TRACE FORMULAE IN FINITE VON NEUMANN ALGEBRAS

A Dissertation
presented to
the Faculty of the Graduate School
University of Missouri-Columbia

In Partial Fulfillment
Of the Requirements for the Degree

Doctor of Philosophy

by

ANNA SKRIPKA

Dr. Konstantin A. Makarov, Dissertation Supervisor

MAY 2007

The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled.

TRACE FORMULAE IN FINITE VON NEUMANN ALGEBRAS

Presented by Anna Skripka

A candidate for the degree of Doctor of Philosophy

And hereby certify that in their opinion it is worthy of acceptance.

Professor Konstantin A. Makarov

Professor Fritz Gesztesy

Professor Yuri Latushkin

Professor Marius Mitrea

Professor David G. Retzloff

To the memory of

my grandmothers

Maria and Olena

ACKNOWLEDGMENTS

First, I wish to thank the faculty members of the Department of Mathematics at the University of Missouri–Columbia for contributing to my education in mathematics and for helping me to develop as an independent thinker.

In particular, I would like to express my gratitude to my advisor, Konstantin A. Makarov, for leading me into this fascinating area of mathematics and generously sharing his ideas all along.

With great pleasure, I thank my husband and friends for being a constant source of support and inspiration. I am very grateful to my elder friends for their wise advice and help.

Finally, I thank everyone who made me stronger as a mathematician and individual in every way.

TABLE OF CONTENTS

Acknowledgements **ii**

Abstract **v**

Introduction **1**

1 Averaged spectral characteristics of an operator **4**

1.1 The averaged argument of a dissipative operator 4

1.1.1 The averaged logarithm of a dissipative operator 5

1.1.2 Properties of the averaged argument 12

1.2 The averaged argument as the integrated density of states 20

1.2.1 The spectrum distribution function 20

1.2.2 The integrated density of states 21

2 The ξ-index and its properties **29**

2.1 Analogy with the ξ-function theory 30

2.1.1 The ξ-index, the ξ-function, and the index of two projections . 30

2.1.2 The ξ-index and the perturbation determinant 35

2.1.3 A spectral averaging formula 43

2.2 The Birman-Schwinger principle and its consequences 48
2.2.1 The Birman-Schwinger principle 48
2.2.2 Consequences of the Birman-Schwinger principle 55

A Basic facts on von Neumann algebras 61

A.1 A trace and the relative dimension 61
A.2 The Chain Rule 65

B Basic facts on operator theory 69

B.1 General operator theory 69
B.2 A self-adjoint dilation of a dissipative operator 70

C On the Birman-Schwinger principle and scattering theory 75

C.1 The classical Birman-Schwinger principle 75
C.2 Scattering theory 78

Bibliography 81

Vita 88

TRACE FORMULAE IN FINITE VON NEUMANN ALGEBRAS

Anna Skripka

Dr. Konstantin A. Makarov, Dissertation Supervisor

ABSTRACT

The dissertation is devoted to some aspects of spectral perturbation theory in the context of finite von Neumann algebras. The central results are analogs of the Birman-Schwinger principle and the Birman-Krein formula for the ξ-index, a spectral parameter independent counterpart of Krein's spectral shift function.

The proofs of the main results are based on diverse properties of the operator logarithm and argument averaged with respect to a normal tracial state that are derived in this work. In addition, formula representations for the ξ-index and the ξ-function are obtained and the concept of the ξ-index is related to that of the spectrum distribution function for some random operators.

Introduction

In this dissertation, we explore the spectral characteristic of a dissipative operator in a finite von Neumann algebra which is obtained by averaging the operator argument with respect to a normal tracial state. Pairing such spectral characteristics for two operators creates a parameter independent finite von Neumann algebra analog of the spectral shift function playing an important role in perturbation theory. Diverse properties of the spectral shift function obtained by pairing the averaged spectral measures of self-adjoint operators are also contemplated.

This work consists of two substantial chapters concentrated on different aspects of the subject and three appendices.

In Chapter 1, we study various properties of the averaged argument of a dissipative operator including its similarity to the numerical argument and connection with the de la Harpe-Skandalis determinant of an operator. We also discuss examples of operators in von Neumann factors of type II_1 as well as relate the concept of the averaged argument to that of the spectrum distribution function. The spectrum distribution function is known to reflect important spectral properties of an operator. In particular, the spectrum distribution function carries over "non-random" characteristics of some random operators, for instance, the integrated density of states.

Chapter 2 is devoted to the ξ-index, the ξ-function, and the trace formulae associated with these objects. The cornerstone result for the ξ-index is a finite W^* analog of the Birman-Schwinger principle that "extends" now from a spectral gap to the spectrum of the operators involved and implies an analog of the Birman-Krein formula. Another interesting consequence of the new Birman-Schwinger principle is the monotonicity of an averaged function of an operator provided the initial scalar function is monotone. "Inserting" a real parameter in the ξ-index reveals further properties of this object, including an analog of the Birman-Solomyak formula.

Appendices A-C collect standard facts on finite von Neumann algebras, operator theory, the traditional Birman-Schwinger principle, and scattering theory. Moreover, the appendices include proofs of some auxiliary results employed in the dissertation and not found in the literature.

In part, Sections 1.1, 2.1, and 2.2 are based on paper [39] and work [40].

In this work, $\mathcal{H}$ denotes a separable Hilbert space, $\mathcal{B}(\mathcal{H})$ the space of linear bounded operators on $\mathcal{H}$, $\mathcal{A}$ a finite von Neumann algebra of operators in $\mathcal{B}(\mathcal{H})$, and $\tau(\cdot)$ a normal tracial state on $\mathcal{A}$ (cf. Appendix A). In the case when $\mathcal{A}$ is a finite factor, $\mathrm{Dim}(\cdot)$ denotes the relative dimension associated with $\mathcal{A}$. The symbol $E_H(\cdot)$ stands for the spectral measure of a normal operator H, $\mathbb{R}_-$ for the set $(-\infty, 0)$, and $\mathbb{C}_\pm$ for the set $\{z \in \mathbb{C} : \mathrm{Im}(z) \lessgtr 0\}$. Given an operator $H \in \mathcal{B}(\mathcal{H})$, as usual, $\sigma(H)$ denotes the spectrum and $\rho(H)$ the resolvent set of H. Symbols s-lim and n-lim indicate limits in the strong and norm operator topologies, respectively. Throughout what follows, ε denotes a positive number and $\lim_{\varepsilon \downarrow 0}$ the limit as ε approaches 0 from the right.

Chapter 1

Averaged spectral characteristics of an operator

1.1 The averaged argument of a dissipative operator

Let $\mathcal{D}_{\mathcal{A}}$ be the set of the boundedly invertible dissipative (cf. Appendix B) operators in $\mathcal{A}$.

Let $\log(H)$ denote the principal branch of the operator logarithm of $H \in \mathcal{D}_{\mathcal{A}}$ with the cut along the negative imaginary semi-axis provided by the Riesz functional calculus (cf. [22]). We remark that a useful representation for the operator logarithm of H is given by the norm-convergent Riemann integral

$$\log(H) = -\mathrm{i}\int_0^\infty \big((H + \mathrm{i}\lambda I)^{-1} - (1 + \mathrm{i}\lambda)^{-1} I\big)\, d\lambda. \tag{1.1.1}$$

If the operator H is normal, then the logarithm $\log(H)$ can be defined by the Spectral Theorem; as follows:

$$\log(H) = \int_{\sigma(H)} \log(z)\, dE_H(z). \tag{1.1.2}$$

By analogy with the self-adjoint case, we call $\mathrm{Im}(\log(H))$ the argument of an operator $H \in \mathcal{D}_{\mathcal{A}}$ and denote it $\arg(H)$. We recall (cf. [31, Lemma 2.7]) that for an operator $H \in \mathcal{D}_{\mathcal{A}}$, one has $0 \leq \arg(H) \leq \pi I$. The averaged logarithm $\tau[\log(H)]$ and the averaged argument $\tau[\arg(H)]$ play important roles in the spectral analysis of a dissipative operator H.

The first section in this chapter is devoted to properties of the averaged logarithm and argument of a dissipative operator. In the second section, some examples of finite factors (cf. Appendix A) are given and the averaged argument of a self-adjoint operator is related to the integrated density of states.

1.1.1 The averaged logarithm of a dissipative operator

In this subsection, we discuss some properties of the averaged logarithm resembling the laws of logarithm for numbers.

Theorem 1.1.1 (Law of the logarithm)**.** *Let $A, B \in \mathcal{D}_{\mathcal{A}}$ such that $AB \in \mathcal{D}_{\mathcal{A}}$. Assume, in addition, that the numerical range (cf. Appendix B) of the dissipative operator $A + B$ is contained in the sector*

$$C_{\gamma_+} = \{z \,:\, 0 \leq \arg(z) \leq \pi\gamma\}, \quad 0 \leq \gamma < 1. \tag{1.1.3}$$

Then,

$$\tau\big[\log(AB)\big] = \tau[\log(A)] + \tau[\log(B)]. \tag{1.1.4}$$

Proof. We choose some $\beta \in \big(0, \min\big\{1 - \gamma, \frac{1}{2}\big\}\big]$ and set

$$X(z) = AB + z^{\beta}(A + B) + z^{2\beta} I, \quad z \in \mathbb{C}_+,$$

with z^{β} such branch of the root that $\operatorname{Im}(z^{\beta}) \geq 0$. Then $X(z)$ is a Herglotz operator. Applying the Dixmier-Fuglede-Kadison differentiation formula (cf. Appendix A) to

$\log(X(z))$ and cyclicity of τ yields

$$\begin{aligned}\frac{d}{dz}\tau\big[\log(X(z))\big] &= \tau\left[\big(AB + z^{\beta}(A+B) + z^{2\beta}I\big)^{-1}\beta z^{\beta-1}\big(A+B+2z^{\beta}I\big)\right] \\ &= \tau\left[((A+z^{\beta}I)(B+z^{\beta}I))^{-1}\beta z^{\beta-1}(A+z^{\beta}I+B+z^{\beta}I)\right] \\ &= \beta z^{\beta-1}\tau\left[(A+z^{\beta}I)^{-1}(A+z^{\beta}I+B+z^{\beta}I)(B+z^{\beta}I)^{-1}\right] \\ &= \beta z^{\beta-1}\tau\left[(A+z^{\beta}I)^{-1}\right] + \beta z^{\beta-1}\tau\left[(B+z^{\beta}I)^{-1}\right] \\ &= \frac{d}{dz}\tau\left[\log(A+z^{\beta}I)\right] + \frac{d}{dz}\tau\left[\log(B+z^{\beta}I)\right].\end{aligned}$$

Integrating the former equalities implies

$$\tau\big[\log(AB + z^{\beta}(A+B) + z^{2\beta}I)\big] = \tau\big[\log(A+z^{\beta}I)\big] + \tau\big[\log(B+z^{\beta}I)\big] + C, \tag{1.1.5}$$

where C is some constant. Combining the asymptotic expansions

$$\begin{aligned}\tau\big[\log(AB + z^{\beta}(A+B) + z^{2\beta}I)\big] &= \log(\mathrm{i}y) + \mathcal{O}\left(\frac{1}{y}\right), \\ \tau\big[\log(A+z^{\beta}I)\big] &= \log(\mathrm{i}y) + \mathcal{O}\left(\frac{1}{y}\right), \\ \tau\big[\log(B+z^{\beta}I)\big] &= \log(\mathrm{i}y) + \mathcal{O}\left(\frac{1}{y}\right)\end{aligned}$$

as $y \to +\infty$, we infer that the constant C in (1.1.5) equals zero.

Therefore, if we prove

$$\lim_{\varepsilon\downarrow 0}\tau\big[\log(AB + (\mathrm{i}\varepsilon)^{\beta}(A+B) + (\mathrm{i}\varepsilon)^{2\beta}I)\big] = \tau[\log(AB)], \tag{1.1.6}$$

$$\lim_{\varepsilon\downarrow 0}\tau\big[\log(A + (\mathrm{i}\varepsilon)^{\beta}I)\big] = \tau[\log(A)], \tag{1.1.7}$$

$$\lim_{\varepsilon\downarrow 0}\tau\big[\log(B + (\mathrm{i}\varepsilon)^{\beta}I)\big] = \tau[\log(B)], \tag{1.1.8}$$

then it will imply (1.1.4). We will only verify (1.1.6) as (1.1.7) and (1.1.8) can be checked in a very similar way. One estimates

$$\begin{aligned}
&\left\|\log\left(AB+(\mathrm{i}\varepsilon)^{\beta}(A+B)+(\mathrm{i}\varepsilon)^{2\beta}I\right)-\log(AB)\right\| \\
&=\left\|\int_0^\infty\left((AB+(\mathrm{i}\varepsilon)^{\beta}(A+B)+(\mathrm{i}\varepsilon)^{2\beta}I+\mathrm{i}\lambda I)^{-1}-(AB+\mathrm{i}\lambda I)^{-1}\right)d\lambda\right\| \\
&\le\int_0^\delta\left\|(AB)^{-1}\right\|\Big(1-(\varepsilon^{2\beta}+\lambda)\left\|(AB)^{-1}\right\|^{-1}-\varepsilon^{\beta}\left\|(A+B)(AB)^{-1}\right\|\Big)^{-1}d\lambda \\
&+\int_0^\delta\left\|(AB)^{-1}\right\|\left(1-\lambda\left\|(AB)^{-1}\right\|\right)^{-1}d\lambda+\varepsilon^{\beta}\int_\delta^\infty\lambda^{-2}d\lambda \\
&\le\frac{2\delta}{\left\|(AB)^{-1}\right\|^{-1}-(\varepsilon^{2\beta}+\delta)-\varepsilon^{\beta}\left\|(AB)^{-1}(A+B)\right\|\left\|(AB)^{-1}\right\|^{-1}}+\frac{\varepsilon^{\beta}}{\delta},
\end{aligned}$$

for an arbitrary $0<\delta<\frac{1}{2\|(AB)^{-1}\|}$ and ε such that $0<\varepsilon^{2\beta}+\delta<\left\|(AB)^{-1}\right\|^{-1}$ and

$$\varepsilon^{3\beta}+\varepsilon^{2\beta}\left\|(AB)^{-1}(A+B)\right\|\left\|(AB)^{-1}\right\|^{-1}+\varepsilon^{\beta}\delta<\delta^2.$$

Passing to the limit as $\varepsilon\downarrow 0$ yields (1.1.6). □

Remark 1.1.2. *For normal operators A and B in $\mathcal{D}_{\mathcal{A}}$, with AB a dissipative operator, we also have*

$$\tau[\log(AB)]=\tau[\log(A)]+\tau[\log(B)], \tag{1.1.9}$$

provided the spectrum of $A+B$ is contained in the sector (1.1.3) *or A and B commute. In the first case,* (1.1.9) *follows from Theorem 1.1.1 and Theorem B.1.4 and, in the second one, upon applying the Spectral Theorem to $A=f(H)$, $B=g(H)$, with H a self-adjoint operator and f, g continuous, non-zero on the spectrum of H functions (cf. Corollary A.1.18).*

Our next result reduces computation of the averaged logarithm of a matrix defined on $\mathcal{H}\oplus\mathcal{H}$ with entries in $\mathcal{A}$ to the one of the averaged logarithms of operators on $\mathcal{H}$.

First, we recall that the set $\mathcal{A}\overline{\otimes}M_2$ of 2×2 matrices in $\mathcal{B}(\mathcal{H}\oplus\mathcal{H})$ with entries in $\mathcal{A}$ is a finite von Neumann algebra (cf. [36, Example 11.2.2 and Theorem 11.2.20]) and $\mathcal{T}(\cdot)$ given by

$$\mathcal{T}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{1}{2}\tau(A) + \frac{1}{2}\tau(D), \quad \text{for } \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{A}\overline{\otimes}M_2,$$

is a normal tracial state on it.

Hypothesis 1.1.3. *Let $K \in \mathcal{A}$, and $M, N \in \mathcal{D}_{\mathcal{A}}$. Suppose, in addition, that the operator $M - K^*N^{-1}K$ is boundedly invertible.*

Theorem 1.1.4. *Assume Hypothesis 1.1.3. Then, $\mathbf{M} = \begin{pmatrix} M & K^* \\ K & N \end{pmatrix} \in \mathcal{D}_{\mathcal{A}\overline{\otimes}M_2}$ and*

$$2\mathcal{T}\big[\log(\mathbf{M})\big] = \tau\big[\log(M - K^*N^{-1}K)\big] + \tau\big[\log(N)\big]. \tag{1.1.10}$$

Proof. We introduce an auxiliary Herglotz operator-valued function

$$z \to \mathbf{M}(z) = \begin{pmatrix} M + zI & K^* \\ K & N + zI \end{pmatrix}, \quad z \in \mathbb{C}_+,$$

with values in $\mathcal{A}\overline{\otimes}M_2$. The values $\mathbf{M}(z)$, $z \in \mathbb{C}_+$, are boundedly invertible operators in $\mathcal{A}\overline{\otimes}M_2$ and the diagonal entries of $\mathbf{M}^{-1}(z)$ are the Schur complements of the matrix $\mathbf{M}(z)$ or, equivalently, the inverses of

$$\mathcal{M}(z) = M + zI - K^*(N + zI)^{-1}K,$$

$$\mathcal{N}(z) = N + zI - K(M + zI)^{-1}K^*.$$

By the Dixmier-Fuglede-Kadison differentiation formula and the equality

$$\frac{d}{dz}\mathbf{M}(z) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

we obtain

$$\frac{d}{dz}\mathcal{T}\big[\log(\mathbf{M}(z))\big] = \mathcal{T}\big[\mathbf{M}^{-1}(z)\big] = \frac{1}{2}\tau\big[\mathcal{M}^{-1}(z)\big] + \frac{1}{2}\tau\big[\mathcal{N}^{-1}(z)\big]. \tag{1.1.11}$$

By direct computations, we get

$$\mathcal{N}^{-1}(z) = (N+zI)^{-1} + (N+zI)^{-1}K\mathcal{M}^{-1}(z)K^*(N+zI)^{-1}. \tag{1.1.12}$$

Employing the additivity and cyclicity of $\tau(\cdot)$ and representation (1.1.12) yields

$$\begin{aligned}
&\tau\big[\mathcal{M}^{-1}(z)\big] + \tau\big[\mathcal{N}^{-1}(z)\big] \\
&\quad = \tau\big[\mathcal{M}^{-1}(z)\big] + \tau\big[\mathcal{M}^{-1}(z)K^*(N+zI)^{-2}K\big] + \tau\big[(N+zI)^{-1}\big] \\
&\quad = \tau\big[\mathcal{M}^{-1}(z)(I+K^*(N+zI)^{-2}K)\big] + \tau\big[(N+zI)^{-1}\big]
\end{aligned} \tag{1.1.13}$$

Expression (1.1.13) equals

$$\frac{d}{dz}\big(\tau[\log(\mathcal{M}(z))] + \tau[\log(N+zI)]\big) \tag{1.1.14}$$

which together with (1.1.11) gives

$$\frac{d}{dz}\big(2\mathcal{T}[\log(\mathbf{M}(z))]\big) = \frac{d}{dz}\big(\tau[\log(\mathcal{M}(z))] + \tau[\log(N+zI)]\big). \tag{1.1.15}$$

Integrating (1.1.15) implies

$$2\mathcal{T}[\log(\mathbf{M}(z))] = \tau[\log(\mathcal{M}(z))] + \tau[\log(N+zI)] + C, \tag{1.1.16}$$

with C a constant. By direct computations, upon applying representation (1.1.1),

$$\begin{aligned}
C =& -\mathrm{i}\tau\left[\int_0^\infty \big(\mathcal{M}^{-1}(\mathrm{i}y+\mathrm{i}t) - (\mathcal{M}(\mathrm{i}y)+\mathrm{i}tI)^{-1}\big)\,dt\right] \\
& -\mathrm{i}\tau\left[\int_0^\infty \big(\mathcal{N}^{-1}(\mathrm{i}y+\mathrm{i}t) - (N+\mathrm{i}yI+\mathrm{i}tI)^{-1}\big)\,dt\right] \\
=& \mathcal{O}\left(\frac{1}{y}\right), \quad y\to\infty,
\end{aligned}$$

and, hence, $C = 0$. Computing the normal boundary values in (1.1.16) leads to (1.1.10). □

Corollary 1.1.5. *Assume Hypothesis 1.1.3 and let* $\mathbf{M} = \begin{pmatrix} M & K^* \\ K & N \end{pmatrix}$. *Let* U *and* W *be isometries from* $\mathcal{H}$ *into* $\mathcal{H} \oplus \mathcal{H}$ *such that* $U^*\mathbf{M}U = M$ *and* $W^*\mathbf{M}W = N$. *Then,*

$$\begin{aligned} 2\mathcal{T}[\arg(\mathbf{M})] &= \tau\left[\arg\left(\left(W^*\mathbf{M}^{-1}W\right)^{-1}\right)\right] + \tau[\arg(U^*\mathbf{M}U)] \\ &= \tau\left[\arg\left(\left(U^*\mathbf{M}^{-1}U\right)^{-1}\right)\right] + \tau[\arg(W^*\mathbf{M}W)]. \end{aligned} \tag{1.1.17}$$

Remark 1.1.6. *In the* I_∞ *setting, a relation similar to* (1.1.17) *has been recently derived in [19] for self-adjoint operators.*

Interchanging the roles of M and N, K and K^*, respectively, we arrive at the dual representation for $2\mathcal{T}\big[\log(\mathbf{M})\big]$.

Corollary 1.1.7. *Assume Hypothesis 1.1.3. Then,*

$$2\mathcal{T}[\log(\mathbf{M})] = \tau\left[\log\left(N - KM^{-1}K^*\right)\right] + \tau[\log(M)]. \tag{1.1.18}$$

Proof. One has the equality

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} M & K^* \\ K & N \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}^{-1} = \begin{pmatrix} N & K \\ K^* & M \end{pmatrix}.$$

It is straightforward to check that, for any unitary operator $U \in \mathcal{A}\overline{\otimes}M_2$ and any $H \in \mathcal{D}_{\mathcal{A}\overline{\otimes}M_2}$,

$$\log\left(UHU^{-1}\right) = U\left(\log(H)\right)U^{-1}, \tag{1.1.19}$$

which along with the invariance of $\mathcal{T}(\cdot)$ with respect to unitary transformations yields

$$\mathcal{T}\left[\log\left(UHU^{-1}\right)\right] = \mathcal{T}[\log(H)]. \tag{1.1.20}$$

Finally, Theorem 1.1.4 along with (1.1.20) implies (1.1.18). □

Another version of the law of averaged logarithm with dismissed restriction on the numerical range is implied by Theorem 1.1.4.

Corollary 1.1.8. *Assume Hypothesis 1.1.3. Then*

$$\tau\left[\log\left(M-K^*N^{-1}K\right)\right]-\tau\left[\log\left(N-KM^{-1}K^*\right)\right]=\tau[\log(M)]-\tau[\log(N)], \quad (1.1.21)$$

or equivalently,

$$\tau\left[\log\left(M-K^*N^{-1}K\right)\right]-\tau[\log(M)]=\tau\left[\log\left(N-KM^{-1}K^*\right)\right]-\tau[\log(N)]. \quad (1.1.22)$$

Remark 1.1.9. *The law of logarithm*

$$\tau(\log|AB|)=\tau(\log|A|)+\tau(\log|B|),$$

for boundedly invertible $A, B \in \mathcal{B}(\mathcal{H})$, follows from the multiplicativity of the Fuglede-Kadison determinant [28].

Evaluating the imaginary parts in (1.1.21) and (1.1.22), one obtains invariance of the averaged argument with respect to certain perturbations.

Corollary 1.1.10. *Assume Hypothesis 1.1.3. Then,*

$$\tau\big[\arg\left(N-KM^{-1}K^*\right)\big]-\tau\big[\arg\left(M-K^*N^{-1}K\right)\big]=\tau\big[\arg(N)\big]-\tau\big[\arg(M)\big], \quad (1.1.23)$$

or equivalently,

$$\tau\big[\arg\left(M-K^*N^{-1}K\right)\big]-\tau\big[\arg(M)\big]=\tau\big[\arg\left(N-KM^{-1}K^*\right)\big]-\tau\big[\arg(N)\big]. \quad (1.1.24)$$

Remark 1.1.11. *We note that a differential analog of representation* (1.1.24) *holds. The Gateaux derivative of the averaged logarithm* $\tau[\log(\cdot)]$ *at the point* M *in the direction* $-K^*N^{-1}K$ *coincides with the Gateaux derivative of* $\tau[\log(\cdot)]$ *at the point* N *in the direction* $-KM^{-1}K^*$*, that is,*

$$\frac{d}{dt}\tau\left[\log\left(M - tK^*N^{-1}K\right)\right]\Big|_{t=0} = \frac{d}{dt}\tau\left[\log\left(N - tKM^{-1}K^*\right)\right]\Big|_{t=0}. \qquad (1.1.25)$$

Proof. *Using the Dixmier-Fuglede-Kadison differentiation formula yields*

$$\frac{d}{dt}\tau\left[\log\left(M - tK^*N^{-1}K\right)\right]\Big|_{t=0} = -\tau\big(M^{-1}K^*N^{-1}K\big),$$
$$\frac{d}{dt}\tau\left[\log\left(N - tKM^{-1}K^*\right)\right]\Big|_{t=0} = -\tau\big(N^{-1}KM^{-1}K^*\big),$$

and then equality (1.1.25) *follows from the equality*

$$\tau\left(M^{-1}K^*N^{-1}K\right) = \tau\left(N^{-1}KM^{-1}K^*\right).$$

1.1.2 Properties of the averaged argument

Geometric properties of the phase. Our first result in this paragraph relates the averaged argument $\tau[\arg(A+\mathrm{i}B)]$ with the trace of the arctangent of the symmetrized version of the ratio "B versus A".

Theorem 1.1.12. *Assume that* A *is a boundedly invertible and* B *a non-negative self-adjoint operators in* $\mathcal{A}$*, with* $A + \mathrm{i}B$ *in* $\mathcal{D}_{\mathcal{A}}$*. Then,*

$$\tau[\arg(A + \mathrm{i}B)] = \pi\tau[E_A(\mathbb{R}_-)] + \tau\Big[\arctan\left(B^{1/2}A^{-1}B^{1/2}\right)\Big]. \qquad (1.1.26)$$

Proof. First, we observe that $A + \mathrm{i}B$ is a boundedly invertible dissipative operator and $A + \mathrm{i}B = A - B^{1/2}(\mathrm{i}I)^{-1}B^{1/2}$. Employing Corollary 1.1.10 implies

$$\tau[\arg(A + \mathrm{i}B)] - \tau[\arg(A)] = \tau\Big[\arg\left(\mathrm{i}I - B^{1/2}A^{-1}B^{1/2}\right)\Big] - \tau[\arg(\mathrm{i}I)], \qquad (1.1.27)$$

which by the Spectral Theorem applied to $B^{1/2}A^{-1}B^{1/2}$ equals

$$\begin{aligned}\int_{\mathbb{R}} \big(\operatorname{Im}\log(\mathrm{i}-\lambda) - \operatorname{Im}\log \mathrm{i}\big) d\tau[E_{B^{1/2}A^{-1}B^{1/2}}(\lambda)] \\ = \int_{\mathbb{R}} \operatorname{Im}\log(1+\mathrm{i}\lambda) d\tau[E_{B^{1/2}A^{-1}B^{1/2}}(\lambda)] \\ = \tau\big[\arctan\big(B^{1/2}A^{-1}B^{1/2}\big)\big].\end{aligned}$$

Finally, the Spectral Theorem implies that $\operatorname{Im}(\log(A)) = \pi E_A(\mathbb{R}_-)$ and, therefore, $\tau[\arg(A)] = \pi\tau[E_A(\mathbb{R}_-)]$. □

The next result of this section reveals an intimate connection between the averaged argument $\tau[\arg(H)]$ and the operator angle of inclination of the subspace $\mathcal{H}$ with respect to the subspace $E_L(\mathbb{R}_-)\mathcal{K}$, where H is a dissipative operator in $\mathcal{A}$ and L its minimal self-adjoint dilation (cf. Appendix B) in the Hilbert space $\mathcal{K}$.

We recall the concept of an operator angle of inclination of one subspace of a Hilbert space with respect to another one. Let (P, Q) be a pair of orthogonal projections in a Hilbert space $\mathcal{K}$. Following [43], we define the angle Θ of inclination of the subspace $\operatorname{Ran}(P)$ with respect to the subspace $\operatorname{Ran}(Q)$ in the space $\mathcal{K}$ to be the operator on $\operatorname{Ran}(P)$ satisfying the equality

$$\Theta = \arccos(PQ|_{\operatorname{Ran}(P)})^{\frac{1}{2}}. \tag{1.1.28}$$

Hypothesis 1.1.13. *Let H be a dissipative operator in $\mathcal{A}$ and L the minimal self-adjoint dilation of H in the Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. Denote by $P_{\mathcal{H}}$ the orthogonal projection in $\mathcal{K}$ onto $\mathcal{H}$.*

Theorem 1.1.14. *Assume Hypothesis 1.1.13. Let Θ be the operator angle of inclination of $\mathcal{H}$ with respect to $E_L(\mathbb{R}_-)\mathcal{K}$. Assume, in addition, that H is boundedly* *invertible. Then,*

$$\arg(H) = \pi \cos^2(\Theta). \tag{1.1.29}$$

Proof. Definition (1.1.28) implies

$$P_{\mathcal{H}} E_L(\mathbb{R}_-)\big|_{\mathcal{H}} = \cos^2(\Theta). \tag{1.1.30}$$

Then (1.1.29) follows from the representation

$$\operatorname{Im}(\log(H)) = \pi P_{\mathcal{H}} E_L(\mathbb{R}_-)\big|_{\mathcal{H}} \tag{1.1.31}$$

(cf. [31, Lemma 2.7]). □

By means of explicit (independent) computations, the following example links representations (1.1.26) and $\tau[\arg(H)] = \pi\tau\big[\cos^2(\Theta)\big]$ (cf. Theorem 1.1.14) for the averaged argument of a dissipative operator with boundedly invertible real part that acts on a one-dimensional Hilbert space. Also, this example shows that representation (1.1.29) naturally extends (1.1.26) when the real part of H is not boundedly invertible.

Example 1.1.15. Let $f(\lambda) = \frac{1}{\lambda - a + \mathrm{i}b}$, $\lambda \in \mathbb{R}$, and H be the operator acting on the subspace $\mathcal{H} = \operatorname{span}\{f\}$ of the space $\mathcal{K} = L^2(\mathbb{R}, d\lambda)$ as the multiplication by a number $a + \mathrm{i}b$, with $a, b \in \mathbb{R}$, $b > 0$. Representation (1.1.26) is an immediate consequence of the trivial observation that the argument of a complex number $a + \mathrm{i}b$ is $\arg(a)$ plus the arctangent of the quotient b/a.

The minimal self-adjoint dilation of the operator H is the operator L acting as the multiplication by an independent variable on the space $\mathcal{K}$. The latter follows from the equality

$$\big\langle (L + \mathrm{i}yI)^{-1} f, f \big\rangle = \big\langle (H + \mathrm{i}yI)^{-1} f, f \big\rangle, \quad y > 0,$$

that can be established by means of direct computations. The spectral measure of L acts as the multiplication by the characteristic function of the Borel sets of $\mathbb{R}$. It is straightforward to check that

$$\pi P_{\mathcal{H}} E_L(\mathbb{R}_-)\big|_{\mathcal{H}} = \left(\arctan\left(\frac{-a}{b}\right) + \frac{\pi}{2}\right) I. \tag{1.1.32}$$

Since $\tan\left(\arctan\left(\frac{-a}{b}\right) + \frac{\pi}{2}\right) = -\cot\left(\arctan\left(\frac{-a}{b}\right)\right) = \frac{b}{a}$, one has for $a > 0$ that

$$\pi P_{\mathcal{H}} E_L(\mathbb{R}_-)\big|_{\mathcal{H}} = \arctan\left(\frac{b}{a}\right),$$

and if $a < 0$, then

$$\pi P_{\mathcal{H}} E_L(\mathbb{R}_-)\big|_{\mathcal{H}} = \arctan\left(\frac{b}{a}\right) + \pi.$$

If $a = 0$, then (1.1.32) entails

$$\pi P_{\mathcal{H}} E_L(\mathbb{R}_-)\big|_{\mathcal{H}} = \frac{\pi}{2}.$$

We remark that in the latter case representation (1.1.26) is no longer available for $\tau[\arg(H)]$.

If H is a dissipative non-invertible operator in $\mathcal{A}$, the averaged argument $\tau[\arg(H)]$ is not well-defined. Nevertheless, the following representation for the normal boundary value of $\tau[\arg(H + \mathrm{i}\varepsilon I)]$, $\varepsilon \in \mathbb{R}_+$, $\varepsilon \downarrow 0$, is available.

Theorem 1.1.16. *Assume Hypothesis 1.1.13. Let Θ be the operator angle of inclination of the subspace $\mathcal{H}$ with respect to the subspace $E_L(\mathbb{R}_-)\mathcal{K}$. Then,*

$$\lim_{\varepsilon \downarrow 0} \tau[\arg(H + \mathrm{i}\varepsilon I)] = \pi\tau\left[\cos^2(\Theta) + \frac{1}{2} P_{\mathcal{H}} E_L(\{0\})\big|_{\mathcal{H}}\right]. \tag{1.1.33}$$

If, in addition, 0 is not an eigenvalue of a self-adjoint part of H, then the second summand in representation (1.1.33) *equals* 0.

Proof. Let $\Xi(A)$ be the Ξ operator (cf. Appendix B) associated with a dissipative operator A, that is,

$$\Xi(A) = P_{\mathcal{H}} E_{L_A}(\mathbb{R}_-)\big|_{\mathcal{H}},$$

with L_A its minimal self-adjoint dilation. In view of Lemma B.2.1,

$$s\text{-}\lim_{\varepsilon\downarrow 0} \Xi(H + \mathrm{i}\varepsilon I) = \Xi(H) + \frac{\pi}{2} P_{\mathcal{H}} E_L(\{0\})\big|_{\mathcal{H}}, \tag{1.1.34}$$

with $\frac{\pi}{2} P_{\mathcal{H}} E_L(\{0\})\big|_{\mathcal{H}} = 0$ when 0 is not an eigenvalue of a self-adjoint part of H. From (1.1.28) it follows that $\Xi(H) = \cos^2(\Theta)$, where Θ is the operator angle of inclination of the subspace $\mathcal{H}$ with respect to the subspace $E_L(\mathbb{R}_-)\mathcal{K}$. Applying τ on both sides of (1.1.34) yields (1.1.33). □

Remark 1.1.17. *If H is a self-adjoint operator, relation* (1.1.33) *simplifies to*

$$\lim_{\varepsilon\downarrow 0} \tau[\arg(H + \mathrm{i}\varepsilon I)] = \pi\tau[E_H(\mathbb{R}_-)] + \frac{\pi}{2}\tau[E_H(\{0\})] \tag{1.1.35}$$

(cf. Lemma B.2.3).

Theorem 1.1.18. *Assume that K, M, $N \in \mathcal{A}$, $\operatorname{Im} M$, $\operatorname{Im} N \geq 0$. Then,*

$$\lim_{\varepsilon\downarrow 0} \tau\left[\arg\left(M + \mathrm{i}\varepsilon I - K^*(N + \mathrm{i}\varepsilon I)^{-1}K\right)\right] \text{ exists.} \tag{1.1.36}$$

If, in addition, $0 \in \rho(M)$, then

$$\lim_{\varepsilon\downarrow 0} \tau\left[\arg\left(M - K^*(N + \mathrm{i}\varepsilon I)^{-1}K\right)\right] \text{ exists.} \tag{1.1.37}$$

Proof. Let $\mathbf{M} = \begin{pmatrix} M & K^* \\ K & N \end{pmatrix}$. Corollary 1.1.7 implies

$$\tau\left[\arg\left(N + \mathrm{i}\varepsilon I - K(M + \mathrm{i}\varepsilon I)^{-1}K^*\right)\right] = 2\,\mathcal{T}[\arg(\mathbf{M} + \mathrm{i}\varepsilon\mathbf{I})] - \tau[\arg(M + \mathrm{i}\varepsilon I)]. \tag{1.1.38}$$

The limit on the right-hand side of (1.1.38) as $\varepsilon \downarrow 0$ exists because of Theorem 1.1.16, proving (1.1.36).

Observe that invertibility of M implies that of the operator $M - K^*(N + i\varepsilon I)^{-1}K$ for any $\varepsilon > 0$. Indeed, the Herglotz operator-valued function

$$z \mapsto M - K^*(N + zI)^{-1}K$$

in the upper-half plane is invertible for $|\mathrm{Im} z|$ large enough and, therefore, it is invertible for all $z \in \mathbb{C}_+$ (cf. [29, Lemma 2.3]). It follows from Corollary 1.1.8 that

$$\begin{aligned}\tau&\left[\arg\left(M - K^*(N + i\varepsilon I)^{-1}K\right)\right] \\ &= \tau\Big[\arg\left(N + i\varepsilon I - KM^{-1}K^*\right)\Big] + \tau\Big[\arg(M)\Big] - \tau\Big[\arg(N + i\varepsilon I)\Big].\end{aligned}$$

Passing to the limit $\varepsilon \downarrow 0$ in the latter equality concludes the proof of (1.1.37). □

Remark 1.1.19. *Under the stronger requirement that $M = M^{-1} = M^*$ and $N = N^*$, the existence of the limit in* (1.1.37) *follows from the existence of the limit*

$$s\text{-}\lim_{\varepsilon\downarrow 0} \arg\left(M - K(N + i\varepsilon I)^{-1}K^*\right)$$

(cf. [18, Remark 3.1]).

Stability results. Our next goal is to show that the averaged argument is stable along a continuous path of boundedly invertible self-adjoint operators.

Lemma 1.1.20. *Assume that $[0, 1] \ni t \mapsto H_t$ is a C^1-path of boundedly invertible self-adjoint operators in $\mathcal{A}$. Then,*

$$\tau[\arg(H_1)] = \tau[\arg(H_0)].$$

Proof. Applying the Dixmier-Fuglede-Kadison differentiation formula and taking into account that $\text{Im}\,(\tau[AB]) = 0$ whenever A and B are self-adjoint operators in $\mathcal{A}$, we obtain

$$\tau[\arg(H_1)] - \tau[\arg(H_0)] = \int_0^1 \text{Im}\left(\tau\left[H_t^{-1}\dot{H}_t\right]\right) dt = 0.$$

□

Using this result it is not hard to establish that the averaged argument of a self-adjoint boundedly invertible operator is stable under small self-adjoint perturbations.

Lemma 1.1.21. *Assume that H_0 and H_1 are self-adjoint operators in $\mathcal{A}$. Assume, in addition, that H_0 is boundedly invertible and*

$$\|H_1 - H_0\| < \text{ dist}(\sigma(H_0), 0).$$

Then,

$$\tau[\arg(H_1)] = \tau[\arg(H_0)].$$

Proof. This is a direct consequence of Lemma 1.1.20 applied to the smooth path

$$[0, 1] \ni t \mapsto H_t = H_0 + t(H_1 - H_0).$$

□

Now we are ready to establish a criterium for the averaged arguments of two boundedly invertible self-adjoint operators in a factor of type II_1 to coincide.

Theorem 1.1.22. *(i) Let $t \mapsto H_t$, $t \in [0,1]$, be a continuous path of self-adjoint boundedly invertible operators in $\mathcal{A}$. Then,*

$$\tau[\arg(H_1)] = \tau[\arg(H_0)].$$

(ii) If in addition, $\mathcal{A}$ is a factor of type II_1, the equality $\tau[\arg(H_1)] = \tau[\arg(H_0)]$ implies the existence of a continuous path $[0,1] \ni t \mapsto H_t$ of self-adjoint boundedly invertible operators in $\mathcal{A}$ that connects H_0 and H_1.

Proof. (i) Since the path $[0,1] \ni t \mapsto H_t$ is continuous, there exists a natural number N such that

$$\left\| H_{\frac{k+1}{N}} - H_{\frac{k}{N}} \right\| < \inf_{t\in[0,1]} \operatorname{dist}(\sigma(H_t), 0), \quad k = 0, 1, ..., N-1.$$

By Lemma 1.1.21,

$$\tau\left[\arg\left(H_{\frac{k+1}{N}}\right)\right] = \tau\left[\arg\left(H_{\frac{k}{N}}\right)\right], \quad k = 0, 1, ..., N-1,$$

and hence $\tau[\arg(H_1)] = \tau[\arg(H_0)]$, completing the proof of part (i).

(ii) Since for every boundedly invertible self-adjoint operator H in $\mathcal{A}$, the continuous path of boundedly invertible self-adjoint operators given by

$$[0,1] \ni s \mapsto J_s = s(H - \operatorname{sign}(H)) + \operatorname{sign}(H),$$

connects H and $\operatorname{sign}(H)$, the signature operator given by

$$\operatorname{sign}(H) = (-E_H(\mathbb{R}_-)) \oplus E_H((0,\infty)),$$

it is sufficient to prove the assertion in the particular case of $H_j = \operatorname{sign}(H_j)$, $j = 0, 1$. Assume that this is the case.

Since $\tau[\arg(H_1)] = \tau[\arg(H_0)]$, the projections $E_{H_0}(\mathbb{R}_-)$ and $E_H(\mathbb{R}_-)$ are equivalent relative to $\mathcal{A}$. Therefore, there exists a unitary operator U, $U \in \mathcal{A}$, such that

$$E_H(\mathbb{R}_-) = U E_{H_0}(\mathbb{R}_-) U^{-1},$$

and hence,

$$\operatorname{sign}(H_1) = U \operatorname{sign}(H_0)\, U^{-1}.$$

Let $U = e^{\mathrm{i}A}$ for some self-adjoint bounded operator A with spectrum in the interval $[0, 2\pi]$ such that the point 2π is not an eigenvalue of A. Then,

$$[0,1] \ni t \mapsto S_t = e^{\mathrm{i}At}[\operatorname{sign}(H_0)]e^{-\mathrm{i}At}$$

is a continuous path of self-adjoined boundedly invertible operators connecting the operators $\operatorname{sign}(H_0)$ and $\operatorname{sign}(H_1)$. □

1.2 The averaged argument as the integrated density of states

In this section, only self-adjoint elements of the algebra $\mathcal{A}$ are considered and an additional assumption that the state $\tau(\cdot)$ is faithful is made. We survey examples of operators in factors of type II_1 and discuss "non-random" characteristics of some random operators.

1.2.1 The spectrum distribution function

Definition 1.2.1. The function

$$\mathbb{R} \ni \lambda \mapsto n_H(\lambda) = \tau\big[E_H((-\infty, \lambda))\big]$$

is called the spectrum distribution function (SDF) of a self-adjoint operator H in $\mathcal{A}$.

The lemma below (cf. [58]) shows that the spectrum of a self-adjoint operator is the set of all points of growth of the SDF, with the eigenvalues the jump points of the SDF.

Lemma 1.2.2. *Let $H = H^* \in \mathcal{A}$. Then,*

$$\sigma(H) = \{\lambda \in \mathbb{R} \,:\, n_H(\lambda + \varepsilon) - n_H(\lambda - \varepsilon) > 0,\ \forall \varepsilon > 0\}$$

and

$$\tau\big[E_H\{\lambda\}\big] = \lim_{\varepsilon \downarrow 0} \big[n_H(\lambda + \varepsilon) - n_H(\lambda - \varepsilon)\big].$$

It follows from the faithfulness of the state that the SDF is the distribution function for a measure in the maximal spectral type of the corresponding operator.

Lemma 1.2.3. *The measure $\tau\big[E_H(\cdot)\big]$ is in the maximal spectral type of $H = H^* \in \mathcal{A}$.*

For some classes of random as well as non-random operators, the SDF is known to coincide with the integrated density of states. The integrated density of states (IDS), whenever it can be defined, is a volume independent function that counts the average number of the energy levels of a system per unit volume (cf. [16]). Rigorous definitions will be given for some particular models described in the next subsection. The IDS has a physical importance as it can be measured experimentally in some cases (e.g., for crystals). In mathematics, estimates for the IDS were crucial in technical results on spectral properties, in particular, localization, of random Hamiltonians.

1.2.2 The integrated density of states

Discrete magnetic Laplacians. Let $\mathcal{H} = l^2\left(\mathbb{Z}^2\right)$ and $\alpha \notin \mathbb{Q}$. Define the operators

$$(U_\alpha u)_{m,n} = e^{-\mathrm{i}\pi\alpha n} u_{m+1,n}, \quad (V_\alpha u)_{m,n} = e^{\mathrm{i}\pi\alpha m} u_{m,n+1}.$$

It was proved in [58] that the W^* algebra $\mathcal{A} = \overline{\left\{\sum_{i=-m}^{m}\sum_{j=-n}^{n} c_{ij}U_\alpha^i V_\alpha^j\right\}}^{w}$ generated by the operators U_α and V_α is a factor of type II_1 with the relative trace (cf. Appendix A) given by the formula $\tau(H) = \langle H\delta_0, \delta_0\rangle$.

The discrete magnetic Laplacian for a lattice electron in an irrational magnetic field is given by the formula

$$\Delta_{\alpha,\mu} = U_\alpha + U_\alpha^* + \mu V_\alpha + \mu V_\alpha^*, \quad \mu \in \mathbb{R}.$$

Let $\{\Lambda_n\}$ be a blowing sequence of finite sets in $\mathbb{Z}^2$, $H_{\Lambda_n} = \chi_{\Lambda_n}\Delta_{\alpha,\mu}\big|_{l^2(\Lambda_n)}$, and $N_{H_{\Lambda_n}}(\lambda)$ the number of eigenvalues, counting multiplicity, of H_{Λ_n} that do not exceed λ. For any $R \geq 1$, let

$$(\partial\Lambda_n)_R = \{x \,:\, x \in \Lambda_n,\ \mathrm{dist}(x, \mathbb{Z}^2 - \Lambda_n) \leq R\}.$$

We write $\Lambda_n \nearrow \infty$ if for any fixed $R \geq 1$,

$$\lim_{n\to\infty} \frac{\mathrm{card}\big((\partial\Lambda_n)_R\big)}{\mathrm{card}(\Lambda_n)} = 0.$$

Definition 1.2.4. The IDS for $\Delta_{\alpha,\mu}$ is defined by

$$N_{\Delta_{\alpha,\mu}}(\lambda) = \lim_{\Lambda_n \nearrow \infty} \frac{N_{H_{\Lambda_n}}(\lambda)}{\mathrm{card}(\Lambda_n)}$$

with the limit independent of choice $\Lambda_n \nearrow \infty$.

The existence of the limit in the definition of the IDS for $\Delta_{\alpha,\mu}$ was proved in [58]. Moreover, the IDS can be expressed in terms of the relative trace τ.

Theorem 1.2.5 ([58, Theorem 3.2])**.** $N_{\Delta_{\alpha,\mu}}(\lambda) = n_{\Delta_{\alpha,\mu}}(\lambda), \quad \lambda \in \mathbb{R}.$

Random operators indexed by Delone sets. Delone sets are used to model the long-range aperiodic order of quasicrystals, with the points interpreted as the positions of the atoms of the quasicrystals.

Definition 1.2.6. $\omega \subset \mathbb{R}^d$ is called a Delone set if $\exists\, 0 < r_0 < r_1$ such that $\forall p \in \mathbb{R}^d$, $\operatorname{card}\big(B_{r_0}(p) \cap \omega\big) \leq 1$ and $\operatorname{card}\big(B_{r_1}(p) \cap \omega\big) \geq 1$, with $B_{r_0}(p)$ denoting the ball of radius r_0 centered at p.

Let Ω be a closed set in the *natural topology* (cf. [44] for definitions and details) of Delone sets ω in $\mathbb{R}^d$ invariant under translations, that is, $\omega \in \Omega$ implies $T_t\omega = \omega + t \in \Omega$, for all $t \in \mathbb{R}^d$. The pair $(\Omega, \{T_t\}_t)$ is called a Delone dynamical system.

Definition 1.2.7. A Delone dynamical system is said to be of finite type if

$$\operatorname{card}\big((\omega - \omega) \cap B_R(p)\big) < \infty, \quad \forall R > 0, p \in \mathbb{R}^d, \omega \in \Omega,$$

and is said to be aperiodic if

$$\omega + t = \omega \Rightarrow t = 0.$$

Assume, in addition, that the Delone dynamical system is uniquely ergodic, that is, there is a unique probability measure P on the Borel subsets of Ω invariant with respect to $\mathbb{R}^d$-translations. Consider the Hilbert space $\mathcal{H} = \int_\Omega \oplus\, l^2(\omega)\, dP(\omega)$ and the factor $\mathcal{A} = \big\{\mathbb{H} = \int_\Omega \oplus H_\omega\, dP(\omega)\big\}$ of type II_1 (cf. [45]), with $\{H_\omega\}$ a weakly-measurable essentially bounded family of operators satisfying the covariance condition

$$H_{\omega+t} = U_t H_\omega U_t^*, \text{ with } U_t : l^2(\omega) \to l^2(\omega + t) \text{ induced by the translation } t \mapsto \omega + t.$$

Theorem 1.2.8 ([45, Proposition 2.1])**.** *Let* $\mathbb{H} \in \mathcal{A}$. *Then,*

$$\operatorname{supp} \tau\big[E_{\mathbb{H}}(\cdot)\big] = \sigma(H_\omega) \quad \text{for a.e. } \omega \in \Omega.$$

Definition 1.2.9 ([45, Definition 3.1])**.** An operator $\mathbb{H} \in \mathcal{A}$ is said to be an operator of finite range if there exists $R > 0$ such that

$$\langle H_\omega \delta_x, \delta_y \rangle = 0 \text{ whenever } x, y \in \omega \text{ and } |x - y| \geq R,$$

$$\langle H_{\omega+t} \delta_{x+t}, \delta_{y+t} \rangle = \langle H_{\widetilde{\omega}} \delta_x, \delta_y \rangle \text{ for } \omega \cap B_R(x+t) = \widetilde{\omega} \cap B_R(x) + t \text{ and } x, y \in \widetilde{\omega} \in \Omega.$$

Operators of finite range in $\mathcal{A}$ generate a C^* subalgebra $\mathcal{A}_{fr}$ of the W^* algebra $\mathcal{A}$ (cf. [45]).

Let $\{\Lambda_n\}$ be a blowing sequence of bounded sets in $\mathbb{R}^d$. For any $R \geq 1$, let

$$(\partial \Lambda_n)_R = \{x \,:\, x \in \mathbb{R}^d,\ \operatorname{dist}(x, \partial \Lambda_n) \leq R\}.$$

We write $\Lambda_n \nearrow \infty$ if for any fixed $R \geq 1$,

$$\lim_{n \to \infty} \frac{\operatorname{card}\big((\partial \Lambda_n)_R\big)}{\operatorname{card}(\Lambda_n)} = 0.$$

The sequence $\{\Lambda_n\}$ is called a van Hove sequence.

Let H_{ω,Λ_n} be the restriction of a self-adjoint operator H_ω onto $l^2(\Lambda_n \cap \omega)$ and $N_{H_{\omega,\Lambda_n}}(\lambda)$ the number of the eigenvalues of H_{ω,Λ_n}, counting multiplicity, that do not exceed λ.

Definition 1.2.10. Let $\{H_\omega\}_\omega$ be a family of operators such that the self-adjoint operator $\mathbb{H} = \int_\Omega \oplus H_\omega dP(\omega)$ is of finite range. The IDS is defined for a.e. operator H_ω by

$$N_{H_\omega}(\lambda) = \lim_{\Lambda_n \nearrow \infty} \frac{N_{H_{\omega,\Lambda_n}}(\lambda)}{\operatorname{card}(\Lambda_n)},$$

with the limit independent of the choice of a van Hove sequence $\Lambda_n \nearrow \infty$.

The limit defining IDS for H_ω is the same for almost all $\omega \in \Omega$.

Theorem 1.2.11 ([45, Theorem 3.2])**.** *Let $(\Omega, \{T_t\}_t)$ be a uniquely ergodic aperiodic Delone dynamical system of finite type and $\mathbb{H} \in \mathcal{A}_{fr}$ a self-adjoint operator. Then,*

$$N_{H_\omega}(\lambda) = n_{\mathbb{H}}(\lambda), \quad \lambda \in \mathbb{R}, \quad \text{for a.e. } \omega \in \Omega.$$

Remark 1.2.12 ([45, Theorem 3.3])**.** *If, in addition, the dynamical system $(\Omega, \{T_t\}_t)$ is minimal, that is, $\overline{\{\omega + t \,:\, t \in \mathbb{R}^d\}} = \Omega$, for all $\omega \in \Omega$, then the limit defining the IDS is uniform in ω.*

A measure-theoretic example (cf. [48]). Let $(\Omega, \mathfrak{B}, P)$ be a measure space with a non-atomic probability measure P. Assume that $(\Omega, \mathfrak{B}, P)$ is countably separated, that is, there is a sequence $\{E_1, E_2, \dots\}$ of non-empty sets $\in \mathfrak{B}$ such that whenever $\omega, \omega' \in \Omega$, $\omega \neq \omega'$, there is an integer j for which $\omega \in E_j$, $\omega' \notin E_j$.

Let $G = \{g_1, g_2, \dots\}$ be a countable group acting by measure-preserving automorphisms $\{T_g\}_{g\in G}$ ergodically on $(\Omega, \mathfrak{B}, P)$. Suppose that G acts freely on Ω, that is, $\{\omega \in \Omega \,:\, T_g(\omega) = \omega \text{ for all } g \in G\}$ is a null set.

An example of a pair $(\Omega, \mathfrak{B}, P)$, $\{T_g\}_{g\in G}$ with the properties described above is the set $[0,1)$ with the Lebesgue measure on the σ-algebra of the Borel subsets of $[0,1)$ and the group of all rational translations, modulo 1, of $[0,1)$, that is, $T_g\omega = \{\omega + g\}$, where $g \in \mathbb{Q} \cap [0,1)$ and $\{a\}$ denotes the fractional part of the real number a.

Let us consider the Hilbert space $\mathcal{H} = \int_\Omega \oplus l^2(G)\, dP(\omega)$. Define the bounded linear operators U_g, V_ϕ, X_g, Y_ϕ on $\mathcal{H}$ for any $g \in G$ and bounded Borel measurable function ϕ on Ω by

$$U_g u = \int_\Omega \sum_{h\in G} u_{h-g}(\omega)\, dP(\omega), \quad V_\phi u = \int_\Omega \sum_{h\in G} \phi(T_{-h}\omega) u_h(\omega)\, dP(\omega),$$

$$X_g u = \int_\Omega \sum_{h\in G} u_{h+g}(T_g\omega)\, dP(\omega), \quad Y_\phi u = \int_\Omega \phi(\omega) \sum_{h\in G} u_h(\omega)\, dP(\omega).$$

On the strength of [48, Lemma I.12.3.3], the W^* algebra generated by all X_g, Y_ϕ is a commutant of the W^* algebra generated by all U_g, V_ϕ; denote the first algebra by $\mathcal{A}'$ and the second one by $\mathcal{A}$. The algebras $\mathcal{A}$ and $\mathcal{A}'$ are factors (cf. [48, Lemma I.12.3.4]) of type II_1 (cf. [48, Lemma I.13.1.2]). From [48, Lemma I.12.4.3] and [48, Theorem I.II], it follows that there is $e \in \mathcal{H}$ such that $\tau(\mathbb{H}) = \langle \mathbb{H}e, e\rangle$ for any $\mathbb{H} \in \mathcal{A}$. Moreover, it follows from [48, Lemma I.12.3.2 and Lemma I.12.4.1], that $e_g(\omega) \equiv \delta_0(g)$.

Lemma 1.2.13. *Let $\mathbb{H} = \int_\Omega \oplus H_\omega dP(\omega)$ be a self-adjoint operator in $\mathcal{A}$. Then,*

$$\tau\big[E_{\mathbb{H}}(\cdot)\big] = \mathbb{E}_\omega \big\langle E_{H_\omega}(\cdot)\delta_0, \delta_0\big\rangle_{l^2(G)},$$

with $\mathbb{E}_\omega$ the mathematical expectation in ω.

Proof. It is straightforward to see that

$$\tau[E_{\mathbb{H}}(\cdot)] = \langle E_{\mathbb{H}}(\cdot)e, e\rangle_{\mathcal{H}} = \int_\Omega \sum_{g\in G} \langle E_{H_\omega}(\cdot)\delta_0, \delta_0\rangle_{l^2(G)}\; dP(\omega) = \mathbb{E}_\omega \left\langle E_{H_\omega}(\cdot)\delta_0, \delta_0\right\rangle_{l^2(G)}.$$

□

An ergodic Schrödinger operator. Let $\{\phi(g,\cdot)\}_{g\in\mathbb{Z}^d}$ be an independent identically distributed sequence of bounded (real-valued) random variables, $\{g_i\}_{i=1}^d$ the set of canonical generators of $G = \mathbb{Z}^d$, $(\Omega, \mathfrak{B}, P)$ the space $\Omega = \mathbb{R}^{\mathbb{Z}^d}$ with the σ-algebra $\mathfrak{B} = (Cyl)_\sigma$ generated by the algebra of cylinders on $\mathbb{R}^{\mathbb{Z}^d}$ and $P = \otimes_{g\in\mathbb{Z}^d} P_{\phi(g,\cdot)}$ the compactly supported product measure of distributions of $\phi(g,\cdot)$, $g \in \mathbb{Z}^d$. Let

$$U_g u = \int_\Omega \sum_{h\in G} u_{h-g}(\omega)\, dP(\omega), \quad Mu = \int_\Omega \sum_{h\in G} \omega_h u_h(\omega)\, dP(\omega),$$

$$\mathbb{H} = \sum_{i=1}^{d} (U_{g_i} + U_{g_i}^*) + M. \tag{1.2.1}$$

These operators fall in the framework of the previous example. Indeed, the coordinate translations $\{\omega_h\} \mapsto T_g\{\omega_h\} = \{\omega_{h-g}\}$ on $\mathbb{R}^{\mathbb{Z}^d}$ act ergodically and freely on the measure space $(\mathbb{R}^{\mathbb{Z}^d}, (Cyl)_\sigma, \otimes_{g\in\mathbb{Z}^d} P_{\phi(g,\cdot)})$ and $M = V_\phi$, with $\phi(\omega) = \omega_0$.

Let $\{\Lambda_n\}$ be a sequence of blowing cubes in $\mathbb{Z}^d$ centered at the origin. Denote by H_{ω,Λ_n} the restriction $\chi_{\Lambda_n} H_\omega\big|_{l^2(\Lambda_n)}$ of an operator H_ω onto $l^2(\Lambda_n)$. Let $N_{H_{\omega,\Lambda_n}}(\lambda)$ be the number of the eigenvalues of H_{ω,Λ_n}, counting multiplicity, that do not exceed λ.

Definition 1.2.14. The IDS for a.e. H_ω in the direct integral decomposition of the operator $\mathbb{H} = \int_\Omega \oplus H_\omega dP(\omega)$ in (1.2.1) is defined by

$$N_{H_\omega}(\lambda) = \lim_{\Lambda_n \nearrow \infty} \frac{N_{H_{\omega,\Lambda_n}}(\lambda)}{\text{volume}(\Lambda_n)},$$

with the limit independent of choice of a sequence $\Lambda_n \nearrow \infty$.

Lemma 1.2.15. *Let $\{H_\omega\}_\omega$ be such family of operators that $\mathbb{H} = \int_\Omega \oplus H_\omega\, dP(\omega)$ satisfies* (1.2.1)*. Then, for a.e. $\omega \in \Omega$,*

$$N_{H_\omega}(\lambda) = n_{\mathbb{H}}(\lambda), \quad \lambda \in \mathbb{R}.$$

Proof. Applying Lemma 1.2.13 implies

$$n_{\mathbb{H}}(\lambda) = \mathbb{E}_\omega \big\langle E_{H_\omega}((-\infty, \lambda))\delta_0, \delta_0 \big\rangle_{l^2(G)}.$$

It follows from [25] (also, cf. [52, Theorem 4.8 and Theorem 4.11]) that the latter equals $N_{H_\omega}(\lambda)$ for a.e. $\omega \in \Omega$. □

Chapter 2

The ξ-index and its properties

Let H_0 and H be dissipative elements in $\mathcal{A}$. We define the ξ-index associated with the pair (H_0, H) by

$$\xi(H_0, H) = \lim_{\varepsilon \downarrow 0} \left(\frac{1}{\pi} \tau[\arg(H + \mathrm{i}\varepsilon I)] - \frac{1}{\pi} \tau[\arg(H_0 + \mathrm{i}\varepsilon I)] \right).$$

We note that the limit exists in view of Theorem 1.1.16. In particular, for self-adjoint operators H_0 and H in $\mathcal{A}$,

$$\xi(H_0, H) = \tau[E_H(\mathbb{R}_-)] - \tau[E_{H_0}(\mathbb{R}_-)] + \frac{1}{2}\tau\big[E_H(\{0\})\big] - \frac{1}{2}\tau\big[E_{H_0}(\{0\})\big]$$

(cf. Remark 1.1.17). Moreover, if $0 \in \rho(H_0) \cap \rho(H)$, then the ξ-index associated with the pair (H_0, H) equals

$$\xi(H_0, H) = \frac{1}{\pi}\tau[\arg(H)] - \frac{1}{\pi}\tau[\arg(H_0)].$$

We will widely use without explicit mention the properties of the ξ-index collected in the lemma below.

Lemma 2.0.16. *Let H, H_0, and H_1 be dissipative operators in $\mathcal{A}$. Then,*

- $\xi(H_0, H) \in [-1, 1]$,

- $\xi(H_0, H) = -\xi(H, H_0)$,
- $\xi(H_0, H) = \xi(H_0, H_1) + \xi(H_1, H)$.

Proof. Statement (i) follows from [31, Lemma 2.7]. Properties (ii) and (iii) are straightforward. □

The first section in this chapter concerns representations and properties of the ξ-index and ξ-function (Lifshits-Krein spectral shift function) in the finite von Neumann algebra setting that strongly resemble those for the traditional ξ-function (cf. Appendix C) collected in [11] and do not seem to follow from the law of averaged logarithm (invariance principle) (1.1.22). The second section discusses properties of the ξ-index that are based on the invariance principles (1.1.22) and (1.1.24).

2.1 Analogy with the ξ-function theory

2.1.1 The ξ-index, the ξ-function, and the index of two projections

The spectral shift function. For a pair of self-adjoint operators in $\mathcal{A}$, the spectral shift function (the ξ-function) is defined as the function satisfying the theorem below.

Theorem 2.1.1 ([5, Theorem 3.1]). *Let H_0 and H be self-adjoint elements in $\mathcal{A}$. Then, there exists a unique function $\xi(\cdot, H_0, H) \in L^1(\mathbb{R})$ such that for each*

$$f(\lambda) = const + \int_{\mathbb{R}} \frac{e^{\mathrm{i}t\lambda} - 1}{\mathrm{i}t} \, d\omega(t),$$

where $\omega(t)$ is a complex-valued function of bounded variation and $\int d|\omega(t)| < \infty$,

$$\tau[f(H) - f(H_0)] = -\int_{\mathbb{R}} f'(\lambda)\xi(\lambda, H_0, H) \, d\lambda. \qquad (2.1.1)$$

Remark 2.1.2. *Formula* (2.1.1) *is called Krein's trace formula. Originally, it was proved in [41] for self-adjoint operators* H_0 *and* H*, with* $H - H_0$ *a trace class operator. In the context of a semi-finite von Neumann algebra, Theorem 2.1.1 was proved in [18] for* H_0 *and* H *in the algebra and extended in [5] to* H_0 *and* H *affiliated with the algebra. It has been observed in [5] that Theorem 2.1.1 holds for an absolutely continuous function* f *such that* $f' \in L^1(\mathbb{R})$ *and self-adjoint operators* H_0 *and* H *affiliated with* $\mathcal{A}$*.*

Corollary 2.1.3. *Let* H_0 *and* H *be self-adjoint elements in* $\mathcal{A}$*. Then,*

$$\int_{\mathbb{R}} \xi(\lambda, H_0, H)\, d\lambda = \tau(H - H_0).$$

Proof. The result follows upon applying Krein's trace formula to a smooth function f satisfying $f(\lambda) = \begin{cases} \lambda, & \lambda \in [a,b] \supset \sigma(H_0) \cup \sigma(H), \\ 0, & \lambda \in \mathbb{R} \setminus [a,b]. \end{cases}$ □

Lemma 2.1.4. *Let* H_0 *and* H *be self-adjoint elements in* $\mathcal{A}$ *and* f *a function of bounded variation on* $[a,b]$*, where* $[a,b]$ *contains* $\sigma(H_0) \cup \sigma(H)$*. Then,*

$$\tau\left[f(H) - f(H_0)\right] = -\int_a^b \Big(\tau\big[E_H((-\infty,\lambda))\big] - \tau\big[E_{H_0}((-\infty,\lambda))\big]\Big)\, df(\lambda).$$

Moreover, if $f \in C^1[a,b]$*, then*

$$\tau\left[f(H) - f(H_0)\right] = -\int_a^b f'(\lambda)\Big(\tau\big[E_H((-\infty,\lambda))\big] - \tau\big[E_{H_0}((-\infty,\lambda))\big]\Big)\, d\lambda.$$

Proof. Applying the Spectral Theorem leads to the equality

$$\tau\left[f(H) - f(H_0)\right] = \int_a^b f(\lambda)\, d\Big(\tau\big[E_H((-\infty,\lambda))\big] - \tau\big[E_{H_0}((-\infty,\lambda))\big]\Big). \tag{2.1.2}$$

As it follows from [21, Theorem 17.4], the Stieltjes integral

$$\int_a^b \Big(\tau\big[E_H((-\infty,\lambda))\big] - \tau\big[E_{H_0}((-\infty,\lambda))\big]\Big)\, df(\lambda) \tag{2.1.3}$$

exists. By the integration by parts argument (cf. [21, Theorem 17.8]), (2.1.3) equals

$$-\int_a^b f(\lambda)\, d\Big(\tau\big[E_H((-\infty,\lambda))\big] - \tau\big[E_{H_0}((-\infty,\lambda))\big]\Big). \tag{2.1.4}$$

Combining (2.1.2) - (2.1.4) proves the first claim of the lemma. For $f \in C^1[a,b]$, (2.1.3) equals

$$\int_a^b f'(\lambda)\Big(\tau\big[E_H((-\infty,\lambda))\big] - \tau\big[E_{H_0}((-\infty,\lambda))\big]\Big)\, d\lambda. \tag{2.1.5}$$

Thus, combining (2.1.2), (2.1.3), and (2.1.5) completes the proof of the lemma.

□

Computation of the ξ-function associated with pairs of operators in a finite von Neumann algebra can be reduced to the one of the spectrum distribution functions of the operators, as suggested in [46].

Lemma 2.1.5. *Let H_0 and H be self-adjoint elements in $\mathcal{A}$. Then,*

$$\xi(\lambda, H_0, H) = \tau\big[E_H((-\infty,\lambda))\big] - \tau\big[E_{H_0}((-\infty,\lambda))\big] \quad \textit{for a.e. } \lambda \in \mathbb{R}.$$

Proof. For f satisfying Theorem 2.1.1, one has

$$\tau\left[f(H) - f(H_0)\right] = -\int_{\mathbb{R}} f'(\lambda)\xi(\lambda, H_0, H)\, d\lambda, \tag{2.1.6}$$

which along with Lemma 2.1.4 and the uniqueness of the ξ-function implies the result.

□

It follows immediately from Lemma 2.1.5 that the ξ-index and the ξ-function are closely related objects.

Corollary 2.1.6. *For H_0 and H self-adjoint elements in $\mathcal{A}$,*

$$\xi(\lambda, H_0, H) = \xi(H_0 - \lambda I, H - \lambda I) \quad \textit{for a.e. } \lambda \in \mathbb{R}.$$

The Fredholm index for a pair of orthogonal projections. Let P and Q be two orthogonal projections in $\mathcal{A}$. Denote

$$\begin{aligned}
\mathfrak{M}_{pq} &:= \left\{ f \in \mathfrak{H} \middle| Pf = pf,\ Qf = qf \right\}, \quad p, q = 0, 1, \\
\mathfrak{M}'_0 &:= \operatorname{Ran}(P) \ominus\ (\mathfrak{M}_{10}\ \oplus\ \mathfrak{M}_{11}), \\
\mathfrak{M}'_1 &:= \operatorname{Ran}\left(P^{\perp}\right) \ominus\ (\mathfrak{M}_{00}\ \oplus\ \mathfrak{M}_{01}), \\
\mathfrak{M}' &:= \mathfrak{M}'_0 \oplus \mathfrak{M}'_1, \\
P' &:= P|_{\mathfrak{M}'}, \\
Q' &:= Q|_{\mathfrak{M}'}.
\end{aligned}$$

The space $\mathcal{H}$ admits the canonical orthogonal decomposition (cf. [34])

$$\mathcal{H} = \mathfrak{M}_{00}\ \oplus\ \mathfrak{M}_{01}\ \oplus\ \mathfrak{M}_{10}\ \oplus\ \mathfrak{M}_{11}\ \oplus\ \mathfrak{M}'. \tag{2.1.7}$$

With respect to this decomposition, the projections P and Q read

$$\begin{aligned}
P &= 0 \oplus \quad 0 \quad \oplus I_{\mathfrak{M}_{10}} \oplus I_{\mathfrak{M}_{11}} \oplus P', \\
Q &= 0 \oplus I_{\mathfrak{M}_{01}} \oplus \quad 0 \quad \oplus I_{\mathfrak{M}_{11}} \oplus Q'.
\end{aligned}$$

Definition 2.1.7. For P and Q, orthogonal projections in $\mathcal{A}$, the τ-Fredholm index is defined by

$$\mathrm{ind}_\tau(P,Q) = \tau(I_{\mathfrak{M}_{01}} - I_{\mathfrak{M}_{10}}).$$

Remark 2.1.8. *The quantity*

$$\mathrm{ind}_\tau(P,Q) \tag{2.1.8}$$

contains the essence of the spectral flow. Indeed, if the projections P and Q commute, then (2.1.8) *coincides with*

$$\tau(Q - PQ) - \tau(P - PQ)$$

which, informally, means "the amount of nonnegative spectrum gained minus the amount of non-negative spectrum lost".

The notion of the spectral flow in the case of a I_∞ factor was introduced by M. F. Atiyah, V. K. Patodi, and I. M. Singer [1, 2, 3]. According to their definition, the spectral flow "is the number of eigenvalues counted with multiplicities which pass through 0 *in the positive direction minus the number of those which pass through* 0 *in the negative direction as one moves from the initial point of the path to the final one". An extensive exposition of the spectral flow in the II_∞ setting can be found in [7] and literature cited therein. Relation of the ξ-function to the spectral flow is discussed in [4].*

Below, we obtain a convenient representation for the Fredholm index of orthogonal projections in $\mathcal{A}$ and relate the Fredholm index to the ξ-index.

Lemma 2.1.9. *For P and Q, orthogonal projections in $\mathcal{A}$,*

$$\mathrm{ind}_\tau(P,Q) = \tau(Q-P).$$

Proof. It follows from [14, Propositions 3.4 and 4.1] that P' and Q' are unitarily equivalent with the corresponding unitary operator in $\mathcal{A}$. Therefore,

$$\tau(Q'-P') = 0$$

and

$$\tau(Q-P) = \tau(I_{\mathfrak{M}_{01}} - I_{\mathfrak{M}_{10}}).$$

□

Theorem 2.1.10. *Given a pair of self-adjoint elements H_0 and H in $\mathcal{A}$,*

$$\xi(H_0,H) = \mathrm{ind}_\tau(E_{H_0}([0,\infty)), E_H([0,\infty))) + \frac{1}{2}\mathrm{ind}_\tau\big(E_H(\{0\}), E_{H_0}(\{0\})\big).$$

Proof. It follows from Lemma 2.1.9 and direct computations that

$$\begin{aligned}
&\mathrm{ind}_\tau(E_{H_0}([0,\infty)), E_H([0,\infty))) + \frac{1}{2}\mathrm{ind}_\tau\big(E_H(\{0\}), E_{H_0}(\{0\})\big)\\
&\quad = \tau\big[E_{H_0}([0,\infty)) - E_H([0,\infty))\big] + \frac{1}{2}\tau\big[E_H(\{0\}) - E_{H_0}(\{0\})\big]\\
&\quad = \tau\big[E_H(\mathbb{R}_-) - E_{H_0}(\mathbb{R}_-)\big] + \frac{1}{2}\tau\big[E_H(\{0\}) - E_{H_0}(\{0\})\big]\\
&\quad = \xi(H_0,H).
\end{aligned}$$

□

2.1.2 The ξ-index and the perturbation determinant

Determinants. Firstly, we recall the concept of a determinant introduced by P. de la Harpe and G. Skandalis in [35].

Let $GL^0(\mathcal{A})$ be the set of boundedly invertible elements in $\mathcal{A}$. Given a C^1-path of operators $[0,1] \ni t \mapsto H_t \in GL^0(\mathcal{A})$, the de la Harpe-Skandalis determinant associated with the path $t \mapsto H_t$ is defined by

$$\Delta(t \mapsto H_t) = \exp\left(\int_0^1 \tau\left[\dot{H}_t H_t^{-1}\right] dt\right). \tag{2.1.9}$$

Some important properties of the de la Harpe-Skandalis determinant are listed in the lemma below. The proofs of these facts can be found in [35, Lemma 1 and Proposition 2].

Lemma 2.1.11. *Suppose that $[0,1] \ni t \mapsto H_t$ is a C^1-path of operators in $GL^0(\mathcal{A})$.*

(i) The determinant $\Delta(t \mapsto H_t)$ is invariant under fixed endpoint homotopies.

(ii) The absolute value of the determinant $\Delta(t \mapsto H_t)$ is path-independent. Moreover,

$$|\Delta(t \mapsto H_t)| = \Delta\left(H_1 H_0^{-1}\right),$$

where $\Delta(A) = \exp(\tau[\log(\sqrt{A^*A})])$ denotes the Fuglede-Kadison determinant (cf. [28]) of a boundedly invertible operator $A \in \mathcal{A}$.

(iii) If $\|H_t - I\| < 1$ for all $t \in [0,1]$, then

$$\Delta(t \mapsto H_t) = \exp(\tau[\log(H_1)] - \tau[\log(H_0)]), \tag{2.1.10}$$

where the operator logarithm $\log(H_j)$, $j = 1, 2$, in (2.1.10) is understood as the norm convergent series

$$\log(H_j) = -\sum_{k=1}^{\infty} \frac{(I - H_j)^k}{k}, \quad j = 0, 1.$$

(iv) Let $H_t^{(j)} : [0,1] \to GL^0(\mathcal{A})$, $j = 1, 2$, *be* C^1*-paths. Then,*

$$\Delta\left(t \mapsto H_t^{(1)} H_t^{(2)}\right) = \Delta\left(t \mapsto H_t^{(1)}\right) \Delta\left(t \mapsto H_t^{(2)}\right).$$

The following result reduces the computation of the determinant for paths of operators in either $\mathcal{D}_{\mathcal{A}}$ or

$$\mathcal{U}_{\mathcal{A}} = \{U \, : \, U = (\mathrm{i}I - H)(\mathrm{i}I + H)^{-1} \quad \text{for some} \quad H = H^* \in \mathcal{A}\}$$

to that of the state τ of the operator logarithm.

Lemma 2.1.12. *(i) For a* C^1*-path of operators* $[0,1] \ni t \mapsto H_t \in \mathcal{D}_{\mathcal{A}}$ *with* $H_0 = I$,

$$\Delta(t \mapsto H_t) = \exp(\tau[\log(H_1)]). \tag{2.1.11}$$

(ii) For a C^1*-path of operators* $[0,1] \ni t \mapsto U_t \in \mathcal{U}_{\mathcal{A}}$ *with* $U_0 = I$,

$$\Delta(t \mapsto U_t) = \exp\left(\tau\left[\widetilde{\log}(U_1)\right]\right),$$

where $\widetilde{\log}(\cdot)$ *is the principal branch of the operator logarithm of* U_1 *with the cut along the negative real semi-axis provided by the Spectral Theorem.*

Proof. (i) One notices that $\tau\left[\dot{H}_t H_t^{-1}\right] = \frac{d}{dt}\tau[\log(H_t)]$. Integrating the latter expression from 0 to 1 and comparing the result with (2.1.9) implies (2.1.11). The proof of (ii) goes along the same lines as that of (i). □

The following theorem offers a computation of the determinant of a 2×2 block operator matrix analogous to the result by L. G. Brown for the Fuglede-Kadison determinant (cf. [15, Lemma 4.5]).

Theorem 2.1.13.

$$\det\begin{pmatrix} M & K^* \\ K & N \end{pmatrix} = \det\begin{pmatrix} M - K^*N^{-1}K & 0 \\ 0 & I \end{pmatrix} \cdot \det\begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix} \tag{2.1.12}$$

$$= \det\begin{pmatrix} M - K^*N^{-1}K & 0 \\ 0 & N \end{pmatrix}. \tag{2.1.13}$$

Proof. From Theorem 1.1.4, one derives

$$\left(\det\begin{pmatrix} M & K^* \\ K & N \end{pmatrix}\right)^2 = \det(M - K^*N^{-1}K) \cdot \det N. \tag{2.1.14}$$

It is straightforward to check that $\tau(\log(N)) = 2\mathcal{T}\left(\log\begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix}\right)$ and, therefore, $\det N = \left(\det\begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix}\right)^2$. Applying the same argument to the first factor in (2.1.14) completes the proof of (2.1.12).

The matrices in (2.1.12) and (2.1.13) are dissipative, so the determinants of these matrices coincide with the de la Harpe-Skandalis determinants of the related paths of operators in $\mathcal{C}_0(\mathcal{A}\overline{\otimes}M_2)$. Applying multiplicativity of the de la Harpe-Skandalis determinant (Lemma 2.1.11 (iv)) implies equality (2.1.13). □

Remark 2.1.14. *The proof of the representation in [15, Lemma 4.5], which is similar to* (2.1.12)*, is based on the multiplicativity of the Fuglede-Kadison determinant for products of arbitrary (non-dissipative) operators while our main tool is a Dixmier-Fuglede-Kadison differentiation formula and no multiplicativity properties of the determinant are involved in the proof of* (2.1.12)*.*

Supported by the results of Lemma 2.1.12, we call

$$\det(H) = \exp(\tau[\log(H)]) \tag{2.1.15}$$

the determinant of a dissipative operator $H \in \mathcal{D}_{\mathcal{A}}$ and

$$\det(U) = \exp\left(\tau\left[\widetilde{\log}(U)\right]\right) \tag{2.1.16}$$

the determinant of a unitary operator $U \in \mathcal{U}_{\mathcal{A}}$.

Below, we provide an independent proof of the fact that the determinant (2.1.15) of an operator in $\mathcal{D}_{\mathcal{A}}$ differs from its Fuglede-Kadison determinant by a unimodular factor.

Lemma 2.1.15. *For an operator H in $\mathcal{D}_{\mathcal{A}}$, $|\det(H)| = \Delta(H)$.*

Proof. Let $A = \log(H)$. Upon applying the Dixmier-Fuglede-Kadison differentiation formula to the function

$$t \to \tau\big[\log\left(e^{tA^*}e^{tA}\right)\big], \quad 0 \le t \le 1,$$

and then integrating the result from 0 to 1, we obtain (cf. [28])

$$\tau\big[\log\big(e^{A^*}e^{A}\big)\big] = \tau(A^*) + \tau(A). \tag{2.1.17}$$

In view of [31, Lemma 2.6 (v)], we have

$$H = e^{\log(H)} = e^{A}, \quad H^* = \left(e^{A}\right)^* = e^{A^*}. \tag{2.1.18}$$

Combining (2.1.17) and (2.1.18) yields

$$\tau\big[\log(H^*H)\big] = 2\tau\big[\mathrm{Re}(\log(H))\big]. \tag{2.1.19}$$

We observe that the left-hand side of (2.1.19) equals $2\log(\Delta(H))$, while the right-hand side of (2.1.19) equals $2\log|\det(H)|$. Therefore, $\Delta(H) = |\det(H)|$. $\square$

We link the averaged argument of a dissipative operator in $\mathcal{D}_{\mathcal{A}}$ to the phase of its determinant.

Theorem 2.1.16. *Assume that* $H \in \mathcal{D}_{\mathcal{A}}$. *Then,*

$$\det(H) = \exp(\mathrm{i}\pi\tau[\arg(H)]) \cdot \Delta(H),$$

with $\Delta(\cdot)$ *the Fuglede-Kadison determinant.*

Proof. As any complex number, $\det(H)$ can be written in polar form

$$\det(H) = \exp(\mathrm{i}\mathrm{Im}(\log[\det(H)])) \cdot |\det(H)|. \tag{2.1.20}$$

We recall that $\det(H) = \exp(\tau[\log(H)])$ (cf. (2.1.15)) and Lemma 2.1.15 implies that $|\det(H)| = \Delta(H)$. Combining the latter representations with (2.1.20), one gets

$$\det(H) = \exp(\mathrm{i}\pi\mathrm{Im}(\tau[\log(H)])) \cdot \Delta(H). \tag{2.1.21}$$

By positivity of the state τ, one concludes that $\tau \circ \mathrm{Im} = \mathrm{Im} \circ \tau$, and hence the right-hand side of (2.1.21) equals $\exp(\mathrm{i}\pi\tau[\mathrm{Im}(\log(H))]) \cdot \Delta(H)$. □

The ξ-index via the perturbation determinant. Now we discuss the connection between the perturbation determinant and the ξ-index.

Definition 2.1.17. Let (H, H_0) be a pair of self-adjoint operators in $\mathcal{A}$. We define the perturbation determinant for the pair (H, H_0) by

$$\det{}_{H/H_0}(z) = \frac{\det\left((H_0 - zI)^{-1}\right)}{\det\left((H - zI)^{-1}\right)},$$

for $z \in \rho(H) \cap \rho(H_0) \setminus \mathbb{C}_-$.

Under certain additional assumptions, the perturbation determinant admits a representation analogous to the one built in the definition of the perturbation determinant in the traditional case.

Lemma 2.1.18. *Suppose* H_0 *and* H *are commuting self-adjoint operators in* $\mathcal{A}$ *with non-negative perturbation* $V = H - H_0$*. Then,*

$$\det_{H/H_0}(z) = \det\big((H - zI)(H_0 - zI)^{-1}\big),$$

for $z \in \rho(H_0) \cap \rho(H) \setminus \mathbb{C}_-$*.*

Proof. By standard reasoning (cf. [13, Section 2]),

$$(H - \cdot I)(H_0 - \cdot I)^{-1} = I + V(H_0 - \cdot I)^{-1}.$$

By the commutativity of H_0 and H, the latter equals

$$I + \sqrt{V}(H_0 - \cdot I)^{-1}\sqrt{V},$$

and hence, it is a Herglotz (operator-valued) function. In view of [31, Lemma 2.6 (v)], for z in $\rho(H_0) \cap \rho(H) \setminus \mathbb{C}_-$,

$$\begin{aligned} \exp\left[\log\left((H_0 - zI)^{-1}\right)\right] &= (H_0 - zI)^{-1}, \\ \exp\left[-\log\left((H - zI)^{-1}\right)\right] &= \left(\exp\left[\log\left((H - zI)^{-1}\right)\right]\right)^{-1} = (H - zI), \\ \exp\left[\log\left((H - zI)(H_0 - zI)^{-1}\right)\right] &= (H - zI)(H_0 - zI)^{-1}. \end{aligned}$$

The commutativity of $(H - zI)^{-1}$ and $(H_0 - zI)^{-1}$ implies

$$\begin{aligned} \exp\left[\log\left((H - zI)(H_0 - zI)^{-1}\right)\right] &= (H - zI)(H_0 - zI)^{-1} \\ &= \exp\left[\log\left((H_0 - zI)^{-1}\right)\right]\exp\left[-\log\left((H - zI)^{-1}\right)\right]. \end{aligned} \tag{2.1.22}$$

Applying Corollary A.1.18, the Spectral Theorem, and (2.1.22) yields

$$\log\left((H - zI)(H_0 - zI)^{-1}\right) = \log\left((H_0 - zI)^{-1}\right) - \log\left((H - zI)^{-1}\right) + 2\pi\mathrm{i}n(z)I,$$

with $n(z) \in \mathbb{Z}$. Hence, for $z \in \rho(H_0) \cap \rho(H) \setminus \mathbb{C}_-$,

$$\begin{aligned}&\exp\left(\tau\left[\log\left((H - zI)(H_0 - zI)^{-1}\right)\right]\right)\\&\quad= \exp\left(\tau\left[\log\left((H_0 - zI)^{-1}\right) - \log\left((H - zI)^{-1}\right)\right]\right),\end{aligned}$$

completing the proof. □

The next lemma is straightforward.

Lemma 2.1.19. *Let H, H_0, H_1 be self-adjoint operators in $\mathcal{A}$. Then, the chain rule for perturbation determinants*

$$\det{}_{H/H_1}(z)\det{}_{H_1/H_0}(z) = \det{}_{H/H_0}(z),$$

holds for $z \in \rho(H) \cap \rho(H_0) \cap \rho(H_1) \setminus \mathbb{C}_-$.

As an immediate consequence of Remark (1.1.17) and the definition of the ξ-index we obtain the following analog of the representation of the ξ-function via M. G. Krein's perturbation determinant [41, 42].

Lemma 2.1.20. *For self-adjoint operators H_0, H in $\mathcal{A}$, the representation*

$$\xi(H_0 - \lambda I, H - \lambda I) = \lim_{\varepsilon\downarrow 0} \frac{1}{\pi}\mathrm{Im}\left(\log\left(\det{}_{H/H_0}(\lambda + \mathrm{i}\varepsilon)\right)\right) \tag{2.1.23}$$

holds for a.e. $\lambda \in \mathbb{R}$.

Proof. Krein's trace formula applied to $f(\lambda) = \log \frac{1}{\lambda - z}$, $z \in \mathbb{C}_+$, yields

$$\tau\big[\log\big((H - zI)^{-1}\big)\big] - \tau\big[\log\big((H_0 - zI)^{-1}\big)\big] = \int_{\mathbb{R}} \frac{\xi(H_0 - \lambda I, H - \lambda I)\, d\lambda}{\lambda - z}.$$

Along with the definition of the perturbation determinant, the latter implies

$$\det_{H/H_0}(z) = \exp\left[\int_{\mathbb{R}} \frac{\xi(H_0 - \lambda I, H - \lambda I)\, d\lambda}{\lambda - z}\right].$$

By the inversion formula, this results in (2.1.23). □

2.1.3 A spectral averaging formula

In this subsection, we obtain a representation for the SSF via the integral over the coupling constant by analogy with the Birman-Solomyak formula [12]. We perform this using two different techniques, one in the case of a linear dependence and the other in the general case of a non-linear dependence of a perturbation on the parameter.

Theorem 2.1.21 (The Birman-Solomyak formula)**.** *Let H_0, V be self-adjoint operators in $\mathcal{A}$, $H_t = H_0 + tV$, $\quad 0 \le t \le 1$, and $\mathfrak{B}(\mathbb{R})$ an algebra of Borel subsets in $\mathbb{R}$. Then,*

$$\int_0^1 \tau[V E_{H_t}(G)]\, dt = \int_G \xi(H_1 - \lambda I, H_0 - \lambda I)\, d\lambda, \quad G \in \mathfrak{B}(\mathbb{R}).$$

Proof. First we apply the Dixmier-Fuglede-Kadison differentiation formula to the functions $t \to \tau[f_s(H_t)]$, where $f_s(z) = e^{isz}$, $s \in \mathbb{R}$. This formula is applicable since $\{H_t\}_{0 \le t \le 1}$ is a differentiable family of operators in $\mathcal{A}$, the set $\bigcup_{0 \le t \le 1} \sigma(H_t)$ is contained in $[-a, a] \subset \mathbb{R}$, with $a = \max_{0 \le t \le 1} \|H_t\|$ (as $t \to \|H_t\|$ is a continuous function), and for all $s \in \mathbb{R}$, the function f_s is analytic on any domain enclosing $\bigcup_{0 \le t \le 1} \sigma(H_t)$. We have

$$\frac{d}{dt}\big(\tau[f_s(H_t)]\big) = \tau[f_s'(H_t)V] = \tau\left[\int_{\mathbb{R}} f_s'(\lambda)\, dE_{H_t}(\lambda)V\right] \tag{2.1.24}$$

$$= \mathrm{i}s \int_{\mathbb{R}} e^{\mathrm{i}s\lambda}\, d\tau\big[E_{H_t}(\lambda)V\big]. \tag{2.1.25}$$

Integrating from 0 to 1 on both sides of (2.1.24), we obtain

$$\tau[f_s(H_1)] - \tau[f_s(H_0)] = \mathrm{i}s \int_0^1 dt \int_{\mathbb{R}} e^{\mathrm{i}s\lambda}\, d\tau[E_{H_t}(\lambda)V].$$

Since $\lambda \to e^{\mathrm{i}s\lambda}$ is a bounded function and $\int_{\mathbb{R}} d\tau[E_{H_t}(\lambda)V]$ is uniformly bounded in t (namely, equals $\tau(V)$), changing the order of integration yields

$$\tau[f_s(H_1)] - \tau[f_s(H_0)] = \mathrm{i}s \int_{\mathbb{R}} e^{\mathrm{i}s\lambda}\, d\mu(\lambda),$$

where Borel measure μ is defined by $\mu(\cdot) = \int_0^1 \tau[E_{H_t}(\cdot)V]dt$.

From Krein's trace formula, we have

$$\begin{aligned}\tau[f_s(H_1)] - \tau[f_s(H_0)] &= -\int_{\mathbb{R}} f_s'(\lambda)\xi(H_0 - \lambda I, H_1 - \lambda I)\, d\lambda \\ &= -\mathrm{i}s \int_{\mathbb{R}} e^{\mathrm{i}s\lambda}\xi(H_0 - \lambda I, H_1 - \lambda I)\, d\lambda.\end{aligned}$$

Thus, for any $s \in \mathbb{R}$,

$$\int_{\mathbb{R}} e^{is\lambda}\xi(H_1 - \lambda I, H_0 - \lambda I)\, d\lambda = \int_{\mathbb{R}} e^{is\lambda}\, d\mu(\lambda),$$

and hence,

$$\int_G \xi(H_1 - \lambda I, H_0 - \lambda I)\, d\lambda = \int_0^1 \tau[E_{H_t}(G)V]\, dt,$$

for any $G \in \mathfrak{B}$. □

Corollary 2.1.22. *Under the assumptions of Theorem 2.1.21,*

$$\xi(H_1 - \lambda I, H_0 - \lambda I) = \frac{d}{d\lambda}\int_0^1 \tau\big[E_{H_t}(\lambda)V\big]\,dt.$$

The following lemma demonstrates an alternative way of computing $\xi(H_0, H)$ whenever the operators H_0 and H are connected by a smooth path of dissipative elements in $\mathcal{A}$.

Lemma 2.1.23. *Let $[0,1] \ni t \mapsto H_t$ be a C^1-path of dissipative elements in $\mathcal{A}$ connecting the end-points H_0 and H_1. Then, $\xi(H_0, H)$ admits the representation*

$$\begin{aligned}\xi(H_0, H_1) &= \lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\mathrm{Im}\big(\log\big(\Delta(t\mapsto H_t + \mathrm{i}\varepsilon I)\big)\big)\\ &= \lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_0^1 \mathrm{Im}\,\tau\big[(H_t + \mathrm{i}\varepsilon I)^{-1}\dot{H}_t\big]\,dt.\end{aligned}$$

Proof. We obtain directly from the definition of the ξ-index that

$$\xi(H_0, H_1) = \frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\big(\tau[\arg(H_1 + \mathrm{i}\varepsilon I)] - \tau[\arg(H_0 + \mathrm{i}\varepsilon I)]\big). \tag{2.1.26}$$

On the other hand,

$$\tau[\arg(H_1 + \mathrm{i}\varepsilon I)] - \tau[\arg(H_0 + \mathrm{i}\varepsilon I)] = \int_0^1 \mathrm{Im}\left(\tau\left[(H_t + \mathrm{i}\varepsilon I)^{-1}\dot{H}_t\right]\right)dt \tag{2.1.27}$$

since the dissipative boundedly invertible operators $H_0 + \mathrm{i}\varepsilon I$ and $H_1 + \mathrm{i}\varepsilon I$, $\varepsilon > 0$, are connected by the C^1-path $H_t + \mathrm{i}\varepsilon I$, $t \in [0,1]$. Combining (2.1.26) and (2.1.27) completes the proof of the lemma. □

Theorem 2.1.24. *Let $[0,1] \ni t \mapsto H_t$ be a C^1-path of self-adjoint operators in $\mathcal{A}$ connecting H_0 and H_1. The identity*

$$\int_0^1 \tau\left[E_{H_t}(G)\dot{H}_t\right]dt = \int_G \xi(H_1 - \lambda I, H_0 - \lambda I)\,d\lambda \tag{2.1.28}$$

holds for any $G \in \mathfrak{B}(\mathbb{R})$.

Proof. From Lemma 2.1.23, it follows that for all $\alpha < \beta$ in $\mathbb{R}$,

$$\begin{aligned} &\int_\alpha^\beta \xi(H_0 - \lambda I, H_1 - \lambda I)\, d\lambda \\ &\quad = \int_\alpha^\beta \lim_{\varepsilon \downarrow 0} \int_0^1 \operatorname{Im}\left(\tau\left[\dot{H}_t (H_t + \mathrm{i}\varepsilon I - \lambda I)^{-1}\right]\right) dt\, d\lambda \\ &\quad = \int_\alpha^\beta \lim_{\varepsilon \downarrow 0} \int_0^1 \tau\left[\dot{H}_t \operatorname{Im}\big((H_t + \mathrm{i}\varepsilon I - \lambda I)^{-1}\big)\right] dt\, d\lambda. \end{aligned} \tag{2.1.29}$$

Note that

$$\int_\alpha^\beta \tau\left[\dot{H}_t \operatorname{Im}\big((H_t + \mathrm{i}\varepsilon I - \lambda I)^{-1}\big)\right] d\lambda = \tau\left[\dot{H}_t \operatorname{Im}\left(\int_\alpha^\beta (H_t + \mathrm{i}\varepsilon I - \lambda I)^{-1}\, d\lambda\right)\right].$$

Since by the Spectral Theorem

$$\int_\alpha^\beta (H_t + \mathrm{i}\varepsilon I - \lambda I)^{-1} d\lambda = \log(H_t + \mathrm{i}\varepsilon I - \alpha I) - \log(H_t + \mathrm{i}\varepsilon I - \beta I),$$

we have that for any $\varepsilon > 0$,

$$\left\| \operatorname{Im}\left(\int_\alpha^\beta (H_t + \mathrm{i}\varepsilon I - \lambda I)^{-1}\, d\lambda\right)\right\| \leq 2\pi.$$

By this observation using Lebesgue's dominated convergence theorem and Fubini's theorem, we can write the right hand side of (2.1.29) as

$$\begin{aligned} &\lim_{\varepsilon \downarrow 0} \int_\alpha^\beta \int_0^1 \tau\left[\dot{H}_t \operatorname{Im}\big((H_t + \mathrm{i}\varepsilon I - \lambda I)^{-1}\big)\right] dt\, d\lambda \\ &\quad = \lim_{\varepsilon \downarrow 0} \int_0^1 \int_\alpha^\beta \tau\left[\dot{H}_t \operatorname{Im}\big((H_t + \mathrm{i}\varepsilon I - \lambda I)^{-1}\big)\right] d\lambda\, dt. \end{aligned}$$

Applying once more Lebesgue's dominated convergence theorem, we get

$$\int_\alpha^\beta \xi(H_0 - \lambda I, H_1 - \lambda I)\, d\lambda = \int_0^1 \lim_{\varepsilon \downarrow 0} \int_\alpha^\beta \tau\left[\dot{H}_t \operatorname{Im}\big(H_t + \mathrm{i}\varepsilon I - \lambda I)^{-1}\big)\right] d\lambda\, dt.$$

Since $\tau(\cdot)$ is continuous in the strong operator topology, it follows from Stone's formula (B.1) that

$$\int_\alpha^\beta \xi(H_0 - \lambda I, H_1 - \lambda I)\, d\lambda = -\frac{1}{2}\int_0^1 \tau\left[\dot{H}_t\big(E_{H_t}([\alpha,\beta]) + E_{H_t}((\alpha,\beta))\big)\right] dt. \quad (2.1.30)$$

Taking the limit $\alpha \to \beta$ in (2.1.30), we get

$$\int_0^1 \tau\left[\dot{H}_t E_{H_t}(\{\beta\})\right] dt = 0.$$

Similarly, $\int_0^1 \tau\left[\dot{H}_t E_{H_t}(\{\alpha\})\right] dt = 0$, which proves (2.1.28) for G being an arbitrary (open, closed, semi-open) interval and, hence, for any $G \in \mathfrak{B}(\mathbb{R})$. $\square$

Remark 2.1.25. *The original proof of the traditional Birman-Solomyak formula for a trace class perturbation linearly dependent on the parameter as well as the proof of its counterpart in the context of semi-finite W^* algebras were based on the Sieltjes double integration (cf. [12] and [4], respectively). The Chain Rule for the derivative of the trace of a composite function was employed in [59] in the proof of the Birman-Solomyak formula for a general (not only linearly dependent on a parameter) trace class perturbation with non-negative derivative. For the proof in the case of an arbitrary trace class perturbation one can consult [31, Theorem 4.3]. Additional details on account of the Birman-Solomyak formula can be found in [30] and references cited therein.*

2.2 The Birman-Schwinger principle and its consequences

2.2.1 The Birman-Schwinger principle

As an immediate application of (1.1.24), we obtain a finite W^* analog of the Birman-Schwinger principle in an arbitrary gap.

Theorem 2.2.1 (The Birman-Schwinger principle in a gap)**.** *Let $K = K^*$, $M = M^*$, and $N = N^*$ be self-adjoint operators in $\mathcal{A}$ such that M, N, $M - K^*N^{-1}K$, and $N - KM^{-1}K^*$ are boundedly invertible. Then,*

$$\xi\left(M, M - K^*N^{-1}K\right) = \xi\left(N, N - KM^{-1}K^*\right). \tag{2.2.1}$$

Remark 2.2.2. *The operator $M - K^*N^{-1}K$ is boundedly invertible if and only if $N - KM^{-1}K^*$ is boundedly invertible. In particular, if* 0 *is in a joint spectral gap of the operator M and its perturbation $M - K^*N^{-1}K$, then* $0 \in \rho\,(N - KM^{-1}K^*)$.

Remark 2.2.3. *The invertibility of the four operators in* (2.2.1) *was either required or followed from the other assumptions in the traditional version of the Birman-Schwinger principle in a gap (cf. Theorem C.1.7).*

Similarly to the traditional case, we have a finite W^* analog of the Birman-Schwinger principle in a semi-infinite gap. If, in addition, the algebra $\mathcal{A}$ is a factor of type II_1, then the Birman-Schwinger principle can be expressed as the equality of the relative dimensions of certain spectral subspaces of the operators involved.

Corollary 2.2.4. *Assume that $\mathcal{A}$ is a factor of type II_1 and τ the relative trace on $\mathcal{A}$. Let H_0 and V be positive self-adjoint operators in $\mathcal{A}$, with H_0 boundedly invertible* *such that* 0 *is in the resolvent set of* $H_0 - V$*. Then,*

$$\operatorname{Dim}\big[E_{H_0-V}(\mathbb{R}_-)\mathcal{H}\big] = \operatorname{Dim}\big[E_{V^{1/2}H_0^{-1}V^{1/2}}((1,\infty))\mathcal{H}\big]. \tag{2.2.2}$$

Proof. With $M = H_0$, $N = I$, and $K = V^{1/2}$ in (2.2.1), one obtains

$$\operatorname{Dim}\big[E_{H_0-V}(\mathbb{R}_-)\mathcal{H}\big] = \operatorname{Dim}\big[E_{I-V^{1/2}H_0^{-1}V^{1/2}}(\mathbb{R}_-)\mathcal{H}\big],$$

which equals $\operatorname{Dim}\big[E_{V^{1/2}H_0^{-1}V^{1/2}}((1,\infty))\mathcal{H}\big]$ by Stone's formula (B.1). □

The following theorem suggests a recipe for the computation of the ξ-index associated with an off-diagonal perturbation problem for a 2×2 operator matrix.

Theorem 2.2.5. *Under the hypothesis of Theorem 2.2.1,*

$$2\,\xi\left[\begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}, \begin{pmatrix} M & K^* \\ K & N \end{pmatrix}\right] = \xi(M, M - K^*N^{-1}K).$$

Proof. It follows from the definition of the ξ-index and Theorem 1.1.4 that

$$\begin{aligned} &2\,\xi\left[\begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}, \begin{pmatrix} M & K^* \\ K & N \end{pmatrix}\right] \\ &\quad = \tau[\arg(M - K^*N^{-1}K)] + \tau[\arg(N)] - \big(\tau[\arg(M)] + \tau[\arg(N)]\big) \\ &\quad = \xi(M, M - K^*N^{-1}K). \end{aligned}$$

□

Our next goal is to obtain an extension of the basic invariance principle stated in Theorem 2.2.5 by relaxing the invertibility hypotheses.

Theorem 2.2.6. *Let* M*,* $N \in \mathcal{A}$ *be dissipative and* K *an arbitrary operator in* $\mathcal{A}$*. Then, the following assertions hold.*

(i)

$$\lim_{\varepsilon\downarrow 0}\xi\left(M+\mathrm{i}\varepsilon I, M+\mathrm{i}\varepsilon I-K^*(N+\mathrm{i}\varepsilon I)^{-1}K\right)$$
$$=\lim_{\varepsilon\downarrow 0}\xi\left(N+\mathrm{i}\varepsilon I, N+\mathrm{i}\varepsilon I-K(M+\mathrm{i}\varepsilon I)^{-1}K^*\right).$$

(ii) Assume that N has a bounded inverse. Then

$$\xi\left(M, M-K^*N^{-1}K\right)=\lim_{\varepsilon\downarrow 0}\xi\left(N, N-K(M+\mathrm{i}\varepsilon I)^{-1}K^*\right). \qquad (2.2.3)$$

(iii) If, in addition, the limit

$$K(M+\mathrm{i}0I)^{-1}K^*=n\text{-}\lim_{\varepsilon\downarrow 0}K(M+\mathrm{i}\varepsilon I)^{-1}K^*$$

exists and $N-K(M+\mathrm{i}0I)^{-1}K^*$ has a bounded inverse, then

$$\xi\left(M, M-K^*N^{-1}K\right)=\xi\left(N, N-K(M+\mathrm{i}0I)^{-1}K^*\right). \qquad (2.2.4)$$

Proof. (i) Theorem 2.2.5 guarantees that

$$\xi\left(M+\mathrm{i}\varepsilon I, M+\mathrm{i}\varepsilon I-K^*(N+\mathrm{i}\varepsilon I)^{-1}K\right)$$
$$=\xi\left(N+\mathrm{i}\varepsilon I, N+\mathrm{i}\varepsilon I-K(M+\mathrm{i}\varepsilon I)^{-1}K^*\right), \quad \varepsilon>0.$$

Therefore, to prove the claim, it is sufficient to establish the existence of the limit

$$\lim_{\varepsilon\downarrow 0}\xi\left(M+\mathrm{i}\varepsilon I, M+\mathrm{i}\varepsilon I-K^*(N+\mathrm{i}\varepsilon I)^{-1}K\right)$$
$$=\frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\Big(\tau\left[\arg\left(M+\mathrm{i}\varepsilon I-K^*(N+\mathrm{i}\varepsilon I)^{-1}K\right)\right]-\tau[\arg(M+\mathrm{i}\varepsilon I)]\Big).$$

By Theorem 1.1.16 and Theorem 1.1.18, the limits

$$\lim_{\varepsilon\downarrow 0}\tau[\arg(M+\mathrm{i}\varepsilon I)] \quad \text{and} \quad \tau\Big[\arg\left(M+\mathrm{i}\varepsilon I-K^*(N+\mathrm{i}\varepsilon I)^{-1}K\right)\Big]$$

exist.

(ii) By Theorem 2.2.5, we obtain that the equality

$$\xi\left(M + \mathrm{i}\varepsilon I, M - K^*N^{-1}K + \mathrm{i}\varepsilon I\right) = \xi\left(N, N - K(M + \mathrm{i}\varepsilon I)^{-1}K^*\right) \tag{2.2.5}$$

holds for all $\varepsilon > 0$. Passing to the limit as $\varepsilon \downarrow 0$ in (2.2.5) and making use of Theorem 1.1.16 and Theorem 1.1.18 implies (2.2.3).

(iii) Since by hypothesis $N - K(M + \mathrm{i}0I)^{-1}K^*$ has a bounded inverse, using continuity of the operator logarithm and that of the state τ, we obtain

$$\lim_{\varepsilon\downarrow 0} \tau\left[\arg\left(N - K(M + \mathrm{i}\varepsilon I)^{-1}K^*\right)\right] = \tau\left[\arg\left(N - K(M + \mathrm{i}0I)^{-1}K^*\right)\right].$$

Now the claim follows from (ii). □

Our next goal is to relate the ξ-index associated with a perturbation problem for non-invertible self-adjoint operators to the argument of the abstract scattering operator associated with the same perturbation problem. To achieve this goal, we need the following technical lemma.

Lemma 2.2.7. *Assume that $A, B \in \mathcal{A}$ with $A = A^*$ and $B = B^* \geq 0$. Assume, in addition, that A and $A + \mathrm{i}B$ are boundedly invertible. Then,*

$$\tau\big[\arctan\left(B^{1/2}A^{-1}B^{1/2}\right)\big] = \frac{1}{2}\arg(U),$$

where

$$U = I - 2\mathrm{i}B^{1/2}(A + \mathrm{i}B)^{-1}B^{1/2}.$$

Here

$$\arg(U) = \int_{|z|=1} \arg(z)\, dE_U(z), \quad \arg z \in (-\pi, \pi], \quad z \in \mathbb{C} \setminus \{0\},$$

with the cut along the negative real semi-axis.

Proof. Note that

$$U = (\mathrm{i}I - H)(\mathrm{i}I + H)^{-1},$$

where $H = B^{1/2}A^{-1}B^{1/2}$. Introduce a smooth path

$$[0,1] \ni t \mapsto U_t = (\mathrm{i}I - tH)(\mathrm{i}I + tH)^{-1},$$

of unitaries which links I and U. Applying the Dixmier-Fulgede-Kadison differentiation formula and the properties of the normal tracial state implies

$$\begin{aligned}
&\tau\left[\dot{U}_t U_t^{-1}\right] \\
&\quad = -\tau\left[(H(\mathrm{i}I + tH)^{-1} + (\mathrm{i}I - tH)(\mathrm{i}I + tH)^{-2}H)(\mathrm{i}I + tH)(\mathrm{i}I - tH)^{-1}\right] \\
&\quad = -\tau\left[H(\mathrm{i}I + tH)^{-1}\right] - \tau\left[(\mathrm{i}I - tH)^{-1}H\right] \\
&\quad = -\frac{d}{dt}\tau[\log(\mathrm{i}I + tH)] + \frac{d}{dt}\tau[\log(\mathrm{i}I - tH)].
\end{aligned} \tag{2.2.6}$$

Integrating (2.2.6) from 0 to 1 yields

$$\begin{aligned}
\int_0^1 \tau\left[\dot{U}_t U_t^{-1}\right] dt &= \tau\big[\log(\mathrm{i}I - H)\big] - \tau\big[\log(\mathrm{i}I)\big] \\
&\quad - \left(\tau\big[\log(\mathrm{i}I + H)\big] - \tau\big[\log(\mathrm{i}I)\big]\right).
\end{aligned} \tag{2.2.7}$$

Upon evaluating the imaginary parts in (2.2.7) and subsequently applying the Spectral

Theorem, we obtain

$$\begin{aligned}
\arg(U) &= \frac{1}{\mathrm{i}} \int_0^1 \tau \left[\dot{U}_t U_t^{-1}\right] dt \\
&= \int_{\mathbb{R}} \left[\operatorname{Im}\left(\log\left(\frac{\mathrm{i}-\lambda}{\mathrm{i}}\right)\right) - \operatorname{Im}\left(\log\left(\frac{\mathrm{i}+\lambda}{\mathrm{i}}\right)\right)\right] d\tau[E_H(\lambda)] \\
&= \int_{\mathbb{R}} \big(\arg(1+\mathrm{i}\lambda) - \arg(1-\mathrm{i}\lambda)\big) d\tau[E_H(\lambda)] \\
&= 2\int_{\mathbb{R}} \arg(1+\mathrm{i}\lambda) d\tau[E_H(\lambda)] \\
&= 2\tau[\arctan(H)],
\end{aligned}$$

completing the proof. □

Corollary 2.2.8. *Under the assumptions of Lemma 2.2.7,*

$$\arg(A+\mathrm{i}B) = \pi\tau[E_A(\mathbb{R}_-)] + \frac{1}{2}\arg(U), \tag{2.2.8}$$

with

$$U = \left(\mathrm{i}I - B^{1/2}A^{-1}B^{1/2}\right)\left(\mathrm{i}I + B^{1/2}A^{-1}B^{1/2}\right)^{-1}.$$

It is convenient to collect the further assumptions in the following hypothesis.

Hypothesis 2.2.9. *Suppose that $H_0 = H_0^*$ and $V = V^*$ are elements in $\mathcal{A}$ and $H = H_0 - V$. Assume that V is factored in the form $V = -K^*N^{-1}K$, where $K \in \mathcal{A}$ and $N = N^*$ is a boundedly invertible element in $\mathcal{A}$. Assume that the limit*

$$K^*(H_0 + \mathrm{i}0I)^{-1}K = n\text{-}\lim_{\varepsilon\downarrow 0} K^*(H_0 + \mathrm{i}\varepsilon I)^{-1}K$$

exists in the norm operator topology and both the operators $\mathcal{N} = N - K^*(H_0 + \mathrm{i}0I)^{-1}K$ and $\operatorname{Re}(\mathcal{N})$ have bounded inverses.

Let S_{H,H_0} be the abstract S-operator associated with the pair (H_0, H), that is,

$$S_{H,H_0} = I - 2\pi \mathrm{i}\Big(\frac{\mathrm{Im}(\mathcal{N})}{\pi}\Big)^{1/2} N^{-1}\big(I + (-K^*(H_0 + \mathrm{i}0I)^{-1}K)N^{-1}\big)^{-1}\Big(\frac{\mathrm{Im}(\mathcal{N})}{\pi}\Big)^{1/2}, \tag{2.2.9}$$

which is unitary and equal to

$$S_{H,H_0} = I - 2\mathrm{i}(\mathrm{Im}(\mathcal{N}))^{1/2}\mathcal{N}^{-1}(\mathrm{Im}(\mathcal{N}))^{1/2}.$$

We note that the operator S_{H,H_0} coincides with the value of the Lifshits characteristic function of $\mathcal{N}$ at the spectral point $\lambda = 0$ (see, e.g., [32, Section IV.6]). One can check directly that S_{H,H_0} equals

$$\begin{aligned} S_{H,H_0} = &\big(\mathrm{i}I - (\mathrm{Im}(\mathcal{N}))^{1/2}(\mathrm{Re}(\mathcal{N}))^{-1}(\mathrm{Im}(\mathcal{N}))^{1/2}\big) \\ &\times \big(\mathrm{i}I + (\mathrm{Im}(\mathcal{N}))^{1/2}(\mathrm{Re}(\mathcal{N}))^{-1}(\mathrm{Im}(\mathcal{N}))^{1/2}\big)^{-1}. \end{aligned} \tag{2.2.10}$$

Theorem 2.2.10. *Assume Hypothesis 2.2.9 and let S_{H,H_0} be the abstract S-operator associated with the pair (H_0, H). Then,*

$$\xi(H_0, H) = \xi(N, \mathrm{Re}(\mathcal{N})) + \frac{1}{2\pi}\arg\left(S_{H,H_0}\right). \tag{2.2.11}$$

Proof. Applying Theorem 2.2.6 (iii) yields

$$\xi(H_0, H) = \xi(N, \mathcal{N}) = \xi(N, \mathrm{Re}(\mathcal{N})) + \xi(\mathrm{Re}(\mathcal{N}), \mathcal{N}).$$

By Lemma 2.2.7, one has

$$\xi(\mathrm{Re}(\mathcal{N}), \mathcal{N}) = \frac{1}{2\pi}\tau[\arg\left(S_{H,H_0}\right)],$$

completing the proof of (2.2.11). □

2.2.2 Consequences of the Birman-Schwinger principle

An analog of the Birman-Krein formula. In the finite W^* algebra setting, an analog of the Birman-Krein formula (cf. Appendix C) that originally connected the scattering matrix with the spectral shift function takes place. A new Birman-Krein formula involves, however, an additional term since the ξ-index may attain any real value falling in the range $[-1, 1]$ while the spectral shift function in the traditional setting would take on only integer values under similar requirements. Moreover, the new Birman-Krein formula follows from the Birman-Schwinger principle as distinct from the traditional setting.

Theorem 2.2.11 (The Birman-Krein formula)**.** *Assume Hypothesis 2.2.9 and let* $S_{H_0,H}$ *be the abstract* S*-operator associated with the pair* (H_0, H)*. Then,*

$$\det(S_{H,H_0}) = \exp\big[-2\pi \mathrm{i}\xi(H, H_0) - 2\pi \mathrm{i}\xi(N, N + A)\big].$$

Proof. The theorem follows upon evaluating exponents in identity (2.2.11) multiplied by $2\pi\mathrm{i}$ and then applying Lemma 2.1.11 (ii) and Lemma 2.1.12 (ii), recalling definition (2.1.16). □

An analog of the Pushnitski formula. Under some additional assumptions (compared to the traditional case), a finite W^* algebra analog of the Pushnitski formula [54, 55] that initially represented the spectral shift function as the integral of the eigenvalue counting function is also available.

Theorem 2.2.12 (The Pushnitski formula)**.** *Assume Hypothesis 2.2.9. Assume, in addition, that the operators* $\mathrm{Re}(\mathcal{N})$ *and* $\mathrm{Im}(\mathcal{N})$ *commute. Denote the operators* $\mathrm{Re}(\mathcal{N})$

and $\operatorname{Im}(\mathcal{N})$ *by* A *and* B*, respectively. Then,*

$$\xi(H_0, H) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\xi(N, A + tB)}{1 + t^2}\, dt.$$

Remark 2.2.13. *The operators* $\operatorname{Re}(\mathcal{N})$ *and* $\operatorname{Im}(\mathcal{N})$ *commute if and only if* $\mathcal{N}$ *is normal.*

Proof of Theorem 2.2.12. Theorem 2.2.6 and Lemma 2.2.7 imply

$$\xi(H_0, H) = \xi(N, A) + \frac{1}{\pi}\tau\big[\arctan\big(B^{1/2}A^{-1}B^{1/2}\big)\big]. \tag{2.2.12}$$

Employing the Spectral Theorem and then integrating by parts yields

$$\begin{aligned}&\frac{1}{\pi}\tau\big[\arctan\big(B^{1/2}A^{-1}B^{1/2}\big)\big]\\ &\quad = \frac{1}{\pi}\int_{-\infty}^{\infty} \arctan(\mu)\, d\tau\big[E_{B^{1/2}A^{-1}B^{1/2}}((-\infty, \mu))\big]\\ &\quad = \frac{1}{2} - \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\tau\big[E_{-\mu I + B^{1/2}A^{-1}B^{1/2}}(\mathbb{R}_-)\big]}{1+\mu^2}\, d\mu.\end{aligned} \tag{2.2.13}$$

For any $\varepsilon > 0$, Theorem 2.2.6 (ii) implies

$$\xi\left(-\mu I, -\mu I - B^{1/2}(-A)^{-1}B^{1/2}\right) = \lim_{\varepsilon\downarrow 0} \xi\left(-A, -A + \frac{1}{\mu - \mathrm{i}\varepsilon}B\right). \tag{2.2.14}$$

Recall that

$$\begin{aligned}\xi\left(-\mu I, -\mu I - B^{1/2}(-A)^{-1}B^{1/2}\right) =& \lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\tau\left[\arg\left(-\mu I + \mathrm{i}\varepsilon I - B^{1/2}(-A)^{-1}B^{1/2}\right)\right]\\ &- \lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\tau\left[\arg\left(-\mu I + \mathrm{i}\varepsilon I\right)\right].\end{aligned} \tag{2.2.15}$$

In view of Remark 1.1.17,

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\tau\left[\arg\left(-\mu I + \mathrm{i}\varepsilon I - B^{1/2}(-A)^{-1}B^{1/2}\right)\right]\\ &\quad = \tau\left[E_{-\mu I + B^{1/2}A^{-1}B^{1/2}}(\mathbb{R}_-)\right] + \tau\left[E_{B^{1/2}A^{-1}B^{1/2}}(\{\mu\})\right]\end{aligned} \tag{2.2.16}$$

and

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\tau\left[\arg(-\mu I+\mathrm{i}\varepsilon I)\right]=\tau[E_{-\mu I}(\mathbb{R}_-)],\quad \mu\neq 0. \tag{2.2.17}$$

By the Spectral Theorem and commutativity of $N+A$ and B (cf. Corollary A.1.18),

$$\lim_{\varepsilon\downarrow 0}\tau\left[\arg\left(-A+\frac{1}{\mu-\mathrm{i}\varepsilon}B\right)\right]=\tau\left[E_{-A+\frac{1}{\mu}B}((-\infty,0])\right]. \tag{2.2.18}$$

Combining together (2.2.14)-(2.2.18) yields

$$\begin{aligned}\tau\left[E_{-\mu I-B^{1/2}(-A)^{-1}B^{1/2}}(\mathbb{R}_-)\right]&=\tau\left[E_{-A+\frac{1}{\mu}B}((-\infty,0])\right]-\tau[E_{-A}(\mathbb{R}_-)]\\&\quad+\tau[E_{-\mu I}(\mathbb{R}_-)]-\tau\left[E_{B^{1/2}A^{-1}B^{1/2}}(\{\mu\})\right].\end{aligned} \tag{2.2.19}$$

By Stone's formula (B.1),

$$E_{-A+\frac{1}{\mu}B}((-\infty,0])=E_{A-\frac{1}{\mu}B}([0,-\infty))=I-E_{A-\frac{1}{\mu}B}(\mathbb{R}_-)$$

and

$$E_{-A}(\mathbb{R}_-)=E_A((0,\infty))=I-E_A(\mathbb{R}_-),$$

which together with (2.2.19) give

$$\begin{aligned}\tau\left[E_{-\mu I-B^{1/2}(-A)^{-1}B^{1/2}}(\mathbb{R}_-)\right]&=-\tau\left[E_{A-\frac{1}{\mu}B}(\mathbb{R}_-)\right]+\tau[E_A(\mathbb{R}_-)]\\&\quad+\tau[E_{-\mu I}(\mathbb{R}_-)]-\tau\left[E_{B^{1/2}A^{-1}B^{1/2}}(\{\mu\})\right].\end{aligned} \tag{2.2.20}$$

It follows from (2.2.13) and (2.2.20) that

$$\begin{aligned}&\frac{1}{\pi}\tau\left[\arctan\left(B^{1/2}A^{-1}B^{1/2}\right)\right]\\&\quad=\frac{1}{2}-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\tau\left[E_{-\mu I}(\mathbb{R}_-)\right]}{1+\mu^2}\,d\mu+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\tau\left[E_{A-\frac{1}{\mu}B}(\mathbb{R}_-)\right]}{1+\mu^2}\,d\mu\\&\qquad-\tau[E_A(\mathbb{R}_-)]-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\tau\left[E_{B^{1/2}A^{-1}B^{1/2}}(\{\mu\})\right]}{1+\mu^2}\,d\mu.\end{aligned} \tag{2.2.21}$$

It is easy to see that

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\tau\big[E_{-\mu I}(\mathbb{R}_-)\big]}{1+\mu^2}\,d\mu=\frac{1}{2}.$$

Next, equalities (2.2.12) and (2.2.21) imply

$$\begin{aligned}\xi(H_0,H)&=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\tau\left[E_{A-\frac{1}{\mu}B}(\mathbb{R}_-)\right]-\tau[E_N(\mathbb{R}_-)]}{1+\mu^2}\,d\mu\\&\quad-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\tau\big[E_{B^{1/2}A^{-1}B^{1/2}}(\{\mu\})\big]}{1+\mu^2}\,d\mu\\&=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\tau\left[E_{A-\frac{1}{\mu}B}(\mathbb{R}_-)\right]-\tau[E_N(\mathbb{R}_-)]}{1+\mu^2}\,d\mu.\end{aligned}\tag{2.2.22}$$

Making the substitution $t=-\frac{1}{\mu}$ in (2.2.22) concludes the proof of the theorem. □

Monotonicity of a function with operator argument. We start with a result showing that the ξ-index preserves the (partial) order relation $\leq$ of self-adjoint elements in $\mathcal{A}$.

Lemma 2.2.14. *Let H_1 and H_2 be self-adjoint operators in $\mathcal{A}$ with $H_1\leq H_2$. Then, $\xi(H_1,H_2)\leq 0$.*

Proof. Applying the Birman-Schwinger principle yields

$$\begin{aligned}&-\xi(H_1,H_2)\\&\quad=\xi(H_2,H_1)=\xi\big(H_2,H_2-(H_2-H_1)\big)\\&\quad=\lim_{\varepsilon\downarrow 0}\xi\big(I,I-(H_2-H_1)^{1/2}(H_2+\mathrm{i}\varepsilon I)^{-1}(H_2-H_1)^{1/2}\big)\\&\quad=\lim_{\varepsilon\downarrow 0}\tau\big[\mathrm{Im}\left(\log\left(I-(H_2-H_1)^{1/2}(H_2+\mathrm{i}\varepsilon I)^{-1}(H_2-H_1)^{1/2}\right)\right)\big]\geq 0.\end{aligned}$$

□

Corollary 2.2.15. *Let H_0, H_1, and H_2 be self-adjoint operators in $\mathcal{A}$ with $H_1 \leq H_2$. Then,*

$$\xi(H_0, H_1) \geq \xi(H_0, H_2).$$

Proof. One has

$$\xi(H_0, H_2) = \xi(H_0, H_1) + \xi(H_1, H_2),$$

which along with Lemma 2.2.14 implies the result. □

As an application of the monotonicity of the ξ-index and Krein's trace formula, we obtain monotonicity of a smooth monotone function (as a scalar function of a scalar argument) defined on self-adjoint elements in $\mathcal{A}$.

Theorem 2.2.16. *Let H_1 and H_2 be self-adjoint operators in $\mathcal{A}$ with $H_1 \leq H_2$. Assume that f is a measurable function on $\mathbb{R}$ such that f increases on $[a, b]$, where $[a, b]$ contains $\sigma(H_0) \cup \sigma(H_1)$. Assume, in addition, that either f is absolutely continuous or both H_0 and H_1 have no eigenvalues. Then,*

$$\tau[f(H_1)] \leq \tau[f(H_2)].$$

Proof. Applying Krein's trace formula to an absolutely continuous f implies

$$\tau[f(H_2)] - \tau[f(H_1)] = -\int_a^b f'(\lambda)\xi(H_1 - \lambda I, H_2 - \lambda I)\, d\lambda,$$

from which the result follows because of Corollary 2.1.6 and Lemma 2.2.14.

If both H_0 and H_1 have no eigenvalues, then

$$\tau\big[E_{H_2}((-\infty, \lambda))\big] - \tau\big[E_{H_1}((-\infty, \lambda))\big] = \xi(H_1 - \lambda I, H_2 - \lambda I),$$

for every $\lambda \in \mathbb{R}$, and the claim of the theorem follows from the representation

$$\tau[f(H_2)] - \tau[f(H_1)] = -\int_a^b \Big(\tau\big[E_{H_2}((-\infty,\lambda))\big] - \tau\big[E_{H_1}((-\infty,\lambda))\big]\Big)\, df(\lambda)$$

(cf. Lemma 2.1.4) and Lemma 2.2.14. □

Remark 2.2.17. *Monotonicity of the composition $\tau \circ f$ for an arbitrary (non-smooth) monotone f was proved in [53] by continuous variational techniques [23, 24].*

Appendix A

Basic facts on von Neumann algebras

Most of the material in this section is standard and discussed in many books on operator algebras (cf. [56] for a brief exposition). The emphasis in this appendix is put on *finite* von Neumann algebras.

A.1 A trace and the relative dimension

Definition A.1.1. A set $\mathcal{A}$ of bounded operators on $\mathcal{H}$ containing identity such that $A \in \mathcal{A}$, $B \in \mathcal{A}$, and $\alpha \in \mathbb{C}$ altogether imply AB, $A+B$, $\alpha A \in \mathcal{A}$ is called an operator algebra. If it is also true of $\mathcal{A}$ that $A \in \mathcal{A}$ implies $A^* \in \mathcal{A}$, then $\mathcal{A}$ is called a self-adjoint operator algebra.

Definition A.1.2. A self-adjoint operator algebra closed in the weak operator topology is called a W^* or von Neumann algebra.

Definition A.1.3. A W^*-algebra $\mathcal{A}$ is called a factor if

$$\mathcal{A} \cap \mathcal{A}' = \{\alpha I \, : \, \alpha \in \mathbb{C}\},$$

where $\mathcal{A}'$ is the set of bounded linear operators commuting with all elements in $\mathcal{A}$.

Given a factor $\mathcal{A}$, let $\mathfrak{A}$ denote the set of those closed subspaces $\mathfrak{M}$ of $\mathcal{H}$ for which $P_{\mathfrak{M}}$, the orthogonal projection onto $\mathfrak{M}$, is an element in $\mathcal{A}$.

Definition A.1.4. Let $\mathfrak{M}$ and $\mathfrak{N}$ be elements in $\mathfrak{A}$. The subspace $\mathfrak{M}$ is said to be equivalent to $\mathfrak{N}$ relative to a factor $\mathcal{A}$ ($\mathfrak{M} \sim \mathfrak{N}$) if there exists $U \in \mathcal{A}$ satisfying

$$U^*U = P_{\mathfrak{M}}, \quad UU^* = P_{\mathfrak{N}}.$$

Remark A.1.5. *If two subspaces are equivalent relative to a factor, then their (standard) dimensions are equal.*

Definition A.1.6. A subspace $\mathfrak{M} \in \mathfrak{A}$ is said to be finite relative to $\mathcal{A}$ if it is not equivalent to any of its proper subspaces relative to $\mathcal{A}$.

Theorem A.1.7. *There exits a relative dimension* $\mathrm{Dim}(\cdot) : \mathfrak{A} \to \overline{\mathbb{R}}_+$ (*unique up to a constant factor*) *such that for* $\mathfrak{M}, \mathfrak{N} \in \mathfrak{A}$,

- $\mathfrak{M} \sim \mathfrak{N} \Leftrightarrow \mathrm{Dim}(\mathfrak{M}) = \mathrm{Dim}(\mathfrak{N})$,
- $\mathfrak{M} \perp \mathfrak{N} \Rightarrow \mathrm{Dim}(\mathfrak{M} \oplus \mathfrak{N}) = \mathrm{Dim}(\mathfrak{M}) + \mathrm{Dim}(\mathfrak{N})$.
- $\mathfrak{M}$ *is finite relative to* $\mathcal{A} \Leftrightarrow \mathrm{Dim}(\mathfrak{M}) < \infty$.

Definition A.1.8. All factors are classified according to the values of the relative dimension function as shown in the table below.

I_n	$\mathrm{Ran}(\mathrm{Dim}(\cdot)) = \{1, 2, \cdots, n\}$
I_∞	$\mathrm{Ran}(\mathrm{Dim}(\cdot)) = \{1, 2, \cdots\}$
II_1	$\mathrm{Ran}(\mathrm{Dim}(\cdot)) = [0, 1]$
II_∞	$\mathrm{Ran}(\mathrm{Dim}(\cdot)) = [0, \infty]$
III	$\mathrm{Ran}(\mathrm{Dim}(\cdot)) = \{0, \infty\}$

In the context of a factor of type II_1, by $\mathrm{Dim}(\cdot)$ we mean the relative dimension function which is normalized so that $\mathrm{Ran}(\mathrm{Dim}(\cdot)) = [0, 1]$.

Definition A.1.9. Let $\mathcal{A}$ be a factor of type II_1. The relative trace $\tau(\cdot)$ of a self-adjoint element $A \in \mathcal{A}$ is defined by

$$\tau(A) = \int_{\mathbb{R}} \lambda \, d\, \mathrm{Dim}[E_A(\lambda)\mathcal{H}]$$

and of an arbitrary $A \in \mathcal{A}$ by

$$\tau(A) = \tau[\mathrm{Re}(A)] + \mathrm{i}\, \tau[\mathrm{Im}(A)].$$

Remark A.1.10. $\tau(P_{\mathfrak{M}}) = \mathrm{Dim}(\mathfrak{M})$ *for* $\mathfrak{M} \in \mathfrak{A}$.

Theorem A.1.11. *The relative trace on a factor $\mathcal{A}$ of type II_1 satisfies the following properties.*

(i) τ *is linear,*

(ii) $\tau(I) = 1$,

(iii) $\tau(A) \geq 0$ *if* $0 \leq A \in \mathcal{A}$,

(iv) $\mathcal{A} \ni A \geq 0$ *and* $\tau(A) = 0$ *imply* $A = 0$,

(v) $\tau(AB) = \tau(BA)$ *for* $A, B \in \mathcal{A}$,

(vi) $\tau(A_n) \to \tau(A)$ *if the sequence* $\{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$ *converges to* $\mathcal{A}$ *in the strong operator topology.*

Theorem A.1.12. *Let $\mathcal{A}$ be a W^*-algebra in a separable Hilbert space $\mathcal{H}$. Then, there exists a direct integral decomposition*

$$\mathcal{H} = \int_{\mathbb{R}} \oplus \mathcal{H}(\lambda)\mu(d\lambda)$$

of the Hilbert space $\mathcal{H}$ relative to which $\mathcal{A}$ is decomposable as a direct integral

$$\mathcal{A} = \int_{\mathbb{R}} \oplus \mathcal{A}(\lambda)\mu(d\lambda),$$

where all the W^*-algebras $\mathcal{A}(\lambda)$ are factors and μ is a finite positive Borel measure.

Definition A.1.13. A W^*-algebra $\mathcal{A}$ is said to be of pure type I_n (II_1) if μ-almost all of the factors in the direct integral decomposition of $\mathcal{A}$ are of type I_n (II_1).

Definition A.1.14. A W^*-algebra $\mathcal{A}$ is said to be finite if it admits decomposition $\mathcal{A} = \mathcal{A}_{I_n} \oplus \mathcal{A}_{II_1}$, with the W^*-algebras $\mathcal{A}_{I_n}$ and $\mathcal{A}_{II_1}$ of pure types I_n and II_1, respectively.

Remark A.1.15. *Equivalently, a W^* algebra $\mathcal{A}$ is called finite if the Hilbert space $\mathcal{H}$ is finite relative to $\mathcal{A}$.*

Theorem A.1.16. *Given a finite W^*-algebra $\mathcal{A}$, there exists a functional $\tau(\cdot)$ on $\mathcal{A}$ satisfying assertions (i)-(v) of Theorem A.1.11 and*

(vi′) $\tau(A_\alpha) \to \tau(A)$ for each monotone increasing net of operators $\{A_\alpha\} \subset \mathcal{A}$ with the least upper bound A.

In addition, any such functional $\tau(\cdot)$ is representable (cf. [36, Theorem 7.1.12]) in the form $\tau = \sum_{n=1}^{\infty} \tau_{x_n}$, with $\{x_n\}_{n=1}^{\infty}$ an orthogonal family of vectors in $\mathcal{H}$ and $\tau_{x_n}(A) = \langle Ax_n, x_n\rangle$, for $A \in \mathcal{A}$, and $|\tau(A)| \leq \|A\|$.

A functional is called a *state* if it satisfies assertions (i)-(iii) of Theorem A.1.11, *faithful* if it satisfies assertion (iv), *tracial* if it satisfies assertion (v), and *normal* if satisfies (vi$'$). It follows from [36, Theorem 7.1.12] that for a state on $\mathcal{A}$ normality implies property (vi).

Theorem A.1.17. *Let $\mathcal{A}$ be an abelian W^*-algebra. Then there exists a self-adjoint $H \in \mathcal{A}$ such that $\mathcal{A}$ is identical with the collection of all bounded Borel functions of $\mathcal{A}$, or equivalently, with the W^*-algebra generated by H.*

Corollary A.1.18. *Let A and B be normal commuting operators in a W^* algebra $\mathcal{A}$. Then, there exists a self-adjoint operator H such that A and B are bounded Borel functions of H.*

Proof. It follows from the commutativity of A and B that the operators A and B^*, A^* and B also commute (cf. [26, Theorem I]). Therefore, A and B generate an abelian W^*-algebra in $\mathcal{A}$, and thus, A and B are functions of some self-adjoint operator in this subalgebra. □

A.2 The Chain Rule

For an averaged composite function of an operator in $\mathcal{A}$, the Chain Differentiation Rule is available.

Lemma A.2.1 (Dixmier-Fuglede-Kadison differentiation formula)**.** *Let f be a function analytic in a domain Ω bounded by a rectifiable curve Γ in the complex plane and $X : \mathfrak{K} \to \mathcal{A}$ a differentiable (in the norm operator topology) operator-valued function* *such that the spectrum of each $X(z)$ lies in Ω, with $\mathfrak{K}$ a bounded domain in $\mathbb{C}$ or $\mathbb{R}$. Then, $f \circ X$ is differentiable in $\mathfrak{K}$ and*

$$\tau\left[\frac{d}{dz} f(X(z))\right] = \tau\big[g(X(z))X'(z)\big], \tag{A.1}$$

with $g(\lambda) = \frac{d}{d\lambda} f(\lambda)$ and $X'(z) = \frac{d}{dz} X(z)$, where the function of an operator is understood in the sense of the Riesz functional calculus.

Remark A.2.2. *Originally, formula* (A.1) *was proved for $\mathfrak{K} \subset \mathbb{R}$ in the case of a factor of type II_1 [28] and then extended to a general finite W^* algebra in [20]. In the case $\mathfrak{K} \subset \mathbb{C}$, the proof can be repeated along the same lines.*

Lemma A.2.3. *Let $X(\cdot)$ is an operator-valued function analytic in a bounded domain $\mathfrak{K}$, $\overline{\mathfrak{K}} \subset \mathbb{C}_+$, with values in $\mathcal{A}$ such that the spectra of all $X(z)$ are contained in a compact subset Ω of $\mathbb{C}_+$. Then, $\tau[\log(X(\cdot))]$ is analytic on $\mathfrak{K}$ and*

$$\tau\left[\frac{d}{dz} \log(X(z))\right] = \tau\big[X^{-1}(z)X'(z)\big], \tag{A.2}$$

with the logarithm given by (1.1.1).

Proof. Since the spectra of all $X(z)$ are contained in a compact subset of $\mathbb{C}_+$, there exists such number $a > 0$ that $\mathrm{Im}(X(z)) \geq aI$ for all $z \in \mathfrak{K}$.

The inequalities $\|(\mathrm{i}\lambda I + X(z))^{-1}\| \leq \frac{1}{\lambda}$ and $\|(\mathrm{i}\lambda I + X(z))^{-1}\| \leq \frac{1}{a}$ imply

$$\int_0^\infty \big\|(\mathrm{i}\lambda I + X(z))^{-2} X'(z)\big\| \, d\lambda$$
$$= \|X'(z)\| \left(\int_0^\alpha + \int_\alpha^\infty\right) \big\|(\mathrm{i}\lambda I + X(z))^{-2}\big\| \, d\lambda \leq \|X'(z)\| \left(\frac{\alpha}{a^2} + \frac{1}{\alpha}\right), \quad \alpha > 0.$$

Therefore, the integral $\int_0^\infty \tau\big[(\mathrm{i}\lambda I + X(z))^{-2} X'(z)\big] d\lambda$ converges uniformly on every compact subset of $\mathfrak{K}$ and one can differentiate the integral representing $\tau[\log(X(z))]$.

Moreover, by the second resolvent identity, the cyclicity of $\tau(\cdot)$, and the Dixmier-Fuglede-Kadison differentiation formula applied to $z \to \tau\big[(\mathrm{i}\lambda I + X(z))^{-1}\big]$, we have

$$\frac{d}{dz}\tau[\log(X(z))] = \mathrm{i}\int_0^\infty \tau\big[(\mathrm{i}\lambda I + X(z))^{-2}X'(z)\big]\,d\lambda. \tag{A.3}$$

Applying Lemma A.2.1 to the function $\lambda \to \tau[(\mathrm{i}\lambda I + X(z))^{-1}]$ and evaluating the integral in (A.3) complete the proof of the lemma. □

Theorem A.2.4. *The averaged operator logarithm* (1.1.1) *equals the averaged principal branch of the logarithm of $H \in \mathcal{D}_{\mathcal{A}}$, with the cut along the negative imaginary semi-axis, provided by the Riesz functional calculus.*

Proof. It follows from Lemma A.2.3 that for $z \in \mathbb{C}_+$,

$$\tau\left[\frac{d}{dz}\log(H + zI)\right] = \tau\big[(H + zI)^{-1}\big].$$

Therefore, the averaged operator logarithm (1.1.1) coincides up to a constant summand with the averaged principal branch of the logarithm with the cut along the negative imaginary semi-axis provided by the Riesz functional calculus for operators $H + zI$, $z \in \mathbb{C}_+$. By direct computations, these two logarithms coincide on the operators $\mathrm{i}aI$, with $a > 0$, and, hence, coincide on all operators $H + zI$, $z \in \mathbb{C}_+$. Employing the continuity of the Riesz operator logarithm and the property $\lim_{\varepsilon\downarrow 0}\log(H + \mathrm{i}\varepsilon I) = \log(H)$ (cf. [31, Lemma 2.6 (iv)]), one has that the two averaged logarithms of H coincide. □

Remark A.2.5. *In fact, a statement stronger than the one in Theorem A.2.4 is true, but it is not needed for our purpose. More precisely, the operator logarithm*

(1.1.1) *equals the principal branch of the logarithm of a boundedly invertible dissipative operator* H *on* $\mathcal{H}$ *with the cut along the negative imaginary semi-axis provided by the Riesz functional calculus. The proof of this fact is rather technical.*

Appendix B

Basic facts on operator theory

B.1 General operator theory

We recall that the real and imaginary parts of an operator $H \in \mathcal{B}(\mathcal{H})$ are defined by $\mathrm{Re}(H) = \frac{H+H^*}{2}$ and $\mathrm{Im}(H) = \frac{H-H^*}{2\mathrm{i}}$, respectively.

Definition B.1.1. An operator $H \in \mathcal{B}(\mathcal{H})$ is called dissipative if its imaginary part is non-negative.

Definition B.1.2. The numerical range $W(H)$ of an operator $H \in \mathcal{B}(\mathcal{H})$ is the subset of $\mathbb{C}$ given by

$$W(H) = \{\langle Hx, x\rangle \,:\, x \in \mathcal{H},\ \|x\| = 1\}.$$

Theorem B.1.3. *[33, Theorem 1.2-1] The spectrum of an operator is contained in the closure of its numerical range.*

Theorem B.1.4. *[33, Theorem 1.4-4] The closure of the numerical range of a normal operator is the convex hull of its spectrum.*

Theorem B.1.5 (Stone's formula)**.** *Let $H = H^* \in \mathcal{B}(\mathcal{H})$. Then, for any interval*

$(a, b) \subset \mathbb{R}$,

$$\begin{aligned}
&1/2\left[E_H\big([a,b]\big) + E_H\big((a,b)\big)\right] \\
&\quad = s\text{-}\lim_{\varepsilon\downarrow 0} \frac{1}{2\pi\mathrm{i}} \int_a^b \left[(H - \lambda I - \mathrm{i}\varepsilon I)^{-1} - (H - \lambda I + \mathrm{i}\varepsilon I)^{-1}\right] d\lambda. \qquad \text{(B.1)}
\end{aligned}$$

B.2 A self-adjoint dilation of a dissipative operator

In this section, we study some properties of the Ξ operator [31] (cf. also [17])

$$\Xi(H) = P_{\mathcal{H}} E_L(\mathbb{R}_-)\big|_{\mathcal{H}},$$

associated with the dissipative operator H, where L is the minimal self-adjoint dilation of H in the Hilbert space $\mathcal{K}$.

We recall that the minimal self-adjoint dilation of a dissipative operator H is the self-adjoint operator L on the Hilbert space $\mathcal{K}$ satisfying

$$(H + \mathrm{i}\lambda I)^{-1} = P_{\mathcal{H}}(L + \mathrm{i}\lambda I)^{-1}\big|_{\mathcal{H}}, \quad \lambda > 0,$$

and $\mathcal{K} = \operatorname{span}\{(L + \mathrm{i}\lambda I)^{-1}\mathcal{H} \,:\, \lambda > 0\}$.

For a boundedly invertible dissipative operator H, $\Xi(H) = \arg(H)$ (cf. [31, Lemma 2.7]).

Theorem B.2.1. *Let H be a bounded dissipative operator. Then,*

$$s\text{-}\lim_{\varepsilon\downarrow 0} \Xi(H + \mathrm{i}\varepsilon I) = \Xi(H) + \frac{1}{2} P_{\mathcal{H}} E_L(\{0\})|_{\mathcal{H}}. \qquad \text{(B.1)}$$

If, in addition, 0 is not an eigenvalue of a self-adjoint part of H, then the second summand in equality (B.1) *is 0.*

Proof. The central tool in the proof of this lemma is the operator logarithm of a bounded invertible dissipative operator.

For a bounded invertible dissipative operator $H + \mathrm{i}\varepsilon I$, $\varepsilon > 0$, [31, Lemma 2.7] implies

$$
\begin{aligned}
\Xi(H+\mathrm{i}\varepsilon I) &= \frac{1}{\pi}\mathrm{Im}\big(\log(H+\mathrm{i}\varepsilon I)\big) \\
&= -\frac{1}{\pi}\int_0^\infty \mathrm{Re}((H+\mathrm{i}\varepsilon I+\mathrm{i}\lambda I)^{-1} - (1+\mathrm{i}\lambda)^{-1}I_{\mathcal{H}})d\lambda \\
&= -\frac{1}{\pi}\int_0^t \mathrm{Re}((H+\mathrm{i}\varepsilon I+\mathrm{i}\lambda I)^{-1} - (1+\mathrm{i}\lambda)^{-1}I_{\mathcal{H}})d\lambda \\
&\quad - \frac{1}{\pi}P_{\mathcal{H}}\int_t^\infty \mathrm{Re}((L+\mathrm{i}\varepsilon I+\mathrm{i}\lambda I)^{-1} - (1+\mathrm{i}\lambda)^{-1}I_{\mathcal{K}})\big|_{\mathcal{H}}d\lambda \\
&= -\frac{1}{\pi}\int_0^t \mathrm{Re}((H+\mathrm{i}\varepsilon I+\mathrm{i}\lambda I)^{-1} - (1+\mathrm{i}\lambda)^{-1}I_{\mathcal{H}})d\lambda \\
&\quad - \frac{1}{\pi}P_{\mathcal{H}}\int_t^\infty (L(L^2+(\lambda+\varepsilon)^2 I)^{-1} - (1+\lambda^2)^{-1}I_{\mathcal{K}})\big|_{\mathcal{H}}d\lambda. \qquad \text{(B.2)}
\end{aligned}
$$

Next, we prove that

$$
\lim_{\varepsilon\downarrow 0}\lim_{t\downarrow 0}\left\|\int_0^t \mathrm{Re}((H+\mathrm{i}\varepsilon I+\mathrm{i}\lambda I)^{-1} - (1+\mathrm{i}\lambda)^{-1}I_{\mathcal{H}})d\lambda\right\| = 0. \qquad \text{(B.3)}
$$

For any $\varepsilon > 0$ and $0 < \lambda < \|(H+\mathrm{i}\varepsilon I)^{-1}\|^{-1}$, one gets the series of inequalities

$$
\begin{aligned}
&\big\|(H+\mathrm{i}\varepsilon I+\mathrm{i}\lambda I)^{-1} - (1+\mathrm{i}\lambda)^{-1}I_{\mathcal{H}}\big\| \\
&\quad \le \big\|(H+\mathrm{i}\varepsilon I)^{-1}\big\| \frac{1}{|1+\mathrm{i}\lambda\,\|(H+\mathrm{i}\varepsilon I)^{-1}\|\,|} + 1 \\
&\quad \le \big\|(H+\mathrm{i}\varepsilon I)^{-1}\big\| \frac{1}{1-\lambda\,\|(H+\mathrm{i}\varepsilon I)^{-1}\|} + 1.
\end{aligned}
$$

Thus, the norm of the integral in (B.3) does not exceed

$$
-\log\big(1 - t\,\big\|(H+\mathrm{i}\varepsilon I)^{-1}\big\|\big) + t
$$

for $0 < t < \|(H+\mathrm{i}\varepsilon I)^{-1}\|^{-1}$, and the proof of (B.3) is complete.

Now we check that

$$s\text{-}\lim_{\varepsilon\downarrow 0}\lim_{t\downarrow 0}\frac{1}{\pi}\int_t^\infty L(L^2+(\lambda+\varepsilon)^2 I)^{-1}d\lambda = \frac{1}{2}\big(E_L((0,\infty)) - E_L((-\infty,0))\big). \tag{B.4}$$

The family of operators $\frac{1}{\pi}\int_t^\infty (L(L^2+(\lambda+\varepsilon)^2 I)^{-1}d\lambda$ is uniformly bounded in $t>0$, and therefore, it suffices to check the convergence (B.4) on a dense set in $\mathcal{K}$. A natural candidate for this set is

$$\left\{\bigcup_{\delta>0} E_L(\mathbb{R}\setminus(-\delta,\delta))f \bigcup E_L(\{0\})f \ :\ f\in\mathcal{K}\right\}.$$

Convergence (B.4) on $\{\bigcup_{\delta>0} E_L(\mathbb{R}\setminus(-\delta,\delta))f \ :\ f\in\mathcal{K}\}$ is checked in [31], so it remains to check the convergence on $\{E_L(\{0\})f \ :\ f\in\mathcal{K}\}$.

Below, we apply [6, Theorem 8 and Lemma 3b] to interchange the limits and integrals (with respect to the measure $E_L(\cdot)f$) and the Fubini theorem to interchange the integrals (first, for scalar spectral measures $\langle E(\cdot)f,g\rangle$, and then for operator-valued spectral measures concluding the equality of the operators from the coincidence of their quadratic forms). For $f\in E_L((-\delta,\delta))\mathcal{H}$,

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\lim_{t\downarrow 0}\frac{1}{\pi}\int_t^\infty L(L^2+(\lambda+\varepsilon)^2 I)^{-1}d\lambda f\\
&= \lim_{\varepsilon\downarrow 0}\lim_{t\downarrow 0}\frac{1}{\pi}\int_t^\infty d\lambda\int_{-\delta}^{\delta}\frac{\mu}{\mu^2+(\lambda+\varepsilon)^2}dE_L(\mu)f\\
&= \lim_{\varepsilon\downarrow 0}\lim_{t\downarrow 0}\frac{1}{\pi}\int_t^\infty d\lambda\lim_{\gamma\downarrow 0}\left(\int_{-\delta}^{-\gamma}+\int_\gamma^\delta\right)\frac{\mu}{\mu^2+(\lambda+\varepsilon)^2}dE_L(\mu)f\\
&= \lim_{\varepsilon\downarrow 0}\lim_{t\downarrow 0}\lim_{\gamma\downarrow 0}\frac{1}{\pi}\int_t^\infty d\lambda\left(\int_{-\delta}^{-\gamma}+\int_\gamma^\delta\right)\frac{\mu}{\mu^2+(\lambda+\varepsilon)^2}dE_L(\mu)f\\
&= \lim_{\varepsilon\downarrow 0}\lim_{t\downarrow 0}\lim_{\gamma\downarrow 0}\frac{1}{\pi}\left(\int_{-\delta}^{-\gamma}+\int_\gamma^\delta\right)\left(\frac{\pi}{2}-\arctan\left(\frac{\varepsilon+t}{|\mu|}\right)\right)dE_L(\mu)f\\
&= \lim_{\varepsilon\downarrow 0}\lim_{t\downarrow 0}\frac{1}{\pi}\int_{-\delta}^{\delta}h_t(\mu)\,dE_L(\mu)f = \frac{1}{2}E_L((-\delta,\delta)\setminus\{0\})f,
\end{aligned}\tag{B.5}$$

where

$$h_t(\mu) = \begin{cases} \frac{\pi}{2} - \arctan\left(\frac{\varepsilon+t}{|\mu|}\right) & \text{if } \mu \neq 0, \\ 0 & \text{if } \mu = 0. \end{cases}$$

Applying (B.5) to $f = E_L(\{0\})g$, $g \in \mathcal{K}$, completes the proof of (B.4).

In view of (B.2), (B.3), and (B.4),

$$\begin{aligned} s\text{-}\lim_{\varepsilon\downarrow 0} \Xi(H + \mathrm{i}\varepsilon I) &= \frac{1}{2} P_{\mathcal{H}}\big(- E_L((0,\infty)) + E_L((-\infty,0)) + I_{\mathcal{K}}\big)\big|_{\mathcal{H}} \\ &= P_{\mathcal{H}}\big(E_L((-\infty,0)) + \frac{1}{2}E_L(\{0\})\big)\big|_{\mathcal{H}}, \end{aligned}$$

which completes the proof of (B.1).

Now we proceed to the proof of the second assertion of the theorem. First of all, we notice that the Caley transform of H is a contraction. According to [51, Theorem I.3.3.2], every contraction S on the Hilbert space $\mathcal{H}$ corresponds to a decomposition of $\mathcal{H}$ into an orthogonal sum $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ of two reducing subspaces of S such that the part of S on $\mathcal{H}_0$ is unitary, and the part of S on $\mathcal{H}_1$ is completely non-unitary. The minimal unitary dilation of a completely non-unitary contraction has absolutely continuous spectrum (see [50]), and, hence, in particular, it does not have the eigenvalue -1. Therefore, the kernel of the minimal self-adjoint dilation of H is trivial. □

Corollary B.2.2. *Let H be a dissipative operator in $\mathcal{A}$. Then the normal boundary value $s\text{-}\lim_{\varepsilon\downarrow 0} \Xi(H + \mathrm{i}\varepsilon I)$ is an element in $\mathcal{A}$.*

For a normal operator, strong convergence in Theorem B.2.1 can be substituted by norm convergence.

Lemma B.2.3. *Let H be a self-adjoint operator in $\mathcal{B}(\mathcal{H})$. Then,*

$$n\text{-}\lim_{\varepsilon\downarrow 0} \Xi(H + \mathrm{i}\varepsilon I) = E_H(\mathbb{R}_-) + \frac{1}{2}E_H(\{0\}).$$

Proof. Applying the Spectral Theorem yields

$$n\text{-}\lim_{\varepsilon\downarrow 0} \Xi(H + \mathrm{i}\varepsilon I) = n\text{-}\lim_{\varepsilon\downarrow 0} \frac{1}{\pi}\mathrm{Im}(\log(H + \mathrm{i}\varepsilon I)) = \frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\int_{\sigma(H)} \arg(\lambda + \mathrm{i}\varepsilon) dE_H(\lambda).$$

By [6, Theorem 8] which is applicable in view of [6, Lemma 3b],

$$n\text{-}\lim_{\varepsilon\downarrow 0}\int_{\sigma(H)} \arg(\lambda + \mathrm{i}\varepsilon) dE_H(\lambda) = \int_{\sigma(H)} \left(\lim_{\varepsilon\downarrow 0} \arg(\lambda + \mathrm{i}\varepsilon)\right) dE_H(\lambda).$$

The function $\lim_{\varepsilon\downarrow 0} \arg(\lambda + \mathrm{i}\varepsilon)$ can be represented as the sum of g and h defined by

$$g(\lambda) = \begin{cases} \arg(\lambda) & \text{if } \lambda \neq 0, \\ 0 & \text{if } \lambda = 0, \end{cases}$$

$$h(\lambda) = \begin{cases} \frac{\pi}{2} & \text{if } \lambda = 0, \\ 0 & \text{if } \lambda \neq 0. \end{cases}.$$

Thus,

$$n\text{-}\lim_{\varepsilon\downarrow 0} \Xi(H + \mathrm{i}\varepsilon I) = \frac{1}{\pi}g(H) + \frac{1}{2}E_H(\{0\}),$$

completing the proof. □

Appendix C

On the Birman-Schwinger principle and scattering theory

C.1 The classical Birman-Schwinger principle

In 1961, M. Sh. Birman [8] and J. Schwinger [57] independently introduced a method to control the number of negative eigenvalues of Schrödinger operators. In the abstract operator-theoretic setting, the classical Birman-Schwinger principle (in its simplest form) states (see, e.g., [60]): *Given a self-adjoint strictly positive operator H_0 and a non-negative self-adjoint compact operator V on a Hilbert space $\mathcal{H}$, the number of negative eigenvalues (counting multiplicity) of the operator $H = H_0 - V$ coincides with the number of eigenvalues greater than one of the Birman-Schwinger operator $V^{1/2}H_0^{-1}V^{1/2}$.* That is,

$$\dim[E_H(\mathbb{R}_-)\mathcal{H}] = \dim\left[E_{I-V^{1/2}H_0^{-1}V^{1/2}}(\mathbb{R}_-)\mathcal{H}\right], \tag{C.1}$$

The case of a semi-infinite gap. In this paragraph, we state two versions of the Birman-Schwinger principle in the case of a semi-infinite gap.

Theorem C.1.1 ([8, Lemma 1.3]). *Let $H_0 > 0$ and $V \geq 0$ be symmetric operators* *with coinciding domains. Assume that the quadratic form* $H_0[u,u]$ *of the operator* H_0 *satisfies*

$$H_0[u,u] > 0 \textit{ for } u \neq 0 \textit{ in its domain } \mathrm{dom}[H_0].$$

Assume, in addition, that

$$\inf_{u \in \mathrm{dom}(H_0), \|u\|=1} \langle (H_0 - V)u, u \rangle > -\infty$$

and the form $V[\cdot,\cdot]$ *considered on* $\mathrm{dom}[H_0]$ *admits a closure* $\overline{V}[\cdot,\cdot]$ *in the Hilbert space* $\mathcal{H}_{H_0}$ *which is the completion of* $\mathrm{dom}[H_0]$ *in the metric induced by* $H_0[\cdot,\cdot]$*. Let* $\widetilde{V}$ *be the self-adjoint operator in* $\mathcal{H}_{H_0}$ *generated by the form* $\overline{V}[\cdot,\cdot]$ *and* H *the Friedrichs extension (cf. [8] or [27]) of* $H_0 - V$*. Then,*

$$\dim\big[E_H((-\infty,0))\mathcal{H}\big] = \dim\big[E_{\widetilde{V}}((1,\infty))\mathcal{H}_{H_0}\big].$$

Remark C.1.2 ([8, Theorem 1.4])**.** *Under the assumptions of Theorem C.1.1, the total multiplicity of the negative spectrum of the operator* H *is finite if and only if the operator* $\widetilde{V}$ *is compact.*

Theorem C.1.3 ([38, Theorem 2 and Corollary])**.** *Assume that* H_0 *is a positive self-adjoint operator with* $0 \in \rho(H_0)$ *and* K *a densely defined, closed operator with domain* $\mathrm{dom}(H_0)$ *a subset in* $\mathrm{dom}(K)$*. Assume, in addition, that for some (and hence for any)* $z \in \rho(H_0)$*, the operator* $KR_0(z)K^*$ *defined on* $\mathrm{dom}(K^*)$ *has a bounded extension* $Q(z)$ *which is a compact operator for all* $z \in \rho(H_0)$*. Finally, assume that there exists the limit* $\lim_{\lambda\to 0^-} Q(\lambda) =: Q(0)$ *in the uniform operator topology and for all* $\alpha \in (0,1]$*, there exists* $z \in \rho(H_0)$ *such that* $1 \in \rho(\alpha Q(z))$*. Let* H *be the operator* H_0"$-$"K^*K

understood in the sense of Kato (cf. [37]). Then,

$$\dim[E_H(\mathbb{R}_-)\mathcal{H}] = \dim[E_{Q(0)}((1,\infty))\mathcal{H}].$$

The case of an arbitrary gap and a sign-indefinite perturbation. The Birman-Schwinger principle in case of a sign-indefinite operator and sign-indefinite perturbation was formulated in terms of the spectral shift function. The spectral shift function (the ξ-function) was introduced by I. M. Lifshits in 1952 [46]. The base for the comprehensive mathematical ξ-theory was set up by M. G. Krein in his famous paper [41]. On account of the further development of the ξ-theory in the classical setting, one can consult [11], [13], [60], [61] for survey exposition and references.

Definition C.1.4. Let H_0 and H be self-adjoint operators with coinciding domains. Assume that $H - H_0$ is in the trace class. Then, the ξ-function $\xi(\lambda, H_0, H)$ associated with the pair of operators (H_0, H) is uniquely defined for a.e. $\lambda \in \mathbb{R}$ by

$$\xi(\cdot, H_0, H) \in L^1(\mathbb{R}, d\lambda),$$

$$\operatorname{tr}\big[(H - zI)^{-1} - (H_0 - zI)^{-1}\big] = \int_{\mathbb{R}} \frac{\xi(\lambda, H_0, H)}{(\lambda - z)^2} d\lambda.$$

Remark C.1.5. *The ξ-function can also be defined under more general conditions (cf. [42]), for example, under the condition*

$$(H - zI)^{-1} - (H_0 - zI)^{-1} \text{ is in the trace class for all } z \in \rho(H_0) \cap \rho(H).$$

Remark C.1.6. *For almost all λ in a joint spectral gap (a, b) of operators H_0 and H,*

$$\xi(\lambda, H_0, H) = \operatorname{tr}\big[E_H((-\infty, \lambda)) - E_{H_0}((-\infty, \lambda))\big] = n \in \mathbb{Z}.$$

We may and will choose a representative from the equivalence class of the SSF that equals n on the whole interval (a, b).

Theorem C.1.7 ([29, Remark 5.13])**.** *Let H_0 be a self-adjoint operator and $V = V^*$ a trace class operator. Let $J = \operatorname{sgn}(V)$, with $\operatorname{sgn}(x) = \begin{cases} 1 & \text{if} \quad x \geq 0, \\ -1 & \text{if} \quad x < 0. \end{cases}$ Then, for λ in a joint spectral gap (a, b) of H_0 and H,*

$$\operatorname{tr}\big[E_{H_0+V}((-\infty, \lambda)) - E_{H_0}((-\infty, \lambda))\big] = \operatorname{tr}\big[E_J(\mathbb{R}_-) - E_{J+|V|^{1/2}(H_0-\lambda I)^{-1}|V|^{1/2}}(\mathbb{R}_-)\big],$$

or equivalently,

$$\xi(\lambda, H_0, H_0 + V) = \xi\big(0, J + |V|^{1/2}(H_0 - \lambda I)^{-1}|V|^{1/2}, J\big).$$

Remark C.1.8. *If λ is in a joint spectral gap of the operators H_0 and $H_0 + V$, then the operator $J + |V|^{1/2}(H_0 - \lambda I)^{-1}|V|^{1/2}$ is boundedly invertible.*

C.2 Scattering theory

Hypothesis C.2.1. *Let H_0 be a self-adjoint operator, $V = V^*$ a trace class operator, and $H = H_0 + V$.*

Let P_0 be the projection onto the absolutely continuous subspace of H_0. Then, the wave operators

$$W_\pm = s\text{-}\lim_{t\to\pm\infty} \exp(\mathrm{i}tH)\exp(-\mathrm{i}tH_0)P_0$$

exist (see, e.g., [61]). The scattering operator $S_{H,H_0} = W_+^* W_-$ is unitary in $P_0\mathcal{H}$ and commutes with H_0. Let $P_0\mathcal{H}$ be decomposed into a direct integral

$$P_0\mathcal{H} = \int_{\mathbb{R}} \oplus \mathcal{H}(\lambda) d\lambda, \tag{C.1}$$

where H_0 acts as multiplication by λ. Then, S_{H,H_0} in decomposition (C.1) acts as multiplication by the operator-valued function $S_{H,H_0}(\lambda)$, called the scattering matrix. The operators $S_{H,H_0}(\lambda)$ are unitary in $\mathcal{H}(\lambda)$ and $S_{H,H_0}(\lambda) - I_{\mathcal{H}_\lambda}$ is in the trace class for a.e. $\lambda \in \mathbb{R}$.

Theorem C.2.2 (The Birman-Krein formula [9, 10])**.** *Assume Hypothesis C.2.1. Then, the Fredholm determinant of the scattering matrix equals*

$$\det S_{H,H_0}(\lambda) = \exp\big[-2\pi \mathrm{i}\xi(\lambda, H, H_0)\big] \quad \text{for a.e. } \lambda \in \mathbb{R}.$$

Under some additional assumptions the scattering matrix admits the following representation.

Theorem C.2.3 ([61, Section 5.7])**.** *Assume Hypothesis C.2.1. Let $J = \operatorname{sgn}(V)$ (cf. Theorem C.1.7) and $\Lambda \subset \sigma_{ac}(H_0)$. Assume, in addition, that for a.e. $\lambda \in \Lambda$, the limit*

$$A(\lambda + \mathrm{i}0) + \mathrm{i}B(\lambda + \mathrm{i}0) = n\text{-}\lim_{\varepsilon\downarrow 0} |V|^{1/2}\big(H_0 - (\lambda + \mathrm{i}\varepsilon)I\big)^{-1}|V|^{1/2}$$

exists and that $0 \in \rho\big(J + A(\lambda + \mathrm{i}0) + \mathrm{i}B(\lambda + \mathrm{i}0)\big)$, with $A(\lambda + \mathrm{i}0)$ and $B(\lambda + \mathrm{i}0)$ self-adjoint. Then, the scattering matrix $S_{H,H_0}(\lambda)$ is unitarily equivalent to the operator

$$I - 2\mathrm{i}B(\lambda + \mathrm{i}0)^{1/2}(N + A(\lambda + \mathrm{i}0) + \mathrm{i}B(\lambda + \mathrm{i}0))^{-1}B(\lambda + \mathrm{i}0)^{1/2}$$

for a.e. $\lambda \in \Lambda$.

Remark C.2.4. *If $M(\cdot) - I$ is a Herglotz operator valued function with values in the trace class, then the normal boundary values $M(\lambda + \mathrm{i}0)$ exist for a.e. $\lambda \in \mathbb{R}$ in the norm of the Schatten-von Neumann class γ_p, $p > 1$, and hence, in the operator norm* *(cf. [49]). If, in addition,* $M(z_0)$ *is boundedly invertible for some (and hence for all)* $z_0 \in \mathbb{C}_+$*, then* $M(\lambda + \mathrm{i}0)$ *is boundedly invertible for a.e.* $\lambda \in \mathbb{R}$.

Bibliography

[1] M. F. Atiyah, V. K. Patodi, I. M. Singer, *Spectral asymmetry and Riemannian geometry. I,* Math. Proc. Cambridge Philos. Soc. **77** (1975), 43–69.

[2] M. F. Atiyah, V. K. Patodi, I. M. Singer, *Spectral asymmetry and Riemannian geometry. II,* Math. Proc. Cambridge Philos. Soc. **78** (1975), no. 3, 405–432.

[3] M. F. Atiyah, V. K. Patodi, I. M. Singer, *Spectral asymmetry and Riemannian geometry. III,* Math. Proc. Cambridge Philos. Soc. **79** (1976), no. 1, 71–99.

[4] N. A. Azamov, A. L. Carey, P. G. Dodds, F. A. Sukochev, *Operator integrals, spectral shift, and spectral flow,* Canad. J. Math., to appear.

[5] N. A. Azamov, P. G. Dodds, F. A. Sukochev, *The Krein spectral shift function in semifinite von Neumann algebras*, Integr. Equ. Oper. Theory **55** (2006), 347 – 362.

[6] R. G. Bartle, *A general biliniar vector integral,* Studia Math. **15** (1956), 337–352.

[7] M.-T. Benameur, A. L. Carey, J. Phillips, A. Rennie, F. A. Sukhochev, K. P. Wojciechowski, *An analytic approach to spectral flow in von Neumann algebras,* to appear in "Spectral and Geometric Analysis on Manifolds - Papers in Honour of K.P. Wojiechowski", World Scientific.

[8] M. Sh. Birman, *On the spectrum of singular boundary-value problems,* Mat. Sbornik N.S. **55 (97)** (1961), 125–174; English. transl. in AMS Trans. **53** (1966), 23–80.

[9] M. Sh. Birman, M. G. Krein, *On the theory of wave operators and scattering operators,* Dokl. Akad. Nauk SSSR **144** (1962), 475–478; English transl. in Soviet Math. Dokl. **3** (1962), 740–744.

[10] M. Sh. Birman, M. G. Krein, Some topics of the theory of the wave and scattering operators, in *Outlines Joint Sympos. Partial Differential Equations (Novosibirsk, 1963)*, 39–45, Acad. Sci. USSR Siberian Branch, Moscow.

[11] M. Sh. Birman, A. B. Pushnitski, *Spectral shift function, amazing and multi-faceted,* Integral Equations Operator Theory **30** (1998), no. 2, 191–199.

[12] M. Sh. Birman, M. Z. Solomyak, *Remarks on the spectral shift function,* Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **27** (1972), 33–46; English transl. in J. Soviet Math. **3** (1975), no. 4, 408–419.

[13] M. Sh. Birman, D. R. Yafaev, *The spectral shift function. The work of M. G. Krein and its further development,* Algebra i Analiz **4** (1992), no. 5, 1–44; English transl. in St. Petersburg Math. J. **4** (1993) 833–870.

[14] S. Borac, *On the algebra generated by two projections,* J. Math. Phys. **36** (1995), no. 2, 863–874.

[15] L. G. Brown, *Lidskiĭ's theorem in the type* II *case,* Geometric methods in operator algebras (Kyoto, 1983), 1–35, Pitman Res. Notes Math. Ser., 123, Longman Sci. Tech., Harlow, 1986.

[16] R. Carmona, J. Lacroix, *Spectral theory of random Schrödinger operators*, Birkhäuser Boston, Boston, MA, 1990.

[17] R. W. Carey, *A unitary invariant for pairs of self-adjoint operators,* Reine Angew. Math. **283/284** (1976), 294 – 312.

[18] R. W. Carey, J. D. Pincus, *Mosaics, principal functions, and mean motion in von Neumann algebras,* Acta Math. **138** (1977), no. 3-4, 153–218.

[19] S. Cuccagna, D. Pelinovsky, and V. Vougalter, *Spectra of positive and negative energies in the linearized NLS problem,* Comm. Pure Appl. Math. **58** (2005), 1 – 29.

[20] J. Dixmier, *Von Neumann algebras*, Translated from the second French edition by F. Jellett, North-Holland, Amsterdam, 1981.

[21] H. R. Dowson, *Spectral theory of linear operators, LMS Monographs,* **12**, Academic Press, New York, 1978.

[22] N. Dunford, J. T. Schwartz, *Linear Operators. I. General Theory.* With the assistance of William G. Bade and Robert G. Bartle. Reprint of the 1958 original. John Wiley & Sons, Inc., New York, 1988.

[23] T. Fack, *Sur la notion de valeur caractéristique,* J. Operator Theory **7** (1982), 307–333.

[24] T. Fack, H. Kosaki, *Generalized s-numbers of τ-measurable operators,* Pacific J. Math. **123** (1986), no. 2, 269–300.

[25] A. L. Figotin, Ergodic properties and essential selfadjointness of random matrix operators, in *Operators in function spaces and problems in function theory*

(Russian), 13–27, 143, "Naukova Dumka", Kiev.

[26] B. Fuglede, *A commutativity theorem for normal operators*, Proc. Nat. Acad. Sci. U.S.A. **36** (1950), 35–40.

[27] K. Friedrichs, *Spektraltheorie halbbeschränkter Operatoren und Anwendung auf die Spektralzerlegung von Differentialoperatoren,* Math. Ann. **109** (1934), no. 4-5, 465–487; 685–713.

[28] B. Fuglede, R. V. Kadison, *Determinant theory in finite factors,* Ann. of Math. (2) **55** (1952), 520–530.

[29] F. Gesztesy, K. A. Makarov, *The* Ξ *operator and its relation to Krein's spectral shift function,* J. Anal. Math. **81** (2000), 139–183.

[30] F. Gesztesy, K. A. Makarov, $\mathrm{SL}_2(\mathbb{R})$*, exponential representation of Herglotz functions, and spectral averaging,* Algebra i Analiz **15** (2003), no. 3, 104–144; English transl. in St. Petersburg Math. J. **15** (2004), no. 3, 393–418.

[31] F. Gesztesy, K. A. Makarov, S. N. Naboko, *The spectral shift operator*, Oper. Theory Adv. Appl. **108** (1999), 59–90, Birkhäuser, Basel.

[32] I. C. Gohberg and M. G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators in Hilbert Space,* Amer. Math. Soc. Transl. of Math. Monographs 18, Providence, RI, 1969.

[33] K. E. Gustafson and D. K. M. Rao, *Numerical range*, Springer, New York, 1997.

[34] P. R. Halmos, *Two subspaces,* Trans. Amer. Math. Soc. **144** (1969), 381–389.

[35] P. de la Harpe, G. Skandalis, *Déterminant associé à une trace sur une algèbre de Banach,* Ann. Inst. Fourier (Grenoble) **34** (1984), no. 1, 241–260.

[36] R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras Vol. II,* Academic Press, Orlando, FL, 1986.

[37] T. Kato, *Wave operators and similarity for some non-selfadjoint operators,* Math. Ann. **162** (1965/1966), 258–279.

[38] R. Konno, S. T. Kuroda, *On the finiteness of perturbed eigenvalues,* J. Fac. Sci. Univ. Tokyo Sect. I **13** (1966), 55–63 (1966).

[39] V. Kostrykin, K. A. Makarov, A. Skripka, *The Birman-Schwinger principle in von Neumann algebras of finite type,* J. Funct. Anal., to appear.

[40] V. Kostrykin, K. A. Makarov, A. Skripka, *Some applications of the perturbation determinant in finite von Neumann algebras,* in preparation.

[41] M. G. Krein, *On the trace formula in perturbation theory,* Mat. Sbornik N.S. **33(75)** (1953), 597–626.

[42] M. G. Krein, *On perturbation determinants and a trace formula for unitary and self-adjoint operators,* Dokl. Akad. Nauk SSSR **144** (1962), 268–271; English transl. in Soviet Math. Dokl. **3** (1962), 707–710.

[43] M. G. Krein, M. A. Krasnosel'skiĭ, D. P. Milman, *On defect numbers of linear operators in a Banach space and some geometric questions,* Sbornik Trudov Inst. Mat. Akad. Nauk USSR, **11** (1948), 97–112.

[44] J. C. Lagarias, P. A. B. Pleasants, *Repetitive Delone sets and quasicrystals,* Ergodic Theory Dynam. Systems **23** (2003), no. 3, 831–867.

[45] D. Lenz, P. Stollmann, *Quasicrystals, aperiodic order, and groupoid von Neumann algebras,* C. R. Math. Acad. Sci. Paris **334** (2002), no. 12, 1131–1136.

[46] I. M. Lifshits, *On a problem of the theory of perturbations connected with quantum statistics,* Uspehi Matem. Nauk (N.S.) **7** (1952), no. 1(47), 171–180. (Russian)

[47] V. Mathai, S. Yates, *Approximating spectral invariants of Harper operators on graphs,* J. Funct. Anal. **188** (2002), no. 1, 111–136.

[48] F. J. Murray, J. Von Neumann, *On rings of operators,* Ann. of Math. (2) **37** (1936), no. 1, 116–229.

[49] S. N. Naboko, On the conditions for existence of wave operators in the nonselfadjoint case, in *Wave propagation. Scattering theory*, Amer. Math. Soc. Transl. (2) **157** (1993), 127–149, Amer. Math. Soc., Providence, RI.

[50] B. Sz.-Nagy and C. Foias, *Sur les contractions de l'espace de Hilbert. IV,* Acta Sci. Math. Szeged **21** (1960), 251–259.

[51] B. Sz.-Nagy and C. Foias, *Harmonic analysis of operators on Hilbert space*, Translated from the French and revised, North-Holland, Amsterdam, 1970.

[52] L. Pastur, A. Figotin, *Spectra of random and almost-periodic operators*, Springer, Berlin, 1992.

[53] D. Petz, *Spectral scale of self-adjoint operators and trace inequalities,* J. Math. Anal. Appl. **109** (1985), 74–82.

[54] A. B. Pushnitski, *A representation for the spectral shift function in the case of perturbations of fixed sign,* Algebra i Analiz **9** (1997), no. 6, 197–213; English transl. in St. Petersburg Math. J. **9** (1998), no. 6, 1181–1194.

[55] A. Pushnitski, *The spectral shift function and the invariance principle,* J. Funct. Anal. **183** (2001), no. 2, 269–320.

[56] J. T. Schwartz, *W^*-Algebras*, Notes on Mathematics and its Applications, New York, Gordon and Breach, 1967.

[57] J. Schwinger, *On the bound states of a given potential,* Proc. Nat. Acad. Sci. U.S.A. **47** (1961), 122–129.

[58] M. A. Shubin, *Discrete magnetic Laplacian,* Comm. Math. Phys. **164** (1994), no. 2, 259–275.

[59] B. Simon, *Spectral averaging and the Krein spectral shift,* Proc. Amer. Math. Soc. **126** (1998), no. 5, 1409–1413.

[60] B. Simon, *Trace Ideals and Their Applications*, Cambridge Univ. Press, Cambridge, 1979.

[61] D. R. Yafaev, *Mathematical scattering theory*, Translated from Russian by J. R. Schulenberger, Amer. Math. Soc., Providence, RI, 1992.

VITA

Anna Skripka was born March 5, 1979, in Zaporizhzhya, Ukraine. After graduating from a Mathematics and Physics oriented public school, Anna left Zaporizhzhya to pursue her studies in Kharkiv. She completed the Full Higher Education in "Mathematics" and "Mathematics and Computer Science Education" at Kharkiv National University in 2001. The same year, Anna entered the graduate school at the University of Missouri-Columbia. Her PhD dissertation is based on research done during the years 2005-2006. In 2007, Anna will move to Texas to begin her three-year Visiting Assistant Professor position at the Texas A&M University.