# NONLINEAR MECHANICS AND TESTING OF HIGHLY FLEXIBLE ONE-DIMENSIONAL STRUCTURES USING A CAMERA-BASED MOTION ANALYSIS SYSTEM 

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## By

Jiazhu Hu
Dr. P. Frank Pai, Dissertation Supervisor

The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

# NONLINEAR MECHANICS AND TESTING OF HIGHLY <br> FLEXIBLE ONE-DIMENSIONAL STRUCTURES USING A CAMERA-BASED MOTION ANALYSIS SYSTEM 

Presented by JIAZHU HU
A candidate for the degree of Doctor of Philosophy
And hereby certify that in their opinion it is worthy of acceptance.

Professor P. Frank Pai

| Professor Douglas E. Smith |
| :---: |
| Professor Steven P. Neal |
| Professor Uee Wan Cho Zhen Chen |

# DEDICATED 

To<br>My parents, Hu, Yunxi (father) and Ouyang, Juan (mother), and also to<br>my brothers, Jialin and Jiazhen, and sisters, Jiaqiong and Jianhua, for their love, without which I could not get to this point.

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# NONLINEAR MECHANICS AND TESTING OF HIGHLY FLEXIBLE ONE-DIMENSIONAL STRUCTURES USING A CAMERA-BASED MOTION ANALYSIS SYSTEM 

Jiazhu Hu<br>Dr. P. Frank Pai, Dissertation Supervisor


#### Abstract

Recent developments in aerospace exploration have stimulated extensive research interest in Highly Flexible Structures (HFSs), such as dish antennas, space telescopes, and solar collectors due to their prospectively wide applications and involved challenging mechanics problems. As the basic structural elements for these HFSs, strings, cables and beams play an important role in the high precision design of these structures. Large elastic deformation and nonlinear dynamics of these one-dimensional structures need to be fully understood in order to design such structures and to control them when they are in service.

In this dissertation, exact equations of motion for finite-amplitude vibration of strings were derived based on a fully nonlinear string model. The Method of Multiple Scales (MMS) was used to solve the weakly nonlinear governing differential equations of strings subjected to a harmonic base-excitation. Two different ways of using the MMS were followed and the results were compared. Bifurcations of solutions due to variations of system parameters (e.g., the frequency detuning, excitation amplitude and damping coefficient) were studied in detail using the obtained modulation equations in both polar xvii


and Cartesian forms. Frequency response curves, trajectories of various orbits, frequency spectrum, bifurcation structures, and bifurcation diagrams were used for a detailed qualitative as well as quantitative analysis of dynamic responses.

A 3D motion analysis system was used to perform dynamic testing on string and cable vibrations. Strings with different tensions and cables with different sag to span ratios were tested and the results were comparatively analyzed. A modal decomposition method based on the use of the first four linear mode shapes was used to extract timeand space-varying modal coordinates to reveal modal coupling caused by internal resonance and other nonlinear phenomena. Experimental frequency response curves of physical points on a string were obtained and compared with the theoretical ones. For cables, experimental frequency response curves of modal coordinates were used for analysis. Frequency spectra of responses of one marker and four modal coordinates, phase relations between participating modes, and Poincare sections were used to characterize vibrations. Linear and nonlinear modal couplings, resulting in isolated and simultaneous internal resonances were observed in cable vibration at the first crossover point. The concept of nonlinear normal mode was questioned.

To understand the packaging of 1-D structures, we also performed largedeformation analysis of a triangular frame using a geometrically exact beam theory that accounts for large displacements, large rotations, initial curvatures, extensionality, and transverse shear strains. The problem was presented as a boundary value problem described by a set of first-order ordinary differential equations. The multiple-shooting method was used to solve this two-point boundary value problem. Numerically exact deformed geometries at different stages of packaging were obtained. Appropriate loading
schemes and corresponding efficient and controllable deployment schemes were also discussed.

## CHAPTER 1

## INTRODUCTION

### 1.1 Motivation

Today's space exploration missions ask for larger and larger space structures. Air Force Research Laboratory, NASA Langley Research Center, and the Jet Propulsion Laboratory in conjunction with the DARPA Special Program Office are exploring the development of large space structures (e.g., structures having a diameter of 200-300 m) for a variety of future space-based intelligence, surveillance and reconnaissance (ISR)


Fig. 1.1 Boeing's large space station (http://www.abo.fi).


Fig. 1.2 Extremely large communication satellites (http://www.abo.fi).
missions. Figs. 1.1 and 1.2 show two representative future space structures of extreme scales. The cost of putting a satellite in orbit using an expendable launch vehicle, however, is extremely expensive - about \$18,000 per kilogram. Highly Flexible

Structures (HFSs), having light weight and high packaging efficiency, are thus gaining more and more applications in various fields, especially in aerospace engineering. For example, instead of using electro-mechanical types of deploying systems, NASA's recent space exploration missions use more and more inflatable structures and highly flexible mechanism-free deployable structures in order to reduce the stowed volume and weight, minimize extra vehicular activities in space, and/or decrease the operation time and cost as well.

Highly flexible structures are mostly consisted of one-dimensional (1D) structural members (i.e., strings, cables and flexible beams) because it is easy to pack a 1D structure into a small volume. The difference between a cable and a string is whether there is initial sag, and the difference between a cable or string and a flexible beam is whether there is the bending stiffness. The importance of sag and bending stiffness changes for different analysis cases. For example, in a study of high-frequency vibration of a cable with small sag, the influence of sag may be negligible, and, a study of lowfrequency vibration of a long flexible beam, the influence of bending stiffness may be negligible also. So, the behavior depends on the geometry, material as well as loading, and it is meaningful to investigate the vibrations of strings, cables and flexible beams together. These 1D structural members are used in many highly flexible space structures and are also widely used in other engineering fields. Fig.1.3 is an experiment conducted by NASA Dryden Flight Research Center (1998) to support a Phase II Small Business Innovation Research (SBIR) contract between U.S. Air Force Research Labs and Kelly Space \& Technology (KST). The idea is to use an aerotowed reusable launch vehicle to put small satellites into low orbits around the Earth because towing a launch vehicle to


Fig. 1.3 The C-141A airplane towed the QF-106A airplane in flight tests. The airplanes were connected by a tow rope $1,000 \mathrm{ft}$ ( 305 m ) long.


Fig. 1.4 A cable/string suspended "flying" models (NASA Langley Research Center)
altitude should increases the payload and decreases the launch cost. A C-141A was used as the towing airplane, and a QF-106A (a modified F-106) was used as the towed airplane. To successfully realize this innovative idea, large deformation analysis, static and dynamics stability analysis, and experimental verification of ground taxiing and flight dynamics need to be conducted in order to understand this tow-dynamics and to design the ropes. For example, the tow rope was assumed to be a straight line before the test. However, significant sail of the rope from the airflow was observed during the experiment. This sail is believed to have an important effect on the trim of the aircraft relative to the predications from the simulation, the angle at the point where the rope meets the QF-106A, and other safety and controllability indexes. To have a good control of these important parameters for the test, the influence of the flight speed, the wind (airflow) speed, the weights of aircrafts, the length and material properties of the rope and their relation need be understood. Obviously, first of all, the aerodynamics around a rope and the behaviors under differential tension and sail need to be clarified.

There are some other important applications of cables and strings in aerospace engineering. Fig.1.4 shows an F/A-18 E/F cable-mount model for flying test in the Langley Transonic Dynamics Tunnel (TDT). The purpose is to determine the effects of ground-wind loads on launch vehicles and to provide steady and unsteady aerodynamic pressure data to support computational aeroelasticity and computational fluid dynamics code development and validation. Strings are used to mount the flying model and undergo drastic vibration when they are subjected wind loads and reaction forces of the


Fig. 1.5 An artistic conception of a satellite with a tether (cable). (http://www.biologydaily.com)


Fig. 1.6 One concept for the space elevator tethered to a mobile seagoing platform. (http://www.biologydaily.com)
mounted vehicle. Fig. 1.5 shows an artistic conception of a satellite propelled by a tether (cable). Tether propulsion uses long, strong cables to change the orbits of spacecraft. It has the potential to make space travel significantly cheaper. One way to use tethers for propulsion, for example, is Electrodynamic Tether. It conducts electricity to act against the magnetic field of the earth and can be used either to accelerate or brake an orbiting spacecraft. As the electrodynamic tether cuts the earth's magnetic field, its magnetic field
interacts with the magnetic and gravitational fields of the earth and vibrations may happen due to variations of the magnetic and gravitational fields of the earth. Unless they are damped somehow, the vibrations may grow large enough to cause stresses so high that the tether fails from mechanical stress. A tether has many modes of vibration which can be sensed by radio beacons on the tether, or inertial and tension sensors on the endpoints. Thus, control of the vibrations is possible with a full understanding of the cable dynamics. Fig.1.6 shows the concept of the space elevator tethered to a mobile seagoing platform which may be built in the coming decades. A space elevator allows human send objects and astronauts to space much more often, and at costs only a fraction of those associated with current means. Normally, a space elevator design includes a base station, a cable, climbers, and a counterweight. Among various possibilities that may cause failure and safety issues, the cable vibration could not be paid enough attention. For example, if the cable is excited at its resonant frequencies by the climber or aerodynamic loads, the vibrational energy could build up to a dangerous level beyond the cable's tensile strength. This problem is similar to the vibration of elevator cables for high-rise buildings.

Strings and cables are not only used in aerospace engineering. They are also widely used in mechanical, structural and ocean engineering. Fig.1.7 shows a cablesuspended robot. Compared with conventional serial and parallel robots, the cable driven systems have a variety of advantages, including a large work volume, capability of handling with 6 degrees of freedom, enhanced crane capabilities (no sway and rotation found in an existing single cable crane), and reconfigurability. Fig. 1.8 shows the

conceptual design of the Ruck-A-Chucky Bridge, crossing the middle fork of the American River in California. In order to provide a vertical clearance of 50 ft above high reservoir water level (with water depth 450 ft ), the bridge would have a length of $1,300 \mathrm{ft}$ between the hillsides which have a slope of 40 degrees to the horizontal. Conventional bridges were found to be completely unsatisfactory and hence a hanging arc, with curved girders suspended by cables was proposed. The design has an ideal stress distribution with small bending and torsional moments. Fig. 1.9 shows a typical ocean towed cable system. In this system, the tether or umbilical cable connect as well as transmits the ship (floating system) motion along the submerged cable, and it excites the towed body (underwater system) to vibrate. Depending on the geometric configuration and operational conditions, the cable may be taut or slacken. When the cable is slacken, the towed body cannot follow the umbilical oscillations due to high inertia and drag forces, and thus the motions of the ship and the towed body are decoupled and they oscillate independently with different amplitudes and phases. When
the cable becomes taut, large snap loads and erratic motions of the towed body occur


Fig. 1.9 A typical ocean towed cable system
resulting in severe damage of the umbilical cable. The resulting degradation reduces the life span of the umbilical cable and endangers the recovery of the suspended or towed subsea module. For the assessment and planning of future towing and tethered marine operations, non-linear cable dynamics need to be understood.

In the theoretical definition, cables and strings are one-dimensional structures that can only sustain longitudinal tension loads because they have negligible flexural, torsional, and shear rigidities and have zero buckling loads. A taut string is a straight cable under pretension with a sag-to-span ratio close to zero, and a cable is a string with sag. In the real situation, however, the flexural, torsional, and shear rigidities of cables and strings can never be zero. Depending on the geometry, loading and boundary conditions, a real cable may need to be considered as a flexible beam or rod. For example, the cable in Fig.1.9 may undergo a dynamic buckling process, resulting in the
hoop form shown in Fig. 1.10 due to a significant decrease of tension when the system is under rest. Obviously, the cable undergoes a dynamic buckling process in which torsional strain energy is converted to bending strain energy, which can only explained by modeling the cable as a flexible beam. This idea is also used to study the dynamic formation of DNA loops and supercoils. The crystal structure of the lac repressor (Fig.1.11) is formed with all four of the DNA-binding portions pointing in one direction. When all four subunits bind at the same time, the DNA crystal structure will be twisted into a small loop. These deformation can be studied be modeling DNA structure as flexible beam with bending and torsion stiffness.


Fig. 1.10 A low tension cable may form loops and tangles on the sea floor.

There are many challenging issues in the modeling and analysis of highly flexible structural members. To develop packaging methods, fully nonlinear structural theories need to be derived and large static deformation analysis needs to be successfully and exactly performed. In order to control a 1D highly flexible structure in service, the
nonlinear dynamic characteristics of strings, cables and beams need to be understood and experimentally verified during the design. This research work is to partially but substantially fulfill these objectives.

### 1.2 Overview of the dissertation

The dissertation presents theoretical studies of nonlinear string vibrations, experimental studies of string and cable vibrations, and large deformation packaging analysis of highly flexible beams. In the second chapter, we present a detailed literature review of string dynamics. It begins with an overview of the history of equations of motion of strings and then a summary of dynamical phenomena of strings observed in the literature.

In the third chapter, finite-amplitude vibrations of a string subjected to a harmonic base-excitation are investigated. Exact equations of motion are derived based on a fully nonlinear model. Two different approaches are followed when the method multiple scales is applied to this weakly nonlinear problem. In the direct approach, the excitation displacement is assumed to be linearly distributed along the string when the discretization is applied to the governing differential equations and the boundary conditions are considered as a restriction to the solvability condition. The modulation equations obtained from the two approaches are identical except a small difference in the forcing terms caused by small damping. Frequency responses are obtained by solving the obtained algebraic equations. Influences of system parameters, e.g., damping values and excitation amplitudes, on the dynamic responses are investigated. Bifurcations of limit cycle solutions are studied using numerical integration.

In the fourth chapter, we describe in detail the usage of a 3D motion analysis system to characterize nonlinear dynamics of strings. It begins with a detailed description of the system and data acquisition devices. Then the experimental procedure is presented and discussed. Frequency responses curves were obtained for three strings that had different but small sag-to-span ratios. The string with richer responses was specially studied by focusing on modal analysis of various typical responses.

The experiments presented in this chapter reveal planar and non-planar (e.g., whirling) vibrations, as predicted by the theory. Branches of planar responses represent a transition from one mode to the next mode; the amplitude of the lower one decreases and that of the other increases when the excitation frequency increases. Branches of nonplanar responses are motions mainly composed of one single mode. However, for some cases, small-amplitude vibrations of other modes participate in both planar and nonplanar vibrations, making the vibration profile plot more complex and irregular. The hysteretic phenomenon is revealed by the sudden lose of stability of one branch when the excitation frequency is swept up and down. Small sag does not make the string far from being a cable at the first cross over (Chapter 5), but it causes the observed smallamplitude 1:2( $\Omega: \omega)$ resonance. Modal decomposition analysis of responses to higherfrequency excitations showed that the use of four modes may not be enough for accurate modal decomposition.

Attempts to quantitatively match the experimental results with the theoretical ones were not very successful. The difficulty in making such comparisons is mostly due to following reasons. One of them is the inability to precisely measure the system parameters such as axial stiffness, longitudinal wave speeds, and damping. The other one
is the discrepancy originating from the nature of the external excitation. In theoretical analysis, the excitation is applied to one of the two ends of the string. However, to avoid the longitudinal vibration, the excitation is applied to a location very close to one of the two fixed boundaries and only the vibration between the attachment point of the shaker and the other end is considered. This arrangement, however, constrained the out-of-plane motion at the shaker attachment point and decoupled the cable vibration into two components for out-of-plane response. Additionally, we used a level to adjust the position of the shaker and tried to make the excitation vertical. This may not be as good as desired. Moreover, the theoretical prediction is accurate for weakly nonlinear problems only. Our studies, however, were out of the weakly nonlinear range because the vibration amplitudes were quite large and the frequency bandwidth was so wide that its limits were possibly beyond the natural frequencies of the next mode or the previous mode. Theoretically predicted coexisting clockwise and counterclockwise whirling motions were not observed. This may be due to the intrinsic asymmetry of the experimental setup.

In the fifth chapter, we begin with a review of the nonlinear dynamics of cables. The methods for and observations from numerical simulations and experiments are summarized. Then the linear theory of cable vibration is introduced. The crossover phenomena between natural frequencies of in-plane and out-of-plane and symmetric and anti-symmetric modes are presented. In our experiments, theoretically predicted isolated and simultaneous internal resonances were observed for the cable around the first crossover. A theoretically unpredicted simultaneous internal resonance were also observed and explained.

The sixth chapter presents the derivation of a geometrically exact beam theory for highly flexible beams undergoing large deformations. The exact beam theory accounts for large displacements, large rotations, initial curvatures, extensionality and transverse shear strains. The concepts of local displacements, Jaumann stress and strain measures, and orthogonal virtual rotations are used to derive the geometrically exact beam theory. The extended Hamilton principle is used to derive fully nonlinear governing equations. The geometrically exact beam theory is used in the numerically exact packaging analysis of a highly flexible triangular frame using a multiple shooting method.

In the last chapter, important tasks for future research are listed and discussed.

## CHAPTER 2

## LITERATURE REVIEW OF STRING DYNAMICS

We present a comprehensive literature review of string dynamics in this chapter. It begins with an overview of the history of equations of motion of strings and the review is followed by a summary of dynamical phenomena observed and presented in the literature.

### 2.1 Introduction

The research on string vibration has a long history. As people know more and more about string dynamics, the topics changed from planar linear vibrations to planar nonlinear vibrations, then to non-planar nonlinear vibrations; from periodic to quasiperiodic vibrations, then to chaotic vibrations and from stable vibrations to modulated and unstable vibrations. In the case of free vibration with sufficiently small initial displacements, the variation of tension in a string is so small that it is reasonable to assume the tension to be constant. However, the most important non-linear phenomena like whirling motion (i.e., ballooning or galloping) and jump phenomena in the frequency response curve (FRC) are not expected in numerical simulations based on this assumption. In the case of forced vibration, the variation of tension due to large amplitude vibration would be significant no matter how small the driving amplitude
might be, especially if the driving frequency is close to a resonance one. Moreover, the excited planar vibration may become unstable due to the modal coupling between the vibrations in two transverse polarizations and then circular (whirling, ballooning or tabular) motions are expected. The derivation of nonlinear equations of motion is the ground work for the investigation of vibrating strings and hence one of the major topics in the history of research on string vibrations. Early studies on various topics based on inaccurate equations of motion were often questioned by following studies that presented better and/or more accurate ones, normally having more reasonable nonlinear terms or parameters. So, it is quite meaningful to present a summary of historical assumptions and derivations of equations of motion in a dissertation specializing in nonlinear vibration and dynamics of strings.

### 2.2 Equations of Motion

In this chapter, $\bar{x}$ represents the longitudinal coordinate of a point of the string under pretension. $d x_{0}, d \bar{x}$ and $d \tilde{x}$ represent infinitesimal lengths of the string without pretension, under pretension and during vibration, respectively. $u, v$ and $w$ represent displacements of the string in the longitudinal direction $x_{1}$ and two transverse directions $x_{2}$ and $x_{3}$ respectively. The undeformed length of the string is $L_{0}$, the deformed length of the string under pretension $\bar{T}$ is $\bar{L}$, and the strain under pretension $e_{0}$. The tension during vibration is represented by $\tilde{T}$, and the strain during vibration is $e . m$ represents the mass per unit length of the string under pretension. $c_{1}$ and $c_{2}$ represent the longitudinal and transverse wave speeds, respectively. ()$_{\bar{x}}=\partial() / \partial \bar{x}$ and ()$_{t}=\partial() / \partial t . E$ is the
elasticity modulus and $A$ is the cross-sectional area of the string. For convenience, we replace $\bar{x}$ by $x$ and $\bar{T}$ by $T$ in our following presentation.

### 2.2.1 Planar Linear Vibration

Early research interests in vibrating strings were due to its wide applications in musical instruments. At that stage, linear models were used in the research of string vibration. Only some basic characteristics of string vibrations were obtained. In a linear model, the displacement gradient $|\tan (\theta, t)|=|\partial w(x, t) / \partial x| \ll 1$ is assumed to be small, which ensures the transverse displacement $w(x, t)$ to be small compared with the length of the string. The tension is assumed to have no variation along the length and so there is no longitudinal vibration. Also, as the vibration is small, the tension has no variation with respect to time. So, the tension is constant (i.e., $\tilde{T}(x, t)=T$ ) in both temporal and spatial domains. Some basic results were obtained based on above assumptions. The first of them is the relation between the free undamped frequency and the tension $T$, the length $\bar{L}$ of the string under pretension, and the mass per unit length $m$, in the following form (Taylor, 1713):

$$
\begin{equation*}
f=\frac{1}{2 \bar{L}} \sqrt{\frac{T}{m}} \tag{2.2.1}
\end{equation*}
$$

Another one is the linear wave equation (D'Alembert, 1749)

$$
\begin{equation*}
c_{w}^{2} w_{x x}=w_{t t} \tag{2.2.2}
\end{equation*}
$$

with a solution (D'Alembert, 1751)

$$
\begin{equation*}
w(x, t)=f\left(x-c_{w} t\right)+g\left(x+c_{w} t\right) \tag{2.2.3}
\end{equation*}
$$

where $c_{w}=\sqrt{T / m}$ is the transverse wave propagation velocity and $f(x, t)$ and $g(x, t)$ are functions determined by initial and boundary conditions. Apparently, transverse displacements $v$ and $w$ are uncoupled in this model.

### 2.2.2 Planar Non-linear Vibration

Experimentally observed string dynamics shows complex nonlinear phenomena which cannot be explained by linear models. There are several causes for the non-linear large-amplitude vibrations. Some of the most important ones are the variation of tension during vibration, the cross-section contraction, and the longitudinal vibration, which may even make small-amplitude vibrations behave nonlinearly. The relation between the fundamental transverse frequency $\omega_{w}$ and longitudinal one $\omega_{u}$ was found by Poisson (1829) to be

$$
\begin{equation*}
\frac{\omega_{u}}{\omega_{w}}=\sqrt{\frac{\Delta \bar{L}}{\bar{L}}} \tag{2.2.4}
\end{equation*}
$$

where $\Delta \bar{L}$ is the elongation due to $\bar{T}$. For planar vibration, the formula for the strain of an initially tensioned vibrating string is

$$
\begin{equation*}
e=\frac{d \tilde{x}-d x}{d x}=\sqrt{\left(1+u_{x}\right)^{2}+\left(w_{x}\right)^{2}}-1 \tag{2.2.5}
\end{equation*}
$$

The equations of motion are

$$
\begin{align*}
& (\tilde{T} \cos \theta)_{x}=\rho A u_{t t}  \tag{2.2.6}\\
& (\tilde{T} \sin \theta)_{x}=\rho A w_{t t} \tag{2.2.7}
\end{align*}
$$

Following above equations, Carrier $(1945,1969)$ obtained the following equations governing both longitudinal and transverse vibrations:

$$
\begin{align*}
& \frac{\partial^{2}}{\partial \xi^{2}}[(1+\tau) \varphi]=\frac{\partial^{2}}{\partial \eta^{2}}\left[\left(1+\alpha^{2} \tau\right) \varphi\right]  \tag{2.2.8}\\
& \frac{\partial^{2}}{\partial \xi^{2}}\left[(1+\tau)\left(1-\varphi^{2}\right)^{1 / 2}\right]=\frac{\partial^{2}}{\partial \eta^{2}}\left[\left(1+\alpha^{2} \tau\right)\left(1-\varphi^{2}\right)^{1 / 2}\right] \tag{2.2.9}
\end{align*}
$$

where non-dimensional quantities

$$
\alpha^{2}=T / E A, \tau=(\tilde{T}-T) / T, \xi=\pi x / \bar{L}, \eta=t \sqrt{\pi / \bar{L}(T / \rho A)}
$$

are used and $\varphi=\sin (\theta(x, t))=w_{x} /\left(1+\alpha^{2} \tau\right)$. Neglecting the longitudinal vibration and assuming

$$
e=(d \tilde{x}-d x) / d x=\sqrt{\left(1+u_{x}\right)^{2}+w_{x}^{2}}-1 \approx w_{x}^{2} / 2
$$

and

$$
\tilde{T}=T+w_{x}^{2} / 2, \sin \theta=w_{x}^{2} /\left(w_{x}^{2}+1\right) \approx w_{x}^{2}
$$

Lee (1957) obtained the following governing equation for planar non-linear vibrating strings

$$
\begin{equation*}
\left(T+\frac{3}{2} E A w_{x}^{2}\right) w_{x x}=\rho A w_{t t} \tag{2.2.10}
\end{equation*}
$$

where the tension is a function of both spatial and temporal variables. Lee also carried out an experiments to check the frequency response. The hardening effect and discontinuities of amplitudes (i.e., jump phenomena) under frequency sweeping were detected. If the tension throughout the length of the string is considered to be the same and equal to the
average over the string with the longitudinal vibration $u$ being neglected (Oplinger, 1960),

$$
\begin{equation*}
\tilde{T}=T+(E A / 2 L) \int_{0}^{\bar{L}} w_{x}^{2} d x \tag{2.2.11}
\end{equation*}
$$

and the following Kirchhoff (1883) nonlinear partial differential equation describing the transverse vibration of strings can be obtained as

$$
\begin{equation*}
\left[T+(E A / 2 \bar{L}) \int_{0}^{\bar{L}} w_{x}^{2} d x\right] w_{x x}=\rho A w_{t t} \tag{2.2.12}
\end{equation*}
$$

### 2.2.3 Non-planar Non-linear Vibration

The first successful experiment of non-planar periodic motion (i.e., whirling) of strings was performed by Harrison (1948). Using the Kirchhoff equation, Oplinger (1960) numerically investigated frequency response and repeated Harrison’s experiment. The experimental and numerical frequency response curves showed a good agreement between branches that represent planar small-amplitude responses. Out-of-plane vibrations happen only in a small range around the resonance frequency, and it was suggested that the stability theory of Mathieu and Hill be used to analyze the observed out-of-plane motion. Using the energy method, Murthy and Ramakrishna (1965) defined the potential energy as

$$
\begin{equation*}
P=\int_{0}^{\bar{L}}\left[T\left(v_{x}^{2}+w_{x}^{2}\right) / 2+(E A-T)\left(v_{x}^{2}+w_{x}^{2}\right)^{2} / 8+f_{w}(x, t) w\right] d x \tag{2.2.13}
\end{equation*}
$$

and the kinetic energy as

$$
\begin{equation*}
K=\frac{\rho A}{2} \int_{0}^{\bar{L}}\left(v_{t}^{2}+w_{t}^{2}\right) d x \tag{2.2.14}
\end{equation*}
$$

Defining the Lagrangian $L \equiv K-P$ and using

$$
\begin{align*}
& \frac{\partial L}{\partial v}=\frac{\partial}{\partial t} \frac{\partial L}{v_{t}}-\frac{\partial}{\partial x} \frac{\partial L}{v_{x}}=\frac{\partial}{\partial t} \frac{\partial K}{v_{t}}+\frac{\partial}{\partial x} \frac{\partial P}{v_{x}}  \tag{2.2.15}\\
& \frac{\partial L}{\partial w}=\frac{\partial}{\partial t} \frac{\partial L}{w_{t}}-\frac{\partial}{\partial x} \frac{\partial L}{w_{x}} \tag{2.2.16}
\end{align*}
$$

the equations of motion can be obtained using the Euler-Lagrange equations to be

$$
\begin{align*}
& v_{t t}-c_{2}^{2} v_{x x}-\frac{3}{2} c_{1}^{2} v_{x x} v_{x}^{2}-\frac{1}{2} c_{1}^{2} \frac{\partial}{\partial x}\left(v_{x} w_{x}^{2}\right)=0  \tag{2.2.17}\\
& w_{t t}-c_{2}^{2} w_{x x}-\frac{3}{2} c_{1}^{2} w_{x x} w_{x}^{2}-\frac{1}{2} c_{1}^{2} \frac{\partial}{\partial x}\left(w_{x} v_{x}^{2}\right)=\frac{f_{w}}{\rho A} \tag{2.2.18}
\end{align*}
$$

where $c_{2}^{2}=T / \rho A$ and $c_{1}^{2}=E A / \rho A$ are the squares of the transverse and longitudinal wave speeds, respectively. Akulenko (1996) obtained equivalent equations in the following form

$$
\begin{align*}
& \rho A v_{t t}=T v_{x x}+(E A-T)\left[\left(v_{x}^{2}+1 / 2 h^{2}\right) v_{x x}+v_{x} w_{x} w_{x x}\right]  \tag{2.2.19}\\
& \rho A w_{t t}=T w_{x x}+(E A-T)\left[\left(w_{x}^{2}+1 / 2 h^{2}\right) w_{x x}+w_{x} v_{x} v_{x x}\right]+f_{w} \tag{2.2.20}
\end{align*}
$$

where $h^{2}=\left(v_{x}^{2}+w_{x}^{2}\right)$. The last terms on the left-hand sides of equations (2.2.19) and (2.2.20) represent the coupling effect between $w$ and $v$ components of the vibration. The equation of Lee (1957) appears as a special case of above equations when the out-ofplane component $v$ is ignored. Moreover, these equations reduce to linear ones if the $\operatorname{term}(E A-T)\left(v_{x}^{2}+w_{x}^{2}\right)^{2} / 8$ which indicates the variation of potential energy $P$ due to the change of tension, is ignored in equation (2.2.13). Apparently, whether this term can be ignored or not depends on the ratio

$$
\begin{equation*}
T\left(v_{x}^{2}+w_{x}^{2}\right) / 2 /\left[E A\left(v_{x}^{2}+w_{x}^{2}\right)^{2} / 8\right]=4 T /\left[E A\left(v_{x}^{2}+w_{x}^{2}\right)\right] \tag{2.2.21}
\end{equation*}
$$

which in turn depends on the amplitude indicator $\left(v_{x}^{2}+w_{x}^{2}\right)$. Attacking (2.2.17) and (2.2.18) using the method of harmonic balance, Murthy and Ramakrishna (1965) numerically investigated the frequency response and compared with experimental results. Whirling motion and jump phenomena were recorded for the string subjected to external planar excitations.

Instead of formulating the potential and kinetic energies using actual displacements $v$ and $w$, Miles (1965) represented the transverse displacement and excitation forces using non-dimensionalized Fourier series as

$$
\begin{align*}
& \{v, w\}=(\varepsilon L / \pi)\left\{\alpha_{n}(t), \beta_{n}(t)\right\} \sin (n \pi x / L)  \tag{2.2.22}\\
& \left\{f_{v}, f_{w}\right\}=(\mu L / \pi)\left\{A_{n}(t), B_{n}(t)\right\} \sin (n \pi x / L) \tag{2.2.23}
\end{align*}
$$

where both $\varepsilon$ and $\mu$ are dimensionless scale parameters to be determined. Keeping only the dominant terms and substituting them into the Lagrangian, the governing equations were obtained to be in the following different form:

$$
\begin{equation*}
\left[\delta^{-1}\left(D^{2}+1\right)+2 / 3\left(\alpha^{2}+\beta^{2}\right)\right]\{\alpha, \beta\}=\{1,0\} \cos (\omega t) \tag{2.2.24}
\end{equation*}
$$

where $\delta=\sqrt{9 \mu^{2} / 16 e^{2}}$ and $e$ is the strain under pretension. For cases with only the first mode being considered $\left\{A_{1}(t), B_{1}(t)\right\}=\{1,0\} \cos (\omega t)$.

In above models, the longitudinal motion $u=0$ was assumed under transversely excited vibrations. Narashimha (1968) examined this assumption and declared that it was neither necessary nor justifiable. Actually, he found out that the couplings between the longitudinal motion and two transverse ones were through a parameter which was a product of a large quantity (the ratio of longitudinal to transverse wave speeds) and a
small one (essentially proportional to the amplitude of forcing), and it may be important even if the amplitude is small. Knowing this important but previously neglected fact, he obtained the following equations

$$
\begin{equation*}
p_{t t}+2 \delta \omega_{1} p_{t}-\left[c_{2}^{2}+\frac{c_{1}^{2}}{2 L} \int_{0}^{L}\left(v_{x}^{2}+w_{x}^{2}\right) d x\right] v_{x x}=F(x, t) / m \tag{2.2.25}
\end{equation*}
$$

where $p \equiv(v, w)$, and $\delta$ and $\omega_{1}$ are damping ratios for the fundamental mode and the fundamental frequency of transverse linear vibration, respectively. These equations were adopted by following researchers like Miles (1984), Johnson and Bajaj (1989), and Bajaj and Johanson (1992).

Considering the longitudinal motion and its effect on the variation of tension, Anand (1969) defined the tension as

$$
\begin{equation*}
\tilde{T}=T+E A\left(\frac{d \tilde{x}-d x}{d x}\right) d x=T+E A\left(u_{x}+v_{x}^{2} / 2+w_{x}^{2} / 2\right) \tag{2.2.26}
\end{equation*}
$$

Starting from equation (2.2.26), he obtained the following nonlinear equations of motion for vibrating strings

$$
\begin{align*}
& u_{t t}-c_{1}^{2} u_{x x}-\left(c_{1}^{2}-c_{2}^{2}\right) / 2(\partial / \partial x)\left(v_{x}^{2}+w_{x}^{2}\right)=0  \tag{2.2.27}\\
& v_{t t}+\mu_{v} v_{t}-c_{2}^{2} v_{x x}-\left(c_{1}^{2}-c_{2}^{2}\right)(\partial / \partial x)\left[v_{x}\left(u_{x}+\left(v_{x}^{2}+w_{x}^{2}\right) / 2\right)\right]=0  \tag{2.2.28}\\
& w_{t t}+\mu_{w} w_{t}-c_{2}^{2} w_{x x}-\left(c_{1}^{2}-c_{2}^{2}\right)(\partial / \partial x)\left[w_{x}\left(u_{x}+\left(v_{x}^{2}+w_{x}^{2}\right) / 2\right)\right]=0 \tag{2.2.29}
\end{align*}
$$

where $c_{2}=(T / \rho A)^{1 / 2}$ and $c_{1}=(E A / \rho A)^{1 / 2}$ are the transverse and longitudinal wave speeds in the linear theory for string vibration. Because $c_{1}^{2} / c_{2}^{2}=E A / T \gg 1$ for typical metallic strings, we have the following simplified equations

$$
\begin{align*}
& u_{t t}-c_{1}^{2} u_{x x}-c_{1}^{2} / 2(\partial / \partial x)\left(v_{x}^{2}+w_{x}^{2}\right)=0  \tag{2.2.30}\\
& v_{t t}+\mu_{v} v_{t}-c_{2}^{2} v_{x x}-c_{1}^{2}(\partial / \partial x)\left[v_{x}\left(u_{x}+\left(v_{x}^{2}+w_{x}^{2}\right) / 2\right)\right]=0  \tag{2.2.31}\\
& w_{t t}+\mu_{w} w_{t}-c_{2}^{2} w_{x x}-c_{1}^{2}(\partial / \partial x)\left[w_{x}\left(u_{x}+\left(v_{x}^{2}+w_{x}^{2}\right) / 2\right)\right]=0 \tag{2.2.32}
\end{align*}
$$

If the longitudinal inertia term $u_{t t}$ is neglected, then we have

$$
\begin{equation*}
u_{x x}=-\frac{1}{2}(\partial / \partial x)\left(v_{x}^{2}+w_{x}^{2}\right) \tag{2.2.33}
\end{equation*}
$$

and equations (2.2.31) and (2.2.32) can be further simplified to

$$
\begin{align*}
& v_{t t}+\mu_{v} v_{t}-c_{2}^{2} v_{x x}-\frac{c_{1}^{2}}{2 \bar{L}} v_{x x} \int_{0}^{\bar{L}}\left(v_{x}^{2}+w_{x}^{2}\right) d x=0  \tag{2.2.34}\\
& w_{t t}+\mu_{w} w_{t}-c_{2}^{2} w_{x x}-\frac{c_{1}^{2}}{2 \bar{L}} w_{x x} \int_{0}^{\bar{L}}\left(v_{x}^{2}+w_{x}^{2}\right) d x=f_{w}(x, t) \tag{2.2.35}
\end{align*}
$$

which are the most widely used equations for string vibration analysis. Using above equations, Anand (1969) studied the inter-modal coupling, oscillatory characters of amplitudes, and energy transfer between vibrations in two directions.

Looking at the history of research on string vibrations, we can see that at the beginning, it focused on planar vibrations using linear equations with the tension being assumed to be constant along the string during the vibration. This was followed by planar vibrations using equations with nonlinear terms accounting for the variation of tension. Then, out-of-plane vibration was considered and nonlinear equations for non-planar vibrations were derived. The longitudinal motion was not considered to be influential on string dynamics until Narashimha (1968) proposed the model that accounted for the coupling between two transverse vibrations and the coupling between transverse and longitudinal vibrations. This model was used by many following researchers.

The deformation-induced stretch that results in a tension varying with time and location was the main reason for various nonlinear phenomena experimentally observed and theoretically predicted. Usually, it is appropriate to assume the cross section is constant during vibration for most strings. However, for a string with a small elasticity modulus, its vibration is accompanied sectional contraction. Consequently, Poisson’s effect becomes another source of nonlinearity. We derive fully nonlinear string equations including Poisson's effect in Chapter 3 using the principle of mass conservation. Leisa (1994) did not consider Poisson's effect (i.e., setting $A \equiv A_{0}$ ) but used a varying mass density. Because of the conservation of mass, we have

$$
\begin{equation*}
\rho d \tilde{x}=\rho_{0} d x_{0}=\rho_{0} \frac{d x_{0}}{d x} d x=\frac{\rho_{0}}{1+e_{0}} d x=\bar{\rho} d x \tag{2.2.36}
\end{equation*}
$$

Substituting Eq. (2.2.36) and $T=E A e$ into the equations of motion obtained by applying Newton's second law to the differential element yields

$$
\begin{align*}
& \frac{\partial}{\partial x}[T(x, t) \cos (\theta(x, t))] d x=\rho A(x, t) d \tilde{x} \frac{\partial^{2} u(x, t)}{\partial t^{2}}  \tag{2.2.37}\\
& \frac{\partial}{\partial x}[T(x, t) \sin (\theta(x, t))] d x=\rho A(x, t) d \tilde{x} \frac{\partial^{2} v(x, t)}{\partial t^{2}} \tag{2.2.38}
\end{align*}
$$

The following coupled, nonlinear partial differential equations describing the largeamplitude planar longitudinal and transverse free vibration of elastic strings are obtained

$$
\begin{equation*}
E A_{0}\left(1+e_{0}\right)^{2} v_{x x}-E A\left(1-e_{0}\right) \frac{\left(1+u_{x}\right)^{2} v_{x x}-v_{x}\left(1+u_{x}\right) u_{x x}}{\left[\left(1+u_{x}\right)^{2}+v_{x}^{2}\right]^{3 / 2}}=\rho_{0} v_{t t} \tag{2.2.35}
\end{equation*}
$$

$$
\begin{equation*}
E A\left(1+e_{0}\right)^{2} u_{x x}-E A\left(1+e_{0}\right) \frac{u_{x x} v_{x}^{2}-v_{x}\left(1+u_{x}\right) u_{, x x}}{\left[\left(1+u_{x}\right)^{2}+v_{x}^{2}\right]^{3 / 2}}=\rho_{0} u_{t t} \tag{2.2.35}
\end{equation*}
$$

### 2.3 Nonlinear Dynamics of Strings

In this section, we review the string dynamics experimentally and theoretically observed and presented in the literature. We started with review of out-of-plane vibrations. Murthy and Ramakrishna (1964) were the first to successfully predict jump (both upward and downward) and hysteresis phenomena of vibration using a model without longitudinal vibration. It was pointed out that the out-of-plane vibration is basically due to internal resonance and it may happen even if the driving force is small.

Based on the inaccurate equations of motion in which the longitudinal displacement is inappropriately neglected, Miles (1965) studied the stability of nonlinear response of a string subjected to a transverse harmonic excitation with a frequency around the first natural frequency. Three different bifurcation frequencies were detected. They are now known to be the forward pitchfork point $\sigma_{1}$, where the stable planar vibration is transformed to (or from) the stable non-planar vibration when the excitation frequency increases (or decreases); the reversed pitchfork bifurcation point $\sigma_{5}$, where the non-planar vibration is transferred into the planar one when the excitation frequency increases; and the saddle node bifurcation point $\sigma_{2}$, where the planar lower-amplitude vibration is transferred into the non-planar larger-amplitude one when the excitation frequency decreases. Using the first-order approximation, he investigated the inter-modal coupling and focused on the modal coupling between the first mode and the third one. In
the doubt of the existence of nonlinear coupling among the normal modes of two perpendicular planes shown by Miles using inaccurate equations of motion, Feng(1995) investigated the inter-coupling of planar vibration modes using the amplitude modulation equations obtained by attacking the equations of motion derived by Narasimha (1968) using the method of multiple scales. The results showed that the one-to-three inter-modal coupling caused by cubic nonlinearity will not actually take place because there are no such coupling terms in the amplitude equations.

Anand (1966) investigated nonlinear resonant vibrations of stretched strings by including viscous damping. It was found that the resonant frequency is a function of the driving force and it increases with the driving force. For a fixed damping, the response is qualitatively related to the driving force. Non-planar vibration and jump phenomena are possible only if the driving force is large and the responses are linear for cases of small excitations.

Narasimha (1968) carefully examined the assumption $u=0$ used in previous studies. It was argued that this assumption was neither necessary nor justifiable. The exact equations of motion were first formulated and approximate solutions were provided for cases with small strains. The responses in the absence of damping were studied. It was shown that planar response was unstable at sufficiently large amplitude and the critical stable amplitude fell to zero if the string was excited at one of its linear natural frequencies or their sub-harmonics.

Anand (1969a) rederived the equations of motion to include the coupling of longitudinal and transverse vibrations. It was shown that temporal and spatial variables were separable. The time-dependent parts of the equations were solved by the method of
variable amplitude and phase. It was found that there was a continuous exchange of energy between two mutually perpendicular transverse components. The path of one point was traced in polar coordinates and shown to be an ellipse with slowly rotating and shrinking axes.

Using the method of Hill and van der Pol, Anand (1969b) investigated the stability of damped forced vibrations and undamped free vibrations based the equations he derived before (Anand, 1969a). The regions of stability for the forced vibrations were plotted. As a continuation of previous work, Anand (1973) studied the negative resistance phenomenon, i.e., the amplitude of response increase as the driving force is reduced and vice versa.

Miles (1984) was the first one to construct the widely used averaged equations in a four-dimensional phase space, in which the coordinates are the slowly varying amplitudes of a sinusoidal motion of the dominant mode at the driving frequency. Using the four autonomous first-order non-linear ordinary differential equations, he performed an exhaustive study on the fixed-point solutions and their stability. It was found that the average equations lost (gained) stability through a forward (reverse) Hopf bifurcation and therefore suggested the possible existence of strange attractors in the branch of modulated motion. However, he did not observe any chaotic response for admissible parameter values of damping and resonant offset because the solutions always terminated at the lower planar fixed point solution.

Yasuda and Torii $(1985,1986)$ studied multiple-mode planar responses to excitations near the second resonant frequency (1985) and the first, third and fourth resonant frequencies (1986). They performed an experiment study on a thin steel strip to
validate their theoretical results. Both numerical results and experimental validations showed that possible responses were pure harmonics; harmonics with sub-harmonics, super-harmonics and super-sub-harmonics, and the so-called summed and differential (modulated) motion. Their time-deflection (amplitude) plots and spectrum analysis results showed that the participation of each mode was different at different locations of the strip and varied under different excitations. The results were clearly and easily observed in our experiments as shown later in Chapter 4.

Tufillaro (1989) simulated the nonlinear string vibration using a simple singlemode model of a mass-spring system. For planar motion, the governing equation was of the forced Duffing form. By constructing the bifurcation diagrams for both planar and whirling motions, various nonlinear phenomena were predicted, including periodic, quasi-periodic and chaotic motions. Because many modes may be excited under complex vibrations, multi-mode analysis was recommended.

In addition to a detailed summary of the analysis of the amplitude modulation equations, Johnson and Bajaj (1989) and Bajaj and Johnson (1992) performed extensive bifurcation analyses of nonlinear string vibrations and predicted the existence of quasiperiodic torus-doubling bifurcations and chaotic motions as well as the phenomena of boundary crisis. It was shown that the averaged equations with small enough damping possessed a solution branch that begin with planar response became non-planar through pitchfork bifurcation, and then became modulated motion through a Hopf bifurcation. Limit cycle solutions of the Hopf branch exhibited several period-doubling bifurcations and then merged with the planar periodic branch as the parameters varied. No chaotic motion was observed in the period doubling process of the Hopf branch. For smaller
damping, an isolated limit-cycle branch that included two sub-branches (one stable periodic solution and one unstable periodic solution) was created by a global saddle node bifurcation. The unstable isolated branch eventually merged with the stable Hopf branch via a saddle-node bifurcation. While the stable isolated branch went through a cascade of period-doubling bifurcations and resulted in the formation of a Rossler type attractor. As the damping was decreased further, another isolated branch was created and the unstable part of this branch merged with the stable part of the previous isolated branch in exactly the same manner as the first isolated branch was created and the unstable isolated branch was merged with the Hopf branch. The creation and merge of the isolated branch culminated in the formation of a homoclinic orbit originating from a saddle focus. The eigen-value structure of the saddle focus and Sil'nikov's theorem were used to interpret the bifurcation behavior. Away from the homoclinicity frequency, a series of bifurcations resulted in the formation of a Lorenz type chaotic attractor. At low values of damping, the Lorenz type attractor abruptly disappeared in a frequency interval. This phenomenon was explained using the concept of boundary crisis.

Molteno and Tufillaro (1990) were the first to report experimental observations of torus-doubling bifurcations leading to chaotic vibration of a string. They identifed the types of non-linear motions by using the Poincare sections of trajectories. Correlation dimensions were calculated to confirm the existence of chaotic attractors. Other nonlinear behaviors like hysteresis, period-doubling and chaotic transience were recorded also. Molteno (1994) and Molteno and Tufillaro (2004) did a more detailed experimental study of string dynamics by examining different values of parameters. Interesting non-linear phenomena like boundary crisis and intermittent transition to chaos, which were
predicted by Johnson and Bajaj (1989) and Bajaj and Johnson (1992), were observed. It was shown, that for large damping, the experimental results agreed well with the prediction of averaged equations with small damping. Tufillaro, et al. (1995) performed a topological time series analysis of experiments on string dynamics using a synchronized model, in which the experimental data were used to estimate the topological parameter values in order to quantitatively characterize the chaotic attractor.

Using the four known coupled, autonomous ordinary differential equations, O’Reilly and Homles (1992) and O’Reilly (1992) performed local bifurcation analysis, especially which of chaotic responses, of nonlinear vibration of a string subjected to planar harmonic excitations. The global bifurcations and mechanisms that lead to chaotic motions in non-Hamiltonian systems, the averaged equations in the presence of forcing and damping, and Hamiltonian systems without damping but with forcing were all investigated and compared. It was stated that "For the non-Hamiltonian system, the mechanism is a pair of homoclinic orbits to a fixed point of saddle-focus type, and, for the integrable Hamiltonian system, the mechanism is a pair of (nearly) homoclinic orbits to a fixed point of saddle-center type". Comparisons of the numerical and experimental results showed good qualitative agreements but poor quantitative agreements.

In all of these studies, strings were excited transversely. Melde (1895) fixed one end of a string and attached the other end to a large tuning fork so that the motion of the tuning fork was parallel to the axis of the string, causing a parametric excitation. He observed that the string could be made to oscillate transversely although the force was along the axis of the string. Nayfeh, Nayfeh and Mook (1995) studied the nonlinear response of a taut string subjected to an excitation having components both parallel and
perpendicular to its axis. The method of multiple scales was applied directly to the two governing partial differential equations (and corresponding boundary conditions) obtained by neglecting the longitudinal inertia term of the three-equation model (Nayfeh and Mook, 1979). A continuation method was then employed to determine the equilibrium solutions of the modulation equations and their stability. An experimental study was conducted and the results were found to be in good agreement with the theoretical predictions both qualitatively and quantitatively.

## CHAPTER 3

# NUMERICAL DYNAMIC CHARACTERISTICS OF STRINGS 

In this chapter, finite but small-amplitude vibrations of strings subject to boundary excitations are investigated. Exact equations of motion are derived based on a fully nonlinear model. Two different approaches are followed when the method multiple scales is applied to this weakly nonlinear problem. The excitation displacement is assumed to be linearly distributed along the string when the discretization approach is applied to attack the governing differential equations and the boundary conditions are considered as constraints on the solvability condition in the direct approach. The modulation equations obtained from the two approaches are identical except the forcing terms, which are different but almost equal due to small damping. Frequency responses curves are obtained by solving the obtained algebraic equations describing modulation. The effects of different parameters (e.g. damping and forcing) on the dynamics of the system are investigated. Bifurcations of limit-cycle solutions of the modulated branch are investigated in detail.

### 3.1 Modeling of Taut Strings

### 3.1.1 Fully Nonlinear Model

Fig. 3.1(a) shows the deformed configuration of a taut, linearly elastic, uniform and homogeneous string and the inertial coordinate system $x_{1} x_{2} x_{3}$. Moreover $\rho\left(x_{0}\right)$ denotes the mass density of the unloaded string, $A_{0}\left(x_{0}\right)$ denotes the unloaded cross-section area, and ()$_{x_{0}}=\partial() / \partial x_{0}$. The coordinates of $\bar{P}$ are $(\bar{x}, 0,0)$ with $\bar{x}=x_{0}+\int_{0}^{x_{0}} e_{0} d x_{0}, e_{0}$ denotes the static axial strain due to pretension, and $\Delta L=\int_{0}^{L_{0}} e_{0} d x_{0}$. The coordinates of $\tilde{P}$ are $\left(x_{1}, x_{2}, x_{3}\right)$, and the dynamic displacements of $\tilde{P}$ are $u, v$ and $w$ along the axes $x_{1}, x_{2}$, and $x_{3}$, respectively. Thus,

$$
\begin{equation*}
x_{1}=\bar{x}+u, x_{2}=v, x_{3}=w \tag{3.1.1}
\end{equation*}
$$



Fig. 3.1: A string: (a) the deformed configuration of a taut string, where $x_{1} x_{2} x_{3}$ is a Cartesian, inertial coordinate system, and (b) the free-body diagram of a differential string element. $P_{0}$ : the position of the observed particle when the string is not loaded; $\bar{P}$ : the deformed position of $P_{0}$ under a static pretension; $\tilde{P}$ : the deformed position of $P_{0}$ under the static pretension and dynamics loads; $x_{0}$ : the un-deformed length measured from support $S_{A}$ to the observed particle; $\bar{x}\left(x_{0}\right)$ : the corresponding statically deformed length under pretension; $\tilde{x}\left(x_{0}\right)$ : the corresponding dynamically deformed arc-length; $L_{0}$ : un-deformed total length; $\Delta L$ : extension of the string due to pretension; and $T$ : tension of the string during vibration.

It follows from Eq. (3.1.1) and Fig. 3.1(b) that the axial strain during vibration, $e$, defined with respect to the initial un-tensioned length is given by

$$
\begin{equation*}
e=\frac{(1+e) d x_{0}-d x_{0}}{d x_{0}}=\sqrt{\left(1+e_{0}+u_{x_{0}}\right)^{2}+v_{x_{0}}^{2}+w_{x_{0}}^{2}}-1 \tag{3.1.2}
\end{equation*}
$$

If the initial tensioned state $(\bar{x})$ is used to define $e, e$ has form (Leissa, 1994):

$$
\begin{align*}
& e=\sqrt{\left[1+e_{0}+\left(1+e_{0}\right) u_{\bar{x}}\right]^{2}+\left(1+e_{0}\right)^{2} v_{\bar{x}}^{2}+\left(1+e_{0}\right)^{2} w_{\bar{x}}^{2}}-1  \tag{3.1.3}\\
& =\sqrt{\left(1+u_{\bar{x}}\right)^{2}+v_{\bar{x}}^{2}+w_{\bar{x}}^{2}}\left(1+e_{0}\right)-1
\end{align*}
$$

where

$$
\begin{equation*}
d \bar{x}=\left(1+e_{0}\right) d x_{0} \text { or } \bar{x}=x_{0}+\int_{0}^{x_{0}} e_{0} d x_{0} \tag{3.1.4}
\end{equation*}
$$

If Poisson's effect is considered, the tension during vibration is

$$
\begin{equation*}
T\left(x_{0}, t\right)=E A e=E A_{0}(1-v e)^{2} e \tag{3.1.5}
\end{equation*}
$$

where $A=A_{0}(1-v e)^{2}$ is the contracted cross section. Using Newton's second law, we get the following equation

$$
\begin{align*}
& m_{0}\left(u_{t t} \mathbf{i}_{1}+v_{t t} \mathbf{i}_{2}+w_{t t} \mathbf{i}_{3}\right)+\left(\tilde{\mu}_{u} u_{t} \mathbf{i}_{1}+\tilde{\mu}_{v} v_{t} \mathbf{i}_{2}+\tilde{\mu}_{w} w_{t} \mathbf{i}_{3}\right) \\
& =\frac{\partial}{\partial x_{0}}\left\{T /(1+e)\left[\left(1+e_{0}+u_{x_{0}}\right) \mathbf{i}_{1}+v_{x_{0}} \mathbf{i}_{2}+w_{x_{0}} \mathbf{i}_{3}\right]\right\}+\tilde{f}_{u} \mathbf{i}_{1}+\tilde{f}_{v} i_{2}+\tilde{f}_{w} \mathbf{i}_{3} \tag{3.1.6}
\end{align*}
$$

where $m_{0}=\rho_{0} A_{0} ;()_{t}=\partial() / \partial t ; \mathbf{i}_{\mathbf{k}}(k=1,2,3)$ are the base vectors of $x_{1}, x_{2}$ and $x_{3}$ axes respectively; $\tilde{\mu}_{u}, \tilde{\mu}_{v}, \tilde{\mu}_{w}$ are the damping coefficients per unit length of the un-deformed string along $x_{1}, x_{2}, x_{3}$ directions; and $\tilde{f}_{u}, \tilde{f}_{v}, \tilde{f}_{w}$ denote the distributed dynamic loads per unit length of the un-deformed string along $x_{1}, x_{2}, x_{3}$ directions. Setting each coefficient of the base vector equal to zero, we obtain the equations of motion as

$$
\begin{align*}
& m_{0} u_{t t}+\tilde{\mu}_{u} u_{t}=\frac{\partial}{\partial x_{0}}\left\{\frac{\left(1+e_{0}+u_{x_{0}}\right) T}{1+e}\right\}+\tilde{f}_{u}  \tag{3.1.7}\\
& m_{0} v_{t t}+\tilde{\mu}_{v} v_{t}=\frac{\partial}{\partial x_{0}}\left\{\frac{v_{x_{0}} T}{1+e}\right\}+\tilde{f}_{v}  \tag{3.1.8}\\
& m_{0} w_{t t}+\tilde{\mu}_{w} w_{t}=\frac{\partial}{\partial x_{0}}\left\{\frac{w_{x_{0}} T}{1+e}\right\}+\tilde{f}_{w} \tag{3.1.9}
\end{align*}
$$

Because either the displacements or the forces along the $x_{1}, x_{2}$ and $x_{3}$ directions are known at the ends, the boundary conditions are to specify

$$
\begin{align*}
& u \text { or } \frac{\left(1+e_{0}+u_{x_{0}}\right) T}{1+e}  \tag{3.1.10}\\
& v \text { or } \frac{v_{x_{0}} T}{1+e}  \tag{3.1.11}\\
& w \text { or } \frac{w_{x_{0}} T}{1+e} \tag{3.1.12}
\end{align*}
$$

at $x_{0}=0$ and $L_{0}$, equations (3.1.7) - (3.1.12) are fully non-linear and they account for Poisson's effect and pretension.

### 3.1.2 Approximate Equations of Motion without Poisson's Effect

Neglecting Poisson's effect (i.e. $v=0$ ), we find from (3.1.5) that

$$
\begin{equation*}
T\left(x_{0}, t\right)=E A e=E A_{0} e(1-v e)^{2}=E A_{0} e \tag{3.1.13}
\end{equation*}
$$

and $m=\rho A=\rho_{0} A_{0}=m_{0}$. Hence (3.1.7) - (3.1.9) become

$$
\begin{equation*}
\rho_{0} A_{0} u_{t t}+\tilde{\mu}_{u} u_{t}=\frac{\partial}{\partial x_{0}}\left\{\frac{\left(1+e_{0}+u_{x_{0}}\right)}{1+e} E A_{0} e\right\}+\tilde{f}_{u} \tag{3.1.14}
\end{equation*}
$$

$$
\begin{align*}
& \rho_{0} A_{0} v_{t t}+\tilde{\mu}_{v} v_{t}=\frac{\partial}{\partial x_{0}}\left\{\frac{v_{x_{0}}}{1+e} E A_{0} e\right\}+\tilde{f}_{v}  \tag{3.1.15}\\
& \rho_{0} A_{0} w_{t t}+\tilde{\mu}_{w} w_{t}=\frac{\partial}{\partial x_{0}}\left\{\frac{w_{x_{0}}}{1+e} E A_{0} e\right\}+\tilde{f}_{w} \tag{3.1.16}
\end{align*}
$$

The boundary conditions are to specify the following primary or secondary variables:

$$
\begin{align*}
& u \text { or } \frac{\left(1+e_{0}+u_{x_{0}}\right)}{1+e} E A_{0} e  \tag{3.1.17}\\
& v \text { or } \frac{v_{x_{0}}}{1+e} E A_{0} e  \tag{3.1.18}\\
& w \text { or } \frac{w_{\chi_{0}}}{1+e} E A_{0} e \tag{3.1.19}
\end{align*}
$$

at $x_{0}=0$ and $L_{0}$. Knowing $\bar{x}=x_{0}+\int_{0}^{x_{0}} e_{0} d x_{0}$, we define $\alpha \equiv \frac{d \bar{x}}{d x_{0}}=\left(1+e_{0}\right)$. So (3.1.14) (3.1.16) can be changed from the description based on the static un-deformed state $x_{0}$ to that on the static equilibrium state $\bar{x}$. For convenience, we replace $\bar{x}$ by $x$ and obtain from (3.1.14) - (3.1.16) that

$$
\begin{align*}
& \rho_{0} A_{0} u_{t t}+\tilde{\mu}_{u} u_{t}=\alpha \frac{\partial}{\partial x}\left\{\frac{\alpha\left(1+u_{x}\right)}{1+e} E A_{0} e\right\}+\tilde{f}_{u}  \tag{3.1.20}\\
& \rho_{0} A_{0} v_{t t}+\tilde{\mu}_{v} v_{t}=\alpha \frac{\partial}{\partial x}\left\{\frac{\alpha E A_{0} e}{1+e} v_{x}\right\}+\tilde{f}_{v}  \tag{3.1.21}\\
& \rho_{0} A_{0} w_{t t}+\tilde{\mu}_{w} w_{t}=\alpha \frac{\partial}{\partial x}\left\{\frac{\alpha E A_{0} e}{1+e} w_{x}\right\}+\tilde{f}_{w} \tag{3.1.22}
\end{align*}
$$

Next, we approximate (3.1.20) - (3.1.22) for small but finite-amplitude vibrations. It follows from (3.1.3) that

$$
\begin{equation*}
e=\alpha \sqrt{\left(1+u_{x}\right)^{2}+v_{x}^{2}+w_{x}^{2}}-1 \tag{3.1.23}
\end{equation*}
$$

Taylor's expansion of (3.1.23) yields

$$
\begin{equation*}
\frac{\alpha e}{1+e}=\alpha-1+u_{x}-u_{x}^{2}+\frac{1}{2}\left(v_{x}^{2}+w_{x}^{2}\right)+u_{x}^{3}-\frac{3}{2} u_{x}\left(v_{x}^{2}+w_{x}^{2}\right)+\cdots \tag{3.1.24}
\end{equation*}
$$

Substituting (3.1.24) into (3.1.20) - (3.1.22) and assuming that $\alpha, E$ and $A_{0}$ are constant, we obtain the following approximate equations:

$$
\begin{gather*}
\rho_{0} A_{0} u_{t t}+\tilde{\mu}_{u} u_{t}=\alpha^{2} E A_{0} u_{x x}+\alpha E A_{0} \frac{\partial}{\partial x}\left[\left(\frac{1}{2}-u_{x}\right)\left(v_{x}^{2}+w_{x}^{2}\right)\right]+\tilde{f}_{u}  \tag{3.1.25}\\
\rho_{0} A_{0} v_{t t}+\tilde{\mu}_{v} v_{t}=\alpha(\alpha-1) E A_{0} v_{x x}+\alpha E A_{0} \frac{\partial}{\partial x}\left\{v_{x}\left(u_{x}-u_{x}^{2}+\frac{1}{2}\left(v_{x}^{2}+w_{x}^{2}\right)\right)\right\}+\tilde{f}_{v}  \tag{3.1.26}\\
\rho_{0} A_{0} w_{t t}+\tilde{\mu}_{w} w_{t}=\alpha(\alpha-1) E A_{0} w_{x x}+\alpha E A_{0} \frac{\partial}{\partial x}\left\{w_{x}\left(u_{x}-u_{x}^{2}+\frac{1}{2}\left(v_{x}^{2}+w_{x}^{2}\right)\right)\right\}+\tilde{f}_{w} \tag{3.1.27}
\end{gather*}
$$

As we know, the density changes even if Poisson's effect is not considered. Updating the density from the static un-deformed state $\rho_{0}$ to the dynamics state $\rho$ by following the mass conservation, we have

$$
\begin{equation*}
\frac{\rho_{0}}{\rho}=\frac{A_{0}(1+e)}{A_{0}}=(1+e) \rightarrow \rho_{0}=\rho(1+e) \tag{3.1.28}
\end{equation*}
$$

and $T\left(x_{0}, t\right)=E A_{0} e_{0}=E A_{0}(\alpha-1)$. Hence we get the final form of the approximate equations as

$$
\begin{align*}
& u_{t t}+\mu_{u} u_{t}-c_{1}^{2} u_{x x}=\left(c_{1}^{2}-c_{2}^{2}\right) \frac{\partial}{\partial x}\left\{\left(\frac{1}{2}-u_{x}\right)\left(v_{x}^{2}+w_{x}^{2}\right)\right\}+f_{u}  \tag{3.1.29}\\
& v_{t t}+\mu_{v} v_{t}-c_{2}^{2} v_{x x}=\left(c_{1}^{2}-c_{2}^{2}\right) \frac{\partial}{\partial x}\left\{v_{x}\left[u_{x}-u_{x}^{2}+\frac{1}{2}\left(v_{x}^{2}+w_{x}^{2}\right)\right]\right\}+f_{v} \tag{3.1.30}
\end{align*}
$$

$$
\begin{equation*}
w_{t t}+\mu_{w} w_{t}-c_{2}^{2} w_{x x}=\left(c_{1}^{2}-c_{2}^{2}\right) \frac{\partial}{\partial x}\left\{w_{x}\left[u_{x}-u_{x}^{2}+\frac{1}{2}\left(v_{x}^{2}+w_{x}^{2}\right)\right]\right\}+f_{w} \tag{3.1.31}
\end{equation*}
$$

where $\mu_{u}=\frac{\tilde{\mu}_{u}}{\rho_{0} A_{0}}$ and $f_{u}=\frac{\tilde{f}_{u}}{\rho_{0} A_{0}}$ for $u$ and the same relations for $v$ and $w$, and

$$
\begin{gather*}
c_{1}^{2}=\frac{\alpha^{2} E A_{0}}{\rho_{0} A_{0}}=\frac{\alpha^{2} E}{\rho_{0}}  \tag{3.1.32}\\
c_{2}^{2}=\frac{\alpha T}{\rho_{0} A_{0}}=\frac{\alpha(\alpha-1) E A_{0}}{\rho_{0} A_{0}}=\frac{\alpha(\alpha-1) E}{\rho_{0}} \tag{3.1.33}
\end{gather*}
$$

Equations (3.1.29) - (3.1.31) are the same as (7.5.9) - (7.5.11) of Nayfeh and Mook (1979). However, the wave speed $c_{1}$ in the axial (longitudinal) direction given by (3.1.32) is slightly different from (7.5.12) of Nafeh and Mook. The difference is due to the fact that the whole strain (strain due to initial pretension and vibration) is defined based on the un-deformed length whereas the deformed length under initial pretension was used by Nayfeh and Mook to define the strain during vibration. This difference is small for typical metallic strings in the elastic range. Using (3.1.24), the boundary conditions shown in (3.1.17)-(3.1.19) can be expressed as

$$
\begin{align*}
& u \text { or } E A_{0}\left[\alpha u_{x}+\left(\frac{1}{2}-u_{x}\right)\left(v_{x}^{2}+w_{x}^{2}\right)\right]  \tag{3.1.34}\\
& v \text { or } E A_{0} v_{x}\left[\alpha-1+u_{x}-u_{x}^{2}+\frac{1}{2}\left(v_{x}^{2}+w_{x}^{2}\right)\right]  \tag{3.1.35}\\
& w \text { or } E A_{0} w_{x}\left[\alpha-1+u_{x}-u_{x}^{2}+\frac{1}{2}\left(v_{x}^{2}+w_{x}^{2}\right)\right] \tag{3.1.36}
\end{align*}
$$

### 3.1.3 The Two-Equation Model

In this section, we condense the three-equation model (3.1.29) - (3.1.31) into a two-equation model. To accomplish this, first we consider the linear undamped free oscillation problem. Neglecting the damping, forcing, and nonlinear terms in (3.1.29) (3.1.31), we obtain the following uncoupled equations

$$
\begin{align*}
& u_{t t}-c_{1}^{2} u_{x x}=0  \tag{3.1.37}\\
& v_{t t}-c_{2}^{2} v_{x x}=0  \tag{3.1.38}\\
& w_{t t}-c_{2}^{2} w_{x x}=0 \tag{3.1.39}
\end{align*}
$$

Next, we consider the boundary conditions:

$$
\begin{equation*}
u=v=w=0 \text { at } x=0 \text { and } x=L \tag{3.1.40}
\end{equation*}
$$

It follows from (3.1.37) and (3.1.40) that the mode shapes of the axial motion are given by

$$
\begin{equation*}
u=\sin \frac{n \pi x}{L} \tag{3.1.41}
\end{equation*}
$$

corresponding to the natural frequencies

$$
\begin{equation*}
\omega_{n}=\frac{n \pi c_{1}}{L} \tag{3.1.42}
\end{equation*}
$$

where $n$ is a positive integer. Similarly, the mode shapes of the transverse motion in the $x_{2}$ and $x_{3}$ directions are given by

$$
\begin{equation*}
v=\sin \frac{m \pi x}{L} \text { and } w=\sin \frac{k \pi x}{L} \tag{3.1.43}
\end{equation*}
$$

corresponding to the natural frequencies

$$
\begin{equation*}
\omega_{m}=\frac{m \pi c_{2}}{L} \text { and } \omega_{k}=\frac{k \pi c_{2}}{L} \tag{3.1.44}
\end{equation*}
$$

where $m$ and $k$ are positive integers.

It follows from (3.1.32), (3.1.33) and the fact $\alpha=1+e_{0}$ that

$$
\begin{equation*}
\frac{c_{1}}{c_{2}}=\sqrt{\frac{E A_{0} \alpha}{T_{0}}}=\sqrt{\frac{E A_{0}\left(1+e_{0}\right)}{E A_{0} e_{0}}}=\sqrt{\frac{1+e_{0}}{e_{0}}} \tag{3.1.45}
\end{equation*}
$$

which is very large (several hundreds) for typical metals in the elastic range. Hence, for a given frequency order (i.e., $m=k=n$ ), the transverse frequencies are much less than the longitudinal frequencies according to (3.1.42) and (3.1.44). Consequently, if the excitation frequencies are much smaller than the fundamental longitudinal frequency $\omega_{1}\left(=\pi c_{1} / L\right)$, the longitudinal inertia in (3.1.29) can be neglected, as shown below. Hence, (3.1.30) and (3.1.31) describe the two-equation model for strings. Next, we give a formal derivation of the two-equation model. To analyze the nonlinear response of the string to an excitation having a frequency close to the $m t h$ transverse linear natural frequency and being applied at the right end, we normalize the displacements and time by using $L$ and $L / m \pi c_{2}$ as

$$
\xi \equiv \frac{x}{L}, \tau \equiv \frac{m \pi c_{2} t}{L}, U \equiv \frac{u}{L}, V \equiv \frac{v}{L}, W \equiv \frac{w}{L}
$$

Hence (3.1.29) becomes

$$
\begin{equation*}
\left(\frac{m \pi c_{2}}{c_{1}}\right)^{2} U_{\tau \tau}+\bar{\mu}_{u} \frac{m \pi c_{2} L}{c_{1}^{2}} U_{\tau}-U_{\xi \xi}=\frac{\left(c_{1}^{2}-c_{2}^{2}\right)}{c_{1}^{2}} \frac{\partial}{\partial \xi}\left[\left(\frac{1}{2}-U_{\xi}\right)\left(V_{\xi}^{2}+W_{\xi}^{2}\right)\right] \tag{3.1.46}
\end{equation*}
$$

where the distributed longitudinal excitation $\bar{f}_{u}(t)$ is assumed to be zero. For typical metallic strings in the elastic range, $\left(c_{2} / c_{1}\right)^{2} \ll 1$ and hence the terms proportional to $U_{\tau \tau}$ and $U_{\tau}$ can be neglected in (3.1.46). Hence we have

$$
\begin{equation*}
U_{\xi \xi}=-\frac{\partial}{\partial \xi}\left[\left(\frac{1}{2}-U_{\xi}\right)\left(V_{\xi}^{2}+W_{\xi}^{2}\right)\right] \tag{3.1.47}
\end{equation*}
$$

Integrating (3.1.47) once with respect to $\xi$ yields

$$
\begin{equation*}
U_{\xi}=-\left(\frac{1}{2}-U_{\xi}\right)\left(V_{\xi}^{2}+W_{\xi}^{2}\right)+b(\tau) \tag{3.1.48}
\end{equation*}
$$

where $b(\tau)$ depends on the boundary conditions. Equation (3.1.48) shows that $U_{\xi}$ is of $\operatorname{order} O\left(V_{\xi}^{2}\right)=O\left(W_{\xi}^{2}\right)$. Hence, to the order $O\left(V_{\xi}^{2}, W_{\xi}^{2}\right)$, (3.1.48) becomes

$$
\begin{equation*}
U_{\xi}=-\frac{1}{2}\left(V_{\xi}^{2}+W_{\xi}^{2}\right)+b(\tau) \tag{3.1.49}
\end{equation*}
$$

Because the force is assumed to be imposed at the right end

$$
\begin{equation*}
U(0, \tau)=0 \text { and } U(1, \tau) \equiv L F(\tau) \tag{3.1.50}
\end{equation*}
$$

Integrating (3.1.49) once and using (3.1.50) yields

$$
\begin{equation*}
U=-\frac{1}{2} \int_{0}^{\xi}\left(V_{\xi}^{2}+W_{\xi}^{2}\right) d \xi+b(\tau) \xi \tag{3.1.51}
\end{equation*}
$$

where

$$
\begin{equation*}
b(\tau)=\frac{1}{2} \int_{0}^{1}\left(V_{\xi}^{2}+W_{\xi}^{2}\right) d \xi+L F(\tau) \tag{3.1.52}
\end{equation*}
$$

In terms of dimensional variables, (3.1.49) and (3.1.52) becomes

$$
\begin{align*}
& u_{x}=-\frac{1}{2}\left(v_{x}^{2}+w_{x}^{2}\right)+b(t)  \tag{3.1.53}\\
& b(t)=L F(t)+\frac{1}{2 L} \int_{0}^{L}\left(v_{x}^{2}+w_{x}^{2}\right) d x \tag{3.1.54}
\end{align*}
$$

Hence

$$
\begin{equation*}
u_{x}=-\frac{1}{2}\left(v_{x}^{2}+w_{x}^{2}\right)+L F(t)+\frac{1}{2 L} \int_{0}^{L}\left(v_{x}^{2}+w_{x}^{2}\right) d x \tag{3.1.55}
\end{equation*}
$$

Substituting (3.1.55) into (3.1.30) and (3.1.31), neglecting $c_{2}^{2}$ in $c_{1}^{2}-c_{2}^{2}$, and keeping terms up to cubic nonlinearity, we obtain the following two-equation model:

$$
\begin{align*}
& v_{t t}+\mu_{v} v_{t}-c_{2}^{2} v_{x x}=h(t) v_{x x}+\frac{c_{1}^{2}}{2 L} v_{x x} \int_{0}^{L}\left(v_{x}^{2}+w_{x}^{2}\right) d x+f_{v}(x, t)  \tag{3.1.56}\\
& w_{t t}+\mu_{w} w_{t}-c_{2}^{2} w_{x x}=h(t) w_{x x}+\frac{c_{1}^{2}}{2 L} w_{x x} \int_{0}^{L}\left(v_{x}^{2}+w_{x}^{2}\right) d x+f_{w}(x, t) \tag{3.1.57}
\end{align*}
$$

where $h(t)=c_{1}^{2} L F(t)$ is due to the longitudinal excitation. Narasimha (1968) carefully argued that motions in which $u=0$ when $f_{u}=0$ are generally not possible and the most significant nonlinear problem arises when $u=O\left(v^{2}, w^{2}\right)$, as employed in deriving (3.1.56) and (3.1.57). If $u=0$ and $f_{u}=0$, then it follows from (3.1.29) that $\left(v_{x}^{2}+w_{x}^{2}\right)$ is independent of $x$. This is possible only if the string is straight or it is helical all the time, including when $t=0$. Clearly such a motion cannot exit except under very special initial conditions and/or excitations. Therefore, in general $u \neq 0$. Moreover, whenever the motion is planar (i.e., $w=0), v_{x}^{2}$ cannot be independent of $x$ if $v$ is to be zero at the ends unless $v \equiv 0$, i.e., the string is straight. However, experiments (Harrison, 1948; Murthy and Ramakrishna, 1965; Molteno and Tufillaro, 1990; S, Nayfeh, Nayfeh. And Mook, 1995) show that, when a string is excited by forces acting in a given plane below a critical frequency, then the motion is also in the same plane and hence $u$ cannot be zero below that frequency. Moreover, there is no reason whatsoever to expect that with or without the onset of whirling motions that $u$ will vanish.

### 3.2 Multiple-Scale Analysis

To analyze the coupled nonlinear equations (3.1.56) and (3.1.57), we employ the method of multiple scales (Nayfeh, 1979). There are two approaches of using the method of multiple scales to attack the problem. For the discretization approach, the multiple scale analysis is applied to the ordinary differential equations obtained by discretizing the governing partial differential equations using spatial eigen-functions of the linear problem. For the direct treatment approach, the multiple scale analysis is directly applied to the governing partial differential equations. In our analysis, the excitation displacement is assumed to be linearly distributed along the string when the discretization approach was applied. In the direct approach the boundary conditions were considered as a restriction to the solvability condition. We discuss the differences between the results obtained from the two approaches.

### 3.2.1 Discretized Model

Using the Galerkin method, the spatial dependence can be eliminated from the equations of motion. Partial differential equations are discretized into ordinary differential equations in time and so the spatial coordinates and the temporal variables are separated. For a string only transversely excited, $h(t)=0$ and equations (3.1.56) and (3.1.57) become

$$
\begin{align*}
& v_{t t}+\mu_{v} v_{t}-c_{2}^{2} v_{x x}=\frac{c_{1}^{2}}{2 L} v_{x x} \int_{0}^{L}\left(v_{x}^{2}+w_{x}^{2}\right) d x+f_{v}(x, t)  \tag{3.2.1}\\
& w_{t t}+\mu_{w} w_{t}-c_{2}^{2} w_{x x}=\frac{c_{1}^{2}}{2 L} w_{x x} \int_{0}^{L}\left(v_{x}^{2}+w_{x}^{2}\right) d x+f_{w}(x, t) \tag{3.2.2}
\end{align*}
$$

The string with one end fixed and the other being periodically excited in the $w$ direction has the following non-homogeneous boundary conditions

$$
\begin{gather*}
v(0, t)=0 \text { and } v(L, t)=0  \tag{3.2.3}\\
w(0, t)=0 \text { and } w(L, t)=b(t)=B \cos \Omega t \tag{3.2.4}
\end{gather*}
$$

If the base displacement is assumed to be linearly distributed along the string

$$
\begin{equation*}
w(x, t)=\bar{w}(x, t)+\frac{x}{L} b(t) \tag{3.2.5}
\end{equation*}
$$

where $b(t)=B \cos (\Omega t)$ denotes the base motion with a frequency $\Omega$ and an amplitude $B \cdot \bar{w}(x, t)$ is the relative displacement of the string with respect to rigidly displaced position $x_{3}=\frac{x}{L} b(t)$. It follows from (3.2.5) that

$$
\begin{align*}
& w_{t}=\bar{w}_{t}+\frac{x}{L} b_{t}=\bar{w}_{t}-\frac{x}{L} B \Omega \sin (\Omega t) \\
& w_{t t}=\bar{w}_{t t}+\frac{x}{L} b_{t t}=\bar{w}_{t t}-\frac{x}{L} B \Omega^{2} \cos (\Omega t)  \tag{3.2.6}\\
& w_{x}=\bar{w}_{x}+\frac{b}{L} \\
& w_{x x}=\bar{w}_{x x}
\end{align*}
$$

The boundary conditions become

$$
\begin{align*}
& v(0, t)=0, v(L, t)=0  \tag{3.2.7}\\
& \bar{w}(0, t)=0, \bar{w}(L, t)=0 \tag{3.2.8}
\end{align*}
$$

Substituting above equations into the governing equations (3.2.1) and (3.2.2) and assuming the damping coefficients for vibrations in the two planes are the same (i.e. $\mu_{v}=\mu_{w}=\mu$ ), for the same mode, the governing equations read

$$
\begin{equation*}
v_{t t}+\mu v_{t}-c_{2}^{2} v_{x x}=\frac{c_{1}^{2}}{2 L} v_{x x} \int_{0}^{L}\left(v_{x}^{2}+w_{x}^{2}\right) d x \tag{3.2.9}
\end{equation*}
$$

$$
\begin{equation*}
w_{t t}+\mu w_{t}-c_{2}^{2} w_{x x}=\frac{c_{1}^{2}}{2 L} w_{x x} \int_{0}^{L}\left(v_{x}^{2}+w_{x}^{2}\right) d x+\frac{c_{1}^{2}}{2 L} w_{x x} \int_{0}^{L}\left(2 w_{x} \frac{b}{L}+\left(\frac{b}{L}\right)^{2}\right) d x+f_{w}(x, t) \tag{3.2.10}
\end{equation*}
$$

where $\bar{w}$ is replaced by $w$ for convenience and $f_{w}(x, t)=\frac{x}{L} B \Omega^{2} \cos (\Omega t)+\mu \frac{x}{L} B \Omega \sin (\Omega t)$
As we know, the linear normal modes for transverse vibrations are

$$
\begin{equation*}
v_{m}(x)=\sin \frac{m \pi x}{L} \text { and } w_{k}(x)=\sin \frac{k \pi x}{L} \tag{3.2.11}
\end{equation*}
$$

Hence, following the Galerkin procedure, we can represent the solution of the nonlinear equations (3.2.9) and (3.2.10) as

$$
\begin{equation*}
v(x, t)=\sum_{k=1}^{\infty} \eta_{k}(t) \sin \frac{k \pi x}{L}, w(x, t)=\sum_{k=1}^{\infty} \zeta_{k}(t) \sin \frac{k \pi x}{L} \tag{3.2.12}
\end{equation*}
$$

which satisfy exactly the transformed boundary conditions given in (3.2.7) and (3.2.8). Substituting (3.2.12) into (3.2.9) and (3.2.10) yields

$$
\begin{align*}
& \begin{aligned}
& \sum_{k=1}^{\infty}\left(\ddot{\eta}_{k}+\omega_{k}^{2} \eta_{k}\right) \sin \frac{k \pi x}{L}+\mu \sum_{k=1}^{\infty} \dot{\eta}_{k} \sin \frac{k \pi x}{L} \\
&=-\frac{c_{1}^{2}}{4}\left(\frac{\pi}{L}\right)^{4}\left[\sum_{k=1}^{\infty} k^{2} \eta_{k} \sin \frac{k \pi x}{L}\right]\left[\sum_{k=1}^{\infty} k^{2}\left(\eta_{k}^{2}+\zeta_{k}^{2}\right)\right] \\
& \sum_{k=1}^{\infty}\left(\ddot{\zeta}_{k}+\omega_{k}^{2} \zeta_{k}\right) \sin \frac{k \pi x}{L}+\mu \sum_{k=1}^{\infty} \dot{\zeta}_{k} \sin \frac{k \pi x}{L} \\
&=-\frac{c_{1}^{2}}{4}\left(\frac{\pi}{L}\right)^{4}\left[\sum_{k=1}^{\infty} k^{2} \zeta_{k} \sin \frac{k \pi x}{L}\right]\left[\frac{2 b^{2}}{\pi^{2}}+\sum_{k=1}^{\infty} k^{2}\left(\eta_{k}^{2}+\zeta_{k}^{2}\right)\right]+f_{w}(x, t)
\end{aligned} \tag{3.2.13}
\end{align*}
$$

Multiplying (3.2.13) and (3.2.14) with $\sin \frac{n \pi x}{L}$ and integrating the results from $x=0$ to $x=L$, we obtain

$$
\begin{equation*}
\ddot{\eta}_{n}+\omega_{n}^{2} \eta_{n}+2 \mu_{n} \dot{\eta}_{n}=-\frac{c_{1}^{2}}{4}\left(\frac{\pi}{L}\right)^{4} n^{2} \eta_{n} \sum_{k=1}^{\infty} k^{2}\left(\eta_{k}^{2}+\zeta_{k}^{2}\right) \tag{3.2.15}
\end{equation*}
$$

$$
\begin{equation*}
\ddot{\zeta}_{n}+\omega_{n}^{2} \zeta_{n}+2 \mu_{n} \dot{\zeta}_{n}=-\frac{c_{1}^{2}}{4}\left(\frac{\pi}{L}\right)^{4} n^{2} \zeta_{n}\left[\frac{2 b^{2}}{\pi^{2}}+\sum_{k=1}^{\infty} k^{2}\left(\eta_{k}^{2}+\zeta_{k}^{2}\right)\right]+F_{w n}(t) \tag{3.2.16}
\end{equation*}
$$

where

$$
\begin{align*}
& \mu_{n} \equiv \frac{1}{L} \int_{0}^{L} \mu \sin ^{2} \frac{n \pi x}{L} d x=\frac{\mu}{2}  \tag{3.2.17}\\
& F_{w n}(t) \equiv \frac{2}{L} \int_{0}^{L} f_{w}(x, t) \sin \frac{n \pi x}{L} d x \tag{3.2.18}
\end{align*}
$$

Therefore, the distributed-parameter problem consisting of (3.2.1) - (3.2.4) has been transformed into an infinite system of nonlinearly coupled ordinary differential equations. For the case of primary resonance of the mth mode in the $w$ direction, we define the following order for the forcing term so that it counters the effect of nonlinear terms.

$$
\begin{equation*}
b(t)=B \cos (\Omega t)=\varepsilon^{3} B^{*} \cos \Omega t, \Omega \equiv \omega_{\mathrm{m}}+\varepsilon^{2} \sigma \tag{3.2.19}
\end{equation*}
$$

where $\varepsilon$ is a small dimensionless measure of the amplitude used as a bookkeeping device. Correspondingly

$$
\begin{equation*}
F_{w n}=\varepsilon^{3} F_{w n}^{*} \cos (\Omega t) \tag{3.2.20}
\end{equation*}
$$

Moreover, we order the damping to be $\mathrm{O}\left(\varepsilon^{2}\right)$ so that its influence balances the influence of nonlinearity and resonance. Thus, we replace $\mu_{n}$ with $\varepsilon^{2} \mu_{n}$. To determine the secondorder approximate solution of (3.2.15) and (3.2.16) for this case, we use the method of multiple scales and assume that

$$
\begin{align*}
& \eta_{n}(t)=\varepsilon \eta_{n 1}\left(T_{0}, T_{2}\right)+\varepsilon^{3} \eta_{n 3}\left(T_{0}, T_{2}\right)+\cdots  \tag{3.2.21}\\
& \zeta_{n}(t)=\varepsilon \zeta_{n 1}\left(T_{0}, T_{2}\right)+\varepsilon^{3} \zeta_{n 3}\left(T_{0}, T_{2}\right)+\cdots \tag{3.2.22}
\end{align*}
$$

Substituting (3.2.21) and (3.2.22) into (3.2.15) and (3.2.16) and equating coefficients of like powers of $\varepsilon$, we obtain

Order $\varepsilon$ :

$$
\begin{align*}
& D_{0}^{2} \eta_{n 1}+\omega_{n}^{2} \eta_{n 1}=0  \tag{3.2.23}\\
& D_{0}^{2} \zeta_{n 1}+\omega_{n}^{2} \zeta_{n 1}=0 \tag{3.2.24}
\end{align*}
$$

Order $\varepsilon^{3}$ :

$$
\begin{align*}
& D_{0}^{2} \eta_{n 3}+\omega_{n}^{2} \eta_{n 3}=-2 D_{0} D_{2} \eta_{n 1}-2 \mu_{n} D_{0} \eta_{n 1}-\frac{c_{1}^{2} n^{2} \pi^{4}}{4 L_{0}^{4}} \eta_{n 1}\left[\sum_{k=1}^{\infty} k^{2}\left(\eta_{k 1}^{2}+\zeta_{k 1}^{2}\right)\right]  \tag{3.2.25}\\
& D_{0}^{2} \zeta_{n 3}+\omega_{n}^{2} \zeta_{n 3}= \\
& -2 D_{0} D_{2} \zeta_{n 1}-2 \mu_{n} D_{0} \zeta_{n 1}-\frac{c_{1}^{2} n^{2} \pi^{4}}{4 L_{0}^{4}} \zeta_{n 1}\left[\sum_{k=1}^{\infty} k^{2}\left(\zeta_{k 1}^{2}+\eta_{k 1}^{2}\right)\right]+F_{3 n}^{*} \cos \left(\Omega T_{0}\right) \tag{3.2.26}
\end{align*}
$$

The solutions of (3.2.23) and (3.2.24) can be expressed as

$$
\begin{align*}
& \eta_{n 1}=A_{v n}\left(T_{2}\right) e^{i \omega_{m} T_{0}}+c c  \tag{3.2.27}\\
& \zeta_{n 1}=A_{w n}\left(T_{2}\right) e^{i \omega_{m} T_{0}}+c c \tag{3.2.28}
\end{align*}
$$

Substituting (3.2.27) and (3.2.28) into (3.2.25) and (3.2.26) yields

$$
\begin{align*}
& D_{0}^{2} \eta_{n 3}+\omega_{n}^{2} \eta_{n 3}=-2 i \omega_{m}\left(A_{v n}^{\prime}+\mu_{n} A_{v n}\right) e^{i \omega_{n} T_{0}}-\frac{c_{1}^{2} n^{2} \pi^{4}}{4 L_{0}^{4}}\left(A_{v n} e^{i \omega_{n} T_{0}}+\bar{A}_{v n} e^{-i \omega_{n} T_{0}}\right)  \tag{3.2.29}\\
& \times \sum_{k=1}^{\infty} k^{2}\left[\left(A_{v k}^{2}+A_{w k}^{2}\right) e^{2 i \omega_{n} T_{0}}+A_{v k} \bar{A}_{v k}+A_{w k} \bar{A}_{w k}\right]+c c \\
& D_{0}^{2} \zeta_{n 3}+\omega_{n}^{2} \zeta_{n 3}=-2 i \omega_{m}\left(A_{w n}^{\prime}+\mu_{n} A_{w n}\right) e^{i \omega_{m} T_{0}}-\frac{c_{1}^{2} n^{2} \pi^{4}}{4 L_{0}^{4}}\left(A_{w n} e^{i \omega_{n} T_{0}}+\bar{A}_{w n} e^{-i \omega_{m} T_{0}}\right)  \tag{3.2.30}\\
& \times \sum_{k=1}^{\infty} k^{2}\left[\left(A_{v k}^{2}+A_{w k}^{2}\right) e^{2 i \omega_{m} T_{0}}+A_{v k} \bar{A}_{v k}+A_{w k} \bar{A}_{w k}\right]+\frac{1}{2} F_{3 n}^{*} e^{i \Omega T_{0}}+c c
\end{align*}
$$

Using (3.2.19) in (3.2.20) and setting the secular terms in (3.2.29) and (3.2.30) to zero yields

$$
\begin{equation*}
2 i \omega_{n}\left(A_{v n}^{\prime}+\mu_{n} A_{v n}\right)+\frac{c_{1}^{2} n^{2} \pi^{4}}{4 L^{4}}\left[2 A_{v n} \sum_{k=1}^{\infty} k^{2}\left(A_{v k} \bar{A}_{v k}+A_{w k} \bar{A}_{v k}\right)+n^{2}\left(A_{v n}^{2}+A_{w n}^{2}\right) \bar{A}_{v n}\right]=0 \tag{3.2.31}
\end{equation*}
$$

$$
\begin{align*}
& 2 i \omega_{n}\left(A_{w n}^{\prime}+\mu_{w n} A_{w n}\right) \\
& +\frac{c_{1}^{2} n^{2} \pi^{4}}{4 L^{4}}\left[2 A_{n} \sum_{k=1}^{\infty} k^{2}\left(A_{v k} \bar{A}_{v k}+A_{w k} \bar{A}_{w k}\right)+n^{2}\left(A_{v n}^{2}+A_{w n}^{2}\right) \bar{A}_{w n}\right]+\frac{1}{2} F_{w n}^{*} \delta_{n m} e^{i \sigma T_{2}}=0 \tag{3.2.32}
\end{align*}
$$

Expressing the $A_{v n}$ and $A_{w n}$ in the polar form as

$$
\begin{equation*}
A_{v n}=\frac{1}{2} a_{v n} e^{i \beta_{v n}}, A_{w n}=\frac{1}{2} a_{w n} e^{i \beta_{w n}} \tag{3.2.33}
\end{equation*}
$$

we separate (3.2.31) and (3.2.32) into real and imaginary parts. If $n \neq m$, the imaginary parts of the secular equations are given by

$$
\begin{align*}
& \left(a_{v n}^{\prime}+\mu_{n} a_{v n}\right)+\frac{c_{1}^{2} n^{3} \pi^{3}}{32 L^{3} c_{2}} a_{w n}^{2} a_{v n} \sin \left(2 \beta_{w n}-2 \beta_{v n}\right)=0  \tag{3.2.34}\\
& \left(a_{w n}^{\prime}+\mu_{n} a_{w n}\right)+\frac{c_{1}^{2} n^{3} \pi^{3}}{32 L^{3} c_{2}} a_{v n}^{2} a_{w n} \sin \left(2 \beta_{v n}-2 \beta_{w n}\right)=0 \tag{3.2.35}
\end{align*}
$$

when $n \neq m$. Adding $a_{v n}$ times (3.2.34) to $a_{w n}$ times (3.2.35) yields

$$
\begin{equation*}
a_{v n} a_{v n}^{\prime}+a_{w n} a_{w n}^{\prime}+\mu_{n} a_{v n}^{2}+\mu_{n} a_{w n}^{2}=0 \tag{3.2.36}
\end{equation*}
$$

which can be written as

$$
\begin{equation*}
\frac{d}{d T_{2}}\left(a_{v n}^{2}+a_{w n}^{2}\right)+\mu_{n} a_{v n}^{2}+\mu_{n} a_{w n}^{2}=0 \tag{3.2.37}
\end{equation*}
$$

Because $\mu_{n}$ are positive, $a_{v n}^{2}+a_{w n}^{2} \rightarrow 0$ as $T_{2}$ or $t \rightarrow \infty$ because the derivatives of $a_{v n}^{2}+a_{w n}^{2}$ w.r.t. $T_{2}$ is negative. Therefore, $a_{v n} \rightarrow 0$ and $a_{w n} \rightarrow 0$ as $t \rightarrow \infty$ for $n \neq m$. The long-term non-linear response of the string is governed by (3.2.31) and (3.2.32) with $A_{v_{n}}$ and $A_{w n}$ being equal to zero if $\mathrm{n} \neq \mathrm{m}$. In other words, the $\mathrm{m} t h$ mode, and the response of the string is governed by

$$
\begin{align*}
& 2 i\left(A_{v}^{\prime}+\mu_{m} A_{v}\right)+\frac{c_{1}^{2} m^{3} \pi^{3}}{4 L^{3} c_{2}}\left[2 A_{v}\left(A_{v} \bar{A}_{v}+A_{w} \bar{A}_{w}\right)+\left(A_{v}^{2}+A_{w}^{2}\right) \bar{A}_{v}\right]=0  \tag{3.2.38}\\
& 2 i\left(A_{w}^{\prime}+\mu_{m} A_{w}\right)+\frac{c_{1}^{2} m^{3} \pi^{3}}{4 L^{3} c_{2}}\left[2 A_{w}\left(A_{v} \bar{A}_{v}+A_{w} \bar{A}_{w}\right)+\left(A_{v}^{2}+A_{w}^{2}\right) \bar{A}_{w}\right]=-\frac{L F_{w m}^{*} e^{i \sigma T_{2}}}{2 m \pi c_{2}} \tag{3.2.39}
\end{align*}
$$

where the subscript $m$ on $A_{v m}$ and $A_{w m}$ has been dropped. Hence, the long term nonlinear response of the string to the primary resonance of the $m$ th mode is governed by a two-degree-of-freedom model, which can be obtained from (3.2.15) and (3.2.16) by setting $\eta_{n}=\zeta_{n}=0$ for $n \neq m$. The result is

$$
\begin{align*}
& \ddot{\eta}_{m}+\omega_{m}^{2} \eta_{m}+2 \mu_{m} \dot{\eta}_{m}=-\frac{c_{1}^{2} \pi^{4} m^{4}}{4 L^{4}} \eta_{m}\left(\eta_{m}^{2}+\zeta_{m}^{2}\right)  \tag{3.2.40}\\
& \ddot{\zeta}_{m}+\omega_{m}^{2} \zeta_{m}+2 \mu_{m} \dot{\zeta}_{m}=-\frac{c_{1}^{2} \pi^{4} m^{4}}{4 L^{4}} \zeta_{m}\left(\eta_{m}^{2}+\zeta_{m}^{2}\right)+F_{w m}^{*}(x, t) \tag{3.2.41}
\end{align*}
$$

Moreover, the transformed end excitation can be written as

$$
\begin{equation*}
f_{w}(x, t)=\frac{x}{L} B \Omega^{2} \cos (\Omega t)+\mu_{m} \frac{x}{L} B \Omega \sin (\Omega t)=\frac{x}{L} B \Omega \sqrt{\Omega^{2}+\mu_{m}^{2}} \cos (\Omega t-\alpha) \tag{3.2.42}
\end{equation*}
$$

where $\tan (\alpha)=\mu_{m} / \Omega$. Substituting $f_{w}$ into (3.2.18) and integrating yields

$$
\begin{equation*}
F_{w m}(t)=\frac{2}{m \pi} B \Omega \sqrt{\Omega^{2}+\mu_{m}^{2}} \cos (\Omega t-\alpha)(-1)^{m-1} \tag{3.2.43}
\end{equation*}
$$

Because $b(t)=B \cos (\Omega t)=\varepsilon^{3} B^{*} \cos (\Omega t)$, we have

$$
\begin{equation*}
F_{w m}=\varepsilon^{3} F_{w m}^{*} \cos (\Omega t-\alpha) \text { with } F_{w m}^{*}=\frac{2}{m \pi} B^{*} \Omega \sqrt{\Omega^{2}+\mu_{m}^{2}}(-1)^{m-1} \tag{3.2.43}
\end{equation*}
$$

Correspondingly,

$$
\begin{equation*}
f_{w m}=-\frac{L F_{w m}^{*}}{2 m \pi c_{2}}=\frac{L}{2 m \pi c_{2}} \frac{2}{m \pi} B^{*} \Omega \sqrt{\Omega^{2}+\mu_{m}^{2}}(-1)^{m-1}=\frac{B^{*}}{m \pi} \sqrt{\Omega^{2}+\mu_{m}^{2}}(-1)^{m} \tag{3.2.45}
\end{equation*}
$$

$$
\begin{align*}
& \text { If } \mu_{m} \ll \Omega, \\
& \qquad f_{w m} \approx \frac{B^{*}}{m \pi} \Omega(-1)^{m}=\frac{B^{*}}{m \pi} \frac{m \pi c_{2}}{L}(-1)^{m-1}=\frac{B^{*} c_{2}}{L}(-1)^{m} \tag{3.2.46}
\end{align*}
$$

where $B^{*}$ is the excitation amplitude. The governing modulation equations (3.2.38) and (3.2.39) become

$$
\begin{align*}
& i\left(\mu A_{v}+2 A_{v}^{\prime}\right)+\frac{c_{1}^{2} m^{3} \pi^{3}}{4 L^{3} c_{2}}\left[2 A_{v}\left(A_{v} \bar{A}_{v}+A_{w} \bar{A}_{w}\right)+\left(A_{v}^{2}+A_{w}^{2}\right) \bar{A}_{v}\right]=0  \tag{3.2.47}\\
& i\left(\mu A_{w}+2 A_{w}^{\prime}\right)+\frac{c_{1}^{2} m^{3} \pi^{3}}{4 L^{3} c_{2}}\left[2 A_{w}\left(A_{v} \bar{A}_{v}+A_{w} \bar{A}_{w}\right)+\left(A_{v}^{2}+A_{w}^{2}\right) \bar{A}_{w}\right]=\frac{(-1)^{m} B^{*} c_{2}}{L} e^{i \sigma T_{2}} \tag{3.2.48}
\end{align*}
$$

where $2 \mu_{m}$ becomes $\mu_{m}$ because of (3.2.17) and, for convenience, $\mu_{m}$ is replaced by $\mu$. We note that $\mu_{m} \ll \Omega$ is assumed.

### 3.2.2 Direct Treatment

In this approach, the method of multiple scales is applied directly to the non-linear integral-partial differential equations of motion and associated boundary conditions to determine the approximate solutions. We follow the derivation procedure of Nayfeh (1995). The damping and forcing terms are ordered similarly for the same reason as in the discretized approach, i.e.,

$$
\begin{equation*}
b(t)=\varepsilon^{3} B \cos (\Omega t), \mu=\varepsilon^{2} \mu \tag{3.2.49}
\end{equation*}
$$

Assuming asymptotic solutions for $v$ and $u$ as

$$
\begin{align*}
& v(x, t)=\varepsilon v_{1}\left(x, T_{0}, T_{2}\right)+\varepsilon^{3} v_{3}\left(x, T_{0}, T_{2}\right)+\cdots  \tag{3.2.50}\\
& w(x, t)=\varepsilon w_{1}\left(x, T_{0}, T_{2}\right)+\varepsilon^{3} w_{3}\left(x, T_{0}, T_{2}\right)+\cdots \tag{3.2.51}
\end{align*}
$$

we rewrite the boundary conditions as
$o(\varepsilon):$

$$
\begin{gather*}
v_{1}(0, t)=0, v_{1}(L, t)=0  \tag{3.2.52}\\
w_{1}(0, t)=0, w_{1}(L, t)=0  \tag{3.2.53}\\
o\left(\varepsilon^{3}\right): \quad \\
 \tag{3.2.54}\\
 \tag{3.2.55}\\
\\
\\
\\
\\
\\
\\
\\
\\
\\
\end{gather*}(0, t)=0, v_{3}(L, t)=0, w_{3}(L, t)=\varepsilon^{3} b(t)=B \cos \Omega t
$$

Without separating the temporal and spatial variables, we substitute (3.2.50) and (3.2.51) into governing equations (3.2.1) and (3.2.2) and equating coefficients of like power of $\varepsilon$ and obtain
$o(\varepsilon):$

$$
\begin{align*}
& D_{0}^{2} v_{1}-c_{2}^{2} \frac{\partial^{2} v_{1}}{\partial x^{2}}=0  \tag{3.2.56}\\
& D_{0}^{2} w_{1}-c_{2}^{2} \frac{\partial^{2} w_{1}}{\partial x^{2}}=0 \tag{3.2.57}
\end{align*}
$$

$o\left(\varepsilon^{3}\right):$

$$
\begin{align*}
& D_{0}^{2} v_{3}-c_{2}^{2} \frac{\partial^{2} v_{3}}{\partial x^{2}}=-\mu D_{0} v_{1}-2 D_{2} D_{0} v_{1}+\frac{c_{1}^{2}}{2 L} \frac{\partial^{2} v_{1}}{\partial x^{2}} \int_{0}^{L}\left(\left(\frac{\partial v_{1}}{\partial x}\right)^{2}+\left(\frac{\partial w_{1}}{\partial x}\right)^{2}\right) d x  \tag{3.2.58}\\
& D_{0}^{2} w_{3}-c_{2}^{2} \frac{\partial^{2} w_{3}}{\partial x^{2}}=-\mu D_{0} w_{1}-2 D_{2} D_{0} w_{1}+\frac{c_{1}^{2}}{2 L} \frac{\partial^{2} w_{1}}{\partial x^{2}} \int_{0}^{L}\left(\left(\frac{\partial v_{1}}{\partial x}\right)^{2}+\left(\frac{\partial w_{1}}{\partial x}\right)^{2}\right) d x \tag{3.2.59}
\end{align*}
$$

Applying boundary condition (3.2.52) and (3.2.53) to equations (3.2.56) and (3.2.57), we obtain the general solution of the first-order problem as

$$
\begin{align*}
& v_{1}=\sum_{n=1}^{\infty}\left(A_{v n} e^{i \omega_{n} T_{0}}+\bar{A}_{v n} e^{-i \omega_{n} T_{0}}\right) \sin \left(\frac{\omega_{n} X}{c_{2}}\right)  \tag{3.2.60}\\
& w_{1}=\sum_{n=1}^{\infty}\left(A_{w n} e^{i \omega_{n} T_{0}}+\bar{A}_{w n} e^{-i \omega_{n} T_{0}}\right) \sin \left(\frac{\omega_{n} X}{c_{2}}\right) \tag{3.2.61}
\end{align*}
$$

where the $A_{v n}$ and $A_{w n}$ are unknown functions of $T_{2}$ at this order of approximation. They are to be determined by imposing the so-called solvability or consistency conditions at the second order approximation by requiring the third-order problems, shown by equations (3.2.58) and (3.2.59) and boundary conditions (3.2.54) and (3.2.55), to have solutions and yield uniform expansions.

The excitation frequency is near the $m t h$ natural frequency $\omega_{m}$ of the string, i.e.,

$$
\begin{equation*}
\Omega=\omega_{m}+\varepsilon^{2} \sigma \tag{3.2.62}
\end{equation*}
$$

Substituting (3.2.60) and (3.2.61) into (3.2.59), we obtain

$$
\begin{align*}
& D_{0}^{2} w_{3}-c_{2}^{2} \frac{\partial^{2} w_{3}}{\partial x^{2}} \\
& =-\mu \sum_{n=1}^{\infty} i \omega_{n}\left(A_{w n} e^{i \omega_{n} T_{0}}-\bar{A}_{w n} e^{-i \omega_{n} T_{0}}\right) \sin \left(\frac{\omega_{n} x}{c_{2}}\right)-2 \sum_{n=1}^{\infty} i \omega_{n}\left(A_{w n}^{\prime} e^{i \omega_{n} T_{0}}-\bar{A}_{w n}^{\prime} e^{-i \omega_{n} T_{0}}\right) \sin \left(\frac{\omega_{n} x}{c_{2}}\right)  \tag{3.2.63}\\
& \quad \frac{-c_{1}^{2}}{4 c_{2}^{4}} \sum_{n=1}^{\infty} \omega_{n}^{2}\left(A_{w n} e^{i \omega_{n} T_{0}}+\bar{A}_{w n} e^{-i \omega_{n} T_{0}}\right) \sin \left(\frac{\omega_{n} x}{c_{2}}\right) \\
& \quad \sum_{n=1}^{\infty} \omega_{n}^{2}\left[\left(A_{v n}^{2}+A_{w n}^{2}\right) e^{2 i \omega_{n} T_{0}}+2\left(A_{v n} \bar{A}_{v n}+A_{w n} \bar{A}_{w n}\right)+\left(\bar{A}_{v n}^{2}+\bar{A}_{w n}^{2}\right) e^{-2 i \omega_{n} T_{0}}\right]
\end{align*}
$$

Equation (3.2.63) and the boundary conditions, (3.2.54) and (3.2.55) constitute an inhomogeneous part whose homogeneous part is the same as the first-order problem shown by equation (3.2.56) and (3.2.57). Because the latter problem has a nontrivial solution, the inhomogeneous problem given by equations (3.2.63) has a solution only if the solvability condition is satisfied. To determine this solvability condition, we convert equation (3.2.63) into a system of ordinary differential equations by assuming that

$$
\begin{equation*}
w_{3}=\sum_{m=1}^{\infty}\left[\psi_{m}\left(x, T_{2}\right) e^{i \omega_{m} T_{0}}+\bar{\psi}_{m}\left(x, T_{2}\right) e^{-i \omega_{m} T_{0}}\right] \tag{3.2.64}
\end{equation*}
$$

where $\psi_{m}\left(x, T_{2}\right)$ is to be determined. Substituting equation (3.2.64) into equation (3.2.63), multiplying both sides by $e^{-i \omega_{s} T_{0}}$, and integrating the result with respect to $T_{0}$ from 0 to $2 \pi /\left(\pi c_{2} / L\right)=2 L / c_{2}$, we obtain

$$
\begin{equation*}
\frac{\omega_{s}^{2}}{c_{2}^{2}} \psi_{s}\left(x, T_{2}\right)+\frac{\partial^{2} \psi_{s}}{\partial x^{2}}=\zeta_{s}\left(x, T_{2}\right) \tag{3.2.65}
\end{equation*}
$$

where

$$
\begin{align*}
& \zeta_{s}\left(x, T_{2}\right)=\frac{c_{1}^{2}}{4 c_{2}^{6}} \sum_{k=1}^{\infty} \omega_{k}^{2} \sin \left(\frac{\omega_{k} X}{c_{2}}\right) \\
& \quad \times \sum_{n=1}^{\infty} \omega_{n}^{2}\left[A_{w k}\left(A_{v n}^{2}+A_{w n}^{2}\right) \delta(2 n-s+k)+A_{w k}\left(\bar{A}_{v n}^{2}+\bar{A}_{w n}^{2}\right) \delta(-2 n-s+k)\right.  \tag{3.2.66}\\
& \left.\quad+\bar{A}_{w k}\left(A_{v n}^{2}+A_{w n}^{2}\right) \delta(2 n-s-k)\right] \\
& \quad+\sin \left(\frac{\omega_{s} X}{c_{2}}\right)\left\{\frac{c_{1}^{2}}{4 c_{2}^{6}} \omega_{s}^{2} \sum_{n=1}^{\infty} \omega_{n}^{2}\left[2 A_{w s}\left(A_{v n} \bar{A}_{v n}+A_{w n} \bar{A}_{w n}\right)\right]+\frac{\mu i \omega_{s} A_{w s}+2 i \omega_{s} A_{w s}^{\prime}}{c_{2}^{2}}\right\}
\end{align*}
$$

and

$$
\delta(m)= \begin{cases}0 & m \neq 0  \tag{3.2.67}\\ 1 & m=0\end{cases}
$$

Substituting equation (3.2.64) into boundary conditions (3.2.55), multiplying both sides by $e^{i \omega_{s} T_{0}}$ and integrating the results with respect to $T_{0}$ from 0 to $2 L / c_{2}$, we obtain the boundary conditions on $\psi_{s}$ as

$$
\begin{equation*}
\psi_{s}\left(0, T_{2}\right) \equiv 0, \psi_{s}\left(L, T_{2}\right) \equiv \frac{B}{2} \delta\left(\omega_{m}-\omega_{s}\right) e^{i \sigma T_{2}} \tag{3.2.68}
\end{equation*}
$$

We wish to determine the conditions for the existence of solution for the boundary-value problem given by equations (3.2.65) and (3.2.68). Using the method of variation of parameters, we find that the general solution of equation (3.2.65) can be expressed as

$$
\begin{align*}
\psi_{s}\left(x, T_{2}\right) & =\left[\frac{c_{2}}{\omega_{s}} \int_{0}^{x} \zeta_{s}\left(z, T_{2}\right) \cos \left(\frac{\omega_{s} z}{c_{2}}\right) d z+e_{1}\left(T_{2}\right)\right] \sin \left(\frac{\omega_{s} z}{c_{2}}\right) \\
& +\left[\frac{c_{2}}{\omega_{s}} \int_{0}^{x} \zeta_{s}\left(z, T_{2}\right) \sin \left(\frac{\omega_{s} z}{c_{2}}\right) d z+e_{2}\left(T_{2}\right)\right] \cos \left(\frac{\omega_{s} z}{c_{2}}\right) \tag{3.2.69}
\end{align*}
$$

where $e_{1}$ and $e_{2}$ are arbitrary functions of $T_{2}$. Application of the first boundary condition to equation (3.2.68) yields $e_{2} \equiv 0$. Then applying the secondary boundary condition to equation (3.2.68), we obtain

$$
\begin{equation*}
\left[\frac{c_{2}}{s} \int_{0}^{L_{0}} \zeta_{s}\left(z, T_{2}\right) \sin \left(\frac{\omega_{s} z}{c_{2}}\right) d z\right] \cos (s \pi)=\frac{B}{2} \delta\left(\omega_{m}-\omega_{s}\right) e^{i \sigma T_{2}} \tag{3.2.70}
\end{equation*}
$$

which is the solvability condition. We note that $e_{1}\left(T_{2}\right)$ is arbitrary. It can be chosen to be zero so that the amplitudes of motion are uniquely defined by the $A_{w s}$. Substituting the expression (3.2.66) into the solvability condition and assuming $m=s$, we obtain the modulation equation governing $A_{w s}$

$$
\begin{align*}
& i\left(\mu A_{w_{m}}+2 A_{w m}^{\prime}\right)+ \\
& \frac{c_{1}^{2}}{4 c_{2}^{4}} \omega_{m}\left\{A_{w m} \sum_{n=1}^{\infty} \omega_{n}^{2}\left[2\left(A_{w n} \bar{A}_{w n}+A_{w n} \bar{A}_{w n}\right)\right]+\omega_{m}^{2} \bar{A}_{w m}\left(A_{w m}^{2}+A_{w m}^{2}\right)\right\}=(-1)^{m} \frac{B c_{2}}{L} e^{i \sigma T_{2}} \tag{3.2.71}
\end{align*}
$$

Thus, if the forcing frequency is near that of the sth mode, the sth mode is externally excited. Otherwise, the sth mode may occur only through internal resonance when the natural frequencies are commensurate or nearly commensurate. Similarly, we obtain the modulation equation governing $A_{v s}$ as

$$
i\left(\mu A_{v m}+2 A_{v m}^{\prime}\right)+\frac{c_{1}^{2}}{4 c_{2}^{4}} \omega_{m}\left\{A_{v m} \sum_{n=1}^{\infty} \omega_{n}^{2}\left[2\left(A_{v n} \bar{A}_{v n}+A_{v n} \bar{A}_{v n}\right)\right]+\omega_{m}^{2} \bar{A}_{v m}\left(A_{v m}^{2}+A_{v m}^{2}\right)\right\}=0 \text { (3.2.72) }
$$

Similarly, one can prove that both in-plane and out-of-plane modes not directly excited decay to zero and equations (3.2.71) and (3.2.72) become

$$
\begin{align*}
& i\left(\mu A_{w}+2 A_{w}^{\prime}\right)+\frac{m^{3} \pi^{3} c_{1}^{2}}{4 L^{3} c_{2}}\left\{2 A_{w}\left(A_{w} \bar{A}_{w}+A_{w} \bar{A}_{w}\right)+\bar{A}_{w}\left(A_{w}^{2}+A_{w}^{2}\right)\right\}=(-1)^{m} \frac{B c_{2}}{L} e^{i \sigma T_{2}}  \tag{3.2.73}\\
& i\left(\mu A_{v}+2 A_{v}^{\prime}\right)+\frac{m^{3} \pi^{3} c_{1}^{2}}{4 L^{3} c_{2}}\left\{2 A_{v}\left(A_{v} \bar{A}_{v}+A_{v} \bar{A}_{v}\right)+\bar{A}_{v}\left(A_{v}^{2}+A_{v}^{2}\right)\right\}=0 \tag{3.2.74}
\end{align*}
$$

which are exactly the same as equations (3.2.47) and (3.2.48).

### 3.3 The Cartesian Form of the Averaged Equations

The complex amplitudes $A_{v m}$ and $A_{w m}$ of the mth mode can be written in the polar form as

$$
\begin{equation*}
A_{v m}=\frac{1}{2} a_{1} e^{i \beta_{1}} \quad A_{w m}=\frac{1}{2} a_{2} e^{i \beta_{2}} \tag{3.3.1}
\end{equation*}
$$

Substituting (3.3.1) into (3.2.47) and (3.2.48) and separating real and imaginary parts yields

$$
\begin{align*}
& a_{1}^{\prime}+\mu_{2} a_{1}+k a_{2}^{2} a_{1} \sin \left(\gamma_{1}\right)+f \sin \left(\gamma_{2}\right)=0  \tag{3.3.2}\\
& k\left(3 a_{1}^{3}+2 a_{2}^{2} a_{1}+a_{2}^{2} a_{1} \cos \left(\gamma_{1}\right)\right)+f \cos \left(\gamma_{2}\right)=\beta_{1}^{\prime} a_{1}  \tag{3.3.3}\\
& a_{2}^{\prime}+\mu_{2} a_{2}-k a_{1}^{2} a_{2} \sin \left(\gamma_{1}\right)=0  \tag{3.3.4}\\
& k\left(3 a_{2}^{3}+2 a_{1}^{2} a_{2}+a_{1}^{2} a_{2} \cos \left(\gamma_{1}\right)\right)=a_{2} \beta_{2}^{\prime} \tag{3.3.5}
\end{align*}
$$

where $f_{w m}$ is replaced by $f$,

$$
\begin{equation*}
\gamma_{1} \equiv 2 \beta_{2}-2 \beta_{1}, \gamma_{2} \equiv \sigma T_{2}-\beta_{1}, f \equiv(-1)^{m} \frac{p c_{2}}{L} e^{i \sigma T_{2}}=\frac{L F_{3 m}^{*}}{2 m \pi c_{2}}, k \equiv \frac{c_{1}^{2} m^{3} \pi^{3}}{32 L^{3} c_{2}} \tag{3.3.6}
\end{equation*}
$$

To obtain the steady-state response which corresponds to $a_{1}^{\prime}=a_{2}^{\prime}=\gamma_{1}^{\prime}=\gamma_{2}^{\prime}=0$, we use (3.3.6) and rewrite equations (3.3.2) - (3.3.5) as

$$
\begin{align*}
& a_{1}^{\prime}=-\mu_{2} a_{1}-k a_{2}^{2} a_{1} \sin \left(\gamma_{1}\right)-f \sin \left(\gamma_{2}\right)  \tag{3.3.7}\\
& a_{2}^{\prime}=-\mu_{2} a_{2}+k a_{1}^{2} a_{2} \sin \left(\gamma_{1}\right)  \tag{3.3.8}\\
& \gamma_{2}^{\prime}=\sigma-k\left(3 a_{1}^{2}+2 a_{2}^{2}+a_{2}^{2} \cos \left(\gamma_{1}\right)\right)-\frac{f \cos \left(\gamma_{2}\right)}{a_{1}}  \tag{3.3.9}\\
& \gamma_{1}^{\prime}=\left(2 k a_{1}^{2}-2 k a_{2}^{2}\right)\left(\cos \left(\gamma_{1}\right)-1\right)-\frac{2 f \cos \left(\gamma_{2}\right)}{a_{1}} \tag{3.3.10}
\end{align*}
$$

Frequency responses curves can be obtained by solving equations (3.3.7) - (3.3.10) for fixed point solutions. The stability of these fixed point solutions can be determined by the eigenvalues of the Jacobian matrix of the right-hand side of Eqs. (3.3.7) - (3.3.10). A given fixed point is stable if and only if the real parts of all eigenvalues are less than or equal to zero. If there is a pair of complex values having positive real parts, amplitudeand phase-modulated motions are expected. Moreover, we have

$$
\begin{equation*}
A_{1}=\frac{1}{2} a_{1} e^{i \beta_{1}}=\frac{1}{2} a_{1} e^{i \sigma T_{2}} e^{-i \gamma_{2}}=\frac{1}{2}\left(a_{1} \cos \gamma_{2}-i a_{1} \sin \gamma_{2}\right) e^{i \sigma T_{2}}=\frac{1}{2}\left(p_{1}-i q_{1}\right) e^{i \sigma T_{2}} \tag{3.3.11}
\end{equation*}
$$

where $p_{1} \equiv a_{1} \cos \gamma_{2}, q_{1} \equiv a_{1} \sin \gamma_{2}$ and

$$
\begin{equation*}
A_{2}=\frac{1}{2} a_{2} e^{i \beta_{2}}=\frac{1}{2} a_{2} e^{\sigma T_{2}} e^{-\gamma_{3}}=\frac{1}{2}\left(a_{2} \cos \gamma_{3}-i a_{2} \sin \gamma_{3}\right) e^{\sigma T_{2}}=\frac{1}{2}\left(p_{2}-i q_{2}\right) e^{i \sigma T_{2}} \tag{3.3.12}
\end{equation*}
$$

Here $p_{2} \equiv a_{2} \cos \gamma_{3}, q_{2} \equiv a_{2} \sin \gamma_{3}$ and $\gamma_{3} \equiv \gamma_{2}-\frac{\gamma_{1}}{2}$. Substituting above equations into (3.2.71) and (3.2.72), we obtain the following modulation equations in the Cartesian
form, which are exactly the same as those obtained by Miles (1984) using particular scaling and non-dimensionalization.

$$
\begin{align*}
& p_{1}^{\prime}=-\sigma q_{1}-\mu p_{1}+3 k q_{1} E+\frac{k p_{2}}{4} M  \tag{3.3.13}\\
& q_{1}^{\prime}=\sigma p_{1}-\mu q_{1}-3 k p_{1} E+\frac{q_{2} k}{4} M-f  \tag{3.3.14}\\
& p_{2}^{\prime}=-\sigma q_{2}-\mu p_{2}+3 k q_{2} E-\frac{k p_{1}}{4} M  \tag{3.3.15}\\
& q_{2}^{\prime}=\sigma p_{2}-\mu q_{2}-3 k p_{2} E-\frac{q_{1} k}{4} M \tag{3.3.16}
\end{align*}
$$

where

$$
\begin{equation*}
E \equiv p_{1}^{2}+q_{1}^{2}+p_{2}^{2}+q_{2}^{2}, M \equiv p_{1} q_{2}-p_{2} q_{1}, k=\frac{c_{1}^{2} m^{3} \pi^{3}}{32 L_{0}^{3} c_{2}} \tag{3.3.17}
\end{equation*}
$$

Again, note that $f=(-1)^{m} \frac{B^{*} c_{2}}{L}$ by the direct approach or $f=\frac{B^{*}}{m \pi} \sqrt{\Omega^{2}+\mu^{2}}(-1)^{m}$ by the discretized approach in which the displacement of excitation is assumed to be linearly distributed along the string. It is important to note here that the above sets of ordinary differential equations, either in polar coordinates (3.3.13)-(3.3.16) or in Cartesian coordinates (3.3.7)-(3.3.10), describe the amplitude and phase modulations of the first-order solutions specified in equations (3.2.21) and (3.2.22) or (3.2.50) and (3.3.51). These equations are invariant to the transformation $\left(p_{2}, q_{2}\right) \rightarrow-\left(p_{2}, q_{2}\right)$ which means that there is a reflective symmetry in the $\left(p_{1}, q_{1}, p_{2}, q_{2}\right)$ state space about the $\left(p_{1}, q_{1}\right)$ plane. Thus, if we find a steady-state solution, another distinct solution for the same parameters can be obtained by using this
mirror-image transformation, unless the solution itself is symmetric. There are other systems have similar amplitude equations with different coefficients that represent the physical parameters of the system. These systems include the weakly resonant motions of a spherical pendulum (Miles and Maewal, 1984b), of surface waves in a circular cylinder (Miles, 1984c), as well as the motions of structural components such as an elastic beam (Maewal, 1986) and an axis-symmetric shell (Maewal, 1987), etc.

### 3.4 Frequency Responses

### 3.4.1 The Effects of Damping

Damping plays an important role in the possible appearance of various nonlinear phenomena, especially chaotic motions, in the dynamics of strings. In our investigation, the damping is assumed to be the same for all modes of vibrations in the two directions. We adopt the parameters, except the damping, from the string selected by Neyfeh and Pai (2004). The effects of damping on the dynamics of strings are studied in detail. Experimental verifications and comparisons will be presented in Chapter 4. The parameters of the string we studied are

$$
\begin{align*}
& L=2.13 \mathrm{~m}, f^{*}=5.39 \times 10^{-5} \mathrm{~m} / \mathrm{sec}, \mu^{*}=0.03411 / \mathrm{sec}  \tag{3.4.1}\\
& c_{1}=55.7 \mathrm{~m} / \mathrm{s}, c_{2}=10.5 \mathrm{~m} / \mathrm{s}, \varepsilon=\pi c_{2}^{2} / c_{1}^{2}=0.351
\end{align*}
$$

We chose to study the vibration composed of the sixth mode, i.e., response to a harmonic base excitation at sixth natural frequency ( $m=6$ and $\omega_{6}=6 \pi c_{2} / L$ ). Fig.3.2 presents the frequency response curves with different value for $\mu$. For a large enough damping $\mu$, all responses are planar vibrations. The frequency response curves with $\mu=1.4 \mu^{*}$ are
plotted in Fig. 3.2(a). All steady-state responses are periodic vibrations in the plane of excitation, which means all input energy is absorbed by the in-plane vibration. There are two stable planar solutions in the frequency range $\left(\Delta_{2}, \Delta_{6}\right)$; one has large amplitude and the other has small amplitude. There is only one solution for other frequency ranges. For a small $\mu$, the planar solution is not always stable. The vibration loses or gains stability when the frequency detuning $\Delta=\sigma \varepsilon^{2}$ changes. The frequency response curves with $\mu=1.2 \mu^{*}$ are plotted in Fig. 3.2(b). A non-planar solution, corresponding to a periodic whirling motion, arises after the pitchfork bifurcation at $\Delta_{1}$, where the excitation frequency is slightly beyond the linear natural frequency. Then the whirling motion loses stability at the reverse pitch fork bifurcation point $\Delta_{5}$. For the whirling motion of this non-planar branch, we have the following observations summarized from the literature. As the excitation frequency increases beyond $\Delta_{1}$, the amplitude of in-plane vibration does not increase as fast as when $\Delta<\Delta_{1}$. For some damping values, it even keeps constant and shows the saturation phenomenon, which is indicated by the horizontal section of the response curve shown in Fig. 3.2(b). The out-of-plane response, however, has a rapid increase of amplitude, which means the input energy mostly goes into the out-of-plane vibration. After the out-of-plane response reaches its maximum, the amplitude of in-plane response increases again while that of out-of-plane response decreases as the excitation frequency increases. At $\Delta_{5}$, the out-of-plane response disappears (i.e., $a_{2}=0$ ) and the vibration becomes stable planar one again. Because the model is symmetric about the excitation plane, the non-planar (whirling) motions can have different directions of whirling (clockwise or counterclockwise). These two solutions can not be distinguished
from the trajectory plotted on the $a_{1}-a_{2}$, but they can be distinguished from the trajectory plotted on the $p_{2}-q_{2}$ plane. The solutions appear as fixed points in either the upper right (the first) or the lower left (the third) quadrant of the $p_{2} \sim q_{2}$ plane according to the direction of the whirling (Reilly, 1992).


Fig. 3.2: The in-plane ( $a_{1}$ ) and out-of-plane ( $a_{2}$ ) frequency response curves with different values for $\mu$ and a fixed excitation amplitude $f=f^{*}$ : (a) $\mu=1.4 \mu^{*}$, (b) $\mu=1.2 \mu^{*}$, (c) $\mu=1.09 \mu^{*}$, (d) $\mu=1.015 \mu^{*}$, (e) $\mu=1.01 \mu^{*}$, (f) $\mu=1.00 \mu^{*}$, (g) $\mu=0.98 \mu^{*}$, (h) $\mu=0.96 \mu^{*}$, (i) $\mu=0.85 \mu^{*}$, (j) $\mu=0.80 \mu^{*}$. Solid line: planar stable solutions; dash-dotdash line non-planar stable solutions; dashed point line unstable solutions; dots modulated solutions. $\Delta_{1}$ : the forward pitchfork bifurcation; $\Delta_{5}$ : the reverse pitchfork bifurcation point; $\Delta_{2}, \Delta_{6}$ and $\Delta_{3}, \Delta_{4}$ : the tuning points of the planar and non-planar solution branches, respectively; $\Delta_{1}^{*}$ and $\Delta_{2}^{*}$ : the forward and reverse Hopf bifurcation points, respectively.


Fig. 3.2 (Continued)






Fig. 3.2 (Continued)


Fig. 3.2 (Continued)


Fig. 3.3: Zoom-in views of : (a) Fig.3.2 (f), (b) Fig.3.2 (i), and (c) Fig.3.2 (i)

Fig .3.2(c) shows that, with a smaller damping, the stable non-planar solution becomes unstable due to Hopf bifurcations at points $\Delta_{1}^{*}$ and $\Delta_{2}^{*}$. When the excitation increases, a pair of complex conjugate eigenvalues cross the imaginary axis from the left half plane to the right half plane and the solution loses its stability at $\Delta_{1}^{*}$, or from the right half plane to the left half plane and the solution regains its stability at $\Delta_{2}^{*}$. The frequency band $\Delta_{1}^{*}<\Delta<\Delta_{2}^{*}$ is within the frequency band defined by the two pitchfork bifurcation points $\Delta_{1}$ and $\Delta_{5}$. Responses under other frequency bands are the same as those under
larger dampings. Steady state solutions obtained by direct numerical integration of (3.3.7)-(3.3.10) are found to be essentially the same as the fixed point solutions obtained by solving the modulation equations (3.3.7)-(3.3.10) with $a_{1}^{\prime}=a_{2}^{\prime}=\gamma_{1}^{\prime}=\gamma_{2}^{\prime}=0$. The difference between these two solutions is small and is due to neglecting the second-order term of the excitation during the averaging. We have more detailed discussions on this topic in Chapter 7. For the fixed points on the Hopf branch, the responses are amplitude and phase modulated limit cycle solutions, i.e., whirling or ballooning motions according to the Hopf bifurcation theorem. With $\mu=1.09 \mu^{*}$ (Fig. 3.2(c)), there are no further bifurcations and the limit-cycle solutions of the Hopf branch are stable over the entire detuning interval, which means all responses are continuous periodic motions. Miles (1984a) performed some numerical integrations around the frequency band $\left(\Delta_{1}^{*}, \Delta_{2}^{*}\right)$, and his results showed that all solutions converged to the lower planar solutions with small amplitudes. We conjecture that it is probably due to the initial conditions he selected were too close to fixed points of the lower planar branch.

As shown in Figs. 3.2(c) - 3.2(j), the distance between the two Hopf bifurcation points increases when $\mu$ decreases. The variations of the Hopf bifurcation points on the $\Delta-\mu$ plane and the $\Delta-f$ plane are plotted in Figs. 3.4(a) and 3.4(b), respectively. They reveal that, as the damping decreases or the forcing increases, the Hopf bifurcation range increases. It is obvious that the upper bifurcation point changes more than the lower one when the parameter changes. Between the two Hopf bifurcation points, The limit-cycle solution may becomes unstable and undergoes period doubling bifurcations.


Fig. 3.4: Variations of the Hopf bifurcation set $\left(\Delta_{1}^{*}, \Delta_{2}^{*}\right)$ on: (a) the $\Delta \sim \mu$ plane and (b)the $\Delta \sim f$ plane. $\Delta_{1}^{*}$ and $\Delta_{2}^{*}$ : the forward (lower one with dots) and reverse (upper one with stars) Hopf bifurcation points.


Fig. 3.5: Solutions that reveal the invariance of the modulation in: (a) Cartesian coordinates, and (b) polar coordinates, where $\Delta=0.108$ and $\mu=1.015 \mu^{*}$. Solid dots: planar solutions; $\nabla$ : unstable planar solutions; circle: limit-cycle solutions.

Fig. 3.2(d) shows the frequency response curves for both in-plane and out-of-plane vibrations when $\mu=1.015 \mu^{*}$ and $f=f^{*}$. Figs. 3.5(a) and 3.5(b) show the invariance of the modulated solutions of the governing ordinary differential equations of both Cartesian and polar coordinates. Due to the trigonometric relations between the amplitudes of vibration and Cartesian coordinates, there are trajectories in all four quadrants of the Cartesian coordinates while only the first and third quadrants of the polar coordinates have trajectories. Fig. 3.6 lists a series of phase plots showing the sequence of bifurcation of this case. The corresponding frequency spectrums are also included. The bifurcation sequence of the limit cycle we obtained is the same as those obtained by Johnson and Bajaj (1989) and Bajaj and Johnson (1992). Fig. 3.7 shows the bifurcation structure of the whole Hopf branch. The amplitude of the limit cycle increases as the detuning increases and it reaches its maximum around the midpoint of the interval, and then it shrinks to zero as the detuning approaches $\Delta_{2}^{*}$ (see Fig. 3.2(d)). Our numerical investigation shows that the closer the detuning is to the two bifurcation points, the longer it takes for the integration to converge. Moreover, from the 2-D $\Delta-q_{2}$ view of the bifurcation structure, we observe a discontinuous increase or decrease of the amplitude $q_{2}$ when the period-doubling happens. Although the bifurcation is rich, no chaotic motions are observed.


Fig. 3.6: Bifurcation sequence (phase plots and frequency spectrum) of the Hopf branch with $\mu=1.015 \mu^{*}$ and $f=f^{*}:$ (a) $\Delta=0.0940$, (b) $\Delta=0.1020$, (c) $\Delta=0.1120$, (d) $\Delta=0.1240$,
(e) $\Delta=0.1340$, (f) $\Delta=0.1440$, and (g) $\Delta=0.1520$, where $|A(f)|=\sqrt{a_{n}^{2}+b_{n}^{2}}$ and $a_{n}$ and $b_{n}$ are spectral coefficients for $q_{2}(t)$ from FFT analysis.


Fig. 3.6 (continued)


Fig. 3.6 (continued)


Fig. 3.7: Bifurcation structure of the Hopf branch: (a) 3-D view of the structure, and (b) 2-D $\Delta-q_{2}$ view of the structure, where $\mu=1.015 \mu^{*}$ and $f=f^{*}$.


Fig. 3.7 (continued)

Decreasing the damping to $\mu=1.01 \mu^{*}$, Fig. 3.2(e) shows that the interval $\left(\Delta_{1}^{*}, \Delta_{2}^{*}\right)$ expands with the lower bound $\Delta_{1}^{*}$ being almost unchanged while the upper bound $\Delta_{2}^{*}$ being increased. Through a series of period-doubling bifurcations, the Hopf branch becomes unstable, and another coexisting branch (the isolated branch) is created through a global saddle node bifurcation. In other words a stable and a unstable branch of limit cycle solutions are created simultaneously. Together with the stable planar branch, there are three possible solutions within the frequency range. The frequency response curves of Fig. 3.2(e) are basically the same as those of Fig. 3.2(d) except that the Hopf bifurcation range is larger. Figs. 3.8(a)-(k) show a series of representative trajectories with different detuning values for this case. Figs. 3.8(a)-(d) show HP1, HP2, HP4, and

HP8 solutions, respectively. Here HPn denotes a Hopf branch solution having n periods. We note that the HP8 solution was not obtained by Johnson and Bajaj (1989) and Bajaj and Johnson (1992). Generally speaking, this case has the same bifurcation sequence as the case with $\mu=1.015 \mu^{*}$ and $f=f^{*}$, except that, around the middle part of the interval, the bifurcation shrinks and the oscillator has smaller amplitude. It is a conjecture that the forward and reverse bifurcation sequences of the Hopf branch in the frequency interval where two branches coexist is due to the constraints or stabilizing influence of the isolated branch. Plots in Fig. 3.9 show us the properties of the isolated branch with $\mu=1.01 \mu^{*}$ and $f=f^{*}$. At a damping of this level, the isolated branch undergoes no period-doubling and so the amplitude for the whole branch is almost constant (Fig. 3.9 (b)). Fig. 3.9 (c) shows the positions and geometries of Hopf branch solution and the isolated branch solution on the $p_{2}-q_{2}$ plane when $\Delta=0.1280$. The major difference between them is that the isolated branch solution has a cusp at a location close to the planar fixed point at $\left(p_{2}, q_{2}\right)=(0,0)$. The cusp becomes more obvious when the damping is further reduced because the isolated branch solution is attracted more by the planar fixed point. The upside-down triangle represents the unstable planar solutions whose $a_{2}$ is zero and so are $p_{2}$ and $q_{2}$.


Fig. 3.8: Bifurcation sequence of the Hopf branch with $\mu=1.01 \mu^{*}$ and $f=f^{*}$ : (a) $\Delta=0.0980$, (b) $\Delta=0.1040$, (c) $\Delta=0.1080$, (d) $\Delta=0.1120$, (e) $\Delta=0.1160$, (f) $\Delta=0.1240$, (g) $\Delta=0.1340$, (g) $\Delta=0.1385$, (i) $\Delta=0.1400$, (j) $\Delta=0.1480$, and (k) $\Delta=0.1540$.


Fig. 3.8 (Continued)


Fig. 3.8 (Continued)


Fig. 3.8 (Continued)


Fig. 3.9: The isolated branch with $\mu=1.01 \mu^{*}$ and $f=f^{*}$ : (a) 3-D view of the phase plots, (b) 2-D view, and (c) coexistence of the isolated branch and the Hopf branch at $\Delta=0.1280$. Solid dots: stable planar solutions; $\nabla$ : unstable planar solutions; circle: limit-cycle solutions.


Fig. 3.9 (Continued)

As the damping is further decreased to $\mu=1.005 \mu^{*}$, the bifurcation of the Hopf branch solution has no much difference from those of previous damping cases. The isolated branch, however, undergoes a period-doubling transition to chaos, resulting in a Rossler type chaotic attractor which encircles only one unstable (modulated) non-planar fixed point. Fig. 3.10 shows the bifurcation of the isolated branch solution with $\mu=1.005 \mu^{*}$ and $f=f^{*}$. There are iP1 (iPn means a period-n solution of the isolated branch), iP2, iP4, and chaotic attractor iPC attractor (iPc means a chaotic solutions of the isolated branch) of the Rossler type.


Fig. 3.10: Bifurcation of the isolated branch with $\mu=1.005 \mu^{*}$ and $f=f^{*}$ :
(a) $\Delta=0.1120$, (b) $\Delta=0.1140$, (c) $\Delta=0.1155$, (d) $\Delta=0.1165$, (e) $\Delta=0.1220$, (f) $\Delta=0.1330$, (g) $\Delta=0.1345$, and (h) $\Delta=0.1360$.


Fig. 3.10 (Continued)


Fig. 3.10 (Continued)
For the case with $\mu=\mu^{*}$ and $f=f^{*}$ (see Fig. 3.2(f)), Fig. 3.11 shows that the isolated branch has iP1 and iP2 solutions and periodic solutions reappear (Fig. 3.11(h)) after the Rossler type attractor (Fig. 3.11(e)). According to Bajaj and Johnson (1992), this isolated branch is attributed to a global saddle node bifurcation. For a damping of this level, the isolated branch is truly isolated - not connected with any other branches of solutions. When the damping is further decreased, the unstable part of this second isolated branch mergs with the stable part of the first isolated branch by exactly the same
way the first isolated branch merges with the Hopf branch. For the Hopf branch of this case, the bifurcation is the same as those of previous cases except that a steady state HP1 solution is obtained at the middle part of the frequency range where only HP2 solutions were predicted in the literature. It is reasonable to say that this is due to a more influential constraint (stabilizing effect) on the bifurcation of the Hopf branch by the two isolated branches together. Depending on the stabilizing effect, the detuning interval of the HP1 solution might be quite small and easy to miss. Similarly, the second isolated branch has the same stabilizing effect on the first isolated branch, forcing the Rossler type attractor to undergo reverse bifurcation leading to periodic solutions as the excitation frequency increases.



Fig. 3.11: Bifurcation of the isolated branch with $\mu=1.00 \mu^{*}$ and $f=f^{*}$ : (a) $\Delta=0.1105$; (b) $\Delta=0.1110$, (c) $\Delta=0.1120$, (d) $\Delta=0.1125$, (e) $\Delta=0.1150$, (f) $\Delta=0.1200$, (g) $\Delta=0.1250$,
(h) $\Delta=0.1300$,
(i) $\Delta=0.1325$, (j) $\Delta=0.1340$,
(k) $\Delta=0.1360$, (l) $\Delta=0.1390$, (m) $\Delta=0.1395$, and (n) $\Delta=0.1415$.


Fig. 3.11 (Continued)


Fig. 3.11 (Continued)


Fig. 3.11 (Continued)

The frequency response curves shown in Fig. 3.2(g) with $\mu=0.98 \mu^{*}$ and $f=f^{*}$. show that there is a frequency gap between the stable non-planar branch and the modulation branch. The frequency increment we used for the scanning is pretty small, but the gap never disappears even when the increment is further decreased. Actually, for a smaller damping, the whole modulation branch is discontinuous as shown in Figs. 3.2(h) - (j). Further work is necessary to determine the possible reasons. For the Hopf branch of Fig. 3.2(g), the stable limit cycle solutions do not exist continuously around the middle part of the frequency interval. This is consistent with the conjecture of Johnson and Bajaj (1989) and Bajaj and Johnson (1992) that the stable Hopf branch breaks and merges with the unstable solutions of the isolated branch, leading to a saddle type bifurcation. Again, this is due to the stabilizing effect of upper isolated branches, preventing the Hopf branch to period - double to infinity in a straightforward manner. As more and more isolated branches appear when the damping decreases, the effects become more and more influential and finally disconnect the Hopf branch. Similarly, the bifurcations of former isolated branches are influenced by following isolated branches. At the middle part of the detuning interval, two symmetric trajectories of the newly created isolated branch, located at the first and third quadrants of the polar coordinate, become connected, creating a new trajectory which undergoes period-doubling bifurcation and results into a Lorenz type chaotic attractor, as shown in Figs. 3.12(a) - (n). As the excitation frequency increases, the attractor undergoes period-doubling, becomes a Rossler type attractor (Fig. 3.12(c)), changes to a Lorenz type attractor (Fig. 3.12(h)), and then shrinks back to a Rossler type attractor (Fig. 3.12(1)). At somewhere within the first half of the detuning interval, the isolated trajectories become more and more attracted by the lower planar
solution and two symmetric trajectories are connected at the planar fixed point (see Fig. 3.12 (e)), creating the homoclinic orbit. When the two symmetric attractors are connected, the dynamics becomes chaotic although the trajectories themselves seem quite clear. This is due to the dynamic balance between the attractions from two symmetric modulated attractors which makes the indeterminate vibration go to the first quadrant for some time and the third quadrant for some other time. This phenomenon is more clearly shown in Fig. 3.12 (k) with a zoomed plot. As the detuning increases beyond (Fig. 3.12(e)), the connected two symmetric Rossler type trajectories become a Lorenz type enclosing the two symmetric non-planar fixed points. For further detuning increase, the trajectory departs from the planar fixed point and undergoes period-doubling bifurcation, leading to the Lorenz type chaotic attractor (Fig. 3.12 (h)). Figs. 3.13 (a) and (b) are the 3-D and 2-D views of the bifurcation structure for the whole modulated branch.


Fig. 3.12: Bifurcation of the isolated branch with $\mu=0.98 \mu^{*}$ and $f=f^{*}$ : (a) $\Delta=0.1040$, (b) $\Delta=0.1050$, (c) $\Delta=0.1120$, (d) $\Delta=0.1135$, (e) $\Delta=0.1140$, (f) $\Delta=0.1150$, (g) $\Delta=0.1180$, (h) $\Delta=0.1230$, (i) $\Delta=0.1400$, (j) $\Delta=0.1460$, (k) $\Delta=0.1475$, (l) $\Delta=0.1530$, (m) $\Delta=0.1575$, and (n) $\Delta=0.1580$.


Fig. 3.12 (Continued)


Fig. 3.12 (Continued)


Fig. 3.12 (Continued)


Fig. 3.12 (Continued)


Fig. 3.12 (Continued)


Fig. 3.13: The bifurcation structure of the isolated branch: (a) 3-D view of the structure, and (b) 2-D $\Delta-q_{2}$ view of the structure for the case with $\mu=0.980 \mu^{*}$ and $f=f^{*}$.

As the damping is further reduced to $\mu=0.96 \mu^{*}$ (Fig. 3.2(h)), the Hopf branch becomes more discontinuous no matter how small the detuning increment is used for the scanning (Fig. 3.2 (h)). Fig. 3.14 (a) - (h) are representative phase plots of trajectories at this level of damping level. We see period-doubling bifurcations of the Hopf branch and the isolated branch, the coexistence of solutions of two branches (Figs. 3.14(b) and (c)), and the transition of solutions from one branch to the other one (Fig. 3.14 (e)). The most distinctive feature of responses at this level of damping is that the Lorenz type attractors abruptly disappear over a frequency interval and the only stable solution found in this interval is the lower planar fixed point solution. This can be explained by the boundary crisis or the hetero-clinic bifurcation in which the chaotic attractor becomes tangent to and then intersects with the stable manifold of the saddle-type unstable planar solution existing between the two turning points on the planar solution branch. Figs. 3.14(g) and (h) are trajectories showing the boundary crisis of the isolated branch for two different detuning values.



Fig. 3.14: Bifurcation and crisis of the isolated branch with $\mu=0.96 \mu^{*}$ and $f=f^{*}$ : (a) $\Delta=0.089$, (b) $\Delta=0.102$, Hopf branch, (c) $\Delta=0.102$, isolated branch, (d) $\Delta=0.1025$, (e) $\Delta=0.1055$, (f) $\Delta=0.1140$, and (g) $\Delta=0.1190$ and (h) $\Delta=0.1525$, chaotic attractors
of the isolated branch destroyed by the boundary crisis (transient chaos). Solid dots: stable planar solutions; $\nabla$ : unstable planar solutions; circle: limit-cycle solutions.


Fig. 3.14 (Continued)


Fig. 3.14 (Continued)


Fig. 3.14 (Continued)

Decreasing the damping further to $\mu=0.85 \mu^{*}$, Fig. 3.2(i) shows that the modulation branch of the frequency response curve becomes even more discrete. The zoomed plots of frequency response curves around the area of the first Hopf bifurcation (Figs. 3.3 (b) and (c)) show that the two intervals $\left(\Delta_{3}, \Delta_{4}\right)$ and $\left(\Delta_{1}^{*}, \Delta_{3}\right)$ have different responses. The responses to excitations in the interval $\left(\Delta_{3}, \Delta_{4}\right)$ are unstable non-planar vibrations. The responses to excitations in the interval $\left(\Delta_{1}^{*}, \Delta_{3}\right)$ (a quite small interval) are stable non-planar vibrations. Consequently, there is a small interval $\left(\Delta_{1}^{*}, \Delta_{3}\right)$ in which there are two stable non-planar responses. Fig. 3.2 (j) shows the frequency response curves for the case with $\mu=0.80 \mu^{*}$ and $f=f^{*}$. Figs. 3.15(a) - (j) show representative trajectories of the modulated branch. The trajectory of the Hopf branch becomes distorted and, surprisingly, we obtain a chaotic attractor (Fig. 3.15(c)) following the perioddoubled solutions (Fig. 3.15 (a, b)), which was not reported by Johnson and Bajaj (1989) and Bajaj and Johnson (1992). The chaotic attractor is transformed into a distinct limit
cycle attractor as the detuning increases further (Fig. 3.15(d)). This limit cycle has a new geometry and undergoes period doubling bifurcation and results into chaotic attractors (Fig. 3.15(i)), showing a transition to the isolated branch. For further larger excitation frequencies, the bifurcation has the same property as those of previous cases.


Fig. 3.15: Bifurcation and Crisis of the isolated branch for the case with $\mu=0.80 \mu^{*}$ and $f=f^{*}$ : (a) $\Delta=0.092$; (b) $\Delta=0.0925$, (c) $\Delta=0.0950$, (d) $\Delta=0.0970$, (e) $\Delta=0.0980$, (f) $\Delta=0.0985$, (g) $\Delta=0.0995$, (h) $\Delta=0.1010$, (i) $\Delta=0.1020$, and (j) $\Delta=0.1040$.


Fig. 3.15 (Continued)


Fig. 3.15 (Continued)


Fig. 3.15 (Continued)

### 3.4.2 Characteristics of the Responses and the Hysteresis

Here we summarize the characteristics of dynamic responses of strings by considering a typical case. The responses with $\mu=0.85 \mu^{*}$ and $f=f^{*}$ (see Figs. 3.2(i)) have all types of solutions we discussed. Sweeping the frequency monotonically, increasingly or decreasingly, we have the following observations:

As shown in Fig. 3.2(i), when the excitation frequency $\Omega$ is far below the natural frequency (i.e., $\Delta=0$ ) or the pitchfork bifurcation point (i.e., $\Delta=\Delta_{1}>0$ ), the amplitude
of the planar response exhibits a steady increase with the driving frequency. For $\Delta_{1}<\Delta<\Delta_{4}$, the increase rate of the in-plane amplitude with $\Omega$ is lower than that for $\Delta<\Delta_{1}$. The amplitude of out-of-plane vibration (perpendicular to the plane of the driving force), however, has a sharp rise as the driving frequency increases. Depending on the damping and excitation, the amplitude of out-of-plane response may reach a maximum value which is less than but pretty close to the in-plane vibration amplitude at $\Delta=\Delta_{4}$. The vibration around $\Delta=\Delta_{4}$ for this part is whirling (or tubular or ballooning) and can be easily observed in experiments. The beginning frequency of the modulated motions, $\Delta_{1}^{*}$, may be either equal to or less than $\Delta_{4}$, depending on the damping value. When the damping is small, $\Delta_{1}^{*}<\Delta_{4}$ and there are multiple non-planar solutions. One is stable and periodic and the other is modulated, and there is an unstable non-planar branch connecting $\Delta_{1}^{*}$ and $\Delta_{4}$. Lower damping divides the frequency range between $\Delta_{1}^{*}$ and $\Delta_{4}$ into two intervals having different responses. The first one is a non-planar unstable branch $\left(\Delta_{3}, \Delta_{4}\right)$ and the following one $\left(\Delta_{3}, \Delta_{1}^{*}\right)$, usually a pretty small interval, is a nonplanar stable one. $\Delta_{3}$ is a local minimum detuning for the sectional Z-shape in-plane curve. Figs. 3.3 (b) and (c) are the zoomed plots of the planar and non-planar response curves, respectively. For the interval of modulated motions, the response increases as the detuning increases. The detuning $\Delta_{2}^{*}$ ends the modulated branch, and the modulated solution gains stability and becomes periodic. The reverse pitchfork bifurcation happens at $\Delta_{5}$, where the unstable planar branch and the stable non-planar branch (between $\Delta_{2}^{*}$ and $\Delta_{5}$ ) merge. This situation is clearer in the zoomed plot shown in Fig. 3.3
(a) for the case with $\mu=1.00 \mu^{*}$ and $f=f^{*}$. Increasing the excitation frequency beyond $\Delta_{5}$, the response again becomes stable planar vibration. At $\Delta_{2}^{*}$, there is a collapse (jump) of the planar response, and the solution suddenly loses its stability. In other words, a small change of the control parameter causes the trajectory to go out of the immediate neighborhood. Then stable planar response of small amplitude is left for larger detuning.


Fig. 3.16: The hysteresis phenomena of the case with $\mu=\mu^{*}$ and $f=1.2 f^{*}$.

If the excitation frequency is monotonically decreased, the path the response follows is much simpler. The amplitude of the response monotonically increases slowly along the stable planar branch (the lower one) until the frequency reaches $\Delta_{2}$, where the jump phenomenon happens again following a saddle-node bifurcation. Between the
saddle node bifurcation points $\Delta_{2}$ and $\Delta_{6}$, the bi-stability phenomenon (two stable attractors separated or connected by unstable attractors) results in a hysteresis loop as the parameter (the frequency detuning $\Delta$ in this case) varies. Fig. 3.16 is a frequency response curve with $\mu=1.2 \mu^{*}$ and $f=f^{*}$ (Fig. 3.2(b)), where arrows indicate the change of the response when the detuning is increasingly and decreasingly swept. The response is simple but enough to show the so called hysteresis phenomenon. Apparently, the hysteresis shows the lack of reversibility as the parameter is varied. If the response (amplitude) increases when the excitation frequency increases, it is called a hardening hysteresis and, reversely, it is a softening type.

### 3.4.3 The Effect of Excitation

The excitation amplitude and damping are two similar parameters as far as the change of vibration characteristics of the response to the change of parameter is considered. There are similar detuning intervals in which similar responses can be obtained by varying the forcing or the damping. Let's consider the case that the damping is fixed to be $\mu=\mu^{*}=0.03411 /$ sec and other parameters are kept the same as before. Varying the forcing amplitude $f$, we have following observations for the response. For a small enough excitation amplitude, the steady-state response is periodic and planar. Fig. 3.17(a) is the response curve for the case with $f=0.6 f^{*}$. Increasing the forcing amplitude, the frequency response curves have the properties shown in Fig. 3.17(b) for the case with $f=0.75 f^{*}$. Out-of-plane vibration appears at the forward pitchfork bifurcation point $\Delta_{1}$ and it disappears at the reverse pitchfork bifurcation point $\Delta_{5}$.

Increasing the excitation amplitude further, we observe a Hopf bifurcation that results in amplitude and phase-modulated motions. For the detuning interval $\left(\Delta_{1}, \Delta_{5}\right)$, pitchfork, Hopf, reverse Hopf and reverse pitchfork bifurcation appear in succession at $\Delta_{1}, \Delta_{1}^{*}$, $\Delta_{2}^{*}$ and $\Delta_{5}$. Fig. 3.17(c) shows the response with $f=1.4 f^{*}$. It has response similar to the case with $\mu=0.85 \mu^{*}$ and $f=f^{*}$ (see Fig. 3.2(i)).


Fig. 3.17: The in-plane (a1) and out-of-plane (a2) frequency response curves under $\mu=\mu^{*}$ and values of different $f$ : (a) $f=0.6 f^{*}$, (b) $f=0.75 f^{*}$, and (c) $f=1.4 f^{*}$. Solid line: stable planar solutions; dash-dot-dash line: stable non-planar solutions; dashed line: unstable solutions; dots: modulated solutions. $\Delta_{1}$ : the forward pitchfork bifurcation; $\Delta_{5}$ : the reverse pitchfork bifurcation point; $\Delta_{2} \& \Delta_{6}$ and $\Delta_{3} \& \Delta_{4}$ : the tuning points of the planar and non-planar solution branches, respectively. $\Delta_{1}^{*}$ and $\Delta_{2}^{*}$ : the forward and reverse Hopf bifurcation points, respectively.


Fig. 3.17 (Continued)

### 3.5. Conclusions and Discussions

Periodic, non-periodic and planar and non-planar responses of strings subjected to external periodic excitations were investigated. The stability was determined by the eigenvalues of the Jocobian matrix of the modulation equations obtained using the method of multiple scales. The rich responses and bifurcations of the modulated branch within the detuning interval $\left(\Delta_{1}^{*}, \Delta_{2}^{*}\right)$ were investigated in detail. In addition to the results
obtained by Johnson and Bajaj (1989), Bajaj and Johnson (1992), O’Reilly (1990), Timothy (1994), and Timothy et al. (2004), some new phenomena were observed and a more detailed study of bifurcations was performed. Next, we summarized all the results we obtained as well as those in the literature.

For a small damping, the modulated branch is continuous and the solution has a single modulation period. For a smaller damping, the modulated solution undergoes period doubling bifurcation. We draw the bifurcation diagrams in Fig. 3.18 by plotting the maximum and minimum values of the modulated amplitudes of in-plane responses. Fig. 3.18(a) corresponds to the case with $\mu=1.015 \mu^{*}$ and $f=f^{*}$ (Fig. 3.2(d)). It is shown that as the detuning increases, the modulated vibration undergoes forward period doubling and then shrinks back to a single-period attractor after the midpoint of the interval. The representative trajectory plots are presented in Fig. 3.6. Fig. 3.19 (a) is the qualitative bifurcation diagram of the modulated solution with $\mu=1.015 \mu^{*}$.

For a smaller damping $\mu=1.010 \mu^{*}$, an isolated branch appears due to saddle node bifurcation around the midpoint of the modulated branch. For a damping at this level, the Hopf branch undergoes period doubling bifurcation while the isolated branch keeps periodic. The curves of the maximum and minimum of $a_{1}$ of the isolated branch are smooth, indicating no sudden change of amplitude due to bifurcation or new creation of isolated branches. The bifurcation diagram of a representative case is shown in Fig.3.18 (b) for the case with $\mu=1.010 \mu^{*}$ and $f=f^{*}$. The bifurcation of the Hopf branch has the sequence $H P(1-2-4-8-4-2-4-2-1)$, whose phase plane plots are presented in Fig. 3.8. The qualitative bifurcation diagram is presented in Fig. 3.19(b).

Fig.3.18(c) shows the bifurcation by plotting the maximum and minimum inplane amplitude for both the Hopf branch and the isolated branch for the case with $\mu=1.00 \mu^{*}$ and $f=f^{*}$. In this case, the stable part of the isolated branch undergoes a cascade of period doubling which leads to the Rossler type attractor, i.e., trajectories encircle only one unstable non-planar fixed point. The Hopf branch undergoes period doubling bifurcation with the sequence $H P(1-2-4-8-4-2-1-2-4-2-1)$ for the whole modulated branch. We did not present the phase plot for this case because the trajectories are almost the same as those of previous cases except there is one HP1 solution. Examining the bifurcation diagram in Fig. 3.18(c), we see that the bifurcation is not symmetric, because the period doubling bifurcation in the second half interval does not bifurcate into HP8 as in the first half. Moreover, we see that the bifurcation shrinks back to the $H P 1$ instead of the $H P 2$ solution. The nonlinear curve of the isolated branch shown in Fig. 3.18(c) tells us there is period doubling bifurcation happened. The phase plots of the bifurcation are presented in Fig. 3. 11. We see there are more than one Rossler type chaotic attractors and there are periodic limit cycles between them. Johnson and Bajaj (1989) and Bajaj, and Johnson (1992) stated that these are newly created isolated branches whose unstable part are merged with the stable part of previous one by saddle node bifurcation, exactly the same mechanism by which the first isolated branch is created and merged with the Hopf branch. Moreover, they stated that the newly created branch seems to have a stabilizing effect on the previous branch. For example, the second isolated branch forces the Rossler type chaotic attractor arising from the first isolated branch to reversely bifurcate back to the periodic solution. Consequently, new attractors are created and they merge with previous ones, followed by reverse scenes in the
modulation branch. The stabilizing or constraint effect basically is due to the attraction of oscillators of the isolated branches. Being stabilized, reverse bifurcation or even disconnection of a branch is along with the increase of the attraction. Fig. 3.19(c) presents the qualitative bifurcation diagram of this situation.

For a smaller damping $\mu=0.98 \mu^{*}$, the Hopf branch becomes discontinuous. There is no stable limit cycle of Hopf type in the disconnected frequency range. Integrations for all detuning value in this interval lead to stable limit cycles of the isolated branch. So, for the frequency interval where the Hopf branch is discontinuous, there are two attractors. One is the limit cycle of the isolated branch and the other is the lower planar one. The representative bifurcation diagram is presented in Fig. 3.18(d) for the case with $\mu=0.98 \mu^{*}$ and $f=f^{*}$, corresponding to the phase plane plots shown in Fig. 3. 12. There is a corresponding unstable part of the isolated branch which will merge with the stable Hopf branch when it becomes disconnected, creating a saddle node bifurcation. Fig. 3.19(d) presents the qualitative bifurcation diagram describing these situations.

For a even smaller damping $\mu=0.96 \mu^{*}$, as the detuning increases, a cascade of attractors of isolated branches appears via saddle-node bifurcation. The unstable part merges with the stable part of previous isolated branch in exactly the same manner as the first isolated branch is created and the unstable isolated branch is merged with the stable Hopf branch. This sequence ends with the creation of a homoclinic orbit, a trajectory that asymptotically approaches a saddle type fixed points as $t \rightarrow+\infty$ and $t \rightarrow-\infty$. And the fixed point has eigen-values which satisfy Shilnikov's inequality. Lorenz type chaotic attractors are found in the neighborhood of this homoclinic orbit. Then the isolated branch will disappear following the Lorenz type attractor being destroyed by the so called
boundary crisis, the sudden disappearance of the attractor. For the decreasing sweeping of detuning, there is another boundary crisis for detuning at the second half of the interval. For the detuning values between the two frequency ranges where the boundary crisis happens, there is only one stable solution, the lower planar solution. Fig. 3.18(e) shows the bifurcation for the case with $\mu=0.96 \mu^{*}$ and $f=f^{*}$ and Fig. 3.14 shows the corresponding phase plane plots. No limit cycle solutions are obtained by numerical integration for detuning between the two boundary crises. Fig. 3.19(e) presents the qualitative bifurcation diagram of this situation.

Our numerical investigation for further smaller damping values, making the system close to a Hamiltonian one, shows similar bifurcation structures. However, for a small enough damping, the Hopf branch does bifurcate into chaotic attractors before it is merged with the isolated branch, and the trajectories are highly twisted, as shown in Fig. 3.15 for the case with $\mu=0.80 \mu^{*}$ and $f=f^{*}$.





Fig. 3.18: The bifurcation diagrams of the modulated solutions: $\mu=1.010 \mu^{*}$ and $f=f^{*}$, the appearance of the isolated branch (dashed line) ; (c) $\mu=1.00 \mu^{*}$ and $f=f^{*}$; (d) $\mu=0.98 \mu^{*}$ and $f=f^{*}$; and (e) $\mu=0.96 \mu^{*}$ and $f=f^{*}$, the appearance of boundary crisis. Circled points correspond to detunings $\Delta$ where period doubling bifurcations of the Hopf branch happen. The bifurcations of the isolated branch are not clear.


Stable solution --- Unstable solution


Stable solution --- Unstable solution


Fig. 3.19: Qualitative bifurcation diagrams of the modulated solutions: (a) $\mu=1.015 \mu^{*}$ and $f=f^{*}$; (b) $\mu=1.010 \mu^{*}$ and $f=f^{*}$ the appearance of the isolated branch; (c) $\mu=1.00 \mu^{*}$ and $f=f^{*}$; (d) $\mu=0.98 \mu^{*}$ and $f=f^{*}$; and (e) $\mu=0.96 \mu^{*}$ and $f=f^{*}$, the appearance of boundary crisis.

## CHAPTER 4

# EXPERIMENTAL DYNAMIC CHARACTERISTICS OF STRINGS 

In this chapter we describe in detail the usage of a 3D motion analysis system to characterize nonlinear dynamics of strings. We begin with a detailed description of the experimental set-up and the data acquisition devices. Then the experimental procedure is presented and discussed. At the end, frequency responses curves are obtained for the vibrations of three strings which have different but small sag-to-span ratios. The string with richer responses are carefully studied by focusing on the modal analysis of various typical responses.

### 4.1 The 3D Motion Analysis System

The lightweight nature of highly flexible structures (HFSs) precludes the attachment of measurement sensors (e.g., accelerometers) to the structures because as the mass of sensors may significantly influence the static and dynamic structural properties. So, a non-contacting vibration measurement device is needed for experimental investigation of mechanics of HFSs. Fig. 4.1(a) shows a typical set-up of an Eagle-500 digital real-time motion analysis system for non-contact measurement of large static and/or dynamic deformations of HFSs. This system uses several (6 in our study) high-
resolution CMOS (complementary metal-oxide-semiconductor) cameras to capture images of a structure when the visible red LED


Fig. 4.1: EAGLE-500 digital real-time motion analysis system: (a) typical set-up of a the system, (b) an Eagle digital camera, and (c) the Eagle hub.
strobes light up the retro-reflective markers stuck on the structure. Using triangulation techniques and the known focal lengths (after calibrations using an L-frame with four markers and a T-wand with three markers) of the cameras and the known coordinates of the bright points (caused by the retro-reflective markers) on the 2D images inside the cameras, the Eagle real-time software EVaRT 4.1 automatically computes and records the instant 3D coordinates of the center of each retro-reflective marker that is seen by at least two cameras. Hence, 3D time traces of all markers are available for performing dynamic animation using stick figures and showing pop-up figures of displacements, velocities and accelerations, and they can be output to other programs for signal processing. The recording time length is effectively infinite and up to 600 markers can be
simultaneously traced because of the use of a large computer memory and a 100Mbit data upload rate. Because the 3D coordinates of each marker are checked and calibrated when more than two cameras see the marker, the measurement accuracy is pretty high. For example, the measurement error is far less than 1.0 mm when the measurement volume is $2 \times 2 \times 2 m^{3}$. The system's measurement power, ease of operation, simply to set up, and extreme accuracy have made the system a new standard for motion capture.

### 4.1.1 Eagle Digital Camera

The Eagle digital camera shown in Fig. 4.1(b) has a resolution of 1.3 million pixels. It runs with the $1280 \times 1024$ full resolution up to 480 frames per second (FPS), with a $1280 \times 512$ resolution at 1000 FPS, and with a $1280 \times 256$ resolution at 2000 FPS. And it has a processing rate of 600 million pixels per second. The camera revolutionizes the motion capture industry with its extreme resolution, unprecedented high frame rate, upgradeable functionality, and ease of use. Signals from an Eagle digital camera go directly to the tracking computer via an Ethernet connection. The signal processing is embedded in the camera. This streamlined system of motion capture from camera to computer means less hardware and less potential for equipment problems. The FPGA (Field Programmable Gate Array) built into the Eagle is software and firmware upgradeable via the internet. The features of Eagle cameras are listed below:

- $1-2000 \mathrm{~Hz}$ selectable frame rates
- Built-in zoom provides more visual options for ease of set-up
- High quality 35 mm lenses for low optical distortion
- Separate zoom, iris and focus settings independent of ring light
- Available with visible red, near red, or infrared ring lights
- LED display panel for camera identification and status
- 237 LED's for brighter and better light uniformity
- Strobe ringlight with camera body heat sink

Camera placement is the most important aspect of setting up the motion capture volume. If properly done, good camera placement will be rewarded with highly accurate results and greatly reduced editing time. There are several factors to be considered when deciding the number of cameras to be used.

1) There should be sufficient number of cameras to insure that, at all times, all markers will be visible by at least two cameras, and preferably the more the better. In general, the number of cameras must be increased when the motion of the subject becomes less restrained and the capture volume increases
2) As more cameras are used, each camera should view only a portion of the capture volume to achieve higher accuracy, and care should be taken to prevent too many cameras seeing a marker. The only requirement is that all 4 markers on the calibration square should be visible to at least $1 / 2$ of the cameras used. It is noted that, when more than 5 or 6 cameras see the same marker, the accuracy of tracking is not increased but the computation time increases.
3) Camera views should not include areas outside the capture volume to ensure the highest possible spatial resolution.

The motion analysis system uses a dynamic linearization technique that is currently available and capable of producing precise and accurate calibration. First, an L-
frame with four markers is used for defining the XYZ axes. A 500mm wand (for large capture volumes) or a 150 mm wand (for small capture volumes) is then used for establishing camera linearization parameters.

### 4.1.2 Eagle Hub

The Eagle hub shown in Fig. 4.1(c) consists of multi-port Ethernet switch (100 Mbps ) and provides power for the cameras. A single Ethernet Cat 5 cable is used for all signals and power between a camera and the Eagle hub.

### 4.1.3 EVaRT

The EVa real-time software (EVaRT) provides the user with a simple and powerful interface. Under a single software environment the user can set up, calibrate, capture motion in real-time, capture motion for post processing, and edit and save data in a chosen format.

### 4.1.4 Triangulation Measurement

The principle of triangulation measurement using two cameras is based on photogrammetry, as shown in Fig, 4.2. Here $(x, y, z)$ are the coordinates of the measurement point with respect to the xyz coordinate system defined by the four markers on the Lframe, $\left(x_{c}, y_{c}, z_{c}\right)$ are the coordinates of the lens center, $(\xi, \eta, \zeta)$ are the coordinates of the measurement point with respect to the $\xi \eta \zeta$ coordinate system defined by the sensor plane (i.e., the $\eta \zeta$ plane) and the optical axis (i.e., the $\xi$ axis), $f$ is the focus length of the
camera, and $v$ and $w$ are the image plane coordinates of the measurement point image on the image plane inside the camera. Here $v$ and $w$ represent corrected image plane


Fig. 4.2 Triangulation measurement using two cameras coordinates obtained by using, for example, the following distortion model to correct perspective and optical distortion:

$$
\begin{gather*}
\Delta v=v r^{2} K_{1}+v r^{4} K_{2}+v r^{6} K_{3}+\left(r^{2}+2 v^{2}\right) P_{1}+2 v w P_{2} \\
\Delta w=w r^{2} K_{1}+w r^{4} K_{2}+w r^{6} K_{3}+\left(r^{2}+2 w^{2}\right) P_{1}+2 v w P_{1}  \tag{4.1.1}\\
r \equiv \sqrt{v^{2}+w^{2}}
\end{gather*}
$$

where the point of symmetry for distortion has already been subtracted from $v$ and $w$. If the image plane is perpendicular to the optical axis, the point of symmetry and the photogrammetric principle point coincide. $K_{i}$ and $P_{j}$ are camera parameters necessary for conversion from pixels to the corrected image plane coordinates (i.e., $v$ and $w$ ), and they need to be determined by performing a calibration test. Typically, the third-order radial distortion $K_{1}$ is the dominant term. In addition, the asymmetrical terms $P_{1}$ and $P_{2}$ are small and projectively coupled to the point of symmetry and external orientation of the camera.

We let $\boldsymbol{i}_{x}, \boldsymbol{i}_{y}$ and $\boldsymbol{i}_{z}$ denote the unit vectors of the xyz coordinate system and $\boldsymbol{i}_{1}, \boldsymbol{i}_{2}$ and $\boldsymbol{i}_{3}$ denote the unit vectors of the $\xi \eta \zeta$ coordinate system. Then the two coordinate system are related by a transformation matrix $[T]$ as

$$
\left\{\begin{array}{l}
\mathrm{i}_{1}  \tag{4.1.2}\\
\mathrm{i}_{2} \\
\mathrm{i}_{3}
\end{array}\right\}=[T]\left\{\begin{array}{l}
\mathrm{i}_{x} \\
\mathrm{i}_{y} \\
\mathrm{i}_{z}
\end{array}\right\},\left\{\begin{array}{l}
\xi \\
\eta \\
\zeta
\end{array}\right\}=[T]\left\{\begin{array}{l}
x-x_{c} \\
y-y_{c} \\
z-z_{c}
\end{array}\right\}
$$

If the pointing direction of the optical axis $\xi$ is determined by three consecutive Euler angles $\alpha, \beta$ and $\gamma$ with respect to the axes $x, y$ and $z$ respectively, we have

$$
\begin{align*}
& {[T]=} {\left[\begin{array}{ccc}
\cos \gamma & \sin \gamma & 0 \\
-\sin \gamma & \cos \gamma & 0 \\
0 & 0 & 1
\end{array}\right]\left[\begin{array}{ccc}
\cos \beta & 0 & -\sin \beta \\
0 & 1 & 0 \\
\sin \beta & 0 & \cos \beta
\end{array}\right]\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \alpha & \sin \alpha \\
0 & -\sin \alpha & \cos \alpha
\end{array}\right] } \\
&= {\left[\begin{array}{cc}
\cos \beta \cos \gamma & \cos \alpha \sin \gamma+\sin \alpha \sin \beta \cos \gamma \\
-\cos \beta \sin \gamma & \cos \alpha \cos \gamma-\sin \alpha \sin \beta \sin \gamma \\
\sin \beta & -\sin \alpha \cos \beta
\end{array}\right.}  \tag{4.1.3}\\
&\left.\quad \begin{array}{c}
\sin \alpha \sin \gamma-\cos \alpha \sin \beta \cos \gamma \\
\sin \alpha \cos \gamma+\cos \alpha \sin \beta \sin \gamma \\
\cos \alpha \cos \beta
\end{array}\right]
\end{align*}
$$

The pointing direction of the optical axis $\xi$ can also be determined by two Euler angles $\alpha$ and $\phi$ (Nayfeh and Pai, 2004).

From the similarity of two triangles and (4.1.2) we obtain the following collinearity equations:

$$
\begin{align*}
& v=f \frac{\eta}{\xi}=f \frac{T_{21}\left(x-x_{c}\right)+T_{22}\left(y-y_{c}\right)+T_{23}\left(z-z_{c}\right)}{T_{11}\left(x-x_{c}\right)+T_{12}\left(y-y_{c}\right)+T_{13}\left(z-z_{c}\right)}  \tag{4.1.4}\\
& w=f \frac{\zeta}{\xi}=f \frac{T_{31}\left(x-x_{c}\right)+T_{32}\left(y-y_{c}\right)+T_{33}\left(z-z_{c}\right)}{T_{11}\left(x-x_{c}\right)+T_{12}\left(y-y_{c}\right)+T_{13}\left(z-z_{c}\right)}
\end{align*}
$$

To determine the eight (or seven) unknowns

$$
\begin{gather*}
f, x_{c}, y_{c}, z_{c}, T_{11}, T_{12}, T_{13}, \phi \\
\left(f, x_{c}, y_{c}, z_{c}, \alpha, \beta, \gamma\right) \tag{4.1.5}
\end{gather*}
$$

of each camera before the motion analysis system can be used for actual measurements one needs to perform measurements using the L-frame with four markers having known coordinates:

Marker \#0: $(x, y, z)=(0,0,0)$
Marker \#1: $(x, y, z)=\left(x_{1}, 0,0\right)$
Marker \#2: $(x, y, z)=\left(x_{2}, 0,0\right)$
Marker \#3: $(x, y, z)=\left(0, y_{3}, 0\right)$
Substituting (4.1.6) and the measured image plane coordinates $v_{i}$ and $w_{i}(i=0,1,2,3)$ of the four markers into (4.1.4) yields eight nonlinear algebraic equations that can be solved for the eight unknowns of each camera by iteration.

After the calibration procedure, the eight parameters of each camera are known. To determine the 3D coordinates of an arbitrary marker seen by two cameras one can substitute the eight camera parameters of each camera and the image plane coordinates $v$ and $w$ measured by each of the two cameras into (4.1.4) to obtain four linear algebraic equations in the three unknowns $x, y$ and $z$. One can perform the pseudo inverse of the $4 \times 3$ constant matrix to obtain the three unknowns, which is equivalent to the linear leastsquares curve-fitting processing.

The Eagle digital system provides simultaneous viewing of up to four different panels, including 3D display of different views and angles, 2D display of digital grayscale and threshold images, color video display (avi), XYZ graphs, and analog graphs.

### 4.1.5 Experimental Setup



Fig. 4.3: The experimental setup.
Figure 4.3 shows the experimental setup for measuring vibration of a string subjected to harmonic base excitations provided by a shaker. Due to the small mass of the string, it can be mounted either horizontally to avoid the effect of gravity on the tension or vertically to avoid the effect of gravity on the initial curvature. The string was clamped using set-screws to the two rigid supports $A$ and $B$ which were horizontally apart by 1.475 m and were mounted on heavy weights to ensure mechanical isolation. The shaker head was attached to the string at $x=0.05 m$ from the left support. Forty-eight $4 \mathrm{~mm} \times 15 \mathrm{~mm}$ rectangle retro-reflective markers were adhered by winding to the string and were distributed equally between the shaker and the other fixed support. Markers \#1 and \#48 were put close to the end points. The motion analysis system shown in Fig. 4.1 makes it possible to record the dynamic response of the whole string instead of just measuring one point using the experiment set-ups used by Nayfeh, Nayfeh, and Mook(1995), O’Reilly (1990), and Molteno et al. (2004).

In our experiments, only six Eagle cameras were used and placed at different locations, approximately symmetric with respect to the string with three on each side.

The location of camera were determined to make sure that the cameras had good view angles and at least 2 cameras could see all the 48 markers on the vibrating string all the time. The capture volume was set to be $1.5 \times 1.5 \times 1.5 \mathrm{~m}^{3}$. Then the calibration of the system was done using an L -frame and then a T -wand to define the coordinate system XYZ .

The string was excited using a Ling Dynamics LDS V408 shaker, which had a maximum output force of 196 N and a frequency range of 5 to 9000 Hz , to supply the oscillatory boundary condition at one of the two ends. The shaker-head motion was monitored by means of an accelerometer mounted to the shaker head. The axis of the shaker was vertical and perpendicular to the line connecting the two end markers of the string so that the string was only transversely excited. The excitation was controlled by using a PCB J35CB03 shear ICP accelerometer attached to the head of the shaker for feed back. The accelerometer fed back the base motion to a DSC4-CE shaker controller that modified the AC voltage accordingly to keep the base motion harmonic. The EVaRT software in the computer system controlled the entire measuring system. It tracked and identified the markers, and provided the XYZ coordinates of all the markers at consecutive instants of time. The data were exported in the ASCII format and then were processed using our post process codes written in MATLAB.

### 4.2 Experimental Procedure

We performed temporal and spatial-domain data acquisitions of the vibrating string excited at a chosen frequency for each test. The coordinates of all the 48 markers on the string were recorded. Because the nonlinear vibration may depend on initial
conditions, the tests were carried out by continuously varying, either increasing and then decreasing, or decreasing and then increasing the excitation parameter (either the excitation frequency or the amplitude) through the desired range of the parameter. After the data were collected by continuously increasing the parameter value, a second set of data were collected by continuously decreasing the parameter value. However, such kind of sweeping tests may mistake nonlinear transient vibration as a steady-state one, and only one type of vibration for each parameter value can be obtained. In our experiments, we stopped each test before changing the parameter (frequency or amplitude) for the next test. Each test started with zero initial conditions, and hence the tests were direction independent of the changing direction of the parameter. After one steady state was recorded, we disturbed the vibration and then obtained other possible steady state solutions. By this way we simulated vibrations starting with different initial conditions and hopefully would not miss possible multiple solutions.

First of all, we did tests by varying the excitation frequency while the excitation amplitude was held constant. For every parameter change, either increasing or decreasing, it was necessary to make sure the transient motion died out and so the steadystate was reached before the response was captured. In general, a steady state was attained in one to two minutes after the excitation reached the specified amplitude. Normally, it takes more time for the response to reach a steady state for tests with parameters around where multiple solutions exist. For example, it took more time to reach a steady state around where the planar response branch and the non-planar branch were connected. Theoretically, the step size of increase or decrease of the parameter depends on the closeness of the excitation frequency to the bifurcation frequencies. When
it is close to the resonant frequency, the increase step needs to be small because various nonlinear phenomena exist in a small frequency range. In our experiments, we increased the excitation frequency by 1 Hz or more when the frequency was far from the resonant frequency and 0.5 Hz or even smaller when it was close to the resonant frequency in order to observe possible important phenomena like period-doubling, quasi-periodic, and chaotic motions. Actually, appropriate increase or decrease of the step size practically depends on experience. Usually, a rough first screening will give us information about an appropriate step size. The total number of excitation frequency increments and the number of excitation amplitude increments depend on the frequency range we want to investigate and also the capacity of our experimental apparatus.

### 4.3 Experimental Parameters

Our experiments were performed on a 1.425 m (effective length) steel wire whose diameter is $0.4572 \mathrm{~mm}(0.018 \mathrm{in})$. The string was stretched with a pretension $T$, which was provided by attaching weights to the string through a pulley before the string is fixed. Due to friction, the true tension in the string was smaller than that was provided by the weights. Although not accurate, it should be around the value with a deviation less than $1 l b$ as observed. The tension needs to be low to ensure that no yielding occurs in the string but needs to be high enough so that the small bending stiffness will not affect the vibration much. Also, the maximum vibratory strain, which can be determined by the maximum vibrating amplitude, in the string can not be too large to be out of the linearly elastic range of the material. As steel wires were used and only low-frequency vibrations
were tested in our experiments, we did not have above problems. For steel strings, the longitudinal wave speed $c_{u}$ is much higher than the transverse ones $c_{v}$ and $c_{w}$. This was assumed in the theoretical analysis.

To reduce the inertia effect of the retro-reflective markers on the string vibration, the diameter of the markers is necessary to be as small as possible but visible to the cameras. A smaller marker diameter results in more accurate experimental results. Large markers cause image distortion and reduce the accuracy of the measured center locations (i.e., the measured or recorded center position of each marker by averaging different views from cameras). In our experiments, the diameter of the markers is about 1.5 mm .

As far as the dynamic parameters are concerned, first of all, the natural frequencies need to be determined in order to have a general idea about the dynamics of the string. The theoretical method of using the material stiffness, mass density and crosssectional geometry of string to determine the natural frequencies is easy but can be used for reference only because of the inaccuracy caused nonlinearities and uncertain pretension force. Experimental approaches were preferred. There are different methods to determine the natural frequencies, including swept-sine or stepped-sine testing, white noise and chirp signal excitation, etc. Using different techniques to check the results from different method makes the obtained system parameter values more reliable. In the swept sine method, the forcing amplitude is fixed and the forcing frequency is varied slowly but continuously through the range of interest - a narrow band around the interest natural frequency. The sweeping needs to be sufficiently slow to ensure a steady state is attained. An excessive sweep rate leads to distortion of the frequency response curve. More accurate results can be obtained by doing the measurement twice, sweeping down and up,
and trying to make these two results close. Then, the swept-sine spectral analysis can be applied to get the frequency response curve. We were not able to do swept sine tests because of the incapability of our experimental device. Different from the swept-sine method, stepped-sine testing does tests at discrete frequencies. In addition to that the density of points needs to be appropriate. It is necessary to ensure that steady-sate conditions have been attained before the measurements are made. In the low-amplitude white noise (random noise) method, the input to the system is broad band noise and the output is the measured (vertical) displacement. The process needs to repeat several times and an average frequency response function is obtained from the FFT analysis. Note that swept-sine testing, stepped-sine testing, and white noise method are intended for linear systems with natural frequencies independent of the input forcing frequencies and amplitudes. So, a small input level should be kept during this process so that the response remains in an essentially linear regime. Using a chirp signal, the frequency contents can be precisely chosen between the starting and finishing frequencies of the fast sine sweep. The frequency resolution is determined by the number of FFT lines and the sweep range. Usually, the spatially (all different positions) averaged frequency response function is used to identify natural frequencies and mode shapes.

The fundamental frequency of transverse vibration can be determined by FFT analysis of the decaying free oscillation of the string disturbed by a small initial transverse displacement. We plucked the string at the middle point and hence the initial shape of the string was a triangle consisting of many harmonics. As the high frequency harmonics died out quickly and only the primary harmonic was left for a longer time, the fundamental frequency was obtained by analyzing the later part of the time domain data.

Due to the initial curvature, possible uneven gravity distribution, unrealistic pluck by a finger and other reasons, the string was observed to undergo non-planar vibration. In our experiments, we plucked the string many times until an approximately pure planar vibration was obtained and captured. The recorded data we used for the determination of the primary frequency and damping ratio is shown in Fig.4.4. We see the decay of the vibration is smooth and the vibration is mostly planar. After the fundamental frequency is obtained, other natural frequencies can be roughly deduced by using the linear string vibration theory, the tension force, and the mass density.


Fig. 4.4: The experimental data used for damping calculation.
The longitudinal wave speed can be determined by detecting the longitudinal motions of the markers. As suggested by Nayfeh, Nayfeh, and Mook (1995), a piece of tape can be attached to the string so that the longitudinal motion can be determined. However, the longitudinal wave speed of the steel wires, $4000 \sim 6000 \mathrm{~m} / \mathrm{s}$, is very high and beyond the measurement capability of the motion analysis system. By assuming two frames to be captured when the elastic wave starts at one end and arrives at the other end, the maximum wave velocity that can be measured by the motion analysis system, $\bar{c}_{u}$, is

$$
\bar{c}_{u}=L /(1 / 2000 \mathrm{sec})=1.425 \times 2000 \mathrm{~m} / \mathrm{s}
$$

which is lower than the wave speed of the steel wires. Here $L$ is the length of the string. Hence this approach is just theoretically possible but practically impossible for our studies. On the other hand, the pulse-echo technique is a simple, quick, and accurate method for measuring the speed of waves in solids. It sends an electric or ultrasonic pulse produced by instant connection of a circuit or ultrasound transducer and receives the echo of the wave pulse. The longitudinal wave speed then can be calculated as:

$$
c_{u}=\frac{2 L}{\Delta t}
$$

where $2 L$ is the wave traveling distance and $\Delta t$ is the time from sending to receiving the pulse wave.

The damping ratio $\zeta$ was determined using the logarithmic decrement method (Meirovitch, 1986). The logarithmic decrement $\delta_{i j}$ between the $i$ th and $j$ th peaks of the time trance (e.g., Fig. 4.4) is given by

$$
\delta_{i j}=\frac{1}{j-i} \ln \frac{x_{i}}{x_{j}}
$$

Several $\delta_{i j}$ with different values of $j-i$ need to be calculated and an averaged logarithmic decrement $\delta$ is adopted, and then the damping ratio $\zeta$ is determined as

$$
\zeta=\frac{\delta}{\sqrt{(2 \pi)^{2}+\delta^{2}}} \approx \frac{\delta}{2 \pi}
$$

The system parameters identified using above methods for our experiments are listed in following table:

Table 4.1 Parameters of the Steel Wire

| Parameter | Symbol | Measured/calculated Values |
| :---: | :---: | :---: |
| Damping ratio | $\zeta$ (assuming $\zeta_{v}=\zeta_{w}=\zeta$ ) | $0.005 \sim 0.01$ |
| Length | $L$ | 1.474 m |
| Diameter | $d$ | 0.4572 mm |
| The 1st natural frequency | $\omega_{1}$ | $17.5 \mathrm{~Hz}, 25 \mathrm{~Hz}, 17.5 \mathrm{~Hz}$ for |
| the $1^{\text {st }}, 2^{\text {nd }}$, and 3 ${ }^{\text {rd }}$ cases |  |  |
| Tension force | $T_{0}$ | $\approx 2 \mathrm{lbf}$ |
| Young's modulus | $E$ | $200 \times 10^{9} \mathrm{~N} / \mathrm{mm}^{2}$ |
| Mass per unit length | $m$ | $0.0013 \mathrm{~kg} / \mathrm{m}$ |

### 4.4 Frequency Response

The excitation amplitude of the shaker was fixed to be 1 mm for frequency sweeping. The in-plane and out-of-plane response amplitudes of each marker were the lengths of the major and minor axes of the elliptic trajectories obtained by curve fitting. The frequency response curve for a marker was obtained by plotting the excitation frequency with respect to the response amplitude of the chosen marker. Although the response curves of different markers are different due to their different positions, they show the same dynamic characteristics of the string. The frequency response curve of each marker should show the same hysterestic phenomena if it exists. The frequency response spectrum of each marker's time trace was obtained by FFT analysis.

We did experiment for the string with three different sag-to-span ratios. The sag-to-span ratios were small, and hence the wire behaved more like strings. The lower and upper limits of the excitation frequency range are limited by the experimental system. Lower or higher excitation frequencies made the system out of control when a closed loop excitation was used. This was due to the influences of large variation of the tension force on the shaker during the vibration. The first string had a sag-to-span ratio of 1/801 and its static equilibrium configuration is shown in Fig 4.5 (a). For this case, we only examined the responses to harmonic excitations around the first natural frequency and the scanning range was from 12 Hz to 35 Hz only. The in-plane and out-of-plane frequency response curves of marker 25 are shown in Figs 4.5(b) and (c), respectively. The second string is tightly tensioned and the sag-to-span ratio was less than $1 / 1000$. Fig. 4.6(a) shows its static equilibrium configuration. The frequency range was from 15 Hz to 53.5 Hz . The in-plane and out-of-plane frequency response curves of marker 25 are shown in Fig. 4.6(b) and (c), respectively. The third string had a sag-to-span ratio of $1 / 756$, and was considered as a string because its elasto-geometric parameter showen later in Chapter 5 was determined to be 0.2171 , which is much less than 2 . Hence it behaved more like a string than a cable although it had small sag. The frequency range was from 15 Hz to 53.5 Hz also. The static equilibrium configuration is shown in Fig. 4.7(a). The in-plane and out-of-plane frequency responses of marker 9 are shown in Figs. 4.7(b) and (c), respectively. Marker 9 was chosen because, at this location, the thirdmode vibration had a peak value. It was practically impossible for the out-of-plane response to be completely zero even under planar vibration because the markers had nonzero sizes. Consequently, solid dots corresponding to small amplitudes in the plots often
appear in the out-of-plane response curves. Figs. 4.6(b) and (c) reveal that the second string had a tension force larger than that of the first and third string (i.e., $T_{0} \approx 2 \mathrm{lbf}$ ) because the first natural frequency of the second string was about 25 Hz ( $>17.5 \mathrm{~Hz}$ )


Fig. 4.5: The first string with a sag-to-span ratio of $1 / 801$ : (a) the static equilibrium configuration plotted using the curve fitted data and the original data, (b) the in-plane frequency response curves, and (c) the out-of-plane frequency response curve of marker 25 . Dots represent out-of-plane response and circles represent in-plane response.


Fig. 4.6: The second string with a sag-to-span ratio of $1 / 1000$ : (a) the static equilibrium configuration; (b) the in-plane frequency response curves, and (c) the out-of-plane frequency response curve of marker 25 . Dots represent out-of-plane response and circles represent in-plane response.

(a)



Fig. 4.7: The third string with a sag-to-span ratio of $1 / 756$ : (a) the static equilibrium configuration; (b) the in-plane frequency response curves, and (c) the out-of-plane frequency response curves of marker 9 . Dots represent out-of-plane response and circles represent in-plane response.

Because the third string had richer responses, we concentrated our study on this string. Based on the major vibration types, we divide the responses into seven branches that are marked as B1- B7 in Figs. 4.7 (b) and (c). For convenience of following analysis, each continuous response curve is divided into planar branch and non-planar branches. The first branch B1 is the one whose response starts from a translation motion (under a low-frequency excitation) to a planar vibration of the first mode (under an excitation close to the first natural frequency). The second, fourth and sixth branches represent nonplanar vibrations mainly composed of the first, second, and third modes, respectively. The third, fifth, and seventh branches represent transitions of a planar vibration from the first to the second, the second to the third, and the third to the fourth mode, respectively. Based on the number of different types of responses to one excitation frequency, we divide the excitation frequency range into five subintervals. The subintervals are marked as S1- S5 in Fig. 4.7(b) and (c). We are going to do moda1 analysis of typical types of
responses of each branch in Section 4.5. In this section, we concentrate on different types of responses of each subinterval. By this way, we investigate the losing and gaining stability of one branch of response, the appearance or disappearance of one type of response as the excitation frequency changes.

The first subinterval S 1 from 10 Hz to 17.5 Hz is. The $\mathrm{B} 1(10 \mathrm{~Hz}$ to 14.2 Hz$)$ is a stable planar and the $\mathrm{B} 2(14.5 \mathrm{~Hz}$ to 17.5 Hz$)$ is a stable non-planar vibration of the first mode. Starting from 18 Hz , the second subinterval S2 has two kinds of stable responses. The first one is the B 2 , which is the non-planar first-mode vibration. The response amplitude of this branch increases with the excitation frequency. The second one starts with a planar vibration consisting of the first and second modes (B3). The amplitude of the first mode decreases and that of the second mode increases when the excitation frequency increases. This type of vibration belongs to the third branch (B3), which represents the transition of vibration from the first mode to the second mode. Starting from 28.5 Hz , a small amplitude in-plane vibration consisting of the first and second modes loses stability, and the vibration becomes a non-planar one mainly consisting of the second mode (B4). The second subinterval lasts to 32 Hz . The third subinterval S3 starts from 33 Hz and lasts to 37 Hz . These are three different types of response for each excitation frequency in this interval. Except the two non-planar solutions (B2 and B4) that also exist in the second subinterval, the third one is a planar response (B5) which consists of the second and third modes. At the end of this subinterval, the response of non-planar vibration composed of the first mode (B2) loses stability and disappears. The frequency around which the hysteresis phenomenon

|  | Branch \＃1／2 | Branch \＃3／4 | Branch \＃5／6 | Branch \＃7／8 |
| :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  |
|  | （b） |  |  |  |
| N | （c） | （f） |  |  |
|  | （d） | （g） |  |  |
| 華 | （e） | （h） | （I） |  |
| $\begin{aligned} & \text { \# } \\ & \text { \# } \\ & \text { d } \end{aligned}$ |  | （i） | （m） |  |
| 苞 |  | （j） | （n） |  |
| 品 |  | （k） | （o） | （p） |

Fig 4.8: The typical trajectories of responses of each subinterval. The labels and scales for all figures are shown in (a). The excitation frequency $\Omega(\mathrm{Hz})$, the major and minor axes (mm) $r_{1}$ and $r_{2}$ of the fitted ellipse, and the frames per excitation period (fpp) for each case are given in Table 4.2.

Table 4.2: Parameters values for Fig.4.8.

| Fig. | $\Omega(\mathrm{Hz})$ | $r_{1}(\mathrm{~mm})$ | $r_{2}(\mathrm{~mm})$ | $f p p$ | Fig. | $\Omega(\mathrm{Hz})$ | $r_{1}(\mathrm{~mm})$ | $r_{2}(\mathrm{~mm})$ | $f p p$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| (a) | 14.2 | 0.53 | 2.84 | 36 | (i) | 40 | 4.74 | 5.26 | 14 |
| (b) | 17.5 | 6.16 | 7.29 | 30 | (j) | 49 | 6.39 | 6.81 | 11 |
| (c) | 28 | 14.79 | 15.67 | 20 | (k) | 53 | 7.20 | 7.45 | 10 |
| (d) | 32 | 17.67 | 18.43 | 17 | (l) | 37 | 0.31 | 1.16 | 15 |
| (e) | 37 | 20.94 | 21.61 | 15 | (m) | 40 | 0.32 | 1.54 | 14 |
| (f) | 28 | 0.30 | 2.19 | 20 | (n) | 49 | 2.08 | 2.54 | 11 |
| (g) | 32 | 2.62 | 3.29 | 17 | (o) | 50 | 2.29 | 2.61 | 11 |
| (h) | 37 | 4.05 | 4.60 | 15 | (p) | 53 | 0.29 | 0.93 | 10 |

happens when the excitation frequency increases is 37 Hz . The fourth subinterval S4 starts from 38 Hz and has two types of stable responses. From 38 Hz to 41 Hz , they are the planar vibration of B5 and the non-planar vibration of B4. From 42 Hz to 50 Hz , the planar response that mainly consists of the second and third modes (B5), becomes nonplanar vibration mainly consisting of the third mode (B6). The responses are similar to those of the second subinterval except that the contributing modes are different. The last available subinterval S 5 is from 50 Hz to 53 Hz . A type of planar response mainly consisting of the third and fourth modes is added to the two types of responses of subinterval S4. This interval is similar to the third subinterval.

In general, we can summarize the responses under the increasing frequency scanning as follows. Initially, there is no out-of-plan vibration and the motion is strictly planar. As the excitation frequency is increased, the planar vibration becomes unstable at
excitation corresponding to $\Delta_{1}$ in the theoretical prediction (see Fig. 3.2), which is just a little bit above the natural frequency. The in-plane vibration loses stability and the out-ofplane vibration is not zero any more. As the whirling or ballooning motion appears, the trajectory changes from a vertical line to an ellipse as shown in Fig. 4.8. At this frequency, the response curve is not second-order continuous because the slope decreases (see Fig. 4.5(b)), indicating that the input energy is no more completely fed into the planar vibration only. As the frequency is increased further, the major and minor axes of the ellipse, representing the amplitudes of in-plane and out-of-plane vibrations, increase with the minor axis $r_{2}$ growing at a little bit faster rate than the major axis $r_{1}$. Just as the ellipse is about to become a circle (i.e., $r_{1}=r_{2}$ ), the ballooning motion loses stability and the hysteresis phenomenon happens. The response to higher frequency excitations actually depends on the properties of the string. After the hysteretic frequency, corresponding to $\Delta_{6}$ in the theoretical prediction shown in Chapter 3, the only stable response would be a planar vibration of small amplitude as presented in the literature. However, there are two types of responses in our case, one is a planar vibration (B5) and the other is a non-planar one (B4). This is because the natural frequencies of a string with a small tension are closely spaced and the damping is small. Thus, the effective frequency range of each natural mode becomes wider. These planar and non-planar responses persist and the amplitudes increase as the excitation is increased further.

Although we did not actually decreasingly sweep the frequency, we used different disturbances to simulate different initial conditions, by which the responses supposed to be obtained by sweeping decreasingly were also captured. The planar response was often
obtained by exciting the initially still string, and the non-planar response was obtained by giving a disturbance to the planar vibration. Plotting the obtained planar and non-planar response amplitudes versus the detuning, one can see the hysteretic phenomenon as shown in Figs. 4.7(b) and (c). When $\Omega$ decreases the planar vibration mainly consisting of two neighboring modes loses stability at theoretically predicted $\Delta_{2}$ and jumps to a large-amplitude ballooning motion at a frequency close to the natural frequency as shown in Fig. 3.2. For a frequency below $\Delta_{2}$ but larger than $\Delta_{1}$ (where non-planar vibration starts), only ballooning vibration mainly consisting of one mode exists. The typical trajectories of responses at each subinterval are plotted in Fig. 4.8. Reading vertically, we can observe the appearing and disappearing (losing stability) of each branch. Reading horizontally, we can see the different responses to excitations within that frequency subinterval. We did not observe non-linear vibrations of period-doubled, quasi-periodic, torus-doubled, or chaotic type. This is probably due to the large damping of our test strings because these phenomena happen only for strings of pretty small damping (O’Reilly, 1990). All the responses of the three tested strings were periodic which was verified by their frequency spectra.

### 4.5 Modal Analysis

Although our experimental string responses were periodic, most vibrations consisted of several modes due to linear and nonlinear modal couplings. Depending on the closeness of the excitation frequency to the neighboring natural frequencies, the contributing modes with natural frequencies close to the excitation frequency carry different weights. In our low frequency tests, the vibrations were mainly composed of the
first four modes. Hence, to investigate the contributions of the first four linear normal modes to the experimental operational deflection shapes at $t=t_{k}$, we assume

$$
\begin{equation*}
w\left(x, t_{k}\right)=b\left(t_{k}\right)+\sum_{i=1}^{4} a_{i}\left(t_{k}\right) \phi_{i}(x) \tag{4.5.1}
\end{equation*}
$$

where $b(t)$ is the shaker's z-direction displacements during excitation and $a_{i}$ denotes the modal displacement of the ith linear mode. In our following analysis, $C Z_{i}(t)(i=1,2,3,4)$ and $C Y_{i}(t)(i=1,2,3,4)$ are used to represent the modal displacements for in-plane and out-of-plane responses, respectively. To obtain the value of $a_{i}\left(t_{k}\right)$ by least-squares fitting, we define a spatial-domain error function Ex as

$$
\begin{equation*}
E x \equiv \sum_{m=1}^{n m}\left[w\left(x_{m}, t_{k}\right)-\hat{w}\left(x_{m}, t_{k}\right)\right]^{2} \tag{4.5.2}
\end{equation*}
$$

where $\hat{w}$ denote experimental data. $t_{k}(k=1, \cdots \cdots N)$ are the recorded time instants and $N$ is chosen to be $2^{9}$ or $2^{10}$ for convenience in FFT analysis of the time domain data. $n m$ is the number of markers in this study. The equations for determining $a_{i}\left(t_{k}\right)$ are

$$
\begin{equation*}
\frac{\partial E x}{\partial a_{i}} \equiv \sum_{m=1}^{n m} 2\left[w\left(x_{m}, t_{k}\right)-\hat{w}\left(x_{m}, t_{k}\right)\right] \phi_{i}\left(x_{m}\right)=0, \quad i=1,2,3,4 \tag{4.5.3}
\end{equation*}
$$

The standard deviation $S D$ of the fitted displacement profile at each time instant can be calculated as

$$
\begin{equation*}
S D \equiv \sqrt{E x / n m} \tag{4.5.4}
\end{equation*}
$$

Moreover, one can check the displacement of the marker at $x=x_{m}$ to see how much its curve-fitted displacement $w\left(x_{m}, t_{k}\right)$ from Eq. (4.5.1) matches with its experimental one $\hat{w}\left(x_{m}, t_{k}\right)$, and this can be quantified using the following time-domain error function

$$
\begin{equation*}
E t=\sum_{k=1}^{1024}\left[w\left(x_{m}, t_{k}\right)-\hat{w}\left(x_{m}, t_{k}\right)\right]^{2} \tag{4.5.5}
\end{equation*}
$$

Considering the linear combination of two modes vibrating at the same frequency

$$
\begin{align*}
& \bar{w}=\bar{\phi}_{2}(x) \sin (\Omega t)+\bar{\phi}_{1}(x) \sin (\Omega t+\alpha)=\Phi(x) \sin (\Omega t+\bar{\alpha})  \tag{4.5.6}\\
& \bar{\phi}_{1} \equiv a_{1 \max } \phi_{1}, \quad \bar{\phi}_{2} \equiv a_{2 \max } \phi_{2}
\end{align*}
$$

where

$$
\Phi(x) \equiv \sqrt{\bar{\phi}_{2}^{2}+2 \bar{\phi}_{1} \bar{\phi}_{2} \cos (\alpha)+\bar{\phi}_{1}^{2}}, \quad \bar{\alpha} \equiv \tan ^{-1} \frac{\bar{\phi}_{1} \sin (\alpha)}{\bar{\phi}_{2}+\bar{\phi}_{1} \cos (\alpha)}
$$

If the phase difference $\alpha$ between these two harmonics is $\alpha \neq 0^{\circ}$ or $180^{\circ}(\pi)$, the phase for the combined vibration, $\bar{\alpha}$, is a function of $x$ and hence it is called a complex mode. A complex mode is one in which each point of the structure has its own amplitude and phase. Consequently, each point of the structure will reach its own maximum deflection at a different time instant in the vibration cycle. The zero deflection positions will be reached at different time instants also.

Next, we perform typical response studies of each branch. The first branch (B1) in Figs. 4.7(b) and (c) is the response to a low-frequency excitation. The vibration under the low-frequency excitation was mostly dominated by the movement of the end support because the string was vibrating with small amplitude. The vibration could not be well measured because the response amplitude and the magnitude of the marker ( $1 \sim 2 \mathrm{~mm}$ ) were close to the excitation amplitude ( 1 mm ) . Fig. 4.9(a) shows the 52 curve-fitted consecutive vibration profiles when the string was excited at 10 Hz . As expected, the frequency spectrum of the vibration of marker 25 around the midpoint of the string is complicated because of a lose signal-to-noise ratio and multiple-mode vibration.


Fig. 4.9: Response to an excitation at $\Omega=10 \mathrm{~Hz}$ : (a) 52 curve-fitted consecutive vibration profiles $(0.098 \mathrm{sec})$, and (b) the frequency spectra of marker 25.

For a higher excitation frequency at 14.2 Hz , we had a better capture of the motion because it had larger amplitude, as shown in Fig.4.10 (a). The frequency spectrum of marker 25 (Fig. 4.10 (b)) is much clearer with one major harmonic at the excitation frequency. Although the string was still not fully excited, complex behaviors of the vibration was obvious. Although the largest deformation was around the midpoint of the string, close examination shows that profiles at different times have different locations of maximum deflections. We see traveling waves (e.g., the dotted one) in Fig. 4.10 (a), which are profiles having more than one local maximum. This irregular vibration shape is the result of the interference of the incident sine wave started by the excitation with a reflected sine wave in a rather non-sequenced and untimely manner.


Fig. 4.10: Response to an excitation at $\Omega=14.2 \mathrm{~Hz}$ : (a) 37 curve-fitted consecutive vibration profiles ( 0.069 sec ), and (b) the corresponding frequency spectra of marker 25.

The B2 branch in Figs. 4.7(b) and (c) is the whirling motion mainly consisting of the first mode, and there is no node except the two support ends. As shown in Fig. 4.7, the amplitude of vibration increases with the excitation frequency. The curve-fitted vibration profiles shown in Fig. 4.11(a) are clearly separated, whcih indicates that the string came back to the same position after some time and so it was periodic. Close examination of the vibration profiles, we find that there are exactly 16 profiles $(=560(F P S) / 35(H z))$. This is verified by the trajectory of marker 25 shown in Fig. 4.11(b), which has 16 discrete dots indicating that the string came back to the same position after 16 frames (i.e., one period). Fig. 4.11(c) and (d) show that the periodic harmonic vibrations in both planes are exactly at the excitation frequency. Fig. 4.11(a) also shows that there are crossovers between the profiles at different time instants. This tells us the vibration was composed of more than one mode. The unsymmetrical crossovers indicate obvious participation of anti-symmetric modes. The modal coordinates and corresponding frequency spectra presented in Figs. 4.11(e) - (h) show the
existence of the second mode (first anti-symmetric mode) vibrating at both one time $(\Omega)$ and two times $(2 \Omega)$ of the excitation frequency for both XY- and XZ- plane vibrations. The $\Omega$ component is caused by linear coupling because the forcing function has a spatial distribution non-orthogonal all linear modes and the $2 \Omega$ component is due to a $\omega_{1}: \omega_{2}=1: 2$ internal resonance. This phenomenon cannot be explained by a string model in which only cubic nonlinearity is considered. It reveals that the complicate unexpected situations may happen at real experiments and a more accurate theoretical predication asks for a more accurate theoretical model capable of simulating real vibrations. Moreover, close examination of Fig. 4. 11(a) shows the crossovers between profiles of the XY-plane vibration is more serious than that of the XZ-plane vibration. This is because the amplitude of the $2 \Omega$ harmonic of the XY-plane vibration is much larger than that of the $\Omega$ harmonic (Fig. 4. 11(h)). For the XZ-plane vibration, the amplitudes of the $\Omega$ and $2 \Omega$ harmonics have about the same magnitude (Fig. 4. 11(f)). This is true for most of our tests of this branch and we believe it is because the XZ-plane vibrations were influenced by the gravity. The time domain error function $\operatorname{Et}(x)$ of marker 25, the spatial domain error function $E x(t)$, and the standard deviation $S D(t)$ of the modal decompositions are shown in Figs. 4.11(i) and (j), respectively. The small values of error functions indicate that four modes are enough for an accurate modal decomposition for this response.

(e)










Fig. 4.11: Response to an excitation at $\Omega=35 \mathrm{~Hz}$ : (a) 32 curve-fitted consecutive vibration profiles $(0.055 \mathrm{sec})$, (b) the trajectory of marker 25 , (c) the frequency spectrum of the Z-direction vibration of marker 25, (d) the frequency spectrum of the Y-direction vibration of marker 25 , (e) the modal coordinates of the Z -vibration; (f) the frequency spectra of the modal coordinates of the Z-vibration, (g) the modal coordinates of the Yvibration, (h) the frequency spectra of the modal coordinates for the Y-vibration, (i) error functions of the modal decomposition of the Z-vibration, and ( j ) error functions of the modal decomposition of the Y-vibration.

The B3 branch in Figs. 4.7(b) and (c) represents the transition from the first-mode planar vibration to the second-mode one. Fig. 4.12 shows the responses when the string was excited at $\Omega=17.75 \mathrm{~Hz}$. Fig. 4.13 shows the response when the string was excited at $\Omega=22 \mathrm{~Hz}$. Fig. 4.14 shows the response when the string was excited at $\Omega=28 \mathrm{~Hz}$. Figs. 4.12(a), 4.13(a), and 4.14(a) show the obvious transition of the vibration from the
one mainly consisting of the first mode to the one mainly consisting of the second mode. From the modal coordinates and frequency spectra (Figs. 4.12 (b) and (c), Figs. 4.13(b) and (c), and Figs. 4.14(b) and (c)), we see that the first and second modes are the major component and are always in phase or have a phase difference of $\pi$ for these three cases. Comparing the amplitudes of the first and second modes of these cases, we see that the first-mode amplitude decreases and the second-mode increases, indicating a gaining participation of the second mode. Due to linear coupling, some higher modes had small but nontrivial amplitudes and vibrated at the excitation frequency, making the vibration complicated.


Fig. 4.12: Response to an excitation at $\Omega=17.75 \mathrm{~Hz}$ : (a) 64 curve-fitted consecutive
vibration profiles ( 0.111 sec ), (b) the modal coordinates of the Z-vibration, (c) the frequency spectra of the modal coordinates of the Z-vibration of marker 25, and (d) error functions for the modal decomposition of the Z -vibration.


Fig. 4.13: Response to an excitation at $\Omega=22 \mathrm{~Hz}$ : (a) 50 curve-fitted consecutive vibration profiles $(0.089 \mathrm{sec})$, (b) the modal coordinates of the Z -vibration, (c) the frequency spectra of the modal coordinates of the Z-vibration of marker 13, and (d) error functions of the modal decomposition of the Z-vibration.

The B4 branch in Fig. 4.7(b) and (c) represents a whirling motion mainly composed of the second mode and so there is one node at the middle of the string. This branch has similar properties as the B2 branch. Figs. 4.15 (a) - (j) are representative plots of the response when the string was excited at $\Omega=29.5 \mathrm{~Hz}$. The clearly separated curve-
fitted vibration profiles (Fig. 4.15 (a)) tell us that the vibration was periodic. The trajectory of marker 13 (Fig. 4.15 (b))


Fig. 4.14: Response to an excitation at $\Omega=28 \mathrm{~Hz}$ : (a) 40 curve-fitted consecutive vibration profiles $(0.07 \mathrm{sec})$, (b) the modal coordinates of the Z -vibration, (c) the frequency spectra of the modal coordinates of the Z-vibration of marker 13, and (d) error functions of the modal decomposition of the Z-vibration.
has exactly 18 points, which indicates the vibration come back to the same position after one excitation period. The frequency spectra of the vibrations of marker 13 (Fig. 4.15 (c, d)) show us the vibrations were periodic at the excitation frequency also. Figs. 4.15 (e) (h) are the modal coordinates and corresponding frequency spectra of the in XY- and XZ-
plane vibrations, respectively. They show that the vibrations were dominated by the second mode. However, there are modes other than the second mode. Actually, it was the participation of the symmetric modes vibrating at the excitation frequency made the inner


(c)












Fig. 4.15: Response to an excitation at $\Omega=29.5 \mathrm{~Hz}$ : (a) 36 curve-fitted consecutive vibration profiles ( 0.066 sec ), (b) the trajectory of marker 13, (c) the frequency spectra of the Z-direction vibration of marker 13, (d) the frequency spectra of the Y-direction vibration of marker 13, (e) the modal coordinates of the Z-vibration, (f) the frequency spectra of the modal coordinates of the Z-vibration; (g) the modal coordinates of the Yvibration; (h) the frequency spectra of the modal coordinates of the Y-vibration, (i) error functions of the modal decomposition of the Z-vibration, and (j) error functions of the modal decomposition of the Y -vibration.
node unclear. And, participation of modes different from the second mode caused the crossover between different profiles shown in Fig. 4.15(a).

The B5 branch in Figs. 4.7(b) and (c) represents a transition from the secondmode planar vibration to the third-mode planar vibration. It has similar properties as the

B3 branch. Fig. 4.16 shows the results when the string was excited at $\Omega=33 \mathrm{~Hz}$. Fig. 4.17 are plots of responses when the string was excited at $\Omega=35 \mathrm{~Hz}$. Fig. 4.18 shows the results when the string was excited at $\Omega=37 \mathrm{~Hz}$. Fig.4.19 shows the results when the string was excited at $\Omega=40 \mathrm{~Hz}$. First of all, from the modal coordinates shown in Figs. 4.16(b) - 4.19(b) we see the second and third modes are always in phase for all cases. Comparing the amplitudes of the second and third modes, we see that the second-mode amplitude


Fig. 4.16: Response to an excitation at $\Omega=33 \mathrm{~Hz}$. (a) 32 curve-fitted consecutive vibration profiles ( 0.059 sec ), (b) the modal coordinates of the Z-vibration, (c) the frequency spectra of the modal coordinates of the Z-vibration, and (d) error functions of the modal decomposition of the Z-vibration.


Fig. 4.17: Response to an excitation at $\Omega=35 \mathrm{~Hz}$ : (a) 32 curve-fitted consecutive vibration profiles $(0.055 \mathrm{sec})$, (b) the modal coordinates of the Z -vibration, (c) the frequency spectra of the modal coordinates of the Z-vibration, and (d) error functions of the modal decomposition of the Z-vibration.
decreases and that of the third mode increases, indicating a gaining participation of the third mode. Due to linear modal coupling, the first mode is not negligible when $\Omega=33 \mathrm{~Hz}$ and $\Omega=35 \mathrm{~Hz}$, and the fourth mode is not negligible when $\Omega=37 \mathrm{~Hz}$ and $\Omega=40 \mathrm{~Hz}$. Traveling waves are evident due to the participations of modes with phase differences being neither 0 nor $\pi$ for all cases. Each profile seems to have a different location of maximum deflection.


Fig. 4.18: Response to an excitation at $\Omega=37 \mathrm{~Hz}$ : (a) 30 curve-fitted consecutive vibration profiles ( 0.052 sec ), (b) the modal coordinates of the Z-vibration, (c) the frequency spectra of the modal coordinates of the Z-vibration, and (d) error functions of the modal decomposition of the Z-vibration.


Fig. 4.19: Response to an excitation at $\Omega=40 \mathrm{~Hz}$ : (a) 28 curve-fitted consecutive vibration profiles $(0.048 \mathrm{sec})$, (b) the modal coordinates of the Z -vibration, (c) the frequency spectra of the modal coordinates of the Z-vibration, and (d) error functions of the modal decomposition of the Z-vibration.

The B6 branch in Figs. 4.7(b) and (c) has non-planar responses similar to those of B2 and B4 branches, but they are mainly composed of the third mode. The amplitudes of these higher-mode vibrations are smaller than those of lower-mode vibrations of B2 and B4. Consequently, the measured vibration profiles have lower relative precision. Fig. 4.20 shows the responses when the string was excited at $\Omega=41 \mathrm{~Hz}$. Fig. 4.21(a) shows the vibration profiles when the string was excited at $\Omega=50 \mathrm{~Hz}$. It is a typical response for branch B6. It is obvious that the major components of the vibrations in both planes



Fig. 4.20: Response to an excitation at $\Omega=41 \mathrm{~Hz}$ : (a) curve-fitted consecutive vibration profiles ( 0.047 sec ), (b) the trajectory of marker 8 , (c) the frequency spectra of the Zdirection vibration of marker 8 , (d) the frequency spectra of the Y -vibration of marker 8 , (e) the modal coordinates of the Z-vibration, (f) the frequency spectra of the modal coordinates of the Z-vibration, (g) the modal coordinates of the Y-vibration, (h) the frequency spectra of the modal coordinates of the Y-vibration, (i) error functions of the modal decomposition of the Z-vibration, and (j) error functions of the modal decomposition of the Y-vibration.
are the third mode. As the excitation frequency increases, the fourth mode may be excited by linear modal coupling due to the closeness of the excitation frequency to the fourth natural frequency. Fig.4. 22 (a) shows the very complex vibration when the string was excited at $\Omega=53 \mathrm{~Hz}$. For the out-of-plane vibration (Fig.4. 22 (g) and (h)), the amplitude of the third mode is much larger than other contributing modes so both nodes



Fig. 4.21: Response to an excitation at $\Omega=50 \mathrm{~Hz}$ : (a) 22 curve-fitted consecutive vibration profiles ( 0.038 sec ), (b) the trajectory of marker 8 , (c) the modal coordinates of the Z-vibration, (d) the frequency spectra of the modal coordinates of Z-vibration, (e) the modal coordinates of the Y-vibration, (f) the frequency spectra of the modal coordinates of the Y-vibration, (g) error functions of the modal decomposition of the Z-vibration, and (h) error functions of the modal decomposition of the Y-vibration.
are clear. For the in-plane vibration (Fig. 4.22(e) and (f)), the amplitudes of the third and fourth modes are of smaller difference and so the right node is not as clear. The other reason for the unclear node is that the right node is close to the excitation support and so the influence of the support displacement on this node is more serious. This is especially serious for higher-mode vibrations because they have small amplitudes. Figs. 4.22(e) and (g) show that the phase difference between the two major linear components, the third and fourth modes, of the XY- and XZ-plane vibrations. Note that the phase difference between the third and fourth modal coordinates changes with the excitation frequency, indicating these two linear modes move independently and cannot be combined into one nonlinear normal mode. Figs. 4.22(c) and (d) show the frequency spectra of the XZ- and XY-plane vibrations of marker 8. Both contain frequency harmonics with small but not negligible amplitudes. Consequently, four modes may not be enough for an accurate modal decomposition of this high frequency vibration.



Fig. 4.22: Response to an excitation at $\Omega=53 \mathrm{~Hz}$ : (a) curve-fitted consecutive vibration profiles ( 0.036 sec ), (b) the trajectory of marker 8 , (c) the frequency spectra of Zvibration of marker 8 , (d) the frequency spectra of the Y- vibration of marker 8 , (e) the modal coordinates of the Z-vibration, (f) the frequency spectra of the modal coordinates of Z-vibration, (g) the modal coordinates of the Y-vibration, (h) the frequency spectra of the modal coordinates of the Y-vibration, (i) error functions of the modal decomposition of the Z-vibration, (j) error functions of the modal decomposition of the Y-vibration.

The solutions of B7 were not studied in detail because the shaker was often out of control for excitations at this frequency range. We only got responses for three different excitations. Figs. 4.23 and 4.24 show the results when the string was excited at $\Omega=50 \mathrm{~Hz}$ and $\Omega=53 \mathrm{~Hz}$, respectively.


Fig. 4.23: Response to an excitation frequency at $\Omega=50 \mathrm{~Hz}$. (a) curve-fitted consecutive vibration profiles $(0.038 \mathrm{sec})$, (b) the modal coordinates for the Z-vibration, (c) the frequency spectra of the modal coordinates for the Z-vibration, and (d) error functions of the modal decomposition of the Z-vibration.

The curve-fitted profiles shown in Fig. 4.23(a) are clearly separated because the vibration was periodic. The number of separated discrete profiles of the left half of the string is different from that of the right half, indicating that the vibration consists of two or more modes. Figs. 4.23(b) and (c) show that the displacement profiles are mainly dominated by the third and fourth linear normal modes and these two major two components are in phase.


Fig. 4.24: Response to an excitation at $\Omega=53 \mathrm{~Hz}$ : (a) 20 curve-fitted consecutive vibration profiles $(0.036 \mathrm{sec})$, (b) the modal coordinates of the Z -vibration, (c) the frequency spectra of the modal coordinates of the Z-vibration, and (d) error functions of the modal decomposition of the Z-vibration.

The second linear mode's contribution to the vibration is small and its phase is $180^{0}$ different from that of the third and fourth modes. The contribution of the first linear normal mode is mainly due to the gravity-induced sag. It seems that the characteristics of responses of this branch to higher frequency excitations are similar to the corresponding low-frequency branches. For higher-mode vibrations, the influence of gravity and initial curvature on the natural frequency decreases. This can be explained by the crossover of
the natural frequencies of cables as shown later in Chapter 5 . For strings with non-zero small sags (i.e., cables with a small elasto-geometric parameter), only natural frequencies of low-order in-plane symmetric modes are affected by the sag while those of high-order modes keep almost unchanged. The commensurable relations between natural frequencies of higher in-plane and out-of-plane modes still exist and it is possible for the various resonances to happen. Hence, more linear normal modes are necessary for accurate decomposition of high frequency vibrations.

The strings we considered are not strings under the strict definition of a string. However, because their sap-to-span ratios and hence elasto-geometric parameters are small, it is reasonable and appropriate to consider them as strings. As expected, the experiments showed more properties of a string than those of a cable. There are two types of responses. The first type is planar vibration mainly composed of two neighboring modes and represents the transition phase from the lower mode to the next one. As the excitation frequency increases, the amplitude of the lower one decreases while that of the other increases. These two linearly coupled modes are always in phase or out of phase. Lower and higher modes may be excited due to linear coupling, making the vibration more complicated. The second type is non-planar vibration mainly composed of a single mode for both in-pane and out-of-plane responses. For some cases, linear coupling happens and makes the vibration complex. For most cases, the phase differences between different modes change with the excitation frequency, which indicates the modes move independently and so cannot be combined into one non-linear normal mode. The concept of non-linear normal mode is thus questionable for modal decomposition of string vibration. The string we tested was not as straight and tensioned as assumed in the
theoretical model. So, the theoretically impossible 1:2 internal resonance was experimentally observed, but the magnitude was small.

### 4.6 Conclusion and Discussion

Our experiments successfully recorded planar and non-planar/whirling vibrations, hardening effect, and modal coupling predicted by theoretical investigations. Branches of planar responses represent a transition between two neighboring modes, from the lower one to the higher one. The amplitude of the lower one decreases and that of the other one increases for increasing excitation frequency sweeping. Branches of non-planar responses are motions mainly composed of one single mode. However, for some cases, other modes may participate in both planar and non-planar vibrations, making the vibration profile more complex and irregular although the other modes have relatively small amplitudes. The hysteretic phenomenon was also verified by the sudden lose of stability of one branch and jumping to another branch for increasing and decreasing frequency sweepings. A pretty small sag, small enough to make the string far from being a cable at the first cross over (see later in Chapter 5), accounts for the observed small amplitude 1:2 internal resonance, which may happen theoretically only in cables. The modal decompositions of responses under higher-frequency excitations show four modes may not be enough for accurate modal decomposition.

Attempts to quantitatively match the experimental results with the theoretical results were not fully successful. The difficulty in making such comparisons is mostly due to following reasons. One of them is the inability to precisely measure the parameters like axial stiffness, longitudinal wave speeds, and damping, etc, of the tested string. The
other one accounts for the discrepancy originates from the nature of the external excitation. In theoretical analysis, the excitation is applied to one of the two ends of the string. However, to avoid the longitudinal vibration, the excitation is applied at a location very close to one of the two fixed boundaries, and only the vibration between the attachment point of the shaker and the other end is considered. This arrangement, however, constraints the out-of-plane motion at the shaker attachment point and tends to decouple the cable response into two components, at least that of the out-of-plane response. Additionally, we used a level to adjust the position of the shaker and tried to make the excitation vertical. This may not be as good as desired. Moreover, the theoretical prediction is accurate for weakly nonlinear problems only. Our studies, however, are out of the weakly nonlinear range because the vibration amplitudes are quite large and the frequency detuning interval is so wide that its limits are possibly beyond the natural frequencies of neighboring modes. Theoretically predicted coexisting clockwise and counterclockwise whirling motions (O’Reilly, 1990) were not observed. This may be due to the intrinsic asymmetry of the experimental set-up.

## CHAPTER 5

# EXPERIMENTAL DYNAMIC CHARACTERISTICS OF CABLES 

In this chapter, we begin with a review of nonlinear dynamics of cables. The methods for and observations from numerical investigation and experimental testing are summarized. Then the linear theory of cable vibration is introduced. The crossover phenomenon of natural frequencies of in plane and out-of-plane, and symmetric and antisymmetric modes are presented. In our experiments, theoretically predicted isolated and simultaneous internal resonances were observed for the cable around the first frequency crossover. A theoretically unpredicted simultaneous internal resonance was also observed.

### 5.1 Review of Nonlinear Cable Oscillations

Cables have some distinct mechanic properties such as flexibility, lightweight, and zero buckling strength. Due to these special characteristics, on one hand, cables are widely used in mechanical, civil, ocean and aerospace engineering. Most popular applications are suspension bridges and cable-stayed bridges, power transmission lines, aerial tramways, mooring cables, cable nets, etc. On the other hand, because of their flexibility and lightweight, the oscillations of cables subjected to external loads seriously
impair the performances of the devices or structures in which cables are used. So, it is meaningful to do thoroughly theoretical and experimental investigations on cable dynamics. The 3D motion analysis system has distinct features as stated before in Chapter 4, and the system makes it possible for us to obtain more insightful dynamic characteristics of cables. In this section, we give a historical review of cable dynamics first.

Strings are tensioned cables. Cables are loose strings. Increasing the sag of a string from zero, the behavior of a string decreases and that of a cable increases. Physically speaking, cables are similar to strings and hence dynamics of string is similar to that of cables of small sag-to-span ratios. Considering the sag as a parameter, the dynamics of cables of zero sag is that of strings. However, early linear cable theories, in which inextensibility was assumed, failed to reconcile with the taut string theory. Noticing this discrepancy, Irvine and Caughey (1974) did a thorough investigation about the transverse horizontal vibration, including anti-symmetric and symmetric in-plane vibrations. It was shown that, to the first order, the transverse horizontal motion (both symmetric and anti-symmetric out-of-plane vibrations) and the anti-symmetric in-plane motion, which consist of an anti-symmetric vertical component and a symmetric longitudinal component, introduce no additional tension during the vibration. The symmetric in-plane mode, however, introduces additional tension due to the elastogeometry parameter that accounts for the effects of cable geometry and elasticity. The frequency of the symmetric in-plane mode varies with the elasto-geometry parameter and there are cross-over points where the frequencies of the symmetric in-plane modes and corresponding anti-symmetric modes are equal.

Triantafyllou (1984a, 1984b, 1987, and 1991) did a serial review of research conducted on the linear and non-linear dynamics of cables and other mooring systems. Triantafyllou (1984a, 1984b) investigated the effect of the inclination angle on the first two natural frequencies of an elastic, inclined cable with a small sag. The phenomenon he termed "frequency avoidance", by which he meant the natural frequencies do not cross over due to the inclination, was introduced.

Luongo, Rega, and Vestroni (1982) studied the changes of frequencies and amplitudes of mono-frequency oscillations, both of the in-plane extensional type and the out-of-plane pendulum type, due to the variation of the elasto-geometric parameter in the presence of nonlinear coupling but in the absence of internal resonance. For the in-plane type motion, the drift of the midpoint of the trajectories on the state plane (symmetry breaking), one of the signatures of nonlinearities, was observed. The effects of quadratic and cubic nonlinearities on the frequency response functions were studied respectively. For the out-of-plane pendulum vibration, the study was focused on those of cables around the first cross-over point.

Luongo, Rega, and Vestroni (1984) investigated the planar non-linear free vibrations of an elastic cable. It was shown that the behavior was initially hardening for a cable with a small elasto-geometric parameter (i.e., a string), due to the prevailing cubic nonlinearities. As the value of the parameter increased, the response was softening for low-amplitude vibrations and became hardening as the amplitude increased, corresponding to a transition from a vibration dominated by quadratic nonlinearity due to initial curvatures to a vibration dominated by cubic nonlinearity due to the stretching effect. The drift of the midpoint of oscillation was evidenced in the temporal law of
motion, and this was especially apparent for cables vibrating with larger amplitudes. The non-dimensionalization with respect to the sag or span was discussed. Two different procedures to discretize the partial differential equations were discussed. In the first approach, the longitudinal inertial force was neglected and so the additional cable tension was a function of time only and could be derived from the longitudinal equation of motion. Applying this relation to the equation of transverse vibration, the unique partial integro-differential equation was obtained. In this procedure, only one shape function was necessary to describe the transverse vibration when applying Galerkin discretization and neglecting the longitudinal inertia force. The nonlinear relations between the displacements of longitudinal and transverse directions were exactly treated. In the second approach, two eigen-functions, one for the longitudinal displacement and the other for the transverse displacement, was necessary. However, these two components did not exactly satisfy the nonlinear relation describing the longitudinal equation of motion but did in an average sense via an integral method. These two approaches resulted in different coefficients for cubic terms, but not for the quadratic terms. Another approach was proposed in which the perturbation method was applied to the equations first and then the Galerkin procedure was then applied to each obtained equation. Similar to the second approach, two eigen-functions were necessary and the nonlinear relation between the longitudinal and transverse displacements was asymptotical. Rega, Vestroni, Benedettini (1984) studied the dependence of frequencies of pre-stressed and slack cables on the amplitude of oscillation of one particular mode, the first and second in-plane symmetric modes and the first anti-symmetric one, respectively, by varying the mechanical and geometrical properties of the cable. The dependence of hardening or
softening on the amplitude was verified. The larger the sag-to-span ratio and the higher the mode number, the stronger were the effects of nonlinearities. Moreover, it was shown that the crossover frequency was no more fixed which was the same as the linear cases due to the amplitude-dependent frequencies. So there were different crossover frequencies and the number of crossover points was infinite instead of just one as predicted in the linear theory. Benedettini, and Rega, Vestroni (1986) investigated the influence of modal coupling between in-plane and out-of-plane modes in the absence of internal resonance for the free cable oscillation when different initial conditions were applied. It was shown that the responses to even small initial amplitudes differed notably from the linear one due to the strong the modal coupling of a slack cable. Benedettini and Rega (1987) studied the nonlinear dynamics of an elastic cable subject to a planar excitation with the assumption that the deformed geometry was parabolic. A high-order perturbation analysis was accomplished to determine the frequency response equations, the time law, and the region of instability of steady solutions. Numerical results were presented for a nearly taut string and a suspended cable. It was shown that the string had purely hardening response because of prevailing cubic nonlinearity due to stretching and the cable has dominantly softening response under small-amplitude vibration and hardening response under large-amplitude vibration because of the additional quadratic nonlinearity due to the curvature. The dependence of responses on the initial conditions and multiple solutions was investigated also.

Takahashi and Konishi (1987a) investigated the non-linear non-planar free vibration of cables. The intra-planar coupling was studied and it was found that the geometrical nonlinearity was generally a hardening type but a softening type for some
particular sag-to-span ratios. Takahashi and Konishi (1987b) studied the non-planar vibration of cables under in-plane sinusoidal forcing using a multiple-degree-of-freedom model. Using the harmonic balance method and considering two out-of-plane modes, the unstable regions were constructed and compared with responses having simple parametric resonance, with only one out-of-plane mode.

Al-Noury and Ali (1985) studied the influence of geometrical non-linearity (function of the sag-to-span ratio) on the responses of parabolic cables. The spatial problem was solved using the Galerkin method and the temporal problem was attacked using a perturbation method. The investigations were presented for responses to a transverse excitation, a vertical excitation, and a transverse excitation with interactions between vertical and transverse modes having closely spaced frequencies.

Rao and Iyengar (1991) studied the vibration with 1:2 internal resonance between the first symmetric in-plane and out-of-plane modes of a cable dominated by quadratic nonlinearity using a quasi-static model in which the influence of the longitudinal inertia was neglected while that of the longitudinal vibration was retained. The external excitation was a combination of an in-plane harmonic component and a uniform lateral load one.

Pakdemirli, Nayfeh, and Nayfeh (1995) considered the one-to-one internal resonance between the in-plane and out-of-plane modes as well as primary resonance of in-plane modes. They focused the investigation on the difference between two approaches used for the problem. The discretization approach applied the method of multiple scales to the ordinary differential equations obtained by discretizing the partial differential equations using the spatial eigen-functions of the linear problem. The direct
approach directly applied the method of multiple scales to the governing partial differential equations. It was shown that the discretization approach was inaccurate and yielded predictions qualitatively different from those by the direct approach. In the discretization approach, the spatial variation of the quadratic terms was erroneously assumed to be the same as that of the linear terms and hence quadratic terms in the original system were eliminated during the linear discretization process because they are orthogonal to the linear mode shape.

Using statistics methods and Monte Carlo simulations, Chang, Ibrahim and Afaneh (1996) investigated the single 2:1 internal resonance response between the first in-plane and out-of-plane symmetric modes of a suspended cable excited by an in-plane random force generated numerically from a normal distribution using an inverse Cumulative Distribution Function (CDF) technique. Chang and Ibrahim (1997) and Ibrahim and Chang (1999) investigated the multiple internal resonances (2:1:2) between the first in plane and the first two out-of-plane modes of a cable excited by a random inplane loading. The on-off intermittency of auto-parametrically excited modes, the energy exchange between coupled modes in the neighborhood of internal resonance and the saturation phenomenon were observed. It was found that both the excitation level and the internal damping had significant effects on the region of modal interaction.

Benedettini and Rega (1994), and Benedettini, Rega and Alaggio (1995) considered primary and sub-harmonic resonances of an initial parabola cable using a four-degree-of-freedom model, accounting for the planar and non-planar symmetric and anti-symmetric motions. Possible (single and multiple) internal resonance solutions, unimodal, bimodal, tri-modal and complete solutions were discussed, especially considering
the dependence on the relative damping ratios between different modes. Rega, Alaggio and Benedettini (1997) and Benedettini and Rega (1997) experimentally investigated the nonlinear response of an elastic cable made of nylon wire carrying eight equally spaced concentrated masses and hanged at two supports which were given either in-phase or out-of-phase vertical harmonic excitations. The cable was adjusted to be at the first crossover so internal resonances might happen between the first planar and non-planar symmetric and anti-symmetric modes. Responses to the variation of the control parameters, (i.e., the excitation frequency and forcing) were measured and analyzed. The meaningful ranges of the control parameters include those having internal resonances (e.g., the one half sub-harmonics), the primary and super-harmonic resonances of the anti-symmetric modes, and the corresponding one fourth sub-harmonic, one half subharmonic and primary resonances of the out-of-plane symmetric mode. The local analysis, focusing on the modal interaction, participation, competition and bifurcation for different small ranges of the control parameter spaces, and the global analysis, focusing on a large region of the control parameter space, were the major works of these two papers.

Lee and Perkins (1992) studied the existence and stability of weakly nonlinear, periodically forced cable oscillation containing a two-to-one internal resonance between the symmetric in-plane and out-of-plane modes. The governing differential equations were derived using the Hamilton's principle and discretized using a two-degree-offreedom model. Both first order and second order perturbation analyses were applied to the discrete model. The steady state solutions obtained by the first-order and secondorder expansions, and direct numerical integration starting from differential initial
conditions were compared for three cases with different damping, frequency and forcing levels. It was shown that the in-plane vibration saturated for the first-order expansion but actually was disrupted by the second-order corrections. The stable and unstable branches of in-plane response were degenerated in the first-order expansion but actually were split by the second-order corrections. The asymmetry and jump phenomenon of the frequency response were observed also. Perkins (1992) considered a suspended cable subjected to a very small tangential oscillation at one support in the presence of a two-to-one internal resonance between the symmetric in-plane and out-of-plane modes. The quasi-static assumption in which the longitudinal inertial force was neglected was adopted to combine the longitudinal equation of motion with transverse ones. The obtained governing differential equations had variable coefficients for both in-plane and out-ofplane vibrations, and inhomogeneous terms for the in-plane vibration only. This told us that the support oscillations led to parametric excitation for the out-of-plane vibration and simultaneous parametric and external excitations for the in-plane vibration. The amplitude-forcing and amplitude-frequency response for various internal and external detuning cases were investigated. The disruption of the saturation phenomenon due to the parametric resonance was shown in the amplitude-forcing plots. The separation of parametric and external resonances due to internal detuning was shown in the amplitudefrequency plots. An experiment was carried out for a cable around the cross over region and the results were in well qualitative agreement with theoretical predictions. Lee and Perkins (1995) studied the simultaneous resonances of a suspended cable using a three-degree-of-freedom model (symmetric in-plane, and symmetric and asymmetric out-ofplane modes). A second-order perturbation method was adopted and the detuning for two
modes excited by different internal resonances were ordered differently so that the $1: 1$ resonance appeared with cubic nonlinearities at the second order and the $2: 1$ resonance appeared with quadratic nonlinearities at the first order. Four classes of solutions were found: pure in-plane response, 1:1 internally resonant response, 2:1 internally resonant response and simultaneous $1: 1$ and 2:1 internally resonant response. The stabilities of these solutions were determined with respect to perturbations in all three modes by linearizing the autonomous equations about each singular point. Pitchfork, saddle node and Hopf bifurcations were observed. The Poincare section of the quasi-periodic vibration between Hopf bifurcations $f$ and $g$ represents a cross section of a torus attractor. It was shown that the stable and unstable 2:2:1 periodic solutions defined the transition between stable periodic solutions dominated by quadratic nonlinearities (2:1) and those dominated by cubic nonlinearities (2:2). Lee and Perkins (1995a) experimentally studied the isolated and simultaneous internal resonances for a suspended cable driven by harmonic, planar excitations. Two types of response tests were conducted with one of the parameters (excitation frequency and amplitude) hold constant while the other one varied. Isolated 2:1 and 1:1 resonances and simultaneous 2:2:1 resonance was observed for cables with some particular curvatures.

Luongo and Piccardo (1998) investigated the galloping motion of a cable at the first cross-over point under a transverse wind flow excitation. Both analytical solutions by perturbation analysis and numerical ones by numerical integration were developed and compared. As well as aerodynamic non-linearities, geometric nonlinearities (the major factors responsible for the coupling phenomena) were considered for this aero-elastic oscillation problem. The critical values of the mean wind speed for bifurcations
(appearance and disappearance of mono-modal and bimodal gallopings) were detected and 1:2 internal resonance was verified.

Studies of Benedettini and Rega (1994), Benedettini, Rega and Alaggio (1995), and Lee and Perkins (1995) were conducted by applying the method of multiple scales to a set of second-order ordinary-differential equations obtained by discretization using the Galerkin procedure. Rega, Lacarbonara, Nayfeh and Chin $(1997,1999)$ investigated the multiple resonance of a suspended cable near the first crossover by applying the method of multiple scales directly to the second-order governing differential equations of motion and associated boundary conditions (direct approach) and to the four-degree-of-freedom Galerkin discretized model (discretizatioin approach). In order to render their reconstituted modulation equations derivable from a Lagrangian, it was necessary to include the homogeneous solutions of the second-order problem. The results were inconsistent because there were some coefficients depending on arbitrary constants. Moreover, the frequency response curves from the two approaches showed quantitative as well as qualitative differences even with the same choice of equation constants. This was due to the fact that the influence of nonlinearities on the responses represented by the coefficients of the modulation equations were not accurately evaluated because the coefficients, supposed to be determined by infinite eigen-modes, were actually determined by only the first several ones assumed in the dicretization approach. To overcome the inconsistencies, Nayfeh, Arafat, Chin and Lacarbonara (2002) firstly rewrote the two governing partial differential equations as a system of four first-order (in time) ones. Then, the method of multiple scales was applied directly to these first order equations instead of second order ones as before, and the second-order uniform
asymptotic expansions of the solution were obtained. It was shown that the modulation equations, obtained by the method of reconstitution (Nayfeh, 1985) and governing the dynamics of the amplitudes and phases of the four interacting modes, satisfied all of the symmetry conditions, and therefore they were derivable form a Lagrangian. Furthermore, all of the coefficients of the non-linear terms in the modulation equations were determined uniquely, and hence the solution was systematic and consistent.

Arafat and Nayfeh (2003) investigated the nonlinear response of shallow suspended cables subjected to primary resonance excitations. The influence of the number of terms retained in the application of the discretization approach on the accuracy of the predicted effective non-linearity was especially investigated. The results from direct treatment and discretization approach were compared for responses to four resonant excitations - harmonic excitations at the resonant frequencies of the first symmetric and anti-symmetric in-plane and out-of-plane modes. It was assumed that the directly excited mode was not involved in an auto-parametric resonance with any of other modes. It was shown that only the symmetric in-plane modes contributed to the effective non-linearity. Discretization of single-mode, or even two-mode and three-mode for some cases, might result in qualitative error by predicting the effective non-linearity as hardening while in fact it was softening. The solutions converged to the direct approach solutions once enough modes were retained in the discretization procedure.

### 5.2 Summary of Theories

### 5.2.1 Linear Theory for the Free Vibration of Suspended Cables

Cables and strings are one-dimensional structures that can only sustain longitudinal tension loads because they have negligible flexural, torsional, and shear rigidities and zero buckling loads. A taut string is a straight cable under pretension with a sag-to-span ratio close to zero. A cable is a string with sag. The commonly used governing partial differential equations for cable vibrations were derived by Lee and Perkins (1995) as

$$
\begin{gather*}
{\left[c_{t}^{2}+c_{l}^{2} g(t)\right] w_{s \mathrm{~s}}+\frac{c_{l}^{2}}{c_{t}^{2}} g(t)+\hat{F}(s) \cos (\Omega t)=w_{t t}+\hat{\mu}_{w} w_{t}}  \tag{5.2.1}\\
{\left[c_{t}^{2}+c_{l}^{2} g(t)\right] v_{s s}=v_{t t}+\hat{\mu}_{v} v_{t}} \tag{5.2.2}
\end{gather*}
$$

where $g(t)=\int_{0}^{t}\left[-v / c_{2}^{2}+\left(w_{s}^{2}+v_{s}^{2}\right) / 2\right] d s, w$ is the dynamic in-plane displacement, $v$ is the dynamic out-of-plane displacement, $\hat{\mu}_{w}$ is the damping coefficient of the in-plane vibration, $\hat{\mu}_{v}$ the damping coefficient of the out-of-plane vibration, $c_{l}$ is the longitudinal wave speed, and $c_{t}$ is the transverse wave speed. We see there are quadratic and cubic terms which are due to the initial curvature and the stretching during the vibration, respectively. Increasing the sag, the quadratic nonlinearity effect is expected to increase because the curvature increases and the cubic nonlinearity effect is expected to decrease because the corresponding possible stretching decreases. When the cubic nonlinearity dominates over the quadratic nonlinearity, the behavior of taut strings is dominant due to the stretching during vibration. When the quadratic nonlinearity dominates over the cubic nonlinearity, the behavior of cable is more evident due to the prominent influence of the
initial curvature. As a result, just like what was shown in many experimental observations, there are much discrepancy between the dynamics of cables and that of strings though they are similar as far as the geometry is concerned. Basically, this is because the frequencies of the in-plane symmetric modes of cables are dependent on the elasto-geometry parameter, which results in the well known cross-over phenomenon, while those of other modes are kept unchanged.

Let's start from an important parameter for the dynamics of cables and sagged "strings". The elasto-geometric parameter $\kappa$, defined as

$$
\begin{equation*}
\kappa^{2} \approx(8 d / L)^{3} E A /\left[m g L\left(1+8(d / L)^{2}\right)\right] \tag{5.2.3}
\end{equation*}
$$

describes the geometry and material of a cable and governs the nature of solutions to the linear governing differential equation of cables. In most practical problems, it is the sag-to-span ratio $d / L$, rather than the cable elasticity term $E A /\left[m g L\left(1+8(d / L)^{2}\right)\right]$, that dictates the value of $\kappa^{2}$ and so the dynamics. We discussed in previous chapters that for a taut string, the elasto-geometric parameter is close to zero due to zero or small sag $(d \approx 0)$. The natural frequencies of a taut string are $\omega_{n}=\lambda_{n}=n \pi c_{v} / L$ with $n=1,2,3, \cdots$ for in-plane and out-of-plane modes, where $c_{v}=T / \rho A$ is the transverse wave velocity, $T$ the tension, and $\rho A$ the mass per unit length. Keep in mind that $n$ is an odd number for symmetric modes and $n$ is even for anti-symmetric modes. Increasing the sag and so the elasto-geometric parameter from zero, a string will behave more like a cable. For a cable, the normalized out-of-plane mode shapes are given by $\chi_{n}=\sqrt{2} \sin (n \pi s)$ corresponding to the natural frequencies $\lambda_{n}=n \pi$ with $n=1,2,3, \cdots$. The
anti-symmetric in-plane modes have mode shapes $\phi_{n}=\sqrt{2} \sin (n \pi s)$ corresponding natural frequencies $\omega_{n}=n \pi$ with $n=2,4,6, \cdots$ (Nayfeh and Pai, 2004). They are independent of the sag $d$ and have the same properties as those of a taut string. The reason is that no additional tension is induced during vibrations that are composed of these modes. The mode shapes and natural frequencies of symmetric in-plane modes are different. For example, with the increase of sag from zero, the frequency of the first symmetric in-plane mode (i.e., $\phi_{1}=\sqrt{2} \sin (\pi s)$ ) grows from $\omega_{1}=\pi$ to a value close to but less than the natural frequency of the third out-of-plane mode, $\lambda_{3}=3 \pi$, and the corresponding mode shape changes from $\sin (\pi s)$ (no inner node) towards $\sin (3 \pi s)$ (two inner nodes), as shown in Fig. 5.1. During


Fig. 5.1: The first symmetric in-plane mode of a horizontal, small-sag elastic cable with the elastic-geometric parameter around the first crossover ( $\kappa / \pi=2$ ).
the process, there is a point called (the first) cross-over point when a specific sag makes $\kappa=2 \pi$, and hence the frequency of the first symmetric mode and the frequency of the first anti-symmetric mode are equal, i.e., $\omega_{1}=\omega_{2}=2 \pi$. The mode shape of the first symmetric mode at the first crossover is tangential to the horizontal line at the supports, as shown in Fig. 5.1. Considering the tension variation, the natural frequencies of the symmetric in-plane modes are the roots $\beta l$ of a non-linear equation (Irvine and Caughey, 1974)

$$
\begin{equation*}
\tan (\beta l / 2)=(\beta l / 2)-4 / \kappa^{2}(\beta l / 2)^{3} \tag{5.2.4}
\end{equation*}
$$

Obviously, the solutions are closely dependent on the elasto-geometric properties of the cable. The graphical solution for the first non-zero root is shown in Fig. 5.2. The first root is always larger than $\lambda_{1}$ but smaller than $\lambda_{3}$ for $d, \kappa \neq 0$. From above analysis, we can see the frequency of the first out-of-plane symmetric mode is the lowest natural frequency of any given cable. Except when the internal resonance exists between the in-plane and out-of-plane modes, the cable may have free out-of-plane vibration because the out of plane motion induces no additional tension to any first-order cable stretching and so less energy is necessary for the cable to be excited. We note here that additional tension is induced in the nonlinear range (Srinil, Rega, and Chucheepsakul, 2003). As a common sense, it is geometrically impossible for an inextensible (i.e., no stretch) cable to oscillate in a symmetric mode because stretching is required for such deformations. For $\kappa / \pi<2$, the frequency of the first symmetric in-plane mode $\omega_{1}$ is less than the frequency of the first anti-symmetric in-plane mode and the relation between the first few natural frequencies
is $\omega_{1}<\omega_{2}=\lambda_{2}=2 \lambda_{1}$. When $\kappa^{2}$ is very small, equation (5.2.4) is reduced to $\tan (\beta l / 2)=-\infty$, whose roots are equal to the natural frequencies of out-of-plane


Fig. 5.2: Graphical solution for the first non-zero root of equation (5.2.4), which describes the relation between the natural frequency ( $\beta l$ ) of the first symmetric in-plane mode of a horizontal, small sag elastic cable with its elastic-geometric parameter around the first crossover $(\kappa / \pi=2)$.
symmetric modes $\omega_{n},(\beta l)_{n}=(2 n-1) \pi$ where $n=1,2,3, \cdots$. At $\kappa / \pi \approx 2$, called the first avoidance or cross-over point, the frequency of the first symmetric in-plane mode (S-I), anti-symmetric in-plane mode (A-I) and the first anti-symmetric out-of-plane mode (A$\mathbf{O}$ ) are equal and are twice the natural frequency of the first symmetric out-of-plane mode (S-O), and the relation between them is $\omega_{1} \approx \omega_{2}=\lambda_{2}=2 \lambda_{1}$. Due to the commensurable relation between these frequencies, various internal resonances may happen. For $\kappa / \pi>2$, the frequency of the first symmetric in-plane mode is larger than the
frequency of the first anti-symmetric in-plane mode $\omega_{1}>\omega_{2}=\lambda_{2}=2 \lambda_{1}$. When $\kappa^{2}$ is very large, $\omega_{1}$ is almost equal to $\lambda_{3}$ and the dynamics of the cable will not much influenced by the change of $\kappa$. This is the reason why the assumption of inextensibility (i.e., neglecting the stretching) is appropriate for a cable with a large sag-to-span ratio. Similar properties exhibit for a cable around the second and other avoidance or cross over points where $\kappa / \pi \approx 2 n, n=2,3 \cdots$, each corresponding to the natural frequency relation $\omega_{2 n-1} \approx \omega_{2 n}=\lambda_{2 n}=2 \lambda_{n}$ and marking a transition of the nth symmetric in-plane mode from a typical response of "elastic and taut cables" to one of "stiff and sagged cables" with two inner nodes added and more dynamic tension induced. The first three crossover points are plotted in Fig. 5.3.


Fig.5.3: Variation of the first few non-dimensional natural frequencies with the parameter $\kappa / \pi$. A-O/I: Anti-symmetric (Out of/In) plane mode, S-O/I: Symmetric (Out of/In) plane mode.

### 5.2.2 Resonances and Modal Interactions

For a cable at the neighborhood of the first crossover point (the most studied case), isolated and simultaneous resonances may happen. This is because the frequencies of symmetric and anti-symmetric in-plane and out-of-plane modes are related as $\omega_{2 n-1} \approx \omega_{2 n}=\lambda_{2 n}=2 \lambda_{n}, n=1,2,3, \cdots$, which makes the response of the excited mode (the first symmetric in-plane mode) become a parametric excitation to other three modes through quadratic and cubic nonlinearities. At the first cross over point $\kappa / \pi \approx 2, \omega_{1} \approx \omega_{2}=\lambda_{2}=2 \lambda_{1}$ and theoretically possible isolated resonances between the first two in-plane and out-of-plane modes are internal resonance between the $1^{\text {st }}$ S-I and $1^{\text {st }}$ A-I modes, $\omega_{1}: \omega_{2} \approx 2: 2$ internal resonance between the $1^{\text {st }}$ S-I and $1^{\text {st }} \mathrm{A}-\mathrm{O}$ modes, $\omega_{1}: \lambda_{2} \approx 2: 2$ internal resonance between the $1^{\text {st }} \mathrm{A}-\mathrm{I}$ and $1^{\text {st }} \mathrm{A}-\mathrm{O}$ mode, $\omega_{2}: \lambda_{2}=2: 2$ internal resonance between the $1^{\text {st }} \mathrm{S}$-I and $1^{\text {st }} \mathrm{S}$-O modes $\omega_{1}: \lambda_{1} \approx 2: 1$ internal resonance between the $1^{\text {st }} \mathrm{A}$-I and $1^{\text {st }} \mathrm{S}$-O modes $\omega_{2}: \lambda_{1}=2: 1$ internal resonance between the $1^{\text {st }} \mathrm{A}-\mathrm{O}$ and $1^{\text {st }} \mathrm{S}-\mathrm{O}$ modes $\lambda_{2}: \lambda_{1}=2: 1$. Possible simultaneous internal resonances may happen between the symmetric in-plane and anti-symmetric and symmetric out-of-plane modes became $\omega_{1}: \lambda_{2}: \lambda_{1} \approx 2: 2: 1$ and between anti-symmetric in-plane and anti-symmetric and symmetric out-of-plane modes became $\omega_{2}: \lambda_{2}: \lambda_{1} \approx 2: 2: 1$. For these cases, 1:2 super-harmonic resonance, in which the resonant frequency is two times of the excitation frequency, and 2:1 sub-harmonic resonance, in which the resonant frequency is a half of the excitation frequency, may
happen. Actually, multiple resonance $\left\langle\Omega=\omega_{4}\right\rangle: \omega_{2}: \lambda_{4}: \lambda_{2}=4: 2: 4: 2$ is possible if the excitation is close to the natural frequency of the second anti-symmetric mode. These phenomena, not recorded before, were observed in our experiments.

### 5.3 Experimental Observations

Loosing the string studied in Chapter 4, we obtained a cable with initial curvatures. As the cable was made of steel and had a length only about 1.4 m and a diameter less than 0.5 mm , the change of the pretension due to loosing did not change the length much. So, it is appropriate for the cable to use the same spatial distribution of markers as the string. Except the curvature which is dependent on the elasto-geometrical parameter $\kappa$, parameters of the cable (i.e., the length, diameter, and mass density) are kept unchanged.

### 5.3.1 Experimental Identifications

Following the same procedure of frequency scanning we did for the string testing in Chapter4, we obtained the displacements of all markers in the time domain using the 3D motion analysis system. Based on these time domain data, we perform the vibration analysis. There are many qualitative and quantitative measures that can be used to analyze the experimental data. We may use one or some of these measures to identify the dynamic characteristics of the vibration, depending on the properties of typical responses.

The frequency spectrum obtained by FFT analysis is by far one of the most widely used measures. The spectrum of a periodic, a period-doubled, a quasi-periodic,
and a chaotic signal has one spike, one primary spike with rationally related sidebands representing sub-harmonic and super-sub-harmonics, two spikes not rationally related, and a broad band of frequencies, respectively. For beating signals, the spectrum has two spikes with slightly different frequencies and no sidebands. Autocorrelation function is another tool for signal processing related to FFT. It is a time domain function measuring how much a signal resembles a delayed version of itself. For a periodic signal, the autocorrelation is one at zero and other integral times of the period and zero at odd integral times of one half of the period. For other instants, it is between one and zero. For a chaotic signal, it is one at zero time delay and essentially zero at other delays. Due to the possible presence of a principle frequency component, the auto-correlation function may decay to zero after some time instead of immediately. So, the functions are periodic of a unit amplitude for periodic signals and non-periodic of a non-zero amplitude for quasi-periodic signals. Pincare sections can be obtained by sampling the position of any marker on the cable once per period (stroboscopic map) or a chosen time interval. For a periodic signal, the Poincare map has only one point because the vibration comes back to the same position after one period. For period-doubled, period-4, and period-8 signals, the map has two, four, and eight points, respectively. For a quasi-periodic signal, it has many points along the trajectory and the dimension of the map is one, which means the vibration is possible at anywhere of the trajectory during the vibration. For a chaotic signal, the map is an attractor and has a dimension between one and two for a two dimension flow. The correlation/fractal dimension calculated from experimental data helps verify the existence of a strange attractor, as well as provides a measure of its fractal structure. The exponential growth rate of a dynamic system responding to an
infinitesimal perturbation is called the Lyapunov exponent. For regular motions, the Lyapunov exponent $\lambda \leq 0$ and for chaotic motion $\lambda>0$. Showing the probability of the chaotic trajectory visiting some particular region of the attractor, the Probability Distribution Functions (PDFs) provides a statistical measure of a chaotic dynamics for the probability or invariant distribution of the attractor.

### 5.3.2 Experimental Results

The tested cable had a sag-to-span ratio1/321 before the test and $1 / 378$ after the test and the elasto-geometric parameters $\kappa / \pi$ were 2.84 and 1.7441 , correspondingly. Fig. 5.4 shows the XY and YZ projections of the cable before the test. The main reason for the change of sags was that the geometry was essentially changed. Although the cable is flexible, there is still small but influential bending stiffness and hence the static equilibrium configuration was not only determined by its weight. After the high speed oscillation, the geometry of the cable was not exactly the same as the one before the vibration tests. Although this change of geometry was small, the change of parameter was not small because the material was steel and the span was so small (less than 2 m ) according to (5.2.3). Another reason was that the markers were not firmly stuck to the cable. So the effective positions of the markers and hence the sag might be changed due to high speed rotation. Moreover, although the camera precision was pretty high ( 0.01 mm ), the measurement accuracy was lower because of the large size of markers (more than 1 mm ). Moreover, pretty small rotation or deformation happened to the markers might change the lights reflected back to the cameras and the recorded positions of the markers in the camera system. Then the measured or recorded positions of markers
at two different times might be quite different. Fortunately, the parameter $\kappa / \pi$ was around 2 before and after the test, which means the cable was around the first crossover during the test. And consequently, it is still valid to use this cable to investigate the nonlinear properties, especially those internal resonances that may happen only to cables at the crossovers. The excitation frequency range we tested was from 10 Hz to 53 Hz .


Fig. 5.4: The static equilibrium configuration of the cable ( $\mathrm{X}-\mathrm{Y}$ and $\mathrm{X}-\mathrm{Z}$ views) before the test with a sag-to-span ratio $1 / 321$, corresponding to an elasto-geometric parameter $\kappa / \pi=2.84$.

The modal coordinates which indicate the participations of modes are more meaningful for the investigation of cable dynamics dominated by modal coupling. The frequency response curves are plotted as detuning versus the modal coordinates of the responses, instead of detuning versus the response of one marker as those in Chapter 4 for strings. Because there are four modal coordinates for one response and there are possible
multiple responses for one excitation, one figure with four responses will be too complicated. To present the results clearly, we plot the response in two different ways. The first is to plot the contributions of some particular linear normal mode to all tested excitations. Fig. 5.5 shows plots of contributions of the first four linear normal modes to the responses in the X-Z plane to all tested excitations. Fig.5.6 are plots of contributions of the first four linear normal modes to the responses in the $\mathrm{X}-\mathrm{Y}$ plane to all tested excitations. From these plots, we see some resonances that will be explained later. Based on the number of different responses obtained, the scanned frequency range can be divided into five subintervals, S1, S2, S3, S4 and S5, as marked in the plots. Based on the major contributing mode for the vibration, the recorded frequency responses can be named as several different continuous Branches (B1, B2, B3 and B4). Note that we did not present the frequency response curve for one marker for all excitations and hence mark the branch divisions in the curves as we did for string experiment. The second type of plot is the contributions of all four linear normal modes to responses of a branch versus all excitations of this branch. Fig. 5.7 are plots of contributions of the first four linear normal modes to the responses in the $\mathrm{X}-\mathrm{Z}$ plane for different branches, and Fig. 5. 8 are plots of contributions of the first four linear normal modes to the responses in the $\mathrm{X}-\mathrm{Y}$ plane for different branches.

From 10 Hz to 12 Hz is the first subinterval (S1). The only response to an excitation frequency within this subinterval is the first-mode vibration. From 10 Hz to 11 Hz , the response is planar and changes from incompletely excited to fully excited. From 11.5 Hz to 12 Hz , the response becomes non-planar. All responses to excitations within this subinterval belong to B1.


Fig. 5.5: Contributions of the first four linear normal modes to the responses in the $\mathrm{X}-\mathrm{Z}$ plane: (a) the first mode, (b) the second mode, (c) the third mode, and (d) the fourth mode.


Fig. 5.6: Contributions of the first four linear normal modes to the responses in the $\mathrm{X}-\mathrm{Y}$ plane: (a) the first mode, (b) the second mode, (c) the third mode, (d) the fourth mode.


Fig. 5.7: Contributions of the first four linear normal modes to the responses in the $\mathrm{X}-\mathrm{Z}$ plane for different branches: (a) Branch\#1, (b) Branch\#2, (c) Branch\#3, and (d) Branch\#4.



Fig. 5.8: Contributions of the first four linear normal modes to the responses in the $\mathrm{X}-\mathrm{Y}$ plane: (a) Branch\#1, (b) Branch\#2, (c) Branch\#3, and (d) Branch\#4.

The second subinterval (S2) is from 12.5 Hz to 21 Hz . Within this interval, two different responses were obtained at each excitation frequency. The non-planar vibration of the first mode (B1) was stable and its amplitude increased as the frequency increased. The other one started to appear at 12.5 Hz . It was a vibration transition from the first mode to the second mode (B2), and from planar to non-planar. Actually, as the second mode participated more and more, the vibration experienced period-doubling before it became fully non-planar and the trajectory was one full ellipse. The third subinterval (S3) starts from 22 Hz to 34 Hz and has three different responses. Besides the two responses (B1 and B2) shown in S2, the third mode came into the response starting from 22 Hz (B3). The newly appeared response was a transition from the second mode to the third mode. At the beginning, the response was planar. The stable region for this planar response was so small that for some excitations, we did not obtain any stable response. The response always jumped to a non-planar vibration of the second mode. At the end of this interval, the response became a non-planar vibration of the third mode. At 35 Hz , the beginning of the fourth subinterval, vibrations representing a transition from the third
mode to the fourth mode appear (i.e., B4). The other three types of responses, non-planar vibration of the first, second and third modes (B1, B2 and B3) remained. Four different responses existed within this excitation interval. Due to the capability of the motion analysis system, we did not get very accurate measurements of these small-amplitude higher-mode vibrations. This interval lasted to 38 Hz . At 39 Hz , the non-planar vibration of the first mode (B1) lost stability and disappeared. The left three types of responses remained in the frequency range from 39 Hz to 53 Hz , which is the fifth subinterval (S5). Table 5.1 shows our records for of the experimental responses.

The first branch (B1) represents both planar and non-planar vibrations mainly composed of the first mode. This is verified by Fig. 5.7 (a) in which the contribution of the first linear normal mode is larger than other modes. This branch starts from 10 Hz with a planar vibration, becomes non-planar at 12 Hz , and lasts till 38 Hz at which the non-planar vibration of the first mode loses stability and disappears. Although the cable is at the first crossover and so various modal couplings may happen, internal resonances are not very evident in the response plots. This is because the directly excited planar vibration of the first mode has a amplitude larger than those of other modes. Fig. 5.9 shows plots of one of the two typical responses when the string was excited at $\Omega=13.5 \mathrm{~Hz}$. The frame rate was $513 F P S$ and the recording time length was 2 seconds. Correspondingly, the total number of the captured frames is $N F=1026$ and the number of frame per excitation period is $513 / 13.5=38$ frames. Fig. 5.9 (a) shows the first 38 curve fitted consecutive vibration profiles. The total used time $T=(38-1) / 513=0.07212 \mathrm{sec} \approx 0.072 \mathrm{sec}$, which was the rounded value shown
Table5:1: $\lambda_{i}$ are frequencies of excited out-of-plane modes. $\omega_{i}$ are frequencies of excited in-plane
modes. $I P / O P$ means in-plane/out-of-plane response. $\uparrow / \downarrow$ means the amplitude of the response increases or
decreases as the frequency increases. Mode $\# 1>M o d e \# 2$ means the amplitude of $M o d e \# 1$ is larger than that of
Mode $\# 2$ for responses in this branch excited at a frequency in this interval. [ ] indicates the number of
different responses for this subinterval. ( ) represents the type of the response. $\rangle$ is the excitation frequency.

|  |  | Freq.(Hz) | Branch\#1 | Branch\#2 | Branch\#3 | Branch\#4 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 1[1] | 10~11; | (1)-IP: Mode\#1 $\uparrow$ |  |  |  |
|  |  | 11.5~12; |  |  |  |  |
|  |  | 12.5~15; |  | $\text { (3) }-I P \text { : }$ <br> Mode\# $2 \uparrow$ Mode\#1 $\downarrow$ |  |  |
|  | 2[2] | 16~21; | $\text { (2) }-O P \text { : }$ <br> Mode\#1 $\uparrow$ Mode\# $2 \downarrow$ <br> Mode\#1>Mode\#2 $\left\langle\omega_{1}\right\rangle: \lambda_{1}: \omega_{2}: \lambda_{2}$ | (4)-OP: <br> Mode\# $2 \uparrow$ Mode\#1 $\uparrow \downarrow$ <br> Mode\#2 > Mode \#1 $\left\langle\omega_{2}\right\rangle: \lambda_{2}: \lambda_{1}=2: 2: 1$ |  |  |
|  | 3[3] | 22~27; | $=1: 1: 2: 2$ |  | $\text { (6) }-I P \text { : }$ <br> Mode\#3 $\uparrow$ Mode\# $2 \downarrow$ |  |
|  |  | 28~34; |  |  |  |  |
|  | 4[4] | 35~38; |  | $\text { (5) }- \text { OP: Mode\# } 2 \uparrow$ |  | $\text { (9) }-O P \text { : }$ <br> Mode\#4 $\uparrow$ Mode\#3 $\downarrow$ |
|  | 5[3] | 39~53; |  | $\left\langle\omega_{2}\right\rangle: \lambda_{2}=2: 2$ | $\begin{aligned} & \text { (8)-OP: Mode \#3 } \uparrow \\ & \left\langle\omega_{3}\right\rangle: \lambda_{3}=3: 3 \end{aligned}$ | $\text { (9) }-O P \text { : }$ <br> Mode\#4 $\uparrow$ Mode $\# 2 \rightarrow$ $\begin{aligned} & \left\langle\omega_{4}\right\rangle: \omega_{2}: \lambda_{4}: \lambda_{2} \\ & =4: 2: 4: 2 \end{aligned}$ |


(b)


(d)

(e)


Fig. 5.9: Response to an excitation at $\Omega=13.5 \mathrm{~Hz}$ : (a) 38 consecutive vibration profiles ( 0.072 sec ), (b) trajectories of marker ( $2,6,10,15,25$ ), (c) trajectories of marker ( 25,30 , $35,40,44,47$ ), (d) the frequency spectra of the Z-vibration of marker 25 , (e) the frequency spectra of the Y-vibration of marker 25, (f) the frequency spectra of the Zvibration of marker 10, (g) the frequency spectra of the Y-vibration of marker 10, (h) the
modal coordinates of the X-Z plane vibration, (i) the frequency spectra of the modal coordinates of the $\mathrm{X}-\mathrm{Z}$ plane vibration, $(\mathrm{j})$ the modal coordinates of the $\mathrm{X}-\mathrm{Y}$ plane vibration, and (k) the frequency spectra of the modal coordinates of the $\mathrm{X}-\mathrm{Y}$ plane vibration.


Fig. 5.9 (Continued)
in the experimental output data file. The vibration profiles on the $\mathrm{X}-\mathrm{Y}$ and $\mathrm{X}-\mathrm{Z}$ planes are not symmetric and there are intersections between different profiles. This means the vibration waves are traveling or propagating, which is due to the participations of modes (mainly the second mode) other than the first one. Moreover, the vibration profiles in the two planes tilt to different directions. This is due to different phases of the two major vibration components (the first and second linear normal mode) for vibrations in the two polarized planes. This can be seen in from the time-domain traces of the modal coordinates shown in Fig. 5.9(h) and (j). Fig. 5.9(h) and (j) also show that the major modes are the first and second modes that have amplitudes about 10 mm and 2 mm , respectively. Higher modes are negligible compared with these two. The frequency spectra of the modal coordinates show that the first mode vibrated at the excitation frequency. The second mode vibration contained two harmonics. The first one is $\Omega$ with a small amplitude and the other one is $2 \Omega$ with a larger amplitude. The first one is due to linear coupling because the excitation frequency is not close enough to the natural frequency of the first or the second linear normal mode. The second one is due to $1: 2$ internal resonance. So, internal resonances $\left\langle\Omega=\omega_{1}\right\rangle: \lambda_{1}: \omega_{2}: \lambda_{2}=1: 1: 2: 2$ between the first symmetric and anti-symmetric in-plane and out-of-plane modes happened. Here $\left\langle\omega_{1}=\Omega\right\rangle$ means the in-plane symmetric mode is directly excited and has a frequency $\omega_{1}$ equal to the excitation frequency $\Omega$. This type of simultaneous internal resonances was not theoretically predicted and studied before. We conjecture that this is because the cross-over and corresponding commensurable relations between frequencies does not really exist though the cable is at the first crossover based on the linear theory.

The cable behaved more like a string and the commensurable relations of frequencies of a string dominated the vibration. However, the quadratic nonlinearity due to initial curvature exists and consequently the 1:2 simultaneous resonances happened. Figs. 5.9(d)-(g) show the frequency spectra of in-plane and out-of-plane vibrations of marker 25 (approximately the midpoint of the cable) and marker 10 (about $1 / 4$ span of the cable). Again, the major harmonics were $13.5406 \mathrm{~Hz} \approx \Omega$ and $27.08 \mathrm{~Hz} \approx 2 \Omega$ for vibrations in both planes. We note that amplitudes of the $2 \Omega$ components in two planes of marker 10 are much larger than those of marker 25 . This is consistent with the fact that the $1 / 4$ location was at the peak of the second mode shape.

Figs. 5.9 (b) and (c) are trajectories of markers 2, 6, 10, 15, and 25 and markers $25,30,35,39$, and 43 plotted in one figure, respectively. It is interesting to see the shapes of the trajectories, which was not shown or observed by previous researchers because their experimental could measure only one point. Using the major harmonics and their amplitudes of the responses in two directions, we numerically simulated the vibration and obtained trajectories almost the same as above experimental results. The number of points in each trajectory reveals that the vibration was periodic and had a frequency $\Omega$ because each maker came back to the same position after exactly one excitation period. At the end of this branch, responses were fully excited first-mode vibrations. Fig. 5.10 show the vibration profiles when the cable was excited at $\Omega=38 \mathrm{~Hz}$. The major harmonic is the first symmetric mode for both in-plane and out-of-plane vibrations. The second mode (the first anti-symmetric mode), though with a small amplitude, are not negligible and accounts for the unobvious but discernible crossovers between different profiles.


Fig. 5.10: Response to an excitation at $\Omega=38 \mathrm{~Hz}$ : (a) 28 consecutive vibration profiles ( 0.051 sec ), and (b) trajectory of marker 25 .

The second branch (B2) started at 12.5 Hz with planar vibrations mainly composed of the first and second modes. It existed in the whole frequency range of our test. At the beginning, the multi-mode planar vibrations were basically due to linear coupling because the excitation was not close to the natural frequencies of neighboring linear normal modes. Also small damping makes modes with natural frequencies close to the excitation frequency to be excited. As the excitation frequency increased, the second mode participated more and more and finally took over the first mode, and became the predominant component of the non-planar vibration. As the excitation frequency was close to the natural frequency of the second mode, multi-mode resonances due to nonlinear coupling happened.

Fig. 5.11 show the response of the cable excited at $\Omega=18 \mathrm{~Hz}$, a typical non-planar response of the second branch. The frame rate used was 522FPS and the recording time length was 2 seconds. Correspondingly, the total number of captured frames was $N F=1044$ and the number of frame per excitation period was 522/18 $=29$.

Figs. 5.11(a) and (b) show 29 and 58 curve-fitted consecutive vibration profiles respectively. The time length is $T=(29-1) / 522=0.054 \mathrm{sec}$ for Fig. 5.11(a) and 0.109 sec for Fig. 5.11(b). These profiles indicate that the vibration as periodic because the cable came back to the same position and had the same shape after some time. The 29 asymmetric profiles shown in Fig. 5.11(a) indicate that the period of the vibration was longer than the excitation one because the vibration did not come back to the same position after one excitation period. The 58 symmetric profiles shown in Fig. 5.11(b) indicate that the period was doubled because the cable came back to the same position after two excitation periods. These profiles sow that the wave was traveling and different parts of the cable passed through their zero or maximum deflection positions at different time instants, which is an important characteristic of complex vibration.


Fig. 5.11


Fig. 5.11 (Continued)
(h)

(j)




(I)





(i)

(k)



(m)





Fig. 5.11 (Continued)


Fig. 5.11: Response to an excitation at $\Omega=18 \mathrm{~Hz}$ : (a) 29 consecutive vibration profiles ( 0.054 sec ), (b) 58 consecutive vibration profiles ( 0.109 sec ), (c) trajectory of marker 15 , (d) trajectory of marker 25, (e) trajectory of marker 35, (f) Poincare section of marker 15 by sampling once per excitation period, (g) Poincare section of marker 15 by sampling once per two times excitation period, (h) the frequency spectra of the vibration in the $\mathrm{X}-\mathrm{Z}$ plane of marker 15, (i) the frequency spectra of the vibration in the $\mathrm{X}-\mathrm{Y}$ plane of marker 15 , (j) the modal coordinates of the X-Z plane vibration, ( k ) the frequency spectra of the modal coordinates of the $\mathrm{X}-\mathrm{Z}$ plane vibration, (l) the modal coordinates of the $\mathrm{X}-\mathrm{Y}$ plane vibration, (m) the frequency spectra of the modal coordinates of the X-Y plane vibration. ( n ) error functions of the modal decomposition of the $\mathrm{X}-\mathrm{Z}$ plane vibration, (o) error functions of the modal decomposition of the $\mathrm{X}-\mathrm{Y}$ plane vibration.

Figs. 5.12 and 5.13 show instant profiles of the in-plane and out-of-plane vibrations during two excitation periods ( 58 frames in 0.109 seconds). It is obvious that, during the 0.109 seconds, the vibration in the $\mathrm{X}-\mathrm{Y}$ plane finished only one period while the vibration in the X-Z plane finished two periods. In Fig. 5.12, frames\#1-\#26 belong to the first period, and frames from \#30 to \#58 belong to the second period but they have the same shapes as frame \#1 - \#26. In Fig. 5.13, the out-of-plane vibration, however, shows distinctive shapes for all time instants, except the beginning one and ending one.


Fig. 5.12: The instant profiles of the in-plane ( $\mathrm{X}-\mathrm{Z}$ ) vibration during two excitation periods ( 58 frames). The solid line denotes the vibrating profile and the dashed line denotes the static equilibrium configuration.


Fig. 5.13: The instant profiles of the out-of-plane ( $\mathrm{X}-\mathrm{Y}$ ) vibration during two excitation periods ( 58 frames). The solid line denotes the vibrating profile and the dashed line denotes the static equilibrium configuration.

Figs. $5.11(\mathrm{j})-(\mathrm{m})$ show the time-domain plots of the modal coordinates of the XY- and XZ-plane vibrations and their corresponding frequency spectra. The X-Z plane vibration was mainly composed of the first anti-symmetric mode vibrating at the excitation frequency $\Omega\left(=\omega_{2}\right)$ with an amplitude about 2 mm . The modal coordinate of the first mode had a negative shift about 4 mm which is equal to the initial sag of the cable. The $\mathrm{X}-\mathrm{Y}$ plane vibration was mainly composed of the first symmetric mode vibrating at $\Omega / 2\left(=\lambda_{1}\right)$ and the first anti-symmetric mode vibrating at $\Omega\left(=\lambda_{2}\right)$. These two modes had approximately the same amplitude of 1.6 mm . The participations of these two modes caused the cross-over of profiles, and hence the node was not clear. Vibrations of higher modes are negligible compared with these two. The $\omega_{2}: \lambda_{2}: \lambda_{1}=2: 2: 1$ resonances between the first in-plane anti-symmetric mode and the first out-of-plane anti-symmetric and symmetric modes happened. Fig. 5.11(e) shows that the phase difference between the two major modes is not 0 or $\pi$, which is another character of complex vibration. Thus the concept of nonlinear normal mode is questioned. Fig. 5.11(j) show that the phases of the first harmonic (although negligible) and the second one are the same. The $\mathrm{X}-\mathrm{Z}$ plane vibration was not complex and we see no traveling waves in Fig. 5.11 (a).

Figs. 5.11-(c) - (e) show trajectories (Y-Z views) of markers 15,25 and 35 for the whole test period. Compared with that of marker15, the trajectory of marker 35 is an upside-down version of the trajectory of marker 15, which is expected because the vibration profiles were anti-symmetric with respect to the mid-point of the cable. The trajectory of marker 15 was two ellipses with a finite number (58) of dots evenly
distributed along the two ellipses, indicating it is a periodic motion. The trajectory of marker 25 has a V shape because it is at the middle position where the semi-minor axis of the ellipse shrinks to almost zero while the semi-major axis of the ellipse does not. The arrows indicate the vibration route. The number of dots is two times of the number of frames captured during one excitation period, which means the vibration period is doubled. The dots in Fig.5.11- (f) and (g) are Poincare section plots of marker 15, obtained by sampling the position of the targeted marker once per period and once per two periods respectively. There are two dots in Fig. 5.11(f) and only one dot in Fig. 5.11(g), which means marker 15 came back to the same position after two periods instead of one. Period-doubling is verified again. The excitation period is $T=1 / \Omega$ , $T=1 / 25 \mathrm{~Hz}=0.04 \mathrm{~s}$ in this case. So the total recorded number of frames in the Poincare section for the two seconds is $1000 /(500 \times 0.04)=50$. Figs. 5.11 (c) and (e) show that the vibrations of markers 15 and 35 consisted of two components, one made the ellipse and the other accounted for the sway of the ellipse. This is consistent with the results shown Fig. 5.11 (a) and (b), which reveals that there are multi-mode vibrations in the X-Y plane. For markers closer to the middle point, the minor and major axes of the trajectory will be smaller. We note that the swing of the ellipse still exists. Figs. 5.11 (h, i) are frequency spectra of vibrations of markers 15 in the X-Y and X-Z planes, respectively. We choose to analyze the time domain data of marker 15 because the second mode had a peak around this marker. The amplitudes of the first and second modes at marker 15 are closer compared with those at other marker locations and so the participation of the second mode is more discernible. For the $\mathrm{X}-\mathrm{Y}$ plane vibration, there are two major harmonics as shown in Fig. 5.11(i). One is $\lambda_{1}=9.18 \mathrm{~Hz} \approx \Omega / 2$ with a magnitude about 1.2 mm and the
other is $\lambda_{2}=18.75 \mathrm{~Hz} \approx \Omega$ with an amplitude about 1.5 mm verifying that marker 15 is around the peak position of the second mode. For the X-Z plane vibration shown Fig.5.11 (h), there is only one major harmonic which has a frequency $\omega_{2}=18.75 \mathrm{~Hz} \approx \Omega$. It took two excitation periods for the first symmetric mode vibrating at frequency $\lambda_{1}$ to finish one complete period, making the period of vibration doubled. The frequency spectrum of marker 25 is of a chaotic type. This is because of the inaccurate measurement of the vibration due to the small vibration amplitude, and hence the modes can not be distinctively decomposed. Simultaneous internal resonances $\left\langle\omega_{2}\right\rangle: \lambda_{2}: \lambda_{1}=2: 2: 1$, where $\left\langle\omega_{2}\right\rangle: \lambda_{2}=2: 2$ is a primary resonance and $\left\langle\omega_{2}\right\rangle: \lambda_{1}=2: 1$ is a $1 / 2$ sub-harmonic resonance, happened between the first anti-symmetric in-plane mode, the first antisymmetric and the first symmetric out-of-plane mode. The frequency spectra of vibrations of marker 35 in the $\mathrm{X}-\mathrm{Y}$ and $\mathrm{X}-\mathrm{Z}$ planes have the same properties as those of marker 15. Figs.5. 11(n, o) are spatial and temporal domain errors of the decomposition, Ex, Et and SD of vibrations in the X-Y and X-Z planes. The pretty small values of the error functions indicate that four modes are enough for an accurate decomposition of this response.

At the end of the second branch, it was a fully excited first anti-symmetric mode vibration. Fig. 5.14(a) shows the curve-fitted vibration profiles when the cable was excited at 53 Hz . The exclusively dominant harmonic is the first anti-symmetric mode for both in-plane and out-of-plane responses. Other participating modes, though with small amplitudes, are not completely negligible and account for the asymmetry of the vibration profiles. Additionally, we point out here that the local coordinate system XYZ
defined specific markers might not be really vertical and horizontal, and the Z direction might not be exactly the excitation direction. This might cause the artificial crossovers between different profiles. Comparing Fig. 5.14(a) with Fig. 5.11(a), and with other responses of this branch, we see that the amplitude of the first symmetric out-of-plane mode increased first due to the existence of the $1 / 2$ sub-harmonic resonance and then it became almost zero, indicating the disappearance of resonance. The amplitudes of the first anti-symmetric mode of the in-plane and out-of-plane vibrations increased as the excitation frequency increased.


Fig. 5.14: Response to an excitation at $\Omega=53 \mathrm{~Hz}$ : (a) 20 consecutive vibration profiles ( 0.036 sec ), (b) trajectory of marker 12.

The third branch (B3) started with a planar vibration mainly composed of the second and third modes. As the excitation frequency increased, the third mode participated more and more and finally became the predominant component. Under some low-frequency excitations, we did not obtain stable planar response, which might be due to the stable frequency interval was too small. Under higher-frequency excitations, the responses were mainly composed of the second symmetric mode for the in-plane and out-
of-plane vibrations. Fig. 5.15 shows the vibration profiles when the cable was excited at $\Omega=53 \mathrm{~Hz}$. It is obvious that the major mode is the second symmetric mode for both inplane and out-of-plane vibrations.


Fig. 5.15: Response to an excitation at $\Omega=53 \mathrm{~Hz}$ : (a) 20 consecutive vibration profiles ( 0.036 sec ), and (a) trajectory of marker 8.

The fourth branch (B4) started at 35 Hz with a non-planar vibration consisting of multiple modes. Theoretically predicted planar vibration representing the transition from the third mode to the fourth mode was not observed. It might be due to the closeness of the excitation to integral times of the natural frequencies of all neighboring modes and hence both linear and nonlinear couplings happened. Small damping makes the effective response interval of each mode wider and so neighboring modes can be more easily excited by linear coupling. Because the cable was at the first crossover point, some modes might be excited by nonlinear modal coupling. Figs. 5.16 (a) and (b) show 15 and 30 consecutive vibration profiles respectively when the cable was excited at 35 Hz . The vibration completed a cycle in two excitation periods. Figs. 5.16(g)-(j) show the modal coordinates and corresponding frequency spectra of the XY- and XZ-plane vibrations.

The third (the second symmetric $\omega_{3}$ ) and fourth (the second anti-symmetric $\omega_{4}$ ) modes were excited simultaneously through linear modal coupling because the excitation frequency was close to the third and fourth natural frequencies. The second mode (the first anti-symmetric one $\omega_{2}$ ) was excited by the $1 / 2$ sub-harmonic resonance $\omega_{4}: \omega_{2}=2: 1$ due to quadratic nonlinearity. Vibrations in both planes had the same harmonics but their vibration profiles were different. This is due to different contributions from participating modes. As the excitation frequency increased, the fourth in-plane mode (the second anti-symmetric in-plane mode $\omega_{4}$ ) was more and more directly excited. The third mode (the second symmetric in-plane mode $\omega_{3}$ ), was excited due to linear coupling and initially had a small amplitude, but its amplitude decreased as the excitation frequency increased. The second mode (the first anti-symmetric mode $\omega_{2}$ ) was excited by the $1 / 2$ sub-harmonic resonance because of quadratic nonlinearity, and was excited more and more in both planes.

Figs.5.16 (c)-(f) show frequency spectra of markers 10 and 20 in the $\mathrm{X}-\mathrm{Y}$ and $\mathrm{X}-\mathrm{Z}$ planes. They confirm above conclusions. Figs. 5.17 (a) and (b) show 13 and 26 consecutive vibration profiles respectively when the cable was excited at $\Omega=40 \mathrm{~Hz}$. First of all, the vibration completed a cycle in two excitation periods, as shown in Fig. 5.17(b). We see that the major vibrating modes were the second mode (the first anti-symmetric mode) and the fourth mode (the second anti-symmetric mode) for both in-plane and out-of-plane vibrations. Figs. 5.17 (m)-(p) show the modal coordinates and corresponding spectra for the XY- and XZ-plane vibrations. The two major components with large amplitudes were the second and fourth modes for vibrations in both planes and they
vibrated at $\Omega / 2$ and $\Omega$, respectively. Simultaneous resonances $\omega_{4}: \omega_{2}: \lambda_{4}: \lambda_{2}=2: 1: 2: 1$ happened. The fourth mode $\lambda_{4}$ was excited by the $1: 1$ resonance. The second mode, both the first anti-symmetric in-plane and out-of-plane mode with frequencies $\omega_{2}$ and $\lambda_{2}$, were excited by the 2:1 internal resonance because of the quadratic nonlinearity. Figs. 5.17 (i)-(l) show the frequency spectra of the XY- and XZ-plane vibrations of marker 7 (close to the peak of the fourth mode) and marker 12 (close to the peak of the second mode). It is shown that, at marker 7, the amplitude of the fourth mode had a larger amplitude, and the second mode had a larger amplitude at marker 2. Figs. 5.17 (c)-(f) show the trajectories of markers 8, 19, 32, and 44, respectively. Due to the contributions of different modes at different locations were not the same and the asymmetry of two major modes, the trajectories were not symmetric and had preferred directions. They all had two close loops. They were about symmetric with respect to the middle node of the cable between the trajectories of markers 8 and 32 and markers 19 and 44, and they were anti-symmetric with respect to the $1 / 4$ point of the cable between the trajectories of markers 8 and 19 and markers 32 and 44 . This can be explained by the anti-symmetry of the two main contributing modes. Moreover, because markers 32 and 44 are close to the excitation end, their responses were more influenced by the support-end vibration, which was not small compared with the small amplitudes of the excited higher modes. The Poincare sections of marker 32 shown in Figs. 5.17(g) and (h) reveal that there were two points if sampling once per period and only one point if sampling once every two periods. This tells us the period of vibration was doubled. Simultaneous resonances $\omega_{4}: \omega_{2}: \lambda_{4}: \lambda_{2}=2: 1: 2: 1$ are verified again. Fig. 5.18 shows
plots of responses of the cable excited at $\Omega=53 \mathrm{~Hz}$. They verified our above observations on the responses of Branch\#4 again.

### 5.4. Conclusion and Discussion

The forced vibration of a cable at the first crossover, with the elasto-geometric parameter $\kappa / \pi \approx 2$, was experimentally investigated using a 3-D motion analysis system. Isolated and simultaneous resonances were observed. These include
(1) $\left\langle\Omega=\omega_{i}\right\rangle: \lambda_{i}=1: 1, i=1,2,3,4$ (happen on branches B1, B2, B3, and B4)
(2) $\left\langle\Omega=\omega_{1}\right\rangle: \lambda_{1}: \omega_{2}: \lambda_{2}=1: 1: 2: 2$ (happen on branch B1).
(3) $\left\langle\Omega=\omega_{2}\right\rangle: \lambda_{2}: \lambda_{1}=2: 2: 1$ (happen on branch B2)
(4) $\left\langle\Omega=\omega_{4}\right\rangle: \omega_{2}: \lambda_{4}: \lambda_{2}=4: 2: 4: 2$ (happen on branch B4)
where $\lambda_{1}, \lambda_{2}, \lambda_{3}$, and $\lambda_{4}$ are the frequencies of the first and second symmetric and antisymmetric out-of-plane modes, respectively. $\omega_{1}, \omega_{2}, \omega_{3}$, and $\omega_{4}$ are the frequencies of the first and second symmetric and anti-symmetric in-plane modes respectively. Whenever there is an excited mode vibrating at one half of the excitation frequency, the vibration period is doubled (i.e., cases (3) and (4)). Quasi-periodic and chaotic motions were not observed. This might be due to the use of a large frequency increment $(1 \mathrm{~Hz})$ or the frequency interval of these responses were too small so that they were missed. The natural frequencies were changed by the quadratic and cubic nonlinearities.


Fig. 5.16


Fig. 5.16: Response to an excitation at $\Omega=35 \mathrm{~Hz}$ : (a) 15 consecutive vibration profiles ( 0.027 sec ), (b) 30 consecutive vibration profiles ( 0.055 sec ), (c) the frequency spectra of the vibration in the X-Z plane vibration of marker 10, (d) the frequency spectra of the XY plane vibration of marker 10, (e) the frequency spectra of the X-Z plane vibration of marker 20, (f) the frequency spectra of the X-Y plane vibration of marker 20, (g) the modal coordinates of the X-Z plane vibration, (h) the frequency spectra of the modal coordinates of the $\mathrm{X}-\mathrm{Z}$ plane vibration, (i) the modal coordinates of the $\mathrm{X}-\mathrm{Y}$ plane vibration, and ( j ) the frequency spectra of the modal coordinates of the $\mathrm{X}-\mathrm{Y}$ plane vibration.


Fig. 5.17


Fig. 5.17 (Contined)


Fig. 5.17: Response to an excitation at $\Omega=40 \mathrm{~Hz}$ : (a) 13 vibration consecutive profiles ( 0.023 sec ), (b) 26 consecutive vibration profiles ( 0.048 sec ), (c) trajectory of marker 8 , (d) trajectory of marker 19, (e) trajectory of marker 32, (f) trajectory of marker 44, (g) Poincare section of marker 32 by sampling once per excitation period, (h) Poincare section of marker 32 by sampling once every two excitation periods, (i) the frequency spectra of the $\mathrm{X}-\mathrm{Z}$ plane vibration of marker 7, ( j ) the frequency spectra of the $\mathrm{X}-\mathrm{Y}$ plane vibration of marker $7,(\mathrm{k})$ the frequency spectra of the $\mathrm{X}-\mathrm{Z}$ plane vibration of marker 12, (l) the frequency spectra of the $\mathrm{X}-\mathrm{Y}$ plane vibration of marker 12, (m) the modal coordinates of the $\mathrm{X}-\mathrm{Z}$ plane vibration, ( n ) the frequency spectra of the modal coordinates of the X-Z plane vibration, (o) the modal coordinates of the X-Y plane vibration, and (p) the frequency spectra of the modal coordinates of the $\mathrm{X}-\mathrm{Y}$ plane vibration.


Fig. 5.18 (Contined)


Fig. 5.18: Response to an excitation at $\Omega=53 \mathrm{~Hz}$ : (a) 10 consecutive vibration profiles ( 0.017 sec ), (b) 20 consecutive vibration profiles ( 0.036 sec ), (c) the frequency spectra of the $\mathrm{X}-\mathrm{Z}$ plane vibration of marker 10 , (d) the frequency spectra of the $\mathrm{X}-\mathrm{Y}$ plane vibration of marker 10, (e) the frequency spectra of the X-Z plane vibration of marker 18, (f) the frequency spectra of the $\mathrm{X}-\mathrm{Y}$ plane vibration of marker 18, (g) the modal coordinates of the X-Z plane vibration, (h) the frequency spectra of the modal coordinates of the $\mathrm{X}-\mathrm{Z}$ plane vibration, (i) the modal coordinates of the $\mathrm{X}-\mathrm{Y}$ plane vibration, and ( j ) the frequency spectra of the modal coordinates of the $\mathrm{X}-\mathrm{Y}$ plane vibration.

## CHAPTER 6

## PACKAGING ANALYSIS OF HIGHLY FLEXIBLE TRIANGULAR FRAMES

This chapter presents the derivation of a geometrically exact beam theory for highly flexible beams undergoing large deformations by following Nayfeh and Pai (2004). The theory fully accounts for large displacements, large rotations, initial curvatures, extensionality and transverse shear strains. The concepts of local displacements, Jaumann stress and strain measures, and orthogonal virtual rotations are used to derive the geometrically exact beam theory. The extended Hamilton principle is used to derive fully nonlinear governing equations. Then the geometrically exact beam theory is presented in terms of first-order ordinary differential equations and is applied to the packaging analysis of a highly flexible triangular frame using multiple-shooting method.

### 6.1 Reference Line Deformation



Fig. 6.1 A rotating clamped-free beam.

A naturally curved and twisted beam, as shown in Fig. 6.1, is considered. Three coordinate systems are used in order to model the large deformations of beams. The system $x y z$ is an orthogonal curvilinear coordinate system, where the axis x denotes the undeformed reference line of the beam and $s$ is the undeformed arc length from the root of the beam to the reference point on the observed cross section. The system $a b c$ is a rectangular coordinate system attached to the beam root and is used for reference in calculating initial curvatures. The system $\xi \eta \zeta$ is a local orthogonal curvilinear coordinate system, where the axes $\xi$ represents the deformed reference line and the axes $\eta$ and $\zeta$ represent the deformed configurations of the axes $y$ and $z$. Moreover, $\mathbf{i}_{x}, \mathbf{i}_{y}$ and $\mathbf{i}_{z}$ are the unit vectors along the axes $x, y$ and $z$, respectively; $\mathbf{i}_{a}, \mathbf{i}_{b}$ and $\mathbf{i}_{c}$ are the unit vectors along the axes $a, b$ and $c$, respectively; and $\mathbf{i}_{1}, \quad \mathbf{i}_{2}$ and $\mathbf{i}_{3}$ are the unit vectors along the axes $\xi, \eta$ and $\zeta$, respectively.

The undeformed position vector $\mathbf{R}$ of the reference point of the observed cross section is represented by

$$
\begin{equation*}
\mathrm{R}=A(s) \mathbf{i}_{a}+B(s) \mathbf{i}_{b}+C(s) \mathbf{i}_{c} \tag{6.1.1}
\end{equation*}
$$

Also, the undeformed angles $\theta_{21}, \theta_{22}$, and $\theta_{23}$ of the axis $y$ with respect to the $a b c$ system are assumed to be known and given by

$$
\begin{equation*}
\theta_{21}=\cos ^{-1}\left(\mathbf{i}_{y} \cdot \mathbf{i}_{a}\right), \quad \theta_{22}=\cos ^{-1}\left(\mathbf{i}_{y} \cdot \mathbf{i}_{b}\right), \quad \theta_{23}=\cos ^{-1}\left(\mathbf{i}_{y} \cdot \mathbf{i}_{c}\right) \tag{6.1.2}
\end{equation*}
$$

where $\theta_{2 i}$ are functions of $s$ only and $0 \leq \theta_{2 i} \leq 180^{\circ}$. Differentiating Eq. (6.1.1) with respect to s yields

$$
\begin{equation*}
\mathbf{i}_{x}=\mathrm{R}^{\prime}=\frac{\partial A}{\partial \mathrm{~s}} \mathbf{i}_{a}+\frac{\partial B}{\partial \mathrm{~s}} \mathbf{i}_{b}+\frac{\partial C}{\partial \mathrm{~s}} \mathbf{i}_{c}=A^{\prime} \mathbf{i}_{a}+B^{\prime} \mathbf{i}_{b}+C^{\prime} \mathbf{i}_{c} \tag{6.1.3}
\end{equation*}
$$

where ()$^{\prime} \equiv \partial() / \partial s$. Because of Eqs. (6.1.2) and (6.1.3) and $\mathbf{i}_{z}=\mathbf{i}_{x} \times \mathbf{i}_{y}$, we obtain

$$
\left\{\begin{array}{l}
\mathbf{i}_{x}  \tag{6.1.4}\\
\mathbf{i}_{y} \\
\mathbf{i}_{z}
\end{array}\right\}=\left[T^{x}\right]\left\{\begin{array}{l}
\mathbf{i}_{a} \\
\mathbf{i}_{b} \\
\mathbf{i}_{c}
\end{array}\right\}
$$

where the transformation matrix $\left[T^{x}\right]$ is given by

$$
\left[T^{x}\right]=\left[\begin{array}{ccc}
A^{\prime} & B^{\prime} & C^{\prime}  \tag{6.1.5}\\
\cos \theta_{21} & \cos \theta_{22} & \cos \theta_{23} \\
B^{\prime} \cos \theta_{23}-C^{\prime} \cos \theta_{22} & C^{\prime} \cos \theta_{21}-A^{\prime} \cos \theta_{23} & A^{\prime} \cos \theta_{22}-B^{\prime} \cos \theta_{21}
\end{array}\right]
$$

Using Eqs. (6.1.4) and (6.1.5), the orthonomality property of $\mathbf{i}_{x}, \mathbf{i}_{y}$, and $\mathbf{i}_{z}$, (e.g., $\mathbf{i}_{x}^{\prime} \cdot \mathbf{i}_{x}=0, \mathbf{i}_{x}^{\prime} \cdot \mathbf{i}_{y}=-\mathbf{i}_{y}^{\prime} \cdot \mathbf{i}_{x}$ ) and the identity $\left[\mathrm{T}^{\mathrm{x}}\right]^{-1}=\left[\mathrm{T}^{\mathrm{x}}\right]^{\mathrm{T}}$ (because $\left[\mathrm{T}^{\mathrm{x}}\right]$ is a unitary matrix), we obtain

$$
\begin{align*}
& \frac{d}{d s}\left\{\begin{array}{l}
\mathbf{i}_{x} \\
\mathbf{i}_{y} \\
\mathbf{i}_{z}
\end{array}\right\}=\left[\begin{array}{ccc}
0 & \frac{d \mathbf{i}_{x}}{d s} \cdot \mathbf{i}_{y} & \frac{d \mathbf{i}_{x}}{d s} \cdot \mathbf{i}_{z} \\
\frac{d \mathbf{i}_{y}}{d s} \cdot \mathbf{i}_{x} & 0 & \frac{d \mathbf{i}_{y}}{d s} \cdot \mathbf{i}_{z} \\
\frac{d \mathbf{i}_{z}}{d s} \cdot \mathbf{i}_{x} & \frac{d \mathbf{i}_{z}}{d s} \cdot \mathbf{i}_{y} & 0
\end{array}\right]=[k]\left\{\begin{array}{l}
\mathbf{i}_{x} \\
\mathbf{i}_{y} \\
\mathbf{i}_{z}
\end{array}\right\}  \tag{6.1.6}\\
& {[k]=\left[P\left(k_{1}, k_{2}, k_{3}\right)\right] \equiv\left[\begin{array}{ccc}
0 & k_{3} & -k_{2} \\
-k_{3} & 0 & k_{1} \\
k_{2} & -k_{1} & 0
\end{array}\right]} \tag{6.1.7}
\end{align*}
$$

where $k_{1}, k_{2}$, and $k_{3}$ are the initial curvatures with respect to the axes $\mathrm{x}, \mathrm{y}$, and z , respectively, and they are functions of $s$ and are given by

$$
\begin{equation*}
k_{1}=\frac{d \mathbf{i}_{y}}{d s} \cdot \mathbf{i}_{z}=\frac{d T_{2 i}^{x}}{d s} T_{3 i}^{x}, \quad k_{2}=-\frac{d \mathbf{i}_{x}}{d s} \cdot \mathbf{i}_{z}=\frac{-d T_{1 i}^{x}}{d s} T_{3 i}^{x}, \quad k_{3}=\frac{d \mathbf{i}_{x}}{d s} \cdot \mathbf{i}_{y}=\frac{d T_{1 i}^{x}}{d s} T_{2 i}^{x} \tag{6.1.8}
\end{equation*}
$$

Following Alkire (1984), we use two sequential Euler angles $\alpha$ and $\phi$ to describe the rotation of the observed element from the undeformed position to the deformed one, as shown in Fig. 6.2. The angle $\alpha$ represents the bending rotation of the element about the axis $\boldsymbol{n}$. The system $x y z$ is rigidly translated and then rotated by an angle $\alpha$ about the axis $\boldsymbol{n}$ to produce the intermediate system $\xi \mathrm{y}_{1} \mathrm{z}_{1}$. The transformation relating the unit vectors of these two systems is

$$
\left\{\begin{array}{l}
\mathbf{i}_{1}  \tag{6.1.9}\\
\mathbf{i}_{2} \\
\mathbf{i}_{\hat{3}}
\end{array}\right\}=[B(\alpha)]\left\{\begin{array}{l}
\mathbf{i}_{x} \\
\mathbf{i}_{y} \\
\mathbf{i}_{z}
\end{array}\right\}
$$

where $\mathbf{i}_{\hat{\mathbf{2}}}$ and $\mathbf{i}_{\hat{\mathbf{3}}}$ are unit vectors along the $\mathrm{y}_{1}$ and $z_{1}$, respectively. The transformation matrix $[B(\alpha)]$ is due to the bending rotation $\alpha$, which rotates the axis x to the axis $\xi$, the axis $y$ to the axis $y_{1}$, and the axis $z$ to the axis $z_{1}$. The angle between $y$ and $y_{1}$, and the angle between z and $z_{1}$ are not equal to $\alpha$ because the planes $y y_{1}$ and $z z_{1}$ are not perpendicular to the axis $n$. The intermediate system is then rotated by an angle $\phi$ about the axis $\xi$ to produce the system $\xi \eta \zeta$. The transformation matrix that relates the unit vectors along the axes of these two systems is given by

$$
\left\{\begin{array}{l}
\mathbf{i}_{1}  \tag{6.1.10}\\
\mathbf{i}_{2} \\
\mathbf{i}_{3}
\end{array}\right\}=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \phi & \sin \phi \\
0 & -\sin \phi & \cos \phi
\end{array}\right]\left\{\begin{array}{l}
\mathbf{i}_{1} \\
\mathbf{i}_{\hat{2}} \\
\mathbf{i}_{\hat{3}}
\end{array}\right\}
$$

The second rotation $\phi$ is related to the torsional motion about the bent reference axis $\xi$.

Hence, the transformation which relates the undeformed coordinate system $x y z$ to the deformed coordinate system $\xi \eta \zeta$ is

$$
\left\{\begin{array}{l}
\mathbf{i}_{1}  \tag{6.1.1}\\
\mathbf{i}_{2} \\
\mathbf{i}_{3}
\end{array}\right\}=[T]\left[\begin{array}{l}
\mathbf{i}_{x} \\
\mathbf{i}_{y} \\
\mathbf{i}_{z}
\end{array}\right\} \quad, \quad[T]=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \phi & \sin \phi \\
0 & -\sin \phi & \cos \phi
\end{array}\right][B(\alpha)]
$$

where $[T]^{-1}=[T]^{T}$ since $[T]$ is a unitary matrix.


Fig. 6.2: Two successive Euler angle rotations of a differential beam element.

Next we represent $[B(\alpha)]$ in terms of the displacements $u, v$, and $w$ of the reference point of the observed cross section. First, we relate $[B(\alpha)]$ to $\alpha$ and the components $n_{1}, n_{2}$, and $n_{3}$ of the unit vector $\mathbf{n}\left(=n_{1} \mathbf{i}_{x}+n_{2} \mathbf{i}_{y}+n_{3} \mathbf{i}_{z}\right)$. To accomplish this, we derive an expression for the transformation of an arbitrary vector $\mathbf{r}$ undergoing a rotation by an angle $\alpha$ about an axis $n$. In Fig. 6.3, we show a plate $\mathbf{O A B}$ rotated by an angle $\alpha$ about $\overline{O A}$. The line $\overline{O A}$ is perpendicular to $\overline{O B}$, and $\mathbf{n}, \mathbf{j}_{a}, \mathbf{j}_{b}$, and $\mathbf{j}_{\hat{a}}$ are unit vector. It follows from Fig. 6.3 that

$$
\begin{equation*}
\mathbf{r}=r \cos \theta \quad \mathbf{n}+r \sin \theta \quad \mathbf{j}_{a} \tag{6.1.12}
\end{equation*}
$$

$$
\begin{align*}
& \hat{\mathbf{r}}=r \cos \theta \quad \mathbf{n}+r \sin \theta \quad \mathbf{j}_{\hat{a}}  \tag{6.1.23}\\
& \mathbf{j}_{\hat{a}}=\cos \alpha \quad \mathbf{j}_{a}+\sin \alpha \quad \mathbf{j}_{b} \tag{6.1.34}
\end{align*}
$$

It follows from Eq. (6.1.12) and Fig. 6.3 that

$$
\begin{equation*}
\mathbf{j}_{b}=\frac{\mathbf{n} \times \mathbf{r}}{|\mathbf{n} \times \mathbf{r}|}=\frac{1}{r \sin \theta} \mathbf{n} \times \mathbf{r} \tag{6.1.45}
\end{equation*}
$$

Moreover, it follows from Eq. (6.1.15) and Fig. 6.3 that

$$
\begin{equation*}
\mathbf{j}_{a}=\mathbf{j}_{b} \times \mathbf{n}=\frac{1}{r \sin \theta}(\mathbf{n} \times \mathbf{r}) \times \mathbf{n}=\frac{1}{r \sin \theta}[\mathbf{r}-(\mathbf{r} \cdot \mathbf{n}) \mathbf{n}] \tag{6.1.56}
\end{equation*}
$$

Substituting Eqs. (6.1.14)-(6.1.16) into Eq. (6.1.13) and using the identity $r \cos \theta=\mathbf{r} \cdot \mathbf{n}$ yields

$$
\begin{equation*}
\hat{\mathbf{r}}=(1-\cos \alpha)(\mathbf{r} \cdot \mathbf{n}) \mathbf{n}+\cos \alpha \mathbf{r}+\sin \alpha \mathbf{n} \times \mathbf{r} \tag{6.1.67}
\end{equation*}
$$

which shows the relation between the arbitrary vector $\mathbf{r}$ and its rotated vector $\hat{\mathbf{r}}$. In Fig. 6.2, because $\mathbf{i}_{x}$ is transformed into $\mathbf{i}_{1}$ by the rotation $\alpha$, it follows from Eq. (6.17) that

$$
\begin{equation*}
\mathbf{i}_{1}=(1-\cos \alpha) n_{1} \mathbf{n}+\cos \alpha \mathbf{i}_{x}+\sin \alpha\left(n_{3} \mathbf{i}_{y}-n_{2} \mathbf{i}_{z}\right) \tag{6.1.78}
\end{equation*}
$$

Similarly, because $\mathbf{i}_{y}$ is transformed into $\mathbf{i}_{y 1}=\mathbf{i}_{\hat{2}}$ and $\mathbf{i}_{z}$ is transformed into $\mathbf{i}_{z 1}=\mathbf{i}_{\hat{3}}$, we have

$$
\begin{align*}
& \mathbf{i}_{\hat{2}}=(1-\cos \alpha) n_{2} \mathbf{n}+\cos \alpha \mathbf{i}_{y}+\sin \alpha\left(-n_{3} \mathbf{i}_{x}+n_{1} \mathbf{i}_{z}\right)  \tag{6.1.89}\\
& \mathbf{i}_{\hat{3}}=(1-\cos \alpha) n_{3} \mathbf{n}+\cos \alpha \mathbf{i}_{z}+\sin \alpha\left(n_{2} \mathbf{i}_{x}-n_{1} \mathbf{i}_{y}\right) \tag{6.1.20}
\end{align*}
$$

It follows from Eqs. (6.1.18) - (6.1.20) and (6.1.9) that

$$
[B(\alpha)]=\left[\begin{array}{ccc}
n_{1}^{2} & & \text { sym. } \\
n_{1} n_{2} & n_{2}^{2} & \\
n_{1} n_{3} & n_{2} n_{3} & n_{3}^{2}
\end{array}\right](1-\cos \alpha)+\left[\begin{array}{ccc}
0 & n_{3} & -n_{2} \\
-n_{3} & 0 & n_{1} \\
n_{2} & -n_{1} & 0
\end{array}\right] \sin \alpha+[I] \cos \alpha(6.1 .91)
$$

where $[I]$ is a $3 \times 3$ identity matrix.

Next, we relate $n_{i}$ and $\alpha$ to $u, v$, and $w$. To this end, we show in Fig. 6.4 the relationship between the reference line and the Euler angles $\alpha$ and $\phi$. It follows from Fig. 6.4 and Eqs. (6.1.6) and (6.1.7) that the displacement vectors of points $p$ and $q$ are

$$
\begin{align*}
p: \mathbf{D}_{1}=u \mathbf{i}_{x} & +v \mathbf{i}_{y}+w \mathbf{i}_{z}  \tag{6.1.102}\\
& q: \quad \mathbf{D}_{2}=\mathbf{D}_{1}+\frac{\partial \mathbf{D}_{1}}{\partial s} d s \\
& =\mathbf{D}_{1}+\left[\left(u^{\prime}-v k_{3}+w k_{2}\right) \mathbf{i}_{x}+\left(v^{\prime}+u k_{3}-w k_{1}\right) \mathbf{i}_{y}+\left(w^{\prime}-u k_{2}+v k_{1}\right) \mathbf{i}_{z}\right] d s \tag{6.1.113}
\end{align*}
$$



Fig. 6.3: The rotation of a vector $\mathbf{r}$ through an angle $\alpha$ with respect to the axis $\mathbf{n}$.

Thus, the vector from the deformed reference point $\hat{p}$ to the deformed reference point $\hat{q}$ is

$$
\begin{align*}
\overrightarrow{\hat{p} \hat{q}} & =d s \mathbf{i}_{x}+\mathbf{D}_{2}-\mathbf{D}_{1}  \tag{6.1.124}\\
& =\left[\left(1+u^{\prime}-v k_{3}+w k_{2}\right) \mathbf{i}_{x}+\left(v^{\prime}+u k_{3}-w k_{1}\right) \mathbf{i}_{x}+\left(w^{\prime}-u k_{2}+v k_{1}\right) \mathbf{i}_{z}\right] d s
\end{align*}
$$

Therefore, it follows from Eqs. (6.1.24) and (6.1.11) that

$$
\begin{equation*}
\mathbf{i}_{1}=\frac{\overrightarrow{\hat{p} \hat{q}}}{(1+e) d s}=T_{11} \mathbf{i}_{x}+T_{12} \mathbf{i}_{y}+T_{13} \mathbf{i}_{z} \tag{6.1.135}
\end{equation*}
$$

where $e$ denotes the axial strain along the deformed reference line and

$$
\begin{equation*}
T_{11}=\frac{1+u^{\prime}-v k_{3}+w k_{2}}{1+e}, T_{12}=\frac{v^{\prime}+u k_{3}-w k_{1}}{1+e}, \quad T_{13}=\frac{w^{\prime}-u k_{2}+v k_{1}}{1+e} \tag{6.1.146}
\end{equation*}
$$

It follows from Eq. (6.1.24) and Fig. 6.4 that the relationship between the axial strain $e$ and the displacements is

$$
\begin{align*}
e & =\frac{\overline{\hat{p} \hat{q}}-d s}{d s}  \tag{6.1.157}\\
& =\sqrt{\left(1+u^{\prime}-v k_{3}+w k_{2}\right)^{2}+\left(v^{\prime}+u k_{3}-w k_{1}\right)^{2}+\left(w^{\prime}-u k_{2}+v k_{1}\right)^{2}}-1
\end{align*}
$$

A rotation axis $\boldsymbol{n}$ and a rotation angle $\alpha$ about the $\boldsymbol{n}$ axis are used to define the bending rotation. As shown in Fig. 6.4, the axis $n$ is chosen to be

$$
\begin{equation*}
\mathbf{n} \equiv \frac{\mathbf{i}_{x} \times \mathbf{i}_{1}}{\left|\mathbf{i}_{x} \times \mathbf{i}_{1}\right|}=n_{1} \mathbf{i}_{x}+n_{2} \mathbf{i}_{y}+n_{2} \mathbf{i}_{z} \tag{6.1.168}
\end{equation*}
$$

Substituting for $\mathbf{i}_{1}$ from Eq. (6.1.25) into Eq. (6.1.28) yields

$$
\begin{equation*}
n_{1}=0, \quad n_{2}=\frac{-T_{13}}{\sqrt{T_{12}^{2}+T_{13}^{2}}}, \quad n_{3}=\frac{T_{12}}{\sqrt{T_{12}^{2}+T_{13}^{2}}} \tag{6.1.179}
\end{equation*}
$$

Substituting Eq. (6.1.29) into Eq. (6.1.21), assuming $0 \leq \alpha<180^{\circ}$, and using the relationship

$$
\begin{equation*}
T_{11}^{2}+T_{12}^{2}+T_{13}^{2}=1, \cos \alpha=\mathbf{i}_{1} \cdot \mathbf{i}_{x}=T_{11}, \sin \alpha=\left|\mathbf{i}_{1} \times \mathbf{i}_{x}\right|=\sqrt{T_{12}^{2}+T_{13}^{2}} \tag{6.1.30}
\end{equation*}
$$

we obtain

$$
[B(\alpha)]=\left[\begin{array}{ccc}
T_{11} & T_{12} & T_{13}  \tag{6.1.181}\\
-T_{12} & T_{11}+T_{13}^{2} /\left(1+T_{11}\right) & -T_{12} T_{13} /\left(1+T_{11}\right) \\
-T_{13} & -T_{12} T_{13} /\left(1+T_{11}\right) & T_{11}+T_{12}^{2} /\left(1+T_{11}\right)
\end{array}\right]
$$

Therefore from Eq. (6.1.11) we have

$$
[T]=\left[\begin{array}{ccc}
1 & 0 & 0  \tag{6.1.32}\\
0 & \cos \phi & \sin \phi \\
0 & -\sin \phi & \cos \phi
\end{array}\right]\left[\begin{array}{ccc}
T_{11} & T_{12} & T_{13} \\
-T_{12} & T_{11}+T_{13}^{2} /\left(1+T_{11}\right) & -T_{12} T_{13} /\left(1+T_{11}\right) \\
-T_{13} & -T_{12} T_{13} /\left(1+T_{11}\right) & T_{11}+T_{12}^{2} /\left(1+T_{11}\right)
\end{array}\right]
$$



Fig. 6.4: Relationship between the reference line and the Euler angles.

Moreover, $\phi$ is the Euler angle related to the twisting angle of the observed cross section with respect to the deformed reference axis $\xi$ and e is the axial strain along the axis $\xi$. It follows from Eq. (6.1.32) that $T_{2 i}$ and $T_{3 i}$ can be represented in terms of $T_{11}, T_{12,} T_{13}$ and $\phi$ as

$$
\begin{align*}
& T_{21}=-\cos \phi T_{12}-\sin \phi T_{13} \\
& T_{22}=\cos \phi\left(T_{11}+\frac{T_{13}^{2}}{1+T_{11}}\right)-\sin \phi \frac{T_{12} T_{13}}{1+T_{11}}  \tag{6.1.33}\\
& T_{23}=\sin \phi\left(T_{11}+\frac{T_{12}^{2}}{1+T_{11}}\right)-\cos \phi \frac{T_{12} T_{13}}{1+T_{11}} \\
& T_{31}=\sin \phi T_{12}-\cos \phi T_{13} \\
& T_{32}=-\sin \phi\left(T_{11}+\frac{T_{13}^{2}}{1+T_{11}}\right)-\cos \phi \frac{T_{12} T_{13}}{1+T_{11}}  \tag{6.1.33}\\
& T_{33}=\cos \phi\left(T_{11}+\frac{T_{12}^{2}}{1+T_{11}}\right)+\sin \phi \frac{T_{12} T_{13}}{1+T_{11}}
\end{align*}
$$

Because $[T]$ is a unitary matrix, we have the identity

$$
\begin{equation*}
[T]^{T}=[T]^{-1} \tag{6.1.34}
\end{equation*}
$$

Differentiating Eq. (6.1.11) with respect to $s$ yields

$$
\frac{\partial}{\partial s}\left\{\begin{array}{l}
\mathbf{i}_{1}  \tag{6.1.35}\\
\mathbf{i}_{2} \\
\mathbf{i}_{3}
\end{array}\right\}=[\mathrm{K}]\left\{\begin{array}{l}
\mathbf{i}_{1} \\
\mathbf{i}_{2} \\
\mathbf{i}_{3}
\end{array}\right\}
$$

where

$$
[\mathrm{K}]=\left[\begin{array}{ccc}
0 & \rho_{3} & -\rho_{2}  \tag{6.1.36}\\
-\rho_{3} & 0 & \rho_{1} \\
\rho_{2} & -\rho_{1} & 0
\end{array}\right]=[T]^{\prime}[T]^{T}+[T][k][T]^{T}
$$

where $\rho_{1}$ is the deformed twisting curvature and $\rho_{2}$ and $\rho_{3}$ are the deformed bending curvatures. We note that $\rho_{i}$ are not real curvatures because the differentiation is with respect to the undeformed differential length $\mathrm{d} s$, instead of the deformed length $(1+\mathrm{e}) \mathrm{ds}$. Post-multiplying Eq. (6.1.36) by $[T]$ and using Eq. (6.1.34) yields

$$
\left[\begin{array}{ccc}
T_{11}^{\prime} & T_{12}^{\prime} & T_{13}^{\prime}  \tag{6.1.37}\\
T_{21}^{\prime} & T_{22}^{\prime} & T_{23}^{\prime} \\
T_{31}^{\prime} & T_{32}^{\prime} & T_{33}^{\prime}
\end{array}\right]=[K][T]-[T][k]
$$

Using Eqs. (6.1.35), (6.1.36), (6.1.11) and (6.1.4), one can show that

$$
\begin{align*}
\rho_{1}= & \mathbf{i}_{3} \cdot \mathbf{i}_{2}^{\prime} \\
= & \left(T_{31} \mathbf{i}_{x}+T_{32} \mathbf{i}_{y}+T_{33} \mathbf{i}_{z}\right) \cdot\left[\begin{array}{r}
T_{21}^{\prime} \mathbf{i}_{x}+T_{22}^{\prime} \mathbf{i}_{y}+T_{23}^{\prime} \mathbf{i}_{z}+T_{21}\left(k_{3} \mathbf{i}_{y}-k_{2} \mathbf{i}_{z}\right) \\
\\
\\
+T_{22}\left(k_{1} \mathbf{i}_{z}-k_{3} \mathbf{i}_{x}\right)+T_{23}\left(k_{2} \mathbf{i}_{x}-k_{1} \mathbf{i}_{y}\right)
\end{array}\right] \\
= & T_{31} T_{21}^{\prime}+T_{32} T_{22}^{\prime}+T_{33} T_{23}^{\prime}  \tag{6.1.38}\\
& +\left(T_{33} T_{22}-T_{32} T_{23}\right) k_{1}+\left(T_{31} T_{23}-T_{33} T_{21}\right) k_{2}+\left(T_{32} T_{21}-T_{31} T_{22}\right) k_{3} \\
= & T_{31} T_{21}^{\prime}+T_{32} T_{22}^{\prime}+T_{33} T_{23}^{\prime}+T_{11} k_{1}+T_{12} k_{2}+T_{13} k_{3}
\end{align*}
$$

where we used the identities $\mathbf{i}_{1} \times \mathbf{i}_{2}=\mathbf{i}_{3}, \quad \mathbf{i}_{2} \times \mathbf{i}_{3}=\mathbf{i}_{1}$ and $\mathbf{i}_{3} \times \mathbf{i}_{1}=\mathbf{i}_{2}$. Substituting Eq. (6.1.33) in Eq. (6.1.38), one can show that

$$
\begin{equation*}
\phi^{\prime}=\rho_{1}-\frac{1}{1+T_{11}}\left(T_{13} T_{12}^{\prime}-T_{12} T_{13}^{\prime}\right)-T_{11} k_{1}-T_{12} k_{2}-T_{13} k_{3} \tag{6.1.39}
\end{equation*}
$$

Equation (6.1.39) shows that $\phi^{\prime} \neq \rho_{1}-k_{1}$ because of the initial curvatures $k_{i}$ and the deformation induced $T_{1 i}$. In other words, $\phi$ is not the actual twisting angle.

### 6.2 Constitutive Equation and Strain-Displacement Relation

Jaumann strains $B_{i j}$ are chosen for this study because they are fully nonlinear objective strains and are work conjugate to Jaumann stresses $J_{i j}$. For beams consisting of laminated orthotropic layers, Jaumann stresses and strains are related as

$$
\left\{\begin{array}{l}
J_{11}  \tag{6.2.1}\\
J_{12} \\
J_{13}
\end{array}\right\}=[\bar{Q}]\left\{\begin{array}{l}
B_{11} \\
B_{12} \\
B_{13}
\end{array}\right\}, \quad[\bar{Q}\rceil \equiv\left[\begin{array}{lll}
\bar{Q}_{11} & \bar{Q}_{16} & \bar{Q}_{15} \\
\bar{Q}_{61} & \bar{Q}_{66} & \bar{Q}_{65} \\
\bar{Q}_{51} & \bar{Q}_{56} & \bar{Q}_{55}
\end{array}\right]
$$

where $[\bar{Q}]$ is the transformed, reduced material stiffness matrix and is symmetric. If $[Q]$ is the $6 \times 6$ material stiffness matrix of the ith layer with respect to the layer's material direction, the transformed material stiffness matrix $\mid \hat{Q}\rfloor$ can be obtained by coordinate transformation using the angle between the material direction and the structural coordinate $x$. Then the $6 \times 6$ matrix $\mid \hat{Q}]$ can be reduced to the $3 \times 3$ matrix $[\bar{Q}]$ in Eq. (6.2.1) by assuming $J_{22}=J_{33}=J_{23}=0$.

For an isotropic material, one can assume that

$$
[\bar{Q}]=\left[\begin{array}{ccc}
E & 0 & 0  \tag{6.2.2}\\
0 & G & 0 \\
0 & 0 & G
\end{array}\right]
$$

where E is Young's modulus, $\mathrm{G}(=E /(1+v))$ is shear modulus, and $v$ is Poisson's ratio.
Because rigid-body displacements do not result in any strain energy, to calculate the elastic energy we only need to deal with the strainable, local displacement field $\mathbf{U}$. The local displacement field of a beam can be assumed to be $\mathbf{U}=u_{1} \mathbf{i}_{1}+u_{2} \mathbf{i}_{2}+u_{3} \mathbf{i}_{3}$
$u_{1}(s, y, z, t)=u_{1}^{0}(s, t)+z \bar{\theta}_{2}(s, t)-y \bar{\theta}_{3}(s, t)+\bar{\rho}_{1}(s, t) g_{11}(y, z)+\gamma_{5}(s, t) g_{15}(y, z)+\gamma_{6}(s, t) g_{16}(y, z)$
$u_{2}(s, y, z, t)=u_{2}^{0}(s, t)-z \bar{\theta}_{1}(s, t)+\bar{\rho}_{2}(s, t) g_{22}(y, z)+\bar{\rho}_{3}(s, t) g_{23}(y, z)+e(s, t) g_{24}(y, z)$
$u_{3}(s, y, z, t)=u_{3}^{0}(s, t)+y \bar{\theta}_{1}(s, t)+\bar{\rho}_{2}(s, t) g_{32}(y, z)+\bar{\rho}_{3}(s, t) g_{33}(y, z)+e(s, t) g_{34}(y, z)$
where t is the time; $u_{1}, u_{2}$ and $u_{3}$ are local strainable displacements with respect to the axes $\xi, \eta$ and $\zeta$, respectively; $u_{i}^{0}(s, t) \equiv u_{i}(s, 0,0, t) ; \quad \overline{\theta_{i}} \equiv \theta_{i}-\theta_{i 0} ; \quad \theta_{1}, \theta_{2}$ and $\theta_{3}$ are the rotation angles of the observed cross section with respect to the axes $\xi, \eta$, and $\zeta$, respectively; $\theta_{10}, \theta_{20}$ and $\theta_{30}$ are the initial rotation angles (after the rigid-body
displacements $\mathrm{u}, \mathrm{v}, \mathrm{w}$ and $\phi$ ) of the observed cross section with respect to the axes $\xi, \eta$, and $\zeta$, respectively. Furthermore, $\bar{\rho}_{i}=\rho_{i}-k_{i} ; \rho_{1}, \rho_{2}$ and $\rho_{3}$ are the deformed curvatures with respect to the axes $\xi, \eta$, and $\zeta$, respectively; $k_{1}, k_{2}$ and $k_{3}$ are the initial curvatures with respect to the axes $x$, $y$, and $z$, respectively. Moreover, $e$ is the extensional strain of the reference line, and $\gamma_{5}$ and $\gamma_{6}$ are the shear rotation angles at the reference point and with respect to the axes y and -z , respectively. $g_{11}$ is the torsion induced out-of-plane warping function; $g_{15}$ and $g_{16}$ are shear-induced out-of-plane warping functions; $g_{22}, g_{23}, g_{32}$ and $g_{33}$ are bending-induced in-plane warping functions; and $g_{24}$ and $g_{34}$ are extension-induced in-plane warping functions. Because $u_{i}^{0}$ are defined as $u_{i}^{0}(s, t) \equiv u_{i}(s, 0,0, t),\left.\quad g_{i j}\right|_{(y, z)=(0,0)}=g_{i j}(0,0)=0$.

Because the system $\xi \eta \zeta$ is a local coordinate system attached to the observed cross section and the unit vector $\mathbf{i}_{1}$ is tangent to the deformed reference axis, we have

$$
\begin{gather*}
u_{1}^{0}=u_{2}^{0}=u_{3}^{0}=\theta_{10}=\theta_{20}=\theta_{30}=\theta_{1}=\theta_{2}=\theta_{3}=\frac{\partial u_{2}^{0}}{\partial \mathrm{~s}}=\frac{\partial u_{3}^{0}}{\partial \mathrm{~s}}=0  \tag{6.2.4}\\
e \equiv \frac{\partial u_{1}^{0}}{\partial \mathrm{~s}}, \quad \rho_{i} \equiv \frac{\partial \theta_{i}}{\partial \mathrm{~s}}, \quad k_{i} \equiv \frac{\partial \theta_{i 0}}{\partial \mathrm{~s}}, \quad i=1,2,3
\end{gather*}
$$

It follows from Eqs. (6.2.3), (6.2.4), (6.1.35) and (6.1.36) that

$$
\begin{aligned}
\frac{\partial \mathbf{U}}{\partial s}= & {\left[e+z \bar{\rho}_{2}-y \bar{\rho}_{3}+\bar{\rho}_{1}^{\prime} g_{11}+\gamma_{5}^{\prime} g_{15}+\gamma_{6}^{\prime} g_{16}\right] \mathbf{i}_{1}+\left[\rho_{2}\left(\bar{\rho}_{2} g_{32}+\bar{\rho}_{3} g_{33}+e g_{34}\right)\right.} \\
& \left.-\rho_{3}\left(\bar{\rho}_{2} g_{22}+\bar{\rho}_{3} g_{23}+e g_{24}\right)\right] \mathbf{i}_{1}+\left[-z \bar{\rho}_{1}+\bar{\rho}_{2}^{\prime} g_{22}+\bar{\rho}_{3}^{\prime} g_{23}+e^{\prime} g_{24}\right] \mathbf{i}_{2} \\
& +\left[\rho_{3}\left(\bar{\rho}_{1} g_{11}+\gamma_{5} g_{15}+\gamma_{6} g_{16}\right)-\rho_{1}\left(\bar{\rho}_{2} g_{32}+\bar{\rho}_{3} g_{33}+e g_{34}\right)\right] \mathbf{i}_{2}+\left[y \bar{\rho}_{1}+\bar{\rho}_{2}^{\prime} g_{32}+\bar{\rho}_{3}^{\prime} g_{33}+e^{\prime} g_{34}\right] \mathbf{i}_{3} \\
& +\left[\rho_{1}\left(\bar{\rho}_{2} g_{22}+\bar{\rho}_{3} g_{23}+e g_{24}\right)-\rho_{2}\left(\bar{\rho}_{1} g_{11}+\gamma_{5} g_{15}+\gamma_{6} g_{16}\right)\right] \mathbf{i}_{3}
\end{aligned}
$$

$$
\begin{align*}
\frac{\partial \mathbf{U}}{\partial y}= & {\left[\bar{\rho}_{1} g_{11 y}+\gamma_{5} g_{15 y}+\gamma_{6} g_{16 y}\right] \mathbf{i}_{1}+\left[\bar{\rho}_{2} g_{22 y}+\bar{\rho}_{3} g_{23 y}+e g_{24 y}\right] \mathbf{i}_{2} }  \tag{6.2.5}\\
& +\left[\bar{\rho}_{2} g_{32 y}+\bar{\rho}_{3} g_{33 y}+e g_{34 y}\right] \mathbf{i}_{3} \\
\frac{\partial \mathbf{U}}{\partial z}= & {\left[\bar{\rho}_{1} g_{11 z}+\gamma_{5} g_{15 z}+\gamma_{6} g_{16 z}\right] \mathbf{i}_{1}+\left[\bar{\rho}_{2} g_{22 z}+\bar{\rho}_{3} g_{23 z}+e g_{24 z}\right] \mathbf{i}_{2} } \\
& +\left[\bar{\rho}_{2} g_{32 z}+\bar{\rho}_{3} g_{33 z}+e g_{34 z}\right] \mathbf{i}_{3}
\end{align*}
$$

Without performing any complex polar decomposition, Jaumann strains $B_{i j}$ can be derived using the local displacement field as

$$
\begin{align*}
B_{11}= & \frac{\partial \mathbf{U}}{\partial s} \cdot \mathbf{i}_{1} \\
= & e+z \bar{\rho}_{2}-y \bar{\rho}_{3}+\bar{\rho}_{1}^{\prime} g_{11}+\gamma_{5}^{\prime} g_{15}+\gamma_{6}^{\prime} g_{16}+\rho_{2}\left(\bar{\rho}_{2} g_{32}+\bar{\rho}_{3} g_{33}+e g_{34}\right) \\
& \quad-\rho_{3}\left(\bar{\rho}_{2} g_{22}+\bar{\rho}_{3} g_{23}+e g_{24}\right) \\
B_{12}= & \frac{\partial \mathbf{U}}{\partial s} \cdot \mathbf{i}_{2}+\frac{\partial \mathbf{U}}{\partial y} \cdot \mathbf{i}_{1} \\
= & -z \bar{\rho}_{1}+\bar{\rho}_{2}^{\prime} g_{22}+\bar{\rho}_{3}^{\prime} g_{23}+e^{\prime} g_{24}+\bar{\rho}_{1} g_{11 y}+\gamma_{5} g_{15 y}+\gamma_{6} g_{16 y}  \tag{6.2.6}\\
& +\rho_{3}\left(\bar{\rho}_{1} g_{11}+\gamma_{5} g_{15}+\gamma_{6} g_{16}\right)-\rho_{1}\left(\bar{\rho}_{2} g_{32}+\bar{\rho}_{3} g_{33}+e g_{34}\right) \\
B_{13}= & \frac{\partial \mathbf{U}}{\partial s} \cdot \mathbf{i}_{3}+\frac{\partial \mathbf{U}}{\partial z} \cdot \mathbf{i}_{1} \\
=y & \bar{\rho}_{1}+\bar{\rho}_{2}^{\prime} g_{32}+\bar{\rho}_{3}^{\prime} g_{33}+e^{\prime} g_{34}+\bar{\rho}_{1} g_{11 z}+\gamma_{5} g_{15 z}+\gamma_{6} g_{16 z} \\
& +\rho_{1}\left(\bar{\rho}_{2} g_{22}+\bar{\rho}_{3} g_{23}+e g_{24}\right)-\rho_{2}\left(\bar{\rho}_{1} g_{11}+\gamma_{5} g_{15}+\gamma_{6} g_{16}\right)
\end{align*}
$$

The $\overline{\rho_{i}^{\prime}}$ and $e^{\prime}$ in Eq. (6.2.6) will be neglected because these secondary effects are important only in the study of in-plane and torsional warping restraint effects around the two boundary points of a thick beam. Furthermore, the nonlinear terms $\rho_{i} \bar{\rho}_{j}, \rho_{i} \gamma_{j}$ and $\rho_{i} e$ are secondary effects due to the coupling of curvatures and warpings. For thick beams, the deformed curvatures $\rho_{i}$ cannot change very much from the undeformed curvatures $k_{i}$ before structural failure. For thin beams, the warpings are
negligible. Hence, $\rho_{i}$ will be replaced with $k_{i}$ in these nonlinear terms. Therefore, without significant loss of accuracy, Eq. (6.2.6) can be simplified as

$$
\begin{align*}
& B_{11}=e+z \bar{\rho}_{2}-y \bar{\rho}_{3}+\bar{\rho}_{1}^{\prime} g_{11}+\gamma_{5}^{\prime} g_{15}+\gamma_{6}^{\prime} g_{16}+k_{2}\left(\bar{\rho}_{2} g_{32}+\bar{\rho}_{3} g_{33}+e g_{34}\right)-k_{3}\left(\bar{\rho}_{2} g_{22}+\bar{\rho}_{3} g_{23}+e g_{24}\right) \\
& B_{12}=-z \bar{\rho}_{1}+\bar{\rho}_{1} g_{11 y}+\gamma_{5} g_{15 y}+\gamma_{6} g_{16 y}+k_{3}\left(\bar{\rho}_{1} g_{11}+\gamma_{5} g_{15}+\gamma_{6} g_{16}\right)-k_{1}\left(\bar{\rho}_{2} g_{32}+\bar{\rho}_{3} g_{33}+e g_{34}\right)  \tag{6.2.7}\\
& B_{13}=y \bar{\rho}_{1}+\bar{\rho}_{1} g_{11 z}+\gamma_{5} g_{15 z}+\gamma_{6} g_{16 z}+k_{1}\left(\bar{\rho}_{2} g_{22}+\bar{\rho}_{3} g_{23}+e g_{24}\right)-k_{2}\left(\bar{\rho}_{1} g_{11}+\gamma_{5} g_{15}+\gamma_{6} g_{16}\right)
\end{align*}
$$

These strains can be put in matrix form as

$$
\begin{equation*}
\{\varepsilon\}=[S]\{\psi\} \tag{6.2.8}
\end{equation*}
$$

where

$$
\{\varepsilon\} \equiv\left\{B_{11}, B_{12}, B_{13}\right\}^{T}
$$

and

$$
\begin{gather*}
\{\psi\} \equiv\left\{e, \gamma_{6}, \gamma_{5}, \bar{\rho}_{1}, \bar{\rho}_{2}, \bar{\rho}_{3}, \gamma_{6}^{\prime}, \gamma_{5}^{\prime}\right\}^{T} \\
{[S] \equiv\left[\begin{array}{ccc}
1+k_{2} g_{34}-k_{3} g_{24} & -k_{1} g_{34} & k_{1} g_{24} \\
0 & g_{16 y}+k_{3} g_{16} & g_{16 z}-k_{2} g_{16} \\
0 & g_{15 y}+k_{3} g_{15} & g_{15 z}-k_{2} g_{15} \\
0 & g_{11 y}-z+k_{3} g_{11} & g_{11 z}+y-k_{2} g_{11} \\
z+k_{2} g_{32}-k_{3} g_{22} & -k_{1} g_{32} & k_{1} g_{22} \\
-y+k_{2} g_{33}-k_{3} g_{23} & -k_{1} g_{33} & k_{1} g_{23} \\
g_{16} & 0 & 0 \\
g_{15} & 0 & 0
\end{array}\right]} \tag{6.2.9}
\end{gather*}
$$

The variation of the elastic energy V can be obtained from Eqs. (6.2.1), (6.2.2), (6.2.8) and (6.2.9) as

$$
\begin{align*}
\delta V & \left.=\int_{0}^{L} \int_{A}\{\delta \varepsilon\}^{T}\{\sigma\} d A d s=\int_{0}^{L} \int_{A}\{\delta \psi\}^{T}[S]^{T}[\bar{Q}] S\right]\{\psi\} d A d s  \tag{6.2.10}\\
& =\int_{0}^{L}\{\delta \psi\}^{T}\{\hat{F}\} d s
\end{align*}
$$

where $A$ is the cross-sectional area, $L$ is the curvilinear beam length, $\{\sigma\}=\left\{J_{11}, J_{12}, J_{13}\right\}^{T}$ and

$$
\left.\{\widehat{F}\}=\left\{\begin{array}{l}
F_{1}  \tag{6.2.11}\\
\lambda_{2} \\
\lambda_{3} \\
M_{1} \\
M_{2} \\
M_{3} \\
-m_{3} \\
m_{2}
\end{array}\right\}=[D]\right\}\left\{\begin{array}{c}
e \\
\gamma_{6} \\
\gamma_{5} \\
\rho_{1}-k_{1} \\
\rho_{2}-k_{2} \\
\rho_{3}-k_{3} \\
\gamma_{6}^{\prime} \\
\gamma_{5}^{\prime}
\end{array}\right\}, \quad[D] \equiv \int_{A}[S]^{T}[\bar{Q}][S] d A
$$

Here $[D]$ is a $8 \times 8$ symmetric matrix; $F_{1}, \lambda_{2}$ and $\lambda_{3}$ are stress resultants workconjugate to $e, \gamma_{6}$ and $\gamma_{5}$, respectively; $M_{i}$ are stress moments work-conjugate to $\rho_{1}, \rho_{2}$ and $\rho_{3}$, respectively; and $m_{2}$ and $m_{3}$ are stress moments work-conjugate to $\gamma_{5}^{\prime}$ and $\gamma_{6}^{\prime}$, respectively. Inverting Eq. (6.2.11) yields

$$
\left\{\begin{array}{c}
e  \tag{6.2.12}\\
\gamma_{6} \\
\gamma_{5} \\
\rho_{1} \\
\rho_{2} \\
\rho_{3} \\
\gamma_{6}^{\prime} \\
\gamma_{5}^{\prime}
\end{array}\right\} \equiv[\hat{D}]\left\{\begin{array}{c}
F_{1} \\
\lambda_{2} \\
\lambda_{3} \\
M_{1} \\
M_{2} \\
M_{3} \\
-m_{3} \\
m_{2}
\end{array}\right\}+\left\{\begin{array}{c}
0 \\
0 \\
0 \\
k_{1} \\
k_{2} \\
k_{3} \\
0 \\
0
\end{array}\right\}, \quad[\hat{D}] \equiv\left[\begin{array}{c}
{\left[\hat{D}_{1}\right]} \\
{\left[\hat{D}_{2}\right]}
\end{array}\right] \equiv[D]^{-1}
$$

where $\left[\hat{D}_{1}\right]$ is a $6 \times 8$ matrix, and $\left[\hat{D}_{2}\right]$ is a $2 \times 8$ matrix. We note that substituting the $[S]$ in Eq. (6.2.9) and the $[\bar{Q}]$ in Eq. (6.2.1) into Eq. (6.2.11) yields a full $8 \times 8$ matrix [D] for composite beams.

In beam theories, the influence of warping functions is nothing but modifications of the stiffness matrix $[D]$, as shown in Eqs. (6.2.11) and (6.2.9). For thin beams, most of the influences of warpings are negligible except that the decrease of torsional stiffness due to torsional warping and the decrease of flexural rigidity due to transverse shear
warpings are significant and cannot be neglected. If in-plane and torsional warpings are neglected and Timoshenko's beam theory (i.e., the first-order shear theory) is adopted to account for transverse shear strains, we have

$$
\begin{equation*}
g_{22}=g_{33}=g_{23}=g_{24}=g_{32}=g_{34}=g_{11}=0, \quad g_{15}=z, \quad g_{16}=y \tag{6.2.13}
\end{equation*}
$$

The strain-displacement relations in Eqs. (6.2.7) and (6.2.9) become

$$
\begin{gather*}
B_{11}=e+z \bar{\rho}_{2}-y \bar{\rho}_{3}+z \gamma_{5}^{\prime}+y \gamma_{6}^{\prime} \\
B_{12}=-z \bar{\rho}_{1}+\left(1+y k_{3}\right) \gamma_{6}+z k_{3} \gamma_{5} \\
B_{13}=y \bar{\rho}_{1}+\left(1-z k_{2}\right) \gamma_{5}-y k_{2} \gamma_{6} \\
\{\varepsilon\}=[S]\{\psi\},\{\psi\} \equiv\left\{e, \gamma_{6}, \gamma_{5}, \bar{\rho}_{1}, \bar{\rho}_{2}, \bar{\rho}_{3}, \gamma_{6}^{\prime}, \gamma_{5}^{\prime}\right\}^{T}  \tag{6.2.14}\\
{[S]=\left[\begin{array}{cccccccc}
1 & 0 & 0 & 0 & z & -y & y & z \\
0 & 1+y k_{3} & z k_{3} & -z & 0 & 0 & 0 & 0 \\
0 & -y k_{2} & 1-z k_{2} & y & 0 & 0 & 0 & 0
\end{array}\right]}
\end{gather*}
$$

Here $\gamma_{5}$ and $\gamma_{6}$ are energy-averaged shear rotation angles, and the actual shear rigidities accounting for non-uniform shear strains and the actual torsional rigidity accounting for torsional warping can be accounted for by modifying $[D]$ and $\{\hat{F}\}$ as

$$
\begin{array}{r}
{[D]=\left[\begin{array}{cccccccc}
D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} & D_{17} & D_{18} \\
D_{21} & c_{2} D_{22} & c_{3} D_{23} & D_{24} & D_{25} & D_{26} & c_{2} D_{27} & c_{3} D_{28} \\
D_{31} & c_{3} D_{32} & c_{1} D_{33} & D_{34} & D_{35} & D_{36} & c_{3} D_{37} & c_{1} D_{38} \\
D_{41} & D_{42} & D_{43} & c_{4} D_{44} & D_{45} & D_{46} & D_{47} & D_{48} \\
D_{51} & D_{52} & D_{53} & D_{54} & D_{55} & D_{56} & D_{57} & D_{58} \\
D_{61} & D_{62} & D_{63} & D_{64} & D_{65} & D_{66} & D_{67} & D_{68} \\
D_{71} & c_{2} D_{72} & c_{3} D_{73} & D_{74} & D_{75} & D_{76} & c_{2} D_{77} & c_{3} D_{78} \\
D_{81} & c_{3} D_{82} & c_{1} D_{83} & D_{84} & D_{85} & D_{86} & c_{3} D_{87} & c_{1} D_{88}
\end{array}\right]}  \tag{6.2.15}\\
\qquad \begin{array}{c}
\{\hat{F}\}=\left\{F_{1}, F_{2}, F_{3}, M_{1}, M_{2}, M_{3},-\hat{m}_{3}, \hat{m}_{2}\right\}^{T}=[D]\{\psi\}
\end{array} .=\left[\begin{array}{lll} 
\\
0
\end{array}\right)
\end{array}
$$

where $c_{1}$ and $c_{2}$ are shear correction factors accounting for non-uniform shear strains over the cross section, and $c_{3}$ is the shear coupling factor accounting for the effect of coupling between $\gamma_{6}$ and $\gamma_{5}$. Moreover, $c_{4}$ is used to account for the influence of torsional warping on the torsional rigidity. We note that $F_{2}, F_{3}, \hat{m}_{2}$ and $\hat{m}_{3}$ are workconjugate to the energy-averaged shear deformations $\gamma_{6}, \gamma_{5}, \gamma_{5}^{\prime}$ and $-\gamma_{6}^{\prime}$, respectively, and $F_{2}$ and $F_{3}$ also represent the actual transverse shear forces acting on the cross section.

If the material is isotropic and homogeneous, if the cross section is rectangular, and if the origin of xyz is the area centroid, one can obtain from Eqs. (6.2.11), (6.2.13) and (6.2.14) and Fig. 6.1 that

$$
\begin{gather*}
{[D]=\left[\begin{array}{cccccccc}
E A & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & G A_{2} & 0 & -k_{2} G I_{33} & 0 & 0 & 0 & 0 \\
0 & 0 & G A_{3} & -k_{3} G I_{22} & 0 & 0 & 0 & 0 \\
0 & -k_{2} G I_{33} & -k_{3} G I_{22} & G I_{11} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & E I_{22} & 0 & 0 & E I_{22} \\
0 & 0 & 0 & 0 & 0 & E I_{33} & -E I_{33} & 0 \\
0 & 0 & 0 & 0 & 0 & -E I_{33} & c_{2} E I_{33} & 0 \\
0 & 0 & 0 & 0 & E I_{22} & 0 & 0 & c_{1} E I_{22}
\end{array}\right]}  \tag{6.2.16}\\
I_{22}=\frac{1}{12} b h^{3}, I_{33}=\frac{1}{12} b^{3} h \\
A_{2}=c_{2}\left[A+\left(k_{2}^{2}+k_{3}^{2}\right) I_{33}\right], A_{3}=c_{1}\left[A+\left(k_{2}^{2}+k_{3}^{2}\right) I_{22}\right] \\
c_{1}=c_{2}=\frac{10(1+v)}{12+11 v} \\
I_{11}=c_{4}\left(I_{22}+I_{33}\right)=\frac{1}{3} b h^{3}\left(1-\frac{192 h}{\pi^{5} b} \sum_{n=1,3, \ldots}^{\infty} \frac{1}{n^{5}} \tanh \frac{n \pi b}{2 h}\right)
\end{gather*}
$$

The shear correction factors $c_{1}, c_{2}$, and $c_{3}$ can be calculated using an energy-equivalent
first-order shear-deformation theory (Pai and Schulz, 1999), and $c_{4}$ is obtained using the theory of elasticity (Timoshenko and Goodier, 1970).

### 6.3 Governing Equations

Variations of the unit vectors $\mathbf{i}_{j}$ are due to virtual rigid-body rotations of the coordinate system $\xi \eta \zeta$ and are given by

$$
\left\{\begin{array}{l}
\delta \mathbf{i}_{1}  \tag{6.3.1}\\
\delta \mathbf{i}_{2} \\
\boldsymbol{i}_{3}
\end{array}\right\}=\left[\begin{array}{ccc}
0 & \delta \theta_{3} & -\delta \theta_{2} \\
-\delta \theta_{3} & 0 & \delta \theta_{1} \\
\delta \theta_{2} & -\delta \theta_{1} & 0
\end{array}\right]\left\{\begin{array}{l}
\mathbf{i}_{1} \\
\mathbf{i}_{2} \\
\mathbf{i}_{3}
\end{array}\right\}
$$

where $\delta \theta_{1}, \delta \theta_{2}$, and $\delta \theta_{3}$ are virtual rigid-body rotations with respect to the axes $\xi, \eta$ and $\zeta$, respectively. We note that $\delta \theta_{i}$ are infinitesimal rotations and hence they are vector quantities. Moreover, $\delta \theta_{i}$ are along three perpendicular directions and hence they are mutually independent.

Using the extended Hamilton principle, Jaumann stress and strain measures, the first-order shear theory, and the concept of orthogonal virtual rotations $\delta \theta_{i}$, one can show that (Pai and Nayfeh, 2002)

$$
\begin{align*}
\int_{0}^{t} \int_{0}^{L} & {[ } \\
\partial x & \left(\left\{F_{1}, F_{2}, F_{3}\right\}[T]\right)\{\delta u v w\}+\left\{F_{1}, F_{2}, F_{3}\right\}[T][k]\{\delta u v w\} \\
& -\left\{m \ddot{u}-q_{1}, m \ddot{v}-q_{2}, m \ddot{w}-q_{3}\right\}\{\delta u v w\} \\
& +\left(M_{1}^{\prime}+M_{3} \rho_{2}-M_{2} \rho_{3}+q_{4}\right) \delta \theta_{1}  \tag{6.3.2}\\
& +\left(M_{2}^{\prime}-M_{3} \rho_{1}+M_{1} \rho_{3}-(1+e) F_{3}+q_{5}\right) \delta \theta_{2} \\
& +\left(M_{3}^{\prime}+M_{2} \rho_{1}-M_{1} \rho_{2}+(1+e) F_{2}+q_{6}\right) \delta \theta_{3} \\
& \left.-\left(-\hat{m}_{2}^{\prime}+F_{3}\right) \delta \gamma_{5}-\left(\hat{m}_{3}^{\prime}+F_{2}\right) \delta \gamma_{6}\right] d x d t \\
& -\int_{0}^{4}\left[\left\{\begin{array}{l}
M_{1} \\
M_{2} \\
M_{3}
\end{array}\right\}^{T}\left\{\begin{array}{l}
\delta \theta_{1} \\
\delta \theta_{2} \\
\delta \theta_{3}
\end{array}\right\}+\left\{\begin{array}{l}
F_{1} \\
F_{2} \\
F_{3}
\end{array}\right\}[T]\left\{\begin{array}{l}
\delta u \\
\delta v \\
\delta w
\end{array}\right\}+\hat{m}_{2} \delta \gamma_{5}-\hat{m}_{3} \delta \gamma_{6}\right]_{0}^{L} d t=0
\end{align*}
$$

where $q_{1}, q_{2}$, and $q_{3}$ are distributed forces per unit length along the axes $x, y$, and $z$, respectively; $q_{4}, q_{5}$, and $q_{6}$ are distributed moments per unit length along the axes $\xi, \eta$, and $\zeta$, respectively; and $\{\delta u v w\} \equiv\{\delta u, \delta v, \delta w\}^{T}$. The rotary inertias of highly flexible beams are negligibly small and are neglected here. The equations governing fully nonlinear deformation of beams become

$$
\begin{align*}
& \frac{\partial\{F\}}{\partial x}+[K]^{T}\{F\}+[T]\left\{\begin{array}{c}
q_{1} \\
q_{2} \\
q_{3}
\end{array}\right\}=[T]\left\{\begin{array}{l}
m \ddot{u} \\
m \ddot{v} \\
m \ddot{w}
\end{array}\right\}  \tag{6.3.3}\\
& \frac{\partial\{M\}}{\partial x}+[K]^{T}\{M\}+\left\{\begin{array}{c}
0 \\
-(1+e) F_{3} \\
(1+e) F_{2}
\end{array}\right\}+\left\{\begin{array}{l}
q_{4} \\
q_{5} \\
q_{6}
\end{array}\right\}=\left\{\begin{array}{l}
0 \\
0 \\
0
\end{array}\right\} \\
& \hat{m}_{2}^{\prime}=F_{3} \\
& \hat{m}_{3}^{\prime}=-F_{2}
\end{align*}
$$

where $\{F\} \equiv\left\{F_{1}, F_{2}, F_{3}\right\}^{T}$ and $\{M\} \equiv\left\{M_{1}, M_{2}, M_{3}\right\}^{T}$. The boundary conditions are of the form

$$
\begin{array}{lll}
\delta u=0 & \text { or } & F_{x}=\text { constant } \\
\delta v=0 & \text { or } & F_{y}=\text { constant } \\
\delta w=0 & \text { or } & F_{z}=\text { constant } \\
\delta \theta_{1}=0 & \text { or } & M_{1}=\text { constant }  \tag{6.3.4}\\
\delta \theta_{2}=0 & \text { or } & M_{2}=\text { constant } \\
\delta \theta_{3}=0 & \text { or } & M_{3}=\text { constant } \\
\delta \gamma_{5}=0 & \text { or } & \hat{m}_{2}=\text { constant } \\
\delta \gamma_{6}=0 & \text { or } & \hat{m}_{3}=\text { constant }
\end{array}
$$

where $F_{x}, F_{y}$, and $F_{z}$ are the projections of stress resultants along the axes $x, y$, and $z$, respectively, and $\delta \theta_{1}, \delta \theta_{2}$, and $\delta \theta_{3}$ are virtual rotations with respect to the axes $\xi, \eta$, and $\zeta$, respectively. They are given by

$$
\begin{align*}
& F_{x}=F_{1} T_{11}+F_{2} T_{21}+F_{3} T_{31} \\
& F_{y}=F_{1} T_{12}+F_{2} T_{22}+F_{3} T_{32}  \tag{6.3.5}\\
& F_{z}=F_{1} T_{13}+F_{2} T_{23}+F_{3} T_{33} \\
& \delta \theta_{1}=T_{31} \delta T_{21}+T_{32} \delta T_{22}+T_{33} \delta T_{23} \\
& \delta \theta_{2}=-\left(T_{31} \delta T_{11}+T_{32} \delta T_{12}+T_{33} \delta T_{13}\right)  \tag{6.3.6}\\
& \delta \theta_{3}=T_{21} \delta T_{11}+T_{22} \delta T_{12}+T_{23} \delta T_{13}
\end{align*}
$$

Equations (6.3.3) and (6.3.4) are the equations of motion and boundary conditions for beams undergoing large deformations.

It follows from equations (6.3.3) and (6.1.36) that

$$
\begin{align*}
& F_{1}^{\prime}=\rho_{3} F_{2}-\rho_{2} F_{3}+T_{11}\left(m \ddot{u}-q_{1}\right)+T_{12}\left(m \ddot{v}-q_{2}\right)+T_{13}\left(m \ddot{w}-q_{3}\right)  \tag{6.3.7a}\\
& F_{2}^{\prime}=\rho_{1} F_{3}-\rho_{3} F_{1}+T_{21}\left(m \ddot{u}-q_{1}\right)+T_{22}\left(m \ddot{v}-q_{2}\right)+T_{23}\left(m \ddot{w}-q_{3}\right)  \tag{6.3.7b}\\
& F_{3}^{\prime}=\rho_{2} F_{1}-\rho_{1} F_{2}+T_{31}\left(m \ddot{u}-q_{1}\right)+T_{32}\left(m \ddot{v}-q_{2}\right)+T_{33}\left(m \ddot{w}-q_{3}\right)  \tag{6.3.7c}\\
& M_{1}^{\prime}=\rho_{3} M_{2}-\rho_{2} M_{3}-q_{4}  \tag{6.3.7d}\\
& M_{2}^{\prime}=\rho_{1} M_{3}-\rho_{3} M_{1}+(1+e) F_{3}-q_{5}  \tag{6.3.7e}\\
& M_{3}^{\prime}=\rho_{2} M_{1}-\rho_{1} M_{2}-(1+e) F_{2}-q_{6}  \tag{6.3.7f}\\
& \hat{m}_{2}^{\prime}=F_{3}  \tag{6.3.7~g}\\
& \hat{m}_{3}^{\prime}=-F_{2} \tag{6.3.7h}
\end{align*}
$$

Also we obtain from Eqs. (6.1.37), (6.1.36) and (6.1.7) that

$$
\begin{align*}
& T_{11}^{\prime}=\rho_{3} T_{21}-\rho_{2} T_{31}+T_{12} k_{3}-T_{13} k_{2}  \tag{6.3.7i}\\
& T_{12}^{\prime}=\rho_{3} T_{22}-\rho_{2} T_{32}+T_{13} k_{1}-T_{11} k_{3}  \tag{6.3.7j}\\
& T_{13}^{\prime}=\rho_{3} T_{23}-\rho_{2} T_{33}+T_{11} k_{2}-T_{12} k_{1} \tag{6.61k}
\end{align*}
$$

Substituting Eq. (6.3.7j) and Eq. (6.3.7k) and into Eq. (6.1.39) one can show that

$$
\begin{align*}
\phi^{\prime}= & \rho_{1}-T_{11} k_{1}-T_{12} k_{2}-T_{13} k_{3}-\frac{T_{13}}{1+T_{11}}\left(\rho_{3} T_{22}-\rho_{2} T_{32}+T_{13} k_{1}-T_{11} k_{3}\right)  \tag{6.3.71}\\
& +\frac{T_{12}}{1+T_{11}}\left(\rho_{3} T_{23}-\rho_{2} T_{33}+T_{11} k_{2}-T_{12} k_{1}\right)
\end{align*}
$$

Moreover, it follows from Eq. (6.1.26) that

$$
\begin{align*}
& u^{\prime}=-1+v k_{3}-w k_{2}+(1+e) T_{11}  \tag{6.3.7m}\\
& v^{\prime}=w k_{1}-u k_{3}+(1+e) T_{12}  \tag{6.3.7n}\\
& w^{\prime}=u k_{2}-v k_{1}+(1+e) T_{13} \tag{6.3.7o}
\end{align*}
$$

And, it follows from Eqs. (6.2.12) and (6.2.15) that

$$
\left\{\begin{array}{l}
\gamma_{6}^{\prime}  \tag{6.3.7p,q}\\
\gamma_{5}^{\prime}
\end{array}\right\}=\left[\hat{D}_{2}\right]\left\{F_{1}, F_{2}, F_{3}, M_{1}, M_{2}, M_{3},-\hat{m}_{3}, \hat{m}_{2}\right\}^{T}
$$

Eqs. (6.3.7a-q) are the seventeen governing differential equations. The seventeen unknown dependent variables are

$$
\begin{equation*}
F_{1}, F_{2}, F_{3}, M_{1}, M_{2}, M_{3}, \hat{m}_{2}, \hat{m}_{3}, T_{11}, T_{12}, T_{13}, \phi, u, v, w,\left(\gamma_{6}, \gamma_{5}\right) \tag{6.3.8}
\end{equation*}
$$

It can be seen from Eq. (6.3.4) that there are only sixteen boundary conditions, and hence the order of the system is sixteen. Consequently, there are only sixteen of the seventeen unknown variables are independent and one of the differential Eqs. (6.3.7a-q) is redundant, which is because $\mathbf{i}_{1}$ is a unit vector and hence

$$
\begin{equation*}
T_{11}^{2}+T_{12}^{2}+T_{13}^{2}=1 \tag{6.3.9}
\end{equation*}
$$

In other words, $T_{11}$ is known when $T_{12}$ and $T_{13}$ are specified. However, using the seventeen equations instead of using sixteen equations makes the programming easier and the numerical results from the redundant equation can be used to double-check the results, especially to check whether $T_{11}^{2}+T_{12}^{2}+T_{13}^{2}=1$ is satisfied. We note that Eqs.
$(6.3 .7 \mathrm{~g}, \mathrm{~h})$, and $(6.3 .7 \mathrm{p}, \mathrm{q})$ govern the shear deformations $\gamma_{5}$ and $\gamma_{6}$, and Eqs. (6.3.7a-f) and (6.3.7i-o) govern the bending and torsional deformations, and they are the same as the 13 equations of the 3D Euler-Bernoulli beam theory (Pai and Palazotto, 1996).

Because $\gamma_{5}$ and $\gamma_{6}$ do not appear in Eqs. (6.3.7a-o) and they are usually unknown at boundaries, it is not necessary to integrate Eq. (6.3.7p,q). Hence, we can integrate only Eqs. (6.3.7a-o). After $F_{i}, M_{i}$ and $\hat{m}_{i}$ are obtained from the multiple shooting process, one can obtain $\gamma_{5}$ and $\gamma_{6}$ using Eq. (6.2.12), and $\gamma_{5}^{\prime}$ and $\gamma_{6}^{\prime}$ using Eq. (6.3.7p,q).

Equations (6.2.11) and (6.2.16) show that, for isotropic beams, $M_{2}=\hat{m}_{2}$ if $\gamma_{5}^{\prime}=0$ and/or $c_{1}=1$, and $M_{3}=\hat{m}_{3}$ if $\gamma_{6}^{\prime}=0$ and / or $c_{2}=1$. If $\gamma_{5}^{\prime}$ and $\gamma_{6}^{\prime}$ are negligibly small and $/$ or $c_{1} \approx c_{2} \approx 1$, one can replace $\hat{m}_{2}$ and $\hat{m}_{3}$ with $M_{2}$ and $M_{3}$ in Eq. (6.2.12) and the e and $\rho_{i}$ needed in Eqs. (6.3.7a-f) and (6.3.7i-o) can be represented in terms of only $F_{i}$ and $M_{i}$. Consequently, one needs to solve the 13 equations shown in Eqs. (6.3.7af) and (6.3.7i-o). After $F_{i}$ and $M_{i}$ are obtained from the multiple shooting method, one can obtain $\hat{m}_{2}$ and $\hat{m}_{3}$ by integrating Eqs. (6.3.7g-h), $\gamma_{5}$ and $\gamma_{6}$ using Eq. (6.2.12), and $\gamma_{5}^{\prime}$ and $\gamma_{6}^{\prime}$ using Eq. (6.3.7p,q).

### 6.4 Eigenvalue Analysis

To derive linear natural frequencies of vibrations with respect to a nonlinearly deformed static configuration we assume

$$
\begin{equation*}
u=\bar{u}+\widetilde{u}, \quad v=\bar{v}+\widetilde{v}, \quad w=\bar{w}+\widetilde{w} \tag{6.4.1}
\end{equation*}
$$

where $\bar{u}, \bar{v}$, and $\bar{w}$ denote large static displacements, and $\widetilde{u}, \widetilde{v}$, and $\widetilde{w}$ denote small dynamic displacements. If the dynamic displacements are assumed to be harmonic at a natural frequency $\omega$, we have

$$
\begin{equation*}
\ddot{u}=-\omega^{2} \widetilde{u}, \quad \ddot{v}=-\omega^{2} \widetilde{v}, \quad \ddot{w}=-\omega^{2} \widetilde{w} \tag{6.4.2}
\end{equation*}
$$

Substituting Eqs. (6.4.1) and (6.4.2) into Eqs. (6.3.7a-o) and using Taylor's expansions yields the following first-order expansions:

$$
\begin{align*}
\widetilde{F}_{1}^{\prime}= & \bar{\rho}_{3} \widetilde{F}_{2}+\widetilde{\rho}_{3} \bar{F}_{2}-\bar{\rho}_{2} \widetilde{F}_{3}-\widetilde{\rho}_{2} \bar{F}_{3} \\
& -\bar{T}_{11} m \widetilde{u} \omega^{2}-\widetilde{T}_{11} q_{1}-\bar{T}_{12} m \widetilde{v} \omega^{2}-\widetilde{T}_{12} q_{2}-\bar{T}_{13} m \widetilde{w} \omega^{2}-\widetilde{T}_{13} q_{3}  \tag{6.4.3a}\\
\widetilde{F}_{2}^{\prime}= & \bar{\rho}_{1} \widetilde{F}_{3}+\widetilde{\rho}_{1} \bar{F}_{3}-\bar{\rho}_{3} \widetilde{F}_{1}-\widetilde{\rho}_{3} \bar{F}_{1}  \tag{6.4.3b}\\
& -\bar{T}_{21} m \widetilde{u} \omega^{2}-\widetilde{T}_{21} q_{1}-\bar{T}_{22} m \widetilde{v} \omega^{2}-\widetilde{T}_{22} q_{2}-\bar{T}_{23} m \widetilde{w} \omega^{2}-\widetilde{T}_{23} q_{3} \\
\widetilde{F}_{3}^{\prime}= & \bar{\rho}_{2} \widetilde{F}_{1}+\widetilde{\rho}_{2} \bar{F}_{1}-\bar{\rho}_{1} \widetilde{F}_{2}-\widetilde{\rho}_{1} \bar{F}_{2}  \tag{6.4.3c}\\
& -\bar{T}_{31} m \widetilde{u} \omega^{2}-\widetilde{T}_{31} q_{1}-\bar{T}_{32} m \widetilde{v} \omega^{2}-\widetilde{T}_{32} q_{2}-\bar{T}_{33} m \widetilde{w} \omega^{2}-\widetilde{T}_{33} q_{3} \\
\widetilde{M}_{1}^{\prime}= & \bar{\rho}_{3} \widetilde{M}_{2}+\widetilde{\rho}_{3} \bar{M}_{2}-\bar{\rho}_{2} \widetilde{M}_{3}-\widetilde{\rho}_{2} \bar{M}_{3}  \tag{6.4.3d}\\
\widetilde{M}_{2}^{\prime}= & \bar{\rho}_{1} \widetilde{M}_{3}+\widetilde{\rho}_{1} \bar{M}_{3}-\bar{\rho}_{3} \widetilde{M}_{1}-\widetilde{\rho}_{3} \bar{M}_{1}+(1+\bar{e}) \widetilde{F}_{3}+\widetilde{e} \bar{F}_{3}  \tag{6.4.3e}\\
\widetilde{M}_{3}^{\prime}= & \bar{\rho}_{2} \widetilde{M}_{1}+\widetilde{\rho}_{2} \bar{M}_{1}-\bar{\rho}_{1} \widetilde{M}_{2}-\widetilde{\rho}_{1} \bar{M}_{2}-(1+\bar{e}) \widetilde{F}_{2}-\widetilde{e} \bar{F}_{2}  \tag{6.4.3f}\\
\widetilde{m}_{2}^{\prime}= & \widetilde{F}_{3}  \tag{6.4.3g}\\
\widetilde{m}_{3}^{\prime}= & -\widetilde{F}_{2}  \tag{6.4.3h}\\
\widetilde{T}_{11}^{\prime}= & \bar{\rho}_{3} \widetilde{T}_{21}+\widetilde{\rho}_{3} \bar{T}_{21}-\bar{\rho}_{2} \widetilde{T}_{31}-\widetilde{\rho}_{2} \bar{T}_{31}+\widetilde{T}_{12} k 3-\widetilde{T}_{13} k 2  \tag{6.4.3i}\\
\widetilde{T}_{12}^{\prime}= & \bar{\rho}_{3} \widetilde{T}_{22}+\widetilde{\rho}_{3} \bar{T}_{22}-\bar{\rho}_{2} \widetilde{T}_{32}-\widetilde{\rho}_{2} \bar{T}_{32}+\widetilde{T}_{13} k-\widetilde{T}_{11} k 3 \tag{6.4.3j}
\end{align*}
$$

$$
\begin{align*}
& \widetilde{T}_{13}^{\prime}= \bar{\rho}_{3} \widetilde{T}_{23}+\widetilde{\rho}_{3} \bar{T}_{23}-\bar{\rho}_{2} \widetilde{T}_{33}-\widetilde{\rho}_{2} \bar{T}_{33}+\widetilde{T}_{11} k_{2}-\widetilde{T}_{12} k_{1}  \tag{6.4.3k}\\
& \widetilde{\phi}^{\prime}= \widetilde{\rho}_{1}-\widetilde{T}_{11} k_{1}-\widetilde{T}_{12} k_{2}-\widetilde{T}_{13} k_{3} \\
&-\frac{\bar{T}_{13}}{1+\bar{T}_{11}}\left(\bar{\rho}_{3} \widetilde{T}_{22}+\widetilde{\rho}_{3} \bar{T}_{22}-\bar{\rho}_{2} \widetilde{T}_{32}-\widetilde{\rho}_{2} \bar{T}_{32}+\widetilde{T}_{13} k_{1}-\widetilde{T}_{11} k_{3}\right) \\
&-\left(\frac{\widetilde{T}_{13}}{1+\bar{T}_{11}}-\frac{\bar{T}_{13} \widetilde{T}_{11}}{\left(1+\bar{T}_{11}\right)^{2}}\right)\left(\bar{\rho}_{3} \bar{T}_{22}-\bar{\rho}_{2} \bar{T}_{32}+\bar{T}_{13} k_{1}-\bar{T}_{11} k_{3}\right)  \tag{6.4.31}\\
&+\frac{\bar{T}_{12}}{1+\bar{T}_{11}}\left(\bar{\rho}_{3} \widetilde{T}_{23}+\widetilde{\rho}_{3} \bar{T}_{23}-\bar{\rho}_{2} \widetilde{T}_{33}-\widetilde{\rho}_{2} \bar{T}_{33}+\widetilde{T}_{11} k_{2}-\widetilde{T}_{12} k_{1}\right) \\
&+\left(\frac{\widetilde{T}_{12}}{1+\bar{T}_{11}}-\frac{\bar{T}_{12} \widetilde{T}_{11}}{\left(1+\bar{T}_{11}\right)^{2}}\right)\left(\bar{\rho}_{3} \bar{T}_{23}-\bar{\rho}_{2} \bar{T}_{33}+\bar{T}_{11} k_{2}-\bar{T}_{12} k_{1}\right) \\
& \widetilde{u}^{\prime}=\widetilde{v} k_{3}-\widetilde{w} k_{2}+(1+\bar{e}) \widetilde{T}_{11}+\widetilde{e} \bar{T}_{11}  \tag{6.4.3m}\\
& \widetilde{v}^{\prime}=\widetilde{w} k_{1}-\widetilde{u} k_{3}+(1+\bar{e}) \widetilde{T}_{12}+\widetilde{e} \bar{T}_{12}  \tag{6.4.3n}\\
& \widetilde{w}^{\prime}= \widetilde{u} k_{2}-\widetilde{v} k_{1}+(1+\bar{e}) \widetilde{T_{13}}+\widetilde{e} \bar{T}_{13}  \tag{6.4.3o}\\
& \omega^{\prime}= 0 \tag{6.4.3p}
\end{align*}
$$

where

$$
\begin{gathered}
\widetilde{T}_{21}=-\cos \bar{\phi} \widetilde{T}_{12}+\widetilde{\phi} \sin \bar{\phi} \bar{T}_{12}-\sin \bar{\phi} \widetilde{T}_{13}-\widetilde{\phi} \cos \bar{\phi} \bar{T}_{13} \\
\widetilde{T}_{22}=\cos \bar{\phi}\left(\widetilde{T}_{11}+\frac{2 \bar{T}_{13}}{1+\bar{T}_{11}} \widetilde{T}_{13}-\frac{\bar{T}_{13}^{2}}{\left(1+\bar{T}_{11}\right)^{2}} \widetilde{T}_{11}\right)-\widetilde{\phi} \sin \bar{\phi}\left(\bar{T}_{11}+\frac{\bar{T}_{13}^{2}}{1+\bar{T}_{11}}\right) \\
-\sin \bar{\phi}\left(\frac{\bar{T}_{13}}{1+\bar{T}_{11}} \widetilde{T}_{12}+\frac{\bar{T}_{12}}{1+\bar{T}_{11}} \widetilde{T}_{13}-\frac{\bar{T}_{12} \bar{T}_{13}}{\left(1+\bar{T}_{11}\right)^{2}} \widetilde{T 1}\right)-\widetilde{\phi} \cos \bar{\phi} \frac{\bar{T}_{12} \bar{T}_{13}}{1+\bar{T}_{11}} \\
\widetilde{T}_{23}=\sin \bar{\phi}\left(\widetilde{T}_{11}+\frac{2 \bar{T}_{12}}{1+\bar{T}_{11}} \widetilde{T}_{12}-\frac{\bar{T}_{12}^{2}}{\left(1+\bar{T}_{11}\right)^{2}} \widetilde{T}_{11}\right)+\widetilde{\phi} \cos \bar{\phi}\left(\bar{T}_{11}+\frac{\bar{T}_{12}^{2}}{1+\bar{T}_{11}}\right) \\
-\cos \bar{\phi}\left(\frac{\bar{T}_{13}}{1+\bar{T}_{11}} \widetilde{T}_{12}+\frac{\bar{T}_{12}}{1+\bar{T}_{11}} \widetilde{T}_{13}-\frac{\bar{T}_{12} \bar{T}_{13}}{\left(1+\bar{T}_{11}\right)^{2}} \widetilde{T}_{11}\right)+\widetilde{\phi} \sin \bar{\phi} \frac{\bar{T}_{12} \bar{T}_{13}}{1+\bar{T}_{11}}
\end{gathered}
$$

$$
\begin{aligned}
\tilde{T}_{31}= & \sin \bar{\phi} \tilde{T}_{12}+\tilde{\phi} \cos \bar{\phi} \bar{T}_{12}-\cos \bar{\phi} \tilde{T}_{13}+\tilde{\phi} \sin \bar{\phi} \bar{T}_{13} \\
\tilde{T}_{32}= & -\sin \bar{\phi}\left(\tilde{T}_{11}+\frac{2 \bar{T}_{13}}{1+\bar{T}_{11}} \tilde{T}_{13}-\frac{\bar{T}_{13}^{2}}{\left(1+\bar{T}_{11}\right)^{2}} \tilde{T}_{11}\right)-\tilde{\phi} \cos \bar{\phi}\left(\bar{T}_{11}+\frac{\bar{T}_{13}^{2}}{1+\bar{T}_{11}}\right) \\
& -\cos \bar{\phi}\left(\frac{\bar{T}_{13}}{1+\bar{T}_{11}} \tilde{T}_{12}+\frac{\bar{T}_{12}}{1+\bar{T}_{11}} \tilde{T}_{13}-\frac{\bar{T}_{12} \bar{T}_{13}}{\left(1+\bar{T}_{11}\right)^{2}} \tilde{T}_{11}\right)+\tilde{\phi} \sin \bar{\phi} \frac{\bar{T}_{12} \bar{T}_{13}}{1+\bar{T}_{11}} \\
\tilde{T}_{33}= & \cos \bar{\phi}\left(\tilde{T}_{11}+\frac{2 \bar{T}_{12}}{1+\bar{T}_{11}} \tilde{T}_{12}-\frac{\bar{T}_{12}^{2}}{\left(1+\bar{T}_{11}\right)^{2}} \tilde{T}_{11}\right)-\tilde{\phi} \sin \bar{\phi}\left(\bar{T}_{11}+\frac{\bar{T}_{12}^{2}}{1+\bar{T}_{11}}\right) \\
& +\sin \bar{\phi}\left(\frac{\bar{T}_{13}}{1+\bar{T}_{11}} \tilde{T}_{12}+\frac{\bar{T}_{12}}{1+\bar{T}_{11}} \tilde{T}_{13}-\frac{\bar{T}_{12} \bar{T}_{13}}{\left(1+\bar{T}_{11}\right)^{2}} \tilde{T}_{11}\right)+\tilde{\phi} \cos \bar{\phi} \frac{\bar{T}_{12} \bar{T}_{13}}{1+\bar{T}_{11}}
\end{aligned}
$$

Equations (6.4.3a-p) are the sixteen govern differential equations. The sixteen unknown dependent variables are

$$
\begin{equation*}
\widetilde{F}_{1}, \widetilde{F}_{2}, \widetilde{F}_{3}, \widetilde{M}_{1}, \tilde{M}_{2}, \tilde{M}_{3}, \widetilde{m}_{2}, \widetilde{m}_{3}, \widetilde{T}_{11}, \widetilde{T}_{12}, \widetilde{T}_{13}, \widetilde{\phi}, \widetilde{u}, \widetilde{v}, \widetilde{w}, \omega \tag{6.4.4}
\end{equation*}
$$

Equation (6.4.3p) is based on the fact that the natural frequency $\omega$ is the same for every point of the beam.

### 6.5 Packaging Analysis of a Triangular Frame

We investigate the packaging deformation of a triangular frame in this section. The packaging is a large deformation process which can be depicted in Fig. 6.5. The frame has a equilateral triangle geometry. Each side is a flexible slender beam. Fig. 6.6 shows the actual geometry and the coordinate system used for geometrically exact modeling and analysis. The beam has following material properties and dimensions:

$$
E=144 G P a, \quad v=0.32, \quad l=1 \mathrm{ft}, \quad b=0.25 \mathrm{in}, \quad h=1 / 12 \mathrm{in}
$$

To make the numerical iteration smooth, the triangular frame is modeled as three straight flexible beam connected by three circular sections. The joints have a length $r \pi / 3$ and a curvature $k_{2}=1 / r$. The twisting angle $\theta$ always increases from 0 to $180^{\circ}$ no matter


Fig. 6.5: The packaging of a triangular frame.


Fig. 6.6: The geometry of a triangular frame
what packaging scheme is used. The value of twisting moment $\hat{M}_{3}$, however, does not increase monotonically when $\theta$ increases from 0 to $180^{\circ}$. Thus the deformation could be stable and controllable if $\theta$ instead of $M_{3}$ is used as the control parameter. Note that the twisted angle with relative to its original position of the top branch, or the bottom branch, is $\theta$ and the total twisted angle of the frame is $2 \theta$. As the triangular frame and the deformation of the packaging process are both symmetric with respect to the axis, it is appropriate to analyze only one half of the triangular frame. Then, the packaging process
is modeled as a two-point boundary value problem describing the deformation of the curved beam from point A and the middle point of B . The problem can be solved by directly attacking the corresponding differential equations (6.3.7a-q). The solutions are uniquely determined by the specified boundary conditions and the coefficients of the equations. The boundary conditions are

$$
\begin{aligned}
& \text { At } s=0: u=v=w=\phi=T_{13}=0, T_{12}=\sin \theta, T_{11}=\cos \theta \\
& \text { At } s=L: u=v=F_{z}=\phi=T_{13}=0, T_{12}=\sin \theta, T_{11}=\cos \theta
\end{aligned}
$$

The multiple shooting method (Stoer 1980) is an efficient numerical method which provides results as accurate as necessary for BVPs and the obtained solutions are numerically exact. It has been successfully applied for many large deformation problems of highly flexible structures, including a cantilever beam subjected to an end moment; first, second, and third mode buckling of a cantilever; a cantilever subjected to transverse end load; a cantilever subjected to uniformly distributed transverse load; a fixed-free half circular ring subjected to a tangential end load; a circular ring subjected to twisting; a clamped-free beam rotating at a constant angular velocity $\Omega$; a cantilever L-frame subjected to a concentrated force at one of the two corners at the tip of the L-frame; a circular arch subjected a concentrated load; and a helical spring subjected to an axial displacement (Pai, 1996, 2002). We are going to use the multiple shooting method to do the packaging analysis of the triangular frame. Fig. 6.7 shows the three dimensional views and 2D projections of the deformed geometries before the penetration happens at $\theta<2 \pi / 3$. The penetration is the cross over between two branches, which happens only in the numerical simulation. Figs. 6.7(a)-(h) are solutions from the multiple shooting analysis, and Figs. 6.7(i)-(p) are experimental results obtained using the 3D motion
system introduced in Chapter 4. We note that the experimental frame has the following material properties and dimensions, which are different from those of the numerical model:

$$
E=200 G P a, \quad v=0.32, \quad L=1.25 \mathrm{~m}, \quad b=2.5 \mathrm{~mm}, \quad h=1.25 \mathrm{~mm}
$$







Fig.6.7 Three-dimensional views and 2D projections of the deformed geometries when $\theta=\pi / 12: \pi / 12: 23 \pi / 36$, where (a)-(h) are numerical solutions, and (i)-(p) are experimental results.


Fig.6.7 (Continued)


Fig. 6.7 (Continued)

The two frames have different material properties and dimensions but they show similar deformation process during packaging. We did not obtain numerically exact simulation results for the deployment process after penetration. Fig. 6.8 show the experimental three-dimensional views and 2D projections of the deformed geometries after penetration happened at $\theta>2 \pi / 3$. In our experiments, the contact of two branches began when the twisting angle was close to $\theta_{0} \approx 2 \pi / 3$. The exact value of $\theta_{0}$ depends on the bending and twisting stiffness of the frame, which is in turn dependent on the geometry and material of the frame. For the frame we studied, $\theta_{0} \approx 117 \pi / 180$.


Fig. 6.8: Experimental three-dimensional views and 2 D projections of the deformed geometry when $\theta>\theta_{0} \approx 2 \pi / 3$.

Practically speaking, the contact can be modeled by applying either a concentrated force at the contacting point or a distributed force at the contacting area.

The force should be perpendicular to the branch, and it should neither be too large so that the two branches are obviously detached nor too small so that penetration happens in the numerical simulation. Normally, distributed-load (represented by the $q_{1}, q_{2}$ and $q_{3}$ in the governing equations) is adopted in the numerical simulation as it guarantees a better convergence for our program. In real situation, the frame may be packaged by applying twisting moments. However, depending on the smoothness of the surfaces of the frame, the friction force due to the contact may prevent a smooth packaging. Also, because the direction and magnitude of the contact force can not be exactly determined, this approach cannot strictly simulate the real contact and the packaging process. Moreover, unsmooth deployment is not allowed for high precision space structures. However, to certain extent, it provides us knowledge about the deformed geometry during the packaging process. Fig. 6.9 shows the deformed geometry obtained using the contact approach.


Fig. 6.9: The deformed geometry obtained by simulating the contact when $\theta=8 \pi / 9$.

### 6.6 Conclusion and Discussion

We were not fully successful in simulating the packaging of the highly flexible triangular frame. The main reason is that in the packaging scheme by applying twisting moments, we did not find an efficient numerical algorithm to simulate the contact yet or mathematically saying, an algorithm to solve the two-point boundary value problem with internal restriction. However, we believe there are exact solutions and we will keep working on it. Meanwhile, one may consider other deployment schemes for packaging. During our experiments, we observed that the packaging process could be smooth and no contact between two branches if deformation paths of points N1 and N3 (see Fig.6.6) are perpendicular to the plane defined by the two connecting sections at N1 and N3 at every point of the deformation path. This can be realized by dividing the deformation into many steps and restricting the loads to be perpendicular to the deformed plane defined by the two connecting sections at every step. The finite element method is a more suitable method for analyzing this problem.

## CHAPTER 7

## RECOMMENDATIONS FOR FUTURE WORK

In this dissertation, we performed both theoretical and experimental studies of vibrations of strings and cables, and packaging analysis of highly flexible triangular frames. Here we recommend some tasks in each area for future research.

### 7.1 Theoretical Study of Nonlinear Vibration of Strings

As a classical problem in structural dynamics, nonlinear string vibration continuously inspires the interests of researchers. Many articles on meaningful topics that are related to various applications have been presented in the literature. Based on the work done in this dissertation, a more insightful investigation on the string vibration is possible. For this purpose, it is necessary to do more detailed studies, including trying cases with damping and forcing at various levels (especially those with rich bifurcations) and frequency scanning using smaller increment (especially around the bifurcation points). The bifurcations and the creation and vanish of the isolated and Hopf branches need to be studied in detail. Study by direct integration is a useful approach because the asymptotic solutions from perturbation analysis neglect higher-order nonlinear effects, which may be important when the vibration amplitude increases. Another important issue in string dynamics we did not studied in this dissertation is the corresponding relations between the solutions of the averaged system
(equations (3.3.7) - (3.3.10)) and those of the original non-autonomous system (equations (3.1.56) - (3.1.57)) . The original non-autonomous system can be directly solved by numerical integration. The stability of the periodic solutions is determined by the engenvalues (Floquent multipliers) of the associated monodromy matrix. Generally speaking, between the averaged system (the first-order approximation) and the original system, there are some connections that can be determined by using the following theory of integral manifolds:
(1) A hyperbolic (with eigenvalues away from the imaginary axis) fixed-point constant solution corresponds to a hyperbolic periodic solution of the original system;
(2) A steady-state periodic limit-cycle solution of the averaged system corresponds to an amplitude modulated motion with the basic period of natural time of the original system. The period of the modulation is determined by the slow time scale;
(3) A hyperbolic periodic orbit corresponds to an invariant closed curve in the Poincare map and a hyperbolic invariant torus. In other words, limit-cycle solutions of the averaged system imply a 2-torus for the original system. And, a cascade of period-doubling bifurcations implies a series of torus-doubling bifurcation in the corresponding nonautonomous system.

Above observations are not always true. Firstly, the correspondences are valid only for cases with small enough excitation amplitude. If the higher-order (second order) terms are not neglected in the averaging, then its disturbing effect on the system may change the connection quantitatively or even qualitatively. Consequently, the averaging results may not be valid for problems of physical interest. Secondly, the correspondence may not be accurate for solutions close to the bifurcation points because there is shifting between the bifurcation-
points of two systems although the frequency response agrees qualitatively if the excitation is small enough. As the excitation strength reduces, the difference becomes smaller. If the excitation strength increases, the shifting of the bifurcation points may become pretty large so that even the bifurcation sequence changes. Moreover, the shifting is not only in the detuning but also in the damping as well, which means the same bifurcation sequence, if not shifted in the detuning, exist in the averaged system and the original system having different damping values.

For more complex solutions like aperiodic, asymptotic (in time), and chaotic behavior, the connections between two systems are to be determined.

### 7.2 Experimental Study of Nonlinear Vibration of Strings

More experimental tests of strings and cables using the 3D motion analysis system are meaningful and contributive to the understanding of dynamics of strings and cables. It is expected that the experimental results can be improved by testing strings and cables made of materials less stiff and by adjusting the set-up and the sag-to-span ratio. Complex nonlinear phenomena such as period-doubled, quasi- periodic and chaotic vibrations may become observable. Moreover, we recommend analyzing the experimental nonlinear time series using the Hilbert-Huang transformation (HHT). HHT provides the instantaneous frequency and amplitude of the vibration. Together with other contributive properties, HHT is extremely efficient for analyzing nonlinear and non-stationary data to extract different nonlinearities, hardening and softening effects, etc...

### 7.3 Experimental Study of Nonlinear Vibration of Cables

There are much more experimental studies needed for cable vibrations because it has more complex dynamic phenomena as well as wider industrial applications. As we know, the crossover phenomenon, the basis for resonant vibration of cables in current nonlinear studies, is valid for linear theory only. And so it cannot exactly predict and explain the nonlinear resonant vibrations as observed in our experiments. By doing more extensive vibration tests, the relation between the elasto-geometry parameter (describing the relation between the tension, geometry (span, sag), and material) and the excitations can be better understood better. Also, the influence of cable inclination on its dynamics can be effectively studied using the 3D motion analysis system.

### 7.3 Packaging Analysis of Highly Flexible Triangular Frames

Our experiment for the packaging of a highly flexible triangular frame is successful. However, we did not finish the numerical simulation of the packaging due to convergence issue of the algorithm. Finite element procedure, famous for its robust convergence and convenience for force and displacement control for solid mechanics problems, may help us attack this large deformation problem. It is a problem involving both large displacements and rotations. Pai (1989, 1990 and 2004) developed a geometrically exact beam theory which is powerful for the analysis of large deformation problems and can be used to develop the beam element needed for our analysis. The necessary future work is to develop a finite element iteration procedure that can model and determine appropriate loadings to guarantee a smooth and controllable packaging.

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## VITA

Jiazhu Hu was born on November 19, 1977 in Ganzhou, Jiangxi Province, People’s Republic of China. He graduated with a bachelor degree in structural engineering from Anhui University of Science and Technology in 1999. He started his graduate study in structural engineering at Southeast University in 2000, and received his M.S. degree in March 2003. He came to the University of Missouri-Columbia in August 2003 and received his Ph.D degree from the Department of Mechanical and Aerospace Engineering in 2006.

