APPLICATIONS OF THE FOURIER TRANSFORM TO CONVEX GEOMETRY

A Dissertation
presented to
the Faculty of the Graduate School
University of Missouri-Columbia

In Partial Fulfillment
of the Requirements for the Degree

Doctor of Philosophy

by
VLADYSLAV YASKIN

Dr. Alexander Koldobsky, Dissertation Supervisor

MAY 2006

The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

APPLICATIONS OF THE FOURIER TRANSFORM
TO CONVEX GEOMETRY

presented by Vladyslav Yaskin

a candidate for the degree of Doctor of Philosophy

and hereby certify that in their opinion it is worthy of acceptance.

[illegible]

David G. Retzloff

[illegible]

[illegible]

Alex Iosevich

ACKNOWLEDGEMENTS

I would like to express my gratitude to all those who gave me the possibility to complete this thesis. There are many people without whom this dissertation would not have seen the light of day.

First, I would like to thank my advisor Alexander Koldobsky for the guidance and encouragement he provided over the past few years. Everything I know about Convex Geometry and Fourier Analysis I learned from him. I am grateful for his understanding, endless patience and caring.

I am also grateful to Yu. A. Aminov and Stamatis Dostoglou for raising me as a mathematician and cultivating my love of Geometry.

I would like to thank my committee members for taking the time to read the dissertation and giving many valuable suggestions.

I also wish to thank my co-authors N. J. Kalton, M. Yaskina, A. Pajor for fruitful collaboration and discussions on topics related to this thesis. I would also like to thank Artem Zvavitch and Dmitry Ryabogin for critical reading of our manuscripts and inspiring conversations.

I am thankful to the Department of Mathematics of the University of Missouri for creating a friendly atmosphere and productive working environment.

Last, but not least, I am grateful to my wife and parents for their love and constant support, and to my little daughter for being a source of inspiration. To

them this work is dedicated.

APPLICATIONS OF THE FOURIER TRANSFORM

TO CONVEX GEOMETRY

Vladyslav Yaskin

Dr. Alexander Koldobsky, Dissertation Supervisor

ABSTRACT

The thesis is devoted to the study of various problems arising from Convex Geometry and Geometric Functional Analysis using tools of Fourier Analysis.

In chapters two through four we consider the Busemann-Petty problem and its different modifications and generalizations. We solve the Busemann-Petty problem in hyperbolic and spherical spaces, and the lower dimensional Busemann-Petty problem in the hyperbolic space. In the Euclidean space we modify the assumptions of the original Busemann-Petty problem to guarantee the affirmative answer in all dimensions.

In chapter five we introduce the notion of embedding of a normed space in L_0, investigate the geometry of such spaces and prove results confirming the place of L_0 in the scale of L_p spaces.

Chapter six is concerned with the study L_p-centroid bodies associated to symmetric convex bodies and generalization of some known results of Lutwak and Grinberg, Zhang to the case $-1 < p < 1$.

In chapter seven we discuss Khinchin type inequalities and the slicing problem.

We obtain a version of such inequalities for $p > -2$ and as a consequence we prove the slicing problem for the unit balls of spaces that embed in L_p, $p > -2$.

Contents

Acknowledgements **ii**

Abstract **iv**

1 Preliminaries **1**

1.1 Functional and Fourier Analysis 1

1.2 Differential Geometry 7

2 The Busemann-Petty problem in hyperbolic and spherical spaces 12

2.1 Introduction 12

2.2 Proofs of main results 14

3 The lower dimensional Busemann-Petty problem in the hyperbolic space **28**

3.1 Introduction 28

3.2 Proof of main result 29

4 Modified Busemann-Petty problem **38**

4.1 Introduction 38

4.2 Distributions of the form $|x|_2^{-r}\|x\|_K^{-s}$ 41

4.3 The proof of the main result 47

5 The geometry of L_0 **57**

5.1 Introduction 57

5.2 The definition of embedding in L_0. 59

5.3 A Fourier analytic characterization of subspaces of L_0 62

5.4 A geometric characterization of subspaces of L_0. 67

5.5 Addition in L_0 72

5.6 Confirming the place of L_0 in the scale of L_p-spaces. 80

6 Centroid bodies and comparison of volumes **84**

6.1 Introduction 84

6.2 Centroid bodies for $-1 < p < 1$, $p \neq 0$. 88

6.3 Centroid bodies for $p = 0$. 97

7 Khinchin type inequalities and sections of L_p-balls, $p > -2$. **107**

7.1 Introduction 107

7.2 Subspaces of L_p with $p > 2$. 109

7.3 Subspaces of L_p with $p < 0$ 112

7.4 Khinchin type inequalities 114

7.5 Slicing problem for L_p-balls, $-2 < p < 0$. 119

7.6 Appendix 120

Bibliography **123**

Vita **132**

Chapter 1

Preliminaries

1.1 Functional and Fourier Analysis

The **Minkowski functional** of a star-shaped origin-symmetric body $K \subset \mathbb{R}^n$ is defined as

$$\|x\|_K = \min\{a \geq 0 : x \in aK\}.$$

The **radial function** of K is given by $\rho_K(x) = \|x\|_K^{-1}$. If $x \in S^{n-1}$ then the radial function $\rho_K(x)$ is the Euclidean distance from the origin to the boundary of K in the direction of x.

Writing the volume of K in polar coordinates, one can express the volume in terms of the Minkowski norm:

$$\mathrm{vol}_n(K) = \frac{1}{n} \int_{S^{n-1}} \|\theta\|_K^{-n} d\theta. \tag{1.1}$$

We say that a body K is infinitely smooth if its radial function ρ_K restricted to the unit sphere S^{n-1} belongs to the space $C^\infty(S^{n-1})$ of infinitely differentiable functions on the unit sphere.

We say that a closed bounded set K in $\mathbb{R}^n$ is a **star body** if for every $x \in K$ each point of the interval $[0, x)$ is an interior point of K, and $\|x\|_K$, the Minkowski functional of K, is a continuous function on $\mathbb{R}^n$.

The **radial metric** on the set of all origin symmetric star bodies is defined by

$$\rho(K, L) = \max_{x \in S^{n-1}} |\rho_K(x) - \rho_L(x)|.$$

One of the most important tools in this work is the Fourier transform of distributions. Let ϕ be a function from the Schwartz space $\mathcal{S}$ of rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$. We define the Fourier transform of ϕ by

$$\hat{\phi}(\xi) = \int_{\mathbb{R}^n} \phi(x) e^{-i\langle x, \xi\rangle} dx, \quad \xi \in \mathbb{R}^n.$$

The Fourier transform of a distribution f is defined by $\langle \hat{f}, \phi\rangle = \langle f, \hat{\phi}\rangle$ for every test function ϕ from the space $\mathcal{S}$.

We say that a distribution f is **positive definite**, if its Fourier transform is a positive distribution, in the sense that $\langle \hat{f}, \phi\rangle \geq 0$ for every non-negative test function ϕ.

If a distribution f acts on test functions in the same way as a continuous function g then we write that $f(x) = g(x)$ pointwise. This is just notation meaning that f and g coincide on all test functions. In particular if $\hat{f} = g$ on test functions we write $\hat{f}(x) = g(x)$ pointwise, where in the left-hand side we do not mean the convergent Fourier integral, but understand this as equality of distributions.

Let ϕ be an integrable function on $\mathbb{R}^n$ that is also integrable on hyperplanes, let $\xi \in S^{n-1}$, and let $t \in \mathbb{R}^n$. Then

$$\mathcal{R}\phi(\xi; t) = \int_{(x,\xi)=t} \phi(x) dx$$

is the **Radon transform of ϕ in the direction ξ at the point t**. A simple connection between the Fourier and Radon transforms is that for every fixed $\xi \in$

$\mathbb{R}^n \setminus \{0\}$

$$\hat{\phi}(s\xi) = \left(\mathcal{R}\phi(\xi;t)\right)^{\wedge}(s), \quad \forall s \in \mathbb{R} \tag{1.2}$$

where in the right hand side we have the Fourier transform of the function $t \to \mathcal{R}\phi(\xi;t)$, see e.g. [Ko12, Lemma 2.11].

The **spherical Radon transform** $R : C(S^{n-1}) \to C(S^{n-1})$ is defined by

$$Rf(\xi) = \int_{S^{n-1}\cap\xi^{\perp}} f(x)dx.$$

The following Lemma gives a relation between the spherical Radon transform and the Fourier transform. (See Koldobsky [Ko3, Lemma 2.5], or Semyanistyi [Se] for more general results.)

Lemma 1.1.1. *Let $g(x)$ be an even homogeneous function of degree $-n+1$ on $\mathbb{R}^n \setminus \{0\}$, $n > 1$, so that $g(x)|_{S^{n-1}} \in C(S^{n-1})$ then*

$$Rg(\xi) = \frac{1}{\pi}\hat{g}(\xi), \qquad \forall \xi \in S^{n-1}.$$

The latter equality means that $\hat{g}$ is a homogeneous function of degree -1 on $\mathbb{R}^n$, whose values on S^{n-1} are equal to Rg.

Let f be an integrable continuous function on $\mathbb{R}$, m-times continuously differentiable in some neighborhood of zero, $m \in \mathbb{N}$. For a number $q \in (m-1, m)$ the **fractional derivative** of the order q of the function f at zero is defined by

$$f^{(q)}(0) = \frac{1}{\Gamma(-q)} \int_0^{\infty} t^{-1-q}\Big(f(t) - f(0) - tf'(0) - \cdots - \\ - \frac{t^{m-1}}{(m-1)!} f^{(m-1)}(0)\Big)dt.$$

Note that without dividing by $\Gamma(-q)$ the expression for the fractional derivative represents an analytic function in the domain $\{q \in \mathbb{C}, -1 < \text{Re } q < m\}$ not including integers and has simple poles at non-negative integers. The function $\Gamma(-q)$ is analytic in the same domain and also has simple poles at non-negative integers. Therefore, after division we get an analytic function in the whole domain $\{q \in \mathbb{C}, -1 < \text{Re } q < m\}$, which also defines fractional derivatives of integer orders. Moreover, computing the limit as $q \to k$, where k is a non-negative integer and $k < m$, we see that the fractional derivatives of integer orders coincide with usual derivatives up to a sign:

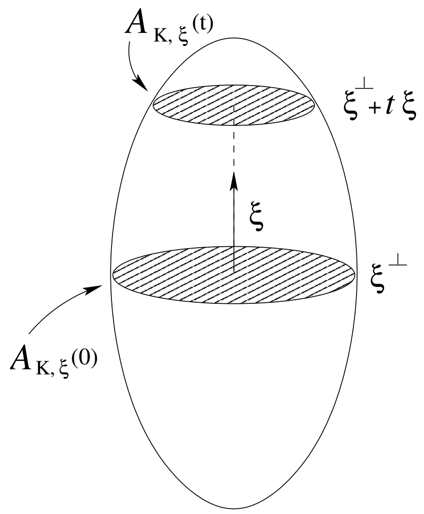
$$f^{(k)}(0) = (-1)^k \frac{d^k}{dt^k} f(t)|_{t=0}.$$

More details on fractional derivatives may be found in [Ko12, Section 2.6].

Let K be a star body and $\xi \in S^{n-1}$, the **parallel section function** of K is defined as follows:

$$A_{K,\xi}(z) = \text{vol}_{n-1}(K \cap \{\langle x, \xi \rangle = z\}).$$

(We also assume that $K \cap \{\langle x, \xi \rangle = z\}$ is star-shaped for small z).



Recall the following fact:

Theorem 1.1.2. ([GKS], Theorem 1) *Let K be an origin-symmetric star body in $\mathbb{R}^n$ with C^∞ boundary, and let $k \in \mathbb{N} \setminus \{0\}$, $k \neq n-1$. Suppose that $\xi \in S^{n-1}$, and let A_ξ be the corresponding parallel section function of K: $A_\xi(z) = \int_{K \cap \langle x, \xi \rangle = z} dx$. (We also assume that $K \cap \{\langle x, \xi \rangle = z\}$ is star-shaped for small z).*

(a) If k is even, then

$$(\|x\|^{-n+k+1})^\wedge(\xi) = (-1)^{k/2}\pi(n-k-1)A_\xi^{(k)}(0).$$

(b) If k is odd, then

$$(\|x\|^{-n+k+1})^\wedge(\xi) = (-1)^{(k+1)/2} 2(n-1-k)k! \times$$
$$\times \int_0^\infty \frac{A_\xi(z) - A_\xi(0) - A''_\xi(0)\frac{z^2}{2} - \cdots - A_\xi^{(k-1)}(0)\frac{z^{k-1}}{(k-1)!}}{z^{k+1}} dz,$$

where $A_\xi^{(k)}$ stands for the derivative of the order k and the Fourier transform is considered in the sense of distributions.

In particular, it follows that for infinitely smooth bodies the Fourier transform of $\|x\|^{-n+k+1}$ restricted to the unit sphere is a continuous function (see also [Ko12, Section 3.2])

We will also need the following version of Parseval's formula on the sphere

Lemma 1.1.3. (Koldobsky, [Ko6]) *If K and L are origin symmetric infinitely smooth bodies in $\mathbb{R}^n$ and $0 < p < n$, then $(\|x\|_K^{-p})^\wedge$ and $(\|x\|_L^{-n+p})^\wedge$ are continuous functions on S^{n-1} and*

$$\int_{S^{n-1}} \left(\|x\|_K^{-p}\right)^\wedge(\xi) \left(\|x\|_L^{-n+p}\right)^\wedge(\xi) d\xi = (2\pi)^n \int_{S^{n-1}} \|x\|_K^{-p} \|x\|_L^{-n+p} dx.$$

We will also be using the following version of this Lemma, see [Ko6, Corollary 1].

Corollary 1.1.4. *Let f and g be functions on $\mathbb{R}^n$, continuous on S^{n-1} and homogeneous of degree -1 and $-n+1$ respectively. Suppose that f represents a positive definite distribution. Then there exists a measure γ_0 on S^{n-1} such that*

$$\int_{S^{n-1}} \widehat{g}(\theta)\, d\gamma_0(\theta) = (2\pi)^n \int_{S^{n-1}} f(\theta)\, g(\theta)\, d\theta.$$

Here we do not assume that f is an infinitely differentiable function, so its Fourier transform is not necessarily a function, but merely a measure.

The following result was also proved in [Ko6].

Lemma 1.1.5. *Let L be an origin symmetric star body with C^∞ boundary in $\mathbb{R}^n$. Then for every $(n-k)$-dimensional subspace H of $\mathbb{R}^n$ we have*

$$(2\pi)^k \int_{S^{n-1}\cap H} \|\theta\|_L^{-n+k} d\theta = \int_{S^{n-1}\cap H^\perp} (\|x\|_L^{-n+k})^\wedge(\theta) d\theta.$$

The preceding two lemmas were formulated for Minkowski functionals, but in fact they are true for arbitrary infinitely differentiable even functions on the sphere extended to $\mathbb{R}^n \setminus \{0\}$ as homogeneous functions of corresponding degrees. (Indeed, any such function of degree $-p$ can be obtained as the difference of Minkowski functionals raised to the power $-p$).

Let K be an origin symmetric star body in $\mathbb{R}^n$. We denote by $(\mathbb{R}^n, \|\cdot\|_K)$ the Euclidean space equipped with the Minkowski functional of the body K. Clearly, $(\mathbb{R}^n, \|\cdot\|_K)$ is a normed space if and only if the body K is convex.

A well-known result of P.Lévy, see for example [Ko12, Section 6.1], is that a space $(\mathbb{R}^n, \|\cdot\|)$ embeds into L_p, $p > 0$ if and only if there exists a finite Borel measure μ on the unit sphere so that, for every $x \in \mathbb{R}^n$,

$$\|x\|^p = \int_{S^{n-1}} |(x,\xi)|^p d\mu(\xi). \tag{1.3}$$

If p is not an even integer, this condition is equivalent to the fact that $(\Gamma(-p/2)\|x\|^p)^\wedge$ is a positive distribution outside of the origin, see [Ko12, Theorem 6.10].

The concept of embedding in L_p with $-n < p < 0$ was introduced in [Ko7] by extending formula (6.6) analytically to negative values of p. Namely, a space $(\mathbb{R}^n, \|\cdot\|)$ embeds into L_p, $-n < p < 0$ if there exists a finite Borel measure on S^{n-1} so that for every test function $\phi \in \mathcal{S}$

$$\int_{\mathbb{R}^n} \|x\|_K^p \phi(x)dx = \int_{S^{n-1}} \left(\int_{\mathbb{R}} |z|^{-p-1} \hat{\phi}(z\theta)dt \right) d\mu(\theta).$$

It was also proved that, as for positive p, there is a Fourier analytic characterization for such embeddings, namely a space $(\mathbb{R}^n, \|\cdot\|)$ embeds in L_{-p} if and only if the Fourier transform of $\|\cdot\|^{-p}$ is a positive distribution in $\mathbb{R}^n$.

1.2 Differential Geometry

Let $\mathbb{S}^n$ be the unit sphere in $\mathbb{R}^{n+1}$. Using the stereographic projection (from the north pole onto the hyperplane containing the equator) we can think of it as $\mathbb{R}^n$ equipped with the metric of constant curvature $+1$:

$$ds^2 = 4\frac{dx_1^2 + \cdots + dx_n^2}{(1 + (x_1^2 + \cdots + x_n^2))^2},$$

where x_1,..., x_n are the standard Euclidean coordinates in $\mathbb{R}^n$. (See [DFN, §9, §10], and [R, §4.5] for details about the spherical and hyperbolic spaces). It is well-known that geodesic lines on the sphere are great circles. Later on, in order to define convexity, we will need the uniqueness property of geodesics joining given 2 points. But this is not the case on the sphere. However if we restrict ourselves to an open hemisphere, then for any two points there exists a unique geodesic segment connecting them. Under the stereographic projection the open south hemisphere gets mapped onto the open unit ball B^n in $\mathbb{R}^n$. This is the model we will be working in. The geodesics in this model are arcs of the circles intersecting the boundary of the ball B^n in antipodal points and straight lines through the origin.

Also it is well-known that the hyperbolic space $\mathbb{H}^n$ can be identified with the interior of the unit ball in $\mathbb{R}^n$ with the metric:

$$ds^2 = 4\frac{dx_1^2 + \cdots + dx_n^2}{(1 - (x_1^2 + \cdots + x_n^2))^2}.$$

This is the Poincaré model of the hyperbolic space in the ball. Note that it can be also obtained from the pseudeosphere in the Lorentzian space via the stereographic projection. The geodesic lines in this model are arcs of the circles orthogonal to the boundary of the ball B^n and straight lines through the origin.

Since both geometries are defined in the unit ball in $\mathbb{R}^n$, we will treat them simultaneously, considering the open ball $B^n \subset \mathbb{R}^n$ with the metric

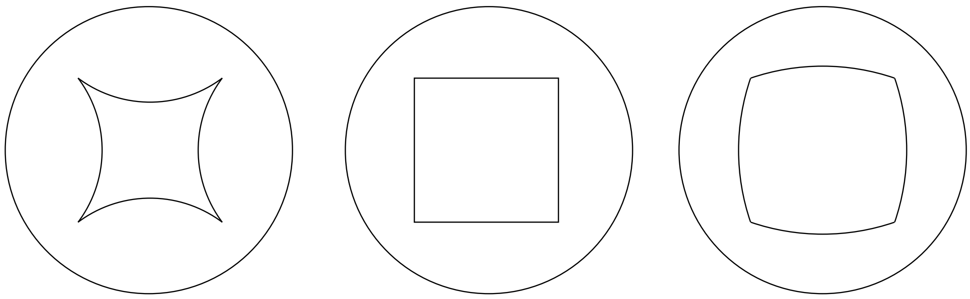
$$ds^2 = 4\frac{dx_1^2 + \cdots + dx_n^2}{(1 + \delta\ (x_1^2 + \cdots + x_n^2))^2}, \tag{1.4}$$

where $\delta = -1$ for the hyperbolic case, $+1$ for the spherical space. In addition if we consider $\delta = 0$ we get the original case of the Euclidean space.

The definition of convexity in hyperbolic and spherical spaces (recall that we work in an open hemisphere) is analogous to that in the Euclidean space (see [Po, Chapter I, §12]). A body K (compact set with non-empty interior) is called **convex** if for every pair of points in K the geodesic segment joining them also belongs to the body K. For our definition of convexity in $\mathbb{S}^n$ it is crucial that we work in an open hemisphere, since in this case we have a unique geodesic segment through any two points.

Let K be a body in the open unit ball B^n. In order to distinguish between different types of convexity we will adopt the following system of notations. The body K is called s-convex (or $+1$-convex), if it is convex in the spherical metric defined in the ball B^n. Similarly it is called h-convex (or -1-convex) if it is convex with respect to the hyperbolic metric. e-convex bodies (or 0-convex) are the bodies convex in the usual Euclidean sense. Analogously s-(h-,e-)geodesics are the straight lines of the spherical (hyperbolic, Euclidean) metric. (In this terminology we follow [MeP]. Note that in the literature there are other definitions of h-convexity or δ-convexity which have absolutely different meaning).

Shown below are some examples of convex hulls of 4 points with respect to hyperbolic, Euclidean and spherical metrics correspondingly.



Clearly, any s-convex body containing the origin is also e-convex and any e-convex body containing the origin is h-convex. (See for example [MeP]).

A submanifold $\mathcal{F}$ in a Riemannian space $\mathcal{R}$ is called **totally geodesic** if every geodesic in $\mathcal{F}$ is also a geodesic in the space $\mathcal{R}$. In the Euclidean space the totally geodesic submanifolds are Euclidean planes, on the sphere they are great subspheres. In the Poincaré model of the hyperbolic space described above the totally geodesic submanifolds are represented by the spheres orthogonal to the boundary of the unit ball B^n. In a sense, totally geodesic submanifolds are analogs of Euclidean planes in Riemannian spaces. For elementary properties of totally geodesic submanifolds see [A, Chap.5, §5].

Since origin-symmetric h-convex and s-convex bodies are star bodies in $\mathbb{R}^n$, we can apply tools of Functional analysis. In particular, the definition of the Minkowski functional makes sense.

For a centrally-symmetric δ-convex body $K \in B^n$ $(\delta = 0, 1, -1)$ consider the section of K by the hypersurface $\xi^\perp = \{\langle x, \xi \rangle = 0\}$, where $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product. Clearly such a hypersurface is a totally geodesic hyperplane in the metric (1.4) for any $\delta = 0, 1, -1$. This hyperplane passes through the origin with the normal vector ξ.

The volume element of the metric (1.4) equals

$$d\mu_n = 2^n \frac{dx_1 \cdots dx_n}{(1 + \delta\ (x_1^2 + \cdots + x_n^2))^n} = 2^n \frac{dx}{(1 + \delta\ |x|^2)^n}.$$

Therefore the volume of a body K is given by the formula:

$$\mathrm{vol}_n(K) = \int_K d\mu_n = 2^n \int_K \frac{dx}{(1 + \delta\ |x|^2)^n}.$$

Note that in polar coordinates the latter formula looks as follows:

$$\mathrm{vol}_n(K) = 2^n \int_{S^{n-1}} \int_0^{\|\theta\|_K^{-1}} \frac{r^{n-1}}{(1 + \delta\ r^2)^n} dr\ d\theta. \tag{1.5}$$

Similarly the volume element of the hypersurface $\xi^\perp$ is

$$d\mu_{n-1} = 2^{n-1}\frac{dx}{(1+\delta\ |x|^2)^{n-1}},$$

therefore the $(n-1)$-volume of the section of K by the hyperplane $\xi^\perp$ is given by the formula:

$$S_K(\xi) = \int_{K\cap\langle x,\xi\rangle=0} d\mu_{n-1} = 2^{n-1}\int_{K\cap\langle x,\xi\rangle=0}\frac{dx}{(1+\delta\ |x|^2)^{n-1}}.$$

In general, if H is a k-dimensional totally geodesic plane through the origin (as mentioned above, in our model this is just a k-dimensional Euclidean plane through the origin), then the volume element of H in the metric (1.4) is

$$d\mu_k = 2^k\frac{dx}{(1+\delta\ |x|^2)^k},$$

therefore k-volume of the section of K by H in the δ-metric is given by the formula:

$$\mathrm{vol}_k(K\cap H) = \int_{K\cap H} d\mu_k = 2^k\int_{K\cap H}\frac{dx}{(1+\delta\ |x|^2)^k},$$

or in polar coordinates:

$$\mathrm{vol}_k(K\cap H) = 2^k\int_{S^{n-1}\cap H}\int_0^{\|\theta\|_K^{-1}}\frac{r^{k-1}}{(1+\delta\ r^2)^k}dr\ d\theta. \tag{1.6}$$

Chapter 2

The Busemann-Petty problem in hyperbolic and spherical spaces

2.1 Introduction

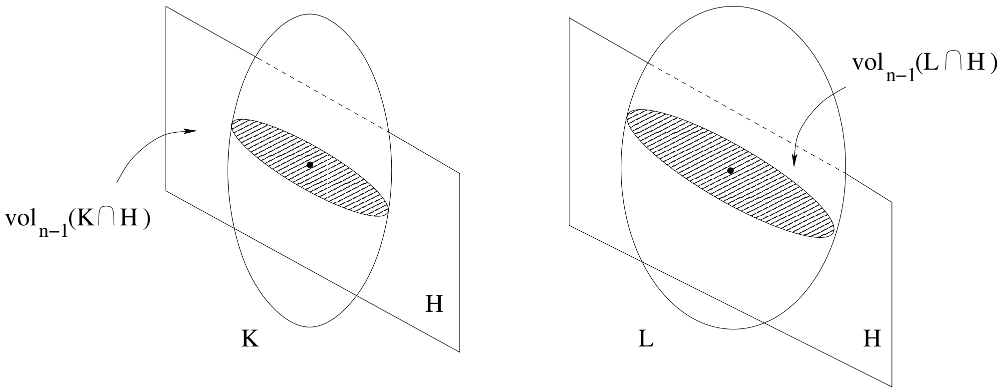
The classical Minkowski's uniqueness theorem states that an origin-symmetric star body in $\mathbb{R}^n$ is uniquely determined by the volumes of its central hyperplane sections in all directions, see for example [Ko12, Corollary 3.9]. This result provides a strong intuition towards an affirmative answer in the following Busemann-Petty problem [BP]: given two convex origin-symmetric bodies K and L in $\mathbb{R}^n$ such that

$$\mathrm{vol}_{n-1}(K \cap H) \leq \mathrm{vol}_{n-1}(L \cap H)$$

for every central hyperplane H in $\mathbb{R}^n$, does it follow that

$$\mathrm{vol}_n(K) \leq \mathrm{vol}_n(L)?$$

The answer to this problem in $\mathbb{R}^n$ is known to be affirmative if $n \leq 4$ and negative if $n \geq 5$. The solution appeared as the result of work of many mathematicians (Larman and Rogers [LR], Ball [Ba1], Giannopoulos [Gi], Bourgain [Bou2], Gardner [Ga1], Papadimitrakis [Pap], Gardner [Ga2], Zhang [Zh2], Gardner, Koldobsky, Schlumprecht [GKS]).



In this chapter we consider the Busemann-Petty problem in hyperbolic and spherical spaces in place of the Euclidean space. We present the results from [Y1], where we prove the following.

Theorem 2.1.1. *Let K and L be centrally symmetric convex bodies in the spherical space $\mathbb{S}^n$, $n \leq 4$ (more precisely in a hemisphere) such that*

$$\mathrm{vol}_{n-1}(K \cap H) \leq \mathrm{vol}_{n-1}(L \cap H) \tag{2.1}$$

for every central totally-geodesic hyperplane H in $\mathbb{S}^n$. Then

$$\mathrm{vol}_n(K) \leq \mathrm{vol}_n(L).$$

On the other hand, if $n \geq 5$ there are convex symmetric bodies K, $L \subset \mathbb{S}^n$ that satisfy (4.5) *but* $\mathrm{vol}_n(K) > \mathrm{vol}_n(L)$.

So, the answer to the Busemann-Petty in $\mathbb{S}^n$ is exactly the same as in the Euclidean space. However, the situation in the hyperbolic space is different. Trivially, the answer is affirmative if $n = 2$, since the condition (4.5) in this case is equivalent to $K \subseteq L$, but for higher dimensions we have the following:

Theorem 2.1.2. *There are convex centrally symmetric bodies K, $L \subset \mathbb{H}^n$, $n \geq 3$ that satisfy the condition*

$$\mathrm{vol}_{n-1}(K \cap H) \leq \mathrm{vol}_{n-1}(L \cap H)$$

for every central totally-geodesic hyperplane H in $\mathbb{H}^n$, but $\mathrm{vol}_n(K) > \mathrm{vol}_n(L)$.

The idea to find analogs of known results in non-Euclidean spaces is not new. For example in [GHS] the authors study intrinsic volumes in hyperbolic and spherical spaces. The Brunn-Minkowski inequality in different spaces is discussed in [Ga4]. Also a number of papers is concerned with other generalizations of the Busemann-Petty problem. In our proof we will be using results from [Zv], where Zvavitch studied the Busemann-Petty problem for arbitrary measures. For other generalizations of the Busemann-Petty problem see [BZ], [Ko6], [Ko8], [Ko9], [Ko10], [RZ], [KYY].

2.2 Proofs of main results

First we derive a formula for the function $S_K(\xi)$ using the Fourier transform, similar to [Zv]. For $\delta = 0$ this is the formula from [Ko6].

Lemma 2.2.1. *Let K be an origin-symmetric δ-convex body in B^n with Minkowski functional $\|\cdot\|_K$. Let $\xi \in S^{n-1}$ and $\xi^\perp$ be the hyperplane through the origin orthogonal to ξ. Then the volume of the section of the body K by the hyperplane $\xi^\perp$ in the metric (1.4) equals*

$$S_K(\xi) = \frac{2^{n-1}}{\pi} \left(|x|_2^{-n+1} \int_0^{\frac{|x|}{\|x\|_K}} \frac{r^{n-2}}{(1+\delta\, r^2)^{n-1}}\, dr \right)^{\wedge} (\xi).$$

Proof. Passing to spherical coordinates we get:

$$S_K(\xi) = 2^{n-1} \int_{\xi^\perp} \chi(\|x\|_K) \frac{dx}{(1+\delta\ |x|^2)^{n-1}} =$$
$$= 2^{n-1} \int_{S^{n-1}\cap\xi^\perp} \int_0^{\|\theta\|_K^{-1}} \frac{r^{n-2}dr}{(1+\delta\ r^2)^{n-1}}\ d\theta.$$

We can rewrite the integral above as follows (note that $|x|=1$, since $x \in S^{n-1}$):

$$S_K(\xi) = 2^{n-1} \int_{S^{n-1}\cap\xi^\perp} |x|^{-n+1} \int_0^{|x|/\|x\|_K} \frac{r^{n-2}dr}{(1+\delta\ r^2)^{n-1}}\ dx.$$

The function under the spherical integral is a homogeneous function of x of degree $-n+1$ and therefore by Lemma 1.1.1:

$$S_K(\xi) = \frac{2^{n-1}}{\pi} \left(|x|_2^{-n+1} \int_0^{\frac{|x|}{\|x\|_K}} \frac{r^{n-2}}{(1+\delta\ r^2)^{n-1}}\ dr \right)^\wedge (\xi). \quad \square$$

Now we construct counterexamples to the Busemann-Petty problem in $\mathbb{H}^n$ and $\mathbb{S}^n$ for $n \geq 5$.

Theorem 2.2.2. *There exist convex origin-symmetric bodies K and L in $\mathbb{S}^n$ (or $\mathbb{H}^n$), $n \geq 5$ such that*

$$\mathrm{vol}_{n-1}(K \cap H) \leq \mathrm{vol}_{n-1}(L \cap H)$$

for every central hyperplane, but $\mathrm{vol}_n(K) > \mathrm{vol}_n(L)$.

Proof. We will show the proof only for the case of the spherical space, the hyperbolic case is similar. The idea here is to use the property that any Riemannian space locally is "almost" Euclidean.

Let K and L be convex origin-symmetric bodies in $\mathbb{R}^n$ that give a counterexample to the original Busemann-Petty problem (see for example [Ko12, Section 5.1]). That is

$$\mathrm{EVol}_{n-1}(K \cap H) \leq \mathrm{EVol}_{n-1}(L \cap H) \tag{2.2}$$

for every central hyperplane H, but

$$\mathrm{EVol}_n(L) < \mathrm{EVol}_n(K). \tag{2.3}$$

(Here we denote the usual Euclidean volume by EVol to avoid confusion with the spherical volume.)

In fact, since the inequality (2.3) is strict, we can dilate one of the bodies a little to make the inequality (2.2) strict. Recall also, that in the original counterexample the body L was strictly convex, and the body K was obtained from the body L by small perturbations. Note that K can also be made strictly convex.

In view of the latter remarks, we will assume that K and L are strictly convex origin-symmetric bodies that satisfy the strict version of (2.2). Moreover, the function $\mathrm{EVol}_{n-1}(K \cap H)/\mathrm{EVol}_{n-1}(L \cap H)$ is a continuous function of $\xi \in S^{n-1}$, where ξ is the normal vector to the hyperplane H. Since this function is strictly less than 1, there exists an $\epsilon > 0$ such that

$$\mathrm{EVol}_{n-1}(K \cap H) < (1-\epsilon)\mathrm{EVol}_{n-1}(L \cap H)$$

for all H and

$$\mathrm{EVol}_n(L) < (1-\epsilon)\mathrm{EVol}_n(K).$$

Clearly, any dilations αK and αL also provide a counterexample. We can take α so small that both bodies K and L lie in a ball of radius r that satisfies the inequality:

$$1-\epsilon \leq \frac{1}{(1+r^2)^n} < 1.$$

Now the volumes of the bodies K and L in the spherical metric are related by the inequality:

$$\begin{aligned}
\mathrm{vol}_n(L) &= 2^n \int_L \frac{dx}{(1+|x|^2)^n} \leq 2^n \int_L dx \\
&= 2^n \mathrm{EVol}_n(L) < (1-\epsilon) 2^n \mathrm{EVol}_n(K) \\
&= (1-\epsilon)\ 2^n \int_K dx \leq 2^n \int_K \frac{dx}{(1+|x|^2)^n} \\
&= \mathrm{vol}_n(K).
\end{aligned}$$

Analogously, for the volumes of sections we have

$$\begin{aligned}
\mathrm{vol}_{n-1}(K\cap\xi^\perp) &= 2^{n-1} \int_{K\cap\langle x,\xi\rangle=0} \frac{dx}{(1+|x|^2)^{n-1}} \\
&\leq 2^{n-1} \int_{K\cap\langle x,\xi\rangle=0} dx \\
&< (1-\epsilon) 2^{n-1} \int_{L\cap\langle x,\xi\rangle=0} dx \\
&\leq 2^{n-1} \int_{L\cap\langle x,\xi\rangle=0} \frac{dx}{(1+|x|^2)^{n-1}} = \mathrm{vol}_{n-1}(L\cap\xi^\perp).
\end{aligned}$$

To finish the proof we only need to show that if K is a strictly e-convex body, then αK is s-convex for sufficiently small α. Consider the boundary of the body K. Define

$$k = \min\{k_i(x) : x \in \partial K,\ i = 1, ..., n-1\},$$

where $k_i(x)$, $i = 1$,..., $n-1$, are the principal curvatures at the point x on the boundary of K. Since K is strictly e-convex the quantity defined above is strictly positive: $k > 0$. For the body αK it is equal to k/α. On the other hand in a small neighborhood of the origin the totally geodesic s-planes are the spheres with almost zero curvature (from the Euclidean point of view). Consider all the spheres, which are totally geodesic in the spherical metric and tangent to the body αK, and let R be the smallest radius of all such spheres. We can choose an α so small that

$$k/\alpha > 1/R$$

and therefore the body αK lies on one side with respect to any tangent totally geodesic s-hyperplane. Hence αK is s-convex.

The situation in the hyperbolic space is even easier since every e-convex body containing the origin is also h-convex. $\square$

In 1988 Lutwak [Lu1] introduced the concept of intersection body and proved that the Busemann-Petty problem has affirmative answer if the body with smaller sections is an intersection body. Later, in [Ko5] Koldobsky proved that a body K is an intersection body if and only if $\|x\|_K^{-1}$ is a positive definite distribution. Then in [Ko6] Koldobsky generalized Lutwak's connection using a version Parseval's formula on the sphere.

Later, Zvavitch ([Zv]) solved the Busemann-Petty problem for arbitrary measures. Namely, let $f_n(x)$ be a locally integrable function on $\mathbb{R}^n$, and $f_{n-1}(x)$ a function on $\mathbb{R}^n$, locally integrable on central hyperplanes. Then let μ_n be the measure on $\mathbb{R}^n$ with density $f_n(x)$ and μ_{n-1} be the $(n-1)$-dimensional measure on central hyperplanes with density $f_{n-1}(x)$ such that $t\frac{f_n(tx)}{f_{n-1}(tx)}$ is an increasing function of t for any fixed x. Then if

$$\|x\|_K^{-1}\frac{f_n(\frac{x}{\|x\|_K})}{f_{n-1}(\frac{x}{\|x\|_K})}$$

is a positive definite distribution on $\mathbb{R}^n$ then the Busemann-Petty problem for these measures has affirmative answer, i.e. $\mu_{n-1}(K\cap\xi^\perp)\le\mu_{n-1}(L\cap\xi^\perp)$ implies $\mu_n(K)\le\mu_n(L)$. Our next result is a particular case of Zvavitch's theorem, but for the sake of completeness we include a proof.

Theorem 2.2.3. *Let K and L be $\delta-$convex origin-symmetric bodies in B^n such that $\dfrac{\|x\|_K^{-1}}{1+\delta\ (\frac{|x|}{\|x\|_K})^2}$ is a positive definite distribution. If*

$$\mathrm{vol}_{n-1}(K\cap H)\le\mathrm{vol}_{n-1}(L\cap H)$$

for every totally geodesic hyperplane through the origin, then

$$\mathrm{vol}_n(K)\le\mathrm{vol}_n(L).$$

Proof. Let us first prove the following elementary inequality (cf. Zvavitch, [Zv]). For any $a,b\in(0,1)$

$$\frac{a}{1+\delta\ a^2}\int_a^b\frac{r^{n-2}}{(1+\delta\ r^2)^{n-1}}dr\le\int_a^b\frac{r^{n-1}}{(1+\delta\ r^2)^n}dr.$$

Indeed, since the function $\dfrac{r}{1+\delta\ r^2}$ is increasing on the interval $(0,1)$ we have the following

$$\begin{aligned}\frac{a}{1+\delta\ a^2}\int_a^b\frac{r^{n-2}}{(1+\delta\ r^2)^{n-1}}dr &= \int_a^b\frac{r^{n-1}}{(1+\delta\ r^2)^n}\frac{a}{1+\delta\ a^2}\left(\frac{r}{1+\delta\ r^2}\right)^{-1}dr\\ &\le \int_a^b\frac{r^{n-1}}{(1+\delta\ r^2)^n}dr.\end{aligned}$$

Note that latter inequality does not require that $a\le b$.

Using the previous inequality with $a = \|x\|_K^{-1}$ and $b = \|x\|_L^{-1}$ we get

$$\int_{S^{n-1}} \frac{\|x\|_K^{-1}}{1+\delta\ \|x\|_K^{-2}} \int_{\|x\|_K^{-1}}^{\|x\|_L^{-1}} \frac{r^{n-2}}{(1+\delta\ r^2)^{n-1}} dr\ dx \leq$$
$$\leq \int_{S^{n-1}} \int_{\|x\|_K^{-1}}^{\|x\|_L^{-1}} \frac{r^{n-1}}{(1+\delta\ r^2)^{n}} dr\ dx.$$

Suppose we can show that the left-hand side is non-negative, then it will follow that

$$\int_{S^{n-1}} \int_0^{\|x\|_K^{-1}} \frac{r^{n-1}}{(1+\delta\ r^2)^{n}} dr\ dx \leq \int_{S^{n-1}} \int_0^{\|x\|_L^{-1}} \frac{r^{n-1}}{(1+\delta\ r^2)^{n}} dr\ dx,$$

that is $\mathrm{vol}_n(K) \leq \mathrm{vol}_n(L)$, see the polar formula (1.5).

So we only need to show that

$$\int_{S^{n-1}} \frac{\|x\|_K^{-1}}{1+\delta\ \|x\|_K^{-2}} \int_0^{\|x\|_K^{-1}} \frac{r^{n-2}}{(1+\delta\ r^2)^{n-1}} dr\ dx \leq$$
$$\leq \int_{S^{n-1}} \frac{\|x\|_K^{-1}}{1+\delta\ \|x\|_K^{-2}} \int_0^{\|x\|_L^{-1}} \frac{r^{n-2}}{(1+\delta\ r^2)^{n-1}} dr\ dx.$$

But this follows from the assumption of the theorem, the Parseval's formula on the sphere (Corollary 1.1.4) and formula for the volume of central sections (Lemma 2.2.1). Indeed, let γ_0 be the measure from Corollary 1.1.4 corresponding to the Fourier transform of the positive definite distribution $\frac{\|x\|_K^{-1}}{1+\delta\ (\frac{|x|}{\|x\|_K})^2}$, then

$$(2\pi)^n \int_{S^{n-1}} \frac{\|x\|_K^{-1}}{1+\delta\ \|x\|_K^{-2}} \int_0^{\|x\|_K^{-1}} \frac{r^{n-2}}{(1+\delta\ r^2)^{n-1}} dr\, dx =$$
$$= \int_{S^{n-1}} \left(\frac{\|x\|_K^{-1}}{1+\delta\ (\frac{|x|}{\|x\|_K})^2} \right) \cdot \left(|x|^{-n+1} \int_0^{\frac{|x|}{\|x\|_K}} \frac{r^{n-2}}{(1+\delta\ r^2)^{n-1}} dr \right) dx =$$
$$= \int_{S^{n-1}} \left(|x|^{-n+1} \int_0^{\frac{|x|}{\|x\|_K}} \frac{r^{n-2}}{(1+\delta\ r^2)^{n-1}} dr \right)^{\wedge} (\theta)\, d\gamma_0(\theta) =$$
$$= \int_{S^{n-1}} \frac{\pi}{2^{n-1}} S_K(\theta)\, d\gamma_0(\theta) \leq \int_{S^{n-1}} \frac{\pi}{2^{n-1}} S_L(\theta)\, d\gamma_0(\theta) =$$

$$= \int_{S^{n-1}} \left(|x|^{-n+1} \int_0^{\frac{|x|}{\|x\|_L}} \frac{r^{n-2}}{(1+\delta\, r^2)^{n-1}} dr \right)^{\wedge} (\theta)\, d\gamma_0(\theta) =$$

$$= (2\pi)^n \int_{S^{n-1}} \frac{\|x\|_K^{-1}}{1+\delta\ \|x\|_K^{-2}} \int_0^{\|x\|_L^{-1}} \frac{r^{n-2}}{(1+\delta\, r^2)^{n-1}} dr. \quad \square$$

Remark 2.2.4. Since $\|x\|_K^{-1}$ is positive definite for any convex origin-symmetric body in $\mathbb{R}^n$, $n \le 4$ (see [GKS]), the previous theorem implies the affirmative part of the original Busemann-Petty problem in $\mathbb{R}^n$.

Now we investigate for which classes of bodies $\dfrac{\|x\|_K^{-1}}{1+\delta\ (\frac{|x|}{\|x\|_K})^2}$ is a positive definite distribution.

Proposition 2.2.5. *Let K be an origin-symmetric body in B^n, $n \le 4$.*

i) If K is h-convex then $\dfrac{\|x\|_K^{-1}}{1+(\frac{|x|}{\|x\|_K})^2}$ is positive definite.

ii) If K is s-convex then $\dfrac{\|x\|_K^{-1}}{1-(\frac{|x|}{\|x\|_K})^2}$ is positive definite.

Proof. i) Consider a h-convex origin-symmetric body $K \subset B^n$, $n \le 4$. Define a body M by the formula:

$$\|x\|_M^{-1} = \frac{\|x\|_K^{-1}}{1+(\frac{|x|}{\|x\|_K})^2}.$$

It is enough to show that M is e-convex. If we pass to polar coordinates then the map

$$(r, \theta) \mapsto \left(\frac{r}{1+r^2}, \theta \right)$$

transforms the body K into the body M.

Take two points in K and connect them by a hyperbolic segment. This segment belongs to K since K is h-convex. Consider the 2-dimensional plane through the origin and these 2 points. The section of the body K by this plane is a 2-dimensional h-convex body. Introduce polar coordinates on this plane and (without loss of generality) assume that the h-geodesic segment has the equation $r^2 - a\, r\cos\phi + 1 = 0$. Applying the above transformation one can see that this h-segment gets mapped into an e-segment given by the equation $r = \dfrac{1}{a\cos\phi}$. Therefore the body M is e-convex and $(\|x\|_M^{-1})^\wedge$ is positive in dimensions $n \le 4$ (see [GKS]).

ii) Similar to (i). Take a s-geodesic given by the equation $r^2 + a\, r\cos\phi - 1 = 0$. The image of this geodesic under the map

$$(r,\theta) \mapsto \left(\frac{r}{1-r^2}, \theta\right) \tag{2.4}$$

is an e-geodesic $r = \dfrac{1}{a\cos\phi}$. $\square$

Since every s-convex body containing the origin is h-convex, we have the following

Corollary 2.2.6. $\dfrac{\|x\|_K^{-1}}{1+(\frac{|x|}{\|x\|_K})^2}$ *is positive-definite for every origin-symmetric s-convex body K in dimension $n \le 4$.*

This fact combined with Theorem 2.2.3 implies the affirmative answer to the spherical Busemann-Petty problem for $n \le 4$.

However not every h-convex body is s-convex and this idea will be used in constructing counterexamples to the hyperbolic Busemann-Petty problem.

Proposition 2.2.7. *There exist h-convex origin-symmetric bodies in B^n, $n \ge 3$ that give a counterexample to the hyperbolic Busemann-Petty problem.*

Proof. In view of Theorem 2.2.2 we are interested only in the cases $n = 3$ and 4. First we construct a body L for which $\dfrac{\|x\|_L^{-1}}{1-(\frac{|x|}{\|x\|_L})^2}$ is not positive definite.

Let L be a circular cylinder of radius $\sqrt{2}/2$ with x_1 being its axis of revolution. (See Fig.2.) To the top and bottom of the cylinder attach spherical caps, that are totally geodesic in the spherical metric. Clearly the body L constructed this way is e-convex and therefore h-convex. Using the formula

$$\|x\|_M^{-1} = \frac{\|x\|_L^{-1}}{1-(\frac{|x|}{\|x\|_L})^2} \tag{2.5}$$

we define a body M.

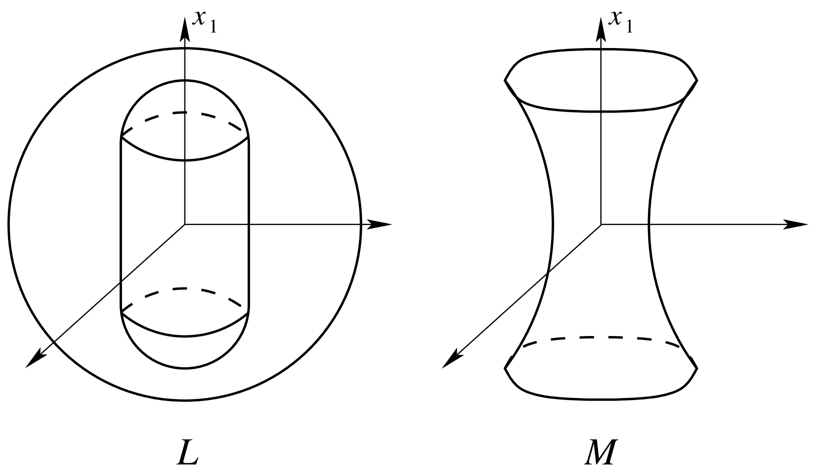


Figure 2

Clearly the body M is the image of L under the map (2.4). It can be checked directly that the cylinder is mapped into the surface of revolution obtained by rotating the hyperbola $x_2 = \dfrac{1}{2}\left(\sqrt{2}+\sqrt{2+4x_1^2}\right)$ about the x_1-axis, and the top and bottom spherical caps are mapped into flat disks.

In fact the body L constructed above is not smooth. But we can approximate it by infinitely smooth e-convex bodies that differ from L only in a small neighborhood of the edges. Since the body M is obtained from L by (2.5), and the denominator in (2.5) is never equal to zero, the body M is also infinitely smooth. (Now that the bodies L and M are smooth, Figure 2 might be confusing, but we wanted to make it as simple as possible, just to emphasize the idea).

Now that we defined the body M, we can explicitly compute its parallel section function $A_{M,\xi}$ in the direction of the x_1-axis.

$$A_{M,\xi}(t) = \begin{cases} \pi \left(\dfrac{\sqrt{2}+\sqrt{2+4t^2}}{2} \right)^2, & \text{in dimension } n = 3, \\ \dfrac{4\pi}{3} \left(\dfrac{\sqrt{2}+\sqrt{2+4t^2}}{2} \right)^3, & \text{in dimension } n = 4. \end{cases}$$

Since M is an infinitely smooth body, $(\|x\|_M^{-1})^\wedge$ is a function. Applying Theorem 1.1.2 with $n = 3$ and $q = 1$ we get

$$(\|x\|_M^{-1})^\wedge(\xi) = -2 \int_0^\infty \frac{A_{M,\xi}(t) - A_{M,\xi}(0)}{t^2} dt.$$

Let the height of the cylindrical part of L be equal to $\sqrt{2} - 2\epsilon$ and the hight of its image under (2.4) equal to N. If ϵ tends to zero, the top and bottom parts of the body L get closer to the sphere $x_1^2 + \cdots + x_n^2 = 1$. Recalling the definition of the radial function of M:

$$\rho_M(x) = \frac{\rho_L(x)}{1 - \rho_L(x)^2}, \quad \forall x \in S^{n-1},$$

one can see that the the body M becomes larger in the direction of x_1 as $\epsilon \to 0$, and therefore its height N approaches infinity.

Since in dimension $n = 3$ the section function can be written as

$$A_{M,\xi}(t) = \pi \left(1 + t^2 + \sqrt{1 + 2t^2} \right)$$

for $-N \le t \le N$, we get:

$$\begin{aligned}(\|x\|_M^{-1})^\wedge(\xi) &= -2\pi \int_0^N \frac{1+t^2+\sqrt{1+2t^2}-2}{t^2}dt - 2\pi \int_N^\infty \frac{(-2)}{t^2}dt \le \\ &\le -2\pi \int_0^N dt + 4\pi \int_N^\infty \frac{1}{t^2}dt = -2\pi N + \frac{4\pi}{N} < 0\end{aligned}$$

for N large enough.

If $n=4$ and $q=2$ Theorem 1.1.2 implies

$$(\|x\|_M^{-1})^\wedge(\xi) = -\pi A''_{M,\xi}(0) < 0,$$

since the second derivative of the function $A_{M,\xi}$ in dimension $n = 4$ equals: $A''_{M,\xi}(0) = 8\sqrt{2}\cdot\pi$.

Thus we have proved that $\left(\dfrac{\|x\|_L^{-1}}{1-(\frac{|x|}{\|x\|_L})^2}\right)^\wedge(\xi) = (\|x\|_M^{-1})^\wedge(\xi)$ is negative for some direction ξ.

Now apply a standard argument to construct another body K which along with the body K provides a counterexample to the hyperbolic Busemann-Petty problem (cf. [Ko6, Theorem 2] or [Zv, Theorem 2]). By continuity of $(\|x\|_M^{-1})^\wedge$ there is a neighborhood of ξ where this function is negative. Let

$$\Omega = \{\theta \in S^{n-1} : (\|x\|_M^{-1})^\wedge(\theta) < 0\}.$$

Choose a non-positive infinitely-smooth even function v supported on Ω. Extend v to a homogeneous function $r^{-1}v(\theta)$ of degree -1 on $\mathbb{R}^n$. By Lemma 5 from [Ko6] we know that the Fourier transform of $r^{-1}v(\theta)$ is equal to $r^{-n+1}g(\theta)$ for some infinitely smooth function g on S^{n-1}.

To construct a counterexample to the Busemann-Petty problem, define another body K as follows:

$$\int_0^{\|\theta\|_K^{-1}} \frac{r^{n-2}}{(1-r^2)^{n-1}}dr = \int_0^{\|\theta\|_L^{-1}} \frac{r^{n-2}}{(1-r^2)^{n-1}}dr + \epsilon g(\theta)$$

for some $\epsilon > 0$ small enough (to guarantee that K is still convex in hyperbolic sense). Indeed, define a function $\alpha_\epsilon(\theta)$ such that

$$\int_0^{\|\theta\|_L^{-1}} \frac{r^{n-2}}{(1-r^2)^{n-1}}dr + \epsilon v(\theta) = \int_0^{\|\theta\|_L^{-1}+\alpha_\epsilon(\theta)} \frac{r^{n-2}}{(1-r^2)^{n-1}}dr,$$

then

$$\|\theta\|_K^{-1} = \|\theta\|_L^{-1} + \alpha_\epsilon(\theta).$$

Note that in our construction L is e-convex, but we can perturb it a little (by adding $\alpha|\theta|_2$ to the norm $\|\theta\|_L$ with $\alpha > 0$ small enough), so we can assume that L is strictly e-convex. Therefore one can choose ϵ small enough such that K is also e-convex (for details see [Zv, Proposition 2]). Hence we can assume that both L and K are h-convex.

Using Lemma 2.2.1 we get

$$\begin{aligned} \mathrm{vol}_{n-1}(K \cap \xi^\perp) &= \frac{2^{n-1}}{\pi}\left(|x|^{-n+1}\int_0^{|x|/\|x\|_K} \frac{r^{n-2}}{(1-r^2)^{n-1}}dr\right)^\wedge(\xi) = \\ &= \frac{2^{n-1}}{\pi}\left(|x|^{-n+1}\int_0^{|x|/\|x\|_L} \frac{r^{n-2}}{(1-r^2)^{n-1}}dr\right)^\wedge(\xi) + \epsilon v(\xi) \le \\ &\le \frac{2^{n-1}}{\pi}\left(|x|^{-n+1}\int_0^{|x|/\|x\|_L} \frac{r^{n-2}}{(1-r^2)^{n-1}}dr\right)^\wedge(\xi) = \\ &= \mathrm{vol}_{n-1}(L \cap \xi^\perp). \end{aligned}$$

Proceeding as in the proof of Theorem 2.2.3 we can show the opposite inequality for volumes. Since the body L is infinitely smooth, one can use the Parseval's formula in the form of Lemma 1.1.3:

$$
\begin{aligned}
&(2\pi)^n \int_{S^{n-1}} \frac{\|x\|_L^{-1}}{1-\ \|x\|_L^{-2}} \int_0^{\|x\|_K^{-1}} \frac{r^{n-2}}{(1-\ r^2)^{n-1}} dr dx = \\
&= \int_{S^{n-1}} \left(\frac{\|x\|_L^{-1}}{1-\ (\frac{|x|}{\|x\|_L})^2} \right)^{\wedge} (\theta) \left(|x|^{-n+1} \int_0^{\frac{|x|}{\|x\|_K}} \frac{r^{n-2}}{(1-\ r^2)^{n-1}} dr \right)^{\wedge} (\theta) d\theta = \\
&= \int_{S^{n-1}} \left(\frac{\|x\|_L^{-1}}{1-\ (\frac{|x|}{\|x\|_L})^2} \right)^{\wedge} (\theta) \left(|x|^{-n+1} \int_0^{\frac{|x|}{\|x\|_L}} \frac{r^{n-2}}{(1-\ r^2)^{n-1}} dr \right)^{\wedge} (\theta) d\theta + \\
&+ \int_{S^{n-1}} \left(\frac{\|x\|_L^{-1}}{1-\ (\frac{|x|}{\|x\|_L})^2} \right)^{\wedge} (\theta) \cdot \epsilon v\,(\theta)\, d\theta > \\
&> (2\pi)^n \int_{S^{n-1}} \frac{\|x\|_L^{-1}}{1-\ \|x\|_L^{-2}} \int_0^{\|x\|_L^{-1}} \frac{r^{n-2}}{(1-\ r^2)^{n-1}} dr dx. \quad \square
\end{aligned}
$$

Chapter 3

The lower dimensional Busemann-Petty problem in the hyperbolic space

3.1 Introduction

The lower dimensional Busemann-Petty problem (LDBP) in $\mathbb{R}^n$ asks the same question as the original Busemann-Petty problem with k-dimensional subspaces in place of hyperplanes. Bourgain and Zhang [BZ] proved that this problem has a negative answer if $3 < k < n$, see [Ko8] for another solution. The cases $k = 2, 3$ are still open in dimensions $n > 4$.

In this chapter we study the lower dimensional Busemann-Petty problem in the hyperbolic space. Namely, let $1 \leq k < n$, and K, L be origin-symmetric convex bodies in $\mathbb{H}^n$, $n \geq 3$, such that

$$\mathrm{vol}_k(K \cap H) \leq \mathrm{vol}_k(L \cap H)$$

for every k-dimensional totally geodesic plane through the origin. Does it follow that

$$\mathrm{vol}_n(K) \leq \mathrm{vol}_n(L)?$$

For the case $k=1$ the answer is trivially affirmative, since in all directions the radius of K does not exceed the radius of L. In this chapter we prove that the answer to the hyperbolic lower dimensional Busemann-Petty problem is negative for every $2 \leq k < n$. This chapter is based on the results from [Y2].

3.2 Proof of main result

The next lemma is a Fourier analytic version of a result of Zhang [Zh1, Lemma 2].

Lemma 3.2.1. *Let k be an integer, $1 \leq k \leq n-1$, and let f be an infinitely differentiable even function on the sphere S^{n-1}, such that $f(x/|x|)|x|^{-k}$ is not a positive definite distribution on $\mathbb{R}^n$, where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^n$. Then there exists an even function $g \in C^\infty(S^{n-1})$ such that*

$$\int_{S^{n-1}} f(x)g(x)dx > 0 \tag{3.1}$$

and

$$\int_{S^{n-1}\cap H} g(x)dx \leq 0, \tag{3.2}$$

for any $(n-k)$-dimensional plane H through the origin.

Proof. Since f is infinitely differentiable, by [Ko12, Section 3.2] we have that $(f(x/|x|)|x|^{-k})^\wedge$ is a continuous function on $\mathbb{R}^n \setminus \{0\}$. By our assumption there exists $\xi \in S^{n-1}$ such that $(f(x/|x|)|x|^{-k})^\wedge(\xi) < 0$. By continuity of $(f(x/|x|)|x|^{-k})^\wedge$ there is a neighborhood of ξ where this function is negative. Let

$$\Omega = \{\theta \in S^{n-1} : (f(x/|x|)|x|^{-k})^\wedge(\theta) < 0\}.$$

Choose a non-positive infinitely-smooth even function v supported in Ω. Extend v to a homogeneous function $|x|^{-n+k}v(x/|x|)$ of degree $-n+k$ on $\mathbb{R}^n$. By [Ko12, Section 3.2], the Fourier transform of $|x|^{-n+k}v(x/|x|)$ is equal to $|x|^{-k}g(x/|x|)$ for some infinitely smooth function g on S^{n-1}.

By Parseval's formula on the sphere (Lemma 1.1.3) we have

$$\begin{aligned}
&\int_{S^{n-1}} f(x)g(x)dx = \int_{S^{n-1}} \left(f(x/|x|)|x|^{-k}\right) \left(g(x/|x|)|x|^{-n+k}\right) dx \\
&= \frac{1}{(2\pi)^n}\int_{S^{n-1}} \left(f(x/|x|)|x|^{-k}\right)^{\wedge}(\theta) \left(g(x/|x|)|x|^{-n+k}\right)^{\wedge}(\theta)d\theta \\
&= \frac{1}{(2\pi)^n}\int_{S^{n-1}} \left(f(x/|x|)|x|^{-k}\right)^{\wedge}(\theta)v(\theta)d\theta > 0,
\end{aligned}$$

since v is non-positive and supported in the set where $\left(f(x/|x|)|x|^{-k}\right)^{\wedge}$ is negative.

Secondly, by Lemma 1.1.5 we have

$$\begin{aligned}
&(2\pi)^k\int_{S^{n-1}\cap H} g(x)dx = (2\pi)^k\int_{S^{n-1}\cap H} g(x/|x|)|x|^{-n+k}dx \\
&= \int_{S^{n-1}\cap H^{\perp}} \left(g(x/|x|)|x|^{-n+k}\right)^{\wedge}(\theta)d\theta = \int_{S^{n-1}\cap H^{\perp}} v(\theta)d\theta \le 0,
\end{aligned}$$

since v is non-positive.

□

Proposition 3.2.2. *Let $1 \le k \le n-2$. There exists an infinitely smooth origin symmetric strictly e-convex body L in the unit ball $B^n \subset \mathbb{R}^n$, so that*

$$\frac{\|x\|_L^{-k}}{(1-(\frac{|x|}{\|x\|_L})^2)^k} \tag{3.3}$$

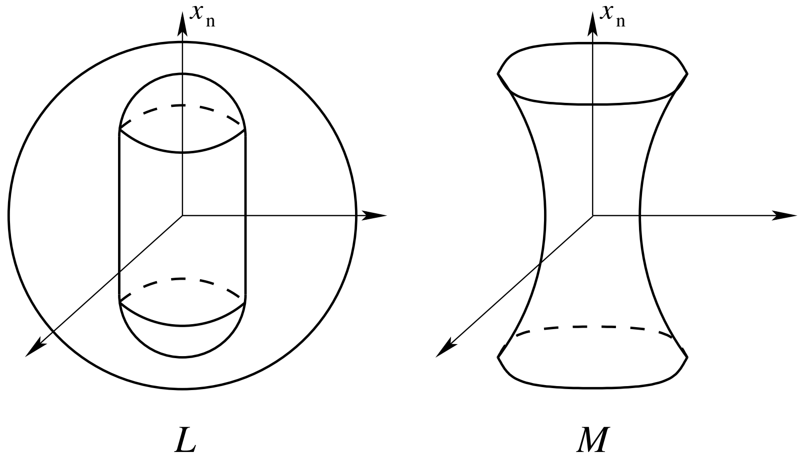
is not a positive definite distribution on $\mathbb{R}^n$.

Proof. First, we consider the cases $k = n-2$ and $n-3$. We will use a construction similar to [Y1, Proposition 3.9]. Let L be a circular cylinder of radius $\sqrt{2}/2$ with

x_n being its axis of revolution. To the top and bottom of the cylinder attach spherical caps, that are totally geodesic in the spherical metric. Clearly the body L constructed this way is e-convex and therefore h-convex. Using the formula

$$\|x\|_M^{-1} = \frac{\|x\|_L^{-1}}{1 - (\frac{|x|}{\|x\|_L})^2} \tag{3.4}$$

we define a body M. (Note, that M is well-defined, since L lies entirely in the unit ball B^n and the denominator in the latter formula is never equal to zero).



Clearly the body M is the image of L under the map:

$$(r, \theta) \mapsto \left(\frac{r}{1 - r^2}, \theta\right) \tag{3.5}$$

It can be checked directly that the cylinder is mapped into the surface of revolution obtained by rotating the hyperbola $x_1 = \frac{1}{2}\left(\sqrt{2} + \sqrt{2 + 4x_n^2}\right)$ about the x_n-axis, and the top and bottom spherical caps are mapped into flat disks. The latter follows from the fact that (3.5) maps s-geodesics into e-geodesics. Indeed, without loss of generality we may consider a s-geodesic given by the equation: $r^2 + a\, r \cos\phi - 1 = 0$ in some 2-dimensional plane. The image of this s-geodesic under the map (3.5) is an e-geodesic $r = \dfrac{1}{a \cos\phi}$.

The body L constructed above is not smooth. But we can approximate it by infinitely smooth e-convex bodies that differ from L only in a small neighborhood of the edges. Since the body M is obtained from L by (3.4), and the denominator in (3.4) is never equal to zero, the body M is also infinitely smooth.

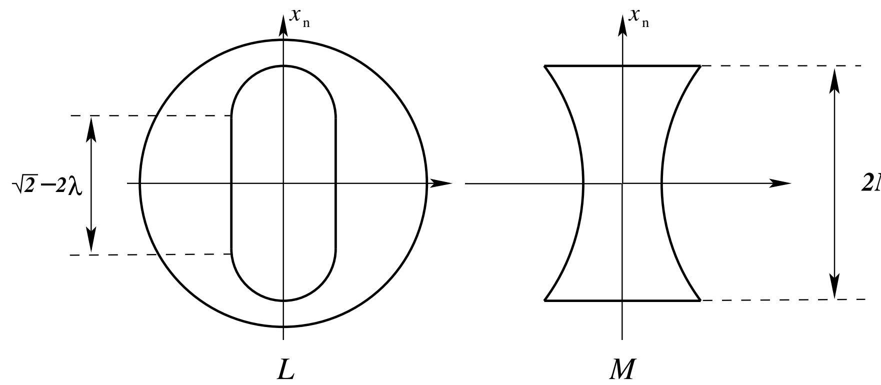
Now that we have defined the body M, we can explicitly compute its parallel section function $A_{M,\xi}$ in the direction of the x_n-axis.

$$A_{M,\xi}(t) = C_n \left(\sqrt{2} + \sqrt{2+4t^2}\right)^{n-1}.$$

Let the height of the cylindrical part of L be equal to $\sqrt{2} - 2\lambda$ and the height of its image under (3.5) equal to $2N$ (see the picture below). Since the radius of the cylinder equals $\sqrt{2}/2$, when λ tends to zero the top and bottom parts of the body L get closer to the sphere $x_1^2 + \cdots + x_n^2 = 1$. Recalling the definition of the radial function of M:

$$\rho_M(x) = \frac{\rho_L(x)}{1 - \rho_L(x)^2}, \quad \forall x \in S^{n-1},$$

one can see that the height $2N$ of the body M approaches infinity as $\lambda \to 0$.



Since M is an infinitely smooth body, $(\|x\|_M^{-n+k+1})^\wedge$ is a function. Applying

Theorem 1.1.2 with $k = 1$ we get

$$\begin{aligned}
(\|x\|_M^{-n+2})^\wedge(\xi) &= -2(n-2)\int_0^\infty \frac{A_{M,\xi}(t) - A_{M,\xi}(0)}{t^2}dt \\
&= -2(n-2)C_n \int_0^N \frac{\left(\sqrt{2}+\sqrt{2+4t^2}\right)^{n-1} - (2\sqrt{2})^{n-1}}{t^2}dt + \\
&\quad +2(n-2)C_n \int_N^\infty \frac{(2\sqrt{2})^{n-1}}{t^2}dt.
\end{aligned}$$

To estimate the first integral we use the binomial theorem,

$$\begin{aligned}
\left(\sqrt{2}+\sqrt{2+4t^2}\right)^{n-1} &= (\sqrt{2})^{n-1} + (n-1)(\sqrt{2})^{n-2}\sqrt{2+4t^2} + \\
&\quad + \frac{(n-1)(n-2)}{2}(\sqrt{2})^{n-3}(2+4t^2) + \cdots \\
&\geq (2\sqrt{2})^{n-1} + 2(n-1)(n-2)(\sqrt{2})^{n-3}t^2,
\end{aligned}$$

where the last inequality was obtained by putting $t = 0$ in all the terms of the binomial expansion, except for the third term. Therefore, for some positive constants C_n' and C_n'' we have

$$(\|x\|_M^{-n+2})^\wedge(\xi) \leq -C_n' \int_0^N dt + C_n'' \int_N^\infty \frac{1}{t^2}dt = -C_n'N + C_n''\frac{1}{N} < 0$$

for N large enough.

Therefore the body M, corresponding to this N, is not a $(n-2)$-intersection body in the Euclidean sense, which implies that

$$\frac{\|x\|_L^{-n+2}}{(1 - (\frac{|x|}{\|x\|_L})^2)^{n-2}} = \|x\|_M^{-n+2} \tag{3.6}$$

is not a positive definite distribution.

Similarly we can show that M is not a $(n-3)$-intersection body. Indeed, if $k = 2$ Theorem 1.1.2 implies

$$(\|x\|_M^{-n+3})^\wedge(\xi) = -\pi(n-3)A''_{M,\xi}(0) < 0,$$

since the second derivative of the function $A_{M,\xi}$ equals:

$$A''_{M,\xi}(0) = C_n(n-1)(2\sqrt{2})^{n-1} > 0.$$

Next we handle the case when $1 \leq k < n-3$. For this we use a different construction. Let M be an infinitely smooth origin symmetric e-convex body in $\mathbb{R}^n$, for which $\|x\|_M^{-k}$ is not positive definite. (For example, the unit ball of the space ℓ_4^n, see [Ko4]). Dilate this body M, if needed, to make sure that it lies in the unit Euclidean ball. Let $\rho_M(x)$ be the radial function of this body. Define a body L as follows:

$$\rho_L(x) = \frac{-1+\sqrt{1+4(\rho_M(x))^2}}{2\rho_M(x)}, \qquad \text{for } x \in S^{n-1}.$$

One can check that

$$\rho_M(x) = \frac{\rho_L(x)}{1-\rho_L(x)^2}, \qquad \text{for } x \in S^{n-1}.$$

Clearly, M is the image of L under the transformation (3.5). Since (3.5) maps s-geodesics into e-geodesics, L is a s-convex body, and therefore e-convex.

Thus we have proved that for $1 \leq k < n-3$,

$$\frac{\|x\|_L^{-k}}{\left(1-(\frac{|x|}{\|x\|_L})^2\right)^k} = \|x\|_M^{-k}$$

is not positive definite.

To finish the proof, note that in our construction L is not necessarily strictly e-convex. But one can replace L with L_ϵ, defined by

$$\|\theta\|_{L_\epsilon}^{-1} = \|\theta\|_L^{-1} + \epsilon|\theta|^{-1}.$$

One can choose $\epsilon > 0$ small enough, so that L_ϵ is strictly e-convex, and so that $\frac{\|x\|_{L_\epsilon}^{-k}}{\left(1-(\frac{|x|}{\|x\|_{L_\epsilon}})^2\right)^k}$ is still not positive definite (see, for example, the approximation argument in [Ko12, Lemma 4.10]). $\square$

Theorem 3.2.3. *Let $1 \le k < n-1$. There are origin-symmetric convex bodies K and L in $\mathbb{H}^n$, $n \ge 3$, such that*

$$\mathrm{vol}_{n-k}(K \cap H) \le \mathrm{vol}_{n-k}(L \cap H)$$

for every $(n-k)$-dimensional totally geodesic plane through the origin, but

$$\mathrm{vol}_n(K) > \mathrm{vol}_n(L).$$

Proof. Let L be an infinitely smooth origin symmetric e-convex body from Proposition 3.2.2, for which $\frac{||x||_L^{-k}}{\left(1-(\frac{|x|}{||x||_L})^2\right)^k}$ is not positive definite.

By Lemma 3.2.1 there exists an even function $g \in C^\infty(S^{n-1})$ such that

$$\int_{S^{n-1}} \frac{||x||_L^{-k}}{\left(1-(\frac{|x|}{||x||_L})^2\right)^k} g(x)dx > 0 \tag{3.7}$$

and

$$\int_{S^{n-1}\cap H} g(x)dx \le 0, \qquad \text{for all } H. \tag{3.8}$$

Now apply a standard argument to construct another body K which along with the body L provides a counterexample to the hyperbolic LDBP problem (cf. [Ko6], Theorem 2 or [Zv], Theorem 2). Define a new body K as follows:

$$\int_0^{\|\theta\|_K^{-1}} \frac{r^{n-k-1}}{(1-r^2)^{n-k}}dr = \int_0^{\|\theta\|_L^{-1}} \frac{r^{n-k-1}}{(1-r^2)^{n-k}}dr + \epsilon g(\theta) \tag{3.9}$$

for $\theta \in S^{n-1}$ and some $\epsilon > 0$ small enough (to guarantee that K is still convex in hyperbolic sense). Indeed, define a function $\alpha_\epsilon(\theta)$ such that

$$\int_0^{\|\theta\|_L^{-1}} \frac{r^{n-k-1}}{(1-r^2)^{n-k}} dr + \epsilon v(\theta) = \int_0^{\|\theta\|_L^{-1}+\alpha_\epsilon(\theta)} \frac{r^{n-k-1}}{(1-r^2)^{n-k}} dr,$$

then

$$\|\theta\|_K^{-1} = \|\theta\|_L^{-1} + \alpha_\epsilon(\theta).$$

The function $\alpha_\epsilon(\theta)$ and its first and second derivatives converge uniformly to zero as $\epsilon \longrightarrow 0$ (cf. [Zv, Proposition 2]), therefore since L is strictly e-convex, there exists ϵ small enough, so that K is also strictly e-convex, and hence h-convex.

Let H be an $(n-k)$-plane through the origin. Integrating (3.9) over $S^{n-1} \cap H$ and using inequality (3.8), we get

$$\int_{S^{n-1}\cap H} \int_0^{\|\theta\|_K^{-1}} \frac{r^{n-k-1}}{(1-r^2)^{n-k}} dr d\theta \leq \int_{S^{n-1}\cap H} \int_0^{\|\theta\|_L^{-1}} \frac{r^{n-k-1}}{(1-r^2)^{n-k}} dr d\theta,$$

which, by formula (1.6), is equivalent to

$$\mathrm{vol}_{n-k}(K \cap H) \leq \mathrm{vol}_{n-k}(L \cap H).$$

On the other hand, multiplying both sides of (3.9) by $\left(\frac{\|x\|_L^{-1}}{1-\|x\|_L^{-2}}\right)^k$ and integrating over the sphere S^{n-1} we get

$$\int_{S^{n-1}} \left(\frac{\|x\|_L^{-1}}{1-\|x\|_L^{-2}}\right)^k \int_0^{\|x\|_K^{-1}} \frac{r^{n-k-1}}{(1-r^2)^{n-k}} dr dx =$$
$$= \int_{S^{n-1}} \left(\frac{\|x\|_L^{-1}}{1-\|x\|_L^{-2}}\right)^k \int_0^{\|x\|_L^{-1}} \frac{r^{n-k-1}}{(1-r^2)^{n-k}} dr dx + \epsilon \int_{S^{n-1}} \left(\frac{\|x\|_L^{-1}}{1-\|x\|_L^{-2}}\right)^k g(x) dx.$$

From (3.7) it follows that

$$\int_{S^{n-1}} \left(\frac{\|x\|_L^{-1}}{1-\|x\|_L^{-2}}\right)^k \int_0^{\|x\|_K^{-1}} \frac{r^{n-k-1}}{(1-r^2)^{n-k}} dr dx >$$
$$> \int_{S^{n-1}} \left(\frac{\|x\|_L^{-1}}{1-\|x\|_L^{-2}}\right)^k \int_0^{\|x\|_L^{-1}} \frac{r^{n-k-1}}{(1-r^2)^{n-k}} dr dx$$

Therefore,

$$0 < \int_{S^{n-1}} \left(\frac{\|x\|_L^{-1}}{1 - \|x\|_L^{-2}} \right)^k \int_{\|x\|_L^{-1}}^{\|x\|_K^{-1}} \frac{r^{n-k-1}}{(1-r^2)^{n-k}} dr dx \tag{3.10}$$

Next we need the following elementary inequality (cf. Zvavitch, [Zv]). For any $a, b \in (0,1)$

$$\frac{a^k}{(1-a^2)^k} \int_a^b \frac{r^{n-k-1}}{(1-r^2)^{n-k}} dr \leq \int_a^b \frac{r^{n-1}}{(1-r^2)^n} dr.$$

Indeed, since the function $\frac{r^k}{(1-r^2)^k}$ is increasing on the interval $(0,1)$ we have the following

$$\begin{aligned} \frac{a^k}{(1-a^2)^k} \int_a^b \frac{r^{n-k-1}}{(1-r^2)^{n-k}} dr &= \int_a^b \frac{r^{n-1}}{(1-r^2)^n} \frac{a^k}{(1-a^2)^k} \left(\frac{r^k}{(1-r^2)^k} \right)^{-1} dr \\ &\leq \int_a^b \frac{r^{n-1}}{(1-r^2)^n} dr. \end{aligned}$$

Note that in the latter inequality it does not matter whether $a \leq b$ or $a \geq b$.

Applying the elementary inequality to (3.10) with $a = \|x\|_L^{-1}$ and $b = \|x\|_K^{-1}$, we get

$$\begin{aligned} 0 &< \int_{S^{n-1}} \left(\frac{\|x\|_L^{-1}}{1 - \|x\|_L^{-2}} \right)^k \int_{\|x\|_L^{-1}}^{\|x\|_K^{-1}} \frac{r^{n-k-1}}{(1-r^2)^{n-k}} dr dx \\ &\leq \int_{S^{n-1}} \int_{\|x\|_L^{-1}}^{\|x\|_K^{-1}} \frac{r^{n-1}}{(1-r^2)^n} dr dx. \end{aligned}$$

Hence

$$\int_{S^{n-1}} \int_0^{\|x\|_L^{-1}} \frac{r^{n-1}}{(1-r^2)^n} dr dx < \int_{S^{n-1}} \int_0^{\|x\|_K^{-1}} \frac{r^{n-1}}{(1-r^2)^n} dr dx,$$

that is $\mathrm{vol}_n(L) < \mathrm{vol}_n(K)$. $\square$

Chapter 4

Modified Busemann-Petty problem

4.1 Introduction

Since the answer to the Busemann-Petty problem in $\mathbb{R}^n$ is affirmative only if the dimension $n \leq 4$, and it is negative if $n \geq 5$, it is natural to ask what information about the volumes of central hyperplane sections of two bodies does allow to compare the volumes of these bodies in all dimensions. Our main result of this chapter suggests an answer to this question. Here we present our results from [KYY].

For an origin-symmetric convex body K in $\mathbb{R}^n$, consider the section function

$$S_K(\xi) = \mathrm{vol}_{n-1}(K \cap \xi^\perp), \qquad \xi \in S^{n-1},$$

where $\xi^\perp$ is the central hyperplane in $\mathbb{R}^n$ orthogonal to ξ. We extend S_K from the sphere to the whole $\mathbb{R}^n$ as a homogeneous function of degree -1. Our goal is to find a condition in terms of the section functions of two bodies only that allows to compare the n-dimensional volumes of these bodies. We prove in this chapter that, for two origin-symmetric smooth bodies K, L in $\mathbb{R}^n$ and $\alpha \in \mathbb{R}$, $\alpha \geq n-4$, the inequalities

$$(-\Delta)^{\alpha/2} S_K(\xi) \leq (-\Delta)^{\alpha/2} S_L(\xi), \qquad \forall \xi \in S^{n-1} \tag{4.1}$$

imply that $\mathrm{vol}_n(K) \leq \mathrm{vol}_n(L)$, while for $\alpha < n-4$ this is not necessarily true. Here Δ is the Laplace operator on $\mathbb{R}^n$, and the fractional powers of the Laplacian are defined by

$$(-\Delta)^{\alpha/2} f = \frac{1}{(2\pi)^n} (|x|_2^\alpha \hat{f}(x))^\wedge,$$

where the Fourier transform is considered in the sense of distributions, and $|x|_2$ stands for the Euclidean norm in $\mathbb{R}^n$. Of course, if α is an even integer and f is an even distribution we get the Laplacian applied $\alpha/2$ times. The fact that both sides of (4.1) represent continuous functions of the variable ξ follows from [Ko12, Lemma 3.16].

This result means that one has to differentiate the section functions at least $n-4$ times in order to compare the n-dimensional volumes. The case $\alpha = 0$ corresponds to the original Busemann-Petty problem, so our result can also be considered as a "continuous" generalization of the problem. Other generalizations of the Busemann-Petty problem and related open questions can be found in [BZ], [Ko6], [Ko8], [Ko11], [MiP], [RZ], [Y1], [Zv].

Let us briefly outline the idea of the proof. As shown in [Ko6], the section function can be expressed in terms of the Fourier transform, as follows:

$$S_K(\xi) = \frac{1}{\pi(n-1)} (\|x\|_K^{-n+1})^\wedge(\xi), \tag{4.2}$$

so the condition (4.1) can be written as

$$(|x|_2^\alpha \|x\|_K^{-n+1})^\wedge \leq (|x|_2^\alpha \|x\|_K^{-n+1})^\wedge. \tag{4.3}$$

Now let us write the volume in polar coordinates and use a spherical version of Parseval's formula from [Ko6], which allows to remove the Fourier transforms of homogeneous functions in the integrals over the sphere under the condition that the degrees of homogeneity of these functions add up to $-n$:

$$n\ \mathrm{vol}_n(K) = \int_{S^{n-1}} \|x\|_K^{-n}\ dx = \int_{S^{n-1}} |x|_2^{-\alpha}\|x\|_K^{-1}|x|_2^{\alpha}\|x\|_K^{-n+1}\ dx$$
$$= \frac{1}{(2\pi)^n}\int_{S^{n-1}} \left(|x|_2^{-\alpha}\|x\|_K^{-1}\right)^{\wedge}(\xi)\left(|x|_2^{\alpha}\|x\|_K^{-n+1}\right)^{\wedge}(\xi)\ d\xi.$$

Suppose that the distribution $|x|_2^{-\alpha}\|x\|_K^{-1}$ is positive definite, so its Fourier transform is non-negative. Then the latter equality combined with (4.3) implies that

$$n\ \mathrm{vol}_n(K) \leq \int_{S^{n-1}} \|x\|_K^{-1}\|x\|_L^{-n+1}\ dx,$$

and applying Hölder's inequality to the right-hand side we get that $\mathrm{vol}_n(K) \leq \mathrm{vol}_n(L)$. On the other hand, if $|x|_2^{-\alpha}\|x\|_K^{-1}$ is not positive definite one can construct a counterexample using a more or less standard perturbation procedure.

Thus, the problem is essentially reduced to the question, for which α is the distribution $|x|_2^{-\alpha}\|x\|_K^{-1}$ positive definite, for every origin-symmetric convex body K in $\mathbb{R}^n$. Note that for $\alpha = 0$ this happens only if the dimension $n \leq 4$, as proved in [GKS]. We prove that this function is positive definite for $\alpha \geq n-4$ and any symmetric convex body K in $\mathbb{R}^n$ by an argument modifying the proof from [GKS]. If $\alpha < n-4$ we construct examples of bodies for which this distribution is not positive definite. The latter requires a substantial technical effort.

4.2 Distributions of the form $|x|_2^{-r}\|x\|_K^{-s}$

For $\xi \in S^{n-1}$, consider a function $A_{K,\xi,p}$ on $\mathbb{R}$

$$A_{K,\xi,p}(t) = \int_{K\cap\langle x,\xi\rangle=t} |x|_2^{-p}dx,$$

where $p < n-1$.

In this section we establish some regularity properties of the function $A_{K,\xi,p}$ and express its fractional derivatives in terms of the Fourier transform. We assume that K is an infinitely smooth body.

For a real number q define the ceiling function $\lceil q \rceil$, which gives the smallest integer greater than or equal to q.

Lemma 4.2.1. *Let $\xi \in S^{n-1}$, $k \in \mathbb{N}$, $0 \le p < n-k-1$. Then the function $A_{K,\xi,p}$ is k-times continuously differentiable (uniformly with respect to ξ) in some neighborhood of zero.*

For fixed $q \in \mathbb{C}$, the fractional derivative $A_{K,\xi,p}^{(q)}(0)$ is a continuous function of the variable $\xi \in S^{n-1}$, and, for fixed $\xi \in S^{n-1}$, it is an analytic function of q in the domain $\{q \in \mathbb{C}\colon\ -1 < \lceil Re\ q \rceil < n-p-1\}$, with convergence in the derivatives by q being uniform with respect to ξ.

The proof is similar to that of [Ko12, Lemma 2.4]. The only difference is that in our case the function is differentiable only up to a certain order. To explain this, write the function in the form

$$A_{K,\xi,p}(t) = \int_{S_t^{n-2}} \left(\int_0^{\rho_{K\cap H_t}(\theta)} r^{n-2}(r^2+t^2)^{-p/2}dr \right) d\theta,$$

where $\rho_{K\cap H_t}(\theta)$ is the radial function of the body $K \cap H_t$ and S_t^{n-2} is the unit sphere in $H_t = \{x \in \mathbb{R}^n \colon \langle x, \xi\rangle = t\}$. If we differentiate by t too many times the integral stops being convergent when $t = 0$, which is why we have restrictions on k and q.

The following Lemma is a generalization of Theorem 2 from [GKS].

Lemma 4.2.2. *Let K be an infinitely smooth origin-symmetric convex body in $\mathbb{R}^n$, $q > -1$, $q \neq n-p-1$ and $0 \leq p < n - \lceil q \rceil - 1$. Then for every $\xi \in S^{n-1}$,*

$$A^{(q)}_{K,\xi,p}(0) = \frac{\cos\frac{\pi q}{2}}{\pi(n-p-q-1)}(\|x\|_K^{-n+p+q+1} \cdot |x|_2^{-p})^\wedge(\xi).$$

Proof. We simply write $\|\cdot\|$ for $\|\cdot\|_K$. By [Ko12, Lemma 3.16], $(\|x\|^{-n+p+q+1} \cdot |x|_2^{-p})^\wedge$ is a continuous function on $\mathbb{R}^n \setminus \{0\}$.

Suppose first that $-1 < q < 0$. The function

$$A_{K,\xi,p}(t) = \int_{K\cap\langle x,\xi\rangle=t} |x|_2^{-p} dx = \int_{\langle x,\xi\rangle=t} \chi(\|x\|)|x|_2^{-p} dx$$

is even. Applying Fubini's theorem and passing to spherical coordinates, we get

$$\begin{aligned}
A^{(q)}_{K,\xi,p}(0) &= \frac{1}{\Gamma(-q)}\int_0^\infty t^{-q-1}A_{K,\xi,p}(t)dt \\
&= \frac{1}{2\Gamma(-q)}\int_{-\infty}^\infty |t|^{-q-1}A_{K,\xi,p}(t)dt \\
&= \frac{1}{2\Gamma(-q)}\int_{-\infty}^\infty |t|^{-q-1}\int_{\langle x,\xi\rangle=t}\chi(\|x\|)|x|_2^{-p}dxdt \\
&= \frac{1}{2\Gamma(-q)}\int_{\mathbb{R}^n} |\langle x,\xi\rangle|^{-q-1}\chi(\|x\|)|x|_2^{-p}dx \\
&= \frac{1}{2\Gamma(-q)}\int_{S^{n-1}} |\langle \theta,\xi\rangle|^{-q-1}\int_0^\infty r^{-q-1}\chi(r\|\theta\|)r^{-p}r^{n-1}drd\theta \\
&= \frac{1}{2\Gamma(-q)}\int_{S^{n-1}} |\langle \theta,\xi\rangle|^{-q-1}\int_0^{\frac{1}{\|\theta\|}} r^{n-p-q-2}drd\theta \\
&= \frac{1}{2\Gamma(-q)(n-p-q-1)}\int_{S^{n-1}} |\langle \theta,\xi\rangle|^{-q-1}\|\theta\|^{-n+p+q+1}d\theta.
\end{aligned}$$

Now we extend $A^{(q)}_{K,\xi,p}(0)$ to $\mathbb{R}^n$ as a homogeneous function of ξ of degree $-1-q$. Then for every even test function $\phi \in \mathcal{S}$,

$$\langle A^{(q)}_{K,\xi,p}(0), \phi(\xi)\rangle = \frac{1}{2\Gamma(-q)(n-p-q-1)} \times \\ \times \int_{S^{n-1}} \|\theta\|^{-n+p+q+1} \int_{\mathbb{R}^n} |\langle\theta,\xi\rangle|^{-q-1}\phi(\xi)d\xi d\theta.$$

Using Lemma 5 from [GKS]

$$\begin{aligned} &= \frac{-1}{4\Gamma(-q)\Gamma(1+q)(n-p-q-1)\sin\frac{q\pi}{2}} \times \\ &\qquad \times \int_{S^{n-1}} \|\theta\|^{-n+p+q+1} \int_{-\infty}^{\infty} |t|^q \hat{\phi}(t\theta)dtd\theta \\ &= \frac{-\sin(-\pi q)}{2\pi(n-p-q-1)\sin\frac{q\pi}{2}} \langle (\|x\|^{-n+p+q+1}\cdot|x|_2^{-p})^\wedge(\xi), \phi(\xi)\rangle. \end{aligned}$$

The latter follows from the fact that $\Gamma(-q)\Gamma(q+1) = -\pi/\sin(q\pi)$ and the calculation

$$\begin{aligned} &\langle (\|x\|^{-n+p+q+1}\cdot|x|_2^{-p})^\wedge(\xi), \phi(\xi)\rangle \\ &= \int_{R^n} \|x\|^{-n+p+q+1}\cdot|x|_2^{-p}\hat{\phi}(x)dx \\ &= \int_{S^{n-1}} \|\theta\|^{-n+p+q+1} \int_0^\infty t^{-n+p+q+1}t^{-p}t^{n-1}\hat{\phi}(t\theta)dtd\theta \\ &= \int_{S^{n-1}} \|\theta\|^{-n+p+q+1} \int_0^\infty t^q\hat{\phi}(t\theta)dtd\theta. \end{aligned}$$

We have proved that

$$\langle A^{(q)}_{K,\xi,p}(0), \phi(\xi)\rangle = \frac{\cos\frac{\pi q}{2}}{\pi(n+p-q-1)} \langle (\|x\|^{-n+p+q+1}\cdot|x|_2^{-p})^\wedge(\xi), \phi(\xi)\rangle$$

for $-1 < q < 0$. Since both $A^{(q)}_{K,\xi,p}(0)$ and $(\|x\|^{-n+p+q+1}\cdot|x|_2^{-p})^\wedge(\xi)$ are continuous functions of $\xi \in \mathbb{R}^n \setminus \{0\}$, we get the statement of the Lemma for $-1 < q < 0$.

To prove the Lemma for other values of q we use the fact that for every even test function ϕ the functions

$$q \mapsto \langle A^{(q)}_{K,\xi,p}(0), \phi(\xi) \rangle$$

and

$$q \mapsto \frac{\cos\frac{\pi q}{2}}{\pi(n-p-q-1)} \langle (\|x\|^{-n+p+q+1} \cdot |x|_2^{-p})^\wedge(\xi), \phi(\xi) \rangle$$

are analytic in the domain $\{q \in \mathbb{C}\colon\ -1 < \lceil \mathrm{Re}\, q \rceil < n-p-1\}$. (The fact, that $(\|x\|^{-n+p+q+1} \cdot |x|_2^{-p})^\wedge(\xi)$ is analytic with respect to q, can be seen from the argument of [Ko12, Lemma 2.22]). The result of the Lemma follows, since these analytic functions coincide for $q \in (-1, 0)$, ϕ is arbitrary and, by Lemma 4.2.1, the fractional derivative is a continuous function of ξ outside of the origin.

$\square$

Lemma 4.2.3. *Let K be an origin-symmetric convex body in $\mathbb{R}^n$. Assume that $q \in (-1, 2]$ and $0 \le p < n - \lceil q \rceil - 1$, then $\|x\|_K^{-n+p+q+1} \cdot |x|_2^{-p}$ is a positive definite distribution on $\mathbb{R}^n$.*

Proof. First we prove that

$$A_{K,\xi,p}(t) \le A_{K,\xi,p}(0), \quad \text{for all } t \ge 0 \tag{4.4}$$

If $p = 0$, it follows from Brunn's theorem (see [Ko12, Theorem 2.3]) that the central hyperplane section of an origin-symmetric convex body has maximal volume among all hyperplane sections orthogonal to a given direction. If $p > 0$ one can see that

$$|x|_2^{-p} = p \int_0^\infty \chi(z|x|_2) z^{p-1} dz,$$

therefore

$$\begin{aligned}
A_{K,\xi,p}(t) &= \int_{K\cap\langle x,\xi\rangle=t} |x|_2^{-p}dx \\
&= p\int_{K\cap\langle x,\xi\rangle=t}\int_0^\infty \chi(z|x|_2)z^{p-1}dzdx \\
&= p\int_0^\infty z^{p-1}\int_{K\cap\langle x,\xi\rangle=t}\chi(z|x|_2)dxdz \\
&= p\int_0^\infty z^{p-1}\int_{B_{1/z}\cap K\cap\langle x,\xi\rangle=t} dxdz \\
&\le p\int_0^\infty z^{p-1}\int_{B_{1/z}\cap K\cap\langle x,\xi\rangle=0} dxdz = A_{K,\xi,p}(0)
\end{aligned}$$

by Brunn's theorem applied to the convex origin-symmetric body $B_{1/z}\cap K$, where $B_{1/z}$ is a ball of radius $1/z$.

Now consider $q\in(1,2)$. Here $\cos\dfrac{q\pi}{2}$ is negative, therefore we need to prove that $A^{(q)}_{K,\xi,p}(0)\le 0$. Using inequality (4.4), the formula for fractional derivatives for $q\in(1,2)$ and the fact that $A'(0)=0$ we get

$$\begin{aligned}
A^{(q)}_{K,\xi,p}(0) &= \frac{1}{\Gamma(-q)}\int_0^\infty t^{-q-1}(A(t)-A(0)-tA'(0))dt \\
&= \frac{1}{\Gamma(-q)}\int_0^\infty t^{-q-1}(A(t)-A(0))dt \le 0
\end{aligned}$$

since $\Gamma(-q)$ is positive.

If $q\in(0,1)$ then $\cos\frac{q\pi}{2}$ is positive and

$$A^{(q)}_{K,\xi,p}(0)=\frac{1}{\Gamma(-q)}\int_0^\infty t^{-q-1}(A(t)-A(0))dt\ge 0$$

since $\Gamma(-q)<0$ for these values of q.

Finally if $q\in(-1,0)$ then $\cos\frac{q\pi}{2}$ is positive, $\Gamma(-q)$ is also positive and

$$A^{(q)}_{K,\xi,p}(0)=\frac{1}{\Gamma(-q)}\int_0^\infty t^{-q-1}A(t)dt\ge 0$$

We still have to prove the Lemma for $q=0,1,2$.

When $q = 0$, $\cos \frac{\pi q}{2} = 1$ and

$$A^{(0)}_{K,\xi,p}(0) = (-1)^0 A_{K,\xi,p}(0) \geq 0.$$

When $q = 2$, $\cos \frac{\pi q}{2} = -1$ and

$$A^{(2)}_{K,\xi,p}(0) = (-1)^2 A''_{K,\xi,p}(0) \leq 0,$$

since $A_{K,\xi,p}(t)$ has maximum at 0.

When $q = 1$, take small $\varepsilon > 0$. By what we just proved for non-integer q, for any non-negative test function ϕ,

$$\langle (|x|_2^{-p} \|x\|_K^{-n+p+2+\varepsilon})^\wedge, \phi \rangle \geq 0.$$

Since $\|x\|_K \leq C|x|_2$ for some C, it follows that

$$\|x\|_K^{-n+p+2+\varepsilon} |x|_2^{-p} \leq \tilde{C}|x|_2^{-n+2+\varepsilon} \leq \tilde{C}|x|_2^{-n+1},$$

the latter being a locally-integrable function on $\mathbb{R}^n$.

Set $g(x) = \tilde{C}|x|_2^{-n+1}|\hat{\phi}(x)|$ for $|x|_2 < 1$ and $g(x) = \tilde{C}|\hat{\phi}(x)|$ for $|x|_2 > 1$. The function $g(x)$ is integrable on $\mathbb{R}^n$ and for small ε we have that $\|x\|_K^{-n-p+2+\varepsilon}|x|_2^p \hat{\phi}(x) \leq g(x)$. Therefore by the Lebesgue dominated convergence theorem,

$$\langle (\|x\|_K^{-n+p+2} |x|_2^{-p})^\wedge, \phi \rangle = \int_{\mathbb{R}^n} \|x\|_K^{-n+p+2} |x|_2^{-p} \hat{\phi}(x) dx =$$

$$= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \|x\|_K^{-n+p+2+\varepsilon} |x|_2^{-p} \hat{\phi}(x) dx = \lim_{\varepsilon \to 0} \langle (\|x\|_K^{-n+p+2+\varepsilon} |x|_2^{-p})^\wedge, \phi \rangle \geq 0$$

□

4.3 The proof of the main result

Theorem 4.3.1. *Let $\alpha \in [n-4, n-1)$, K and L be origin-symmetric infinitely smooth convex bodies in $\mathbb{R}^n$, $n \geq 4$, so that for every $\xi \in S^{n-1}$*

$$(-\Delta)^{\alpha/2} S_K(\xi) \leq (-\Delta)^{\alpha/2} S_L(\xi). \tag{4.5}$$

Then

$$\mathrm{vol}_n(K) \leq \mathrm{vol}_n(L).$$

On the other hand, for any $\alpha \in [n-5, n-4)$ there are convex origin-symmetric bodies $K, L \in \mathbb{R}^n$, $n \geq 5$ that satisfy (4.5) *for every $\xi \in S^{n-1}$ but* $\mathrm{vol}_n(L) < \mathrm{vol}_n(K)$.

Proof of the affirmative part. Let $S_K(\xi) = \mathrm{vol}_{n-1}(K \cap \xi^\perp)$, $\xi \in S^{n-1}$, the central section function defined in the Introduction. Then, as proved in [Ko6]

$$S_K(\xi) = \frac{1}{\pi(n-1)} (\|x\|_K^{-n+1})^\wedge(\xi). \tag{4.6}$$

Extending $S_K(\xi)$ to $\mathbb{R}^n$ as a homogeneous function of degree -1 and using the definition of fractional powers of the Laplacian we get

$$(-\Delta)^{\alpha/2} S_L(\theta) = \frac{1}{\pi(n-1)} (|x|_2^{\alpha} \|x\|_L^{-n+1})^\wedge(\theta),$$

therefore

$$\begin{aligned}
(2\pi)^n & \int_{S^{n-1}} \|x\|_K^{-1} \|x\|_L^{-n+1} dx = \\
= \quad & (2\pi)^n \int_{S^{n-1}} (|x|_2^{-\alpha} \|x\|_K^{-1})(|x|_2^{\alpha} \|x\|_L^{-n+1}) dx \\
= \quad & \int_{S^{n-1}} (|x|_2^{-\alpha} \|x\|_K^{-1})^\wedge(\theta) (|x|_2^{\alpha} \|x\|_L^{-n+1})^\wedge(\theta) d\theta \\
= \quad & \pi(n-1) \int_{S^{n-1}} (|x|_2^{-\alpha} \|x\|_K^{-1})^\wedge(\theta) (-\Delta)^{\alpha/2} S_L(\theta) d\theta
\end{aligned}$$

Here we used Parseval's formula on the sphere (1.1.3) and formula (4.6).

By Lemma 4.2.3 with $p = \alpha$ and $q = n - \alpha - 2$, $(|x|_2^{-\alpha} \|x\|_K^{-1})^\wedge$ is a non-negative function on S^{n-1}, therefore using the condition of the theorem and repeating the above calculation in the opposite order, we get

$$\int_{S^{n-1}} \|x\|_K^{-1} \|x\|_K^{-n+1} dx \leq \int_{S^{n-1}} \|x\|_K^{-1} \|x\|_L^{-n+1} dx.$$

Then by Hölder's inequality and the polar formula for the volume (1.1),

$$n \ \mathrm{vol}_n(K) \leq \left(\int_{S^{n-1}} \|\theta\|_K^{-n} d\theta \right)^{1/n} \left(\int_{S^{n-1}} \|\theta\|_L^{-n} d\theta \right)^{(n-1)/n} =$$

$$n(\mathrm{vol}_n(K))^{1/n} (\mathrm{vol}_n(L))^{(n-1)/n},$$

which yields the statement of the positive part of the theorem.

Proof of the negative part. Let $\alpha \in [n-5, n-4)$. We need to construct two convex origin-symmetric bodies $K, L \in \mathbb{R}^n$, $n \geq 5$ such that for every $\xi \in S^{n-1}$

$$(-\Delta)^{\alpha/2} S_K(\xi) \leq (-\Delta)^{\alpha/2} S_L(\xi),$$

but

$$\mathrm{vol}_n(L) < \mathrm{vol}_n(K).$$

First let us prove the following Lemma.

Lemma 4.3.2. *Let $\alpha \in [n-5, n-4)$. There exists an infinitely smooth origin-symmetric convex body L with positive curvature, so that*

$$\|x\|_L^{-1} \cdot |x|_2^{-\alpha}$$

is not a positive definite distribution.

Proof. First assume that $\alpha \in (n-5, n-4)$. Put $q = n - \alpha - 2$, so $q \in (2,3)$. Our goal is to construct a body L so that there is a $\xi \in S^{n-1}$ satisfying

$$\int_0^\infty t^{-q-1} \left(A_{L,\xi,\alpha}(t) - A_{L,\xi,\alpha}(0) - A''_{L,\xi,\alpha}(0)\frac{t^2}{2} \right) dt < 0. \tag{4.7}$$

If we construct such a body L the result of this lemma immediately follows from Lemma 4.2.2 and the definition of fractional derivatives.

Consider the function

$$f(t) = \left(1 - t^2 - Nt^4\right)^{\frac{1}{n-\alpha-1}}$$

Let a_N be the positive real root of the equation $f(t) = 0$. Define the body $L \in \mathbb{R}^n$ as follows.

$$L = \left\{ (x_1, ..., x_n) \in \mathbb{R}^n \colon x_n \in [-a_N, a_N] \text{ and } \left(\sum_{i=1}^{n-1} x_i^2 \right)^{1/2} \leq f(x_n) \right\},$$

which is a strictly convex infinitely differentiable body.

Take ξ to be the unit vector in the direction of the x_n-axis. Then for $t \in [0, a_N]$,

$$\begin{aligned} A_{L,\xi,\alpha}(t) &= \int_{S^{n-1}} \int_0^{f(t)} (t^2 + r^2)^{-\alpha/2} r^{n-2} dr \; d\theta \\ &= C_n \int_0^{f(t)} (t^2 + r^2)^{-\alpha/2} r^{n-2} dr \end{aligned}$$

where $C_n = |S^{n-1}|$, and for $t > a_N$ we have $A_{L,\xi,\alpha}(t) = 0$.

One can compute:

$$A_{L,\xi,p}(0) = \frac{C_n}{n - \alpha - 1},$$

and

$$A''_{L,\xi,p}(0) = -C_n \left[\frac{\alpha}{n - \alpha - 3} + \frac{2}{n - \alpha - 1} \right].$$

In order to estimate the integral (4.7), we split it into three parts: over $[0, b_N]$, $[b_N, a_N]$ and $[a_N, \infty)$, where b_N is the positive real root of the equation $1 - t^2 - Nt^4 = t^{q+1}$. Recall that a_N was defined as the positive real root of the equation $1 - t^2 - Nt^4 = 0$. It is easy to check that $a_N \simeq b_N \simeq N^{-1/4}$ for large N. Also note that on $[0, a_N]$ we have $f(t) \geq 0$, and $f(t) \geq t$ if and only if $t \in [0, b_N]$.

First consider the interval $[0, b_N]$. For all t from this interval we have $t \leq f(t)$. Then we can break the integral:

$$\int_0^{f(t)} (t^2 + r^2)^{-\alpha/2} r^{n-2} dr = I_1 + I_2$$

into two parts, where the first one can be estimated as follows

$$I_1 = \int_0^t (t^2 + r^2)^{-\alpha/2} r^{n-2} dr \leq \int_0^t (r^2)^{-\alpha/2} r^{n-2} dr = \frac{t^{n-\alpha-1}}{n - \alpha - 1}$$

and for the second one we will use the inequality:

$$(1 + x)^{-\gamma} \leq 1 - \gamma x + \frac{\gamma(\gamma + 1)}{2} x^2, \quad \text{for } \gamma > 0 \text{ and } 0 < x < 1.$$

Then

$$\begin{aligned}
I_2 &= \int_t^{f(t)} (t^2 + r^2)^{-\alpha/2} r^{n-2} dr \\
&= \int_t^{f(t)} \left(1 + \frac{t^2}{r^2}\right)^{-\alpha/2} r^{n-\alpha-2} dr \leq \\
&\leq \int_t^{f(t)} \left(1 - \frac{\alpha}{2}\frac{t^2}{r^2} + \frac{\frac{\alpha}{2}\left(\frac{\alpha}{2} + 1\right)}{2}\frac{t^4}{r^4}\right) r^{n-\alpha-2} dr \\
&= \left[\frac{r^{n-\alpha-1}}{n - \alpha - 1} - \frac{\alpha}{2}\frac{t^2 r^{n-\alpha-3}}{n - \alpha - 3} + \frac{\frac{\alpha}{2}\left(\frac{\alpha}{2} + 1\right)}{2}\frac{t^4 r^{n-\alpha-5}}{n - \alpha - 5}\right]_t^{f(t)} \\
&= \frac{f^{n-\alpha-1}(t)}{n - \alpha - 1} - \frac{\alpha}{2}\frac{t^2}{n - \alpha - 3} f^{n-\alpha-3}(t) + \\
&\quad + \frac{\frac{\alpha}{2}\left(\frac{\alpha}{2} + 1\right)}{2}\frac{t^4}{n - \alpha - 5} f^{n-\alpha-5}(t) + C t^{n-\alpha-1}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{f^{n-\alpha-1}(t)}{n-\alpha-1} - \frac{\alpha}{2}\frac{t^2}{n-\alpha-3}f^{n-\alpha-3}(t) + Ct^{n-\alpha-1} \\
&= \frac{1-t^2-Nt^4}{n-\alpha-1} - \frac{\alpha}{2}\frac{t^2}{n-\alpha-3}(1-t^2-Nt^4)^{\frac{n-\alpha-3}{n-\alpha-1}} + Ct^{n-\alpha-1}
\end{aligned}
$$

for some constant C. The last inequality follows from $f(t) \geq 0$ on $[0, b_N]$ and $\alpha \in (n-5, n-4)$.

Using the inequality:

$$
(1-x)^\gamma \geq 1 - \gamma x(1-x)^{\gamma-1}, \quad \text{for } 0 < \gamma < 1 \text{ and } 0 < x < 1,
$$

applied to the second term in the previous expression, we get

$$
\begin{aligned}
I_2 &\leq \frac{1-t^2-Nt^4}{n-\alpha-1} - \frac{\alpha}{2}\frac{t^2}{n-\alpha-3} \times \\
&\times \left(1 - \frac{n-\alpha-3}{n-\alpha-1}(1-t^2-Nt^4)^{\frac{n-\alpha-3}{n-\alpha-1}-1}(t^2+Nt^4)\right) + Ct^{n-\alpha-1} \\
&= \frac{1-t^2-Nt^4}{n-\alpha-1} - \frac{\alpha}{2}\frac{t^2}{n-\alpha-3} + \\
&\qquad + C_1\frac{t^4+Nt^6}{(1-t^2-Nt^4)^{\frac{2}{n-\alpha-1}}} + Ct^{n-\alpha-1}
\end{aligned}
$$

Now using the estimates for I_1 and I_2 we get

$$
\begin{aligned}
&\int_0^{b_N} t^{-q-1}\left(A_{L,\xi,\alpha}(t) - A_{L,\xi,\alpha}(0) - A''_{L,\xi,\alpha}(0)\frac{t^2}{2}\right) dt \leq \\
&\leq C_n \int_0^{b_N} t^{-q-1}\Big(\frac{1-t^2-Nt^4}{n-\alpha-1} - \frac{\alpha}{2}\frac{t^2}{n-\alpha-3} + C_1\frac{t^4+Nt^6}{(1-t^2-Nt^4)^{\frac{2}{n-\alpha-1}}} \\
&\qquad + Ct^{n-\alpha-1} - \frac{1}{n-\alpha-1} + \left[\frac{\alpha}{n-\alpha-3} + \frac{2}{n-\alpha-1}\right]\frac{t^2}{2}\Big)dt \\
&= C_n \int_0^{b_N} t^{-q-1}\left(\frac{-Nt^4}{n-\alpha-1} + C_1\frac{t^4+Nt^6}{(1-t^2-Nt^4)^{\frac{2}{n-\alpha-1}}} + Ct^{n-\alpha-1}\right) dt
\end{aligned}
$$

Now one can estimate each term of the last integral separately. Since $b_N \simeq N^{-1/4}$, we get that

$$
\int_0^{b_N} t^{-q-1}\frac{-Nt^4}{n-\alpha-1}dt \simeq -C_2 N^{q/4}
$$

for a positive constant C_2.

For the second term, we change the variable of integration: $u = N^{1/4}t$. Then

$$\begin{aligned}
&\int_0^{b_N} t^{-q-1}\frac{t^4+Nt^6}{(1-t^2-Nt^4)^{\frac{2}{n-\alpha-1}}}dt \\
&\qquad = N^{q/4}\int_0^{b_N N^{1/4}} u^{-q-1}\frac{u^4N^{-1}+u^6N^{-1/2}}{(1-N^{-1/2}u^2-u^4)^{\frac{2}{n-\alpha-1}}}du \\
&\qquad \le N^{(q-2)/4}\int_0^{b_N N^{1/4}} u^{-q-1}\frac{u^4+u^6}{(1-N^{-1/2}u^2-u^4)^{\frac{2}{n-\alpha-1}}}du \\
&\qquad \le C_3 N^{(q-2)/4},
\end{aligned}$$

since $b_N N^{1/4} \to 1$ as $N \to \infty$, and the integral

$$\int_0^1 u^{-q-1}\frac{u^4+u^6}{(1-u^4)^{\frac{2}{n-\alpha-1}}}du$$

converges both at 0 and 1.

And finally the integral of the last term is small for large values of N, since $n-\alpha-1 = q+1$. From what we have obtained one can see that the integral over $[0, b_N]$ will be negative for large values of N since the leading term is $-C_2N^{q/4}$.

Now consider the integral over $[b_N, a_N]$. The expression

$$A_{L,\xi,\alpha}(t) - A_{L,\xi,\alpha}(0) - A''_{L,\xi,\alpha}(0)t^2/2$$

can be estimated from above by a constant C, not depending on N. Indeed, $A_{L,\xi,\alpha}(t) \le A_{L,\xi,\alpha}(0)$, $A''_{L,\xi,\alpha}(0)$ is a constant independent of N, and $t \le a_N \simeq N^{-1/4} \le 1$ for N large enough. Therefore

$$\begin{aligned}
&\int_{b_N}^{a_N} t^{-q-1}\left(A_{L,\xi,\alpha}(t) - A_{L,\xi,\alpha}(0) - A''_{L,\xi,\alpha}(0)\frac{t^2}{2}\right)dt \le \\
&\quad \le C\int_{b_N}^{a_N} t^{-q-1}dt \le C\int_{b_N}^{a_N}(b_N)^{-q-1}dt = C\frac{a_N-b_N}{(b_N)^{q+1}}
\end{aligned}$$

Recalling that a_N and b_N satisfy the equations

$$1 - a_N^2 - Na_N^4 = 0 \quad \text{and} \quad 1 - b_N^2 - Nb_N^4 = b_N^{q+1}$$

we conclude that

$$b_N^{q+1} = (a_N^2 - b_N^2)(1 + N(a_N^2 + b_N^2)).$$

Therefore

$$C \int_{b_N}^{a_N} t^{-q-1} dt \leq \frac{C}{(a_N + b_N)(1 + N(a_N^2 + b_N^2))} \simeq CN^{-1/4}.$$

Finally, the integral over $[a_N, \infty)$ can be computed as follows

$$\int_{a_N}^{\infty} t^{-q-1} \left(-A_{L,\xi,\alpha}(0) - A''_{L,\xi,\alpha}(0)\frac{t^2}{2} \right) dt \ \simeq \ -D_1 N^{q/4} + D_2 N^{(q-2)/4}$$

where $D_1 > 0$. Therefore, this integral is negative for N large enough.

Combining all the integrals one can see that for N large enough the desired integral (4.7) is negative. This means that for some direction $\xi \in S^{n-1}$ the function $(\|x\|_L^{-1} \cdot |x|_2^{-\alpha})^\wedge(\xi)$ is negative, if $\alpha \in (n-5, n-4)$.

If $\alpha = n-5$, both sides of the equality in the statement of Lemma 4.2.2 vanish, therefore we need to apply the argument from [GKS] (see the proof of Theorem 1). Then

$$\begin{aligned}(\|x\|_L^{-1} \cdot |x|_2^{-n+5})^\wedge(\xi) &= \\ &= C \int_0^{\infty} t^{-4} \left(A_{L,\xi,\alpha}(t) - A_{L,\xi,\alpha}(0) - A''_{L,\xi,\alpha}(0)\frac{t^2}{2} \right) dt\end{aligned}$$

for a positive constant C. Considering the same body as before, we get that $(\|x\|_L^{-1} \cdot |x|_2^{-n+5})^\wedge(\xi)$ is also negative at some point ξ.

$\square$

Now we are ready to finish the proof of the negative part. Apply Lemma 4.3.2 to construct an infinitely smooth origin-symmetric body L with positive curvature for which $(\|x\|_L^{-1} \cdot |x|_2^{-\alpha})^\wedge(\xi) < 0$ for some direction ξ. By Lemma 4.2.2, the function $(\|x\|_L^{-1} \cdot |x|_2^{-\alpha})^\wedge$ is continuous on the sphere S^{n-1}, hence there is a neighborhood of ξ where it is negative.

Let

$$\Omega = \{\theta \in S^{n-1} \colon (\|x\|_L^{-1} \cdot |x|_2^{-\alpha})^\wedge(\theta) < 0\}.$$

Choose a non-positive infinitely differentiable even function v supported on Ω. Extend v to a homogeneous function $r^{-\alpha-1}v(\theta)$ of degree $-\alpha-1$ on $\mathbb{R}^n$. By [Ko12, Lemma 3.16], the Fourier transform of $|x|_2^{-\alpha-1}v(x/|x|_2)$ is equal to $|x|_2^{-n+\alpha+1}g(x/|x|_2)$ for some infinitely differentiable function g on S^{n-1}.

Define a body K by

$$\|x\|_K^{-n+1} = \|x\|_L^{-n+1} + \varepsilon |x|_2^{-n+1} g(x/|x|_2)$$

for some small ε so that the body K is convex (see for example [Ko12, Theorem 5.3] for this standard perturbation argument). Multiply both sides by $\dfrac{1}{\pi(n-1)}|x|_2^\alpha$ and apply the Fourier transform:

$$(-\Delta)^{\alpha/2} S_K = (-\Delta)^{\alpha/2} S_L + \frac{\varepsilon(2\pi)^n}{\pi(n-1)} |x|_2^{-\alpha-1} v(x/|x|_2) \le (-\Delta)^{\alpha/2} S_L,$$

since v is non-positive.

On the other hand,

$$\int_{S^{n-1}} (\|x\|_L^{-1} \cdot |x|_2^{-\alpha})^\wedge(\theta)(-\Delta)^{\alpha/2} S_K d\theta =$$

$$= \int_{S^{n-1}} (\|x\|_L^{-1} \cdot |x|_2^{-\alpha})^\wedge(\theta)(-\Delta)^{\alpha/2} S_L d\theta$$

$$+\varepsilon \frac{(2\pi)^n}{\pi(n-1)} \int_{S^{n-1}} (\|x\|_L^{-1} \cdot |x|_2^{-\alpha})^\wedge(\theta) v(\theta) d\theta$$

$$> \int_{S^{n-1}} (\|x\|_L^{-1} \cdot |x|_2^{-\alpha})^\wedge(\theta)(-\Delta)^{\alpha/2} S_L d\theta.$$

Repeating the argument from the proof of the affirmative part we get:

$$\mathrm{vol}_n(L) < \mathrm{vol}_n(K).$$

□

Remarks. (i) The negative part is formulated only for $q \in [n-5, n-4)$, because we wanted this to work for $n = 5$. In fact, for bigger n one can take $q \in [0, n-4)$. Also the condition (1) can be written in terms of the Fourier transforms so that no smoothness of the bodies is required.

(ii) In the case where $q = n-4$ and n is an even integer, the result of Theorem 4.3.1 was proved in [Ko11] using an induction argument. The proof from [Ko11] can not be extended to other values of q and n and does not produce any results in the negative direction.

(iii) Shephard's problem (see, for example, [Ko12, Section 8.4]) asks whether convex origin-symmetric bodies with smaller projections necessarily have smaller n-dimensional volume. As proved independently by Petty [Pe] and Schneider [Sc], the answer to this problem is affirmative only in dimension $n = 2$, so one may try to modify Shephard's problem to guarantee the affirmative answer in all dimensions.

However, attempts to repeat the proof of Theorem 4.3.1 for Shephard's problem fail, since the section function $A_{K,\xi,p}$ may not be sufficiently differentiable.

Chapter 5

The geometry of L_0

5.1 Introduction

In this chapter we present our results from [KKYY]. Suppose that we have the unit Euclidean ball in $\mathbb{R}^n$ and are allowed to construct new bodies using three operations – linear tranformations, multiplicative summation and closure in the radial metric. The **multiplicative sum** $K +_0 L$ of star bodies K and L is defined by

$$\|x\|_{K+_0L} = \sqrt{\|x\|_K \|x\|_L}. \tag{5.1}$$

What class of bodies do we get from the unit ball by means of these three operations?

We are going to prove that in dimension $n = 3$ we get all origin-symmetric convex bodies, while in dimension 4 and higher this is no longer the case. However, the class of bodies that we get in arbitrary dimension also has a clear interpretation. We introduce the concept of embedding in L_0 and show that the bodies that we get by means of these three operations are exactly the unit balls of spaces that embed in L_0.

The idea of this interpretation comes from a similar result for L_p-spaces with $p \in [-1,1],\ p \neq 0$. Namely, if we replace the multiplicative summation by p-summation

$$\|x\|_{K+_pL} = (\|x\|_K^p + \|x\|_L^p)^{1/p} \tag{5.2}$$

then we get the unit balls of all spaces that embed in L_p. The case $p = 1$ is well-known (see [Ga3, Corollary 4.1.12]) and the unit balls of subspaces of L_1 have a clear geometric meaning - these are the polar projection bodies (see [Bol]). On the other hand, it was proved by Goodey and Weil [GW] that if $p = -1$ (this case corresponds to the radial summation) then we get the class of intersection bodies in $\mathbb{R}^n$. As shown in [Ko8], intersection bodies are the unit balls of spaces that embed in L_{-1}. The concept of embedding in $L_p,\ p < 0$ was introduced in [Ko7] as an analytic extension of the same property for $p > 0$, see [KK2] for related results. The result of Goodey and Weil can easily be extended to $p \in (-1,1),\ p \neq 0$. Note that this construction provides a continuous (except for $p = 0$) path from polar projection bodies to intersection bodies, which is important for understanding the duality between projections and sections of convex bodies. One of the goals of this chapter is to fill the gap in this scheme at $p = 0$ and better understand the geometry of this intermediate case.

Another interesting similarity of our result with other values of p is that for $p = 1$ the procedure defined above gives all origin-symmetric convex bodies only in dimension 2. This follows from a result of Schneider [Sc] that every origin-symmetric convex body is a polar projection body only in dimension 2. When

$p = -1$ we get all origin-symmetric convex bodies only in dimensions 4 and lower, because, by results from [Ga2], [Zh2], [GKS], only in these dimensions every origin-symmetric convex body is an intersection body. The transition between the dimensions 2 and 3 in the case $p = 1$ and the transition between the dimensions 4 and 5 in the case $p = -1$ directly correspond to the transition between the affirmative and negative answers in the Shephard and Busemann-Petty problems, respectively. It would be interesting to find a similar geometric result corresponding to the transition between dimensions 3 and 4 in the case $p = 0$. We refer the reader to the book [Ko12, Chapter 6] for more details and history of the connection between convex geometry and the theory of L_p-spaces.

5.2 The definition of embedding in L_0.

Recall a well-known result of P.Lévy, see [BL, p. 189] or [Ko12, Section 6.1], that a space $(\mathbb{R}^n, \|\cdot\|)$ embeds into L_p, $p > 0$ if and only if there exists a finite Borel measure μ on the unit sphere so that, for every $x \in \mathbb{R}^n$,

$$\|x\|^p = \int_{S^{n-1}} |(x,\xi)|^p d\mu(\xi). \tag{5.3}$$

On the other hand, the definition of embedding in L_p with $p < 0$ from [Ko7] implies that a space $(\mathbb{R}^n, \|\cdot\|)$ embeds into L_p, $p \in (-n, 0)$ if and only if there exists a finite symmetric measure μ on the sphere S^{n-1} so that for every test function ϕ,

$$\int_{\mathbb{R}^n} \|x\|^p \phi(x) dx = \int_{S^{n-1}} d\mu(\xi) \int_{\mathbb{R}} |t|^{-p-1} \hat{\phi}(t\xi) dt. \tag{5.4}$$

Both representations (5.3) and (5.4) are invariant with respect to p-summation. This gives an idea of defining embedding in L_0 by means of a representation that is invariant with respect to multiplicative summation. Note that the multiplicative summation is the limiting case of p-summation as $p \to 0$.

Definition 5.2.1. *We say that a space $(\mathbb{R}^n, \|\cdot\|)$ embeds in L_0 if there exist a finite Borel measure μ on the sphere S^{n-1} and a constant $C \in \mathbb{R}$ so that, for every $x \in \mathbb{R}^n$,*

$$\ln \|x\| = \int_{S^{n-1}} \ln |(x, \xi)| d\mu(\xi) + C. \tag{5.5}$$

While being similar to (5.3) and (5.4), this definition has its unique features. First, the measure μ must be a probability measure on S^{n-1}. In fact, put $x = ky$, $k > 0$ in (5.5). Then

$$\ln k + \ln \|y\| = \int_{S^{n-1}} \ln k \; d\mu(\xi) + \int_{S^{n-1}} \ln |(y, \xi)| d\mu(\xi) + C$$

and, again by (5.5) with $x = y$, we get $\ln k = \int_{S^{n-1}} \ln k \; d\mu(\xi)$, so $\int_{S^{n-1}} d\mu(\xi) = 1$.

Secondly, the constant C depends on the norm and can be computed precisely. In order to compute this constant, integrate the equality (5.5) over the uniform measure on the unit sphere. We get

$$\begin{aligned} C \cdot |S^{n-1}| &= \int_{S^{n-1}} \ln \|x\| dx - \int_{S^{n-1}} \int_{S^{n-1}} \ln |(x, \theta)| \, d\mu(\theta) dx \\ &= \int_{S^{n-1}} \ln \|x\| dx - \int_{S^{n-1}} \int_{S^{n-1}} \ln |(x, \theta)| \, dx \, d\mu(\theta) \\ &= \int_{S^{n-1}} \ln \|x\| dx - \int_{S^{n-1}} \ln |(x, \theta)| \, dx, \end{aligned}$$

since $\int_{S^{n-1}} \ln |(x, \theta)| \, dx$ is rotationally invariant and, therefore, is a constant for $\theta \in S^{n-1}$, and μ is a probability measure.

To compute the latter integral, use the well-known formula (see [Ko12, Section

6.4])

$$\int_{S^{n-1}} |(x,\theta)|^p \, dx = \frac{2\pi^{(n-1)/2}\Gamma((p+1)/2)}{\Gamma((n+p)/2)}.$$

Differentiating with respect to p and letting $p = 0$ we get

$$\int_{S^{n-1}} \ln|(x,\theta)| \, dx = \pi^{(n-1)/2}\left[\frac{\Gamma'(1/2)}{\Gamma(n/2)} - \sqrt{\pi}\frac{\Gamma'(n/2)}{\Gamma^2(n/2)}\right]$$

Note that

$$|S^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)},$$

so

$$C = \frac{1}{|S^{n-1}|}\int_{S^{n-1}} \ln\|x\| dx - \frac{1}{2\sqrt{\pi}}\Gamma'(1/2) + \frac{1}{2}\frac{\Gamma'(n/2)}{\Gamma(n/2)}.$$

Let us remark that Definition 5.2.1 is equivalent to the following. A finite-dimensional normed space $X = (\mathbb{R}^n, \|\cdot\|)$ embeds into L_0 if and only if there is a probability space (Ω,μ) and a linear map $T : X \to \mathcal{M}(\Omega,\mu)$ (where $\mathcal{M}(\Omega,\mu)$ denotes the space of $\mu-$measurable functions on Ω) such that

$$\int_\Omega \ln|Tx(\omega)| d\mu(\omega) < \infty, \qquad x \in X$$

and

$$\ln\|x\| = \int_\Omega \ln|Tx(\omega)| d\mu(\omega), \qquad x \in X.$$

Indeed if such an operator T exists we can write it in the form

$$Tx(\omega) = h(\omega)(x, \xi(\omega)), \qquad x \in X$$

where $h : \Omega \to \mathbb{R}^+$ and $\xi : \Omega \to S^{n-1}$ are measurable. Then for each $\omega \in \Omega$

$$\int_{S^{n-1}} \ln|(x,\xi(\omega))| dx > -\infty$$

so that it follows for some $x \in S^{n-1}$, $\omega \to \ln |(x, \xi(\omega)|$ is $\mu-$integrable. Hence so is $\ln h$ and further

$$\ln \|x\| = \int \ln h(\omega)\, d\mu(\omega) + \int \ln |(x, \xi(\omega))| d\mu(\omega).$$

Now we can induce a probability measure μ' on S^{n-1} by $\mu'(B) = \mu\{\omega : \xi(\omega) \in B\}$ and we have the same situation as Definition 2.1.

On the other hand, if X satisfies Definition 5.2.1, we may take $\Omega = S^{n-1}$ and μ is a probability measure. If we define $Tx(\xi) = e^C(x, \xi)$ then $T : X \to \mathcal{M}(S^{n-1}, \mu)$ satisfies our conditions.

One advantage of this viewpoint is that we can make sense of the statement that an infinite-dimensional Banach space embeds into L_0.

5.3 A Fourier analytic characterization of subspaces of L_0

The fact that the Fourier transform is useful in the study of subspaces of L_p has been known for a long time. A well-known result of P.Levy is that a finite dimensional normed space $(\mathbb{R}^n, \|\cdot\|)$ embeds isometrically in $L_p,\ 0 < p \leq 2$ if and only if $\exp(-\|\cdot\|^p)$ is a positive definite function on $\mathbb{R}^n$. It was proved in [Ko2] that a space $(\mathbb{R}^n, \|\cdot\|)$ embeds isometrically in $L_p,\ p > 0, p \notin 2\mathbb{N}$ if and only if the Fourier transform of the function $\Gamma(-p/2)\|x\|^p$ (in the sense of distributions) is a positive distribution outside of the origin. If $-n < p < 0$ a similar fact was proved in [Ko7]: a space $(\mathbb{R}^n, \|\cdot\|)$ embeds in L_p if and only if the Fourier transform of $\|\cdot\|^p$ is a positive distribution in the whole $\mathbb{R}^n$. These characterizations have proved to be useful in the study of subspaces of L_p and intersection bodies, see [Ko12, Chapter 6]. In this section we prove a similar characterization of spaces that embed in L_0.

Theorem 5.3.1. *Let K be an origin symmetric star body in $\mathbb{R}^n$. The space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0 if and only if the Fourier transform of $\ln \|x\|_K$ is a negative distribution outside of the origin in $\mathbb{R}^n$.*

Proof. First, assume that $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0. Let ϕ be a non-negative even test function with compact support outside of the origin. By the definition of embedding in L_0, formula (1.2)(note that $\hat{\hat{\phi}} = (2\pi)^n\phi$ for even ϕ) and the Fubini theorem,

$$\begin{aligned}
\langle (\ln \|x\|)^\wedge, \phi \rangle &= \langle \ln \|x\|, \hat{\phi}(x) \rangle \\
&= \int_{S^{n-1}} \int_{\mathbb{R}^n} \ln |(x,\xi)| \hat{\phi}(x)\ dx\ d\mu(\xi) + C \int_{\mathbb{R}^n} \hat{\phi}(x) dx \\
&= \int_{S^{n-1}} \left\langle \ln |t|, \int_{(x,\xi)=t} \hat{\phi}(x)\ dx \right\rangle d\mu(\xi) \\
&= (2\pi)^{-1} \int_{S^{n-1}} \left\langle (\ln |t|)^\wedge(z), \Big(\int_{(x,\xi)=t} \hat{\phi}(x)\ dx \Big)^\wedge (z) \right\rangle d\mu(\xi) \\
&= (2\pi)^{n-1} \int_{S^{n-1}} \int_{\mathbb{R}} (\ln |t|)^\wedge (z) \phi(z\xi)\ dz\ d\mu(\xi) \qquad (5.6)
\end{aligned}$$

since $\int_{\mathbb{R}^n} \hat{\phi}(x)dx = (2\pi)^n\phi(0) = 0$. Now, the formula for the Fourier transform of $\ln |t|$ from [GS, p.362] implies that

$$(\ln |t|)^\wedge (z) = -\pi |z|^{-1} < 0 \qquad (5.7)$$

outside of the origin, so (5.6) is negative (recall that ϕ is non-negative with support outside of the origin). This means that $(\ln \|x\|)^\wedge$ is a negative distribution.

To prove the other direction, note that, by [Ko12, Section 2.6], a distribution that is positive outside of the origin coincides with a finite Borel measure on every set of the form

$$A \times (a,b) = \{x \in \mathbb{R}^n : x = t\theta, t \in (a,b), \theta \in A\},$$

where A is an open subset of S^{n-1} and $0 < a < b < \infty$.

Denote by $\mu = -(\ln\|x\|)^\wedge$. This distribution coincides with a finite Borel measure on each set $A \times (a,b)$, as above, so for any test function ϕ supported outside of the origin

$$\langle -(\ln\|x\|)^\wedge, \phi\rangle = \langle \mu, \phi\rangle = \int_{\mathbb{R}^n} \phi(x) d\mu(x).$$

Now for every test function ϕ with support outside of the origin and $t > 0$, we have $\left(\phi(x/t)\right)^\wedge(z) = t^n \hat{\phi}(tz)$, so

$$\begin{aligned}
\langle \mu(x), \phi(x/t)\rangle &= -\langle (\ln\|x\|)^\wedge(x), \phi(x/t)\rangle \\
&= -\int_{\mathbb{R}^n} \ln\|z\| \hat{\phi}(tz) t^n dz \\
&= -\int_{\mathbb{R}^n} \hat{\phi}(\tilde{x}) \ln\|\frac{1}{t}\tilde{x}\| d\tilde{x} \\
&= -\int_{\mathbb{R}^n} \hat{\phi}(\tilde{x}) \ln\|\tilde{x}\| d\tilde{x} + \ln|t| \int_{\mathbb{R}^n} \hat{\phi}(\tilde{x}) d\tilde{x} \\
&= -\int_{\mathbb{R}^n} \hat{\phi}(\tilde{x}) \ln\|\tilde{x}\| d\tilde{x} \\
&= \langle \mu(x), \phi(x)\rangle. \qquad (5.8)
\end{aligned}$$

Let $\chi_{A\times(a,b)}$ be the indicator of the set $A \times (a,b)$. Approximating $\chi_{A\times(a,b)}$ by test functions and using (5.8), we get for any $(a,b) \subset (0,\infty)$ and $A \subset S^{n-1}$

$$\begin{aligned}
\mu(A \times (a,b)) &= \int_{\mathbb{R}^n} \chi_{A\times(a,b)}(x) d\mu(x) \\
&= \int_{\mathbb{R}^n} \chi_{A\times(1,b/a)}(x/a) d\mu(x) \\
&= \int_{\mathbb{R}^n} \chi_{A\times(1,b/a)}(x) d\mu(x) \\
&= \mu(A \times (1, b/a)).
\end{aligned}$$

Applying this formula n times,

$$\mu\left(A \times (1, a^n)\right) = n\mu\left(A \times (1, a)\right) \tag{5.9}$$

for $n \in \mathbb{N}$. Moreover, we can extend formula (5.9) to $n \in \mathbb{R}$. So, for any $a \in (0, \infty)$, $A \subset S^{n-1}$

$$\mu\left(A \times [1, a]\right) = \mu\left(A \times [1, e^{\ln a}]\right) = \ln a \cdot \mu\left(A \times [1, e]\right)$$

Now for every $(a, b) \subset (0, \infty)$ and $A \subset S^{n-1}$ we have

$$\begin{aligned}
\mu\left(A \times (a, b)\right) &= \mu\left(A \times (1, b/a)\right) \\
&= \ln\left(\frac{b}{a}\right) \mu\left(A \times (1, e)\right) \\
&= \left(\ln(b) - \ln(a)\right) \mu(A \times (1, e)).
\end{aligned}$$

Define a measure μ_0 on S^{n-1} by

$$\mu_0(A) = \frac{\mu\left(A \times (a, b)\right)}{\left(\ln(b) - \ln(a)\right)} = \mu(A \times (1, e))$$

for every Borel set $A \subset S^{n-1}$. We have

$$\begin{aligned}
\int_{S^{n-1}} d\mu_0(\theta) \int_0^\infty |t|^{-1} \chi_{A \times (a,b)}(t\theta) dt &= \left(\ln(b) - \ln(a)\right) \mu_0(A) \\
&= \mu(A \times (a, b)) \\
&= \int_{\mathbb{R}^n} \chi_{A \times (a,b)}(x) d\mu(x)
\end{aligned} \tag{5.10}$$

Therefore, for an arbitrary even test function ϕ supported outside of the origin,

$$\begin{aligned}
\langle \mu, \phi \rangle &= \int_{S^{n-1}} d\mu_0(\theta) \int_0^\infty |t|^{-1} \phi(t\theta) dt \\
&= \frac{1}{2} \int_{S^{n-1}} d\mu_0(\theta) \int_{\mathbb{R}} |t|^{-1} \phi(t\theta) dt
\end{aligned} \tag{5.11}$$

since A, a, b are arbitrary in (5.10).

Using $\mu = -\left(\ln\|x\|\right)^{\wedge}$, we get

$$\langle \left(\ln\|x\|\right)^{\wedge}(\xi), \phi\rangle = -\frac{1}{2}\int_{S^{n-1}} d\mu_0(\theta)\int_{\mathbb{R}} |t|^{-1}\phi(t\theta)dt.$$

Define a new measure $\tilde{\mu}_0 = (2\pi)^n\mu_0$. By (5.7), (5.11) and the connection between the Fourier and Radon transforms

$$\begin{aligned}
\langle \ln\|x\|, \hat{\phi}(x)\rangle &= -\frac{1}{2(2\pi)^n}\int_{S^{n-1}} d\tilde{\mu}_0(\theta)\int_{\mathbb{R}} |t|^{-1}\phi(t\theta)dt \\
&= \int_{S^{n-1}} \langle \ln|z|, \mathcal{R}\hat{\phi}(\theta; z)\rangle d\tilde{\mu}_0(\theta) \\
&= \int_{S^{n-1}} d\tilde{\mu}_0(\theta)\int_{\mathbb{R}} \ln|z| \left(\int_{(x,\theta)=z} \hat{\phi}(x)dx\right) dz \\
&= \int_{S^{n-1}} d\tilde{\mu}_0(\theta)\int_{\mathbb{R}^n} \ln|(x,\theta)|\,\hat{\phi}(x)dx
\end{aligned}$$

Thus, we have proved that for any even test function ϕ supported outside of the origin

$$\langle (\ln\|x\|)^{\wedge}, \phi\rangle = \left\langle \left(\int_{S^{n-1}} \ln|(x,\theta)|\, d\tilde{\mu}_0(\theta)\right)^{\wedge}, \phi\right\rangle.$$

Therefore the distributions $\ln\|x\|$ and $\int_{S^{n-1}} \ln|(x,\theta)|\, d\tilde{\mu}_0(\theta)$ can differ only by a polynomial. Clearly, this polynomial cannot contain terms homogeneous of degree different from zero, so it is a constant.

□

Remark 5.3.2. *Let K be an infinitely smooth body. From the proof of the previous theorem it follows that the measure μ from Definition 5.2.1 is equal to restriction of the Fourier transform of $\ln\|x\|_K$ to the sphere. In the next section we are going to prove that this is a function, therefore*

$$d\mu(\xi) = -\frac{1}{(2\pi)^n}\left(\ln\|x\|_K\right)^{\wedge}(\xi)d\xi.$$

In particular, since μ is a probability measure, for any infinitely smooth body K we get

$$-\frac{1}{(2\pi)^n}\int_{S^{n-1}} \left(\ln \|x\|_K\right)^\wedge (\xi) d\xi = 1.$$

5.4 A geometric characterization of subspaces of L_0.

If K has an infinitely smooth boundary then the function $A_{K,\xi}(t)$ is an infinitely differentiable function of t in some neighborhood of zero and as was shown in [GKS] the fractional derivatives of $A_{K,\xi}(t)$ can be computed in terms of the Fourier transform of the Minkowski functional raised to certain powers. Namely, for $q \in \mathbb{C}$, $q \neq n-1$,

$$A_{K,\xi}^{(q)}(0) = \frac{\cos\frac{q\pi}{2}}{\pi(n-q-1)} \left(\|x\|_K^{-n+q+1}\right)^\wedge (\xi), \tag{5.12}$$

and, in particular, $\left(\|x\|_K^{-n+q+1}\right)^\wedge$ is a continuous function on $\mathbb{R}^n \setminus \{0\}$. Here we extend $A_{K,\xi}^{(q)}(0)$ from the sphere to the whole $\mathbb{R}^n$ as a homogeneous function of the variable ξ of degree $-q-1$. Note that $\langle A_{K,\xi}^{(q)}(0), \phi\rangle$ is an analytic function of q for any fixed test function ϕ.

Since the right-hand side of formula (5.12) is not defined for $q = n-1$, in our next Theorem we use a limiting argument to extend this formula to the case $q = n-1$.

Let $\mathcal{D}$ be an open set in $\mathbb{R}^n$, f, g two distributions. We say that $f = g$ on $\mathcal{D}$ if $\langle f, \phi\rangle = \langle g, \phi\rangle$ for any test function ϕ with compact support in $\mathcal{D}$.

Theorem 5.4.1. *Let K be an infinitely smooth origin symmetric star body in*

$\mathbb{R}^n$. *Extend* $A_{K,\xi}^{(n-1)}(0)$ *to a homogeneous function of degree* $-n$ *of the variable* $\xi \in \mathbb{R}^n \setminus \{0\}$. *Then* $(\ln \|\cdot\|_K)^\wedge$ *is a continuous function on* $\mathbb{R}^n \setminus \{0\}$ *and*

$$A_{K,\xi}^{(n-1)}(0) = -\frac{\cos(\pi(n-1)/2)}{\pi} \left(\ln \|\cdot\|_K\right)^\wedge (\xi), \tag{5.13}$$

as distributions (of the variable ξ*) acting on test functions with compact support outside of the origin. In particular,*

i) if n *is odd*

$$\left(\ln \|x\|_K\right)^\wedge (\xi) = (-1)^{(n+1)/2} \pi A_{K,\xi}^{(n-1)}(0), \quad \xi \in \mathbb{R}^n \setminus \{0\}$$

ii) if n *is even, then for* $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$\left(\ln \|x\|_K\right)^\wedge (\xi) = a_n \int_0^\infty \frac{A_\xi(z) - A_\xi(0) - A_\xi''(0)\frac{z^2}{2} - \dots - A_\xi^{n-2}(z)\frac{z^{n-2}}{(n-2)!}}{z^n} dz,$$

where $a_n = 2(-1)^{n/2+1}(n-1)!$

Proof. Let us start with the case where n is odd. Let ϕ be a test function supported outside of the origin.

Using formula (5.12) for q close to $n-1$, we have

$$\begin{aligned}
\langle A_{K,\xi}^{(q)}(0), \phi(\xi)\rangle &= \frac{\cos(\pi q/2)}{\pi(n-q-1)} \langle \left(\|x\|^{-n+q+1}\right)^\wedge (\xi), \phi(\xi)\rangle \\
&= \frac{\cos(\pi q/2)}{\pi(n-q-1)} \langle \|x\|^{-n+q+1}, \hat{\phi}(x)\rangle \\
&= \frac{\cos(\pi q/2)}{\pi(n-q-1)} \int_{\mathbb{R}^n} \|x\|^{-n+q+1} \hat{\phi}(x) dx \\
&= \frac{\cos(\pi q/2)}{\pi(n-q-1)} \int_{\mathbb{R}^n} \left(\|x\|^{-n+q+1} - 1\right) \hat{\phi}(x) dx \\
&\quad + \frac{\cos(\pi q/2)}{\pi(n-q-1)} \int_{\mathbb{R}^n} \hat{\phi}(x) dx \\
&= \frac{\cos(\pi q/2)}{\pi} \int_{\mathbb{R}^n} \frac{\|x\|^{-n+q+1} - 1}{n-q-1} \hat{\phi}(x) dx,
\end{aligned}$$

since $\int_{\mathbb{R}^n} \hat{\phi}(x)dx = (2\pi)^n\phi(0) = 0$. Taking the limit of both sides as $q \to n-1$, we get

$$\langle A_{K,\xi}^{(n-1)}(0), \phi(\xi)\rangle = \Big\langle -\frac{\cos(\pi(n-1)/2)}{\pi} \left(\ln \|x\|\right)^\wedge(\xi), \phi(\xi)\Big\rangle$$

since

$$\begin{aligned}\lim_{q\to n-1} \int_{\mathbb{R}^n} \frac{\|x\|^{-n+q+1}-1}{n-q-1}\hat{\phi}(x)dx &= -\int_{\mathbb{R}^n} \ln\|x\|\hat{\phi}(x)dx \\ &= \langle -\left(\ln\|x\|\right)^\wedge(\xi), \phi(\xi)\rangle.\end{aligned}$$

When n is odd the formula of i) follows immediately.

When n is even, both sides of (5.13) are equal to zero, and we repeat the reasoning from Theorem 1 in [GKS]. Divide both sides of (5.12) by $\cos(\frac{\pi q}{2})$

$$\Big\langle \frac{\left(\|x\|_K^{-n+q+1}\right)^\wedge(\xi)}{(n-q-1)}, \phi(\xi)\Big\rangle = \pi\Big\langle \frac{A_{K,\xi}^{(q)}(0)}{\cos\frac{\pi q}{2}}, \phi(\xi)\Big\rangle$$

and take the limit of both sides when $q \to n-1$.

We have already proved that

$$\lim_{q\to n-1} \Big\langle \frac{\left(\|x\|_K^{-n+q+1}\right)^\wedge(\xi)}{(n-q-1)}, \phi(\xi)\Big\rangle = \langle -\left(\ln\|x\|\right)^\wedge(\xi), \phi(\xi)\rangle$$

for any test function ϕ supported outside of the origin.

To compute the limit of $\dfrac{A_{K,\xi}^{(q)}(0)}{\cos\frac{q\pi}{2}}$ we use the definition of fractional derivatives in exactly the same way as it was done in [GKS, Theorem 1].

$$\lim_{q\to n-1} \Gamma(-q)A_{K,\xi}^{(q)}(0) = \int_0^\infty \frac{A_\xi(z) - A_\xi(0) - A_\xi''(0)\frac{z^2}{2} - \ldots - A_\xi^{n-2}(z)\frac{z^{n-2}}{(n-2)!}}{z^n}dz$$

and

$$\lim_{q\to n-1} \Gamma(-q)\sin\frac{(q+1)\pi}{2} = \frac{\pi}{2}(-1)^{n/2}\frac{1}{(n-1)!}.$$

Combining these two formulas we get the formula in the statement ii) of the Theorem. $\square$

An immediate application of Theorem 5.4.1 is

Corollary 5.4.2. *Let K be an infinitely smooth body in $\mathbb{R}^n$. Then*

i) if n is odd, $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0 if and only if

$$(-1)^{(n-1)/2} A_{K,\xi}^{(n-1)}(0) \geq 0, \quad \forall \xi \in S^{n-1};$$

ii) if n is even, $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0 if and only if, for every $\xi \in S^{n-1}$,

$$(-1)^{(n+2)/2} \int_0^\infty \frac{A_\xi(z) - A_\xi(0) - A_\xi''(0)\frac{z^2}{2} - \ldots - A_\xi^{n-2}(z)\frac{z^{n-2}}{(n-2)!}}{z^n} dz \geq 0.$$

Corollary 5.4.3. *Every 3-dimensional normed space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0.*

Proof. The unit ball K of a normed space is an origin-symmetric convex body. First assume that K is infinitely smooth. By Brunn's theorem the central section of a convex body has maximal volume among all sections perpendicular to a given direction. Therefore, for any ξ the function $A_{K,\xi}(t)$ attains its maximum at $t = 0$, hence $A''_{K,\xi}(0) \leq 0$. So, by Theorem 5.4.1, for smooth convex bodies in $\mathbb{R}^3$ the distribution $-(\ln \|x\|)^\wedge$ is positive outside of the origin, and our result follows from Theorem 5.3.1. For general convex bodies the result follows from the facts that any convex body can be approximated by smooth convex bodies and that positive definiteness is preserved under limits. In fact, let $\{K_i\}$ be a sequence of infinitely smooth convex bodies that approach K in the radial metric. Then for any non-negative test function ϕ supported outside of the origin we have

$$-\int_{\mathbb{R}^n} \ln \|x\|_{K_i} \hat{\phi}(x) dx = \langle -\ln \|x\|_{K_i}, \hat{\phi}(x) \rangle = \langle -(\ln \|x\|_{K_i})^\wedge(\xi), \phi(\xi) \rangle \geq 0$$

Since K_i approximate K there is a constant $C > 0$, such that

$$|\ln \|x\|_{K_i}| \leq C + |\ln |x|_2| \,,$$

therefore the functions $|\ln \|x\|_{K_i} \hat{\phi}(x)|$ are majorated by an integrable function $(C + |\ln |x|_2|)|\hat{\phi}(x)|$ and by the Lebesgue Dominated Convergence Theorem we get

$$\begin{aligned} -\lim_{i\to\infty} \int_{\mathbb{R}^n} \ln \|x\|_{K_i} \hat{\phi}(x) dx &= -\int_{\mathbb{R}^n} \ln \|x\|_K \hat{\phi}(x) dx \\ &= \langle -(\ln \|x\|_K)^\wedge(\xi), \phi(\xi)\rangle \geq 0 \end{aligned}$$

$\square$

Our next result shows that that the previous statement is no longer true in $\mathbb{R}^n$, $n \geq 4$.

Theorem 5.4.4. *There exists an origin-symmetric convex body K in $\mathbb{R}^n$, $n \geq 4$ so that the space $(\mathbb{R}^n, \|\cdot\|_K)$ does not embed in L_0.*

Proof. It is enough to construct a convex body for which the distribution $-(\ln \|x\|)^\wedge$ is not positive. The construction will be similar to that from [GKS].

Define $f_N(x) = (1 - x^2 - Nx^4)^{1/3}$, let $a_N > 0$ be such that $f_N(a_N) = 0$ and $f_N(x) > 0$ on the interval $(0, a_N)$. Define a body K in $\mathbb{R}^4$ by

$$K = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \, x_4 \in [-a_N, a_N] \text{ and } \sqrt{x_1^2 + x_2^2 + x_3^2} \leq f_N(x_4)\}.$$

The body K is strictly convex and infinitely smooth. By Theorem 5.4.1,

$$-\left(\ln \|x\|_K\right)^\wedge(\xi) = 12 \int_0^\infty \frac{A_\xi(z) - A_\xi(0) - A''_\xi(0)\frac{z^2}{2}}{z^4} dz.$$

The function $A_{K,\xi}$ can easily be computed:

$$A_{K,\xi}(x) = \frac{4\pi}{3}(1 - x^2 - Nx^4).$$

We have

$$\int_0^\infty \frac{A_\xi(z) - A_\xi(0) - A''_\xi(0)\frac{z^2}{2}}{z^4} dz = \frac{4\pi}{3}(-Na_N + \frac{1}{a_N} - \frac{1}{3a_N^3}).$$

The latter is negative for N large enough, because $N^{1/4} \cdot a_N \to 1$ as $N \to \infty$.

$\square$

5.5 Addition in L_0

It is clear from the definition that the class of bodies K for which $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0 is closed with respect to multiplicative summation, i.e. if two spaces $(\mathbb{R}^n, \|\cdot\|_{K_1})$ and $(\mathbb{R}^n, \|\cdot\|_{K_2})$ embed in L_0 and $K = K_1 +_0 K_2$, then $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0. In this section we are going to prove that the unit ball of every space $(\mathbb{R}^n, \|\cdot\|_K)$ that embeds in L_0 can be obtained from the Euclidean ball by means of multiplicative summation, linear transformations and closure in the radial metric, i.e. it can be approximated in the radial metric by multiplicative sums of ellipsoids.

Consider the set of bodies K for which $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0. As mentioned above, this set is closed with respect to multiplicative summation, also from the proof of Corollary 5.4.3 it follows that this set is closed with respect to limits in the radial metric. Let us show that it is closed with respect to linear transformations. Suppose that $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0. By Theorem 5.3.1 $(\ln \|x\|_K)^\wedge$ is a negative distribution outside of the origin. Let T be an invertible linear transformation in $\mathbb{R}^n$, then for any non-negative test function ϕ with support outside of the origin, we have

$$\begin{aligned}
\langle (\ln \|Tx\|_K)^\wedge, \phi \rangle &= \langle \ln \|Tx\|_K, \hat{\phi}(x) \rangle \\
&= \int_{\mathbb{R}^n} \ln \|Tx\|_K \hat{\phi}(x) dx \\
&= |\det T|^{-1} \int_{\mathbb{R}^n} \ln \|x\|_K \hat{\phi}(T^{-1}x) dx \\
&= \int_{\mathbb{R}^n} \ln \|x\|_K \left(\phi(T^* y)\right)^\wedge (x) dx \\
&= \langle \ln \|x\|_K, \left(\phi(T^* y)\right)^\wedge (x) \rangle, \\
&= \langle (\ln \|x\|_K)^\wedge (y), \phi(T^* y) \rangle \leq 0.
\end{aligned}$$

So $(\ln \|Tx\|_K)^\wedge$ is a negative distribution outside of the origin. By Theorem 5.3.1, $(\mathbb{R}^n, \|\cdot\|_{TK})$ embeds in L_0.

Moreover, if $(\ln \|x\|)^\wedge$ is a function, then

$$(\ln \|Tx\|)^\wedge (y) = |\det T|^{-1} (\ln \|x\|)^\wedge ((T^*)^{-1} y). \tag{5.14}$$

To prove the main result of this section we need a few lemmas. For a fixed $x \in S^{n-1}$, let $E_{a,b}(x)$ be an ellipsoid with the norm

$$\|\theta\|_{E_{a,b}(x)} = \left(\frac{(x,\theta)^2}{a^2} + \frac{1-(x,\theta)^2}{b^2} \right)^{1/2}, \quad \text{for } \theta \in S^{n-1}.$$

Lemma 5.5.1. *For all $\theta \in S^{n-1}$,*

$$\left(\ln \|\xi\|_{E_{a,b}(x)}\right)^\wedge_\xi (\theta) = -\frac{2^{n-1}\pi^{n/2}\Gamma(n/2)}{a^{n-1}b} \|\theta\|^{-n}_{E_{b,a}(x)}.$$

Proof. For $-n < \lambda < 0$ the following formula holds (see [GS, p.192]):

$$\left(|x|_2^\lambda\right)^\wedge (\xi) = 2^{\lambda+n}\pi^{n/2} \frac{\Gamma((\lambda+n)/2)}{\Gamma(-\lambda/2)} |\xi|_2^{-\lambda-n}.$$

Dividing both sides by λ, using the formula $x\Gamma(x) = \Gamma(1+x)$ and sending $\lambda \to 0$ we get

$$(\ln |x|_2)^\wedge (\xi) = -2^{n-1}\pi^{n/2}\Gamma(n/2)|\xi|_2^{-n},$$

as distributions outside of the origin. Note that, by rotation, it is enough to prove Lemma for the ellipsoids $E_{a,b}(x)$ with $x = (0, 0, \dots, 0, 1)$.

$$\|\xi\|_{E_{a,b}(x)} = \left(\frac{\xi_n^2}{a^2} + \frac{\xi_1^2 + \cdots + \xi_{n-1}^2}{b^2}\right)^{1/2}.$$

Since this norm can be obtained from the Euclidean norm by an obvious linear transformation, one can use formula (5.14) to get

$$\begin{aligned}\left(\ln \|\xi\|_{E_{a,b}(x)}\right)_\xi^\wedge (\theta) &= -2^{n-1}\pi^{n/2}\Gamma(n/2)ab^{n-1}\|\theta\|_{E_{1/a,1/b}(x)}^{-n} \\ &= -\frac{2^{n-1}\pi^{n/2}\Gamma(n/2)}{a^{n-1}b}\|\theta\|_{E_{b,a}(x)}^{-n}.\end{aligned}$$

$\square$

Lemma 5.5.2. *Let K be a star body, then $\ln \|x\|_K$ can be approximated in the space $C(S^{n-1})$ by the functions of the form*

$$f_{a,b}(x) = \frac{1}{|S^{n-1}|a^{n-1}b}\int_{S^{n-1}} \ln \|\theta\|_K \|\theta\|_{E_{b,a}(x)}^{-n} d\theta, \tag{5.15}$$

as $a \to 0$ and b is fixed.

Proof. The proof is similar to that of [GW, Lemma 2]. First, note that the space $\mathbb{R}^n$ with the Euclidean norm embeds in L_0, so $(\mathbb{R}^n, \|\cdot\|_E)$ embeds in L_0 for any ellipsoid E with center at the origin. Therefore, by Remark 5.3.2 and Lemma 5.5.1 we get

$$\int_{S^{n-1}} \frac{1}{|S^{n-1}|a^{n-1}b}\|\theta\|_{E_{b,a}(x)}^{-n} d\theta = 1,$$

for all values of a and b. From now on b will be fixed.

We have

$$\begin{aligned}
&\left|\ln\|x\|_K - \frac{1}{|S^{n-1}|a^{n-1}b}\int_{S^{n-1}}\ln\|\theta\|_K\|\theta\|^{-n}_{E_{b,a}(x)}d\theta\right| \\
&\leq \frac{1}{|S^{n-1}|a^{n-1}b}\int_{S^{n-1}}\left|\ln\|x\|_K - \ln\|\theta\|_K\right|\|\theta\|^{-n}_{E_{b,a}(x)}d\theta \\
&= \frac{1}{|S^{n-1}|a^{n-1}b}\int_{|(x,\theta)|\geq\delta}\left|\ln\|x\|_K - \ln\|\theta\|_K\right|\|\theta\|^{-n}_{E_{b,a}(x)}d\theta \\
&+\frac{1}{|S^{n-1}|a^{n-1}b}\int_{|(x,\theta)|<\delta}\left|\ln\|x\|_K - \ln\|\theta\|_K\right|\|\theta\|^{-n}_{E_{b,a}(x)}d\theta \\
&= I_1 + I_2.
\end{aligned}$$

For the first integral I_1 use the uniform continuity of $\ln\|x\|_K$ on the sphere. For any given $\epsilon > 0$ there exists $\delta \in (0,1)$, δ close to 1, so that $|(x,\theta)| \geq \delta$ implies $\left|\ln\|x\|_K - \ln\|\theta\|_K\right| < \epsilon/2$. Therefore

$$\begin{aligned}
I_1 &= \frac{1}{|S^{n-1}|a^{n-1}b}\int_{|(x,\theta)|\geq\delta}\left|\ln\|x\|_K - \ln\|\theta\|_K\right|\|\theta\|^{-n}_{E_{a,b}(x)}d\theta \\
&\leq \frac{\epsilon}{2}\left[\frac{1}{|S^{n-1}|a^{n-1}b}\int_{|(x,\theta)|\geq\delta}\|\theta\|^{-n}_{E_{a,b}(x)}d\theta\right] \leq \frac{\epsilon}{2}.
\end{aligned}$$

Now fix δ chosen above and estimate the integral I_2 as follows

$$\begin{aligned}
I_2 &= \frac{1}{|S^{n-1}|a^{n-1}b}\int_{|(x,\theta)|<\delta}\left|\ln\|x\|_K - \ln\|\theta\|_K\right|\|\theta\|^{-n}_{E_{b,a}(x)}d\theta \\
&\leq \frac{C(n,b,K)}{a^{n-1}}\int_{|(x,\theta)|<\delta}\|\theta\|^{-n}_{E_{b,a}(x)}d\theta,
\end{aligned}$$

where

$$C(n,b,K) = \frac{2\max_{S^{n-1}}|\ln\|x\|_K|}{|S^{n-1}|b}.$$

For the latter integral we use an elementary formula (see e.g. [Ko12, Section 6.4])

$$\int_{|(x,\theta)|<\delta} f((x,\theta))d\theta = |S^{n-2}|\int_{-\delta}^{\delta}(1-t^2)^{(n-3)/2}f(t)dt, \qquad \text{for } x \in S^{n-1}.$$

Now,

$$\begin{aligned}
I_2 &\leq \frac{C(n,b,K)|S^{n-2}|}{a^{n-1}} \int_{-\delta}^{\delta} (1-t^2)^{(n-3)/2} \left(\frac{t^2}{b^2} + \frac{1-t^2}{a^2}\right)^{-n/2} dt \\
&\leq \frac{C(n,b,K)|S^{n-2}|}{a^{n-1}} \int_{-\delta}^{\delta} (1-t^2)^{(n-3)/2} \left(\frac{1-t^2}{a^2}\right)^{-n/2} dt \\
&= a \cdot C(n,b,K)|S^{n-2}| \int_{-\delta}^{\delta} (1-t^2)^{-3/2} dt \\
&\leq a \cdot C(n,b,K)|S^{n-2}| \frac{2\delta}{(1-\delta^2)^{3/2}}.
\end{aligned}$$

Now we can choose a so small that $I_2 \leq \epsilon/2$. □

Lemma 5.5.3. *If μ is a probability measure on S^{n-1} and $a, b > 0$, then the function*

$$f(x) = \int_{S^{n-1}} \ln \|\xi\|_{E_{a,b}(x)} d\mu(\xi)$$

can be approximated in $C(S^{n-1})$ by the sums of the form

$$\sum_{i=1}^{m} \frac{1}{p_i} \ln \|x\|_{E_i},$$

where E_1,...,E_m are ellipsoids and $1/p_1 + \cdots + 1/p_m = 1$.

Proof. Let $\sigma > 0$ be a small number and choose a finite covering of the sphere by spherical σ-balls $B_\sigma(\eta_i) = \{\eta \in S^{n-1} : |\eta - \eta_i| < \sigma\}$, $\eta_i \in S^{n-1}$, $i = 1, \dots, m = m(\delta)$. Define

$$\widetilde{B}_\sigma(\xi_1) = B_\sigma(\xi_1)$$

and

$$\widetilde{B}_\sigma(\xi_i) = B_\sigma(\xi_i) \setminus \bigcup_{j=1}^{i-1} B_\sigma(\xi_j), \qquad \text{for } i = 2, ..., m.$$

Let $1/p_i = \mu(\widetilde{B}_\sigma(\xi_i))$. Clearly, $1/p_1 + \cdots + 1/p_m = 1$.

Let $\rho(E_{a,b}(\xi), x)$ be the radial function of the ellipsoid $E_{a,b}(\xi)$, that is

$$\rho(E_{a,b}(\xi), x) = \|x\|^{-1}_{E_{a,b}(\xi)}.$$

Note that $\rho(E_{a,b}(\xi),x)=\rho(E_{a,b}(x),\xi)$, therefore

$$|\rho(E_{a,b}(\xi),x)-\rho(E_{a,b}(\theta),x)|\leq C_{a,b}|\xi-\theta|,$$

with a constant $C_{a,b}$ that depends on a and b. Also note that, since we consider a close to zero and b fixed, we may assume

$$a\leq\rho(E_{a,b}(\xi),x)\leq b,\qquad x\in S^{n-1}.$$

Then,

$$\begin{aligned}
&\left|\int_{S^{n-1}}\ln\rho(E_{a,b}(\xi),x)d\mu(\xi)-\sum_{i=1}^{m}\frac{1}{p_i}\ln\rho(E_{a,b}(\xi_i),x)\right|=\\
&=\left|\sum_{i=1}^{m}\left(\int_{\widetilde{B}_\sigma(\xi_i)}\ln\rho(E_{a,b}(\xi),x)d\mu(\xi)-\int_{\widetilde{B}_\sigma(\xi_i)}\ln\rho(E_{a,b}(\xi_i),x)d\mu(\xi)\right)\right|\leq\\
&\leq\sum_{i=1}^{m}\int_{\widetilde{B}_\sigma(\xi_i)}\left|\ln\frac{\rho(E_{a,b}(\xi),x)}{\rho(E_{a,b}(\xi_i),x)}\right|d\mu(\xi)\leq\\
&\leq\sum_{i=1}^{m}\int_{\widetilde{B}_\sigma(\xi_i)}\left|\ln\frac{\rho(E_{a,b}(\xi_i),x)+[\rho(E_{a,b}(\xi),x)-\rho(E_{a,b}(\xi_i),x)]}{\rho(E_{a,b}(\xi_i),x)}\right|d\mu(\xi)\leq\\
&\leq\sum_{i=1}^{m}\int_{\widetilde{B}_\sigma(\xi_i)}\left|\ln(1\pm C'_{a,b}|\xi-\xi_i|)\right|d\mu(\xi)\leq\\
&\leq\left|\ln(1\pm C'_{a,b}\sigma)\right|,
\end{aligned}$$

and the result follows since σ is arbitrarily small.

$\square$

Now we are ready to prove the following

Theorem 5.5.4. *Let K be an origin symmetric star body in $\mathbb{R}^n$. The space $(\mathbb{R}^n,\|\cdot\|_K)$ embeds in L_0 if and only if $\|x\|_K$ is the limit (in the radial metric) of finite products $\|x\|_{E_1}^{1/p_1}\cdots\|x\|_{E_m}^{1/p_m}$, where E_1,...,E_m are ellipsoids and $1/p_1+\cdots+1/p_m=1$.*

Proof. The "if" part is a consequence of the fact that L_0 is closed with respect to the three operations as discussed above.

The proof of "only if" part easily follows the Lemmas we have proved.

Suppose that $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0 with the corresponding probability measure μ on S^{n-1} and constant C. By Remark 5.3.2, $(\mathbb{R}^n, \|\cdot\|_{E_{a,b}(x)})$ embeds in L_0 with the measure $-\frac{1}{(2\pi)^n}\left(\ln\|x\|_E\right)^\wedge(\theta)d\theta$ and some constant $C_{E_{a,b}}$. Note, this constant does not depend on x. We have

$$\begin{aligned}
&\int_{S^{n-1}} \ln\|\xi\|_{E_{a,b}(x)} d\mu(\xi)\\
=\;& \int_{S^{n-1}}\int_{S^{n-1}} \ln|(\xi,\theta)| \left(-\frac{1}{(2\pi)^n}\right)\left(\ln\|x\|_{E_{a,b}(x)}\right)^\wedge(\theta)d\theta d\mu(\xi) + C_{E_{a,b}}\\
=\;& \int_{S^{n-1}}\left[\int_{S^{n-1}} \ln|(\xi,\theta)|d\mu(\xi) + C_K\right]\left(-\frac{1}{(2\pi)^n}\right)\left(\ln\|x\|_{E_{a,b}(x)}\right)^\wedge(\theta)d\theta\\
&+C_{E_{a,b}} - C_K\\
=\;& \int_{S^{n-1}} \ln\|\theta\|_K\left(-\frac{1}{(2\pi)^n}\right)\left(\ln\|x\|_{E_{a,b}(x)}\right)^\wedge(\theta)d\theta + C_{E_{a,b}} - C_K\\
=\;& \int_{S^{n-1}} \ln\|\theta\|_K\left(-\frac{1}{(2\pi)^n}\right)\left(\ln\|x\|_{E_{a,b}(x)}\right)^\wedge(\theta)d\theta + C_{E_{a,b}} - C_K\\
=\;& \frac{1}{|S^{n-1}|a^{n-1}b}\int_{S^{n-1}} \ln\|\theta\|_K\|\theta\|^{-n}_{E_{b,a}(x)}d\theta + C_{E_{a,b}} - C_K
\end{aligned}$$

In Lemma 5.5.2 we proved that $\ln\|x\|_K$ can be uniformly approximated by the integrals of the form

$$\frac{1}{|S^{n-1}|a^{n-1}b}\int_{S^{n-1}} \ln\|\theta\|_K\|\theta\|^{-n}_{E_{b,a}(x)}d\theta,$$

as $a \to 0$. Therefore, using the previous calculations, one can see that $\ln\|x\|_K$ can be uniformly approximated by

$$\int_{S^{n-1}} \ln\|\xi\|_{E_{a,b}(x)}d\mu(\xi) + C'.$$

Hence, by Lemma 5.5.3, $\ln \|x\|_K$ can be uniformly approximated by the sums

$$\sum_{i=1}^{m} \frac{1}{p_i} \ln \|x\|_{E_i} + C'.$$

Replacing E_1 by another ellipsoid E_1' given by $\|x\|_{E_1'}^{1/p_1} = e^{C'} \|x\|_{E_1}^{1/p_1}$, we get the statement of the Theorem.

□

Corollary 5.5.5. *Any convex body in $\mathbb{R}^3$ can be obtained from the Euclidean unit ball by means of three operations: linear transformations, multiplicative addition and closure in the radial metric.*

Proof. As was proved in Theorem 5.5.4, any convex body can be approximated by the finite products of the type $\|x\|_{E_1}^{1/p_1} \cdots \|x\|_{E_m}^{1/p_m}$. Since any number $1/p$ can be approximated by the sums

$$\frac{1}{2^{i_1}} + \frac{1}{2^{i_2}} + \cdots + \frac{1}{2^{i_k}},$$

the result follows.

□

A proof similar to that of Theorem 5.5.4 can be used to show that the previous theorem holds for p-summation with $-1 < p < 1$, $p \neq 0$, in place of the multiplicative summation.

Theorem 5.5.6. *Let K be an origin symmetric star body in $\mathbb{R}^n$. The space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_p, $-1 < p < 1$, $p \neq 0$ if and only if $\|x\|_K^p$ is the limit (in the radial topology) of finite sums $\|x\|_{E_1}^p + \cdots + \|x\|_{E_m}^p$, where $E_1,\ldots,E_m$ are ellipsoids.*

5.6 Confirming the place of L_0 in the scale of L_p-spaces.

In this section we establish the relations between embedding in L_0 and in L_p with $p \neq 0$, which confirm the place of L_0 between L_p with $p > 0$ and $p < 0$. We are going to use the following result from [Ko7, Theorem 1]:

Theorem 5.6.1. *An n-dimensional homogeneous space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_{-p}, $p \in (0, n)$ if and only if $\|x\|_K^{-p}$ is a positive definite distribution.*

We also use a well-known result of P.Levy (see [BL, p.189], also [BDK] for the infinite dimensional case):

Theorem 5.6.2. *A space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_p, $p \in (0, 2]$ if and only if the function $\exp(-\|x\|_K^p)$ is positive definite.*

Now we are ready to prove

Theorem 5.6.3. *Let K be an origin symmetric star body in $\mathbb{R}^n$. If the space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0 then it also embeds in L_{-p}, $0 < p < n$.*

Proof. By Theorem 5.5.4, $\|x\|_K$ is the limit of finite products $\|x\|_{E_1}^{1/p_1} \cdots \|x\|_{E_m}^{1/p_m}$. Consider $\|x\|_K^{-p}$ for $0 < p < n$. It is the limit of the products of the form $\|x\|_{E_1}^{-p/p_1} \cdots \|x\|_{E_m}^{-p/p_m}$. Using the formula

$$\|x\|^{-p} = \frac{2}{\Gamma(p/2)} \int_0^\infty t^{p-1} \exp(-t^2\|x\|^2) dt,$$

we get

$$\|x\|_{E_1}^{-p/p_1}\cdots\|x\|_{E_m}^{-p/p_m} = C\int_0^\infty\cdots\int_0^\infty t_1^{p/p_1-1}\cdots t_m^{p/p_m-1}\times$$
$$\times\exp(-t_1^2\|x\|_{E_1}^2-\cdots-t_m^2\|x\|_{E_m}^2)dt_1\cdots dt_m,$$

where

$$C=\frac{2^m}{\Gamma(p/2p_1)\cdots\Gamma(p/2p_m)}.$$

Therefore, for any non-negative test function ϕ we have

$$\langle(\|x\|_{E_1}^{-p/p_1}\cdots\|x\|_{E_m}^{-p/p_m})^\wedge(\xi),\phi(\xi)\rangle=\langle\|x\|_{E_1}^{-p/p_1}\cdots\|x\|_{E_m}^{-p/p_m},\hat{\phi}(x)\rangle=$$
$$=C\int_0^\infty\cdots\int_0^\infty t_1^{p/p_1-1}\cdots t_m^{p/p_m-1}\times$$
$$\times\langle\exp(-t_1^2\|x\|_{E_1}^2-\cdots-t_m^2\|x\|_{E_m}^2),\hat{\phi}(x)\rangle dt_1\cdots dt_m=$$
$$=C\int_0^\infty\cdots\int_0^\infty t_1^{p/p_1-1}\cdots t_m^{p/p_m-1}\times$$
$$\times\langle(\exp(-t_1^2\|x\|_{E_1}^2-\cdots-t_m^2\|x\|_{E_m}^2))^\wedge(\xi),\phi(\xi)\rangle dt_1\cdots dt_m.$$

We claim that the latter expression is non-negative. Indeed, $(\mathbb{R}^n,\|x\|_E)$ embeds in L_2 for any ellipsoid, therefore the 2-sum of ellipsoids $t_1^2\|x\|_{E_1}^2+\cdots+t_m^2\|x\|_{E_m}^2$ embeds in L_2, and hence by Theorem 5.6.2, the function $\exp(-t_1^2\|x\|_{E_1}^2-\cdots-t_m^2\|x\|_{E_m}^2)$ is positive definite. Now the fact that $\langle(\|x\|_K^{-p})^\wedge,\phi\rangle\geq 0$ follows by an approximation argument, as in Corollary 5.4.3.

□

Theorem 5.6.4. *Let K be an origin symmetric star body in $\mathbb{R}^n$. If the space $(\mathbb{R}^n,\|\cdot\|_K)$ embeds in L_{-p} for every $p\in(0,\epsilon)$, then it also embeds in L_0.*

Proof. The space $(\mathbb{R}^n,\|\cdot\|_K)$ embeds in L_{-p}, so by Theorem 5.6.1 the distribution $\|x\|^{-p}$ is positive definite. Then for every non-negative test function ϕ supported outside of the origin,

$$\begin{aligned} -\int_{\mathbb{R}^n} \ln\|x\|\hat{\phi}(x)dx &= \lim_{p\to 0}\frac{1}{p}\int_{\mathbb{R}^n}(\|x\|^{-p}-1)\hat{\phi}(x)dx \\ &= \lim_{p\to 0}\frac{1}{p}\int_{\mathbb{R}^n}\|x\|^{-p}\hat{\phi}(x)dx \geq 0. \end{aligned}$$

The result follows from Theorem 5.3.1.

□

Theorem 5.6.5. *There are normed spaces that embed in L_0, but do not embed in L_p for $p > 0$.*

Proof. As proved above, every 3-dimensional normed space embeds in L_0, hence l_q^3 with $q > 2$ does. On the other hand, l_q^3, $q > 2$ does not embed in L_p for $0 < p \leq 2$ (see [Ko1]).

□

Let us also mention that one can use the approach of [KK1] to produce examples in the same spirit. It follows from [KK1], Proposition 3.5 that $\mathbb{R}\oplus_2 \ell_1$ does not embed isometrically into L_p for $p > 0$; hence neither does $\mathbb{R}\oplus_2 \ell_1^n$ for large enough n.

Proposition 5.6.6. *For any $n \in \mathbb{N}$ the space $\mathbb{R}\oplus_2 \ell_1^n$ embeds in L_0.*

Proof. Let $(f_n)_{n=1}^\infty$ be a sequence of functions on some probability space which are independent and 1-stable symmetric, so that $\mathbb{E}(e^{itf_j}) = e^{-|t|}$ (i.e. the f_j have the Cauchy distribution). Then it is clear that

$$\mathbb{E}\ln|\sum_{j=1}^n a_j f_j| = \ln\sum_{j=1}^n |a_j|.$$

Indeed this follows from the fact that

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\ln|x|}{1+x^2}dx=0.$$

On the other hand if $f=\sum_{j=1}^{n}a_jf_j$ where $\sum_{j=1}^{n}|a_j|=1$ then f has the Cauchy distribution and so has the same distribution as g_1/g_2 where g_1, g_2 are independent normalized Gaussians. Hence

$$\begin{aligned}\mathbb{E}\ln|a+bf| &= \mathbb{E}(\ln|ag_2+bg_1|-\ln|g_2|)\\ &= \ln(a^2+b^2)^{\frac{1}{2}}.\end{aligned}$$

Now for any $a_0, a_1, \ldots, a_n \in \mathbb{R}$ we have

$$\mathbb{E}|a_0+\sum_{j=1}^{n}a_jf_j| = \ln\left(|a_0|^2+(\sum_{j=1}^{n}|a_j|)^2\right)^{\frac{1}{2}}.$$

This shows (using the remarks at the end of section 5.2) that $\mathbb{R}\oplus_2 \ell_1^n$ embeds into L_0 for every n. □

Theorem 5.6.7. *Let K be an origin symmetric star body in $\mathbb{R}^n$. If the space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_{p_0} , $0<p_0\leq 2$, then it also embeds in L_0.*

Proof. Since $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_{p_0}, $0<p_0\leq 2$, by [Ko7, Theorem 2] it also embeds in L_{-p} for any $p\in(0,n)$ and hence, by Theorem 5.6.4, it embeds in L_0.

□

Chapter 6

Centroid bodies and comparison of volumes

6.1 Introduction

This section is based on our results from [YY]. Let K be a star body in $\mathbb{R}^n$, then the centroid body of K is a convex body ΓK defined by its support function:

$$h_{\Gamma K}(\xi) = \frac{1}{\mathrm{vol}(K)} \int_K |(x,\xi)| dx, \quad \xi \in \mathbb{R}^n.$$

Let K and L be two origin-symmetric star bodies in $\mathbb{R}^n$ such that $\Gamma K \subset \Gamma L$, what can be said about the volumes of K and L? Lutwak [Lu2] proved that, if L is a polar projection body then $\mathrm{vol}(K) \leq \mathrm{vol}(L)$. On the other hand, if K is not a polar projection body, then there is a body L, so that $\Gamma K \subset \Gamma L$, but $\mathrm{vol}(K) > \mathrm{vol}(L)$. Since in $\mathbb{R}^2$ every convex body is a polar projection body [Sc], the results of Lutwak imply the following:

Suppose that K and L are two origin-symmetric convex bodies in $\mathbb{R}^n$ such that $\Gamma K \subset \Gamma L$. If $n = 2$, then we necessarily have $\mathrm{vol}(K) \leq \mathrm{vol}(L)$, while this is no longer true if $n \geq 3$.

Let K be a star body in $\mathbb{R}^n$ and $p \geq 1$, then the p-centroid body of K is the body $\Gamma_p K$ defined by:

$$h_{\Gamma_p K}(\xi) = \left(\frac{1}{\mathrm{vol}(K)} \int_K |(x,\xi)|^p dx \right)^{1/p}, \quad \xi \in \mathbb{R}^n. \tag{6.1}$$

Clearly, $h_{\Gamma_p K}$ is a homogeneous function of degree 1, and if $p \geq 1$, then this function is convex, and, therefore, $\Gamma_p K$ is well-defined. The polar of $\Gamma_p K$ is called the polar p-centroid body of K and denoted by $\Gamma_p^* K$. Since the support function of a body is the norm of its polar, $h = \|\cdot\|_*$, the polar p-centroid body of K is given by

$$\|\xi\|_{\Gamma_p^* K} = \left(\frac{1}{\mathrm{vol}(K)} \int_K |(x,\xi)|^p dx \right)^{1/p}, \quad \xi \in \mathbb{R}^n. \tag{6.2}$$

The p-centroid bodies and their polars have recently been studied by different authors, see e.g. [CG], [GZ], [Lu2], [LYZ1], [LZ]. In [GZ] Grinberg and Zhang generalized the results of Lutwak discussed in the beginning of this section. Namely, let K and L be two origin-symmetric star bodies in $\mathbb{R}^n$ such that for $p \geq 1$

$$\Gamma_p K \subset \Gamma_p L.$$

They prove that if the space $(\mathbb{R}^n, \|\cdot\|_L)$ embeds in L_p, then we necessarily have

$$\mathrm{vol}(K) \leq \mathrm{vol}(L).$$

On the other hand, if $(\mathbb{R}^n, \|\cdot\|_K)$ does not embed in L_p, then there is a body L so that $\Gamma_p K \subset \Gamma_p L$, but $\mathrm{vol}(K) \leq \mathrm{vol}(L)$.

Note, that if $p = 1$ the positive answer holds for all convex bodies in $\mathbb{R}^2$, while if $p > 1$ there is no dimension where this is always true. The preceding remark suggests considering $p < 1$ in order to make the answer affirmative in higher dimensions.

If $p < 1$, then the function $h_{\Gamma_p K}(\xi)$ in (6.1) is not necessarily convex, therefore it is not a support function, but the definition of the polar p-centroid body still makes sense, even though these bodies may be non-convex. So for all $p > -1$, $p \neq 0$ we define the polar p-centroid body of a star body K by the formula:

$$\|\xi\|_{\Gamma_p^* K} = \left(\frac{1}{\mathrm{vol}(K)} \int_K |(x,\xi)|^p dx \right)^{1/p}, \quad \xi \in \mathbb{R}^n. \tag{6.3}$$

For $p = 0$, this definition looks as follows (if we send $p \to 0$):

$$\|\xi\|_{\Gamma_0^* K} = \exp\left(\frac{1}{\mathrm{vol}(K)} \int_K \ln |(x,\xi)| dx \right), \quad \xi \in \mathbb{R}^n. \tag{6.4}$$

Now we can ask the question discussed above for all $p > -1$. Namely, suppose that

$$\Gamma_p^* L \subset \Gamma_p^* K, \tag{6.5}$$

for origin-symmetric star bodies K and L. Does it follow that we have an inequality for the volumes of K and L? In this paper we show that if $(\mathbb{R}^n, \|\cdot\|_L)$ embeds in L_p, $p > -1$, then we have $\mathrm{vol}(K) \le \mathrm{vol}(L)$. However if $(\mathbb{R}^n, \|\cdot\|_K)$ does not embed in L_p, we construct counterexamples to the latter result.

These results can also be reformulated as follows:

(i) If $0 < p < 1$, then in $\mathbb{R}^2$ the condition (6.5) implies that $\mathrm{vol}(K) \le \mathrm{vol}(L)$, while this is no longer true in dimensions $n \ge 3$.

(ii) If $-1 < p \le 0$, (6.5) implies that $\mathrm{vol}(K) \le \mathrm{vol}(L)$ if and only if $n \le 3$.

Clearly the integral in (6.3) diverges if $p \le -1$, but still we can make sense of this integral considering fractional derivatives. Indeed, if $-1 < p < 0$

$$\begin{aligned}
\frac{1}{\mathrm{vol}(K)} \int_K |(x,\xi)|^p dx &= \frac{1}{\mathrm{vol}(K)} \int_{-\infty}^{\infty} |z|^p \int_{(x,\xi)=z} \chi(\|x\|_K) dx\, dz \\
&= \frac{1}{\mathrm{vol}(K)} \int_{-\infty}^{\infty} |z|^p A_{K,\xi}(z) dz \\
&= \frac{2\Gamma(p+1)}{\mathrm{vol}(K)} A_{K,\xi}^{(-p-1)}(0),
\end{aligned}$$

where $A_{K,\xi}(z)$ is the parallel section function of K, and $A_{K,\xi}^{(-p-1)}(0)$ is its fractional derivative at zero. So, in such terms our problem can be written as follows:

Suppose K and L are two origin-symmetric star bodies, so that for all $\xi \in S^{n-1}$:

$$\frac{A_{K,\xi}^{(-p-1)}(0)}{\mathrm{vol}(K)} \le \frac{A_{L,\xi}^{(-p-1)}(0)}{\mathrm{vol}(L)}.$$

Do we necessarily have an inequality for the volumes of K and L?

Note that Koldobsky already considered such inequalities (see e.g. [Ko10]) without dividing by volumes. So, for $-1 < p < 0$ the positive part of our results can also be obtained from the results of Koldobsky, but we give our own proof. The case $p = -1$ leads to the following modification of the Busemann-Petty problem. Let K and L be two convex origin-symmetric bodies in $\mathbb{R}^n$ such that

$$\frac{\mathrm{vol}_{n-1}(K \cap \xi^{\perp})}{\mathrm{vol}(K)} \le \frac{\mathrm{vol}_{n-1}(L \cap \xi^{\perp})}{\mathrm{vol}(L)}.$$

Does this imply an inequality for the volumes of K and L?

It is easy to show that in dimensions $n \le 4$ we have $\mathrm{vol}(L) \le \mathrm{vol}(K)$. The proof is almost identical to that of the original solution of the Busemann-Petty problem from [GKS]. The counterexamples in dimensions $n \ge 5$ from [GKS] also work in this situation.

In view of all these remarks one can consider our results as a certain bridge between the results of Lutwak-Grinberg-Zhang about p-centroid bodies and the results of Busemann-Petty type obtained by Koldobsky.

6.2 Centroid bodies for $-1 < p < 1$, $p \neq 0$.

The **support function** of a convex body K in $\mathbb{R}^n$ is defined by

$$h_K(x) = \max_{\xi \in K}(x, \xi), \quad x \in \mathbb{R}^n.$$

If K is origin-symmetric, then h_K is the Minkowski norm of the polar body K^*.

Recall a result P.Lévy, (see [BL, p. 189] or [Ko12, Section 6.1]), that a space $(\mathbb{R}^n, \|\cdot\|)$ embeds into L_p, $p > 0$ if and only if there exists a finite Borel measure μ on the unit sphere so that, for every $x \in \mathbb{R}^n$,

$$\|x\|^p = \int_{S^{n-1}} |(x, \xi)|^p d\mu(\xi). \tag{6.6}$$

On the other hand, this can be considered as the definition of embedding in L_p, $-1 < p < 0$ (cf. [Ko7]).

It was proved in [Ko2] that a space $(\mathbb{R}^n, \|\cdot\|)$ embeds isometrically in L_p, $p > 0$, $p \notin 2\mathbb{N}$ if and only if the Fourier transform of the function $\Gamma(-p/2)\|x\|^p$ (in the sense of distributions) is a positive distribution outside of the origin. If $-n < p < 0$ a similar fact was proved in [Ko7]: a space $(\mathbb{R}^n, \|\cdot\|)$ embeds in L_p if and only if the Fourier transform of $\|\cdot\|^p$ is a positive distribution in the whole $\mathbb{R}^n$.

Now we are ready to prove our first result.

Theorem 6.2.1. *Let $-1 < p < 1$, $p \neq 0$. Let K and L be origin-symmetric convex* *bodies in* $\mathbb{R}^n$*, so that* $(\mathbb{R}^n, \|\cdot\|_K)$ *embeds in* L_p *and*

$$\Gamma_p^* K \subset \Gamma_p^* L. \tag{6.7}$$

Then $\mathrm{vol}(L) \le \mathrm{vol}(K)$.

Proof. First let us prove the case $0 < p < 1$. Since $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_p, there exists a measure μ_K on the unit sphere S^{n-1} such that

$$\|x\|_K^p = \int_{S^{n-1}} |(x,\xi)|^p d\mu_K(\xi).$$

Note that (6.7) can be written as

$$\frac{1}{\mathrm{vol}(L)} \int_L |(x,\xi)|^p dx \le \frac{1}{\mathrm{vol}(K)} \int_K |(x,\xi)|^p dx. \tag{6.8}$$

Integrating both sides of the last inequality over S^{n-1} with the measure μ_K, we get

$$\frac{1}{\mathrm{vol}(L)} \int_{S^{n-1}} \int_L |(x,\xi)|^p dx \; d\mu_K(\xi) \le \frac{1}{\mathrm{vol}(K)} \int_{S^{n-1}} \int_K |(x,\xi)|^p dx \; d\mu_K(\xi).$$

Applying Fubini's Theorem,

$$\frac{1}{\mathrm{vol}(L)} \int_L \|x\|_K^p dx \le \frac{1}{\mathrm{vol}(K)} \int_K \|x\|_K^p dx. \tag{6.9}$$

Note that

$$\begin{aligned} \int_K \|x\|_K^p dx &= \int_{S^{n-1}} \left(\int_0^{\|\theta\|_K^{-1}} \|r\theta\|_K^p \, r^{n-1} dr \right) d\theta \\ &= \frac{1}{n+p} \int_{S^{n-1}} \|\theta\|_K^{-n} d\theta = \frac{n}{n+p} \mathrm{vol}(K). \end{aligned}$$

Therefore, (6.9) can be rewritten as

$$\frac{1}{\mathrm{vol}(L)} \int_L \|x\|_K^p dx \le \frac{n}{n+p}.$$

Using the inequality

$$\frac{1}{\mathrm{vol}(L)}\int_L \|x\|_K^p dx \geq \frac{n}{n+p}\left(\frac{\mathrm{vol}(L)}{\mathrm{vol}(K)}\right)^{p/n} \tag{6.10}$$

from [MiP, Section 2.2], we get

$$\frac{n}{n+p} \quad \geq \quad \frac{1}{\mathrm{vol}(L)}\int_L \|x\|_K^p dx \geq \frac{n}{n+p}\left(\frac{\mathrm{vol}(L)}{\mathrm{vol}(K)}\right)^{p/n},$$

therefore $\mathrm{vol}(L) \leq \mathrm{vol}(K)$, which proves the theorem for $0 < p < 1$.

Now consider $-1 < p < 0$. In this case (6.7) is equivalent to

$$\frac{1}{\mathrm{vol}(L)}\int_L |(x,\xi)|^p dx \geq \frac{1}{\mathrm{vol}(K)}\int_K |(x,\xi)|^p dx. \tag{6.11}$$

Since $(\mathbb{R}^n, \|\cdot\|_K)$ embeds into L_p, $p > -1$, there exists a measure μ_K on the unit sphere such that

$$\|x\|_K^p = \int_{S^{n-1}} |(x,\xi)|^p d\mu_K(\xi).$$

Integrating both sides of (6.11) over S^{n-1} with the measure μ_K and using the same argument as in the first part of the proof, we get

$$\frac{1}{\mathrm{vol}(L)}\int_L \|x\|_K^p dx \geq \frac{n}{n+p}. \tag{6.12}$$

Passing to spherical coordinates and applying Hölder's inequality

$$\begin{aligned}
\int_L \|x\|_K^p dx &= \int_{S^{n-1}}\left(\int_0^{\|\theta\|_L^{-1}} r^{n+p-1}\|\theta\|_K^p dr\right) d\theta \\
&= \frac{1}{n+p}\int_{S^{n-1}} \|\theta\|_L^{-n-p}\|\theta\|_K^p d\theta \\
&\leq \frac{1}{n+p}\left(\int_{S^{n-1}} \|\theta\|_L^{-n} d\theta\right)^{(n+p)/n}\left(\int_{S^{n-1}} \|\theta\|_K^{-n} d\theta\right)^{-p/n} \\
&= \frac{n}{n+p}\left(\mathrm{vol}(L)\right)^{(n+p)/n}\left(\mathrm{vol}(K)\right)^{-p/n}.
\end{aligned}$$

So (6.12) can be written as

$$\begin{aligned} 1 &\leq \frac{1}{\mathrm{vol}(L)} \left(\mathrm{vol}(L)\right)^{(n+p)/n} \left(\mathrm{vol}(K)\right)^{-p/n} \\ &= \left(\mathrm{vol}(L)\right)^{p/n} \left(\mathrm{vol}(K)\right)^{-p/n}. \end{aligned}$$

Therefore, using the fact that $p < 0$, we get $\mathrm{vol}(L) \leq \mathrm{vol}(K)$.

□

Since all 2-dimensional spaces embed in L_1, and therefore in L_p with $-2 < p < 1$ (see e.g. [Ko12, Chapter 6]), and all 3-dimensional spaces embed in L_0, and therefore in L_p with $-3 < p < 0$ (see [KKYY]), we have the following

Corollary 6.2.2. *Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$, so that $\Gamma_p^* K \subset \Gamma_p^* L$. Then*

i) if $0 < p < 1$, we necessarily have $\mathrm{vol}(L) \leq \mathrm{vol}(K)$ in dimension $n = 2$,

ii) if $-1 < p < 0$, we necessarily have $\mathrm{vol}(L) \leq \mathrm{vol}(K)$ in dimensions $n = 2$ and 3.

In order to show a negative counterpart of Theorem 6.2.1, we need some lemmas. The following Lemma is [Ko12, Corollary 3.15] with $k = 0$ and $p = -q - 1$.

Lemma 6.2.3. *Let $-1 < p < 1$, $p \neq 0$. For an origin-symmetric convex body K in $\mathbb{R}^n$ we have*

$$\left(\|x\|_K^{-n-p}\right)^\wedge(\xi) = -\frac{\pi}{2\Gamma(p+1)\sin(\pi p/2)} \int_{S^{n-1}} |(\theta,\xi)|^p \, \|\theta\|_K^{-n-p} d\theta.$$

We will use this formula in the following form:

$$\left(\|x\|_K^{-n-p}\right)^\wedge(\xi) = -\frac{\pi(n+p)}{2\Gamma(p+1)\sin(\pi p/2)} \int_K |(x,\xi)|^p dx.$$

Also we can write this formula in terms of fractional derivatives of the parallel section function of K. Recall that the parallel section function of a an origin-symmetric star body K is defined by

$$A_{K,\xi}(z) = \int_{(x,\xi)=z} \chi(\|x\|_K)dx.$$

For $-1 < q < 0$ the fractional derivative of this function at zero is defined by

$$A^{(q)}_{K,\xi}(0) = \frac{1}{2\Gamma(-q)} \int_{-\infty}^{\infty} |z|^{-1-q} A_{K,\xi}(z)dz = \frac{1}{2\Gamma(-q)} \int_K |(x,\xi)|^{-1-q}dx.$$

In fact one can see that this is analytically extendable to $q < -1$. Therefore Lemma 6.2.3 can be reformulated as follows. Let $-1 < p < 1$, $p \neq 0$, then

$$\left(\|x\|_K^{-n-p}\right)^\wedge(\xi) = -\frac{\pi(n+p)}{\sin(\pi p/2)} A^{(-p-1)}_{K,\xi}(0).$$

Note, that for $-1 < p < 0$ this formula was proved in [GKS].

Now recall a version of Parseval's formula on the sphere proved by Koldobsky [Ko6].

Lemma 6.2.4. *If K and L are origin-symmetric infinitely smooth bodies in $\mathbb{R}^n$ and $0 < p < n$, then $(\|x\|_K^{-p})^\wedge$ and $(\|x\|_L^{-n+p})^\wedge$ are continuous functions on S^{n-1} and*

$$\int_{S^{n-1}} \left(\|x\|_K^{-p}\right)^\wedge(\xi) \left(\|x\|_L^{-n+p}\right)^\wedge(\xi)d\xi = (2\pi)^n \int_{S^{n-1}} \|x\|_K^{-p}\|x\|_L^{-n+p}dx.$$

Remark 6.2.5. A proof of this formula via spherical harmonics was given in [Ko10]. Repeating this proof word by word and using the above definition of the fractional derivative of order $q < -1$, one can easily extend this result to $-1 < p < 0$.

Now we prove a negative counterpart of Theorem 6.2.1.

Theorem 6.2.6. *Let L be an infinitely smooth origin-symmetric strictly convex body in $\mathbb{R}^n$, for which $(\mathbb{R}^n, \|\cdot\|_L)$ does not embed in L_p, $-1 < p < 1$, $p \neq 0$. Then there exists an origin-symmetric convex body K in $\mathbb{R}^n$ such that*

$$\Gamma_p^* K \subset \Gamma_p^* L.$$

but

$$\mathrm{vol}(L) > \mathrm{vol}(K).$$

Proof. First consider $0 < p < 1$. Since $(\mathbb{R}^n, \|\cdot\|_L)$ does not embed in L_p, there exists a $\xi \in S^{n-1}$ such that $(\|x\|_L^p)^\wedge(\xi)$ is positive; for more details see [Ko2]. Because $(\|x\|_L^p)^\wedge(\theta)$ is a continuous function on S^{n-1}, there exists a neighborhood of ξ where it is positive. Define

$$\Omega = \{\theta \in S^{n-1} : (\|x\|_L^p)^\wedge(\theta) > 0\}.$$

Choose a non-positive infinitely-smooth even function v supported in Ω. Extend v to a homogeneous function $|x|_2^{-n-p} v(x/|x|_2)$ of degree $-n-p$ on $\mathbb{R}^n$. By [Ko12, Lemma 3.16], the Fourier transform of $|x|_2^{-n-p} v(x/|x|_2)$ is equal to $|x|_2^p\, g(x/|x|_2)$ for some infinitely smooth function g on S^{n-1}.

Define a body K by

$$\|x\|_K^{-n-p} = \|x\|_L^{-n-p} + \epsilon |x|_2^{-n-p} g(x/|x|_2)$$

for some small ϵ so that the body K is convex (see e.g. the perturbation argument from [Ko12, p.96]). Applying the Fourier transform to both sides we get

$$\left(\|x\|_K^{-n-p}\right)^\wedge(\xi) = \left(\|x\|_L^{-n-p}\right)^\wedge(\xi) + \epsilon(2\pi)^n|\xi|_2^p v(\xi/|\xi|_2).$$

So using the formula from Lemma 6.2.3

$$\left(\|x\|_K^{-n-p}\right)^\wedge(\xi) = \Gamma(-p)\sin\left(\frac{\pi(p+1)}{2}\right)\int_K |(x,\xi)|^p dx$$

we have

$$\int_L |(x,\xi)|^p dx < \int_K |(x,\xi)|^p dx. \tag{6.13}$$

Consider the integral

$$\begin{aligned}
&\int_{S^{n-1}} (\|x\|_L^p)^\wedge(\xi)\left(\|x\|_K^{-n-p}\right)^\wedge(\xi)d\xi \\
&= \int_{S^{n-1}} (\|x\|_L^p)^\wedge(\xi)\left(\|x\|_L^{-n-p}\right)^\wedge(\xi)d\xi + \epsilon(2\pi)^n\int_{S^{n-1}} (\|x\|_L^p)^\wedge(\xi)v(\xi)d\xi \\
&< \int_{S^{n-1}} (\|x\|_L^p)^\wedge(\xi)\left(\|x\|_L^{-n-p}\right)^\wedge(\xi)d\xi \\
&= (2\pi)^n\int_{S^{n-1}} \|x\|_L^p\|x\|_L^{-n-p}dx = (2\pi)^n n\mathrm{vol}(L).
\end{aligned} \tag{6.14}$$

Here we used a version of Parseval's formula (Lemma 6.2.4 and Remark 6.2.5) and the fact that v is negative on Ω.

On the other hand, again using Parseval's formula and (6.10)

$$\begin{aligned}
&\int_{S^{n-1}} (\|x\|_L^p)^\wedge(\xi)\left(\|x\|_K^{-n-p}\right)^\wedge(\xi)d\xi = (2\pi)^n\int_{S^{n-1}} \|x\|_L^p\|x\|_K^{-n-p}dx \\
&= (2\pi)^n(n+p)\int_K \|x\|_L^p dx \geq (2\pi)^n n\mathrm{vol}(K)\left(\frac{\mathrm{vol}(L)}{\mathrm{vol}(L)}\right)^{p/n}.
\end{aligned} \tag{6.15}$$

Combining (6.14) and (6.15) we get

$$\mathrm{vol}(K) < \mathrm{vol}(L). \tag{6.16}$$

Now from (6.16) and (6.13) it follows that

$$\frac{1}{\mathrm{vol}(L)}\int_L |(x,\xi)|^p dx \leq \frac{1}{\mathrm{vol}(K)}\int_K |(x,\xi)|^p dx,$$

which is equivalent to

$$\Gamma_p^* K \subset \Gamma_p^* L.$$

Now consider the case $-1 < p < 0$. Since $(\mathbb{R}^n, \|\cdot\|_L)$ does not embed in L_p, there exists a $\xi \in S^{n-1}$ such that $(\|x\|_L^p)^\wedge(\xi)$ is negative, see [Ko7, Theorem 1]. Define

$$\Omega = \{\theta \in S^{n-1} : (\|x\|_L^p)^\wedge(\theta) < 0\}$$

and choose $v(\theta)$ the same way as in the first part.

Define a body K by

$$\frac{\|x\|_K^{-n-p}}{\mathrm{vol}(K)} = \frac{\|x\|_L^{-n-p}}{\mathrm{vol}(L)} + \epsilon |x|_2^{-n-p} g(x/|x|_2)$$

for some small ϵ so that the body K is convex. Applying Fourier transform to both sides we get

$$\frac{1}{\mathrm{vol}(K)} \left(\|x\|_K^{-n-p}\right)^\wedge(\xi) = \frac{1}{\mathrm{vol}(L)} \left(\|x\|_L^{-n-p}\right)^\wedge(\xi) + \epsilon(2\pi)^n |\xi|_2^p v(\xi/|\xi|_2).$$

Again using the formula from Lemma 6.2.3 and the fact that $v(\theta)$ is non-positive, we have

$$\frac{1}{\mathrm{vol}(K)} \int_K |(x,\xi)|^p dx < \frac{1}{\mathrm{vol}(L)} \int_L |(x,\xi)|^p dx,$$

which is the same as $\Gamma_p^* K \subset \Gamma_p^* L$, since $-1 < p < 0$.

Consider the integral

$$\frac{1}{\mathrm{vol}(K)} \int_{S^{n-1}} (\|x\|_L^p)^\wedge(\xi) \left(\|x\|_K^{-n-p}\right)^\wedge(\xi) d\xi$$

$$= \frac{1}{\mathrm{vol}(L)} \int_{S^{n-1}} (\|x\|_L^p)^\wedge(\xi) \left(\|x\|_L^{-n-p}\right)^\wedge(\xi) d\xi + \epsilon(2\pi)^n \int_{S^{n-1}} (\|x\|_L^p)^\wedge(\xi) v(\xi) d\xi$$

$$> \frac{1}{\mathrm{vol}(L)} \int_{S^{n-1}} \left(\|x\|_L^p\right)^\wedge (\xi) \left(\|x\|_L^{-n-p}\right)^\wedge (\xi) d\xi = (2\pi)^n n. \tag{6.17}$$

Here we used Parseval's formula and the fact that v is negative on Ω.

On the other hand, again using Parseval's formula and Hölder's inequality

$$\begin{aligned}
\int_{S^{n-1}} \left(\|x\|_L^p\right)^\wedge (\xi) \left(\|x\|_K^{-n-p}\right)^\wedge (\xi) d\xi &= (2\pi)^n \int_{S^{n-1}} \|x\|_L^p \|x\|_K^{-n-p} dx \\
&\leq (2\pi)^n \left(\int_{S^{n-1}} \|x\|_L^{-n} dx \right)^{-p/n} \left(\int_{S^{n-1}} \|x\|_K^{-n} dx \right)^{(n+p)/n} \\
&= (2\pi)^n n \left(\mathrm{vol}(L)\right)^{-p/n} \left(\mathrm{vol}(K)\right)^{(n+p)/n}.
\end{aligned} \tag{6.18}$$

So combining (6.17) and (6.18) we get $\mathrm{vol}(L) > \mathrm{vol}(K)$.

□

The result of Theorem 6.2.6 can be formulated as follows:

Corollary 6.2.7. *i) Let $-1 < p < 0$. There exist origin-symmetric convex bodies K and L in $\mathbb{R}^4$, so that $\Gamma_p^* K \subset \Gamma_p^* L$, but* $\mathrm{vol}(L) > \mathrm{vol}(K)$.

ii) Let $0 < p < 1$. There exist origin-symmetric convex bodies K and L in $\mathbb{R}^3$, so that $\Gamma_p^* K \subset \Gamma_p^* L$, but $\mathrm{vol}(L) > \mathrm{vol}(K)$.

Proof. Consider only the case $-1 < p < 0$, the other case is similar. In view of the previous theorem it is enough to construct an origin-symmetric infinitely smooth convex body $L \in \mathbb{R}^4$ for which the distribution $(\|x\|_L^p)^\wedge$ is not positive. The construction will be similar to that from [GKS].

Define $f_N(x) = (1 - x^2 - Nx^4)^{1/3}$; let $a_N > 0$ be such that $f_N(a_N) = 0$ and $f_N(x) > 0$ on the interval $(0, a_N)$. Define a body L in $\mathbb{R}^4$ by

$$L = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \, x_4 \in [-a_N, a_N] \, \text{and} \, \sqrt{x_1^2 + x_2^2 + x_3^2} \leq f_N(x_4)\}.$$

The body L is strictly convex and infinitely smooth.

By the formula

$$A_{L,\xi}^{(q)}(0) = \frac{\cos\frac{\pi q}{2}}{\pi(n-q-1)}\left(\|x\|_L^{-n+q+1}\right)^\wedge(\xi)$$

from [GKS] and the definition of fractional derivatives, we get

$$\begin{aligned}\left(\|x\|_L^p\right)^\wedge(\xi) &= \frac{\pi p}{\cos\frac{\pi(3+p)}{2}} A_{L,\xi}^{(3+p)}(0) \\ &= \frac{\pi p}{\Gamma(-3-p)\cos\frac{\pi(3+p)}{2}} \int_0^\infty \frac{A_{L,\xi}(z) - A_{L,\xi}(0) - A''_{L,\xi}(0)\frac{z^2}{2}}{z^{4+p}} dz.\end{aligned}$$

Note that the coefficient in the latter formula is positive, therefore it is enough to show that the integral is negative.

The function $A_{L,\xi}$ can easily be computed:

$$A_{L,\xi}(x) = \frac{4\pi}{3}(1 - x^2 - Nx^4).$$

We have

$$\begin{aligned}&\int_0^\infty \frac{A_\xi(z) - A_\xi(0) - A''_\xi(0)\frac{z^2}{2}}{z^{4+p}} dz = \\ &= \frac{4\pi}{3}\left(-\frac{1}{1+p}Na_N^{1+p} + \frac{1}{(1+p)a_N^{(1+p)}} - \frac{1}{(3+p)a_N^{3+p}}\right).\end{aligned}$$

The latter is negative for N large enough, because $N^{1/4}\cdot a_N \to 1$ as $N \to \infty$.

□

6.3 Centroid bodies for $p = 0$.

In this section we extend the results of the previous section to $p = 0$. Recall that a space $(\mathbb{R}^n, \|\cdot\|)$ embeds in L_0 if there exist a finite Borel measure μ on the sphere S^{n-1} and a constant $C \in \mathbb{R}$ so that, for every $x \in \mathbb{R}^n$,

$$\ln\|x\| = \int_{S^{n-1}} \ln|(x,\xi)| d\mu(\xi) + C. \tag{6.19}$$

In our next Lemma we prove that a representation similar to (6.19) holds for all infinitely smooth bodies, with μ being a signed measure.

Lemma 6.3.1. *Let K be an infinitely smooth origin-symmetric star body in $\mathbb{R}^n$. Then*

$$\ln \|x\|_K = -\frac{1}{(2\pi)^n} \int_{S^{n-1}} \ln |(x,\xi)| \left(\ln \|x\|_K\right)^\wedge (\xi) d\xi + C_K, \tag{6.20}$$

where C_K is the constant from the definition of embedding in L_0.

Proof. Since the body K is infinitely smooth, by Theorem 5.4.1, $\left(\ln \|x\|_K\right)^\wedge (\xi)$ is a continuous homogeneous function of degree $-n$ on $\mathbb{R}^n \setminus \{0\}$.

Let ϕ be an even test function supported outside of the origin, then

$$\left\langle \left(\int_{S^{n-1}} \ln |(x,\xi)| \left(\ln \|x\|_K\right)^\wedge (\xi) d\xi \right)^\wedge, \phi \right\rangle$$

$$= \left\langle \int_{S^{n-1}} \ln |(x,\xi)| \left(\ln \|x\|_K\right)^\wedge (\xi) d\xi, \hat{\phi}(x) \right\rangle$$

$$= \int_{\mathbb{R}^n} \left[\int_{S^{n-1}} \ln |(x,\xi)| \left(\ln \|x\|_K\right)^\wedge (\xi) d\xi \right] \hat{\phi}(x) dx$$

$$= \int_{S^{n-1}} \left[\int_{\mathbb{R}^n} \ln |(x,\xi)| \hat{\phi}(x) dx \right] \left(\ln \|x\|_K\right)^\wedge (\xi) d\xi$$

Now compute the inner integral using Fubini's theorem and the connection between the Radon and Fourier transforms (1.2):

$$\int_{\mathbb{R}^n} \ln |(x,\xi)| \hat{\phi}(x) dx = \int_{\mathbb{R}} \ln |t| \int_{(x,\xi)=t} \hat{\phi}(x) dx dt$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} (\ln |t|)^\wedge(z) \left(\int_{(x,\xi)=t} \hat{\phi}(x) dx \right)^\wedge (z) dz = -\frac{1}{2} \int_{\mathbb{R}} |z|^{-1} \hat{\hat{\phi}}(z\xi) dz$$

$$= -2^{n-1}\pi^n \int_{\mathbb{R}} |z|^{-1} \phi(z\xi) dz = -(2\pi)^n \int_0^\infty z^{-1} \phi(z\xi) dz$$

Here we used the formula for the Fourier transform of $\ln|t|$ (see [GS, p.362])

$$(\ln|z|)^\wedge(t) = -\pi|t|^{-1} \tag{6.21}$$

outside of the origin. Therefore, passing from polar to Euclidean coordinates and recalling from Theorem 5.4.1, that $(\ln\|x\|_K)^\wedge$ is a homogeneous function of degree $-n$ on $\mathbb{R}^n \setminus \{0\}$, we get

$$\langle\left(\int_{S^{n-1}} \ln|(x,\xi)|\,(\ln\|x\|_K)^\wedge(\xi)d\xi\right)^\wedge, \phi\rangle$$

$$= -(2\pi)^n \int_{S^{n-1}} \left[\int_0^\infty z^{-1}\phi(z\xi)dz\right](\ln\|x\|_K)^\wedge(\xi)d\xi$$

$$= -(2\pi)^n \int_{\mathbb{R}^n} \phi(y)\,(\ln\|x\|_K)^\wedge(y)dy = -(2\pi)^n\langle(\ln\|x\|_K)^\wedge, \phi\rangle.$$

It follows that

$$\left(\int_{S^{n-1}} \ln|(x,\xi)|\,(\ln\|x\|_K)^\wedge(\xi)d\xi\right)^\wedge = -(2\pi)^n\,(\ln\|x\|_K)^\wedge$$

as distributions outside of the origin. Hence, the functions $-(2\pi)^n\ln\|x\|_K$ and $\int_{S^{n-1}} \ln|(x,\xi)|\,(\ln\|x\|_K)^\wedge(\xi)d\xi$ may differ only by a polynomial. But

$$\frac{1}{(2\pi)^n}\int_{S^{n-1}} \ln|(x,\xi)|\,(\ln\|x\|_K)^\wedge(\xi)d\xi + \ln\|x\|_K$$

is a homogeneous function of degree zero, therefore this polynomial is some constant C, which is exactly the constant from Definition 5.2.1, as computed in [KKYY].

□

Now we need a version of Parseval's formula for L_0. How does the formula of Lemma 6.2.4 look if we pass to the limit as $p \to 0$? The answer to this question is given in our next Lemma. Even though in the proof we use an argument based on

Lemma 6.3.1, one can obtain the following Lemma by taking the limit in Parseval's formula.

Lemma 6.3.2. *Let K and L be infinitely smooth origin-symmetric star bodies in $\mathbb{R}^n$. Then*

$$-\frac{1}{(2\pi)^n}\int_{S^{n-1}}\left[\int_L \ln|(x,\xi)|dx\right](\ln\|x\|_K)^\wedge(\xi)d\xi = \int_L (\ln\|x\|_K - C_K)dx.$$

Proof. By Lemma 6.3.1 we have

$$-\frac{1}{(2\pi)^n}\int_{S^{n-1}}\ln|(x,\xi)|\,(\ln\|x\|_K)^\wedge(\xi)d\xi = \ln\|x\|_K - C_K.$$

Integrating this equality over the body L we get the statement of the Lemma.

□

Now we prove the main result of this section.

Theorem 6.3.3. *Let K and L be two origin-symmetric star bodies in $\mathbb{R}^n$ such that $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0 and*

$$\Gamma_0^* K \subset \Gamma_0^* L \tag{6.22}$$

for every $\xi \in S^{n-1}$. Then

$$\mathrm{vol}(L) \le \mathrm{vol}(K).$$

Proof. Since $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0, there exist a probability measure μ_K on S^{n-1} (which is the restriction of the Fourier transform of $\ln\|x\|_K$ to the unit sphere) and a constant C_K from Definition 5.2.1.

Rewrite inequality (6.22) as follows:

$$\frac{\int_L \ln|(x,\xi)|dx}{\mathrm{vol}(L)} \le \frac{\int_K \ln|(x,\xi)|dx}{\mathrm{vol}(K)},$$

and integrate it over S^{n-1} with respect to μ_K to get

$$\int_{S^{n-1}} \frac{\int_L \ln|(x,\xi)|dx}{\mathrm{vol}(L)} d\mu_K(\xi) \le \int_{S^{n-1}} \frac{\int_K \ln|(x,\xi)|dx}{\mathrm{vol}(K)} d\mu_K(\xi).$$

Using the Fubini theorem and the definition of embedding in L_0, we get

$$\frac{1}{\mathrm{vol}(L)} \int_L (\ln\|x\|_K - C_K)dx \le \frac{1}{\mathrm{vol}(K)} \int_K (\ln\|x\|_K - C_K)dx.$$

Therefore

$$\frac{1}{\mathrm{vol}(L)} \int_L \ln\|x\|_K dx \le \frac{1}{\mathrm{vol}(K)} \int_K \ln\|x\|_K dx = -\frac{1}{n},$$

where the latter equality follows from the formula

$$\frac{1}{\mathrm{vol}(K)} \int_K \|x\|_K^p dx = \frac{n}{n+p},$$

that we had earlier, after differentiating and letting $p = 0$.

Now use the following inequality from Milman and Pajor [MiP, Section 2.2]:

$$\frac{1}{\mathrm{vol}(L)} \int_L \ln\|x\|_K dx \ge -\frac{1}{n} + \frac{1}{n}[\ln(\mathrm{vol}(L)) - \ln(\mathrm{vol}(K))]. \tag{6.23}$$

Therefore

$$\mathrm{vol}(L) \le \mathrm{vol}(K).$$

$\square$

Remark 6.3.4. *Since every three dimensional normed space embeds in L_0 (see [KKYY, Corollary 4.3]), the previous theorem holds for all convex bodies in $\mathbb{R}^3$.*

To prove our next Theorem we need the following Lemma.

Lemma 6.3.5. *Let K be an origin-symmetric star body in $\mathbb{R}^n$, then the Fourier transform of $\|x\|_K^{-n}$ is a continuous function on $\mathbb{R}^n \setminus \{0\}$ and equals*

$$\begin{aligned}(\|x\|_K^{-n})^\wedge(\xi) = & - \quad n\int_K \ln|(x,\xi)|dx + \\ & + \quad (n\Gamma'(1)-1)\mathrm{vol}(K) - \int_{S^{n-1}} \|\theta\|_K^{-n}\ln\|\theta\|_K d\theta.\end{aligned}$$

Proof. Let ϕ be an even test function. Using the definition of the action of a homogeneous function of degree $-n$ (see [GS, p.303]) we get

$$\begin{aligned}
&\langle(\|x\|_K^{-n})^\wedge,\phi\rangle = \langle\|x\|_K^{-n},\hat\phi(x)\rangle\\
&= \int_{B_1(0)}\|x\|_K^{-n}(\hat\phi(x)-\hat\phi(0))dx + \int_{\mathbb{R}^n\setminus B_1(0)}\|x\|_K^{-n}\hat\phi(x)dx\\
&= \int_{S^{n-1}}\int_0^1 r^{-1}\|\theta\|_K^{-n}(\hat\phi(r\theta)-\hat\phi(0))drd\theta + \int_{S^{n-1}}\int_1^\infty r^{-1}\|\theta\|_K^{-n}\hat\phi(r\theta)drd\theta\\
&= \int_{S^{n-1}}\|\theta\|_K^{-n}\left(\int_0^1 r^{-1}(\hat\phi(r\theta)-\hat\phi(0))dr + \int_1^\infty r^{-1}\hat\phi(r\theta)dr\right)d\theta\\
&= \frac{1}{2}\int_{S^{n-1}}\|\theta\|_K^{-n}\langle|r|^{-1},\hat\phi(r\theta)\rangle d\theta\\
&= \frac{1}{2}\int_{S^{n-1}}\|\theta\|_K^{-n}\langle 2\Gamma'(1)-2\ln|t|,\int_{(\theta,\xi)=t}\phi(\xi)d\xi\rangle d\theta\\
&= \langle\int_{S^{n-1}}\|\theta\|_K^{-n}\left(\Gamma'(1)-\ln|(\theta,\xi)|\right)d\theta,\phi(\xi)\rangle.
\end{aligned}$$

Here we used the formula for the Fourier transform of $|r|^{-1}$ from [GS, p.361]:

$$(|r|^{-1})^\wedge(t) = 2\Gamma'(1) - 2\ln|t|.$$

Thus we have proved that

$$(\|x\|_K^{-n})^\wedge(\xi) \quad = \quad \int_{S^{n-1}}\|\theta\|_K^{-n}\Big(\Gamma'(1)-\ln|(\theta,\xi)|\Big)d\theta. \tag{6.24}$$

Next, let us compute the following:

$$\begin{aligned}\int_K \ln|(x,\xi)|dx &= \int_{S^{n-1}}\int_0^{\|\theta\|_K^{-1}} r^{n-1}\ln|(r\theta,\xi)|drd\theta\\ &= \int_{S^{n-1}}\int_0^{\|\theta\|_K^{-1}} r^{n-1}\ln r dr d\theta + \int_{S^{n-1}}\ln|(\theta,\xi)|\int_0^{\|\theta\|_K^{-1}} r^{n-1}drd\theta\\ &= -\frac{1}{n}\int_{S^{n-1}}\left(\|\theta\|_K^{-n}\ln\|\theta\|_K + \frac{1}{n}\|\theta\|_K^{-n}\right)d\theta + \frac{1}{n}\int_{S^{n-1}}\|\theta\|_K^{-n}\ln|(\theta,\xi)|d\theta.\end{aligned}$$

Therefore

$$\begin{aligned}&\int_{S^{n-1}}\|\theta\|_K^{-n}\ln|(\theta,\xi)|d\theta =\\ &\qquad = n\int_K \ln|(x,\xi)|dx + \int_{S^{n-1}}\left(\|\theta\|_K^{-n}\ln\|\theta\|_K + \frac{1}{n}\|\theta\|_K^{-n}\right)d\theta.\end{aligned}$$

Combining this formula with the formula (6.24), we get

$$\begin{aligned}(\|x\|_K^{-n})^\wedge(\xi) = &- n\int_K \ln|(x,\xi)|dx +\\ &+ (n\Gamma'(1)-1)\mathrm{vol}(K) - \int_{S^{n-1}}\|\theta\|_K^{-n}\ln\|\theta\|_K d\theta.\end{aligned}$$

□

Theorem 6.3.6. *There are convex bodies K and L in $\mathbb{R}^n$, $n \geq 4$ such that*

$$\Gamma_0^* K \subset \Gamma_0^* L$$

for every $\xi \in S^{n-1}$, but

$$\mathrm{vol}(K) < \mathrm{vol}(L).$$

Proof. Let L be a strictly convex infinitely smooth body in $\mathbb{R}^n$, $n \geq 4$, for which $-(\ln\|x\|_L)^\wedge$ is not positive everywhere. (See [KKYY, Theorem 4.4] for an explicit construction of such a body.)

Let $\xi \in S^{n-1}$ be such that $-(\ln\|x\|_L)^\wedge(\xi) < 0$. By continuity of the function $(\ln\|x\|_L)^\wedge(\theta)$ on the sphere there is a neighborhood of ξ where this function is negative. Let

$$\Omega = \{\theta \in S^{n-1} : -(\ln \|x\|_L)^\wedge(\theta) < 0\}.$$

Choose an infinitely smooth body D whose Minkowski norm $\|x\|_D$ is equal to 1 outside of Ω and $\|x\|_D < 1$ for $x \in \Omega$. Let v be a homogeneous function of degree 0 on $\mathbb{R}^n \setminus \{0\}$, defined as follows:

$$v(x) = \ln \|x\|_D - \ln |x|_2.$$

Clearly $v(x) < 0$ if $x \in \Omega$ and $v(x) = 0$ if $x \in S^{n-1} \setminus \Omega$.

In view of Theorem 5.4.1, the Fourier transforms of $\ln \|x\|_D$ and $\ln |x|_2$ outside of the origin are some homogeneous functions of degree $-n$, therefore the Fourier transform of $v(x)$ outside of the origin is equal to $|x|_2^{-n} g(x/|x|_2)$ for some infinitely smooth function g on S^{n-1}. Since by Remark 5.3.2

$$\int_{S^{n-1}} (\ln \|x\|_D)^\wedge(\theta) d\theta = \int_{S^{n-1}} (\ln |x|_2)^\wedge(\theta) d\theta = -(2\pi)^n,$$

we have

$$\int_{S^{n-1}} g(\theta) d\theta = 0. \tag{6.25}$$

Define a body K by the formula:

$$\frac{\|x\|_K^{-n}}{\mathrm{vol}(K)} = \frac{\|x\|_L^{-n}}{\mathrm{vol}(L)} + n(2\pi)^{-n} \epsilon |x|_2^{-n} g(x/|x|_2). \tag{6.26}$$

Note that formula (6.25) validates this definition, since integrating the last equality over the unit sphere we get the same quantity in both sides. Also, since L is strictly convex, there is an ϵ small enough, so that K is also convex (see e.g. the perturbation argument from [Ko12, p.96]). From now on we fix such an ϵ.

Now we will show that K together with L constructed above satisfy the assumptions of the theorem. Apply the Fourier transform to both sides of (6.26). Note, that the Fourier transform of $|x|_2^{-n} g(x/|x|_2)$ is equal to $(2\pi)^n v$ on test functions, whose Fourier transform is supported outside of the origin. Such distributions can differ only by a polynomial, which must be a constant in this case, since both functions cannot grow faster than a logarithm (see Lemma 6.3.5). So

$$\left(|x|_2^{-n} g(x/|x|_2)\right)^\wedge = (2\pi)^n (v + \alpha),$$

for some constant α whose value has no significance for us. Hence, by Lemma 6.3.5, the Fourier transform of (6.26) looks as follows:

$$-\frac{n\int_K \ln|(x,\xi)|dx}{\mathrm{vol}(K)} = -\frac{n\int_L \ln|(x,\xi)|dx}{\mathrm{vol}(L)} + n\epsilon \cdot v(\xi) + C, \tag{6.27}$$

where the constant C equals

$$C = \frac{\int_K \|\theta\|_K^{-n} \ln\|\theta\|_K d\theta dx}{\mathrm{vol}(K)} - \frac{\int_L \|\theta\|_L^{-n} \ln\|\theta\|_L d\theta dx}{\mathrm{vol}(L)} + n\epsilon \cdot \alpha.$$

Since the bodies L and D are fixed, dilating the body K we can make this constant equal to zero. Indeed, multiply the Minkowski functional of K by a positive constant λ, then

$$\begin{aligned} C &= \frac{\int_K (\lambda\|\theta\|_K)^{-n} \ln\lambda\|\theta\|_K d\theta dx}{\lambda^{-n}\mathrm{vol}(K)} - \frac{\int_L \|\theta\|_L^{-n} \ln\|\theta\|_L d\theta dx}{\mathrm{vol}(L)} + n\epsilon \cdot \alpha \\ &= \frac{\int_K \|\theta\|_K^{-n}\left[\ln\lambda + \ln\|\theta\|_K\right] d\theta dx}{\mathrm{vol}(K)} - \frac{\int_L \|\theta\|_L^{-n} \ln\|\theta\|_L d\theta dx}{\mathrm{vol}(L)} + n\epsilon \cdot \alpha \\ &= n\ln\lambda + \frac{\int_K \|\theta\|_K^{-n} \ln\|\theta\|_K d\theta dx}{\mathrm{vol}(K)} - \frac{\int_L \|\theta\|_L^{-n} \ln\|\theta\|_L d\theta dx}{\mathrm{vol}(L)} + n\epsilon \cdot \alpha. \end{aligned}$$

One can choose a $\lambda > 0$ so that $C = 0$. Therefore from (6.27) we get

$$\frac{\int_K \ln|\langle x,\xi\rangle|dx}{\mathrm{vol}(K)} = \frac{\int_L \ln|\langle x,\xi\rangle|dx}{\mathrm{vol}(L)} - \epsilon\, v(\xi) \geq \frac{\int_L \ln|\langle x,\xi\rangle|dx}{\mathrm{vol}(L)}, \tag{6.28}$$

since v is non-positive. Therefore

$$\Gamma_0^* K \subset \Gamma_0^* L.$$

Now using Parseval's formula and inequality (6.28) we get

$$\begin{aligned}
&\frac{1}{\mathrm{vol}(K)} \int_K (\ln \|x\|_L - C_L) dx = \\
&= -\frac{1}{(2\pi)^n} \frac{1}{\mathrm{vol}(K)} \int_{S^{n-1}} \left[\int_K \ln |\langle x, \xi \rangle| dx \right] (\ln \|x\|_L)^\wedge(\xi) d\xi \\
&= -\frac{1}{(2\pi)^n} \int_{S^{n-1}} \left[\frac{1}{\mathrm{vol}(L)} \int_L \ln |\langle x, \xi \rangle| dx - \epsilon v(\xi) \right] (\ln \|x\|_L)^\wedge(\xi) d\xi \\
&= -\frac{1}{(2\pi)^n} \frac{1}{\mathrm{vol}(L)} \int_{S^{n-1}} \left[\int_L \ln |\langle x, \xi \rangle| dx \right] (\ln \|x\|_L)^\wedge(\xi) d\xi \\
&\quad + \frac{1}{(2\pi)^n} \frac{1}{\mathrm{vol}(L)} \int_{S^{n-1}} \epsilon v(\xi) (\ln \|x\|_L)^\wedge(\xi) d\xi \\
&< -\frac{1}{(2\pi)^n} \frac{1}{\mathrm{vol}(L)} \int_{S^{n-1}} \left[\int_L \ln |\langle x, \xi \rangle| dx \right] (\ln \|x\|_L)^\wedge(\xi) d\xi \\
&= \frac{1}{\mathrm{vol}(L)} \int_L (\ln \|x\|_L - C_L) dx,
\end{aligned}$$

where the inequality follows from the fact that v is non-positive and supported on the set where $-(\ln \|x\|_L)^\wedge(\xi) < 0$.

Recalling the inequality (6.23)

$$-\frac{1}{n} \geq \frac{1}{\mathrm{vol}(K)} \int_K \ln \|x\|_L dx \geq -\frac{1}{n} + \frac{1}{n} [\ln(\mathrm{vol}(K)) - \ln(\mathrm{vol}(L))],$$

we get

$$\mathrm{vol}(K) < \mathrm{vol}(L).$$

□

Chapter 7

Khinchin type inequalities and sections of L_p-balls, $p > -2$.

7.1 Introduction

In this section we study Khinchin type inequalities and their application to the slicing problem. These results are from [KPY]. A simple version of the Khinchin inequality states that for a convex origin symmetric body K in $\mathbb{R}^n$ with $\mathrm{vol}(K) = 1$ and $p > q > 0$, we have for all $\xi \in \mathbb{R}^n$

$$\left(\int_K |(x,\xi)|^p dx \right)^{\frac{1}{p}} \leq C \left(\int_K |(x,\xi)|^q dx \right)^{\frac{1}{q}},$$

where C depends only on p and q.

A result of Guédon [Gu] implies that this inequality holds for $q > -1$. In this article we extend it further to $q > -2$ and as an application we prove the slicing problem for the unit balls of spaces that embed in L_p, $p > -2$.

Recall that an origin symmetric convex body $K \subset \mathbb{R}^n$ is called **isotropic** with **constant of isotropy** L_K if $\mathrm{vol}_n(K) = 1$ and

$$\int_K (x,\theta)^2 dx = L_K^2, \qquad \text{for all } \theta \in S^{n-1}.$$

For every convex origin symmetric body K there exists a linear isomorphism T

of $\mathbb{R}^n$, such that TK is isotropic, and we define the constant of isotropy of K by $L_K = L_{TK}$.

Recall that the slicing problem asks the following question. Does there exist a universal constant C such that, for every convex origin symmetric body K in any dimension, we have $L_K < C$?

An equivalent formulation of this problem is whether there exists a universal constant C_1 such that for every origin symmetric convex body in $\mathbb{R}^n$ the following inequality holds

$$(\mathrm{vol}(K))^{(n-1)/n} \leq C_1 \max_{\xi \in S^{n-1}} \mathrm{vol}(K \cap \xi^\perp), \tag{7.1}$$

where $\xi^\perp$ is the central hyperplane orthogonal to ξ. In other words, does there exist a universal constant such that every convex origin symmetric body of volume one has a hyperplane section of volume greater than this universal constant?

The problem remains open, the best known estimate up to date is $L_K \leq O(n^{1/4} \log n)$, as proved by Bourgain [Bou1]. However there are many classes of bodies for which the slicing problem holds true with a constant independent of dimension (see e.g [Ba2], [BKM], [KMP], [MiP]). In particular the slicing problem is solved for the unit balls of quotients of L_p, $p > 1$ by Junge [J] and later for the unit balls of subspaces of L_p, $p \geq 1$ by E. Milman [M]. As $p \to \infty$ the latter would have solved the problem, hadn't the constant behaved at infinity as $\sqrt{p}$.

We try a different approach, considering negative values of p. The concept of embedding in L_{-p} with $0 < p < n$ was introduced in [Ko7], and it was proved that a space $(\mathbb{R}^n, \|\cdot\|)$ embeds in L_{-p} if and only if the Fourier transform of $\|\cdot\|^{-p}$ is a positive distribution in $\mathbb{R}^n$. We will call unit balls of such spaces L_{-p}-*balls.* For example, L_{-1}-balls are intersection bodies and L_{-k} balls are k-intersection bodies, see [Ko8].

We would like to know whether the statement of the slicing problem is true for L_p-balls with p negative. Of course, if one could show this for $p \in (-n, -n+3]$, then one would solve the slicing problem completely, since for any convex body $K \in \mathbb{R}^n$, the space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_p for such values of p, see [Ko12, Section 4.2]. In this paper we employ Khinchin type inequalities, discussed above, to show that the slicing problem is true for L_p-balls, $p > -2$.

For other results on the slicing problem we refer the reader to [Bou3], [D], [Kl], [MiP], [Pao].

7.2 Subspaces of L_p with $p > 2$.

In this section we give a different proof of the result of E. Milman mentioned in the introduction. Note that if $0 \le p \le 2$ then the unit ball of the finite-dimensional subspace of L_p is an intersection body (see [Ko7] for $0 < p \le 2$ and [KKYY] for $p = 0$), and the slicing problem for such bodies follows from the positive part of the Busemann-Petty problem. This problem asks the following question. Let K and L be two origin-symmetric convex bodies in $\mathbb{R}^n$, such that $\mathrm{vol}_{n-1}(K \cap H) \le \mathrm{vol}_{n-1}(L \cap H)$ for every central hyperplane H. Is it true that $\mathrm{vol}_n(K) \le \mathrm{vol}_n(L)$? The connection between intersection bodies and the Busemann-Petty problem was found by Lutwak [Lu1]. The answer to the problem is affirmative if K is an intersection body and L is any origin symmetric star body. Hence, in order to prove the slicing problem for intersection bodies it is enough to take L to be the Euclidean ball of the same volume as K, see [MiP, Proposition 5.5].

In view of the previous remarks it is enough to consider $p > 2$.

Theorem 7.2.1. *Let $p > 2$, there exists a constant $C(p)$ depending only on p such that $L_K \leq C(p)$ for the unit ball K of any finite-dimensional subspace of L_p. Moreover, $C(p) = O(\sqrt{p})$, as $p \to \infty$.*

Proof. According to a theorem of Lewis [Le] (see also [LYZ2, Theorem 8.2] for the following formulation), if $(\mathbb{R}^n, \|\cdot\|_K)$ is a subspace of L_p, $p \geq 1$, then there exists a finite Borel measure μ on S^{n-1} such that for all $x \in \mathbb{R}^n$

$$\|x\|_K^p = \int_{S^{n-1}} |(x,u)|^p \, d\mu(u), \tag{7.2}$$

and

$$|x|^2 = \int_{S^{n-1}} |(x,u)|^2 \, d\mu(u). \tag{7.3}$$

On the other hand, for any body K one has (see [MiP, Section 1.6])

$$L_K^2 \leq \frac{1}{n(\mathrm{vol}(K))^{1+2/n}} \int_K |x|^2 \, dx. \tag{7.4}$$

Using formula (7.3), applying Hölder's inequality twice and then using formula (7.2) we get

$$\int_K |x|^2 \, dx = \int_K \int_{S^{n-1}} |(x,u)|^2 \, d\mu(u) \, dx$$
$$\leq (\mathrm{vol}(K))^{1-2/p} \int_{S^{n-1}} \left(\int_K |(x,u)|^p \, dx \right)^{2/p} d\mu(u)$$
$$\leq (\mathrm{vol}(K))^{1-2/p} \left(\int_{S^{n-1}} \int_K |(x,u)|^p \, dx \, d\mu(u) \right)^{2/p} \left(\int_{S^{n-1}} d\mu(u) \right)^{1-2/p}$$

$$= (\mathrm{vol}(K))^{1-2/p} \left(\int_K \|x\|_K^p \, dx\right)^{2/p} \left(\int_{S^{n-1}} d\mu(u)\right)^{1-2/p}.$$

Passing to polar coordinates one can easily check that

$$\int_K \|x\|_K^p \, dx = \frac{n}{n+p}\mathrm{vol}(K),$$

therefore the previous computations combined with inequality (7.4) yield

$$\begin{aligned} L_K^2 &\leq \frac{1}{n}(\mathrm{vol}(K))^{-2/n}\Big(\frac{n}{n+p}\Big)^{2/p} \left(\int_{S^{n-1}} d\mu(u)\right)^{1-2/p} \\ &\leq \frac{1}{n}(\mathrm{vol}(K))^{-2/n} \left(\int_{S^{n-1}} d\mu(u)\right)^{1-2/p}. \end{aligned} \tag{7.5}$$

Let us estimate from below the volume of the body K. Let σ be the normalized Haar measure on the sphere.

$$\int_{S^{n-1}} \|x\|_K^p \, d\sigma(x) = \int_{S^{n-1}} \int_{S^{n-1}} |(x,u)|^p \, d\mu(u) \, d\sigma(x) =$$

$$= \int_{S^{n-1}} |x_1|^p \, d\sigma(x) \cdot \int_{S^{n-1}} d\mu(u) \leq \left(\frac{Cp}{n+p}\right)^{p/2} \int_{S^{n-1}} d\mu(u),$$

where C is an absolute constant. The latter estimate follows, for example, from [Ko12, Lemma 3.12] and Stirling's formula.

We get

$$\frac{Cp}{n+p} \left(\int_{S^{n-1}} d\mu(u)\right)^{2/p} \geq \Big(\int_{S^{n-1}} \|x\|_K^p \, d\sigma(x)\Big)^{2/p} \geq$$

$$\geq \Big(\int_{S^{n-1}} \|x\|_K^{-n} \, d\sigma(x)\Big)^{-2/n} = \big(\mathrm{vol}(K)/\mathrm{vol}(B_2^n)\big)^{-2/n} \sim \frac{1}{n}(\mathrm{vol}(K))^{-2/n},$$

since $\mathrm{vol}(B_2^n)^{1/n} \sim n^{-1/2}$, meaning that $\mathrm{vol}(B_2^n)^{1/n} n^{1/2}$ approaches a non-zero constant, as $n \to \infty$, see e.g. [Ko12, p.32].

Therefore inequality (7.5) implies

$$L_K^2 \leq \frac{Cp}{n+p} \int_{S^{n-1}} d\mu(u), \tag{7.6}$$

where C is an absolute constant (possibly different from the one used above).

Finally let us compute the measure of S^{n-1} with respect to μ. Integrating equation (7.3) with respect to σ we get

$$1 = \int_{S^{n-1}} |x|^2 \, d\sigma(x) = \int_{S^{n-1}} \int_{S^{n-1}} (x, u)^2 \, d\mu(u) \, d\sigma(x)$$
$$= \int_{S^{n-1}} |x_1|^2 \, d\sigma(x) \cdot \int_{S^{n-1}} d\mu(u) = \frac{1}{n} \int_{S^{n-1}} d\mu(u).$$

This equality together with (7.6) implies

$$L_K \leq C\sqrt{p}.$$

□

7.3 Subspaces of L_p with $p < 0$

Recall that Koldobsky [Ko7] proved that a space $(\mathbb{R}^n, \|\cdot\|)$ embeds in L_{-p} if and only if the Fourier transform of $\|\cdot\|^{-p}$ is a positive distribution in $\mathbb{R}^n$. We will call unit balls of such spaces **p-intersection bodies** or **L_{-p}-balls**.

Lemma 7.3.1. *Let K be an infinitely smooth origin symmetric convex body in $\mathbb{R}^n$. If K is a p-intersection body, $0 < p < n$, then*

$$(\mathrm{vol}(K))^{(n-p)/n} \leq C(n,p) \max_{\xi \in S^{n-1}} (\|x\|_K^{-n+p})^\wedge(\xi),$$

where

$$C(n,p) = \frac{2^{1-p} \ \pi^{-p/2}}{\Gamma(\frac{p}{2})} \frac{1}{n^{(n-p)/n}} \frac{\Gamma(\frac{n-p}{2})}{2\pi^{(n-p)/2}} |S^{n-1}|^{(n-p)/n}.$$

Proof. Using the formula for the volume in polar coordinates and Parseval's formula

$$\mathrm{vol}(K) = \frac{1}{n} \int_{S^{n-1}} \|x\|_K^{-n} dx = \frac{1}{n} \int_{S^{n-1}} \|x\|_K^{-p} \|x\|_K^{-n+p} dx$$

$$= \frac{1}{(2\pi)^n n} \int_{S^{n-1}} (\|x\|_K^{-p})^\wedge(\xi) (\|x\|_K^{-n+p})^\wedge(\xi) d\xi.$$

If K is a $p-$intersection body, then $(\|x\|_K^{-p})^\wedge(\xi) \geq 0$, therefore

$$\mathrm{vol}(K) \leq \frac{1}{(2\pi)^n n} \int_{S^{n-1}} (\|x\|_K^{-p})^\wedge(\xi) d\xi \cdot \max_{\xi \in S^{n-1}} (\|x\|_K^{-n+p})^\wedge(\xi).$$

Using that (see [GS, p.192]):

$$(|x|_2^{-n+p})^\wedge(\xi) = 2^p \pi^{n/2} \frac{\Gamma(\frac{p}{2})}{\Gamma(\frac{n-p}{2})} |\xi|_2^{-p},$$

and applying Parseval's formula again, we get

$$\mathrm{vol}(K) \leq \frac{2^{-p}\pi^{-n/2}}{(2\pi)^n n} \frac{\Gamma(\frac{n-p}{2})}{\Gamma(\frac{p}{2})} \int_{S^{n-1}} (\|x\|_K^{-p})^\wedge(\xi) (|x|_2^{-n+p})^\wedge(\xi) d\xi \times$$

$$\times \max_{\xi \in S^{n-1}} (\|x\|_K^{-n+p})^\wedge(\xi)$$

$$= \frac{2^{-p}\pi^{-n/2}}{n} \frac{\Gamma(\frac{n-p}{2})}{\Gamma(\frac{p}{2})} \int_{S^{n-1}} \|x\|_K^{-p} dx \cdot \max_{\xi \in S^{n-1}} (\|x\|_K^{-n+p})^\wedge(\xi)$$

$$\leq \frac{2^{-p}\pi^{-n/2}}{n} \frac{\Gamma(\frac{n-p}{2})}{\Gamma(\frac{p}{2})} \left(\int_{S^{n-1}} \|x\|_K^{-n} dx \right)^{p/n} \cdot |S^{n-1}|^{(n-p)/n} \cdot \max_{\xi \in S^{n-1}} (\|x\|_K^{-n+p})^\wedge(\xi)$$

$$= C(n,p) (\mathrm{vol}(K))^{p/n} \cdot \max_{\xi \in S^{n-1}} (\|x\|_K^{-n+p})^\wedge(\xi).$$

□

Lemma 7.3.2. *Let $0 \leq p < n$ and $C(n,p)$ as defined above, then*

$$C(n,p) \cdot (n-p) \leq \frac{2^{1-p}\ \pi^{-p/2}}{\Gamma(\frac{p}{2})}.$$

Proof. We need to show that

$$\frac{(n-p)}{n^{(n-p)/n}} \frac{\Gamma(\frac{n-p}{2})}{2\pi^{(n-p)/2}} |S^{n-1}|^{(n-p)/n} \leq 1.$$

The left-hand side is equal to

$$\frac{(n-p)}{n^{(n-p)/n}} \frac{\Gamma(\frac{n-p}{2})}{2\pi^{(n-p)/2}} \left(\frac{2\pi^{n/2}}{\Gamma(n/2)} \right)^{(n-p)/n} = \frac{\Gamma(\frac{n-p}{2}+1)}{(\Gamma(n/2+1))^{(n-p)/n}}.$$

Since the function $\log(\Gamma(x))$ is convex [Ko12, p.30], we have

$$\frac{\log(\Gamma(n/2+1)) - \log(\Gamma(1))}{n/2} \geq \frac{\log(\Gamma((n-p)/2+1)) - \log(\Gamma(1))}{(n-p)/2},$$

therefore

$$(\Gamma(n/2+1))^{n/2} \geq (\Gamma((n-p)/2+1))^{(n-p)/2}.$$

$\square$

From the result of Lemma 7.3.1 it follows that one can obtain inequalities of type (7.1) by finding a good upper estimate for $(\|x\|_K^{-n+p})^\wedge(\xi)$ in terms of the section function. This will be the main objective of the next section.

7.4 Khinchin type inequalities

Let K be an origin symmetric convex body. For $\xi \in S^{n-1}$, consider the parallel section function $A_{K,\xi}$ on $\mathbb{R}$ defined by

$$A_{K,\xi}(t) = \mathrm{vol}_{n-1}\left(K \cap \{(x,\xi) = t\}\right).$$

In this section we are interested in Khinchin-type inequalities that would give an upper bound for $(\|x\|_K^{-n+p})^\wedge(\xi)$ in terms of $A_{K,\xi}(0)$, the central section.

Recall that if K has an infinitely smooth boundary then the fractional derivatives of the function $A_{K,\xi}$ can be computed in terms of the Fourier transform of the Minkowski functional raised to certain powers. Namely, for $p > 0$, $p \neq n$ we have

$$A_{K,\xi}^{(-1+p)}(0) = \frac{\sin(\pi p/2)}{\pi(n-p)} \left(\|x\|_K^{-n+p}\right)^\wedge(\xi). \tag{7.7}$$

In particular, $\left(\|x\|_K^{-n+p}\right)^\wedge(\xi)$ is a continuous function on the sphere S^{n-1}, see also [Ko12, Section 3.3].

Note also that from the definition of fractional derivative it follows that for $0 < p < 1$

$$A_{K,\xi}^{(-1+p)}(0) = \frac{1}{\Gamma(1-p)} \int_0^\infty t^{-p} A_{K,\xi}(t) dt \tag{7.8}$$

and for $1 < p < 2$

$$A_{K,\xi}^{(-1+p)}(0) = \frac{1}{\Gamma(1-p)} \int_0^\infty t^{-p} (A_{K,\xi}(t) - A_{K,\xi}(0)) dt. \tag{7.9}$$

Lemma 7.4.1. *Let K be an origin symmetric convex infinitely smooth body in $\mathbb{R}^n$. Then for $p \in (0,1)$ we have*

$$\left(\|x\|_K^{-n+p}\right)^\wedge(\xi) \le \frac{2^{p-1}\pi(n-p)}{\Gamma(2-p)\sin(\pi p/2)} \left(\mathrm{vol}(K)\right)^{(1-p)} \left(A_{K,\xi}(0)\right)^p.$$

Proof. From [MiP, p.76] it follows that

$$F(q) = \left((q+1)\int_0^\infty t^q \frac{A_{K,\xi}(t)}{A_{K,\xi}(0)} dt\right)^{1/(1+q)}$$

is an increasing function of q on $(-1, \infty)$.

Therefore, taking $q = -p$ with $0 < p < 1$ and using $F(-p) \le F(0)$ we get

$$\left((1-p)\int_0^\infty t^{-p} \frac{A_{K,\xi}(t)}{A_{K,\xi}(0)} dt\right)^{1/(1-p)} \le \int_0^\infty \frac{A_{K,\xi}(t)}{A_{K,\xi}(0)} dt = \frac{\mathrm{vol}(K)}{2\, A_{K,\xi}(0)}.$$

Using formulas (7.7), (7.8) and applying the previous inequality , we get

$$\begin{aligned}
\left(\|x\|_K^{-n+p}\right)^\wedge(\xi) &= \frac{\pi(n-p)}{\sin(\pi p/2)} A_{K,\xi}^{(-1+p)}(0) \\
&= \frac{\pi(n-p)}{\Gamma(1-p)\sin(\pi p/2)} \int_0^\infty t^{-p} A_{K,\xi}(t) dt \\
&\le \frac{2^{p-1}\pi(n-p)}{(1-p)\Gamma(1-p)\sin(\pi p/2)} \left(\mathrm{vol}(K)\right)^{(1-p)} \left(A_{K,\xi}(0)\right)^p \\
&= \frac{2^{p-1}\pi(n-p)}{\Gamma(2-p)\sin(\pi p/2)} \left(\mathrm{vol}(K)\right)^{(1-p)} \left(A_{K,\xi}(0)\right)^p.
\end{aligned}$$

$\square$

Note, that Khinchin type inequalities for negative exponents were also studied by Guédon [Gu], and the previous Lemma could be also derived from his results.

Next we will be interested in Khinchin type inequalities for $p \in (1,2)$.

Lemma 7.4.2. *Let K be an origin symmetric convex body in $\mathbb{R}^n$. Then for $p \in (1,2)$ we have*

$$\left(\|x\|_K^{-n+p}\right)^\wedge(\xi) \leq \frac{2^{p-1}\pi(n-p)}{\sin(\pi p/2)} A_{K,\xi}(0)^p \left(\mathrm{vol}(K)\right)^{1-p}.$$

Proof. What follows is similar to [MiP, Section 2.6]. Consider the function

$$G(p) = \left(\frac{\int_0^\infty t^{-p} \frac{A_{K,\xi}(0) - A_{K,\xi}(t)}{A_{K,\xi}(0)} dt}{\int_0^\infty t^{-p}(1-e^{-t})dt} \right)^{\frac{1}{1-p}}.$$

We want to show that it is increasing on $(1,2)$.

Let $\Phi(t) = \log A_{K,\xi}(0) - \log A_{K,\xi}(t)$. By Brunn's theorem, $\Phi(t) \geq 0$ and it is increasing and convex on the support of $A_{K,\xi}(t)$. Now

$$G(p) = \left(\frac{\int_0^\infty t^{-p}(1-e^{-\Phi(t)})dt}{\int_0^\infty t^{-p}(1-e^{-t})dt} \right)^{\frac{1}{1-p}}.$$

Let $\alpha = 1/G(p)$, then it is not hard to check that

$$\int_0^\infty t^{-p}(1-e^{-\alpha t})dt = \int_0^\infty t^{-p}(1-e^{-\Phi(t)})dt.$$

Consider the function

$$H(x) = \int_x^\infty t^{-p}(e^{-\Phi(t)} - e^{-\alpha t})dt.$$

We want to show that $H(x) \leq 0$ for $x \in [0,\infty)$. Since $H(0) = H(\infty) = 0$, it suffices to show that $H(x)$ is first decreasing and then increasing.

Indeed,

$$H'(x) = -x^{-p}(e^{-\Phi(x)} - e^{-\alpha x}).$$

Since $\Phi(x)$ is increasing and convex, there is a point x_0, such that $\Phi(x) \le \alpha x$ for $0 < x < x_0$ and $\Phi(x) \ge \alpha x$ for $x > x_0$. Therefore $H'(x) \le 0$ if $0 < x < x_0$ and $H'(x) \ge 0$ if $x > x_0$. So, we have proved that $H(x) \le 0$, which means that for every $x > 0$

$$\int_x^\infty t^{-p}(1 - e^{-\Phi(t)})dt \ge \int_x^\infty t^{-p}(1 - e^{-\alpha t})dt.$$

Now let $1 < q < p < 2$, we have

$$\int_0^\infty t^{-q}(1 - e^{-\Phi(t)})dt = (p-q)\int_0^\infty x^{p-q-1}\int_x^\infty t^{-p}(1 - e^{-\Phi(t)})dt$$
$$\ge (p-q)\int_0^\infty x^{p-q-1}\int_x^\infty t^{-p}(1 - e^{-\Phi(t)})dt = \int_0^\infty t^{-q}(1 - e^{-\alpha t})dt$$
$$= \alpha^{q-1}\int_0^\infty t^{-q}(1 - e^{-t})dt.$$

Therefore, using the definition of α, we get

$$\frac{\int_0^\infty t^{-q}(1 - e^{-\Phi(t)})dt}{\int_0^\infty t^{-q}(1 - e^{-t})dt} \ge G(p)^{1-q}$$

or

$$G(q) \le G(p).$$

So, $G(p)$ is increasing on $(1, 2)$. One can also check that (see Appendix)

$$\lim_{p \to 1^+} G(p) = \exp\Big(\int_0^\infty t^{-1}(e^{-\Phi(t)} - e^{-t})dt\Big).$$

Since, $G(p)$ is increasing on $(1, 2)$, we get

$$G(p) \ge \exp\Big(\int_0^\infty t^{-1}(e^{-\Phi(t)} - e^{-t})dt\Big)$$

for $p \in (1, 2)$.

If we extend the function $G(p)$ to $p \in (0, 1)$ by the formula:

$$G(p) = \left(\frac{\int_0^\infty t^{-p} e^{-\Phi(t)} dt}{\int_0^\infty t^{-p} e^{-t} dt} \right)^{\frac{1}{1-p}},$$

then according to [MiP, p.81], this function is increasing on $(0, 1)$ and therefore

$$G(p) \geq G(0) = \frac{\mathrm{vol}(K)}{2A_{K,\xi}(0)}.$$

One can show that $\lim_{p \to 1^+} G(p) = \lim_{p \to 1^-} G(p)$, therefore for $p \in (1, 2)$

$$G(p) \geq \frac{\mathrm{vol}(K)}{2A_{K,\xi}(0)}.$$

So, for $p \in (1, 2)$

$$\left(\frac{\int_0^\infty t^{-p} \frac{A_{K,\xi}(0) - A_{K,\xi}(t)}{A_{K,\xi}(0)} dt}{\int_0^\infty t^{-p} (1 - e^{-t}) dt} \right)^{\frac{1}{1-p}} \geq \frac{\mathrm{vol}(K)}{2A_{K,\xi}(0)}$$

or

$$\frac{1}{\Gamma(1-p)} \int_0^\infty t^{-p} \frac{A_{K,\xi}(t) - A_{K,\xi}(0)}{A_{K,\xi}(0)} dt \leq \left(\frac{\mathrm{vol}(K)}{2A_{K,\xi}(0)} \right)^{1-p}.$$

Using formulas (7.7), (7.9) and applying the previous inequality , we get

$$\begin{aligned} \left(\|x\|_K^{-n+p} \right)^\wedge (\xi) &= \frac{\pi(n-p)}{\sin(\pi p/2)} A_{K,\xi}^{(-1+p)}(0) \\ = \frac{\pi(n-p)}{\sin(\pi p/2)\Gamma(1-p)} &\int_0^\infty t^{-p} (A_{K,\xi}(t) - A_{K,\xi}(0)) dt \\ &\leq \frac{2^{p-1}\pi(n-p)}{\sin(\pi p/2)} A_{K,\xi}(0)^p \left(\mathrm{vol}(K)\right)^{1-p}. \end{aligned}$$

$\square$

7.5 Slicing problem for L_p-balls, $-2 < p < 0$.

In this section we combine the results of previous sections to prove the slicing for L_p-balls, $-2 < p < 0$. Note that it is enough to consider only infinitely smooth bodies, since every L_p-ball can be approximated in the radial metric by infinitely smooth L_p-balls.

Theorem 7.5.1. *Let $0 < p < 1$, if K is an infinitely smooth convex p-intersection body, then*

$$(\mathrm{vol}(K))^{(n-1)/n} \le C(p) \max_{\xi \in S^{n-1}} A_{K,\xi}(0),$$

where

$$C(p) = \left(\frac{\pi^{1-p/2}}{\Gamma(p/2)\Gamma(2-p)\sin(\pi p/2)} \right)^{1/p}.$$

Proof. Combining the result of Lemma 7.4.1 with the inequality

$$(\mathrm{vol}(K))^{(n-p)/n} \le C(n,p) \max_{\xi \in S^{n-1}} (\|x\|_K^{-n+p})^\wedge(\xi)$$

from Lemma 7.3.1 we get

$$(\mathrm{vol}(K))^{(pn-p)/n} \le \frac{C(n,p)2^{p-1}\pi(n-p)}{\Gamma(2-p)\sin(\pi p/2)} \max_{\xi \in S^{n-1}} (A_{K,\xi}(0))^p$$

or

$$(\mathrm{vol}(K))^{(n-1)/n} \le \left(\frac{C(n,p)2^{p-1}\pi(n-p)}{\Gamma(2-p)\sin(\pi p/2)} \right)^{1/p} \max_{\xi \in S^{n-1}} A_{K,\xi}(0).$$

Invoking Lemma 7.3.2, we get the result.

□

Now we are interested in the values $1 < p < 2$.

Theorem 7.5.2. *Let $1 < p < 2$, if K is an infinitely smooth convex p-intersection body, then*

$$(\mathrm{vol}(K))^{(n-1)/n} \leq C(p) \max_{\xi \in S^{n-1}} A_{K,\xi}(0),$$

where

$$C(p) = \left(\frac{\pi^{1-p/2}}{\Gamma(p/2)\sin(\pi p/2)} \right)^{1/p}.$$

Proof. Using Lemmas 7.3.1 and 7.4.2 we get

$$(\mathrm{vol}(K))^{(np-p)/n} \leq C(n,p) \frac{2^{p-1}\pi(n-p)}{\sin(\pi p/2)} \max_{\xi \in S^{n-1}} A_{K,\xi}(0)^p.$$

Again recalling Lemma 7.3.2 we get the result.

□

7.6 Appendix

Here we compute the limit that we used in Lemma 7.4.2.

Lemma 7.6.1. *Let*

$$G(p) = \left(\frac{\int_0^\infty t^{-p} \frac{A_{K,\xi}(0) - A_{K,\xi}(t)}{A_{K,\xi}(0)} dt}{\int_0^\infty t^{-p}(1-e^{-t})dt} \right)^{\frac{1}{1-p}}.$$

Then

$$\lim_{p \to 1^+} G(p) = \exp\left(\int_0^\infty t^{-1} \left(\frac{A_{K,\xi}(t)}{A_{K,\xi}(0)} - e^{-t} \right) dt \right).$$

Proof. Denote $\Phi(t) = \log A_{K,\xi}(0) - \log A_{K,\xi}(t)$. We have

$$\lim_{p \to 1^+} \left(\frac{\int_0^\infty t^{-p}(1-e^{-\Phi(t)})dt}{\int_0^\infty t^{-p}(1-e^{-t})dt} \right)^{\frac{1}{1-p}}$$

$$= \lim_{p \to 1^+} \exp\left(\frac{1}{1-p} \left[\log\left(\int_0^\infty t^{-p}(1-e^{-\Phi(t)})dt \right) - \log\left(\int_0^\infty t^{-p}(1-e^{-t})dt \right) \right] \right)$$

$$= \lim_{p\to 1^+} \exp\Big(\frac{1}{1-p}\Big[\log\Big(\int_0^1 t^{-p}(1-e^{-\Phi(t)})dt - \frac{1}{1-p} - \int_1^\infty t^{-p}e^{-\Phi(t)}dt\Big)$$
$$- \log\Big(\int_0^1 t^{-p}(1-e^{-t})dt - \frac{1}{1-p} - \int_1^\infty t^{-p}e^{-t}dt\Big)\Big]\Big)$$

Applying l'Hospital's rule, we get

$$= \lim_{p\to 1^+} \exp\Big(-\frac{\int_0^1 t^{-p}\log t\,(1-e^{-\Phi(t)})dt - \frac{1}{(1-p)^2} - \int_1^\infty t^{-p}\log t\, e^{-\Phi(t)}dt}{\int_0^\infty t^{-p}(1-e^{-\Phi(t)})dt}$$
$$+\frac{\int_0^1 t^{-p}\log t\,(1-e^{-t})dt - \frac{1}{(1-p)^2} - \int_1^\infty t^{-p}\log t\, e^{-t}dt}{\int_0^\infty t^{-p}(1-e^{-t})dt}\Big)$$

$$= \lim_{p\to 1^+} \exp\Big(-\frac{\Big(\int_0^1 t^{-p}\log t\,(1-e^{-\Phi(t)})dt - \frac{1}{(1-p)^2} - \int_1^\infty t^{-p}\log t\, e^{-\Phi(t)}dt\Big)}{\int_0^\infty t^{-p}(1-e^{-\Phi(t)})dt\int_0^\infty t^{-p}(1-e^{-t})dt}\times$$
$$\times\int_0^\infty t^{-p}(1-e^{-t})dt+$$
$$+\frac{\Big(\int_0^1 t^{-p}\log t\,(1-e^{-t})dt - \frac{1}{(1-p)^2} - \int_1^\infty t^{-p}\log t\, e^{-t}dt\Big)}{\int_0^\infty t^{-p}(1-e^{-\Phi(t)})dt\int_0^\infty t^{-p}(1-e^{-t})dt}\times$$
$$\times\int_0^\infty t^{-p}(1-e^{-\Phi(t)})dt\Big)$$

Now multiplying both the numerator and denominator by $(1-p)^2$ and using that

$$\lim_{p\to 1^+}(p-1)\int_0^\infty t^{-p}(1-e^{-\Phi(t)})dt =$$
$$= \lim_{p\to 1^+}(p-1)\left[\int_0^1 t^{-p}(1-e^{-\Phi(t)})dt + \int_1^\infty t^{-p}dt - \int_1^\infty t^{-p}e^{-\Phi(t)}dt\right]$$
$$= \lim_{p\to 1^+}(p-1)\left[\int_0^1 t^{-p}(1-e^{-\Phi(t)})dt + \frac{1}{p-1} - \int_1^\infty t^{-p}e^{-\Phi(t)}dt\right] = 1$$

and similarly

$$\lim_{p\to 1^+}(p-1)\int_0^\infty t^{-p}(1-e^{-t})dt = 1$$

we get

$$\lim_{p\to 1^+} G(p) = \lim_{p\to 1^+} \exp\Big(\int_0^\infty t^{-p}(1-e^{-t})dt - \int_0^\infty t^{-p}(1-e^{-\Phi(t)})dt\Big)$$

$$= \exp\Big(\int_0^\infty t^{-1}(e^{-\Phi(t)} - e^{-t})dt\Big)$$

$\square$

Bibliography

[A] YU. A. AMINOV, *The geometry of submanifolds*, Gordon and Breach Science Publishers, Amsterdam, 2001.

[Ba1] K. BALL, *Some remarks on the geometry of convex sets*, in: Geometric Aspects of Functional Analysis, ed. by J. Lindenstrauss and V. D. Milman, Lecture Notes in Math., **1317**, Springer, Heidelberg, 1988, pp. 224–231.

[Ba2] K. BALL, *Normed spaces with a weak-Gordon-Lewis property*, Functional analysis (Austin, TX, 1987/1989), 36–47, Lecture Notes in Math., **1470**, Springer, Berlin, 1991.

[BDK] J. BRETAGNOLLE, D. DACUNHA-CASTELLE AND J. L. KRIVINE, *Lois stables et espaces L_p*, Ann. Inst. H.Poincaré Probab. Statist. **2** (1966) 231–259.

[BKM] J. BOURGAIN, B. KLARTAG, V. MILMAN, *Symmetrization and isotropic constants of convex bodies*, Geometric aspects of functional analysis, 101–115, Lecture Notes in Math., **1850**, Springer, Berlin, 2004.

[BL] Y. Benyamini and J. Lindenstrauss, *Geometric nonlinear functional analysis*, Colloqium publications, vol.48, American Mathematical Society, 2000.

[Bol] E. D. Bolker, *A class of convex bodies*, Trans. Amer. Math. Soc., **145** (1969), 323–345.

[Bou1] J. Bourgain, *On the distribution of polynomials on high dimensional convex sets,* Geometric aspects of functional analysis (1989-90), 127–137, Lecture Notes in Math., **1469**, Springer, Berlin.

[Bou2] J. Bourgain, *On the Busemann-Petty problem for perturbations of the ball,* Geom. Func. Anal. **1** (1991), 1–13.

[Bou3] J. Bourgain, *On the isotropy-constant problem for "psi-2"-bodies*, Geometric aspects of functional analysis, 114–121, Lecture Notes in Math., **1807**, Springer, Berlin, 2003.

[BP] H. Busemann, C. M. Petty, *Problems on convex bodies*, Math. Scand. **4** (1956), 88–94.

[BZ] J. Bourgain, Gaoyong Zhang, *On a generalization of the Busemann-Petty problem*, Convex geometric analysis (Berkeley, CA, 1996), 65-76, Math. Sci. Res. Inst. Publ., **34**, Cambriddge Univ.Press, Cambridge, 1999.

[CG] S. Campi, P. Gronchi , *The L_p-Busemann-Petty centroid inequality*, Adv. Math. **167** (2002), 128–141.

[D] S. Dar, *Remarks on Bourgain's problem on slicing of convex bodies*, Geometric aspects of functional analysis (Israel, 1992–1994), 61–66, Oper. Theory Adv. Appl., **77**, Birkhauser, Basel, 1995.

[DFN] B. A. Dubrovin, A. T. Fomenko, S. P. Novikov, *Modern geometry - methods and applications. Part I. The geometry of surfaces, transformation groups, and fields. Second edition*, Springer-Verlag, New York, 1992.

[Ga1] R. J. Gardner, *Instersection bodies and the Busemann-Petty problem*, Trans. Amer. Math. Soc. **342** (1994), 435–445.

[Ga2] R. J. Gardner, *A positive answer to the Busemann-Petty problem in three dimensions*, Annals of Math. **140** (1994), 435–447.

[Ga3] R. J. Gardner, *Geometric tomography*, Cambridge University Press, New York, 1995.

[Ga4] R. J. Gardner, *The Brunn-Minkowski inequality*, Bulletin (New Series) of the American Mathematical Society, Vol. 39 (2002), Number 3, 355-405.

[GHS] F. Gao, D. Hug, R. Schneider, *Intrinsic volumes and polar sets in spherical space*, Math. Notae, Año XLI (2001/02), 159–176 (2003).

[Gi] A. Giannopoulos, *A note on a problem of H. Busemann and C. M. Petty concerning sections of symmetric convex bodies*, Mathematika, **37** (1990), 239–244.

[Gu] O. GUÉDON, *Kahane-Khinchine type inequalities for negative exponent*, Mathematika, **46** (1999), 165–173.

[GKS] R. J. GARDNER, A. KOLDOBSKY, T. SCHLUMPRECHT, *An analytic solution to the Busemann-Petty problem on sections of convex bodies*, Annals of Math. **149** (1999), 691–703.

[GS] I. M. GELFAND, G. E. SHILOV, *Generalized Functions, vol.1 Properties and Operetaions*, Academic Press, New York and London, 1964.

[GV] I. M. GELFAND, N. YA. VILENKIN, *Generalized functions, vol.4. Applications of harmonic analysis*, Academic Press, New York, 1964.

[GW] P. GOODEY AND W. WEIL, *Intersection bodies and ellipsoids*, Mathematika, **42** (1995), 295–304.

[GZ] E. GRINBERG, G. ZHANG, *Convolutions, transforms and convex bodies*, Proc. London Math. Soc. (3)**78** (1999), 77–115.

[J] M. JUNGE, *Hyperplane conjecture for quotient spaces of L_p*, Forum Math. **6** (1994), no. 5, 617–635.

[KK1] N. J. KALTON AND A. KOLDOBSKY, *Banach spaces embedding isometrically into L_p when $0 < p < 1$*, Proc. Amer. Math. Soc. **132** (2003) 67–76.

[KK2] N. J. KALTON AND A. KOLDOBSKY, *Intersection bodies and L_p-spaces*, Advances in Math. **196** (2005), no. 2, 257–275.

[KKYY] N. J. Kalton, A. Koldobsky, V. Yaskin and M. Yaskina, *The geometry of L_0*, Canadian J. Math., to appear.

[Kl] B. Klartag, *An isomorphic version of the slicing problem*, J. Funct. Anal. **218** (2005), no. 2, 372–394.

[Ko1] A. Koldobsky, *Schoenberg's problem on positive definite functions*, English translation, St.Petersburg Math.J., **3** (1992), 563–570; Algebra i Analiz, **3** (1991), 78–85 (Russian).

[Ko2] A. Koldobsky, *Generalized Lévy representation of norms and isometric embeddings into L_p-spaces*, Ann. Inst. H.Poincaré Sér. B **28** (1992), 335–353.

[Ko3] A. Koldobsky, *Inverse formula for the Blashke-Levy representation*, Houston J. Math. **23** (1997), 95–108.

[Ko4] A. Koldobsky, *Intersection bodies in $\mathbb{R}^4$*, Advances in Math. **136** (1998), 1–14.

[Ko5] A. Koldobsky, *Intersection bodies, positive definite distributions and the Busemann-Petty problem*, Amer. J. Math, **120** (1998), no.4, 827–840.

[Ko6] A. Koldobsky, *A generalization of the Busemann-Petty problem on sections of convex bodies*, Israel J. Math. **110** (1999), 75–91.

[Ko7] A. Koldobsky, *Positive definite distributions and subspaces of L_{-p} with applications to stable processes*, Canad. Math. Bull., **42** (1999), no.3, 344–353.

[Ko8] A. Koldobsky, *A functional analytic approach to intersection bodies*, Geom. Funct. Anal. **10**(2000),1507-1526.

[Ko9] A. Koldobsky, *On the derivatives of x-ray functions*, Arch. Math. **79** (2002), 216–222.

[Ko10] A. Koldobsky, *The Busemann-Petty problem via spherical harmonics*, Advances in Math. **177** (2003), 105–114.

[Ko11] A. Koldobsky, *Comparison of volumes of convex bodies by means of areas of central sections*, Adv. Appl. Math., **33** (2004), no. 4, 728–732.

[Ko12] A. Koldobsky, *Fourier Analysis in Convex Geometry*, Mathematical Surveys and Monographs, American Mathematical Society, Providence RI, 2005.

[KMP] H. König, M. Meyer, A. Pajor, *The isotropy constants of the Schatten classes are bounded*, Math. Ann. **312** (1998), no. 4, 773–783.

[KPY] A. Koldobsky, A. Pajor, V. Yaskin, *Khinchin type inequalities and sections of L_p-balls, $p > -2$*, preprint.

[KYY] A. Koldobsky, V. Yaskin, M. Yaskina, *Modified Busemann-Petty problem on sections of convex bodies*, Israel J. Math., to appear.

[Le] D. R. Lewis, *Finite dimensional subspaces of L_p*, Studia Math. **63** (1978), 207–212.

[Lu1] E. Lutwak, *Intersection bodies and dual mixed volumes*, Advances in Math. **71** (1988), 232–261.

[Lu2] E. Lutwak, *Centroid bodies and dual mixed volumes*, Proc. London Math. Soc. (3)**60** (1990), 365–391.

[LR] D. G. Larman and C. A. Rogers, *The existance of a centrally symmetric convex body with central sections that are unexpectedly small*, Mathematika. **22** (1975), 164–175.

[LYZ1] E. Lutwak, D. Yang and G. Zhang, *L_p affine isoperimetric inequalities*, J.Diff.Geom. **56** (2000), 111–132.

[LYZ2] E. Lutwak, D. Yang, and G. Zhang, *L_p John ellipsoids*, Proceedings of the London Mathematical Society, **90** (2005), 497-520.

[LZ] E. Lutwak, G. Zhang, *Blaschke-Santaló inequalities*, J.Diff.Geom. **47** (1997), 1–16.

[M] E. Milman, *Dual mixed volumes and the slicing problem*, Advances in Math., to appear.

[MeP] D. Mejía, Ch. Pommerenke, *On spherically convex univalent functions*, Michigan Math. J., **47** (2000), 163–172.

[MiP] V. D. Milman, A. Pajor, *Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space*, in: Geometric Aspects of Functional Analysis, ed. by J.Lindestrauss and V.D.Milman, Lecture Notes in Mathematics **1376**, Springer, Heidelberg, 1989, 64–104.

[Pao] G. Paouris, *On the isotropic constant of non-symmetric convex bodies*, Geometric aspects of functional analysis, 239–243, Lecture Notes in Math., **1745**, Springer, Berlin, 2000.

[Pap] M. Papadimitrakis, *On the Busemann-Petty problem about the convex, centrally symmetric bodies in* $\mathbb{R}^n$, Mathematika **39** (1992), 258–266.

[Pe] C. M. Petty, *Projection bodies*, Proc. Coll. Convexity (Copenhagen 1965), Kobenhavns Univ. Mat. Inst., 234-241.

[Po] A. V. Pogorelov, *Extrinsic geometry of convex surfaces*, Translations of Mathematical Monographs, vol. 35, American Mathematical Society, Providence, R.I., 1973.

[R] J. G. Ratcliffe, *Foundations of hyperbolic manifolds*, Springer-Verlag, New York, 1994.

[RZ] B. Rubin, Gaoyong Zhang, *Generalizations of the Busemann-Petty problem for sections of convex bodies*, J. Funct. Anal., **213** (2004), 473–501.

[Sc] R. Schneider, *Zur einem Problem von Shephard uber die Projektionen konvexer Korper*, Math. Z. **101** (1967), 71–82 (German).

[Se] V. I. Semyanistyi, *Some integral transformations and integral geometry in an elliptic space*, Trudy Sem. Vector. Tenzor. Anal., **12** (1963), 397–441 (in Russian).

[Y1] V. Yaskin, *The Busemann-Petty problem in hyperbolic and spherical spaces*, Advances in Math., to appear.

[Y2] V. Yaskin, *A solution to the lower dimensional Busemann-Petty problem in hyperbolic space*, preprint.

[YY] V. Yaskin, M. Yaskina *Centroid bodies and comparison of volumes*, Indiana Univ. Math. J., to appear.

[Zh1] Gaoyong Zhang, *Sections of convex bodies* , Amer. J. Math. **118** (1996), 319–340.

[Zh2] Gaoyong Zhang, *A positive answer to the Busemann-Petty problem in four dimensions*, Annals of Math. **149** (1999), 535–543.

[Zv] A. Zvavitch, *The Busemann-Petty problem for arbitrary measures*, Math. Ann., **331** (2005), 867–887.

VITA

Vladyslav Yaskin was born January 27, 1974, in Poltava, Ukraine. After attending public schools in Ukraine, he entered undergraduate studies at Kharkov State University (Ukraine), from where Vladyslav received his Master's Degree in Mathematics in 1997. Presently, he is a Ph.D. student at the Mathematics Department of the University of Missouri-Columbia. Vlad is married to Maryna Yaskina, their daughter Maria was born in 2005.