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LORENTZIAN WARPED PRODUCTS
AND STATIC SPACE-TIMES

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ABSTRACT

Let (M,g) be a Lorentzian manifold, (H,h) a Riemannian manifold, and let $f: H \to (0,\infty)$ be an arbitrary smooth function. Then the product manifold $M \times H$ with Lorentzian metric $\bar{g} = (f^2 g) \oplus h$ is called a Lorentzian warped product and denoted by $M_f \times H$. In the case $(a,b)_f \times H$, $-\infty \leq a < b \leq \infty$, with metric $\bar{g} = (-f^2 dt^2) \oplus h$, the metric is static and the warped product is called a standard static space-time. Examples of such manifolds include Minkowski space-time, Schwarzschild space-time, universal anti-de Sitter space-time, and the Einstein static universe.

Geodesic completeness in products of the form $M_f \times H$ is considered. A standard static space-time $(a,b)_f \times H$ with a or b finite is timelike, null, and spacelike geodesically incomplete. However, in $\mathbb{R}_f \times H$ the geodesic completeness depends on the warping function $f: H \to (0,\infty)$ and completeness of the Riemannian space (H,h). We say that f satisfies the K-growth condition if for all $\alpha > 0$, there is a compact set K in H such that $f(x) \geq \alpha$ for all $x \in H \backslash K$. Then a sufficient condition for timelike geodesic complete-

ness of $\mathbb{R}_f \times H$ is that $f: H \to (0,\infty)$ satisfy the K-growth condition. For null geodesic completeness of $\mathbb{R}_f \times H$, it is sufficient that (1) (H,h) be a complete Riemannian manifold and (2) that $f: H \to [m,\infty)$ be bounded from below by $m > 0$.

We investigate conditions on a standard static space-time which guarantee that the strong energy, null convergence, and generic conditions are satisfied. If the warping function f is convex and $\mathrm{Ric}(v,v) \geq 0$ for all $v \in TH$, then $(a,b)_f \times H$ satisfies the strong energy condition. If, in addition, f is strictly convex, then $(a,b)_f \times H$ also satisfies the generic condition.

Also, we study causality and other elementary properties of doubly warped products. If $f: H \to (0,\infty)$ and $e: M \to (0,\infty)$ are smooth functions, then a Lorentzian doubly warped product is the manifold $M \times H$ equipped with the metric $\bar{g} = f^2 g \oplus e^2 h$.

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CHAPTER 1

Introduction

Lorentzian manifolds have been studied for their importance in the theory of general relativity since the early part of this century. Interest has increased since the late 1960's, especially as a result of the singularity theorems of Hawking and Penrose (1970). Techniques developed to study Riemannian geometry - employing methods from topology and differential geometry - have been used fruitfully to study Lorentzian manifolds as well as the more general pseudo-Riemannian manifolds. In addition, techniques developed more specifically for Lorentzian manifolds and motivated by the physical applications to general relativity, e.g. causal structure (cf. Chapter 2), have led to much progress in understanding these manifolds. In this paper we will be studying properties of a large class of pseudo-Riemannian manifolds called doubly warped products. Much of our attention will be on a class of Lorentzian warped products: the standard static space-times (Definition 2.3).

If (M,g) and (H,h) are two pseudo-Riemannian manifolds, there is a natural metric g_0 defined on the product $M \times H$ so that $(M \times H, g_0)$ is a pseudo-Riemannian manifold. Warped product manifolds are a larger class which are created by "warping" the natural metric g_0 on $M \times H$ by using warping functions defined on M and H. That is, if $e\colon M \to (0,\infty)$ and $f\colon H \to (0,\infty)$ are smooth functions, then $M_f \times {}_eH$ is the manifold $M \times H$ with the doubly warped metric $\bar{g} = f^2 g \oplus e^2 h$. Bishop and O'Neill (1969) studied the case when both (M,g) and (H,h) are Riemannian manifolds. Using warped products they were able to construct a wide variety of complete Riemannian manifolds with everywhere negative sectional curvature. Several authors have subsequently studied warped products in the Lorentzian and pseudo-Riemannian case [Beem and Ehrlich (1981), Beem and Powell (1982), Kemp (1981), Kobayashi and Obata (1980), O'Neill (1983), Powell (1982)]. Here they have turned out to be a useful and unifying concept. We will primarily focus on Lorentzian singly warped products which have the warping function f defined on the Riemannian factor space. If the interval (a,b) is given the negative definite metric $-dt^2$ and (H,h) is Riemannian, $f: H \to (0,\infty)$ a smooth function, then the singly warped product $(a,b)_f \times H$ with metric $-f^2\, dt^2 \oplus h$ is a static space-time, called a standard static space-time. Examples of such manifolds include the Schwarzschild space-time and universal anti-de Sitter space-time (cf. section 2.12).

Causality in a Lorentzian manifold refers to the general question of which points can be joined by nonspacelike curves. Relativistically, causality refers to which events can influence (or be influenced by) a given event. The definition of chronological future and causal future (and chronological past and causal past) are fundamental to the concept of causality. First, in a time-oriented connected Lorentzian manifold (M,g), the time orientation divides all null and timelike tangent vectors into two separate classes, called future and past-directed. Then the chronological future $I^+(p)$ is defined to be the set of all points in M which can be reached by a future-directed timelike curve from p. The causal future $J^+(p)$ is the set of all points in M which can be reached by a future-directed nonspacelike curve from p. The corresponding past versions are defined similarly using past-directed curves.

Various conditions which restrict the causal structure, and thereby rule out certain "causality violations", have been defined in the literature. For example, one might require that a Lorentzian manifold (M,g) not contain any closed timelike curves (i.e., $p \notin I^+(p)$ for all $p \in M$); for a closed timelike curve would allow a person to travel to his or her own past and this would lead to paradoxes. A more restrictive causality condition is that of global hyperbolicity. A Lorentzian manifold is said to be globally hyperbolic if it is strongly causal and if for each pair of points $p,q \in M$, the set $J^+(p) \cap J^-(q)$ is compact (see section 2.7 for the definition of strongly causal). In chapter 3 we discuss how the causal structure of Lorentzian doubly warped products depends on the factor spaces and the warping functions.

A basic result along these lines is that a Lorentzian doubly warped product $M_f \times {}_eH$ is globally hyperbolic if and only if (1) (M,g) is a globally hyperbolic space-time and (2) $(H,f^{-2}h)$ is a complete Riemannian manifold [Beem and

Powell (1982)]. Theorem 3.16 gives sufficient conditions for $M_f \times_e H$ to be globally hyperbolic that are more straightforward to calculate. If (M,g) is a globally hyperbolic space-time or else $(\mathbb{R}, -dt^2)$, (H,h) is a complete Riemannian manifold, and $f: H \to (0,\infty)$ and $e: M \to (0,\infty)$ are smooth functions, then $M_f \times_e H$ is globally hyperbolic if the integral $\int_0^\infty \frac{dr}{F(r)}$ diverges; where $F: (0,\infty) \to (0,\infty)$ is defined as the maximum value of $f(y)$ for points y in the closed ball of radius r about a fixed point p_0 in H. In particular, $M_f \times_e H$ is globally hyperbolic if $f: H \to (0,L]$ is bounded above by some $L < \infty$.

On the other hand, we state the following criterion for determining when $M_f \times_e H$ fails to be globally hyperbolic. Suppose (M,g) is a space-time or else $(\mathbb{R}, -dt^2)$ and (H,h) is a noncompact complete Riemannian manifold. Define $G: (0,\infty) \to (0,\infty)$ to be the minimum value of $f(y)$ for all points y that are a distance r from a fixed $p_0 \in H$. If $\int_0^\infty \frac{dr}{G(r)}$ is finite, then $M_f \times_e H$ is <u>not</u> globally hyperbolic. This result applies to universal anti-de Sitter space-time to show that it is not globally hyperbolic.

The singularity theorems of Hawking and Penrose were both an impetus to and a large step toward the understanding of the global properties of the space-times used as models in general relativity. The existence of singularities in many important space-times has been known for a long time but their existence was often thought to be due to certain unrealistic assumptions in the model, such as symmetries [cf. Tipler, Clarke, and Ellis (1980)]. Futhermore, to precisely define what one means by a singularity poses a problem [cf. Geroch (1968), Hawking and Ellis (1973)]. A minimum condition is certainly that the space-time be timelike or null geodesically incomplete. The theorems of Hawking and Penrose prove that large classes of physically realistic space-times are timelike or null geodesically incomplete, and hence are "singular".

The above considerations motivate us to study geodesic completeness in Lorentzian warped products in Chapter 4. It is shown that standard static space-times of the form $(a,b)_f \times H$ with (a,b) an interval and a or b finite are always timelike, null, and spacelike geodesically incomplete, independent of the warping function $f: H \to (0,\infty)$ (Propositions 4.2, 4.4, and 4.5). However, timelike and null geodesic completeness in a standard static space-time $\mathbb{R}_f \times H$ depends on the warping function. We say that a function $f: H \to (0,\infty)$ satisfies the K-growth condition if for all $\alpha > 0$ there is a compact set K in H such that $f(x) \geq \alpha$ for all $x \in H\backslash K$. Then a sufficient condition for timelike geodesic completeness of $\mathbb{R}_f \times H$ is that $f: H \to (0,\infty)$ satisfies the K-growth condition (Theorem 4.7). Here we require no condition of completeness on the Riemannian factor space (H,h). On the other hand, our most easily stated sufficient condition for null geodesic completeness is (1) completeness of (H,h) and (2) that $f: H \to [m,\infty]$ be bounded from below by a constant $m > 0$ (Theorem 4.12).

In chapter 5 we discuss conditions on a standard static space-time which guarantee that certain of the energy conditions are satisfied. In particular, we discuss the strong energy, null convergence and generic conditions. The strong energy condition is the requirement that the Ricci curvature $Ric(v,v)$ be nonnegative for all timelike tangent vectors v. The null convergence condition is that $Ric(v,v)$ be nonnegative for all null tangent vectors v. A precise definition of the generic condition is stated in section 5.1. Any physically realistic space-time should satisfy these conditions. For example, the strong energy condition says that on the average, gravity attracts. The conditions are important due to their role in proving the singularity theorems of Hawking and Penrose.

In section 5.2 we show that if the warping function $f: H \to (0,\infty)$ is convex and the Ricci curvature of the Riemannian manifold (H,h) is nonnegative, then the standard static space-time $(a,b)_f \times H$ satisfies the strong energy condition. If, in addition, f is strictly convex, then $(a,b)_f \times H$ also satisfies the generic condition. In case (H,h) is Ricci flat, we derive the interesting result that $(a,b)_f \times H$ satisfies the strong energy condition if and only if it satisfies the null convergence condition.

Chapter 2

Preliminaries

In this chapter we will state the basic definitions and background material. The basic references are [Beem and Ehrlich (1981), Hawking and Ellis (1973), O'Neill (1983)].

2.1 Pseudo-Riemannian Manifolds

In this paper a manifold M will be connected Hausdorff and paracompact. For $p \in M$, T_pM denotes the set of tangent vectors to M at p. The tangent bundle

$$TM = \bigcup_{p \in M} T_pM$$

is the set of all tangent vectors of M. A metric tensor g on M is a symmetric nondegenerate (0,2) tensor field on M of constant index. In other words, at each point $p \in M$, g smoothly assigns a scalar product g_p on the tangent space T_pM and the index of g_p is the same for all $p \in M$. A pseudo-Riemannian manifold (M,g) is a smooth manifold furnished with a metric tensor g. If the index of g is zero then g_p is a positive definite inner product and we say (M,g) is a Riemannian manifold. If the index of g is 1 and the dimension of M is ≥ 2 then we say (M,g) is a Lorentzian manifold. In this case, the sign convention $(-,+,+,\cdots,+)$ will be used.

Let (M,g) be a Lorentzian manifold. A nonzero tangent vector $v \in TM$ is timelike (resp. nonspacelike, null, spacelike) if $g(v,v) < 0$ (resp. ≤ 0, $= 0$, > 0). The category in which a vector falls is said to determine its causal character. A smooth section X of TM is a vector field on M. A continuous vector field X on M is timelike if $g(X(p),(X(p))$ < 0 for each $p \in M$. If M admits a timelike vector field $X:M \to TM$ then M is said to be time-oriented by X. The vector field X divides the set of nonspacelike tangent vectors at a point $p \in M$ into two classes; those that are future-directed and those that are past-directed. A nonspacelike tangent vector $v \in T_pM$ is said to be future-directed (resp. past-directed) if $g(X,v) < 0$ (resp. $g(X,v) > 0$).

Now we can state the following basic definition: A space-time (M,g) is a connected smooth Hausdorff manifold of dimension ≥ 2 which has a countable basis, a Lorentzian metric g with signature $(-,+,+,\cdots,+)$, and a time-orientation.

2.2 Geodesics

Let ∇ denote the Levi-Civita connection on a pseudo-Riemannian manifold (M,g). A smooth curve in (M,g) is said to be timelike (resp. nonspacelike, null, spacelike) if its tangent vector is always timelike (resp. nonspacelike, null, spacelike). A geodesic is a smooth curve $c: (a,b) \to M$ that moves by parallel displacement, that is, the geodesic differential equation $\nabla_{c'}c'(t) = 0$ is satisfied for all $t \in (a,b)$. Using linearity and the compatibility of ∇ with the metric g, we obtain the following:

$$\frac{d}{dt} g(c'(t),c'(t)) = 2g(\nabla_{c'}c'(t),c'(t)) = 0$$

for a geodesic c. Hence, $g(c'(t),c'(t))$ = constant for all $t \in (a,b)$ and thus, the tangent vector $c'(t)$ has the same causal character for all t. Geodesics are said to be timelike, nonspacelike, null, or spacelike depending on the causal character of $c'(t)$ for some t. A curve $\gamma:(a,b) \to M$ which can be reparameterized so that $\nabla_{\gamma'}\gamma'(t) = 0$ is said to be a pre-geodesic. A parameter t for a pre-geodesic γ for which $\nabla_{\gamma'}\gamma'(t) = 0$ is said to be an affine parameter. For timelike and spacelike geodesics, affine parameters correspond to a constant length parameterization of the geodesic.

A curve $\gamma: (a,b) \to M$ is inextendible to $t = b$ if the limit of $\gamma(t)$ does not exist as $t \to b-$. An inextendible geodesic is sometimes called a maximally extended geodesic.

2.3 Geodesic Completeness

In the Riemannian case, the Hopf-Rinow Theorem [Choquet, DeWitt, and Dillard (1982), Gromoll, Klingenberg, and Meyer (1968), Kobayashi and Nomizu (1963)] implies the equivalence of metric completeness and geodesic completeness. This theorem fails in the Lorentzian case, and consequently, the consideration of geodesic completeness is an important and more subtle task. Let (M,g) be an arbitrary Lorentzian manifold. A geodesic c in (M,g) with affine parameter t is said to be complete if it can be defined for $-\infty < t < \infty$. A past and future inextendible geodesic is said to be incomplete if it cannot be extended to arbitrarily large positive and negative values of an affine parameter. The Lorentzian manifold (M,g) is said to be timelike (resp. nonspacelike, null, spacelike) geodesically complete if all timelike (resp. nonspacelike, null, spacelike) inextendible geodesics are complete. (M,g) is geodesically complete if all inextendible geodesics are complete. On the other hand, (M,g) is said to be timelike (resp. nonspacelike, null, spacelike) geodesically incomplete if one timelike (resp. nonspacelike, null spacelike) inextendible geodesic is incomplete. Finally, a nonspacelike incomplete space-time is said to be a geodesically singular space-time [Geroch 1968]. It is interesting to note that the three conditions, spacelike completeness, null completeness and timelike completeness, are independent [Beem and Ehrlich (1981)].

2.4 Exponential Map

Now the exponential map $\exp_p: T_pM \to M$ will be defined. Given $v \in T_pM$, let $c_v(t)$ denote the unique geodesic in M with $c_v(0) = p$ and $c_v'(0) = v$. Then $\exp_p(v)$ is defined by $\exp_p(v) = c_v(1)$, provided $c_v(1)$ exists.

Let $v_1, \ldots, v_n$ be any basis for T_pM. For sufficiently small $(x^1, x^2, \ldots x^n) \in \mathbb{R}^n$, the map

$$x^1v_1 + \ldots + x^nv_n \to \exp_p(x^1v_1 + x^2v_2 + \ldots + x^nv_n)$$

is a diffeomorphism of a neighborhood of the origin in T_pM onto a neighborhood U(p) of p in M. So a coordinate chart for M can be defined by assigning coordinates $(x^1, x^2, \ldots, x^n)$ to the point $\exp_p(x^1v_1 + \ldots + x^nv_n)$ in U(p). These coordinates are called normal coordinates based at p. The set U(p) is called a convex normal neighborhood of p if any two points in U(p) can be joined by a unique geodesic segment of (M,g) lying entirely in U(p) and if for any $q \in U(p)$

there are normal coordinates based at q containing U(p) [Hicks (1965, p. 133-136), Hawking and Ellis (1973)].

2.5 Operators

On a pseudo-Riemannian manifold, there are natural generalizations of the familiar differential operators of vector calculus on $\mathbb{R}^3$: gradient, divergence, and Laplacian. The gradient grad f of a function f: $M \to \mathbb{R}$ is the vector field corresponding to the (0,1) tensor field df on M. Thus, $Y(f) = df(Y) = g(\text{grad } f, Y)$ for an arbitrary vector field Y. In local coordinates, grad f is represented by

$$\text{grad } f = \sum_{i,j=1}^{n} g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j} .$$

For a tensor A the contraction of the new covariant slot in its covariant differential DA with one of its original slots is called a divergence div A of A. In two special cases the divergence is uniquely defined. For a vector field X, $\text{div } X = \sum_{i=1}^{n} g(E_i, E_i) g(\nabla_{E_i} X_i, E_i)$, where $E_1, E_2, \ldots, E_n$ is a frame field. Recall that a frame field is a set of n = dim M mutually orthogonal unit vector fields. The second case is that of a symmetric (0,2) tensor A. Then div A will be a one-form, or a (0,1) tensor field on M. For a frame field $E_1, E_2, \ldots, E_n$,

$$(\text{div } A)(X) = \sum_{i=1}^{n} g(E_i, E_i)(\nabla_{E_i} A)(E_i, X) .$$

The Hessian Hess(f) of a function $f: M \to \mathbb{R}$ is the second covariant differential of f. Equivalently, the Hessian Hess(f) of f is the symmetric (0,2) tensor field such that $\mathrm{Hess}(f)(X,Y) = g(\nabla_X \mathrm{grad}\, f, Y)$ for vector fields X,Y on M.

Finally, the Laplacian Δf of a function $f: M \to \mathbb{R}$ is the divergence of its gradient: $\Delta f = \mathrm{div}\,\mathrm{grad}\, f$.

2.6 Curvature

Let (M,g) be a pseudo-Riemannian manifold with Levi-Civita connection ∇. The Riemannian curvature tensor R is a (1,3) tensor field on M defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for vector field X,Y,Z where [,] denotes the Lie bracket. The trace of the Riemannian curvature tensor is the Ricci curvature Ric. Ric is a symmetric (0,2) tensor, and is given relative to a frame field by

$$\mathrm{Ric}(X,Y) = \sum_{i=1}^{n} g(E_i,E_i)\, g(R(E_i,Y)X,E_i).$$

The scalar curvature τ of M is the trace of its Ricci tensor. Thus, if $e_1,\ldots,e_n$ is a frame at p, i.e. an orthonormal basis of T_pM, then

$$\tau = \sum_{i=1}^{n} g(e_i,e_i)\,\mathrm{Ric}(e_i,e_i).$$

2.7 Causality

Causality in a Lorentzian manifold refers to the question of which points in the manifold can be joined by nonspacelike curves. The notation $p \ll q$ is used if there is a smooth future-directed timelike curve from p to q, and $p \leq q$ is used if either $p = q$ or there is a smooth future-directed nonspacelike curve from p to q. The <u>chronological future</u> $I^+(p)$ of p is the set $I^+(p) = \{q \in M : p \ll q\}$ and the <u>chronological past</u> $I^-(p) = \{q \in M : q \ll p\}$. The <u>causal future</u> $J^+(p)$ of p is $J^+(p) = \{q \in M : p \leq q\}$ and the <u>causal past</u> is $J^-(p) = \{q \in M : q \leq p\}$.

A number of causality conditions have been defined in general relativity. Some of these conditions will now be discussed in order of increasing strength. If a space-time (M,g) contains no closed timelike curves, then we say (M,g) is <u>chronological</u>. A space-time with no closed nonspacelike curves is said to be <u>causal</u>. An open set U in a space-time is said to be <u>causally convex</u> if no nonspacelike curve intersects U in a disconnected set. Given $p \in M$, the space-time (M,g) is said to be <u>strongly causal at p</u> if p has arbitrarily small causally convex neighborhoods. A <u>strongly causal</u> space-time is one that is strongly causal at each point.

Before continuing our discussion of causality conditions, we will briefly discuss the fine C^r topologies on the space Lor(M) of all Lorentzian metrics on M. The <u>fine C^r topologies</u> on Lor(M) may be defined by using a fixed countable locally finite covering $B = \{B_i\}$ of M by coordinate neighborhoods. Let $\delta: M \to (0,\infty)$ be a continuous function. Then $g_1, g_2 \in \text{Lor}(M)$ are said to be $\delta: M \to (0,\infty)$ close in the C^r topology, written $|g_1 - g_2|_r < \delta$, if for each $p \in M$ all of the corresponding coefficients and derivatives up to order r of the metric tensors g_1 and g_2 are $\delta(p)$ close at p when calculated in the fixed coordinates of all $B_i \in B$ which contain p. A basis for the fine C^r topology on Lor(M) consists of the sets $\{g_1 \in \text{Lor}(M) : |g_1 - g_2|_r < \delta\}$ for $g_2 \in \text{Lor}(M)$ arbitrary and $\delta: M \to (0,\infty)$ an arbitrary continuous function.

A space-time (M,g) is said to be <u>stably causal</u> if there is a fine C^0 neighborhood U(g) of g in Lor(M) such that each $g_1 \in U(g)$ is causal. A C^0 function $f: M \to \mathbb{R}$ is a <u>global time function</u> if f is strictly increasing along each future-directed timelike curve. A space-time is stably causal if and only if it admits a global time function [Hawking (1968), Seifert (1977)]. The final and strongest causality condition that we will discuss is global hyperbolicity. A strongly causal space-time (M,g) is said to be <u>globally hyperbolic</u> if for each pair of points $p,q \in M$, the set $J^+(p) \cap J^-(q)$ is compact. Globally hyperbolic space-times may be characterized using Cauchy surfaces. A subset of M which every inextendible nonspacelike curve intersects exactly once is called a <u>Cauchy surface</u>. A space-time is globally hyperbolic if and only if it admits a Cauchy surface [Hawking and Ellis (1973, p. 211-212)]. An important property of globally hyperbolic space-times is that any pair of causally related points may be joined by a nonspacelike geodesic segment of maximal length. [Avez (1963), Seifert (1967)].

2.8 Lorentzian Distance

A distance function can be defined for Lorentzian manifolds in an analogous fashion to the Riemannian distance. Let (M,g) be a Lorentzian manifold of dimension ≥ 2. Given $p,q \in M$, with $p \leq q$, let $\Omega_{p,q}$ be the path space of all future-directed nonspacelike curves $\gamma\colon [0,1] \to M$ with $\gamma(0) = p$, $\gamma(1) = q$. The Lorentzian are length functional $L = L_g\colon \Omega_{p,q} \to \mathbb{R}$ is then defined as follows: Given a piecewise smooth curve $\gamma \in \Omega_{p,q}$, choose a partition $0 = t_o < t_1 < \cdots < t_{n-1} < t_n = 1$ such that $\gamma|(t_i,t_{i+1})$ is smooth for each $i = 0,1,2,\ldots,n-1$. Define

$$L(\gamma) = L_g(\gamma) = \sum_{i=0}^{n-1} \int_{t=t_i}^{t_{i+1}} \sqrt{-g(\gamma'(t),\gamma'(t))}\,dt.$$

Now we can define the distance function $d\colon M \times M \to \mathbb{R}$. Given $p \in M$, if $q \notin J^+(p)$, set $d(p,q) = 0$. If $q \in J^+(p)$ set $d(p,q) = \sup\{L_g(\gamma) : \gamma \in \Omega_{p,q}\}$. The appearance of "sup" rather than "inf" in the preceding definition results in a duality of results on minimality in a Riemannian setting and maximality in the Lorentzian case.

Many properties of the Lorentzian distance function may be found in Beem and Ehrlich (1981). Here we mention only two. In globally hyperbolic space-times, the Lorentzian distance function is finite and continuous. In arbitrary Lorentzian manifolds, a reverse triangle inequality is satisfied. Explicitly, if $p \leq q \leq r$, then $d(p,r) \geq d(p,q) + d(q,r)$.

2.9 Submanifolds and the Induced Connection

Let N be an immersed submanifold of a pseudo-Riemannian manifold (M,g). If $i: N \to M$ denotes the inclusion map, by identifying $i_{*p}(T_pN)$ with T_pN, we may regard T_pN as being a subspace of T_pM. Let $g_0 = i^*g$ denote the pullback of the metric g for M to a symmetric tensor field on N. Under the identification of T_pN and $i_{*_p}(T_pN)$, we may also identify g_o at p and $g|T_pN \times T_pN$ for all $p \in N$. This identification will be used throughout this section. The tensor field g_o will be a metric on N if and only if each T_pN is a non-degenerate subspace of T_pM relative to g (i.e. for each $p \in N$ and nonzero $v \in T_pN$, there exists some $w \in T_pN$ with $g(v,w) \neq 0$) and the index of T_pN is the same for all p. In this case, we say N is a <u>nondegenerate submanifold</u> of (M,g). If in addition, $g|T_pN \times T_pN$ is positive definite for all $p \in N$, then N is said to be a <u>spacelike submanifold</u>. If $g|T_pN \times T_pN$ is a Lorentzian metric for each $p \in N$, then N is said to be a <u>timelike submanifold</u>.

For the rest of this section we will assume that N is a nondegenerate submanifold of (M,g). Then for each $p \in N$ there is a subspace $T_p^{\perp}N$ of T_pM given by

$$T_p^{\perp}N = \{v \in T_pM\colon g(v,w) = 0 \text{ for all } w \in T_pN\}$$

such that $T_pN \cap T_p^{\perp}N = \{0\}$. Consequently, we can define an orthogonal projection map $P\colon T_pM \to T_pN$. If ∇ is the Levi-Civita connection for (M,g), we can define $\nabla^0_X Y = P(\nabla_X Y)$ for vector field X,Y tangent to N. It can be verified that ∇^0 is actually the Levi-Civita connection for (N,g_0). [Hicks (1963), Cheeger and Ebin (1975, p. 22)] That is, ∇^0 is the unique torsion free connection on (N,g_0) satisfying

$$X(g_o(Y,Z)) = g_o(\nabla^0_X Y, Z) + g_o(Y, \nabla^0_X Z)$$

for all vector fields X,Y,Z on N.

The second fundamental form measures the difference between ∇ and ∇^0. Specifically, if $x,y \in T_pN$, extend them to local vector fields X,Y tangent to N and define $S(x,y) = \nabla_X Y - \nabla^0_X Y$. Then $S\colon T_pN \times T_pN \to T_p^{\perp}N$ is a symmetric, bilinear vector-valued tensor called the second fundamental form tensor or the shape tensor. Given $n \in T_p^{\perp}N$, define the second fundamental form $S_n\colon T_pN \times T_pN \to \mathbb{R}$ in the direction n by

$$S_n(x,y) = g(S(x,y),n) = g(\nabla_X Y|_p, n) = g(\nabla_X Y|_p - \nabla^0_X Y|_p, n).$$

It may be checked that the definition of S and S_n are independent of the extensions X,Y for $x,y \in T_pN$ and also that $S_n\colon T_pN \times T_pN \to \mathbb{R}$ is a symmetric bilinear form.

A nondegenerate submanifold N of (M,g) is said to be <u>totally geodesic</u> if the second fundamental form tensor $S = 0$ on N. The following result is well-known [Cheeger and Ebin (1975, p. 23), Beem and Ehrlich (1981, p. 55)].

Proposition 2.1

Let N be a nondegenerate submanifold of a pseudo-Riemannian manifold (M,g). The following are equivalent.

1) N is totally geodesic in M
2) Each geodesic γ of (M,g) with $\gamma(0) = p$ and $\gamma'(0) \in T_pN$ is contained in N in some neighborhood of p.
3) Every geodesic of N is also a geodesic of M.

A point $p \in N \subset M$ is said to be <u>umbilic</u> provided there is a normal vector $n \in T_p^{\perp}N$ such that $S(v,w) = g(v,w)n$ for all $v,w \in T_pN$. A nondegenerate submanifold N of (M,g) is <u>totally umbilic</u> provided every point of N is umbilic.

2.10 Warped Products

If (M,g) and (H,h) are two pseudo-Riemannian manifolds, there is a natural metric g_0 defined on the product manifold $M \times H$ so that $(M \times H, g_0)$ is a pseudo-Riemannian manifold. A larger class of manifolds, called warped products, have been studied by several authors. Bishop and O'Neill (1969) studied the case with (M,g) and (H,h) Riemannian manifolds. Using warped products they were able to construct a wide variety of complete Riemannian manifolds of everywhere negative sectional curvature. Several authors have subsequently studied warped products in the Lorentzian case [Beem and Ehrlich (1981), Beem, Ehrlich, and Powell (1982), Beem and Powell (1982), Kemp (1981), Kobayashi and Obata (1980), O' Neill (1983), Powell (1982)].

Definition 2.2:

Let (M,g) and (H,h) be pseudo-Riemannian manifolds and let $e: M \to (0,\infty)$ and $f: M \to (0,\infty)$ be smooth functions. Let $\pi: M \times H \to M$ and $\eta: M \times H \to H$ be the projections. The doubly warped product $M_f \times {}_eH$ is the product manifold $M \times H$ furnished with the metric tensor $\bar{g}$ defined by

$$\bar{g}(v,w) = (f\circ\eta)^2(p)g(\pi_* v,\pi_* w) + (e\circ\pi)^2(p)h(\eta_* v,\eta_* w)$$

for $v,w \in T_p(M \times H)$.

In classical notation the line element for $M_f \times {}_eH$ is

$$ds^2 = f(\eta)^2 d\sigma_1^2 + e(\pi)^2 d\sigma_2^2$$

where $d\sigma_1^2$ and $d\sigma_2^2$ are the line elements on M and H, respectively. The functions $f: M \to (0,\infty)$ and $e: M \to (0,\infty)$ are called warping functions. If either $e \equiv 1$ or $f \equiv 1$, but not both, then we obtain a singly warped product. If both $e \equiv 1$ and $f \equiv 1$ then we have a product manifold.

If (M,g) and (H,h) are both Riemannian manifolds then $M_f \times {}_e H$ is also Riemannian. On the other hand, in order for $M_f \times {}_e H$ to be a Lorentzian manifold, we will take (M,g) to be Lorentzian or else a one-dimensional manifold with a negative definite metric $-dt^2$ and take (H,h) to be a Riemannian manifold. In this paper we will primarily be interested in Lorentzian singly warped products of the form $M_f \times H$, with the warping function defined on the Riemannian factor H. We will also state some results for doubly warped products.

2.11 Static Space-Times:

A <u>static</u> metric for a space-time is one which admits a timelike Killing vector field K which is orthogonal to a family of spacelike surfaces [Hawking and Ellis (1973 , p.72)].

<u>Definition 2.3</u>:

Let H be a given Riemannian manifold, (a,b) an open interval, $-\infty \leq a < b \leq \infty$, and $f: H \to (0,\infty)$ a smooth function. Let t and η be the projections of (a,b) × H onto (a,b) and H. The <u>standard static space-time</u> $(a,b)_f \times H$ is the manifold (a,b) × H with the line element $ds^2 = -f(\eta)^2 dt^2 + d\sigma^2$ where $d\sigma^2$ is the line element of H.

Note that a standard static space-time is a special case of a warped product. O'Neill (1983 p. 361) proves that any static space-time is locally isometric to a standard static space-time. The space-times described in the next section are examples of standard static space-times.

2.12 Examples

The simplest example of a space-time is Minkowski space-time L^n, the Lorentzian analogue of euclidean space. L^n is the manifold $\mathbb{R}^n$ with the metric

$$ds^2 = -dx_1^2 + \sum_{i=2}^{n} dx_i^2.$$

L^n is the space-time of special relativity and the geometry induced on each fixed tangent space of an arbitrary Lorentzian manifold. Minkowski space-time L^n can be viewed as a warped product via $\mathbb{R}_f \times {}_e\mathbb{R}^{n-1}$ with $f \equiv 1$, $e \equiv 1$, $\bar{g} = -dt^2 \oplus h$, where h is the usual euclidean metric on $\mathbb{R}^{n-1}$.

Schwarzschild space-time is the relativistic model of a universe containing a single star which is assumed to be static and spherically symmetric and to be the only source of gravitation for the space-time [Misner, Thorne, and Wheeler (1973), Weinberg (1972)]. Let $\mathbb{R}^4$ be given the coordinates (t,r,θ,ϕ) where (r,θ,ϕ) are the usual spherical coordinates on $\mathbb{R}^3$. The exterior Schwarzschild space-time is defined on the subset $r > 2m$ of $\mathbb{R}^4$ where m is a positive constant. Topologically this subset is $\mathbb{R}^2 \times S^2$ and is given the metric

$$ds^2 = -(1 - \frac{2m}{r})dt^2 + (1 - \frac{2m}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

The exterior Schwarzschild space-time can be viewed as a singly warped product in two distinct ways. First, we can consider the warped product of $(\mathbb{R}, -dt^2)$ and the Riemannian manifold $(2m,\infty) \times S^2$ with the metric

$$ds^2 = (1 - \frac{2m}{r})^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \; r > 2m.$$

The warping function $f\colon \mathbb{R}^2 \times S^2 \to (0,\infty)$ in the Riemannian factor is defined by $f(r,\theta,\phi) = (1 - \frac{2m}{r})^{1/2}$. In this fashion the space-time is written as $\mathbb{R}_f \times ((2m,\infty) \times S^2)$. The alternate method is to consider $M = \{(t,r) \in \mathbb{R}^2\colon r > 2m\}$ with the Lorentzian metric $g = -(1 - \frac{2m}{r})dt^2 + (1 - \frac{2m}{r})^{-1}dr^2$ and let $H = S^2$ be given in the usual Riemannian metric h of constant sectional curvature 1 induced by the inclusion $S^2 \to \mathbb{R}^3$. Now let the warping function $e\colon M \to (0,\infty)$ on the Lorentzian factor be defined by $e(t,r) = r$. Thus, the exterior Schwarzschild space-time is written as $(M \times_e H, g \oplus e^2 h)$. In this paper we will usually prefer the first view of the exterior Schwarzschild space-time with the warping function defined on the Riemannian factor space.

Universal anti-de Sitter space-time is the Lorentzian analogue of the Riemannian hyperbolic space of constant negative sectional curvature. The hyperboloid

$$H_1^n = \{x \in \mathbb{R}^{n+1}\colon -x_1^2 - x_2^2 + x_3^2 + \dots + x_{n+1}^2 = -r^2\},$$

for constant $r > 0$, with the pseudo-euclidean metric

$$ds^2 = -(dx_1)^2 - (dx_2)^2 + (dx_3)^2 + \dots + (dx_{n+1})^2$$

is called anti-de Sitter space-time. This space contains closed timelike curves. However the universal covering space

$\tilde{H}_1^n$ of anti-de Sitter space-time does not contain any closed timelike lines. This space is the one we shall subsequently refer to as (universal) anti-de Sitter space-time $\tilde{H}_1^n$ [cf. Wolf (1974), Penrose (1968)]. It has the topology of $\mathbb{R}^n$. In the case dimension = 4 and curvature k = -1, $\tilde{H}_1^4$ may be given coordinates (t,r,θ,ϕ) for which the metric has the form

$$ds^2 = -\cosh^2 r\, dt^2 + dr^2 + \sinh^2 r(d\theta^2 + \sin^2\theta d\phi^2).$$

Note that universal anti-de Sitter space-time may be viewed as a warped product of the form $(\mathbb{R}_r \times H, -f^2dt^2 \oplus h)$. We define $f: H \to (0,\infty)$ to be $f(r,\theta,\phi) = \cosh r$ and let h be the complete Riemannian metric of constant negative sectional curvature on hyperbolic 3-space $H = \mathbb{R}^3$.

The final example we will discuss in this section is the <u>Einstein static universe</u>. Consider $\mathbb{R}$ with the negative definite metric $-dt^2$ and let $H = S^{n-1}$ with the standard spherical Riemannian metric. The Lorentzian product manifold $\mathbb{R} \times H$ is called the n-dimensional Einstein static universe.

Chapter 3

Causality in Lorentzian Warped Products

In this chapter we will dicuss the causal structure of Lorentzian warped products. We will begin with a discussion of some of the elementary properties of pseudo-Riemannian doubly warped products.

3.1 Basic Properties

The following properties follow directly from the definition 2.1 of a pseudo-Riemannian doubly warped product $M_f \times {}_eH$ [cf. Beem and Ehrlich (1981, Remark 2.44), Kemp (1981, p. 37-38), O'Neill (1983, p. 205)]. Recall that a <u>homothetic map</u> $F : (M,g_1) \to (M,g_2)$ is a diffeomorphism such that $F^*(g_2) = cg_1$ for some constant c.

<u>Remark 3.1</u>

Let $M_f \times {}_eH$ be a pseudo-Riemannian doubly warped product with metric $\bar{g} = f^2g \oplus e^2h$. Let $\pi : M \times H \to M$ and $\eta : M \times H \to H$ be projections.

(a) For each $q \in H$, the restrction $\pi|\eta^{-1}(q): \eta^{-1}(q) \to M$ is a homothetic map with homothetic factor $1/f(q)^2$.

(b) For each $p \in M$, the restriction $\eta|\pi^{-1}(p): \pi^{-1}(p) \to H$ is a homothetic map with homothetic factor $1/e(p)^2$.

(c) For each $(p,q) \in M \times H$, the factor $\eta^{-1}(q) = M \times q$ and the leaf $\pi^{-1}(p) = p \times H$ are orthogonal at (p,q)

(d) For each $(p,q) \in M \times H$ the submanifolds $\pi^{-1}(p) = p \times H$ and $\eta^{-1}(q) = M \times q$ are non-degenerate.

(e) If $\phi : H \to H$ is an isometry such that $f \circ \phi = f$ then the map $\Phi = 1_m \times \phi \colon M_f \times {}_eH \to M_f \times {}_eH$ given by $\Phi(p,q) = (p,\phi(q))$ is an isometry of $M_f \times {}_eH$.

(f) If $\psi : M \to M$ is an isometry such that $e \circ \psi = e$ then the map $\Psi = \psi \times 1_H \colon M_f \times {}_eH \to M_f \times {}_eH$ given by $\Psi(p,q) = (\psi(p),q)$ is an isometry of $M_f \times {}_eH$.

Vectors tangent to leaves will be called horizontal. Vectors tangent to fibers will be called vertical.

To relate the calculus of a (topological) product manifold $M \times H$ to that of its factors we use the notion of lifting. If $\psi : M \to \mathbb{R}$ is a smooth map, the lift $\bar{\psi}$ of ψ to $M \times H$ is the map $\bar{\psi} = \psi \circ \pi \colon M \times H \to \mathbb{R}$. If $v \in T_pM$ and $q \in H$ then the lift $\bar{v}$ of v to $(p,q) \in M \times H$ is the unique vector in $T_{(p,q)}(M \times H)$ such that $\pi_*\bar{v} = v$ and $\eta_*\bar{v} = 0_q \in T_qH$. If X is a vector field on M then the lift of X to $M \times H$ is the vector field $\bar{X}$ such that $\bar{X}_{(p,q)}$ is the lift of X_p to (p,q). Functions, tangent vectors, and vector fields on H can be lifted to $M \times H$ using the projection $\eta \colon M \times H \to H$. Ehrlich [1974, p. 139] noted the following fact.

Lemma 3.2

Let $\bar{M} = M_f \times {}_eH$ be a pseudo-Riemannian doubly warped product. If $\psi : M \to \mathbb{R}$, then the gradient of the lift $\psi \circ \pi$ of ψ to $\bar{M}$ is $\frac{1}{\bar{f}^2} = \frac{1}{(f \circ \eta)^2}$ times the lift to $\bar{M}$ of the gradient of ψ on M. That is,

$$\pi_*(\operatorname{grad}_{\bar{M}}(\psi \circ \pi)) = \frac{1}{\bar{f}^2} \operatorname{grad}_M \psi .$$

Similarly, if $\phi : H \to \mathbb{R}$ the

$$\eta_*(\operatorname{grad}_{\bar{M}}(\phi \circ \pi)) = \frac{1}{\bar{e}^2} \operatorname{grad}_H \phi = \frac{1}{(e \circ \pi)^2} \operatorname{grad}_H \phi .$$

<u>Proof</u>:

Let $\psi : M \to \mathbb{R}$ be a smooth function and suppose x is the lift of a tangent vector on H to M × H, i.e. X is a horizontal tangent vector to M × H. Then

$$\bar{g}(\operatorname{grad}_{\bar{M}}(\psi \circ \pi), x) = x(\psi \circ \pi) = \pi_*(x)\psi = 0$$

since $\pi_*(x) = 0$. Thus, $\operatorname{grad}_{\bar{M}}(\psi \circ \pi)$ is a vertical vector. Let v be any vertical tangent vector to $\bar{M}$ at $(p,q) \in \bar{M}$. Then since π is a homothety we obtain

$$\begin{aligned} g(\pi_*(\operatorname{grad} \psi \circ \pi), \pi_* v)|_{(p,q)} &= \frac{1}{f(q)^2} \bar{g}(\operatorname{grad} \psi \circ \pi, v)|_{(p,q)} \\ &= \frac{1}{f(q)^2} v(\psi \circ \pi)|_{(p,q)} \\ &= \frac{1}{f(q)^2} \pi_*(v)\psi|_p \\ &= \frac{1}{f(q)^2} g(\operatorname{grad}_M \psi, \pi_* v)|_p . \end{aligned}$$

Hence, at each point

$$\pi_*(\mathrm{grad}_{\bar{M}}(\psi\circ\pi)) = \frac{1}{\bar{f}^2}\,\mathrm{grad}_M\psi$$

A similar calculation yields the second part of the lemma.

□

Give vector fields X_1 and Y_1 on M and vector fields X_2 and Y_2 on H, we may lift them to vector fields $X = (X_1,0) + (0,X_2) = (X_1,X_2)$ and $Y = (Y_1,0) + (0,Y_2) = (Y_1,Y_2)$. Now we will proceed to determine the Levi-Civita connection ∇ for a doubly warped product $(M_f \times {}_eH, \bar{g})$ on X and Y as above.

Let ∇^1 denote the Levi-Civita connection for (M,g) and ∇^2 denote the Levi-Civita connection for (H,h). Recall that we denote the lifts of f and e to $M \times H$ by $\bar{f} = f\circ\eta$ and $\bar{e} = e\circ\pi$. The connection ∇ for $(M_f \times {}_eH,\ f^2g \oplus e^2h)$ is related to the metric $\bar{g} = f^2g \oplus e^2h$ by the Koszul formula

$$2\bar{g}(\nabla_XY,Z) = X\bar{g}(Y,Z) + Y\bar{g}(X,Z) - Z\bar{g}(X,Y) + \bar{g}([X,Y],Z)$$
$$- \bar{g}([X,Y],Y) - \bar{g}([Y,Z],X)$$

[cf. Cheeger and Ebin (1975, p.2)]. A calculation yields the following formula for the Levi-Civita connection ∇ for $M_f \times {}_eH$.

$$(1)\quad \nabla^1_X Y = \nabla^1_{X_1} Y_1 + \nabla^2_{X_2} Y_2 + \frac{X_2(f)}{f} Y_1 + \frac{Y_2(f)}{f} X_1 + \frac{X_1(e)}{e} Y_2 + \frac{Y_1(e)}{e} X_2 - \bar{g}(X_1,Y_1) \frac{\operatorname{grad}_{\bar{M}} \bar{f}}{\bar{f}} - \bar{g}(X_2,Y_2) \frac{\operatorname{grad}_{\bar{M}} \bar{e}}{\bar{e}} .$$

Here we are identifying the vector $\nabla^1_{X_1} Y_1 \in T_p M$ with the vector $(\nabla^1_{X_1} Y_1|_p, 0_q) \in T_{(p,q)}(M \times H)$, etc.

From formula (1) we can immediately deduce the following:

<u>Proposition 3.3</u>

In a doubly warped product $\bar{M} = M_f \times {}_e H$ each leaf $\pi^{-1}(p) = p \times H$ and fiber $\eta^{-1}(q) = M \times q$ is totally umbilic.

<u>Proof</u>:

From the symmetry in (1) between M and H it is clear that the proofs for fibers and leaves are similar, so we will only prove the assertion for leaves $\pi^{-1}(p) = p \times H$.

Let X,Y be horizontal fields tangent to $p \times H$. By formula (1), $\nabla_X Y = \nabla^2_X Y - \bar{g}(X,Y) \dfrac{\operatorname{grad}_{\bar{M}} \bar{e}}{\bar{e}}$, since $\pi_* X = \pi_* Y = 0$, $\eta_* X = X$, $\eta_* Y = Y$. Hence the second fundamental form tensor $S(X,Y) = \nabla_X Y - \nabla^0_X Y = -\bar{g}(X,Y) \dfrac{\operatorname{grad}_{\bar{M}} \bar{e}}{\bar{e}}$.

By the definition of totally umbilic (Section 2.9) we need only to show that $\operatorname{grad}_{\bar{M}} \bar{e}$ is a vertical vector, i.e.

$\text{grad}_{\bar{M}} \bar{e}|_{(p,q)} \in T^{\perp}_{(p,q)}(p \times H)$.

We have already established this fact in the proof of Lemma 3.2.

□

In the case of a singly warped product of the form $M_f \times H$ we should note the following sharper statement [cf. Beem and Ehrlich (1981, Lemma 2.34), O'Neill (1983, Corollary 7.36)].

Proposition 3.4:

For a pseudo-Riemannian warped product of the form $M_f \times H$, each leaf $\pi^{-1}(p) = p \times H$ is totally geodesic and each fiber $\eta^{-1}(q) = M \times q$ is totally umbilic.

Proof:

Fibers are totally umbilic by Proposition 3.3. To show that leaves are totally geodesic, let X,Y be horizontal vector fields tangent to the leaf $\pi^{-1}(p) = p \times H$. From formula (1) with $e \equiv 1$ we obtain $\nabla_X Y = \nabla^2_X Y$. Hence, the second fundamental form tensor $S(X,Y) = 0$ and the leaf is totally geodesic by definition.

□

Now we specialize to the case where (M,g) is an n-dimensional manifold ($n \geq 1$) with a signature $(-,+,\ldots,+)$ and (H,h) is a Riemannian manifold. As usual let $e\colon M \to (0,\infty)$ and $f\colon H \to (0,\infty)$ be smooth functions. Then $(M_f \times {}_eH),\ f^2g \oplus e^2h)$ becomes a Lorentzian doubly warped product. We first note the following elementary properties [cf. Beem and Ehrlich (1981, Remark 2.44), Kemp (1981, p. 37-38)].

Remark 3.5

Let $M_f \times {}_eH$ be a doubly warped product with a Lorentzian metric $\bar{g} = f^2 g \oplus e^2 h$. Let $\pi: M \times H \to M$ and $\eta: M \times H \to H$ be the projections.

(a) If $v \in T_{(p,q)}(M \times H)$ then $f(q)^2 g(\pi_* v, \pi_* v) \le \bar{g}(v,v)$. Thus $\pi_*: T_{(p,q)}(M \times H) \to T_p M$ maps nonspacelike vectors to nonspacelike vectors and π maps nonspacelike curves of $M_f \times {}_eH$ to nonspacelike curves of M.

(b) The map π is length nondecreasing on nonspacelike curves if $f : H \to (0,1]$, since $v \in T(M \times H)$,

$$|g(\pi_* v, \pi_* v)| \ge \frac{|\bar{g}(v,v)|}{f(\eta)^2} \ge |\bar{g}(v,v)|.$$

3.2 Causality

In order to discuss causality, we first need to discuss a time orientation for $M_f \times {}_eH$.

Lemma 3.6:

The Lorentzian doubly warped product $M_f \times {}_eH$ is time-orientable if and only if (M,g) is time-orientable (if dim M $\ge$ 2) or (M,g) is a one-dimensional manifold with a negative definite metric.

Proof:

Suppose that $M_f \times {}_e H$ is time-oriented by a continuous timelike vector field X on $M_f \times {}_e H$. If the dimension of M is 1 then we are done by the definition of Lorentzian warped product, so consider the case dim $M \geq 2$. Fix a point $q \in H$ and let $\bar{p} = (p,q) \in M \times H$. Since h is positive definite and X is timelike, we have

$$f(q)^2 g(\pi_* X_{\bar{p}}, \pi_* X_{\bar{p}}) \leq f(q)^2 g(\pi_* X_{\bar{p}}, \pi_* X_{\bar{p}}) + e(p)^2 h(\eta_* X_{\bar{p}}, \eta_* X_{\bar{p}})$$

$$= \bar{g}(X_{\bar{p}}, X_{\bar{p}}) < 0.$$

Thus, $g(\pi_* X_{\bar{p}}, \pi_* X_{\bar{p}}) < 0$ for each $\bar{p} \in M \times H$ showing that $\pi_* | (M \times q)(X)$ is a continuous timelike vector field on $M \times q$. The vector field defined on M by $\bar{Y}_m = \pi_* | (M \times q)(X)_{(m,q)}$ will time-orient (M,g).

Conversely, suppose dim $M \geq 2$ and (M,g) is time oriented by a vector field V. Lift V to a vector field $\bar{V}$ on $M_f \times {}_e H$ such that $\pi_* \bar{V} = V$, $\eta_* \bar{v} = 0$. Then for $\bar{p} = (p,q) \in M \times H$

$$\bar{g}(V_{\bar{p}}, V_{\bar{p}}) = f(q)^2 g(v_p, v_p) + e(p)^2 h(0_q, 0_q) < 0$$

since $g(v_p, v_p) < 0$. Thus V time orients $M_f \times {}_e H$.

Now consider the case dim $M = 1$ so that M is diffeomorphic to S^1 or $\mathbb{R}$. In either case let T be a smooth vector field on M with $g(T,T) = -1$. The lift $\bar{T}$ to $M_f \times {}_e H$ time orients $M_f \times {}_e H$.

□

In our discussion of causality we begin by considering the case when M is a one-dimensional manifold with a negative definite metric $-dt^2$. As noted above, M is diffeomorphic to S^1 or $\mathbb{R}$. In case $M \cong S^1$ the integral curves of the time orienting vector field $\bar{T}$ in the proof of Lemma 3.6 are closed timelike curves. Hence $S^1_f \times {}_e H$ is never chronological.

In the case $M \cong \mathbb{R}$ we have the following proposition.

Proposition 3.7

Let (H,h) be an arbitrary Riemannian manifold and let $M = (a,b)$ for $-\infty \leq a < b \leq \infty$ be given the negative definite metric $-dt^2$. Let $f : H \to (0,\infty)$ and $e: (a,b) \to (0,\infty)$ be arbitrary smooth functions. Then the Lorentzian doubly warped product $\bar{M} = (a,b)_f \times {}_e H$ with metric $\bar{g} = -f^2\, dt^2 \oplus e^2 h$ is stably causal, and consequently, strongly causal, distinguishing, causal, and chronological.

Proof:

To show stable causality it suffices to show $\bar{M} = (a,b)_f \times {}_e H$ admits a time function. Let $\bar{F} : \bar{M} \to \mathbb{R}$ be defined by $\bar{F}(t,x) = t$ for $(t,x) \in (a,b) \times H$. We need to show $\bar{g}(\text{grad } \bar{F}, \text{grad } \bar{F}) < 0$. Note that $\bar{F}$ is the lift on (a,b) to $(a,b)_f \times {}_e H$ of the identity function $F: (a,b) \to \mathbb{R}$ defined by $F(t) = t$. The gradient of F on (a,b) is given by $\text{grad } F = -\frac{\partial}{\partial t}$. By Lemma 3.2 the gradient on $\bar{M}$ of the lift $\bar{F} = F \circ \pi$ is $1/f(\eta)^2$ times the lift to $\bar{M}$ of grad F. Thus,

$$\text{grad}_{\bar{M}} \bar{F} = -\frac{1}{f(\eta)^2} \frac{\partial}{\partial t}.$$

Hence,

$$\bar{g}(\text{grad } \bar{F}|_{(t,x)}, \text{grad } \bar{F}|_{(t,x)}) = \bar{g}(-\frac{1}{f(x)^2}\frac{\partial}{\partial t}, -\frac{1}{f(x)^2}\frac{\partial}{\partial t})$$

$$= \frac{1}{f(x)^4}\,\bar{g}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t})$$

$$= -\frac{1}{f(x)^2}$$

$$< 0.$$

By the hierarchy of causality conditions $\bar{M}$ is also chronological, causal, distinguishing and strongly causal.

<u>Corollary 3.8</u>

Every standard static space-time (Definition 2.2) is stably causal.

Now, causality in a Lorentzian warped product $M_f \times_e H$ with metric $f^2 g \oplus e^2 h$ where (M,g) is a space-time of $\dim M \geq 2$ will be discussed.

<u>Lemma 3.9</u>

Let $p = (p_1, p_2)$, $q = (q_1, q_2)$ be two points in $M_f \times_e H$. Then

(a) $p \ll q$ implies $p_1 \ll q_1$

(b) $p \leq q$ implies $p_1 \leq q_1$

Proof:

If $p << q$ then there is a future-directed timelike curve γ in $M_f \times {}_eH$ from p to q. Then

$$\begin{aligned} f(\eta(\gamma(t)))^2 g(\pi_*\dot\gamma,\pi_*\dot\gamma) &\le f(\eta(\gamma(t)))^2 g(\pi_*\dot\gamma,\pi_*\dot\gamma) \\ &\quad + e(\pi(\gamma(t)))^2 h(\eta_*\dot\gamma,\eta_*\dot\gamma) \\ &= \bar g(\dot\gamma(t),\dot\gamma(t)) \\ &< 0. \end{aligned}$$

Thus $g(\pi_*\dot\gamma,\pi_*\dot\gamma) < 0$. Since $\pi_*\dot\gamma(t) = (\pi\circ\gamma)'(t)$, $\pi\circ\gamma$ is a timelike curve in M from p_1 to q_1.

Thus $p_1 << q_1$. The second implication follows similarly. □

Lemma 3.10

Let $p_1, q_1 \in M$ and $b \in H$. Set $p = (p_1,b)$ and $q = (q_1,b)$. Then

(a) $p_1 << q_1$ implies $p << q$

(b) $p_1 \le q_1$ implies $p \le q$.

Proof:

Let γ_1: $[0,1] \to M$ be a future-directed timelike curve from p_1 to q_1. Define γ: $[0,1] \to M_f \times {}_eH$ by $\gamma(t) = (\gamma_1(t),b)$ for $0 \le t \le 1$. Then γ is a future-directed timelike curve from p to q, and thus $p << q$. The second implication is proved similarly.

□

Lemmas 3.9 and 3.10 imply that each leaf $\eta^{-1}(q) = M \times q$ has the same chronology and causality as (M,g). In fact, we can state the following result.

Proposition 3.11

Let (M,g) be a space-time and (H,h) an arbitrary Riemannian manifold. The Lorentzian doubly warped product $(M_f \times {}_eH,\bar{g})$ is causal (resp. chronological) if and only if the Lorentzian manifold (M,g) is causal (resp. chronological).

Proof:

We will prove the causal assertion. The chronological assertion has a similar proof. Suppose $(M_f \times {}_eH,\bar{g})$ is not causal. Then there are points $p,q \in M_f \times {}_eH$ with $p \leq q \leq p$. But Lemma 3.9 implies $\pi(p) \leq \pi(q) \leq \pi(p)$. Therefore, (M,g) is not causal.

Now suppose (M,g) is not causal. That is, assume there are points $p_1,q_1 \in M$ such that $p_1 \leq q_1 \leq p_1$. Define $p = (p_1,b)$ and $q = (q_1,b)$ for some $b \in H$. Lemma 3.10 implies $p \leq q \leq p$ so $M_f \times {}_eH$ is not causal.

□

Similar propositions can be stated for strong causality and stable causality.

Proposition 3.12

Let (M,g) be a space-time and (H,h) an arbitrary Riemannian manifold. Then $(M_f \times {}_eH,\bar{g})$ is strongly causal if and only if (M,g) is strongly causal.

Proof:

Assume (M,g) is not strongly causal at $p_1 \in M$. Let $b \in H$ and define $p = (p_1,b) \in M \times H$. Since (M,g) is not strongly causal at p_1, there is an open neighborhood U_1 of p_1 in M and a sequence $\{\gamma_k : [0,1] \to M\}$ of future-directed nonspacelike curves with $\gamma_k(0) \to p_1$, $\gamma_k(1) \to p_1$ as $k \to \infty$ but $\gamma_k(\frac{1}{2}) \notin U_1$ for all k. Define $\sigma_k\colon [0,1] \to M \times H$ by $\sigma_k(t) = (\gamma_k(t),b)$. Let V_1 be any open neighborhood of b in H and set $U = U_1 \times V_1$ in $M \times H$. Then U is an open neighborhood of $p = (p_1,b)$ in $M \times H$ and $\{\sigma_k\}$ is a sequence of future-directed nonspacelike curves in $M_f \times {}_eH$ with $\sigma_k(0) \to p$ and $\sigma_k(1) \to p$ as $k \to \infty$ but $\sigma_k(\frac{1}{2}) \notin U$ for all k. Thus $(M_f \times {}_eH,\bar{g})$ is not strongly causal.

To show the converse, assume $(M_f \times {}_eH,\bar{g})$ is not strongly causal at $p = (p_1,q_1)$. Let $(x_1,\cdots,x_i)$ be local coordinates on M near p_1 such that f^2g has the form diag $\{-1,1,\ldots,1\}$ at p_1 and let $\{x_{i+1},\cdots,x_n\}$ be local coordinates on H near q_1 such that h has the form diag $\{1,1,\cdots,1\}$ at q_1. Then $(x_1,\cdots,x_i,x_{i+1},\cdots,x_n)$ are local coordinates for $M_f \times {}_eH$ near p. Futhermore, $F_1 = x_1$ and $F_2 = x_1 \circ \pi$ are (locally defined) time functions for (M,f^2g) near p_1 and $(M_f \times {}_eH,\bar{g})$ near p, resp. Since $M_f \times {}_eH$ is not strongly causal at p by assumption, there is a sequence $\gamma_k\colon [0,1] \to M \times H$ of future-directed nonspacelike curves with $\gamma_k(0) \to p$, $\gamma_k(1) \to p$ as $k \to \infty$ but $F_2(\gamma_k(\frac{1}{2})) \geq \epsilon > 0$ for all k and some parameterization of γ_k. Now choose a neighborhood W of p_1 in M such that W is covered by local coordinates $(x_1,\cdots,x_i)$ and such that sup $\{F_1(r) : r \in W\} \leq \epsilon/2$. This is possible since the coordinates at p_1 are $(0,0,\cdots,0)$ and $F_1 = x_1$. The projection $\bar{\gamma}_k = \pi \circ \gamma_k$ of the curve γ_k in $M \times H$ to M will be future-directed and nonspacelike for each k and $\bar{\gamma}_k(0) \to p_1$, $\bar{\gamma}_k(1) \to p_1$ as $k \to \infty$. Futhermore, $\bar{\gamma}_k(\frac{1}{2}) \notin W$ for all k since if $\bar{\gamma}_k(\frac{1}{2}) \in W$ then $F_1(\bar{\gamma}_1(\frac{1}{2})) \leq \epsilon/2$. But the last inequality contradicts $F_1(\bar{\gamma}_1(\frac{1}{2})) = F_1(\pi \circ \gamma_k(\frac{1}{2})) = F_2(\gamma_k(\frac{1}{2})) \geq \epsilon$ for all k. Therefore, (M,g) is not strongly causal at p_1.

□

Before stating the result on stable causality, we recall that a global time function is a continuous function $f : M \to \mathbb{R}$ that is strictly increasing along future-directed nonspacelike curves. A space-time is (M,g) stably causal if and only if it admits a global time function [Hawking (1968)].

Proposition 3.13

Let (M,g) be a space-time and let (H,h) be an arbitrary Riemannian manifold. The Lorentzian doubly warped product $M_f \times {}_eH$ is stably causal if and only if (M,g) is stably causal.

Proof:

Assume (M,g) is stably causal with global time function $\psi : M \to \mathbb{R}$. To show $(M_f \times {}_eH, \bar{g})$ is stably causal, it suffices to show that the lift $\Psi = \psi \circ \pi : M \times H \to \mathbb{R}$ of ψ to $M_f \times {}_eH$ is a global time function. So consider a future-directed nonspacelike curve $\gamma : (a,b) \to M \times H$ with nonvanishing tangent vector $\dot{\gamma}(t)$. Let $\alpha = \pi \circ \gamma : (a,b) \to M$ be the projection of γ into M. Then

$$f(\eta(\gamma(t)))^2 g(\dot{\alpha}(t),\dot{\alpha}(t)) \le f(\eta(\gamma(t)))^2 g(\dot{\alpha}(t),\dot{\alpha}(t)) + e(\alpha(t))^2 h(\eta_* \dot{\gamma}(t),\eta_* \dot{\gamma}(t))$$
$$= \bar{g}(\dot{\gamma}(t),\dot{\gamma}(t))$$
$$\le 0$$

since $\dot{\gamma}(t)$ is a future-directed nonspacelike tangent vector for all t. Thus, $g(\dot{\alpha}(t),\dot{\alpha}(t) \le 0$ and α is a nonspacelike curve in M. It is easy to check that α is future-directed and hence, for $t_1 < t_2$, $\psi(\alpha(t_1)) < \psi(\alpha(t_2))$ since ψ is a global time function. Thus,

$$\Psi(\gamma(t_1)) = \psi \circ \pi(\gamma(t_1)) = \psi(\alpha(t_1)) < \psi(\alpha(t_2))$$
$$= \psi \circ \pi(\gamma(t_2)) = \Psi(\gamma(t_2)) \qquad \text{for } t_1 < t_2.$$

Therefore, Ψ is a global time function on $M_f \times {}_eH$.

Conversely, suppose $M_f \times {}_eH$ is stably causal and let $\Psi : M \times H \to \mathbb{R}$ be a global time function. Fix $q \in H$ and define $\psi : M \to \mathbb{R}$ by $\psi(m) = \Psi(m,q)$. To show (M,g) is stably causal we only need to show ψ is a global time function on M. Suppose $\gamma : (a,b) \to M$ is a future-directed nonspacelike curve in M and define $\bar{\gamma} : (a,b) \to M \times H$ by $\bar{\gamma}(t) = (\gamma(t),q)$. Then $\bar{g}(\dot{\bar{\gamma}}(t),\dot{\bar{\gamma}}(t)) = f(b)^2 g(\dot{\gamma}(t),\dot{\gamma}(t)) + e(\gamma(t))^2 h(0_q,0_q) \le 0$ since $\dot{\gamma}(t)$ is nonspacelike. Thus, $\bar{\gamma}$ is a nonspacelike curve in $M_f \times {}_eH$. Again it is easy to check that $\bar{\gamma}$ is future-directed. So for $t_1 < t_2$ we have $\Psi(\bar{\gamma}(t_1)) < \Psi(\bar{\gamma}(t_2))$.

Also, $\psi(\gamma(t_1)) = \Psi(\gamma(t_1),q) = \Psi(\bar{\gamma}(t_1)) < \Psi(\bar{\gamma}(t_2))$

$$= \Psi(\gamma(t_2),q) = \psi(\gamma(t_2)) \quad \text{if } t_1 < t_2.$$

Therefore, ψ is a global time function for (M,g) and consequently, (M,g) is stably causal.

□

To summarize what we have shown so far, the elementary causality of a Lorentzian doubly warped product $(M_f \times {}_eH, \bar{g})$ is determined by the causality of the space-time factor (M,g). The most restrictive causality condition that we defined in Chapter 2 was global hyperbolicity and we now turn our attention to this important condition in doubly warped products.

Recall that two pseudo-Riemannian metrics g_1 and g_2 on M are conformal if there exists a smooth function $\Omega: M \to (0,\infty)$ such that $g_1 = \Omega g_2$. It can be shown that two strongly causal Lorentzian metrics g_1 and g_2 on M determine the same past and future sets if and only if the two metrics are conformal [cf. Beem and Ehrlich (1981, p.6)]. Hence the causal structure of a space-time depends only on its conformal class.

For a singly Lorentzian warped product of the form $M \times {}_eH$ (dim $M \geq 2$), Beem and Ehrlich (1981, Theorem 2.55) have proved that $(M \times {}_eH,\ g \oplus e^2h)$ is globally hyperbolic if and only if (M,g) is globally hyperbolic and (H,h) is a complete Riemannian manifold. In case $M = (a,b) \subseteq \mathbb{R}$, $((a,b) \times {}_eH,\ -dt^2 \oplus e^2h)$ is globally hyperbolic if and only if (H,h) is complete.

Using the results above on singly warped products, the dependency of causal structure on conformal classes, and the observation that $(M \times H, f^2 g \oplus e^2 h)$ is conformal to $(M \times H,\ g \oplus e^2(f^{-2}h))$, Beem and Powell (1982) proved the following necessary and sufficient conditions for a doubly warped product to be globally hyperbolic.

Theorem 3.14

Let (M,g) be a space-time and (H,h) a Riemannian manifold. The doubly warped product $M_f \times {}_e H$ is globally hyperbolic if and only if both of the following conditions are satisfied:

(1) (M,g) is globally hyperbolic

(2) $(H,f^{-2}h)$ is complete.

By a similar argument the following theorem can be established.

Theorem 3.15

Let (H,h) be a Riemannian manifold. The Lorentzian doubly warped product $\mathbb{R}_f \times {}_e H$ with metric $\bar{g} = -f^2\, dt^2 \oplus e^2 h$ is globally hyperbolic if and only if $(H,f^{-2}h)$ is complete.

Thus the question of global hyperbolicity of a standard static space-time $(a,b)_f \times H$ is reduced to the question of whether $(H,f^{-2}h)$ is a complete Riemannian manifold.

The following proposition gives conditions on the warping function $f : H \to (0,\infty)$ which ensure that $M_f \times {}_e H$ is globally hyperbolic. This sufficient condition is more straight-forward and easier to calculate then the condition of completeness of $(H,f^{-2}h)$ which is used in Theorems 3.14 and 3.15.

Theorem 3.16

Let (M,g) be a globally hyperbolic space-time ($\dim M \geq 2$) or else $M \cong \mathbb{R}^1$ with metric $-dt^2$. Let (H,h) be a complete Riemannian manifold. Let $f : H \to (0,\infty)$ and $e : M \to (0,\infty)$ be smooth functions. For a fixed $p_o \in H$ define $D^n(p_o,r) = \{x \in H : d_h(p_o,x) \leq r\}$ to be the ball in (H,h) about p_o of radius r. Define $F : (0,\infty) \to (0,\infty)$ by $F(r) = \max\{f(y) : y \in D^n(p_o,r)\}$. If $\int_0^\infty \frac{dr}{F(r)} = \infty$ then the Riemannian manifold $(H,f^{-2}h)$ is complete and the Lorentzian doubly warped product $M_f \times {}_e H$ is globally hyperbolic.

Proof:

By the Hopf-Rinow Theorem, in order to show $(H,f^{-2}h)$ is complete it sufficies to show that it satisfies finite compactness. That is, denoting the distance function associated with the metric $f^{-2}h$ by $\bar{d} : H \times H \to (0,\infty)$, we must show every $\bar{d}$ - bounded subset of H has compact closure.

Consider $K_\alpha = \{p \in H : \bar{d}(p_o,p) \le \alpha\}$

and
$$B_\beta = \{p \in H : d_h(p_o,p) \le \beta\}.$$

We claim that given $\alpha > 0$, there exists a $\beta > 0$ such that $K_\alpha \subset B_\beta$. From this assertion we can show $(H,f^{-2}h)$ is complete as follows. (H,h) is complete, and hence, finitely compact so that $\bar{B}_\beta$ is compact for every $\beta > 0$. So $\bar{K}_\alpha$ is compact for $K_\alpha \subset B_\beta$. Now let K be an $\bar{d}$-bounded subset of H. Since $K \subset K_\alpha \subset B_\beta$ for some $\alpha > 0$ and $\beta > 0$, $\bar{K}$ is compact. Therefore $(H,f^{-2}h)$ is complete and by Theorem 3.14, $M_f \times {}_eH$ is globally hyperbolic.

It remains to prove that given $\alpha > 0$, there exists a $\beta > 0$ such that $K_\alpha \subset B_\beta$. Assuming $\int_0^\infty \frac{dr}{F(r)} = \infty$, we can choose β so that $\int_0^\beta \frac{dr}{F(r)} > \alpha + 1$. Suppose $x_0 \in K_\alpha \backslash B_\beta$ and let $\gamma : [0,\delta] \to H$ be any unit speed piecewise smooth curve with $\gamma(0) = p_0$, $\gamma(\delta) = x_0$. We can assume $h(\dot{\gamma},\dot{\gamma}) = 1$ by taking a strictly increasing reparametrization, if necessary. Since $x_0 \notin B_\beta$, $\beta < d_h(p_0,x_0) \le \delta$. Also for any $t \ge 0$, $d(p_o,\gamma(t)) \le t$ so $\gamma(t) \in D^n(p_o,t)$. Hence, $F(t) \ge f(\gamma(t))$ for all $t \ge 0$. Now consider the length of γ with respect to the metric $f^{-2}h$.

$$L_{f^{-2}h}(\gamma) = \int_0^\delta \sqrt{\frac{h(\dot{\gamma}(t),\dot{\gamma}(t))}{f(\gamma(t))^2}}\,dt \ge \int_0^\delta \frac{1}{F(t)} \ge \int_0^\beta \frac{1}{F(t)}\,dt > \alpha+1.$$

Since γ was an arbitrary curve (up to parameterization) from p_0 to x_0, $\bar{d}(p_0,x_0) = \inf\{L_{f^{-2}h}(\sigma) : \sigma$ is a piecewise smooth path from p_0 to $x_0\} > \alpha$. This last inequality contradicts the choice of $x_0 \in K_\alpha$. Thus, $K_\alpha \subset B_\beta$.

□

Corollary 3.17

Let (M,g) be a globally hyperbolic space-time or else $M = \mathbb{R}^1$ with metric $-dt^2$ and let (H,h) be a complete Riemannian manifold. If $e: M \to (0,\infty)$ is smooth and $f : H \to (0,b)$ is a smooth bounded function then $(M_f \times {}_eH, \bar{g})$ is globablly hyperbolic.

Proof:

If $0 < f(x) < L < \infty$ for all $x \in H$ then $F(r) = \max\{f(y) : y \in D^n(p_0,r)\} \leq L$ where p_0 is any point in H. Thus, $\int_0^\infty \frac{dr}{F(r)} \geq \int_0^\infty \frac{dr}{L} = \infty$. Hence, $M_f \times {}_eH$ is globally hyperbolic by Theorem 3.14 or 3.15.

□

It should be noted that [Kemp (1981, p. 47-51)] stated Corollary 3.17 for the case $e \equiv 1$. Also, the converse of 3.17 fails as is shown by the following counterexample.

Example 3.18

Let $(M,g) = (\mathbb{R}, -dt^2)$ and $(H,h) = (\mathbb{R}, \frac{1}{1+x^4}\,dx^2)$. Define $f : H \to (0,1]$ by $f(x) = \frac{1}{\sqrt{1+x^4}}$ and $e \equiv 1$.

Then $(M_f \times {}_e H,\ -f^2dt^2 \oplus \frac{dx^2}{1+x^4})$ is globally hyperbolic and f is bounded but (H,h) fails to be complete.

Proof:

$(H, f^{-2}h) = (\mathbb{R}, dx^2)$ which is complete. Thus, $M_f \times {}_e H$ is globally hyperbolic by Theorem 3.15. Let the pregeodesic $\gamma : (-\infty,\infty) \to H$ be defined by $\gamma(s) = s$. Then

$$L_h(\gamma) = \int_{-\infty}^{\infty} \sqrt{h(\dot\gamma(s),\dot\gamma(s))}\ ds = \int_{-\infty}^{\infty} \frac{ds}{\sqrt{1+s^4}} < \infty$$

shows γ has finite length and hence, (H,h) is incomplete.

We can also state the following criterion for determining the failure of $M_f \times {}_e H$ to be globally hyperbolic.

Theorem 3.19

Let (M,g) be a space-time or else $M \cong \mathbb{R}^1$ with a negative definite metric $-dt^2$ and let (H,h) be a noncompact complete Riemannian manifold. Let $e : M \to (0,\infty)$ and $f : H \to (0,\infty)$ be smooth functions. For a fixed $p_0 \in H$, define $G(r) = \min\{f(y): d(p_o,y) = r\}$. If $\int_0^{\infty} \frac{dr}{G(r)} < \infty$ then $(H, f^{-2}h)$ is incomplete and $(M_f \times {}_e H, \bar{g})$ is not globally hyperbolic.

Proof:

Suppose that $(H, f^{-2}h)$ is complete. Let $\gamma : (0,\infty) \to H$ be a geodesic ray issuing from p_0 having unit speed. That is, $h(\dot\gamma,\dot\gamma) = 1$ and γ realizes the Riemannian distance between every pair of its points. If $(H, f^{-2}h)$ is complete then γ

must have infinite length with respect to the metric $f^{-2}h$. But this length is given by the following:

$$L_{f^{-2}h}(\gamma) = \int_0^{\infty} \sqrt{f(\gamma(s))^{-2} h(\dot{\gamma}(s), \dot{\gamma}(s))} \; ds = \int_0^{\infty} \frac{ds}{f(\gamma(s))}$$

Since γ is a unit speed ray, $d_h(p_0,\gamma(s)) = s$ for all $s \geq 0$ and thus, $\gamma(s) \in \{y: d_h(p_0,y) \leq s\}$ for all $s \geq 0$. Hence, $G(s) = \min\{f(y): d_h(p_0,y) \leq s\} \leq f(\gamma(s))$ for all $s \geq 0$. Therefore,

$$\infty = L_{f^{-2}h}(\gamma) = \int_0^{\infty} \frac{ds}{f(\gamma(s))} \leq \int_0^{\infty} \frac{ds}{G(s)} .$$

So $\int_0^{\infty} \frac{dr}{G(r)} < \infty$ implies $(H, f^{-2}h)$ is incomplete and $(M_f \times {}_e H, \bar{g})$ is not globally hyperbolic by Theorems 3.14 or 3.15.

□

We end this chapter with an application to universal anti-de Sitter space-time $\tilde{H}_1^4$. Recall that $\tilde{H}_1^4$ may be viewed as a warped product of the form $(\mathbb{R}_f \times H, -f^2\, dt^2 \oplus h)$ with h the complete Riemannain metric of constant negative sectional curvature on hyperbolic 3-space $H = \mathbb{R}^3$ and $f: H \to (0,\infty)$ is defined by $f(r,\theta,\phi)$ on H. Using the notation of Theorem 3.18, fix $0 \in \mathbb{R}^3$ so that the function $G: H \to (0,\infty)$ is given by $G(r) = \cosh r$. Then the elementary computation

$$\int_0^{\infty} \frac{dr}{G(r)} = \int_0^{\infty} \frac{dr}{\cosh r} = \int_0^{\infty} \frac{2dr}{e^r + e^{-r}} \leq \int_0^{\infty} \frac{2dr}{e^r} = 2 < \infty$$

yields the well known fact that universal anti-de Sitter space-time is not globally hyperbolic.

Chapter 4

Geodesic Completeness in Lorentzian Warped Products of the form $M_f \times H$

Completeness in a Riemannian warped product $M_f \times H$ with (M,g) and (H,h) both Riemannian manifolds was considered by Bishop and O'Neill (1969). They proved the Riemannian manifold $M_f \times H$ is complete if and only if both (M,g) and (H,h) are complete. Subsequently, several authors have studied nonspacelike geodesic completeness in Lorentzian warped products of the form $(M \times {}_e H,\ g \oplus e^2 h)$ [Beem and Ehrlich (1981, Section 2.6), Beem and Powell (1982), Powell (1982)], primarily for the special case $(a,b) \times {}_e H$, $-\infty \le a \le b \le \infty$. The results differ markedly from the Riemannian case as geodesic completeness is seen to depend on the warping function $e: M \to (0,\infty)$. Kemp (1981) has considered geodesic completeness in Lorentzian warped products of the type $(a,b)_f \times H$, $-\infty \le a \le b \le \infty$, where the warping function $f: H \to (0,\infty)$ is on the Riemannian factor space H. In this chapter we will give some new results on geodesic completeness in standard static space-time $\bar{M} = (a,b)_f \times H$. A few results on $\bar{M} = M_f \times H$, $\dim M \ge 2$ are discussed in the last section.

4.1 Geodesic Completeness in $(a,b)_f \times H$

We begin this section with a discussion of geodesics in a standard static space-time $\bar{M} = (a,b)_f \times H$, $-\infty \le a \le b \le \infty$, with (H,h) a Riemannian manifold, $f: H \to (0,\infty)$ a smooth function, and metric $\bar{g} = -f^2\, dt^2 \oplus h$. We will always assume that $\frac{\partial}{\partial t}\Big|_{(t,x)}$ is the time orientation on $\bar{M}$.

Following Kobayashi and Obata (1980) we can give the Christoffel symbols $\bar{\Gamma}^k_{ij}$ on $(\bar{M},\bar{g})$ in terms of those on (H,h). In the coordinate system $(t,x^1,x^2,\ldots,x^n)$ on $\bar{M}$ we set $x^0 = t$. Let Γ^k_{ij} represent the Christoffel symbols for (H,h).

(4.1) $$\bar{\Gamma}^0_{i0} = f^{-1} \frac{\partial f}{\partial x^i} \qquad 1 \le i \le n$$

$$\bar{\Gamma}^k_{00} = f \sum_{a=1}^{n} h^{ak} \frac{\partial f}{\partial x^a} \qquad 1 \le k \le n$$

$$\bar{\Gamma}^k_{ij} = \Gamma^k_{ij} \qquad 1 \le i,j,k \le n$$

$$\bar{\Gamma}^0_{00} = \bar{\Gamma}^0_{ij} = \bar{\Gamma}^k_{i0} = 0 \qquad 1 \le i,j,k \le n$$

Now suppose γ is a geodesic on $\bar{M}$ and let $\beta = \eta \circ \gamma$ be its projection into H. By the definition of a geodesic, γ satisfies the geodesic differential equation

(4.2) $$\nabla_{\dot{\gamma}} \dot{\gamma} = 0$$

Rewriting this equation using local coordinates and (4.1) for $\gamma(s) = (x^0(s),\, x^1(s),\ldots,x^n(s))$ we obtain

(4.3) $$\frac{d^2x^0}{ds^2} + 2(f^{-1})\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \frac{dx^i}{ds}\right) \frac{dx^0}{ds} = 0$$

(4.4) $$\frac{d^2x^k}{ds^2} + \sum_{i,j=1}^{n} \Gamma^k_{ij} \frac{dx^j}{ds} \frac{dx^i}{ds} = -(f)\left(\sum_{a=1}^{n} h^{ak} \frac{\partial f}{\partial x^a}\right) \left(\frac{dx^0}{ds}\right)^2$$

for $1 \le k \le n$.

For any geodesic $\gamma(s) = (x^0(s), \beta(s))$ with $\frac{dx^0}{ds} \neq 0$, we can obtain

(4.5) $\quad \frac{dx^0}{ds} = cf^{-2}(\beta(s)),\ c = \text{constant} > 0,$ using (4.3).

Then from (4.4) and (4.5) we obtain

(4.6) $$\nabla^2_{\dot\beta(s)} \dot\beta(s) = \frac{c^2}{2} \text{ grad } f^{-2}.$$

Now it is straight-forward to prove the following result of Kobayashi and Obata (1980, p. 1336-1337).

<u>Proposition 4.1</u>

Let $\bar{M} = (a,b)_f \times H$ with static metric $\bar{g} = -f^2\, dt^2 \oplus h$. Let $\gamma(s)$ be a timelike geodesic in $\bar{M}$ with $\gamma(s) = (x^0(s), \beta(s))$. Then

(1) f = constant on $\bar{M}$ implies β is a geodesic H

(2) If $f \neq$ constant on $\bar{M}$ then $\beta(s) \in \{x \in H\colon f(x) \leq c\}$ for some constant c. Furthermore, β is a geodesic in H if and only if it is an orthogonal trajectory of f.

□

If α is a curve in a pseudo-Riemannian manifold whose tangent vector never vanishes, then there is a strictly increasing reparameterization function h such that $\beta = \alpha \circ h$ has unit speed, i.e. $|\beta'| = 1$. In particular, a spacelike or timelike pregeodesic $\alpha\colon [0,b) \to M$ in a Lorentzian manifold (M,g) has a unit speed reparameterization. Since the unit speed reparameterization is a geodesic on the interval $[0,L(\alpha))$, it follows that a spacelike or timelike pregeodesic α is complete (to the right) if and only if it has infinite length $L(\alpha)$. This fact is used in the following proposition to establish the timelike geodesic incompleteness of standard static space-times $(a,b)_f \times H$ with a or b finite.

Proposition 4.2

Let $\bar{M} = (a,b)_f \times H$ be a standard static space-time. If $b < \infty$ (resp. $a > -\infty$) then all future-directed timelike geodesics are future (resp. past) incomplete.

Proof:

Suppose $b < \infty$ and let $\gamma_0\colon [0,d) \to \bar{M}$ be an arbitrary future-directed inextendible timelike geodesic. Reparameterize γ_0 to an (inextendible) timelike pregeodesic of the form $\gamma(t) = (t,\beta(t))$ where $\gamma\colon [w_0,w_1) \to \bar{M}$, $a \le w_0 \le w_1 \le b < \infty$, and $\beta = \eta\circ\gamma$ is the projection of γ into H. Then

$$L(\gamma) = \lim_{t\to w_1} \int_{w_0}^{t} \sqrt{-\bar{g}(\gamma'(s),\gamma'(s))}\; ds$$

$$= \lim_{t\to w_1} \int_{w_0}^{t} \sqrt{(f\circ\beta)^2(s) - h(\beta'(s),\beta'(s))}\; ds$$

$$\le \lim_{t\to w_1} \int_{w_0}^{t} (f\circ\beta)(s)\; ds.$$

By Proposition 4.1, $\eta \circ \gamma_0(s)$, and hence also $\beta(s)$, remains in $\{x \in H : f(x) \leq c\}$ for some constant c. Since $w_1 \leq b < \infty$ and $f \circ \beta(s) \leq c$ for all s, $L(\gamma)$ is finite and therefore, γ_0 is future incomplete. The case $a > -\infty$ is similar.

□

In contrast, $\bar{M} = (a,b)_f \times H$ with a or b finite can contain null inextendible complete geodesics, as is shown by the following example.

Example 4.3

Let $H = (0,\infty)$ and $(a,b) = (0,\infty)$. Let $f\colon H \to (0,\infty)$ be $f(x) = \frac{1}{x}$. So $ds^2 = -\frac{1}{x^2}dt^2 + dx^2$ on the space $\bar{M} = (0,\infty)_f \times (0,\infty)$. Then the curve $\gamma\colon \mathbb{R} \to \bar{M}$ defined by $\gamma(s) = (\frac{1}{2}e^{2s}, e^s)$ is a complete null geodesic.

However, if a or b is finite, $\bar{M} = (a,b)_f \times H$ will always contain at least one incomplete null geodesic.

Proposition 4.4

Let $\bar{M} = (a,b)_f \times H$ be a standard static space-time. If $b < \infty$ (resp. $a > -\infty$), then $\bar{M}$ is future (resp. past) null geodesically incomplete.

Proof:

We will prove the case $b < \infty$. The case $a > -\infty$ is similar. Let γ be a null future-directed inextendible geodesic in $\bar{M}$. If γ is incomplete we are done so assume γ is defined on the interval $[0,\infty)$. Let $\alpha = \pi \circ \gamma$ and $\beta = \eta \circ \gamma$ be the projections of γ into (a,b) and H, resp. Define a new curve $\bar{\gamma}$ in $\bar{M}$ by $\bar{\gamma}(s) = (\alpha(s) + b - \alpha(1), \beta(s))$. Clearly $\bar{\gamma}$ is also a null geodesic. But $\lim_{s \to 1^-} \bar{\gamma}(s)$ does not exist so $\bar{\gamma}$ is future incomplete.

The argument in the above proposition applies as well to any spacelike geodesic not of the form $\gamma(t) = (t_0, \beta(t))$ for a fixed $t_0 \in (a,b)$. To see this we recall that (4.5) holds for any geodesic not of the form $\gamma(t) = (t_0, \beta(t))$. Thus a geodesic which is not of this form at one value of t will always fail to be of this form.

We can summarize the above discussion with the following:

Proposition 4.5

Let $\bar{M} = (a,b)_f \times H$ be a standard static space-time. If $a > -\infty$ or $b < \infty$ then $\bar{M}$ is timelike, null, and spacelike geodesically incomplete.

We have seen that standard static space-times $(a,b)_f \times H$ with a or b finite are timelike geodesically incomplete independent of the warping function $f\colon H \to (0,\infty)$. Now we will discuss timelike geodesic completeness in space-times of the form $\mathbb{R}_f \times H$. Here the warping function will be important.

Definition 4.6

If for all $\alpha > 0$ there is a compact set $K \subset H$ such that $f(x) \geq \alpha$ for all $x \in H \backslash K$, then $f\colon H \to (0,\infty)$ is said to satisfy the K-growth condition.

Now we can state the following sufficient condition for $\bar{M} = \mathbb{R}_f \times H$ to be timelike geodesically complete.

Theorem 4.7

Let $\bar{M} = \mathbb{R}_f \times H$ be a standard static space-time with metric $\bar{g} = -f^2\, dt^2 \oplus h$. If $f\colon H \to (0,\infty)$ satisfies the K-growth condition then $\bar{M}$ is timelike geodesically complete.

Proof:

Let γ be a timelike future-directed inextendible geodesic in $\bar{M}$ and assume without loss of generality that $\bar{g}(\dot{\gamma},\dot{\gamma}) = -1$. If we let $x^0 = \pi \circ \gamma$ and $\beta = \eta \circ \gamma$ be the projections into $\mathbb{R}$ and H, then we have

$$(4.6) \qquad \frac{dx^0}{ds}(s) = \frac{c}{f(\beta(s))^2} \quad \text{for } c = \text{constant} > 0$$

and $f(\beta(s)) \leq c$ for all s by Proposition 4.1.

Together these imply

$$(4.7) \qquad 0 < \beta_1 \leq \frac{dx^0}{ds} \quad \text{for some constant } \beta_1.$$

On the other hand, by the K-growth condition there is a compact set $K \subset H$ such that $f(x) \geq 1$ for all $x \in H \backslash K$. Then we can find an m, $o < m < \infty$, such that $f(x) \geq m$ for all $x \in K$ since K is compact. So $\beta(s) \in K$ implies $f(\beta(s)) \geq m$ and $\beta(s) \in H \backslash K$ implies $f(\beta(s)) \geq 1$. In any case, $\frac{dx^0}{ds}(s) = \frac{c}{f(\beta(s))^2}$ is bounded above. This fact, together with (4.6), gives

$$(4.8) \qquad 0 < \beta_1 \leq \frac{dx^0}{ds}(s) \leq \beta_2 < \infty$$

for all s for constants β_1 and β_2.

We need to show $\gamma(s)$ is defined for arbitrarily large positive and negative values of s. Suppose by way of contradiction that $\gamma(s)$ is only defined for $-\infty \leq a \leq s \leq b < \infty$. Fix $s_0 \in (a,b)$. By an integration of (4.8), we can find constants B_1 and B_2 such that $-\infty < B_1 \leq x^0(s) \leq B_2 < \infty$ for $s_0 \leq s < b$. Furthermore, we can find a compact $K_0 \subset H$ such that $\beta(s) \in K_0$ for all s. To choose K_0, recall that $f(\beta(s)) \leq c$ for some constant c and use the K-growth condition to choose K_0 such that $f(x) \geq c + 1$ for all $x \in H \backslash K_0$. Then $\beta(s) \in K_0$, for otherwise, $f(\beta(s)) \geq c + 1$ which is a contradiction. So we have shown $\gamma(s) = (x^0(s), \beta(s)) \in [B_1, B_2] \times K_0$ for all s. Since the set $[B_1, B_2] \times K_0$ is compact, we have shown that γ is an inextendible timelike geodesic curve which is future imprisoned in a compact set. This is impossible in a strongly causal space-time [Hawking and Ellis (1973, p. 194)].

A similar contradiction is obtained if $\gamma(s)$ cannot be defined for arbitrarily large negative values. Since γ was arbitrary, $\bar{M}$ must be timelike geodesically complete.

□

From Proposition 4.1 we can deduce that if f is a constant function, the incompleteness of (H,h) implies that $(\mathbb{R}_f \times H, -f^2\, dt^2 \oplus h)$ is timelike geodesically incomplete. Kobayashi and Obata (1981) posed the following problem: Given timelike geodesic completeness of $(\mathbb{R}_f \times H, -f^2\, dt^2 \oplus h)$, find (other) conditions on f such that (H,h) is necessarily complete. Theorem 4.7 indicates that if the warping function f satisifes the K-growth condition, then $\mathbb{R}_f \times H$ is timelike geodesically complete independent of the completeness of (H,h). So if f satisfies the K-growth condition, then the Kobayashi and Obata problem will fail to have a solution.

One approach to studying null geodesic completeness is to parallel the treatment of Beem, Ehrlich and Powell (1980) in their study of Lorentzian warped products of the form $M \times_e H$.

<u>Theorem 4.8</u>

Let $\bar{M} = (a,b)_f \times H$, $-\infty \leq a < b \leq \infty$, be a standard static space-time. If γ_0 is a future-directed inextendible null geodesic in $(\bar{M},\bar{g})$, we will denote by γ the reparameterization of γ_0 to an inextendible null pregeodesic of the form $\gamma(t) = (t,c(t))$ defined on the interval (w_0,w_1) with $a \leq w_0 < w_1 \leq b$. Then γ_0 is past [resp. future] incomplete if and only if

$$\lim_{t\to w_0+}\int_{t_0}^{t}(f\circ c)^2(u)\,du = \lim_{t\to w_0+}\int_{t_0}^{t} h(c'(u),c'(u))\,du > -\infty$$

$$[\text{resp.}\ \lim_{t\to w_1-}\int_{t_0}^{t}(f\circ c)^2(u)\,du = \lim_{t\to w_1-}\int_{t_0}^{t} h(c'(u),c'(u))\,du < \infty]$$

for a fixed $t_0 \in (w_0,w_1)$.

Proof:

Let γ_0 and γ be as in the statement of the theorem. Since γ is a null pregeodesic, there is a $G(t)$, such that

$$(4.9)\quad \bar{\nabla}_{\gamma'(t)}\gamma'(t) = G(t)\gamma'(t) = G(t)\frac{\partial}{\partial t}\Big|_t + g(t)c'(t)$$

[cf. Hawking and Ellis 1973, p. 33)].

By the connection formula (3.1)

$$(4.10)\quad \nabla_{\gamma'(t)}\gamma'(t) = \nabla^2_{c'(t)}c'(t) + \frac{2c'(f)}{f\circ c}\frac{\partial}{\partial t}\Big|_t - \bar{g}\left(\frac{\partial}{\partial t},\frac{\partial}{\partial t}\right)\frac{\text{grad}_{\bar{M}}\bar{f}}{\bar{f}}\ .$$

Since $\text{grad}_{\bar{M}}\bar{f}$ will be a horizontal vector, if we equate terms in (4.9) and (4.10) with a $\frac{\partial}{\partial t}$ component, we obtain

$$G(t) = \frac{2\ c'(f)(t)}{(f\circ c)(t)} = \frac{2(f\circ c)'(t)}{(f\circ c)(t)} = \frac{d}{dt}\ (\ln(f\circ c)^2(t)).$$

Let $S(t) = (f\circ c)^2(t)$. Then note that $S(t) > 0$ and

$\frac{dS}{dt}(t) = S(t)G(t)$. Define $p(t) = \int_{t_0}^{t} S(u)du$ for a fixed $t_0 \in (w_0,w_1)$. Then $p'(t) = S(t) > 0$. Hence, p^{-1} exists. If we let $\gamma_1(t) = \gamma \circ p^{-1}(t) = (p^{-1}(t), c \circ p^{-1}(t))$, then clearly γ_1 is a null curve. Moreover, one can show using the classical theory of projective transformation that γ_1 is a a null geodesic [cf. Spivak (1970, p. 273-275)].

Let $A = \lim_{t \to w_0+} p(t)$ and $B = \lim_{t \to w_1-} p(t)$. Since p is monotone increasing, $p\colon (w_0,w_1) \to (A,B)$ is a bijection so $\gamma_1 = \gamma \circ p^{-1}$ will be defined on the interval (A,B).

Then γ_1 is past [resp. future] incomplete if and only if $A > -\infty$ [resp. $B < \infty$]. Since γ_1 is a geodesic reparameterization of γ_0, the reparameterization h must be of the form $h(t) = \alpha t + \beta$ for α, β constants [O'Neill (1983, p. 69)]. Thus, γ_0 is past [resp. future] incomplete if and only if $A > -\infty$ [resp. $B < \infty$]. Finally, since $\gamma(t) = (t, c(t))$ is a null curve, we have

$$0 = \bar{g}(\gamma'(t),\gamma'(t)) = -(f \circ c)^2(t) + h(c'(t),c'(t)).$$

Thus, $$p(t) = \int_{t_0}^{t} (f \circ c)^2(u)du = \int_{t_0}^{t} h(c'(u),c'(u))du.$$

□

Theorem 4.8 can be reformulated in terms of the energy of the projection $c = \eta \circ \gamma$ of the null geodesic γ into the Riemannian manifold (H,h). The functional defined by $E_c(t) = \int_{t_0}^{t} h(c'(u),c'(u))du$ is classically known as the

<u>energy</u> function of the Riemannian curve $c|_{t_0}^{t}$. If $L_c(t) = L(c|_{t_0}^{t})$ denotes the arc length of c, then the Cauchy-Schwarz inequality yields

$$L_c^2(t) \le (t - t_0)E_c(t).$$

Here equality holds if and only if $h(c'(t),c'(t))$ is constant. By Theorem 4.8 a future-directed null pregeodesic $\gamma : [t_0,w_1) \to \bar{M}$ of the form $\gamma(t) = (t,c(t))$ is future incomplete if and only if the energy $E_c(w_1)$ of $c|_{t_0}^{w_1}$ is finite.

The above approach to studying null geodesic completeness in $\mathbb{R}_f \times H$ is difficult to apply since the integrals depend on the parameterization of the null geodesic in question. In fact, this parameterization is not even the affine parameterization. To avoid this difficulty we offer the following approach to obtain sufficient conditions for null geodesic completeness in $\mathbb{R}_f \times H$.

We begin with the statement of a well-known result [cf. Beem and Ehrlich (1981, p. 64)] which is needed in the proof of Lemma 4.11.

<u>Lemma 4.10</u>

Let (H,h) be a complete Riemannian manifold. If $\gamma : [0,1) \to H$ is a curve of finite length in (H,h), then there exists $p \in H$ such that $\gamma(t) \to p$ as $t \to 1-$.

Lemma 4.11

Let $\bar{M} = \mathbb{R}_f \times H$ be a standard static space-time with (H,h) a complete Riemannian manifold. If $\gamma : [0,d) \to \bar{M}$ is a smooth inextendible (to the right) null geodesic with $\pi \circ \gamma = \alpha$, $\eta \circ \gamma = \beta$ then the Riemannian length of $\beta|_0^d$ is infinite.

Proof:

By the geodesic differential equation

$$\frac{d\alpha}{ds} = \frac{c}{f(\beta(s))^2} \qquad \text{for } c = \text{constant} > 0. \tag{4.5}$$

Thus, $\alpha : [0,d) \to \mathbb{R}$ is an increasing function and consequently, either $\lim_{s \to d-} \alpha(s)$ exists or $\lim_{s \to d-} \alpha(s) = +\infty$. We consider the two cases separately.

Case 1:

$$\lim_{s \to d-} \alpha(s) = t_0 < \infty .$$

Suppose $L(\beta|_0^d) < \infty$. Then by Lemma 4.10, there exists a $p_0 \in H$ such that $\beta(s) \to p_0$ as $s \to d-$. Then since we have the product topology on $\mathbb{R} \times H$, $\gamma(s) \to (t_0, p_0) \in \mathbb{R} \times H$ as $s \to d-$. But this contradicts the inextendibility of γ. Thus, $L(\beta|_0^d) = \infty$.

Case 2:

$$\lim_{s \to d-} \alpha(s) = \infty.$$

Again, we suppose $L(\beta|_0^d) < \infty$ so there exists a $p_0 \in H$ such that $\beta(s) \to p_0$ as $s \to d-$. Since f is smooth, $f(\alpha(s)) \to f(p_0) > 0$ as $s \to d-$. Now we will show that $d = \infty$.

By (4.5) $\frac{d\alpha}{ds}(s) = \frac{c}{f(\beta(s))^2}$, so

$$\lim_{s \to d-} \int_0^s d\alpha = \lim_{s \to d-} \int_0^s \frac{c}{f(\beta(t))^2}\, dt.$$

Hence, $\infty = \lim_{s \to d-} \alpha(s) - \alpha(0) = \lim_{s \to d-} \int_0^s \frac{c}{f(\beta(t))^2}\, dt.$

Since $f(\beta(s)) \to f(\gamma_0)$ as $s \to d-$, we can find an s_0 such that

$\frac{c}{f(\beta(s))^2} \leq \frac{2c}{f(p_0)^2}$ for $s_0 < s \leq d$.

Therefore,

$$\infty = \lim_{s \to d-} \int_0^s \frac{c}{f(\beta(t))^2}\, dt \leq \int_0^{s_0} \frac{c}{f(\beta(t))^2}\, dt + \int_{s_0}^d \frac{2c}{f(p_0)^2}\, dt.$$

But this inequality is possible only if $d = \infty$. Now we can derive a contradiction. Note that

$$0 = \bar{g}(\dot{\gamma}(s), \dot{\gamma}(s)) = - f(\beta(s))^2 \left[\frac{c}{f(\beta(s))^2} \right]^2 + h(\dot{\beta}(s), \dot{\beta}(s)).$$

Also, we can find an N such that $f(\beta(s)) \leq 2f(p_0)$ for $s \geq N$.

Thus,

$$L(\beta|_0^d) = \int_0^\infty \sqrt{h(\dot\beta(s),\dot\beta(s))}\,ds = \int_0^\infty \frac{c}{f(\beta(s))}\,ds$$

$$\geq \int_0^N \frac{c}{f(\beta(s))}\,ds + \int_N^\infty \frac{c}{2f(p_0)}\,ds$$

$$\geq \int_N^\infty \frac{c}{2f(p_0)}\,ds$$

$$= \infty.$$

So, in either case, we must have $L(\beta|_0^d) = \infty$. □

It is clear that a similar lemma can be proven in case a null geodesic $\gamma: (d,0] \to \bar{M}$ is inextendible to the left, in which case $\eta \circ \gamma = \beta|_d^0$ would have infinite length.

Now we can state sufficient conditions for $\mathbb{R}_f \times H$ to be null geodesically complete.

Theorem 4.12.

Let $\bar{M} = \mathbb{R}_f \times H$ be a standard static space-time with (H,h) a complete Riemannian manifold. If $0 < m \leq f(x)$ for some constant m and all $x \in H$, then $(\bar{M},\bar{g})$ is null geodesically complete.

Proof:

To show $(\bar{M},\bar{g})$ must be future null geodesically complete, we let $\gamma: [0,A) \to \bar{M}$ be an arbitrary null inextendible (to the right) geodesic. Let $\alpha = \pi \circ \gamma$, $\beta = \eta \circ \gamma$. We must show γ

is defined for arbitrarily large values of s, so suppose not, i.e. suppose $A < \infty$.

By (4.5) we have

$$\frac{d\alpha}{ds}(s) = \frac{c}{f(\beta(s))^2} \quad \text{for some constant } c > 0.$$

Also, γ is null so

$$0 = \bar{g}(\dot{\gamma}(s),\dot{\gamma}(s)) = -f(\beta(s))^2\left(\frac{d\alpha}{ds}(s)\right)^2 + h(\dot{\beta}(s),\dot{\beta}(s)).$$

Hence, $h(\dot{\beta}(s),\dot{\beta}(s)) = \dfrac{c^2}{f(\beta(s))^2}$ for all s.

Now compute the length of $\beta|_0^A$.

$$L(\beta|_0^A) = \int_0^A \sqrt{h(\dot{\beta}(s),\dot{\beta}(s))}\, ds = \int_0^A \frac{c}{f(\beta(s))}\, ds$$

$$\leq \int_0^A \frac{c}{m}\, ds = \frac{cA}{m} < \infty$$

since $m > 0$, $c > 0$, and we are assuming $A < \infty$. But by Lemma 4.11, $L(\beta|_0^A) = \infty$. So γ must be future complete. A similar argument with an arbitrary null geodesic $\gamma : (-A,0] \to \bar{M}$ shows $(\bar{M},\bar{g})$ is past null geodesically complete.

□

To show that the hypothesis of completeness is needed in the above Theorem, we consider the following example. Let $\bar{M} = \mathbb{R}_f \times (c,d)$ with either c or d finite, $f \equiv 1$, and metric $d\bar{s}^2 = -dt^2 + dx^2$. Then $(\bar{M},\bar{g})$ is null geodesically incomplete, but of course, $((c,d), dx^2)$ is not complete as a

Riemannian manifold.

On the other hand, incompleteness of (H,h) does not imply null geodesic incompleteness of $\mathbb{R}_f \times H$.

Example 4.13

Let $\bar{M} = \mathbb{R}_f \times (-1,1)$ with metric $d\bar{s}^2 = -f(x)^2 dt^2 + dx^2$. Let $f : (-1,1) \to (0,\infty)$ be defined $f(x) = \dfrac{1}{1 - x^2}$. Then $((-1,1), dx^2)$ is an incomplete Riemannian manifold and $(\bar{M}, d\bar{s}^2)$ is null geodesically complete. The latter claim can be established as follows. If we consider $\bar{M} = \mathbb{R} \times (-1,1)$ with the metric $-d\bar{s}^2 = f(x)^2\, dt^2 - dx^2$, we have a Lorentzian warped product of the type studied by Beem, Ehrlich, and Powell (1980, p. 51). Using Remark 4.2 from that article, we need to show that $\lim\limits_{t \to 1-} \displaystyle\int_{w_0}^{t} \frac{ds}{1 - s^2} = \infty$, for a fixed $w_0 \in (-1,1)$, in order to show $(\bar{M}, -ds^2)$ is null geodesically complete. But

$$\lim_{t \to 1-} \int_0^t \frac{ds}{1 - s^2} = \lim_{t \to 1-} (\tanh^{-1} t) = \infty .$$

Combining Theorems 4.7 and 4.12 we can prove the following.

Corollary 4.14.

Let $\bar{M} = \mathbb{R}_f \times H$ be a standard static space-time with (H,h) a complete Riemannian manifold. If $f : H \to (0,\infty)$ satisfies the K-growth condition then $(\bar{M}, \bar{g})$ is nonspacelike geodesically complete.

Proof:

The K-growth condition forces $(\bar{M},\bar{g})$ to be timelike geodesically complete by Theorem 4.7. To show null geodesic completeness we need to show f is bounded away from zero. Now we know that there is a compact $K \subset H$ such that $f(x) \geq 1$ for all $x \in H\backslash K$. Also, thre is an $m > 0$ such that $f(x) \geq m$ for all $x \in K$ by the compactness of K. Hence $f(x) \geq \min\{1,m\}$ for all $x \in H$.

□

To end this section we will apply the above results to universal anti-de Sitter space-time $\tilde{H}_1^4$. Recall that $\tilde{H}_1^4$ may be viewed as a standard static space-time $\mathbb{R}_f \times H$ with the complete Riemannian metric of constant negative sectional curvature -1 on hyperbolic 3 space $H = \mathbb{R}^3$. Define $f: H \to (0,\infty)$ by $f(r,\theta,\phi) = \cosh r$. Since (H,h) is complete and $\cosh r$ satisfies the K-growth condition, $\tilde{H}_1^4$ is nonspacelike geodesically complete.

4.2 Geodesic Completeness in $M_f \times H$ (dim $M \geq 2$)

In this section we will discuss some results on geodesic completeness in a Lorentzian warped product of the form $\bar{M} = M_f \times H$ where (M,g) is a space-time (dim $M \geq 2$) and (H,h) is a Riemannian manifold and we use the Lorentzian metric $\bar{g} = f^2 g \oplus h$ on $\bar{M}$.

Let $\pi: M \times H \to M$ and $\eta: M \times H \to H$ be the projections. For vector fields $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ on $\bar{M}$ and using ∇^1 (resp. ∇^2) to denote the Levi-Civita connection on (M,g) (resp. (H,h)), we recall from (3.1) that the Levi-Civita connection ∇ on $(\bar{M},\bar{g})$ is given by

$$(4.11)\quad \nabla_X Y = \nabla^1_{X_1} Y_1 + \nabla^2_{X_2} Y^2 + \frac{X_2(f)}{f} Y_1 + \frac{Y_2(f)}{f} X_1 - \bar{g}(X_1, Y_1)\frac{\text{grad}_{\bar{M}} \bar{f}}{\bar{f}} .$$

In addition, we note that by Lemma 3.2, we can write $\text{grad}_H f$ in place of $\text{grad}_{\bar{M}} \bar{f}$.

Our first lemma in this section is due to Ehrlich:

<u>Lemma 4.15</u>

If γ is a geodesic of $\bar{M} = M_f \times H$, then $\pi \circ \gamma$ is a pre-geodesic of M.

<u>Proof</u>:

Let $\alpha = \pi \circ \gamma$, $\beta = \eta \circ \gamma$. By (4.11) we have

$$0 = \nabla_{\gamma'} \gamma' = \nabla^1{}_{\alpha'} \alpha' + \nabla^2{}_{\beta'} \beta' + \frac{2\beta'(f)}{f} \alpha' - \bar{g}(\alpha', \alpha') \frac{\text{grad}_H f}{f}$$

Projecting onto TM we obtain

$$0 = \nabla^1_{\alpha'} \alpha' + \frac{2\beta'(f)}{f} \alpha'.$$

Thus, $\nabla^1_{\alpha'} \alpha'$ is a multiple of α', and so α' can be reparameterized to a geodesic. □

Lemma 4.16

Let α be a geodesic in M with $\alpha(0) = p$. For any $\bar{p} = (p,q) \in M \times H$ and for $\bar{v} = (\alpha'(0),v) \in T_{\bar{p}}(M \times H)$ there exists a pregeodesic γ such that $\pi \circ \gamma = \alpha$ and $\gamma'(0) = \bar{v}$.

Proof:

Let γ_1 be the unique geodesic in $M_f \times H$ with $\gamma_1'(0) = \bar{v}$, $\gamma_1(0) = \bar{p}$. Now $\pi \circ \gamma_1$ is a pregeodesic of M by Lemma 4.15, and $\pi_*(\gamma_1')(0) = \alpha'(0)$, $\pi \circ \gamma_1(0) = p$. By uniqueness of geodesics in M, $\pi \circ \gamma_1$ can be reparameterized to α. This same reparameterization applied to γ_1 results in the desired pregeodesic γ.

□

Now we can state sufficient conditions for timelike geodesic completeness of $M_f \times H$ to imply timelike geodesic completeness of M.

Theorem 4.17

Let (M,g) be a space-time and (H,h) be a Riemannian manifold. Let $f: H \to (0,N]$ be a bounded smooth function. If $\bar{M} = M_f \times H$ is timelike geodesically complete, then (M,g) is also timelike geodesically complete.

Proof:

Suppose (M,g) is timelike geodesically incomplete. Let $\alpha: [0,b) \to M$ be a unit speed inextendible timelike geodesic with $b < \infty$, $\alpha(0) = p \in M$. Fix $q \in H$ and let

$\bar{v} = (\alpha'(0), 0_q) \in T_{(p,q)}(M \times H)$. Using Lemma 4.16 we can find a pregeodesic γ in $\bar{M}$ with $\gamma'(0) = \bar{v}$ and $\pi \circ \gamma = \alpha$. Let $\beta = \eta \circ \gamma$. Since $\bar{g}(\gamma'(0), \gamma'(0)) = f(\beta(0))^2 g(\alpha'(0), \alpha'(0)) < 0$, γ is timelike.

Now γ can be maximally extended to some interval $[0, b')$ with $b' \leq b$. To show $\bar{M}$ is timelike geodesically incomplete, it suffices to show that the length of the pregeodesic $\gamma|_0^{b'}$ is finite.

$$
\begin{aligned}
L(\gamma|_0^{b'}) &= \int_0^{b'} \sqrt{-\bar{g}(\gamma'(s), \gamma'(s))}\, ds \\
&= \int_0^{b'} \sqrt{f(\beta(s))^2 - h(\beta'(s), \beta'(s))}\, ds \\
&\leq \int_0^{b'} f(\beta(s))\, ds \\
&\leq Nb' \\
&< \infty
\end{aligned}
$$

□

Interestingly, the analogous theorem for null geodesic completeness does not require any conditions on the warping function.

Theorem 4.18

Let (M, g) be a space-time and (H, h) a Riemannian manifold. If the Lorentzian warped product $\bar{M} = M_f \times H$ is null geodesically complete, then (M, g) is also null geodesically complete.

Proof:

Suppose (M,g) is null geodesically incomplete. So there exists an inextendible null geodesic $\alpha : [0,b) \to M$ with $b < \infty$. Define a curve γ in $\bar{M}$ by $\gamma(s) = (\alpha(s),q)$ for a fixed $q \in H$. Since

$$\bar{g}(\gamma'(s),\ \gamma'(s)) = f(q)^2 g(\alpha'(s),\alpha'(s)) + h(0_q,0_q) = 0,$$

γ is a null curve. The connection formula (4.11) yields

$$\nabla_{\gamma'(s)} \gamma'(s) = \nabla^1_{\alpha'(s)} \alpha'(s) - \frac{1}{2}\bar{g}(\alpha'(s),\alpha'(s))\ \frac{\mathrm{grad}_H f}{f} = 0$$

since α is a null geodesic. Hence, γ is a null geodesic in $\bar{M}$. But, γ: $[0,b) \to \bar{M}$ is not extendible to b. If it were, then we would be able to extend $\alpha = \pi \circ \gamma$ to b, contradicting the definition of b. Hence, $(\bar{M},\bar{g})$ is null geodesically incomplete. □

Chapter 5

Curvature and Energy Conditions in $(a,b)_f \times H$

In general relativity the geometry of the space-time (M,g) is related to the energy-momentum content of the matter in the space-time via Einstein's equations,

$$\text{(5.1)} \qquad \mathrm{Ric} - \frac{1}{2}\tau\, g = 8\pi\, T,$$

where τ = scalar curvature and T = energy-momentum tensor [cf. Misner, Thorne, and Wheeler (1973, Chapter 12), Weinberg (1972, Chapter 7)]. The exact form of the energy-momentum tensor T is difficult to determine since it will depend on the contributions of a large number of matter fields. However, there are certain inequalities which it is physically realistic to assume T satisfies. [Hawking and Ellis (1973, Sections 4.3-4.4)]. These inequalities include the energy conditions and the generic condition. Via Einstein's equations, these conditions on T become conditions on the curvature tensor of the space-time.

The importance of these conditions has been their role in proving certain singularity theorems in general relativity [Hawking and Penrose (1970)]. In particular, the strong energy condition and generic condition guarantee that if a nonspacelike geodesic γ in a space-time of dimension ≥ 3 is complete, then γ contains a pair of conjugate points. Then with additional geometric or physical assumptions such as that the space-time contains a closed trapped set or is causally disconnected, it can be shown that the space-time is nonspacelike geodesically incomplete.

In this chapter we will discuss conditions on the warping function f which guarantee that a standard static space-time $(a,b)_f \times H$ satisfies certain of the energy conditions and the generic conditions.

In particular, if $f : H \to (0,\infty)$ is convex and the Ricci curvature of H is nonnegative, then $(a,b)_f \times H$ satisfies the strong energy condition, (Corollary 5.7). If, in addition, $f : H \to (0,\infty)$ is strictly convex, then $(a,b)_f \times H$ also satisfies the generic condition.

In preparation, we begin by defining the energy conditions and the generic condition. In section 2 we recall the formula for the Ricci curvature tensor for a singly warped product $M_f \times H$ and then state our results on space-times of the form $(a,b)_f \times H$.

5.1 The generic and energy conditions

Let (M,g) be a space-time. (M,g) is said to satisfy the <u>generic condition</u> if for each inextendible nonspacelike geodesic γ with tangent vector $W = \gamma'$, there is some point $\gamma(t_0)$ along γ such that

$$W^c W^d {}_{[a}R_{b]cd[e}W_{f]} \neq 0 \quad \text{at the point } \gamma(t_0).$$

[Hawking and Ellis (1973 p. 101)]. An invariant definition of the generic condition ca be found in Beem and Ehrlich (1981, p. 353). We note the following well-known fact proved in Beem and Ehrlich (1981, Appendix B).

Proposition 5.1

Let (M,g) be a space-time. If $Ric(v,v) > 0$ for all nonzero nonspacelike vectors $v \in TM$ then (M,g) satisfies the generic condition.

A space-time is said to satisfy the strong energy condition if $Ric(v,v) \geq 0$ for all nonspacelike tangent vectors $v \in TM$. By continuity, the strong energy condition is equivalent to the timelike convergence condition [Hawling and Ellis (1973, p.95)] that $Ric(v,v) \geq 0$ for all timelike tangent vectors $v \in TM$.

A space-time is said to satisfy the null convergence condition if $Ric(v,v) \geq 0$ for all null tangent vectors $v \in TM$.

The physical significance of these above conditions is discussed in Hawking and Ellis (1973, Chapter 4).

5.2 Curvature and Energy Conditions on $(a,b)_f \times H$

We will be interested in conditions on the warping function and on the Riemannian space (H,h) such that $\bar{M} = (a,b)_f \times H$ satisfies the energy conditions.

First we recall the formula for the Ricci curvature tensor of a singly warped product $M_f \times H$ [Kemp (1981, p.33), O'Neill (1983, p.211)]. Recall that grad f denotes the gradient of $f : H \to (0,\infty)$, Hess(f) denotes the Hessian of f, Δf denotes the Laplacian of f on the Riemannian manifold (H,h). Let Ric (resp. Ric_M, Ric_H) denote the Ricci curvature tensor on $\bar{M}$ (resp. M,H). As in section 3.1 we decompose tangent vectors $x \in T_{\bar{p}}(M \times H)$ as $x = (x_1, x_2)$. Also, we write $\nabla f = \text{grad } f$, $\|\nabla f\|^2 = h(\nabla f, \nabla f)$, and $m = \dim M$. Then the formula for the Ricci curvature of $\bar{M}$ is given by

$$\begin{aligned}(5.2)\quad \text{Ric}(x,y) &= \text{Ric}_M(x_1,y_1) + \text{Ric}_H(x_2,y_2) \\ &\quad - \bar{g}(x_1,y_1)\{\Delta f(q)/f(q) - (m-1)\|\nabla f(q)\|^2/f(q)^2\} \\ &\quad - m\,\text{Hess}(f)(x_2,y_2)/f(q)\end{aligned}$$

where $x = (x_1,y_1)$, $y = (y_1,y_2) \in T_{(p,q)}(M \times H)$.

In the special case of a standard static space-time $\bar{M} = (a,b)_f \times H$, formula (5.2) reduces to the formula

$$\begin{aligned}(5.3)\qquad \text{Ric}(x,y) &= \text{Ric}_H(x_2,y_2) - f(q)^{-1}[\Delta f(q)\bar{g}(x_1,y_1) \\ &\quad + \text{Hess}(f)(x_2,y_2)].\end{aligned}$$

We begin our study of the energy conditions on $(a,b)_f \times H$ by noting the following necessary condition.

<u>Proposition 5.2</u>

If $\bar{M} = (a,b)_f \times H$ satisfies the strong energy condition, then $\Delta f \geq 0$.

Proof:

From formula (5.3) we have Ric(x,y) equals

$$\mathrm{Ric}_H(x_2,y_2) - f^{-1}\bar{g}(x_1,y_1)\Delta f - f^{-1}\mathrm{Hess}(f)(x_2,y_2)$$

for $x,y \in T\,\bar{M}$, $x_1 = \pi_* x$, $x_2 = \eta_* x$, etc.

Let $v = \frac{\partial}{\partial t} \in T\,\bar{M}$. Since v is timelike we have $\mathrm{Ric}(v,v) \geq 0$ by the strong energy condition. But $\mathrm{Ric}(v,v) = f\Delta f$. Since $f > 0$, it follows that $\Delta f \geq 0$.

□

Corollary 5.3

Let $\bar{M} = (a,b)_f \times H$ with (H,h) a compact Riemannian manifold. If $\Delta f \geq 0$ then f = constant. In particular, if $(\bar{M},\bar{g})$ satisfies the strong energy condition then f = constant.

Proof:

If $\Delta f \geq 0$ on a Riemannian manifold then f satisfies a maximum principle, [Yano and Bochner (1953),p.26)]. That is, f has no maximum on H unless f is constant. But H is compact so f assumes its maximum on H. Therefore, f = constant.

□

In the special case dimension H = 1, $\Delta f \geq 0$ is necessary and sufficient for the strong energy condition on $(a,b)_f \times H$.

Proposition 5.4

Assume dim H = 1. Let $\bar{M} = (a,b)_f \times H$. Then $\text{Ric}\,(v,v) \geq 0$ for all nonspacelike $v \in T\bar{M}$ if and only if $\Delta f \geq 0$.

Proof:

One direction is simply Proposition 5.2. Assume $\Delta f \geq 0$ and let $v \in T_{(p,q)}\bar{M}$ be a nonspacelike tangent vector. Taking a coordinate x on H such that $h(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = 1$, we can write $v = \alpha\frac{\partial}{\partial t}\Big|_p + \beta\frac{\partial}{\partial x}\Big|_q$, and then,

(5.4) $\quad \bar{g}(v,v) \leq 0$ implies $\alpha^2 f(q)^2 \geq \beta^2$.

Also, $\Delta f = \text{Hess}(f)(\frac{\partial}{\partial x}\,\frac{\partial}{\partial x}) = \dfrac{\partial^2 f}{\partial x^2}$ for $\frac{\partial}{\partial x} \in T\,H$.

So we compute Ric (v,v):

$$\begin{aligned}\text{Ric}(v,v) &= -f(q)^{-1}\bar{g}(\alpha\tfrac{\partial}{\partial t}, \alpha\tfrac{\partial}{\partial t})\,\Delta f - f(q)^{-1}\text{Hess}(f)(\beta\tfrac{\partial}{\partial x}, \beta\tfrac{\partial}{\partial x}) \\ &= f(q)^{-1}[\alpha^2 f(q)^2\,\Delta f - \beta^2\,\text{Hess}(f)(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial x})] \\ &\geq f(q)^{-1}[\beta^2\,\Delta f - \beta^2\text{Hess}(f)(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial x})] \quad \text{using (5.4)} \\ &= 0.\end{aligned}$$

□

In singularity theory the assumption that $\dim \bar{M} \geq 3$ is made because null conjugate points never exist in two-dimensional Lorentzian manifolds. We now turn our attention to this case.

Theorem 5.5

Let $\bar{M} = (a,b)_f \times H$ be a standard static space-time with dim $H \geq 2$. If $Ric_H(w,w) \geq 0$ for all $w \in T H$ and $h(w,w)\,\Delta f - Hess(f)(w,w) \geq 0$ for all $w \in T H$ then $Ric(v,v) \geq 0$ for all nonspacelike $v \in T\bar{M}$.

Proof:

We first show that $f \geq 0$. Fix $p \in H$ and let U be a normal neighborhood of p. Let $(e_1,\ldots,e_n)$ be an orthonormal basis for T_pH and $(x^1,\ldots,x^n)$ be the corresponding normal coordinate system. Then

$$\Delta f = \text{trace} \circ \text{Hessian} = \sum_{i=1}^{n} h(e_i,e_i)\text{Hess}(f)(e_i,e_i)$$

$$= \sum_{i=1}^{n} \frac{\partial^2 f}{\partial {x^i}^2} .$$

Now we have by hypothesis

$$0 \leq h(e_i,e_i)\Delta f - \text{Hess}(f)(e_i,e_i) = \sum_{j\neq i} \frac{\partial^2 f}{\partial {x^j}^2} \quad \text{for } i=1,\ldots n.$$

Adding these n equations we obtain

$$(n - 1)\sum_{i=1}^{n} \frac{\partial^2 f}{\partial {x^i}^2} \geq 0.$$

Hence, for $n \geq 2$ we obtain

$$\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial {x^i}^2} \geq 0.$$

Now to prove the theorem let $v \in T_{(t,p)}\bar{M}$ be a nonspacelike vector. We can write $v = (c \frac{\partial}{\partial t}|_t, v_2)$ where $v_2 \in T_pH$. The nonspacelike causal character of v implies

$$0 \geq \bar{g}(v,v) = -f(p)^2 c^2 + h(v_2,v_2).$$

Thus,

$$h(v_2,v_2) \leq f(p)^2 c^2. \tag{5.5}$$

Now using (5.3) we have

$$\begin{aligned} Ric(v,v) &= Ric_H(v_2,v_2) - f(p)^{-1}\Delta f_{|p}\bar{g}(c \tfrac{\partial}{\partial t}, c \tfrac{\partial}{\partial t}) \\ &\qquad - f(p)^{-1}Hess(f)(v_2,v_2) \\ &\geq c^2 f(p)\Delta f_{|p} - f(p)^{-1}Hess(f)(v_2,v_2) \end{aligned}$$

since $Ric_H(w,w) \geq 0$ for all $w \in T H$.

Using (5.5) and the inequality $\Delta f \geq 0$ we then have

$$\begin{aligned} Ric(v,v) &\geq f(p)^{-1}[h(v_2,v_2)\ \Delta f - Hess(f)(v_2,v_2)] \\ &\geq 0 \end{aligned}$$

□

Recall that a function $\psi : (M,g) \to \mathbb{R}$ on a pseudo-Riemannian manifold (M,g) is said to be convex if $Hess(\psi)(v,v) \geq 0$ for all $v \in TM$.

Corollary 5.6

Let $\bar{M} = (a,b)_f \times H$ be a standard static space-time. If $Ric_H(w,w) \geq 0$ for all $w \in TH$ and $f: H \to (0,\infty)$ is a convex function then $Ric(v,v) \geq 0$ for all nonspacelike vectors $v \in T\bar{M}$.

Proof:

In case dim $H = 1$ we let $e_1 \in T_pH$ be a unit vector. So $\Delta f(p) = Hess(f)(e_1,e_1)|_p \geq 0$. Hence, $Ric\ (v,v) \geq 0$ for all nonspacelike $v \in T\bar{M}$ by Proposition 5.4

Now in the case dim $H \geq 2$, by Theorem 5.5 it suffices to show that the convexity of f implies $h(w,w)\Delta f - Hess(f)(w,w) \geq 0$ for all $w \in TH$. Fix a $p \in H$ and let $w \in T_pH$ be a nonzero tangent vector. Let $e_1 = \frac{w}{\|w\|}, e_2, \ldots, e_n$ be an orthonormal basis of T_pH.

Then $\Delta f(p) = \sum_{i=1}^{n} Hess(f)(e_i,e_i)|_p$ by the definition of the Laplacian. Thus,

$$\Delta f(p)h(w,w) - Hess(f)(w,w)|_p = h(w,w)\left[\Delta f(p) - Hess(f)\left(\frac{w}{\|w\|},\frac{w}{\|w\|}\right)|_p\right]$$

$$= h(w,w)\left[\sum_{i=1}^{n} Hess(f)(e_i,e_i)|_p - Hess(f)(e_1,e_1)|_p\right]$$

$$= h(w,w)\left[\sum_{i=2}^{n} Hess(f)(e_i,e_i)|_p\right]$$

$$\geq 0$$

if f is convex. □

The following example shows that convexity of the warping function f is not a necessary condition for the strong energy condition to be satisfied by $\bar{M}$, with $Ric_H \geq 0$.

Example 5.7

Let $\bar{M} = \mathbb{R}_f \times \mathbb{R}^3$ with metric $\bar{g} = -f^2\, dt^2 \oplus h$, where h denotes the euclidean metric on $\mathbb{R}^3$. With the usual cartesian coordinates x_1, x_2, x_3 on H, define $f\colon H \to (0,\infty)$ by $f(x_1,x_2,x_3) = \sin x_1 + x_2^2 + x_3^2 + 2$. So $Ric_H \equiv 0$ and f is not convex since

$$\mathrm{Hess}(f)\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}\right) = \frac{\partial^2 f}{\partial x_1^2} = -\sin x_1.$$

Note that $\Delta f(x_1,x_2,x_3) = 4 - \sin x_1$ and for

$$w = w_1 \frac{\partial}{\partial x_1} + w_2 \frac{\partial}{\partial x_2} + w_3 \frac{\partial}{\partial x_3} \in T_pH, \quad p = (p_1,p_2,p_3) \in H,$$

a calculation yields

$$\begin{aligned} \Delta f\, h(w,w) - \mathrm{Hess}(f)(w,w) &= 4w_1^2 + 2w_2^2 + 2w_3^2 - \sin p_1 (w_2^2 + w_3^2) \\ &\geq 4w_1^2 + w_2^2 + w_3^2 \\ &\geq 0. \end{aligned}$$

Thus, by Theorem 5.10 and Corollary 5.11, $Ric(v,v) \geq 0$ for all nonspacelike $v \in T\bar{M}$.

The conditions of nonnegative Ricci curvature on (H,h) and $h(w,w)\,\Delta f - \mathrm{Hess}(f)(w,w) \geq 0$ for all tangent vectors $w \in TH$ do not imply that $Ric\,(v,v) \geq 0$ for all <u>spacelike</u> vectors $v \in T\bar{M}$. For example, let $\bar{M} = \mathbb{R}_f \times \mathbb{R}^3$ with euclidean metric on $\mathbb{R}^3$. In the usual cartesian coordinates (x_1, x_2, x_3) on $\mathbb{R}^3$, let $f: \mathbb{R}^3 \to (0,\infty)$ be defined by $f(x_1,x_2,x_3) = x_1^2 + b$, $b > 0$. Let $v \in T\,\bar{M}$ be the spacelike vector $v = \frac{\partial}{\partial x_1}$. Since $\Delta f = \text{Hess}(f)(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}) = \frac{\partial^2 f}{\partial x_1^2} = 2$, we can calculate

$$\begin{aligned} \text{Ric}\,(v,v) &= \text{Ric}_H(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}) - f^{-1}\,\bar{g}(0_t, 0_t)\Delta f - f^{-1}\text{Hess}(f)(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}) \\ &= -\frac{2}{x_1^2 + b} \\ &< 0. \end{aligned}$$

Since $\mathbb{R}^3$ is flat in the euclidean metric and for $w = w_1 \frac{\partial}{\partial x_1} + w_2 \frac{\partial}{\partial x_2} + w_3 \frac{\partial}{\partial x_3} \in T\,H$, $h(w,w)\,\Delta f - \text{Hess}(f)(w,w) = 2(w_2^2 + w_3^2) \geq 0$, we can conclude that $\text{Ric}\,(v,v) \geq 0$ for all nonspacelike $v \in T\,\bar{M}$ by Theorem 5.5.

Corollary 5.8

Let $\bar{M} = (a,b)_f \times H$ be a standard static space-time with $\dim H \geq 2$. If $\text{Ric}_H(w,w) \geq 0$ for all $w \in T\,H$ and

$$h(w,w)\Delta f - \text{Hess}(f)(w,w) > 0 \qquad (5.6)$$

for all $w \in T\,H$ then $\text{Ric}(v,v) > 0$ for all nonzero nonspacelike vectors $v \in T\,\bar{M}$. Hence, both the generic and strong energy conditions are satisfied on $\bar{M}$.

Proof:

We only need to reexamine the proof of Theorem 5.5 in the light of the strict inequality (5.6). Repeating the computations in the previous proof, we obtain $\Delta f > 0$. Hence, if $v \in T\bar{M}$ is of the form $v = c\frac{\partial}{\partial t}$, $c \neq 0$, then

$$\text{Ric}\,(v,v) = c^2 f \Delta f > 0.$$

If $v = c\frac{\partial}{\partial t}\big|_t + v_2 \in T_{(t,p)}(\mathbb{R} \times H)$ is nonspacelike, $v_2 \neq 0$ then repeating the computation in the proof of Theorem 5.5, we obtain

$$\begin{aligned}\text{Ric}\,(v,v) &\geq f(p)^{-1}[h(v_2,v_2)\Delta f - \text{Hess}(f)(v_2,v_2)] \\ &> 0 \qquad \text{by (5.6).}\end{aligned}$$

The final claim is simply Proposition 5.1.

□

Remark 5.9

From the proof of Corollary 5.7 for the case dim $H \geq 2$, we see that strictly convexity of f implies (5.6). Thus, $\text{Ric}_H \geq 0$ and strict convexity of f imply both the generic and strong energy conditions on $\bar{M} = (a,b)_f \times H$ with dim $H \geq 2$.

Now we consider the case with (H,h) a Ricci flat Riemannian manifold of dim ≥ 2.

Theorem 5.10

Let $\bar{M} = (a,b)_f \times H$ be a standard static space-time with (H,h) a Ricci flat Riemannian manifold, dim $H \geq 2$. Then $h(w,w)\Delta f - \text{Hess}(f)(w,w) \geq 0$ for all $w \in TH$ if and only if

$\bar{M}$ satisfies the null convergence condition (i.e., $\text{Ric}\,(v,v) \geq 0$ for all null $v \in T\,\bar{M}$).

Proof:

By Theorem 5.5, $h(w,w)\Delta f - \text{Hess}(f)(w,w) \geq 0$ implies the strong energy condition, which in turn certainly implies the null convergence condition. So assume $\text{Ric}\,(v,v) \geq 0$ for all null $v \in T\,\bar{M}$. Let $p \in H$ and let $w \in T_pH$ be a nonzero vector. Define $v \in T_{(t,p)}(\mathbb{R} \times H)$ by

$$v = \frac{\sqrt{h(w,w)}}{f(p)} \left.\frac{\partial}{\partial t}\right|_t + w.$$

Then $\bar{g}(v,v) = -f(p)^2 \left(\frac{\sqrt{h(w,w)}}{f(p)}\right)^2 + h(w,w) = 0,$

so v is a null vector. Using (5.1),

$$\begin{aligned} 0 \leq \text{Ric}\,(v,v) &= \text{Ric}_H(w,w) \\ &\quad -f(p)^{-1}\Delta f(p)\,\bar{g}\left(\frac{\sqrt{h(w,w)}}{f(p)}\frac{\partial}{\partial t}, \frac{\sqrt{h(w,w)}}{f(p)}\frac{\partial}{\partial t}\right) \\ &\quad - f(p)^{-1}\text{Hess}(f)_{|p}(w,w) \\ &= f(p)^{-1}[h(w,w)\Delta f_{|p} - \text{Hess}(f)_{|p}(w,w)]. \end{aligned}$$

Thus, $h(w,w)\Delta f - \text{Hess}(f)(w,w) \geq 0$, as desired. □

Corollary 5.11

Let $\bar{M} = (a,b)_f \times H$ be a standard static space-time with (H,h) Ricci flat and $\dim H \geq 2$. Then $\bar{M}$ satisfies the null convergence condition if and only if it satisfies the strong energy condition.

Proof:

Suppose $\bar{M}$ satisfies the null convergence condition. Then by Theorem 5.10,

$$h(w,w)\Delta f - \text{Hess}(f)(w,w) \geq 0 \quad \text{for all } w \in T\,H.$$

So Theorem 5.5 implies $\bar{M}$ satisfies the strong energy condition. □

It is interesting to compare Theorem 5.5 and Corollary 5.6 with a result of Steen Markvorsen (personal communication, 1984) on Lorentzian warped products of the form $\bar{M} = (a,b) \times_e H$. Here $e : (a,b) \to (0,\infty)$ is a smooth function on the interval (a,b) and $\bar{M}$ is given the metric $\bar{g} = -dt^2 \oplus e^2 h$. By considering the Ricci curvature tensor on a warped product of this type, Markvorsen has shown that if $\text{Ric}_H \geq 0$, then concavity of the function e on the interval (a,b) implies that $\text{Ric}\,(v,v) \geq 0$ for all nonspacelike vectors $v \in T\,\bar{M}$.

So far in this chapter we have discussed conditions on a standard static space-time which guarantee that certain energy conditions are satisfied. In the remainder of this chapter we will discuss some standard static space-times which cannot satisfy certain energy conditions.

Theorem 5.12

Let (H,h) be a compact Riemannian manifold of dimension ≥ 2. Suppose for some point $p \in H$ there is a tangent vector $w \in T_pH$ with $Ric_H(w,w) < 0$. Then the standard static space-time $\bar{M} = (a,b)_f \times H$ cannot satisfy the strong energy condition, i.e. $Ric\,(v,v) < 0$ for some nonspacelike vector $v \in T\bar{M}$.

Proof:

Suppose by way of contradiction that the strong energy condition is satisfied by $\bar{M}$. Then by Corollary 5.3, $f : H \to (0,\infty)$ must be a constant function. Thus, $\Delta f \equiv Hess(f) \equiv 0$. If $w \in T_pH$ is chosen as in the hypothesis, let $v \in T_{(t,p)}\bar{M}$ be defined by

$$v = \frac{\sqrt{h(w,w)}}{f(p)} \left.\frac{\partial}{\partial t}\right|_t + w.$$

So v is a null tangent vector and $Ric\,(v,v) \geq 0$ since we are assuming the strong energy condition. But by (5.3),

$$Ric\,(v,v) = Ric_H(w,w) < 0.$$

The contradiction implies that $\bar{M}$ cannot satisfy the strong energy condition.

□

With some additional hypotheses, Theorem 5.12 can be extended to noncompact, finite volume Riemannian manifolds. In the proof we use some of the techniques that Bishop and O'Neill (1969) used to show finite volume complete Riemannian manifolds do not admit nonconstant convex functions. Before stating the result we state a well-known lemma.

Lemma 5.13

Let (H,h) be a complete Riemannian manifold and let X be a vector field of bounded length on H. Then X is a complete vector field.

Theorem 5.14

Let $\bar{M} = (a,b)_f \times H$ be a standard static space-time with (H,h) a complete, finite volume Riemannian manifold. Suppose the gradient of $f : H \to (0,\infty)$ never vanishes on H. Then for some $p \in H$

$$(5.7) \quad \Delta f|_p h(\text{grad } f, \text{grad } f)|_p - \text{Hess}(f)(\text{grad } f, \text{grad } f)|_p < 0.$$

If, in addition,

$$(5.8) \quad \text{Ric}_H(\text{grad } f, \text{grad } f) \leq 0 \quad \text{at the point } p \in H,$$

then $(\bar{M},\bar{g})$ cannot satisfy the null convergence condition.

Proof:

We begin by establishing (5.7). Define a unit length vector field U on H by

$$U(x) = \frac{\text{grad}(f)}{\|\text{grad}(f)\|}.$$

Then by Lemma 5.13, U is a complete vector field on H; that is, the flow transformations $\{\phi_t\}$ of U are defined for all $t \in \mathbb{R}$. Equivalently, the integral curves α_x of U such that $\alpha_x(0) = x \in H$ and $\alpha_x'(t) = U_{\alpha_x(t)}$, are defined on all of $\mathbb{R}$.

Since grad $f \neq 0$ and h is a positive definite metric, we can compute

$$Uf = h(U,\text{grad } f) = \|\text{grad } f\|^{-1} h(\text{grad } f,\text{grad } f)$$

$$= \|\text{grad } f\| > 0.$$

Then $(f \circ \alpha_x)'(t) = \alpha_x'(t) f = U_{\alpha_x(t)} f > 0$, so $f \circ \alpha_x : \mathbb{R} \to (0,\infty)$ is a strictly increasing function for all $x \in H$.

Now let $m \in (0,\infty)$ be a number such that $0 \leq \inf_{x \in H} f(x) < m < \sup_{x \in H} f(x) \leq \infty$ and define the set $c = \{x \in H : f(x) \geq m\}$. Since $f \circ \alpha_x$ is a strictly increasing function, the flow transformation ϕ_t, for $t > 0$, will map C properly into a set of smaller volume. Thus the divergence of U must be negative somewhere in U, say at $p \in H$.

Now consider a local orthonormal basis $E_1 = U, E_2, \ldots, E_n$ for T_pH. Let $G = \text{grad } f$. Then

$$\text{div } U = \sum_{i=1}^{n} h(\nabla_{E_i} U, E_i) = \sum_{i=1}^{n} h(\nabla_{E_i} \frac{G}{\|G\|}, E_i)$$

$$\text{div } U = \sum_{i=1}^{n} [h(E_i(\|G\|^{-1})G + \|G\|^{-1} \nabla_{E_i} G, E_i)]$$

$$= E_1(\|G\|^{-1})h(G,E_1) + \|G\|^{-1} \sum_{i=1}^{n} h(\nabla_{E_i} G, E_i)$$

$$= E_1(\|G\|^{-1})\|G\| + \|G\|^{-1} \Delta f.$$

But

$$E_1(\|G\|^{-1}) = U(\|G\|^{-1})$$

$$= \|G\|^{-3} h(\nabla_U G, G)$$

$$= \|G\|^{-2} \text{Hess}(f)(U,U).$$

Thus, $\text{div } U = \|G\|^{-1}[\Delta f - \text{Hess}(f)(U,U)]$

$$= \|G\|^{-3}[h(G,G)\Delta f - \text{Hess}(f)(G,G)].$$

Since $\text{div } U_{|p} < 0$ and $\|G\| > 0$, we conclude that

$$h(\text{grad } f, \text{grad } f)_{|p} \, \Delta f_{|p} - \text{Hess}(f)(\text{grad } f, \text{grad } f)_{|p} < 0.$$

With (5.7) established, we assume (5.8) and show $\bar{M} = (a,b)_f \times H$ cannot satisfy the null convergence condition. For fixed $p \in H$ as above for some $t \in (a,b)$, define a null tangent vector $v \in T_{(t,p)}\bar{M}$ by

$$v = c \frac{\partial}{\partial t}_{|p} + \text{grad } f_{|p} \quad , \quad \text{where}$$

$$c = \frac{\sqrt{h(\text{grad } f, \text{grad } f_{|p})}}{f(p)} \quad .$$

Then using (5.3) we compute

$$\text{Ric}\,(v,v) = \text{Ric}_H(\text{grad } f, \text{grad } f) - f^{-1}\,\bar{g}(c\,\frac{\partial}{\partial t}, c\,\frac{\partial}{\partial t})\Delta f$$

$$- f^{-1}\text{Hess}(f)(\text{grad } f, \text{grad } f).$$

Using (5.7) and (5.8), we have

$$\text{Ric}\,(v,v) \le f^{-1}(p)\,[h(\text{grad } f, \text{grad } f)_{|p}\ \ \Delta f_{|p}$$

$$- \text{Hess}(f)(\text{grad } f, \text{grad } f)_{|p}]$$

$$< 0$$

□

Corollary 5.15

Let $\bar{M} = (a,b)_f \times H$ be a standard static space-time with (H,h) a complete finite volume Riemannian manifold with nonpositive Ricci curvature. Suppose the gradient of $f : H \to (0,\infty)$ never vanishes on H. Then $(\bar{M},\bar{g})$ cannot satisfy the null convergence condition.

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AND STATIC SPACE-TIMES

presented by Dean Edwin Allison

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