COMPLEX AND ALMOST-COMPLEX STRUCTURES ON SIX DIMENSIONAL MANIFOLDS

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by
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The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

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presented by James Ryan Brown

a candidate for the degree of Doctor of Philosophy

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ABSTRACT

We investigate the properties of hypothetical exotic complex structures on three dimensional complex projective space $\mathbb{C}\mathrm{P}^3$. This is motivated by the long standing question in differential geometry of whether or not the six sphere S^6 admits an integrable almost-complex structure. An affirmative answer to this question would imply the existence of many exotic complex structures on $\mathbb{C}\mathrm{P}^3$. It is known that $\mathbb{C}\mathrm{P}^3$ admits many topologically different almost-complex structures, but it is unknown whether or not $\mathbb{C}\mathrm{P}^3$ admits an integrable almost-complex structure other that the standard Kähler structure. In this manuscript we give lower bounds on the Hodge numbers of hypothetical exotic structures on $\mathbb{C}\mathrm{P}^3$ and a necessary condition for the Frölicher spectral sequence to degenerate at the second level. We also give topological constraints on the classes of hypothetical exotic complex structures which are $\mathbb{C}^*$-symmetric. We give restrictions on the fixed point sets of such $\mathbb{C}^*$ actions.

Was sich überhaupt sagen lässt, lässt die Worte fassen; und wovon man nicht reden kann, darüber mus man schweigen.

-Ludwig Wittenstein, *Tractatus Logico-Philosophicus*

What can be said at all can be said clearly; and whereof one cannot speak thereof one must be silent.

-C.K. Odgen, English translation

Chapter 1

Preliminaries

A famous long-standing question in differential geometry is whether or not the six sphere S^6 admits a complex structure. Hirzebruch and Yau have each listed this problem among the fundamental open problems in differential geometry in [36, Problem 13] and [72, Problem 52], respectively. Borel and Serre [14] have shown that the only spheres which admit almost-complex structures are the two sphere S^2 and the six sphere S^6, see appendix A for constructions of almost-complex structures on S^2 and S^6 modeled on the sets of quaternions and octonions, respectively. By deforming a given almost-complex structure one obtains many almost-complex structures on S^2 and S^6. It is well-known that S^2 admits a unique complex structure. It is unknown, however, whether or not S^6 admits any integrable almost-complex structure.

Over the past fifty years none of the assertions of resolution of this question has withstood close scrutiny. There have been several published and unpublished manuscripts claiming to show that S^6 does not admit a complex structure, e.g. [2] and [39], but none has gained acceptance among experts. On the other hand there is also a recent preprint [28] which asserts that S^6 is indeed a complex manifold, but this was found to have a gap in the proof of the main theorem and was quickly withdrawn by its author, leaving the question unanswered still. In addition to having no definitive answer to this existence question, there are relatively few results which give an indication of what the actual answer is.

In this chapter we list and summarize the prerequisite material and previous results in this field. In section 1.1 we define complex and almost-complex structures on manifolds and give a classical condition which is both necessary and sufficient for an almost-complex structure to be complex. In section 1.2 we list some of the advances on the six sphere question. In section 1.3 we describe how this question may be reframed in the context of projective space followed by a summary of results on projective space in section 1.4. We conclude in section 1.5 with an announcement of the results to be found in this dissertation.

1.1 Almost-complex structures and torsion

In this section we follow [45] and [46] to describe briefly complex and almost-complex structures on manifolds, and we state a classical theorem giving a necessary and sufficient condition for an almost-complex structure to be integrable.

Recall that a smooth n-dimensional manifold X is a second countable Hausdorff space which admits a smooth atlas $\{U_i, \varphi_i\}$, i.e., $\{U_i\}$ is an open covering of X, $\varphi_i : U_i \to \varphi_i(U_i) \subset \mathbb{R}^n$ is a homeomorphism, and $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$

is smooth. A complex manifold X of complex dimension n is a smooth $2n$-dimensional manifold with the additional conditions $\varphi_i(U_i)$ is homeomorphic to a subset of $\mathbb{C}^n$ and $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ is holomorphic.

We would like to study manifolds X which are not necessarily complex, but retain a particularly important feature of complex manifolds, that is, the tangent bundle TX of X is a complex vector bundle. We digress for a moment into a discussion of complex vector spaces and complex structures on real vector spaces.

Definition 1.1.1. *A* **complex structure** J *on a real vector space* V *is an endomorphism* $J : V \to V$ *with the property* $J^2 = -\mathrm{I}$ *where* I *is the identity map.*

A real vector space V with a complex structure J can be given the structure of a complex vector space by defining complex scalar multiplication by

$$(\alpha + i\beta)v = \alpha v + \beta J(v),$$

where α, $\beta \in \mathbb{R}$ and $v \in V$. Conversely, if V is a complex vector space of complex dimension n, then define and endomorphism of J of V by

$$J(v) := iv.$$

If we consider V as a real vector space of real dimension $2n$, then J is a complex structure on V.

At each point x on a smooth $2n$-dimensional manifold X the tangent space T_xX is a real $2n$-dimensional vector space. If we can endow each tangent space T_xX with a complex structure J_x so that this structure varies smoothly over the manifold, then we have an almost-complex structure on X.

Definition 1.1.2. *Let X be a differentiable manifold of dimension $2n$, and suppose J is a differentiable vector bundle isomorphism*

$$J : TX \to TX$$

such that $J_x : T_xX \to T_xX$ is a complex structure for T_xX, i.e. $J^2 = -\mathrm{I}$ where I is the identity vector bundle isomorphism. Then J is called an **almost-complex structure** *for the differentiable manifold X. A manifold with a fixed almost-complex structure is called an* **almost-complex manifold**.

If X is a complex manifold then X carries a canonical almost-complex structure. Let $\mathbb{C}^n$ be the set of n-tuples of complex numbers $(z_1, \ldots, z_n)$ with $z_j = x_j + iy_j$. With respect to the coordinate system $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ on $\mathbb{C}^n$ define an almost-complex structure J by

$$\begin{aligned} J\left(\frac{\partial}{\partial x_j}\right) &= \frac{\partial}{\partial y_j} \\ J\left(\frac{\partial}{\partial y_j}\right) &= -\frac{\partial}{\partial x_j}. \end{aligned}$$

To define an almost-complex structure on a complex manifold transfer this almost-complex structure on $\mathbb{C}^n$ to the manifold via holomorphic charts. Since a mapping f of $\mathbb{C}^n$ to $\mathbb{C}^m$ preserves an almost-complex structure if and only if f is holomorphic, it follows that the almost-complex structure on a manifold is chart independent.

The set of complex manifolds is a proper subset of the set of almost-complex manifolds, that is, a complex manifold is naturally an almost-complex manifold, but an almost-complex manifold need not be a complex manifold, see [69]. To determine whether or not a given almost-complex structure is induced by a complex structure we introduce torsion.

Definition 1.1.3. *The* **torsion** N *of an almost-complex structure* J *is a tensor field of type* $(1,2)$ *defined by*

$$N(X,Y) = 2\{[JX,JY] - [X,Y] - J[X,JY] - J[JX,Y]\}$$

for any two vector fields X *and* Y. *An almost-complex structure is said to be* **integrable** *if it has no torsion.*

Let $(x_1, \ldots, x_{2n})$ be a local coordinate system. The torsion tensor is bilinear, for if $X = \frac{\partial}{\partial x_j}$ and $Y = \frac{\partial}{\partial x_k}$ are vector fields and J^i_j are the components of J, then by direct calculation the i^{th} component of the torsion tensor is given by

$$N\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right)^i = N^i_{jk} = 2\sum_{h=0}^{2n}(J^h_j \partial_h J^i_k - J^h_k \partial_h J^i_j - J^i_h \partial_j J^h_k + J^i_h \partial_k J^h_j), \tag{1.1}$$

where ∂_h denotes partial differentiation $\frac{\partial}{\partial x_h}$. In chapter 2 we give an equivalent characterization of the failure of an almost-complex structure to be integrable.

In case an almost-complex manifold is real-analytic it is not hard to show that an integrable almost-complex structure is complex as an application of the Frobenius theorem, see [25], [26], and [30]. Without the real-analyticity assumption, however, this result is highly nontrivial and is due to Newlander and Nirenberg [55].

Theorem 1.1.4 (Newlander-Nirenberg, 1957)**.** *An almost-complex structure is complex if and only if it is integrable.*

1.2 Summary of known results on S^6

In the introduction to this chapter we stated that there are relatively few results on S^6. In this section we summarize some of these results. Suppose J is an almost-complex structure on S^6, and g is the standard "round" metric on S^6. J is called **orthogonal** if $g(X,Y) = g(JX,JY)$ for any vector fields X and Y. The almost-complex structure arising from the octonions is orthogonal. LeBrun shows in [51] that S^6 has no integrable orthogonal almost-complex structure. In [12] Bor and Hernández-Lamoneda generalize this result to metrics in a certain neighborhood of the standard metric on S^6.

Huckleberry, Kebekus, and Peternell in [41] and Brunella in [21] study the automorphism group of a complex S^6. Suppose $X = S^6$ carries a complex structure and let $\mathrm{Aut}_{\mathcal{O}}(X)$ denote the group of holomorphic automorphisms of X. A classical result of Bochner and Montgomery [11] is that the group of holomorphic automorphisms of a compact complex manifold is a finite dimensional complex Lie group. We can then use tools from the study of complex Lie groups to study X. We introduce the following terminology and notation.

Definition 1.2.1. *A closed subgroup H of a complex Lie group G is called a* **real**

form *of* G *if*

$$\mathfrak{g} = \mathfrak{h} \otimes \mathbb{C} = \mathfrak{h} \oplus i\mathfrak{h},$$

where $\mathfrak{g}$ *and* $\mathfrak{h}$ *denote the Lie algebras of* G *and* H*, respectively.*

We restrict our attention to the case that the complex Lie group G is connected, though reductive is also well-defined for complex Lie groups which are not connected.

Definition 1.2.2. *A connected complex Lie group* G *is called* **reductive** *if it has a compact real form.*

The group $\mathbb{C}^*$ is a reductive group and its real form is S^1. We can understand a $\mathbb{C}^*$ action on a manifold X by understanding the action of S^1 on X. Complex Lie groups are generally noncompact so that it frequently occurs that a complex Lie group action on a manifold has an open orbit.

Definition 1.2.3. *A complex manifold* X *is called* **almost homogeneous** *if the group of holomorphic automorphisms* $\mathrm{Aut}_{\mathcal{O}}(X)$ *has an open orbit on* X*.*

Theorem 1.2.4 (HKP1, 2000)**.** *Suppose* S^6 *has a complex structure* X*.* X *is not almost homogeneous. In other words, the automorphism group* $\mathrm{Aut}_{\mathcal{O}}(X)$ *does not have an open orbit.*

To prove this theorem the authors rely on an earlier result in [22].

Theorem 1.2.5 (CDP, 1998)**.** *Let* X *be a compact complex manifold homeomorphic to* S^6*. Then there are no nonconstant meromorphic functions on* X*.*

The authors also rely on the following linearization theorem.

Theorem 1.2.6 (Linearization)**.** *Assume that a reductive complex Lie group G acts holomorphically on a complex manifold X, and assume that $x \in X$ is an G-fixed point. Let H be a maximal compact subgroup of G and let $T(g) : T_xX \to T_xX$ be the tangential map. Then there exist neighborhoods U of x and V of $0 \in T_xX$ and an isomorphism $\phi : U \to V$ such that $\phi \circ h = T(h) \circ \phi$ for all $h \in H$.*

Moreover, if W is a neighborhood of H and $U' \subset U$ open so that $WU' \subset U$, then $(T(w) \circ \phi)(x) = (\phi \circ w)(x)$ for all $x \in U'$.

U is called a linearizing neighborhood of x. We have this theorem as a result of being able to average over a compact group. See [40] for more information on linearization. This linearization theorem yields very useful information about the fixed point set of a $\mathbb{C}^*$ action on X.

Proposition 1.2.7 (HKP2, 2000)**.** *Suppose $\mathbb{C}^*$ acts holomorphically on $X = S^6$. Let F denote the fixed point set. Then either $F \cong \mathbb{C}\mathrm{P}^1$ or F consists of two isolated points.*

We present a proof of this proposition in chapter 3. To prove 1.2.4 the authors assume the following: G is a connected, simply connected complex Lie group that acts holomorphically and almost effectively on X and has an open orbit $\Omega := G \cdot x_0$. The automorphism group $\mathrm{Aut}_{\mathcal{O}}(X)$ is a complex Lie group that acts holomorphically on X. Although it may not be simply connected, its universal cover G is also a Lie group that acts holomorphically. Since we are willing to admit discrete ineffectivity, $I = \{g \in G : g(x) = x, \forall x \in X\}$, the assumptions above are equivalent to assuming that $\mathrm{Aut}_{\mathcal{O}}(X)$ has an open orbit.

The proof of theorem 1.2.4 is long and quite involved, so we only sketch the strategy of the proof. The proof proceeds by contradiction. A consequence of theorem 1.2.5 is that $\dim G = 3$. The authors first argue that G is either semisimple or solvable. Since $\dim G = 3$, if G is semisimple then $G \cong \mathrm{SL}(2)$. The authors exclude this possibility via a relatively simple argument, thus showing that if G exists, it must be solvable. They use proposition 1.2.7 and check several cases given by the restrictions on the fixed point set of a $\mathbb{C}^*$ action and eliminate the possibility that G is solvable as well. This gives that G cannot exist, showing that $\mathrm{Aut}_{\mathcal{O}}(X)$ does not have an open orbit.

In related work [21] Brunella continues the study of the automorphism group of X. In his paper he works under in the more general setting that X is a complex homology sphere, that is, a complex three dimensional manifold with the same homology ring as S^6. A consequence of 1.2.5 and 1.2.4 is that $\dim_{\mathbb{C}} \mathrm{Aut}_{\mathcal{O}}(X) \leq 2$. Let $\mathrm{Aut}_0(X)$ be the connected component of the identity of $\mathrm{Aut}_{\mathcal{O}}(S)$. Brunella arrives at the following result.

Theorem 1.2.8. *Let X be a complex homology sphere. If $\mathrm{Aut}_0(X)$ contains a $\mathbb{C}^*$ action, then $\mathrm{Aut}_0(X)$ is abelian.*

He proves this by first showing that there are restrictions on the weights on the tangent spaces of fixed points of any holomorphic $\mathbb{C}^*$ action on S^6. From this he concludes using 1.2.7 that if $\mathbb{C}^*$ acts on X fixing two isolated points, then there is a $\mathbb{C}^*$ action on S^6 fixing a smooth rational curve whose normal bundle N splits as $N = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. He then works in a tubular neighborhood of the fixed point set to obtain his theorem, invoking the deep classification results of surfaces given by Kodaira.

1.3 Almost-complex structures on $\mathbb{C}\mathrm{P}^3$

Until now the main body of work on this problem has been focused on the six sphere. At least as early as [36] Hirzebruch had known that blowing up an almost-complex S^6 at a point gives an almost-complex manifold X which is diffeomorphic to the complex three dimensional projective space $\mathbb{C}\mathrm{P}^3$, but whose almost-complex structure is different from the standard (integrable) almost-complex structure on $\mathbb{C}\mathrm{P}^3$, thus reframing the existence question on S^6 in terms of an existence question on $\mathbb{C}\mathrm{P}^3$. We will call the resulting almost-complex structure on $\mathbb{C}\mathrm{P}^3$ *exotic* because its Chern classes are different from the Chern classes of the standard almost-complex structure on $\mathbb{C}\mathrm{P}^3$.

Let $x \in H^2(\mathbb{C}\mathrm{P}^3, \mathbb{Z})$ denote the standard generator of $H^*(\mathbb{C}\mathrm{P}^3, \mathbb{Z})$. The Chern classes of standard almost-complex structure on $\mathbb{C}\mathrm{P}^3$ are the following.

$$c_1 = 4x, \quad c_2 = 6x^2, \quad c_3 = 4x^3$$

The blow up of S^6 at a point has the following Chern classes.

$$c_1 = -2x, \quad c_2 = 0, \quad c_3 = 4x^3$$

Since the cohomology $H^k(S^6, \mathbb{Z})$ of S^6 is trivial for $k \neq 0, 6$, the Chern numbers are zero except the top Chern number which equals the Euler number of S^6 which is 2.

In [66] E. Thomas partially answers another question posed by Hirzebruch [36, Problem 14] by characterizing the topological types of almost-complex structures on manifolds X which are diffeomorphic to $\mathbb{C}\mathrm{P}^n$ for $n = 1, 2, 3, 4$.

Since each X has the same differentiable structure as $\mathbb{C}\mathrm{P}^n$, the Pontrjagin classes of X are the same as the Pontrjagin classes of $\mathbb{C}\mathrm{P}^n$. Neither S^6 nor X has torsion. This means that we can consider the cohomology with real coefficients or integer coefficients as we please without losing information. By considering the cohomology of X with real coefficients we may realize the Pontrjagin and Chern classes of X as differential forms, see Appendix B.1, and they are related by the following theorem.

Theorem 1.3.1. *The Chern classes $c_i \in H^{2i}(X, \mathbb{R})$ of the almost-complex manifold X are related to the Pontrjagin classes $p_i \in H^{4i}(X, \mathbb{R})$ of X (regarded as a differentiable manifold) by the equation*

$$\sum_{i=0}^{\infty} (-1)^i p_i = \sum_{i=0}^{\infty} c_i \sum_{k=0}^{\infty} (-1)^k c_k.$$

Suppose X is an almost-complex manifold diffeomorphic to $\mathbb{C}\mathrm{P}^n$, and as above let $x \in H^2(X, \mathbb{Z})$ denote the standard generator of $H^*(X, \mathbb{Z})$. The total Chern class of

$\mathbb{C}\mathrm{P}^n$ is given by

$$c(\mathbb{C}\mathrm{P}^n) = (1+x)^{n+1}.$$

Theorem 1.3.1 gives a necessary condition on the Chern classes of X.

Theorem 1.3.2 (Thomas, 1967). *Consider a manifold X diffeomorphic to the complex projective space $\mathbb{C}\mathrm{P}^n$, for $n = 1, 2, 3, 4$. The following cohomology classes, and only these, occur as the total Chern class of an almost-complex structure on X.*

$$\begin{aligned}
n = 1 \ &: \ 1 + 2x \\
n = 2 \ &: \ 1 + 3x + 3x^2; \ 1 - 3x + 3x^2 \\
n = 3 \ &: \ 1 + 2jx + 2(j^2 - 1)x^2 + 4x^3, \ j \in \mathbb{Z} \\
n = 4 \ &: \ 1 + \epsilon x - 2x^2 + \epsilon x^3 + 5x^4, \\
&\quad\ 1 + 5\epsilon x + 10x^2 + 10\epsilon x^3 + 5x^4, \\
&\quad\ 1 + 25\epsilon x + 60x^2 + 1922\epsilon x^3 + 5x^4, \\
&\quad\ \textit{where}\ \epsilon = 1\ \textit{or}\ \epsilon = -1.
\end{aligned}$$

For $n = 1$ the statement is clear. We know already that there exists an almost-complex structure on $\mathbb{C}\mathrm{P}^1$ with $c(\mathbb{C}\mathrm{P}^1) = 1 + 2x$ and that this is the only class since the top Chern number is also the Euler number of the manifold.

From theorem 1.3.1 if $n = 2$ then we have the restriction $c_1^2 = 9x^2$ which gives $c_1 = \pm 3x$. If $n = 3$ then we have the restriction $2c_2 - c_1^2 = 4$. We have a further constraint on $\mathbb{C}\mathrm{P}^3$ involving the Steifel-Whitney classes which are cohomology classes defined for $\mathbb{Z}/2\mathbb{Z}$ coefficients. The restriction is $w_2 \equiv c_1 \pmod 2$. Since $c_1 \equiv 0$ (mod

2), we have $c_1 = 2jx$ and $c_2 = 2(j^2 - 1)x^2$ for $j \in \mathbb{Z}$. For $n = 4$ the same necessary conditions apply and Thomas also uses Steenrod squaring operations.

Our primary interest is in $\mathbb{C}\mathrm{P}^3$ whose set of almost-complex structures is indexed by $j \in \mathbb{Z}$. We denote by X_j an almost-complex manifold diffeomorphic to $\mathbb{C}\mathrm{P}^3$ whose total Chern class is given as in theorem 1.3.2. In particular notice that the standard almost-complex structure has $j = 2$ and the blown-up S^6 has $j = -1$.

1.4 Summary of known results on projective space

Suppose now that S^6 carries a complex structure. If we blow up S^6 at any point p, the result is a complex manifold X^p whose Chern classes are given in theorem 1.3.2 which correspond to $j = -1$. If two points p and q lie in different orbits of $\mathrm{Aut}_{\mathcal{O}}(S^6)$, the resulting manifolds X^p and X^q each correspond to $j = -1$ in theorem 1.3.2, but X^p and X^q are not biholomorphic to each other. Since theorem 1.2.4 gives that $\mathrm{Aut}_{\mathcal{O}}(S^6)$ does not have an open orbit, we have the following corollary.

Corollary 1.4.1 (HKP, 2000)**.** *If S^6 admits a complex structure, then there is a 1-dimensional family of complex structures on $\mathbb{C}\mathrm{P}^3$.*

This corollary stands in marked contrast to the classic theorem of Hirzebruch and Kodaira in [38] and the nondeformability result of Siu in [60].

Theorem 1.4.2 (Hirzebruch and Kodaira, 1957)**.** *Let X be an n-dimensional compact Kähler manifold which is diffeomorphic to $\mathbb{C}\mathrm{P}^n$. In case n is odd, X is biholomorphic* *to* $\mathbb{C}\mathrm{P}^n$*. Let* $x = [\omega] \in H^2(X, \mathbb{R})$ *denote the cohomology class of the Kähler form on* X*. In case* n *is even,* X *is biholomorphic to* $\mathbb{C}\mathrm{P}^n$ *if* $c_1(X) \neq -(n+1)x$*.*

Theorem 1.4.3 (Siu, 1989)**.** *Let* $\pi : M \to \Delta$ *be a holomorphic family of compact complex manifolds (where* $\Delta = \{t \in \mathbb{C} : |t| < 1\}$*) such that* $M_t := \pi^{-1}(t)$ *is biholomorphic to* $\mathbb{C}\mathrm{P}^n$ *for* $t \in \Delta - 0$*. Then* M_0 *is biholomorphic to* $\mathbb{C}\mathrm{P}^n$*.*

In [56] Peternell gives a similar rigidity theorem on $\mathbb{C}\mathrm{P}^3$.

Theorem 1.4.4 (Peternell, 1985)**.** *Let* X *be a compact complex manifold of complex dimension 3 which carries 3 algebraically independent meromorphic functions. Assume that* X *is homeomorphic to* $\mathbb{C}\mathrm{P}^3$*. Then* X *is biholomorphically equivalent to* $\mathbb{C}\mathrm{P}^3$*.*

In dimension 2, Yau [71] gives a much stronger result.

Theorem 1.4.5 (Yau, 1977)**.** *Every complex surface that is homotopic to the complex projective plane* $\mathbb{C}\mathrm{P}^2$ *is biholomorphic to* $\mathbb{C}\mathrm{P}^2$*.*

1.5 Statement of results

It is the purpose of this dissertation to sharpen and constrain the known results on S^6 and $\mathbb{C}\mathrm{P}^3$. In trying to determine whether or not $\mathbb{C}\mathrm{P}^3$ admits exotic complex structures we investigate the properties exotic structures would have. In chapter 2 we obtain lower bounds on the hodge numbers, which depend on j, in theorems 2.2.2 and 2.3.5. We also study the Frölicher spectral sequence to determine at what level it degenerates. We give a j-independent necessary condition for the spectral sequence to degenerate at the second level in corollary 2.3.4.

After considering the properties of hypothetical exotic complex structures on $\mathbb{C}P^3$ we investigate properties of hypothetical exotic complex structures on $\mathbb{C}P^3$ which are invariant under $\mathbb{C}^*$ actions in chapter 3. We find give the following nonexistence result. The only X_j which could admit $\mathbb{C}^*$-symmetric complex structures have $-1 \leq j \leq 2$. Additionally we give restrictions on both the fixed point set of a holomorphic $\mathbb{C}^*$ action and the weights of such an action on the tangent spaces of the fixed points.

Chapter 2

Properties of hypothetical complex structures on $\mathbb{CP}^3$

In this chapter we assume that $\mathbb{CP}^3$ admits complex structures whose Chern classes are different from the Chern classes of the standard Kähler structure. Theorem 1.3.2 gives a specific description of the Chern classes of these structures. Since the classes of structures are indexed by $j \in \mathbb{Z}$ we denote by X_j a complex structure whose total Chern class is given in theorem 1.3.2. As the standard Kähler structure has $j = 2$, we restrict our attention to X_j with $j \neq 2$. We investigate the properties of hypothetical exotic structures on X_j. In particular we investigate properties of the Dolbeault cohomology groups of X_j and examine the Hodge numbers of X_j. This leads further to a study of the Frölicher spectral sequence of X_j.

We give lower bounds on the Hodge numbers of a hypothetical exotic complex structures on X_j in theorems 2.2.2 and 2.3.5, employing methods used in [32]. In section 2.3 we give a necessary condition for the Frölicher spectral sequence to degenerate at the second level in corollary 2.3.4 by applying the methods of [68] more broadly.

2.1 Dolbeault cohomology and the Frölicher spectral sequence

In this section we recall Dolbeault cohomology groups and some general facts about the Frölicher spectral sequence of a complex manifold.

Suppose X is a differentiable manifold and let $(x_1, \ldots, x_n)$ be local coordinates on X. A smooth differential m-form φ on X may be expressed as

$$\varphi = \sum a_{i_1 \ldots i_m} dx_{i_1} \wedge \ldots \wedge dx_{i_m}$$

where $a_{i_1 \ldots i_m}$ is a smooth real-valued function on X. Denote by Ω^m the space of smooth m-forms on X. Let $d : \Omega^m \to \Omega^{m+1}$ denote the exterior derivative. The de Rham cohomology groups of X are defined as follows.

$$H^m_{dR}(X, \mathbb{R}) = \frac{(\ker d) \cap \Omega^m}{(\operatorname{im} d) \cap \Omega^m}$$

The total de Rham cohomology $H^*(X, \mathbb{R}) = \oplus H^m(X, \mathbb{R})$ has a graded ring structure, see Appendix B.1. The de Rham cohomology of X is a topological invariant of X.

Suppose now that X is a complex manifold of complex dimension n and let $(z_1, \ldots, z_n)$ be local holomorphic coordinates on X. A differential form of type (p, q) on X is a complex differential form φ which can be written as

$$\varphi = \sum a_{i_1 \ldots i_p j_1 \ldots j_q} dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge \ldots \wedge d\overline{z}_{j_q}.$$

where $a_{i_1 \dots i_p j_1 \dots j_q}$ is smooth complex-valued (not necessarily holomorphic) function on X. Let $\Omega^{p,q}$ denote the space of smooth (p, q) forms on X. We can decompose Ω^m as follows.

$$\Omega^m = \bigoplus_{p+q=m} \Omega^{p,q}$$

If X is almost-complex, d may not preserve the bigrading. In general

$$d : \Omega^{p,q} \longrightarrow \Omega^{p+q+1}$$

and

$$d = \partial + \overline{\partial} + \dots$$

where

$$\partial(\Omega^{p,q}) \subset \Omega^{p+1,q}$$

and

$$\overline{\partial}(\Omega^{p,q}) \subset \Omega^{p,q+1}.$$

The de Rham complex is a complex in the sense that $d^2 = 0$, but if X is not complex then it is not necessarily the case that $\overline{\partial}^2 = 0$. An equivalent characterization of an integrable almost-complex structure is that d preserves the bigrading and that we may express d as

$$d = \partial + \overline{\partial},$$

so that on a complex manifold X

$$d(\Omega^{p,q}) \subset \Omega^{p+1,q} \oplus \Omega^{p,q+1}.$$

In this case $\overline{\partial}^2 = 0$, so we have the following complex, called the Dolbeault complex.

$$0 \longrightarrow \Omega^{p,0} \xrightarrow{\overline{\partial}} \Omega^{p,1} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \Omega^{p,n} \longrightarrow 0$$

The torsion given in equation 1.1 is precisely the measure of the extent to which the above sequence fails to be a complex. Since on a complex manifold $\overline{\partial}^2 = 0$, this complex has well-defined cohomology. Define the Dolbeault cohomology groups to be the cohomology of this complex.

$$H^{p,q}(X) = \frac{(\ker \overline{\partial}) \cap \Omega^{p,q}}{(\mathrm{im}\ \overline{\partial}) \cap \Omega^{p,q}}.$$

Unlike the de Rham cohomology the Dolbeault cohomology is not a topological invariant, though it does carry a ring structure. The Dolbeault cohomology depends very much on the particular complex structure a manifold carries. Define $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}(X)$ to be the Hodge numbers of X. In our analysis we use the well-known duality theorem of Serre.

Lemma 2.1.1 (Serre Duality)**.** *Let X be a compact complex manifold of complex dimension n. Then*

$$H^{p,q}(X) = H^{n-p,n-q}(X).$$

We also use a result of Gray in [32], though the statement and proof we give below vary from the original presentation.

Lemma 2.1.2. *Let X be a compact complex manifold of complex dimension n. There exists a natural injective map*

$$i : H^{n,0}(X) \hookrightarrow H^n_{dR}(X).$$

Proof. Since $(\text{im } \overline{\partial}) \cap \Omega^{n,0} = 0$, we have $H^{n,0}(X) = (\ker \overline{\partial}) \cap \Omega^{n,0}$. In addition we have $(\ker d) \cap \Omega^{n,0} = (\ker \overline{\partial}) \cap \Omega^{n,0}$ which gives a natural map $i : H^{n,0}(X) \to H^n_{dR}(X)$. We only need to show that this map is injective.

Suppose that $\beta \in \Omega^*$ is such that $d\beta \in \Omega^{n,0}$. Then

$$\int_X d\beta \wedge \overline{d\beta} = \int_X d(\beta \wedge \overline{d\beta}) = 0,$$

by Stokes' theorem. Write $d\beta$ locally as $d\beta = f dz_1 \wedge \ldots \wedge dz_n$. Then

$$\begin{aligned} d\beta \wedge \overline{d\beta} &= |f|^2 dz_1 \wedge \ldots \wedge dz_n \wedge d\overline{z_1} \wedge \ldots \wedge d\overline{z_n} \\ &= (-1)^{(1/2)n(n-1)} |f|^2 dz_1 \wedge d\overline{z_1} \wedge \ldots \wedge dz_n \wedge d\overline{z_n} \\ &= (-1)^{(1/2)n(n-1)} |f|^2 dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n, \end{aligned}$$

where $z_j = x_j + \sqrt{-1} y_j$, $j = 1, \ldots, n$. The vanishing of the integral shows that $d\beta = 0$ which gives the injectivity of i. □

Corollary 2.1.3 (Gray, 1997). *Let X be a compact complex manifold of complex dimension n such that $b_n(X) = 0$. For any complex structure on X, we have*

$$h^{n,0} = h^{0,n} = 0.$$

Proof. The previous lemma gives that $H^{n,0}(X) \hookrightarrow H^n_{dR}(X)$, and since $b_n(X) = 0$ we have that $h^{n,0} = 0$. $h^{0,n} = 0$ follows by Serre duality. □

A powerful tool used in algebraic topology is the spectral sequence, which is a generalization of a long exact sequence. Since we assumed that X is complex we can use a particular spectral sequence called the Frölicher spectral sequence. We now turn to the Frölicher spectral sequence. For a further details see [17] and [33]. As above we form the associated de Rham complex (Ω^*, d) from the double complex $(\Omega^{*,*}, \partial, \overline{\partial})$.

$$\Omega^m = \bigoplus_{p+q=m} \Omega^{p,q}$$

$$d = \partial + \overline{\partial}$$

There are two filtrations on (Ω^*, d) given by

$${}'F^p\Omega^m = \bigoplus_{\substack{p'+q=m \\ p'\geq p}} \Omega^{p',q}$$

$${}''F^q\Omega^m = \bigoplus_{\substack{p+q''=m \\ q''\geq q}} \Omega^{p,q''}.$$

Associated with each filtration is a spectral sequence $\{{}'E_r\}$ and $\{{}''E_r\}$ both of which abut to $H^*_{dR}(X)$. The first filtration ${}'F^p\Omega^m$ gives the Frölicher spectral sequence, for in this case ${}'E_1^{p,q}$ is given by

$${}'E_1^{p,q} = H^q_{\overline{\partial}}(X, \Omega^p) = H^{p,q}(X),$$

the Dolbeault cohomology groups of X. Henceforth we will drop this prime notation, denoting ${}'E_r^{p,q}$ by $E_r^{p,q}$. $E_r^{*,*}$ also has a ring structure, but the spectral sequence does not respect the ring structure from one level to the next.

Here we note that if X is a Kähler manifold, then the Frölicher spectral sequence degenerates at the E_1 level and we have the Hodge decomposition

$$H^m(X) = \bigoplus_{p+q=m} H^{p,q}(X)$$

as well as

$$H^{p,q}(X) = \overline{H^{q,p}}(X).$$

As above we let $h^{p,q} = \dim H^{p,q}(X) = \dim E_1^{p,q}$, and we also define $h_r^{p,q} = \dim E_r^{p,q}$ where

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

and

$$E_{r+1}^{p,q} = \frac{(\ker\ d_r) \cap E_r^{p,q}}{(\mathrm{im}\ d_r) \cap E_r^{p,q}}.$$

For each p, let

$$\chi_p(X) = \sum_{q=0}^{n} (-1)^q h^{p,q}.$$

Observe that $h_{r+1}^{p,q} \leq h_r^{p,q}$, and that if $p = 0$, then following Hirzebruch [37], $\chi_0(X)$ is the familiar arithmetic genus. In [68] Ugarte gives the following useful proposition.

Proposition 2.1.4 (Ugarte, 2000)**.** *Let X be a compact complex manifold of complex dimension n. If there are no holomorphic n-forms on X, that is if $H^{n,0}(X) = 0$, then $E_n \cong E_\infty$.*

Proof. Recall that we obtain $E_{n+1}^{p,q}$ by taking the cohomology of the following sequence.

$$\cdots \to E_n^{p-n,q+n-1} \xrightarrow{d_n} E_n^{p,q} \xrightarrow{d_n} E_n^{p+n,q-n+1} \to \ldots \tag{2.1}$$

Since the complex dimension of X is n we have that $E_r^{p,q} = \{0\}$ for all $p, q < 0$ and $p, q > n$ and all r. Then the sequence 2.1 reduces to

$$0 \to E_n^{p,q} \to 0$$

for all p, q except

$$0 \to E_n^{0,n-1} \xrightarrow{d_n} E_n^{n,0} \to 0 \tag{2.2}$$

and

$$0 \to E_n^{0,n} \xrightarrow{d_n} E_n^{n,1} \to 0. \tag{2.3}$$

The holomorphic n-forms are by definition $\Omega^{n,0} \cap (\ker \overline{\partial})$ which by the proof of lemma (2.1.2) is $H^{n,0}(X)$. Corollary 2.1.3 gives that $h^{n,0} = h^{0,n} = 0$ for any complex structure on X. Since $h_r^{p,q} \geq h_{r+1}^{p,q}$ we have that $E_n^{n,0} = \{0\}$ and $E_n^{0,n} = \{0\}$. Thus the sequences 2.2 and 2.3 are

$$0 \to E_n^{0,n-1} \to 0$$

and

$$0 \to E_n^{n,1} \to 0.$$

Hence $E_n^{p,q} \cong E_{n+1}^{p,q} \cong \cdots \cong E_\infty^{p,q}$. □

2.2 Cohomology relations for hypothetical exotic complex structures

In this section we consider the relations among the Hodge numbers for hypothetical exotic complex structures on $\mathbb{C}\mathrm{P}^3$. We employ the Hirzebruch-Riemann-Roch theorem as it appears in [8] and [37]. See Appendix B for a discussion of index theorems.

Suppose X is a compact complex manifold of complex dimension n. The Riemann-

Roch-Hirzebruch theorem gives the following relationship

$$\chi_0(X) = \sum_{q=0}^{n} h^{0,q} = \mathrm{Td}(X)[X],$$

where $\mathrm{Td}(X)[X]$ is the Todd class of X evaluated on the fundamental class of X. If $n = 3$, then $\chi_0(X)$ has an expression in terms of the Chern classes of X.

$$\chi_0(X) = \frac{1}{12}c_1c_2[X] = \frac{1}{12}\int_X c_1c_2$$

In the special case of $X = S^6$ we have a theorem of Gray [32] for hypothetical complex structure on X.

Theorem 2.2.1 (Gray, 1997)**.** *Any complex structure on S^6 has the property that*

$$h^{0,1}(S^6) \geq 1.$$

Proof. Any complex structure on S^6 satisfies

$$\chi_0(S^6) = \frac{1}{24}c_1c_2[X].$$

Since the cohomology $H^k(X)$ vanishes for all $k \neq 0, 6$ we have $h^{0,3} = 0$ and $\frac{1}{24}c_1c_2[X] = 0$ so that

$$1 - h^{0,1} + h^{0,2} = 0,$$

which gives

$$h^{0,1} = 1 + h^{0,2} \geq 1.$$

□

We can extend this result to hypothetical exotic complex structures on X_j. Since the standard Kähler structure on $\mathbb{C}\mathrm{P}^3$ has $j = 2$, we only investigate the Hodge numbers of hypothetical complex structures on X_j with $j \neq 2$.

Theorem 2.2.2. *Suppose X_j is a complex manifold diffeomorphic to $\mathbb{C}\mathrm{P}^3$ whose total Chern class is given by $c(X_j) = 1 + 2jx + 2(j^2 - 1)x^2 + 4x^3$, where x generates $H^2(X_j, \mathbb{Z})$.*

a. *If $j < 2$, then*

$$h^{0,1}(X_j) \geq 1, \qquad \text{and} \qquad h^{1,1} + h^{2,0} \geq 2.$$

b *If $j > 2$, then*

$$h^{0,2}(X_j) \geq 3, \qquad \text{and} \qquad h^{1,0} + h^{1,2} \geq 2.$$

Remark. *If $j \neq 2$, then X_j is not Kähler because this is inconsistent with Hodge decomposition. The results of [38] imply this as well. We can also see that see that if $j \neq 2$, then X_j is not Kähler since the Frölicher spectral sequence lives at least to E_2. We will explore this further in section 2.3.*

Proof. From theorem 1.3.2 for each $j \in \mathbb{Z}$, the total Chern class of X_j is given by

$$c(X_j) = 1 + 2jx + 2(j^2 - 1)x^2 + 4x^3.$$

As above

$$\chi_0(X_j) \quad = \quad 1 - h^{0,1}(X_j) + h^{0,2}(X_j)$$

since $h^{3,0}(X_j) = 0$. Combining this with the index theorem gives

$$1 - h^{0,1}(X_j) + h^{0,2}(X_j) = \frac{j(j^2-1)}{6},$$
$$h^{0,1}(X_j) \geq 1 - \frac{j(j^2-1)}{6} \geq 1, \quad \text{for} \quad j < 2,$$
$$h^{0,2}(X_j) \geq \frac{j(j^2-1)}{6} - 1 \geq 3, \quad \text{for} \quad j > 2.$$

Additionally, the topological Euler characteristic may be expressed

$$\begin{aligned} \chi_{Top}(X_j) &= \sum_{p=0}^{3}\sum_{q=0}^{3}(-1)^{p+q}h^{p,q} \\ &= 2\left(\sum_{q=0}^{3}(-1)^q h^{0,q} - \sum_{q=0}^{3}(-1)^q h^{1,q}\right) \\ &= 2(\chi_0 - \chi_1). \end{aligned}$$

In particular, $\chi_1 = \chi_0 - 2$. This expression for χ_1 along with Serre duality give

$$\chi_1 = h^{1,0} - h^{1,1} + h^{1,2} - h^{2,0} = \frac{j(j^2-1)}{6} - 2,$$

so that

$$h^{1,1} + h^{2,0} \geq 2 - \frac{j(j^2-1)}{6} \geq 2 \quad \text{for} \quad j < 2,$$
$$h^{1,0} + h^{1,2} \geq \frac{j(j^2-1)}{6} - 2 \geq 2 \quad \text{for} \quad j > 2.$$

□

In section 2.3 we prove a sharper inequality for $h^{1,2}$ using the Frölicher spectral sequence.

2.3 Frölicher Spectral Sequence Computations

Since $b_1(X_j) = 0$ and $b_2(X_j) = 1$, it follows from theorem 2.2.2 that if $j \neq 2$, the Frölicher spectral sequence lives at least to E_2. Since $b_3(X_j) = 0$ we also have by lemma 2.1.4 that $E_3 \cong E_\infty$. It is a natural then to ask the following question. Under what conditions does the Frölicher spectral sequence of X_j degenerate at E_2? For a compact complex manifold X of complex dimension three, consider the dimension grids below.

E_1

0	$h^{1,3}$	$h^{2,3}$	1
$h^{0,2}$	$h^{1,2}$	$h^{2,2}$	$h^{3,2}$
$h^{0,1}$	$h^{1,1}$	$h^{2,1}$	$h^{3,1}$
1	$h^{1,0}$	$h^{2,0}$	0

E_2

0	$h_2^{1,3}$	$h_2^{2,3}$	1
$h_2^{0,2}$	$h_2^{1,2}$	$h_2^{2,2}$	$h_2^{3,2}$
$h_2^{0,1}$	$h_2^{1,1}$	$h_2^{2,1}$	$h_2^{3,1}$
1	$h_2^{1,0}$	$h_2^{2,0}$	0

E_3

0	$h_3^{1,3}$	$h_3^{2,3}$	1
$h_3^{0,2}$	$h_3^{1,2}$	$h_3^{2,2}$	$h_3^{3,2}$
$h_3^{0,1}$	$h_3^{1,1}$	$h_3^{2,1}$	$h_3^{3,1}$
1	$h_3^{1,0}$	$h_3^{2,0}$	0

We recall two facts about the dimension grids above: First, each entry $h_r^{p,q}$ is a non-negative integer, and second, $\dim H^n_{dR}(X) = \sum_{p+q=n} h_\infty^{p,q} = \sum_{p+q=n} h_3^{p,q}$. The computations in the subsections below use the following basic homological algebra fact. If

$$A : 0 \to A_0 \xrightarrow{\delta} A_1 \xrightarrow{\delta} \cdots \xrightarrow{\delta} A_n \to 0$$

is a complex of finite dimensional vector spaces, i.e. $\delta \circ \delta = 0$, then

$$\sum_{i=0}^{n} (-1)^i \dim A_i = \sum_{i=0}^{n} (-1)^i \dim H^i(A).$$

2.3.1 The Frölicher Spectral Sequence for a hypothetical complex S^6

We recall some of L. Ugarte's main results in [68], since we know that $\dim H^n_{dR}(S^6) = 0$ for all $n \neq 0, 6$ we have $h_3^{p,q} = 0$ for all pairs (p, q) except $(0, 0)$ and $(3, 3)$, so that the E_3 term becomes:

E_3

0	0	0	1
0	0	0	0
0	0	0	0
1	0	0	0

Since the E_3 term comes from the following sequences

$$0 \longrightarrow E_2^{p,q} \xrightarrow{d_2} E_2^{p+2,q-1} \longrightarrow 0, \tag{2.4}$$

and $E_2^{p,q} = 0$ for all $p, q < 0$, $p, q > 3$, and $(p, q) = (0, 3), (3, 0)$ we know that

$$h_2^{1,0} = h_2^{2,3} = h_2^{1,1} = h_2^{2,2} = h_2^{3,0} = h_2^{0,3} = 0.$$

We also know that for the cohomology of the complex (2.4) to vanish we need $E_2^{p,q} \cong E_2^{p+2,q-1}$ hence we have

$$h_2^{0,1} = h_2^{2,0}$$

$$h_2^{0,2} = h_2^{2,1}$$

$$h_2^{1,2} = h_2^{3,1}$$

$$h_2^{1,3} = h_2^{3,2}.$$

On the other hand the entries of the E_2 term arise from the following sequences

$$0 \to E_1^{p,q} \xrightarrow{d_1} E_1^{p+1,q} \xrightarrow{d_1} E_1^{p+2,q} \xrightarrow{d_1} E_1^{p+3,q} \to 0, \tag{2.5}$$

so that

$$h_2^{0,q} - h_2^{1,q} + h_2^{2,q} - h_2^{3,q} = h^{0,q} - h^{1,q} + h^{2,q} - h^{3,q}.$$

By Serre duality we know that $h^{p,q} = h^{3-p,3-q}$. Then we have

$$1 + h_2^{2,0} = 1 - h^{1,0} + h^{2,0} = 1 - h^{2,3} + h^{1,3} = 1 + h_2^{1,3},$$

which gives

$$h_2^{0,1} = h_2^{2,0} = h_2^{1,3} = h_2^{3,2}.$$

We also have

$$\begin{aligned} h_2^{0,1} + h_2^{2,1} - h_2^{3,1} &= h^{0,1} - h^{1,1} + h^{2,1} - h^{3,1} \\ &= h^{3,2} - h^{2,2} + h^{1,2} - h^{0,2} \\ &= h_2^{3,2} + h_2^{1,2} - h_2^{0,2} \\ &= h_2^{0,1} + h_2^{3,1} - h_2^{2,1}, \end{aligned}$$

which gives

$$h_2^{0,2} = h_2^{1,2} = h_2^{2,1} = h_2^{3,1}.$$

Let $a = h_2^{0,1} = \dim((\ker\, d_1) \cap H^{0,1}(S^6))$ and $b = h_2^{0,2} = \dim((\ker\, d_1) \cap H^{0,2}(S^6))$. Then the E_2 term is

E_2

0	a	0	1
b	b	0	a
a	0	b	b
1	0	a	0

Proposition 2.3.1 (Ugarte). *If $X = S^6$ is a complex manifold, then either*

a. $H^{1,1}(X) \neq 0$, *or*

b. $H_2^{2,0}(X) \neq 0$ *and* $E_1 \ncong E_2 \ncong E_3 \cong E_\infty$.

2.3.2 The Frölicher Spectral Sequence for hypothetical exotic structures

Consider now the case $X = X_j$. Since $b_0 = b_2 = b_4 = b_6 = 1$ and $b_1 = b_3 = b_5 = 0$ we have

$$h_3^{0,0} = h_3^{3,3} = 1$$

$$h_3^{0,1} = h_3^{1,0} = h_3^{0,3} = h_3^{1,2} = h_3^{2,1} = h_3^{3,0} = h_3^{2,3} = h_3^{3,2} = 0$$

$$h_3^{0,2} + h_3^{1,1} + h_3^{2,0} = 1$$

$$h_3^{1,3} + h_3^{2,2} + h_3^{3,1} = 1,$$

so the E_3 term becomes

E_3

0	$h_3^{1,3}$	0	1
$h_3^{0,2}$	0	$h_3^{2,2}$	0
0	$h_3^{1,1}$	0	$h_3^{3,1}$
1	0	$h_3^{2,0}$	0

Unlike the case of S^6 we cannot determine all of the entries of the E_3 term exactly, but we do know that either $h_3^{0,2}, h_3^{1,1}$, or $h_3^{2,0}$ is 1, and $h_3^{1,3}, h_3^{2,2}$, or $h_3^{3,1}$ is 1. This observation allows us to regard the nine cases of E_3 individually. Before we do this we can make some general observations.

Since

$$h_3^{0,1} = h_3^{1,0} = h_3^{0,3} = h_3^{1,2} = h_3^{2,1} = h_3^{3,0} = h_3^{2,3} = h_3^{3,2} = 0,$$

we can conclude that

$$h_2^{0,3} = h_2^{1,0} = h_2^{2,3} = h_2^{3,0} = 0.$$

By Serre Duality at the E_1 level we have

$$h_2^{1,3} = h_2^{2,0}.$$

We can also conclude

$$\begin{aligned}
h_2^{1,1} &= h_3^{1,1} \\
h_2^{2,2} &= h_3^{2,2} \\
h_3^{0,2} &= h_2^{0,2} - h_2^{2,1} \\
h_3^{2,0} &= h_2^{2,0} - h_2^{0,1} \\
h_3^{1,3} &= h_2^{1,3} - h_2^{3,2} \\
h_3^{3,1} &= h_2^{3,1} - h_2^{1,2} \\
h_2^{0,1} - h_2^{1,1} + h_2^{2,1} - h_2^{3,1} &= h_2^{3,2} - h_2^{2,2} + h_2^{1,2} - h_2^{0,2}.
\end{aligned}$$

In all of the cases that follow let $a = h_2^{0,1} = \dim((\ker d_1) \cap H^{0,1}(X_j))$ and $b = h_2^{0,2} = \dim((\ker d_1) \cap H^{0,2}(X_j))$.

Case 1: $h_3^{0,2} = 1$ and $h_3^{1,3} = 1$.

E_3

0	1	0	1
1	0	0	0
0	0	0	0
1	0	0	0

From equations 2.6 it follows that the E_2 term for all $j \in \mathbb{Z}$ is given by the following.

E_2

0	a	0	1
b	b	0	$a-1$
a	0	$b-1$	b
1	0	a	0

From this we conclude that $a, b > 0$ so that

i. $H^{0,1}(X_j) \neq 0$, $H^{0,2}(X_j) \neq 0$ and

ii. This spectral sequence lives to E_3.

Case 2: $h_3^{0,2} = 1$ and $h_3^{2,2} = 1$

E_3

0	0	0	1
1	0	1	0
0	0	0	0
1	0	0	0

Then the E_2 term becomes for all $j \in \mathbb{Z}$:

E_2

0	a	0	1
b	b	1	a
a	0	$b-1$	b
1	0	a	0

from which we conclude that $b > 0$ so that

i. $H^{0,2}(X_j) \neq 0$ and

ii. This spectral sequence lives to E_3.

Case 3: $h_3^{0,2} = 1$ and $h_3^{3,1} = 1$

E_3

0	0	0	1
1	0	0	0
0	0	0	1
1	0	0	0

Then the E_2 term becomes for all $j \in \mathbb{Z}$:

E_2

0	a	0	1
b	$b-1$	0	a
a	0	$b-1$	b
1	0	a	0

from which we conclude that $b > 0$ so that

i. $H^{0,2}(X_j) \neq 0$ and

ii. $E_2 \cong E_\infty$ if and only if $a = 0$ and $b = 1$.

Case 4: $h_3^{1,1} = 1$ and $h_3^{1,3} = 1$

E_3

0	1	0	1
0	0	0	0
0	1	0	0
1	0	0	0

Then the E_2 term becomes for all $j \in \mathbb{Z}$:

E_2

0	a	0	1
b	b	0	$a-1$
a	1	b	b
1	0	a	0

from which we conclude that $a > 0$ so that

i. $H^{0,1}(X_j) \neq 0$ and

ii. This spectral sequence lives to E_3.

Case 5: $h_3^{1,1} = 1$ and $h_3^{2,2} = 1$

E_3

0	0	0	1
0	0	1	0
0	1	0	0
1	0	0	0

Then the E_2 term becomes for all $j \in \mathbb{Z}$:

E_2

0	a	0	1
b	b	1	a
a	1	b	b
1	0	a	0

from which we conclude

i. $E_2 \cong E_\infty$ if and only if $a = b = 0$.

Case 6: $h_3^{1,1} = 1$ and $h_3^{3,1} = 1$

E_3

0	0	0	1
0	0	0	0
0	1	0	1
1	0	0	0

Then the E_2 term becomes for all $j \in \mathbb{Z}$:

E_2

0	a	0	1
b	$b-1$	0	a
a	1	b	b
1	0	a	0

from which we conclude that $b > 0$ so that

i. $H^{0,2}(X_j) \neq 0$ and

ii. This spectral sequence lives to E_3.

Case 7: $h_3^{2,0} = 1$ and $h_3^{1,3} = 1$

E_3

0	1	0	1
0	0	0	0
0	0	0	0
1	0	1	0

Then the E_2 term becomes for all $j \in \mathbb{Z}$:

E_2

0	$a+1$	0	1
b	b	0	a
a	0	b	b
1	0	$a+1$	0

from which we conclude

i. $E_2 \cong E_\infty$ if and only if $a = b = 0$.

Case 8: $h_3^{2,0} = 1$ and $h_3^{2,2} = 1$

E_3

0	0	0	1
0	0	1	0
0	0	0	0
1	0	1	0

Then the E_2 term becomes for all $j \in \mathbb{Z}$:

E_2

0	$a+1$	0	1
b	b	1	$a+1$
a	0	b	b
1	0	$a+1$	0

from which we conclude

i. This spectral sequence lives to E_3.

Case 9: $h_3^{2,0} = 1$ and $h_3^{3,1} = 1$

E_3

0	0	0	1
0	0	0	0
0	0	0	1
1	0	1	0

Then the E_2 term becomes for all $j \in \mathbb{Z}$:

E_2

0	$a+1$	0	1
b	$b-1$	0	$a+1$
a	0	b	b
1	0	$a+1$	0

from which we conclude that $b > 0$ so that

i. $H^{0,2}(X_j) \neq 0$ and

ii. This spectral sequence lives to E_3.

2.3.3 General descriptions of the terms of the Frölicher spectral sequence

We combine the preceding nine cases to make some general case-independent observations about when the spectral sequence lives to E_3, and when it degenerates at the E_2 level. For the remaining statements we make no assumptions on the vanishing of specific terms at the various levels of the spectral sequence.

Proposition 2.3.2. *If $E_2 \cong E_\infty$, then $h_2^{p,q} = h_2^{3-p,3-q}$.*

Proposition 2.3.3. *$h_2^{p,q} = h_2^{3-p,3-q}$ if and only if $h_3^{p,q} = h_3^{3-p,3-q}$.*

Combining these together gives a necessary condition for the degeneration of the Frölicher spectral sequence at the second level.

Corollary 2.3.4. *If* $h_3^{p,q} \neq h_3^{3-p,3-q}$, *then* $E_1 \ncong E_2 \ncong E_3$.

We complement Theorem 2.2.2 with the following.

Theorem 2.3.5. *Let* X_j *be a complex manifold diffeomorphic to* $\mathbb{C}\mathrm{P}^3$ *whose total Chern class is given by* $c(X_j) = 1 + 2jx + 2(j^2 - 1)x^2 + 4x^3$, *where* x *generates* $H^2(X_j, \mathbb{Z})$. *If* $j > 2$, *then* $h^{1,2} = h^{2,1} \geq 2$. *Moreover, if* $h_3^{0,2} \neq 1$ *or* $h_3^{3,1} \neq 1$, *then* $h^{1,2} \geq h^{0,2} \geq 3$.

Proof. Observe that in all nine cases above either $h_2^{1,2} = h_2^{0,2}$ or $h_2^{1,2} = h_2^{0,2} - 1$. Let us suppose $h_2^{1,2} = h_2^{0,2}$. To simplify the notation we consider the complex

$$0 \to E_1^{0,2} \xrightarrow{\alpha} E_1^{1,2} \xrightarrow{\beta} \dots$$

where α and β are the maps d_1. We know $h_2^{0,2} = \dim(\ker\alpha)$ and $h_2^{1,2} = \dim(\ker\beta) - \mathrm{rank}(\alpha)$, thus giving

$$\begin{aligned} h^{0,2} &= \dim(\ker\alpha) + \mathrm{rank}(\alpha) \\ &= h_2^{1,2} + \mathrm{rank}(\alpha) \\ &= \dim(\ker\beta) - \mathrm{rank}(\alpha) + \mathrm{rank}(\alpha) \\ &\leq h^{1,2}. \end{aligned}$$

We assumed that $h_2^{1,2} = h_2^{0,2}$, but suppose instead that $h_2^{1,2} = h_2^{0,2} - 1$. If this occurs, then unless $h_3^{0,2} = h_3^{3,1} = 1$, we have $h^{3,1} = h^{2,1}$. We can repeat the above argument for $h^{3,1}$ and $h^{2,1}$. Serre duality again gives

$$h^{0,2} = h^{3,1} \leq h^{2,1} = h^{1,2}.$$

In case $h_3^{0,2} = h_3^{3,1} = 1$ we have $h^{2,1} = h^{1,2} = h^{0,2} - 1$. The same arguments go through except that now we have

$$h^{0,2} \leq h^{1,2} + 1.$$

$\square$

Chapter 3

Symmetric complex structures

In this chapter we assume that X_j is complex and that X_j admits a holomorphic $\mathbb{C}^*$ action. We study the properties of hypothetical symmetric exotic complex structures on X_j. We rely on the fixed point theorems derived from equivariant versions of the Atiyah-Singer index theorem. In Appendix B we follow the presentations in [4], [5], [6], [7], and [8] as well as [59] to give a summary of the results that we use.

We begin this chapter with a reformulation of the G-index theorem for the action of a compact Lie group G on X_j due to Kosniowski in [48]. We then state the Bredon-Su fixed point theorem for projective space which gives restrictions on the fixed point sets of S^1 actions on generalized projective space. We use this to obtain a nonexistence result for $\mathbb{C}^*$-invariant complex structures on the manifolds X_j, giving restrictions on the Chern numbers of hypothetical $\mathbb{C}^*$-symmetric exotic complex structures. We also give restrictions on the fixed point sets of such $\mathbb{C}^*$ actions and restrictions on the weights of such actions on the tangent spaces of the fixed points.

3.1 Index theorems

In this section we summarize the main results of [48]. These are related to earlier and more familiar results of Bott [16] and [15] on the zeros of holomorphic vector fields on compact complex manifolds. Suppose X is a compact complex manifold of complex dimension n. Recall from section 2 the definition of $\chi_p(X)$.

$$\chi_p(X) = \sum_{q=0}^{n} (-1)^q h^{p,q}$$

Following [37] we introduce the $\chi_y(X)$ characteristic,

$$\chi_y(X) = \sum_{p=0}^{n} \chi_p(X) y^p$$

where $y \in \mathbb{C}$ varies. Notice that if $y = -1$, then $\chi_y(X) = \chi_{Top}(X)$ the Euler characteristic of X.

If A is a holomorphic vector field with simple isolated zeros, then at each zero x of A, there is an induced linear endomorphism $L_x(A)$ of T_xX, which is nonsingular at all of the zeros of A. The eigenvalues of A are in general nonzero complex numbers. Let c be a complex number such that for all eigenvalues θ of $L(A)$,

$$\text{Real}(\theta/c) \neq 0.$$

Define $s(x,c)$ to be the number of eigenvalues of $L_x(A)$ for which $\text{Real}(\theta/c)$ is positive.

Theorem 3.1.1 (Kosniowski, 1970). *If X is a compact complex manifold and A a holomorphic vector field with simple isolated zeros then*

$$\chi_y(X) = \sum_{x \in Zero(A)} (-y)^{s(x,c)},$$

where the sum is over all of the zeros x of A.

In [48], Kosniowski gives a further relationship among the eigenvalues of $L(A)$.

Theorem 3.1.2 (Kosniowski, 1970)**.** *The eigenvalues of $L(A)$ at the zeros of a holomorphic vector field A, with simple isolated zeros, can be paired such that the sum of the elements in any pair is zero.*

The pairing in the theorem occurs among the set of all of the eigenvalues. It says nothing about the eigenvalues at any particular zero.

Extend theorem 3.1.1 to arbitrary fixed point sets by using the holomorphic Lefschetz fixed point formula of Atiyah-Segal in theorem B.4.1. It is required that the one parameter group of the vector field lies in a compact group. We call such a vector field a compact vector field. Decompose the normal bundle N^A of the zero set of A as

$$N^A = \sum_\theta N^A(\theta),$$

where $N^A(\theta)$ is the sub-bundle of N^A on which $\exp(A)$ acts as $\exp(i\theta)$, θ being a real number. Then define $s(k,+)$ (resp $s(k,-)$) as the number of θ's at a component X_k^A of the zero set X^A which are positive (resp negative).

Theorem 3.1.3 (Kosniowski, 1970)**.** *Let X be a compact complex manifold and A a compact holomorphic vector field, then*

$$\begin{aligned}\chi_y(X) &= \sum_k (-y)^{s(k,+)} \chi_y(X_k^A) \\ &= \sum_k (-y)^{s(k,-)} \chi_y(X_k^A)\end{aligned}$$

where X_k^A is a component of the zero set X^A of A.

These theorems together give necessary conditions for a vector field to be a holomorphic vector field on X. We will use these to determine necessary conditions for X_j to admit a holomorphic $\mathbb{C}^*$ action.

Since this analysis involves studying the fixed point sets of holomorphic vector fields on X_j, in the next section we give restrictions on the fixed point sets of actions on X_j.

3.2 Actions of $\mathbb{C}^*$

We begin this section by stating the Bredon-Su fixed point theorem of [19], [18], and [64] in a special case.

Theorem 3.2.1 (Bredon-Su, 1964). *Suppose S^1 acts on a smooth manifold X diffeomorphic to $\mathbb{C}\mathrm{P}^n$. Let F denote the fixed point set of the action. Then F has at most $n+1$ components F_i, $i = 1, \ldots, m$, and each F_i has the same cohomology ring of complex projective k_i-space with*

$$\sum_{i=1}^{m} k_i = n - m + 1.$$

This theorem holds in the more general case where X is an integral cohomology complex projective space, that is a topological space with the same cohomology ring as $\mathbb{C}\mathrm{P}^n$ with integer coefficients. We will apply this theorem to the almost-complex manifolds X_j which are diffeomorphic to $\mathbb{C}\mathrm{P}^3$ and hence are integral cohomology complex projective spaces.

Suppose $\mathbb{C}^*$ acts effectively on X_j, i.e. the only which fixes all of X_j is the identity $e \in \mathbb{C}$. First we restate and prove proposition 1.2.7 from chapter 1.1 which gives a restriction on the fixed point set of a holomorphic $\mathbb{C}^*$ action on S^6.

Proposition 3.2.2 (HKP2, 2000)**.** *Suppose $\mathbb{C}^*$ acts holomorphically on $X = S^6$. Let F denote the fixed point set of I. Then F is either biholomorphic to $\mathbb{C}\mathrm{P}^1$ or F consists of two isolated points.*

Proof. A result of the linearization theorem is that the fixed point set of the $\mathbb{C}^*$ action is a complex submanifold of X. The fixed point set of the $\mathbb{C}^*$ action coincides with the fixed point set of the S^1 action contained in the $\mathbb{C}^*$ action. A classical result of Borel [13, IV 5.9] is that the fixed point set of an S^1 action on an integral homology sphere is itself an even dimensional integral homology sphere, so the fixed point set of a $\mathbb{C}^*$ action on X has the integral homology of S^0, S^2, or S^4. The first case gives two isolated points and the second case gives $\mathbb{C}\mathrm{P}^1$. To exclude the case of S^4, note that $b_2(X) = b_4(X) = 0$. It follows by the adjunction formula that any compact complex surface S has Euler characteristic 0.

$$c_2(S) = c_2(X) \cdot S - c_1(S) \cdot c_1(\mathcal{O}_X(S)|_S) = 0.$$

Hence S cannot have the integral homology of S^4. □

We now restrict our attention to X_j. We know that the standard Kähler structure on $\mathbb{C}\mathrm{P}^3$ has $j = 2$. We also know that a linear $\mathbb{C}^*$ action on $\mathbb{C}^4$ descends to a linear $\mathbb{C}^*$ action on $\mathbb{C}\mathrm{P}^3$ under which the Kähler structure is invariant. We would like to know whether or not X_j, $j \neq 2$, could admit a $\mathbb{C}^*$-symmetric complex structure. The answer to this question is negative except for $j = -1, 0, 1$.

Theorem 3.2.3. *Suppose* $\mathbb{C}^*$ *acts holomorphically on* X_j*, then* $-1 \leq j \leq 2$.

Proof. The fixed point set F of the $\mathbb{C}^*$ action coincides with the fixed point set of the S^1 action contained in the $\mathbb{C}^*$. According to theorem 3.2.1, F has at most four connected components F_1, F_2, F_3, F_4 with $H^*(F_i) \cong H^*(\mathbb{C}\mathrm{P}^{n_i})$. In addition each F_i is a complex submanifold.

As a consequence of Serre duality, lemma 2.1.1, $\chi_3(X_j) = -\chi_0(X_j)$ and $\chi_2(X_j) = -\chi_1(X_j)$. We then have the following expression for $\chi_y(X_j)$.

$$\chi_y = \chi_0 + \chi_1 y - \chi_1 y^2 - \chi_0 y^3. \tag{3.1}$$

Suppose F has two connected components F_1 and F_2. The relationship $n_1+n_2 = 2$ gives two cases: (1) $F_1 = x$, $F_2 = S$ where x is an isolated point and S is a complex surface with the integral cohomology of $\mathbb{C}\mathrm{P}^2$, or (2) $F_1 = \mathbb{C}\mathrm{P}^1$, $F_2 = \mathbb{C}\mathrm{P}^1$. In the first case we have from theorem 3.1.3

$$\chi_y(X_j) \quad = \quad (-y)^{s(1,+)} + (-y)^{s(2,+)}(\chi_0(S) + \chi_1(S)y + \chi_0(S)y^2) \tag{3.2}$$

where $s(1,+) = 0,1,2,3$ and $s(2,+) = 0,1$. If $s(2,+) = 0$ and $s(1,+) \neq 3$, then we have $\chi_0(X_j) = 0$. This is the case exactly for $j = -1,0,1$. If $s(2,+) = 0$ and $s(1,+) = 3$, then we have $\chi_0(X_j) = 1$. This is the case exactly for $j = 2$. Similarly, if $s(2,+) = 1$ and $s(1,+) = 0$, then $\chi_0(X_j) = 1$, and if $s(2,+) = 1$ and $s(1,+) \neq 0$, then $\chi_0(X_j) = 0$.

Suppose now that $F = \mathbb{C}\mathrm{P}^1 \amalg \mathbb{C}\mathrm{P}^1$. This gives

$$\begin{aligned}\chi_y(X_j) &= (-y)^{s(1,+)}(1-y) + (-y)^{s(2,+)}(1-y) \\ &= (-y)^{s(1,+)} + (-y)^{s(1,+)+1} + (-y)^{s(2,+)} + (-y)^{s(2,+)+1}.\end{aligned}$$

This expression for $\chi_y(X_j)$ is consistent with (3.1) if and only if $\chi_0(X_j) = 1$ or $\chi_0(X_j) = 0$. The former is the case for $j = 2$ and the latter is the case for $j = -1, 0, 1$.

Consider now the case F has three connected components. $F = \{x_1\} \amalg \{x_2\} \amalg \mathbb{C}\mathrm{P}^1$. Theorem 3.1.3 gives

$$\chi_y(X_j) = (-y)^{s(1,+)} + (-y)^{s(2,+)} + (-y)^{s(3,+)}(1-y).$$

Suppose $\chi_0(X_j) \neq 0$ in (3.1). Performing the same calculations as above gives that $\chi_0(X_j) = 1$. We see then that this expression is consistent with (3.1) if and only if $\chi_0(X_j) = 0$ or $\chi_0(X_j) = 1$.

Finally if F has four connected components, then each component F_i is a point x_i, and we have by theorem 3.1.1

$$\chi_y(X_j) = (-y)^{a_1} + (-y)^{a_2} + (-y)^{a_3} + (-y)^{a_4},$$

and again this is consistent with (3.1) if and only if $-1 \leq j \leq 2$. □

Theorem 3.2.1 gives a restriction on the fixed point set of a $\mathbb{C}^*$ action on X_j. We restrict the fixed point even further in the following theorem.

Theorem 3.2.4. *Suppose $\mathbb{C}^*$ acts holomorphically on X_j, $j = -1, 0, 1$. Then the fixed point set F is one of the following.*

(i) $F = \mathbb{C}\mathrm{P}^1 \amalg \mathbb{C}\mathrm{P}^1$,

(ii) $F = \{x_1\} \amalg \{x_2\} \amalg \mathbb{C}\mathrm{P}^1$, *or*

(iii) $F = \{x_1\} \amalg \{x_2\} \amalg \{x_3\} \amalg \{x_4\}$, *where x_i are isolated points.*

Proof. We have shown in 3.2.3 that if X_j admits a holomorphic $\mathbb{C}^*$ action, then $-1 \leq j \leq 2$. We now show that if the fixed point set contains a complex surface S, then $j = 2$. Suppose $j = -1, 0, 1$. Then

$$\chi_y(X_j) = -2y + 2y^2,$$

but if $s(2,+) = 0$ in (3.2)s, then $\chi_0(S) = 0$ or $\chi_0(S) = -1$. In either case the coefficients for the two expressions for $\chi_y(X_j)$ do not agree. On the other hand if $s(2,+) = 1$, then $\chi_0(S) = 0$ or $\chi_0(S) = -1$ and the coefficients again do not agree. Hence, if F contains a complex surface then $j = 2$. □

We see then that the only $\mathbb{C}^*$-symmetric exotic complex structures would live on X_{-1}, X_0, or X_1. We study this situation further by considering the weights of the induced actions on the tangent spaces of the fixed points. Before we consider the weights of the induced action on the tangent spaces for X_j we consider the case of S^6. The results which we give for S^6 below can also be found in [21], though his results were brought to our attention after these calculations and our methods differ. We use the G-index theorem B.4.1 from the appendix and its simplification in the case of isolated fixed points.

Suppose X is a compact complex manifold of complex dimension 3. Let $f : \mathbb{C}^* \times X \to X$ be an effective holomorphic $\mathbb{C}^*$ action. Denote by v the infinitesimal generator of f,

$$v = \frac{\mathrm{d}}{\mathrm{dt}} f_t \Big|_{t=1}.$$

v is a compact holomorphic vector field which is $2\pi i$-periodic and its zero set coincides with the fixed point set $Fix(f)$ of f. A consequence of the linearization theorem, theorem 1.2.6, is that v is linearizable near each point in $Fix(f)$. In particular, $Fix(f)$ is a complex submanifold of X.

Let $x \in Fix(f)$, and let n_1, n_2, n_3 be the weights (of the linear part) of v on the tangent space $T_x X$. The effectivity of f gives that

$$\mathrm{GCD}(n_1, n_2, n_3) = 1.$$

Theorem 3.2.5. *Suppose S^6 is a complex manifold and $\mathbb{C}^*$ acts on S^6 fixing two isolated points $\{x_1, x_2\}$. Then the weights (n_1, n_2, n_3) and (m_1, m_2, m_3) of the induced actions on $T_{x_1}X$ and $T_{x_2}X$, respectively, satisfy the following.*

$$m_i = -n_i$$

$$n_1 + n_2 + n_3 = 0.$$

Proof. Since the fixed point set of the action consists of isolated points we use the Atiyah-Bott fixed point theorem.

$$L(f^{0,*}) = \sum_{f(x)=x} \frac{1}{\det_{\mathbb{C}}(1 - df_x)}$$

This gives in our case

$$L(f^{0,*}) = \frac{1}{(1-z^{n_1})(1-z^{n_2})(1-z^{n_3})} + \frac{1}{(1-z^{\pm n_1})(1-z^{\pm n_2})(1-z^{\pm n_3})}$$

where $z \in \mathbb{C}^*$ has $|z| = 1$. $L(f^{o,*})$ is a Laurent polynomial and cannot have a poles on the unit circle. This first gives the restriction that, upto reordering, $m_i = \pm n_i$. We need then to check that for the four cases listed below that $L(f^{0,*})$ is a Laurent polynomial.

Case 1: $m_1 = n_1$, $m_2 = n_2$, $m_3 = n_3$

In this case $L(f^{0,*})$ simplifies to become

$$\begin{aligned} L(f^{0,*}) &= \frac{1}{(1-z^{n_1})(1-z^{n_2})(1-z^{n_3})} + \frac{1}{(1-z^{n_1})(1-z^{n_2})(1-z^{n_3})} \\ &= \frac{2}{(1-z^{n_1})(1-z^{n_2})(1-z^{n_3})} \end{aligned}$$

This has a pole at $z = 1$.

Case 2: $m_1 = n_1$, $m_2 = n_2$, $m_3 = -n_3$

In this case $L(f^{0,*})$ simplifies to become

$$\begin{aligned} L(f^{0,*}) &= \frac{1}{(1-z^{n_1})(1-z^{n_2})(1-z^{n_3})} + \frac{1}{(1-z^{n_1})(1-z^{n_2})(1-z^{-n_3})} \\ &= \frac{1}{(1-z^{n_1})(1-z^{n_2})} \end{aligned}$$

This has a pole at $z = 1$.

Case 3: $m_1 = n_1$, $m_2 = -n_2$, $m_3 = -n_3$

In this case $L(f^{0,*})$ simplifies to become

$$\begin{aligned} L(f^{0,*}) &= \frac{1}{(1-z^{n_1})(1-z^{n_2})(1-z^{n_3})} + \frac{1}{(1-z^{n_1})(1-z^{-n_2})(1-z^{-n_3})} \\ &= -\frac{1+z^{n_1+n_2}}{(1-z^{n_1})(1-z^{n_2})} \end{aligned}$$

This has a pole at $z = 1$.

Case 4: $m_1 = -n_1$, $m_2 = -n_2$, $m_3 = -n_3$

In this case $L(f^{0,*})$ simplifies to become

$$\begin{aligned} L(f^{0,*}) &= \frac{1}{(1-z^{n_1})(1-z^{n_2})(1-z^{n_3})} + \frac{1}{(1-z^{-n_1})(1-z^{-n_2})(1-z^{-n_3})} \\ &= \frac{-1+z^{n_1+n_2+n_3}}{(1-z^{n_1})(1-z^{n_2})} \end{aligned}$$

This has a pole at $z = 1$ unless $n_1 + n_2 + n_3 = 0$. In this case $L(f^{0,*}) = 0$. Hence any holomorphic $\mathbb{C}^*$ action with isolated fixed points has weights which satisfy $m_i = -n_i$ and $n_1 + n_2 + n_3 = 0$. □

We have a similar restriction on the weights if the action fixes $\mathbb{C}\mathrm{P}^1$. There is also additional information about the normal bundle N of $\mathbb{C}\mathrm{P}^1$ in S^6.

Theorem 3.2.6. *Suppose S^6 is a complex manifold and $\mathbb{C}^*$ acts on S^6 fixing $\mathbb{C}\mathrm{P}^1$. Then for any $x \in \mathbb{C}\mathrm{P}^1 \subset S^6$ the weights of the induced action on T_xX are $(1, -1, 0)$. Moreover, the normal bundle splits as follows.*

$$N = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$$

We first prove a lemma.

Lemma 3.2.7. *Suppose* $\mathbb{C}\mathrm{P}^1$ *is embedded in a compact complex manifold* X *of complex dimension* n *with* $b_2(X) = 0$. *The normal bundle* N *of* $\mathbb{C}\mathrm{P}^1$ *in* X *splits into* $n-1$ *line bundles* $N = L_1 \oplus \cdots \oplus L_{n-1}$ *with* $c_1(L_j) = k_j x$ *where* x *generates* $H^2(\mathbb{C}\mathrm{P}^1, \mathbb{Z})$. *Then*

$$\sum_{j=0}^{n-1} k_j = -2 \tag{3.3}$$

Proof of Lemma. This is an application of naturality of Chern classes and the Whitney product formula. Embed $\mathbb{C}\mathrm{P}^1$ holomorphically into X, $i : \mathbb{C}\mathrm{P}^1 \hookrightarrow X$. Then the tangent bundle of X, $T := TX$ pulls back via i to a rank n complex vector bundle $E := i^*T$ over $\mathbb{C}\mathrm{P}^1$ and $E = L_1 \oplus \cdots \oplus L_n$ splits into n line bundles. Identify L_n with the tangent bundle $T\mathbb{C}\mathrm{P}^1$ of $\mathbb{C}\mathrm{P}^1$ and $L_1 \oplus \cdots \oplus L_{n-1}$ with the normal bundle N of $\mathbb{C}\mathrm{P}^1$ in X. The naturality of Chern classes and the Whitney product formula give $i^*(c(T)) = c(i^*(T)) = c(E) = \prod c(L_j)$. Since $c(L_n) = c(\mathbb{C}\mathrm{P}^1)$, we have that $c(L_n) = (1+2x)$ where x is the generator of $H^2(\mathbb{C}\mathrm{P}^1, \mathbb{Z})$, and $\prod c(L_j) = \prod(1+k_j x)$. Then we have $c(E) = 1+(2+k_1+\cdots+k_{n-1})x$, and $i^*(c_1(X)) = (2+k_1+\cdots+k_{n-1})x = c_1(E)$. Since $b_2(X) = \dim H^2(X) = 0$, we know that $c_1(T) = 0$. The result follows. □

Proof of theorem 3.2.6. The normal bundle of $\mathbb{C}\mathrm{P}^1$ in S^6 splits into line bundles L_1 and L_2. Let $k_j x$ denote $c_1(L_j)$. Equation 3.3 gives that $k_1 + k_2 = -2$. Suppose that $\mathbb{C}^*$ acts as z^{m_1} on L_1 and z^{m_2} on L_2. We now use theorem B.4.1, the holomorphic

Lefschetz formula for higher dimensional fixed point sets.

$$L(\overline{\partial}) = \left\{ \frac{\mathrm{Td}(\mathbb{C}\mathrm{P}^1)}{(1-e^{-k_1 x}z^{-m_1})(1-e^{-k_2 x}z^{-m_2})} \right\} [X^g]$$

The Todd class $\mathrm{Td}(\mathbb{C}\mathrm{P}^1)$ is given by $1+x$. Our expression then simplifies to become

$$L(\overline{\partial}) = \frac{1-(1+k_1)z^{-m_1}+(1+k_1)z^{-m_2}-z^{-m_1-m_2}}{(1-z^{-m_1})^2(1-z^{-m_2})^2}$$

Since $L(\overline{\partial})$ is a Laurent polynomial, for $\mathbb{C}^*$ to act on S^6 holomorphically, $L(\overline{\partial})$ cannot have a pole at $z=1$. To check that this is the case, suppose m_1, $m_2 > 0$ and that the numerator does not vanish identically. Consider the following three cases:

Case 1: $n_1 = m_1 \quad n_2 = m_2$

$$\begin{aligned} L(\overline{\partial}) &= \frac{1-(1+k_1)z^{-n_1}+(1+k_1)z^{-n_2}-z^{-n_1-n_2}}{(1-z^{-n_1})^2(1-z^{-n_2})^2} \\ &= \frac{z^{n_1+n_2}(z^{n_1+n_2}-(1+k_1)z^{n_2}+(1+k_1)z^{n_1}-1)}{(1-z^{n_1})^2(1-z^{n_2})^2} \end{aligned}$$

In this case, if $z=1$ is a zero of the numerator, then it has multiplicity at most n_1+n_2, but $z=1$ is clearly a zero of the denominator of multiplicity $2(n_1+n_2)$, hence $z=1$ is a pole.

Case 2: $n_1 = -m_1 \quad n_2 = m_2$

$$\begin{aligned} L(\overline{\partial}) &= \frac{1-(1+k_1)z^{n_1}+(1+k_1)z^{-n_2}-z^{n_1-n_2}}{(1-z^{n_1})^2(1-z^{-n_2})^2} \\ &= \frac{z^{n_2}(z^{n_2}-(1+k_1)z^{n_1+n_2}-(1+k_1)-z^{n_1})}{(1-z^{n_1})^2(1-z^{n_2})^2} \end{aligned}$$

In this case, if $z = 1$ is a zero of the numerator, then it has multiplicity at most $n_1 + n_2$, but again $z = 1$ is a zero of the denominator of multiplicity $2(n_1 + n_2)$, hence $z = 1$ is a pole.

Case 3: $n_1 = -m_1 \quad n_2 = -m_2$

$$L(\overline{\partial}) \quad = \quad \frac{1 - (1 + k_1)z^{n_1} + (1 + k_1)z^{n_2} - z^{n_1+n_2}}{(1 - z^{n_1})^2(1 - z^{n_2})^2}$$

In this case, if $z = 1$ is a zero of the numerator, then it has multiplicity at most $n_1 + n_2$, but again $z = 1$ is a zero of the denominator of multiplicity $2(n_1 + n_2)$, hence $z = 1$ is a pole.

For the three cases we assumed that the numerator did not vanish identically. If the numerator were zero, then $L(\overline{\partial})$ would certainly be a Laurent polynomial. Notice that the numerator is identically zero

$$(1 - z^{n_1+n_2}) + (1 + k_1)(z^{n_1} + z^{n_2}) \equiv 0$$

if and only if both summands are zero, and this relationship is only satisfied if

a. $n_1 = -n_2$ and

b. $k_1 = -1$ which also gives $k_2 = -1$.

Therefore the normal bundle of a $\mathbb{C}^*$-fixed $\mathbb{CP}^1$ splits as

$$N = \mathcal{O}(-1) \oplus \mathcal{O}(-1),$$

and $\mathbb{C}^*$ acts with weights $(n, -n, 0)$. Since the action is effective we have that the weights must be $(1, -1, 0)$. $\square$

3.3 Future directions

In this section we sketch partial results on the restrictions of the weights of $\mathbb{C}^*$ actions on the tangent spaces of fixed points on X_j. We also propose directions for future research beyond this manuscript.

In order to proceed with the analysis of the weights of a holomorphic $\mathbb{C}^*$ action on the tangent spaces of fixed point, we need to introduce global weights, see [58], [57], [65]. Let $\Omega = \{a_0, a_1, a_2, a_3\}$ be a set of distinct integers, and suppose $\mathbb{C}^*$ acts on $\mathbb{C}\mathrm{P}^3$ as follows.

$$\gamma(z_0 : z_1 : z_2 : z_3) = (\gamma^{a_0} z_0 : \gamma^{a_1} z_1 : \gamma^{a_2} z_2 : \gamma^{a_3} z_3)$$

The fixed points of this action are

$$e_i = (0 : \ldots : 1 : \ldots : 0),$$

where 1 is in the i^{th} position. The induced action on the tangent space $T_{e_i}\mathbb{C}\mathrm{P}^3$ has weights $\Omega_i = \{a_j - a_i | j \neq i\}$. The set Ω is called the set of global weights and the sets Ω_i are called sets of local weights. This is an example of a linear action on $\mathbb{C}\mathrm{P}^3$ with isolated fixed points. If in the set Ω one or more of the integers is repeated, the fixed point set contains higher dimensional components.

There are suitable analogs of global and local weights for $\mathbb{C}^*$ actions on X_j involving equivariant cohomology. We omit the details here and direct the reader to [65]. We also can generalize the notion of a linear action to include linear "type" actions. In [58] T. Petrie constructs S^1 actions which are not of linear type.

We have restricted in theorem 3.2.4 the fixed point set of holomorphic $\mathbb{C}^*$ actions on X_j. A theorem of Tsukada and Washiyama in [67] gives that if the fixed point set F consists of three connected components, then the action is of linear type. We have found that if the fixed point set consists of two connected components. As of the writing of this manuscript we have not yet shown that a $\mathbb{C}^*$ action with isolated fixed points is of linear type. We need only exclude the "Petrie" type actions constructed in [58].

In another line of investigation we intend to see whether or not any complex X_j could be almost-homogeneous. In [41] the authors show that any complex structure on S^6 is not almost-homogeneous, theorem 1.2.4. They use a very technical argument partly because the cohomology $H^k(S^6)$ of S^6 vanishes in all dimensions except $k = 0$ and $k = 6$. In particular, the first and second Chern classes $c_1 \in H^2(S^6)$ and $c_2 \in H^4(S^6)$ of S^6 vanish. We hope to exploit the richer cohomological structure of X_j to extract similar information using different techniques.

Appendix A

Octonions and almost-complex structures

In this appendix we construct almost-complex structures on the spheres S^2 and S^6 using the quaternions and octonions as models. The real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$, and octonions $\mathbb{O}$ are the only four normed division algebras.

Definition A.0.1. *A* **normed division algebra** *is a finite dimensional algebra over* $\mathbb{R}$ *with multiplicative unit 1, and equipped with an inner product* $<,>$ *whose associated square norm* $\| \, \|$ *satisfies the multiplicative property*

$$\|xy\| = \|x\|\|y\|.$$

We briefly describe the Cayley-Dickson process which we use to construct new normed division algebras out of old. For a more complete discussion see [34]. Suppose A is a normed division algebra. We define $A(+)$ as follows.

$$A(+) = A \oplus A$$

Addition is defined component wise.

$$(a,b) \pm (c,d) = (a \pm c, b \pm d)$$

Multiplication is defined as follows.

$$(a,b)(c,d) = (ac - \overline{d}b, da + b\overline{c})$$

The complex numbers $\mathbb{C}$ are simply $\mathbb{R}(+)$. Similarly,

$$\mathbb{H} = \mathbb{C}(+)$$

$$\mathbb{O} = \mathbb{H}(+).$$

With each step of this process we lose some characteristic of the previous space. For example, the complex numbers are associative and commutative. In the construction of the quaternions we lose commutativity and in the construction of the octonions we lose associativity. In general if $x, y, z \in \mathbb{O}$, then $(xy)z \neq x(yz)$, but we do have $x(xy) = (xx)y$. This feature will play an important role soon.

Let $K = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. If $z = (a,b) \in K$ then define the real and imaginary parts of z as follows.

$$\operatorname{Re} z = (\operatorname{Re} a, 0)$$

$$\operatorname{Im} z = (\operatorname{Im} a, b)$$

Conjugation is defined by $\overline{(a,b)} = (\overline{a}, -b)$.

Now we consider the cases $K = \mathbb{H}$ and $K = \mathbb{O}$. Define an inner product and cross product on the subset of K consisting of the purely imaginary elements.

$$< z, w >= -\operatorname{Re}(zw)$$

$$z \times w = \operatorname{Im}(zw)$$

If $K = \mathbb{H}$, then as a vector space $\operatorname{Im} K = E^3 \cong \mathbb{R}^3$. Consider the subset of unit vectors in E^3.

$$S^2 = \{u \in E^3 :< u, u >= 1\}$$

We can identify the tangent space $T_u S^2$ at u of S^2 with $u^\perp = \{v \in E^3 | < u, v >= 0\}$. Define a linear map

$$J_u : T_u S^2 \to T_u S^2$$

by

$$J_u(v) = u \times v.$$

This map is well-defined since

$$\begin{aligned} < u, J_u(v) > &= < u, u \times v > \\ &= -\operatorname{Re}(u \operatorname{Im}(uv)) \\ &= -\operatorname{Re}(u(uv - \operatorname{Re}(uv))) \\ &= -\operatorname{Re}((uu)v + < u, v >) \\ &= -\operatorname{Re}(-v) \\ &= 0. \end{aligned}$$

We also have for each u, $J_u^2 = -1$ by the following.

$$\begin{aligned} J_u(J_u(v)) &= u \times (\mathrm{Im}(uv)) \\ &= \mathrm{Im}(u(uv - \mathrm{Re}(uv))) \\ &= \mathrm{Im}(-v) \\ &= -v. \end{aligned}$$

In this way we define an almost-complex structure on S^2. We repeat this process for $K = \mathbb{O}$ to get an almost-complex structure on S^6.

Appendix B

Index theorems

The purpose of this appendix is the provide a catalog of index theorems and to describe the notation used in the statements of the theorems. We begin this with a discussion of the Chern classes of a complex vector bundle. For a more detailed discussion of these topics, see [17, Chapter IV], [54], and [59]. We then state the index theorems. For a full discussion of these topics we highly recommend the beautiful papers of Atiyah-Bott [4] and [5], Atiyah-Singer [7] and [8], and Atiyah-Segal [6].

B.1 Characteristic Classes

Let X be a topological space and let E be a rank r vector bundle over X. Certain cohomology classes measure the "twisting" of E :

Steifel-Whitney classes:	$w_i \in H^i(X, \mathbb{Z}/2\mathbb{Z})$
Chern classes:	$c_i \in H^{2i}(X, \mathbb{Z})$
Pontrjagin classes:	$p_i \in H^{4i}(X, \mathbb{Z})$
the Euler class:	$e \in H^n(X, \mathbb{Z})$

For Steifel-Whitney classes and Pontrjagin classes E must be a real vector bundle. For the Euler class E must be real and orientable. For the Chern classes E must be a complex vector bundle. Since the cohomology of both S^6 and $\mathbb{C}\mathrm{P}^n$ is torsion free, we can work over either real or integral coefficients without loss of generality.

Suppose $E \xrightarrow{\pi} X$ is a smooth complex vector bundle over a smooth compact manifold X. Suppose ∇ is a connection on E and R its curvature. We represent R as a matrix of differential forms. Define the total Chern class $c(E)$ as follows.

$$c(E) = \det\left(\mathrm{I} + \frac{\sqrt{-1}}{2\pi}R\right)$$

$$c(E) = 1 + c_1(E) + c_2(E) + \cdots + c_n(E)$$

Suppose now that E is a smooth real vector bundle over X. We can repeat this to get the Pontrjagin classes.

$$p(E) = \det\left(\left(\mathrm{I} - \left(\frac{R}{2\pi}\right)^2\right)^{1/2}\right)$$

$$p(E) = 1 + p_1(E) + p_2(E) + \cdots + p_k + \ldots$$

There is an intimate relationship between the Chern classes and the Pontrjagin classes of a manifold. Suppose E is a real vector bundle.

$$p_i(E) = c_{2i}(E \otimes \mathbb{C})$$

As our primary interest lies in the Chern classes of a complex vector bundle we restrict our discussion to Chern classes of a complex vector bundle E over a smooth manifold X. The Chern classes of E satisfy naturality, i.e. if $f : Y \to X$, then

$$f^*(c_i(E)) = c_i(f^*(E)),$$

where f^*E denotes the pull-back bundle over Y. Another important property of the Chern classes is the Whitney Product Formula. If E_1 and E_2 are two complex vector bundles over X, then so is $E_1 \oplus E_2$ and we have

$$c(E_1 \oplus E_2) = c(E_1)c(E_2),$$

where $c(E) = 1 + c_1(E) + c_2(E) + \ldots + c_r(E)$ is the total Chern class of E.

Suppose we can write E as a sum of complex line bundles over X.

$$E = L_1 \oplus \ldots \oplus L_r.$$

We could then write the total Chern class of E in terms of the total Chern classes of the L_i.

$$\begin{aligned} c(E) &= c(L_1) \cdots c(L_r) \\ &= (1 + c_1(L_1)) \cdots (1 + c_1(L_r)) \end{aligned}$$

In general we cannot write E as the sum of complex line bundles. We can, however, use the splitting principle to treat E as the sum of complex line bundles in Chern class computations, see [17]. The splitting principle says the following. If X is a smooth manifold, and E is a complex rank r vector bundle over X, then there is a space $F(E)$ and a map $\sigma : F(E) \to X$ so that

(1) the pull-back of E to $F(E)$ splits as the product of line bundles:

$$\sigma^*(E) = L_1 \oplus \ldots \oplus L_r,$$

(2) σ^* embeds $H^*(X)$ in $H^*(F(E))$.

The splitting principle allows us to express the total Chern class of E in the following manner. Let $x_i = L_i$.

$$c(E) = \prod_{i=1}^{r}(1 + x_i)$$

The j^{th} Chern class of E is the j^{th} symmetric function of x_i.

$$\begin{aligned} c_1(E) &= x_1 + x_2 + \ldots + x_r \\ c_2(E) &= x_1x_2 + x_1x_3 + \ldots + x_{r-1}x_r \\ &\vdots \\ c_r(E) &= x_1x_2\cdots x_r \end{aligned}$$

Any cohomology class which can be expressed as a symmetric polynomial in the classes x_i can be expressed in terms of the Chern classes of E.

The Chern character of E, $\mathrm{ch}(E) \in H^*(X, \mathbb{Q})$, is defined as follows.

$$\mathrm{ch}(E) = \sum_{i=1}^{r} e^{x_i} = r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \ldots$$

Since X is finite dimensional, $e^x = 1 + x + x^2/2 + \ldots$ is a finite sum.

The Todd class of E, $\mathrm{Td}(E) \in H^*(x, \mathbb{Q})$, is defined as follows.

$$\mathrm{Td}(E) = \prod_{i=1}^{r} \frac{x_i}{1 - e^{-x_i}} = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_2 + c_1^2) + \frac{1}{24}c_1c_2 + \ldots$$

Let X be a smooth almost-complex manifold and $E = TX$ its tangent bundle. We define the Chern classes (respectively the Chern character and Todd class) of X to be the Chern classes (resp. Chern character and Todd class) of TX.

B.2 The Atiyah-Singer Index Theorem

For a detailed discussion of differential operators on manifolds see [50] and [70]. Let X be a compact smooth oriented manifold and let E and F be smooth complex vector bundles over X. Let $\Gamma(E)$ and $\Gamma(X)$ denote the space of smooth sections of E and F, respectively. Suppose $D : \Gamma(E) \to \Gamma(F)$ is a smooth differential operator. If D is of order k, then the terms of order k defines the leading symbol $\sigma_k(D)$ of D. This is a bundle homomorphism

$$\sigma_k(D) : \pi^* E \to \pi^* F$$

over the cotangent bundle T^*X of X where $\pi : T^*X \to X$ is the projection. D is said to be **elliptic** if $\sigma_k(D)$ is a bundle isomorphism away from the zero section of T^*X.

Suppose now that there is a sequence of smooth vector bundles over X and a sequence of smooth differential operators.

$$\Gamma(E) : \cdots \to \Gamma(E_i) \xrightarrow{D_i} \Gamma(E_{i+1}) \to \cdots \tag{B.1}$$

The sequence is called a complex if $D_{i+1} \circ D_i = 0$. We do not assume that the D_i are of the same order. We form the symbol sequence.

$$\cdots \to \pi^* E_i \xrightarrow{\sigma_{k_i}(D_i)} \pi^* E_{i+1} \to \cdots \tag{B.2}$$

The complex B.1 is said to be elliptic if the symbol sequence B.2 is exact outside the zero section.

Theorem B.2.1 (Atiyah-Singer Index Theorem, 1968)**.** *Let X be a compact smooth manifold of dimension n, let D be an elliptic operator on X, and let $\sigma(D)$ be the symbol of D. Then*

$$\text{Index}(D) = (-1)^n \{ch(\sigma(D))\,\text{Td}(TX \otimes \mathbb{C})\}[TX] \tag{B.3}$$

where $[TX]$ is the fundamental class of TX.

If X is a compact smooth manifold, the de Rham complex

$$\Omega : 0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n \to 0$$

is elliptic, so the index theorem B.2.1 applies. In this case the left hand side is simply the Euler characteristic $\chi(X)$ of X and the right hand side is $e[X]$, the Euler class evaluated on the fundamental class of X.

If X is a complex manifold, the Dolbeault complex

$$\Omega^{0,q} : 0 \to \Omega^{0,0} \xrightarrow{\overline{\partial}} \Omega^{0,1} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \Omega^{0,n} \to 0$$

is elliptic, so the index theorem B.2.1 applies. The left hand side of B.3 gives $\chi_0 = \sum(-1)^q h^{0,q}$ and the right hand side simplifies to $\text{Td}(X)[X]$. This special case of theorem B.2.1 is a reformulation by Hirzebruch of the Riemann-Roch theorem.

We write explicitly the expressions for $\chi_0(X)$ in dimensions 1, 2, 3, and 4 below.

Let n be the complex dimension of X and let $[X]$ denote the fundamental class of X.

$$n = 1: \quad \chi_0 = \frac{1}{2} c_1[X] \tag{B.4}$$

$$n = 2: \quad \chi_0 = \frac{1}{12}(c_2 + c_1^2)[X] \tag{B.5}$$

$$n = 3: \quad \chi_0 = \frac{1}{24} c_1 c_2 [X] \tag{B.6}$$

$$n = 4: \quad \chi_0 = \frac{1}{720}(-c_1^4 + 4c_1^2 c_2 + 3c_2^2 + c_1 c_3 - c_4)[X] \tag{B.7}$$

We saw above that we can regard the Chern classes $c_i \in H^{2i}(X)$ as differential forms. Then by evaluation on the fundamental class of X we mean integration over the manifold X. In the case $n = 1$ above we see that this is a reformulation of the Gauss-Bonnet-Chern Theorem.

B.3 The Atiyah-Bott Fixed Point Formula

Let X be a smooth manifold and let $\Gamma(E)$ be an elliptic complex.

$$\Gamma(E) : \cdots \to \Gamma(E_i) \xrightarrow{D_i} \Gamma_{i+1} \to \cdots$$

An endomorphism of elliptic complexes is a sequence of linear maps

$$T_i : \Gamma(E_i) \to \Gamma(E_i)$$

such that $D_i \circ T_i = T_{i+1} \circ D_i$. Such an endomorphism induces maps $H^i T$ of cohomology $H^i(\Gamma(E))$. Since each $H^i(\Gamma(E))$ is finite dimensional, trace $H^i T$ is well-defined. Define the **Lefschetz number** $L(T)$ as follows.

$$L(T) = \sum_{i=0}^{n} (-1)^n \operatorname{trace} H^i T$$

If $T = \mathrm{I}$, the identity map, then $L(T) = \chi(\Gamma(E))$.

Consider the elliptic de Rham complex.

$$\Omega : 0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n \to 0$$

This complex behaves naturally with respect to smooth maps $f : X \to X$. The differential df maps tangent spaces $T_x X \to T_{f(x)} X$ so that exterior powers of the transpose of df furnish bundle maps.

$$\Lambda^k df^* : \Lambda^k T^* X \to \Lambda^k T^* X$$

If f fixes only isolated points, then the Atiyah-Bott fixed point formula gives us a way to express the Lefschetz number $L(f^*) = \sum_k \operatorname{trace} H^k(f)$ as follows.

$$L(f^*) = \sum_{f(x)=x} \nu(x)$$

where

$$\nu(x) = \sum_k (-1)^k \frac{\operatorname{trace}(\Lambda^k df_x^*)}{|1 - \det df_x|}$$

This expression simplifies to

$$L(f^*) = \sum_{f(x)=x} \frac{\det(1 - df_x)}{|\det(1 - df_x)|} = \sum_x \pm 1.$$

Now suppose X is a compact complex manifold of complex dimension n and $f : X \to X$ is a holomorphic map fixing isolated points. Consider the elliptic Dolbeault complex.

$$\Omega^{p,*} : 0 \to \Omega^{p,0} \xrightarrow{\bar{\partial}} \Omega^{p,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \Omega^{p,n} \to 0$$

Again since f fixes only isolated points, the Atiyah-Bott fixed point formula gives us a way to express the Lefschetz number $L(f^{p,*}) = \sum_q H^{p,q}(f)$.

$$L(f^{p,*}) = \sum_{f(x)=x} \frac{\operatorname{trace}_{\mathbb{C}} \Lambda^p(df_x)}{\det_{\mathbb{C}}(1 - df_x)}$$

Of particular interest is the case $p = 0$ which gives.

$$\chi_0(X) = \sum_{f(x)=x} \frac{1}{\det_{\mathbb{C}}(1 - df_x)}.$$

The Atiyah-Bott fixed point formula is a special case of the Atiyah-Segal G-Index theorem.

B.4 The Atiyah-Segal G-Index Theorem

The Atiyah-Segal G-index theorem is much more general as it is stated in [6], but our primary interest is in calculating the holomorphic Lefschetz number in cases where the function $f : X \to X$ has components of the fixed point set of nonzero dimension. A more detailed discussion is in [8] and [59].

Let X be a compact complex manifold of complex dimension n and let G be a topologically cyclic compact Lie group acting holomorphically on X. As above the holomorphic Lefschetz number is

$$L(\overline{\partial}) = \sum \operatorname{trace} H^{0,q}(g).$$

For $g \in G$ generate G and let X^g denote the fixed point set of $g : X \to X$. Let N^g denote the normal bundle of X^g in X. The complex vector bundle N^g has a decomposition

$$N^g = \bigoplus_{0<\theta<\pi} N^g(\theta)$$

where $N^g(\theta)$ is the sub-bundle on which g acts as $e^{i\theta}$. A special case of the G-index theorem is the following.

Theorem B.4.1. *Let G be a compact Lie group acting holomorphically on the compact complex manifold X, let X^g be the submanifold of points left fixed by $g \in G$, and let $L(g, \overline{\partial})$ denote the holomorphic Lefschetz number of the mapping $g : X \to X$. Then*

$$L(g, \overline{\partial}) = \left\{ \frac{\mathrm{Td}(X^g)}{\prod_{0<\theta<\pi} \prod_j (1 - e^{-x_j - i\theta})(N^g(\theta))} \right\} [X^g]$$

where x_j corresponds to the splitting principle applied to N^g.

If G acts with isolated fixed points then the formula given above reduces to the holomorphic Lefschetz formula from the previous section.

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