], [KM2], [M1], [MZ], [S1], [S2], [SZ], [TW4] where many fundamental results, and relations to other areas of analysis and geometry are presented.

The theory of fully nonlinear equations of Monge-Ampère type which involve the k-Hessian operator $F_k[u]$ was originally developed by Caffarelli, Nirenberg and Spruck, Ivochkina, and Krylov in the classical setting. We refer to [CNS], [GT], [Gu], [Iv], [Kr], [Ur], [Tru3], [TW1] for these and further results. Recent developments concerning the notion of the k-Hessian measure, weak continuity, and pointwise potential estimates due to Trudinger and Wang [TW2]–[TW4], and Labutin [L] are used extensively in this thesis.

We are specifically interested in quasilinear and fully nonlinear equations of

Lane–Emden type:

$$(1.3) \qquad -\Delta_p u = u^q, \quad \text{and} \quad F_k[-u] = u^q, \qquad u \geq 0 \quad \text{in } \Omega,$$

where $p > 1$, $q > 0$, $k = 1, 2, \ldots, n$, and the corresponding nonlinear inequalities:

$$(1.4) \qquad -\Delta_p u \geq u^q, \quad \text{and} \quad F_k[-u] \geq u^q, \qquad u \geq 0 \quad \text{in } \Omega.$$

The latter can be stated in the form of the inhomogeneous equations with measure data,

$$(1.5) \qquad -\Delta_p u = u^q + \mu, \qquad F_k[-u] = u^q + \mu, \qquad u \geq 0 \quad \text{in } \Omega,$$

where μ is a nonnegative Radon measure on Ω.

The difficulties arising in studies of such equations and inequalities with competing nonlinearities are well known. In particular, (1.3) may have singular solutions [SZ]. The existence problem for (1.5) has been open ([BV2], Problems 1 and 2; see also [BV1], [BV3], [Gre]) even for the quasilinear equation $-\Delta_p u = u^q + f$ with good data $f \in L^s(\Omega)$, $s > 1$. Here solutions are generally understood in the renormalized (entropy) sense for quasilinear equations on bounded domains, potential-theoretic sense for quasilinear equations on $\mathbb{R}^n$, and viscosity, or k-convexity sense, for fully nonlinear equations of Hessian type (see [BMMP], [DMOP], [JLM], [KM1], [TW1]–[TW3], [Ur]).

In this thesis, we present a unified approach to (1.3)–(1.5) which makes it possible to attack a number of open problems. It is based on global pointwise

estimates, nonlinear integral inequalities in Sobolev spaces of fractional order, and analysis of dyadic models, along with the Hessian measure and weak continuity results [TW2]–[TW4]. The latter are used to bridge the gap between the dyadic models and partial differential equations. Some of these techniques were developed in the linear case, in the framework of Schrödinger operators and harmonic analysis [ChWW], [Fef], [KS], [NTV], [V1], [V2], and applications to semilinear equations [KV], [VW], [V3].

Our goal is to establish *necessary and sufficient* conditions for the existence of solutions to (1.5), sharp pointwise and integral estimates for solutions to (1.4), and a complete characterization of removable singularities for (1.3). We are mostly concerned with admissible solutions to the corresponding equations and inequalities. However, even for locally bounded solutions, as in [SZ], our results yield new pointwise and integral estimates, and Liouville-type theorems.

In the "linear case" $p = 2$ and $k = 1$, problems (1.3)–(1.5) with nonlinear sources are associated with the names of Lane and Emden, as well as Fowler. Authoritative historical and bibliographical comments can be found in [SZ]. An up-to-date survey of the vast literature on nonlinear elliptic equations with measure data is given in [Ver], including a thorough discussion of related work due to Adams and Pierre [AP], Baras and Pierre [BP], Berestycki, Capuzzo-Dolcetta, and Nirenberg [BCDN], Brezis and Cabré [BC], Kalton and Verbitsky [KV].

It is worth mentioning that related equations with absorption,

$$-\Delta u + u^q = \mu, \qquad u \geq 0 \quad \text{in } \Omega, \tag{1.6}$$

were studied in detail by Bénilan and Brezis, Baras and Pierre, and Marcus and Véron analytically for $1 < q < \infty$, and by Le Gall, and Dynkin and Kuznetsov using probabilistic methods when $1 < q \leq 2$ (see [D], [Ver]). For a general class of semilinear equations

$$-\Delta u + g(u) = \mu, \qquad u \geq 0 \quad \text{in } \Omega, \tag{1.7}$$

where g belongs to the class of continuous nondecreasing functions such that $g(0) = 0$, sharp existence results have been obtained quite recently by Brezis, Marcus, and Ponce [BMP]. It is well known that equations with absorption generally require "softer" methods of analysis, and the conditions on μ which ensure the existence of solutions are less stringent than in the case of equations with source terms.

Quasilinear problems of Lane–Emden type (1.3)–(1.5) have been studied extensively over the past 15 years. Universal estimates for solutions, Liouville-type theorems, and analysis of removable singularities are due to Bidaut-Véron, Mitidieri and Pohozaev [BV1]–[BV3], [BVP], [MP], and Serrin and Zou [SZ]. (See also [BiD], [Gre], [Ver], and the literature cited there.) The profound difficulties in this theory are highlighted by the presence of the two critical exponents,

$$q_* = \tfrac{n(p-1)}{n-p}, \qquad q^* = \tfrac{n(p-1)+p}{n-p}, \tag{1.8}$$

where $1 < p < n$. As was shown in [BVP], [MP], and [SZ], the quasilinear inequality (1.5) does not have nontrivial weak solutions on $\mathbb{R}^n$, or exterior domains, if $q \leq q_*$. For $q > q_*$, there exist $u \in W^{1,p}_{\text{loc}} \cap L^\infty_{\text{loc}}$ which obey (1.4), as well as singular solutions to (1.3) on $\mathbb{R}^n$. However, for the existence of nontrivial solutions $u \in W^{1,p}_{\text{loc}} \cap L^\infty_{\text{loc}}$ to (1.3) on $\mathbb{R}^n$, it is necessary and sufficient that $q \geq q^*$ [SZ]. In the "linear case" $p = 2$, this is classical [GS], [BP], [BCDN].

The following local estimates of solutions to quasilinear inequalities are used extensively in the studies mentioned above (see, e.g., [SZ], Lemma 2.4). Let B_R denote a ball of radius R such that $B_{2R} \subset \Omega$. Then, for every solution $u \in W^{1,p}_{\text{loc}} \cap L^\infty_{\text{loc}}$ to the inequality $-\Delta_p u \geq u^q$ in Ω,

$$\int_{B_R} u^\gamma \, dx \leq C \, R^{n - \frac{\gamma p}{q-p+1}}, \qquad 0 < \gamma < q, \tag{1.9}$$

$$\int_{B_R} |\nabla u|^{\frac{\gamma p}{q+1}} \, dx \leq C \, R^{n - \frac{\gamma p}{q-p+1}}, \qquad 0 < \gamma < q, \tag{1.10}$$

where the constants C in (1.9) and (1.10) depend only on p, q, n, γ. Note that (1.9) holds even for $\gamma = q$ (cf. [MP]), while (1.10) generally fails in this case. In what follows, we will substantially strengthen (1.9) in the end-point case $\gamma = q$, and obtain global pointwise estimates of solutions.

In [PV1], we proved that all compact sets $E \subset \Omega$ of zero Hausdorff measure, $H^{n - \frac{pq}{q-p+1}}(E) = 0$, are removable singularities for the equation $-\Delta_p u = u^q$, $q > q_*$. Earlier results of this kind, under a stronger restriction $\text{cap}_{1, \frac{pq}{q-p+1} + \epsilon}(E) = 0$ for some $\epsilon > 0$, are due to Bidaut-Véron [BV3]. Here $\text{cap}_{1,s}(\cdot)$ is the capacity

associated with the Sobolev space $W^{1,s}$.

In fact, much more is true. We will show below that a compact set $E \subset \Omega$ is a removable singularity for $-\Delta_p u = u^q$ if and only if it has zero fractional capacity: $\mathrm{cap}_{p,\frac{q}{q-p+1}}(E) = 0$. Here $\mathrm{cap}_{\alpha,s}$ stands for the Bessel capacity associated with the Sobolev space $W^{\alpha,s}$ which is defined in Section 2.1. We observe that the usual p-capacity $\mathrm{cap}_{1,p}$ used in the studies of the p-Laplacian [HKM], [KM2] plays a secondary role in the theory of equations of Lane–Emden type. Relations between these and other capacities used in nonlinear PDE are discussed in [AH], [M2], and [V4].

Our characterization of removable singularities is based on the solution of the existence problem for the equation

$$-\Delta_p u = u^q + \mu, \qquad u \geq 0, \tag{1.11}$$

with nonnegative measure μ obtained in Chapter 6. Main existence theorems for quasilinear equations are stated below (Theorems 5.4 and 6.5). Here we only mention the following corollary in the case $\Omega = \mathbb{R}^n$: If (1.11) has an admissible solution u, then

$$\int_{B_R} d\mu \leq C\, R^{n-\frac{pq}{q-p+1}}, \tag{1.12}$$

for every ball B_R in $\mathbb{R}^n$, where $C = C(p,q,n)$, provided $1 < p < n$ and $q > q_*$; if $p \geq n$ or $q \leq q_*$, then $\mu = 0$.

Conversely, suppose that $1 < p < n$, $q > q_*$, and $d\mu = f\,dx$, $f \geq 0$, where

$$\int_{B_R} f^{1+\epsilon}\,dx \leq C\,R^{n-\frac{(1+\epsilon)pq}{q-p+1}} \tag{1.13}$$

for some $\epsilon > 0$. Then there exists a constant $C_0(p,q,n)$ such that (1.11) has an admissible solution on $\mathbb{R}^n$ if $C \leq C_0(p,q,n)$.

The preceding inequality is an analogue of the classical Fefferman–Phong condition [Fef], which appeared in applications to Schrödinger operators. In particular, (1.13) holds if $f \in L^{n(q-p+1)/pq,\,\infty}(\mathbb{R}^n)$. Here $L^{s,\,\infty}$ stands for the weak L^s space. This sufficiency result, which to the best of our knowledge is new even in the L^s scale, provides a comprehensive solution to Problem 1 in [BV2]. Notice that the exponent $s = \frac{n(q-p+1)}{pq}$ is sharp. Broader classes of measures μ (possibly singular with respect to Lebesgue measure) which guarantee the existence of admissible solutions to (1.11) will be discussed in the sequel.

A substantial part of our work is concerned with integral inequalities for nonlinear potential operators, which are at the heart of our approach. We employ the notion of Wolff's potential introduced originally in [HW] in relation to the spectral synthesis problem for Sobolev spaces. For a nonnegative Radon measure μ on $\mathbb{R}^n$, $s \in (1,\,+\infty)$, and $\alpha > 0$, the Wolff's potential $\mathbf{W}_{\alpha,\,s}\,\mu$ is defined by

$$\mathbf{W}_{\alpha,\,s}\,\mu(x) = \int_0^\infty \Big[\frac{\mu(B_t(x))}{t^{n-\alpha s}}\Big]^{\frac{1}{s-1}}\frac{dt}{t}, \qquad x \in \mathbb{R}^n. \tag{1.14}$$

We write $\mathbf{W}_{\alpha,\,s}\,f$ in place of $\mathbf{W}_{\alpha,\,s}\,\mu$ if $d\mu = f dx$, where $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $f \geq 0$. When

dealing with equations in a bounded domain $\Omega \subset \mathbb{R}^n$, a truncated version is useful:

$$\mathbf{W}^r_{\alpha,\,s}\,\mu(x) = \int_0^r \left[\frac{\mu(B_t(x))}{t^{n-\alpha s}}\right]^{\frac{1}{s-1}} \frac{dt}{t}, \qquad x \in \Omega, \tag{1.15}$$

where $0 < r \leq 2\mathrm{diam}(\Omega)$. In many instances, it is more convenient to work with the dyadic version, also introduced in [HW]:

$$\mathcal{W}_{\alpha,\,s}\,\mu(x) = \sum_{Q\in\mathcal{D}} \left[\frac{\mu(Q)}{\ell(Q)^{n-\alpha s}}\right]^{\frac{1}{s-1}} \chi_Q(x), \qquad x \in \mathbb{R}^n, \tag{1.16}$$

where $\mathcal{D} = \{Q\}$ is the collection of the dyadic cubes $Q = 2^i(k + [0,\,1)^n)$, $i \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, and $\ell(Q)$ is the side length of Q.

An indispensable source on nonlinear potential theory is provided by [AH], where the fundamental Wolff's inequality and its applications are discussed. Very recently, an analogue of Wolff's inequality for general dyadic and radially decreasing kernels was obtained in [COV]; some of the tools developed there are employed below.

The dyadic Wolff's potentials appear in the following discrete model of (1.5) studied in Chapter 4:

$$u = \mathcal{W}_{\alpha,\,s}\,u^q + f, \qquad u \geq 0. \tag{1.17}$$

As it turns out, this nonlinear integral equation with $f = \mathcal{W}_{\alpha,\,s}\,\mu$ is intimately connected to the quasilinear differential equation (1.11) in the case $\alpha = 1$, $s = p$, and to its k-Hessian counterpart in the case $\alpha = \frac{2k}{k+1}$, $s = k+1$. Similar discrete

models are used extensively in harmonic analysis and function spaces (see, e.g., [NTV], [St2], [V1]).

The profound role of Wolff's potentials in the theory of quasilinear equations was discovered by Kilpeläinen and Malý [KM2]. They established local pointwise estimates for nonnegative p-superharmonic functions in terms of Wolff's potentials of the associated p-Laplacian measure μ. More precisely, if $u \geq 0$ is a p-superharmonic function in $B_{3R}(x)$ such that $-\Delta_p u = \mu$, then

$$C_1\, \mathbf{W}^R_{1,p}\, \mu(x) \leq u(x) \leq C_2 \inf_{B(x,R)} u + C_3\, \mathbf{W}^{2R}_{1,p}\, \mu(x), \tag{1.18}$$

where C_1, C_2 and C_3 are positive constants which depend only on n and p.

In [TW1], [TW2], Trudinger and Wang introduced the notion of the Hessian measure $\mu[u]$ associated with $F_k[u]$ for a k-convex function u. Very recently, Labutin [L] proved local pointwise estimates for Hessian equations analogous to (1.18), where the Wolff's potential $\mathbf{W}^R_{\frac{2k}{k+1},\, k+1}\, \mu$ is used in place of $\mathbf{W}^R_{1,p}\, \mu$.

In what follows, we will need *global* pointwise estimates of this type. In the case of a k-convex solution to the equation $F_k[u] = \mu$ on $\mathbb{R}^n$ such that $\inf_{x\in\mathbb{R}^n}(-u(x)) = 0$, one has

$$C_1\, \mathbf{W}_{\frac{2k}{k+1},\, k+1}\, \mu(x) \leq -u(x) \leq C_2\, \mathbf{W}_{\frac{2k}{k+1},\, k+1}\, \mu(x), \tag{1.19}$$

where C_1 and C_2 are positive constants which depend only on n and k. Analogous global estimates are obtained below for admissible solutions of the Dirichlet problem for $-\Delta_p u = \mu$ and $F_k[-u] = \mu$ in a bounded domain $\Omega \subset \mathbb{R}^n$ (see Chapter 3).

In the special case $\Omega = \mathbb{R}^n$, our criterion for the solvability of (1.11) can be stated in the form of the pointwise condition involving Wolff's potentials:

$$\mathbf{W}_{1,p}\,(\mathbf{W}_{1,p}\,\mu\,)^q\,(x) \le C\,\mathbf{W}_{1,p}\,\mu(x) < +\infty \quad \text{a.e.,} \tag{1.20}$$

which is necessary with $C = C_1(p, q, n)$, and sufficient with another constant $C = C_2(p, q, n)$. Moreover, in the latter case there exists an admissible solution u to (1.11) such that

$$c_1\,\mathbf{W}_{1,p}\,\mu(x) \le u(x) \le c_2\,\mathbf{W}_{1,p}\,\mu(x), \qquad x \in \mathbb{R}^n, \tag{1.21}$$

where c_1 and c_2 are positive constants which depend only on p, q, n, provided $1 < p < n$ and $q > q_*$; if $p \ge n$ or $q \le q_*$ then $u = 0$ and $\mu = 0$.

The iterated Wolff's potential condition (1.20) is crucial in our approach. As we will demonstrate in Chapter 5, it turns out to be equivalent to the fractional Riesz capacity condition

$$\mu(E) \le C\,\mathrm{Cap}_{p,\frac{q}{q-p+1}}\,(E), \tag{1.22}$$

where C does not depend on a compact set $E \subset \mathbb{R}^n$. Such classes of measures μ were introduced by V. Maz'ya in the early 60's in the framework of linear problems.

It follows that every admissible solution u to (1.11) on $\mathbb{R}^n$ obeys the inequality

$$\int_E u^q\,dx \le C\,\mathrm{Cap}_{p,\frac{q}{q-p+1}}\,(E) \tag{1.23}$$

for all compact sets $E \subset \mathbb{R}^n$. We also prove an analogous estimate in a bounded domain Ω (see Chapter 6). Obviously, this yields (1.9) in the end-point case $\gamma = q$.

In the critical case $q = q_*$, we obtain an improved estimate (see Corollary 6.2):

$$\int_{B_r} u^{q_*}\, dx \le C\, \left(\log(\tfrac{2R}{r})\right)^{\frac{1-p}{q-p+1}}, \tag{1.24}$$

for every ball B_r of radius r such that $B_r \subset B_R$, and $B_{2R} \subset \Omega$. Certain Carleson measure inequalities are employed in the proof of (1.24). We observe that these estimates yield Liouville-type theorems for all admissible solutions to (1.11) on $\mathbb{R}^n$, or in exterior domains, provided $q \le q_*$ (cf. [BVP], [SZ]).

Analogous results will be established in Chapter 7 for equations of Lane–Emden type involving the k-Hessian operator $F_k[u]$. We will prove that there exists a constant $C_1(k, q, n)$ such that, if

$$\mathbf{W}_{\frac{2k}{k+1},k+1}(\mathbf{W}_{\frac{2k}{k+1},k+1}\mu)^q(x) \le C\, \mathbf{W}_{\frac{2k}{k+1},k+1}\mu(x) < +\infty \text{ a.e.}, \tag{1.25}$$

where $0 \le C \le C_1(k, q, n)$, then the equation

$$F_k[-u] = u^q + \mu, \qquad u \ge 0, \tag{1.26}$$

has a solution u so that $-u$ is k-convex on $\mathbb{R}^n$, and

$$c_1\, \mathbf{W}_{\frac{2k}{k+1},k+1}\, \mu(x) \le u(x) \le c_2\, \mathbf{W}_{\frac{2k}{k+1},k+1}\, \mu(x), \qquad x \in \mathbb{R}^n, \tag{1.27}$$

where c_1, c_2 are positive constants which depend only on k, q, n, for $1 \le k < \frac{n}{2}$. Conversely, (1.25) with $C = C_2(k, q, n)$ is necessary in order that (1.26) has a solution u such that $-u$ is k-convex on $\mathbb{R}^n$ provided $1 \le k < \frac{n}{2}$ and $q > q_* = \frac{nk}{n-2k}$; if $k \ge \frac{n}{2}$ or $q \le q_*$ then $u = 0$ and $\mu = 0$.

In particular, (1.25) holds if $d\mu = f\,dx$, where $f \geq 0$ and $f \in L^{n(q-k)/2kq,\infty}(\mathbb{R}^n)$; the exponent $\frac{n(q-k)}{2kq}$ is sharp.

Also in Chapter 7, we will obtain precise existence theorems for equation (1.26) in a bounded domain Ω with the Dirichlet boundary condition $u = \varphi$, $\varphi \geq 0$, on $\partial\Omega$, for $1 \leq k \leq n$. Furthermore, removable singularities $E \subset \Omega$ for the homogeneous equation $F_k[-u] = u^q$, $u \geq 0$, will be characterized as the sets of zero Bessel capacity $\mathrm{cap}_{2k,\frac{q}{q-k}}(E) = 0$, in the most interesting case $q > k$.

The notion of the k-*Hessian* capacity introduced by Trudinger and Wang proved to be very useful in studies of the uniqueness problem for k-Hessian equations [TW3], as well as associated k-polar sets [L]. Comparison theorems for this capacity and the corresponding Hausdorff measures were obtained by Labutin in [L] where it is proved that the $(n-2k)$-Hausdorff dimension is critical in this respect. We will enhance this result (see Theorem 7.14) by showing that the k-Hessian capacity is in fact locally equivalent to the fractional Bessel capacity $\mathrm{cap}_{\frac{2k}{k+1},k+1}$.

The main results presented in this thesis is taken from the paper [PV2] and we remark that our methods provide a promising approach for a wide class of nonlinear problems, including curvature and subelliptic equations, and more general nonlinearities.

Finally, for the convenience of the reader we include at the end of this thesis two appendices. In Appendix A we give a detailed proof of the weak continuity of quasilinear elliptic operators due to Trudinger and Wang [TW4]. In Appendix

B we aslo follow [TW4] with some modifications to give a detailed proof of the pointwise potential estimate (1.18) for p-superharmonic functions originally found by Kilpeläinen and Malý [KM2].

Chapter 2

Preliminaries

2.1 Some notation

Throughout the thesis, $B_r(x)$ stands for an open ball in $\mathbb{R}^n$, $n \geq 2$, with center at x and with radius $r > 0$. We write $A \simeq B$ if there are constants c_1, c_2 such that $c_1 A \leq B \leq c_2 A$. By a Radon measure we mean a (signed) Borel regular (outer) measure which is finite on compact sets. The class of all nonnegative finite (respectively locally finite) Radon measures on an open set Ω is denoted by $\mathcal{M}_B^+(\Omega)$ (respectively $\mathcal{M}^+(\Omega)$). For a Radon measure μ and a Borel set $E \subset \Omega$, we denote by μ_E the restriction of μ to E: $d\mu_E = \chi_E d\mu$ where χ_E is the characteristic function of E. The closure, the boundary and the n-dimensional Lebesgue measure of E are denoted respectively by $\overline{E}$, ∂E and $|E|$. The notation $E \Subset \Omega$ means that $\overline{E}$ is a compact subset of Ω. The space of p-integrable functions on Ω with respect to Lebesgue measure is denoted by $L^p(\Omega)$. A function u belongs to the Sobolev space $W^{1,p}(\Omega)$ if u and all of its first weak partial derivatives belong to $L^p(\Omega)$. Local

versions of $L^p(\Omega)$ and $W^{1,p}(\Omega)$ are denoted respectively by $L^p_{\rm loc}(\Omega)$ and $W^{1,p}_{\rm loc}(\Omega)$. We define the Riesz potential $\mathbf{I}_\alpha$ of order α, $0<\alpha<n$, on $\mathbb{R}^n$ by

$$\mathbf{I}_\alpha\mu(x)=c(n,\alpha)\int_{\mathbb{R}^n}|x-y|^{\alpha-n}d\mu(y),\qquad x\in\mathbb{R}^n,$$

where $\mu\in\mathcal{M}^+(\mathbb{R}^n)$ and $c(n,\alpha)$ is a normalized constant. For $\alpha>0$, $p>1$, such that $\alpha p<n$, the Wolff's potential $\mathbf{W}_{\alpha,p}\mu$ is defined by

$$\mathbf{W}_{\alpha,p}\mu(x)=\int_0^\infty\Big[\frac{\mu(B_t(x))}{t^{n-\alpha p}}\Big]^{\frac{1}{p-1}}\frac{dt}{t},\qquad x\in\mathbb{R}^n.$$

When dealing with equations in a bounded domain $\Omega\subset\mathbb{R}^n$, it is convenient to use the truncated versions of Riesz and Wolff's potentials. For $0<r\le\infty$, $\alpha>0$ and $p>1$, we set

$$\mathbf{I}^r_\alpha\mu(x)=\int_0^r\frac{\mu(B_t(x))}{t^{n-\alpha}}\frac{dt}{t},\qquad \mathbf{W}^r_{\alpha,p}\mu(x)=\int_0^r\Big[\frac{\mu(B_t(x))}{t^{n-\alpha p}}\Big]^{\frac{1}{p-1}}\frac{dt}{t}.$$

Here $\mathbf{I}^\infty_\alpha$ and $\mathbf{W}^\infty_{\alpha,p}$ are understood as $\mathbf{I}_\alpha$ and $\mathbf{W}_{\alpha,p}$ respectively. For $\alpha>0$, we denote by $\mathbf{G}_\alpha$ the Bessel kernel of order α (see [AH], Sec. 1.2.4). The Bessel potential of a measure $\mu\in\mathcal{M}^+(\mathbb{R}^n)$ is defined by

$$\mathbf{G}_\alpha\mu(x)=\int_{\mathbb{R}^n}\mathbf{G}_\alpha(x-y)d\mu(y),\qquad x\in\mathbb{R}^n.$$

Various capacities will be used throughout this work. Among them are the Riesz and Bessel capacities defined respectively by

$$\mathrm{Cap}_{\mathbf{I}_\alpha,s}(E)=\inf\{||f||^s_{L^s(\mathbb{R}^n)}:\mathbf{I}_\alpha f\ge\chi_E,\ 0\le f\in L^s(\mathbb{R}^n)\},$$

and

$$\mathrm{Cap}_{\mathbf{G}_\alpha,\,s}(E) = \inf\{||f||^s_{L^s(\mathbb{R}^n)} : \mathbf{G}_\alpha f \geq \chi_E,\ 0 \leq f \in L^s(\mathbb{R}^n)\}$$

for any set $E \subset \mathbb{R}^n$.

2.2 $\mathcal{A}$-superharmonic functions

In this section, we collect some crucial facts on $\mathcal{A}$-superharmonic functions from [HKM], [KM1], [KM2], and [TW4] for our later use. Let Ω be an arbitrary open set in $\mathbb{R}^n$, and let $p > 1$ though we will mainly be interested in the case where Ω is bounded and $1 < p \leq n$, or $\Omega = \mathbb{R}^n$ and $1 < p < n$. We assume that $\mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a vector valued mapping which satisfies the following structural conditions:

$$\text{the mapping } x \mapsto \mathcal{A}(x,\xi) \text{ is measurable for all } \xi \in \mathbb{R}^n, \tag{2.1}$$

$$\text{the mapping } \xi \mapsto \mathcal{A}(x,\xi) \text{ is continuous for a.e. } x \in \mathbb{R}^n, \tag{2.2}$$

and there are constants $0 < \alpha \leq \beta < \infty$ such that for a.e. x in $\mathbb{R}^n$, and for all ξ in $\mathbb{R}^n$,

$$\mathcal{A}(x,\xi)\cdot\xi \geq \alpha|\xi|^p, \quad |\mathcal{A}(x,\xi)| \leq \beta|\xi|^{p-1}, \tag{2.3}$$

$$[\mathcal{A}(x,\xi_1) - \mathcal{A}(x,\xi_2)]\cdot(\xi_1 - \xi_2) > 0, \quad \text{if } \xi_1 \neq \xi_2, \tag{2.4}$$

$$\mathcal{A}(x,\lambda\xi) = \lambda|\lambda|^{p-2}\mathcal{A}(x,\xi), \quad \text{if } \lambda \in \mathbb{R}\setminus\{0\}. \tag{2.5}$$

For $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, we define the divergence of $\mathcal{A}(x,\nabla u)$ in the sense of distri-

butions, i.e., if $\varphi \in C_0^\infty(\Omega)$, then

$$\mathrm{div}\mathcal{A}(x, \nabla u)(\varphi) = -\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla\varphi \, dx.$$

It is well-known that every solution $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ to the equation

$$-\mathrm{div}\mathcal{A}(x, \nabla u) = 0 \tag{2.6}$$

has a continuous representative. Such continuous solutions are said to be $\mathcal{A}$-*harmonic* in Ω. If $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla\varphi \, dx \geq 0$$

for all nonnegative $\varphi \in C_0^\infty(\Omega)$, i.e., $-\mathrm{div}\mathcal{A}(x, \nabla u) \geq 0$ in the distributional sense, then u is called a *supersolution* to (2.6) in Ω.

A lower semicontinuous function $u : \Omega \to (-\infty, \infty]$ is called $\mathcal{A}$-*superharmonic* if u is not identically infinite in each component of Ω, and if for all open sets D such that $\overline{D} \subset \Omega$, and all functions $h \in C(\overline{D})$, $\mathcal{A}$-harmonic in D, it follows that $h \leq u$ on ∂D implies $h \leq u$ in D.

In the special case $\mathcal{A}(x, \xi) = |\xi|^{p-2}\xi$, $\mathcal{A}$-superharmonicity is referred to as p-superharmonicity. It is worth mentioning that the latter can also be defined equivalently using the language of viscosity solutions (see [JLM]).

We recall here the fundamental connection between supersolutions of (2.6) and $\mathcal{A}$-superharmonic functions [HKM].

Proposition 2.1 ([HKM]). (i) *If v is $\mathcal{A}$-superharmonic on Ω then*

$$v(x) = \operatorname{ess}\liminf_{y\to x} v(y), \qquad x \in \Omega. \tag{2.7}$$

Moreover, if $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ then

$$-\mathrm{div}\mathcal{A}(x, \nabla v) \geq 0.$$

(ii) *If $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is such that*

$$-\mathrm{div}\mathcal{A}(x, \nabla u) \geq 0,$$

then there is an $\mathcal{A}$-superharmonic function v such that $u = v$ a.e.

(iii) *If v is $\mathcal{A}$-superharmonic and locally bounded, then $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and*

$$-\mathrm{div}\mathcal{A}(x, \nabla v) \geq 0.$$

¿From statement (i) of Proposition 2.1 we see that if u and v are two $\mathcal{A}$-superharmonic functions on Ω such that $u \leq v$ a.e. on Ω then $u \leq v$ everywhere on Ω.

Note that an $\mathcal{A}$-superharmonic function u does not necessarily belong to the space $W^{1,p}_{\mathrm{loc}}(\Omega)$, but its truncation $\min\{u, k\}$ does for every integer k by Proposition 2.1(iii). Using this we set

$$Du = \lim_{k\to\infty} \nabla\,[\min\{u, k\}], \tag{2.8}$$

defined a.e. If either $u \in L^\infty(\Omega)$ or $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, then Du coincides with the regular distributional gradient of u. In general, we have the following gradient estimates [KM1] (see also [HKM], [TW4], and Theorem A.2 in Appendix A).

Proposition 2.2 ([KM1]). *Suppose u is $\mathcal{A}$-superharmonic in Ω and $1 \le q < \frac{n}{n-1}$. Then both $|Du|^{p-1}$ and $\mathcal{A}(\cdot, Du)$ belong to $L^q_{\mathrm{loc}}(\Omega)$. Moreover, if $p > 2 - \frac{1}{n}$, then Du is the distributional gradient of u.*

We can now extend the definition of the divergence of $\mathcal{A}(x, \nabla u)$ to those u which are merely $\mathcal{A}$-superharmonic in Ω. For such u we set

$$-\mathrm{div}\mathcal{A}(x, \nabla u)(\varphi) = \int_\Omega \mathcal{A}(x, Du) \cdot \nabla\varphi \, dx$$

for all $\varphi \in C_0^\infty(\Omega)$. Note that by Proposition 2.2 and dominated convergence theorem,

$$-\mathrm{div}\mathcal{A}(x, \nabla u)(\varphi) = \lim_{k\to\infty} \int_\Omega \mathcal{A}(x, \nabla \min\{u, k\}) \cdot \nabla\varphi \, dx \ge 0$$

whenever $\varphi \in C_0^\infty(\Omega)$ and $\varphi \ge 0$.

Since $-\mathrm{div}\mathcal{A}(x, \nabla u)$ is a nonnegative distribution in Ω for an $\mathcal{A}$-superharmonic u, it follows that there is a positive (not necessarily finite) Radon measure denoted by $\mu[u]$ such that

$$-\mathrm{div}\mathcal{A}(x, \nabla u) = \mu[u] \quad \text{in} \quad \Omega$$

in the sense

$$\int_\Omega \mathcal{A}(x, Du) \cdot \nabla\varphi = \int_\Omega \varphi d\mu[u] \tag{2.9}$$

for every $\varphi \in C_0^\infty(\Omega)$.

Conversely, given a nonnegative finite Radon measure μ in a bounded domain Ω, there is an $\mathcal{A}$-superharmonic function u such that $-\mathrm{div}\mathcal{A}(x, \nabla u) = \mu$ in Ω and $\min\{u, k\} \in W_0^{1,p}(\Omega)$ for all integers k (see [KM1]).

Remark 2.3. For any subdomain $\omega \subset \Omega$ with smooth boundary, $\mu[u](\omega)$ depends only on the value of u near $\partial\omega$. Indeed, by (2.9),

$$\begin{aligned} \mu[u](\omega) &= \int_\omega d\mu[u] \\ &= \lim_{\epsilon\to 0} \int_\omega \varphi_\epsilon d\mu[u] \\ &= \lim_{\epsilon\to 0} \int_{\mathrm{supp}(\nabla\varphi_\epsilon)} \mathcal{A}(x, Du)\nabla\varphi_\epsilon, \end{aligned}$$

where $\varphi_\epsilon \in C_0^\infty(\omega)$ such that $\varphi_\epsilon = 1$ in $\{x \in \omega : d(x, \partial\omega) > \epsilon\}$.

The following weak continuity result in [TW4] will be used later to prove the existence of $\mathcal{A}$-superharmonic solutions to quasilinear equations.

Theorem 2.4 ([TW4])**.** *Let $\{u_n\}$ be a sequence of nonnegative $\mathcal{A}$-superharmonic functions in Ω that converges a.e. to an $\mathcal{A}$-superharmonic function u. Then the sequence of measures $\{\mu[u_n]\}$ converges to $\mu[u]$ weakly, i.e.,*

$$\lim_{n\to\infty} \int_\Omega \varphi\, d\mu[u_n] = \int_\Omega \varphi\, d\mu[u]$$

for all $\varphi \in C_0^\infty(\Omega)$.

In [KM2] (see also [Mi], [MZ], [TW4]) the following pointwise potential estimate for $\mathcal{A}$-superharmonic functions was established, which serves as a major tool in our study of quasilinear equations.

Theorem 2.5 ([KM2]). *Suppose $u \geq 0$ is an $\mathcal{A}$-superharmonic function in $B_{3R}(x)$. If $\mu = -\mathrm{div}\mathcal{A}(x, \nabla u)$, then there are positive constants C_1, C_2 and C_3 which depend only on n, p and the structural constants α, β such that*

$$C_1 \, \mathbf{W}^R_{1,p} \, \mu(x) \leq u(x) \leq C_2 \, \mathbf{W}^{2R}_{1,p} \, \mu(x) + C_3 \inf_{B(x,R)} u. \tag{2.10}$$

We will present the proofs of Theorems 2.4 and 2.5 in Appendices A and B below following a recent paper of Trudinger and Wang [TW4].

A consequence of Theorem 2.5 is the following global version of the above potential pointwise estimate.

Corollary 2.6 ([KM2]). *Let u be an $\mathcal{A}$-superharmonic function in $\mathbb{R}^n$ such that $\inf_{\mathbb{R}^n} u = 0$. If $\mu = -\mathrm{div}\mathcal{A}(x, \nabla u)$, then*

$$C_1 \, \mathbf{W}_{1,p}\mu(x) \leq u(x) \leq C_2 \, \mathbf{W}_{1,p}\mu(x)$$

for all $x \in \mathbb{R}^n$, where C_1, C_2 are positive constant depending only on n, p and the structural constants α, β.

2.3 k-convex functions

The notion of k-convex (k-subharmonic) functions associated with the fully nonlinear k-Hessian operator F_k, $k = 1, \ldots, n$, introduced recently by Trudinger and Wang in [TW1]–[TW3] plays a role similar to that of $\mathcal{A}$-superharmonic functions in the quasilinear theory discussed in the previous section.

Let Ω be an open set in $\mathbb{R}^n$, $n \geq 2$. For $k = 1, \ldots, n$ and $u \in C^2(\Omega)$, the k-Hessian operator F_k is defined by

$$F_k[u] = S_k(\lambda(D^2u)),$$

where $\lambda(D^2u) = (\lambda_1, \ldots, \lambda_n)$ denotes the eigenvalues of the Hessian matrix of second partial derivatives D^2u, and S_k is the kth symmetric function on $\mathbb{R}^n$ given by

$$S_k(\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}.$$

Thus $F_1[u] = \Delta u$ and $F_n[u] = \det D^2u$. Alternatively, we may also write

$$F_k[u] = [D^2u]_k,$$

where for an $n \times n$ matrix A, $[A]_k$ is the k-trace of A, i.e., the sum of its $k \times k$ principal minors. Several equivalent definitions of k-convexity were given in [TW2], one of which involves the language of viscosity solutions: An upper-semicontinuous function $u : \Omega \to [-\infty, \infty)$ is said to be k-convex in Ω, $1 \leq k \leq n$, if $F_k[q] \geq 0$ for any quadratic polynomial q such that $u - q$ has a local finite maximum in Ω. Equivalently, an upper-semicontinuous function $u : \Omega \to [-\infty, \infty)$ is k-convex in Ω if, for every open set $\Omega' \Subset \Omega$ and for every function $v \in C^2(\Omega') \cap C^0(\overline{\Omega'})$ satisfying $F_k[v] \geq 0$ in Ω', the following implication holds:

$$u \leq v \text{ on } \partial\Omega' \Longrightarrow u \leq v \text{ in } \Omega',$$

(see [TW2, Lemma 2.1]). Note that a function $u \in C^2(\Omega)$ is k-convex if and only if

$$F_j[u] \geq 0 \text{ in } \Omega \text{ for all } j = 1, \ldots, k.$$

We denote by $\Phi^k(\Omega)$ the class of all k-convex functions in Ω which are not identically equal to $-\infty$ in each component of Ω. It was proven in [TW2] that $\Phi^n(\Omega) \subset \Phi^{n-1}(\Omega) \cdots \subset \Phi^1(\Omega)$ where $\Phi^1(\Omega)$ coincides with the set of all proper classical subharmonic functions in Ω, and $\Phi^n(\Omega)$ is the set of functions convex on each component of Ω.

The following weak convergence result proved in [TW2] is fundamental to potential theory associated with k-Hessian operators.

Theorem 2.7 ([TW2])**.** *For each $u \in \Phi^k(\Omega)$, there exists a nonnegative Radon measure $\mu_k[u]$ in Ω such that*

(i) *$\mu_k[u] = F_k[u]$ for $u \in C^2(\Omega)$, and*

(ii) *if $\{u_m\}$ is a sequence in $\Phi^k(\Omega)$ converging in $L^1_{\text{loc}}(\Omega)$ to a function $u \in \Phi^k(\Omega)$, then the sequence of the corresponding measures $\{\mu_k[u_m]\}$ converges weakly to $\mu_k[u]$.*

The measure $\mu_k[u]$ in the theorem above is called the k-Hessian measure associated with u. Due to (i) in Theorem 2.7 we sometimes write $F_k[u]$ in place of $\mu_k[u]$ even in the case where $u \in \Phi^k(\Omega)$ does not belong to $C^2(\Omega)$. The k-Hessian measure is an important tool in potential theory for $\Phi^k(\Omega)$. It was used by Labutin

to derive pointwise estimates for functions in $\Phi^k(\Omega)$ in terms of the Wolff's potential, which is an analogue of the Wolff's potential estimates for $\mathcal{A}$-superharmonic functions considered in Theorem 2.5.

Theorem 2.8 ([L]). *Let $u \geq 0$ be such that $-u \in \Phi^k(B_{3R}(x))$, where $1 \leq k \leq n$. If $\mu = \mu_k[-u]$ then*

$$C_1 \mathbf{W}^{\frac{R}{8}}_{\frac{2k}{k+1}, k+1}\mu(x) \leq u(x) \leq C_2 \mathbf{W}^{2R}_{\frac{2k}{k+1}, k+1}\mu(x) + C_3 \inf_{B_R(x)} u,$$

where the constants C_1, C_2 and C_3 depend only on n and k.

The following global estimate is deduced from the preceding theorem as in the quasilinear case.

Corollary 2.9. *Let $u \geq 0$ be such that $-u \in \Phi^k(\mathbb{R}^n)$, where $1 \leq k < \frac{n}{2}$. If $\mu = \mu_k[-u]$ and $\inf_{\mathbb{R}^n} u = 0$ then for all $x \in \mathbb{R}^n$,*

$$C_1 \, \mathbf{W}_{\frac{2k}{k+1}, k+1}\mu(x) \leq u(x) \leq C_2 \, \mathbf{W}_{\frac{2k}{k+1}, k+1}\mu(x)$$

for constants C_1, C_2 depending only on n and k.

Chapter 3

Renormalized solutions and global potential estimates

3.1 Global estimates for renormalized solutions

Let Ω be a bounded, open subset of $\mathbb{R}^n$, $n \geq 2$. We denote by $\mathcal{M}_B(\Omega)$ the set of all Radon measures on Ω with bounded total variations. Recall that $\mathcal{M}_B^+(\Omega)$ denotes the set of all nonnegative finite Radon measures on Ω. For a measure μ in $\mathcal{M}_B(\Omega)$, its positive and negative parts are denoted by μ^+ and μ^- respectively. We say that a sequence of measures $\{\mu_n\}$ in $\mathcal{M}_B(\Omega)$ converges in the narrow topology to $\mu \in \mathcal{M}_B(\Omega)$ if

$$\lim_{n\to\infty} \int_\Omega \varphi \, d\mu_n = \int_\Omega \varphi \, d\mu$$

for every bounded and continuous function φ on Ω.

We also denote by $\mathcal{M}_0(\Omega)$ (respectively $\mathcal{M}_s(\Omega)$) the set of all measures in $\mathcal{M}_B(\Omega)$ which are absolutely continuous (respectively singular) with respect to the capacity $\mathrm{cap}_{1,p}(\cdot, \Omega)$. Here $\mathrm{cap}_{1,p}(\cdot, \Omega)$ is the capacity relative to the domain Ω

defined by

$$\text{(3.1)} \qquad \mathrm{cap}_{1,p}(E,\Omega) = \inf\Big\{ \int_\Omega |\nabla\phi|^p dx : \phi \in C_0^\infty(\Omega), \phi \geq 1 \text{ on } E \Big\}$$

for any compact set $E \subset \Omega$. Recall that, for every measure μ in $\mathcal{M}_B(\Omega)$, there exists a unique pair of measures (μ_0, μ_s) with $\mu_0 \in \mathcal{M}_0(\Omega)$ and $\mu_s \in \mathcal{M}_s(\Omega)$, such that $\mu = \mu_0 + \mu_s$. If μ is nonnegative, then so are μ_0 and μ_s (see [FST], Lemma 2.1).

For $k > 0$ and for $s \in \mathbb{R}$ we denote by $T_k(s)$ the truncation

$$T_k(s) = \max\{-k, \min\{k, s\}\}.$$

Recall also from [BBG] that if u is a measurable function on Ω which is finite almost everywhere and satisfies $T_k(u) \in W_0^{1,p}(\Omega)$ for every $k > 0$, then there exists a measurable function $v : \Omega \to \mathbb{R}^n$ such that

$$\nabla T_k(u) = v\chi_{\{|u|<k\}} \quad \text{a.e.} \quad \text{on} \quad \Omega, \quad \text{for all } k > 0.$$

Moreover, v is unique up to almost everywhere equivalence. We define the gradient Du of u as this function v, and set $Du = v$.

In [DMOP], several equivalent definitions of renormalized solutions are given. In what follows, we will need the following ones.

Definition 3.1. Let $\mu \in \mathcal{M}_B(\Omega)$. Then u is said to be a renormalized solution of

$$\text{(3.2)} \qquad \begin{cases} -\mathrm{div}\mathcal{A}(x, \nabla u) = \mu \quad \text{in} \quad \Omega, \\ \qquad u = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

if the following conditions hold:

(a) The function u is measurable and finite almost everywhere, and $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$ for every $k > 0$.

(b) The gradient Du of u satisfies $|Du|^{p-1} \in L^q(\Omega)$ for all $q < \frac{n}{n-1}$.

(c) If w belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and if there exist $w^{+\infty}$ and $w^{-\infty}$ in $W^{1,r}(\Omega) \cap L^\infty(\Omega)$, with $r > n$, such that

$$\begin{cases} w = w^{+\infty} & \text{a.e. on the set } \{u > k\}, \\ w = w^{-\infty} & \text{a.e. on the set } \{u < -k\} \end{cases}$$

for some $k > 0$ then

$$\int_\Omega \mathcal{A}(x, Du) \cdot \nabla w dx = \int_\Omega w d\mu_0 + \int_\Omega w^{+\infty} d\mu_s^+ - \int_\Omega w^{-\infty} d\mu_s^-. \tag{3.3}$$

Definition 3.2. Let $\mu \in \mathcal{M}_B(\Omega)$. Then u is a renormalized solution of (3.2) if u satisfies (a) and (b) in Definition 3.1, and if the following conditions hold:

(c) For every $k > 0$ there exist two nonnegative measures in $\mathcal{M}_0(\Omega)$, λ_k^+ and λ_k^-, concentrated on the sets $\{u = k\}$ and $\{u = -k\}$, respectively, such that $\lambda_k^+ \to \mu_s^+$ and $\lambda_k^- \to \mu_s^-$ in the narrow topology of measures.

(d) For every $k > 0$

$$\int_{\{|u|<k\}} \mathcal{A}(x, Du) \cdot \nabla \varphi dx = \int_{\{|u|<k\}} \varphi d\mu_0 + \int_\Omega \varphi d\lambda_k^+ - \int_\Omega \varphi d\lambda_k^- \tag{3.4}$$

for every φ in $\mathrm{W}_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Remark 3.3. From Remark 2.18 in [DMOP] we see that if u is a renormalized

solution of (3.2) then (the $\mathrm{cap}_{1,p}$-quasi continuous representative of) u is finite $\mathrm{cap}_{1,p}$-quasieverywhere. Therefore, u is finite μ_0-almost everywhere.

Remark 3.4. By (3.4), if u is a renormalized solution of (3.2) then

$$-\mathrm{div}\mathcal{A}(x, \nabla T_k(u)) = \mu_k \quad \text{in} \quad \Omega, \tag{3.5}$$

where

$$\mu_k = \chi_{\{|u|<k\}}\mu_0 + \lambda_k^+ - \lambda_k^-.$$

Since $T_k(u) \in W_0^{1,p}(\Omega)$, by (2.3) we see that $-\mathrm{div}\mathcal{A}(x, \nabla T_k(u))$ and hence μ_k belongs to the dual space $W^{-1,p'}(\Omega)$ of $W_0^{1,p}(\Omega)$. Moreover, by Remark 3.3, $|u| < \infty$ μ_0-almost everywhere and hence $\chi_{\{|u|<k\}} \to \chi_\Omega$ μ_0-almost everywhere as $k \to \infty$. Therefore, by the monotone convergence theorem, μ_k converges to μ in the narrow topology of measures.

Remark 3.5. If $\mu \geq 0$, i.e., $\mu \in \mathcal{M}_B^+(\Omega)$, and u is a renormalized solution of (3.2) then u is nonnegative. To see this, for each $k > 0$ we "test" (3.3) with $w = T_k(\min\{u, 0\})$, $w^{+\infty} = 0$ and $w^{-\infty} = -k$:

$$\int_\Omega \mathcal{A}(x, Du) \cdot \nabla w dx = \int_\Omega w d\mu_0 + \int_\Omega k d\mu_s^- = \int_\Omega w d\mu_0 \leq 0,$$

since $\mu_s^- = 0$ and $w \leq 0$. Thus using (2.3) we get

$$\int_\Omega |\nabla T_k(\min\{u, 0\})|^p dx \leq 0$$

for every $k > 0$. Therefore $\min\{u, 0\} = 0$ a.e., i.e., u is nonnegative.

Remark 3.6. Let $\mu \in \mathcal{M}_B^+(\Omega)$ and let u be a renormalized solution of (3.2). Since $\min\{u, 0\} = 0$ a.e. (by Remark 3.5) and hence $\min\{u, 0\} = 0$ $\mathrm{cap}_{1,p}$-quasi everywhere (see [HKM], Theorem 4.12), in Remark 3.4 we may take $\lambda_k^- = 0$, and thus μ_k is nonnegative. Hence by (3.5) and Proposition 2.1, the functions v_k defined by $v_k(x) = \operatorname{ess\,lim\,inf}_{y \to x} T_k(u)(y)$ are $\mathcal{A}$-superharmonic and increasing. Using Lemma 7.3 in [HKM], it is then easy to see that $v_k \to v$ as $k \to \infty$ everywhere in Ω for some $\mathcal{A}$-superharmonic function v on Ω such that $v = u$ a.e. In other words, v is an $\mathcal{A}$-superharmonic representative of u.

Remark 3.7. When dealing with pointwise values of a renormalized solution u to (3.2) with measure data $\mu \geq 0$, we always identify u with its $\mathcal{A}$-superharmonic representative mentioned in Remark 3.6.

We now establish a comparison principle for renormalized solutions.

Lemma 3.8. *Let μ, $\nu \in \mathcal{M}_B^+(\Omega)$ be such that $\mu \leq \nu$. Suppose that u and v are renormalized solutions of*

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, \nabla u) = \mu & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, \nabla v) = \nu & \text{in} \quad \Omega, \\ v = 0 & \text{on} \quad \partial\Omega \end{cases}$$

respectively. If u is uniformly bounded then $u \leq v$.

Proof. Let $w = \min\{(u-v)^+, k\}$. Then $w = 0$ on the set $\{v > k + M\}$ and $w = k$ on the set $\{v < -k - M\}$, where $M = \sup_\Omega u$. Moreover, $w \in W_0^{1,p} \cap L^\infty(\Omega)$ as $w = \min\{(u - T_{k+M}(v))^+, k\}$. Thus by Definition 3.1 we have

$$\int_\Omega \mathcal{A}(x, Dv) \cdot \nabla w dx = \int_\Omega w d\nu_0. \tag{3.6}$$

On the other hand, since u is bounded (hence belongs to $W_0^{1,p}(\Omega)$) we have

$$\int_\Omega \mathcal{A}(x, Du) \cdot \nabla w dx = \int_\Omega w d\mu. \tag{3.7}$$

From (3.6) and (3.7) we get

$$\int_\Omega [\mathcal{A}(x, Du) - \mathcal{A}(x, Dv)] \cdot \nabla w dx \leq 0.$$

Consequently,

$$\int_{0<u-v<k} [\mathcal{A}(x, Du) - \mathcal{A}(x, Dv)] \cdot (Du - Dv) dx \leq 0,$$

since $\nabla w = \nabla \max\{T_k(u-v), 0\} = D(u-v)\chi_{\{0<u-v<k\}}$. Thus by (2.4) we have $\nabla w = 0$ and hence $w = 0$ a.e. for every $k > 0$, which gives $u \leq v$. □

In the following lemma we drop the assumption that u is uniformly bounded in Lemma 3.8, but claim only the existence of v such that $v \geq u$.

Lemma 3.9. *Let μ, $\nu \in \mathcal{M}_B^+(\Omega)$ be such that $\nu \geq \mu$. Suppose that u is a renormalized solution of*

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, \nabla u) = \mu \quad \text{in} \quad \Omega, \\ \qquad u = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Then there exists $v \geq u$ such that

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, \nabla v) = \nu \quad \text{in} \quad \Omega, \\ v = 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

in the renormalized sense.

Proof. Let $u_k = \min\{u, k\}$ for each $k \in \mathbb{N}$. From Definition 3.2 of renormalized solutions we have

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, \nabla u_k) = \mu_{0\{u<k\}} + \lambda_k^+ \quad \text{in} \quad \Omega, \\ u_k = 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

in the renormalized sense for a sequence of nonnegative measures $\{\lambda_k^+\}$ that converges to μ_s^+ in the narrow topology of measures. Thus by Lemma 3.8 we have $u_k \leq v_k$, where v_k are renormalized solutions of

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, \nabla v_k) = \mu_0 + \lambda_k^+ + \nu - \mu \quad \text{in} \quad \Omega, \\ v_k = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Finally, from the stability results in [DMOP] we can find a subsequence of $\{v_k\}$ that converges a.e. to a required function v. □

We will also need the following variant of Lemma 3.9.

Lemma 3.10. *Suppose that u is a renormalized solution to (3.2) with data $\mu \in \mathcal{M}_B^+(\Omega)$. Let B be a ball that contains Ω. Then there exists a function w on B such that $u \leq w$ on Ω, and*

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, \nabla w) = \mu \quad \text{in} \quad B, \\ w = 0 \quad \text{on} \quad \partial B \end{cases} \tag{3.8}$$

in the renormalized sense.

Proof. Let $u_k = \min\{u, k\}$, $k > 0$, and let $\mu_k = \chi_{\{u<k\}}\mu_0 + \lambda_k^+$ be as in Remark 3.4 (note that $\lambda_k^- = 0$ by Remark 3.6). We see that $u_k \in W_0^{1,p}(\Omega)$ is the unique solution of problem (3.2) with data μ_k. We next extend u_k by zero outside Ω, and set

$$\Psi = \min\{w_k - u_k, 0\} = \min\{\min\{w_k, k\} - u_k, 0\},$$

where w_k, $k > 0$, is a renormalized solution to the problem

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, \nabla w_k) = \mu_0 + \lambda_k^+ & \text{in} \quad B, \\ w_k = 0 \quad \text{on} \quad \partial B. \end{cases}$$

Note that $\Psi \in W_0^{1,p}(\Omega) \cap W_0^{1,p}(B) \cap L^\infty(B)$ since $|\Psi| \leq u_k$. Then using Ψ as a test function we have

$$\begin{aligned} 0 &\geq \int_B \mathcal{A}(x, \nabla w_k) \cdot \nabla\Psi dx - \int_\Omega \mathcal{A}(x, \nabla u_k) \cdot \nabla\Psi dx \\ &= \int_{B\cap\{w_k<u_k\}} \mathcal{A}(x, \nabla w_k) \cdot \nabla\Psi dx - \int_{B\cap\{w_k<u_k\}} \mathcal{A}(x, \nabla u_k) \cdot \nabla\Psi dx \\ &= \int_{B\cap\{w_k<u_k\}} [\mathcal{A}(x, \nabla w_k) - \mathcal{A}(x, \nabla u_k)] \cdot (\nabla w_k - \nabla u_k) dx. \end{aligned}$$

Thus $\nabla w_k = \nabla u_k$ a.e. on the set $B \cap \{w_k < u_k\}$ by hypothesis (2.4) on $\mathcal{A}$. Hence $\Psi = 0$ a.e., i.e.,

$$u_k \leq w_k \quad \text{a.e.} \tag{3.9}$$

Now by the stability results of renormalized solutions established in [DMOP] we can find subsequence $\{w_{k_j}\}$ of $\{w_k\}$ such that $w_{k_j} \to w$ a.e., where w is a

renormalized solution to equation (3.8). By (3.9) we have $u \leq w$ a.e. on Ω, and hence $u \leq w$ everywhere on Ω due to Remark 3.7 and Proposition 2.1. This completes the proof of the lemma. $\square$

We now give a *global* pointwise potential estimates for quasilinear equations on a bounded domain Ω in $\mathbb{R}^n$.

Theorem 3.11. *Suppose that u is a renormalized solution to the equation*

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, \nabla u) = \omega & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.10}$$

with data $\omega \in \mathcal{M}_B^+(\Omega)$. Then there is a constant $K = K(n, p, \alpha, \beta) > 0$ such that, for all x in Ω,

$$u(x) \leq K\, \mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}\omega(x). \tag{3.11}$$

Proof. Let $B = B_{2R}(a)$ where $R = \mathrm{diam}(\Omega)$ and $a \in \Omega$ so that $\Omega \subset B$. Let w be as in Lemma 3.10 with respect to that choice of B. For $x \in \Omega$ we denote by $d(x)$ the distance from x to the boundary ∂B of B. By Theorem 2.5, Lemma 3.10, and the fact that $d(x) \geq R$, we have

$$\begin{aligned} u(x) &\leq w(x) \leq C\, \mathbf{W}_{1,p}^{\frac{2}{3}d(x)}\mu(x) + C \inf_{B_{\frac{1}{3}d(x)}(x)} w \\ &\leq C\, \mathbf{W}_{1,p}^{2R}\mu(x) + CR^{\frac{-n}{p-1}}||w||_{L^{p-1}(B)} \end{aligned} \tag{3.12}$$

Note that for $p < n$ we have

$$||w||_{L^{\frac{n(p-1)}{n-p},\infty}(B)} \leq C\, \mu(\Omega)^{\frac{1}{p-1}}$$

for a constant C independent of R (see [DMOP, Theorem 4.1] or [BBG, Lemma 4.1]). Thus

$$||w||_{L^{p-1}(B)} \leq CR^{\frac{p}{p-1}}\mu(\Omega)^{\frac{1}{p-1}}. \tag{3.13}$$

Inequality (3.13) also holds for $p \geq n$, see for example [Gre, Lemma 2.1]. From (3.12) and (3.13) we get the desired estimate (3.11). □

Remark 3.12. Estimate (3.11) does not hold in general if u is merely a weak solution of (3.10) in the sense of [KM1]. For a counter example, see [Kil], Sec. 2.

3.2 Global estimates for k-Hessian equations

Definition 3.13. A bounded domain Ω in $\mathbb{R}^n$ is said to be uniformly $(k-1)$-convex, $k = 1, ..., n$, if $\partial\Omega \in C^2$ and $H_j(\partial\Omega) > 0$, $j = 1, ..., k-1$, where $H_j(\partial\Omega)$ denotes the j-mean curvature of the boundary $\partial\Omega$.

We first recall an existence result for k-Hessian equations with measure data established in [TW1], [TW2].

Theorem 3.14. *Let Ω be a bounded uniformly $(k-1)$-convex domain. Suppose that $\varphi \geq 0$, $\varphi \in C^0(\partial\Omega)$ and $\nu = \mu + f$ where $\mu \in \mathcal{M}_B^+(\Omega)$ with compact support in Ω and $f \geq 0$, $f \in L^s(\Omega)$ with $s > \frac{n}{2k}$ if $1 \leq k \leq \frac{n}{2}$, and $s = 1$ if $\frac{n}{2} < k \leq n$. Then there exists $u \geq 0$, $-u \in \Phi^k(\Omega)$ be such that u is continuous near $\partial\Omega$ and solves*

$$\begin{cases} \mu_k[-u] = \nu & \text{in} \quad \Omega, \\ u = \varphi & \text{on} \quad \partial\Omega. \end{cases}$$

Theorem 3.15. *Suppose that* $\varphi \geq 0$, $\varphi \in C^0(\partial\Omega)$ *and* $\nu = \mu + f$ *where* $\mu \in \mathcal{M}_B^+(\Omega)$ *with compact support in* Ω *and* $f \geq 0$, $f \in L^s(\Omega)$ *with* $s > \frac{n}{2k}$ *if* $1 \leq k \leq \frac{n}{2}$, *and* $s = 1$ *if* $\frac{n}{2} < k \leq n$. *Let* $u \geq 0$, $-u \in \Phi^k(\Omega)$ *be such that* u *is continuous near* $\partial\Omega$ *and solves*

$$\begin{cases} \mu_k[-u] = \nu & \text{in} \quad \Omega, \\ u = \varphi & \text{on} \quad \partial\Omega. \end{cases}$$

Then there exists a constant $K = K(n,k)$ *such that, for all* $x \in \Omega$,

$$u(x) \leq K \left[\mathbf{W}^{2\mathrm{diam}(\Omega)}_{\frac{2k}{k+1}, k+1} \nu(x) + \max_{\partial\Omega} \varphi \right].$$

Proof. Suppose that the support of μ is contained in Ω' for some open set $\Omega' \Subset \Omega$. Let $M = \sup_{\overline{\Omega} \setminus \Omega'} u$ and $u_m = \min\{u, m\}$ for $m > M$. Then $-u_m \in \Phi^k(\Omega)$, continuous near $\partial\Omega$, solves

$$\begin{cases} \mu_k[-u_m] = \nu_m & \text{in} \quad \Omega, \\ u_m = \varphi & \text{on} \quad \partial\Omega \end{cases}$$

for certain nonnegative Radon measures ν_m in Ω. Since $u_m \to u$ in $L^1_{\mathrm{loc}}(\Omega)$, by Theorem 2.7 we have

$$\mu_m \to \nu = \mu + f \text{ weakly as measures in } \Omega. \tag{3.14}$$

Note that $u_m = u$ in $\overline{\Omega} \setminus \Omega'$ since $m > M$. Thus $\nu_m = \mu_k[u] = f$ in $\Omega \setminus \overline{\Omega'}$ for all $m > M$. Using this and (3.14) it is easy to see that

$$\int_\Omega \phi d\mu_m \to \int_\Omega \phi d\mu + \int_\Omega \phi f dx$$

as $m \to \infty$ for all $\phi \in C^0(\overline{\Omega})$, i.e.,

$$\mu_m \to \nu = \mu + f \text{ in the narrow topology of measures.}$$

We now take a ball $B = B_{2R}(a)$ where $R = \mathrm{diam}(\Omega)$ and $a \in \Omega$ so that $\Omega \subset B$. Consider the solutions $w_m \geq 0$, $-w_m \in \Phi^k(\Omega)$, continuous near $\partial\Omega$, of

$$\begin{cases} \mu_k[-w_m] = \nu_m & \text{in} \quad B, \\ w_m = \max_{\partial\Omega} \varphi & \text{on} \quad \partial B, \end{cases}$$

where $m > M$. Since u_m is bounded in Ω the measure ν_m is absolutely continuous with respect to the capacity $\mathrm{cap}_k(\cdot, \Omega)$, and hence with respect to the capacity $\mathrm{cap}_k(\cdot, B)$ (see [TW3]). Here $\mathrm{cap}_k(\cdot, \Omega)$ is the k-Hessian capacity defined by

$$\mathrm{cap}_k(E, \Omega) = \sup \Big\{ \mu_k[u](E) : u \in \Phi^k(\Omega), -1 < u < 0 \Big\}. \tag{3.15}$$

By a comparison principle (see [TW3, Theorem 4.1]), we have $w_m \geq \max_{\partial\Omega} \varphi$ in B, and hence $w_m \geq u_m$ on $\partial\Omega$. Thus, applying the comparison principle again, we have

$$w_m \geq u_m \quad \text{in} \quad \Omega. \tag{3.16}$$

Since $\nu_m \to \nu$ in the narrow topology of measures in Ω, we see that $\nu_m \to \nu$ weakly as measures in B. Therefore, arguing as in [TW2], Sec. 6 we can find a subsequence $\{w_{m_j}\}$ such that $w_{m_j} \to w$ a.e. for some $w \geq 0$, $-w \in \Phi^k(B)$ such that w is continuous near ∂B and

$$\begin{cases} \mu_k[-w] = \nu & \text{in} \quad B, \\ w = \max_{\partial\Omega} \varphi & \text{on} \quad \partial B. \end{cases}$$

Note that from (3.16), $w \geq u$ a.e. on Ω and hence $w \geq u$ everywhere on Ω. Using this and Theorem 2.8 applied to the function w on $B_{d(x)}(x)$, where $d(x) = \operatorname{dist}(x, \partial B) \geq R$ we have, for $x \in \Omega$,

$$
\begin{aligned}
(3.17) \qquad u(x) &\leq C\,\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}\nu(x) + C \inf_{B_{\frac{1}{3}d(x)}(x)} w \\
&\leq C\,\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}\nu(x) + C\,R^{-n} \int_{B_{\frac{1}{3}d(x)}(x)} w\,dy.
\end{aligned}
$$

Thus it follows from estimate (6.3) in [TW2] that

$$
u(x) \leq C\Big(\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}\nu(x) + \max_{\partial\Omega}\varphi + R^{2-\frac{n}{k}}\nu(\Omega)^{\frac{1}{k}}\Big).
$$

The proof of Theorem 3.15 is then completed by noting that

$$
\int_R^{2R} \Big[\frac{\nu(B_t(x))}{t^{n-2k}}\Big]^{\frac{1}{k}} \frac{dt}{t} \geq CR^{2-\frac{n}{k}}\nu(\Omega)^{\frac{1}{k}}.
$$

□

The following lemma is an analogue of Lemma 3.9. It is needed in the proof of Theorem 7.1 below to construct a solution to Hessian equations.

Lemma 3.16. *Let Ω be a bounded uniformly $(k-1)$-convex domain and let ν, φ and u be as in Theorem 3.15. Suppose that ν' is a measure similar to ν, i.e., $\nu' = \mu' + f'$, where $\mu' \in \mathcal{M}_B^+(\Omega)$ with compact support in Ω, $f' \geq 0$, $f' \in L^s(\Omega)$ with $s > \frac{n}{2k}$ if $1 \leq k \leq \frac{n}{2}$, and $s = 1$ if $\frac{n}{2} < k \leq n$. Then there exists $w \geq u$ such that $-w \in \Phi^k(\Omega)$ and*

$$
\begin{cases} \mu_k[-w] = \nu + \nu' & \text{in} \quad \Omega, \\ \quad w = \varphi & \text{on} \quad \partial\Omega. \end{cases}
$$

Proof. By approximation we may assume that μ' is absolutely continuous with respect to the capacity $\mathrm{cap}_k(\cdot, \Omega)$. Let u_m and ν_m be as in the proof of Theorem 3.15. Then by the comparison principle in [TW3], Theorem 4.1, we have $u_m \leq w_m$ where w_m is the solution of

$$\begin{cases} \mu_k[-w_m] = \nu_m + \nu' & \text{in} \quad \Omega, \\ w_m = \varphi & \text{on} \quad \partial\Omega. \end{cases}$$

Thus arguing as in [TW2], Sec. 6 we obtain a subsequence $\{w_{m_j}\}$ that converges a.e. to a required function w. $\square$

Chapter 4

Discrete models of nonlinear equations

In this chapter, we consider certain nonlinear integral equations with discrete kernels which serve as a model for both quasilinear and Hessian equations treated in Chapters 5–7. Let $\mathcal{D}$ be the family of all dyadic cubes $Q = 2^i(\mathbf{k} + [0, 1)^n)$, $i \in \mathbb{Z}$, $\mathbf{k} \in \mathbb{Z}^n$, in $\mathbb{R}^n$. For a nonnegative locally finite measure ω on $\mathbb{R}^n$, we define the dyadic Riesz and Wolff's potentials respectively by

$$\mathcal{I}_\alpha \omega(x) = \sum_{Q \in \mathcal{D}} \frac{\omega(Q)}{|Q|^{1-\frac{\alpha}{n}}} \chi_Q(x), \tag{4.1}$$

$$\mathcal{W}_{\alpha,p} \omega(x) = \sum_{Q \in \mathcal{D}} \Big[\frac{\omega(Q)}{|Q|^{1-\frac{\alpha p}{n}}} \Big]^{\frac{1}{p-1}} \chi_Q(x). \tag{4.2}$$

We are concerned with nonlinear inhomogeneous integral equations of the type

$$u = \mathcal{W}_{\alpha,p}(u^q) + f, \qquad u \in L^q_{\mathrm{loc}}(\mathbb{R}^n), u \geq 0, \tag{4.3}$$

where $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$, $f \geq 0$, $q > p - 1$, and $\mathcal{W}_{\alpha,p}$ is defined as in (4.2) with $\alpha > 0$ and $p > 1$ such that $0 < \alpha p < n$.

It is convenient to introduce a nonlinear operator $\mathcal{N}$ associated with the equation (4.3) defined by

$$\mathcal{N}f = \mathcal{W}_{\alpha,p}(f^q), \qquad f \in L^q_{\mathrm{loc}}(\mathbb{R}^n), f \geq 0, \tag{4.4}$$

so that (4.3) can be rewritten as

$$u = \mathcal{N}u + f, \qquad u \in L^q_{\mathrm{loc}}(\mathbb{R}^n), u \geq 0.$$

Obviously, $\mathcal{N}$ is monotonic, i.e., $\mathcal{N}f \geq \mathcal{N}g$ whenever $f \geq g \geq 0$ a.e., and $\mathcal{N}(\lambda f) = \lambda^{\frac{q}{p-1}}\mathcal{N}f$ for all $\lambda \geq 0$. Since

$$(a+b)^{p'-1} \leq \max\{1, 2^{p'-2}\}(a^{p'-1} + b^{p'-1}) \tag{4.5}$$

for all $a, b \geq 0$, it follows that

$$\Big[\mathcal{N}(f+g)\Big]^{\frac{1}{q}} \leq \max\{1, 2^{p'-2}\}\Big[(\mathcal{N}f)^{\frac{1}{q}} + (\mathcal{N}g)^{\frac{1}{q}}\Big]. \tag{4.6}$$

4.1 Discrete Wolff type inequalities

Let $1 < s < \infty$, $\Lambda = \{\lambda_Q\}_{Q\in\mathcal{D}}$, $\lambda_Q \in \mathbb{R}^+$, and let σ be a nonnegative locally finite Radon measure on $\mathbb{R}^n$. We define

$$\begin{aligned}
A_1(\Lambda) &= \int_{\mathbb{R}^n} \Big(\sum_{Q\in\mathcal{D}} \frac{\lambda_Q}{\sigma(Q)}\chi_Q(x)\Big)^s d\sigma(x),\\
A_2(\Lambda) &= \sum_{Q\in\mathcal{D}} \lambda_Q \Big(\frac{1}{\sigma(Q)} \sum_{Q'\subset Q} \lambda_{Q'}\Big)^{s-1},\\
A_3(\Lambda) &= \int_{\mathbb{R}^n} \sup_{Q\ni x} \Big(\frac{1}{\sigma(Q)} \sum_{Q'\subset Q} \lambda_{Q'}\Big)^s d\sigma(x),
\end{aligned}$$

where we assume that $\lambda_Q = 0$ whenever $\sigma(Q) = 0$ and follow the convention that $0 \cdot \infty = 0$. The following theorem is taken from [COV], Proposition 2.2.

Theorem 4.1 ([COV], Proposition 2.2)**.** *Let σ be a nonnegative locally finite Radon measure on $\mathbb{R}^n$. Let $1 < s < \infty$. Then there exist constants $C_i > 0, i = 1, 2, 3$, which depend only on s, such that, for any $\Lambda = \{\lambda_Q\}_{Q \in \mathcal{D}}$, $\lambda_Q \in \mathbb{R}^+$,*

$$A_1(\Lambda) \leq C_1 A_2(\Lambda) \leq C_2 A_3(\Lambda) \leq C_3 A_1(\Lambda).$$

Theorem 4.2. *Let μ be a nonnegative locally finite measure on $\mathbb{R}^n$, and let $\alpha > 0$, $p > 1$, and $q > p - 1$. Then there exist constants $C_i > 0, i = 1, 2, 3$, which depend only on n, p, q, α such that for any dyadic cube P,*

$$B_1(P, \mu) \leq C_1 B_2(P, \mu) \leq C_2 B_3(P, \mu) \leq C_3 B_1(P, \mu),$$

where we define

$$\begin{aligned}
B_1(P, \mu) &= \sum_{Q \subset P} \Big[\frac{\mu(Q)}{|Q|^{1 - \frac{\alpha p}{n}}} \Big]^{\frac{q}{p-1}} |Q|, \\
B_2(P, \mu) &= \int_P \Big[\sum_{Q \subset P} \Big(\frac{\mu(Q)}{|Q|^{1 - \frac{\alpha p}{n}}} \Big)^{\frac{1}{p-1}} \chi_Q(x) \Big]^q dx, \\
B_3(P, \mu) &= \int_P \Big[\sum_{Q \subset P} \frac{\mu(Q)}{|Q|^{1 - \frac{\alpha p}{n}}} \chi_Q(x) \Big]^{\frac{q}{p-1}} dx.
\end{aligned}$$

Proof. Let σ be the restriction of Lebesgue measure on the dyadic cube P. For $Q \in \mathcal{D}$ we set $\lambda_Q = \mu(Q)|Q|^{\frac{\alpha p}{n}}$ if $Q \subset P$ and $\lambda_Q = 0$ otherwise. Note that for $Q \subset P$,

$$\sum_{Q' \subset Q} \mu(Q')|Q'|^{\frac{\alpha p}{n}} = C(\alpha, p)\mu(Q)|Q|^{\frac{\alpha p}{n}}.$$

Thus by Theorem 4.1 we have

$$B_1(P,\mu) \simeq B_3(P,\mu).$$

Also, by Theorem 4.1,

$$B_3(P,\mu) \simeq \int_P \Big[\sup_{x\in Q\subset P} \frac{\mu(Q)}{|Q|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{q}{p-1}} dx. \tag{4.7}$$

Since

$$\Big[\sup_{x\in Q\subset P} \frac{\mu(Q)}{|Q|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{1}{p-1}} \le \sum_{Q\subset P} \Big(\frac{\mu(Q)}{|Q|^{1-\frac{\alpha p}{n}}}\Big)^{\frac{1}{p-1}} \chi_Q(x),$$

from (4.7) we obtain $B_3(P,\mu) \le CB_2(P,\mu)$. In addition, for $p \le 2$ we clearly have $B_2(P,\mu) \le B_3(P,\mu) \le CB_1(P,\mu)$. Therefore, it remains to check that, in the case $p > 2$, $B_2(P,\mu) \le CB_1(P,\mu)$ for some $C > 0$ independent of P and μ. For $q > p-1 > 1$ by Theorem 4.1 we have

$$\begin{aligned} B_2(P,\mu) &= \int_P \Big[\sum_{Q\subset P} \frac{\mu(Q)^{\frac{1}{p-1}}}{|Q|^{(1-\frac{\alpha p}{n})\frac{1}{p-1}}} \chi_Q(x)\Big]^q dx \\ &\le C \sum_{Q\subset P} \frac{\mu(Q)^{\frac{1}{p-1}}}{|Q|^{(1-\frac{\alpha p}{n})\frac{1}{p-1}+q-2}} \Big[\sum_{Q'\subset Q} \frac{\mu(Q')^{\frac{1}{p-1}}}{|Q'|^{(1-\frac{\alpha p}{n})\frac{1}{p-1}-1}}\Big]^{q-1}. \end{aligned} \tag{4.8}$$

On the other hand, by Hölder's inequality,

$$\begin{aligned} \sum_{Q'\subset Q} \frac{\mu(Q')^{\frac{1}{p-1}}}{|Q'|^{(1-\frac{\alpha p}{n})\frac{1}{p-1}-1}} &= \sum_{Q'\subset Q} \Big(\mu(Q')^{\frac{1}{p-1}}|Q'|^{\epsilon}\Big)|Q'|^{-(1-\frac{\alpha p}{n})\frac{1}{p-1}+1-\epsilon} \\ &\le \Big(\sum_{Q'\subset Q} \mu(Q')^{\frac{r'}{p-1}}|Q'|^{\epsilon r'}\Big)^{\frac{1}{r'}} \Big(\sum_{Q'\subset Q} |Q'|^{-r(1-\frac{\alpha p}{n})\frac{1}{p-1}+r-r\epsilon}\Big)^{\frac{1}{r}}, \end{aligned}$$

where $r' = p-1 > 1$, $r = \frac{p-1}{p-2}$ and $\epsilon > 0$ is chosen so that $-r(1-\frac{\alpha p}{n})\frac{1}{p-1}+r-r\epsilon > 1$,

i.e., $0 < \epsilon < \frac{\alpha p}{(p-1)n}$. Therefore,

$$\begin{aligned}\sum_{Q'\subset Q} \frac{\mu(Q')^{\frac{1}{p-1}}}{|Q'|^{(1-\frac{\alpha p}{n})\frac{1}{p-1}-1}} &\leq C\mu(Q)^{\frac{1}{p-1}}|Q|^{\epsilon}|Q|^{-(1-\frac{\alpha p}{n})\frac{1}{p-1}+1-\epsilon} \\ &= C\frac{\mu(Q)^{\frac{1}{p-1}}}{|Q|^{(1-\frac{\alpha p}{n})\frac{1}{p-1}-1}}.\end{aligned}$$

Combining this with (4.8) we obtain

$$\begin{aligned}B_2(P,\mu) &\leq C\sum_{Q\subset P} \frac{\mu(Q)^{\frac{1}{p-1}}}{|Q|^{(1-\frac{\alpha p}{n})\frac{1}{p-1}+q-2}}\Big[\frac{\mu(Q)^{\frac{1}{p-1}}}{|Q|^{(1-\frac{\alpha p}{n})\frac{1}{p-1}-1}}\Big]^{q-1} \\ &= C\sum_{Q\subset P} \frac{\mu(Q)^{\frac{q}{p-1}}}{|Q|^{(1-\frac{\alpha p}{n})\frac{q}{p-1}-1}} = CA_1(P,\mu),\end{aligned}$$

which completes the proof of the theorem. □

4.2 Criteria for solvability

We are now in a position to establish the main results of this chapter.

Theorem 4.3. *Let $\alpha > 0$, $p > 1$ be such that $0 < \alpha p < n$, and let $q > p-1$. Suppose $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$, $f \geq 0$, and $d\omega = f^q dx$. Then the following statements are equivalent.*

(i) *The equation*

$$u = \mathcal{W}_{\alpha,p}(u^q) + \epsilon f \tag{4.9}$$

has a solution $u \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$, $u \geq 0$, for some $\epsilon > 0$.

(ii) *The testing inequality*

$$\int_P \Big[\sum_{Q\subset P} \frac{\omega(Q)}{|Q|^{1-\frac{\alpha p}{n}}}\chi_Q(x)\Big]^{\frac{q}{p-1}} dx \leq C\,\omega(P) \tag{4.10}$$

holds for all dyadic cubes P.

(iii) *The testing inequality*

$$\int_P \Big[\sum_{Q\subset P} \frac{\omega(Q)^{\frac{1}{p-1}}}{|Q|^{(1-\frac{\alpha p}{n})\frac{1}{p-1}}}\chi_Q(x)\Big]^q dx \le C\,\omega(P) \tag{4.11}$$

holds for all dyadic cubes P.

(iv) *There exists a constant C such that*

$$\mathcal{W}_{\alpha,p}[\mathcal{W}_{\alpha,p}(f^q)]^q(x) \le C\mathcal{W}_{\alpha,p}(f^q)(x) < \infty \quad \text{a.e.} \tag{4.12}$$

Proof. Note that by Theorem 4.2 we have (ii)$\Leftrightarrow$(iii). Therefore, it is enough to prove (iv)$\Rightarrow$(i)$\Rightarrow$(iii)$\Rightarrow$(iv).

Proof of (iv)$\Rightarrow$(i). The pointwise condition (4.12) can be rewritten as

$$\mathcal{N}^2 f \le C\mathcal{N} f < \infty \quad \text{a.e.},$$

where $\mathcal{N}$ is the operator defined by (4.4). The sufficiency of this condition for the solvability of (4.9) can be proved using simple iterations:

$$u_{n+1} = \mathcal{N}u_n + \epsilon f, \qquad n = 0,\ 1,\ 2, \dots,$$

starting from $u_0 = 0$. Since $\mathcal{N}$ is monotonic it is easy to see that u_n is increasing and that $\epsilon^{\frac{q}{p-1}}\mathcal{N}f + \epsilon f \le u_n$ for all $n \ge 2$. Let $c(p) = \max\{1, 2^{p'-1}\}$, $c_1 = 0$, $c_2 = [\epsilon^{\frac{1}{p-1}}c(p)]^q$ and

$$c_n = \Big[\epsilon^{\frac{1}{p-1}}c(p)(1 + C^{1/q})c_{n-1}^{p'-1}\Big]^q, \qquad n = 3,\ 4, \dots,$$

where C is the constant in (4.12). Here we choose ϵ so that

$$\epsilon^{\frac{1}{p-1}} c(p) = \Big(\frac{q-p+1}{q}\Big)^{\frac{q-p+1}{q}} \Big(\frac{p-1}{q}\Big)^{\frac{p-1}{q}} C^{\frac{1-p}{q^2}}.$$

By induction and using (4.6) we have

$$u_n \le c_n \mathcal{N} f + \epsilon f, \qquad n = 1,\ 2,\ 3, \dots$$

Note that

$$x_0 = \Big[\frac{q}{p-1} \epsilon^{\frac{1}{p-1}} c(p)\, C^{\frac{1}{q}}\Big]^{\frac{q(p-1)}{p-1-q}}$$

is the only root of the equation

$$x = \Big[\epsilon^{\frac{1}{p-1}} c(p)(1 + C^{\frac{1}{q}} x)\Big]^q$$

and thus $\lim_{n\to\infty} c_n = x_0$. Hence there exists a solution

$$u(x) = \lim_{n\to\infty} u_n(x)$$

to equation (4.9) (with that choice of ϵ) such that

$$\epsilon f + \epsilon^{\frac{q}{p-1}} \mathcal{W}_{\alpha,p}(f^q) \le u \le \epsilon f + x_0 \mathcal{W}_{\alpha,p}(f^q).$$

Proof of (i)$\Rightarrow$(iii). Suppose that $u \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$, $u \ge 0$, is a solution of (4.9). Let P be a cube in $\mathcal{D}$ and $d\mu = u^q dx$. Since

$$[u(x)]^q \ge [\mathcal{W}_{\alpha,p}(u^q)(x)]^q \quad \text{a.e.},$$

we have

$$\int_P [\mathcal{W}_{\alpha,p}(u^q)(x)]^q dx \le \int_P [u(x)]^q dx.$$

Thus,

$$\int_P \Big[\sum_{Q\subset P} \frac{\mu(Q)^{\frac{1}{p-1}}}{|Q|^{(1-\frac{\alpha p}{n})\frac{1}{p-1}}}\chi_Q(x)\Big]^q dx \le C\mu(P) \tag{4.13}$$

for all $P \in \mathcal{D}$. By Theorem 4.2, inequality (4.13) is equivalent to

$$\int_P \Big[\sum_{Q\subset P} \frac{\mu(Q)}{|Q|^{1-\frac{\alpha p}{n}}}\chi_Q(x)\Big]^{\frac{q}{p-1}} dx \le C\mu(P)$$

for all $P \in \mathcal{D}$, which in its turn is equivalent to the weak-type inequality

$$||\mathcal{I}_{\alpha p}(g)||_{L^{\frac{q}{q-p+1},\infty}(d\mu)} \le C||g||_{L^{\frac{q}{q-p+1}}(dx)} \tag{4.14}$$

for all $g \in L^{\frac{q}{q-p+1}}(\mathbb{R}^n)$, $g \ge 0$ (see [NTV], [VW]). Note that by (4.9),

$$d\mu = u^q dx \ge \epsilon^q f^q\, dx = \epsilon^q\, d\omega.$$

We now deduce from (4.14),

$$||\mathcal{I}_{\alpha p}(g)||_{L^{\frac{q}{q-p+1},\infty}(d\omega)} \le \frac{C}{\epsilon^{q-p+1}}||g||_{L^{\frac{q}{q-p+1}}(dx)} \tag{4.15}$$

Similarly, by duality and Theorem 4.2 we see that (4.15) is equivalent to the testing inequality (4.11). The implication (i)$\Rightarrow$ (iii) is proved.

Proof of (iii)$\Rightarrow$(iv). We first deduce from the testing inequality (4.11) that

$$\omega(P) \le C|P|^{1-\frac{\alpha pq}{n(q-p+1)}} \tag{4.16}$$

for all dyadic cubes P. In fact, this can be verified by using (4.11) and the obvious estimate

$$\int_P \Big[\frac{\omega(P)}{|P|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{q}{p-1}} dx \le \int_P \Big[\sum_{Q\subset P} \frac{\omega(Q)^{\frac{1}{p-1}}}{|Q|^{(1-\frac{\alpha p}{n})\frac{1}{p-1}}}\chi_Q(x)\Big]^q dx.$$

Following [KV], [V3], we next introduce a certain decomposition of the dyadic Wolff's potential $\mathcal{W}_{\alpha,\,p}\mu$. To each dyadic cube $P \in \mathcal{D}$, we associate the "upper" and "lower" parts of $\mathcal{W}_{\alpha,\,p}\mu$ defined respectively by

$$\mathcal{U}_P\mu(x) = \sum_{Q\subset P} \Big[\frac{\mu(Q)}{|Q|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{1}{p-1}} \chi_Q(x), \tag{4.17}$$

$$\mathcal{V}_P\mu(x) = \sum_{Q\supset P} \Big[\frac{\mu(Q)}{|Q|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{1}{p-1}} \chi_Q(x). \tag{4.18}$$

Obviously,

$$\mathcal{U}_P\mu(x) \le \mathcal{W}_{\alpha,\,p}\mu(x), \qquad \mathcal{V}_P\mu(x) \le \mathcal{W}_{\alpha,\,p}\mu(x),$$

and for $x \in P$,

$$\mathcal{W}_{\alpha,\,p}\mu(x) = \mathcal{U}_P\mu(x) + \mathcal{V}_P\mu(x) - \Big[\frac{\mu(P)}{|P|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{1}{p-1}}.$$

Using the notation just introduced, we can rewrite the testing inequality (4.11) in the form:

$$\int_P [\mathcal{U}_P\omega(x)]^q\, dx \le C\,\omega(P) \tag{4.19}$$

for all dyadic cubes P. Recall that $d\omega = f^q\, dx$. The desired pointwise inequality (4.12) can be restated as

$$\sum_{P\in\mathcal{D}} \Big[\frac{\int_P [\mathcal{W}_{\alpha,\,p}\omega(y)]^q\, dy}{|P|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{1}{p-1}} \chi_P(x) \le C\,\mathcal{W}_{\alpha,\,p}\omega(x). \tag{4.20}$$

Obviously, for $y \in P$,

$$\mathcal{W}_{\alpha,\,p}\omega(y) \le \mathcal{U}_P\omega(y) + \mathcal{V}_P\omega(y),$$

and from the testing inequality (4.19) we have

$$\sum_{P\in\mathcal{D}}\Big[\frac{\int_P[\mathcal{U}_P\omega(y)]^q\,dy}{|P|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{1}{p-1}}\chi_P(x)\le C\,\mathcal{W}_{\alpha,\,p}\omega(x).$$

Therefore, to prove (4.20) it enough to prove

$$\sum_{P\in\mathcal{D}}\Big[\frac{\int_P[\mathcal{V}_P\omega(y)]^q\,dy}{|P|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{1}{p-1}}\chi_P(x)\le C\,\mathcal{W}_{\alpha,\,p}\omega(x). \tag{4.21}$$

Note that, for $y\in P$,

$$\mathcal{V}_P\omega(y)=\sum_{Q\supset P}\Big[\frac{\omega(Q)}{|Q|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{1}{p-1}}=\text{const}.$$

Using the elementary inequality

$$\Big(\sum_{k=1}^{\infty}a_k\Big)^s\le s\sum_{k=1}^{\infty}a_k\Big(\sum_{j=k}^{\infty}a_j\Big)^{s-1}$$

where $1\le s<\infty$ and $0\le a_k<\infty$, we deduce

$$[\mathcal{V}_P\omega(y)]^{\frac{q}{p-1}}\ \le\ C\sum_{Q\supset P}\Big[\frac{\omega(Q)}{|Q|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{1}{p-1}}\Big\{\sum_{R\supset Q}\Big[\frac{\omega(R)}{|R|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{1}{p-1}}\Big\}^{\frac{q}{p-1}-1}.$$

From this we see that the left-hand side of (4.21) is bounded above by a constant multiple of

$$\sum_{P\in\mathcal{D}}|P|^{\frac{\alpha p}{n(p-1)}}\sum_{Q\supset P}\Big[\frac{\omega(Q)}{|Q|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{1}{p-1}}\Big\{\sum_{R\supset Q}\Big[\frac{\omega(R)}{|R|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{1}{p-1}}\Big\}^{\frac{q}{p-1}-1}\chi_P(x).$$

Changing the order of summation, we see that it is equal to

$$\sum_{Q\in\mathcal{D}}\Big[\frac{\omega(Q)}{|Q|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{1}{p-1}}\chi_Q(x)\Big\{\sum_{P\subset Q}|P|^{\frac{\alpha p}{n(p-1)}}\chi_P(x)[\mathcal{V}_Q\omega(x)]^{\frac{q}{p-1}-1}\Big\}.$$

By (4.16), the expression in the curly brackets above is uniformly bounded. Therefore, the proof of estimate (4.21), and hence of (iii)$\Rightarrow$ (iv), is complete. $\square$

Chapter 5

Quasilinear equations on $\mathbb{R}^n$

In this chapter, we study the solvability problem for the quasilinear equation

$$-\mathrm{div}\mathcal{A}(x, \nabla u) = u^q + \omega \tag{5.1}$$

in the class of nonnegative $\mathcal{A}$-superharmonic functions on the entire space $\mathbb{R}^n$, where $\mathcal{A}(x, \xi) \cdot \xi \approx |\xi|^p$ is defined precisely as in Sec. 2.2. Here we assume $1 < p < n$, $q > p - 1$, and ω is a nonnegative locally finite measure on $\mathbb{R}^n$. In this setting, all solutions are understood in the "potential-theoretic" sense, i.e., $u \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$, $u \geq 0$, is a solution to (5.1) if u is $\mathcal{A}$-superharmonic, and for all $\varphi \in C_0^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \mathcal{A}(x, Du) \cdot \nabla\varphi \, dx = \int_{\mathbb{R}^n} u^q \varphi \, dx + \int_{\mathbb{R}^n} \varphi \, d\omega, \tag{5.2}$$

where Du is defined as in (2.8).

5.1 Continuous Wolff type inequalities

We first prove a continuous counterpart of Theorem 4.2. Here we use the well-known argument due to Fefferman and Stein [FS] which is based on the averaging

over shifts of the dyadic lattice $\mathcal{D}$.

Theorem 5.1. *Let μ be a nonnegative locally finite measure on $\mathbb{R}^n$, and let $0 < R \leq +\infty$, $\alpha > 0$, $p > 1$, and $q > p-1$. Then there exist constants $C_i > 0, i = 1, 2, 3, 4$, which depend only on n, p, q, α, such that,*

$$C_1 \int_{\mathbb{R}^n} (\mathbf{W}^R_{\alpha p,\, \frac{q}{q-p+1}} \mu)^q dx \leq \int_{\mathbb{R}^n} (\mathbf{I}^R_{\alpha p} \mu)^{\frac{q}{p-1}} dx \leq C_2 \int_{\mathbb{R}^n} (\mathbf{W}^R_{\alpha p,\, \frac{q}{q-p+1}} \mu)^q dx, \tag{5.3}$$

and

$$C_3 \int_{\mathbb{R}^n} \mathbf{W}^R_{\alpha p,\, \frac{q}{q-p+1}} \mu d\mu \leq \int_{\mathbb{R}^n} (\mathbf{I}^R_{\alpha p} \mu)^{\frac{q}{p-1}} dx \leq C_4 \int_{\mathbb{R}^n} \mathbf{W}^R_{\alpha p,\, \frac{q}{q-p+1}} \mu d\mu. \tag{5.4}$$

Remark 5.2. Inequality (5.4) may be regarded as a version of Wolff's inequality [HW] (see also [AH], Sec. 4.5):

$$C_1 \int_{\mathbb{R}^n} \mathbf{W}_{\alpha,\, s} \mu \, d\mu \leq \int_{\mathbb{R}^n} (\mathbf{I}_\alpha \mu)^{\frac{s}{s-1}} dx \leq C_2 \int_{\mathbb{R}^n} \mathbf{W}_{\alpha,\, s} \mu \, d\mu, \tag{5.5}$$

where $1 < s < +\infty$, $0 < \alpha < \frac{n}{s}$, and C_1, C_2 depend only on α, s and n. Furthermore,

$$\int_{\mathbb{R}^n} (\mathbf{I}_\alpha \mu)^{\frac{s}{s-1}} dx \simeq \int_{\mathbb{R}^n} (\mathcal{I}_\alpha \mu)^{\frac{s}{s-1}} dx \simeq \sum_{Q \in \mathcal{D}} \left[\frac{\mu(Q)}{|Q|^{1-\frac{\alpha}{n}}} \right]^{\frac{s}{s-1}} |Q|. \tag{5.6}$$

The second equivalence in (5.6) is a dyadic form of (5.5) which was also proved in [HW] (see also [COV], [V2]).

Proof of Theorem 5.1. We will prove only inequality (5.3) since inequality (5.4), which is actually a consequence of Theorem 3.6.2 in [AH], can also be deduced by

a similar argument. We first restrict ourselves to the case $R < +\infty$. Observe that there is a constant $C > 0$ such that

$$\int_{\mathbb{R}^n} (\mathbf{I}_{\alpha p}^{2R}\mu)^{\frac{q}{p-1}}\,dx \le C \int_{\mathbb{R}^n} (\mathbf{I}_{\alpha p}^{R}\mu)^{\frac{q}{p-1}}\,dx. \tag{5.7}$$

In fact, since

$$\int_0^{2R} \frac{\mu(B_t(x))}{t^{n-\alpha p}}\frac{dt}{t} \le C\int_0^R \frac{\mu(B_t(x))}{t^{n-\alpha p}}\frac{dt}{t} + C\,\frac{\mu(B_{2R}(x))}{R^{n-\alpha p}},$$

(5.7) will follow from the estimate

$$\int_{\mathbb{R}^n} \Big[\frac{\mu(B_{2R}(x))}{R^{n-\alpha p}}\Big]^{\frac{q}{p-1}}dx \le C\int_{\mathbb{R}^n}\Big[\int_0^R \frac{\mu(B_t(x))}{t^{n-\alpha p}}\frac{dt}{t}\Big]^{\frac{q}{p-1}}dx. \tag{5.8}$$

Note that for a partition of $\mathbb{R}^n$ into a union of disjoint cubes $\{Q_j\}$ such that $\operatorname{diam}(Q_j) = \frac{R}{4}$ we have

$$\begin{aligned}\int_{\mathbb{R}^n} \mu(B_{2R}(x))^{\frac{q}{p-1}}dx &= \sum_j \int_{Q_j} \mu(B_{2R}(x))^{\frac{q}{p-1}}dx \\ &\le C\sum_j \int_{Q_j} \mu(Q_j)^{\frac{q}{p-1}}dx,\end{aligned}$$

where we have used the fact that the ball $B_{2R}(x)$ is contained in the union of at most N cubes in $\{Q_j\}$ for some constant N depending only on n. Thus

$$\begin{aligned}\int_{\mathbb{R}^n}\Big[\frac{\mu(B_{2R}(x))}{R^{n-\alpha p}}\Big]^{\frac{q}{p-1}}dx &\le C\sum_j\int_{Q_j}\Big[\frac{\mu(B_{R/2}(x))}{R^{n-\alpha p}}\Big]^{\frac{q}{p-1}}dx \\ &\le C\sum_j\int_{Q_j}\Big[\int_0^R\frac{\mu(B_t(x))}{t^{n-\alpha p}}\frac{dt}{t}\Big]^{\frac{q}{p-1}}dx,\end{aligned}$$

which gives (5.8).

By arguing as in [COV], Sec. 3, we can find constants a, C and c depending

only on p and n such that

$$\mathbf{W}^R_{\alpha,p}\mu(x) \le CR^{-n}\int_{|z|\le cR}\sum_{\substack{Q\in\mathcal{D}_z\\ \ell(Q)\le 4\frac{R}{a}}}\Big[\frac{\mu(Q)}{|Q|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{1}{p-1}}\chi_Q(x)dz$$

where $\mathcal{D}_z$, $z\in\mathbb{R}^n$, denotes the lattice $\mathcal{D}+z=\{Q=Q'+z : Q'\in\mathcal{D}\}$ and $\ell(Q)$ is the side length of Q. By Theorem 4.1 and arguing as in the proof of Theorem 4.2 we obtain

$$\int_{\mathbb{R}^n}\Big\{\sum_{\substack{Q\in\mathcal{D}_z\\ \ell(Q)\le 4\frac{R}{a}}}\Big[\frac{\mu(Q)}{|Q|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{1}{p-1}}\chi_Q(x)\Big\}^q dx$$
$$\simeq \int_{\mathbb{R}^n}\Big[\sum_{\substack{Q\in\mathcal{D}_z\\ \ell(Q)\le 4\frac{R}{a}}}\frac{\mu(Q)}{|Q|^{1-\frac{\alpha p}{n}}}\chi_Q(x)\Big]^{\frac{q}{p-1}}dx,$$

where the constants of equivalence are independent of μ, r and z. The last two estimates together with the integral Minkowski inequality then give

$$\begin{aligned}
&\int_{\mathbb{R}^n}(\mathbf{W}^R_{\alpha,p}\mu)^q dx\\
&\quad\le CR^{-n}\int_{|z|\le cR}\Big\{\int_{\mathbb{R}^n}\Big(\sum_{\substack{Q\in\mathcal{D}_z\\ \ell(Q)\le 4\frac{R}{a}}}\Big[\frac{\mu(Q)}{|Q|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{1}{p-1}}\chi_Q(x)\Big)^q dx\Big\}^{\frac{1}{q}}dz\\
&\quad\le CR^{-n}\int_{|z|\le cR}\Big[\int_{\mathbb{R}^n}\Big(\sum_{\substack{Q\in\mathcal{D}_z\\ \ell(Q)\le 4\frac{R}{a}}}\frac{\mu(Q)}{|Q|^{1-\frac{\alpha p}{n}}}\chi_Q(x)\Big)^{\frac{q}{p-1}}dx\Big]^{\frac{1}{q}}dz.
\end{aligned}$$

Note that

$$\begin{aligned}
\sum_{\substack{Q\in\mathcal{D}_z\\ \ell(Q)\le 4\frac{R}{a}}}\frac{\mu(Q)}{|Q|^{1-\frac{\alpha p}{n}}}\chi_Q(x) &\le C\sum_{2^k\le 4\frac{R}{a}}\frac{\mu(B(x,\sqrt{n}2^k))}{2^{k(n-\alpha p)}}\\
&\le C\mathbf{I}_{\alpha p}^{\frac{8R\sqrt{n}}{a}}\mu(x),
\end{aligned}$$

where C is independent of z. Thus, in view of (5.7), we obtain the lower estimate in (5.3).

Now by letting $R \to \infty$ in the inequality

$$\int_{\mathbb{R}^n} (\mathbf{W}^R_{\alpha,\,p}\mu)^q dx \le C \int_{\mathbb{R}^n} (\mathbf{I}^R_{\alpha p}\mu)^{\frac{q}{p-1}} dx, \qquad 0 < R < +\infty,$$

we get the lower estimate in (5.3) with $R = \infty$. The upper estimate in (5.3) can be deduced in a similar way. This completes the proof of Theorem 5.1. □

5.2 Criteria for solvability

In the next theorem, we give a sufficient condition for the solvability of quasilinear equations in $\mathbb{R}^n$. Later on we will show that it is necessary as well, and give equivalent simpler characterizations.

Theorem 5.3. *Let ω be a nonnegative locally finite measure on $\mathbb{R}^n$, and let $1 < p < n$, and $q > p - 1$. Suppose that*

$$\mathbf{W}_{1,\,p}(\mathbf{W}_{1,\,p}\omega)^q \le C\,\mathbf{W}_{1,\,p}\omega < \infty \quad \text{a.e.,} \tag{5.9}$$

where

$$C \le \Big(\frac{q-p+1}{qK\max\{1, 2^{p'-2}\}}\Big)^{q(p'-1)}\Big(\frac{p-1}{q-p+1}\Big), \tag{5.10}$$

and K is the constant used in Theorem 3.11. Then there is an $\mathcal{A}$-superharmonic function $u \in L^q_{\rm loc}(\mathbb{R}^n)$ such that

$$\begin{cases} \inf_{x\in\mathbb{R}^n} u(x) = 0, \\ -\mathrm{div}\mathcal{A}(x, \nabla u) = u^q + \omega, \end{cases} \tag{5.11}$$

and

$$c_1 \mathbf{W}_{1,p}\omega(x) \le u(x) \le c_2\, \mathbf{W}_{1,p}\omega(x)$$

for all x in $\mathbb{R}^n$, where the constants c_1, c_2 depend only n, p, q, and the structural constants α, β.

Proof. For each $m \in \mathbb{N}$, let us construct by an induction argument a nondecreasing sequence $\{u_k^m\}_{k\ge 0}$ of $\mathcal{A}$-superharmonic functions on B_{m+1} such that

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, \nabla u_0^m) = \omega_{B_m} \quad \text{in} \quad B_{m+1}, \\ \qquad u_0^m = 0 \quad \text{on} \quad \partial B_{m+1}, \end{cases}$$

and

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, \nabla u_k^m) = (u_{k-1}^m)^q + \omega_{B_m} \quad \text{in} \quad B_{m+1}, \\ \qquad u_k^m = 0 \quad \text{on} \quad \partial B_{m+1} \end{cases}$$

for each $k \ge 1$, in the renormalized sense. Here B_m denotes the ball of radius m and centered at the origin. The renormalized solutions are needed here only to get the following estimates:

$$u_0^m \le K\, \mathbf{W}_{1,p}\omega \quad \text{and} \quad u_k^m \le K\, \mathbf{W}_{1,p}(u_k^q + \omega)$$

for all $k \ge 1$; see Theorem 3.11. Set $c_0 = K$, where K is the constant in Theorem 3.11. From these estimates and (4.5) we get

$$\begin{aligned} u_1^m &\le K \max\{1, 2^{p'-2}\}\Big[\mathbf{W}_{1,p}(u_0^m)^q + \mathbf{W}_{1,p}\omega\Big] \\ &\le K \max\{1, 2^{p'-2}\}(c_0^{q(p'-1)}C + 1)\mathbf{W}_{1,p}\omega \\ &= c_1 \mathbf{W}_{1,p}\omega, \end{aligned}$$

where $c_1 = K\max\{1, 2^{p'-2}\}(c_0^{q(p'-1)}C + 1)$. By induction we can find a sequence $\{c_k\}_{k\geq 0}$ of positive numbers such that $u_k^m \leq c_k \mathbf{W}_{1,p}\omega$, with $c_0 = K$ and $c_{k+1} = K\max\{1, 2^{p'-2}\}(c_k^{q(p'-1)}C + 1)$ for all $k \geq 0$. It is then easy to see that $c_k \leq \frac{K\max\{1,2^{p'-2}\}q}{q-p+1}$ for all $k \geq 0$ as long as (5.10) is satisfied. Thus

$$u_k^m \leq \frac{K\max\{1, 2^{p'-2}\}q}{q-p+1}\mathbf{W}_{1,p}\omega \quad \text{on} \quad B_{m+1}.$$

Now by weak continuity (Theorem 2.4) or stability results for renormalized solutions in [DMOP] we see that $u_k^m \uparrow u^m$ for an $\mathcal{A}$-superharmonic function $u^m \geq 0$ on B_{m+1} such that

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, \nabla u^m) = (u^m)^q + \omega_{B_m} & \text{in} \quad B_{m+1}, \\ u^m = 0 \quad \text{on} \quad \partial B_{m+1}, \end{cases} \tag{5.12}$$

and

$$u^m \leq C\,\mathbf{W}_{1,p}\omega \quad \text{on} \quad B_{m+1}. \tag{5.13}$$

By Theorem 1.17 in [KM1] we can find a subsequence $\{u^{m_j}\}_j$ of $\{u^m\}_m$ and an $\mathcal{A}$-superharmonic function u on $\mathbb{R}^n$ such that $u^{m_j} \to u$ a.e. Thus by (5.12) and weak continuity (Theorem 2.4) we see that u is a solution to the equation $-\mathrm{div}\mathcal{A}(x, \nabla u) = u^q + \omega$ in $\mathbb{R}^n$. On the other hand, from (5.13) we have

$$u \leq C\,\mathbf{W}_{1,p}\omega \quad \text{a.e. on} \quad \mathbb{R}^n,$$

which by Corollary 2.6 gives

$$u \leq C(u - \inf_{\mathbb{R}^n} u).$$

Thus $\inf_{\mathbb{R}^n} u = 0$, which completes the proof of the theorem. $\square$

We can now prove the main theorem of this section which gives existence criteria for quasilinear equations in $\mathbb{R}^n$.

Theorem 5.4. *Let ω be a nonnegative locally finite measure on $\mathbb{R}^n$, and let $1 < p < n$ and $q > p-1$. Then the following statements are equivalent.*

(i) *There exists a nonnegative $\mathcal{A}$-superharmonic solution $u \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ to the equation*

$$\left\{ \begin{array}{c} \inf_{x\in\mathbb{R}^n} u(x) = 0, \\ -\mathrm{div}\mathcal{A}(x, \nabla u) = u^q + \epsilon\,\omega \quad \text{in} \quad \mathbb{R}^n \end{array} \right. \tag{5.14}$$

for some $\epsilon > 0$.

(ii) *The testing inequality*

$$\int_B \Big[\mathbf{I}_p \omega_B(x)\Big]^{\frac{q}{p-1}} dx \le C\omega(B) \tag{5.15}$$

holds for all balls B in $\mathbb{R}^n$.

(iii) *For all compact sets $E \subset \mathbb{R}^n$,*

$$\omega(E) \le C\,\mathrm{Cap}_{\mathbf{I}_p, \frac{q}{q-p+1}}(E). \tag{5.16}$$

(iv) *The testing inequality*

$$\int_B \Big[\mathbf{W}_{1,p}\omega_B(x)\Big]^q dx \le C\,\omega(B) \tag{5.17}$$

holds for all balls B in $\mathbb{R}^n$.

(v) *There exists a constant C such that*

$$\mathbf{W}_{1,p}\,(\mathbf{W}_{1,p}\omega)^q(x) \le C\,\mathbf{W}_{1,p}\omega(x) < \infty \quad a.e. \tag{5.18}$$

Moreover, there is a constant $C_0 = C_0(n,p,q,\alpha,\beta)$ such that if any one of the conditions (5.15)–(5.18) holds with $C \le C_0$, then equation (5.14) has a solution u with $\epsilon = 1$ which satisfies the two-sided estimate

$$c_1\,\mathbf{W}_{1,p}\omega(x) \le u(x) \le c_2\,\mathbf{W}_{1,p}\omega(x), \qquad x \in \mathbb{R}^n, \tag{5.19}$$

where c_1 and c_2 depend only on n,p,q,α,β. Conversely, if (5.14) has a solution u as in statement (i) *with $\epsilon = 1$, then conditions (5.15)–(5.18) hold with $C = C_1(n,p,q,\alpha,\beta)$. Here α and β are the structural constants of $\mathcal{A}$ defined in Section 2.2.*

Proof. It is well-known that that statements (ii) and (iii) in Theorem 5.4 are equivalent (see, e.g., [V2]). Note that (5.15) is also equivalent to the testing inequality (see, e.g., [VW]):

$$\int_{\mathbb{R}^n} \Big[\mathbf{I}_p\omega_B(x)\Big]^{\frac{q}{p-1}}\,dx \le C\,\omega(B).$$

By applying Theorem 5.1 we deduce (ii)⇒(iv). The implication (v)⇒(i) clearly follows from Theorem 5.3. Therefore, it remains to check (i)⇒(ii) and (iv)⇒(v).

Proof of (i)⇒(ii). Let u be a nonnegative solution of (5.14) and let $\mu = u^q + \epsilon\omega$. Then μ is a nonnegative measure such that $\mu \ge u^q$, $\mu \ge \epsilon\omega$ and $u(x) \ge C\,\mathbf{W}_{1,p}\mu(x)$

by Corollary 2.6. Therefore,

$$\begin{aligned}\int_P d\mu &\geq \int_P u^q\,dx \geq C\int_P (\mathbf{W}_{1,\,p}\mu)^q\,dx\\ &\geq C\int_P \Big[\sum_{Q\subset P}\Big(\frac{\mu(Q)}{|Q|^{1-\frac{p}{n}}}\Big)^{\frac{1}{p-1}}\chi_Q(x)\Big]^q\,dx\end{aligned}$$

for all dyadic cubes P in $\mathbb{R}^n$. Using this and Theorem 4.2, we get

$$\sum_{Q\subset P}\Big[\frac{\mu(Q)}{|Q|^{1-\frac{p}{n}}}\Big]^{\frac{q}{p-1}}|Q| \leq C\,\mu(P), \qquad P\in\mathcal{D}.$$

It is known that the preceding condition, which is a dyadic Carleson measure condition, is equivalent to the inequality (see [V1], Sec. 3)

$$||\mathbf{I}_p(f)||_{L^{\frac{q}{q-p+1}}(d\mu)} \leq C\,||f||_{L^{\frac{q}{q-p+1}}(dx)},$$

where C does not depend on $f\in L^{\frac{q}{q-p+1}}(dx)$. Since $\mu\geq\epsilon\,\omega$, from this we have

$$||\mathbf{I}_p(f)||_{L^{\frac{q}{q-p+1}}(d\omega)} \leq \epsilon^{\frac{q-p+1}{-q}}C||f||_{L^{\frac{q}{q-p+1}}(dx)}.$$

Therefore, by duality we obtain the testing inequality (5.15). This completes the proof of (i)⇒(ii).

Proof of (iv)⇒(v). We first claim that (5.17) yields

$$\omega(B_t(x)) \leq C\,t^{n-\frac{pq}{q-p+1}}, \tag{5.20}$$

where C is independent of x and r. Note that for $y\in B_t(x)$ and $\tau\geq 2t$, we have $B_t(x)\subset B_\tau(y)$. Thus,

$$\begin{aligned}\mathbf{W}_{1,\,p}\omega_{B_t(x)}(y) &\geq \int_{2t}^{\infty}\Big[\frac{\omega(B_\tau(y)\cap B_t(x))}{\tau^{n-p}}\Big]^{\frac{1}{p-1}}\frac{d\tau}{\tau}\\ &\geq C\Big[\frac{\omega(B_t(x))}{t^{n-p}}\Big]^{\frac{1}{p-1}}.\end{aligned}$$

Combining this with (5.17) we obtain 5.20.

Next, we introduce a decomposition of the Wolff's potential $\mathbf{W}_{1,p}$ into its "upper" and "lower" parts, which are the continuous analogues of the discrete ones given in (4.17) and (4.18) above:

$$\mathbf{U}_r\mu(x) = \int_0^r \left[\frac{\mu(B_t(x))}{t^{n-p}}\right]^{\frac{1}{p-1}} \frac{dt}{t}, \quad r > 0, \quad x \in \mathbb{R}^n,$$

$$\mathbf{L}_r\mu(x) = \int_r^\infty \left[\frac{\mu(B_t(x))}{t^{n-p}}\right]^{\frac{1}{p-1}} \frac{dt}{t}, \quad r > 0, \quad x \in \mathbb{R}^n.$$

Let $d\nu = (\mathbf{W}_{1,p}\omega)^q dx$. For each $r > 0$ let $d\mu_r = (\mathbf{U}_r\omega)^q dx$ and $d\lambda_r = (\mathbf{L}_r\omega)^q dx$. Then

$$\nu \le C(q)(\mu_r + \lambda_r) \tag{5.21}$$

Let $x \in \mathbb{R}^n$ and $B_r = B_r(x)$. Since $\mathbf{W}_{1,p}(\mathbf{W}_{1,p}\omega)^q = \mathbf{W}_{1,p}\nu$, we have to prove that

$$\mathbf{W}_{1,p}\nu(x) = \int_0^\infty \left[\frac{\nu(B_r)}{r^{n-p}}\right]^{\frac{1}{p-1}} \frac{dr}{r} \le C\,\mathbf{W}_{1,p}\omega(x).$$

For $r > 0$, $t \le r$ and $y \in B_r$ we have $B_t(y) \subset B_{2r}$. Therefore it is easy to see that $\mathbf{U}_r\omega = \mathbf{U}_r\omega_{B_{2r}}$ on B_r. Using this together with (5.17), we have

$$\mu_r(B_r) = \int_{B_r} (\mathbf{U}_r\omega)^q dx = \int_{B_r} (\mathbf{U}_r\omega_{B_{2r}})^q dx \le C\omega(B_{2r}).$$

Hence,

$$\begin{aligned} \int_0^\infty \left[\frac{\mu_r(B_r)}{r^{n-p}}\right]^{\frac{1}{p-1}} \frac{dr}{r} &\le C \int_0^\infty \left[\frac{\omega(B_{2r})}{r^{n-p}}\right]^{\frac{1}{p-1}} \frac{dr}{r} \\ &\le C'\,\mathbf{W}_{1,p}\omega(x). \end{aligned} \tag{5.22}$$

On the other hand, for $y \in B_r$ and $t \geq r$, we have $B_t(y) \subset B_{2t}$, and consequently

$$
\begin{aligned}
\mathbf{L}_r\omega(y) &\leq \int_r^\infty \left[\frac{\omega(B_{2t})}{t^{n-p}}\right]^{\frac{1}{p-1}} \frac{dt}{t} \\
&\leq C \int_{2r}^\infty \left[\frac{\omega(B_t)}{t^{n-p}}\right]^{\frac{1}{p-1}} \frac{dt}{t} \\
&\leq C\, \mathbf{L}_r\omega(x).
\end{aligned} \tag{5.23}
$$

Using (5.23), we obtain

$$
\lambda_r(B_r) = \int_{B_r} (\mathbf{L}_r\omega(y))^q dy \leq C(\mathbf{L}_r\omega(x))^q r^n.
$$

Thus,

$$
\begin{aligned}
\int_0^\infty \left[\frac{\lambda_r(B_r)}{r^{n-p}}\right]^{\frac{1}{p-1}} \frac{dr}{r} &\leq C' \int_0^\infty (\mathbf{L}_r\omega(x))^{\frac{q}{p-1}} r^{\frac{p}{p-1}} \frac{dr}{r} \\
&= C' \int_0^\infty \left[\int_r^\infty \left(\frac{\omega(B_t)}{t^{n-p}}\right)^{\frac{1}{p-1}} \frac{dt}{t}\right]^{\frac{q}{p-1}} r^{\frac{p}{p-1}} \frac{dr}{r} \\
&= C' \frac{q}{p} \int_0^\infty r^{\frac{p}{p-1}} [\mathbf{L}_r\omega(x)]^{\frac{q}{p-1}-1} \left[\frac{\omega(B_r)}{r^{n-p}}\right]^{\frac{1}{p-1}} \frac{dr}{r},
\end{aligned}
$$

where we have used integration by parts in the last equality. It then follows from (5.20) that

$$
\begin{aligned}
\int_0^\infty \left[\frac{\lambda_r(B_r)}{r^{n-p}}\right]^{\frac{1}{p-1}} \frac{dr}{r} &\leq C'' \int_0^\infty \left[\frac{\omega(B_r)}{r^{n-p}}\right]^{\frac{1}{p-1}} \frac{dr}{r} \\
&= C''\, \mathbf{W}_{1,p}\omega(x).
\end{aligned} \tag{5.24}
$$

Combining (5.21), (5.22) and (5.24) gives

$$
\mathbf{W}_{1,p}\nu(x) = \int_0^\infty \left[\frac{\nu(B_r)}{r^{n-p}}\right]^{\frac{1}{p-1}} \frac{dr}{r} \leq C\, \mathbf{W}_{1,p}\omega(x),
$$

for a suitable constant C independent of ω. Thus, (iv) implies (v) as claimed which completes the proof of the theorem. □

In view of condition (5.16) in the above theorem, we can now deduce a simple sufficient condition for the solvability of (5.14) from the known lower estimate of capacity in terms of Lebesgue measure (see, e.g., [AH], p. 39):

$$|E|^{1-\frac{pq}{n(q-p+1)}} \leq C\,\mathrm{Cap}_{\mathbf{I}_p,\,\frac{q}{q-p+1}}(E).$$

Corollary 5.5. *Suppose that $f \in L^{\frac{n(q-p+1)}{pq},\infty}(\mathbb{R}^n)$ and $d\omega = f dx$. If $q > p-1$ and $\frac{pq}{q-p+1} < n$, then equation (5.14) has a nonnegative solution for some $\epsilon > 0$.*

Remark 5.6. The condition $f \in L^{\frac{n(q-p+1)}{pq},\infty}(\mathbb{R}^n)$ in Corollary 5.5 can be relaxed by using the Fefferman–Phong condition [Fef]:

$$\int_{B_R} f^{1+\delta} dx \leq C R^{n-\frac{(1+\delta)pq}{q-p+1}}$$

for some $\delta > 0$, which is known to be sufficient for the validity of (5.15); see, e.g., [KS], [V2].

Chapter 6

Quasilinear equations on bounded domains

6.1 Sharp integral estimates

Let $\mathcal{Q} = \{Q\}$ be a Whitney decomposition of Ω, i.e., $\mathcal{Q}$ is a disjoint subfamily of the family of dyadic cubes in $\mathbb{R}^n$ such that $\Omega = \cup_{Q\in\mathcal{Q}} Q$, where we can assume that $2^5\text{diam}(Q) \leq \text{dist}(\text{Q}, \partial\Omega) \leq 2^7\text{diam}(Q)$. Let $\{\phi_Q\}_{Q\in\mathcal{Q}}$ be a partition of unity associated with the Whitney decomposition of Ω above: $0 \leq \phi_Q \in C_0^\infty(Q^*)$, $\phi_Q \geq 1/C(n)$ on $\overline{Q}$, $\sum_Q \phi_Q = 1$ and $|D^\gamma \phi_Q| \leq A_\gamma(\text{diam}(Q))^{-|\gamma}$ for all multi-indices γ. Here $Q^* = (1+\epsilon)Q$, $0 < \epsilon < \frac{1}{4}$ and $C(n)$ is a positive constant depending only on n such that each point in Ω is contained in at most $C(n)$ of the cubes Q^* (see [St1]).

The following theorem gives local estimates for solutions of quasilinear equations.

Theorem 6.1. *Let ω be a locally finite nonnegative measure on an open (not*

necessarily bounded) set Ω. Let $p > 1$ and $q > p - 1$. Suppose that there exists a nonnegative $\mathcal{A}$-superharmonic function u in Ω such that

$$-\mathrm{div}\mathcal{A}(\mathrm{x}, \nabla \mathrm{u}) = u^q + \omega \quad \text{in} \quad \Omega.$$

Then, for each cube $P \in \mathcal{Q}$ and compact set $E \subset \Omega$,

$$\mu_P(E) \leq C \, \mathrm{Cap}_{\mathbf{I}_p, \frac{q}{q-p+1}}(E) \tag{6.1}$$

if $\frac{pq}{q-p+1} < n$, and

$$\mu_P(E) \leq C(P) \, \mathrm{Cap}_{\mathbf{G}_p, \frac{q}{q-p+1}}(E) \tag{6.2}$$

if $\frac{pq}{q-p+1} \geq n$. Here $d\mu = u^q dx + d\omega$, and the constant C in (6.1) is independent of $P \in \mathcal{Q}$ and $E \subset \Omega$, but the constant $C(P)$ in (6.2) may depend on the side length of P.

Moreover, if $\frac{pq}{q-p+1} < n$ and Ω is a bounded C^∞-domain, then

$$\mu(E) \leq C \, \mathrm{cap}_{p, \frac{q}{q-p+1}}(E, \Omega)$$

for all compact sets $E \subset \Omega$, where $\mathrm{cap}_{p, \frac{q}{q-p+1}}(E, \Omega)$ is a capacity associated with the space $W^{\alpha, s}$, $\alpha = p$, $s = \frac{q}{q-p+1}$, relative to the domain Ω defined by

$$\mathrm{cap}_{\alpha, s}(E, \Omega) = \inf\{||f||^s_{W^{\alpha, s}(\mathbb{R}^n)} : f \in C_0^\infty(\Omega), f \geq 1 \text{ on } E\}. \tag{6.3}$$

Proof. Let P be a fixed dyadic cube in $\mathcal{Q}$. For a dyadic cube $P' \subset P$ we have

$$\mathrm{dist}(P', \partial\Omega) \geq \mathrm{dist}(P, \partial\Omega) \geq 2^5 \mathrm{diam}(P) \geq 2^5 \mathrm{diam}(P').$$

The lower estimate in Theorem 2.5 then yields

$$\begin{aligned} u(x) &\geq C\,\mathbf{W}_{1,p}^{2^3\mathrm{diam}(P')}\mu(x) \\ &\geq C\sum_{k=0}^{\infty}\int_{2^{-k+2}\mathrm{diam}(P')}^{2^{-k+3}\mathrm{diam}(P')}\Big[\frac{\mu(B_t(x))}{t^{n-p}}\Big]^{\frac{1}{p-1}}\frac{dt}{t} \\ &\geq C\sum_{Q\subset P'}\Big[\frac{\mu(Q)}{|Q|^{1-\frac{p}{n}}}\Big]^{\frac{1}{p-1}}\chi_Q(x) \end{aligned}$$

for all $x \in P'$. Thus it follows from Theorem 4.2 that

$$\sum_{Q\subset P'}\Big[\frac{\mu(Q)}{|Q|^{1-\frac{p}{n}}}\Big]^{\frac{q}{p-1}}|Q| \leq C\int_{P'}u^q dx \leq C\mu(P'), \qquad P'\subset P. \tag{6.4}$$

Hence

$$\mu(P') \leq C|P'|^{1-\frac{pq}{n(q-p+1)}}, \qquad P'\subset P. \tag{6.5}$$

To get a better estimate for $\mu(P')$ in the case $\frac{pq}{q-p+1} = n$, we observe that (6.4) is a dyadic Carleson condition. Thus by the dyadic Carleson imbedding theorem (see, e.g., [NTV], [V1]) we obtain, for $\frac{pq}{q-p+1} = n$,

$$\sum_{Q\subset P}\mu(Q)^{\frac{q}{p-1}}\Big[\frac{1}{\mu(Q)}\int_Q f d\mu\Big]^{\frac{q}{p-1}} \leq C\int_P f^{\frac{q}{p-1}}d\mu, \tag{6.6}$$

where $f \in L^{\frac{q}{p-1}}(d\mu_P)$, $f \geq 0$. From (6.6) with $f = \chi_{P'}$, one gets

$$\mu(P') \leq C\Big(\log\frac{2^n|P|}{|P'|}\Big)^{\frac{1-p}{q-p+1}}, \quad P'\subset P, \tag{6.7}$$

if $\frac{pq}{q-p+1} = n$. Now let P' be a dyadic cube in $\mathbb{R}^n$. From Wolff's inequality (5.6) we

have

$$\int_{\mathbb{R}^n} (\mathbf{I}_p \mu_{P'\cap P})^{\frac{q}{p-1}} dx \le C \sum_{Q\in\mathcal{D}} \Big[\frac{\mu_P(P'\cap Q)}{|Q|^{1-\frac{p}{n}}}\Big]^{\frac{q}{p-1}} |Q|$$

$$= C \sum_{Q\subset P'} \Big[\frac{\mu_P(Q)}{|Q|^{1-\frac{p}{n}}}\Big]^{\frac{q}{p-1}} |Q| + C \sum_{P'\subsetneq Q} \Big[\frac{\mu_P(P')}{|Q|^{1-\frac{p}{n}}}\Big]^{\frac{q}{p-1}} |Q|. \tag{6.8}$$

Thus, for $\frac{pq}{q-p+1} < n$, by combining (6.4) and (6.8) we deduce

$$\int_{\mathbb{R}^n} (\mathbf{I}_p \mu_{P'\cap P})^{\frac{q}{p-1}}\, dx \le C\, \mu_P(P'). \tag{6.9}$$

In the case $\frac{pq}{q-p+1} \ge n$, a similar argument using (6.4), (6.5), (6.7) and Wolff's inequality for Bessel potentials:

$$\int_{\mathbb{R}^n} (\mathbf{G}_p \mu_{P'\cap P})^{\frac{q}{p-1}} dx \le C(P) \sum_{Q\in\mathcal{D},\, Q\subset P} \Big[\frac{\mu_P(P'\cap Q)}{|Q|^{1-\frac{p}{n}}}\Big]^{\frac{q}{p-1}} |Q|,$$

(see [AH], Sec. 4.5), also gives

$$\int_{\mathbb{R}^n} (\mathbf{G}_p \mu_{P'\cap P})^{\frac{q}{p-1}} dx \le C(P)\mu_P(P'), \tag{6.10}$$

where the constant $C(P)$ may depend on the side length of P. Note that (6.9) which holds for all dyadic cubes P' in $\mathbb{R}^n$ is the well-known Kerman-Sawyer condition. Therefore by the results of [KS],

$$||\mathbf{I}_p f||_{L^{\frac{q}{q-p+1}}(d\mu_P)} \le C||f||_{L^{\frac{q}{q-p+1}}(dx)}$$

for all $f \in L^{\frac{q}{q-p+1}}(\mathbb{R}^n)$ which is equivalent to the capacitary condition

$$\mu_P(E) \le C\, \mathrm{Cap}_{\mathbf{I}_p,\, \frac{q}{q-p+1}}(E)$$

for all compact sets $E \subset \mathbb{R}^n$. Thus we obtain (6.1). The inequality (6.2) is proved in the same way using (6.10). From (6.1) and the definition of $\mathrm{cap}_{p,\frac{q}{q-p+1}}(\cdot,\Omega)$, we see that, for each cube $P \in \mathcal{Q}$,

$$\mu_P(E) \le C\mathrm{cap}_{p,\frac{q}{q-p+1}}(E \cap P, \Omega)$$

for all compact sets $E \subset \Omega$. Thus

$$\begin{aligned} \mu(E) &\le \sum_{P\in\mathcal{Q}} \mu_P(E) \\ &\le C \sum_{P\in\mathcal{Q}} \mathrm{cap}_{p,\frac{q}{q-p+1}}(E \cap P, \Omega) \\ &\le C\,\mathrm{cap}_{p,\frac{q}{q-p+1}}(E, \Omega), \end{aligned}$$

where the last inequality follows from the quasi-additivity of $\mathrm{cap}_{p,\frac{q}{q-p+1}}(\cdot,\Omega)$, which is considered in the next theorem. □

Let B_R be a ball such that $B_{2R} \subset \Omega$. It is easy to see that there exists a constant $c > 0$ such that $\ell(P) \ge cR$ for any Whitney cube P that intersects B_R. On the other hand, if B_r is a ball in B_R then we can find at most N dyadic cubes P_i with $c\,\frac{r}{4} \le \ell(P_i) < c\,\frac{r}{2}$ that cover B_r, where N depends only on n. Thus we can deduce from (6.7) the following corollary which gives an improved estimate in the critical case $q = \frac{n(p-1)}{n-p}$, $1 < p < n$.

Corollary 6.2. *Let ω, Ω, p, q and u be as in Theorem 6.1. Then in the case $\frac{pq}{q-p+1} = n$ we have*

$$\int_{B_r} u^q\,dx + \omega(B_r) \le C(\log \tfrac{2R}{r})^{\frac{1-p}{q-p+1}}$$

for all balls $B_r \subset B_R$ such that $B_{2R} \subset \Omega$.

Theorem 6.3. *Suppose that Ω is a C^∞-domain in $\mathbb{R}^n$. Then there exists a constant $C > 0$ such that*

$$\sum_{Q \in \mathcal{Q}} \mathrm{cap}_{p, \frac{q}{q-p+1}}(E \cap Q, \Omega) \leq C \mathrm{cap}_{p, \frac{q}{q-p+1}}(E, \Omega)$$

for all compact sets $E \subset \Omega$.

Proof. Obviously, we may assume that $\mathrm{cap}_{p, \frac{q}{q-p+1}}(E, \Omega) > 0$. Then by definition there exists $f \in C_0^\infty(\Omega)$, $f \geq 1$ on E such that

$$2\, \mathrm{cap}_{p, \frac{q}{q-p+1}}(E, \Omega) \geq ||f||^{\frac{q}{q-p+1}}_{W^{p, \frac{q}{q-p+1}}(\mathbb{R}^n)}.$$

By the refined localization principle on the smooth domain Ω for the function space $W^{p, \frac{q}{q-p+1}}$ (see, e.g., [Tri], Theorem 5.14) we have

$$||f||^{\frac{q}{q-p+1}}_{W^{p, \frac{q}{q-p+1}}(\mathbb{R}^n)} \geq C \sum_{Q \in \mathcal{Q}} ||f\phi_Q||^{\frac{q}{q-p+1}}_{W^{p, \frac{q}{q-p+1}}(\mathbb{R}^n)}.$$

Thus

$$\sum_{Q \in \mathcal{Q}} ||f\phi_Q||^{\frac{q}{q-p+1}}_{W^{p, \frac{q}{q-p+1}}(\mathbb{R}^n)} \leq C \mathrm{cap}_{p, \frac{q}{q-p+1}}(E, \Omega). \tag{6.11}$$

Note that for $x \in E \cap \overline{Q}$,

$$f\phi_Q \geq \phi_Q \geq 1/C(n).$$

Hence by definition we have

$$\mathrm{cap}_{p, \frac{q}{q-p+1}}(E \cap \overline{Q}, \Omega) \leq C ||f\phi_Q||^{\frac{q}{q-p+1}}_{W^{p, \frac{q}{q-p+1}}(\mathbb{R}^n)}.$$

From this and (6.11) we deduce the desired inequality. □

6.2 Criteria for solvability

We next give a sufficient condition for the existence of renormalized solutions to quasilinear equations on a bounded domain Ω, which is an analogue of Theorem 5.3 related to the case $\Omega = \mathbb{R}^n$. Its proof is based on stability results for renormalized solutions in place of the weak continuity of measures generated by $\mathcal{A}$-superharmonic functions used in the proof of Theorem 5.3.

Theorem 6.4. *Let $\omega \in \mathcal{M}_B^+(\Omega)$. Let $p > 1$ and $q > p - 1$. Suppose that*

$$\mathbf{W}_{1,p}^{2R}\,(\mathbf{W}_{1,p}^{2R}\omega)^q \le C\,\mathbf{W}_{1,p}^{2R}\omega \quad \text{a.e.},$$

where $R = \operatorname{diam}(\Omega)$,

$$C \le \Big(\frac{q-p+1}{qK\max\{1, 2^{p'-2}\}}\Big)^{q(p'-1)}\Big(\frac{p-1}{q-p+1}\Big),$$

and K is the constant in Theorem 3.11. Then there is a renormalized solution $u \in L^q(\Omega)$ to the Dirichlet problem

$$\begin{cases} -\operatorname{div}\mathcal{A}(x, \nabla u) = u^q + \omega \quad \text{in} \quad \Omega, \\ \qquad\qquad u = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{6.12}$$

such that

$$u(x) \le M\,\mathbf{W}_{1,p}^{2R}\omega(x)$$

for all x in Ω, where the constant M depends only on p, q, n, and the structural constants α and β.

Proof. By Lemma 3.9 we can find a nondecreasing sequence $\{u_k\}_{k\geq 0}$ of renormalized solutions to the following Dirichlet problems:

$$\begin{cases} -\mathrm{div}\mathcal{A}(x,\nabla u_0) = \omega & \text{in} \quad \Omega, \\ u_0 = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{6.13}$$

and

$$\begin{cases} -\mathrm{div}\mathcal{A}(x,\nabla u_k) = u_{k-1}^q + \omega & \text{in} \quad \Omega, \\ u_k = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{6.14}$$

for $k \geq 1$. By Theorem 3.11 we have

$$u_0 \leq K\,\mathrm{W}_{1,p}^{2R}\omega, \qquad u_k \leq K\,\mathrm{W}_{1,p}^{2R}(u_{k-1}^q + \omega).$$

Thus by arguing as in the proof of Theorem 5.3, we obtain a constant $M > 0$ such that

$$u_k \leq M\,\mathrm{W}_{1,p}^{2R}\omega < \infty \quad \text{a.e.}$$

for all $k \geq 0$. Therefore, $\{u_k\}$ converges pointwise to a nonnegative function u for which

$$u \leq M\,\mathrm{W}_{1,p}^{2R}\omega < \infty \quad \text{a.e.},$$

and $u_k^q \to u^q$ in $\mathrm{L}^1(\Omega)$. Finally, in view of (6.14), the stability result in [DMOP, Theorem 3.4] asserts that u is a renormalized solution of (6.12), which proves the theorem. $\square$

Existence results on a bounded domain Ω analogous to Theorems 5.4 are contained in the following two theorems, where Bessel potentials and the corresponding capacities are used in place of respectively Riesz potentials and Riesz capacities.

Theorem 6.5. *Let $\omega \in \mathcal{M}_B^+(\Omega)$ be compactly supported in Ω. Let $p > 1$, $q > p-1$, and let $R = \mathrm{diam}(\Omega)$. Then the following statements are equivalent.*

(i) *There exists a nonnegative renormalized solution $u \in L^q(\Omega)$ to the equation*

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, \nabla u) = u^q + \epsilon\,\omega & \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{6.15}$$

for some $\epsilon > 0$.

(ii) *For all compact sets $E \subset \Omega$,*

$$\omega(E) \le C\,\mathrm{Cap}_{\mathbf{G}_p, \frac{q}{q-p+1}}(E). \tag{6.16}$$

(iii) *The testing inequality*

$$\int_B [\mathbf{W}_{1,p}^{2R}\omega_B(x)]^q\,dx \le C\,\omega(B) \tag{6.17}$$

holds for all balls B such that $B \cap \mathrm{supp}\omega \neq \emptyset$.

(iv) *There exists a constant C such that*

$$\mathbf{W}_{1,p}^{2R}\,(\mathbf{W}_{1,p}^{2R}\omega)^q(x) \le C\,\mathbf{W}_{1,p}^{2R}\omega(x) \quad \text{a.e. on } \Omega. \tag{6.18}$$

Proof. Since ω is compactly supported in Ω, using Theorem 6.1 we have (i)$\Rightarrow$ (ii). Thus we need to show that (ii)$\Rightarrow$ (iii)$\Rightarrow$ (iv) $\Rightarrow$ (i). Note that the capacitary

inequality (6.16) is equivalent to the Kerman–Sawyer condition

$$\int_{\mathbb{R}^n} [\mathbf{G}_p \omega_B(x)]^{\frac{q}{p-1}}\, dx \le C\, \omega(B), \tag{6.19}$$

(see [KS], [V2]). Note also that

$$\int_{\mathbb{R}^n} [\mathbf{G}_p \mu(x)]^{\frac{q}{p-1}}\, dx \simeq \int_{\mathbb{R}^n} \Big[\int_0^{2R} \frac{\mu(B_t(x))}{t^{n-p}} \Big]^{\frac{q}{p-1}} dx, \tag{6.20}$$

where the constants of equivalence are independent of the measure μ, (see [HW], [AH]). Thus from (6.19), (6.20), and Theorem 5.1 we deduce the implication (ii)⇒(iii). By Theorem 6.4 we have (iv)⇒(i). Thus it is left to show that (iii)⇒(iv). In fact, the proof of this implication is similar to the proof of (iv)⇒(v) in Theorem 5.4. We will only sketch some crucial steps here. We define the "lower" and "upper" parts of the truncated Wolff's potential $\mathbf{W}_{1,p}^{2R}$ respectively by

$$\mathbf{L}_r^{2R}\mu(x) = \int_r^{2R} \Big[\frac{\mu(B_t(x))}{t^{n-p}} \Big]^{\frac{1}{p-1}} \frac{dt}{t}, \qquad 0 < r < 2R,\ x \in \mathbb{R}^n$$

and

$$\mathbf{U}_r^{2R}\mu(x) = \int_0^{r} \Big[\frac{\mu(B_t(x))}{t^{n-p}} \Big]^{\frac{1}{p-1}} \frac{dt}{t}, \qquad 0 < r < 2R,\ x \in \mathbb{R}^n.$$

Since $R = \operatorname{diam}(\Omega)$ and $\omega \in M_B^+(\Omega)$, to prove (6.18), it is enough to verify that, for $x \in \Omega$,

$$\int_0^{2R} \Big[\frac{\mu_r(B_r(x))}{r^{n-p}} \Big]^{\frac{1}{p-1}} \frac{dr}{r} \le C\, \mathbf{W}_{1,p}^{2R}\omega(x), \tag{6.21}$$

and

$$\int_0^{2R} \Big[\frac{\lambda_r(B_r(x))}{r^{n-p}} \Big]^{\frac{1}{p-1}} \frac{dr}{r} \le C\, \mathbf{W}_{1,p}^{2R}\omega(x), \tag{6.22}$$

where $d\mu_r = (\mathbf{U}_r^{2R}\omega)^q dx$, $d\lambda_r = (\mathbf{L}_r^{2R}\omega)^q dx$ and $0 < r < 2R$. The proof of (6.21) is the same as before. For the proof of (6.22), we need an estimate similar to (5.20) in the proof of Theorem 5.4. Namely,

$$\omega(B_t(x)) \le C\, t^{n-\frac{pq}{q-p+1}}, \tag{6.23}$$

for $0 < t < \frac{R}{2}$ and $x \in \Omega$. In fact, note that for $0 < t < \frac{R}{2}$ and $y \in B_t(x)$,

$$\begin{aligned} \mathbf{W}_{1,p}^{2R}\omega_{B_t(x)}(y) &\ge \int_{2t}^{2R} \Big[\frac{\omega(B_\tau(y)\cap B_t(x))}{\tau^{n-p}}\Big]^{\frac{1}{p-1}} \frac{d\tau}{\tau} \\ &\ge C(n,p)\Big[\frac{\omega(B_t(x))}{t^{n-p}}\Big]^{\frac{1}{p-1}}. \end{aligned}$$

Thus from this inequality and (6.17) we get (6.23). This completes the proof of (iii)⇒(iv), and hence Theorem 6.5 is proved. □

Remark 6.6. In the case where ω is not compactly supported in Ω, it can be easily seen from the proof of this theorem that any one of the conditions (ii)–(iv) above is still *sufficient* for the solvability of (6.15). Moreover, in the subcritical case $\frac{pq}{q-p+1} > n$, these conditions are redundant since the Bessel capacity $\mathrm{Cap}_{\mathbf{G}_p, \frac{q}{q-p+1}}$ of a single point is positive (see [AH], Sec. 2.6). This ensures that statement (ii) of Theorem 6.5 holds for some constant $C > 0$ provided ω is a finite measure.

Corollary 6.7. *Suppose that $f \in L^{\frac{n(q-p+1)}{pq},\infty}(\Omega)$ and $d\omega = f dx$. If $q > p-1$ and $\frac{pq}{q-p+1} < n$ then the equation (6.15) has a nonnegative renormalized (or equivalently, entropy) solution for some $\epsilon > 0$.*

6.3 Removable singularities of $-\mathrm{div}\mathcal{A}(x, \nabla u) = u^q$

We are now in a position to obtain a characterization of removable singularities for homogeneous quasilinear equations.

Theorem 6.8. *Let E be a compact subset of Ω. Then any solution u to the problem*

$$\begin{cases} u \text{ is } \mathcal{A}\text{-superharmonic in } \Omega \setminus E, \\ u \in L^q_{\mathrm{loc}}(\Omega \setminus E), \quad u \geq 0, \\ -\mathrm{div}\mathcal{A}(x, \nabla u) = u^q \quad \text{in} \quad \mathcal{D}'(\Omega \setminus E), \end{cases} \tag{6.24}$$

is also a solution to a similar problem with Ω in place of $\Omega \setminus E$ if and only if $\mathrm{Cap}_{\mathbf{G}_p, \frac{q}{q-p+1}}(E) = 0$.

Proof. Let us first prove the "only if" part of the theorem. Suppose that

$$\mathrm{Cap}_{\mathbf{G}_p, \frac{q}{q-p+1}}(E) = 0,$$

and u is a solution of (6.24). We have $\mathrm{cap}_{1,p}(E, \Omega) = 0$, where the capacity $\mathrm{cap}_{1,p}(\cdot, \Omega)$ is defined by (3.1). Thus u can be extended so that it is a nonnegative $\mathcal{A}$-superharmonic function in Ω (see [HKM]). Let $\mu[u]$ be the Radon measure on Ω associated with u, and let φ be an arbitrary nonnegative function in $C_0^\infty(\Omega)$. As in [BP, Lemme 2.2], we can find a sequence $\{\varphi_n\}$ of nonnegative functions in $C_0^\infty(\Omega \setminus E)$ such that

$$0 \leq \varphi_n \leq \varphi; \qquad \varphi_n \to \varphi \quad \mathrm{Cap}_{\mathbf{G}_p, \frac{q}{q-p+1}}\text{-quasi everywhere.} \tag{6.25}$$

By Fatou's lemma we have

$$\begin{aligned}\int_\Omega u^q\,\varphi\,dx &\le \liminf_{n\to\infty}\int_\Omega u^q\,\varphi_n\,dx\\ &= \liminf_{n\to\infty}\int_\Omega \varphi_n\,d\mu[u]\\ &\le \int_\Omega \varphi\,d\mu[u] < \infty.\end{aligned}$$

Therefore $u \in L^q_{\mathrm{loc}}(\Omega)$, and $\mu[u] \ge u^q$ in $\mathcal{D}'(\Omega)$. It is then easy to see that

$$-\mathrm{div}\mathcal{A}(x,\nabla u) = u^q + \mu^E \quad \text{in} \quad \mathcal{D}'(\Omega)$$

for some nonnegative measure μ^E such that $\mu^E(A) = 0$ for any Borel set $A \subset \Omega\setminus E$. Moreover, by Theorem 6.1 we have

$$\mu^E(E) \le C(E)\,\mathrm{Cap}_{\mathbf{G}_p,\frac{q}{q-p+1}}(E) = 0.$$

Thus $\mu^E = 0$ and u solves (6.24) with Ω in place of $\Omega \setminus E$.

The "if" part of the theorem is proved in the same way as in the linear case $p = 2$ using the existence results obtained in Theorem 6.5. We refer to [AP] for details. $\square$

Chapter 7

Hessian equations

In this chapter, we study a fully nonlinear counterpart of the theory presented in Chapters 5 and 6.

7.1 Hessian equations on $\mathbb{R}^n$

¿From Lemma 3.16 and Theorem 3.15 along with the weak continuity of Hessian measures (Theorem 2.7) we get the following existence theorem for fully nonlinear equations whose proof, which we will omit, is similar to that of Theorem 5.3 in the quasilinear case.

Theorem 7.1. *Let $\omega \in \mathcal{M}^+(\mathbb{R}^n)$, $1 \leq k < \frac{n}{2}$, and $q > k$. Suppose that*

$$\mathbf{W}_{\frac{2k}{k+1},k+1}\,(\mathbf{W}_{\frac{2k}{k+1},k+1}\omega)^q \leq C\,\mathbf{W}_{\frac{2k}{k+1},k+1}\omega < \infty \quad \text{a.e.},$$

where

$$C \leq \Big(\frac{q-k}{qK}\Big)^{q/k}\frac{k}{q-k},$$

and K is the constant in Theorem 3.15. Then there exists $u \geq 0$, $u \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$, such that $-u \in \Phi^k(\mathbb{R}^n)$ and

$$\begin{cases} \inf_{x\in\mathbb{R}^n} u(x) = 0, \\ F_k[-u] = u^q + \omega. \end{cases}$$

Moreover, u satisfies the two-sided estimate

$$c_1 \mathbf{W}_{\frac{2k}{k+1},\, k+1}\omega(x) \leq u(x) \leq c_2\, \mathbf{W}_{\frac{2k}{k+1},\, k+1}\omega(x)$$

for all x in $\mathbb{R}^n$, where the constants c_1, c_2 depend only on n, k, q.

We are now in a position to establish the main results of this section.

Theorem 7.2. *Let ω be a measure in $\mathcal{M}^+(\mathbb{R}^n)$, $1 \leq k < \frac{n}{2}$, and $q > k$. Then the following statements are equivalent.*

(i) *There exists a solution $u \geq 0$, $-u \in \Phi^k(\Omega) \cap L^q_{\mathrm{loc}}(\mathbb{R}^n)$, to the equation*

$$\begin{cases} \inf_{x\in\mathbb{R}^n} u(x) = 0 \\ F_k[-u] = u^q + \epsilon\,\omega \quad \text{in} \quad \mathbb{R}^n \end{cases} \tag{7.1}$$

for some $\epsilon > 0$.

(ii) *The testing inequality*

$$\int_B [\mathbf{I}_{2k}\omega_B(x)]^{\frac{q}{k}} dx \leq C\,\omega(B) \tag{7.2}$$

holds for all balls B in $\mathbb{R}^n$.

(iii) *For all compact sets $E \subset \mathbb{R}^n$,*

$$\omega(E) \leq C\,\mathrm{Cap}_{\mathbf{I}_{2k},\, \frac{q}{q-k}}(E). \tag{7.3}$$

(iv) *The testing inequality*

$$\int_B [\mathbf{W}_{\frac{2k}{k+1},\,k+1}\omega_B(x)]^q\,dx \le C\,\omega(B) \tag{7.4}$$

holds for all balls B in $\mathbb{R}^n$.

(v) *There exists a constant C such that*

$$\mathbf{W}_{\frac{2k}{k+1},\,k+1}\,(\mathbf{W}_{\frac{2k}{k+1},\,k+1}\omega)^q(x) \le C\,\mathbf{W}_{\frac{2k}{k+1},\,k+1}\omega(x) < \infty \quad a.e. \tag{7.5}$$

Moreover, there is a constant $C_0 = C_0(n,k,q)$ such that if any one of the conditions (7.2)–(7.5) holds with $C \le C_0$, then equation (7.1) has a solution u with $\epsilon = 1$ which satisfies the two-sided estimate

$$c_1\,\mathbf{W}_{\frac{2k}{k+1},\,k+1}\omega(x) \le u(x) \le c_2\,\mathbf{W}_{\frac{2k}{k+1},\,k+1}\omega(x), \qquad x \in \mathbb{R}^n,$$

where c_1 and c_2 depend only on n,k,q. Conversely, if there is a solution u to (7.1) as in statement (i) *with $\epsilon = 1$, then conditions (7.2)–(7.5) hold with $C = C_1(n,k,q)$.*

Proof. The proof of Theorem 7.2 is completely analogous to that of Theorem 5.4 in the quasilinear case using $\mathbf{W}_{\frac{2k}{k+1},\,k+1}$ in place of $\mathbf{W}_{1,\,p}$ and Theorem 7.1 in place of Theorem 5.3. □

Corollary 7.3. *Suppose that $f \in L^{\frac{n(q-k)}{2kq},\,\infty}(\mathbb{R}^n)$ and $d\omega = f dx$. If $q > k$ and $\frac{2kq}{q-k} < n$ then* (7.1) *has a nonnegative solution for some $\epsilon > 0$.*

Remark 7.4. As in Remark 5.6, the condition $f \in L^{\frac{n(q-k)}{2kq},\,\infty}(\mathbb{R}^n)$ in Corollary 7.3 can be relaxed by using the Fefferman–Phong condition [Fef]:

$$\int_{B_R} f^{1+\delta} dx \le C\,R^{n-\frac{(1+\delta)2kq}{q-k}}$$

for some $\delta > 0$.

Since $\mathrm{Cap}_{I_\alpha,\, s}(E) = 0$ in the case $\alpha\, s \geq n$ for all sets $E \subset \mathbb{R}^n$ (see [AH], Sec. 2.6), from Theorems 6.5 and 7.2 we obtain the following Liouville-type theorems for quasilinear and Hessian differential inequalities.

Corollary 7.5. *If $q \leq \frac{n(p-1)}{n-p}$, then the inequality $-\mathrm{div}\mathcal{A}(x, \nabla u) \geq u^q$ admits no nontrivial nonnegative $\mathcal{A}$-superharmonic solutions in $\mathbb{R}^n$. Analogously, if $q \leq \frac{nk}{n-2k}$, then the inequality $F_k[-u] \geq u^q$ admits no nontrivial nonnegative solutions in $\mathbb{R}^n$.*

Remark 7.6. When $1 < p < n$ and $q > \frac{n(p-1)}{n-p}$, the function $u(x) = c|x|^{\frac{-p}{q-p+1}}$ with

$$c = \Big[\frac{p^{p-1}}{(q-p+1)^p}\Big]^{\frac{1}{q-p+1}} [q(n-p) - n(p-1)]^{\frac{1}{q-p+1}},$$

is a nontrivial admissible (but singular) global solution of $-\Delta_p u = u^q$ (see [SZ]). Similarly, the function $u(x) = c'|x|^{\frac{-2k}{q-k}}$ with

$$c' = \Big[\frac{(n-1)!}{k!(n-k)!}\Big]^{\frac{1}{q-k}} \Big[\frac{(2k)^k}{(q-k)^{k+1}}\Big]^{\frac{1}{q-k}} [q(n-2k) - nk]^{\frac{1}{q-k}},$$

where $1 \leq k < \frac{n}{2}$ and $q > \frac{nk}{n-2k}$, is a singular admissible global solution of $F_k[-u] = u^q$ (see [Tso] or [Tru2], formula (3.2)). Thus, we see that the exponent $\frac{n(p-1)}{n-p}$ (respectively $\frac{nk}{n-2k}$) is critical for the homogeneous equation $-\mathrm{div}\mathcal{A}(x, \nabla u) = u^q$ (respectively $F_k[-u] = u^q$) in $\mathbb{R}^n$. The situation is different when we restrict ourselves only to *locally bounded* solutions in $\mathbb{R}^n$ (see [GS], [SZ]).

7.2 Hessian equations on bounded domains

In this section, we consider the following fully nonlinear problem:

$$\begin{cases} F_k[-u] = u^q + \omega & \text{in} \quad \Omega, \\ u = \varphi & \text{on} \quad \partial\Omega \end{cases} \tag{7.6}$$

in the class of nonnegative functions u such that $-u$ is k-convex in a bounded uniformly $(k-1)$-convex domain Ω. Here ω is a nonnegative finite Radon measure which is regular enough near $\partial\Omega$ so that the boundary condition in (7.6) can be understood in the classical sense (see [TW1], [TW2]).

We first prove a theorem on the existence of solutions to Hessian equations with non-homogeneous boundary condition which is analogous to Theorem 6.4. However, due to the inhomogeneity we will need to take care of the boundary term. Moreover, the weak continuity of Hessian measures is used in place of the stability result for renormalized solutions in the quasilinear case.

Theorem 7.7. *Let Ω be a bounded uniformly $(k-1)$-convex domain in $\mathbb{R}^n$. Suppose that $\omega \in \mathcal{M}_B^+(\Omega)$ such that $\omega = \mu + f$, where $\mu \in \mathcal{M}^+(\Omega)$ with compact support in Ω, $0 \le f \in L^s(\Omega)$ with $s > \frac{n}{2k}$ if $1 \le k \le \frac{n}{2}$ and $s = 1$ if $\frac{n}{2} < k \le n$. Let $q > k$, $R = \operatorname{diam}(\Omega)$ and $0 \le \varphi \in C^0(\partial\Omega)$. Suppose that*

$$\mathbf{W}^{2R}_{\frac{2k}{k+1},k+1}(\mathbf{W}^{2R}_{\frac{2k}{k+1},k+1}\omega)^q \le A\mathbf{W}^{2R}_{\frac{2k}{k+1},k+1}\omega, \tag{7.7}$$

and

$$(\max_{\partial\Omega}\varphi)^{\frac{q}{k}-1} \le \frac{B^{\frac{k}{q}}}{2R^2|B_1(0)|^{\frac{1}{k}}}, \tag{7.8}$$

where A, B are positive constants such that

$$A \le \Big(\frac{q-k}{3^{\frac{q-1}{q}} qK}\Big)^{\frac{q}{k}} \Big(\frac{k}{q-k}\Big), \quad \text{and} \quad B \le \Big(\frac{q-k}{3^{\frac{q-1}{q}} qK^{\frac{q}{k}}}\Big)^{\frac{q}{k}} \Big(\frac{k}{q-k}\Big). \tag{7.9}$$

Here K is the constant in Theorem 3.15. Then there exists a function $u \ge 0$, $-u \in \Phi^k(\Omega) \cap L^q(\Omega)$, continuous near $\partial\Omega$ such that

$$\begin{cases} F_k[-u] = u^q + \omega \quad \text{in} \quad \Omega, \\ u = \varphi \quad \text{on} \quad \partial\Omega. \end{cases} \tag{7.10}$$

Moreover, there is a constant $C = C(n,k,q)$ such that

$$u \le C\Big\{ \mathbf{W}^{2R}_{\frac{2k}{k+1},k+1}\omega + \mathbf{W}^{2R}_{\frac{2k}{k+1},k+1}(\max_{\partial\Omega}\varphi)^q + \max_{\partial\Omega}\varphi \Big\}.$$

Proof. First observe by direct calculations that condition (7.8) is equivalent to

$$\mathbf{W}^{2R}_{\frac{2k}{k+1},k+1}\Big[\mathbf{W}^{2R}_{\frac{2k}{k+1},k+1}(\max_{\partial\Omega}\varphi)^q\Big]^q \le B\,\mathbf{W}^{2R}_{\frac{2k}{k+1},k+1}(\max_{\partial\Omega}\varphi)^q. \tag{7.11}$$

From Lemma 3.16 it follows that we can choose inductively a nondecreasing sequence $\{u_m\}$ of nonnegative functions on Ω such that

$$\begin{cases} F_k[-u_0] = \omega \quad \text{in} \quad \Omega, \\ u_0 = \varphi \quad \text{on} \quad \partial\Omega, \end{cases}$$

and

$$\begin{cases} F_k[-u_m] = u_{m-1}^q + \omega \quad \text{in} \quad \Omega, \\ u_m = \varphi \quad \text{on} \quad \partial\Omega, \end{cases} \tag{7.12}$$

for $m \geq 1$. Here for each $m \geq 0$, $-u_m$ is k-subharmonic and is continuous near $\partial\Omega$. By Theorem 3.15 we have

$$\begin{aligned} u_0 &\leq K\,\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}\omega + K\max_{\partial\Omega}\varphi \\ &= a_0\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}\omega + b_0\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}(\max_{\partial\Omega}\varphi)^q + K\max_{\partial\Omega}\varphi, \end{aligned}$$

where $a_0 = K$ and $b_0 = 0$. Thus

$$\begin{aligned} u_1 &\leq K\,\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}(u_0^q + \omega) + K\max_{\partial\Omega}\varphi \\ &\leq K\Big\{(3^{q-1}a_0^q)^{\frac{1}{k}}\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}\,(\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}\omega)^q + \\ &\quad (3^{q-1}b_0^q)^{\frac{1}{k}}\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}\left[\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}(\max_{\partial\Omega}\varphi)^q\right]^q + \\ &\quad K^{\frac{q}{k}}\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}(\max_{\partial\Omega}\varphi)^q + \mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}\omega\Big\} + K\max_{\partial\Omega}\varphi. \end{aligned}$$

Then by (7.7) and (7.11),

$$\begin{aligned} u_1 &\leq K[(3^{q-1}a_0^q)^{\frac{1}{k}}A + 1]\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}\omega + \\ &\quad K[(3^{q-1}b_0^q)^{\frac{1}{k}}B + K^{\frac{q}{k}}]\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}(\max_{\partial\Omega}\varphi)^q + K\max_{\partial\Omega}\varphi \\ &= a_1\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}\omega + b_1\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}(\max_{\partial\Omega}\varphi)^q + K\max_{\partial\Omega}\varphi, \end{aligned}$$

where

$$a_1 = K[(3^{q-1}a_0^q)^{\frac{1}{k}}A + 1], \qquad b_1 = K[(3^{q-1}b_0^q)^{\frac{1}{k}}B + K^{\frac{q}{k}}].$$

By induction we have

$$u_m \leq a_m\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}\omega + b_m\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}(\max_{\partial\Omega}\varphi)^q + K\max_{\partial\Omega}\varphi,$$

where

$$a_{m+1} = K[(3^{q-1}a_m^q)^{\frac{1}{k}}A + 1], \qquad b_{m+1} = K[(3^{q-1}b_m^q)^{\frac{1}{k}}B + K^{\frac{q}{k}}]$$

for all $m \geq 0$. It is then easy to see that

$$a_m \leq \frac{Kq}{q-k}, \quad \text{and} \quad b_m \leq \frac{K^{\frac{q}{k}+1}q}{q-k},$$

provided (7.9) is satisfied. Thus

$$\begin{aligned} u_m \quad \leq \quad & \frac{Kq}{q-k}\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}\omega + \\ & \frac{K^{\frac{q}{k}+1}q}{q-k}\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}(\max_{\partial\Omega}\varphi)^q + K\max_{\partial\Omega}\varphi. \end{aligned} \tag{7.13}$$

Using (7.7) and (7.13) we see that $u_m \uparrow u$ for a function $u \geq 0$ such that $-u$ is k-subharmonic and $u_m^q \to u^q$ in $L^1(\Omega)$. Thus in view of (7.12) and Theorem 2.7 we see that u is a desired solution of (7.10). □

We will omit the proof of the next theorem as it is completely analogous to the proof of Theorem 6.1 in the quasilinear case.

Theorem 7.8. *Let ω be a locally finite nonnegative measure on an open (not necessarily bounded) set Ω. Let $1 \leq k \leq n$ and $q > k$. Suppose that $u \geq 0$, $-u \in \Phi^k(\Omega)$, such that u is a solution to*

$$F_k[-u] = u^q + \omega \qquad \text{in} \quad \Omega.$$

Then for each cube $P \in \mathcal{Q}$, where $\mathcal{Q} = \{Q\}$ is a Whitney decomposition of Ω as before (see Sec. 6.1), we have

$$\mu_P(E) \leq C \operatorname{Cap}_{\mathbf{I}_{2k}, \frac{q}{q-k}}(E), \tag{7.14}$$

if $\frac{2kq}{q-k} < n$, and

$$\mu_P(E) \leq C(P) \operatorname{Cap}_{\mathbf{G}_{2k}, \frac{q}{q-k}}(E), \tag{7.15}$$

if $\frac{2kq}{q-k} \geq n$ for all compact sets $E \subset \Omega$. Here $d\mu = u^q dx + d\omega$, and the constant C in (7.14) does not depend on $P \in \mathcal{Q}$ and $E \subset \Omega$; however, the constant $C(P)$ in (7.15) may depend on the side length of P.

Moreover, if $\frac{2kq}{q-k} < n$, and Ω is a bounded C^∞-domain then

$$\mu(E) \leq C \operatorname{cap}_{2k, \frac{q}{q-k}}(E, \Omega)$$

for all compact sets $E \subset \Omega$, where $\operatorname{cap}_{2k, \frac{q}{q-k}}(E, \Omega)$ is defined by (6.3).

Remark 7.9. Let B_R be a ball such that $B_{2R} \subset \Omega$. Then in the critical case $q = \frac{nk}{n-2k}$, $(k < \frac{n}{2})$, as in Corollary 6.2 we have

$$\mu(B_r) \leq C(\log \tfrac{2R}{r})^{\frac{-k}{q-k}}$$

for all balls $B_r \subset B_R$.

We are now in a position to establish characterizations of solvability for Hessian equations in a bounded domain.

Theorem 7.10. *Let Ω be a uniformly $(k-1)$-convex domain in $\mathbb{R}^n$, and let $\omega \in \mathcal{M}_B^+(\Omega)$ be compactly supported in Ω. Suppose that $1 \le k \le n$, $q > k$, $R = \mathrm{diam}(\Omega)$, and $\varphi \in C^0(\partial\Omega)$, $\varphi \ge 0$. Then the following statements are equivalent.*

(i) *There exists a solution $u \ge 0$, $-u \in \Phi^k(\Omega) \cap L^q(\Omega)$, continuous near $\partial\Omega$, to the equation*

$$\begin{cases} F_k[-u] = u^q + \epsilon\,\omega & \text{in} \quad \Omega, \\ u = \epsilon\,\varphi & \text{on} \quad \partial\Omega \end{cases} \tag{7.16}$$

for some $\epsilon > 0$.

(ii) *For all compact sets $E \subset \Omega$,*

$$\omega(E) \le C\,\mathrm{Cap}_{\mathbf{G}_{2k},\,\frac{q}{q-k}}(E).$$

(iii) *The testing inequality*

$$\int_B [\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}\omega_B(x)]^q\,dx \le C\,\omega(B)$$

holds for all balls B such that $B \cap \mathrm{supp}\omega \ne \emptyset$.

(iv) *There exists a constant C such that*

$$\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}\,(\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}\omega)^q(x) \le C\,\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}\omega(x) \quad \text{a.e. on } \Omega.$$

Proof. The proof of this theorem is analogous to that of Theorem 7.2 in the quasilinear case. One only has to use Theorems 7.7 and 7.8 in place of Theorems 6.4 and 6.1 respectively. □

Remark 7.11. As in Remark 6.6, any one of the conditions (ii)–(iv) in Theorem 7.10 is still *sufficient* for the solvability of (7.16) if $d\omega = d\mu + f\,dx$, where $\mu \in \mathcal{M}_B^+(\Omega)$ is compactly supported in Ω and $f \in L^s(\Omega)$, $f \geq 0$ with $s > \frac{n}{2k}$ if $k \leq \frac{n}{2}$, and $s = 1$ if $k > \frac{n}{2}$. Moreover, in the subcritical case $\frac{2kq}{q-k} > n$ these conditions are redundant.

Corollary 7.12. *Let $d\omega = (f+g)\,dx$, where $f \geq 0$, $g \geq 0$, $f \in L^{\frac{n(q-k)}{2kq},\infty}(\Omega)$ is compactly supported in Ω, and $g \in L^s(\Omega)$ for some $s > \frac{n}{2k}$. If $q > k$ and $\frac{2kq}{q-k} < n$ then (7.16) has a nonnegative solution for some $\epsilon > 0$.*

The next theorem is on removable singularities for Hessian equations, an analogue of Theorem 6.8.

Theorem 7.13. *Let E be a compact subset of Ω. Then any solution u to the problem*

$$\begin{cases} -u \text{ is } k\text{-subharmonic in } \Omega \setminus E, \\ u \in L^q_{\mathrm{loc}}(\Omega \setminus E), \quad u \geq 0, \\ F_k[-u] = u^q \quad \text{in} \quad \mathcal{D}'(\Omega \setminus E), \end{cases} \tag{7.17}$$

is also a solution to a similar problem with Ω in place of $\Omega \setminus E$ if and only if $\mathrm{Cap}_{\mathbf{G}_{2k},\frac{q}{q-k}}(E) = 0$.

Proof. To prove this theorem, we will proceed as in the proof of Theorem 6.8. For the "only if" part, we may assume that $k < \frac{n}{2}$, since otherwise $\frac{2kq}{q-k} > n$, and so $E = \emptyset$. Note that if $\mathrm{Cap}_{\mathbf{G}_{2k},\frac{q}{q-k}}(E) = 0$ then $\mathrm{Cap}_{\mathbf{G}_{\frac{2k}{k+1}},k+1}(E) = 0$ (see [AH], Sec. 5.5), which implies that $\mathrm{cap}_k(E,\Omega) = 0$ due to Theorem 7.14 below. Here

$\mathrm{cap}_k(\cdot,\Omega)$ is the (relative) k-Hessian capacity associated with the domain Ω defined by (3.15). Thus by [L, Theorem 4.2], E is a k-polar set, i.e., it is contained in the $(-\infty)$-set of a k-subharmonic function in $\mathbb{R}^n$. Suppose that u is a solution of (7.17). It is easy to see that the function $\tilde{u}$ defined by

$$\tilde{u}(x) = \begin{cases} u(x), & x \in \Omega \setminus E, \\ \liminf\limits_{y \to x,\, y \notin E} u(y), & x \in E, \end{cases} \tag{7.18}$$

is an extension of u to Ω such that $-\tilde{u} \in \Phi^k(\Omega)$. The rest of the proof is then the same as in the quasilinear case. $\square$

Finally, we prove the local equivalence of the k-Hessian capacity and an appropriate Bessel capacity that is needed in the proof of Theorem 7.13 above.

Theorem 7.14. *Let $1 \le k < \frac{n}{2}$ be an integer. Then there are constants M_1, M_2 such that*

$$M_1 \,\mathrm{Cap}_{\mathbf{G}_{\frac{2k}{k+1}},\, k+1}(E) \le \mathrm{cap}_k(E,\Omega) \le M_2 \,\mathrm{Cap}_{\mathbf{G}_{\frac{2k}{k+1}},\, k+1}(E), \tag{7.19}$$

for any compact set $E \subset \overline{Q}$ with $Q \in \mathcal{Q}$. Furthermore, if Ω is a bounded C^∞-domain then

$$\mathrm{cap}_k(E,\Omega) \le C \,\mathrm{cap}_{\frac{2k}{k+1},\, k+1}(E,\Omega), \tag{7.20}$$

for any compact set $E \subset \Omega$, where $\mathrm{cap}_{\frac{2k}{k+1},\, k+1}(E,\Omega)$ is defined by (6.3) with $\alpha = \frac{2k}{k+1}$ and $s = k+1$.

Proof. Let R be the diameter of Ω. From Wolff's inequality (5.5) it follows that $\mathrm{Cap}_{\mathbf{G}_{\frac{2k}{k+1}},k+1}(E)$ is equivalent to

$$\sup\{\mu(E):\ \mu\in\mathcal{M}^+(E),\quad \mathbf{W}^{4R}_{\frac{2k}{k+1},k+1}\mu\le 1 \text{ on } \mathrm{supp}\mu\},$$

for any compact set $E\subset\Omega$ (see [HW], Proposition 5). To prove the left-hand inequality in (7.19), let $\mu\in\mathcal{M}^+(E)$ such that $\mathbf{W}^{4R}_{\frac{2k}{k+1},k+1}\mu\le 1$ on $\mathrm{supp}\mu$, and let $u\in\Phi^k(B)$ be a nonpositive solution of

$$\begin{cases} F_k[u]=\mu & \text{in} \quad B\\ u=0 & \text{on} \quad \partial B,\end{cases}$$

where B is a ball of radius R containing Ω. By Theorem 3.15 and the boundedness principle for nonlinear potentials (see [AH], Sec. 2.6), we have

$$|u(x)|\le C\,\mathbf{W}^{4R}_{\frac{2k}{k+1},k+1}\mu(x)\le C,\qquad x\in B.$$

Thus

$$\mu(E)=\mu_k[u](E)\le C\,\mathrm{cap}_k(E,\Omega),$$

which shows that

$$\mathrm{Cap}_{\mathbf{G}_{\frac{2k}{k+1}},k+1}(E)\le C\,\mathrm{cap}_k(E,\Omega).$$

To prove the upper estimate in (7.19), we let $Q\in\mathcal{Q}$, and fix a compact set $E\subset\overline{Q}$. Note that for $\mu\in\mathcal{M}^+(E)$ and $x\in E$ we have

$$\mathbf{W}^{4R}_{\frac{2k}{k+1},k+1}\mu(x)=\mathbf{W}^{2\mathrm{diam}(Q)}_{\frac{2k}{k+1},k+1}\mu(x)+\int_{2\mathrm{diam}(Q)}^{4R}\left[\frac{\mu(E)}{t^{n-2k}}\right]^{\frac{1}{k}}\frac{dt}{t}.$$

Thus, for $k < \frac{n}{2}$,

$$\mathbf{W}^{4R}_{\frac{2k}{k+1},\,k+1}\mu(x) \le C\,\mathbf{W}^{2\mathrm{diam}(Q)}_{\frac{2k}{k+1},\,k+1}\mu(x), \qquad x \in E. \tag{7.21}$$

Now for $u \in \Phi^k(\Omega)$ such that $-1 < u < 0$ by Theorem 2.8 we obtain

$$\mathbf{W}^{2\mathrm{diam}(Q)}_{\frac{2k}{k+1},\,k+1}\mu_E(x) \le \mathbf{W}^{2\mathrm{diam}(Q)}_{\frac{2k}{k+1},\,k+1}\mu(x) \le C\,|u(x)| \le C$$

for all $x \in E$, where $\mu = \mu_k[u]$. Thus, we deduce from (7.21) that

$$\mathbf{W}^{4R}_{\frac{2k}{k+1},\,k+1}\mu_E(x) \le C, \qquad x \in E,$$

which implies

$$\mathrm{cap}_k(E,\Omega) \le C\mathrm{Cap}_{\mathbf{G}_{\frac{2k}{k+1}},\,k+1}(E). \tag{7.22}$$

Finally, if Ω is a C^∞-domain in $\mathbb{R}^n$, and $1 \le k < \frac{n}{2}$, then by (7.22) and the quasi-additivity of the capacity $\mathrm{cap}_{\frac{2k}{k+1},\,k+1}(\cdot,\Omega)$ (see Theorem 6.3) we obtain the global upper estimate (7.20) for the k-Hessian capacity. $\square$

Appendix A

Proof of Theorem 2.4

Theorem 2.4 was proved by Trudinger and Wang [TW4]. Here we give a detailed proof with some simplifications and modifications for the convenience of the reader.

Lemma A.1 ([HKM], Lemma 3.57)**.** *Suppose that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a nonnegative supersolution to $-\mathrm{div}\mathcal{A}(x, \nabla u) = 0$ in Ω. If $\eta \in C_0^\infty(\Omega)$, $\eta \geq 0$, and $\epsilon > 0$, then there exists a constant $C > 0$ such that*

$$\int_\Omega |\nabla u|^p u^{-1-\epsilon} \eta^p dx \leq C \int_\Omega u^{p-1-\epsilon} |\nabla \eta|^p dx.$$

Proof. By replacing u with $u + \delta$ for $\delta > 0$, we may assume that the function $v = u^{-\epsilon}\eta^p$ is a nonnegative function in $W_0^{1,p}(\Omega)$ with compact support. Thus from the inequality $\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla v dx \geq 0$ we obtain

$$\epsilon \int_\Omega \mathcal{A}(x, \nabla u) \eta^p u^{-1-\epsilon} \cdot \nabla u dx \leq p \int_\Omega \mathcal{A}(x, \nabla u) \eta^{p-1} u^{-\epsilon} \cdot \nabla \eta dx.$$

Using the structural condition (2.3) along with Hölder's inequality we then get

$$\begin{aligned}\alpha\epsilon\int_\Omega |\nabla u|^p u^{-1-\epsilon}\eta^p dx &\le \beta p\int_\Omega |\nabla u|^{p-1}u^{-\epsilon}\eta^{p-1}|\nabla\eta|dx\\ &\le \beta p\Big(\int_\Omega |\nabla u|^p u^{-1-\epsilon}\eta^p dx\Big)^{\frac{p-1}{p}}\Big(\int_\Omega u^{p-1-\epsilon}|\nabla\eta|^p dx\Big)^{\frac{1}{p}}.\end{aligned}$$

Thus the lemma follows. □

Theorem A.2 ([HKM], Theorem 7.46)**.** *If u is $\mathcal{A}$-superharmonic in Ω, then $u \in L^s_{\mathrm{loc}}(\Omega)$ and $Du \in L^q_{\mathrm{loc}}(\Omega)$ whenever $0 < s < \frac{n(p-1)}{n-p}$ and $0 < q < \frac{n(p-1)}{n-1}$. Moreover, if u is nonnegative then for any ball B_R such that $B_{4R} \subset \Omega$ we have*

$$\Big(\frac{1}{|B_R|}\int_{B_R}|Du|^q dx\Big)^{\frac{1}{q}} \le C\,R^{-1}\operatorname{ess\,inf}_{B_R} u. \tag{A.1}$$

Proof. Let B_R be a ball such that $B_{4R} \subset \Omega$. We may assume that $u \ge 0$ in B_{4R}. Let $u_k = \min\{u, k\}$, k=1,2,... Then u_k is a supersolution in Ω and hence the weak Harnack's inequality [Tru1] implies that

$$\Big(\frac{1}{|B_R|}\int_{B_R} u_k^s dx\Big)^{\frac{1}{s}} \le C\operatorname{ess\,inf}_{B_R} u$$

for $0 < s < \frac{n(p-1)}{n-p}$. Thus letting $k \to \infty$ we obtain $u \in L^s_{\mathrm{loc}}(\Omega)$.

The integrability of Du follows from this result combined with the estimate in Lemma A.1. Indeed, let $0 < q < \frac{n(p-1)}{n-1}$ and $\epsilon > 0$. By Lemma A.1 with an

appropriate choice of η we have

$$\begin{aligned}\int_{B_R}|\nabla u_k|^q dx &= \int_{B_R}|\nabla u_k|^q u_k^{-(1+\epsilon)q/p}u_k^{(1+\epsilon)q/p}dx\\ &\le \Big(\int_{B_R}|\nabla u_k|^p u_k^{-1-\epsilon}dx\Big)^{q/p}\Big(\int_{B_R}u_k^{(1+\epsilon)q/(p-q)}dx\Big)^{(p-q)/p}\\ &\le C\Big(R^{-p}\int_{B_{2R}}u^{p-1-\epsilon}dx\Big)^{q/p}\Big(\int_{B_R}u^{(1+\epsilon)q/(p-q)}dx\Big)^{(p-q)/p}.\end{aligned}$$

Thus if we choose $0<\epsilon<p-1$ such that $\frac{(1+\epsilon)q}{p-q}<\frac{n(p-1)}{n-p}$ we obtain

$$\frac{1}{|B_R|}\int_{B_R}|\nabla u_k|^q dx \le C(R^{-1}\operatorname{ess\,inf}_{B_R} u)^q.$$

Finally, letting $k\to\infty$ we obtain estimate (A.1). □

Lemma A.3 ([TW4], Lemma 3.3)**.** *If u is nonnegative and $\mathcal{A}$-superharmonic in Ω then for any compact set $E\subset\Omega$ we have*

$$\mu[u](E)\le C\Big(\inf_{\Omega_{\delta/3}\setminus\overline{\Omega}_{2\delta/3}} u\Big)^{p-1},$$

where $C=C(n,p,\beta,\delta)$. Here $\delta=\frac{1}{2}\mathrm{dist}(E,\partial\Omega)$ and for $t>0$, $\Omega_t=\{x\in\Omega:$ $\mathrm{dist}(x,\,\partial\Omega)>t\}$.

Proof. Let u^δ be the *balayage* of u relative to $\overline{\Omega}_\delta$ in Ω (see [HKM], Chapter 8). We have u^δ is $\mathcal{A}$-harmonic in $\Omega\setminus\overline{\Omega}_\delta$, $u^\delta\le u$, and $u^\delta=u$ in Ω_δ. Since $u^\delta=u$ in an open neighborhood of E we get

$$\begin{aligned}\mu[u](E) &= \mu[u^\delta](E)\le\int_\Omega\varphi d\mu[u^\delta]\\ &= \int_\Omega\mathcal{A}(x,\,Du^\delta)\cdot\nabla\varphi dx\\ &\le C\Big(\int_{\mathrm{supp}\nabla\varphi}|Du^\delta|^p dx\Big)^{\frac{p-1}{p}},\end{aligned}$$

where $\varphi \in C_0^\infty(\Omega_{\delta/3})$, $\varphi \geq 0$, and $\varphi = 1$ in $\overline{\Omega}_{2\delta/3}$. Since $\mathrm{supp}\nabla\varphi \subset \Omega_{\delta/3} \setminus \overline{\Omega}_{2\delta/3}$ and $\mu[u^\delta] = 0$ in $\Omega \setminus \overline{\Omega}_\delta$, by Caccioppoli type estimate (see [HKM], Lemma 3.32) and Harnack's inequality we have

$$\mu[u](E) \leq C\Big(\sup_{\Omega_{\delta/3}\setminus\overline{\Omega}_{2\delta/3}} u^\delta\Big)^{p-1} \leq C\Big(\inf_{\Omega_{\delta/3}\setminus\overline{\Omega}_{2\delta/3}} u^\delta\Big)^{p-1} \leq C\Big(\inf_{\Omega_{\delta/3}\setminus\overline{\Omega}_{2\delta/3}} u\Big)^{p-1}.$$

This completes the proof of the lemma. □

Proof of Theorem 2.4. Let $E \subset \Omega$ be a compact set. We first prove that there is a subsequence $\{u_{j_k}\}$ of $\{u_j\}$ such that $Du_{j_k} \to Du$ a.e. on E. By truncation and a diagonal process we may assume that u_j, $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$. Fix $\epsilon > 0$. Let

$$h_j = (\mathcal{A}(x, \nabla u_j) - \mathcal{A}(x, \nabla u)) \cdot (\nabla u_j - \nabla u)$$

and let

$$E_j^\epsilon = \{x \in E : h_j(x) > \epsilon\}.$$

We then have

$$|E_j^\epsilon| \leq |E_j^\epsilon \cap \{|u_j - u| > \epsilon^2\}| + \frac{1}{\epsilon}\int_{E_j^\epsilon \cap |u_j - u| \leq \epsilon^2} h_j(x)dx. \tag{A.2}$$

Let

$$w_j = \begin{cases} u_j - u & \text{if} \quad |u_j - u| \leq \epsilon^2, \\ \epsilon^2 \frac{u_j - u}{|u_j - u|} & \qquad \text{otherwise.} \end{cases}$$

For $\eta \in C_0^\infty(\Omega)$, $\eta \geq 0$, and $\eta = 1$ on E, since

$$\int_{E_j^\epsilon \cap |u_j - u| \leq \epsilon^2} h_j(x)dx \leq \int_\Omega (\mathcal{A}(x, \nabla u_j) - \mathcal{A}(x, \nabla u)) \cdot \nabla w_j \eta dx,$$

we have

$$\begin{aligned}
\int_{E_j^\epsilon \cap |u_j - u| \le \epsilon^2} h_j(x)dx &\le \Big| \int_\Omega (\mathcal{A}(x, \nabla u_j) - \mathcal{A}(x, \nabla u)) \cdot \nabla(w_j \eta) dx \Big| \\
&\quad + \Big| \int_\Omega (\mathcal{A}(x, \nabla u_j) - \mathcal{A}(x, \nabla u)) \cdot w_j \nabla \eta dx \Big| \\
&\le \int_\Omega |w_j| \eta (d\mu[u_j] + d\mu[u]) \\
&\quad + C \int_\Omega |w_j| (|\nabla u_j|^{p-1} + |\nabla u|^{p-1}) dx \\
&\le C\epsilon^2,
\end{aligned}$$

where C is independent of ϵ and j due to Theorem A.2 and Lemma A.3. Thus from (A.2) we obtain

$$|E_j^\epsilon| \le |E_j^\epsilon \cap \{|u_j - u| > \epsilon^2\}| + C\epsilon.$$

It follows that $h_j \to 0$ in measure on E and hence there exists a subsequence $\{h_{j_k}\}$ of $\{h_j\}$ such that $h_{j_k} \to 0$ a.e. on E. Thus from the structural condition (2.4) we see that $\nabla u_{j_k} \to \nabla u$ a.e. on E.

To prove that $\mu[u_j] \to \mu[u]$ weakly as measures we let $\varphi \in C_0^\infty(\Omega)$. For any $\epsilon > 0$ we choose a set $F \subset \text{supp}\nabla\varphi$ with $|F| < \epsilon$ such that $Du_{j_k} \to Du$ uniformly

on $\mathrm{supp}\nabla\varphi \setminus F$. Then for k big enough and for $1 < r < \frac{n}{n-1}$ we have

$$\begin{aligned}
&\int_\Omega (\mathcal{A}(x,\, Du_{j_k}) - \mathcal{A}(x,\, Du)) \cdot \nabla\varphi dx \\
&\quad \le \epsilon_1 + C \int_F (|\mathcal{A}(x,\, Du_{j_k})| + |\mathcal{A}(x,\, Du)|) dx \\
&\quad \le \epsilon_1 + C \int_F (|Du_{j_k}|^{p-1} + |Du|^{p-1}) dx \\
&\quad \le \epsilon_1 + C|F|^{(r-1)/r} \int_F (|Du_{j_k}|^{(p-1)r} + |Du|^{(p-1)r}) dx \\
&\quad \le \epsilon_2,
\end{aligned}$$

in view of Theorem A.2. Thus $\mu[u_{j_k}] \to \mu[u]$ weakly as measures. Since the limit does not depend on the subsequence we have $\mu[u_j] \to \mu[u]$ weakly. $\square$

Appendix B

Proof of Theorem 2.5

In this appendix, we give a detailed proof of Theorem 2.5 on the Wolff's potential estimates for $\mathcal{A}$-superharmonic functions which is due originally to Kilpeläinen and Malý [KM2]. Our proof here is based on the approach of Trudinger and Wang [TW4] with some modifications.

For concentric balls $B_r \subset B_R$, $0 < r < R < \infty$, and $t > 0$, we let

$$P^t_{\overline{B}_r,\, B_R} = \inf\{v \geq 0 : v \text{ is } \mathcal{A}\text{-superharmonic in } B_R \text{ and } v \geq t \text{ on } \overline{B}_r\},$$

and let $\hat{P}^t_{\overline{B}_r,\, B_R}$ be the lower semicontinuous regularization of $P^t_{\overline{B}_r,\, B_R}$, i.e.,

$$\hat{P}^t_{\overline{B}_r,\, B_R}(x) = \lim_{r \to 0} \inf_{B_r(x)} P^t_{\overline{B}_r,\, B_R}.$$

Then

$$\hat{P}^t_{\overline{B}_r,\, B_R} = P^t_{\overline{B}_r,\, B_R} = \begin{cases} t & \text{in} \quad \overline{B}_r, \\ h & \text{in} \quad B_R \setminus \overline{B}_r, \end{cases}$$

where h is the unique solution of

$$\begin{cases} -\mathrm{div}\mathcal{A}(x,\, \nabla u) = 0 & \text{in} \quad B_R \setminus \overline{B}_r, \\ u = t & \text{on} \quad \partial\overline{B}_r, \\ u = 0 & \text{on} \quad \partial\overline{B}_R. \end{cases}$$

Note that $\hat{P}^1_{\overline{B}_r,\,B_R}$ is the $\mathcal{A}$-potential of $\overline{B}_r$ in B_R (see [HKM], Chapter 8) thus by Corollary 3.8 in [KM2] and Example 2.12 in [HKM] we have

$$\mu[\hat{P}^1_{\overline{B}_r,\,B_R}](B_R) \simeq \begin{cases} |R^{\frac{p-n}{p-1}} - r^{\frac{p-n}{p-1}}|^{1-p} & \text{if} \quad p \neq n, \\ (\log \frac{R}{r})^{1-n} & \text{if} \quad p = n. \end{cases} \tag{B.1}$$

Moreover, by definition we have $\hat{P}^t_{\overline{B}_{19R/20},\,B_R} = t\hat{P}^1_{\overline{B}_{R/2},\,B_R}$ and thus from (2.5),

$$\mu[\hat{P}^t_{\overline{B}_r,\,B_R}](B_R) = t^{p-1}\mu[\hat{P}^1_{\overline{B}_r,\,B_R}](B_R). \tag{B.2}$$

We next introduce the idea of local smoothing of $\mathcal{A}$-superharmonic functions, which is called the *Poisson modification*. Suppose that u is an $\mathcal{A}$-superharmonic function and that $\omega \Subset \Omega$ is a regular open set. We define the Poisson modification u^ω of u in ω to be the function

$$u^\omega = \begin{cases} u & \text{in} \quad \Omega \setminus \omega, \\ \tilde{u} & \text{in} \quad \omega, \end{cases}$$

where

$$\tilde{u} = \inf\{v : v \text{ is } \mathcal{A}\text{-superharmonic in } \omega,\ v \geq u \text{ on } \partial\omega\}.$$

Lemma B.1 ([HKM], Lemma 7.14)**.** *The Poisson modification u^ω of u in ω is $\mathcal{A}$-superharmonic in Ω, $\mathcal{A}$-harmonic in ω, $u^\omega \leq u$ in Ω and $u^\omega = u$ on $\Omega \setminus \omega$.*

Proof. From the construction, $u^\omega \leq u$ in Ω. Next, choose an increasing sequence $\varphi_i \in C^\infty(\mathbb{R}^n)$ which converges to u in $\overline{\omega}$. Let $h_i \in C(\overline{\omega})$ be the unique $\mathcal{A}$-harmonic function in ω such that $h_i = \varphi_i$ on $\partial\omega$. Since h_i is increasing and $h_i \leq u$, the Harnack's convergence theorem implies that the function

$$h = \lim_{i\to\infty} h_i$$

is $\mathcal{A}$-harmonic in ω. Note that for $y \in \partial\omega$ and $i \in \mathbb{N}$,

$$\liminf_{x\to y} h(x) \geq \liminf_{x\to y} h_i(x) = \varphi_i(y)$$

and hence

$$\liminf_{x\to y} h(x) \geq u(y).$$

It follows that $h \geq u^\omega$ in ω. On the other hand, the comparison principle implies that $h_i \leq u^\omega$ in ω for all i and therefore u^ω is $\mathcal{A}$-harmonic in ω. Finally, by the pasting lemma (see [HKM], Lemma 7.9) we see that u^ω is $\mathcal{A}$-superharmonic in Ω. □

Lemma B.2 ([TW4], page 394)**.** *Let u be $\mathcal{A}$-superharmonic on an open set Ω with smooth boundary such that $u = 0$ on $\partial\Omega$ and $u \geq t > 0$ on a compact set $E \subset \Omega$. Then for any $\mathcal{A}$-superharmonic function v on Ω with $0 \leq v \leq t$, there holds*

$$\mu[v](E) \leq \mu[u](\Omega). \tag{B.3}$$

Proof. To prove (B.3), by replacing u by $(1+\delta)u$, and v by $(1-\delta)(v - \frac{t}{2}) + \frac{t}{2}$ for some $\delta > 0$ small, we may assume that $E \Subset \{u > t\}$ and $0 < v < t$ in $\overline{\Omega}$. Let $w = \min\{u, v\}$. Then $w = v$ in an open neighborhood of E and $w = u$ near $\partial\Omega$. It follows from Remark 2.3 that

$$\mu[v](E) = \mu[w](E) \leq \mu[w](\Omega) = \mu[u](\Omega),$$

which proves the lemma. □

The following lemma was first proved by Kilpeläinen and Malý in [KM1], Lemma 3.5. Here we give a proof due to Trudinger and Wang in [TW4], Lemma 5.1.

Lemma B.3 ([KM1], [TW4])**.** *If u is nonnegative and $\mathcal{A}$-superharmonic in the ball B_{2R}, then*

$$\Big[\frac{\mu[u](B_{9R/10})}{R^{n-p}}\Big]^{\frac{1}{p-1}} \le C \inf_{B_{R/2}} u.$$

Proof. By replacing u with the *balayage* of u relative to $\overline{B}_{9R/10}$ in B_R (see [HKM], Chapter 8), we may suppose that $u = 0$ on ∂B_R and $\mu[u] = 0$ in $B_R \setminus \overline{B}_{9R/10}$. By Harnack's inequality in the shell $B_R \setminus \overline{B}_{9R/10}$ we have

$$\sup_{\partial B_{19R/20}} u \le C \inf_{\partial B_{19R/20}} u.$$

Thus if $\tilde{u}$ is the Poisson modification of u in $B_{19R/20}$ we have

$$\sup_{\overline{B}_{19R/20}} \tilde{u} \le C \inf_{\overline{B}_{19R/20}} \tilde{u}. \tag{B.4}$$

Let $t = \sup_{\overline{B}_{19R/20}} \tilde{u}$. By Lemma B.2 we have

$$\mu[\tilde{u}](B_R) = \mu[\tilde{u}](\overline{B}_{19R/20}) \le \mu[\hat{P}^t_{\overline{B}_{19R/20},\, B_R}](B_R). \tag{B.5}$$

Note that $\mu[u](B_{9R/10}) \le \mu[\tilde{u}](B_R)$ since $u = \tilde{u}$ near ∂B_R, and $t \le C \inf_{B_{R/2}}$ by (B.4). Thus from (B.1), (B.2) and (B.5) we obtain

$$\mu[u](B_{9R/10}) \le C(\inf_{B_{R/2}})^{p-1} R^{n-p},$$

which proves the lemma. □

Lemma B.4. *Let u be nonnegative and $\mathcal{A}$-superharmonic in a ball B_{2R}. If $\mu[u] = 0$ in the set $\omega = (B_{5R/8} \setminus \overline{B}_{3R/8}) \cup (B_{11R/10} \setminus \overline{B}_{9R/10})$ then*

$$\sup_{\partial B_{R/2}} u - \sup_{\partial B_R} u \leq C \Big[\frac{\mu[u](B_R)}{R^{n-p}} \Big]^{\frac{1}{p-1}}. \tag{B.6}$$

Proof. Let $\omega^* = B_{4R/9} \cup (B_{10R/11} \setminus \overline{B}_{5R/9})$ so that $B_R \setminus \omega \subset \omega^*$. Let $u^* = u^{\omega^*}$ be the Poisson modification of u in ω^*. Then u^* is harmonic in $B_{11R/10} \setminus \partial\omega^*$ and hence the restriction of $\mu[u^*]$ to $B_{11R/10}$ is supported on $\partial\omega^*$. We also have $u^* \in W^{1,p}(B_R)$ (because it is locally bounded in $B_{11R/10}$), $\mu[u^*](B_R) = \mu[u](B_R)$ (because $u^* = u$ near ∂B_R), and $u^* = u$ on $\partial B_{R/2}$ (because $\partial B_{R/2} \subset B_{2R} \setminus \omega^*$). Let w be the $\mathcal{A}$-superharmonic function on B_R such that

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, \nabla w) = \mu[u^*] \quad \text{in} \quad B_R, \\ \qquad w = 0 \quad \text{on} \quad \partial B_R. \end{cases}$$

The existence of w is guaranteed by the fact that $\mu[u^*] \in W^{-1,p}(B_R)$ (since $u^* \in W^{1,p}(B_R)$) which allows the monotone operator theory to apply. Hence w satisfies the Harnack's inequality

$$\sup_E w \leq C \inf_E w,$$

where $E = B_{19R/20} \setminus \mathcal{N}_{R/100}(\partial\omega^*)$. Here $\mathcal{N}_\delta$ denotes the δ-neighborhood. Now replacing w by the Poisson modification of w in $\mathcal{N}_{R/100}(\partial\omega^*)$ we can suppose that w satisfies the Harnack's inequality in $B_{19R/20}$.

Since both w and u^* belong to $W^{1,p}(B_R)$, by comparison principle we have

$$w \geq u^* - \sup_{\partial B_R} u^* = u^* - \sup_{\partial B_R} u,$$

which gives

$$\sup_{\partial B_{R/2}} w \geq \sup_{\partial B_{R/2}} u^* - \sup_{\partial B_R} u = \sup_{\partial B_{R/2}} u - \sup_{\partial B_R} u. \tag{B.7}$$

Thus to prove (B.6) it is enough to prove that

$$\sup_{\partial B_{R/2}} w \leq C\Big[\frac{\mu[u^*](B_R)}{R^{n-p}}\Big]^{\frac{1}{p-1}} = C\Big[\frac{\mu[u](B_R)}{R^{n-p}}\Big]^{\frac{1}{p-1}}. \tag{B.8}$$

To this end we let $t = \inf_{\overline{B}_{R/2}} u$. By Lemma B.2 we have

$$\mu[\hat{P}^t_{\overline{B}_{R/2},\, B_R}](B_R) = \mu[\hat{P}^t_{\overline{B}_{R/2},\, B_R}](\overline{B}_{R/2}) \leq \mu[w](B_R). \tag{B.9}$$

Thus in light of (B.1) and (B.2) we get

$$t \leq C\left(\frac{\mu[w](B_R)}{R^{n-p}}\right)^{\frac{1}{p-1}},$$

which gives the desired estimate (B.8) by Harnack's inequality. □

Proof of Theorem 2.5. We first prove the lower estimate in (2.10). For any $0 < r \leq 2R$, let $\omega = B_{9r/8} \setminus \overline{B}_{3r/4}$, let u^ω be the Poisson modification of u in ω, and let $\tilde{u}$ be the Poisson modification of u^ω in $B_{7r/8}$. Since $\tilde{u} - \inf_{B_{9r/8}} \tilde{u} \geq 0$ in $B_{9r/8}$, by Lemma B.3 we have

$$\begin{aligned}\Big[\frac{\mu[\tilde{u}](B_{9r/10})}{r^{n-p}}\Big]^{\frac{1}{p-1}} &\leq C(\inf_{B_{7r/8}} \tilde{u} - \inf_{B_{9r/8}} \tilde{u}) \\ &\leq C(\inf_{B_{5r/8}} \tilde{u} - \inf_{B_{5r/4}} \tilde{u}).\end{aligned}$$

Observe that

$$\mu[u](B_{r/2}) = \mu[u^\omega](B_{r/2}) \leq \mu[u^\omega](B_{9r/10}) = \mu[\tilde{u}](B_{9r/10}),$$

since $u^\omega = u$ on $\overline{B}_{3r/4}$ and $u^\omega = \tilde{u}$ outside $B_{7r/8}$. Thus we obtain

$$\Big[\frac{\mu[u](B_{r/2})}{r^{n-p}}\Big]^{\frac{1}{p-1}} \le C(\inf_{B_{5r/8}} u - \inf_{B_{5r/4}} u), \tag{B.10}$$

where we have used the fact that $\inf_{B_{5r/4}} \tilde{u} = \inf_{\partial B_{5r/4}} \tilde{u} = \inf_{\partial B_{5r/4}} u = \inf_{B_{5r/4}} u$. We now let $R_j = 2^{-j}(2R)$, where $j = 0, 1, \dots,$ and let $r = R_j$ in (B.10). By summing up we obtain

$$\mathbf{W}^R_{1,p}\mu(x) \le C \sum_{j=0}^{\infty} \Big[\frac{\mu[u](B_{R_j/2})}{R_j^{n-p}}\Big]^{\frac{1}{p-1}} \le u(x).$$

To prove the upper estimate in (2.10), we set $R_j = R2^{-j}$, where $j \ge 0$. Let $u_s = u^{B_{R_s}}$, $s \ge 4$, be the Poisson modification of u in B_{R_s}. Then $u_s \uparrow u$ pointwise and hence $u_s(x) \uparrow u(x)$. We now let $u_s^{\omega_s}$, $s \ge 4$, be the Poisson modification of u_s in ω_s, where $\omega_s = \cup_{j=0}^{s}(B_{5R_j/4} \setminus \overline{B}_{3R_j/4})$. Then $u_s^{\omega_s}$ is $\mathcal{A}$-harmonic in $\omega \cup B_{3R_s/4}$ and $u_s^{\omega_s}(x) = u_s(x)$. By Lemma B.4 we have, for $m \ge 1$,

$$\sup_{\partial B_{R_m}} u_s^{\omega_s} - \sup_{\partial B_{R_{m-1}}} u_s^{\omega_s} \le C\Big(\frac{\mu[u_s^{\omega_s}](B_{R_{m-1}})}{R_{m-1}^{n-p}}\Big)^{\frac{1}{p-1}},$$

which by summing up gives for any $j \ge 1$,

$$\begin{aligned}\sup_{\partial B_{R_j}} u_s^{\omega_s} &\le \sup_{\partial B_R} u_s^{\omega_s} + C\sum_{m=1}^{j}\Big(\frac{\mu[u_s^{\omega_s}](B_{R_{m-1}})}{R_{m-1}^{n-p}}\Big)^{\frac{1}{p-1}} \\ &\le \sup_{\partial B_R} u_s^{\omega_s} + C\sum_{m=0}^{s}\Big(\frac{\mu[u_s^{\omega_s}](B_{R_m})}{R_m^{n-p}}\Big)^{\frac{1}{p-1}}.\end{aligned}$$

Since $u_s^{\omega_s}$ is lower semicontinuous, we then obtain

$$\begin{aligned}u_s(x) &= u_s^{\omega_s}(x) \le \liminf_{y\to x} u_s^{\omega_s}(y) \le \lim_{j\to\infty} \sup_{\partial B_{R_j}} u_s^{\omega_s} \\ &\le \sup_{\partial B_R} u_s^{\omega_s} + C\sum_{m=0}^{s}\Big(\frac{\mu[u_s^{\omega_s}](B_{R_m})}{R_m^{n-p}}\Big)^{\frac{1}{p-1}},\end{aligned}$$

which in light of Harnack's inequality gives

$$u_s(x) \le C \inf_{\partial B_R} u_s^{\omega_s} + C \sum_{m=0}^{s} \Big(\frac{\mu[u_s^{\omega_s}](B_{R_m})}{R_m^{n-p}} \Big)^{\frac{1}{p-1}}, \tag{B.11}$$

since $\mu[u_s^{\omega_s}] = 0$ in $B_{5R/4} \setminus \overline{B}_{3R/4}$. Note that $u_s^{\omega_s} \le u_s \le u$ in B_{2R}. Also, for $0 \le m \le s$,

$$\mu[u_s^{\omega_s}](B_{R_m}) \le \mu[u_s^{\omega_s}](B_{\frac{3}{2}R_m}) = \mu[u_s](B_{\frac{3}{2}R_m}) = \mu[u](B_{\frac{3}{2}R_m})$$

since $u = u_s = u_s^{\omega_s}$ near $\partial B_{\frac{3}{2}R_s}$. Thus we conclude from (B.11) that

$$u_s(x) \le C \inf_{\partial B_R} u + C \sum_{m=0}^{s} \Big(\frac{\mu[u](B_{\frac{3}{2}R_m})}{R_m^{n-p}} \Big)^{\frac{1}{p-1}}.$$

Finally, letting $s \to \infty$ we obtain

$$\begin{aligned} u(x) &\le C \inf_{\partial B_R} u + C \sum_{m=0}^{\infty} \Big(\frac{\mu[u](B_{\frac{3}{2}R_m})}{R_m^{n-p}} \Big)^{\frac{1}{p-1}} \\ &\le C \inf_{\partial B_R} u + C\, \mathbf{W}_{1,p}^{2R} \mu(x). \end{aligned}$$

This completes the proof of Theorem 2.5. □

Bibliography

[AH] D. R. Adams and L. I. Hedberg, *Function Spaces and Potential Theory.* Springer-Verlag, Berlin, 1996.

[AP] D. R. Adams and M. Pierre, *Capacitary strong type estimates in semilinear problems*, Ann. Inst. Fourier (Grenoble), **41** (1991), 117–135.

[BP] P. Baras and M. Pierre, *Critère d'existence des solutions positives pour des équations semi-linéaires non monotones*, Ann. I. H. Poincaré **3** (1985), 185–212.

[BBG] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, and J. L. Vázquez, *An L^1 theory of existence and uniqueness of nonlinear elliptic equations*, Ann. Scuola Norm. Sup. Pisa, Cl. Sci. **22** (1995), 241–273.

[BCDN] H. Berestycki, I. Capuzzo-Dolcetta, and L. Nirenberg, *Superlinear indefinite elliptic problems and nonlinear Liouville theorems*, Topol. Methods Nonlin. Anal. **4** (1994), 59–78.

[BMMP] M. F. Betta, A. Mercaldo, F. Murat, and M. M. Porzio, *Existence of renormalized solutions to nonlinear elliptic equations with a lower-order term and right-hand side a measure*, J. Math. Pures Appl. **80** (2003), 90–124.

[BV1] M. F. Bidaut-Véron, *Local and global behavior of solutions of quasilinear equations of Emden-Fowler type*, Arch. Rational Mech. Anal. **107** (1989), 293–324.

[BV2] M. F. Bidaut-Véron, *Necessary conditions of existence for an elliptic equation with source term and measure data involving p-Laplacian*, Proc. 2001 Luminy Conf. on Quasilinear Elliptic and Parabolic Equations and Systems, Electron. J. Differ. Equ. Conf. **8** (2002), 23–34.

[BV3] M. F. Bidaut-Véron, *Removable singularities and existence for a quasilinear equation with absorption or source term and measure data*, Adv. Nonlinear Stud. **3** (2003), 25–63.

[BVP] M. F. Bidaut-Véron and S. Pohozaev, *Nonexistence results and estimates for some nonlinear elliptic problems*, J. Anal. Math. **84** (2001), 1–49.

[BiD] I. Birindelli and F. Demengel, *Some Liouville theorems for the p-Laplacian*, Proc. 2001 Luminy Conf. on Quasilinear Elliptic and Parabolic Equations and Systems, Electron. J. Differ. Equ. Conf., **8** (2002), 35–46.

[BC] H. Brezis and X. Cabré, *Some simple nonlinear PDE's without solutions*, Boll. Unione Mat. Ital. **8**, Ser. 1-B (1998) 223–262.

[BMP] H. Brezis, M. Marcus, and A. Ponce, *Nonlinear equations with measures revisited*, preprint (2004).

[CNS] L. Caffarelli, L. Nirenberg, and J. Spruck, *The Dirichlet problem for nonlinear second-order elliptic equations. III. Functions of the eigenvalues of the Hessian*, Acta Math. **155** (1985), 261–301.

[COV] C. Cascante, J. M. Ortega, and I. E. Verbitsky, *Nonlinear potentials and two weight trace inequalities for general dyadic and radial kernels*, Indiana Univ. Math. J., **53** (2004), 845–882.

[ChWW] S.-Y. A. Chang, J. M. Wilson, and T. H. Wolff, *Some weighted norm inequalities concerning the Schrödinger operators*, Comment. Math. Helv. **60** (1985), 217–246.

[DMOP] G. Dal Maso, F. Murat, A. Orsina, and A. Prignet, *Renormalized solutions of elliptic equations with general measure data*, Ann. Scuol. Norm. Pisa (4) **28** (1999), 741–808.

[D] E. B. Dynkin, *Superdiffusions and Positive Solutions of Nonlinear Partial Differential Equations*. University Lecture Series **34**, Amer. Math. Soc., Providence, RI, 2004.

[Fef] C. Fefferman, *The uncertainty principle*, Bull. Amer. Math. Soc. **9** (1983), 129–206.

[FS] C. Fefferman and E. M. Stein, *Some maximal inequalities*, Amer. J. Math. **93** (1971) 107–115.

[FST] M. Fukushima, K. Sato, and S. Taniguchi, *On the closable part of pre-Dirichlet forms and the fine support of the underlying measures*, Osaka J. Math. **28** (1991), 517–535.

[GS] B. Gidas and J. Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations*, Commun. Pure Appl. Math. **34** (1981), 525–598.

[GT] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*. Reprint of the 1998 edition. Classics in Math. Springer–Verlag, Berlin–Heidelberg–New York, 2001.

[Gre] N. Grenon, *Existence results for semilinear elliptic equations with small measure data*, Ann. Inst. H. Poincaré Anal. Non Linéaire **19** (2002), 1–11.

[Gu] B. Guan, *The Dirichlet problem for a class of fully nonlinear elliptic equations*, Commun. PDE **19** (1994), 399–416.

[HW] L. I. Hedberg and T. Wolff, *Thin sets in nonlinear potential theory*, Ann. Inst. Fourier (Grenoble) **33** (1983), 161–187.

[HKM] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*. Oxford Univ. Press, Oxford, 1993.

[IM] T. Iwaniec and G. Martin, *Geometric Function Theory and Non-Linear Analysis*. Oxford Mathematical Monographs, Clarendon Press, Oxford, 2001.

[Iv] N. M. Ivochkina, *Solution of the Dirichlet problem for certain equations of Monge-Ampère type*, Math. USSR-Sb. **56** (1987), 403–415.

[JLM] P. Juutinen, P. Lindqvist, and J. Manfredi, *On the equivalence of viscosity solutions and weak solutions for a quasi-linear equation*, SIAM J. Math. Anal. **33** (2001), 699–717.

[KV] N. J. Kalton and I. E. Verbitsky, *Nonlinear equations and weighted norm inequalities*, Trans. Amer. Math. Soc. **351** (1999), 3441–3497.

[KS] R. Kerman and E. Sawyer, *The trace inequality and eigenvalue estimates for Schrödinger operators*, Ann. Inst. Fourier (Grenoble) **36** (1987), 207–228.

[Kil] T. Kilpeläinen, *p-Laplacian type equations involving measures*, Proc. ICM, Vol. III (Beijing, 2002), 167–176, Higher Ed. Press, Beijing, 2002.

[KM1] T. Kilpeläinen and J. Malý, *Degenerate elliptic equations with measure data and nonlinear potentials*, Ann. Scuola Norm. Sup. Pisa, Cl. Sci. **19** (1992), 591–613.

[KM2] T. Kilpeläinen and J. Malý, *The Wiener test and potential estimates for quasilinear elliptic equations*, Acta Math. **172** (1994), 137–161.

[Kr] N. V. Krylov, *Lectures on Fully Nonlinear Elliptic Equations*. Rudolph Lipschitz Lectures, Univ. of Bonn, 1994.

[L] D. A. Labutin, *Potential estimates for a class of fully nonlinear elliptic equations*, Duke Math. J. **111** (2002), 1–49.

[MZ] J. Malý and W. P. Ziemer, *Fine Regularity of Solutions of Elliptic Partial Differential Equations*. Math. Surveys Monogr. **51**, Amer. Math. Soc., Providence, RI, 1997.

[M1] V. G. Maz'ya, *On the continuity at a boundary point of solutions of quasilinear elliptic equations*, Vestnik Leningrad Univ. Math. **3** (1976), 225–242.

[M2] V. G. Maz'ya, *Sobolev Spaces*. Springer-Verlag, Berlin, 1985.

[Mi] P. Mikkonen, *On the Wolff's potential and quasilinear elliptic equations involving measures*, Ann. Acad. Sci. Fenn., Ser AI, Math. Dissert. **104** 1996, 1–71.

[MP] E. Mitidieri and S. I. Pokhozhaev, *Absence of positive solutions for quasilinear elliptic problems in* $\mathbb{R}^N$, Proc. Steklov Inst. Math. **227** (1999), 186–216.

[NTV] F. Nazarov, S. Treil, and A. Volberg, *The Bellman functions and two-weight inequalities for Haar multipliers*, J. Amer. Math. Soc. **12** (1999), 909–928.

[PV1] N. C. Phuc and I. E. Verbitsky, *Local integral estimates and removable singularities for quasilinear and Hessian equations with nonlinear source terms*, preprint (2005).

[PV2] N. C. Phuc and I. E. Verbitsky, *Quasilinear and Hessian equations of Lane–Emden type*, to appear in the Annals of Mathematics.

[S1] J. Serrin, *Local behavior of solutions of quasi-linear equations*, Acta. Math. **111** (1964), 247–302.

[S2] J. Serrin, *Isolated singularities of solutions of quasi-linear equations*, Acta. Math. **113** (1965), 219–240.

[SZ] J. Serrin and H. Zou, *Cauchy–Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities*, Acta. Math. **189** (2002), 79–142.

[St1] E. M. Stein, *Singular Integrals and Differentiability of Functions*. Princeton University Press, Princeton, New Jersey, 1970.

[St2] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*. Princeton University Press, Princeton, New Jersey, 1993.

[Tri] H. Triebel, *The Structure of Functions*. Monogr. Math. **95**. Birkhäuser Verlag, Basel–Boston–Berlin, 2001.

[Tru1] N. S. Trudinger, *On Harnack type inequalities and their applications to quasilinear elliptic equations*. Comm. Pure Appl. Math. **20** (1967), 721–747.

[Tru2] N. S. Trudinger, *On new isoperimetric inequalities and symmetrization*, J. reine angew. Math. **488** (1997), 203–220.

[Tru3] N. S. Trudinger, *Weak solutions of Hessian equations*, Commun. PDE **22** (1997), 1251–1261.

[TW1] N. S. Trudinger and X. J. Wang, *Hessian measures I*, Topol. Meth. Nonlin. Anal. **10** (1997), 225–239.

[TW2] N. S. Trudinger and X. J. Wang, *Hessian measures II*, Ann. Math. **150** (1999), 579–604.

[TW3] N. S. Trudinger and X. J. Wang, *Hessian measures III*, J. Funct. Anal. **193** (2002), 1–23.

[TW4] N. S. Trudinger and X. J. Wang, *On the weak continuity of elliptic operators and applications to potential theory*, Amer. J. Math. **124** (2002), 369–410.

[Tso] K. Tso, *On symmetrization and Hessian equations*, J. Anal. Math. **52** (1989), 94–106.

[Ur] J. Urbas, *On the existence of nonclassical solutions for two classes of fully nonlinear elliptic equations*, Indiana Univ. Math. J. **39** (1990), 355–382.

[V1] I. E. Verbitsky, *Imbedding and multiplier theorems for discrete Littlewood-Paley spaces*, Pacific J. Math., **176** (1996), 529–556.

[V2] I. E. Verbitsky, *Nonlinear potentials and trace inequalities*, Oper. Theory, Adv. Appl. **110** (1999), 323–343.

[V3] I. E. Verbitsky, *Superlinear equations, potential theory and weighted norm inequalities*, Nonlinear Analysis, Function Spaces and Applications. Proc. Spring School **5**, Prague, May 31-June 6, 1998, 1–47.

[V4] I. E. Verbitsky, *Relations between Poisson and Bessel capacities*, Appendix B in: "Superdiffusions and Positive Solutions of Nonlinear Partial

Differential Equations" by E. B. Dynkin, University Lecture Series **34**, Amer. Math. Soc., Providence, RI, 2004, 107–112.

[VW] I. E. Verbitsky and R. L. Wheeden, *Weighted norm inequalities for integral operators*, Trans. Amer. Math. Soc. **350** (1998), 3371–3391.

[Ver] L. Véron, *Elliptic equations involving measures*, Stationary Partial Differential Equations, Vol. I. Handbook for Partial Differential Equations, North-Holland, Amsterdam, 2004, 593–712.

VITA

Nguyen Cong Phuc was born on March 12, 1976, in Long An province , Vietnam. After attending the public schools in Ben Luc, Long An, he received a B.S. degree in Mathematics–Informatics from the National University of Ho Chi Minh City, Vietnam in 1998. Then he started his Ph.D. study at the University of Missouri-Columbia in January 2002. He plans to graduate in May 2006 after which he will go to Purdue University for a three-year Research Assistant Professor position.