ACQUISITION, UTILIZATION, AND RETENTION OF FOUNDATIONAL FRACTION CONCEPTS BY MIDDLE GRADE STUDENTS

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ABSTRACT

This study investigated the acquisition, utilization, and retention of five fraction domains by middle grade students. Based on research findings, the five fraction domains that formed the conceptual analysis framework used in this study were: (1) unit fractions, (2) the whole, (3) modeling fractions, (4) comparing and ordering, and (5) benchmarks and estimation. Students who participated in the study had exhibited difficulties in mathematics and consequently were enrolled in a sixth-grade Tier II Response to Intervention (RtI) mathematics course. RtI provides additional instruction for students who have shown gaps in their learning. This RtI course was conducted utilizing a specially designed fraction curriculum.

Data for the study were obtained from written records including independent student work from student booklets, pre-assessment and post-assessment responses, and one-on-one interviews. Five case study students provided the data for the present study. Results from the student booklet, pre-assessment, and post-assessment were presented by student and then by fraction domain. These results provided information about the case study students’ fraction knowledge when they were sixth-graders. Interviews the following year supplied data by fraction domain for the case study students who were now seventh-graders.

Results from the data analysis revealed that students who were struggling with fraction concepts found success utilizing Cuisenaire rods to develop an understanding of unit fractions, the whole, and equivalent fractions. The case study students retained knowledge of the whole, but became confused with the multiple meanings of the whole. When modeling fractions, they understood that the partitions of the model had to be
equal, but were unable to precisely draw equal partitions. This resulted in uncertainty concerning fractional concepts such as comparing and ordering fractions. The notion that fractions can be compared by the size of the partitions created by the denominator was retained over the year period. But, the students also retained the idea that the numerator was not important when comparing and ordering fractions. Finally, benchmarks and estimation were not strongly developed and, consequently, were not often used when operating on fractions.
CHAPTER 1: INTRODUCTION AND RATIONALE FOR STUDY

Statement of the Problem

According to the National Mathematics Advisory Panel (NMAP) (2008), teachers are concerned about their students’ abilities to understand and use fractions. Survey responses gathered by the NMAP indicated that teachers of first-year Algebra believe that, “Difficulty with fractions (including decimals and percents) is pervasive and is a major obstacle to further progress in mathematics, including algebra” (p. xix). ‘Pervasive’ and ‘major obstacle’ are compelling terms that draw attention to the critical issue of student understanding of fractions. Research indicates that students struggle in general with learning and applying concepts of fractions (DeWolf, Grounds, Bassok, & Holyoak, 2014; Gabriel, Szucs, & Content, 2013; Lamon, 2007). For example, Peck and Jencks (1981) investigated childrens’ conceptual understanding of fractions in the middle grades. They found that fewer than 10% of children who participated in the study had a strong conceptual foundation to guide them in computing or solving problems involving fractions.

The Rational Number Project (RNP), funded by the National Science Foundation (NSF) during the years 1979 – 1983, included a series of studies that focused on how students learn rational number concepts (Post, Behr, & Lesh, 1986). Based on the findings, these researchers recommended that operations with fractions at the symbolic level be postponed until sixth grade and that the denominator of the fractions used in the symbolic representation should not be larger than the number 12. Moreover, they stated that during fourth and fifth grades, mathematics instruction of fractions should concentrate on modeling fractions, fraction number sense, comparing and ordering
fractions, and ideas such as the whole and the unit fraction. Bezuk and Cramer (1989) found that in order to develop these ideas, physical representations, models, and manipulatives are crucial to help students develop a mental referent for fractions rather than relying on step-by-step procedures to solve problems involving fractions.

Weak understanding of fraction concepts “has serious long-term consequences” (Jordan et al., 2013, p. 46). The fractional knowledge gained by students in elementary and middle school prepares them for the use of fractions in daily life and impacts their success in high school algebra. Students who do not succeed in algebra often do not attend college or do not choose to specialize in STEM (science, technology, engineering, and mathematics) fields, which may limit their career opportunities. Moreover, young people who do not develop a strong understanding of fraction concepts and subsequently become elementary school teachers will in turn affect their students’ ability to grasp fractional concepts. As Lamon (2012, p. 136) emphasized, teachers must have “fluid and flexible thinking” concerning student understanding of fractions, so that they will be able to distinguish between children’s strategies that are appropriate for the problem and those strategies that are based on deficient reasoning.

The purpose of this study is to describe the reasoning processes used by middle grade students, who have not previously been successful in mathematics and were enrolled in a grade level mathematics course, as well as a Response to Intervention (RtI) Tier II mathematics course (that provides additional support), as they solve increasingly sophisticated fraction tasks. An examination of emerging ideas and the development of fraction competencies was undertaken. Explanations written by students in response to tasks presented in an intervention curriculum specially designed to promote conceptual
understanding of fraction concepts were analyzed. In addition, a sample of the students who were instructed using the intervention curriculum during the 2013–2014 school year, were interviewed to gather information on fraction concepts retained and the use of these ideas as they solve new fraction tasks. This study will shed light on the ideas used by middle grade students who are struggling with mathematics, as they learn about topics in the fraction domain. The knowledge obtained through this investigation will provide educators with information about conceptions of fractions that students who demonstrate academic deficits bring to the mathematics classroom, specific areas in which students are deficient, and consistent use of fraction big ideas as students move from beginning to more advanced fraction topics.

Calls for Attention to the Development of Fraction Concepts and Skills

Since the 1940s, numerous national reports on the status of mathematics education have stressed the importance of students’ development of conceptual understanding, a problem-solving attitude, and accuracy and fluency with computational skills (Carpenter, 1981; Committee on the Function of Mathematics in General Education for the Commission on Secondary School Curriculum of the Progressive Education Association, 1940; Educational Services Inc., 1963; Joint Commission of the MAA and the NCTM, 1940; National Council of Teachers of Mathematics, 1943). These reports suggested that understanding mathematical ideas and an emphasis on applications and problem solving motivates students, aids retention and helps students decide which procedure is most appropriate to use when solving a problem. However, as these reports also indicated, school mathematics instruction has traditionally emphasized procedural
skills, which results in the notion that mathematics is a set of rules to be learned and
applied, rather than a set of concepts to be understood.

In 1943, the United States Army drew attention to this issue with their report on
the minimum mathematics requirements that were essential for military inductees
(National Council of Teachers of Mathematics, 1943). The report noted that many
enlisted men were unable to apply their mathematical knowledge in practical situations.
However, the report stressed that providing remedial instruction focused on solving more
of the same type of computational problems, in the same way as previously instructed,
was ineffective. Instead, deficiencies demonstrated by the soldiers should be diagnosed
and addressed with multiple experiences in practical applications.

In the NCTM (1943) study, the fraction domain was specifically addressed as an
area of concern. Results from a survey, which culminated in the 1943 NCTM report,
indicated that only one out of three enlisted men were able to solve a problem such as
$7 - 5\frac{3}{4}$. The authors suggested that this was due to a failure of the men to understand the
meaning of a fraction and to apply this understanding when solving problems.
Consequently, military leaders recommended that one of the crucial components of
mathematics education should be operations with common and decimal fractions, with an
emphasis on understanding what a fraction is and how to use fractions in real-life
situations.

Subsequent reports emphasized that during the middle school years, students
should also be able to perform the four operations on fractions both accurately and
fluently (Common Core State Standards Initiative, 2010; Conference Board of the
Mathematical Sciences, 1983; Educational Services Inc., 1963; National Council of
Teachers of Mathematics, 1943; National Council of Teachers of Mathematics, 2000). In order for students to acquire this fluency, teachers were encouraged to focus on addition, subtraction, multiplication, and division with common fractions such as: $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}$, using concrete representations first and then moving to abstract symbolization. It was argued that starting with common fractions would allow students to grasp the nature of addition, subtraction, multiplication, and division of fractions, develop algorithms for each operation, and flexibly apply them in new situations.

In the latter part of the 20th century, various mathematics committees described what and how students should learn about fractions (Conference Board of the Mathematical Sciences, 1983; Educational Services Inc., 1963; National Council of Teachers of Mathematics, 2000). These committees suggested that:

1. a greater emphasis should be placed on understanding a fraction as a number, comparing and ordering fractions, and converting a fraction to a decimal.
2. a decreased emphasis on skill development of the four operations on fractions especially with large denominators.
3. comprehension of the magnitude of a fraction and being able to find its approximate location on a number line was essential.
4. cultivation of the ability to select an appropriate method for computing with fractions such as mental computation, paper-and-pencil, calculator, or estimation was important.
5. development of strategies to estimate results of computation and determine the reasonableness of the solution should be a priority.
According to the NMAP (2008), by the time students begin algebra work, these fractional concepts should be firmly in place. With many students beginning formal study of algebra in seventh and eighth grades, the need for a thorough understanding of fractions must begin early. The Common Core State Standards for Mathematics (CCSSM) (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) suggested that the study of fractions commence in the third grade. Students begin to understand the idea of a fraction as a number by modeling fractions, developing a fraction from a unit fraction, understanding the whole on which a fraction is based, and partitioning the whole into equal-sized parts. Between the fourth and sixth grades, students are expected to learn fraction concepts such as equivalence; ordering; addition, subtraction, multiplication, and division of fractions; decimal fractions; and application of these concepts in word problems. Heck, Weiss, and Paisley (2011) indicated that CCSSM represents a substantial shift in emphasis and grade placement of certain content areas. The persistent difficulties students demonstrate with fractions and the shift in what is emphasized in the CCSSM fraction domain at the intermediate and middle grades substantiate the importance of investigating student learning of fractions in the current educational environment.

**Response to Intervention**

Response to Intervention (RtI) is an organizational and instructional approach that has gained acceptance within the education community as a strategy for decreasing achievement gaps. The National Center on Response to Intervention (Glover, 2010) provides the following definition for RtI:
Response to intervention integrates assessment and intervention within a multi-level prevention system to maximize student achievement and to reduce behavioral problems. With RTI, schools use data to identify students at risk for poor learning outcomes, monitor student progress, provide evidence-based interventions and adjust the intensity and nature of those interventions depending on a student’s responsiveness, and identify students with learning disabilities or other disabilities (p. 2).

As noted in Figure 1, there are typically three tiers that form the RtI framework: (1) core instruction for all students, (2) additional support for students experiencing difficulties with specific mathematical concepts, and (3) intensive support for students with learning disabilities. Lembke, Hampton, and Beyers (2012) offered a description of the three tiers that compose the RtI pyramid. Tier I contains the core curriculum that all mathematics students receive. Within a grade level, 80% of the students will only need
the level of instruction to attain the expected proficiencies. Tier II (the focus of this study) is comprised of students who do not attain proficiency at Tier I, but rather struggle with specific mathematics concepts. This level contains approximately 15% of the student population. These students receive the core curriculum as well as additional instruction that concentrate on the mathematical areas in which they are experiencing difficulties. Finally, Tier III contains 5% of mathematics students. This group of students either has not attained proficiency in Tier I and Tier II or their mathematical deficiencies are great enough to warrant intensive intervention. These students often receive the core curriculum and an additional class, which provides very focused and intense instruction on foundational concepts or possibly below grade level topics. In many districts, this level refers to students who have been identified as special education students. However, some schools/districts prefer to add a fourth level to refer to students receiving special education services and consider the third level as providing even more intensive support for students who are experiencing mathematics difficulties.

The RtI framework was developed at the beginning of the 21st century to address deficiencies in the area of reading and is slowly being introduced into K-12 mathematics education. Consequently, research on the effect of RtI in mathematics is just emerging. Clarke, Gersten, and Newman-Gonchar (2010) stated that for every 15 RtI studies in the area of reading, there is only one study in the area of mathematics. Moreover, according to Fuchs, L., Fuchs, D., and Compton (2010), research on RtI has primarily focused on elementary students and is lacking for middle and high school students. This deficiency in research coupled with the fact that students continue to struggle with fractions in
middle school provided the impetus for the importance of researching the combination of RtI and the development of fraction conceptual understanding by middle school students.

**Research Questions**

The general question under investigation is: *How do middle grade students enrolled in an RtI Tier II mathematics course acquire and utilize major concepts of the fraction domain before, during, and following a specially designed unit of instruction?*

Specific research questions include:

1. What prior knowledge do middle school students enrolled in an RtI Tier II mathematics course demonstrate regarding fraction concepts and skills?
2. What misconceptions do RtI Tier II students demonstrate while studying fraction concepts and skills?
3. What concepts and skills do RtI Tier II students demonstrate following a specially designed unit of instruction focused on fraction concepts and skills?
4. To what extent are major concepts of fractions retained one year after the RtI Tier II instructional sequence?

In order to answer the research questions, data were gathered for each question from the following source(s):

- Research question 1 – Pre-Assessment
- Research question 2 – Pre-Assessment and Student Booklet
- Research question 3 – Student Booklet and Post-Assessment
- Research question 4 – One-on-one interview a year later
Theoretical Framework

The theoretical framework guiding this study builds on the notion that the development of conceptual understanding is essential for students (including struggling students) to better understand the rational number system. Moss and Case (1999) suggested that in order to accomplish the development of this conceptual understanding, a view of the rational number system as a whole and how its components are integrated was crucial. They indicated that this has not yet occurred due to an excessive amount of time spent on teaching procedures, not recognizing and affirming children’s attempts to make sense of fractions, and misrepresenting fractions (including the use of circles as models) by students. Carnine, Jitendra, and Silbert (1997) wrote that curriculum, which would drive conceptual understanding, must revolve around big ideas. Big ideas are fundamental ideas and concepts about a topic that will build understanding of more complex concepts. Connections should be made between related concepts and skills and should not be separate and distinct ideas that students must memorize.

Students who struggle with mathematics often: (a) use procedures that younger, typically achieving students use; (b) make frequent errors when executing procedures; and (c) have a poor understanding of concepts that are foundational to performing procedures (Geary, 2004). Without understanding the concepts underlying procedures, students may experience a delay in adopting more complex procedures, thus putting them further behind their counterparts.

Siegler, Fazio, Bailey, and Zhou (2013) asserted that “…the one property that unites all real numbers is that they possess magnitudes that can be ordered on number lines” (p. 13). In addition, a study conducted by Fuchs et al. (2013) demonstrated that
focusing on fraction magnitudes could have a positive effect on students who struggle with fractions. Fraction lessons written for the overall project, from which the data for this study were gathered, were developed based on these ideas. Instruction and tasks focused on using number lines to represent fractions, developing generalizations based on patterns, visualizing fractions, and building conceptual understanding. The follow up interviews delved deeper into these ideas in order to understand students’ reasoning processes.

Method

Demonstration of conceptual understanding and use of strategies and concepts to solve fraction tasks was accomplished through analyses of student records (their written work over the course of studying a unit of instruction). According to Fazio and Siegler (2013), the examination of strategies used by students provides a window into their understanding of a mathematical task. Students actively construct methods to solve problems and at the same time passively look for patterns or associate new learning with current knowledge (Siegler, 2005). During this cycle, existing strategies that produce success will continue to be used while new strategies will be used intermittently. Through an analysis of strategies used by the students as they worked through fraction tasks, a description of how and when particular strategies are implemented was accumulated.

Additional insight into student reasoning and their retention of concepts and skills was obtained through individual student interviews, a year following initial instruction. In these interviews, students were asked to solve a variety of fraction tasks. The interviews offered evidence of whether or not students held on to crucial fraction concepts that were taught to them during the previous school year using the RtI materials.
The individual student interview allowed the researcher to examine the foundational fraction concepts utilized to complete fraction tasks. For example, does the student exhibit knowledge only for low demand situations? Does the student use a rule without the ability to explain it? Is the student able to describe, but not implement a strategy? Is the student able to both solve a problem and also explain how it was solved? (Ginsburg, 1997). Through the individual interview, insight into the student’s thinking and/or struggle to understand was ascertained.

This study investigated an instructional unit’s impact on struggling middle school students. The unit focused on fractions and was conducted within an RtI Tier II setting. Activities in the fraction intervention module were intended to support students in making connections between their preexisting knowledge and new knowledge, using different models to represent fractions, and understanding the relationship between and among fractions. As students advanced through the lessons, the goal was for their understanding of fractions to grow and solidify. According to Kilpatrick, Swafford, and Findell (2001) and the NMAP (2008), operations with rational numbers is an area in which students may need extra support as they enter high school, as it is essential for success in algebra. Consequently, an intervention curriculum that provides middle grade students the means to strengthen their understanding of fractions and their ability to successfully apply fractional knowledge to solve problems is necessary. Moreover, Hunt and Empson (2014) stated that there is a shortage of research on the reasons behind conceptual gaps demonstrated by students who experience difficulties with fractions. Research into strategies used by RtI students as they solve increasingly sophisticated fraction tasks is needed in order for the intervention to address gaps and weaknesses.
CHAPTER 2: REVIEW OF LITERATURE

This investigation was set within a Response to Intervention (RtI) Tier II classroom and concentrated on student learning and understanding before, during, and following a specially designed curriculum unit focusing on fractions. A primary instructional goal of the curriculum unit was to develop conceptual understanding of the whole, unit fractions and non-unit fractions, magnitude of fractions, comparison and order of fractions, physical and symbolic representations of fractions, and operations with fractions. This review of literature includes research on student understanding and facility with fractions, in particular: whole number bias, rational number sense, the importance of the whole or unit in understanding a fraction, unit fractions, models and equal partitioning, comparison and order of fractions, equivalence, and benchmark fractions and estimation. Additionally, research regarding use of the RtI model in mathematics and with students in the middle grades who struggle with mathematical concepts was presented. Moreover, recommendations regarding instruction of fractions based on the research were summarized.

Student Understanding of and Facility with Fractions

A goal of mathematics instruction is the development of number sense (Conference Board of the Mathematical Sciences, 1983; Greeno, 1991; McIntosh, Reys, & Reys, 1992; National Council of Teachers of Mathematics, 1989; Mathematical Sciences Education Board (MSED) and National Research Council, 1989). McIntosh et al. (1992) defined number sense as: “A propensity for and an ability to use numbers and quantitative methods as a means of communicating, processing, and interpreting information. It results in an expectation that numbers are useful and that mathematics has
a certain regularity (makes sense)” (p. 4). According to Greeno (1991), number sense consists of the ability to: flexibly work with numbers and recognize relationships between and among numbers, estimate solutions to computational problems, and consider the meaning of a number. The development of number sense, particularly with rational numbers, requires active involvement of children with concrete models, conversations among students and the teacher about mathematical ideas, facility with multiple representations, procedural knowledge that develops from conceptual understanding, problem-solving and reasoning skills, and accurate and fluent procedural operations (Cramer, Post, & delMas, 2002; Jordan et al., 2013; Post, et al., 1986).

As emphasized in numerous reports over the past century, a critical area in which number sense must be cultivated is fractions.

Of all the topics in the school curriculum, fractions, ratios, and proportions arguably hold the distinction of being the most protracted in terms of development, the most difficult to teach, the most mathematically complex, the most cognitively challenging, the most essential to success in higher mathematics and science, and one of the most compelling research sites. (Lamon, 2007, p. 629)

Mathematics education researchers have described the complexities that exist within the study of fractions (Bezuk & Cramer, 1989; Cramer et al., 2002; DeWolf et al., 2014; Gabriel et al., 2013; Lamon, 2007). At the middle school level, these challenges included confusion about fractions that children brought into the classroom from previous instructional experiences. These earlier experiences may have resulted in firmly entrenched ideas about numbers, specifically whole numbers. For example, the unique format of the numerical representation of a fraction often generated difficulties for
students as they begin to develop their understanding of a fraction as a number (Clarke & Roche, 2009; DeWolf et al., 2014; Gabriel et al. 2013; Peck & Jencks, 1981). These students did not comprehend that a fraction has a magnitude, but rather considered a fraction as two whole numbers written with one number over a line and another number underneath the line. Consequently, many students continued to apply ideas about whole number properties and procedures to the numerator and denominator of a fraction. This was referred to as “whole number bias” and has been shown to negatively affect the acquisition of the concept of a fraction (Behr, Wachsmuth, Post, & Lesh, 1984; Gabriel et al., 2013; Lamon, 2012; Meert, Gregoire, & Noel, 2010; Post, Wachsmuth, Lesh, & Behr, 1985). Lamon (2012) provided an example of whole number bias that was associated with students being taught that “multiplication makes larger and division makes smaller.” While this notion may be true for whole numbers, it is not always the case with the rational numbers. The transition from this whole-number thinking to understanding a fraction can be a slow process as students learn a new symbolic system that includes distinctive symbolic representations, models, and operational procedures (DeWolf et al., 2014; Gabriel et al., 2013).

Notions about fractions, which students bring into the classroom can contribute to the difficulties students experience with fractions. Peck and Jencks (1981) demonstrated that many students remember and can therefore model common fractions such as \( \frac{1}{2} \) and \( \frac{3}{4} \), which they have been exposed to in real-life situations, but they are unable to translate that knowledge to modeling a fraction such as \( \frac{2}{5} \). This presents a challenge in that some students have not yet generalized the concept of a fraction. Multiple interpretations or models of rational numbers such as: part-whole comparisons, ratios, operators, measures,
or quotients (Behr et al., 1984; Kieren, 1976; Lamon, 2007) add another layer of complexity for students struggling to understand and solve problems with fractions.

Another challenge, according to Peck and Jencks (1981), was that over half of students who were asked to draw a representation of simple fractions such as \( \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \), etc. were unable to successfully complete this task, due to the fact that while they knew that the shape needed to be partitioned, they were unaware that the partitions needed to be of equal size. This basic misconception impeded their success in understanding the magnitude of fractions, which in turn affected their ability to compare and order fractions. Subsequently, the acquisition of conceptual understanding of operations on fractions may be compromised.

Kilpatrick et al. (2001) commented that “of all the ways which rational numbers can be interpreted and used, the most basic is the simplest – rational numbers are numbers. That fact is so fundamental that it is easily overlooked” (p. 235). Many middle school students struggle with concepts of fractions, even when they do not have a diagnosed mathematical learning disability (Gabriel et al., 2013; Mazzocco & Devlin, 2008). The foundational elements of rational number sense, which is a prerequisite to rational number computation, include: magnitude, fraction as a number, the ‘whole’, and comparing and ordering fractions. Numerous formal and informal experiences over a period of time are necessary to develop rational number sense (Jordan et al., 2013; Lamon, 2012; Mazzocco & Devlin, 2008). Demonstration of rational number sense includes students’ ability to judge the reasonableness of their answers, estimate, make corrections to their work when needed, move between representations, problem solve, and undertake fraction computations (Lamon, 2007; Lamon, 2012).
The development of rational number sense is affected by the instruction students receive in school. Traditionally, the teaching of fractions emphasized a part-whole model of fractions (Cramer et al., 2002; Lamon, 2012). The curriculum used in these classrooms described the part-whole model as the partitioning of a whole into equal-sized parts and shading a specified number of the parts. However, singular attention given to this particular model of fractions has led to situations in which students do not or cannot comprehend a fraction as a ratio or rate, a quotient, an operator, or as a measure (Lamon, 2012; Lesh, Landau, & Hamilton, 1983). Fractions have been described as a ratio of two integers, a proportional relation between an object part (numerator) and the object whole (denominator), and as numbers that can be represented on a number line (Booth & Newton, 2012; Gabriel et al., 2013; Lamon, 2007; Liu, Xin, Lin, & Thompson, 2013; Meert et al., 2010). In addition, a fraction conveys a single relationship, regardless of the size of the pieces, their color, their shape, their arrangement, or any other physical characteristics (Lamon, 2007). The lack of exposure to these different fractional constructs leads to holes in students’ fraction knowledge. As Lamon (2012) explained, “Student understanding of a very complex structure (the rational number system) is teetering on a small, shaky foundation” (p. 32).

Fraction concepts that have been attributed to greater success with conceptual understanding of fractions include: (a) the measurement interpretation of fractions and tasks involving number lines, (b) the notion that a fraction is represented by a specific notational system, and (c) the idea that a fraction characterizes size and therefore can be compared and ordered (Bailey, Siegler, & Geary, 2014; Fuchs et al., 2013; Jordan et al., 2013). However, the ability of students to compare and order fractions is dependent on
their understanding of requisite fraction ideas. Some studies have shown that while students understand the meaning of the denominator of a fraction, a lack of awareness of the meaning of the numerator hinders their ability to compare fractions (Clarke & Roche, 2009; Peck and Jencks, 1981). In order to compare fractions, students need to understand the meaning of the numerator and denominator and the relationship between the two numbers. Clarke and Roche (2009) suggested that a definition of these two terms might be “In the fraction \( \frac{a}{b} \), \( b \) is the name or size of the part (e.g. fifths have this name because 5 equal parts can fill a whole) and \( a \) is the number of parts of that name or size” (p. 136). In order for students to compare two fractions such as \( \frac{3}{7} \) and \( \frac{3}{5} \), the student has to comprehend that \( \frac{3}{7} \) refers to three one-seventh pieces and \( \frac{3}{5} \) refers to three one-fifth pieces, as well as be able to recognize that one-seventh is smaller than one-fifth (Lamon, 2012; Post et al., 1985).

Within the literature regarding understanding of fractions, the importance of the notion of magnitude or size was emphasized. According to Behr et al. (1984), results from state and national assessments demonstrated that students do not have a strong conception of a fraction as having size, which hindered their quantitative notion of fractions. Bezuk and Cramer (1989) corroborated this finding and stated that in order for students to develop a quantitative notion of a fraction, they must understand the concept of a fraction’s size. For instance, students should be able to reason that the sum of \( \frac{5}{6} \) and \( \frac{1}{4} \) will be a little more than 1 because \( \frac{5}{6} \) is almost 1, \( \frac{1}{4} \) is greater than \( \frac{1}{6} \), and therefore the sum will be more than 1. Additionally, conceptualizing the magnitude of fractions will assist
students in estimation and promote awareness of the reasonableness of answers to fraction operations (Cramer et al., 2002).

According to Greeno (1991) and McIntosh et al. (1992), fraction number sense involves the general understanding of what a fraction is, the ability to use this understanding to make judgments about fractions, and the application of this knowledge in a variety of situations. The attainment of number sense is a process that evolves over time. Students enter school with some idea of what a number is and this knowledge is expanded and increased through interactions and communication with other students and with the teacher as well as through mathematical activity. One of the components of a number sense framework is “Knowledge of and facility with numbers” (McIntosh et al., 1992, p. 4). As shown in Figure 3, this component consists of four elements: (1) sense of orderliness of numbers, (2) multiple representations for numbers, (3) sense of relative and absolute magnitude of numbers, and (4) system of benchmarks. These elements correlate with fractional concepts. In order to develop fraction number sense, students must develop an awareness of the size of fractions, the orderliness of fractions, physical and symbolic representations of fractions, and fraction benchmarks.
The importance of the development of fraction number sense cannot be understated. As Cramer et al. (2002) noted, “We believe that basic concepts and ideas about order, equivalence, and flexibility of unit are fundamental to any significant initial understanding of fractions” (p. 121). This study focused on students in middle school, who have not yet developed a strong understanding of these underlying concepts, as well as concepts of the unit fraction, modeling, and estimation and therefore experience difficulty working through fraction tasks.

Jones et al. (1996) indicated that when planning and developing mathematics instruction, key elements of the content domain and the reasoning processes which students engage in, must be put to use. In the following sections, a summary of the main themes identified in the research literature regarding the development of understanding
and facility with fractions was provided. The organizational framework for this study was developed from these themes. The themes included: (1) the “whole”, (2) unit fractions, (3) modeling and equal partitioning, (4) comparing and ordering fractions, and (5) benchmarks and estimation.

The “whole”

In mathematics instruction, “The role of the unit is addressed inadequately if at all” (Cramer et al., 2002, p. 112). This statement by Cramer and colleagues articulated a dilemma with the basic but crucial idea of the whole in considering a fraction. The whole or unit is a central concept in the fraction domain and comprehension of this idea can lead to increasingly sophisticated mathematical ideas (Lamon, 2007). A fraction must be defined in relation to the whole that it references, so that awareness of a fraction as being a proportional relationship between the part of the whole (the numerator) and the whole (the denominator) will be fostered (Lamon, 2012; Liu et al., 2013). Often, the concept of the whole is implicit, such as when students are asked to compute \( \frac{3}{8} + \frac{2}{3} \), but it may need to be explicitly expressed. Instruction related to the concept of the “whole” or “unit” must undo the idea that a whole can only refer to one object, which is what students learn as they begin counting and attach one number to one object (Confrey et al., 2012; Lamon, 2012). The unit or whole can also refer to a group of objects such as 24 cookies or five pieces of lumber. Moreover, the whole can refer to a continuous quantity such as length or area, which is not often stressed in fraction instruction.

The notion of the “whole” is essential when students compare and order fractions. Clarke and Roche (2009) and Lamon (2012) indicated that students experience difficulty when comparing fractions with respect to the whole. For example, students may decide
that \( \frac{5}{6} \) is greater than \( \frac{6}{8} \) because \( \frac{5}{6} \) is only “1” (or \( \frac{1}{6} \)) away from the whole, whereas \( \frac{6}{8} \) is “2” (or \( \frac{2}{8} \)) away from the whole. They might believe that \( \frac{5}{6} \) is equal to \( \frac{7}{8} \) because both fractions are “1” away from the whole. Moreover, the inverse relationship between the number of partitions and the size of the partitions in the whole is crucial for students to understand in order to compare and order fractions. That is, when the denominator is a larger number, the number of partitions will be greater, but the size of the partitions will be smaller. Interpretation of the words “greater” or “more” as to whether the word refers to the count of partitions or the size of the partitions in the whole also affected understanding of the whole when comparing or ordering (Behr et al., 1984; Post et al., 1985; Post et al., 1986).

**Unit fractions**

According to Olive and Vomvoridi (2006), a prerequisite for understanding that parts of a whole must be equal in order for them to be combined was the comprehension that a unit fraction is one of many equal parts of a whole. Using this idea, students counted the number of partitions that resulted in a whole and linked this count with iterating the unit fraction (Norton & Wilkins, 2012; Olive & Steffe, 2002). Number lines and fraction strips can help students in developing this concept. When students have incorporated these ideas into their existing fraction knowledge, they have developed a partitive unit fraction scheme and can iterate a unit fraction a number of times creating a non-unit fraction. This scheme was built on partitioning a continuous whole and understanding that one of those partitions (the unit fraction) can be iterated a number of times resulting in an equivalent whole or a fractional part of the whole (Steffe, 2002).
Steffe refers to this notion as a *splitting operation*. Understanding the idea of *splitting* and iterating the unit fraction was an essential component of children’s fractional reasoning.

Given a whole with shaded partitions, some students identified the unit fraction based on the number of shaded partitions. For example, a student stated that the unit fraction of \( \frac{3}{4} \) is \( \frac{1}{3} \) because there are 3 shaded pieces in \( \frac{3}{4} \). Part-whole fraction instruction contributed to this misunderstanding, whereas partitive fraction instruction brought about the understanding that \( \frac{3}{4} \) is three iterations of \( \frac{1}{4} \) (Olive & Vomvoridi, 2006).

**Models, Equal Partitioning, and Iterating**

Post et al. (1985) suggested that by accentuating the idea that a non-unit fraction, \( \frac{m}{n} \), is created by \( m \) iterations of \( \frac{1}{n} \), students could model a fraction by partitioning an object or number line into \( n \) equal pieces and then shading \( m \) of them. When modeling a fraction, most children understood that the denominator, \( n \), of a fraction, \( \frac{m}{n} \), indicated how many pieces the model should be partitioned into. However, some children have not internalized the idea that the \( n \) pieces in the model must be of equal size (Peck & Jencks, 1981). This was particularly challenging with odd-valued denominators. While students could partition a shape such as a circle into two, four, or eight equal-size pieces, partitioning a circle into thirds, fifths, or sixths presented a dilemma to many students (Confrey et al., 2012).

Research by Peck and Jencks (1981) and Cramer et al. (2002) revealed that when constructing a model to represent a fraction, most children drew a circle. The use of primarily one model, the circle, could be problematic. Rather than relying solely or primarily on circles, if students were introduced to modeling fractions as lengths, research suggested that they may have more success transitioning to using a number line.
to understand fraction concepts (Jordan et al., 2013; Keijzer & Terwel, 2001; Siegler et al., 2010).

Most middle school students can and do use mental number lines (Liu et al., 2013). They understand that numbers to the right on a number line are greater quantities than numbers on the left. Number lines are a particularly powerful model not only for representing proper fractions, but also for improper fractions and mixed numbers (Confrey et al., 2012). Additionally, Confrey and colleagues discussed the use of number lines to show how fractions with different denominators can be found at the exact same point on the number line and also to illustrate the multiplicative relationship between equivalent fractions.

Findings from a study by Jordan et al. (2013) indicated that students, who were able to use a number line to characterize the magnitude of whole numbers, could use this knowledge to better understand fractions. In fact, they found that one of the six predictors of success in the domain of fractions was the ability of children to use a number line to estimate the location of whole numbers and fractions. In particular, number line estimation made the largest contribution to proficiency with fraction arithmetic procedures. The predictive value of number line estimation was two times as large as that of calculation fluency.

The use of number lines to show magnitude of fractions, relations between fractions and whole numbers, and relationships between fractions, decimals, and percents has been shown to be an effective method to build students’ fraction sense (Siegler et al., 2010). While the number line is an essential model for representing fractions, results from Cramer et al.’s (2002) study showed that it is crucial that students be provided with
opportunities to work with fractions using multiple representations (one of which is the number line) over extended periods of time.

Comparing, Ordering, and Equivalence

Comparing and ordering fractions has been described as a critical prerequisite to understanding and using the operations of addition, subtraction, multiplication, and division on fractions (Cramer et al., 2002). However, according to Meert et al. (2010), there are a limited number of studies investigating the reasoning processes used by K-12th grade students as they compare fractions. Studies that have been conducted have focused on the comparison of fractions with like numerators and unlike denominators ($\frac{x}{a}$ and $\frac{x}{b}$) and fractions with unlike numerators and like denominators ($\frac{a}{x}$ and $\frac{b}{x}$) (Clarke & Roche, 2009; Meert et al., 2010). Results from these studies indicated that students are more successful at comparing fractions with like denominators, than fractions with unlike denominators. Moreover, students were more successful at comparing fractions such as $\frac{2}{9}$ and $\frac{8}{9}$ as opposed to comparing $\frac{2}{9}$ and $\frac{3}{9}$, due to what is known as the distance effect (DeWolf et al., 2014; Gabriel et al., 2013). The distance effect refers to the numeric distance between two numbers.

Researchers found that when the distance between the two numbers was great, students spent a minimal amount of time answering fraction comparison problems. For example, students quickly determined that $\frac{8}{9}$ was greater than $\frac{2}{9}$, because they viewed the fractions as “8” ninths and “2” ninths and they knew that the distance between 8 and 2 was large. On the other hand, the comparison of $\frac{2}{9}$ and $\frac{3}{9}$ presented more difficulty to students, due to the fact that the distance between “2” ninths and “3” ninths was small.
A more complex comparison situation occurs when the numerators of each fraction are different and the denominators are also different, \( \frac{a}{c} \) and \( \frac{b}{d} \). Lamon (2007) found that students struggled with comparing fractions such as \( \frac{4}{6} \) and \( \frac{6}{9} \), but they had an easier time comparing fractions such as \( \frac{3}{12} \) and \( \frac{5}{24} \), because 24 is a multiple of 12. The various combinations of numerators and denominators in fraction comparison situations presented a formidable challenge for many students. In addition, the terminology used when comparing and ordering fractions, as well as general fraction terminology presented additional complications. Issues noted by researchers included observations that: (a) children initially apply properties of whole numbers that have been deeply ingrained in their mathematical understanding when ordering and comparing fractions, (b) language such as “more than” and “greater than” created confusion, and (c) students tended to use a different strategy for each type of problem they encountered depending on the whole numbers that composed the fraction.

In the first few lessons of the *Rational Number Project* teaching experiment, many children gave evidence that they were still struggling to separate their thinking about fractions from their schemas for whole numbers (Behr et al., 1984). One child gave responses reflecting whole-number dominance even though he/she had participated in the entire teaching experiment, which illustrated how children who have a weak understanding of fractions could regress to more primitive strategies in the face of disequilibrium. According to Clarke and Roche (2009), a strategy often used to compare fractions by students who still view a fraction as two unrelated whole numbers is called *gap thinking*. This occurs when students calculate the distance or difference between the whole numbers that make up the numerators or denominators of the fractions being
compared and then compare these differences. The authors found that 21.2% of students in their study used gap thinking to incorrectly compare $\frac{3}{7}$ and $\frac{5}{8}$. The students determined that $\frac{3}{7}$ was greater than $\frac{5}{8}$ because there is a difference of four when subtracting the numerator and denominator of $\frac{3}{7}$, which is greater than a difference of 3 when subtracting the numerator and denominator of $\frac{5}{8}$. Furthermore, 29.4% claimed that $\frac{5}{6}$ and $\frac{7}{8}$ were equivalent based on the numerator and denominator yielding a difference of 1 in each fraction.

On the other hand, students who were successful in comparing fractions used strategies such as residual thinking and benchmarking. Residual thinking refers to when a student thinks about the fractional piece that is needed to build up the fraction to make the whole (Clarke & Roche, 2009; Cramer et al., 2002). For example, when comparing $\frac{6}{7}$ and $\frac{4}{5}$, there is only $\frac{1}{7}$ needed to make the whole for $\frac{6}{7}$, while $\frac{1}{5}$ is needed to make the whole for $\frac{4}{5}$. Since $\frac{1}{7}$ is a smaller piece than $\frac{1}{5}$, $\frac{6}{7}$ is greater than $\frac{4}{5}$. Benchmarking refers to a strategy whereby a student compares each of two fractions to a third fraction such as $\frac{1}{2}$ or 1 (benchmarks) in order to compare the fractions. While the strategies of residual thinking and benchmarking are based on sound reasoning, they were rarely explicitly taught in mathematics classroom instruction (Behr et al., 1984; Clarke & Roche, 2009).

Children’s understanding about ordering whole numbers may adversely affect their initial understanding about ordering fractions. With whole numbers, students could compare the “bigness” of two numbers by matching elements of finite sets or they could make use of the counting sequence, deciding which number was smaller according to which came first in the sequence (Post et al., 1985). Another whole-number bias that
affected students when comparing and ordering fractions occurred as they attempted to compare two fractions by thinking about how far away the fraction was from the whole or one. For example, when comparing $\frac{3}{5}$ and $\frac{6}{7}$, instead of realizing that $\frac{3}{5}$ is two one-fifths away from one and $\frac{6}{7}$ is one one-seventh away from one or the whole, the student only saw that $\frac{3}{5}$ was two away from the whole and $\frac{6}{7}$ was just one away from the whole and therefore $\frac{3}{5}$ was greater than $\frac{6}{7}$. The students were not considering the magnitude of the fractions, but were continuing to apply whole number procedures to compare fractions.

“More than”, “less than”, “greater”, and “fewer” are terms that may produce disequilibrium in students when they are applied to the domain of fraction comparison. Clarke and Roche (2009) and Post et al. (1986) discussed the critical importance of helping students understand the difference between “more” pieces (a count) and “greater sized” pieces or area (a measure). Post et al. (1985) noted that when children were asked to indicate the fraction which was less than another fraction, students asked if they meant which fraction had the least number of parts or which fraction encompassed the least amount of area. This finding led the researchers to suggest that children need to be reminded that when they are comparing fractions, they should focus on the size of the pieces and not the number of pieces (Cramer, Wyberg, & Leavitt, 2009). Lamon (2012) also discussed the use of precise mathematical language when communicating with children about fractions. For example, she suggested that students should be discouraged from giving an answer such as “3 cookies” when describing what part of a box of four cookies has been eaten. Instead, the student should be encouraged to represent the quantity as “three-fourths” of the box of cookies. The importance of using precise mathematical language was also emphasized by Jordan et al. (2013) who found that
language was one of six predictors that contributed to the development of the conceptual knowledge of fraction concepts.

**Benchmarks and Estimation**

Within the fraction domain, a benchmark is defined as a third fraction to which other fractions of interest are compared (Clarke & Roche, 2009). Lamon (2012) described the use of benchmarking as a strategy for understanding the relative size of fractions and comparing fractions with different numerators and different denominators. However, Clarke and Roche (2009) noted that although this is an appropriate strategy for comparing two fractions such as \( \frac{3}{7} \) and \( \frac{5}{8} \), findings from their study indicated that most students did not make use of this strategy. This may be attributed to benchmarking being a more abstract notion requiring conceptual understanding of fraction magnitude (Behr et al., 1984; Clarke & Roche, 2009) or it may be that the strategy is not generally found in curriculum materials.

Understanding the size of a fraction is a prerequisite for estimating and judging the reasonableness of results produced from operations on fractions (Cramer et al., 2002; Lamon, 2007). Moreover, according to Siegler et al. (2012), estimating a fraction’s magnitude can support students as they determine the proper fraction arithmetic procedure to use to solve a problem and reject those procedures that result in an unlikely solution. Developing an awareness of the size of a fraction could be accomplished by visualizing a fraction as a length (Keijzer & Terwel, 2001). Number lines provide an effective model on which to estimate and represent fraction lengths. Fazio, Bailey, Thompson, and Siegler (2014) found that performance on tasks involving estimating the
location of fractions on a number line was strongly correlated with mathematics achievement on a state standardized assessment.

Implications for Teaching

Understanding of and fluency with fractions is a prerequisite for study in other areas such as proportional reasoning, algebra, and probability (Clarke & Roche, 2009; Gabriel et al., 2013). However, teachers’ knowledge and understanding of fractions may be inadequate. A study by Clarke and Roche (2009) indicated that comparing the fractions $\frac{3}{4}$ and $\frac{7}{9}$ proved to not only be difficult for students (11% success rate), but it was also difficult for primary and middle school teachers who participated in a professional development situation. Although the teachers could identify the larger fraction by finding a common denominator and converting to equivalent fractions to compare, they were unable to explain their solution or to consider another way to justify their solution. Additionally, Clarke and colleagues (2009) found that elementary school teachers’ struggles with fractions influenced the amount of time and the nature of their instruction on improper fractions, which in turn likely resulted in students having difficulties with problems such as comparing the fractions $\frac{2}{4}$ and $\frac{4}{2}$.

Lamon (2007) pointed out, “Historically, rational number education has taken adult knowledge of mathematics and simplified it so that it is palatable for children” (p. 646). Instead, research supports moving away from a procedural focus toward a conceptual focus, targeting the numerical magnitude of a fraction, and providing opportunities for students to use multiple representations (Cramer et al., 2002; Jordan et al., 2013; Peck & Jencks, 1981). By creating a classroom environment where students are actively engaged in understanding fraction concepts, Cramer et al., (2002) found that
students advanced in their conceptual understanding of fractions and were able to complete procedural tasks as easily as those students who were not exposed to a conceptual orientation.

Based on a review of the literature on fractions, the following recommendations for teaching fractions were offered by researchers (Behr et al., 1984; Clarke & Roche, 2009; Cramer et al., 2002; Gersten et al., 2009; Jordan et al., 2013; Liu et al., 2013; Peck & Jencks, 1981):

• Introduce fractions by emphasizing unit fractions and the magnitude of fractions.
• Develop the concept of non-unit fractions using iterations of the unit fraction.
• Base instruction on conceptual understanding, not on procedural rules.
• Use the number line to develop the idea that fractions have magnitude.
• Use multiple representations of fractions.
• Emphasize the notion that a fraction can provide a more precise unit of measure than a whole number.
• Emphasize fraction estimation and use of benchmarks to compare fractions.

The middle school years’ are an important time for these fractional concepts to be developed. Fraction understanding is essential for proportional reasoning development, algebra, probability, and other advanced mathematical topics. For students who have not yet developed conceptual and procedural understanding of fractions, interventions are crucial to reduce and eliminate these academic deficits (Clarke & Roche, 2009; Fuchs et al., 2010).
Response to Intervention (RtI)

RtI is both a method of identifying students who may need additional support in various academic areas, and a means to provide struggling students with the instruction that they need. It provides a way to identify students early in their education so that an academic intervention can begin as soon as possible (Fuchs & Fuchs, 2006). According to Fuchs et al. (2010), the RtI framework consists of three levels or tiers: Primary Prevention, Secondary Prevention, and Tertiary Prevention. The Primary Prevention tier (Tier I) refers to the general education that all students receive. The Secondary Prevention level (Tier II) refers to students who have been identified through various assessments as having difficulties in a particular subject area. These students are given instruction that is in addition to their general education. The third tier, Tertiary Prevention, is the most intensive intervention level. Students in this tier receive general mathematics instruction and intensive and focused mathematics instruction. In some schools, students in Tier III have been diagnosed with a learning disability and receive special education services, while in other schools, students with an IEP are placed in a fourth tier.

To support educational systems as mathematics educators implement an RtI model, Gersten et al. (2009) described two recommendations with strong evidence at the Tier II and III level for the educators to consider when incorporating RtI into their instructional system (p. 6).

1. Instruction during the intervention should be explicit and systematic. This includes providing models of proficient problem solving, verbalizing thought processes, guiding student practice, delivering corrective feedback, and
administering frequent cumulative review.

2. Interventions should include instruction on solving word problems that is based on common underlying structures.

These recommendations were considered when developing the intervention curriculum used in the Tier II RtI mathematics class referenced in this study.

Typically, students are placed in a Tier II mathematics class based on results from universal screeners and scores on standardized assessments. Universal screeners are given to all students in a mathematics class in order to efficiently determine which students are struggling and what areas of mathematics are most problematic. A universal screener often used in RtI is curriculum-based measurement (CBM) (Deno, 2003; Fuchs, 2004; Lembke et al., 2012). CBMs were initially developed to identify students who were in need of special education services and monitor the progress of these students. This measurement approach expanded to include universal screening and progress monitoring of both special education and general education students and is often used within an RtI framework.

Measurement items on CBMs assess progress on instruction related to curriculum objectives that students in a particular grade encounter throughout the school year. Tasks on one CBM are representative and equivalent to tasks on every other CBM used throughout the year (see Appendix F). This results in assessments that are reliable and valid and can be used to inform instruction. The multiple sampling of student progress over time is one of the significant features of CBMs (Deno, 2003; Fuchs, 2004). CBMs are typically very short (1-8 minutes), there is a fixed time period in which to take the assessment, and responses are scored as either correct or incorrect. As students take each
CBM, their performance is monitored and as interventions are provided, data are gathered to determine the effectiveness of the intervention.

Research has shown that CBMs are effective for collecting data that can be used to screen students, place students in intervention programs, and monitor their progress in the intervention programs (Deno, 2003; Fuchs, 2004). Through the use of progress monitoring tools, such as CBMs, student growth in areas of deficiency is tracked and used to make decisions on student placement within the tiered instructional levels. Results from the progress monitoring assessments provide teachers with data that will allow them to immediately adjust instruction to meet the needs of students.

While materials such as the IES RtI Practice Guide (Gersten et al., 2009) provided helpful information for mathematics educators, RtI has been implemented primarily in elementary schools in reading/literacy. Implementation of RtI in mathematics is a more recent phenomenon, as is the study of the effectiveness of a specific mathematics intervention course. Fuchs et al. (2013) and VanDerHeyden, McLaughlin, Algina, and Snyder (2012) conducted studies examining the influence of a mathematics intervention on the achievement of struggling fourth and fifth grade students. Fuchs et al. (2013) specifically investigated the impact of an additional mathematics intervention course focused on developing conceptual understanding of fractions using a measurement approach, number lines, and a limited set of denominators. Results showed a decrease in the achievement gap between the at-risk students and the control group. VanDerHeyden et al. (2012) found that lower performing students produced greater gains on the year-end accountability measure after participating in a focused intervention class.
However, the incorporation of RtI in middle and high schools has been much slower than in elementary schools, due to the structure of the middle and high school systems. Fuchs et al., (2010) identified several concerns related to implementing RtI in middle and high schools, including: (a) academic deficits of middle and high school students that have accumulated over the years and consequently are ingrained in their academic knowledge, (b) low motivation and low self-confidence of students, and (c) the need for individual student-based instructional strategies to address particular deficits. The goal of educators in implementing RtI in middle and high schools is to identify students who are struggling, identify the particular deficit/problem area, determine the intervention, conduct it, and monitor student progress. “Given the limited time remaining in these students’ school careers, many older students including those with and without identified special needs, deserve these new or restructured opportunities for decreasing academic deficiencies, with the goal of eliminating this major obstacle toward successful adult life” (Fuchs et al., 2010, p. 26).

Summary

The literature is clear that students struggle with understanding and using fractions. Researchers have made progress in identifying key themes (big ideas) that are essential for fluency with fractions. These include: (a) an understanding of the whole, (b) unit fractions, (c) modeling, (d) comparing and ordering, and (e) benchmarks and estimation. Trajectories for learning fractions have been developed indicating typical pathways students follow as they learn about fractions. The question still remains as to how key themes of fractions are consistently utilized throughout these trajectories.
RtI, a relatively new intervention model that holds promise for addressing deficits in student thinking regarding fractions is being applied at the middle grades level in a current study funded by the IES. It is within this larger study that the study described here was implemented. The goal was to examine student thinking about fractions as it evolved over the course of a unit of RtI instruction on fraction concepts. The work included analyzing: (a) the initial conceptions which middle school students enrolled in a Tier II RtI mathematics course had concerning fractions, (b) fraction concepts and skills demonstrated by students during the intervention curriculum, and (c) the reasoning processes that students utilized a year later as they completed various fraction tasks.
CHAPTER 3: METHODOLOGY

According to Lamon (2007), there is a “crisis” in the amount of research in the area of fractions (p. 632). The complexity of this area of mathematics, interrelationships among fraction concepts, and the number of concepts within the fraction domain contribute to the insufficient quantity of studies pertaining to fractions. This study illustrated how middle grade students studying a unit of instruction on fractions designed for an RtI Tier II mathematics course developed in their understanding and use of fractions. Specifically, development of major domains (big ideas) of fractions and the extent to which that knowledge is retained into the next school grade were examined. Student responses to items on an assessment prior to an instructional unit on fractions provided baseline data regarding students’ understanding of fractions. Written responses derived from individual student workbooks used during the instructional unit provided information about emerging ideas. In addition, a post-unit assessment offered data that could be compared to the pre-assessment. Finally, student written and oral responses to fraction tasks during an interview one year after the unit of study were collected and examined. It was the intent of this study to characterize the understanding and use of foundational fraction concepts of a group of middle grade students who took part in an RtI Tier II mathematics class.

This chapter begins with a discussion of the research tradition utilized by the researcher and the conceptual analysis framework, which provided a structure for analysis of the five key themes. As this study is part of a larger IES research project, an overview of the IES study is given. The methods of data collection and data analysis for student written records, assessment data, and individual interviews is detailed and a
description of how the findings will be reported as well as the trustworthiness of the study will be offered.

**Research tradition/paradigm**

The postpositivist paradigm is based on the idea that reality exists, but is only approximated; it can never be fully comprehended (Hatch, 2002). Within this paradigm, the researcher serves as the data collection instrument. The methodology consists of “rigorously defined qualitative methods, frequency counts, and low-level statistics” (p. 13). Results from studies using the postpositivist paradigm include “generalizations, descriptions, patterns, and grounded theory” (p. 13).

This research employed the postpositivist paradigm. Students’ reasoning processes as evidenced by work found in their student booklet and during the interview process were gathered and qualitatively analyzed using a research-based conceptual framework. Furthermore, data describing student thinking about rational number problems with a particular emphasis on the use of five domains or big ideas were examined to determine if any patterns were evident and if any generalizations could be made. Results from the data analysis were presented through descriptions of student thinking; patterns found in the use of the concepts of the fraction whole, unit fractions, comparing and ordering of fractions, and benchmark fractions; operations on fractions; and the formulation of generalizations about student thinking as it pertains to big ideas of fractions.

**Learning Trajectories for Acquiring Understanding of and Facility with Fractions**

Fraction trajectories are relevant to a study that investigated the impact of a specially designed mathematics curriculum intended for RtI students. The Rational
Number Project (Behr and Post, 1992), Turn on CC Math (Confrey et al., 2012), and CCSSM Progressions (2011) provide trajectories for learning in the domain of fractions. Figure 3 provides a summary of the three trajectories from beginning fraction ideas to the addition and subtraction of fractions. The concepts from each of these trajectories formed the basis for developing the lessons that comprise the fraction intervention curriculum module. Therefore, this study incorporated these trajectories to analyze students’ acquisition and utilization of fundamental fraction concepts and skills, as they progressed along the trajectory.
### Rational Number Project
- Flexibility of the whole, multiple representations for 1/2, 1/3, 1/4, equal partitioning
- Introduce non-unit fractions as the sum of unit fractions, multiple representations, reconstruct the whole given a part
- Move flexibly between a model, fraction word name, and fraction symbol
- Reasoning when comparing and ordering fractions (not finding common denominators)
- Develop the concept of fraction equivalence with concrete models and use of benchmarks
- Use the numerator and denominator to determine a fraction's relative size
- Add and subtract fractions through estimation first, then models, and finally algorithmically

### Turn on CC Math
- Working with unit fractions
- Equi-partitioning, number lines, models
- Model and name non-unit fractions
- Model improper fractions including the use of number lines
- Work with a whole that is greater than 1 object
- Compare \( a/b \) to \( c/b \) and explain reasoning
- Equivalence and comparison of fractions
- Describe reasoning about comparing and ordering of fractions
- Compare and order fractions using models, number lines, benchmarks, and equi-partitioning
- Compare denominators when the numerators are equal or compare numerators when the denominators are equal
- Compare and order fractions with different numerators and different denominators
- Relate the use of models to numeric algorithms
- Generalize mathematical properties
- Operations with fractions
  - Decompose a fraction into additive parts and use models to justify the decomposition
  - Like denominators
    - Add and subtract fractions
    - Solve contextual problems
  - Unlike denominators
    - Reason and model addition and subtraction
    - Move to symbolic representation
    - Solve contextual problems

### CCSSM Progressions
- The meaning of fractions
- Partitioning whole into equal parts
- Unit fractions
- Build fractions from unit fractions
- The number line
- Equivalent fractions
- Comparing fractions with like denominators or like numerators
- Extend idea of the whole to a collection of objects
- Find equivalent fractions using algorithm
- Comparing fractions with unlike denominators and unlike numerators
- Addition and subtraction of fractions with like denominators
- Convert improper fractions to mixed numbers using decomposition into unit fractions
- Word problems
- Addition and subtraction of fractions with unlike denominators using models and then algorithms
- Solve word problems

*Figure 3. Fraction trajectories.*
To understand students’ thinking, Lamon (2007) and Post et al. (1985) suggested that a *conceptual analysis* be conducted prior to studying the reasoning process. A *conceptual analysis* is an approach where the concept to be taught is analyzed to determine the essential mathematics components, the prerequisite knowledge needed, and the reasoning processes that students may use. According to Thompson (2008), a conceptual analysis can be used to generate the instructional goals for a particular concept, develop hypotheses concerning why students may be having difficulties with certain ideas, and describe ways of knowing that might be beneficial or detrimental in different mathematical situations.

Using findings from research on fractions (Bezuk & Cramer, 1989; Clarke & Roche, 2009; Cramer et al., 2002; Lamon, 2007; Peck & Jencks, 1981; Post et al., 1985; Post et al., 1986), a conceptual analysis framework composed of five main domains or big ideas related to fraction concepts was developed. In Table 1, the three elements (prerequisite knowledge, essential components of the theme, student reasoning processes) of this conceptual analysis framework are summarized for each domain/big idea. These elements were used in the analysis of data (as described in a later section of this chapter).
### Table 1

*Conceptual Analysis Framework*

<table>
<thead>
<tr>
<th>Theme</th>
<th>Prerequisite Knowledge</th>
<th>Essential Components</th>
<th>Students’ Faulty Reasoning Processes</th>
</tr>
</thead>
</table>
| The Whole                  | • Name fractions represented by models  
• Draw models  
• Understand part-whole concept with a continuous model and discrete model | • The whole can be more than “one” object  
• Name fractional parts when the whole is varied  
• Reconstruct the whole given a unit or non-unit fraction  
• Understand that the whole must be the same size when comparing, ordering, or operating on fractions | Assuming the whole is one object, one area model, the distance from 0 to 1 on a number line |
| Unit Fractions             | • Internalization that the symbol for a fraction represents a single entity  
• Understanding of the whole  
• Equipartitioning  
• Able to consider \( \frac{1}{n} \) apart from the whole | • Modeling the unit fraction given a whole that can vary  
• Use a unit fraction to create the whole  
• Understand the inverse relationship between the number of pieces represented by the denominator and the size of the equal parts  
• Unit fractions form the basic unit of all fractions | Thinking of a unit fraction based on the number of shaded parts of the whole. For example, the unit fraction of \( \frac{4}{6} \) is \( \frac{1}{4} \) because there are four pieces shaded. |
| Modeling Fractions         | • Meaning of a fraction  
• Understanding of the whole  
• Equipartitioning  
• Meaning of the denominator and numerator  
• Understand the inverse relationship between the number of pieces represented by the denominator and the size of the equal parts | • The partitions in the model must be the same size  
• Understand representation of fractions using multiple models  
• Modeling improper fractions and mixed numbers follow the same requirements as modeling proper fractions  
• Ability to model fractions with denominators greater than 4  
• Model non-unit fractions including those where the whole is varied  
• Use models to denote equivalent fractions and operations on fractions | Students partition a model in halves or fourths and then attempt to partition those sections to create fifths or sixths etc. |
### Comparing and Ordering Fractions

- Meaning of a fraction
- Understanding of the whole
- Equipartitioning
- Modeling unit fractions
- Modeling non-unit fractions
- Understanding of the symbols $<$, $>$, and $=$
- Representing proper fractions, improper fractions, and mixed numbers in multiple ways
- The whole of each fraction being compared or ordered must be the same
- Understanding relative size of fractions
- Consider both the numerator and denominator – see a fraction as a number, not as two individual components
- Understand the relationship between the size and number of equal parts in a whole. Recognize that e.g. 2 one-fifths covers a smaller area than 2 one-thirds
- Ideas of “less than”, “greater than”, and “equal”
- Comparing and ordering all types of fractions with like numerators and different denominators
- Comparing and ordering all types of fractions with like denominators and different numerators
- Fraction equivalence
- Understand the difference between “more” and “greater than” and “fewer” and “less than”
- Understand the relation between two fractions without referring back to a physical model

### Benchmarks and Estimation

- Understanding of the relative size of a fraction
- Understanding that the symbol for a fraction represents a single entity
- Understanding of the whole
- Mental images of common fractions
- Informal ordering schemes
- Use informal ordering schemes to estimate fractions quickly or judge the reasonableness of answers
- Understanding the location of fractions on a number line
- Seeing fractions as lengths
- Applying gap thinking and residual thinking when using benchmark fractions to compare fractions

### Context of the study

A Department of Education IES study, *Project AIM: Algebra-readiness*

*Intervention Modules for At Risk Students* defined the context for the current study.

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Project AIM investigated the effect of specially written mathematics curriculum materials on the mathematics achievement of middle school students who had previously demonstrated difficulties in mathematics (Dougherty, Bryant, Bryant, Darrough, & Pfannenstiel, 2014). The focus of the curricular materials was content deemed to be essential prerequisites for the study of algebra. The goal was to provide support to struggling middle-school students, so that they developed the foundational knowledge that would support them as they progressed to more advanced mathematics courses such as Algebra I.

Project AIM developed eight modules for use in the middle school RtI Tier II mathematics class. The modules focused on the topic areas of:

<table>
<thead>
<tr>
<th>6th Grade</th>
<th>7th Grade</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Decimals</td>
<td>• Integers</td>
</tr>
<tr>
<td>• Fractions</td>
<td>• Expressions and Equations 2</td>
</tr>
<tr>
<td>• Expressions and Equations 1</td>
<td>• Graphing</td>
</tr>
<tr>
<td>• Ratio and Proportion 1</td>
<td>• Ratio and Proportions 2</td>
</tr>
</tbody>
</table>

During the pilot year, each curriculum module contained 24 lessons. This study examined student work from the pilot sixth-grade fraction module. Table 2 lists the titles of the lessons contained in the Fraction module. Objectives for each lesson are detailed in Appendix K.

Table 2

*Lesson Titles in Sixth-Grade Fraction Module*

<table>
<thead>
<tr>
<th>Lesson Number</th>
<th>Lesson Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Unit Fractions</td>
</tr>
<tr>
<td>2</td>
<td>Measuring with Fractions</td>
</tr>
<tr>
<td>3</td>
<td>Unit Fractions and the Whole</td>
</tr>
<tr>
<td>4</td>
<td>Unit Fractions – Mixed and Improper Fractions</td>
</tr>
</tbody>
</table>
Comparing Fractions with Like Numerators
Comparing Fractions Using Benchmark Fractions
Ordering Fractions
Comparing and Ordering Improper Fractions
Comparing and Ordering any Fraction with Like and Unlike Denominators
Comparing and Ordering Improper Fractions Using Number Lines
Comparing and Ordering Mixed Numbers and Improper Fractions
Identify and Make Equivalent Fractions using Concrete and Pictorial Models
Compare Fractions (Use a game to solidify understanding)
Identify and Create Equivalent Fractions Using Multiplication and Division
Simplify Fractions
Simplify Fractions (that are divisible by multiple factors)
Mixed Numbers and Improper Fractions (converting)
Addition and Subtraction of Fractions using Physical Representations
Addition and Subtraction of Fractions with Unlike Denominators
Addition and Subtraction of Fractions using Equivalent Fractions
Addition and Subtraction of Mixed Numbers and Improper Fractions with Like Denominators
Addition and Subtraction of Mixed Numbers and Improper Fractions with Unlike Denominators
Addition and Subtraction of Mixed Numbers
Review of Addition and Subtraction of Fractions (Contextual Problems)

Domains and Dimensions Covered in the Lessons

Tasks in the first four lessons addressed student understanding of the whole, unit fractions, and the use of a variety of models to represent fractions such as fraction strips, Cuisenaire rods, area models, and number lines. Students discussed constructing and using area models, number lines, and Cuisenaire rods to represent fractions. Similarities and differences among these models were considered. As students worked through these tasks, the concepts of the whole, unit fraction, and equal partitioning were emphasized. Each of these representations was also used to model improper fractions. It was important that the idea of “fractions” encompassed proper fractions, improper fractions, and mixed numbers, and that they were not considered as separate and distinct representations.
Lessons 5 through 11 focused on comparing and ordering fractions, improper fractions, and mixed numbers using concrete models, pictorial models, number lines, and benchmark fractions. Generalizations determined from patterns the students found as they compared and ordered fractions were fostered. Students compared and ordered fractions with like numerators and unlike denominators, unlike numerators and like denominators, and unlike numerators and unlike denominators. As the lessons increased in sophistication, students were introduced to equivalent fractions using concrete models and progressed to finding efficient methods for determining equivalent fractions or developing an understanding of why procedures they had learned previously worked. As with Lessons 1 through 4, the concepts of the whole, unit fraction, and equal partitioning were emphasized throughout the lessons.

Lessons 12 through 17 provided instruction on developing a conceptual understanding of equivalent fractions, procedures to find equivalent fractions and why they work, the meaning of simplifying fractions, and the relationship between improper fractions and mixed numbers. Area models, number lines, and Cuisenaire rods were used to concretely display these concepts.

The fraction module concluded with lessons that concentrated on addition and subtraction of fractions. These lessons began with understanding addition and subtraction of fractions by using concrete models and moved towards developing more efficient ways of performing these two operations. As an example, the students began learning about adding fractions with unlike denominators by looking at number line models and determining what fraction was represented by the number line and then thinking about and discussing how the models and the symbolic representations were related.
Format of Lessons

The Tier II intervention lessons were constructed from evidence-based practices for teaching students with mathematics learning difficulties and from strategies that supported students’ mathematics performance. Each lesson consists of five sections (see Appendices A, B, C, D, E for a sample of each section):

1. Warming Up – review of previous lessons’ material or prerequisite knowledge needed for current lesson
2. Learning to Solve – instruction on lesson content, beginning with physical representations and moving towards abstract representations
3. Practicing Together – students work in pairs or groups to solidify what was learned in the lesson and provide teachers with information about possible misconceptions
4. Trying It On Your Own – independent practice for students to individually demonstrate their understanding of the concept in the lesson.
5. Wrapping It Up – activity to help students summarize what was learned or to cause them to think more deeply about the mathematics concept in the lesson.

A team of special education and mathematics educators collaboratively wrote the curriculum material. This created an environment that promoted the inclusion of research-based practices from both fields. Moreover, each lesson was subsequently modified after testing in an authentic mathematics classroom setting, based on comments from teachers and students as well as classroom observations by project researchers. The
lessons were written built on the beliefs that middle grade students who are struggling in mathematics need

- to develop conceptual understanding before they can move on to abstract ideas,
- to make connections between concrete and symbolic representations,
- explicit instruction that guides them through the reasoning process,
- opportunities to communicate with the teacher and other students, and
- appropriate mathematical challenges.

The curriculum materials for each module include a teacher booklet, teacher resource booklet, and student booklet. The teacher booklet consists of all the lessons in the module, directions to the teacher, possible misconceptions that students may have, and notes on mathematically precise knowledge and language that should be emphasized. The teacher resource booklet contains the problems in each section of the lesson in order to facilitate displaying the problems to the class. In addition, answers to the problems are found in the teacher resource booklet. Students use the student booklet to write answers to assigned problems, explain their thinking, work along with the teacher, work within small groups, and work independently. The student booklet provided the primary data source related to student thinking during the fraction instructional module that was analyzed in this study.

Student achievement data were accumulated via an assessment given before instruction on the module began and a post assessment. Each assessment was formatted in the same way. There were 20 multiple-choice items with three choices for students to select from. The same concepts were tested in each assessment and some of the items
were the same across the assessments (see Appendix F for the pre-assessment used with
the fraction module).

In order to capture student thinking beyond what was portrayed in their student
booklet, individual interviews were conducted a year later. The interview took place
during the time frame of one class period. Students completed six tasks that covered five
fraction domains (the whole, unit fractions, modeling, comparing and ordering,
benchmarks and estimation). These domains have been identified as essential for fraction
understanding (see Appendix G for the Interview Protocol). The student’s written work
and oral explanation of what they were thinking as they solved the problem during the
interview was recorded using a camera and was later transcribed.

Participants

Students in sixth, seventh, and eighth grade who were enrolled in a Tier II RtI
mathematics course during the 2013-14 school year, in addition to their core mathematics
course, comprise the target population for the larger IES study. Placement in a Tier II RtI
course is generally based on scores below the 25th percentile on the district or state
mathematics assessment. Typically, these students struggle with weaknesses in working
memory; cognitive difficulties, which includes understanding factual information, rules
and procedures, and understanding relationships; and linguistic difficulties (B. Bryant &
D. Bryant, 2008).

Research sites for the project were based in Texas and Missouri. However, this
investigation focused only on participants in Missouri. This decision was reached due to
ease of access to both student booklets and students who would participate in interviews
during Spring 2015. In Missouri, three middle schools participated in the study. Two of
the schools were in a city setting and the third school was in a rural setting. At the two middle schools located in the city, there were approximately 120 students and four teachers at the sixth-grade level who were part of the study. The rural school contained 15 seventh-grade students and one teacher.

One school (two teachers and approximately 60 students in sixth grade) was selected to provide the data for this investigation. This school demonstrated good fidelity of implementation based on teacher completion of the lessons contained in the fraction module and the feedback they offered to project staff regarding the benefits and shortcomings of each lesson. Thirty students in sixth grade during the 2013 – 2014 school year had given consent for the initial study. Of these 30 students, 29 were still enrolled in the same school and were seventh-grade students in the 2014 – 2015 school year. All of these students were invited to participate in a one-on-one interview with the researcher. Of these 29 students, seven students returned signed consent forms and participated in the interview process. Analysis of data for five students was included in this study. This was due to redundancy in the data offered by the remaining two students, as well as their reluctance to provide explanations for their answers. Four of the five students were enrolled in a general seventh grade mathematics course and one student was enrolled in a mathematics course that encompassed seventh and eighth grade mathematics concepts. Additionally, two of these five students were also enrolled in a Tier II RtI mathematics course. The sample for this study was composed of three females and two males.
Data Collection

Student Booklets

According to Kieren (1976), written records of children’s work can provide data from which to develop a shared set of behaviors that may show a developmental trend in student thinking. Consequently, student work found in the student booklet was examined to identify evidence of student use of the fraction themes summarized earlier. Data from the student booklets consisted of: (a) answers to single answer questions, (b) answers that require students to describe their thinking, (c) answers that are more open-ended and can have more than one correct answer to be correct, e.g., “Write the whole number 10 as a fraction in 3 different ways”, and (d) student generated representations or models of fractions. This data was extracted from the Trying It On Your Own section of each lesson in the student booklet. Every student response, explanation, and diagram written or drawn by a student in response to a problem within this section was recorded in a spreadsheet. This document served as a written record of students’ use of key fraction domains during the sixth grade.

Assessment Responses

An additional source of data was the results of the pre-assessment given to students prior to starting the unit of instruction and the post-assessment at the conclusion of the unit. The questions in each assessment related directly to the material found in the unit. Each question had three possible response choices, one being the correct response. The two distractors were based on common misconceptions students hold concerning fractions. Questions used in the pre- and post-assessment were selected (with cooperation from the University of Oregon) from the easyCBM progress monitoring assessment.
The pre- and post-assessments consisted of 20 multiple-choice questions that addressed the same concepts on each assessment (see Appendix F for a sample fraction assessment). Seven questions were exactly the same in each assessment, while six questions were similar. For example, in the pre-assessment, a question on fraction comparison asked students to decide which fraction, \( \frac{7}{8}, \frac{5}{8}, \) or \( \frac{6}{8} \), was the greatest. In the post-assessment, the same numbered question asked students to decide which fraction, \( \frac{2}{7}, \frac{1}{7}, \) or \( \frac{4}{7} \), was the greatest.

**Student Interviews**

While written records provided documentation of student responses and a glimpse at their reasoning processes, interviews provided opportunities to probe students’ conceptual understanding and identify specific strategies students use to solve problems. The interview is a sophisticated method to obtain detailed information about the particular problem solving processes students make use of (Cookson & Moser, 1980; Cramer et al., 2002; Ginsburg, 1997). Moreover, Clements and Ellerton (1995) maintained that one-on-one interviews provide more beneficial information about student understanding than written assessments.

All of the interviews took place within one middle school located in a suburban area in the Midwest. Interview sessions lasted approximately 45 minutes and were conducted in a conference room during the student’s RtI class period. The student’s voice and written work was videotaped during the interview. Students were individually interviewed one time using a standardized clinical interview (Ginsburg, 1997) and were presented with a series of tasks. The interview protocol contained fraction tasks chosen from interview questions used by the Rational Number Project researchers (K. Cramer,
personal communication, October 6, 2014). Due to the fact that there was only approximately 45 minutes in which to conduct the interview, two or three problems for each fraction domain were selected. These two or three problems were picked based on their similarity to problems experienced by the case study students during the intervention module in the prior school year. A complete set of tasks used in the interview can be found in Appendix G.

The collection of both student written records (answers on the pre- and post-assessments and responses in student booklets) and responses and explanations on the interview tasks, provided a comprehensive view of the case study students’ understanding of five key fraction domains.

Data analysis

Analysis of Student Responses

Student solutions to tasks in their student booklet were analyzed according to the essential components and reasoning processes noted in the conceptual framework for each of the five big ideas as shown in Figure 4.
Figure 4. Analysis of student work framework

Student records from 20 students at the same research site as the target population were randomly selected for preliminary analysis in order to develop coding schemes for the study. Responses to tasks from the Trying It On Your Own section of the student booklet were analyzed to determine initial strategies and concepts employed by the students. An analysis of these responses in conjunction with important fraction domains as suggested by previous research, provided the initial domains to use in the typological analysis method.

This study utilized Hatch’s (2002) description of steps to follow when employing a typological analysis method. The steps and their connection to this investigation are listed below:

1. Identify topic areas to be analyzed
   
   The whole, unit fractions and non-unit fractions, magnitude of fractions, modeling, comparing and ordering fractions

2. Read the data, marking entries related to topics
All problems completed by students were correlated to one of the main topic areas.

3. Read entries by topic, recording main ideas in entries on a summary sheet

   Explanations provided by students as answers to problems in the Trying It On Your Own section of the student booklet were recorded exactly as written.

4. Look for patterns, categories, relationships within topic areas

   Utilizing the conceptual analysis framework described earlier, patterns in the use of the five big ideas were determined.

5. Read data, coding entries according to patterns identified (keep a record of what entries go with what elements in your patterns)

   Student answers were coded according to the strategies used when implementing the fraction big ideas.

6. Look for relationships among the patterns identified

   A visual representation of the categories discovered from analyzing the student work was created in order to make connections among the reasoning processes that students are using.

7. Write the patterns as one-sentence generalizations

   Hatch writes that expressing the patterns seen in the findings as one-sentence generalizations causes the researcher to think concisely about the findings and write them in a concise manner that is easily communicated to others.

8. Select data excerpts to support generalizations
a. Student solutions to problems found in the lessons were provided to illustrate the generalizations generated.

b. Interview data provided additional information to support the generalizations.

Using current research on fractions and trajectories that show the typical development of fraction topics, advancements or retreats from beginning notions of fractions to more advanced understanding could be traced as students progressed through the fraction module. For instance, research shows that students attempt to apply whole number concepts to fractions, but this strategy should fade as the students gain additional fraction knowledge. My analyses of the subsample’s responses to tasks in the student booklets helped shape and refine the procedure used in the study and confirmed the method was useful in documenting thinking and strategy use. Table 3 provides a description of the fraction domain, criteria utilized to include or exclude student responses, and exemplars of each domain.

Table 3

*Coding Scheme*

<table>
<thead>
<tr>
<th>Domain</th>
<th>Description</th>
<th>Inclusion Criteria</th>
<th>Exclusion Criteria</th>
</tr>
</thead>
</table>
| **The whole**| • It is a proportional relationship between the numerator and the denominator
  • Can refer to multiple objects, not just one object | The whole is referenced either explicitly or through the use of models that show the same size whole. | Writing $\frac{3}{4} = 1$ does not indicate an understanding of the whole |

Create the whole given a part.
### Typical Exemplar of the Whole

<table>
<thead>
<tr>
<th><strong>Unit fraction</strong></th>
<th><strong>Magnitude</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Unit Fraction –</td>
<td><strong>Magnitude</strong> –</td>
</tr>
<tr>
<td>• One of many equal parts of a whole</td>
<td>• Understanding that a fraction is a number.</td>
</tr>
<tr>
<td></td>
<td>• Understanding that a fraction has size.</td>
</tr>
</tbody>
</table>

Identifying the unit fraction based on the number of shaded parts of the whole

Unequal partitions in the whole

### Typical Exemplar of the Unit Fraction

1. What is the unit fraction?

   \[ \frac{1}{5} \]

2. How many times is the unit fraction iterated?

   9 times

### Models

<table>
<thead>
<tr>
<th>Models –</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equal partitioning</td>
</tr>
<tr>
<td>Multiple representations – area models, number lines, symbolic (numeric)</td>
</tr>
<tr>
<td>Partitioning</td>
</tr>
<tr>
<td>Iterating</td>
</tr>
</tbody>
</table>

Typical Exemplar of Modeling

The partitions of the model are equal.

Models are not equally partitioned.

Student uses models that are not just circles.
### Comparing and Ordering

- **Gap thinking** – comparing the difference between the whole numbers, which comprise the numerators and denominators.
- **Residual thinking** – thinking about the fractional piece that is needed to build up the fraction to make the whole.
- **Benchmark thinking** – comparing two fractions to a third fraction such as \( \frac{1}{2} \) or 1.

**Typical Exemplar of Comparing and Ordering**

| Gap thinking – Student states that \( \frac{5}{7} < \frac{4}{9} \) because 7 is less than 9. |
| Residual thinking – Student states that \( \frac{6}{7} \) only has 1 more piece to get to the whole. |

### Benchmarks and Estimation

- **Benchmarks** – Comparing fractions to benchmark fractions such as 0, \( \frac{1}{2} \) or 1.

**Typical Exemplar of Benchmarks and Estimation**

| Explicit statements that a fraction was “close to 1” or “near the middle” |
| Stating that a fraction such as \( \frac{3}{10} \) is the smallest because “3 is closest to 0 and far away from 10”. |

### Pre- and Post-Assessments

The standardized assessments created for the AIM project were comprised of multiple-choice items addressing key objectives of the fraction module (see Appendix F).
Pre- and post-assessment items that covered the same material were analyzed to determine if there was a change over the time period of the module. Items were scored as correct/incorrect and an item analysis was conducted to look for patterns when incorrect answers were chosen.

**Follow up Task-Based Interviews**

According to Cramer et al. (2002), interviews assess the depth of student understandings and misunderstandings and provide insight into students’ reasoning. Therefore, task-based interviews with seven students were conducted to probe student understanding, to gain additional insight into student thinking, and to document retention or erosion of student knowledge regarding fractions.

The primary focus of the interview tasks was use of the whole, unit fractions, modeling fractions, comparing and ordering, and benchmarks and estimation. These were the same constructs analyzed in the student written records and provided consistency across the data. Hatch’s (2002) typological analysis as defined in the Analysis of Student Records section was used to analyze the use of the five themes by interviewees as they completed fraction tasks. In addition, as the fraction tasks in the interviews increased in sophistication and the constructs built on each other, an analysis of each interview participant’s progress was possible.

**Report of Findings**

Wolcott (1994) detailed three forms that provide structure to presenting the findings of a study. These forms are: (a) Descriptive, (b) Analysis, and (c) Interpretation. The *descriptive* form depicts the data as described by the participants’ work or voice, as close as possible to their exact words. The *analysis* form allows the researcher to expand
the description of the data by forming generalizations based on an analysis of the data. The interpretation form provides a structure for the creation of understandings and explanations for the results found in the study. Using Wolcott’s structure, data from the student booklets and the interviews were described using their exact words. An analysis of these descriptions took place for the purpose of identifying patterns within and across the group of students. Finally, a description of student understanding based on a review of demonstrations of the five fraction themes was provided.

**Trustworthiness**

Lincoln and Guba (1985) discussed four elements of trustworthiness: (a) “truth value” or credibility, (b) applicability or transferability, (c) consistency or dependability, and (d) neutrality or confirmability. Credibility involves demonstrating that there is a relationship between two variables. The study must be conducted in such a way that readers will trust the findings. One means of increasing credibility is prolonged engagement with the participants/data. In this investigation, the researcher has been involved with the curriculum development, implementation, and collection of data for over two years. A second way to establish credibility is the technique of persistent observation. Persistent observation extends the prolonged engagement by delving deeper into the data and focusing on particular elements of student work. A third method to increase the credibility of a study is through the triangulation of data. In this study, student work from multiple sources (student booklet, pre- and post-assessments, and interviews) provided the researcher different data points with which to evaluate retention of fraction domains.
Transferability is a thorough description of the context of the study by the researcher. It is then the responsibility of the reader to determine if their context is sufficiently similar to allow the findings of the study to be transferred to their situation (Lincoln and Guba, 1985, p. 298). Transferability was accomplished in this study through a detailed description of the data. This included: representations of student work, a comprehensive account of the population, and a thorough report of the method of data analysis and data interpretation.

Consistency refers to the notion that the findings of a study would be similar if the study was conducted with the same or similar participants in the same or similar context (Lincoln and Guba, 1985, p. 290). One way that consistency was addressed in this study was through the dissertation process. Multiple readings of the proposal and the report on the data analysis, results, and implications provided a corroboration of the process and products of the study. Moreover, the inclusion of Project AIM’s principal investigator and co-principal investigator on the dissertation committee allowed for substantiation of the study’s methods, findings, and implications.

Finally, confirmability refers to the idea that the findings of the study are based solely on the data and are not influenced by the biases of the researcher. For this study, confirmability was achieved through the implementation of the methods used to determine credibility, transferability, and consistency.

Summary

The work of students represented in different forms (e.g., student workbook, teacher-directed written assessments, and researcher conducted interview) provided the basis for understanding the impact of a specially designed RtI curriculum that focused on
fractions. Data furnished a picture of students’ initial conceptions of fraction topics, their application of the concepts of the whole, unit fractions, models, comparing and ordering, and benchmarks and estimation as they reasoned through fraction tasks of increasing sophistication.
CHAPTER 4: FINDINGS

This chapter consists of results obtained from implementation of the research design and analysis procedures described in the previous chapter. Data is presented for five students who were enrolled in a sixth-grade Tier II RtI mathematics class where a specially designed intervention curriculum module was used for instruction and who participated in an individual interview during the following school year (seventh grade). The findings are organized by student and are chronologically arranged with an overview of the student’s results from a pre-assessment, a description of their understanding and application of five key fraction domains (the whole, unit fraction, modeling, comparing and ordering, benchmarks and estimation) during the period of instruction, results from a post-assessment, and an account of the retention and use of key fraction domains a year later.

A pre-assessment was given before implementation of the intervention curriculum in the Tier II RtI mathematics class. A post-assessment was given to students at the completion of the fraction module. Only 13 of the 20 multiple-choice questions comprising the post-assessment were exactly the same or similar (slightly modified by changing the numbers in the problem) as the questions found on the pre-assessment. Seven questions were exactly the same on the pre- and post-assessment and six questions were slightly modified. Therefore, only these 13 questions were compared to reveal changes in students’ understanding of fraction concepts represented by these problems.

While the individual interviews were conducted using an interview protocol that addressed key fraction domains, changes to the interview process were made based on the student’s response to the tasks and his/her emotional state as they proceeded through the
questions. According to Ginsburg (1997), this flexibility is crucial to the success of the interview in revealing students’ thinking. The interviewer may need to vary questions asked of the students, alter the tasks, and devise new problems. During the interviews in this study, alterations were made to the number of tasks, fractions used in the tasks, and the order in which the tasks were presented. Additionally, questions asked of the students varied due to their responses and solution strategies for each task.

A description and analysis of the data gathered from the five case study students’ sixth-grade student booklets, pre- and post-assessments, and their interview during seventh grade is presented on the following pages. The data is organized by student to provide an account of their understanding of fraction concepts, misconceptions, and reasoning processes. The students’ own words and diagrams are included so that their way of thinking is brought forth without influence from any possible researcher bias.
BRIANNE

Brianne was a seventh-grade student enrolled in a general seventh-grade mathematics course. She was not assigned to a Tier II RtI mathematic class. During the prior school year, she was enrolled in both a general sixth-grade mathematics course, as well as a Tier II RtI mathematics course.

Pre-Assessment

Results from the pre-assessment indicated that upon beginning the fraction intervention curriculum, Brianne was able to solve a variety of procedural problems involving fractions. She correctly answered 10 of 13 questions (see Appendices H and I). Within the domain of comparing fractions, Brianne successfully compared unit fractions, fractions with like denominators, and two fractions composed of unlike numerators and unlike denominators. However, she incorrectly answered questions on selecting a fraction that was less than \( \frac{5}{6} \), comparing fractions with like numerators and denominators, and identifying a mixed number closest to 5. Under the domain of ordering fractions, she correctly ordered a group of unit fractions and a group of fractions with denominators that were either a multiple or a factor of each other (e.g., \( \frac{2}{8}, \frac{8}{16}, \) and \( \frac{3}{4} \)).

The pre-assessment also contained problems that required students to determine equivalent fractions when provided with both a symbolic representation and a model. Brianne demonstrated mixed results on these problems – correctly answering two of the problems and incorrectly answering one problem.
Intervention Curriculum

Brianne’s written records from her student booklet used during the fraction module provided data on fraction knowledge that she demonstrated during her enrollment in a Tier II RtI classroom setting.

Unit Fraction

During the course of the fraction intervention curriculum module, Brianne correctly modeled unit fractions using an area model, understood that a unit fraction such as $\frac{1}{2}$ could denote different quantities based on the whole, and was able to use her own drawings as well as Cuisenaire rods to create the whole given a unit fraction. Figure 5 displays an example of Brianne’s understanding of unit fractions and her attempt to use the unit fraction concept to order a set of fractions, $\frac{1}{3}, \frac{3}{5}, \frac{3}{8}, \frac{6}{7}$, from least to greatest.

![Figure 5](image)

Brianne correctly named the unit fraction and number of iterations of the unit fraction for each fraction in the set (e.g. $\frac{3}{8}$ is based on $\frac{1}{8}$ with 3 iterations); however, she did not successfully use this strategy to place the fractions in the correct order (her ordering was: $\frac{3}{8}, \frac{1}{3}, \frac{3}{5}, \frac{6}{7}$) and did not provide evidence of how the unit fraction representation helped her think about ordering fractions.

Whole

Brianne’s answers to tasks in the intervention curriculum that required an understanding of the concept of the whole, provided evidence that she was developing an
understanding of the meaning of the whole. For example, she understood that when given a length model representing \( \frac{1}{5} \), she had to iterate the model five times to create the whole. She also successfully determined the whole given a unit fraction that was represented with a Cuisenaire rod. For instance, when given a task to determine what fraction of a black Cuisenaire rod, the length of a white Cuisenaire rod was, Brianne correctly responded \( \frac{1}{7} \). Brianne’s work also indicated that she understood that when comparing fractions, the whole for each fraction must be the same size and that partitioning a whole into a smaller number of pieces results in the individual pieces being larger (see Figure 6).

![Figure 6. Partitioning the whole.](image)

When explicitly asked to identify the whole, Brianne successfully answered these questions. For example, in the problem, “Reese pours \( \frac{7}{6} \) cups of water on the new tree she planted. She pours another \( \frac{7}{6} \) cup of water on the tree to make sure it has enough water. How many cups of water did she pour on the new tree?”, Brianne identified the whole as “1 whole cup.”

**Modeling**

One of the early tasks in the intervention curriculum addressed the representation of \( \frac{1}{8} \) in three different ways using pre-drawn rectangles that were all the same size.
Brianne correctly partitioned each rectangle into eight sections and shaded one of the sections to represent $\frac{1}{8}$ of the rectangle. As the lessons in the fraction module progressed to comparing and ordering fractions, number line models were introduced as another means of representing fractions. For example, students were asked to compare $\frac{11}{8}$ and $\frac{7}{4}$ by modeling the fractions on number lines that were partitioned into eighths. This representation required students to model improper fractions as well as demonstrate that two one-eighth sections were needed to model one one-fourth section. As shown in Figure 7, Brianne correctly modeled $\frac{11}{8}$ and $\frac{7}{4}$ on two similarly partitioned number lines and showed that fractions represent a distance and not just a point on the number line.

![Figure 7. Modeling improper fractions on number lines.](image)

**Comparing and Ordering**

In response to a question found in the fraction intervention curriculum asking which fraction of the group $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{8}$ represented the largest quantity (see Figure 8), Brianne identified $\frac{1}{2}$ as the largest, justifying it by indicating that each of the two sections of the whole would be “a big piece.”
Brianne’s response that “if all these fractions are on an equal amount,” suggested that she understood that in order to compare fractions, the fractions should refer to the same whole and that the fraction with the smallest denominator would have the largest sized partitions.

Brianne successfully answered questions that required her to order fractions that had unlike numerators and like denominators, as well as fractions with like numerators and unlike denominators. In each of these cases, although she wrote a correct response, she did not explain her thinking. However, when comparing and ordering improper fractions, Brianne explained that she determined her answers by converting the improper fractions to mixed numbers (see Figure 9). As shown in Figure 10, Brianne correctly converted $\frac{7}{4}$ to $1\frac{3}{4}$, but incorrectly converted $\frac{8}{5}$ to $1\frac{4}{5}$, which did not prevent her from correctly answering the question.
Benchmarks

Brianne used benchmark fractions to order the fractions $\frac{3}{5}$, $\frac{3}{10}$, $\frac{1}{2}$ from least to greatest. Her solution was correctly ordered and her explanation started with the fact that $\frac{1}{2}$ would be in the middle and that $\frac{3}{10}$ would be “closer to 0.” This was the only instance throughout the curriculum unit in which she indicated that she used benchmark fractions to solve a problem.

Post-Assessment

On the post-assessment, Brianne answered 9 of 13 questions correctly (see Appendices H and J). She successfully compared fractions with unlike numerators and like denominators, compared unit fractions, as well as ordered a group of fractions with unlike numerators and denominators correctly. However, when determining the relationship between fractions with unlike numerators and unlike denominators, Brianne was less successful. In addition, she erroneously selected $\frac{4}{4}$ from $\frac{4}{3}$, $\frac{4}{8}$, $\frac{3}{4}$, as being the closest to 5. Finally, Brianne correctly solved three problems where she determined the equivalence of two fractions when provided with a symbolic representation as well as a model.

Comparison of Pre-Assessment Results to Post-Assessment Results

An analysis of changes in answers selected by Brianne to comparable questions across the pre- and post-assessment revealed that Brianne improved in determining equivalent fractions, but did not improve when comparing fractions with unlike denominators and unlike numerators. Brianne continued to struggle from the pre-assessment to the post-assessment on determining which fraction was less than a given fraction when the denominator was different and determining the mixed number closest
to a given whole number. On the other hand, Brianne remained consistent in determining the correct answer to problems involving: (a) comparing unit fractions and fractions with unlike numerators and like denominators, (b) finding equivalent fractions when provided with a model, and (c) ordering fractions.

**Interview – One Year Later**

One year after the fraction intervention curriculum was implemented in her sixth-grade Tier II RtI mathematics classroom; Brianne was interviewed in a one-to-one setting. The interview consisted of a series of tasks developed to attend to the same topics as those addressed in the intervention curriculum and the key fraction domains that research has determined to be essential in understanding fraction concepts.

**Unit Fraction**

Although Brianne did not use the term “unit fraction” and did not initially understand the meaning when the interviewer used it, she identified the unit fraction in a task found later in the interview. For example, she was asked to draw a model of $\frac{2}{5}$ and to identify what one piece of the model might be called. She responded “one-fifth.” When asked if she recalled the term unit fraction, she said yes, however, when asked what the unit fraction of $\frac{3}{8}$ was, she replied “wouldn’t you do 3 divided by 8?” After referring back to models she had drawn in the previous problem, Brianne connected the term “unit fraction” to its meaning. For example, when explaining why fractions must have a common denominator in order to add them, she determined the unit fraction of $\frac{3}{4}$ to be $\frac{1}{4}$ and compared the size of $\frac{1}{4}$ to the size of the other addend.
Whole

For Brianne, the denominator of a fraction indicated the whole, particularly when she drew an area model. However, when she represented a fraction on a number line, the number “1” represented the whole. That is, for Brianne, the representation (area model or number line) keyed her notion of a whole. She also tended to see the whole, or “1”, as being the largest fraction within a group of fractions. For example, when asked to order three improper fractions \(\frac{10}{9}, \frac{5}{3}, \text{ and } \frac{9}{9}\), Brianne remarked, “Oh, this one would equal one whole because it is \(\frac{9}{9}\). So that would probably be the greatest.”

T: Why do you say \(\frac{9}{9}\) is the biggest?
S: Well \(\frac{9}{9}\) is the biggest because \(\frac{9}{9}\) is just another way to write 1 whole. (writes \(\frac{10}{9}, \frac{5}{3}, \frac{9}{9}\))

T: So these two \(\frac{5}{3}\) and \(\frac{10}{9}\) are they bigger than one whole?
S: No, yeah, well no, they are over one whole. Oooooh. (erases work and writes \(\frac{9}{9}, \frac{10}{9}, \text{ and } \frac{5}{3}\))

In another task, Brianne compared two fractions by drawing rectangular models (which were not the same size) to represent each fraction (see Figure 10). Using these models, she decided that \(\frac{2}{5}\) was not equal to \(\frac{4}{10}\).

S: Because this one (points to the three unshaded part of \(\frac{2}{5}\)) would have a greater amount compared to this (unshaded part of the \(\frac{4}{10}\) model) because these do have like more little pieces, but they are smaller and these (the unshaded part of \(\frac{2}{5}\)) are bigger overall than that one (the unshaded part of \(\frac{4}{10}\)).

Figure 10. Comparing fractions.
When asked if it makes a difference when comparing fractions if one of the wholes is larger than the other, Brianne said, “Oh yeah, it would, cause one is bigger than the other so you can’t tell.” She then drew another model of $\frac{4}{10}$ above the $\frac{2}{5}$ model that had the same size whole (see Figure 11), but again incorrectly stated that $\frac{2}{5}$ was greater than $\frac{4}{10}$.

**S: This one (points to the $\frac{2}{5}$ model) would still be bigger because it has a bigger amount of space (the unshaded area) compared to that one (the unshaded area of the $\frac{4}{10}$ model).**

*Figure 11. Comparing fractions using the same size whole.*

Another example of Brianne’s thinking about the whole is demonstrated by her response to the question, “Mary spent $\frac{1}{4}$ of her allowance. Jose spent $\frac{1}{4}$ of his allowance. Did Mary and Jose spend the same amount of money?”

**S: So, it would be $\frac{1}{4}$ (writes $\frac{1}{4}$ on the paper). So Mary spent $\frac{1}{4}$ and Jose spent $\frac{1}{4}$. I think they both spent the same.**

**T: Why would you say that?**

**S: Well, they both spent the same fraction of their money, but the thing is that if, no I don't know, I don't know how much they had in the beginning.**

**T: So how much money they had in the beginning will make a difference?**

**S: Yeah.**

**S: So, could we say it depends on the whole?**

She understood that, in this situation, the whole must be taken into consideration.

Brianne was successful in creating the whole given a part as evidenced by her response to the following interview question:
If 3 cubes represent \( \frac{1}{5} \) of a tower of cubes, how many cubes are needed for the whole tower?

S: So, he used three cubes to build a fifth of the tower. So there is three total so that is like the whole and there is \( \frac{1}{5} \) of it is built, so... it would be... Could I like draw a picture?
T: You sure can
S: Wouldn't it be like \( \frac{3}{5} \) though, cause it is three cubes for \( \frac{1}{5} \)?
T: (no answer)
S: (drew a rectangle partitioned into five equal parts) So one of them (shades one piece) is out of it. So three cubes like that (makes horizontal marks in the first shaded section to represent the three cubes).
T: So when the question is asking how tall the tower would be, that's really how many cubes will make up the whole tower.
S: Like when it is all complete?
T: Yes
S: Well, if three cubes can fit in that (points to the shaded section), then three cubes can fit in each one, so wouldn't you just do 3 x 5 which equals 15 so 15 cubes.

Modeling

Brianne’s reaction to the first task of the interview (model the fraction \( \frac{2}{5} \)) was to draw a circle and partition it into fourths. She quickly erased the circle and drew a rectangle, which she partitioned equally into fifths and correctly shaded two of the sections. Brianne was comfortable representing \( \frac{2}{5} \) of a rectangle, but less so \( \frac{2}{5} \) of a circle. In fact, she utilized a rectangular area model to represent fractions for the remaining problems presented in the interview. Brianne expressed her awareness that the partitions in a fraction model had to be the same size, but that she had a difficult time creating equal size partitions when she drew some of the models.

Presented with another representation, a number line with only a tic mark at 0 and a tic mark at 1, and asked to model \( \frac{2}{5} \), Brianne immediately began drawing tic marks from
0 to 1 that were evenly spaced apart, but had nothing to do with the fraction \( \frac{2}{5} \) (see Figure 12). She started by making a mark on the line the same distance from 0 to 1 and wrote 0.5 below that mark. Her next step was to count over 2 marks from the 0 and write \( \frac{2}{5} \). This can be seen in Figure 12 by looking at the partitions and labeling that Brianne erased. The interviewer asked if Brianne could apply what she did on the rectangular model to the number line in order to accurately represent \( \frac{2}{5} \). With this prompting, she was able to correctly represent \( \frac{2}{5} \) on the number line.

![Figure 12. Modeling on a number line.](image)

On tasks involving the representation of improper fractions on a number line that only had the whole labeled or had no labeling at all, Brianne struggled with how to partition the whole. For example, to model \( 1\frac{1}{3} \) on an empty number line, she drew three tic marks between 0 and 1 resulting in four partitions. After some reflection, she repartitioned the number line and successfully represented \( 1\frac{1}{3} \).

**Comparing and Ordering**

As noted earlier, when comparing fractions, Brianne’s tendency was to represent them using rectangular models. For example, she drew a rectangular model of \( \frac{2}{5} \) and a rectangular model of \( \frac{1}{3} \) and correctly determined that \( \frac{2}{5} \) was greater than \( \frac{1}{3} \). Her reasoning though was that the model of \( \frac{2}{5} \) contained three unshaded pieces, which was a larger
number than the two unshaded pieces in the model of \( \frac{1}{3} \). In addition to this whole number bias, the rectangular models that Brianne drew were different sizes. However, when asked if it makes a difference if one model (whole) is bigger than the other model when comparing fractions, she replied that it would because when one is bigger than the other, “you can’t tell.”

Brianne was presented with a contextual problem where she was to order three improper fractions, one of them equal to 1. She asked if she could draw a picture to help her solve the problem (see Figure 13). She drew a same size whole for the models of \( \frac{10}{9} \) and \( \frac{5}{3} \), but did not draw a model of \( \frac{9}{9} \), because she already knew that it was “one whole”.

S: Oh, this one would equal one whole because it is \( \frac{9}{9} \). So that would probably be the greatest. (writes \( \frac{9}{9} \) as 9.9 and doesn’t draw a model for it). Uh, it's going to be \( \frac{10}{9} \) and then \( \frac{5}{3} \).

T: which one is bigger, the \( \frac{10}{9} \) or the \( \frac{5}{3} \)?

S: The \( \frac{5}{3} \) I think is the greatest.

T: Why do you think that?

S: Because the \( \frac{5}{3} \), how it is set up the thirds can spread out farther than like ninths, it would have bigger boxes to fill in, and then the \( \frac{10}{9} \) will have smaller pieces compared to the \( \frac{5}{3} \).

Brianne’s first solution to ordering these fractions from least to greatest was \( \frac{10}{9}, \frac{5}{3}, \frac{9}{9} \), which demonstrated her belief of the “whole” always being the largest fraction in a group of fractions. I asked Brianne if she thought that \( \frac{10}{9} \) and \( \frac{5}{3} \) were both greater than one and she thought for a moment and then said, “…no, they are over one whole.” Brianne also described how thirds have a larger area than ninths. She then correctly ordered the fractions from least to greatest.
Finally, Brianne was given a contextual problem, which stated that the whole was the same and asked the student to compare $\frac{4}{5}$ and $\frac{11}{12}$. Brianne struggled with how to determine the relationship between these two fractions, first wanting to draw pictures and then wanting to find a common denominator. After some reflection and prompting, Brianne thought about the size of the unshaded section, understanding that the remaining $\frac{1}{5}$ of the $\frac{4}{5}$ model would have a larger area than the remaining $\frac{1}{12}$ section of the $\frac{11}{12}$ model. Ultimately, Brianne said, “Because if you think about the pieces and how big they are, $\frac{11}{12}$ would have smaller pieces than $\frac{4}{5}$, so if you think about it, Alice ($\frac{4}{5}$) has spent less of her money and he ($\frac{11}{12}$) has spent more.”

**Benchmarks and Estimation**

The interview provided some evidence of the use of benchmarks and estimation as Brianne solved various fraction tasks. For example, when modeling a fraction on a number line that had 0 and 1 marked on it, Brianne first marked $\frac{1}{2}$ in the middle of the distance from 0 to 1 to guide her in representing the fraction. When given a contextual problem, which required her to add two mixed numbers, $2\frac{3}{4} + 3\frac{3}{5}$, Brianne first incorrectly converted the mixed numbers to improper fractions and then struggled to remember the procedure to find a common denominator so that she could perform the addition operation. She was determined to find the sum using learned procedures so the
interviewer guided her to think about the equivalence of a mixed number and an improper fraction. After successfully finding the improper fractions for each mixed number, Brianne then tried to find the common denominator in order to add the fractions.

S: So I think you add these together? (writes $\frac{11}{4} + \frac{18}{5}$). You have to find the same denominator. So you have to think whatever number like can go, the numerator has to go to the denominator? So, would you do, I think you can, I'm not positive. Four doesn't go into 18. I know you have to divide (she is struggling with this). I think you do um, you keep the 4 because 4 can go into 18, it goes 4 times cause that is 16 and then this one (11), it can go in there, but you can't go over, and then this one (5) can go into 18, 3 times and this (5) can go in there (11) 2 times.

Due to her frustration with trying to remember how to find a common denominator, the interviewer asked her to estimate the answer instead. She first added the whole number part of the mixed numbers (2 and 3). Next, she decided that both of the remaining fractional parts ($\frac{3}{4}$ and $\frac{3}{5}$) were each close to 1 and she was able to combine this information with the whole number sum and determine a reasonable estimate of 7.

Three problems during the interview referenced estimating the sum or difference of fractions with unlike denominators and modeling the estimate on a number line (see Figure 14). For the first problem, $\frac{3}{4} + \frac{1}{5}$, Brianne was successful in estimating that $\frac{3}{4}$ would be close to 1, but estimated that $\frac{1}{5}$ was closer to $\frac{1}{2}$ than 0. This resulted in an estimate of $1\frac{1}{2}$. She may not understand that 0 can be used as a benchmark. In the second problem $\frac{7}{8} - \frac{1}{2}$, Brianne decided that $\frac{7}{8}$ was close to 1 because “well $\frac{7}{8}$ is um, if you had $\frac{8}{8}$ that would be 1.” Consequently, her estimate of $\frac{7}{8} - \frac{1}{2}$ was $\frac{1}{2}$. Finally, she estimated $\frac{3}{8}$ as being close to $\frac{1}{2}$ and $\frac{3}{5}$ as almost 2 and arrived at an estimate of $1\frac{1}{2}$ as modeled in the third problem.
Summary of Brianne’s Understanding of Fractions

As a sixth-grade student enrolled in both a general mathematics course and an RtI mathematics course, Brianne demonstrated the ability to correctly model fractions using area models and number lines, understood that the whole must be the same in order to compare fractions, and had a general notion of a unit fraction. When comparing and ordering fractions, she alternated between using rules such as converting improper fractions to mixed numbers and integrating new knowledge learned through the intervention curriculum. Rules and procedures were used most often to find solutions to various fraction tasks.

During her individual interview, Brianne, now in seventh grade, utilized models that were correctly partitioned and were the same size when used for comparison of
fractions or operations with fractions. She was cognizant of the fact that the partitions in a model must be equal size, but struggled to create equal sized pieces when drawing a model herself. Consequently, Brianne was often unable to confidently determine an answer when relying on her own models.

Brianne retained the notion that the denominator has an inverse relationship with the size of the partitions. That is, the larger the denominator, the smaller the equal-sized pieces in the whole. However, the effect of the numerator was not articulated, which leads to the question of whether or not she understands the role that the numerator plays or that a fraction in itself is a number. Lack of comprehension concerning the notion of the whole was evident in Brianne’s explanations of her answers. Interchangeably using “1” and “the whole” created issues for Brianne as she completed various tasks. She wrestled with the notion that a fraction that equals one and is the whole can be less than another fraction. Her perception was that the whole is always the greatest fraction. She also stated that the denominator was the whole.

Tensions between rules and procedures previously learned, terminology, and new conceptual ideas were evident during Brianne’s interview. Often, she would state that she “had” to do something a certain way and many times this was incorrectly accomplished. For example, she responded to the task \((2\frac{3}{4} + 3\frac{3}{5})\) by stating,

S: So you have to turn these into an improper fraction, so you can like solve them better. So you do 4 × 2, which equals 8 and then I think you subtract the 3, which equals 5, and then you keep the denominator (writes \(\frac{5}{4}\) and then does the same to 3 \(\frac{3}{5}\) and writes \(\frac{12}{5}\)). I think that is how you do it.

Brianne stated that she was told that she had to find common denominators to add fractions, but did not know why that was true. Use of rules and procedures are so
dominant in Brianne’s thinking about fractions that she exhibited disequilibrium when asked to utilize conceptual notions about fractions. However, when presented with models and asked to explain her thinking, Brianne was able to draw on some key fraction concepts.
DAVID

David was a seventh-grade student enrolled in a seventh-grade mathematics course called Challenge Math. This course presents seventh- and eighth-grade mathematics concepts to seventh-grade students. The pace of the course is faster than in the “regular” math course offered to students. He was not assigned to a Tier II RtI mathematics class. During the prior school year, David was enrolled in both a general sixth-grade mathematics course as well as a Tier II RtI mathematics course. Due to his success in both of these mathematics courses, David was placed in the Challenge Mathematics course as a seventh-grader. When asked about David being enrolled in a Tier II intervention class as a sixth-grader and then a Challenge mathematics course as a seventh-grader, his teacher responded that David seemed to struggle when taking standardized assessments, which affected his placement in mathematic classes.

Pre-Assessment

Results from the pre-assessment indicated that upon beginning the fraction intervention curriculum, David could identify equivalent fractions and compare two fractions with unlike numerators and denominators. He correctly answered 10 of 13 questions on the pre-assessment (see Appendices H and I). David was not successful on an item that asked him to compare three unit fractions or on an item comparing three fractions with like numerators and unlike denominators. David also had difficulty ordering fractions with unlike numerators and denominators, where the denominators were either a multiple or factor of each other (e.g. $\frac{2}{8}, \frac{8}{16}, \frac{3}{4}$). However, David correctly answered all questions involving equivalent fractions. Each question presented a
symbolic representation of two equivalent fractions as well as a model (circle, rectangle, or number line) of the fractions.

**Intervention Curriculum**

**Unit Fraction**

At the beginning of the fraction intervention curriculum module, David was asked to model the unit fraction, $\frac{1}{8}$, in three different ways using pre-drawn rectangles that were the same size. He successfully partitioned each rectangle into eight sections and shaded one section in each to represent the unit fraction. While students typically shade the first section in a fraction model, David shaded one section in different positions on each model demonstrating his understanding of the unit being $\frac{1}{n}$ of the area of the model and that it is not restricted to being the first $\frac{1}{n}$ of a model (see Figure 15).

![Figure 15. Demonstrating one-eighth unit fraction.](image)

When given a table of unit fractions and a specified number of iterations, David successfully determined the appropriate non-unit fractions.

**Whole**

David demonstrated his understanding of the whole throughout the lessons in the intervention curriculum. For example, as shown in Figure 16, when given a specific Cuisenaire rod to represent a whole and another rod to represent the unit, David found the fractional length of the whole rod represented by the unit rod.
He also found the Cuisenaire rod that represented the whole when provided with a specific rod that represented a fraction of the whole. Additionally, when given the unit fraction $\frac{1}{5}$, David drew a length model representing the whole, as shown in Figure 17.

When comparing and ordering fractions, David drew models of the fractions to be compared with each whole being the same size. Moreover, he was successful in performing these actions with proper fractions, improper fractions, and mixed numbers.

Figure 18 illustrates David’s representation of the improper fractions $\frac{7}{4}$ and $\frac{8}{5}$ using circles of the same size.

Finally, David identified the whole as “1 cup” in a contextual problem involving the addition of improper fractions whose wholes were cups of water.

**Modeling**

David relied mainly on circles to model fractions. The circles were partitioned into the correct number of sections, equally partitioned, and properly shaded. He was
successful in modeling thirds, fifths, and sevenths using a circle area model, which is often difficult for students. However, even though he had demonstrated an understanding that the partitions in a fraction model had to be equal, as shown in Figure 19, using the circle to model tenths proved difficult for him.

Figure 19. Using a circle to model halves, fifths, and tenths.

David used a rectangular area model only once to provide an explanation of why $\frac{7}{4}$ was equal to $1\frac{3}{4}$ (see Figure 20). The two rectangles he drew to model these two fractions were the same size and partitioned equally, demonstrating his knowledge of the requirements of a fraction model. On his model, he labeled both the 1 (the top rectangle) and the $\frac{3}{4}$ (bottom rectangle) as well as the seven one-fourth sections, thus establishing their equality.

Figure 20. Model of $1\frac{2}{4} = \frac{7}{4}$. 
On a problem that required students to represent the fractions $\frac{11}{8}$ and $\frac{7}{4}$ on two separate number lines that were partitioned into eighths, David shaded the distance from 0 to $\frac{11}{8}$ on the first number line. He then partitioned the second number line into fourths, using two of the eighths to represent one-fourth (see Figure 21).

![Image of number lines](image)

*Figure 21. Representing improper fractions on number lines.*

**Comparing and Ordering**

David’s primary strategy to compare and order fractions was procedure driven. For example, Figure 22 shows David’s work when asked to compare two fractions that contained the same numerator. The directions instructed students not to use any of the common procedures for comparing fractions, however, David arrived at his solutions by finding common denominators for the fractions.
David also used the notion of common denominators to find a fraction that was between $\frac{5}{12}$ and $\frac{3}{4}$. He multiplied $\frac{3}{4}$ by $\frac{3}{3}$ resulting in $\frac{9}{12}$ and then selected $\frac{7}{12}$ as the fraction between $\frac{5}{12}$ and $\frac{9}{12}$. Another procedural strategy used by David when comparing fractions was to convert mixed numbers to improper fractions and improper fractions to mixed numbers. For instance, he wrote that $\frac{11}{8} > 1\frac{1}{8}$ and defended his decision by writing $\frac{11}{8} = 1\frac{3}{8}$ and $1\frac{1}{8} = \frac{9}{8}$.

The use of both procedures and conceptual knowledge to solve problems was observed however, in a contextual problem. Within the problem, students were asked to order $\frac{10}{9}, \frac{5}{3}, \text{ and } \frac{9}{9}$ from least to greatest. David converted each improper fraction to its equivalent mixed number and placed the mixed numbers in order ($1, \frac{1}{9}, \frac{1}{3}$). He then explained his solution as shown in Figure 23.
Figure 23. Ordering improper fractions and explanation of solution.

David used the concept that a larger denominator results in smaller partitions as evidenced by his written explanation: “$1\frac{1}{9}$ has a bigger denominator so it has smaller pieces than $1\frac{2}{3}$. But $1\frac{1}{9}$ is bigger than 1.” In addition, he drew a model depicting each fraction. In this situation, David used procedurally converted the improper fractions to mixed numbers, but also demonstrated his conceptual knowledge of the magnitude of each fraction.

**Benchmarks and Estimation**

In one task, students were to represent the fraction $\frac{3}{4}$ on a number line and then place $\frac{3}{5}$ on the number line based on the location of $\frac{3}{4}$. David successfully used $\frac{3}{4}$ as a benchmark fraction in order to estimate the location of $\frac{3}{5}$ on the number line (see Figure 24).

Figure 24. Locating $\frac{3}{4}$ and $\frac{3}{5}$ on a number line.
This was the only instance indicated by David of using benchmarks and/or estimation during independent work in the intervention curriculum.

**Post-Assessment**

On the post-assessment, David again correctly answered 10 of 13 questions (see Appendices H and J). He successfully compared fractions with unlike numerators and like denominators, compared a fraction to a whole number, compared a list of unit fractions, compared fractions with unlike numerators and denominators, and chose the mixed number that was closest to 5. However, David did not correctly answer the problem comparing $\frac{5}{7}$ to $\frac{2}{3}$.

**Comparison of Pre-Assessment Results to Post-Assessment Results**

Results from the pre- and post-assessments indicated that David was more successful comparing fractions and ordering a list of fractions following the intervention curriculum module. When determining the smallest unit fraction in a list on the pre-assessment, David selected the fraction with the smallest denominator. On the post-assessment, he selected the unit fraction with the largest denominator. Similarly, when ordering a set of fractions on the pre-assessment, he based the order on the numerators. On the post-assessment, however, he ordered the same set of fractions based on their magnitude. He continued to do well on procedural problems where he determined equivalent fractions when given both a symbolic representation as well as a model of a circle, rectangle, or number line. Overall, from the pre-assessment and post-assessment data, David demonstrated a stronger understanding of comparing and ordering fractions following the intervention curriculum.
Interview – One Year Later

David relied on rules and procedures throughout the interview, similar to what he demonstrated on his work in the student booklet during the fraction module. He drew models and explained his reasoning when asked, but often struggled with articulating a reason for his actions.

Unit Fraction

The concept of unit fractions was addressed as David began to solve an ordering problem.

T: Before you start going, tell me about the pieces. If you picture $\frac{12}{7}$ in your head, what can you tell me about the size of the pieces?
S: There are 7 equal size pieces, but they are pretty small... but not smaller than a tenth or twelfth. It takes up more room to make a tenth or twelfth than it does to make a seventh.

He had a good grasp of the size of a unit fraction as compared to other unit fractions.

Later in the interview, when asked if he remembered the word, “unit fraction”, David replied, “Yeah, it is the base fraction of any fraction.” Given a mixed number such as $6\frac{1}{3}$, David stated that the unit fraction was $\frac{1}{3}$ and it was iterated 19 times.

Whole

When presented with a question asking if $\frac{4}{6}$ and $\frac{5}{6}$ were equal or if one of the fractions was less than the other, the following conversation ensued:

S: This one ($\frac{4}{6}$) would be less that this one ($\frac{5}{6}$). Bring to this without doing something to this.
T: But how do you know this one is less than this one? I know you said you can't multiply by anything.
S: (draws a circle model for each fraction and puts a less than sign between the two models)
T: Explain that to me
S: Because four has two more parts till it becomes a whole and this one has just one-sixth till it’s a whole.
T: Ok, you called it a whole. Why did you call this a whole?
S: Cause this is the whole (points to denominator) and this would be part of the whole (points to numerator).
T: You drew your circles pretty much the same size. Does it matter? Could one be a bigger circle?
S: No, they have to be around the same size or it kinda messes up the problem.

In this problem, David defined the whole as the denominator. He also indicated in his explanation of this problem, his understanding that when comparing fractions, the wholes must be the same size.

Confusion about the whole was seen again during the interview. One task involved estimating the sum of $\frac{3}{4} + \frac{1}{5}$. David instead found the common denominator and added the fractions. He then related his solution of $\frac{19}{20}$ to the concept of the “whole”. In this situation, David again considered the denominator to be the whole.

S: If I did the problem, these would both have to equal the same number to be added and what you do to the bottom, you gotta do to the top. (multiplied $\frac{3}{4}$ by $\frac{5}{5}$) and (multiplied $\frac{1}{5}$ by $\frac{4}{4}$). That's $\frac{15}{20}$ and $\frac{4}{20}$, so that would be $\frac{19}{20}$. That would almost be 2 wholes if you counted by 10s.
T: Say that again, $\frac{19}{20}$ would be almost 2 wholes?
S: If you were doing it by 10s.
T: If I was doing it by 10s? OK.
S: But if you are doing it just the 20ths, it would be closer to 1.
T: So you said if I were doing it by 10s, it would be closer to 2, but if you were doing it by 20s, it would be close to 1.
S: Yes.
T: What do you think I should do? Should I do it by 20ths or should I do it by 10s?
S: You should probably do it by 20s.

As shown in Figure 25, David typically wrote out or represented a problem symbolically. He was quick to apply an operation (e.g., addition) to a situation involving fractions even though the operation was not called for.
Mary and Jose both have some money to spend. Mary spends $\frac{1}{4}$ of hers and Jose spends $\frac{1}{4}$ of his. Did Mary and Jose spend the same amount of money or a different amount of money?

\[
\frac{1}{4} + \frac{1}{4} = \frac{2}{4} \text{ or } \frac{1}{2}
\]

*Figure 25.* The meaning of $\frac{1}{4}$ and the importance of the whole.

T: The question asked if they spent the same amount. Did they spend the same amount?
S: Yes.
T: Could they have spent a different amount?
S: Not if they both spent a fourth of their money.
T: OK, if they spent a fourth. But what if Mary had $20 and Jose had $10, but they both spent a fourth. Is it still going to be the same?
S: No, it depends on how much money they start out with.

When prompted with a contextual situation, David was able to explain that the size of a fraction could vary based on the magnitude of the whole.

**Modeling**

When modeling fractions, David consistently used a circle except when directed to model on a number line and in one instance when he compared three fractions by drawing an equal sized rectangular model for each fraction. David’s models were partitioned into the correct number of pieces, the partitions were equal sizes, and the shading was correct. He was especially strong in modeling fractions on a number line. David labeled 0 and 1 on the number line, drew the correct number of tic marks to create the equal partitions, and labeled each tic mark (see Figure 26).
Comparing and Ordering

When comparing \( \frac{4}{6} \) and \( \frac{5}{6} \), David explained that \( \frac{4}{6} \) is less than \( \frac{5}{6} \) using the notion of residual thinking.

\begin{itemize}
  \item \textbf{S:} (draws a circle model for each fraction and puts a less than sign between the circles)
  \item T: Explain that to me.
  \item \textbf{S:} Because 4 has two more parts till it becomes a whole and this one (the \( \frac{5}{6} \) model) has just one-sixth till it’s a whole.
\end{itemize}

Within one statement, David described the residual amount to be both parts of a whole (4 has two more parts) and a fractional amount (one-sixth).

David’s reliance on procedures was evident in many of the comparing and ordering of fractions tasks. As shown in Figure 27, when asked if the two fractions are equal, David stated a rule concerning equivalent fractions and remarked that he did not know why that rule worked. After being encouraged to draw a model however, he successfully demonstrated why multiplying the numerator and denominator by the same number results in an equivalent fraction.
Figure 27. Determining equivalent fractions using a procedure.

S: Well they can be equal by doing multiplication or division.
T: OK, what do you mean by that?
S: If I multiply these two numbers (numerator and denominator) by the same number, they will equal this (the $\frac{4}{6}$). Because $2 \times 2 = 4$ and $3 \times 2 = 6$.
T: Why did you multiply them by the same number?
S: Because what you do to the top is what you do to the bottom.
T: Could you draw me a picture and show me why that works? That is a rule that you do, that you follow, but do you know why it works that way?
S: No.
T: Draw $\frac{2}{3}$ for me and let's see if we can figure out why that works.
S: (draws a circle partitioned into thirds and shades 2 sections)
T: How can you show that the $\frac{2}{3}$ is equal to $\frac{4}{6}$?
S: (long pause - draws lines partitioning circle into sixths) and that’s sixths.

Benchmarks and Estimation

David incorporated the idea of $\frac{1}{2}$ as a benchmark fraction when he represented $\frac{2}{5}$ on a number line. Zero and one were already indicated on this number line. David marked $\frac{1}{2}$ on the number line to assist him in partitioning the distance from 0 to 1 in fifths.

He then determined that $\frac{3}{5}$ would be almost one-half. When asked how he knew that, he replied,

S: Five divided by 2 almost equals 3. It equals 2.5.

David proceeded to correctly partition the length from 0 to 1 on the number line and located $\frac{2}{5}$ (see Figure 28).
Estimation presented a challenge to David due to the fact that he was intent on applying addition or subtraction algorithms that led to exact solutions. As shown in Figure 29 and the dialogue below, when asked to estimate the solution to $1\frac{3}{5} - \frac{3}{8}$, David first converted the mixed number to an improper fraction and then found common denominators in order to determine his estimate.

T: Are you trying to convert it?
S: No, I'm trying to turn this number into an improper fraction.
S: $5 \times 1$ is $5$ and $3$ is $\frac{8}{5}$.
T: Where is $\frac{8}{5}$ on the number line?
S: (points to a location close to $1\frac{1}{2}$) So that would be $\frac{8}{5}$, convert that to 40. (he is finding common denominators)
T: So where on the number line would that be about?
S: (puts a mark right past the 1)
Summary of David’s Understanding of Fractions

As a sixth-grade student enrolled in both a general mathematics course and an RtI course, David demonstrated understanding of unit fractions, the whole, and how to correctly model a fraction using different representations. Results from his independent work in the student booklet indicated that he recognized that the whole must be the same when comparing or ordering fractions as well as when performing operations on fractions. In addition, David had the awareness that the number of partitions denoted by the denominator of a fraction was inversely related to the size of each partition. Finally, he was cognizant of the procedures and rules to convert improper fractions to mixed numbers and vice versa and to find equivalent fractions.

During his individual interview, David, now in seventh grade, displayed his reliance on rules and procedures. For example, when presented with the task of placing the following fractions, $\frac{12}{7}, \frac{12}{11}, \frac{12}{13}, \frac{12}{5}, \frac{12}{12}$, in order from greatest to least, instead of thinking about the fact that due to the numerators being the same number, it is only necessary to think about the size of the fractions based on the denominator, he replied,

S: The numerators are the same, but the denominators are all different so I have to find a common denominator for all of them.

David retained the ability to construct models (area models and number lines) to represent proper fractions, improper fractions, and mixed numbers. Figure 30 portrays his area models of $\frac{4}{6}$ and $\frac{5}{6}$ and number line model of $\frac{4}{3}$. David did express his concern that while he knew that the partitions in a model must be of equal size, he was unable to draw them correctly.
David seemed to struggle at times with understanding what various terms meant. For example, when asked if he was trying to convert the mixed number to an improper fraction in order to estimate $1\frac{3}{5} - \frac{3}{8}$, he replied,

*S: Nope I'm trying to turn this number into an improper fraction.*

In another task where David was to order the fractions $\frac{12}{7}$, $\frac{10}{7}$, $\frac{6}{7}$, $\frac{14}{7}$ from greatest to least, the interviewer asked:

*T: Are you converting them in your head?*
*S: No.*
*T: So what are you doing?*
*S: Taking 7 into the numerator and changing them into mixed numbers.*

David was, in fact, converting the improper fraction to a mixed number.
ANGIE

Angie was a seventh-grade student enrolled in a general mathematics course. She was not assigned to a Tier II RtI mathematics course. During the prior school year, she was enrolled in a general sixth-grade mathematics course as well as a Tier II RtI mathematics course.

Pre-Assessment

Angie correctly answered eight out of 13 questions on the pre-assessment (see Appendices H and I). Results from the pre-assessment indicated that prior to the intervention module on fractions, she could compare fractions with unlike numerators and like denominators and find equivalent fractions when provided with a symbolic representation and either an area model or a number line partitioned based on the denominators.

Angie was less successful on other comparing and/or ordering fraction problems. For example, she incorrectly answered questions involving the comparison of two fractions with unlike numerators and denominators. When asked to select the mixed number closest to 5 from the set $4\frac{1}{3}, 4\frac{7}{8}, 4\frac{3}{4}$, she chose $4\frac{3}{4}$. For a task that required Angie to identify the smallest fraction from among a group of unit fractions, she chose the unit fraction with the smallest denominator. Finally, her response to ordering a group of fractions from least to greatest was to order them by the size of their numerators.

Intervention Curriculum

Unit Fraction

At the beginning of the fraction module, Angie successfully depicted the unit fraction $\frac{1}{8}$ in three different ways (see Figure 31).
Angie also accurately determined the unit fraction when provided a Cuisenaire rod that represented the whole and another Cuisenaire rod that represented a fractional length. For example, when asked what fraction of the length of the orange rod does the length of one red rod represent, she answered, \(\frac{1}{5}\). In addition, she found the Cuisenaire rod that represented the whole when given another Cuisenaire rod that represented a specified unit fraction.

Another task that Angie successfully completed in the student booklet involved determining the non-unit fraction that resulted from iterating a unit fraction a specific number of times. In addition, when provided with an improper fraction or mixed number, she indicated the associated unit fraction and the number of iterations of the unit fraction that represented the improper fraction or mixed number.

**Whole**

The initial task completed by Angie in the student booklet is shown in Figure 32.

\[ \text{Figure 32. The meaning of one-half and its relationship to the whole.} \]

Angie initially assumed that a pizza is always sliced into eight pieces, but she also remarked that \(\frac{1}{2}\) of a pizza could be 4 slices or 2 slices and that the pizzas could be
different sizes. She had some notion of the effect of the whole when determining the meaning of a fraction. Another task from the first lesson of the module involved drawing the whole when given a length that represented $\frac{1}{5}$ of the whole. As shown in Figure 33, Angie successfully completed this task.

*Figure 33. Using a unit fractional length to draw the whole.*

Finally, in the following contextual problem: “Amara walked $\frac{1}{3}$ of a mile to get to the park. She then walked $\frac{1}{9}$ of a mile to get to her friend’s house. How far did Amara walk altogether”, Angie identified the whole as 1 mile.

**Modeling**

As shown above in Figure 31, Angie determined three different ways to equally partition a rectangular area model and correctly shaded each model to represent $\frac{1}{8}$. She also correctly represented the improper fractions, $\frac{11}{8}$ and $\frac{7}{4}$ on number lines partitioned into eighths, although she did not demonstrate that the fraction was a distance from 0 (see Figure 34).

*Figure 34. Modeling improper fractions on number lines.*
Angie did not utilize models to determine answers for any other problems in her independent work.

**Comparing and Ordering**

In the student booklet, Angie’s work indicated that she correctly compared five sets of fractions composed of like numerators and unlike denominators. However, no explanation or representation for her answers was given, so her reasoning process was unknown. When comparing the three unit fractions, \( \frac{1}{3}, \frac{1}{2}, \frac{1}{8} \), Angie correctly selected \( \frac{1}{2} \) as the largest fraction because “it has the smallest denominator.” She used this notion to solve additional comparison tasks. For instance, when determining if \( \frac{7}{4} \) or \( \frac{8}{5} \) was the smaller quantity, she remarked that \( \frac{8}{5} \) was smaller “because the bigger the denominator the smaller the piece.” While her answer was correct, it was not evident if she considered the numerator in making the determination.

Angie represented an improper fraction as a mixed number in order to compare fractions (see Figure 35). When ordering improper fractions and mixed numbers, Angie consistently changed the improper fractions to mixed numbers.

*Figure 35. Comparing an improper fraction and a mixed number.*
She also used benchmark fractions as a strategy to order a set of fractions. For example, to order $\frac{3}{5}, \frac{3}{10}, \frac{1}{2}$ from least to greatest, Angie noted that $\frac{3}{5}$ is closest to 1”, “$\frac{3}{10}$ is closest to 0”, and “$\frac{1}{2}$ is $\frac{1}{2}$” (see Figure 36).

**Figure 36.** Using benchmark fractions to order fractions.

**Estimation and Benchmark**

As demonstrated in Figure 36, Angie utilized the concept of benchmark fractions to order a group of fractions. She also used benchmark fractions to estimate the solution to $\frac{1}{3} + \frac{1}{9}$ and determined that $\frac{1}{2}$ was an appropriate estimate. These were the only instances of Angie utilizing the ideas of benchmarks and estimation.

**Post-Assessment**

On the post-assessment, Angie correctly answered 12 of the 13 questions (see Appendix H and J). She successfully compared fractions with unlike numerators and like denominators, a set of unit fractions, and fractions with like numerators and unlike denominators. Additionally, she selected the correct mixed number closest to 5. Angie also found equivalent fractions when given the symbolic representation as well as a model (rectangle or number line). Her only incorrect response was indicating that $\frac{5}{7} < \frac{2}{3}$. 

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Comparison of Pre-Assessment Results to Post-Assessment Results

The increase in the number of correct answers from pre- to post-assessment was found in the domain of comparing and ordering fractions. Following the intervention module, Angie was successful comparing unit fractions, comparing fractions with unlike numerators and denominators, and selecting the mixed number closest to 5, as well as correctly ordering fractions with like numerators and unlike denominators. While ordering a set of fractions from least to greatest based on their numerators on the pre-assessment, Angie correctly ordered the same set of fractions based on their size on the post-assessment.

Interview – One Year Later

During the interview, Angie employed a number of strategies to solve the fraction tasks presented including: (a) recalled rules/procedures, (b) discrete and continuous models, and (c) conceptual notions such as the larger denominator resulting in smaller partitions.

Unit Fraction

When asked if she remembered the term “unit fraction”, Angie responded, “not really.” Later in the interview, Angie was asked what the unit fraction of $\frac{7}{4}$ was. She replied, “Do you flip these?” (referring to the numerator and denominator). Angie was attempting to apply a learned procedure. The interviewer then read the fraction to Angie as “seven-fourths” and asked her what the unit fraction was. Angie replied, “one-fourth” and stated that there were seven one-fourths in $\frac{7}{4}$. When given the fraction $\frac{10}{3}$, Angie correctly identified the unit fraction as $\frac{1}{3}$ and the number of iterations of the unit fraction as 10.
Whole

The following task was presented to Angie: “Jan said, $\frac{3}{4}$ is always equivalent to $\frac{6}{8}$. Do you agree with Jan?” Angie’s first response was, “I agree with her.” She was asked to demonstrate why she agreed with Jan and wrote the equation shown in Figure 37.

Figure 37. Using a proportion to determine equivalency.

The interviewer then presented Angie with a diagram where $\frac{3}{4}$ of a rectangle was shaded and $\frac{6}{8}$ of another (larger) rectangle was shaded (see Figure 38). Angie was again asked if $\frac{3}{4}$ is always equivalent to $\frac{6}{8}$.

Figure 38. Comparing $\frac{3}{4}$ and $\frac{6}{8}$ with different size models.

S: No.
T: Why not?
S: Because $\frac{3}{4}$ is smaller than $\frac{6}{8}$.
T: Why is that?
S: There's more pieces.
T: What do you mean?
S: There's like more shaded in.
Angie was presented with another diagram of $\frac{3}{4}$ and $\frac{6}{8}$ where the rectangular models were the same size and she was again asked if $\frac{3}{4}$ is always equivalent to $\frac{6}{8}$ (see Figure 39).

**Figure 39.** Comparing $\frac{3}{4}$ and $\frac{6}{8}$ with same size models.

S: Yeah.
T: Why?
S: Ummm.
T: Do they cover the same amount of space?
S: Yes, because this little block is the same amount of space as this one.
T: Is $\frac{3}{4}$ always equivalent to $\frac{6}{8}$?
S: Yes, it represents the same amount.

Angie’s confusion about the meaning of the whole is further noted in her response to the question shown in Figure 40. After reading the problem, Angie’s first response was “a fifth” is the whole.

**Figure 40.** Determining the whole.

The interviewer then asked Angie what the whole meant to her and she replied, “like the whole liter.” When asked why she initially said “a fifth” was the whole, she replied, “Cause 2 + 3 and then, maybe it’s one-sixth. Yeah, I think it’s a sixth cause you multiply
$2 \times 3$ and then multiply $1 \times 1$.” (see Figure 40). Angie stated three different ideas about the whole within this one problem.

Another idea of the whole was revealed when Angie was asked to compare $\frac{1}{5}$ and $\frac{1}{3}$. She drew two rectangles and partitioned one into fifths and the other into thirds (see Figure 41).

Figure 41. The use of the whole when comparing fractions.

T: These rectangles kind of look like they are the same size. Did you do that on purpose or did it just happen? Does it matter?
S: Uh, I kind of drew them the same cause if it was the same thing, these (one-fifth pieces) would be smaller than this (one-third pieces).

In this comparison situation, Angie recognized that the wholes should be the same size because that will affect the size of the partitions.

Another example of Angie’s ideas about the whole was shown as she ordered a group of fractions ($\frac{10}{9}, \frac{5}{3}, \frac{9}{9}$). She stated that the whole for the fraction $\frac{10}{9}$ was 9 because “it is the denominator.” Angie held multiple impressions of the whole resulting in confusion when determining or defining the whole.

**Modeling**

Angie’s fraction models were typically equally partitioned and correctly shaded. When given the task to show $\frac{1}{8}$ of the area of a rectangle in three different ways, Angie accurately represented $\frac{1}{8}$ in two of the rectangles, but not the third (see Figure 42). She
recognized that any of the sections within the model could be shaded because “it’s still one-eighth.”

Figure 42. Modeling $\frac{1}{8}$ in three different ways.

After Angie drew her first model of $\frac{1}{8}$ using vertical lines, she was asked if all of the sections had to be an equal size. Angie replied, “I don’t know. I usually try to make them even.” When Angie had completed her third model, she was asked if all the triangles were the same size and if it mattered. She answered that the triangles in the third model were not all the same size and that it did not matter. The notion that partitions of a fraction model must be of equal size was not firmly established in her mind.

When modeling improper fractions, Angie tended to use discrete representations of the fractions. For example, as shown in Figure 43, Angie drew the following models to represent $\frac{10}{9}$, $\frac{5}{3}$, and $\frac{9}{9}$. She circled a number of squares (based on the denominator) in each model to indicate the whole.
Comparing and Ordering

As Angie compared and ordered fractions throughout the interview, she utilized a variety of strategies. She drew models that were the same size, used the idea that larger denominators resulted in smaller sections, stated that she “felt” that one fraction was larger than another, and used rules and procedures that she had learned. For example, when asked to compare $\frac{1}{2}$ and $\frac{1}{3}$, Angie remarked that, “a half is bigger than a third.” When pressed to explain why, she said, “I feel like a third is smaller than a half.” The next problem presented the following task: “Rena wrote $\frac{1}{5}$ is greater than $\frac{1}{3}$ because the number 5 is greater than the number 3. Do you agree with Rena? Why or why not?” Angie answered, “I disagree, because if you cut $\frac{1}{5}$ into pieces, it would be smaller than a third.” These two problems, which occurred in sequence, demonstrate that Angie has some understanding of the meaning of the denominator of a fraction.
Utilizing the notion that the larger denominator will result in smaller sections, Angie attempted to order \( \frac{12}{7}, \frac{12}{11}, \frac{12}{13}, \frac{12}{5}, \frac{12}{12} \) from least to greatest. Her initial order was

\[
\frac{12}{13}, \frac{12}{11}, \frac{12}{7}, \frac{12}{5}.
\]

**S:** I'm debating on this one (\( \frac{12}{12} \)) cause that's a whole of 1. So I don't know if it would go here (points to between \( \frac{12}{13} \) and \( \frac{12}{11} \)) or...

**T:** Well think about what you said about the denominators, that 13 had more pieces, so the pieces were smaller, isn't that what you said? So do you have to think something different, just because you know that's 1 or can you think about that idea?

**S:** I think you still think about it (writes \( \frac{12}{12} \) between \( \frac{12}{13} \) and \( \frac{12}{11} \)).

Angie successfully ordered this group of fractions from least to greatest, but the idea that the whole and 1 are the same concept caused her some discomfort.

**Estimation and Benchmark**

There were not many instances of Angie’s use of benchmarks and estimation during the interview. However, in two comparison tasks, Angie used the idea of \( \frac{1}{2} \) as a benchmark to help in solving a problem. For example, when determining the greatest fraction from the set \( \frac{1}{4}, \frac{1}{2}, \frac{1}{8} \). Angie promptly remarked that \( \frac{1}{2} \) was bigger than the other two. Her explanation was

**S:** Because like if you cut a pizza or something in half, it will still be bigger than if you cut it into fourths or eighths.

**Summary of Angie’s Understanding of Fractions**

As a sixth-grade student, Angie often applied previously learned procedures. She was also successful in identifying the relative size of fractions if they had a common numerator or denominator. However, in a task that involved ordering a group of proper
and improper fractions with common denominators, Angie was not successful (see Figure 44).

![Figure 44. Ordering a set of fractions containing proper and improper fractions.](image)

As shown in Figure 45, Angie understood that $\frac{3}{5}$ was smaller than $\frac{3}{4}$, although she did not correctly represent these two fractions on a number line.

![Figure 45. Comparing $\frac{3}{5}$ and $\frac{3}{4}$ and modeling them on a number line.](image)

In addition, Angie knew that a fraction with a small denominator is partitioned into sections that are large.

She demonstrated an understanding of unit fractions and iterations of unit fractions. As shown in Figure 46, she determined the unit fraction and number of iterations of the unit fraction needed to represent improper fractions and mixed numbers.
Finally, Angie’s work in the student booklet revealed strong procedural abilities. She correctly converted improper fractions to mixed numbers, found equivalent fractions, simplified fractions, and added and subtracted fractions.

During her individual interview, Angie again demonstrated strong conceptual and procedural knowledge of fractions. However, at times she struggled with tasks or with articulating her ideas. For example, when drawing models, Angie created equal partitions. However, when questioned about the importance of equal partitions, she wasn’t certain that it mattered if the sections were equal. She also struggled with modeling improper fractions and generally used discrete models to represent fractions greater than one.

Angie’s concept of the whole as it relates to fractions was still fragile. For instance, she described the whole as “1” and also as the denominator. On the other hand, Angie successfully: (a) indicated the unit fraction and number of iterations of an improper fraction, (b) applied the concept of a smaller denominator resulting in larger
sections to compare and order fractions, and (c) found equivalent fractions using both a procedure and a model (see Figure 47).

Figure 47. Equivalent fractions using a procedure and a model.

From sixth grade to seventh grade, Angie retained knowledge of a fraction as an iteration of unit fractions, although she was not comfortable with the term “unit fraction” when used during the interview. Angie’s idea of the whole was still in flux as it was in the sixth grade. She understood the whole as used in comparison tasks, but confused the ideas of the whole, the number 1, and the denominator. Angie’s models continued to be equally partitioned and correctly shaded. When comparing and ordering fractions, Angie remembered the concept of using the notion of the smaller denominator resulting in larger sections. Converting a proper fraction to a mixed number when ordering a set of proper and improper fractions was still occurring in the seventh grade. Overall, Angie has retained much of the fraction knowledge from sixth grade.
ETHAN

Ethan was a seventh-grade student enrolled in a general mathematics course and a Tier II RtI mathematics course. During the prior school year, Ethan was enrolled in both a general sixth-grade mathematics course and a Tier II RtI mathematics course.

Pre-Assessment

Results from the pre-assessment indicated that prior to beginning the fraction intervention curriculum module, Ethan experienced difficulty with many fraction concepts. He correctly answered six of 13 questions on the pre-assessment (see Appendices H and I). On the pre-assessment, Ethan demonstrated the ability to compare fractions comprised of unlike numerators and like denominators and determine equivalent fractions when provided with a symbolic representation as well as a model (rectangular area model or number line). However, comparison of fraction tasks that were more sophisticated were not answered correctly by Ethan. He incorrectly compared fractions consisting of like numerators and unlike denominators and decided that of a set of mixed numbers \( \frac{4}{3}, \frac{7}{8}, \frac{3}{4} \), \( \frac{3}{4} \) was closest to 5. In addition, Ethan did not successfully order a list of fractions comprised of unlike numerators and unlike denominators. In this instance, he ordered the fractions from least to greatest based on their numerators, which is representative of whole number bias.

Intervention Curriculum

Unit Fraction

Ethan infrequently used or mentioned the unit fraction in his work throughout the module. However, when provided with Cuisenaire rods and given a certain colored rod that represented the whole and another rod that represented the unit fraction, he was
successful in identifying the numeric unit fraction. Moreover, when given a rod representing the unit fraction and the name of the unit fraction, Ethan determined the whole (see Figure 48).

![Figure 48](image)

*Figure 48. Using Cuisenaire rods to determine the unit fraction and the whole.*

**Whole**

The notion that the whole must be the same size when comparing fractions was depicted in the models Ethan drew. His models were not exactly the same size, however, they were similar enough, that the difference in the magnitude of the partitions was apparent. For example, in order to compare $\frac{4}{6}$ and $\frac{5}{6}$, Ethan drew two similarly sized models and correctly determined that $\frac{4}{6}$ was less than $\frac{5}{6}$ (see Figure 49).

![Figure 49](image)

*Figure 49. Draw equal size wholes to compare fractions.*
Modeling

For some tasks in the module Ethan drew circular area models, while in other tasks, he drew rectangular area models. Ethan was more successful in implementing the requirements (correct number of partitions, equal partitions, correct shading) of properly drawn models when he used a rectangle rather than a circle. For example, Figure 50 displays the rectangular area models that Ethan drew to compare \( \frac{2}{5} \) and \( \frac{2}{7} \), and Figure 51 displays the circular area models that Ethan drew to compare \( \frac{1}{6} \) and \( \frac{1}{14} \).

Figure 50. Rectangle models.

![Rectangle models](image)

Figure 51. Circle models.

![Circle models](image)

The rectangular models both contained the appropriate number of partitions, the partitions were equal, and the correct number of partitions was shaded. On the other hand, while his circular model of \( \frac{1}{6} \) had the correct number of partitions and was shaded correctly, the partitions were not equally sized. His circular model of \( \frac{1}{14} \) contained one shaded part, but the number of partitions was incorrect and the size of the partitions were not equal.

Comparing and Ordering

As revealed by his work in the student booklet, Ethan could order and compare fractions. His explanations did not have enough detail though to determine his thinking
processes as he completed the tasks. For example, Figure 52 reveals that Ethan correctly identified a fraction between $\frac{5}{12}$ and $\frac{3}{4}$. His explanation could mean that he has a sense of the size of these three fractions ($\frac{5}{12}$, $\frac{4}{6}$, $\frac{3}{4}$) or that he thought $\frac{4}{6}$ was a correct solution because 4 is between the numerators of 5 and 3, while 6 is between the denominators of 12 and 4.

![Figure 52. Explanation of why $\frac{4}{6}$ is between $\frac{5}{12}$ and $\frac{3}{4}$.](image)

When comparing a group of unit fractions, Ethan reasoned that the larger the denominator, the smaller the size of the piece of the whole. For example, Figure 53 shows Ethan’s response that $\frac{1}{2}$ was the larger amount. In this case, it appears that Ethan understood that the largest fraction would be the one with the smallest denominator, even though his words described two as being the bigger denominator.

![Figure 53. Comparing unit fractions using the denominator concept.](image)

In another case, Ethan incorrectly stated that $\frac{1}{5}$ was greater than $\frac{1}{3}$ because the number 5 is greater than the number 3 (see Figure 54). In this situation, whole number bias was influencing his decision.
Finally, in a later lesson, when determining the largest fraction from a list of fractions \(\frac{1}{4}, \frac{1}{2}, \frac{1}{8}\), Ethan correctly identified \(\frac{1}{2}\) as the largest fraction (see Figure 55). He was making progress towards a more accurate explanation of how to use the denominator to compare fractions.

As the lessons in the module progressed in difficulty, Ethan correctly ordered a list of four fractions with common denominators and a list of five fractions with common numerators. However, he incorrectly ordered the group of fractions \(\frac{8}{7}, \frac{9}{4}, \frac{6}{5}, \frac{10}{6}\) from greatest to least as \(\frac{6}{5}, \frac{8}{7}, \frac{9}{4}, \frac{10}{6}\). There was no explanation for this sequence, so it is difficult to know the fractional concepts or strategies he relied on to determine his answer. However the numerators were in ascending order, which may indicate whole number bias affecting the fraction order.

![Figure 54. Comparing unit fractions using whole number concepts.](image)

![Figure 55. Comparing unit fractions.](image)
Estimation and Benchmark

Only one written record in the student booklet provided evidence of Ethan’s use of benchmark fractions. As shown in Figure 56, he successfully ordered $\frac{3}{5}$, $\frac{3}{10}$, $\frac{1}{2}$ from least to greatest as $\frac{3}{10}$, $\frac{1}{2}$, $\frac{3}{5}$. Based on his written explanation, he reasoned that $\frac{3}{5}$ was closest to 1 and therefore was the greatest.

Post-Assessment

On the fraction module post-assessment, Ethan correctly answered five of 13 questions (see Appendices H and J). He successfully answered questions involving the comparison of unit fractions, comparing mixed numbers to a whole number, and finding equivalent fractions given a symbolic representation and a model. He incorrectly answered questions involving the comparison of fractions with unlike numerators and like denominators and questions involving the comparison or ordering of fractions with unlike numerators and unlike denominators.

Comparison of Pre-Assessment Results to Post-Assessment Results

A positive change from the pre-assessment to the post-assessment was observed on two questions where Ethan determined the smallest fraction from a list of unit fractions and selected the mixed number closest to 5. On the other hand, a negative change was observed on questions involving the comparison of fractions with unlike numerators and like denominators as well as fractions with unlike numerators and unlike denominators.
denominators and determining the equivalent fraction of an improper fraction given the symbolic representation, as well as a model of a number line.

Ethan’s performance on items (pre-assessment and post-assessment) that asked for the least or greatest fraction was inconsistent. However, on the post-assessment, he selected the fraction that was the least from a list of unit fractions, whereas in the pre-assessment, he had not been successful answering this question. On the post-assessment, he also correctly identified \( \frac{7}{8} \) as being closer to 5 (he had incorrectly selected \( \frac{3}{4} \) on the pre-assessment). Ethan’s overall performance on the post-test indicated that he continued to struggle comparing and ordering fractions with unlike numerators and unlike denominators.

**Interview – One Year Later**

Ethan struggled with many of the tasks presented during the interview. Consequently, all the tasks were not completed and the interviewer modified and replaced some of the tasks. Ethan also attempted to apply rules and procedures that he knew related to fractions, but he was often unsuccessful and became frustrated as a result.

**Unit Fraction**

In the “tower problem”, where three cubes represented \( \frac{1}{5} \), the interviewer asked Ethan how many groups of three cubes would be needed to make a whole tower. Ethan’s first reaction to the “tower problem” was “oh man.” The interviewer asked him if it would be helpful to make a drawing, but Ethan stated that he did not know how to draw one-fifths.

T: So how many groups of 3 cubes would you need to make the whole tower?
S: 5
T: How many cubes would that be?
S: 15
T: How did you get 15?
S: I did 5 times 3.

During a task involving the comparison of $\frac{1}{3}$ and $\frac{1}{5}$, the notion of the size of a unit fraction was discussed. Ethan then visualized the size of the part resulting from the larger denominator in a unit fraction as compared to the size of the part of the unit fraction with a smaller denominator.

T: Why do you think that is? If you look at the pictures, why is the $\frac{1}{3}$ bigger than $\frac{1}{5}$?
S: Cause the $\frac{1}{3}$ space is a bigger space and the $\frac{1}{5}$...the more you go up, isn't it the smaller it gets?
T: So, if I had this as one one-hundredth?
S: It would be really small.

When drawing a representation of these two unit fractions ($\frac{1}{3}$ and $\frac{1}{5}$), Ethan wrote “whole” above each of his models (see Figure 58).

Whole

At the beginning of the interview, Ethan expressed an understanding that the sections created when a whole was partitioned must be equal-sized by using the word “fair”.

S: (draws a circle and partitions it into four sections and then inserts a diagonal line to create sixths)
T: Does is matter when you draw these, if the sections inside are the same size?
S: It does matter, cause then uh, if you give somebody the smaller piece and you get the big piece, that wouldn't be like fair.

This idea of “fairness” along with Ethan’s initial modeling of fractions indicated that he was operating at a beginning level of thinking about fractions.

The following examples demonstrate Ethan’s difficulty with the concept of the whole. These models suggest that he understood the whole must be the same size when...
comparing fractions, however, the unequal partitions prevented him from determining which fraction was the least. As shown in Figure 57, Ethan was able to determine though that \( \frac{4}{6} \) was not equal to \( \frac{5}{6} \).

![Figure 57. Equal size wholes.]

When presented with another comparison problem: “\( \frac{1}{5} \) is greater than \( \frac{1}{3} \) because 5 is greater than 3. True or False?”, Ethan wrote: “False, because they are both wholes” (see Figure 58).

![Figure 58. Representing \( \frac{1}{5} \) and \( \frac{1}{3} \).]

The first model is his representation of \( \frac{1}{5} \) and the second model is his representation of \( \frac{1}{3} \).

Ethan’s misinterpretation of the whole and difficulty in equally partitioning the circles resulted in incorrect answers.

**T:** What do you mean they are both wholes?

**S:** Like they both fill up the circle. So they are like equal.
Given the problem, “John drank $\frac{1}{3}$ of a liter of juice and Suzie drank $\frac{1}{2}$ of a liter juice, who drank more juice?” Ethan described the whole in the problem as “like these two (the $\frac{1}{2}$ liter and the $\frac{1}{3}$ liter) combined together.” A final example of Ethan’s ideas about the whole illustrated that his thinking may be influenced by the context of the situation. The task asked Ethan to evaluate the following situation:

Mary and Jose both spent $\frac{1}{4}$ of their money. Do you think they spent the same amount of money or a different amount of money?

Ethan first replied that Mary and Jose spent the same amount of money because “$\frac{1}{4}$ is the same number.”

T: So they both spent a fourth and you said that is the same number. But, if Mary had $20 and she spent a fourth and Jose had $10 and spent a fourth, did they spend the same amount of money?

S: You say they had $10?
T: Jose had $10, but Mary had $20.
S: No they didn't spend the same amount.
T: Why not?
S: Cause 20 and 10 are two different numbers.
T: So, a fourth could be different?
S: Yeah, depending on what you are talking about.

After seeing a connection between this idea and a contextual situation, Ethan could understand circumstances where one-fourth would not be equal to one-fourth.

**Modeling**

The first task in Ethan’s interview asked him to determine if $\frac{2}{3}$ was equal to $\frac{4}{6}$. He incorrectly applied a “cross-multiply” procedure to test for equivalence. When asked to draw a picture of $\frac{2}{3}$, he drew discrete models of $\frac{2}{3}$ and $\frac{4}{6}$ as shown in Figure 59. He maintained his answer that the fractions were not equal.
As Ethan moved to the next task (comparing two fractions with unlike numerators and like denominators), he was prompted to draw an area model to represent the fractions instead of drawing a discrete model. Ethan’s models of $\frac{4}{6}$ and $\frac{5}{6}$ are shown in Figure 57 above.

T: Instead of drawing circles like that (discrete circles) have you ever drawn a big circle or a rectangle to show $\frac{4}{6}$?

S: Oh yeah.

T: Try doing that.

S: (draws a circle and immediately partitions it into 4 pieces and then inserts two lines to make 6 partitions). Then shade 4 and leave 6 not shaded.

T: That means $\frac{4}{6}$ of that circle is shaded. What about the $\frac{5}{6}$ then?

S: Oh, I think I did this one wrong too (points to the first problem). Cause I remember, you have like a half, then you like shade that in, shade that half in and say you have $\frac{1}{8}$, then if you draw that and that side is equal to the other side, then they are equal.

Throughout the interview, Ethan consistently followed the same procedure to model a fraction. First he drew a circle, then he drew a horizontal line and a vertical line through the center partitioning the circle into fourths, and finally he added lines within each partition as needed to show the specified fraction. For example, Figure 60 shows Ethan’s model of $\frac{2}{5}$; a circle partitioned into fourths and a diagonal line in the upper right section, which partitioned that section into two pieces, thus creating five sections in the circle.
A similar process was observed when Ethan was asked to model \( \frac{2}{5} \) on a number line. He first correctly drew \( \frac{1}{2} \) on the number line and then incorrectly modeled \( \frac{2}{5} \) (see Figure 61).

Ethan’s persistence in drawing circular area models and partitioning them into fourths seemed to negatively affect his understanding of the size of fractions. To further explore this issue, the interviewer provided Ethan with three same-sized rectangles that were correctly partitioned to represent three different fractions. The rectangles were not shaded. Ethan correctly shaded each rectangle to represent the appropriate fraction and then used these models to correctly order the fractions (see Figure 62).
Comparing and Ordering

The first two problems of the interview involved comparing equivalent fractions and fractions with unlike numerators and like denominators. Ethan solved each of these problems by applying a procedure often taught to students to compare fractions. This procedure involves cross-multiplying the numerators and denominators from left to right and selecting the smallest product as denoting the smallest fraction i.e. to compare \( \frac{a}{b} \) and \( \frac{c}{d} \), first multiply \( a \) and \( d \) resulting in the product \( ad \) and then multiply \( b \) and \( c \) resulting in the product \( bc \). If \( ad \) is less than \( bc \), then \( \frac{a}{b} < \frac{c}{d} \). This procedure requires students to remember to multiply, to multiply from left to right and not from right to left, and to remember that the fraction next to the smallest product is the smallest fraction. Ethan first attempted to determine the equality of \( \frac{2}{3} \) and \( \frac{4}{6} \) by applying this procedure (note the erased diagonal lines between the fractions).
Figure 63. Comparing $\frac{2}{3}$ and $\frac{4}{6}$ with a discrete model.

S: (long pause, draws a line connecting 4 and 3 diagonal) 4 and 3 equals 6 and 6 and 2 equal 8, so they are not equal?
T: So 4 and 3 equals 6. Do you mean when you add them, when you multiply them?
S: When you add them.
T: So 4 + 3 = 6?
S: Uh huh and then 6 + 2 is 8. So they are not equal.
T: Could you draw me a picture of $\frac{2}{3}$?
S: (draws 2 circles and then 3 circles below the circles to represent $\frac{2}{3}$ and draws 4 circles and then 6 circles below the four circles to represent $\frac{4}{6}$)

Ethan determined that $\frac{2}{3}$ and $\frac{4}{6}$ were not equal based on his incorrect application of the cross-multiplication procedure. He added rather than multiplied the numbers on the diagonal of each fraction, did not correctly add two of the numbers, and did not move from left to right when multiplying (adding in this case). His subsequent discrete models of $\frac{2}{3}$ and $\frac{4}{6}$ also did not help him decide if the fractions were equivalent (see Figure 63).

Later in the interview, Ethan again used this cross-multiplication strategy incorrectly to compare $\frac{4}{6}$ and $\frac{5}{6}$. Finally, he was asked to compare $\frac{1}{2}$ and $\frac{1}{3}$. His response follows:

T: I want to know who drinks more, John who drinks $\frac{1}{2}$ of a liter or Suzie who drinks $\frac{1}{3}$ of a liter?
S: John.
T: Why?
S: Because um, 3 is larger than 2.
T: So is $\frac{1}{3}$ larger than $\frac{1}{2}$?
S: (draws circles) No.
T: Which one is larger?
S: $\frac{1}{2}$
T: So who drank more?
S: Suzie.

In this case, when Ethan was asked to draw a model of each fraction, he was able to successfully determine that $\frac{1}{2}$ was greater than $\frac{1}{3}$.

At this point, Ethan had experienced many examples of comparing and ordering fractions. When asked to order a group of fractions with like numerators and unlike denominators, Ethan quickly placed the fractions from least to greatest (see Figure 64). He used his understanding of the size of the partitions based on the number in the denominator.

![Figure 64. Ordering fractions with like numerators and unlike denominators.](image)

Ethan was asked if his answers would be different if the numerators were not all the same number. He remarked that you don’t have to look at the numerator, “...it only matters about the denominator.” Ethan was viewing a fraction as being composed of two distinct numbers and not as a single numerical representation.

**Estimation and Benchmark**

When asked to compare $\frac{4}{6}$ and $\frac{5}{6}$, Ethan tried a variety of strategies (as noted in previous sections), one of which was considering benchmark fractions. He stated, “$\frac{4}{6}$ is...
bigger than the other, because it is closer to 0, I think.” The use of benchmarks however, did not help him to correctly compare the fractions.

Three of the problems in the interview involved estimating the solution to the addition or subtraction of fractions and locating the estimate on a number line. The number line was partitioned into halves and 0, 1, and 2 were labeled. The first problem Ethan was asked to estimate was: \( \frac{3}{4} + \frac{1}{5} \).

T: What do you think the answer will be? Will it be close to \( \frac{1}{2} \), close to 1?
S: It would be closer to 2.
T: Why do you say closer to 2?
S: Um, I don't really know. Cause, I know what the answer is, but I don't...
T: OK, what is the answer?
S: Uh, this up here equals 4 and down here equals 9. So \( \frac{4}{9} \).
T: Ok, so where would that be on the number line?
S: Right here, or like over here cause down there is not 10. Hold on, or here (marks after 2).

Ethan did not seem to have a conception of fraction size to use when estimating. He also operated on fractions by first operating on the numerators and then operating on the denominators. Moreover, he did not correctly locate his estimate on the number line. Due to Ethan’s frustration with these problems, the interview continued with questions on other fraction topics described earlier.

**Summary of Ethan’s Understanding of Fractions**

As a sixth-grade student enrolled in both a general mathematics class and a Tier II RtI mathematics class, Ethan struggled with many concepts in the fraction domain. He correctly partitioned circle area models to represent a fraction, but did not draw the partitions equally. Ethan held on to particular ideas when modeling, such as partitioning a circle into four parts without considering the fraction being modeled and marking the halfway point on a number line before thinking about how to partition the number line.
When comparing fractions, he often incorrectly determined which fraction was closest to 1 to make his decision. Ethan was not successful in finding the equivalent mixed number for an improper fraction and vice versa.

During the interview, Ethan called upon various strategies to use when thinking about fractions. When comparing and ordering fractions, he explained his answers by: (a) drawing discrete models, (b) drawing continuous models, (c) applying incorrect procedures, and (d) utilizing whole number concepts. In some cases, he provided an answer and simply repeated the answer when asked to explain (see Figure 65).

Figure 65. Explaining by repeating the answer.

In the interview, Ethan mentioned ideas about fractions that were emphasized in the fraction module, but he often misunderstood or incorrectly applied these ideas. For example, he tended to use discrete models to compare fractions, but when reminded about the emphasis in the module on using a circle or rectangle to model fractions, he began using an area model throughout the remaining problems in the interview. However, the area models were often drawn incorrectly.

The following transcript revealed that Ethan was fixated on using numbers such as 10 and 20 to compare fractions and determine fraction locations on a number line. It was not evident why these numbers were important to him.
T: If you didn't have the pictures, do you see a way that you could decide which fraction is greater?

S: Yeah, because like you can use like a number trail and see which one is closer to 10?

T: Why 10?

S: Cause 10 is bigger than 0.

T: How do you know \( \frac{11}{12} \) is bigger than \( \frac{4}{5} \)? You said because if you have \( \frac{11}{12} \), that is bigger than \( \frac{4}{5} \). How do you know, what tells you that?

S: Um, I would say a number line like I said last time. Uh, \( \frac{4}{5} \) is closer to 0 than it is to 20 or a higher number.

S: Uh, this up here equals 4 and down here equals 9. So \( \frac{4}{9} \).

T: Ok, so where would that be on the number line?

S: Right here, or like over here cause down there is not 10.

Rather than rely on conceptual ideas about fractions, Ethan attempted to retrieve from his memory, knowledge about fractions that he believed would help him solve fraction problems.
MAGGIE

Maggie was a seventh-grade student enrolled in a general mathematics course and a Tier II RtI mathematics class. During the prior school year, she was enrolled in a sixth-grade general mathematics course as well as a Tier II RtI mathematics class.

Pre-Assessment

Results from the pre-assessment indicated that Maggie successfully solved fraction problems such as comparing fractions with unlike numerators and like denominators and finding equivalent fractions when provided with a symbolic representation and a model. Maggie correctly answered nine questions out of 13 questions on the pre-assessment (see Appendices H and I).

Maggie chose the correct answer for problems presented in a list or in a word problem, which involved comparing fractions with unlike numerators and like denominators. She also successfully answered questions requiring her to compare two fractions with unlike numerators and unlike denominators, as well as selecting the mixed number that was closest to 5. However, she was not successful in selecting a fraction less than a given fraction, comparing unit fractions, or ordering a list of fractions with unlike numerators and unlike denominators.

Maggie identified equivalent fractions (with denominators of four, eight, and sixteen) represented symbolically and with an area model or number line. However, she incorrectly answered a question involving equivalent fractions with denominators of five and ten and with a number line partitioned into fifths and tenths.

A review of Maggie’s incorrect answer choices revealed that when ordering a list of fractions from least to greatest, she ordered the fractions based on their denominator...
(apparently without considering the numerator). This could be an indication of whole number bias. Finally, when asked to select the fraction that was less than $\frac{5}{6}$ from a list $\left(\frac{7}{6}, \frac{8}{12}, \frac{7}{8}\right)$, Maggie selected $\frac{7}{6}$.

**Intervention Curriculum**

**Unit Fraction**

One of the beginning tasks in the intervention curriculum module asked students to create the whole given a unit fraction. Maggie successfully accomplished this task as shown in Figure 66.

![Figure 66. Given a unit fraction, draw the whole.](image)

In a later lesson, students were given a specific colored Cuisenaire rod that represented a unit fraction and another rod that represented the whole and were asked to determine the numeric unit fraction. In each task, Maggie successfully determined the unit fraction. Another task involved finding the non-unit fraction when given the unit fraction and number of iterations. Maggie correctly answered each of these questions. As Maggie advanced through the lessons, she was asked to determine the unit fraction of an improper fraction, as well as a mixed number. She did not have any difficulty in finding the unit fraction of the improper fraction as well as the number of iterations, but was not successful in determining the unit fraction and number of iterations of a mixed number. She could however, convert the mixed number to an improper fraction. A final example
of Maggie’s work with unit fractions occurred when she was asked what the unit fraction of the fractions in the problem $\frac{4}{3} + \frac{13}{3}$ was. She wrote, “$\frac{1}{6}$.” This indicated that the notion of a unit fraction was not yet ingrained in her mind.

**Whole**

As shown in Figure 67, Maggie’s preconceived notion about the size of a pizza and how it is sliced, influenced her answer when thinking about $\frac{1}{2}$ and the whole. She assumed that a pizza was always sliced into eight pieces.

![Figure 67. Half of a pizza and the whole.](image)

In another situation, when asked what the whole was in two different word problems, Maggie correctly answered, “1 whole mile” and “1 whole cup of water”, which indicated her understanding of the whole in these two contextual situations.

**Modeling**

When asked to model the fraction $\frac{1}{8}$ in three different ways using three provided rectangles, Maggie equally partitioned and shaded $\frac{1}{8}$ of two of the rectangles, but struggled with a third (middle) representation as shown in Figure 68.

![Figure 68. Representing one-eighth in three different ways.](image)
When modeling an improper fraction \( \frac{8}{5} \), Maggie partitioned her model (a circle) into 8 pieces (the numerator) and then shaded 5 pieces, resulting in a representation of \( \frac{5}{8} \) (see Figure 69). This suggested a misunderstanding of the size of an improper fraction, the meaning of an improper fraction, and how to correctly model an improper fraction.

![Figure 69. Modeling an improper fraction.](image)

Similar results were observed in Maggie’s recorded work when she modeled fractions using a number line. When asked to model \( \frac{3}{4} \) on a number line and then indicate \( \frac{3}{5} \), Maggie first represented the whole (marking 1 on the number line) and then modeled \( \frac{3}{4} \) correctly. Although her positioning of \( \frac{3}{5} \) was approximately correct, it is not clear from the recorded work how she determined the location of \( \frac{3}{5} \) (see Figure 70).

![Figure 70. Modeling \( \frac{3}{4} \) and \( \frac{3}{5} \).](image)

On the other hand, when modeling the improper fractions \( \frac{11}{8} \) and \( \frac{7}{4} \), Maggie first changed them to mixed numbers, erased her work, and then incorrectly modeled them on two number lines, showing \( \frac{11}{8} \) as being greater than \( \frac{7}{4} \). She did, however, recognize that
both fractions would be located after the tic mark indicating 1, but was unsuccessful in modeling the fractions accurately.

**Comparing and Ordering**

In much of Maggie’s written work, she determined the correct relationship between two fractions, but did not provide an explanation and therefore, it was not possible to speculate on her reasoning process. When she drew a model to explain her answer to comparison problems involving improper fractions, she sometimes answered correctly and sometimes did not. For example, when comparing $\frac{7}{4}$ and $\frac{8}{5}$, she correctly answered that $\frac{8}{5}$ was the smaller quantity, but she incorrectly modeled $\frac{8}{5}$ and was unable to model $\frac{7}{4}$ due to what seemed to be difficulty in partitioning a circle into sevenths (see Figure 71).

![Figure 71. Comparing two improper fractions.](image)

When ordering a set of fractions, Maggie either provided no explanation for her solutions or indicated that she performed a procedure, such as converting improper fractions to mixed numbers. For example, Maggie was given the contextual problem, “Cara, John, and Ben are working on their science project. Cara has worked for $\frac{10}{9}$ of an hour, John has worked for $\frac{5}{3}$ of an hour, and Ben has worked for $\frac{9}{9}$ of an hour. Put these times in order from least to greatest.” She converted each improper fraction to its...
equivalent mixed number and then correctly placed the fractions in order. Maggie demonstrated success when ordering a list of fractions with different numerators and the same denominator, but was not successful in ordering improper fractions with like numerators and unlike denominators and fractions with unlike numerators and unlike denominators. For example, when asked to order \( \frac{3}{8}, \frac{1}{3}, \frac{3}{5}, \frac{6}{7} \) from least to greatest, Maggie answered \( \frac{3}{8}, \frac{6}{7}, \frac{3}{5}, \frac{1}{3} \). She did not provide an explanation, however, the denominators were in descending order, which may indicate that she was using the notion of the larger denominator indicating smaller sections.

**Estimation and Benchmark**

Maggie did not reference the idea of benchmarks in any of her written explanations. As shown in Figure 72, when explicitly asked to use benchmark fractions to estimate the solution of a contextual problems, she simply solved the problem.

![Figure 72. Benchmarks and estimation.](image)

**Post-Assessment**

On the post-assessment for the fraction intervention curriculum module, Maggie correctly answered 11 of 13 questions (see Appendices H and J). Maggie was successful in determining the correct answer for problems involving the comparison of fractions with unlike numerators and like denominators, selecting the fraction from a list that was less than a given fraction, comparing unit fractions, and ordering fractions with like
numerators and unlike denominators. She did not correctly answer a problem where she was to determine the relationship between $\frac{5}{7}$ and $\frac{2}{3}$, indicating that $\frac{5}{7}$ was less than $\frac{2}{3}$. In addition, she selected $4\frac{3}{4}$ as the mixed number that was closest to 5, instead of $4\frac{7}{8}$. Maggie successfully answered questions on the post-assessment that required her to determine equivalent fractions when given the symbolic representation and a model (rectangular area model and number line).

**Comparison of Pre-Assessment Results to Post-Assessment Results**

Changes in Maggie’s responses from the pre-assessment to the post-assessment indicated some positive growth in understanding, particularly related to comparing and ordering fractions with like numerators and unlike denominators, finding the equivalent fractions when the denominators are five and ten and a partitioned number line is provided, and comparing unit fractions. There was no change in Maggie’s answers on problems where she found equivalent fractions of fractions with denominators of four and sixteen as well as four and eight and comparing fractions with like denominators.

As revealed by Maggie’s responses to questions on the pre- and post-assessment, she was still having difficulties comparing fractions with unlike numerators and unlike denominators. On the other hand, before Maggie began the fraction intervention curriculum, she ordered a list of fractions based on the number in the denominator. When asked the same question at the end of the module, she correctly placed the fractions in order from least to greatest. Additionally, she seemed to understand that when comparing unit fractions, the fraction with the smaller denominator has larger parts, while the fraction with the larger denominator has smaller parts and is therefore the smaller fraction.
Interview – One Year Later

Maggie attempted to solve many of the tasks included in the interview by applying multiple strategies to each individual task. She would quickly switch from one strategy to another when she began to experience uncertainty. For example, one task required her to model a fraction and then utilize the model to compare that fraction to other fractions. On this task, Maggie modeled the fraction on an area model, struggled with modeling the fraction on a number line and ultimately gave up, stated that she was confusing herself, tried to recall the procedure for finding equivalent fractions, and drew another area model. Consequently, she was not able to complete this task. This phenomenon was common across the interview tasks.

Unit Fraction

While solving one of the early interview tasks, Maggie was asked if she recalled what “unit fraction” meant. She replied that she remembered hearing it, however, she did not know what it meant. The interviewer pointed to one section of the circular area model Maggie had drawn to represent $\frac{2}{5}$ and asked what fraction of the circle that piece represented. She replied $\frac{1}{5}$, but did not connect that to the term “unit fraction.” Moreover, later in the interview when she was specifically asked what the unit fraction of $\frac{4}{6}$ was, she could not name it and said, “Well, 6 is the whole and then four out of 6.”

Whole

Maggie attempted to model $\frac{2}{5}$ on a number line and was asked about the whole. She responded that the whole was, “the 1.” While Maggie did not explicitly reference the whole in any of the remaining problems, when she was asked if models need to be the same size when comparing fractions, she replied, “yeah, probably so.” This implies that
she might not be confident of this notion. However, when completing tasks involving the comparison of fractions, Maggie drew area models that were the same size. For example, when comparing \( \frac{4}{6} \) and \( \frac{5}{6} \), as well as comparing \( \frac{3}{4} \) and \( \frac{9}{12} \), she drew same size models (see Figure 73 and Figure 74).

![Figure 73. Modeling \( \frac{4}{6} \) and \( \frac{5}{6} \).](image)

![Figure 74. Modeling \( \frac{3}{4} \) and \( \frac{9}{12} \).](image)

**Modeling**

The first task in the interview consisted of drawing a model of the fraction \( \frac{2}{5} \). As shown in Figure 75, Maggie represented \( \frac{2}{5} \) using a circular area model that was equally partitioned and appropriately shaded.

![Figure 75. Model of two-fifths.](image)

Figures 73 and 74 above provide additional examples of Maggie’s ability to model fractions using an area model. However, Maggie experienced difficulty when asked to
model a fraction on a number line and also to model an improper fraction. As shown in Figure 76, when asked to model $6\frac{1}{3}$, Maggie drew six circles partitioned into three parts, but was unsure of how she should shade the partitions. Her first attempt was to shade one partition in each circle, but then she thought that three whole circles should be shaded.

![Figure 76. Modeling $6\frac{1}{3}$.

**Comparing and Ordering**

After drawing a model of $\frac{2}{5}$, Maggie was asked to compare $\frac{2}{5}$ to three different fractions and was encouraged to look at her model when determining the relationship between the fractions. For instance, the first problem was a comparison between $\frac{2}{5}$ and $\frac{1}{2}$.

T: Is $\frac{2}{5}$ greater than $\frac{1}{2}$? Tell me what you are thinking as you try to decide on your answer.

LONG PAUSE

S: This $(\frac{2}{5})$ is greater because 2 wait, hmmm.

T: Is there anything on this paper that could help you figure it out?

S: This (points to model of $\frac{2}{5}$ she drew).

S: (long pause). This one $(\frac{2}{5})$ is greater.

T: Why do you say that?

S: Because if I draw like what I did right there (the $\frac{2}{5}$ model) and drew a pie and split it into two and I shade one in…

S: Could they be equal? Wait, no, wait. I'm thinking this one $(\frac{2}{5})$ is bigger.

T: Why do you think that?

S: Because a half is bigger than that $(\frac{2}{5}$ model) but this (shading of $\frac{2}{5}$) looks bigger. I keep thinking it's this (points to $\frac{2}{5}$), but I have a feeling it's this (points to $\frac{1}{2}$).

T: If you look back up at this circle (the $\frac{2}{5}$ circle).
S: This one is bigger *(points to her $\frac{1}{2}$ model)*. Cause if I split this one in half *(points to the $\frac{2}{5}$ model), there is only two and a half.*

Maggie first partitioned a circle into two parts and shaded one part to model $\frac{1}{2}$. She then added the two diagonal lines on the left side as she was thinking about fifths (see Figure 77).

*Figure 77. Maggie's model of one-half.*

Consequently, she created three-fifths on the left side of the circle, which resulted in the right side being composed of two-fifths. This model caused her to think that $\frac{2}{5}$ might be equal to $\frac{1}{2}$.

Maggie finally concluded that $\frac{1}{2}$ was greater than $\frac{2}{5}$. She looked back at her model of $\frac{2}{5}$ and drew a line partitioning the model in half (see Figure 78). She observed that the two one-fifth shaded sections would require an additional half of a one-fifth section in order to be the same size as $\frac{1}{2}$. Therefore, she determined that $\frac{1}{2}$ must be larger than $\frac{2}{5}$.

*Figure 78. Maggie's model of two-fifths.*
The difficulties demonstrated by Maggie when attempting to equally partition area models created confusion and doubt in her mind as she solved fraction tasks.

When presented with comparing $\frac{2}{3}$ and $\frac{4}{10}$, Maggie tried to apply a procedure that she had learned:

T: What are you doing here?
S: I'm setting them up next to each other.
T: Why do you set them next to each other?
S: I remember when I was in fifth grade, we set them up like this, but it was back a long time ago, like two years and I'm trying to remember. I remember setting them up like this (she is setting up a proportion) and you divide this, no I don't think...

Due to the fact that she could not remember the procedure, she returned to drawing a model. Maggie drew two equal sized rectangles and partitioned one of the rectangles into tenths and the other into fifths and appropriately shaded each rectangle. She described how two of the one-tenth sections could fit in one one-fifth section and then related this to the procedure of setting up a proportion.

S: Wait a second, if you multiply this by 2 and multiply this (pause), well, that equals, oh, multiply that by 2, so they are (equal).

A final example of Maggie’s thinking while comparing fractions occurred as she compared $\frac{4}{6}$ and $\frac{5}{6}$. She recognized that the fractions were not equal, but struggled with determining which fraction was greater. Maggie drew a representation of $\frac{4}{6}$ and $\frac{5}{6}$ as shown in Figure 79. While her models correctly represented each fraction, the fact that $\frac{4}{6}$ had two unshaded sections and $\frac{5}{6}$ had only one unshaded section created a dilemma for Maggie. She thought that $\frac{4}{6}$ might be greater than $\frac{5}{6}$ because the number “2” (from two unshaded sections) is greater than the number “1” (from one unshaded section), but yet
she was unsure of this answer. When asked which fraction covered the most area, Maggie immediately indicated that \( \frac{5}{6} \) covered the most area and was therefore larger than \( \frac{4}{6} \). By encouraging her to think about the area that the fractions represented, Maggie was able to successfully compare the two fractions.

![Figure 79. Model of \( \frac{4}{6} \) and \( \frac{5}{6} \).](image)

In a later task, Maggie correctly compared \( \frac{3}{4} \) and \( \frac{1}{5} \) using residual thinking.

\[ S: \text{But if you think about it, this number is bigger. This is one step away and this is four steps away. So this is bigger } \left( \frac{3}{4} \right). \]

\[ T: \text{Four steps away from what?} \]

\[ S: \text{So like from 1 to 5 is 4 and from 3 to 4 is 1.} \]

In this situation, she understood that because 3 was only “one step away” from 4, while 1 was “four steps away” from 5, \( \frac{3}{4} \) would be greater than \( \frac{1}{5} \).

**Estimation and Benchmark**

Maggie was given a task that involved estimating the solution of an addition of fractions problem and two subtraction of fractions problems and locate the estimate on a number line. When beginning to work on these problems, she asked if she could solve it in her head. She was encouraged to estimate the answer, rather than finding an exact solution. Maggie then remarked, “I can’t really add them cause you have to have common denominators.”
When asked to show where the sum of $\frac{3}{4} + \frac{1}{5}$ would be found on the number line, Maggie correctly located $\frac{3}{4}$ on the number line and then indicated that the sum would be “a little beyond 1.” In order to complete the two subtraction of fraction estimation problems, Maggie began by marking where each fraction was located on the number line. She then marked the point halfway between the two fractions as the difference instead of considering the length of the fraction being subtracted (see Figure 80).

![Figure 80. Estimating the difference of two fractions.](image)

**Summary of Maggie’s Understanding of Fractions**

As a sixth-grade student enrolled in both a general sixth-grade mathematics course and a Tier II RtI mathematics course, Maggie demonstrated an understanding of the concept of a unit fraction and represented the whole when given a unit fraction. She could also determine a non-unit fraction when given a unit fraction and number of iterations. However, she could not determine the unit fraction and number of iterations of a mixed number. Maggie successfully compared fractions with like numerators and unlike denominators and ordered improper fractions with unlike numerators and like denominators. She struggled however, when comparing and ordering fractions (including improper fractions) with unlike numerators and unlike denominators. Maggie
appropriately modeled fractions using area models, but incorrectly modeled improper fractions.

One year later, Maggie retained the concept of a unit fraction, but not the terminology. It was not evident however, if she utilized the concept to solve any problems during the interview. Maggie’s fraction models were usually equally partitioned and correctly shaded. However, the lack of precisely equal partitions sometimes made the models difficult for her to utilize or interpret. Modeling a fraction, improper fraction, or mixed number on a number line continued to be something Maggie could not consistently do accurately or efficiently. When she drew models to compare fractions, Maggie understood that the models (the whole) must be the same size. Through her models, she also demonstrated an understanding of equivalent fractions. Maggie was often able to utilize her drawn models to reflect on why a rule or procedure worked.

Maggie fluctuated between utilizing learned procedures and conceptual thinking. For example, when comparing fractions, she would first unsuccessfully attempt to utilize a proportion and then she would draw a model. When estimating the solution of $\frac{3}{4} + \frac{1}{5}$, Maggie wanted to add the two fractions, but stated that she could not add them because they did not have a common denominator. When asked why the fractions needed to have a common denominator, she replied, “To add them.” When pressed, she responded, “No, that’s to multiply them I think.” That is, she seemed more intent on finding the appropriate rule than relying on her intuitive notions about fractions.

Maggie often expressed the thought that she was confusing herself. From her remarks during the interview, this confusion seems to be a result of Maggie not having a
deep-seated conceptual understanding of fractions and of her reliance on learned procedures.

**Summary of Student Understanding of Fractions**

The case studies provided an opportunity to explore student development within five domains related to the understanding of fractions over a period of time. The data were triangulated through an analysis that included student responses to a pre-assessment (prior to study using a module developed for a Tier II RtI mathematics classroom), written student work during the implementation of the module, student responses to a post-assessment (at the end of the module), and student responses during a one-on-one interview a year later. A summary of findings viewed across the five cases is provided in the following sections, organized by domain.

**Unit Fraction**

The written student work provided during the implementation of the intervention module indicated that the five case study students understood the concept of a unit fraction, but did not use unit fractions when they solved problems involving fractions. For example, one participant determined the unit fraction and number of iterations of the unit fraction for each fraction in a set when asked to order them. However, the student did not utilize the fact that e.g., three \( \frac{1}{8} \) units would be less than six \( \frac{1}{7} \) units and three \( \frac{1}{5} \) units to correctly order the fractions. When given a unit fraction as represented by a length model or a Cuisenaire rod, all the case study students successfully determined the whole. They were also proficient in establishing the non-unit fraction when provided with the unit fraction and a number of iterations of the unit fraction. Determining the unit fraction of an improper fraction as well as a mixed number was more difficult for these students.
The individual interviews revealed that one year after the module was implemented, most of the students did not recall what the term “unit fraction” meant. Three of the five participants attempted to explain a unit fraction with a rule they had memorized. For example, when asked what the unit fraction of $\frac{3}{8}$ was, one student responded, “Don’t you divide the 3 by 8?” However, after some reflection, the students were able to recall some notions about the concept of a unit fraction. One student recalled the term and referred to it as the “base fraction.” As was the case during the intervention curriculum unit, students struggled with determining the unit fraction and number of iterations of improper fractions and mixed numbers. When asked to create a whole given a unit fraction that was composed of more than one object (e.g., one-fifth is represented by 3 cubes), only one student experienced difficulty with this task.

In general, the case study students did not recall the term “unit fraction”, but after some reflection, they were able to correctly identify the unit fraction of proper fractions. However, identifying the unit fraction of improper fractions and mixed numbers continued to present difficulties for these students. The case study students retained the understanding of creating the whole when given a unit fraction.

**The Whole**

An issue that presented itself early in the intervention curriculum was the use of a pizza as a contextual model for fractions. When a pizza was used to model a whole and students were asked to decide if $\frac{1}{2}$ of a pizza could be represented by four slices and two slices, the commonly held student conception that a pizza always has eight slices emerged and was an obstacle in their thinking. The fraction module revealed that these
students were able to determine the whole when it was presented in a contextual problem. They also understood that when comparing fractions, the wholes had to be the same size.

A year later, students continued to struggle with the idea that the same fraction could represent something different dependent on the size of the whole. However, all of the case study students demonstrated understanding that when comparing fractions or determining equivalency, the wholes must be the same size. During the interviews, an area of difficulty became apparent. The term, “the whole” had several meanings for the students. For some, it was associated with the denominator of a fraction. For others it represented the sum or product of the denominators when comparing two fractions. The number 1 was also referred to as the whole and consequently students believed that it was the largest fraction in a set of fractions.

**Modeling Fractions**

When students created models to represent and solve problems during the implementation of the intervention module, the dominant model was a circle. This was not unexpected as research shows that a circle is the prevailing model selected by students to represent a fraction (Peck & Jencks, 1981; Cramer et al., 2002). While one of the students was successful in equally partitioning a circle into different equal-sized pieces (e.g., fifths, sevenths, and tenths), the other case study students struggled with modeling fractions with circles other than those represented by halves, fourths, or eighths. They knew that when modeling fractions, the sections should be equal, but struggled with drawing them equally, which in turn caused uncertainty about the importance of equal-sized partitions and difficulty in using the models to compare or operate with fractions. However, they did equally partition area models represented by
rectangles and correctly shaded them to represent a fraction. The use of a number line as a model was more problematic. Four students were successful in modeling improper fractions on a number line that was equally partitioned and labeled with 0, 1, and 2. However, they had difficulty when asked to partition an empty number line and represent a particular fraction.

One year later, the consistent use of discrete models to represent fractions by two students was noted. As with the models in the intervention module, the case study students relied mainly on circles when modeling fractions. When prompted to do so, they used a rectangle and were more successful equally partitioning and shading this area model. All the case study students expressed concern about their ability to draw equal partitions and when asked if the partitions should be equal, the response was that they thought the partitions should be equal, but they weren’t sure. Modeling fractions on a number line continued to be problematic for these students. For example, two students indicated $\frac{1}{2}$ on the number line and begin drawing tic marks without considering the fraction which they were modeling.

**Comparing and Ordering Fractions**

During the intervention curriculum, the case study students utilized the notion that the larger the denominator, the smaller the partitions would be, and the smaller the denominator, the larger the partitions would be. This concept allowed them to successfully compare and order unit fractions, fractions with like numerators and unlike denominators, and fractions with unlike numerators and like denominators. However, students did not incorporate the numerator in their decisions and therefore struggled with comparing and ordering fractions with unlike numerators and unlike denominators.
Benchmark fractions, particularly $0, \frac{1}{2},$ and $1,$ were successfully used by two students to compare and order fractions. For example, the fraction $\frac{3}{5}$ was determined to be the greatest fraction in the set $\frac{3}{10}, \frac{3}{5}, \frac{1}{2}$ because it was “closest to 1.” In most cases, a learned procedure (e.g., finding equivalent fractions with common denominators) was used to order a set of fractions, even though common benchmarks would have been a more efficient and obvious strategy. When comparing and ordering improper fractions, the majority of students converted the improper fractions to mixed numbers and then attempted to find equivalent fractions in order to compare them.

Overall, the case study students demonstrated a variety of strategies when asked to compare and order fractions during the individual interviews. For simple fractions (unit fractions), students simply compared denominators. As the fractions became more complex, two students attempted to set up a proportion or cross-multiply (often incorrectly). In most cases, students used models, benchmark fractions, and residual thinking to compare and order fractions. For instance, when comparing $\frac{4}{5}$ and $\frac{11}{12},$ one student thought about the fact that $\frac{1}{5}$ and $\frac{1}{12}$ remained, respectively, and $\frac{1}{5}$ would have larger sections than $\frac{1}{12}.$ These strategies were represented in the written work during the intervention module and were also observed one year later in the interviews.

**Estimation and Benchmark Fractions**

When solving problems during the intervention module, students occasionally used benchmark fractions and estimation to solve problems. For example, when ordering the set of fractions $\frac{3}{10}, \frac{3}{5}, \frac{1}{2},$ all of the case study students saw $\frac{1}{2}$ as the “middle fraction” and determined that $\frac{3}{10}$ was closer to 0 and $\frac{3}{5}$ was closer to 1, correctly ordering the set of
fractions. One problem explicitly asked students to estimate the sum of two fractions with unlike denominators. Three students correctly estimated the sum, while two students added the denominators and then incorrectly estimated the sum.

One year later, students continued to struggle estimating with fractions. When asked to estimate sums or differences and to locate their estimate on a number line, two students immediately attempted to determine the exact sums/differences and resisted encouragement to estimate. Another problem asked students to estimate the difference of $\frac{3}{5} - \frac{3}{8}$. One student estimated $\frac{3}{5}$ as $\frac{2}{2}$ and $\frac{3}{8}$ as $\frac{1}{2}$ resulting in an estimate of $\frac{1}{2}$, which she correctly located on the number line. Another student estimated the location of each fraction $\frac{3}{5}$ and $\frac{3}{8}$ on the number line and then determined that the difference was the midpoint between these two fractions. While this student could locate the approximate location of the fractions on the number line, she was unable to estimate the difference of the two fractions.

**Conclusion**

According to Siegler et al., (2012), fraction knowledge is a strong predictor of success in algebra and overall mathematics achievement. Results from their study indicated that knowledge of fractions contributes more to mathematics achievement than whole-number addition, subtraction, and multiplication. Consequently, it is important that curriculum for middle school students who struggle with fractions focuses on strategies that will help them develop a strong conceptual understanding of fractions before they enter high school. Research has shown that fraction domains such as the whole, unit fractions, modeling fractions, comparing and ordering fractions, and benchmark and estimation are essential for students to understand in order to be successful when
operating on fractions (Behr et al., 1984; Bezuk & Cramer, 1989; Cramer et al., 2009; Fuchs et al., 2013; Lamon, 2007). RtI classes are slowly being established in the middle school system to provide students with additional support in these crucial areas. Intervention curriculum that provides intensive fraction instruction that promotes students’ conceptual understanding and guidance to the teacher with this mode of instruction is emerging. This study was one means to evaluate the impact of the intervention curriculum on student learning.

Findings from this investigation are based on examination of the work of five students prior to, during, and one year following their experience with the intervention curriculum. The data indicated that all five middle school case study students experienced difficulty with integrating conceptual fraction knowledge with procedural knowledge (skills they have practiced and apply unevenly). However, the study demonstrated that when given instruction focused on sense making, the students retained and used some fraction concepts, but disregarded other concepts, one year after instruction. Table 4 provides details of these concepts.

Table 4

<table>
<thead>
<tr>
<th>Fraction Ideas Noted During Intervention Curriculum and One-on-one Interviews</th>
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<tbody>
<tr>
<td>Concept</td>
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| Unit fraction | • Identifying the unit fraction of proper fractions correctly was apparent.  
• Determining the unit fraction of an improper fraction or a mixed number was difficult.  
• Understanding how to use the unit fraction to assist with | • Determining the unit fraction of an improper fraction or a mixed number was difficult.  
• Understanding how to use the unit fraction to assist with solving lower-level or higher-level tasks was not apparent.  
• Recalling the term “unit fraction” in |
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<tr>
<th>Section</th>
<th>Description</th>
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<td>Whole</td>
<td>• Comparing and ordering fractions require same-sized wholes was understood.</td>
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<td></td>
<td>• Comprehending that a fraction can represent different quantities dependent on the whole is not firmly entrenched.</td>
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<tr>
<td>Modeling</td>
<td>• Partitioning models resulted in unequal-sized parts.</td>
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<td></td>
<td>• Modeling using circles was the dominant model.</td>
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<td>• Modeling on a number line was problematic.</td>
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<tr>
<td>Comparing and Ordering</td>
<td>• Understanding that a larger denominator results in smaller partitions of the whole was evident.</td>
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<tr>
<td></td>
<td>• Realizing that the numerator plays a role when comparing and ordering fractions was not evident.</td>
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<td></td>
<td>• Comparing and ordering fractions with unlike numerators and unlike denominators was difficult.</td>
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<td></td>
<td>• Ordering a set of improper fractions or mixed numbers was challenging</td>
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<td></td>
<td>• Comparing fractions with like denominators correctly was observed for all students.</td>
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<td>• Continuing to have difficulties comparing and ordering fractions with unlike numerators and unlike denominators.</td>
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<td></td>
<td>• Ordering a set of improper fractions or mixed numbers was challenging</td>
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<td></td>
<td>• Ordering a set of improper fractions was a dilemma.</td>
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</table>

Solving lower-level or higher-level tasks was not apparent. Prompting during interview was necessary for students to remember how to find the unit fraction.
that included \( \frac{n}{n} \) was perplexing.

Benchmarks and Estimation

- Using both concepts infrequently.
- Using benchmarks to compare fractions and estimation was observed in only two instances.
- Using both concepts infrequently.
- Using benchmarks to compare fractions and estimation was moderately successful.
- Estimating the sum or difference of fractions using a number line was achieved.

All of the case study students, at some point during the module, referred back to learned procedures when asked to explain their solution to a task. This consisted of converting improper fractions to mixed numbers, cross-multiplying to determine fraction equivalence, and finding common denominators. During the one-on-one interviews, students again attempted to use these learned procedures, often incorrectly. Trying to implement these procedures along with applying conceptual knowledge caused anxiety for many of the students.

In Chapter 5, the data presented and summarized in this chapter related to the five fraction domains will be utilized to respond to the research questions that formed the basis for this work.
CHAPTER 5: DISCUSSION AND RECOMMENDATIONS

Chapter 5 presents a summary of the problem, an overview of the method used in the study, discussion of data and findings, limitations of the study, implications, and future research. The discussion of findings is organized based on the research questions.

Summary of the Problem

In 2010, the CCSSM articulated standards or learning expectations for students, grades K – 12, including those related to fractions, which are primarily found in third through sixth grades. During the same year, the U.S. Department of Education published an IES Practice Guide to provide guidance on fraction instruction for students in Kindergarten through eighth grade (Siegler et al., 2010). As the guide states, “A high percentage of U.S. students lack conceptual understanding of fractions, even after studying fractions for several years; this, in turn, limits students’ ability to solve problems with fractions and to learn and apply computational procedures involving fractions” (p. 6). This gap in fraction knowledge has created a need for additional mathematics interventions to rectify the situation, especially in the middle schools. RtI is one method that has been successfully implemented in the elementary schools to address gaps in knowledge. However, RtI has typically been used in the elementary school to attend to gaps in reading. Until recently, it has not been used in the middle school or to address mathematics learning deficiencies.

Research has provided information on the sequence of fraction topics that afford the best opportunities for student learning as well as effective strategies to develop these topics within a classroom instructional setting (Behr & Post, 1992; Confrey et al., 2012). However, there is little evidence of the impact of applying this knowledge within a
mathematics RtI setting. This study addressed mathematics instruction within an RtI setting and the development of fundamental fraction ideas/domains by middle school students who have demonstrated deficiencies in mathematics. By studying the impact of implementation of a specially designed fraction curriculum for middle grade students in a Tier II RtI class, the researcher sought to demonstrate what knowledge related to fractions, students acquire and to what extent this knowledge was retained a year later.

The following general research question guided this study: *How do middle grade students enrolled in an RtI Tier II mathematics course acquire and utilize big ideas of the fraction domain before, during, and following a specially designed unit of instruction?*

Specific research questions include:

1. What prior knowledge do middle school students enrolled in an RtI Tier II mathematics course demonstrate regarding fraction concepts?

2. What misconceptions do RtI Tier II students demonstrate while studying fraction concepts and skills?

3. What concepts and skills do RtI Tier II students demonstrate following a specially designed unit of instruction focused on fraction concepts and skills?

4. To what extent are major concepts of fractions retained one year after the RtI Tier II instructional sequence?

In order to answer each research question, data were gathered for each question from the following source(s):

- Research question 1 – Pre-Assessment
- Research question 2 – Pre-Assessment and Student booklet
- Research question 3 – Student booklet and Post-Assessment
• Research question 4 – One-on-one interview a year later

Method

This investigation into struggling middle grade students’ retention of key fraction domains was conducted using a case study design that included an analysis of written records of student work during an intervention instructional module and a one-on-one interview with students one year following the intervention module. Data were analyzed using a conceptual analytical framework. The framework was created based on five fraction domains identified by research as being essential to a strong foundation for fraction understanding (Clarke & Roche, 2009; Cramer et al., 2002; Lamon, 2007; Post et al., 1985). The domains include: (a) unit fractions, (b) the whole, (c) modeling fractions, (d) comparing and ordering fractions, and (e) benchmarks and estimation.

The coding of data was aligned with the conceptual analytical framework that consisted of the five fraction domains. Data for each case study were organized by student responses on the pre-assessment, independent work from student booklets, student responses on the post-assessment, and interview transcripts. All the independent work and responses from the interview transcripts were then analyzed and situated in one of the domains of the conceptual analytical framework. Finally, themes across the data were revealed.

Participants in this study had been enrolled in an RtI Tier II mathematics course as sixth-graders. Thirty students completed the majority of lessons in the fraction intervention module and had provided consent to participate in the study. In seventh grade, 29 of these students remained enrolled in the same middle school. They were contacted and asked to participate in a one-on-one interview. Ten students consented to
the interviews. Of these ten, seven interviews were conducted due to scheduling issues. The five case study students used in this investigation were selected based on their responses to fraction tasks during the interview.

Discussion of Findings

Research Question 1

What prior knowledge do middle school students enrolled in an RtI Tier II mathematics course demonstrate regarding fraction concepts and skills?

Prior to the intervention module focused on fractions, the pre-assessment revealed that students understand the idea of comparing fractions with unlike numerators and like denominators such as $\frac{7}{8}$, $\frac{5}{8}$, and $\frac{6}{8}$. However, they are generally not successful comparing fractions with like numerators and unlike denominators. For instance, when determining the least fraction from the set of fractions $\frac{1}{3}$, $\frac{1}{5}$, and $\frac{1}{4}$, four of the five case study students select $\frac{1}{3}$ as the smallest fraction. Only two of the five students correctly identify the mixed number closest to 5, when provided with a set of mixed numbers between 4 and 5. In addition, comparing fractions with unlike numerators and unlike denominators presents difficulties for all five students. In fact, only one student determines the correct answer $\left(\frac{8}{12}\right)$ when selecting the fraction less than $\frac{5}{6}$ from the list $\frac{7}{6}$, $\frac{8}{12}$, and $\frac{7}{8}$. The other four students choose $\frac{7}{6}$. However, on a similar problem with the following format, $\frac{5}{7} \quad \frac{2}{3}$, four of the students correctly choose $> \quad$ as the answer. Do they possess the fraction sense to know that $\frac{5}{7}$ is greater than $\frac{2}{3}$ or did they choose $\frac{5}{7}$ because the 5 and 7 are greater than the 2 and 3 in $\frac{2}{3}$?
Within the domain of comparing fractions, the case study students often apply whole number concepts to fractions and they demonstrate a weak sense of fractions. As noted earlier, comparing fractions with like numerators and unlike denominators as well as those with unlike numerators and denominators is difficult for these students. When provided with an area model or a number line as well as a symbolic representation of two equivalent fractions, four of five students are able to determine the equivalent fraction. Three of the five students are more successful on questions involving addition and subtraction of fractions, due to applying a learned procedure to calculate the sums and differences.

**Research Question 2**

*What misconceptions do RtI Tier II students demonstrate while studying fraction concepts and skills?*

As the case study students progressed through the lessons in the intervention fraction module, some common misconceptions and systematic mathematical errors became evident. The first instance occurs when students are asked to determine if four slices of pizza and two slices of pizza can both equal one-half of a pizza. The belief or perception that a pizza is always cut into eight slices led three students to incorrectly state that one-half of a pizza can only have four slices of pizza. Another instance is observed as students determine the unit fraction. While the case study students can determine the unit fraction of a fraction between 0 and 1, they struggle with determining the unit fraction of an improper fraction and a mixed number.

When students are asked to model fractions or when they initiate representing a fraction with a model, a circle is the model most often used. This presents a problem
when students model fifths, sevenths, and tenths. The students struggle with creating equal partitions in the circle model, which in turn affects their understanding of mathematical concepts, such as comparing and ordering. However, equally partitioning any area model or number line is difficult for the students. Moreover, modeling on a number line is especially problematic, except when the number line is already equally partitioned. Three case study students consistently represent a fraction using a point on the number line. It is unclear if they recognized that a fraction is represented by the distance from 0 to the point designated. These students often represent improper fractions by partitioning models based on the numerator and shading the partitions based on the denominator.

Additional misconceptions are revealed in the written explanations found in the student booklets, when students are asked to compare and order fractions. For example, the students seem focused on using the denominator of a fraction as the primary indicator of the size of the fraction. That is, to them, a small denominator results in large partitions and therefore is a large fraction while a large denominator results in small partitions and therefore is a small fraction. They do not consistently consider the numerators in their evaluation of the size of a fraction. Another misconception exhibited by three students dealt with considering the numerator and/or denominator as whole numbers. For example, when comparing the fractions $\frac{5}{8}$ and $\frac{3}{4}$, they identify $\frac{5}{8}$ as greater than $\frac{3}{4}$ due to the number 5 in the numerator of $\frac{5}{8}$ being larger than the number 3 in the numerator of $\frac{3}{4}$.

Finally, the conversion of improper fractions to mixed numbers in order to assist the student in comparing or ordering fractions is often executed inaccurately (using a learned procedure), thus affecting the correctness of the answer.
Research Question 3

What concepts and skills do RtI Tier II students demonstrate following a specially designed unit of instruction focused on fraction concepts and skills?

Following the intervention module, students are successful in demonstrating an understanding of unit fractions. For example, the case study students correctly determine the non-unit fraction that results from a given unit fraction and a specified number of iterations of the unit fraction. Likewise, when given a particular Cuisenaire rod (representing the whole) and a different colored Cuisenaire rod (representing the unit fraction), students determine the numeric unit fraction portrayed by the Cuisenaire rod. Additionally, when provided with a specific Cuisenaire rod and the numeric unit fraction that the length of the Cuisenaire rod represented, students determine the Cuisenaire rod whose length represents the whole. The intervention module utilized Cuisenaire rods in several lessons and it appears that students are successful in using this length model to represent and discuss fractions.

The case study students also successfully indicate the whole in a variety of problems. For example, when given a model of a fractional length and the unit fraction that symbolically represents the model, all of the case study students accurately draw the whole length. They also successfully determine the whole referred to in contextual problems involving fractions.

Following the instructional module, students also demonstrate understanding that a fraction model can be partitioned based on the number in the denominator. They recognize that the partitions should be equal, although they are often unable to draw precisely equal partitions when representing a fraction. Only when asked by the
interviewer do students use a rectangle to model a fraction. A circle is their normal go-to-model, even though they are more accurate when utilizing a rectangle to model a fraction rather than a circle. Representing an improper fraction on a number line is accomplished only when the number line is pre-partitioned and labeled.

With regards to comparing and ordering fractions, students are more successful following the intervention module as demonstrated by responses on the post-assessment. For instance, they correctly compare fractions with like numerators and unlike denominators by applying the concept that the fraction with the largest denominator is partitioned into the smallest sections and the fraction with the smallest denominator is partitioned into the largest sections. They are also able to represent these fractions on a number line. The case study students also successfully order a list of fractions with unlike numerators and like denominators. When the list of fractions is comprised of fractions with unlike numerators and unlike denominators, students utilize benchmark fractions, a focus of the intervention module, to assist them in ordering the fractions.

**Research Question 4**

*To what extent are major concepts of fractions retained one year after the RtI Tier II instructional sequence?*

Within the domain of unit fractions, facility with the term “unit fraction” is not retained by four of the students. Only with prompting by the interviewer are these students able to identify the unit fraction in the follow up interview. One student, who relies heavily on learned procedures throughout the interview, remembered the term, and also referred to the unit fraction as the “base” fraction. This student is successful in determining the unit fraction of proper fractions, improper fractions, and mixed numbers.
When the case study students are given a unit fraction and asked to indicate the whole, they successfully accomplish this task as they had in their written work during the intervention module. This notion is retained over the period of a year even when the fraction model used in the interview is a group of objects rather than a length on the number line.

Based on the interviews, it is evident that the idea of the whole is not solidified in the minds of the students and consequently they have multiple impressions of the whole one-year following the intervention module. For example, depending on the representation, students consider the whole to be \( \frac{n}{n} \) or the number 1, the denominator, a single area model, or the number 1 on a number line. Confusion about the meaning of the whole affects students’ responses to tasks involving comparing and ordering fractions. For instance, when ordering a list of proper and improper fractions, including \( \frac{n}{n} \), all of which have unlike denominators, the belief that \( \frac{n}{n} \) is the largest fraction in the list because it is “1” or “the whole” is expressed.

A task within the intervention curriculum asked students if four slices of pizza and two slices of pizza could both equal \( \frac{1}{2} \) of a pizza. This problem causes some difficulty for students, in part due to the context – a pizza serving as the model. Three of the case study students state that both four slices and two slices cannot represent half of a pizza because “a pizza has eight slices.” A year later, during the interview, students are asked in one problem if \( \frac{1}{4} \) is always equal to \( \frac{1}{4} \) and in another problem if \( \frac{2}{4} \) is always equal to \( \frac{6}{8} \). In both problems, all students first remark that the fractions are always equal. However,
when provided with either a context or models that helps them reflect more deeply on the problems, the students realize that the size of the whole will affect the size of the fraction.

All of the students retain ideas about drawing models such as: (a) the denominator indicates the number of partitions in the model, (b) the numerator indicates how many partitions are shaded, and (c) the partitions have to be equal. As revealed in both the student written records during the intervention curriculum and the interviews, when drawing their own models, students struggle with creating equal size partitions. Questioning during the interviews reveals that students think the partitions should be equal, but at the same time, they are unsure if it is important for the partitions to be exactly equal.

Throughout the interviews, a circle is the model most often chosen by the students to represent fractions. When students struggle with this area model, they are prompted to use a rectangle, which results in greater success. On the other hand, number lines present the most confusion due to preconceived ideas about a number line. Some of these ideas include marking $\frac{1}{2}$ on the number line and drawing partitions without considering the fraction to be modeled, representing the fraction with a point instead of as a distance, and struggling with labeling 0, 1, etc. on the number line. Modeling improper fractions also continues to present a challenge for these students. They model improper fractions with incorrect discrete models and models partitioned based on the numerator and shaded based on the denominator.

In the domain of comparing and ordering fractions, students utilize strategies of modeling, the size of partitions created by the denominator, and residual thinking. During the intervention module and the interviews, all students demonstrate understanding that
when modeling fractions for comparison purposes, the whole must be drawn the same size. Students retain the idea that a large denominator means the sections or partitions of a fraction will be smaller in size. This notion is often used when comparing and ordering fractions with like numerators and unlike denominators. However, these students often do not take the numerator into consideration when comparing fractions, even following the intervention module. As noted in the written student records during the fraction module, comparing and ordering fractions with unlike numerators and unlike denominators is the most difficult task for students during the interviews.

**Implications and Future Research**

This research regarding concepts acquired and retained by middle school students as a result of a fraction intervention module implemented in a Tier II RtI mathematics course has implications for providing targeted fraction instruction for struggling students. The interviews reveal that students have multiple ideas about many fraction topics and multiple strategies to solve fraction problems. Making a determination as to which idea is correct or which strategy to use to solve a problem often causes confusion for the student. It is clear that, for some students, an over-reliance on learned procedures is a barrier to calling upon holistic notions of fractions. Summarized here are recommendations for instruction related to each fraction domain.

After a year, the term “unit fraction” is not remembered by students and only after prompting do they recall and use unit fractions to answer questions. It does not appear that unit fractions are a foundational concept that these students learned or used beyond the intervention module to think about fractions. Other research has noted the importance of the concept of the unit fraction (Norton & Wilkins, 2012; Olive & Steffe, 2002; Olive
& Vomvoridi, 2006). Therefore, more care must be taken to solidify this concept within instruction and curriculum materials.

The case study students retain the idea of the whole and that it is partitioned and shaded to represent a fraction. They also demonstrate understanding that the whole must be the same when comparing or ordering fractions. Student explanations in the individual interviews however, highlight the conflict students have concerning the whole. For example, case study students describe the whole as: (a) the denominator of a fraction, (b) the fraction \( \frac{n}{n} \), (c) the number 1 on a number line, and (d) a single area model.

Solidification of the notion of the whole and its meaning in different contexts is necessary so that advanced topics such as comparing and ordering can build upon these notions. As Lamon (2007) described, the whole is crucial for more sophisticated fraction ideas to develop as well as for students to develop the idea that a fraction is a proportional relationship between the numerator and the denominator.

Students’ written records (pre-assessment, student booklet, and post-assessment) and their interview data one year later reveal that they realize when modeling a fraction, the partitions must be equal and are based on the denominator and the number of partitions shaded are based on the numerator. The circle area model is the model of choice when students were given the option to choose a model. This makes it difficult for students to model fractions such as fifths, sevenths, and tenths. During the interview, when questioned as to the importance of the partitions in a model being equal, students express the idea that they thought the partitions should be equal, but they could not draw them equally. This in turn affects their solutions to a variety of tasks.
Students demonstrate difficulties modeling on number lines and modeling improper fractions, both in the fraction module and in the interview. As Jordan et al. (2013) suggests, using a number line provides a way to not only model fractions, but also to show their relationship to whole numbers and understand them as a magnitude. Based on students’ difficulties with drawing equal parts and the importance of number line models, more attention to this notion is needed within instruction. For example, students might work initially with models such as fraction strips, Cuisenaire rods, and number lines that are already equally partitioned, in order to develop understanding of fractional concepts. Eliminating or decreasing the effect of students’ drawings on their comprehension may lead to greater fractional knowledge. However, other models need to be investigated, including number lines that are not pre-labeled.

Both in the intervention curriculum and in the interviews, students successfully apply the concept that the larger the denominator the smaller the partitions and the smaller the denominator the larger the partitions, as they compare or order fractions. While this is an important idea, students do not address the effect of the numerator on their solution. More careful attention to this idea is needed. A stronger focus on comparing and ordering fractions with unlike numerators and denominators will need to take place as this is the area of most difficulty.

Finally, benchmarks and estimation will need more emphasis within instruction. In the intervention curriculum, students are asked to estimate their answers to addition and subtraction of fraction problems. However, this does not seem to be a major emphasis during implementation. The placement/sequence of this topic within fraction instruction must also be considered. Should it precede work finding exact sums and
differences? What impact might reordering the sequence of this topic have on student ideas about estimation and use of benchmarks? According to Fuchs et al. (2013), students who struggle with fractions need to receive instruction that will help them develop automaticity with the magnitude of fractions through the use of benchmarks.

**Future Research**

Interventions for middle grade students who are not identified as students with disabilities, but are struggling with fraction concepts will continue to need to be a focus of research and development efforts. While this study notes fraction domains that students retain a year after a specially designed fraction module, questions still remain.

- Are the five fraction domains that served as the focus of this study the only essential elements in a framework for curriculum development and future research in the area of fractions? Are there other key domains or other ways to conceptualize foundational concepts of fractions that might be useful?
- Are the concepts that the RtI students continue to struggle with, the same for other students?
- Was the intervention module too brief to have a longer-term effect?
- How might the fraction module be modified to address some of the deficiencies noted in the follow-up interview?
- Should there be a greater focus in the intervention curriculum on a smaller number of fraction ideas?

These questions will require additional research in order to determine instructional strategies that will support not only Tier II RtI students, but all students who struggle with the topic of fractions.
Limitations

The fidelity with which an intervention module is implemented influences the effect of that intervention (VanDerHeyden et al., 2012). Based on feedback from the teachers who participated in the study, several issues affected how the fraction module was implemented. The middle school class schedule presented a major obstacle to the success of RtI courses. While the school had set aside time for a specific RtI class, initially the class period was only 30 minutes. Consequently, it was difficult for the teachers to fully implement the lessons in the module. Additionally, observations of the classroom environment were not conducted. Therefore, it is unclear how closely the intervention module was followed.

Another limitation was the small case study size. While three middle schools participated in the study, only one school completed all the requirements for participation in the study (e.g., completed the full set of materials). Within the school utilized for the study, the number of students who consented to participate as sixth-graders and then provided consent to be interviewed as seventh-graders was minimal (30 of 65 consented to have their written student records reviewed by the research; 7 of the 30 consented to interviews one year after the intervention module). While the study is not generalizable across all struggling middle grade students, the five students who comprised the case study group offer some insight into fraction domains retained over a year.

The final two limitations are related to the student role and researcher role during the interviews. The ability and/or desire of seventh-grade students to explain their thinking was uneven across the interviews. Some students explained their thinking by drawing pictures, using procedures, and writing explanations. Other students explained
their thinking by restating their answer; saying, “I don’t know”; or not responding at all. The researcher was conflicted when determining how much information to supply the students when they were struggling. Additionally, the concern that it was unethical to allow students to leave the interview with misconceptions was a factor.

Closing Statement

In closing, this investigation reveals that after a year, middle grade students who had been enrolled in an RtI Tier II mathematics course retain pieces of fraction concepts learned during the course. For instance, when comparing or ordering fractions, students understand the inverse relationship between the denominator and the size of the partitions. However, they do not include the numerator in making their decisions about fraction size. Another instance of this partial retention of knowledge is shown when students draw their own models to represent fractions. They realize that the partitions of a model must be equally sized, but due to the fact they struggle with drawing the partition equally, they are not confident in this notion, which in turn affects their solutions to various tasks. The quantity of knowledge held by the middle grade students creates chaos in their reasoning process and consequently only fragments of fraction concepts are retained. Providing students with focused and targeted instruction can help alleviate the amount of information students must wade through in order to solve fraction tasks. Moreover, it can strengthen and solidify this knowledge so that students are confident when solving problems using these fractional concepts.
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APPENDIX A

Sample Fraction Lesson from Student Booklet – Warming Up Section

Name: ________________________________

Warming Up:

1. When you are comparing two improper fractions, what are possible ways to determine which fraction is greater? Write two methods you might use.

   a. Change it to a mixed number

   b. number line

2. Which of these methods do you think is the most efficient?
   a

Why?
Because you can find out the proper fraction of both of the and compare them

3. Put the following fractions in order from least to greatest:

   \[
   \frac{6}{7}, \frac{5}{4}, \frac{1\frac{9}{4}}{} 1\frac{1}{2}, \frac{9}{7}, \frac{5}{8}, \frac{8}{4}
   \]

Be able to explain how you determined the order.

By changing the improper fractions into a mixed number compared them together.
APPENDIX B

Sample Fraction Lesson from Student Booklet – Learning to Solve Section

Learning to Solve:

Problem 1

For dinner, you have the option of serving $\frac{2}{3}$ of a pumpkin pie or $\frac{7}{4}$ of a pumpkin pie. Which size of pumpkin pie will provide a larger piece of pie for your guests, if they share the amount equally?

1. Using either an area model or a number line, draw a representation of this word problem in the space below.

![Area Model and Number Line]

2. Be prepared to explain your model.

Think about:

a. What is the whole?

| pie |

b. How did you partition the circle or the number line?

Used cut denominator

c. How did you know how many circles/partitions to make?
Learning to Solve: continued

Problem 1

3. Write a comparison of these two fractions using the symbols <, >, or =.

\[
\frac{7}{3} > \frac{7}{4}
\]

4. You have 3 minutes to complete this.

Thinking about the different models, what is a way to compare improper fractions that have the same numerator but different denominators?

Talk in your group for 2 minutes about a method you might use.

Describe your method below.

\[\text{the bigger the numerator the bigger the fraction}\]
Problem 2

Jill, Bob, John, and Carla are riding their bikes. Jill rides $\frac{1}{4}$ miles, Bob rides $2\frac{1}{8}$ miles, John rides $1\frac{5}{8}$ miles, and Carla rides $2\frac{1}{2}$ miles.

Use the number lines below to model the distance that Jill, Bob, John, and Carla rode.

Answer the following questions:

1. What is the whole?
   
   1 mile

2. What is the order of these fractions from the longest distance to the shortest distance?
   
   $\frac{1}{4}, \frac{1}{2}, \frac{5}{8}, \frac{1}{4}$

3. What are the names of the students in order from the longest distance they rode to the shortest distance they rode?
   
   Carla, Bob, John, Jill

4. If you didn’t draw a diagram, what would be your first step in putting mixed numbers in order? Look at the whole number first

5. What would be your next step?
Name: ________________

**Problem 3**

You are responsible for placing frames in order with the taller ones behind the shorter ones. All the frames are made the same and they are the same width. But, they are different heights.

The frame heights are:

- \( \frac{7}{2} \) feet \( = 3\frac{1}{2} \)
- \( 1\frac{2}{3} \) feet
- \( \frac{5}{6} \) feet
- \( \frac{3}{2} \) feet \( = 1\frac{1}{2} \)

Talk with the other people at your table and decide how you would order these fractions from shortest to tallest using benchmark fractions. Don’t draw diagrams, just think about benchmark fractions such as \( \frac{1}{2} \), 1, and 2.

1. Write the correct order from shortest frame to tallest frame.
   
   \( \frac{5}{6}, 1\frac{2}{3}, 1\frac{1}{2}, 3\frac{1}{2} \)

2. Be prepared to explain how you determined this order.
APPENDIX C

Sample Fraction Lesson from Student Booklet – Practicing Together Section

Name: ____________________________

Practicing Together:

1. Which symbol ( <, >, = ) would you place in the blank?

\[ \frac{2}{2} \quad \frac{7}{3} \quad \frac{2}{3} \]

2. Explain how you decided which symbol to use in problem 1.

We looked at the fraction part only. We knew \( \frac{1}{2} \) is bigger than \( \frac{3}{4} \).

3. Place these fractions in order from greatest to least: \( \frac{6}{8}, \frac{6}{6}, \frac{6}{6}, \frac{6}{6}, \frac{6}{4}, \frac{6}{3}, \frac{6}{3}, \frac{6}{5} \)

4. Explain how you determined the order in problem 3.

If the numerator is the same, you can compare the denominators. Big denominators mean small pieces.

Nancy, Charles, and Seth walk laps on the school track each afternoon. Yesterday, Nancy walked \( \frac{13}{4} \) laps, Charles walked \( \frac{13}{8} \) laps, and Seth walked \( \frac{5}{8} \) laps.

5. Who walked the longest distance? Nancy

6. Who walked the shortest distance? Seth

7. Explain your answer.

We put them in order using mixed numbers.
APPENDIX D

Sample Fraction Lesson from Student Booklet – Trying It On Your Own Section

Trying It On Your Own:

1. Gabby, Henry, and Cody are growing sunflowers. Gabby’s sunflower is \( \frac{13}{12} \) feet high. Henry’s sunflower is \( \frac{1}{3} \) feet high, and Cody’s sunflower is \( \frac{3}{6} \) feet high.

a. Put the heights of the sunflowers in order from greatest to least.

\( \frac{9}{16}, \; \frac{1}{3}, \; \frac{13}{12} \)

b. Explain your answer.

\( \frac{9}{12} = \frac{3}{4} \) and that is equal to \( \frac{1}{2} \) so I compared \( \frac{1}{2} \) to \( \frac{1}{3} \) and \( \frac{13}{12} \) turned out to be the biggest.

c. Whose sunflower is the tallest?

Cody’s \( \frac{13}{12} = 1\frac{1}{2} \)

2. Decide which symbol (>, <, =) to place in the blank: \( \frac{11}{8} \) __ \( \frac{7}{8} \)

a. Describe how you decided.

8 goes into 11 1 with 3 left over so \( \frac{11}{8} = \frac{13}{8} \) as a bigger fraction than \( \frac{7}{8} \)

3. Put the fractions in order from least to greatest: \( \frac{2}{5}, \; \frac{1}{5}, \; \frac{4}{5}, \; \frac{2}{10} \)

\( \frac{2}{5}, \; \frac{1}{5}, \; \frac{4}{5}, \; \frac{2}{10} \)

Explain your answer.
APPENDIX E

Sample Fraction Lesson from Student Booklet – Wrapping It Up Section

Wrapping It Up:

Angela, Marcus, and George each grew a tomato plant in their own backyards. They measured their plants and wanted to compare the height (in feet) of their tomato plants. George drew number lines to compare the tomato plants. He said, “According to my model, all the tomato plants are the same height.”

George’s Model

1. Do you agree with George?
   No

2. Why or why not?
   
   Because Marcus and George are at 1 and Angela is at 2
   
   Angela = 2 \frac{1}{4}
   Marcus = 1 \frac{1}{2}
   George = 1 \frac{3}{4}
APPENDIX F

Fraction Pre-assessment

1. Jill eats $\frac{2}{5}$ of an orange.
   Alan eats $\frac{3}{5}$ of an orange.
   Betty eats $\frac{1}{5}$ of an orange.
   Who eats the least?

   A. Alan
   B. Jill
   C. Betty

2. $\frac{3}{4} - \frac{1}{4} = \_\_

   A. $\frac{2}{4}$
   B. $\frac{2}{4}$
   C. $\frac{1}{4}$

3. Which value is greatest?

   A. $\frac{7}{8}$
   B. $\frac{6}{8}$
   C. $\frac{6}{8}$

4. $\frac{1}{10} = \_\_

   A. $\frac{2}{2}$
   B. $\frac{1}{4}$
   C. $\frac{1}{4}$

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9. Which goes from least to greatest?
   A. $\frac{1}{2}, \frac{1}{3}, \frac{3}{2}$
   B. $\frac{1}{5}, \frac{1}{3}, \frac{1}{2}$
   C. $\frac{1}{3}, \frac{1}{2}, \frac{3}{8}$

10. Joe eats $\frac{1}{4}$ of a pie.
    John eats $\frac{7}{8}$ of a pie.
    How much do they eat in all?
    A. $\frac{4}{7}$
    B. $\frac{3}{4}$
    C. $\frac{1}{7}$

11. $\frac{4}{4} \_ \_ 1$
    A. $<$
    B. $>$
    C. $=$

12. Which value is closest to 5?
    A. $4 \frac{1}{3}$
    B. $4 \frac{7}{8}$
    C. $4 \frac{3}{4}$

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13

\[ \frac{1}{5} = \frac{2}{10} \]
\[ \frac{7}{5} = \text{____} \]

A. \( \frac{12}{10} \)
B. \( \frac{11}{5} \)
C. \( \frac{14}{10} \)

14

Which goes from least to greatest?

A. \( \frac{2}{9} \), \( \frac{6}{10} \), \( \frac{3}{4} \)
B. \( \frac{2}{9} \), \( \frac{3}{4} \), \( \frac{8}{16} \)
C. \( \frac{3}{4} \), \( \frac{2}{9} \), \( \frac{8}{16} \)

15

Which value is least?

A. \( \frac{1}{3} \)
B. \( \frac{1}{2} \)
C. \( \frac{1}{4} \)

16

There is \( \frac{13}{10} \) of a cake.

Ned eats \( \frac{1}{10} \).

About how much is left?

A. \( \frac{1}{10} \)
B. \( \frac{1}{9} \)
C. \( \frac{1}{8} \)

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17. Mac builds a birdhouse.
    He has 1 foot of wood.
    He uses $\frac{1}{4}$ ft.
    How much wood is left?

A. $\frac{2}{3}$ ft.
B. $\frac{1}{3}$ ft.
C. $\frac{1}{2}$ ft.

18. Each bottle of milk has $\frac{1}{2}$ gallon.
    Sara buys 7 bottles.
    How many gallons of milk does she buy in all?

A. 14
B. 7
C. $3 \frac{1}{2}$

19. Which value is least?

A. $\frac{2}{5}$
B. $\frac{1}{2}$
C. $\frac{3}{4}$

20. $\frac{3}{8} + \frac{2}{16} =$ ___

A. $\frac{5}{16}$
B. $\frac{9}{16}$
C. $\frac{8}{16}$

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Notes: Interview will be videotaped. Only student's work will be videotaped, no faces.

I am going to ask you some questions about fractions. I am interested in how you come up with the answers, so it is important for you to tell me what you are thinking and show me what you are doing as you solve the problems. The problems are not graded so you don’t have to worry about wrong answers. I am just interested in learning about what you think as you solve these fraction problems. Are you ready?

**TASK 1: ORDER/COMPARE QUESTIONS**

1. Say: *I am going to show you two fractions.*
   
   Show: \[ \frac{3}{4}, \frac{9}{12} \]
   
   (a) Say: *Are these two fractions equal or is one of the fractions less than the other fraction? Which one is less?*
   
   (b) Say: *Tell me how you know.*

2. Say: *I am going to show you two fractions.*
   
   Show: \[ \frac{4}{6}, \frac{5}{6} \]
   
   (a) Say: *Are these two fractions equal or is one of the fractions less than the other fraction? Which one is less?*
   
   (b) Say: *Tell me how you know.*

**TASK 2: ORDER/COMPARE QUESTIONS**

1. Say: *I am going to read a problem and would like you to answer a question based on the problem.*

Read this problem to the student: *Alice and James both receive the same allowance. Alice spent \( \frac{5}{8} \) of her allowance on a movie. James spent \( \frac{5}{12} \) of his on a new CD. Did they spend the same amount or did one spend less? [If less, ask: Who spent less?]"
Say: *Tell me how you know. Did you picture anything in your mind as you thought about these fractions?*

2. Say: *Draw a model or representation of the fraction \( \frac{2}{5} \).*
   Student will most likely draw an area model – either a circle or a rectangle.

Say: *Now, show me \( \frac{2}{5} \) on a number line.*
Provide students with a number line that has tic marks at 0 and 1.

Say: *How did you decide to put \( \frac{2}{5} \) there?*
Say: *I’m going to ask you a few questions about \( \frac{2}{5} \). After you answer the question, explain what you were thinking to come up with that answer.*

a. Is \( \frac{2}{5} \) greater than \( \frac{1}{2} \)?

b. Is \( \frac{2}{5} \) equivalent to \( \frac{4}{10} \)?

c. Is \( \frac{2}{5} \) greater than \( \frac{1}{3} \)?

d. Is \( \frac{2}{5} \) equivalent to \( \frac{2}{10} \)?

**TASK 3: BENCHMARKS – ESTIMATION AND OPERATIONS**

Say: *I am going to show you an addition or subtraction of fractions problem. Estimate the answer to the problem by deciding about where on the number line the answer would be.*

For example, if you think the answer is between 0 and \( \frac{1}{2} \), then put an X here. (Show this on a sample number line.)

*Tell me what you are thinking as you estimate where the answer would be.*
**TASK 4: THE WHOLE and UNIT FRACTIONS**

1. Say: *I am going to read you a problem and would like you to answer a question about the problem.*

Read this problem to the student:

*Mary and Jose both have some money to spend. Mary spends \( \frac{3}{4} \) of hers and Jose spends \( \frac{1}{4} \) of his. Is it possible that Mary and Jose spent the same amount of money?*

Say: *Tell me what you are thinking.*

2. Say: *Tim is building a tower. He used 3 cubes like these to build \( \frac{1}{5} \) of the tower. How tall will the whole tower be when it’s finished?*

(a) Ask students to talk aloud as they solve the problem.

Say: *Show me how to solve this problem. Talk aloud as you solve the problem and explain each step.*
**TASK 5: EQUAL PARTITIONING AND MODELS**

1. Show student the number line.

Say: *Pedro lives here. (Point to the tick mark) His school is 1 mile from home. He had walked $\frac{3}{8}$ of a mile from his house when he met his friend Monica. Where on the number line does he meet Monica?*

![Number Line Diagram]

(a) Say: *How do you know this is $\frac{3}{8}$?*

(b) Say: *Where is zero on the number line? Why?*

2. Show student the number line.

Say: *Where is the number $\frac{9}{4}$ on the number line?*

Say: *How do you know? Tell me what you are thinking.*

After the student answers the questions, ask: *What is the whole on the number line?*

3. Show student the empty number line.

Say: *Show the fraction $1\frac{1}{3}$ on the number line.*

*Explain your strategy for showing $1\frac{1}{4}$ on the number line. (If the student shows 0 and 1, ask why she/he did this.*)

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**TASK 6: EQUIVALENCE**

Show student the problem.

\[
\frac{3}{8} + \frac{1}{4}
\]

Say: *I want you to find the actual answer to this problem using a model. Explain what you are doing as you create your model.*

Say: *Now show how to solve the problem with symbols.*

Say: *Explain how this represents what you did with the fraction circles.*
APPENDIX H

Answers from Pre-Assessment to Post-Assessment (Correct – C or Incorrect – I)

<table>
<thead>
<tr>
<th>Problem</th>
<th>Angie</th>
<th>Brianne</th>
<th>David</th>
<th>Ethan</th>
<th>Maggie</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pre</td>
<td>Post</td>
<td>Pre</td>
<td>Post</td>
<td>Pre</td>
</tr>
<tr>
<td>1 Compare fractions with like denominators</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
</tr>
<tr>
<td>2 Subtract fractions with like denominators</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>I</td>
</tr>
<tr>
<td>3 Compare fractions with like denominators</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
</tr>
<tr>
<td>4 Equivalent fractions with area model</td>
<td>C</td>
<td>C</td>
<td>I</td>
<td>C</td>
<td>C</td>
</tr>
<tr>
<td>6 Compare fractions with unlike numerators and unlike denominators</td>
<td>I</td>
<td>C</td>
<td>I</td>
<td>I</td>
<td>C</td>
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<tr>
<td>7 Equivalent fractions with number line</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
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<tr>
<td>8 Compare fractions with unlike numerators and unlike denominators</td>
<td>I</td>
<td>I</td>
<td>C</td>
<td>I</td>
<td>C</td>
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<tr>
<td>10 Add fractions with unlike denominators</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>I</td>
<td>C</td>
</tr>
<tr>
<td>12 Mixed number closest to 5</td>
<td>I</td>
<td>C</td>
<td>I</td>
<td>I</td>
<td>C</td>
</tr>
<tr>
<td>13 Equivalent fraction with number line</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
</tr>
<tr>
<td>14 Order fractions with unlike numerators and unlike denominators</td>
<td>I</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>I</td>
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<tr>
<td>15 Compare unit fractions</td>
<td>I</td>
<td>C</td>
<td>C</td>
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<td>I</td>
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<td>16 Subtract fractions</td>
<td>C</td>
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## APPENDIX I

Pre-Assessment Question and Case Study Student Answer

<table>
<thead>
<tr>
<th>Question Number</th>
<th>Question</th>
<th>Answer choices</th>
<th>Angie</th>
<th>Brianne</th>
<th>David</th>
<th>Ethan</th>
<th>Maggie</th>
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<tbody>
<tr>
<td>3</td>
<td>Which value is greatest?</td>
<td>7/8 5/8 6/8</td>
<td>7/8</td>
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<td><img src="image" alt="Fraction Comparison Diagram" /></td>
<td><img src="image" alt="Fraction Comparison Diagram" /></td>
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<tr>
<td>6</td>
<td>5/6 &gt; ____</td>
<td>7/6 8/12 7/8</td>
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<td>8</td>
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<td>= &gt; &lt;</td>
<td>&gt; &gt; &gt; &gt;</td>
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<tr>
<td>12</td>
<td>Which value is closest to 5?</td>
<td>4 1/3 4 7/8 4 3/4</td>
<td>4 3/4</td>
<td>4 3/4</td>
<td>4 7/8</td>
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<td>4 7/8</td>
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<td>12/10</td>
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<tr>
<td>14</td>
<td>Which goes from least to greatest?</td>
<td>2/8, 8/16, 3/4 2/8, 3/4, 8/16 3/4, 2/8, 8/16</td>
<td>2/8, 3/4, 8/16</td>
<td>2/8, 3/4, 8/16</td>
<td>2/8, 3/4, 8/16</td>
<td>3/4, 2/8, 8/16</td>
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<tr>
<td>15</td>
<td>Which value is least?</td>
<td>1/3 1/5 1/4</td>
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<td>1/3</td>
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## APPENDIX J

Post-Assessment Question and Case Study Student Answer

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<th>Brianne</th>
<th>David</th>
<th>Ethan</th>
<th>Maggie</th>
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<tr>
<td>2</td>
<td>6/7 – 2/7 =</td>
<td>8/7, 4/7, 4/14</td>
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<tr>
<td>3</td>
<td>Which value is greatest?</td>
<td>2/7, 1/7, 4/7</td>
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<td>12/16, 4/16, 6/16</td>
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<td>4/16</td>
<td>4/16</td>
</tr>
<tr>
<td>5</td>
<td>5/6 &gt; ____</td>
<td>7/6, 8/12, 7/8</td>
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<td>7/8</td>
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<td>&lt;</td>
<td>&lt;</td>
<td>&lt;</td>
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<tr>
<td>9</td>
<td>Which value is closest to 5?</td>
<td>4 1/3, 4 7/8, 4 3/4</td>
<td>4 7/8</td>
<td>4 3/4</td>
<td>4 7/8</td>
<td>4 7/8</td>
<td>4 3/4</td>
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<tr>
<td>10</td>
<td><img src="image" alt="Fraction Bars" /></td>
<td>4/5, 6/5, 10/10</td>
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<td>6/5</td>
<td>4/5</td>
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<td>11</td>
<td>Which goes from least to greatest?</td>
<td>2/8, 8/16, 3/4, 2/8, 3/4, 8/16, 3/4, 2/8, 8/16, 3/4, 2/8, 8/16, 3/4</td>
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<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>Which value is least?</td>
<td>1/8, 1/4, 1/6</td>
<td>1/8</td>
<td>1/8</td>
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## APPENDIX K

Fraction Lesson Objectives

<table>
<thead>
<tr>
<th>Lesson</th>
<th>Lesson Objectives</th>
<th>Learning to Solve / Practicing Together Objectives</th>
</tr>
</thead>
</table>
| 1      | • Students identify and name unit fractions (1/2, 1, 1/3, 1/4, 1/5)  
• Students use different models/representations (drawing, number line) to represent unit fractions as they relate to the size of the whole. | • Students develop an understanding of the meaning of 1/2.  
• Unit fraction is introduced.  
• Students understand the concept of iteration with fractions. |
| 2      | • Students use different models/representations (e.g., fraction strip) to represent fractions as they relate to the size of the whole. | • Use a fraction strip to model fractions (Students become more precise with their measurements by creating halves, fourths, eighths, and sixteenths.). |
| 3      | • Students identify the whole given a unit fraction.  
• Students understand how a unit fraction changes if the whole is changed.  
• Students compare unit fractions to identify greater than/less than fractions using concrete and area models. | • Given the "whole", students find the fractional part. (Students use Cuisenaire rods for this and understand that a single rod could represent different fractions based on the whole.)  
• Given a fractional unit, students find the whole. (Students use Cuisenaire rods again.)  
• Students compare fractions using an area model. (Students see that when comparing unit fractions, the larger the denominator, the smaller the pieces are.) |
| 4      | • Students write a mixed fraction from an improper fraction and an improper fraction from a mixed fraction.  
• Students indicate mixed and improper fractions on a number line. | • Symbolize a fraction using multiple representations. (Circle, number line, Cuisenaire rods). Students determine the whole for each model. How to create a number line is discussed.  
• Students use Cuisenaire rods to represent improper fractions. By looking at the rods, students are guided towards writing the improper fraction as a mixed number.  
• Represent an improper fraction on a number line. Students are asked what the whole is, how many parts to partition the whole into, how many iterations are needed, and what is another way to write the improper fraction.  
• Students determine how many fractional pieces are in a mixed number (number of unit fractions). |
<table>
<thead>
<tr>
<th>Lesson</th>
<th>Lesson Objectives</th>
<th>Learning to Solve / Practicing Together Objectives</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>• Students compare fractions with like numerators (e.g. 4/8 and 4/5) using concrete and pictorial models. Students compare fractions that equal 1 whole.</td>
<td>• Students compare fractions with like numerators using Cuisenaire rods. They discuss the unit fraction and what a non-unit fraction means i.e. 2/3 is two one-thirds. • Compare improper fractions having like numerators. • Students generalize a method to determine how to compare fractions with like numerators. • Compare fractions with like numerators on a number line. Students determine the whole, unit fraction, number of partitions. Symbolic notation is brought forth.</td>
</tr>
<tr>
<td>6</td>
<td>Students compare fractions using benchmark fractions (0, 1/2, and 1).</td>
<td>• Students use their fraction strip to compare two fractions by using a benchmark fraction. Students determine whether a fraction is closer to 0, 1/2, or 1. Patterns and/or generalizations are discussed.</td>
</tr>
<tr>
<td>7</td>
<td>Students place fractions in order from least to greatest and greatest to least.</td>
<td>• Ordering unit fractions. Students represent unit fractions on a number line in order to order them. Partitioning the whole is discussed. • Order 3 fractions that are not unit fractions using 3 number lines. The whole is discussed. Order fractions without the use of number lines. Students are asked to determine a method to order fractions.</td>
</tr>
<tr>
<td>8</td>
<td>Students compare and order improper fractions with like and unlike denominators.</td>
<td>• Concept of the whole number as a fraction. Unit fraction is discussed. Order improper fractions using benchmark fractions. The whole, number of unit fractions in a non-unit improper fraction (7/6 is 7 one-sixths), using benchmark fractions are used to solve an ordering contextual problem.</td>
</tr>
<tr>
<td>Lesson</td>
<td>Lesson Objectives</td>
<td>Learning to Solve / Practicing Together Objectives</td>
</tr>
<tr>
<td>--------</td>
<td>-------------------</td>
<td>--------------------------------------------------</td>
</tr>
<tr>
<td>9</td>
<td>• Students compare and order fractions with like and unlike denominators using multiple methods.</td>
<td>• Students compare and order fractions using any method, but must use a different method for each problem. The whole, iterations of the unit fraction, and ordering are determined by the students in each problem.</td>
</tr>
<tr>
<td>10</td>
<td>• Students compare and order improper fractions with like and unlike denominators using number lines.</td>
<td>• Compare and order improper fractions using number lines. Students are asked what to keep in mind as they compare and order (know the whole and the whole must be partitioned into equal-sized parts based on the denominator). • Using a contextual problem, students compare/order improper fractions. They answered questions concerning the whole on the number line, how they decided how to partition the number line, where each fraction was located, and the order of the improper fractions.</td>
</tr>
<tr>
<td>11</td>
<td>• Students compare and order mixed numbers and improper fractions with like and unlike denominators.</td>
<td>• Compare improper fractions with like numerators and unlike denominators. • Students represent a contextual problem using an area model or a number line and explain how they created their model (whole, partitioning, number of partitions). Students use symbolic notation to compare and also generalize a method for comparing improper fractions with like numerators and unlike denominators. • Compare and order mixed numbers. Students model a contextual problem using number lines and answer what the whole is, the order, and how to decide order without drawing number lines. • Order mixed numbers and improper fractions using benchmark fractions.</td>
</tr>
<tr>
<td>12</td>
<td>• Students identify and create equivalent fractions using concrete and pictorial models.</td>
<td>• Review the concept of equivalency and equal shares. Students model an equal share contextual problem. • Students use diagrams to see equivalent fractions through partitioning the same model. • Identify equivalent fractions. Students use Cuisenaire rods to show equivalent fractions. The whole is discussed as well as the unit fraction. • Identify and create equivalent fractions. Students solve a contextual problem using a method of their choice.</td>
</tr>
<tr>
<td>13</td>
<td>• Students compare fractions. • Students create generalizations about fraction relationships.</td>
<td>• Students play a game to build and compare fractions. Students discuss the strategies they used in the game.</td>
</tr>
<tr>
<td>Lesson</td>
<td>Lesson Objectives</td>
<td>Learning to Solve / Practicing Together Objectives</td>
</tr>
<tr>
<td>--------</td>
<td>------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------</td>
<td>-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>14</td>
<td>• Students identify and create equivalent fractions using multiplication and division.</td>
<td>• Partition the same whole for the purpose of identifying equivalent fractions. Using a rectangular area model, students find equivalent fractions for a contextual problem. The whole, unit fractions, meaning of the denominator, and equal partitioning are discussed. Equivalent fractions are found and symbolically represented.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Draw a model to find equivalent fractions. The concept of multiplication is brought forth using the model and then students are guided to finding equivalent fractions without drawing a model and using multiplication instead.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Determine if two fractions are equivalent.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Use division to find equivalent fractions. Students use a model to represent equivalent fractions by division and then find equivalent fractions using only division.</td>
</tr>
<tr>
<td>15</td>
<td>• Students simplify fractions.</td>
<td>• Identify fractions equivalent to 1/2 and 3/4. Students determine fractions equivalent to 1/2 and how 1/2 is the simplest form. Models represent this. Students generalize what being in &quot;simplest form&quot; means.</td>
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<td>• Determine how to find a fraction in simplest form. Students look at examples from previous problems and determine how to find a fraction in simplest form without using a model.</td>
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<tr>
<td>16</td>
<td>• Students simplify fractions.</td>
<td>• Simplify fractions that are divisible by multiple factors. Students are shown a rectangular area model of 12/16 and walk through the process of simplifying this fraction. Symbolic notation of the operation begin perform to arrive at the simplest fraction is demonstrated at each step along with the model.</td>
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<td>• Determine how to find a fraction in simplest form. Using the previous problem, students see how dividing by the GCF will result in the simplest fraction more efficiently. Multiplication facts and factors are used to find the simplest fraction.</td>
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<td>17</td>
<td>• Students write a mixed number from an improper fraction and an improper fraction from a mixed number. Students find equivalent fractions for a mixed number or improper fraction.</td>
<td>• Understand the relationship between a mixed number and improper fraction. Convert from a mixed number to an improper fraction. Students look at models, describe the whole, the unit fraction, # of iterations, the mixed number, and the improper fraction for each model. Writing the equivalence, students think about how to find the equivalence without using a model. Students explain why this works.</td>
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<td>• Convert an improper fraction to a mixed number. Using the equivalence from the previous problems, students determine how to convert an improper fraction to a mixed number and explain why that process works.</td>
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<tr>
<td>Lesson</td>
<td>Lesson Objectives</td>
<td>Learning to Solve / Practicing Together Objectives</td>
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| 18     | • Students add and subtract fractions with like denominators using physical representations. | • Model addition of fractions with like denominators. Discussing non-unit fractions in terms of the unit fractions, addition of the fractions is modeled. The whole is emphasized.  
• Model subtraction of fractions with like denominators. Using a model and talking about the unit fraction and the non-unit fractions in terms of iterations of the unit fraction, students subtract fractions.  
Add and subtract improper fractions and mixed numbers with like denominators. Students perform these operations by decomposing the improper fractions into iterations of the unit fraction. |
| 19     | • Students add and subtract fractions with unlike denominators using physical representations.  
• Students estimate the sum or difference of fractions with unlike denominators. | • Model addition of fractions with unlike denominators. Given the problem 1/4 + 1/8 in a contextual problem, students answer what the whole is and estimate the solution using benchmark fractions. The fractions are represented in two rectangular area models showing the fractions partitioned into the same number of units. The symbolic representation is shown along with the physical representation.  
• Model subtraction of fractions. The whole is discussed within the context of the problem. The difference is estimated using benchmark fractions. Students model the problem using their choice of model (Cuisenaire rods, number line, fraction strip, area model). |
| 20     | • Students add and subtract fractions with unlike denominators using equivalent fractions. | • Review the concept of equivalent fractions to prepare them for finding common denominators. Students look at models and symbolic representations of equivalent fractions and addition. The whole and partitioning are discussed. Students determine a method for finding equivalent fractions without drawing a model.  
• Add and subtract fractions with unlike denominators where one denominator is a multiple of the other denominator. Students estimate the solution using benchmark fractions, determine the common denominator, find the equivalent fractions, and perform the operation.  
• Add and subtract fractions with unlike denominators where one denominator is not a multiple of the other denominator. Students look at worked out problems and decide how the equivalent fractions were calculated. Connections between the physical and symbolic representations are made. |
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<th>Lesson</th>
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| 21     | • Students add and subtract mixed numbers and improper fractions with like denominators. | • Add improper fractions and mixed numbers with like denominators. Given contextual situations, students determine the whole, write the symbolic problem, and find the solution.  
• Add mixed numbers with like denominators involving regrouping. Students connect models of mixed numbers with the symbolic representation as an improper fraction and a mixed number. Using the models, students are guided to the idea of regrouping.  
• Subtract improper fractions and mixed numbers with like denominators. Given contextual situations, students determine the whole, write the symbolic problem, and find the solution. |
| 22     | • Students add and subtract mixed numbers and improper fractions with unlike denominators. | • Add mixed numbers with unlike denominators using a physical representation. Students are shown models of mixed number with unlike denominators and discuss adding the wholes and the fractional parts. The whole and partitioning are discussed.  
• Add and subtract mixed numbers with unlike denominators. Without using models, students find equivalent fractions, common denominators, and then find the sum or difference of each problem.  
• Add and subtract improper fractions with unlike denominators. Students look at a model of improper fractions, a contextual problem, and connect it to the symbolic representations. The whole and equal partitioning are discussed. |
| 23     | • Students add and subtract mixed numbers with unlike denominators involving regrouping. | • Review addition of mixed numbers with like denominators where regrouping is involved. Students look at models of mixed numbers and discuss how to add them. They also look at the same models and determine the improper fraction for each and then add them to demonstrate that each method will allow them to arrive at the correct answer.  
• Add mixed numbers with unlike denominators by converting them to improper fractions. Students convert mixed numbers to improper fractions, find the common denominator, find the equivalent fractions, and add the fractions. |
| 24     | • Students add and subtract fractions.  
• Students review the concept map from Lesson 1. | • Add and subtract fractions, mixed numbers, and improper fractions with like and unlike denominators. Students solve contextual problems, reversibility problems, and determine if solutions are correct. |
VITA

Rebecca L. Darrough was born in St. Louis, Missouri. She is the middle child of a family of eight children of which there are six girls and two boys. Education was an important component of her family life and each family member has a Master degree or higher in a variety of fields. These fields include engineering, medicine, law, microbiology, geophysics, history, and education.

Rebecca attended the University of Missouri – Columbia (Mizzou) and earned her bachelor degree in mathematics. After college, she married her husband and moved to Kentucky where he began his military career as an officer in the Army. During the subsequent years, Rebecca earned her teaching certificate in secondary mathematics, taught school, raised two girls, earned her Master degree in Computer Education, worked as a computer programmer, taught computer science, and worked as the technology coordinator for a high school. She then moved to Alaska with her husband and children where she taught high school mathematics and held the position of Data Coordinator for the local school district. While in Alaska, Rebecca earned her Educational Leadership certificate, which allowed her to serve as a K-12 school principal.

After ten years in Alaska, Rebecca returned to her home state (Missouri) and entered the PhD program in Mathematics Education at the University of Missouri – Columbia. The courses she took during this program, internships in a variety of mathematics methods courses, her research assistantship, and her experiences as a 7-12th grade mathematics teacher guided her path in studying the reasoning processes used by
upper elementary and middle grade students who have demonstrated difficulties in mathematics. Rebecca is especially interested in investigating students’ mathematics thought processes in order to provide classroom teachers with instructional strategies that will target the development of conceptual understanding of mathematics. The specific topic of fractions is of great interest as this is an area of mathematics that is critical for students to understand in order to progress to more advanced mathematics. Moreover, supporting the population of students who are struggling with mathematics, but are not diagnosed with a learning disability is a passion of Rebecca’s. She will continue to conduct research and instruct pre-service mathematics teachers as she begins her career as an Assistant Professor at Austin Peay State University.