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MATH 464WI
History of Mathematics with Dr. Richard Delaware

The Impossible Proof:
An Analysis of Adrien-Marie Legendre's Attempts to Prove Euclid's Fifth Postulate

Abstract

Euclid's Fifth Postulate, the controversial Parallel Postulate, has been labeled by mathematical scholars throughout the centuries as unnecessary, as able to be proved from the other four Postulates. Many throughout the centuries tried to give a firm proof for this claim. However, as of 1750, none had been given. Adrien-Marie Legendre, French mathematician who lived from 1752-1833, spent forty years of his life attempting to produce such a definite proof. This paper explores some of Legendre's major attempts, and it demonstrates the reasons why they failed.

Forty years. For forty long years, Adrien-Marie Legendre [1752-1833] [6], the brilliant French mathematician, attempted to prove that the sum of the angles of a triangle is equal to two right angles, or one hundred eighty degrees, using only the first four Postulates and five Common Notions of Euclid. These attempts were scattered throughout the twelve editions of his book *Elements of Geometry* from 1794 to 1823 **[3, 213]**, texts that expanded on Euclid's Elements and were the "leading elementary text on the topic for around 100 years" [6]. The proof of the sum of the angles of a triangle is fairly simple if one accepts Euclid's Fifth Postulate without question. However, Legendre was not satisfied with this. He wanted a proof that did not utilize this postulate. He desired to display to everyone that Euclid's *Elements* were the surest foundation that could be desired for mathematics, that they were completely true even without the controversial Fifth Postulate. This was possible if he could simply construct a proof determining the exact sum of the angles of a triangle using only Euclid's first four Postulates, his five Common Notions, and the first twenty-eight Propositions deduced from them.

The great Greek mathematician, Euclid of Alexandria, lived three centuries before the Common Era, yet his writings on the subject of geometry have remained the primary authority in that field for thousands of years. Euclid began his work by presenting five primary truths, which he called "Postulates", along with five "Common Notions", which we would call rules of logic. From only these ten statements, the remainder of Euclid's work was built by proof. These ten were to be taken as true without the need of justification, for they appeared, for the most part, self- evident. The five Postulates are as follows:

- [It is possible] to draw a straight line from any point to any point.

- [It is possible] to produce a finite straight line continuously in a straight line.

- [It is possible to] describe a circle with any centre and distance [radius].

- That all right angles are equal to one another.

- That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles. **[3, 195-202]**

Observe the Fifth Postulate. The basic idea of it is that, if two continuous lines, such as *AB* and *CD* below, are not parallel, they will, eventually, meet at a point (in the figure, they will meet on the right). If a third line is drawn (such as *EF* below) intersecting the two non-parallel lines, then the sum of the two interior angles that line makes with the non-parallel lines will be less than "two right angles" **[3, 202]**, or one hundred eighty degrees, on the side where the two parallels meet.

This final postulate, also called the "Parallel Postulate", caused much controversy among mathematicians following Euclid **[3, 202]**. As can easily be seen, it is, first of all, much longer than the first four. Moreover, the nature of its truthfulness is not as immediately evident. It seems that even Euclid himself was hesitant about the necessity of this postulate, for, while proceeding with his Propositions he used the first four almost immediately and quite liberally, he waited to use the Parallel Postulate until his twenty-ninth Proposition of Book I **[2, 36]**. Though many mathematicians after Euclid had argued about and stated opinions concerning the necessity of this postulate, none had been able to present a proof showing that it could be discarded. Legendre set himself to the task. Sadly, he was doomed to failure. Even though he was unsuccessful, however, it is interesting to examine his proofs to see where they were exactly correct and how easily one small fault can cause the downfall of a proof.

We will explore Legendre's journey by looking at four proofs demonstrating the core of his research. The first of these establishes, correctly, that the parallel postulate can be proven, given the knowledge that the angles of a triangle sum to two right angles. Second, we will look at Legendre's successful proof that the angles of a triangle must sum to less than or equal to two right angles. Finally we will look at two of his unsuccessful, yet brilliant, attempts at ascertaining, without the Fifth Postulate, that this sum is strictly equal to two right angles. Throughout these proofs, my comments will be inserted in [square brackets]. Unless otherwise noted, I have also drawn all figures in this paper. We will now proceed with our exploration.

As an integral portion of his succession of proofs, Legendre uses the idea of the *Postulate of Archimedes,* which is as follows:

Preliminary: The Postulate of Archimedes [1, 23]

Let A_1 be any point upon a straight line between the arbitrarily chosen points A and B. Take the points A_2, A_3, ... so that A_1 lies between A and A_2, A_2 lies between A_1 and A_3, etc. moreover let the segments AA_1, A_1A_2, A_2A_3,... be all equal. Then among this series of points, there always exists a certain point An such that B lies between A and A_n.

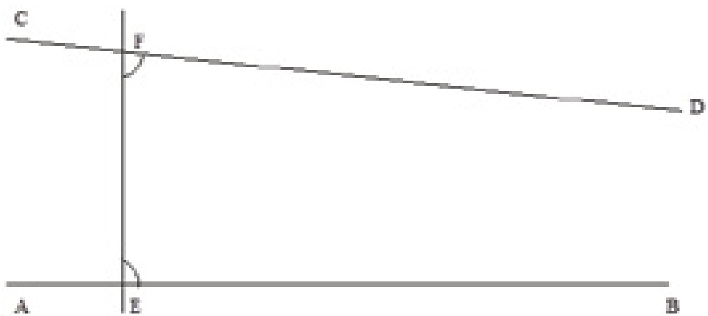
[Thus, for any line segment AB, we are able to choose n sufficiently large so that the sum of the segments AA_1, A_1A_2,... $A_{n-1}A_n$ is greater than the length of the segment AB.]

With this information available to him, Legendre moved towards the first step in his plan. He decided that he would approach the postulate from an indirect route. If it could be known that the angles of a triangle add to two right angles, then the parallel postulate would immediately follow. In Euclid's Elements this is not shown, but rather its converse appears in Proposition I-32. It rested on the assumption that the Fifth Postulate is true, not Legendre's desired goal. Thus, he chose to begin by proving the opposite direction.

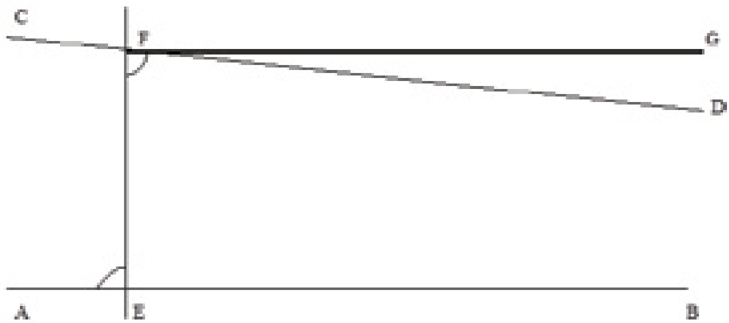
Legendre's First Proof

If the Angles of a Triangle Sum to Two Right Angles, the Fifth Postulate is True [5, 18]

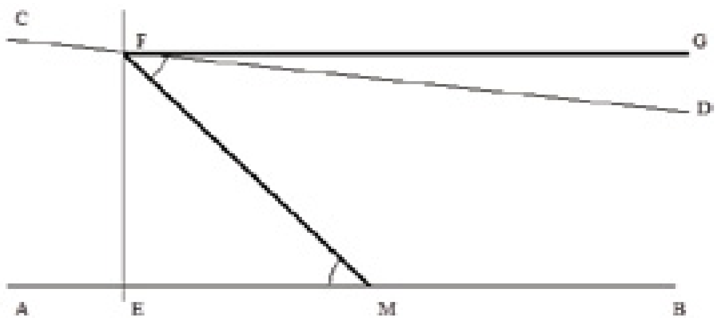
If two straight lines AB, CD, make with a third EF, two interior angles, on the same side, the sum of which is greater or less than two right angles, the lines AB, CD, produced sufficiently far, will meet.



[Proof:] *Demonstration.* Let the sum [of the two interior angles on the right side of *EF*,] *BEF* + *EFD* be less than 2 right angles;



draw FG so as to make the angle EFG equal to AEF; we shall have the sum BEF + EFG equal [because EFG = AEF] to the sum BEF + AEF [which composes a straight line], and consequently equal to two right angles; and, since [the original sum] BEF + EFD is less than two right angles, the straight line DF will be comprehended in [be located within] the angle EFG.



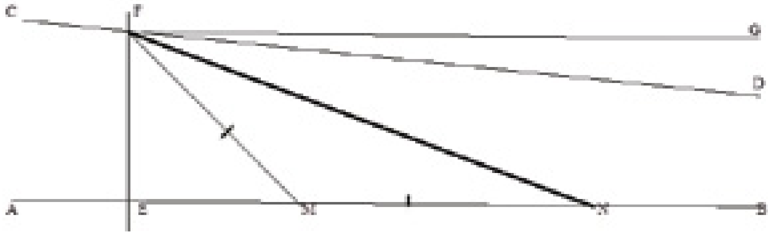
Through the point *F* draw an oblique line *FM*, meeting [line] *AB* in [point] *M*; the angle *AMF* will be equal to *GFM*, since, by adding to each the same quantity *EFM* + *FEM*, the two sums are each equal to two right angles.

[We know from our hypothesis that the angles of a triangle add to two right angles. Therefore, in triangle *FEM*:

$$(EFM + FEM) + AMF = 2 \text{ Right angles}$$

Also,

$$EFM + GFM = EFG = AEF$$
Further, $AEF + FEM = 2$ Right angles.
Thus, (EFM + GFM) + FEM = 2 Right angles].



Take now [on line *AB*] *MN* = *FM*, and join *FN;* the exterior angle *AMF*, of the triangle *FMN*, is equal to the sum of the two opposite interior angles *MFN, MNF*.

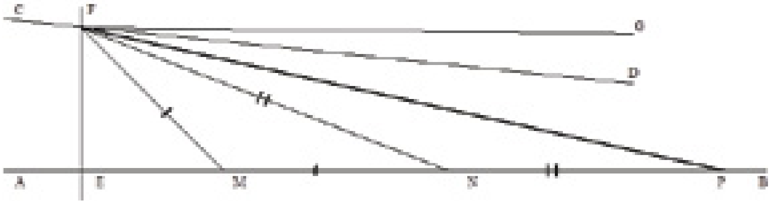
[Again, the angles of a triangle sum to two right angles, by hypothesis. Thus, as follows:

Forming a straight line,	$AMF + FMN = 2$ Right angles
In triangle *FMN*,	$FMN + MFN + MNF = 2$ Right angles
So,	$AMF + FMN = FMN + MFN + MNF$
	Therefore, $AMF = MFN + MNF$.]

[A]nd these [angles, *MFN* and *MNF*] are equal to each other, since they are opposite to the equal sides *MN, FM* [of the isosceles triangle *FMN*]; consequently the angle *AMF*, or its equal *MFG*, is double of [the angle] *MFN*; therefore the straight line *FN* divides into two equal parts the angle *GFM [MFG]*,

$$[MFG = AMF = MFN + FNM = 2MFN.]$$

and [line *FN*] meets the line *AB* in a point *N* situated at a distance *MN* [equal to *FM*].



It follows from the same demonstration, that if we take [on line *AB*] *NP* = *FN*, we determine, upon the line *AB*, the point *P* of the straight line *FP*, which makes the angle *GFP* equal to half the angle *GFN*, or one fourth of the angle *GFM*.

$$[GFP = \tfrac{1}{2}\,GFN = \tfrac{1}{2}\,(\tfrac{1}{2}\,GFM) = \tfrac{1}{4}\,GFM]$$

We are able, therefore, in this manner, to take successively the half, the fourth, the eighth, & [et]c., of the angle *GFM* [which is greater than angle *DFM*], and the lines which form these divisions meet the line *AB* in points more and more distant, but easily determined, since [by construction] $MN = FM$, $NP = FN$, $PQ = PF$, & [et]c. Indeed, it will be remarked that each successive distance of the points of intersection from the fixed point *F*, is not exactly double the distance of the preceding point of intersection; since *FN*, for example, is less than $FM + MN$, or $2FM$ [$FN < 2FM$] [This is from Euclid's Proposition I-20, "In any triangle [such as *FMN*], the sum of any two sides [such as *FM* and *MN*] is greater than the remaining one [*FN*]" [3 286]. In our case, $FM = MN$, so $2FM > FN$]; we have, in like manner, $FP < 2FN$, $FQ < 2\,FP$, & [et]c.

But, by continuing to subdivide the angle *GFM*, in this manner, we shall soon arrive at the angle *GFZ* less than the given angle *GFD* [as shown by the Postulate of Archimedes]; [I]t will nevertheless be true that *FZ* produced will meet *AB* in a determinate [calculable] point [Z]; therefore, for a still stronger reason, the straight line *FD*, comprehended in [contained within] the angle *EFZ*, will meet *AB*.

[End of proof].

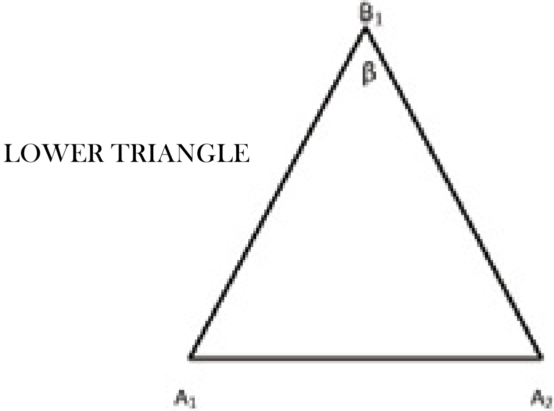
With this portion of his task accomplished, Legendre moved to the next step in his plan: proving that the sum of the angles of a triangle must not be *greater than* two right angles.

We will now proceed with Legendre's second proof.

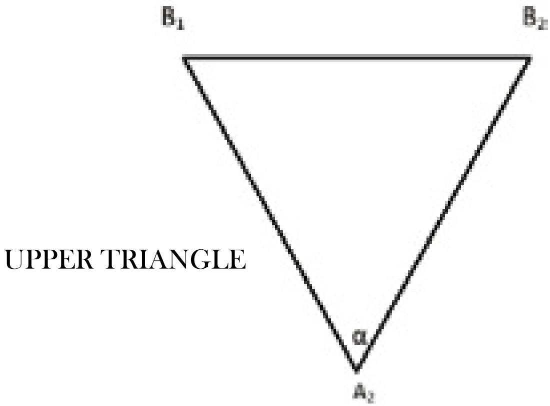
Legendre’s Second Proof

Angles of A Triangle Sum to Less Than or Equal to Two Right Angles [1, 55-56]

[Proof:] Let n equal segments A_1A_2, A_2A_3,…,A_nA_{n+1} be taken one after the other on a straight line



On the same side of the line let n equal [lower] triangles be constructed, having for their third angular points $B_1B_2…B_n$.



The segments B_1B_2, B_2B_3,…,$B_{n-1}B_n$, which join these vertices, are equal [by construction] and can be taken as the bases of n equal [upper] triangles, $B_1A_2B_2$, $B_2A_3B_3$,…,$B_{n-1}A_nB_n$.

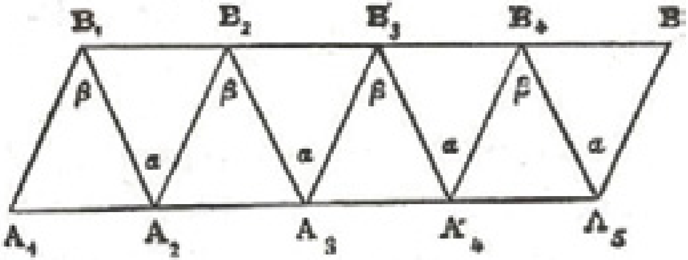


Fig 28.

[1, 56]

[The upper triangles $B_1A_2B_2$, $B_2A_3B_3$,…,$B_{n-1}A_nB_n$ have two sets of sides equal, B_1A_2,…,$B_{n-1}A_n$ and B_2A_2,…,B_nA_n. Further, the angle these sides form, angle α in the figure, is the difference between two right angles (a straight line) and the two lower vertices of the first set of triangles, which are congruent by construction. Thus, each angle α is equal to all other angles α, so the triangles are congruent].

The figure is completed by adding the [upper] triangle $B_n\ A_{n+1}\ B_{n+1}$ which is equal to the others.

Let the angle B_1 of the [lower] triangle $A_1B_1A_2$ be denoted by β, and the angle A_2 of the consecutive [upper] triangle by α.

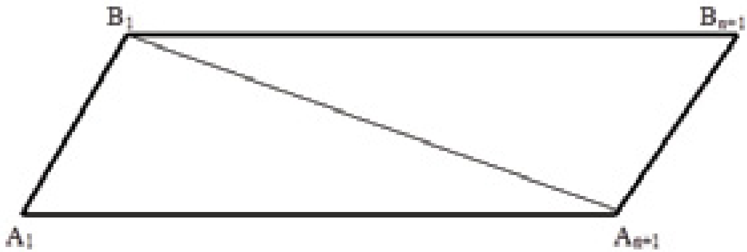
Then [we claim] $\beta \leq \alpha$.

[Proof of claim:] In fact, if [by way of contradiction] $\beta > \alpha$, by comparing the two triangles $A_1B_1A_2$ and $B_1A_2B_2$, which have two equal sides, we would deduce $A_1A_2 > B_1B_2$.

[From Euclid's Proposition I-24, we know that "If two triangles have the two sides equal to the two sides respectively, but have the one of the angles contained by the equal straight lines greater than the other, they will also have the base greater than the base." **[3, 296]** If β is greater than α, the base opposite β, which is A_1A_2, will be greater than the base opposite α, namely B_1B_2].

Further… the broken line $A_1B_1B_2$… $B_{n+1}A_{n+1}$ is greater than the segment A_1A_{n+1}[.]

[A fact that can be shown as follows: by construction, segments A_1A_{n+1} and B_1B_{n+1} are straight lines, which are joined at their endpoints by segments A_1B_1 and $B_{n+1}A_{n+1}$.



By Euclid's proposition I-20, also known as the *triangle inequality*, "In any triangle two sides taken together in any manner are greater than the remaining one" **[3, 286]**. Knowing this,

$$A_1B_1 + A_{n+1}B_1 > A_1A_{n+1} = n\,A_1A_2$$

Hence, certainly it is also true that the sum of three sides of the above figure is greater than the remaining side, A_1A_{n+1}.Therefore, the following inequality is true:]

[Continuing Legendre's proof:]

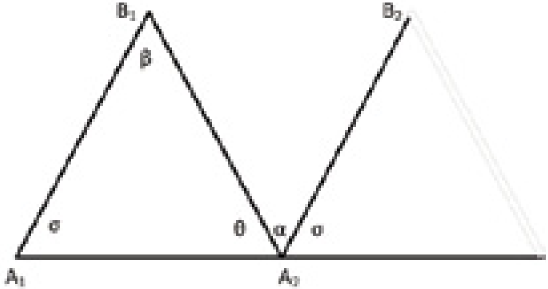
$$A_1B_1 + n\,B_1B_2 + A_{n+1}\,B_{n+1} > n\,A_1A_2$$
$$[A_1B_1 + A_{n+1}\,B_{n+1} > n\,A_1A_2 - n\,B_1B_2]$$
$$\text{ie,}\quad 2A_1B_1 > n\,(A_1A_2 - B_1B_2).$$

But if *n* is taken sufficiently great, this inequality contradicts the *Postulate of Archimedes* [described above].

Therefore A_1A_2 is not greater than B_1B_2, and it follows that it is impossible that $\beta > \alpha$.

Thus, we have $\beta \le \alpha$. **[End of proof of claim.]**

From this it readily follows that the sum of the angles of the triangle $A_1B_1A_2$ is less than or equal to two right angles.



[To see how, as Legendre states, his claim "readily follows", observe the above figure. Let the two angles which form a straight line with α be called θ and σ. Thus, $\alpha + \theta + \sigma = 2$ right angles.

Observe:

$$\beta \leq \alpha$$
$$\beta + \theta \leq \alpha + \theta$$
$$\beta + \theta + \sigma \leq \alpha + \theta + \sigma$$

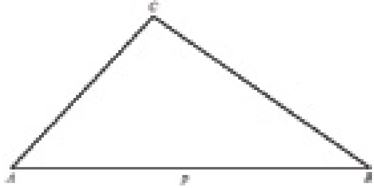
Sum of the angles of triangle $A_1B_1A_2 \leq$ two right angles.]
[End of proof]

Legendre, at this point, had the first portion of his project accomplished. He then had to prove the second half: that the angles of a triangle were not just less than or equal to two right angles, they were strictly "equal to".

Legendre's Third Proof

The First Attempt at Proving that the Angles of a Triangle Sum to Two Right Angles
[4, 205-207]

[Proof] Let us call *p* the side [of the triangle] in question, *A* and *B* the two adjacent angles, *C* the third angle.



The angle C must be entirely determinate [determined], when the angles A and B are known with the side p; for if several angles C could correspond to the three given things A, B, p, there would be as many different triangles, which would have a side and the two adjacent angles of the one equal to a side and the two adjacent angles of the other, which is impossible [by Euclid's Proposition I-26, which states "If any two angles [A and B] equal to two angles respectively, and one side [p] equal to one side, namely... the side adjoining the equal angles..., they will also have the... remaining angle [C, equal] to the remaining angle." **[3, 301]**. This Proposition reveals the Angle-Side-Angle triangle congruency property]; therefore the angle C must be a determinate function of [entirely determined by] the three quantities A, B, p; which may be expressed thus

$$C = \Phi : (A, B, p)$$

[C is a function, which Legendre here calls "Φ", of angle A, angle B, and side p].

Let the right angle be equal to unity [which is 1], then the angles A, B, C, will be numbers comprehended between 0 and 2 [Since each angle is positive, and their sum $A + B + C \leq 2$ by Legendre's second proof above, then $0 < A, B, C < 2$.]; and, since

$$C = \Phi : (A, B, p)$$

[Claim:] we say that the line p does not enter into function Φ [the function Φ does not depend on p].

[Proof of claim:] Indeed we have seen that C must be entirely determined by the data A, B, p, merely, without any other angle or line whatever; but the line p is of a nature heterogeneous to [different than] the numbers [angles] A, B, C; and, if, having any equation whatever among A, B, C, p, we could deduce the value of p in A, B, C, it would follow that p is equal to a number [angle], which is absurd [it is in this statement that Legendre has drawn an unjustified conclusion, as will be discussed below]; therefore, p cannot enter into the function Φ, and we have simply

$$C = \Phi : (A, B)....$$

[End of proof of claim.]

This formula proves already that, if two angles of a triangle are equal to two angles of another triangle, the third must be equal to the third; and, this being supposed, it is easy to arrive at the theorem we have in view.

Legendre continued from this point, but we omit the remainder of his proof, as it was in the noted portion that he made an error that caused the rest of his argument to fail. Here we only remark on his error. In the proof following this one, however, a thorough explanation of that included error will be discussed. Here Legendre seems to conclude that a function, Φ, cannot map the triple of domain elements (line segment *p*, angle *A*, and angle *B*) to the range element angle (*C*), since he could then, by rearranging the function, write or "solve for" the line segment *p* from this "equation" as a combination of angles, forcing it to be an angle too. However, we know from our experience with functions that a general function has no such limitation. For example, trigonometric functions take input values that are angles but produce output values that are not. It is in this false assumption that Legendre's argument fails.

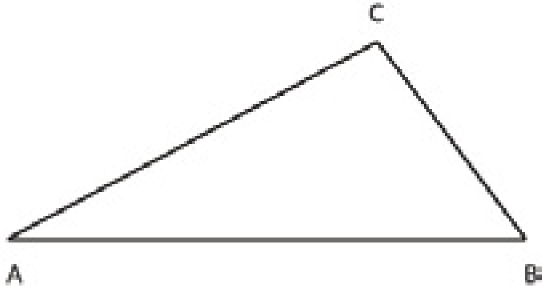
[End of discussion].

Legendre had not succeeded the first time. He was not one to give up easily, however, as his forty years of research proved. In his twelfth and final edition of his *Elements*, another famous attempted proof is found towards the same goal **[3, 215]**. Though logical in most respects, one key flaw can be found which is fatal to the argument.

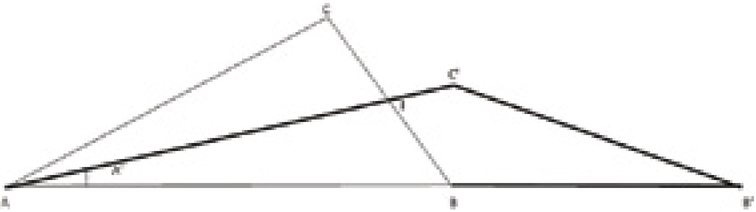
Legendre's Fourth Proof

TThe Second Attempt at Proving the Angles of a Triangle Sum to Two Right Angles. [3, 215]

[Proposition] *57. In any triangle, the sum of the three angles is equal to two right angles.*



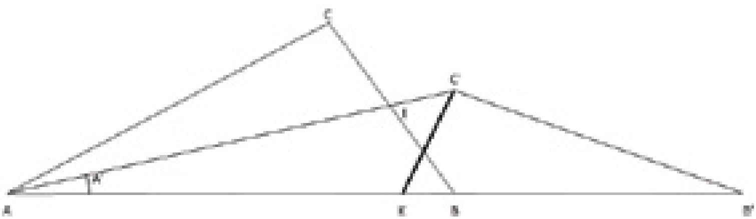
[Proof:] *Demonstration.* Let ABC be the proposed triangle, in which we suppose* that AB is the greatest side, and BC the least, and that, consequently, ACB is the greatest angle, and BAC the least [which is true from Euclid's Proposition I-18, "In any triangle the angle opposite the greater side is greater" **[3, 283]**.]



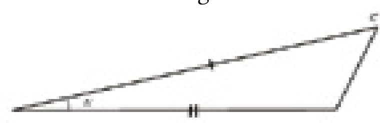
Through the point A, and the middle point I of the opposite side BC, draw the straight line AI, and produce it to C', making $AC' = AB$; produce also AB to B', making AB' double of AI [so, $AB' = 2AI$]. If we designate by $A, B, C,$ the three angles of the triangle ABC, and by A', B', C', the three angles of the triangle $AB'C'$,

[Claim:] we say that [angles] $C' = B + C$, and $A = A' + B'$; from which we deduce [by adding the two equations] $A + B + C = A' + B' + C'$; that is, the sum of the three angles is the same in the two triangles.

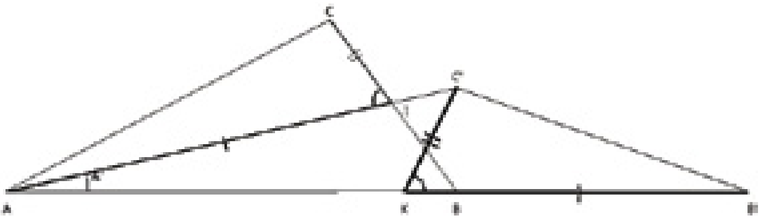
(*This supposition does not exclude the case in which the mean [middle length] side AC [with a length here between that of the greatest side AB and the least side BC] is equal to one of the extremes AB or BC).



[Proof of claim] To prove this, [choose point K on AB' so as to] make $AK = AI$, and join $C'K$; we shall have the triangle $C'AK = BAI$.

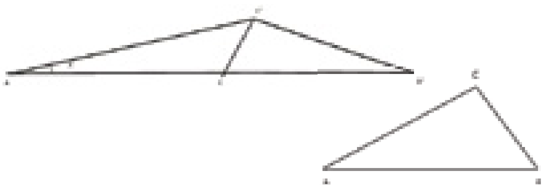


For, in these two triangles, the angle A' is common, and the side $AC' = AB$ [by construction], and $AK = AI$ [by construction]. Therefore [by Euclid’s I-4] the third side $C'K$ is equal to the third BI, and consequently the angle $AC'K=ABC$, and the angle $AKC'=AIB$.



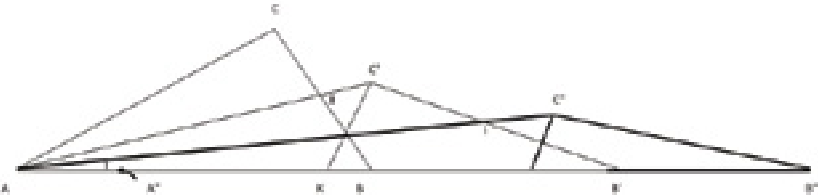
We say now that the triangle $B'C'K$ is equal to the triangle ACI [we will omit the proof of this statement, as it is very similar to the previous demonstration. The above figure displays a diagram of the proof].

[Proof continued] It hence follows [from knowledge of these two sets of triangles],



1. That the angle $AC'B'$, designated by C', is composed of two angles, equal, respectively, to the two angles B [$= AC'K$] and C [$= KC'B$], of the [original] triangle ABC, and that, accordingly, we have $C' = B + C$;
2. That the angle A of the triangle ABC is composed of the angle A', or CAB', which belongs to the triangle $AB'C'$, and the angle CAI, equal to B', of the same triangle, which gives $A = A' + B'$; therefore $A + B + C = A + B' + C'$.

Moreover, since, by hypothesis, we have $AC < AB$, and, [by congruent triangles, $AC = C'B'$ and $AB = AC'$,] consequently, $C'B' < AC'$, it will be seen, that, in the triangle $AC'B'$, the angle at A, designated by A' [opposite from $B'C'$], is less than B' [opposite from AC']; and, as the sum of the two [$A' + B'$] is equal to the angle A of the proposed triangle, it follows that the angle $A' < \frac{1}{2} A$.

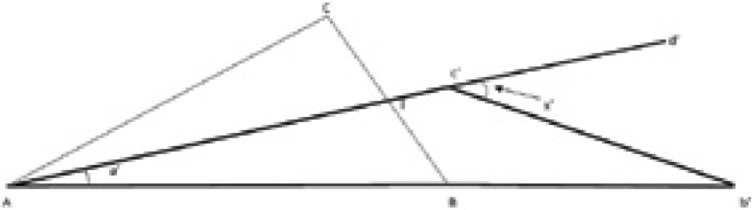


If we apply the same construction to the triangle $AB'C'$, in order to form a third triangle $AC''B''$, designating the angles by A'', B'', C'', respectively, we shall have, in like manner, the two equations $C'' = C' + B'$, and $A' = A'' + B''$, which gives $A' + B' + C' = A'' + B'' + C''$. Thus the sum of the three angles is the same in these three triangles. We have, at the same time, the angle $A'' < \frac{1}{2} A'$, and, consequently, $A'' < \frac{1}{4} A$.

Continuing indefinitely the series of triangles $AC'B'$, $AC''B''$, [et]c., we shall arrive at a triangle *a b c*, in which the sum of the three angles will always be the same as in the proposed triangle *ABC*, and which will have the angle *a* less than any given term of the decreasing progression $\frac{1}{2} A$, $\frac{1}{4} A$, $\frac{1}{8} A$, & [et]c.

We may therefore suppose this series of triangles continued until the angle *a* is less than any given angle…

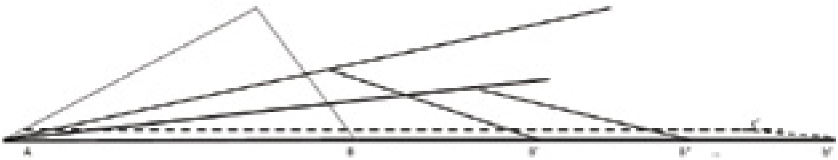
[I]t will hence be seen that the sum of the three angles of the triangle [the final triangle] reduces itself to the single angle *c*' [when *a* is “less than any given angle”].



In order to obtain the exact measure of this sum, let us produce the side $a'c'$ toward d', and designate the exterior angle $b'c'd'$ by x', added to the angle c' of the triangle $a'b'c'$, will make a sum equal to two right angles; thus denoting the right angle by D, we shall have

$$c' = 2D - x' ;$$

therefore the sum of the angles of the triangle $a'c'b'$ will be
$2D + a' + b' - x'$.



But we may imagine the triangle $a'c'b'$ to vary in its angles and sides, so as to represent the successive triangles which are derived ultimately from the same construction [described above], and which approach more and more the limit at which the angles a' and b' are nothing. At this limit, the straight line $a'c'd'$ is confounded [coincides] with $a'b'$, and the three points a', c', b', are in the same straight line [as shall be explained below, it was in this assumption that Legendre made an error];
then the angles b' and x' become nothing at the same time with a', and the quantity $2D + a' + b' - x'$, which is the measure of the three angles of the triangle $a'c'b'$, reduces itself to $2D$; therefore, *in any triangle, the sum of the three angles is equal to two right angles.*

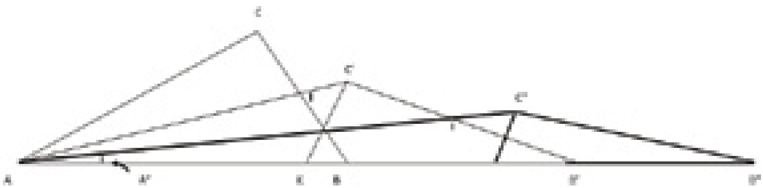
[End of Legendre's proof]

It appeared, at this point, that Legendre was convinced that he had finally reached the end of his quest, for he did not produce another edition of the *Elements*. He had, in his mind, proved that the sum of the angles of a triangle is equal to two right angles. And his demonstration was rather remarkable. Using only what is laid out in Euclid's *Elements* up to Proposition I-28, he had shown how he could create numerous triangles having the same angle sum. But, as had been shown so many times before, the necessity of the Fifth Postulate held. We will now bring certain portions of Legendre's proof into question.

J.P.W. Stein, in the 1824-1825 edition of the Journal "Annales des Mathematiques: Pures et Appliques" [Annals of Mathematics: Pure and Applied], showed that Legendre's proof was faulty.

Below is Stein's argument, shown with all vertex labels changed to match those in Legendre's proof above.

J.P.W. Stein's Demonstration of the Fault in Legendre's Triangle Proof [8]



[As can one can see from reading it, Legendre's proof is] based primarily on the principle of apparent evidence, that when the limit of angle *A* is zero, the limit of the distance to the side *AB* from any point *C* on the side *AC* is equal to zero, which amounts to saying that, at the limit, point *C* will fall on [the line *AB*].

However, it is easy to prove that this principle cannot be admitted.

[Proof] Indeed, divide the angle *BAC* in two successive, four, eight, ..., equal parts by lines *AC', AC"*. [These points] are [successively] lowering *BC* [in a direction] perpendicular to *AC*; we have made *KB*' = *AK*, and being lowered to it is [*BC*, and we continually lower the height of the triangle, cutting the sides successively in half in this manner]. It is easy to demonstrate that, however far we push the operation, we will constantly have [*BI* = *IC*, *B'I'* = *I'C'*, and *AK* = *KB*', etc] Thus, in this construction, the limit of the angles *A, A', A*", etc., is zero, as [demonstrated] above. However, when considering the point *C*, the limit [for it to], fall on *AC* remains rather a constant distance from the right [when the angle $A^{(n)}$ is equal to zero, the limit, the side *B'C*' is not equal to zero, thus has not reached its limit. It still has a "constant distance" remaining to move to the right].

It would be easy, moreover, to imagine a construction in which the decrease of the angles [*A, A', A"*], etc, would be faster than the terms of the progression [of the sides *BC, B'C', B"C"*, etc., which decrease in the manner] 1, 1/2, 1/4, 1/8, etc. or however and point *C* moves constantly [towards the line *AB*], or remain in a distance [from *AB* that is changing by a constant rate] or at least remain a greater distance [than the limit of angle *A*] over a given length.

We must therefore conclude that the demonstration cannot be accepted so far as we have evidence that the distances of the points [*C'*, *C"*, *C'''*], and so on, [from the line *AB*] has [still a value greater than zero] when the correct angle $A^{(n)}$ have [reached the limit of] zero. [We must then conclude] that [proving that the angles of a triangle sum to two right angles] probably could not be done without relying on principles presupposed, that of the theory of parallel already established…The principle [Legendre presented] is based on an obvious falsehood [that both the angle *a'* and the side *c'b'* have the same limit, namely, zero. They, in fact, do not].

[End of proof.]

The task of proving the parallel postulate is, as first shown in 1868, impossible **[3, 219]**. There simply is no way to prove it. Legendre appeared to have come to this opinion following this blundered proof, for it was his final attempt to place the postulate on a more certain foundation. As stated above, it was, in his own lifetime, shown to be flawed. Several years later, in the year before he died, he stated, "It is nevertheless certain that the theorem on the sum of the three angles of the triangle should be considered one of those fundamental truths that are impossible to contest and that are an enduring example of mathematical certitude." **[7]** Legendre, hence, had chosen to content himself with accepting the postulate in its postulate state, without prior proof. In the realm of Euclidean geometry, the Fifth Postulate had stood the test of time. That fact, to Legendre and his research spanning half his lifetime, would have to be enough proof in itself.

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