TEACHERS' KNOWLEDGE OF ALGEBRAIC REASONING:

ITS ORGANIZATION FOR INSTRUCTION

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Doctor of Philosophy

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by

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The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

TEACHERS' KNOWLEDGE OF ALGEBRAIC REASONING:

ITS ORGANIZATION FOR INSTRUCTION

Presented by David Daniel Barker

a candidate for the degree of Doctor of Philosophy

and hereby certify that in their opinion it is worth of acceptance.

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This book is dedicated to my family, who have always been my largest supporter and the foundation that keeps me grounded. They have always believed in me and knew that I had the ability and fortitude to see this project through. Thank you all, we did it!!!

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ABSTRACT

Teacher Knowledge is a critical factor that influences instructional decisions. Elbaz (1983) stated, "the single factor which seems to have the greatest power to carry forward our understanding of the teachers' role is the phenomenon of teachers' knowledge" (p. 45). Hence, if we want to help teachers make wise choices in the classroom we must know and understand the types of knowledge used during the process of instruction. Unfortunately, the mathematics education community still knows too little about the knowledge required for teaching (Ball, Lubienski, & Mewborn, 2001; RAND Mathematics Study Panel, 2003) and "there is no consensus on what critical knowledge is necessary to ensure that students learn mathematics" (Fennema & Franke, 1992, p. 147). This study sought to understand the extent to which, and how, two teachers drew upon their knowledge of mathematics, knowledge of learners, and knowledge of curriculum and instructional strategies in the act of providing instruction in algebraic generalization.

Stimulated recall interviews were conducted for three lessons on algebraic generalization for two teachers, Amanda and Emily. The interviews were transcribed and coded in terms of the type of knowledge used: knowledge of mathematics, knowledge of learners, and knowledge of curriculum and instruction. The transcripts were further segmented into episodes—portions of the transcripts were a single thought was being discussed—and divided into nodes in terms of the knowledge forms used. Each node was analyzed using elements of grounded theory to uncover the types of interactions that occurred among the various forms of knowledge.

The findings provide a description of how often, and in what ways, the teachers drew upon their knowledge of mathematics, learners, and curriculum and instructional

strategies. Both teachers utilized their knowledge of learners prominently in their discussion about teaching. A framework was developed from the data to describe the ways in which the knowledge forms interacted with one another in the process of making instructional decisions. Typically, each teacher drew upon a primary form of knowledge with a second knowledge used in support. This secondary knowledge was either used to explain or justify the primary form of knowledge, or it was used to describe an implication of the primary form of knowledge.

This study developed a framework that describes how different knowledge bases interact in the mind of the teacher during the process of instruction. Further research needs to be conducted to uncover which types of interactions are beneficial to teachers in the act of providing instruction.

CHAPTER 1: INTRODUCTION

Problem Statement

Teaching is a complex endeavor requiring the consideration of multiple factors when making instructional decisions. The classroom environment necessitates that teachers make countless instructional choices, often in a short amount of time and in mentally taxing situations. Before instruction, teachers need to consider questions such as: What concepts do my students need to understand in order to master the objectives for the day? What types of tasks and activities build an understanding of these concepts? How will students think and react to these experiences? During instruction similar questions arise: What concepts are the students understanding and what concepts are they not understanding? What is the cause of these misunderstandings and what adaptations need to be made to address the misconceptions? The knowledge that a teacher brings to the classroom guides her in answering these questions.

Teacher knowledge is the critical factor that influences instructional decisions. Elbaz (1983) stated, "the single factor which seems to have the greatest power to carry forward our understanding of the teachers' role is the phenomenon of teachers' knowledge" (p. 45). Hence, if we want to help teachers make wise choices in the classroom we must know and understand the types of knowledge used during the process of instruction. Unfortunately, the mathematics education community still knows too little about the knowledge required for teaching (Ball, Lubienski, & Mewborn, 2001; RAND Mathematics Study Panel, 2003) and "there is no consensus on what critical knowledge is necessary to ensure that students learn mathematics" (Fennema & Franke, 1992, p. 147).

In addition to identifying what knowledge is used, we must understand how these forms of knowledge are used collectively. The classroom setting requires a mixture of several types of knowledge, and we need to understand the connections that teachers make among these various forms. Fennema and Franke commented that, "(Teacher) knowledge is not monolithic. It is a large, integrated, functioning system with each part difficult to isolate. … Some have studied knowledge as integrated, but most have not" (p. 148). The questions listed in the opening paragraph illustrate the nature of the decisions required by teachers and illuminate how diverse and intertwined this knowledge must be.

An example that illustrates the complexity of teacher knowledge is the selection of a curricular task. A teacher must use his knowledge of mathematics, learners, curriculum and instructional strategies, and other forms of knowledge to decide whether a certain task will deepen student understanding. Mathematical knowledge is needed to ensure that the task illuminates the intended concepts and that students respond in mathematically appropriate ways. Knowledge of learners is necessary in evaluating student understanding, guiding students toward more sophisticated strategies, and predicting and understanding student responses. Knowledge of curriculum and instructional strategies informs the design, adaptation, and implementation of a task to elicit specific mathematical concepts in student thinking. Although I list the roles of each of these types of knowledge separately, in actuality these knowledge bases work collaboratively. For instance, the knowledge of mathematics, learners, and curriculum/instructional strategies work together to inform teachers of the appropriateness of a task to meet a particular instructional goal. Without all three forms of knowledge working together this decision cannot be adequately made.

One difficulty in investigating the knowledge required by teachers is that this knowledge differs depending on the teacher, grade, school district, teacher expectations, etc. This knowledge also varies over time in response to changes in students, policies, teachers, technology, etc. Researchers must understand the nature of teacher knowledge and keep up with the impact these contextual factors have on this complex phenomenon.

For example, one recent change is the recommendation by the National Council of Teachers of Mathematics (NCTM) in the *Principles and Standards for School Mathematics* (NCTM, 2000) for algebra and algebraic reasoning to be introduced in the elementary and middle grades, woven throughout a child's mathematical experience. NCTM expects student understanding to deepen as they "encounter and build proficiency in all [types of reasoning] with increasing sophistication as they move through the curriculum" (p. 59). Unfortunately, student algebraic understanding has often lacked depth due to the traditional focus on symbol manipulation without connection to meaning (e.g. Booth, 1984; Demby, 1997; Kieran, 1992; Lee & Wheeler, 1989; Mason, 1996).

The earlier establishment of algebraic reasoning throughout the K-12 curriculum results in the need for teachers to adjust their knowledge base and creates a need to study the knowledge teachers bring to the instruction of algebraic generalization at the elementary and middle grades. The shift towards a focus on algebraic reasoning creates a ripple effect and establishes the need to understand related concepts such as the representations and justifications used in the development of generalizations. As mentioned before, the complex and integrated ways in which these forms of knowledge are used to develop algebraic reasoning also need to be studied. For these reasons, this study seeks to understand the types and organization of knowledge used by 5th grade teachers in providing instruction on early algebraic reasoning. In particular, this study set out to answer the following research questions:

1) To what extent do two fifth grade teachers draw upon their knowledge of mathematics, their knowledge of learners, and their knowledge of curriculum and instructional strategies when designing, implementing, and reflecting on instruction related to algebraic generalization?
2) How do two fifth grade teachers when designing, implementing, and reflecting on instruction related to algebraic generalization, use these three aspects of pedagogical content knowledge simultaneously?

The answer to these research questions provides an establishment of a base-line description of two 5th grade teachers' knowledge of early algebraic reasoning and how they use this knowledge for instruction. These data will serve as a starting point for developing hypothesis concerning not only the types of knowledge teachers have but how they use this knowledge for instruction. The data also allows for the examination of the research methodology used in this study as a means to measure teacher knowledge.

In order for professional developers to provide support for teachers they must be informed about the structure of teacher knowledge. What knowledge is needed for effective instruction? How is this knowledge structured and interconnected? Understanding the knowledge used by teachers in the classroom informs mathematics teacher educators to the needs of future and current teachers. The ultimate goal of this line of work is to inform the community of the types of knowledge structures that are beneficial to teachers in the act of teaching.

Purpose

Taking the above into consideration, this study seeks to: (a) examine how various forms of teacher knowledge, as they relate to the area of early algebraic reasoning, are integrated in the mind of the teacher; (b) develop and implement research methods that examine the complexities and nature of this knowledge; and (c) uncover relationships among these types of knowledge to inform the professional development community and better prepare teachers.

Theoretical Considerations

Overview

Two critical factors impacting the decisions teachers make in the classroom are the knowledge they bring to instruction and how they use these forms of knowledge during instruction. In this opening chapter, I provide the following: a) A discussion of the concept of teacher knowledge, which includes the different ways it has been studied and issues that arise in its investigation; b) A discussion of pedagogical content knowledge (PCK) and why it is of particular importance when studying the knowledge used in the act of teaching; c) An example of how integrated knowledge could be used in the instruction of early algebraic reasoning; d) A theoretical framework based on my working definition of integrated knowledge to be used for data collection and analysis of the research question. This framework relates the knowledge of mathematics, learners, and curriculum/instructional strategies with the concept of PCK; and e) A discussion of the significance of this study.

Teacher Knowledge

"The category 'teachers' knowledge' is new in the last 20 years, and the nature and development of that knowledge is only beginning to be understood by the present generation of researchers in teaching and teacher education" (Munby, Russell, & Martin, 2001). Despite the relatively recent awareness of the need to understand teacher knowledge, researchers have created a body of knowledge that incorporates many different views and aspects of the term. For instance, researchers have studied and discussed teacher content knowledge (Begle, 1979), pedagogical content knowledge (Wilson, Shulman & Richert, 1987), practical knowledge (Connelly & Clandinin, 1986), subject knowledge for teaching (Hill & Ball, 2004), curriculum knowledge (Behar & Paul, 1994), and knowledge of student thinking (Franke & Kazemi, 2001).

To illustrate these different aspects of teacher knowledge, let us return to the example of choosing a curricular task. If a researcher wanted to use the lens of mathematical content knowledge to study how teachers selected curricular tasks, then she might focus on the mathematics that teachers recognized in the problem or the mathematics recognized in student responses. If knowledge of learners was chosen as the lens for studying the choice of curricular tasks, a researcher might focus on the teachers' ability to predict and recognize student responses, strategies, justifications, misunderstandings, etc. Finally, if the researcher used a pedagogical lens for studying the choice of curricular tasks, the focus might be on how the teacher envisioned implementing the task in the classroom, what types of questions would be asked, or what techniques would be used to engage students in classroom discussions. Each of these perspectives is a valuable contribution to the field; however, they do not account for how a teacher connected these types of knowledge.

Although many studies have focused on a single type of teacher knowledge, an overlap exists among these different knowledge forms. If a researcher wanted to study the various strategies used by a student in solving a particular problem, she must factor in other types of knowledge in addition to the knowledge of learners. For instance, the teacher must understand the mathematics underlying the task in order to gain insight into why a student used such a strategy and to determine whether the strategy was fruitful. Thus the knowledge of mathematics is used to interpret the student thinking that was observed. Furthermore, the teacher must understand the influence of the task, social setting, and instructional technique on student strategies. The way in which a task is presented influences the strategies students are likely to use (Lannin, Barker, Townsend, 2006). Hence, knowledge regarding the context of instruction must be a factor in the discussion of student strategy use. In essence, it is difficult to study any one aspect of teacher knowledge without being cognizant of the others.

As can be seen through these various viewpoints, the idea of teacher knowledge is complex and has many components. Shulman (1985) commented that, "to be a teacher requires extensive and highly organized bodies of knowledge" (p. 47). However, the study of such a large and integrated system of knowledge presents a difficult proposition for educational researchers. Fennema and Franke (1992) contended that:

> Common sense suggests that knowledge is not monolithic. It is a large, integrated, functioning system with each part difficult to isolate. While many have speculated on the components of teacher knowledge, only a few have

> received major attention from researchers: content knowledge, knowledge of learning, knowledge of mathematical representations, and pedagogical knowledge. Some have studied knowledge as integrated, but most have not (p. 148).

To this point, most studies have focused on a single aspect of teacher knowledge, choosing to thoroughly address one aspect of this larger knowledge set by ignoring the added complexity that comes with the interaction with other knowledge forms. However, a few studies have rebuked this trend (Schoenfeld, 1985; Lampert, 1989; Lehrer & Franke, 1992; Ma, 1999) and have examined several aspects of teacher knowledge simultaneously.

In this study, I focused on three components of teacher knowledge and the interactions among them. Attention to a broader view of teacher knowledge was important as it reflected the complex and integrated knowledge that teachers used in the classroom. If teachers rely on integrated knowledge structures when making instructional decisions, then we must attend to this integrated nature when studying teacher knowledge.

Another difficulty in conducting research on teacher knowledge is the development of appropriate research instruments. For many years, researchers have collected data on teacher content knowledge by counting the number of college math courses taken or by administering multiple-choice tests (Darling-Hammond, 1999; Monk, 1994). However, these studies had limited power (Begle, 1979). Fennema and Franke (1992) claim that "perhaps the inadequate measures of knowledge and relatively limited research methodology concealed any relationships that did exist between teachers' knowledge and student learning" (p. 149). To this end, it is imperative that we note methodologies that have been fruitful in uncovering the integrated nature of teacher knowledge and develop new methodologies for obtaining the information needed.

Furthermore, many studies of teacher knowledge have focused on what teachers do not know instead of what teachers *need to know* to be effective (Ma, 1999). Such studies give few examples of the vast knowledge required by teachers to be effective, especially in light of the changes in instruction recommended by the NCTM Standards (1989, 2000). Although knowing what teachers do not know is valuable, it is only useful if we understand the knowledge that is beneficial to the act of teaching. Further research is needed to isolate the knowledge structures used during and for instruction. In order to accomplish this, more studies need to be conducted that look at knowledge used in classroom practice and that have classrooms as their focus of attention.

Two of the tenets of this study are that teacher knowledge should be studied in an integrated fashion and that knowledge actually used in the act of teaching should be the focal point of investigation. With this in mind, the natural starting point is the notion of pedagogical content knowledge discussed in the next section.

Pedagogical Content Knowledge

The previous section illustrated the many types of knowledge used by teachers to provide instruction. However, to conduct a study that investigates the integrated nature of all these forms of knowledge would be overwhelming. With this in mind, I selected a subset of teacher knowledge as the focal point for this investigation. This section provides an argument for the inclusion of pedagogical content knowledge, which is an integrated form of knowledge, as the main focus of this study.

Pedagogical content knowledge is an ever-evolving concept, with several different conceptualizations and adaptations being introduced over the years. Due to its name, PCK is often thought of as the incorporation, or overlap, of subject matter knowledge and knowledge of pedagogy. Although there is truth in this notion, which I discuss in more detail in Chapter 2, there are several aspects of PCK that are essential to building a case for its inclusion as a focal point for this study.

First, pedagogical content knowledge plays an integral role in the act of teaching. Of the various aspects of teacher knowledge, Shulman (1987) stated that:

> Pedagogical content knowledge is of special interest because it identifies the distinctive bodies of knowledge for teaching. … Pedagogical content knowledge is the category most likely to distinguish the understanding of the content specialist from that of the pedagogue. (p. 8)

One of the overarching goals of my research is to identify the types of knowledge that are useful for teachers in providing effective instruction. However, in order for teacher knowledge to make a difference during instruction, it has to be used in the classroom. It does not matter how well a teacher understands a certain mathematical topic; if that knowledge is never translated and implemented into instruction, it does not have an impact on student understanding. In order to ensure that the focus of this study is on knowledge that is relevant to the success of teaching, PCK is a logical place to start.

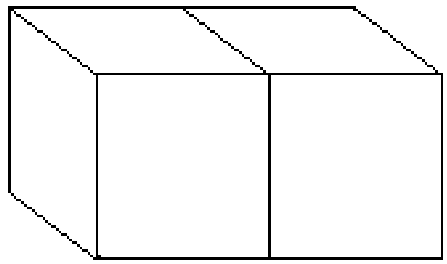
The second reason that pedagogical content knowledge is a plausible choice for this study is that it already is an integrated form of knowledge. Shulman (1987) stated that, "It [PCK] represents the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interest and abilities of learners, and presented for instruction" (p. 8). To understand a teacher's pedagogical content knowledge is to understand how they use their knowledge of content and pedagogy concomitantly to make instructional decisions. This understanding does not emerge from the study of a single knowledge form.

For instance, the study of pedagogy includes general teaching principles; but when this knowledge is applied to make specific instructional decisions in the classroom it is used in collaboration with other knowledge forms. What if a teacher had the goal of guiding her students to generalize certain linear situations? In this case, an understanding of the mathematics and student thinking related to this pattern would influence the teacher to use certain representations, tasks, and instructional techniques to help guide her students toward the instructional goal. This knowledge is indicative of what a teacher needs to know to be effective, fits the definition of pedagogical content knowledge, and has the greatest potential for impacting the preparation of competent teachers. The following section provides a more detailed description of this example.

Example of PCK in Early Algebraic Reasoning

Let us look at the role PCK plays in the instruction of early algebraic reasoning. Figure 1.1 illustrates a task (Lannin, Barker, & Townsend, 2006) that is typical of those found in NSF-funded curricula.

A company makes colored rods by joining cubes in a row and using a sticker machine to place "smiley" stickers on the rods. The machine places exactly one sticker on each exposed face of each cube. Every exposed face of each cube has to have a sticker, so this length two rod would need 10 stickers.



1. How many stickers would you need for rods of length 7? Length 10? Length 20? Length 49? Explain how you determined these values.
2. Explain how you could find the number of stickers needed for a rod of any length. Write a rule that you could use to determine this.

Figure 1.1: Cube Sticker problem

Such a task requires students to examine several particular cases and construct a generalization. For instance, in this task the child is asked to calculate the number of stickers required to cover rods of length 7, 10, 20, and 49. By examining the relationship between the input and output values and/or by examining the context of the problem, the student develops a rule to calculate the number of stickers required for a rod of any length. Thus, the student makes the transition from examining particular cases to generalizing about all cases.

In this task, the knowledge of mathematics, learners, and curriculum/instructional strategies are melded together by the teacher in choosing and implementing this task. Shulman (1987) states that PCK is the, "blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction" (p. 8). Knowledge of pedagogy informs us that students learn best in situations they can relate to and in tasks that promote social discourse. This knowledge of pedagogy could have aided in the selection of this particular task as the task provides a context that students can make connections to and use in developing justifications.

Knowledge of learners, from the research on generalization, informs us that students often use recursive and explicit reasoning when working with contextualized linearly increasing situations (Lannin, Townsend, & Barker, 2006). A student using recursive thinking might state, "I worked out the first few cases and realized that each time I add a cube I need to add four more stickers." A student using an explicit rule might claim, "For each cube I need 4 stickers to surround the cube and 2 more to cover the ends, so there would be $4n+2$ stickers for a rod of length *n."*

The knowledge of learners interacts with the mathematical knowledge required by teachers. For instance, since students use recursive and explicit strategies when solving problems of this type, teachers need to understand the mathematics of both the explicit and recursive rules:

$$\begin{aligned} S_1 &= 6 \\ S_n &= S_{n-1} + 4 \end{aligned} \qquad \text{(Recursive Notation)}$$

$$S(n) = 4n + 2 \text{ where } n \in \mathbb{N} \qquad \text{(Explicit Notation)}$$

In the recursive rule, a relationship between consecutive terms of the sequence is given along with a starting value. Hence, in the above example, if you know that a single cube requires six stickers, then all other rod lengths can be calculated from the first by successively adding fours. In the explicit rule, a direct relationship between the number of cubes and number of stickers is given. If the teacher's goal is for students to think flexibly and to move between forms of reasoning, she must also understand how recursive and explicit rules are related. One important connection is the relationship between adding 4 in the recursive rule and multiplying by 4 in the $4n$ of the explicit rule. The recursive rule describes the relationship between consecutive terms (i.e. that four more stickers are required fro each new cube that you add). This notion of slope is represented by the four in the rule $4n$ and is a critical feature of the pattern. As the value of n increases by one the value of the function increases by 4. This type of content knowledge is critical to the implementation of this task.

Furthermore, if a teacher wanted to elicit a certain type of response from students, he must understand the effects of tasks on student cognition. For example, Lannin, Townsend, and Barker (2006) found that asking students to find the number of stickers required to cover rods of relatively short lengths (less than 20), or rods that were of consecutive lengths, were factors that led students to use recursive reasoning; larger values that were not consecutive generally encouraged students to use explicit reasoning. This example illustrates the complex interaction of knowledge that is required to gain an "understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction" (Shulman, 1986, p.8). Because of the importance of PCK to the practice of teaching, and the fact that it is an integrated form of knowledge, PCK is an important piece of my theoretical framework.

Conceptual Framework

To this point, an argument for the need of research on the integrated nature of teacher knowledge regarding algebraic reasoning at the elementary level has been provided. In order to better describe and study the integrated nature of teachers' knowledge a framework needs to be defined. I begin by discussing several integrated models of teacher knowledge, giving the similarities and differences among them. The framework used in this study draws upon the notions presented in these bodies of work; Shulman's work served as the starting point in the development of my framework.

Integrated Models of Teacher Knowledge

Shulman (1986) proposed a framework that analyzed how teachers integrated three different types of teacher knowledge: subject matter knowledge, pedagogical content knowledge, and curricular knowledge. His framework developed from the need for a more comprehensive model of teacher knowledge that was created because teacher evaluation programs, at that time, tended to focus on a single aspect of teacher knowledge. Shulman defined content knowledge as "the amount and organization of knowledge per se in the mind of the teacher" (p. 9). Within the category of pedagogical content knowledge Shulman included:

> The most regularly taught topics in one subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the ways of representing and formulating a subject that make it comprehensible to others. (p. 9)

He also included, "the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught subjects" under the category of pedagogical content knowledge (p. 9). Finally, he defined curricular knowledge as the “full range of programs designed for the teaching of particular subjects and topics at a given level, the variety of instructional materials available in relation to those programs” (p. 10).

Although Shulman distinguished subject matter knowledge, pedagogical content knowledge, and curricular knowledge as separate bodies of knowledge, his vision of PCK included the integration of several types of knowledge. Shulman and Grossman (1988) further refined the concept of pedagogical content knowledge and developed five subcomponents: knowledge of alternative content frameworks, knowledge of student understanding and misconceptions of a subject, knowledge of curriculum, knowledge of particular content for the purpose of teaching, and knowledge of topic specific pedagogical strategies. If PCK is the aspect of teacher knowledge deemed most important by Shulman (1987) and contains components from other forms of knowledge, it seems reasonable to focus our attention on the interaction of these components within the realm of pedagogical content knowledge. It is important to understand how these various aspects of PCK interact in aiding the decision making process of teachers.

Fennema and Franke (1992) proposed a similar model for the integrated nature of teacher knowledge (see Figure 1.2).

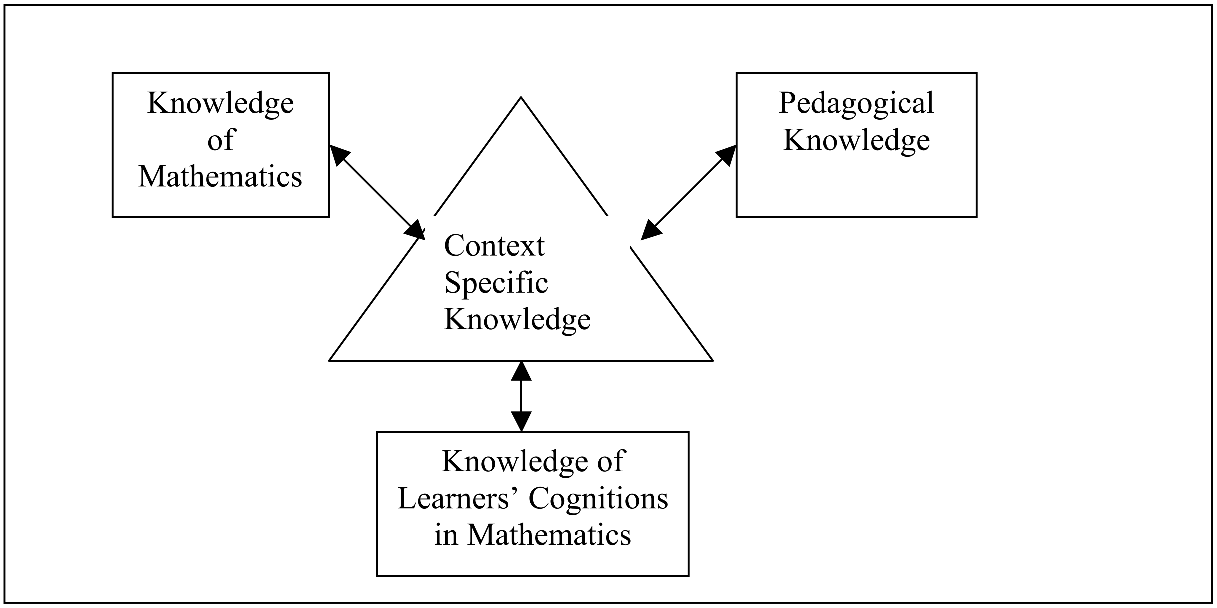


Figure 1.2: Context specific knowledge (Fennema & Franke, 1992)

In their model, they isolate the knowledge of mathematics, pedagogical knowledge, and knowledge of learner's cognitions in mathematics as the three critical pieces of teacher knowledge. The center triangle in their model represents the interaction of these knowledge forms in the teaching context. Fennema and Franke stated,

> The context is the structure that defines the components of knowledge and beliefs that come into play. Within a given context, teachers' knowledge of content interacts with knowledge of pedagogy and students' cognitions and combines with beliefs to create a unique set of knowledge that drives classroom behavior. (p. 162)

The description of context specific knowledge is similar to the concept of PCK described by Shulman and also includes aspects of content, pedagogical knowledge and knowledge of student thinking as they pertain to the act of teaching. Just as in Shulman's model, Fennema and Franke describe three distinct forms of teacher knowledge, but emphasize the importance of the interaction of this knowledge within the context of teaching.

Development of a Framework

The interaction of these distinct forms of teacher knowledge within PCK (Shulman's model) or context specific knowledge (Fennema and Franke's model) is of utmost importance. The mathematics education community needs to view these forms of knowledge as an integrated whole instead of separate entities. Neagoy (1995) supports this position when she stated:

> I object to the place given to pedagogical content knowledge: most all researchers inspired by Shulman's work list pedagogical content knowledge among other essential components of teacher knowledge, such as subject matter and general pedagogy. If we want a keener understanding of the complex phenomenon called teacher knowledge, it is my contention that we begin by putting the pieces back together and reexamining the whole. (p. 19)

Neagoy further elaborated, "Pedagogical content knowledge is not *one* among these sets of knowledge, but rather contains them all. From this position, Neagoy developed the framework illustrated in Figure 1.3.

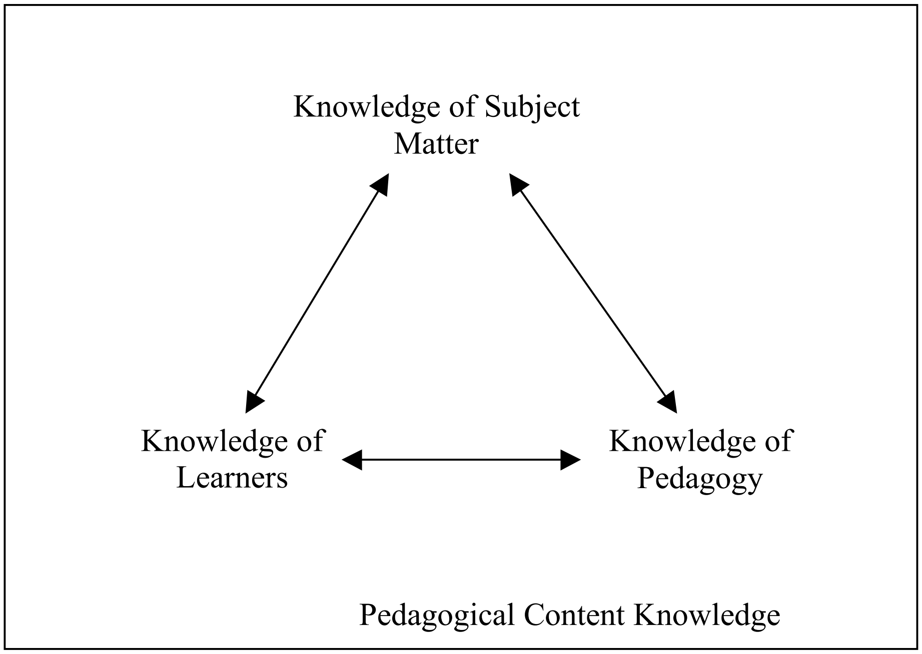


Figure 1.3: Neagoy's view of pedagogical content knowledge

The distinction between Neagoy's model and Shulman's is the central role and importance of pedagogical content knowledge. PCK is not one of several forms of knowledge that are important; it is *the* knowledge of teaching. Whereas the interaction of knowledge in previous models could exist within PCK or between PCK and other forms of knowledge, in Neagoy's model all interactions that occur outside of PCK are ignored. I concur with the sentiment of Neagoy and have developed a theoretical model that addresses these various aspects of teacher knowledge as a collective that resides within pedagogical content knowledge.

In analyzing the various frameworks on integrated knowledge (see Chapter 2), some similarities and differences were found. The framework used in this study, Figure 1.4, contains the three types of knowledge that are drawn from these similarities:

knowledge of mathematics, knowledge of learners, and knowledge of curriculum and instructional strategies.

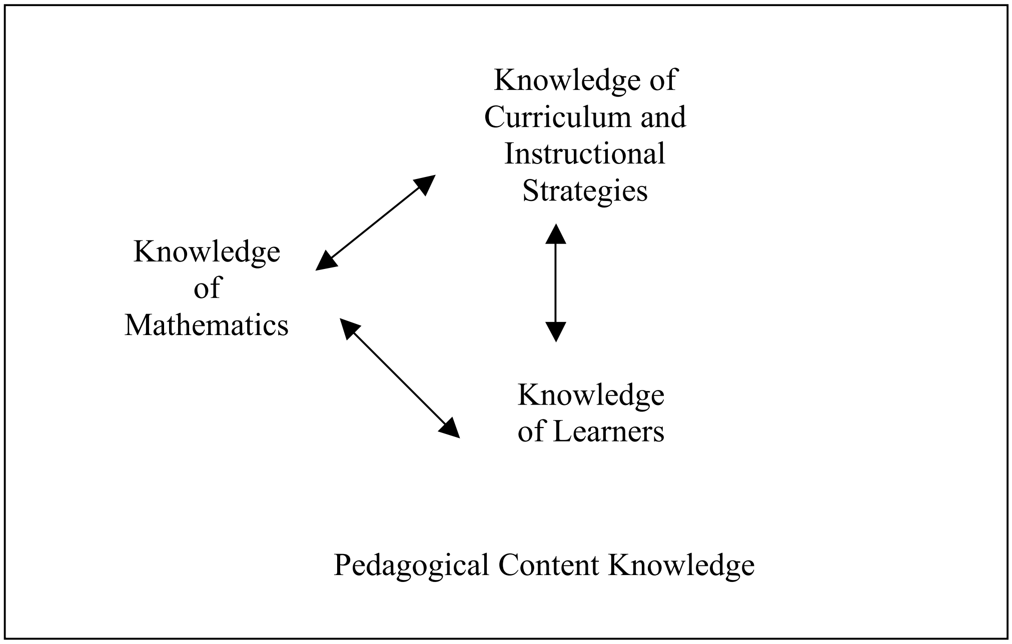


Figure 1.4: Barker's Component Model of PCK

Although these forms of knowledge are not arranged in the exact manner as the models discussed above, these three aspects of teacher knowledge are included in all of them and represent the commonalities that exist among the models. Furthermore, my notion of the knowledge of mathematics, the knowledge learners, and the knowledge curriculum and instructional strategies differs slightly from those presented by Shulman (1987) and Fennnema and Franke (1992). I adjusted my definitions because I am only interested in the aspects of teacher knowledge that aid teachers in making instructional decisions in the classroom or those forms of knowledge that are specific to the instruction of a particular idea. For example, I am only interested in the mathematical content knowledge that is considered a piece of PCK or content specific knowledge. Although Shulman (1987) conceives of PCK as one of several types of teacher knowledge, his conception of PCK developed with Grossman (1988) is similar to my theoretical model in that it brings out these interactions within PCK. Figure 1.4 gives an illustration of this model that I have titled the Component Model of PCK.

Figure 1.5 locates the notion of PCK within the larger set of teacher knowledge. The two critical features of PCK are that it is knowledge used in the act of teaching and it is integrated in nature.

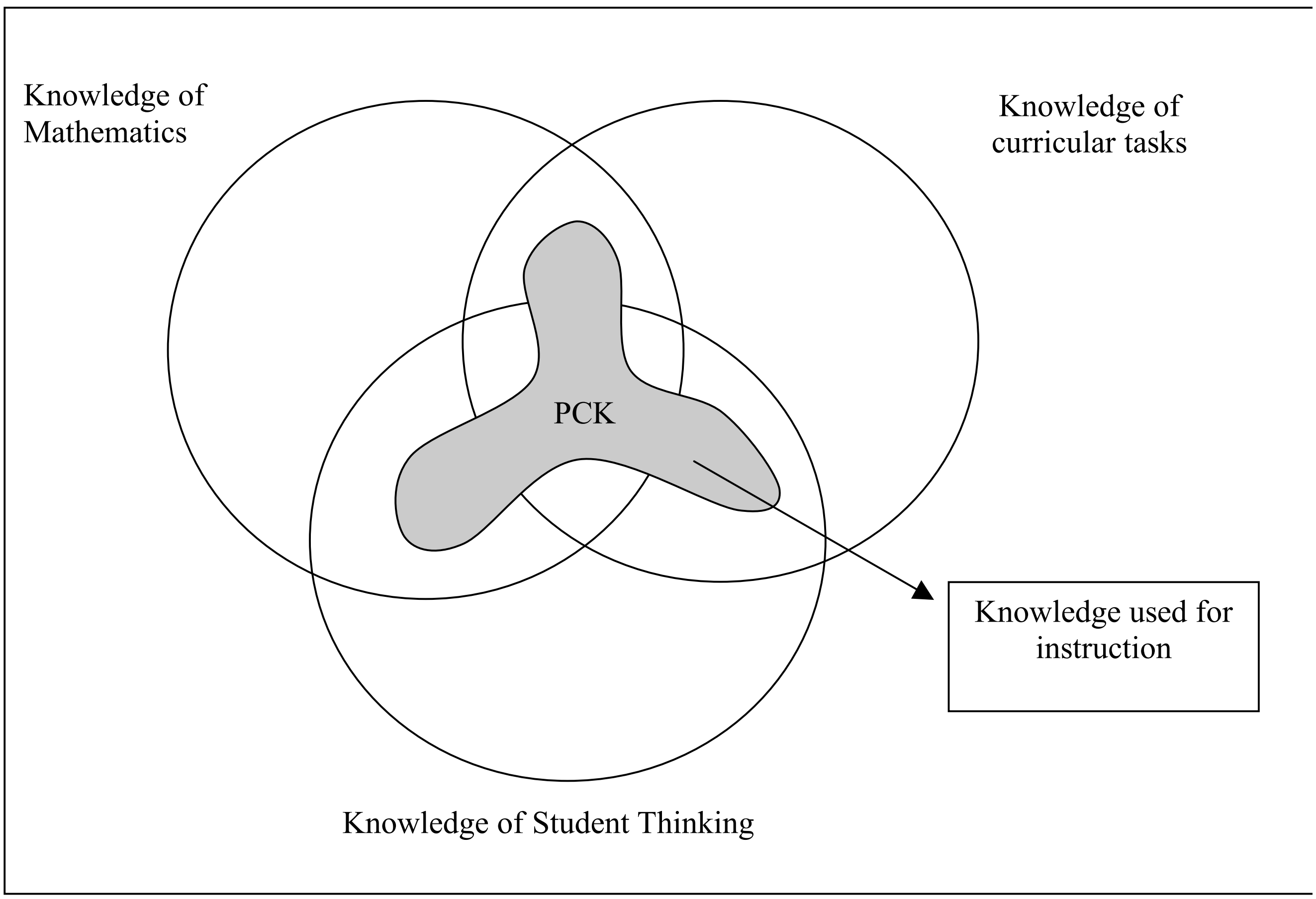


Figure 1.5: PCK in relation to other forms of teacher knowledge

Figure 1.6 represents how a teacher would draw upon these three components of PCK in an integrated fashion during the process of teaching.

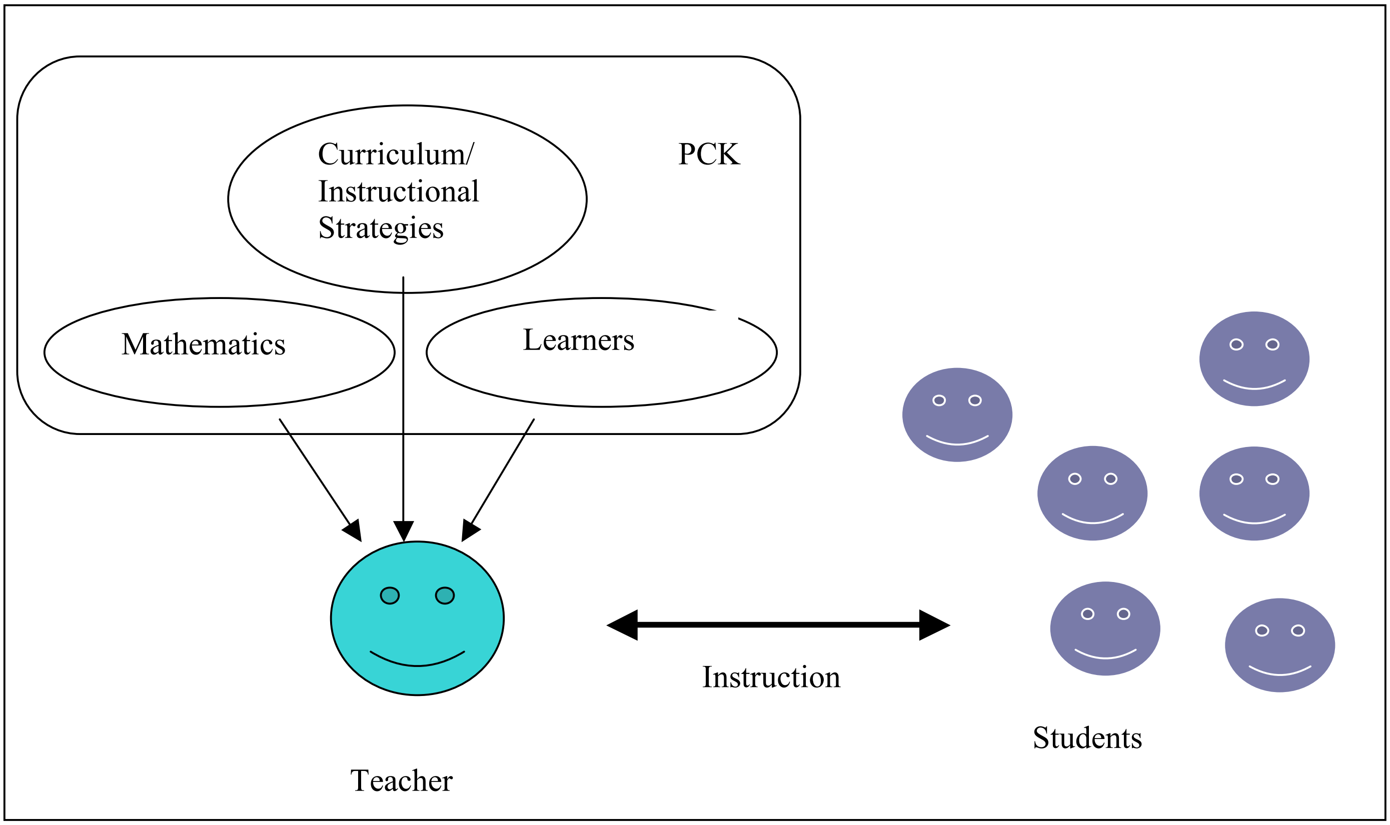


Figure 1.6: Theoretical Framework

In this framework, I focus on the knowledge of mathematics, the knowledge of learners, and the knowledge of curriculum/instructional strategies that a teacher draws upon while designing, implementing, and reflecting upon their lessons. The teacher may draw upon any of the three forms of knowledge at any time and may use different combinations of these knowledge forms. The three arrows going from the three aspects of PCK to the teacher illustrate this. For instance, a particular decision may include a significant amount of knowledge regarding learners, with some knowledge of mathematics to help support and clarify a decision. I expect the teachers to use several different combinations of knowledge forms for a variety of reasons. It must be noted that this model does not assume to be an all-inclusive list; there are other forms of knowledge that could potentially impact instructional decisions that were not included in the analysis.

The arrow between the teacher and the students represents the context in which these three forms of knowledge are being used. The decisions made by the teacher are not occurring in isolation, but within the context of the classroom. Not only are teachers making decisions on what and how to teach, they are interpreting student reactions and performance that factor into this decision-making process. The social dynamic of the classroom plays a critical role in how knowledge is used and ultimately impacts and adjusts the knowledge sets the teachers were drawing upon.

The main motivation for choosing this particular model is that it provides a complex view of teachers' knowledge use in the classroom. Decisions made by teachers in the midst of instruction draw upon several types of knowledge simultaneously. I contend that since this knowledge is interconnected during instruction, it should also be studied in this manner. We must better understand the components of knowledge that teachers use in the act of teaching and apply this understanding to the preparation of current and future teachers.

Definition of Terms

The following terms are used frequently in this study of teacher knowledge; a definition that articulates my usage of the term is provided.

Algebraic Generalization

Generalization is central to mathematical activity (Mason, 1996) and involves extracting from particular instances and examining commonalities across cases (Dienes, 1961; Dörfler, 1991). Algebraic generalization is defined by Kaput (1999) to be the act of:

> Deliberately extending the range of reasoning or communication beyond the case or cases considered, explicitly identifying and exposing commonality across cases, or lifting the reasoning or communication to a level where the focus is no longer on the cases or situation themselves but rather on the patterns, procedures, structures, and the relationships across and among them. (p.136)

Episode

For the purpose of data analysis, I defined an episode to be a portion of a transcript where the teacher was talking about a single thought or idea.

Knowledge Combination

I defined a knowledge combination to be an episode where the teacher used multiple forms of knowledge. For instance, if a teacher talked about the knowledge of mathematics and the knowledge of learners in the same episode then the episode would be a knowledge combination.

Knowledge Interaction

I defined a knowledge interaction as a portion of an episode where one form of knowledge was used to support a second form of knowledge. A knowledge interaction is more specific than a knowledge combination and includes a pairing of two knowledge forms along with the purpose for that combination. For instance, if the knowledge of learners were used to justify an instructional decision then this would be a knowledge interaction between the knowledge of learners and the knowledge of curriculum and instructional strategies. An episode can have multiple knowledge interactions, depending on the number of distinct ways in which the knowledge forms supported one another.

Knowledge of Curriculum and Instructional Strategies

For the purpose of data analysis, I defined the knowledge of curriculum and instructional strategies to be any instance where the teacher mentioned the tasks, curriculum, questions used to further the lesson, or goals of the lesson in relation to algebraic generalization. Comments centered on the implementation of the lesson or decisions made regarding the flow of the lesson were also included in this category.

Knowledge of Learners

For the purpose of data analysis, I defined the knowledge of learners to be any instance where the teacher shared student thinking—what she observed students thinking as well as how she expected students to think about algebraic generalization. This category included conversations about student characteristics, habits, or misunderstandings that may have influenced how they thought about algebraic generalization.

Knowledge of Mathematics

For the purpose of data analysis, I defined the knowledge of mathematics to be any instance where the teacher discussed a mathematical topic connected to the idea of algebraic generalization, including the connections and relationships between these ideas. Included in this category are the ways and means of justifying and providing proof for these ideas. Conversations focused on mathematics, the teacher's view of a mathematical topic—how they think about it-- or the mathematical topics brought up as needed for instruction.

Pedagogical Content Knowledge

Shulman (1987) stated that pedagogical content knowledge “represents the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interest and abilities of learners, and presented for instruction” (p. 8). For the purpose of this study, I have defined PCK to be the blending of the knowledge of mathematics, knowledge of learners, and knowledge of curriculum and instructional strategies into an understanding of how algebraic generalization is organized, represented, and adapted to the diverse interest and abilities of learners, and presented for instruction.

Significance of Study

I opened this chapter by discussing the range of decisions that teachers have to make in the classroom and the knowledge they draw upon when making those decisions. This is the knowledge that is critical; this is the knowledge that can aid teachers in encouraging students to develop a fundamental understanding of mathematics. In order for teachers to be effective in the classroom they must be able to utilize and integrate their knowledge to make appropriate instructional decisions. Our teacher education programs must have information that sheds light into the knowledge that teachers use in the classroom and how this knowledge is structured. Heaton (2000) claims that professional development lacks a curriculum that simultaneously considers the practices of teaching and the content knowledge needed to enact them.

By looking at expert teachers, I hope to ascertain this elusive set of knowledge structures that are developed through experience. In order for professional developers to provide support for teachers they must be informed about the structure of teacher knowledge. What knowledge is needed for effective instruction? How is this knowledge structured and interconnected? Understanding the knowledge used by teachers in the classroom informs mathematics teacher educators to the needs of future and current teachers. Once staff developers understand the relationships between the various aspects of teacher knowledge, professional development programs can be redesigned to better prepare our future teachers to meet the needs of our educational system. The ultimate goal of this line of work is to inform the community of the types of knowledge structures that are beneficial to teachers in the act of teaching.

In addition, this study seeks to understand more that just the knowledge required to teach, but how that knowledge is used in collaboration to make instructional decisions. Eraut (1994) stated that, "the key to unlocking these problems (of teacher education) was my realization that learning knowledge and using knowledge are not separate processes but the same process" (p. 25). If we want to understand the knowledge required for teaching we cannot separate teacher knowledge from teacher practice. This study seeks to explore a broader notion of teacher knowledge. Teacher knowledge is knowledge in practice; it is a process as well as a product.

Summary

Teacher Knowledge is a critical factor that influences instructional decisions in the classroom. In order to meet the needs of our students and to adapt to changes within our schools system we must understand this knowledge in action. As we continue to push for algebraic generalization to be introduced earlier in our school curriculum we must understand how teacher knowledge is used and how it leads to student achievement in this area. To this end, this study seeks to understand how two 5th grade teachers used their knowledge of mathematics, knowledge of learners, and knowledge of curriculum and instructional strategies in the act of teaching algebraic generalization.

The remaining chapters further describe the details of this study. Chapter 2 provides a review of the current literature of teacher knowledge, pedagogical content knowledge, and generalization. Chapter III describes the participants, research instruments, data sources, and methods of analysis. Chapter IV presents the findings that resulted from the analysis and Chapter V provides an interpretation and discussion of these findings as well as the limitations of the study.

CHAPTER 2: REVIEW OF LITERATURE

As teacher educators, one of our goals is to improve the preparation of teachers and thus the education that K-12 students receive. One critical factor in the act of teaching is the knowledge that teachers draw upon when making instructional decisions. Elbaz (1983) states, “the single factor which seems to have the greatest power to carry forward our understanding of the teachers’ role is the phenomenon of teachers’ knowledge” (p. 45). In addition, Brown and Borko (1992) stated, “an individual’s knowledge structures and mental representations of the world play a central role in that individual’s perceptions, thoughts, and actions” (p. 211). The knowledge that teachers have, and how they draw upon that knowledge when making decisions, has a direct impact on student learning. Hence, one question that we as a mathematics education community must answer is, “What knowledge do mathematics teachers need in order to make decisions that maximize student learning and understanding?”

Through the years many documents have tackled this question and provided suggestions (Conference Board of Mathematical Science, 2001; etc.). Unfortunately, according to Fennema and Franke (1992) “there is no consensus on what critical knowledge is necessary to ensure that students learn mathematics” (p. 147). Although the work presented in this study does not intend to address this particular question in its entirety, it does present an approach to this issue within a specific content area and in an integrated fashion that may push the thinking of the mathematics education community as to how to address this important question.

The purpose of the first portion of this chapter is to investigate the challenges that exist in answering the question, “What knowledge do mathematics teachers need to maximize student learning and understanding?" and to make an argument and justification for the inclusion of studies such as my research in answering this question. The study reported here attempts to answer the following research questions:

3) To what extent do two fifth grade teachers draw upon their knowledge of mathematics, their knowledge of learners, and their knowledge of curriculum and instructional strategies when designing, implementing, and reflecting on instruction related to algebraic generalization?
4) How do two fifth grade teachers when designing, implementing, and reflecting on instruction related to algebraic generalization use these three aspects of pedagogical content knowledge simultaneously?

In the pages that follow, I position these questions within the larger mathematics education literature and describe how they are related to the larger question of knowledge needed for teaching. To this end, I provide an overview of research on teacher knowledge, pedagogical content knowledge, and teachers knowledge of the specific area of algebraic reasoning.

Teacher Knowledge

This section provides an overview of research in the area of teacher knowledge and includes a discussion of the history of teacher knowledge, its complexity, the integrated perspective of teacher knowledge, and methodological considerations associated with examining teacher knowledge.

Historical Perspective

The Early Years. The study of teacher knowledge, as a unique from of knowledge, has been examined from a variety of perspectives that have changed through the years.

Dewey (1902) recognized the complexity of subject matter knowledge for teaching when he commented on the plight of a teacher:

> Hence, what concerns him, as teacher, is the way in which that subject may become part of experience; what there is in the child's present that is usable with reference to it; how such elements are used; how his own knowledge of the subject matter may assist in interpreting the child's needs and doings … He is concerned not with subject matter as such, but with the subject matter as a related factor in the total and growing experience. (p. 286).

Dewey viewed teacher knowledge as more than just content knowledge, but as content knowledge in action--knowledge in relation to the act of teaching and learning. Unfortunately, these ideas do not describe a means for measuring teacher knowledge or provide a framework to describe it.

The 1970s. Much of the initial work related to teacher knowledge developed during the 1970s with the prevalence of process-product research that took a quantitative approach toward measuring teacher knowledge (e.g. Rosenshine, 1971; Dunkin & Biddle, 1974; Gage, 1978). These studies related certain teacher characteristics to student performance outcomes.

For example, researchers counted the number of college math courses taken or administered multiple-choice content tests (Darling-Hammond, 1999; Monk, 1994). However, these studies had limited power (Begle, 1968, Eisenberg, 1977). For instance, studies that sought to relate number of mathematics courses taken to student outcomes (Begle, 1979) ultimately found little correlation between these two factors. On the surface we may assume from this conclusion that content courses are not important. However, the lack of correlation can be explained by the fact that counting the number of courses taken is not an accurate measure of a teacher's content knowledge for instruction. Furthermore, it may be difficult to discern the impact of content knowledge given the multitude of factors and knowledge that influence student outcomes. For instance, content knowledge may in fact be an important component of student outcomes, as new research is starting to reveal, but it is difficult to separate this influence from other factors that are present. Frennema and Franke (1992) state, "perhaps the inadequate measure of knowledge and relatively limited research methodology concealed any relationships that did exist between teachers' knowledge and student learning" (p. 149).

In essence, one could claim that these methodologies failed in part because they did not capture the complexity of teacher knowledge. By focusing on one aspect of teacher knowledge, they did not take into account the other forms that were bearing down on teacher decisions or student learning. It is important to remember that instructional decisions made by teachers are not guided by a singular knowledge base, but by a variety of factors and knowledge forms.

The 1980s. Research on teacher knowledge began to change in the early 1980s with the emergence of constructivism as an epistemological foundation for mathematics education. From the constructivist perspective, a learner constructs understanding as they interact with, and attempt to make sense of, the world around them. This process is unique to each student as each student brings different knowledge, experiences, and cognitive strategies to learning. Constructivism slowly became the theoretical foundation of the new vision of mathematics instruction (Cobb & Steffe, 1983; Confrey, 1985;

Davis, 1984; von Glaserfeld, 1983, 1987a, 1987b) that began in the 1980s and extends to the present.

With the adoption of constructivism as a theoretical underpinning, and the idea of mathematics being a socially constructed body of knowledge, changes in the goals and approaches of mathematics education research subsequently followed. No longer were researchers focused only on teacher and student actions (process-product research), researcher were now interested in what students and teachers were thinking. This led to a wave of research in the 1980s that attempted to understand teacher cognitions (Clark and Peterson, 1986). Researchers began to view teachers as learners themselves and began to stress the parallels and connections between students and teachers (Cobb, Yackel, & Wood, 1988).

With the emergence of the constructivist perspective in the early 1980s an expanding body of research has focused on the many different forms of knowledge required by teachers (e.g. Ball, 1988; Clandinin & Connelly, 1996; Darling & Hammond, 1996; Fennema & Franke, 1992; Grossman, Wilson, & Shulman, 1989; Ma, 1999; McNamara, 1991; Shulman, 1986; Shulman & Sykes, 1986). Although many types of teacher knowledge have been studied during this time, much of the work during the 1980s centered on subject matter knowledge and pedagogical knowledge. These two areas were considered the two critical knowledge bases that teachers drew upon. However, the compartmentalization of teacher knowledge led to difficulties, as these domains of knowledge were neither mutually exclusive nor clearly defined. This realization led to further changes in the study of teacher knowledge.

A watershed moment in the study of teacher knowledge occurred when Shulman (1985) introduced the idea of pedagogical content knowledge, which involves a more integrated and complex means of viewing teacher knowledge. The descriptions of pedagogical content knowledge and teacher knowledge provided by Shulman resemble the notions of teacher knowledge provided by Dewey and has led the education community to study teacher knowledge in a more complex and integrated fashion. In addition to developing the idea pedagogical content knowledge, he developed an integrated model of teacher knowledge that has stimulated further research and the development of other models (Shulman, 1986; Peterson, 1988; Fennema & Franke, 1992; Neagoy, 1995). Shulman's notion of PCK, and his models of teacher knowledge, provided a framework for future research that has shaped much of the current literature on the subject.

However, the development of more complicated conceptions of teacher knowledge has resulted in differing perspectives among researchers. These perspectives arose from the fact that different knowledge sets often overlap and are hard to distinguish. Marks (1990) concluded that, "some instances of teacher knowledge clearly represent one or another type, but many fall on the fringes" (p. 8). This difficulty led to changes in the conception of pedagogical content knowledge and the nature of teacher knowledge itself. With this in mind, the following sections provide an overview of pedagogical content knowledge and the integrated nature of teacher knowledge.

Pedagogical Content Knowledge

The development of pedagogical content knowledge. This section traces the development of the construct of pedagogical content knowledge and integrated teacher knowledge in the context of research on teacher knowledge that occurred in the early 1980s. At this time, research on teacher knowledge grew rapidly with new avenues and knowledge forms being investigated—pedagogical knowledge and subject matter knowledge being the most prevalent. Within this milieu of ideas, Shulman and his colleagues recognized the need to develop and organize this growing body of literature. Thus, they developed a framework that categorized the professional knowledge base for teaching. Their model contained seven components: content knowledge, general pedagogical knowledge, curriculum knowledge, pedagogical content knowledge, knowledge of learners, knowledge of educational contexts, and knowledge of educational ends. (Shulman, 1986, 1987; Shulman and Grossman, 1988; and Wilson, Shulman, and Richert, 1987). The authors presented their model as a springboard to prompt further research, discussion, and refinement of the concept of teacher knowledge.

Since the introduction of the categories of professional knowledge base for teaching, pedagogical content knowledge has received extensive attention. This revolutionary idea sparked both discussion and confusion through the years. Many definitions of pedagogical content knowledge begin with Shulman (1986) when he stated that pedagogical content knowledge in one's subject area includes "the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the ways of representing and formulating the subject matter that make it comprehensible to others" (p. 9). However, this definition makes it difficult to place in regards to other forms of knowledge because it appears to be an element of other teacher knowledge domains. Alternatively, pedagogical content knowledge is often viewed as the blending of content knowledge and pedagogy as illustrated by another excerpt from Shulman (1987):

> The key of distinguishing the knowledge base of teaching lies at the intersection of content and pedagogy, in the capacity of a teacher to transform the content knowledge he or she possesses into forms that are pedagogically powerful and yet adaptive to the variations in ability and background presented by the students. (p. 15)

This statement illustrates the connection that pedagogical content knowledge has to both content knowledge and the knowledge of pedagogy. However, pedagogical content knowledge is not simply the overlap of content and pedagogy or the sum of its parts. Pedagogical content knowledge differs from content knowledge it that it is not a required element of content knowledge. The mathematical content that a teacher needs in order to provide instruction on a certain mathematical idea is different than the knowledge that an expert needs to work with that same idea (Putnam, 1985).

For instance, consider generalizing a linear situation; a content expert may be interested in formal explicit equations and the concept of slope. However, a teacher needs an understanding of the different ways in which students approach generalization tasks. Such a perspective would require the knowledge of the mathematics behind recursive, explicit, and proportional reasoning, as well as understanding why proportional reasoning cannot always be applied to linear situations. Experts in the field would not be required to have such large sets of knowledge for instruction (Putnam, 1985). An expert would be required to have an accurate explanation or proof for a concept, but would not necessarily have a compelling example or powerful representation that would connect to a student's prior thinking and facilitate student understanding. Hence, pedagogical content knowledge overlaps, but is not a subset, of subject matter knowledge.

Likewise, pedagogical content knowledge is not a distinct subset of general pedagogical knowledge. Pedagogical content knowledge is distinct in that it is specific to particular content.

The importance of pedagogical content knowledge. For many reasons, the development of pedagogical content knowledge is interesting when situated in the research context of the 1980s. First, of all the forms of knowledge that were investigated at the time, subject matter knowledge and pedagogical knowledge were the most pervasive and were considered most important. Second, in terms of content knowledge, doubts emerged regarding the connection between content knowledge and student performance, or at least doubt in the connection between formal mathematical knowledge (that acquired in college course work) and student learning. Shulman (1986) was critical of teacher evaluation programs that focused primarily on pedagogical knowledge with the exclusion of content. He believed that focusing solely on content, or solely on pedagogy, did not transfer directly into quality teaching. Pedagogical content knowledge was identified as the missing piece of the framework used in prior process-product research on teacher knowledge (Shulman, 1986).

Among the categories of Shulman's framework, pedagogical content knowledge is of special importance because it identifies knowledge unique to the teaching profession. Shulman (1987) stated that it is the most critical component of the framework:

> Pedagogical content knowledge is of special interest because it identifies the distinctive bodies of knowledge for teaching. It represents the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction. Pedagogical content knowledge is the category most likely to distinguish the understanding of the content specialist from that of the pedagogue. (p. 8)

However, disagreements exist as to where this form of knowledge fits into the larger notion of teacher knowledge. I now address some of the changes and different interpretations regarding the notion of pedagogical content knowledge.

The evolution of pedagogical content knowledge. Initially, Shulman defined pedagogical content knowledge to be a component, or subset, of content knowledge, a view still favored by some researchers today. According to Shulman's original model, content knowledge consisted of three categories: subject matter knowledge, pedagogical content knowledge, and knowledge of curriculum. However, as illustrated in the previous example, aspects of pedagogical content knowledge exist that are not included within content knowledge. For this reason there was an argument among some in the education community to view pedagogical content knowledge as a knowledge domain it its own right.

With this in mind, Shulman (1987) shifted his view of pedagogical content knowledge describing it as a "new type of subject matter knowledge that is enriched and enhanced by the other types of knowledge" (Wilson, Shulman, Richert, p. 114). Under this definition pedagogical content knowledge includes aspects of and interaction with other types of knowledge, but would not be totally subsumed under any particular type of knowledge. Although Shulman described it as a type of subject matter knowledge, it is presented as a separate entity.

Once established as a separate form of teacher knowledge, Shulman and Grossman (1988) further refined the concept of pedagogical content knowledge and developed five subcomponents: knowledge of alternative content frameworks, knowledge of student understanding and misconceptions of a subject, knowledge of curriculum, knowledge of particular content for the purpose of teaching, and knowledge of topic specific pedagogical strategies. Hence, pedagogical content knowledge began as a subset of content knowledge and progressed to be a knowledge form in itself with several subcomponents, two of which include elements of content knowledge.

Another interesting aspect of pedagogical content knowledge recognized by Shulman is its changing nature. Pedagogical content knowledge is not a static knowledge set, but one that is ever changing in response to the act of teaching. Shulman (1987) characterized the process of pedagogical reasoning and action to describe how pedagogical content knowledge grows. Of all the knowledge forms, pedagogical content knowledge is the one most impacted by the act of teaching, and the one that is most difficult to impart void of teaching experience. Furthermore, pedagogical content knowledge is influenced by outside forces. For instance, there is a change in pedagogical content knowledge in reaction to modifications of teaching philosophy and content. As more and more teachers begin to embrace a *standards*-based instructional philosophy and as algebraic generalization becomes more incorporated into the elementary grades the pedagogical content knowledge required by teachers will need to be modified.

Future notions of pedagogical content knowledge. At this point, pedagogical content knowledge was viewed as a separate form of knowledge with several subcomponents. However, upon examination of the components provided by Shulman and Grossman (1988), it can be observed that this new conceptualization contains elements of content, pedagogy, student thinking, and curriculum. It appears that pedagogical content knowledge can be viewed as a synergetic interaction and integration of several different forms of knowledge. Furthermore, pedagogical content knowledge is considered by many to be the most important component of knowledge required by teachers. If pedagogical content knowledge is the critical piece of teaching and it contains, in some form, the other forms of knowledge, why is it not *the* focus of attention instead of one of several forms of knowledge? Neagoy (1995) stated:

> I object to the place given to pedagogical content knowledge: most all researchers inspired by Shulman's work list pedagogical content knowledge among other essential components of teacher knowledge, such as subject matter and general pedagogy. If we want a keener understanding of the complex phenomenon called teacher knowledge, it is my contention that we begin by putting the pieces back together and reexamining the whole. (p. 19)

The framework used in this study views pedagogical content knowledge as *the* knowledge of interest. In recognizing the complex and integrated nature of pedagogical content knowledge this study seeks to understand the interaction of three separate components of pedagogical content knowledge: knowledge of mathematics, knowledge of learners, and knowledge of curriculum and instructional strategies. The next section provides a discussion of this complexity.

Complexity

Teaching is a complex endeavor and consequently the knowledge required to teach is equally complex. The history of research on teacher knowledge described in the previous paragraphs alludes to this, as the frameworks and models used to investigate teacher knowledge have become progressively more sophisticated and integrated. Shulman (1983) stated that, "to be a teacher requires extensive and highly organized bodies of knowledge" (p. 47). Not only must teachers have an understanding of a variety of knowledge sets, these knowledge sets must be synthesized in an efficient manner in the act of instruction.

An expanding body of research focused on the different forms of knowledge required by teachers (e.g. Ball, 1988; Clandinin & Connelly, 1996; Darling & Hammond, 1996; Fennema & Franke, 1992; Grossman, Wilson, & Shulman, 1989; Ma, 1999; McNamara, 1991; Shulman, 1986; Shulman & Sykes, 1986). However, research on teacher knowledge is a relatively new area, coupled with the complexity of the phenomenon, implies that this is a relatively unexplored research field. Munby, Russell, & Martin (2001) stated that "the nature of teachers' knowledge' is new in the last 20 years, and the nature and development of that knowledge is only beginning to be understood by the present generation of researchers in teaching and teacher education" (p. 877).

The mathematics education community still knows too little about the knowledge required for teaching (Ball, Lubienski, & Mewborn, 2001; RAND Mathematics Study Panel, 2003). Despite the relatively recent awareness of the need to understand teacher knowledge, researchers have created a body of knowledge that incorporates many different views and aspects of the term. For instance researchers have studied and discussed teacher content knowledge (Begle, 1979), pedagogical content knowledge (Wilson, Shulman, & Reichert, 1987), practical knowledge (Connelly & Clandinin, 1986), subject knowledge for teaching (Hill & Ball, 2004), curricular knowledge (Behar & Paul, 1994), and knowledge of student thinking (Franke & Kazemi, 2001).

One common characteristic of these studies is that they have examined one aspect of teacher knowledge. Although studies that focus on a single type of teacher knowledge are common, and beneficial, an overlap exists among these knowledge types as they are used together by teachers during instruction.

The studies mentioned above provide an illustration of current research, the complexity of teacher knowledge, and the further work that needs to be done to bring these areas of knowledge together into a comprehensive model of teacher knowledge. As can be seen through these various viewpoints, the idea of teacher knowledge is complex and has many components. However, the study of such a large and integrated system of knowledge presents a difficult proposition for education researchers. Fennema and Franke (1992) contend that:

> Common sense suggests that knowledge is not monolithic. It is a large, integrated, functioning system with each part difficult to isolate. While many have speculated on the components of teacher knowledge, only a few have received major attention from researchers: content knowledge, knowledge of learning, knowledge of mathematical representations, and pedagogical knowledge. Some have studied knowledge as integrated, but most have not (p. 148).

To this point, most studies of teacher knowledge have focused on one aspect of this knowledge, to the neglect of the integrated nature of teacher knowledge. However, a few studies have countered this trend and studied several aspects of teacher knowledge simultaneously. The next section provides an overview of the models put forth by researchers to describe the integrated nature of teacher knowledge.

Models of Integrated Teacher Knowledge

To this point, an overview of the research on teacher knowledge has been provided that has led to an argument for the need for research on the integrated nature of teacher knowledge. This section describes the models that have been designed to explain and study this integrated nature. I present four models beginning with the initial model developed by Shulman (1986) and the subsequent adaptations of this model (Peterson, 1988; Fennema & Franke, 1992; and Neagoy, 1995) that bring out and expand upon Shulman's initial notions. Each framework contains different components of teacher knowledge, but these conceptions should not be seen as contradictory as each model was created to bring out a different component of teacher knowledge. For instance, Peterson's (1988) model was adapted to specifically address teacher cognition. Fennema and Franke (1992) state that these models are "not inconsistent, rather they build on each other" (p. 161). Likewise, the framework developed for this study is an adaptation of these models.

Shulman's model of teacher knowledge

Shulman's framework developed from the need for a comprehensive model of teacher knowledge. At the time, teacher evaluation programs tended to focus on a single aspect of teacher knowledge, general pedagogy. In his analysis of teacher tests Shulman found that most teacher entry exams inquired about the knowledge of general teaching methods to the neglect of content knowledge (Shulman, 1986). He stated,

> The emphasis is on how teachers manage their classrooms, organize their activities, allocate time and turns, structure assignments, ascribe praise and blame, formulate the levels of their questions, plan lessons, and judge general student understandings. What we miss are questions about the *content* of the lessons taught, the questions asked, and the explanations offered. (p. 8)

In addition to the neglect of content, Shulman (1986) stated that there has historically been a clear distinction between content knowledge and pedagogical knowledge with respect to understanding teacher knowledge.

Due to these factors, Shulman (1986) proposed a framework that analyzed teachers' knowledge by simultaneously looking at three different types of knowledge: subject matter knowledge, pedagogical content knowledge, and curricular knowledge. Shulman defined content knowledge as "the amount and organization of knowledge per se in the mind of the teacher" (p. 9). Within the category of pedagogical content knowledge Shulman includes:

> The most regularly taught topics in one area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the ways of representing and formulating a subject that make it comprehensible to others (p. 9).

He also include the, "conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught subjects" under the category of pedagogical content knowledge (p. 9). Finally, he defined curricular knowledge as the “full range of programs designed for the teaching of particular subjects and topics at a given level, the variety of instructional materials available in relation to those programs” (p. 10).

Peterson’s model of teacher knowledge

Peterson (1988) adapted Shulman’s model to bring out the role of teacher cognition. Her model describes three kinds of knowledge that are needed in order for teachers to be effective, knowledge of: (a) how students think in specific content areas, (b) how to facilitate growth in students’ thinking, and (c) self-awareness of their own cognitive processes. Although Peterson’s model focuses primarily on teacher cognition it does not ignore mathematical content knowledge. However, content knowledge must be discussed in relationship to teacher or student cognition. If mathematics is not connected to how students thinking or teacher metacognition then it will not be useful in the classroom. If we want to look at knowledge that is useful to teachers it must be connected to the thinking that is occurring in the classroom.

Fennema and Franke’s model of teacher knowledge

Fennema and Franke (1992) proposed a similar model for the integrated nature of teacher knowledge adapting Shulman’s model to better describe this knowledge as it occurs in the context of the classroom (see figure 2.1).

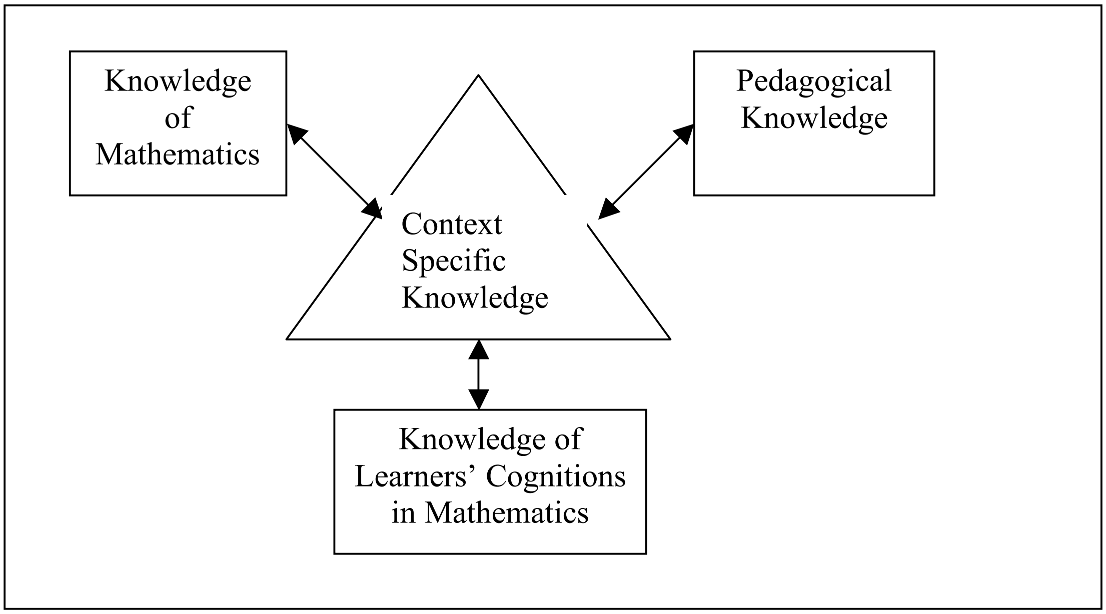


Figure 2.1: Fennnema and Franke's model of teacher knowledge

In their model, they isolate the knowledge of mathematics, pedagogical knowledge, and knowledge of learner's cognitions in mathematics as the three central pieces of teacher knowledge. The central triangle in their model represents the interaction of these knowledge forms in the teaching context. Fennema and Franke state,

> The context is the structure that defines the components of knowledge and beliefs that come into play. Within a given context, teachers' knowledge of content interacts with knowledge of pedagogy and students' cognitions and combines with beliefs to create a unique set of knowledge that drives classroom behavior (p. 162).

The description of context specific knowledge is similar to the concept of PCK described by Shulman and also includes aspects of content, pedagogical knowledge, and knowledge of student thinking as they pertain to the act of teaching. Just as in Shulman's model, Fennema and Franke describe three distinct forms of teacher knowledge, but emphasize the importance of the interaction of this knowledge within the context of teaching.

Neagoy's model of teacher knowledge

Neagoy's (1995) model of pedagogical content knowledge includes three non-disjoint parts that she emphasizes to be indistinguishable: knowledge of subject matter, knowledge of pedagogy, and knowledge of learners. Neagoy has a different organization of pedagogical content knowledge, rather than PCK being one of many forms of teacher knowledge, PCK is teacher knowledge. She stated, "pedagogical content knowledge is not <u>one</u> among these sets of knowledge, but rather, contains them all" (p. 19). In addition, although Neagoy identifies three distinct aspects of pedagogical content knowledge she stated, "over time and with growing experience, I believe that teachers learn to blend these three domains into one indistinguishable construct" (p. 21). She, like Shulman, stresses that this knowledge is evolving and changing in interaction with teaching.

Summary of integrated models of teacher knowledge

The framework developed for this study takes elements and ideas from all the integrated models presented. Like Neagoy (1995), my model views pedagogical content knowledge as *the* type of teacher knowledge of interest, with all other forms of pertinent knowledge being subsumed under PCK. Similar to Neagoy (1995) and Shulman and Grossman (1988), I view pedagogical content knowledge as having several subsets of knowledge that interact together. Peterson (1988) stresses the importance of teacher cognition, while Fennema and Franke (1992) stress the importance of educational contexts. In connecting these frameworks to my study, I am seeking to understand how knowledge impacts teacher decisions making in the context of classroom practice. The next section provides an example of research that used an integrated model.

Example of Research Using an Integrated Model

The work of Lehrer and Franke (1992) provides an interesting example of the importance of an integrated model. In their study they followed an expert first grade teacher over a two-year period, observing the teacher as she switched from a content domain she was comfortable with (addition and subtraction) to one she was not comfortable with (fractions). They found that for the fraction content area the teacher did not use a wide variety of problem types as compared to her lessons on addition and subtraction. One interpretation of this phenomenon is that her lack of content knowledge and knowledge of student thinking in regards to fractions limited her understanding of the types of curricular tasks that would be needed to bring out the subtleties of the content.

Furthermore, Lehrer & Franke found that, although the teacher still asked questions and listened carefully to student responses, she was less likely to follow up on student questions and remarks. Again, it seems likely that her lack of content knowledge and knowledge of student thinking limited her as she responded to the thoughts of her students. This understanding of the links between the knowledge of content, student thinking and curricular tasks would not have been born out if researchers had not used an integrated model of teacher knowledge. By using an integrated model, we can hypothesize that although she still had an adequate knowledge of pedagogy, it was the lack of integration among pedagogy and the other forms of knowledge that limited her ability to prompt students with questions.

Methodological Considerations

The overview of the history of research into teacher knowledge brings up many methodological considerations. How do we measure what teacher knowledge is useful in promoting student learning and understanding? For many years researchers used multiple-choice tests or other easily obtainable measures to get a handle on this phenomenon. Eraut (1994) argued that:

> Most examinations guaranteed only that knowledge they were able to test; and this seldom extended to practical competence. Hence one of the main consequences of their introduction was the transformation of large areas of the professional knowledge base into codified forms, which suited the textbooks needed to prepare students for what were from the outset very traditional examinations. (p. 7).

Eraut (1992) further argued that a result of this codifying process was that much of what was taught in teacher preparation programs did not include the practical aspects of teacher knowledge that were necessary for successful introduction into the teaching community, but rather more formal knowledge. He stated that,

> The norms of higher education tend to favour scientific knowledge rather than professional knowledge, and to encourage research priorities from those likely to be espoused by the professions, thus helping to widen the gap between professional educators and their erstwhile professional colleagues. (p. 9).

This phenomenon in higher education may in fact have its roots in the process-product studies of the 70s. These types of studies may have the unintended result of the exclusion of important practical aspects of teacher knowledge and led to a divide between the presumed experts and the everyday teacher. Leinhardt et al (1991) support this statement when they state, “the knowledge that a teacher needs to have or uses in the course of teaching a particular school-level curriculum in mathematics is different from the knowledge of advanced topics that a mathematician might have” (p. 188).

In continuing with Eraut’s arguments he seems to promote a more integrated view of teacher knowledge, one that brings out the complexity of teaching. He stated:

> Such segmentation and packing of knowledge for credit-based systems seems inappropriate preparation for professional work which involves using several different types of knowledge in an integrated way; and the pedagogic approaches needed for linking book knowledge wit practical experience are almost impossible to implement where there is little continuity in the membership of the student group. (p. 10).

Ball (1988) addresses this issue by developing and using mathematics instruments that probed teachers’ knowledge of mathematics in the context of everyday situations that occur in the course of teaching.

Answering the question, “What knowledge do teachers need in order to promote student learning and understanding?” is a daunting challenge. The complexity of this issue derives from the fact that teaching is a complex endeavor that occurs in an ever-changing environment. Furthermore, it must be noted that different kinds of teaching place different demands on a teachers’ repertoire of teacher knowledge (e.g. Schifter, 1998; Stein, silver, & Smith, 1998). Simply understanding the mathematics presented in a textbook is not sufficient to address all the needs of teachers, other perspectives must be considered. From a mathematical viewpoint, teaching a particular topic well requires "deep" understanding of the underlying mathematical concepts. From a student thinking perspective, we realize the challenges created by the divergent strategies students use, each with mathematical underpinnings that need to be understood by the teacher. "One possible reason for which the problem of what teachers need to know remains unsolved is that people have been looking at [teacher knowledge] from different angles, having available only limited research that concentrated on connecting the knowledge produced by different perspectives" (Stylianides & Ball, 2005, p. 1).

Stylianides and Ball (2005) discuss six different strategies for answering this question, one of which is the study of mathematical practice. We return to Eraut (1994) to support this perspective as he states, "professional knowledge cannot be characterized in a manner that is independent of how it is learned and how it is used" (p. 19). His argument is that if you want to understand professional knowledge you must understand how it is used in practice. Stylianides and Ball (2005) suggest that this approach,

> begins with the study of records of mathematical practice (e.g. videotapes of classroom lessons, transcripts, student work, teacher journal) to uncover and describe in fine mathematical detail the work in which teachers engage in order to understand the mathematical knowledge needed for effective functioning in the course of this work (p. 27).

Unfortunately this approach has not been explored as further described by Stylianides and Ball, "Although a growing number of studies have provided detailed analyses of school mathematics practice with relative emphasis on elementary students' activity in reasoning and proving (e.g. Ball & Bass, 2003; Carpenter et al., 2003; Lampert, 1990, 1992; Maher & Martino, 1996; Reid, 2002; Zack, 1997), only few have presented this work as part of a principal interest “to uncover less exposed mathematical entailments of the work for teachers, seeking useful answer to questions about what teachers need to know and be able to do in order to teach” (Ball & Bass, 2000, p. 198).

When different perspectives are simultaneously considered, a broader and more in depth answer to our ultimate question begins to appear. Stylianides and Ball (2005) suggest that connecting the knowledge produced from these different perspectives may ultimately aid in addressing the question of mathematical content knowledge for teaching. My study on teacher knowledge explores this phenomenon form the perspective of classroom practice.

Generalization

Generalization is central to mathematical activity (Mason, 1996) and involves abstracting from particular instances and examining commonalities across cases (Dienes, 1961; Dörfler, 1991). Generalization is defined by Kaput (1999) to be the act of,

> Deliberately extending the range of reasoning or communication beyond the case or cases considered, explicitly identifying and exposing commonality across cases, or lifting the reasoning or communication to a level where the focus is no longer on the cases or situation themselves but rather on the patterns, procedures, structures, and the relationships across and among them. (p. 136).

The placement of generalization within the school mathematics curriculum has received considerable attention in recent years. Kaput (1995) recommends that algebraic concepts be developed throughout the elementary and middle school grades. Documents from the Australian Educational Council (1994), the National Council of Teachers of Mathematics [NCTM] (2000), and the Department of Education and Skills (2001) of Great Britain concur with Kaput's view. Given this suggested earlier introduction of generalization concepts, it is imperative that research keeps pace; we must understand how students perceive generalization and the strategies teachers use in providing instruction.

Unfortunately, student algebraic understanding has often lacked depth due to the traditional focus on symbol manipulation without connection to meaning (e.g. Booth, 1984; Demby, 1997; Kieran, 1992, Lee & Wheeler, 1989, Mason, 1996). To circumvent this difficulty, generalizing numeric patterns is viewed as a potential vehicle for transitioning students from numeric to algebraic thinking because it offers the potential to establish meaning for algebraic symbols by relating them to a quantitative referent. Representations play a critical role in this transition from numeric to algebraic thinking. As student generalizations increase in sophistication, so do the representations they construct to describe them. Smith (2003) stated that in order for students to create generalization from problem situations they "must progress from idiosyncratic and ad hoc representations of particular problems to conventional, abstract, and general representations that function for a class of problems" (p. 264).

Research on Pedagogical Content Knowledge

In reviewing the literature, I identified 27 studies that investigated the construct of pedagogical content knowledge. Of these 27 studies, sixteen involved the study of preservice teachers (Ball, 1988; Even, 1989, 1990, 1993; Goldman & Barron, 1990; McDiarmid, 1990; Meredith, 1993; Eisenhart et al., 1993; Wilson, 1994; Foss & Kleinsasser, 1996; Quinn, 1997; Van Der Valk & Broekman, 1999; Lowery, 2002; Zevenbergen, 2005), and four studies that involved beginning or first year teachers (Wilson, Shulman, & Richert, 1987; Clermont, 1988; Grossman, 1990; Feiman-Nemser

& Parker, 1990). The large percentage of studies focused on inexperienced teachers was surprising given the integral role that the knowledge of teaching and the knowledge of learners play in the concept of pedagogical content knowledge (Shulman, 1986). Borko et al. (1988) found that the development of pedagogical content knowledge was limited by a lack of knowledge of student thinking. Hence, while studies of preservice teachers pedagogical content knowledge is valuable for improving teacher education programs, they are not the best data for understanding the construction of pedagogical content knowledge.

However, there were seven studies that focused on experienced teachers (Neagoy, 1995; Lehrer & Franke, 1992; Marks, 1990; Carpenter, Fennema, Peterson, & Carey, 1988; Lubinski, 1993; Stump, 1999; Cavey, Whitenack, & Lovin, 2007), four of which focused on experienced elementary mathematics teachers. Two of these involved first grade teachers instruction in problem solving (Carpenter, Fennema, Peterson, & Carey, 1988; Lubinski, 1993), one involved fifth grade teachers understanding of functions (Marks, 1990) and one looked at elementary teachers understanding of fractions (Lehrer & Franke, 1992). None of the studies reviewed studied the pedagogical content knowledge of algebraic generalization.

Summary

One critical factor in the act of teaching is the knowledge that teachers draw upon when making instructional decisions. Of these forms of knowledge, pedagogical content knowledge has been identified as key to the act of teaching. Although the central role of pedagogical content knowledge is generally agreed upon, defining what pedagogical content knowledge is, and where it fits in with the larger concept of teacher knowledge has changed through the years. For the purpose of this study I accept the notion that pedagogical content knowledge is *the* knowledge of teacher knowledge.

As teacher knowledge continues to be explored, the complex interaction of the subsets of this knowledge are being more integrated into research paradigms. Several models of integrated teacher knowledge have been developed to look at this phenomenon in its complexity. These models investigate the complexity of teacher knowledge and each brings out a different feature or characteristic of this knowledge. My theoretical framework draws upon these models and looks at the interaction of three subsets of PCK, knowledge of mathematics, student thinking, and curricular tasks.

I have focused my study on the pedagogical content knowledge of algebraic generalization and the representations used to describe these generalizations. One of the facets of pedagogical content knowledge is that it is ever changing. It changes in response to teacher actions in the classroom, illustrating the amount of teacher learning that takes place in the classroom, and to external forces. One of these external forces is the implementation of generalization earlier into the school curriculum. With this change in curriculum, comes a change in the pedagogical content knowledge of teachers to address this concept. To this point there has not been a study of teachers' pedagogical content knowledge of algebraic generalization at the elementary level, and relatively few studies that have the pedagogical content knowledge of experienced teachers as a focus. With this in mind, the purpose of this study is to look at the pedagogical content knowledge of experienced teachers in regards to algebraic generalization.

CHAPTER 3: METHODOLOGY

In this chapter, I document the methodology used in conducting this study on teachers' knowledge of algebraic reasoning. My goal for this study was to answer the following research questions:

5) To what extent do two fifth grade teachers draw upon their knowledge of mathematics, their knowledge of learners, and their knowledge of curriculum and instructional strategies when designing, implementing, and reflecting on instruction related to algebraic generalization?
6) How do two fifth grade teachers when designing, implementing, and reflecting on instruction related to algebraic generalization use these three aspects of pedagogical content knowledge simultaneously?

The research focused on a sequence of lessons from the *Patterns of Change* unit of the *Investigations in Number, Data, and Space* Curriculum (Tierney, Nemirovsky, Noble, & Clements, 1998). This unit introduced students to algebraic generalizations through connecting students visual descriptions of patterns to number properties. Tierney et. al. (1998) state that the Patterns of Change unit helps students, "Distinguish across time and simultaneous relationships" (p. xvii) and that "fluency in moving back and forth between these two perspectives is central to the mathematical analysis of change" (p. xviii).

An intensive observation and interview process was conducted to ascertain how teachers used their knowledge of algebraic generalization. The following data sources were used: 1) Pre-observation teacher interviews to understand the knowledge used to prepare for the lesson; 2) Three video-taped observations to ascertain how teachers used their knowledge in providing classroom instruction; 3) Three stimulated recall interviews following each lesson to gather information on the knowledge used during the classroom decision-making process; and 4) A follow-up interview conducted a week following the observations to further evaluate the extent of the teachers' knowledge of algebraic generalization. A qualitative methodology was employed to uncover the nature and organization of this knowledge directly from the data.

In the following sections I describe the research design of the project, including a justification for how the design addressed the research questions. Next, I characterize the study participants, detailing the selection process and the context in which data collection occurred. Following this, I describe the data collection instruments, elaborating on why and how they were created and the refinements that were made based on the piloting process. Finally, I give an account of the data analysis techniques used in this study, including the schematics used in organizing the data and describe the limitations of the study.

Research Design

This study on teacher knowledge is a case study of two 5th grade teachers. This section provides the description and purpose for the various aspects of the research design. This includes a general discussion of qualitative research, case studies, and how qualitative methods, along with my theoretical framework, are intertwined to answer the aforementioned research questions. A description of how other researchers have employed these techniques to investigate teacher knowledge is provided in chapter 2.

Qualitative Research

Qualitative research has increased in popularity within the mathematics education research community in the past decade and initially faced challenges of acceptance and credibility. This can partly be explained by the fact that the term ‘qualitative research’ encompasses many different types of research, with confusion and misunderstanding occurring among the nuances that exist in this complicated milieu. Denzin and Lincoln (2000) state that, “Qualitative research is difficult to define clearly. It has no theory or paradigm that is distinctly its own” (p. 2). With this in mind, many have tried to define qualitative research in terms of the common characteristics that these research methodologies share. For instance, Merriam (1998) states that, “Qualitative research is an umbrella concept covering several forms of inquiry that help us understand and explain the meaning of social phenomenon with as little disruption of the natural setting as possible” (p. 5). Others, such as Strauss and Corbin (1998), define it in terms of what it is not. They state, “By the term ‘qualitative research’ we mean any type of research that produces findings not arrived by statistical procedures or other means of quantification” (p. 11). Despite the lack of a clear definition, most researchers agree on several key characteristics of this research paradigm.

Qualitative research has its beginnings in the research philosophy of phenomenology and symbolic interaction (Merriam, 1998). These research paradigms seek to understand a particular phenomenon from the perspective of the research participant. From this stance, the meaning an individual constructs for a particular phenomenon is of utmost importance. Bogdan and Biklen (2003) state, “Meaning is of essential concern to the qualitative approach. Researchers who use this approach are interested in how different people make sense of their lives. In other words, qualitative researchers are concerned with what are called participant perspectives" (p. 7). Merriam (1998) furthers this point,

> The key philosophical assumption upon which all types of qualitative research are based is the view that reality is constructed by individuals interacting with their social worlds. Qualitative researchers are interested in understanding the meaning people have constructed, that is, how they make sense of their world and the experiences they have in the world" (p. 6).

Another distinctive characteristic of qualitative research is the setting in which it takes place. With the objective being to understand a phenomenon from the participant's perspective, data are collected in the natural environment for which the subject experiences the phenomenon. Bogdan and Biklen (2003) state, "Qualitative research has actual settings as the direct source of data, and the researcher is the key instrument. … Qualitative researchers go to the particular setting under study because they are concerned with context" (p. 6). With this requirement, qualitative research usually involves fieldwork with the data collected being descriptive in nature. "The data collected have been termed soft, that is, rich in description of people, places, and conversations, and not easily handled by statistical procedures" (Bogdan & Biklen, 2003, p.2). "Since qualitative research focuses on process, meaning, and understanding, the product of qualitative research is richly descriptive" (Merriam, 1998, p. 8).

Finally, qualitative research employs an inductive strategy. According to Merriam (1998), "this type of research builds abstractions, concepts, hypotheses, or theories rather than testing an existing theory" (p. 7). Bogdan and Biklen state,

> Research questions are not framed by operationalizing variables; rather, they are formulated to investigate topics in all their complexity, in context. While people conducting qualitative research may develop a focus as they collect data, they do not approach the research with specific questions to answer or hypotheses to test" (p. 2).

In essence, qualitative research is used to develop theory rather than to test a hypothesis. With this in mind, qualitative research if often used to explore new areas of research or to advance already established areas by investigating the phenomenon from a new perspective (Stern, 1980).

Justification of Methods

This research study was designed to explain how teachers understand and use knowledge in their classrooms. I sought to understand a particular phenomenon, teacher knowledge used for instruction, from the perspective of the classroom teacher. With this goal, I collected rich, descriptive data from the teachers' classrooms. The teachers were asked to reflect on their own video taped lessons to ascertain the knowledge they deemed important in making decisions. Hence, the data gathered for this study reflects knowledge used by teachers in the classroom and projects their views of the critical components of knowledge needed for successful implementation of the lessons.

The goal of my research questions was to understand and to describe the complexity of teacher knowledge with the hope of building new theory. Even though significant work has been accomplished in the field of teacher knowledge (e.g. Ball, 1988; Fennema & Franke, 1992; Franke & Kazemi, 2001; Hill & Ball, 2004; Ma, 1999; Shulman, 1986, 1987, Wilson, Shulman, & Richert, 1987), these studies vary widely in terms of their focus and perspective. First, many of these studies focus on a single form of knowledge (i.e., mathematical knowledge, pedagogical content knowledge, knowledge of student thinking, etc. in isolation). In order to understand knowledge use from the perspective of classroom teachers we must begin to put the pieces back together, as knowledge used in the classroom is inherently integrated. To this end, I examined the interaction of three different forms of knowledge: (a) mathematics, (b) learners, and (c) curriculum and instructional strategies, as they were used in the act of teaching.

Several studies have investigated teacher knowledge in an integrated fashion (Schoenfeld, 1985; Lampert, 1989; Lehrer & Franke, 1992; Ma, 1999). What distinguished this study from prior work is that I investigated the particular content domain of algebraic generalization within the context of classroom practice. Hence, I investigated a complex topic, from the perspective of the teacher, for a specific content domain. Since my study focused on teacher knowledge from a new perspective, the use of qualitative methods to understand and describe the phenomenon was an appropriate first step. The end product of this work enlightens existing work and aids in the development of theories that can be tested in future research studies. Stylianides and Ball (2004) state that,

> One possible reason for which the problem of what teachers need to know remains unsolved is that people have been looking at it from different angles, having available only limited research that concentrated on connecting the knowledge produced by different perspectives (p. 3).

One goal of this study is to intertwine other perspectives to provide a better understanding of teacher knowledge.

Types of Qualitative Research

According to Merriam (1998) five types of qualitative research are commonly found in education: (a) basic or generic qualitative study, (b) ethnography, (c) phenomenology, (d) grounded theory, and (e) case study. What complicates the identification of a particular research study into one of these five categories is that many studies involve elements of more than one type and often in varying degrees. For instance, an individual could conduct a study that contained elements of both phenomenology and grounded theory. My study is primarily a case study, but draws upon elements of both phenomenology and grounded theory.

Phenomenology is concerned with understanding the essence of a phenomenon. This research paradigm is based on, "the assumption that there is an essence or essences to shared experience. These essences are the core meaning mutually understood through a phenomenon that is commonly experienced" (Patton, 1990, p. 70). In contrast, Strauss and Corbin (1994) pioneered the category Grounded Theory and contend that the major difference between this methodology and other approaches to qualitative research is its emphasis upon theory development. Again, the distinction is one of degree and focus, as one is often interested in developing theory based upon the shared experience of a phenomenon. Since my future goal is to develop theory regarding the knowledge structures used by teachers and to gain a sense of how teachers employ their knowledge in providing instruction in algebraic reasoning, I consider my study to contain elements of both phenomenology and grounded theory. However, the main purpose of this study is to provide a rich, descriptive account of how two teachers used their knowledge in making decisions in the classroom and hence is classified as a case study, which will be more thoroughly discussed in the next section.

Case Studies

Case studies are common in education today and have increased in popularity since the mid-1980s (Merseth, 1996). Case studies are differentiated from other types of qualitative research in that they are intensive descriptions and analyses of a single unit or bounded system (Smith, 1978). Shulman (1987) provided a characterization of case studies:

> These are normally case descriptions of teachers, classrooms, or schools. They do not necessarily claim empirical generalisability. They are presented as instances or exemplars documenting how education was accomplished (or stymied) by a particular group of teachers and students in a particular place. (p. 27)

The purpose of my research was to gain a deep understanding of how two 5^{th} grade teachers used their knowledge when teaching a unit that involved algebraic generalization. The study design allowed for an intensive description of a particular unit and the choice of participants permitted the documentation of exemplar, and highly recommended, teachers.

Case studies are useful when "the boundaries between phenomenon and context are not clearly evident" (Yin, 2003, p. 13). The interaction that occurs among the different forms of teacher knowledge in the classroom is complex. In this environment, teacher decisions are influenced by both the context of the classroom and the knowledge the teacher draws upon. This context must be described and understood to make appropriate judgments about teacher knowledge use. The rich descriptions in case studies can provide the data to discern the complexities of such a situation. Thus, for the purpose of this study a case study methodology was utilized.

During instruction teacher decisions are intimately intertwined with the context of the classroom. Decisions are made based upon student reactions and thinking and guide the direction and pacing of instruction. If student misunderstandings become apparent then adjustments are often made to address those misunderstandings. Knowledge of mathematics, knowledge of learners, and knowledge of curriculum and instructional strategies are used in coordination in making decisions and adjusting to the situation. Hence, the study not only involved 5^{th} grade teachers, but the classrooms in which they operated.

Study Participants

Two fifth-grade teachers participated in this study from an elementary school in a small midwestern city. In this section I discuss the process of participant selection, the characteristics that guided this process, introduce the study subjects, and provide relevant background information.

Participant Characteristics

With the goal of understanding the nature of fifth-grade teachers' knowledge and how they use this knowledge in providing instruction, I chose two fifth-grade teachers to participate in this study. In addition, I chose subjects who: (a) were teaching algebraic generalization using a NSF-funded curriculum and (b) came highly recommended. An additional unintended characteristic of both participating teachers was that they were teaching the *Patterns of Change* (Tierney, Nemirovsky, Noble, & Clements, 1998) unit for the first time. The following sections elaborate on these characteristics.

Use of NSF-funded curricula

During the 1990s, a call for reform began within the mathematics education community that was facilitated by the recommendations of the NCTM *Standards* (1989; 2000). As a result of this call, the National Science Foundation (NSF) funded several grants focused on developing mathematics curriculum at the K-12 level, separated into K-5, 6-8, and 9-12 programs. These curricula were written to promote student involvement and exploration that encouraged student thinking and sense making. This was in stark contrast to the "lecture and practice" oriented style typically found in classrooms (Stigler & Hiebert, 1999).

At the elementary level three particular curricula were developed: *Everyday Mathematics* (Bell et al, 2004); *Investigations in Number, Data and Space* (Tierney, Nemirovsky, Noble, & Clements, 1998); and *Math Trailblazers* (TIMS Project, 2004). The main tenets of these curricula were the following:

1) Curricula should be conceptually oriented, supporting the child's development toward more abstract reasoning.
2) Curricula should actively involve children in doing mathematics.
3) Curricula should include a broad range of content—arithmetic but also measurement, geometry, statistics, probability, and algebra.
4) Curricula should make appropriate use of technology, including calculators and computers. (Alternatives for Rebuilding Curricula Center, 2005).

These curricula were based and developed on an extensive body of research on how students learn and underwent extensive piloting. The curriculum ultimately used for this study was Investigations in *Number, Data, and Space* (Tierney, Nemirovsky, Noble, & Clements, 1998).

In each unit,

> Students explore the central topics in depth through a series of investigations, encountering and using important mathematical ideas. Students actively engage in mathematical reasoning to solve complex, mathematical problems. They represent, explain, and justify their thinking, using mathematical tools and appropriate technology. … The investigations allow significant time for students to think about the problems and to model, draw, write, and talk with peers and the teacher about their mathematical thinking (TERC, 2005).

The participants' use of NSF-funded curriculum aided my study of teacher knowledge in a variety of ways. First, such curricula encourage student interaction and the development and exploration of divergent ideas. The classroom environment intended by this curriculum encourages the teacher to make sense of student thinking and to make quick instructional decisions. This environment provided a unique opportunity to observe how teachers react to dynamic classroom environment and use their knowledge during the process of instruction.

Furthermore, since this curriculum encourages student interaction and problem solving it provides a rich opportunity to examine the participants' understanding of student thinking. Problem exploration combined with student interaction lead to divergent student thinking. This variety of student thought naturally brings out different mathematical perspectives of a concept, encourages the teacher to reflect on the content and how they organize tasks, and forces them to make instructional decisions. This use of a NSF-funded curriculum brought out the different components of the framework chosen for this study.

Teaching Algebraic Generalization

Another component of the framework for this study included teacher content knowledge. In order to obtain a deep and connected understanding of a teacher's content knowledge, the study was limited to the area of early algebraic reasoning. This narrowed focus enhanced the opportunity for rich description, across subject comparisons, and the tracking of content knowledge over the duration of the study. With this in mind, I observed teachers providing instruction on algebraic generalization. This was chosen because of my previous work in the area (Lannin, Barker, & Townsend, 2006) and the fact that algebraic generalization has been recommended for earlier introduction in the elementary schools (Kaput, 1995; NCTM, 2000). With such a recommendation and the subsequent adjustments in elementary curriculum the study of elementary teacher knowledge in this area is very timely. Furthermore, my past work in the area allowed for further interpretation and analysis of the content being displayed in the lessons and interviews.

High-Quality Teachers

To create rich descriptions of how teachers use their knowledge of content, student thinking, and curricular tasks I sought data that focused on these three aspects of knowledge. To this end, I selected teachers who were highly recommended and experienced. This helped in two ways: a) These teachers were more familiar with the issues surrounding content, student thinking, and tasks and they possessed insight into how these aspects related to effective instruction; and b) The inclusion of these teachers minimized the amount of cognitive time and energy devoted to classroom management and general teaching issues that occurred during class so that more observation time could be devoted to the components of my framework. In limiting the search to more experienced teachers, I avoided common first year struggles. One of my participants, Amanda, was a second year teacher. However, she was highly recommended and possessed many of the desired attributes of participant selection.

The purpose of this study was to provide a description of teacher knowledge use. With this in mind, the decision to select teachers who were considered high quality by their principal, district mathematics coordinator, and/or a university faculty member had several other advantages. First, obtaining a description of how expert teachers use their knowledge in providing instruction has the most potential for informing the mathematics education community of the knowledge useful to future teachers of algebraic generalization. Second, I assumed that high-quality teachers would provide important insight into how they drew upon their knowledge of content, student thinking, and tasks. Limiting my search to highly recommended teachers provided a rich source of data.

First Time Teaching Particular Unit

An unintended characteristic was the fact that the participants were teaching the material for the first time. Several advantages existed to this condition. First, an experienced teacher teaching a unit for the first time would be especially focused on content, student thinking, and tasks in relation to this particular content area as these are the aspects of their knowledge that are the least developed. The hypothesis here is that a reflective teacher would focus on the aspects of their teaching that they deemed as areas for improvement. This could include a lack of deep content knowledge, specific areas of student thinking, or tasks related to this content. Second, the initial implementation of a unit provided an opportunity to track growth in the three aspects of knowledge over the course of the unit. If a teacher was especially focused on these aspects of their knowledge, and started with limited knowledge in this area, then more learning and growth should take place. Hence, the fact that the teachers had never taught this unit before led to an interesting data set. In addition to following the growth of knowledge structures related to instruction in generalization over the unit, the data provided insights into the aspects of experience that led to this growth.

Participant Selection

I located subjects by contacting the mathematics coordinator at one of the largest school districts using NSF-funded curricula in the state, Wildwood Public School District (WPSD; a pseudonym, as are all of the people and place names in this document), located in the city of Wildwood. Through the recommendation of the district mathematics coordinator, principals, and a local faculty member I identified two teachers at Pinewood Hills Elementary School (PHE) that were willing to participate. Both of my participants were personally observed by at least two of the three sources of recommendation. The following section provides a detailed description of the two teachers that were the focal point of my study, as well as a description of the school and district they taught at.

During the recruitment process the teachers were given a description of the study and were provided information pertaining to the incentives and time commitments required for the study. A copy of the letter sent to prospective teachers and the overview given to those who decided to participate are provided in appendix A.

Participant Description

The following section introduces the study participants Amanda and Emily. A description of these two teachers, including their professional development and the characteristics of their classrooms, is provided.

Amanda

Amanda was a fifth-grade teacher at Pinewood Hills Elementary School where she completed her student teaching at the fourth-grade level. She was in her second year teaching fifth- grade. Amanda held a Bachelor of Science degree in Elementary Education and completed a Masters of Science in Curriculum and Instruction through a program that presented her the opportunity to teach her first year while simultaneously taking courses. At this point, Amanda had only taught mathematics using the *Investigations in Number, Data, and Space* Curriculum (Tierney, Nemirovsky, Noble, & Clements, 1998).

Professional Development. When asked about her professional development opportunities Amanda talked about her experiences obtaining her Masters' degree, her experience with district coordinated professional development, and what she had learned from her students. Amanda commented on the benefits of attending courses while she was teaching and the connections that were made to what occurred in the classroom. Amanda talked at length about how one of her course instructors influenced her to think differently about student thinking:

> He pushed us to see how we think about it and to figure out how we are thinking about it as students so we could have a better understanding of the students, where they are coming from and kind of compare and contrast. Most of the people have backgrounds where we are given formulas to memorize and given ways to do it, so

> kind of expanding our minds and seeing the differences between the two. (Amanda, Initial Interview)

She felt the adjustment to focusing on student thinking had an impact on her instruction within the classroom and was evident throughout data collection.

Amanda stated that although she benefited from her mathematics methods instruction it was through her district professional development that she learned about the curriculum itself. The Wildwood School District (WSD) provided opportunities for elementary teachers to attend sessions on each particular unit of the *Investigations in Number, Data, and Space* Curriculum (Tierney, Nemirovsky, Noble, & Clements, 1998) being used in the district. She stated that, “Since I have been a teacher, the district has a math meeting time on the different TERC books; that has been the place where I have learned more about the curriculum” (Amanda, Initial Interview). Although Amanda attended several district professional development sessions in her career she had not attended a session on the *Patterns of Change* Unit. She commented that patterns and algebraic generalization had only been touched on briefly at the professional development sessions.

Finally, Amanda stressed the amount of learning that took place by listening to her students as illustrated by her following comment:

> I would actually say that most of my development has been through watching these kids and talking to them. There are many days that I could not prepare for what they would say, I mean they come up with amazing things. (Amanda, Initial Interview)

It appears that her focus on student thinking in her coursework carried over to the classroom and provided insight into her students and their needs.

Classroom characteristics. Amanda had the chairs in her room arranged in groups of four. Although every student had an assigned chair, she would rearrange the students by assigning them specific working groups each day. On a normal day, the class generally spent an hour working on mathematics, starting with a warm-up to work on and review basic calculation skills such as subtraction, multiplication, and division; this time generally lasted about 10 minutes. Amanda expressed the tension she felt between spending time working on skills for state mandated tests and time spent on the conceptual development stressed by the curriculum. She felt there was a lack of procedural competency on the part of students, but at the same time felt the need to develop conceptual understanding.

During instruction Amanda asked questions and emphasized the importance of student reasoning. Many times during lessons she asked her students to slow down and think, providing considerable wait time for them to develop a response. She also established social norms for how students talked and discussed with each other. Conversations occurred both between teacher and students as well as between student and student. Students often said, "I disagree with so and so because". The students appeared comfortable disagreeing with each other respectfully within the classroom. Most, if not all, appeared engaged in the lesson. Amanda varied her class structure; she had her students working individually, in small groups, and in a whole class setting.

Emily

Emily was a colleague of Amanda who also taught fifth grade at Pinewood Hills Elementary School. At the time of this study Emily had taught fifth grade for seven years, fourth grade for one year, and preschool for one year. She held a Bachelor of Science degree in Elementary Education. Since arriving at Pinewood Hills four years ago, she taught fifth grade using the *Investigations in Number, Data, and Space* Curriculum (Tierney, Nemirovsky, Noble, & Clements, 1998). Prior to her time at PHE she taught at a school that utilized a non-NSF funded curriculum. She commented that while she was teaching with the traditional curriculum she created a lot of materials on her own because she did not always like what was provided. Emily was a strong proponent of the *Investigations in Number, Data, and Space* Curriculum (Tierney, Nemirovsky, Noble, & Clements, 1998) and maintained a philosophy that was consistent with the curriculum.

When asked whether she had experience teaching patterns she commented that she taught certain mathematical topics that contained the idea of pattern indirectly. For instance, she stated:

> We do, I don't know if this would be considered patterns, we do mathematical thinking at 5th grade, we do the arrays, I know that when Amanda taught this the arrays came up, like square numbers, so we do teach that. I think that anytime we teach multiplication that is a pattern that is that skip counting pattern. … So it builds, but we don't do a lot in this grade. (Emily, Initial Interview)

Professional development. When asked about her professional development opportunities, Emily discussed the following: (a) district coordinated professional development, (b) leading professional development herself, (c) personal contacts, and (d) Project Construct. These four experiences will be discussed in the paragraphs below.

Similar to Amanda, Emily attended several professional development sessions provided by the Wildwood School District that introduced units of the *Investigations in Number, Data, and Space* Curriculum (Tierney, Nemirovsky, Noble, & Clements, 1998).

Although Emily stated that she benefited from these experiences, she learned more from leading these sessions herself. She noted:

> I have done professional development, taught professional development, which to me is more valuable than … you learn more because you feel like you have to know the stuff inside and out. I actually read a lot of the TERC books this summer, parts of it that I had never read, because I did this whole week workshop. (Emily, Initial Interview)

Emily also mentioned two individuals that have been invaluable to her while in the Wildwood School District--the elementary mathematics coordinator of the district and a person who worked at the local university. Emily made the following statement about her university contact; "She amazes me, her thinking; every time I am with her I learn something new that I didn't know." Emily appears to be someone who loves to discuss teaching with others and has learned a lot through the personal contacts that she has made.

Finally, Emily mentioned Project Construct as having an influence on her teaching. Project Construct was a professional development project developed by the Missouri Department of Elementary and Secondary Education. It was "based on constructivist theory, which states that children construct their own knowledge and values as a result of interactions with the physical and social world" (Project Construct, n.d.). The project developed thirty hour-long institutes, with continuing education made available. The purpose was to "support young children's characteristic ways of learning while at the same time providing teachers, parents, and administrators the information they need to make appropriate decisions" (Project Construct, n.d.). This project influenced Emily in the way she used and developed student thinking within her classroom.

Classroom characteristics. Emily generally taught mathematics for an hour each day. She began her mathematics instruction by collecting student homework and having students discuss any issues they could not understand. She encouraged her students to call her at home if they were having difficulties providing insight into the challenges students encountered. After discussing homework, the class often gathered on the carpet in the front of the room and discussed the activity they would be doing that day.

Emily stressed changing the structure of her lessons periodically to help keep their attention as illustrated by the following:

> I try to get them up a lot, like up and down, and try to switch what they are doing. You have to switch what they are doing every 10 or 15 minutes; something has to change in order to keep their attention. Usually we will do a whole group thing, independent work, or group work and then we will come back to the carpet and we will work. (Emily, Initial Interview)

Emily also asked questions and encouraged students to explain their thinking. During the stimulated recall interview Emily often interjected comments on her role in this process, noting instances where she took over student thinking or gave away answers too quickly.

Emily encouraged student discussion and followed the path of student thought even when it contained faulty logic. A concerted effort was made to investigate student thinking and to make connections to how others were thinking about the problem. Each lesson ended with a discussion of the homework and what her expectations were for the students.

Wildwood Public School District

Wildwood School District is located in a mid-western college town of about 90,000 known for its highly educated workforce. The town has a median household income of $50,000 and a minority population of 20 percent. WSD employed 1500 teachers who provided instruction to over 16,000 students spread over 3 high schools (10-12), 3 Junior Highs (8-9), 3 Middle Schools (6-7), and 20 elementary schools (K-5). The district has a strong state reputation, with many of its schools being nationally recognized for excellence in education.

Pinewood Hills Elementary School

Pinewood Hills Elementary School serves one of the poorer sections of the Wildwood School District. WSD has 31.1 percent of its students in the free and reduced lunch program (Missouri Department of Elementary and Secondary Education); in contrast 43.5 percent of PHE students are enrolled in the program (Columbia Missourian). However, through the years the school has had extensive collaboration with the local university and was considered one of the better schools in the district.

Several years ago school district personnel adopted the *Investigations in Number, Data, and Space* Curriculum (Tierney, Nemirovsky, Noble, & Clements, 1998), which were designed to develop deeper conceptual understanding. Previous instruction focused on following procedures without emphasizing understanding. PHE was a pilot school to test the material and received considerable attention from the district and local university. The fifth graders in this study were the first cohort of students to have been taught using the *Investigations in Number, Data, and Space* Curriculum (Tierney, Nemirovsky, Noble, & Clements, 1998) in grades K-5. Amanda noted that students often, "come up with amazing things, because this class has now had TERC since kindergarten so they have a lot of experience with it and that is how they have been taught to think, with the exception of a few transfers." (Amanda, Initial Interview)

Research Instruments

Two main sources comprise the data for this study: (a) Videotaped classroom observations along with field notes, and (b) transcripts from semi-structured interviews. The following sections describe the observation protocols and the different types of interviews that were conducted.

Interviews

Interviews are the staple of many qualitative studies, including case studies. In this study, interviews were chosen as the primary data source for several reasons. Although much research has been conducted using Shulman's model of integrated teacher knowledge (Shulman, 1987), little is known regarding how it applies to the specific content area of algebraic generalization and the interactions that exists among teachers' knowledge of mathematics, knowledge of learners, and knowledge of curriculum and instructional strategies within this domain. Since little is known about these aspects of teachers' knowledge, it was imperative to obtain rich descriptions of this form of knowledge in the context in which it was used. Stimulated recall interviews are an effective method of gaining an understanding of this form of knowledge. Furthermore, I sought to understand how teachers used and perceived their knowledge in the classroom. Interviews allowed me to probe the thought processes of the teachers and provided focus to the information I collected. During the interview process, interview questions were designed to gather data that pertained to specific aspects of the theoretical framework and particular video episodes were chosen that brought out discussions of mathematics, learners, and curriculum and instructional strategies.

Three distinct interview protocols were used during the study: an initial interview, stimulated recall interview, and a final interview. All three forms were used to gather data on Amanda and Emily. The initial and final interviews were semi-structured in nature, with the stimulated recall interview having components that were semi-structured and others that were open-ended. Further details are provided about these interviews in the following subsections.

Initial interview. The initial interview provided information regarding the participating teachers' teaching background, their philosophy of teaching, experience in teaching algebraic generalization, and the goals they had for the upcoming unit on patterns. This information was used to provide a context for the data collected later in the study. In addition, the initial interview addressed the first research question by prompting for the knowledge of mathematics, knowledge of learners, and the knowledge of curriculum and instructional strategies they expected to use in the upcoming lessons. Sample tasks were provided to encourage this conversation. The initial interview protocol appears in Appendix B.

Stimulated recall interview. The stimulated recall interviews were designed to gather information on the knowledge Amanda and Emily used while making decisions during the act of teaching. At the beginning of each interview they were reminded that the goal of the study was to investigate their knowledge of mathematics, knowledge of learners, and knowledge of curriculum and instructional strategies. They were given the flexibility to discuss anything they deemed important. These interviews consisted of two parts. The first part provided Amanda and Emily the opportunity to make comments concerning a clip from the observation video and was unstructured in nature. Amanda and Emily were allowed to freely make comments, without prompting, on any aspect of the video they believed to be pertinent to their knowledge of mathematics, knowledge of learners, and knowledge of curriculum and instructional strategies. The second aspect of the interview, which was often interspersed with the first, consisted of specific questions posed by me in regard to their knowledge of mathematics, knowledge of learners, and knowledge of curriculum and instructional strategies. Prior to each interview I analyzed the video and developed a list of questions that pertained to how the teacher used her knowledge in the observation.

The stimulated recall interviews were the main source of data for answering both research questions. Having Amanda and Emily reflect and describe how they used their knowledge while making decisions in the classroom directly addresses the research questions. The stimulated recall interview protocol appears in Appendix C.

Final Interview. Approximately one week after the last observation a follow-up interview was conducted. This interview was used to provide supporting data for the research questions and to fill any gaps in knowledge use that I did not obtain with the initial or stimulated recall interviews. The final interview consisted of two parts. The first part asked the participants to reflect on their use of their knowledge of mathematics, knowledge of learners, and knowledge of curriculum and instructional strategies while making instructional decisions during the series of lessons that were observed. This provided Amanda and Emily another opportunity to discuss how they used these three forms of knowledge and allowed for a cumulative perspective on knowledge use instead of one based on specific situations. During the second portion of the interview, Amanda and Emily were given tasks and student work to critique in terms of their knowledge of mathematics, knowledge of learners, and knowledge of curriculum and instructional strategies. This allowed me to gather information regarding the research questions in terms of a common set of questions that were answered by both teachers and based on the same situation. The final interview protocol appears in Appendix D.

Observations

I conducted classroom observations for three class periods for both Amanda and Emily. All observations were video-recorded with the camera focused on the teacher at all times. During each observation, I collected field notes concerning the events in the classroom while a doctoral student videotaped. A zoom microphone was used for instances where the teacher interacted with small groups of students.

The Data

Description of Data

Following completion of data collection, all observations and interviews were transcribed. In addition, a compilation of the stimulated recall and observation transcripts was created so that teacher responses to the stimulated recall interview could be situated within the context of the observation. See Appendix E for a sample portion of a transcript compilation. A timeline of data collection is given in Figure 3.1.

April 14	Amanda – Initial Interview, Observation #1, Stimulated Recall #1
April 15	Amanda – Observation #2, Stimulated Recall #2
April 18	Amanda – Observation #3
	Emily – Observation #1, Stimulated Recall #1
April 19	Amanda – Stimulated Recall #3
April 21	Emily – Observation #2, Stimulated Recall #2
April 22	Emily – Observation #3, Stimulated Recall #3
May 2	Emily – Final Interview
May 3	Amanda – Final Interview
May 4	Jordan – Initial / Final Interview
May 5	Blanche – Initial / Final Interview
May 10	Susan – Initial / Final Interview
May 11	Arlene – Initial / Final Interview

Figure 3.1: Data collection timeline

Data Analysis

Following completion of the data collection and transcription, the data were retrospectively analyzed using a data reduction approach (Miles & Huberman, 1994). The analysis occurred in three separate phases. After each phase of coding categories were developed and analyzed using elements of grounded theory (Strauss & Corbin, 1998) with an eye toward salient themes. During each phase of analysis a constant comparative method (Glaser & Strauss, 1967; Strauss & Corbin, 1990) was applied to test and revise categories. Portions of the transcripts were test coded by outside individuals with all discrepancies being discussed until resolution. A more complete description of these phases is provided below.

Phase 1. Each transcript was initially coded to identify portions of the transcript where algebraic generalization was a topic of conversation. I used Kaput's (1999) definition for algebraic generalization in which he characterized generalization as:

> Deliberately extending the range of reasoning or communication beyond the case or cases considered, explicitly identifying and exposing commonality across cases, or lifting the reasoning or communication to a level where the focus is no longer on the cases or situation themselves but rather on the patterns, procedures, structures, and the relationships across and among them. (p. 136)

Figure 3.2 provides an illustration of an episode coded as involving algebraic generalization. This episode was included in the study because the teacher discussed the struggles of developing a generalized rule.

L *Amanda*: They were having such difficulties coming up with a rule.

Researcher: Do you think the rule from the tower of 3 pattern was easier?

Amanda: I think it was obvious.

Researcher: Here it is a lot more …

L *Amanda*: They want it to be clear, I think even I want it to be. They want to be able to say, "I see what it is doing, there has to be something I can tell you really quick." Which is funny, Emily and I talked about the straw problem for a second--I was so frustrated, "How does this stupid pattern work?" It was frustrating me that we couldn't come up with a formula, because it has to fit, I was trying to make these connections, because there is a one and somehow they are connected. M

Researcher: I see you pointing to the one that is causing you problems.

Amanda: Well we kept saying do you include in the first one (pointing to the extra straw in the straw problem) and make a rule about having four or do you make a rule about the one or do you make that a separate part of your formula? M

Figure 3.2: Excerpt coded as algebraic generalization

The portions of the transcripts that were excluded from consideration dealt mainly with general teaching issues, such as management, or with external forces, such as the impact of state testing.

Phase 2. Following the identification of algebraic generalization within the transcripts, each transcript was further coded in terms of the type of knowledge being discussed: mathematics (M), learners (L), or curriculum and instructional strategies (CI). The original definition for these three terms came from a synthesis of the integrated models of teacher knowledge described in chapter 2. At this point, portions of the transcripts were coded by myself and outside individuals to test the viability of the original definitions with all discrepancies discussed until resolution. This process led to a modification of the definitions that were then used to code the transcripts. Figure 3.3 provides the revised definitions used in the coding of teacher knowledge and Figure 3.4 provides a sample coding.

Knowledge of Mathematics (M): the teacher discussed a mathematical topic connected to the idea of algebraic generalization, including the connections and relationships between these ideas. Included in this category are the ways and means of justifying and providing proof for these ideas. Conversations focused on mathematics, the teacher's view of a mathematical topic—how they think about it-- or the mathematical topics brought up as needed for instruction.

Knowledge of Curriculum and Instructional Strategies (CI): teacher mentioned the tasks, curriculum, questions used to further the lesson, or goals of the lesson in relation to algebraic generalization. Comments centered on the implementation of the lesson or decisions made regarding the flow of the lesson were also included in this category.

Knowledge of Learners (L): the teacher shared student thinking—what she observed students thinking as well as how she expected students to think about algebraic generalization. This category included conversations about student characteristics, habits, or misunderstandings that may have influenced how they thought about algebraic generalization.

Figure 3.3: Code definitions for phase 2 of analysis

CI

E: Well, this is what I am going to do, I want to go back and I want them to start building this (the double staircase), I am just going to do the double staircase for a while just so they, I want to see what they end up coming up with because my thinking, Dan was like, "they are not going to be able to come up with that" and I said, "I don't know," and he said, "maybe a few of them, but I don't think all of them will be able to come up with it."

L

D: Come up with what?

M

E: Like a formula basically, like how do you plug in numbers so that if I give you the 100th step you are going to be able to know, I mean because you are looking for the pattern. I know that I have found some, but I know there have got to be other ones that, because I certainly didn't think of what _____ and _____ thought of. (Referring back to the boys mentioned at the very beginning)

L

D: Lets talk about that, what do you think of their formula?

M

E: Well the only thing, my issue with that, what I kept running into was that they didn't know the total so far to subtract. In hindsight I wish I would have not even said, I think I gave them too much, I wish I would have said, "OK, lets do that, lets plug that in" and we did that to a degree, but I think I was thinking so much myself that I didn't let them grapple with it.

L

CI

Figure 3.4: Sample coding for M, L, and CI

In the example above, the first statement by Emily was coded CI because she discussed what she planed to do in class, “I want to go back and I want them to start building this (the double staircase); I am just going to do the double staircase for awhile.” The CI code was used anytime the teacher was discussing what she was doing, the curricular goals she had for the class, or asking questions. The key here was that her comments were focused on what she, the teacher, was doing. Items coded as mathematics had mathematics as the focus. For instance, when Emily stated, “Like a formula basically, like how do you plug in numbers so that if I give you the 100th step you are going to be able to know, I mean because you are looking for a pattern.” Here the focus is not on the teacher but rather the mathematical relationships themselves. Finally, items were classified as student thinking if the student was the main focus of the comment. When Emily states, “they are not going to be able to come up with that,” she is referring to her students.

Once all transcripts were coded in terms of their knowledge of mathematics, knowledge of learners, and knowledge of curriculum and instructional strategies the first round of analysis determined how often the teachers used these three forms of knowledge in their conversations about teaching. Table 3.5 depicts a sample entry of the percentage of each form of knowledge used. This was calculated by taking the number of lines that were coded as mathematics, student thinking, and curricular tasks respectively and dividing it by the total number of lines identified as dealing with representations.

Interview	Total # coded as representation	M	%M	ST	%ST	TCQ	%TCQ
Amanda Initial	148	33	22.3%	97	65.5%	91	61.5%
Amanda SR1	221	49	22.2%	117	52.9%	100	45.2%

Table 3.5: Percentage of Mathematics, Student Thinking, and Curricular Task

Data from Phase 2 was used to answer research question 1: To what extent do two fifth grade teachers draw upon their knowledge of mathematics, their knowledge of learners, and their knowledge of curriculum and instructional strategies when designing, implementing, and reflecting on instruction related to algebraic generalization?

Phase 3. To further gauge the integrated nature of this knowledge, the data for each interview was divided into episodes where the teacher was discussing a single thought. For each of these episodes I recorded which forms of knowledge were being discussed and then separated and counted the episodes based on the combination of knowledge forms used. Table 3.6 depicts a sample entry from this work.

Interview	Total #	M	ST	TCQ	M & ST	M & TCQ	ST & TCQ	All Three
Amanda Initial	13	0	1	1	2	1	5	3

Table 3.11: Integrated Use of Knowledge

Episodes within the various knowledge combinations (M&L, M&CI, L&CI, M,L,&CI) were then qualitatively analyzed using elements of grounded theory (Strauss & Corbin, 1998) to uncover patterns of knowledge use. For example, were there any similarities among the episodes for which Amanda and Emily combined their knowledge of mathematics with their knowledge of learners? During this part of the analysis a constant comparative method (Glaser & Strauss, 1967; Strauss & Corbin, 1990) was used to test and revise themes within the categories.

Phase 4. The findings from Phase 3 were then used to map out the knowledge interactions of the M-L-CI node resulting in mappings describing how Amanda and Emily used their knowledge. An example mapping is provided in Figure 3.12.

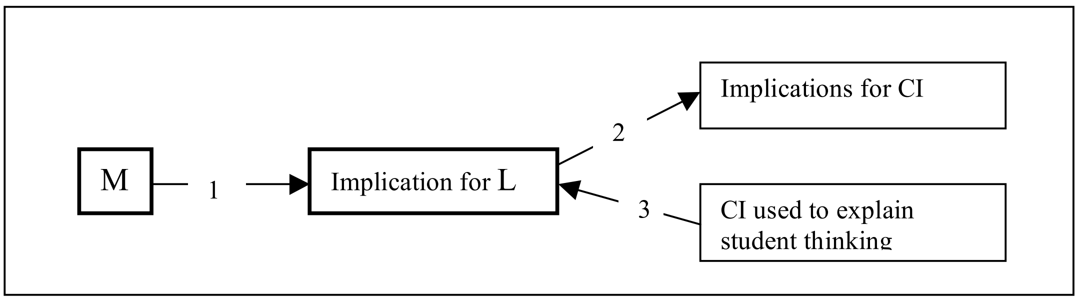


Figure 3.12: Sample mapping of knowledge interaction

Data from phases three and four were used to answer research question 2: How do two fifth grade teachers when designing, implementing, and reflecting on instruction related to algebraic generalization use these three aspects of pedagogical content knowledge simultaneously?

Reliability

The researcher's goal in this study was to understand how teachers use their knowledge while implementing instruction in early algebraic generalization. To increase the trustworthiness of the study, data were collected from multiple sources and triangulated. When themes were identified through observations or interviews they were checked to see if the themes were consistent across the different forms of data and within the same data forms. Furthermore, a fellow doctoral student and a faculty member periodically check coded portions of the transcripts to ensure consistency. In developing the definitions of knowledge of mathematics, knowledge of learners, and knowledge of curriculum and instructional strategies three different individuals coded portions of the transcripts and resolved all discrepancies through adjustment of the definitions.

Summary

Two fifth grade teachers were observed and videotaped as they taught a series of lessons intended to promote algebraic generalization. Following each observation, a stimulated recall interview was conducted to ascertain how the teachers used their knowledge of mathematics, student thinking, and curricular tasks during the lesson. These data were transcribed and coded according to these three knowledge types. For each topic of conversation these three forms of knowledge were analyzed looking for patterns among their usage. Furthermore, each transcript was coded in terms of representation discussed and further analysis was conducted to bring out patterns of knowledge usage for each of the different forms of representation. In the next chapter, I provide my interpretation of the collected data.

CHAPTER 4

DATA INTERPRETATIONS

In this chapter, I document the findings that resulted from this study on how two fifth grade teachers used their pedagogical content knowledge when providing instruction in algebraic generalization. In particular, I am interested in the use and interaction of three subcomponents of PCK: knowledge of mathematics, knowledge of learners, and knowledge of curriculum/instructional strategies. Analyses of stimulated recall, pre, and post interviews were conducted for a series of lessons on algebraic generalization to ascertain how Emily and Amanda used their knowledge during instruction. The purpose of this work was to answer the following research questions:

7) To what extent do two fifth grade teachers draw upon their knowledge of mathematics, their knowledge of learners, and their knowledge of curriculum and instructional strategies when designing, implementing, and reflecting on instruction related to algebraic generalization?
8) How do two fifth grade teachers when designing, implementing, and reflecting on instruction related to algebraic generalization use these three aspects of pedagogical content knowledge simultaneously?

This chapter is organized into two main sections that address the research questions listed above. Within each section I summarize the results of the data analysis that pertain to each question. For the first research question, data analyses were conducted to obtain an overall look as the extent to which the knowledge of mathematics, the knowledge of learners, and the knowledge of curriculum/instructional strategies were used in Amanda and Emily's reflections on their teaching. In particular, I examined: (a) how often each of the subcomponents of PCK were used, (b) how often these knowledge forms were used together in providing instruction, (c) the combinations of knowledge forms that occurred most frequently, and (d) the patterns that emerged in how the teachers used these different forms of knowledge. The first section of this chapter describes the global use of these three forms of knowledge.

The next section of the chapter portrays *how* Amanda and Emily used their knowledge of mathematics, knowledge of learners, and knowledge of curriculum/instructional strategies in providing instruction. I provide results that examine the following questions: What was the purpose for intertwining these various forms of knowledge? What were there differences between how Amanda and Emily integrated these forms of knowledge? In contrast to the first section, which focused on the extent to which the teachers used these forms of knowledge, the second section pertains to *how* these forms of knowledge were used in an integrated fashion. In doing so, the complexity and subtleties that exist within and among these forms of knowledge are described.

In addition to the two main sections of the chapter that correspond to the research questions, an opening section is provided to remind the reader of the theoretical framework utilized for this study. This framework will be referenced in the chapter and serves to organize the discussion of the findings. At the end of the chapter, a summary will be provided that outlines the overall assertions suggested from the findings.

THEORETICAL FRAMEWORK

This study investigated the knowledge used by teachers in the act of teaching algebraic generalization. Pedagogical content knowledge was chosen as the focal point due to its relationship and importance to instruction. I focused on three aspects of PCK:

knowledge of mathematics, knowledge of learners, and knowledge of curriculum/instructional strategies. In addition, I sought to understand how Amanda and Emily used these different aspects of PCK simultaneously when making instructional decisions. Not only did I want to know how often these different forms of knowledge were used, I wanted to understand how they were used together during the process of teaching. Figure 4.1 illustrates the framework used for this study.

In this framework, I focused on the knowledge of mathematics, the knowledge of learners, and the knowledge of curriculum/instructional strategies that a teacher draws upon while designing, implementing, and reflecting upon their lessons. In this model, the teacher may draw upon any of the three forms of knowledge at any time and may use different combinations of these knowledge forms. The three arrows going from the three aspects of PCK to the teacher illustrate this. I expected the teachers to use several different combinations of knowledge forms for a variety of reasons. It must be noted that this model is not an all-inclusive list; there are other forms of knowledge that could potentially impact instructional decisions that were not included in the analysis.

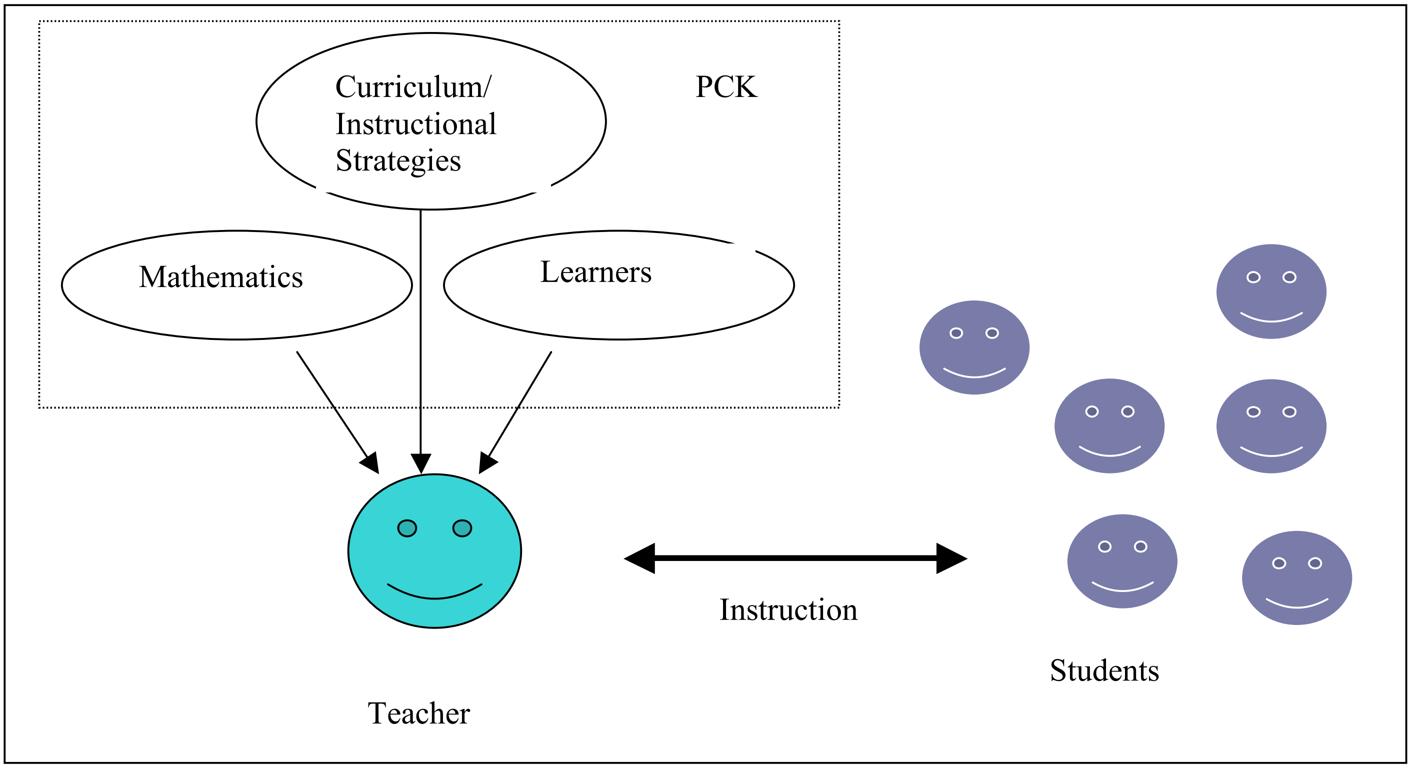


Figure 4.1: Theoretical Framework

The arrow between the teacher and the students represents the context in which these three forms of knowledge are used. The knowledge bases used by the teacher are not applied individually, but rather simultaneously. In addition, teachers are making decisions on what and how to teach as they interpret student reactions that, in turn, factor into and create future decisions. This social dynamic of the classroom plays a critical role in how knowledge is used and ultimately impacts and adjusts the knowledge sets that the teachers were drawing upon. The next section details the extent to which Amanda and Emily used these their knowledge of mathematics, learners, and curriculum/instructional strategies.

RESEARCH QUESTION 1: THE EXTENT OF KNOWLEDGE USE

The data presented in this section of the analysis provide an overall look at the extent to which the knowledge of mathematics, learners, and curriculum/instructional strategies were used. Transcripts of teacher interviews (pre, post, and stimulated recall) were used to determine the extent to which Amanda and Emily drew upon their knowledge of mathematics, knowledge of learners, and knowledge of curriculum and instructional strategies during the implementation of a series of lessons on algebraic generalization. When conducting interviews, questions inquiring about their knowledge of mathematics, learners, and curriculum/instructional strategies were included. In addition, while reacting to the videos during the simulated recall interviews, Amanda and Emily were allowed to add comments they deemed pertinent. I begin by presenting the extent to which they discussed algebraic generalization, followed by the extent for which they used their knowledge of mathematics, learners, and curriculum/instructional strategies, and conclude with the extent to which they used their knowledge in an integrated fashion.

Algebraic Generalization

The initial research question pertains to the extent to which Amanda and Emily used their knowledge of mathematics, knowledge of learners, and knowledge of curriculum/instructional strategies in the act of teaching algebraic generalization. With this in mind, only conversations related algebraic generalization was included for analysis. Table 4.2 depicts the percentage of each transcript that was coded to have included algebraic generalization. The percentages were calculated by taking the number of lines coded as algebraic generalization divided by the total number of lines. SR refers to the stimulated recall interviews.

Interview	Total # of lines	Lines coded as Algebraic Generalization	% of transcript included in analysis
Amanda			
Initial	230	148	64.3%
SR #1	471	221	46.9%
SR #2	613	464	75.6%
SR #3	423	287	67.8%
Final	454	346	76.2%
Amanda Total	**2191**	**1466**	**66.9%**
Emily			
Initial	221	131	59.3%
SR #1	278	204	73.4%
SR #2	331	311	94.0%
SR #3	424	355	83.7%
Final	392	279	71.2%
Total	**1646**	**1280**	**77.8%**
Amanda & Emily			
Total	3837	2746	71.6%

Table 4.2: Percentage of Transcripts Coded as Representation or Generalization

The data reveal that algebraic generalization was a prominent topic of conversation for both Amanda and Emily, occurring 66.9 percent of the time for Amanda and 77.8 percent of the time for Emily. Although this result is not surprising given the nature of the lessons and my interview questions.

In addition, the percentages of transcripts coded as algebraic generalization are roughly equal for the stimulated recall interviews (72.5 percent) and the pre and post interviews (69.7 percent). This suggests that Emily and Amanda focused on algebraic generalization just as often when given the opportunity to freely react to the video as when they were directly prompted.

However, differences existed in the extent to which Amanda and Emily discussed algebraic generalization in their interviews. Amanda discussed algebraic generalization 64.5 percent of the time during her stimulated recall interviews and 72.2 percent of the time during the pre and post interviews. Emily talked about algebraic generalization 84.2 percent of the time during her stimulated recall interviews and 66.9 percent of the time during her pre and post interviews. A 19.7 percent difference existed between the two teachers in relation to the amount of time devoted to algebraic generalization during the stimulated recall interviews.

Mathematics, Learners, and Curriculum/Instructional Strategies

Following the identification of algebraic generalization within the transcripts this subset was further coded in terms of the aspects of pedagogical content knowledge that were used: mathematics (M), learners (L), and curriculum/instructional strategies (CI). (Figure 4.3 provides the definitions for the categories used during the coding process)

Knowledge of Mathematics (M): the teacher discussed a mathematical topic connected to the idea of algebraic generalization, including the connections and relationships between these ideas. Included in this category are the ways and means of justifying and providing proof for these ideas. Conversations focused on mathematics, the teacher's view of a mathematical topic—how they think about it-- or the mathematical topics brought up as needed for instruction.

Knowledge of Curriculum and Instructional Strategies (CI): teacher mentioned the tasks, curriculum, questions used to further the lesson, or goals of the lesson in relation to algebraic generalization. Comments centered on the implementation of the lesson or decisions made regarding the flow of the lesson were also included in this category.

Knowledge of Learners (L): the teacher shared student thinking—what she observed students thinking as well as how she expected students to think about algebraic generalization. This category included conversations about student characteristics, habits, or misunderstandings that may have influenced how they thought about algebraic generalization.

Figure 4.3: Definitions used for coding

After the transcripts were coded in terms of knowledge of mathematics, knowledge of learners, and knowledge of curriculum/instructional strategies, tabulations were made to determine the frequency for which each of the knowledge forms were coded. Table 4.4 depicts the percentage of transcript lines I coded as one of the three forms of knowledge. This was determined by dividing the number of lines coded as mathematics, learners, and curriculum/instructional strategies, respectively, by the total number of lines identified as pertaining to algebraic generalization. Some passages were coded to contain more than one knowledge form, leading to a total sum of more than 100% for some interviews.

Interview	Total # coded as algebraic generalization	M	%M	L	%L	CI	%CI
Amanda							
Initial	148	33	22.3%	97	65.5%	91	61.5%
SR1	221	49	22.2%	117	52.9%	100	45.2%
SR2	464	68	14.7%	181	39.0%	130	28.0%
SR3	287	42	14.6%	157	54.7%	114	39.7%
Final	346	44	12.7%	223	64.5%	91	26.3%
Total	**1466**	**236**	**16.1%**	**775**	**52.9%**	**526**	**35.9%**
Emily							
Initial	131	24	18.3%	33	25.2%	67	51.1%
SR1	204	24	11.8%	83	40.7%	61	29.9%
SR2	311	28	9.0%	135	43.4%	126	40.5%
SR3	355	56	15.8%	136	38.3%	59	16.6%
Final	279	44	15.8%	109	39.1%	93	33.3%
Total	**1280**	**176**	**13.8%**	**496**	**38.8%**	**406**	**31.8%**
Amanda & Emily							
Total	2746	406	14.8%	1310	47.7%	982	35.8%

Table 4.4: Percentage of Mathematics, Student, Thinking, and Curricular Task

Amanda and Emily focused considerable attention on learners as 47.7 percent of their discussion of algebraic generalization was coded to include this form of knowledge. They talked frequently about curriculum/instructional strategies (35.8 percent), and to a lesser extent mathematics (14.8 percent). Both teachers exhibited this ordering of knowledge use (learners, curriculum/instructional strategies, then mathematics) and both used their knowledge of learners and knowledge of curriculum and instructional strategies significantly more than their knowledge of mathematics. This result raises the question as to the role of the knowledge of mathematics in the decision-making process of Amanda and Emily?

However, some differences were observed. For instance, Amanda used her knowledge of learners 52.9 percent of the time as compared to Emily who only used her knowledge of learners 38.8 percent of the time. Hence, it appears that Amanda's reflections on her instruction utilized her knowledge of learners more frequently. This is supported when you take into account that Amanda's use of her knowledge of learners (52.9 percent) is significantly higher than her use of curriculum/instructional strategies (35.9 percent) and mathematics (16.1 percent). In comparison, Emily's use of her knowledge of learners (38.8 percent) is relatively close to her use of curriculum and instructional strategies (31.8 percent). The following sections of this chapter will shed light onto why this difference existed. It is interesting to note that in Amanda's initial interview she commented on a professor who had a profound impact on her teaching. She stated that he "pushed us to have a better understanding of the students." She later commented that "I would say that most of my development has been through watching these kids and talking to them." It appears that Amanda's knowledge of student thinking played a significant role in her conversations about teaching and that this was impacted by her professional development.

Although these data reveal a pattern in Amanda and Emily's overall use of their knowledge of mathematics, knowledge of learners, and knowledge of curriculum and instructional strategies, the data in Table 4.4 do not illustrate the extent to which these knowledge forms were used simultaneously. This topic serves as the focus for the following section where I analyze the frequency for which Amanda and Emily used the difference aspects of PCK simultaneously.

Integrated Nature of Knowledge Use

In order to gauge the integrated nature of Amanda and Emily's use of their knowledge of mathematics, knowledge of learners, and knowledge of curriculum and instructional strategies the data for each interview were segmented into episodes where the teacher discussed a single thought. For each of these episodes, I noted the knowledge types used during the episode. Finally, for each interview, I tallied the number of episodes that included the various combinations of knowledge forms. (M, L, CI, M&L, M&CI, L&CI, and M, L, &CI) The results of this analysis are displayed in Table 4.5.

Interview	Total # of episodes	M	L	CI	M & L	M & CI	L & CI	All Three
Amanda								
Initial	13	0	1	1	2	1	5	3
SR1	21	0	4	2	1	1	5	8
SR2	32	1	7	3	4	0	12	5
SR3	30	2	0	1	6	1	12	8
Final	25	0	4	3	4	0	5	9
Total	**121**	**3**	**16**	**10**	**17**	**3**	**39**	**33**
Emily								
Initial	10	0	0	0	1	2	4	3
SR1	21	0	1	2	5	1	10	2
SR2	31	0	4	2	4	3	15	2
SR3	33	2	7	2	4	1	10	4
Final	17	0	2	3	1	2	7	2
Total	**112**	**2**	**14**	**9**	**15**	**9**	**46**	**13**
Amanda & Emily								
Total	233	5	30	19	32	12	85	46

Table 4.5: Integrated use of knowledge (Raw Numbers)

Table 4.5 presents the number of episodes that included the various combinations of knowledge types. In addition, to further illustrate the use of integrated knowledge, percentages of the episodes using the various combinations of knowledge forms were calculated and are displayed in Table 4.6.

Interview	M	L	CI	M & L	M & CI	L & CI	All Three
Amanda							
Initial	0.0%	7.7%	7.7%	15.4%	7.7%	50.0%	23.1%
SR1	0.0%	19.0%	9.5%	4.8%	4.8%	23.8%	38.1%
SR2	3.1%	21.9%	9.4%	12.5%	0.0%	37.5%	15.6%
SR3	6.7%	0.0%	3.3%	20.0%	3.3%	40.0%	26.7%
Final	0.0%	16.0%	12.0%	16.0%	0.0%	20.0%	45.0%
Total	**2.5%**	**13.2%**	**8.3%**	**14.0%**	**2.5%**	**32.2%**	**27.3%**
Emily							
Initial	0.0%	0.0%	0.0%	10.0%	20.0%	40.0%	30.0%
SR1	0.0%	4.8%	9.5%	23.8%	4.8%	47.6%	9.5%
SR2	0.0%	12.9%	6.5%	12.9%	9.7%	48.4%	6.5%
SR3	6.1%	21.2%	6.1%	12.1%	3.0%	30.3%	12.1%
Final	0.0%	11.8%	17.6%	5.9%	11.8%	41.2%	11.8%
Total	**1.8%**	**12.5%**	**8.0%**	**13.4%**	**8.0%**	**41.1%**	**11.6%**
Amanda & Emily							
Total	2.2%	12.9%	8.2%	13.7%	5.2%	36.5%	19.7%

Table 4.6: Integrated Use of Knowledge (Percentages)

The data confirm that most of Amanda and Emily's episodes involved more than one aspect of PCK. Table 4.7 depicts the number and percentage of the episodes that used a single aspect of PCK and those that utilized multiple aspects of PCK.

Interview	Total #	Single Aspect (Total)	Single Aspect (%)	Multiple Aspects (Total)	Multiple Aspects (%)
Amanda					
Initial	13	2	15.4%	11	84.6%
SR1	21	6	28.6%	15	71.4%
SR2	32	11	34.4%	21	65.6%
SR3	30	3	10.0%	27	90.0%
Final	25	7	28.0%	18	72.0%
Total	**121**	**29**	**24.0%**	**92**	**76.0%**
Emily					
Initial	10	0	0.0%	10	100.0%
SR1	21	3	14.3%	18	85.7%
SR2	31	6	19.4%	24	77.4%
SR3	33	11	33.3%	19	57.6%
Final	17	5	29.4%	12	70.6%
Total	**112**	**25**	**22.3%**	**83**	**74.1%**
Amanda & Emily					
Total	233	50	21.5%	174	74.7%

Table 4.7: Use of Pedagogical Content Knowledge

As seen in Table 4.7, Amanda and Emily had similar percentages of single knowledge use and multiple knowledge use. For Amanda, 76 percent of her episodes involved the integrated use of knowledge, while 24 percent of her episodes contained only a single knowledge form. For Emily, 22.3 percent of her episodes used a single knowledge form, while 74.1 percent used multiple types of knowledge. These results reveal the predominance of integrated knowledge in the conversations that Amanda and Emily had about their instruction.

Comparing Amanda and Emily's use of integrated knowledge (see Table 4.6), they both utilized the combination of knowledge of learners (L) and knowledge of curriculum/instructional strategies (CI) the most (32.2 percent to 41.1 percent respectively). In addition, both Amanda and Emily used the combination of knowledge of mathematics (M) and knowledge of curriculum/instructional strategies (CI) the least (2.5

percent and 8.0 percent respectively). However, the main difference in how they used these integrated forms of knowledge was that Amanda used all three aspects of pedagogical content knowledge 27.3 percent of the time as compared to Emily's 11.6 percent.

In addition, Table 4.6 reaffirms the data from Table 4.4 that Amanda used her knowledge of learners more often when reflecting on instruction as compared to Emily. But Table 4.6 reveals something else about the knowledge of learners. An interesting pattern that can be gleaned from Table 4.6 is the small percentage of episodes that combine the knowledge of mathematics and the knowledge of curriculum and instructional strategies, without the knowledge of learners (5.2 percent for both Amanda and Emily). Given Amanda's propensity to focus on her knowledge of learners this makes sense for her, but this pattern is also evident in Emily's episodes. If we see pedagogical content knowledge as the "blending of content and pedagogy" (Shulman, 1987, p.8) then this finding becomes intriguing as the combination of knowledge of mathematics and the knowledge of curriculum and instructional strategies occurs the least among the different knowledge combinations. This finding deserves further exploration as to why this occurred.

Summary

Amanda and Emily used their knowledge in an integrated fashion during their discussions about teaching. In addition, they used the different aspects of PCK in different proportions as both utilized their knowledge of learners the most and their knowledge of mathematics the least. However, there were distinctions between the two teachers. Amanda had a propensity to use her knowledge of learners that appeared to be influenced by previous professional development. Emily, the more experience teacher, had a more balanced use between her knowledge of learners and her knowledge of curriculum and instruction.

The findings of this first research question raise more questions: Why is the knowledge of learners the predominant form of knowledge, what is the role of knowledge of learners in the use of PCK? If pedagogical content knowledge is the transformation of mathematics into pedagogically powerful representations, why is the knowledge of mathematics used only 15 percent of the time? The one constant is that Amanda and Emily used their knowledge in an integrated fashion and it behooves the educational community to better understand this process of integration.

RESEARCH QUESTION 2: HOW IS PCK USED?

Table 4.5 reveals the extent to which Amanda and Emily used their knowledge of mathematics (M), knowledge of learners (L), and knowledge of curriculum/instructional strategies (CI) in an integrated fashion. The second research question involves further examination of how this knowledge was integrated. In this section I examine the following questions: In what ways did Amanda and Emily draw upon their knowledge of mathematics, knowledge of learners, and knowledge of curriculum/instructional strategies when designing, implementing, and reflecting on instruction related to algebraic generalization? To accomplish this, excerpts identified to contain multiple forms of knowledge were further analyzed using elements of grounded theory (Strauss & Corbin, 1998) to uncover patterns of knowledge use.

The findings and analysis for this research question are divided into two sections. The first section describes how Amanda and Emily used the different combinations of their knowledge of mathematics, knowledge of learners, and knowledge of curriculum/instructional strategies (M&L, L&CI, etc.) while implementing instruction. This was accomplished by separating the episodes according to knowledge types used and then analyzing these nodes for themes. The patterns for each of the different knowledge combinations as well as the differences that existed between how Amanda and Emily drew upon these different aspects of PCK are presented below.

The next section provides a detailed description concerning the patterns that appeared in each of the four different combinations of the Amanda and Emily's knowledge of mathematics, knowledge of learners, and knowledge of curriculum/instructional strategies.

Combinations of Knowledge Forms

In this section, I provide a description of how the knowledge of mathematics, knowledge of learners, and the knowledge of curriculum/instructional strategies were used by Amanda and Emily in support of their teaching decisions. The analysis is divided into four sections that reflect the four different combinations of knowledge forms: (a) knowledge of mathematics (M) and knowledge of learners (L), (b) knowledge of learners (L) and knowledge of curriculum/instructional strategies (CI), (c) knowledge of mathematics (M) and knowledge of curriculum/instructional strategies (CI), and (d) knowledge of mathematics (M), knowledge of learners (L), and knowledge of curriculum/instructional strategies (CI). A summary of the patterns that existed for each knowledge combination along with a discussion of the similarities and differences that existed between Amanda and Emily are provided.

Knowledge of Mathematics and the Knowledge of Learners

Thirty-two episodes included only the knowledge of mathematics and the knowledge of learners. An episode was defined to be a portion of an interview for which the teacher discussed a single thought. Of these episodes, 18 were episodes from Amanda and 14 were from Emily. Two distinct types of interactions occurred in the episodes of both Amanda and Emily: (a) a discussion of learners, with the knowledge of mathematics used to clarify, explain, or support a statement regarding learners. In this instance, Amanda or Emily's knowledge of mathematics was used to supplement their knowledge of learners, and (b) the exploration or discovery of a mathematical idea, usually a concept that the teacher did not fully understand. Student thinking was often the impetus for such explorations or discoveries.

Amanda

Table 4.8 lists the episodes where Amanda used her knowledge of mathematics and her knowledge of learners. A description of how these knowledge types were integrated is included.

Episode (line #s)	Type of Interaction
Amanda Initial (192-212)	Knowledge of M used to support statement about L
Amanda Initial (291-303)	Knowledge of M used to support statement about L
Amanda SR1 (903-926)	Knowledge of M used to support statement about L
Amanda SR2 (262-300)	Knowledge of L prompts exploration of M
Amanda SR2 (427-442)	Knowledge of M used to support statement about L
Amanda SR2 (564-571)	Neither
Amanda SR2 (871-877)	Knowledge of L prompts exploration of M
Amanda SR3 (19-29)	Knowledge of M used to support statement about L
Amanda SR3 (31-46)	Knowledge of M used to support statement about L
Amanda SR3 (165-169)	Knowledge of M used to support statement about L
Amanda SR3 (401-411)	Knowledge of M used to support statement about L
Amanda SR3 (413-428)	Knowledge of M used to support statement about L
Amanda SR3 (634-645)	Knowledge of M used to support statement about L
Amanda Final (448-457)	Knowledge of M used to support statement about L
Amanda Final (270-278)	Knowledge of M used to support statement about L
Amanda Final (280-323)	Knowledge of M used to support statement about L
Amanda Final (560-575)	Knowledge of M used to support statement about L
Amanda Final (599-626)	Knowledge of L prompts exploration of M

Table 4.8: Amanda's use of knowledge of mathematics and knowledge of learners

In 14 of the 18 episodes Amanda used her knowledge of mathematics to clarify, support, or explain a statement she was making about her students. Three of the 18 episodes included an exploration or discovery of a mathematical idea that was prompted by student thinking (knowledge of learners). One episode could not be classified into either of these categories. Although this episode included both the knowledge of mathematics and the knowledge of learners, no clear connection was made between these two forms of knowledge. The following episodes illustrate the different ways in which Amanda integrated her knowledge of mathematics with her knowledge of learners.

Example 1: Knowledge of mathematics used to support knowledge of learners. The following excerpt is an example where Amanda used her knowledge of mathematics to support a statement regarding her knowledge of learners. In this example, Amanda discussed a misconception her students had while working the double staircase problem (see Appendix G). The problem resulted in the pattern 2, 6, 12, 20, 30, … While completing the table for this pattern, the students noticed that the *difference* between consecutive terms of this pattern increased by two, creating a separate pattern of 2, 4, 6, 8, … . This new pattern caused several students to be confused as to what was increasing by two; many believed that it was the original pattern that was increasing by two. The following excerpt describes Amanda's reaction to this misconception.

L { *Amanda*: Well they tried skip counting by 2 because my class got really stuck on it, skip counting by 2, and there was a big misconception there between what skip counting by 2 is and what is adding 2. } { There is a difference between those, and so I think that pattern influenced the next } M one.

(Amanda, Final Interview, 270-275)

Amanda acknowledged the confusion her students had with skip counting (L). After stating the misconception, Amanda attempted to make sense of the misconception by referring to the mathematics of the situation. She stated that skip counting was mathematically different from adding two (M). The students struggled with the distinction between a pattern whose difference between consecutive terms increased by two each time, and a pattern whose difference between consecutive terms was always two. Although Amanda did not elaborate on the difference between the two ideas, she correctly recognized that not understanding the mathematical difference caused student confusion. In this episode, Amanda used her knowledge of mathematics to clarify a student misconception. The main theme of this episode, and the surrounding text, was to make sense of student thinking (knowledge of learners).

Example 2: Knowledge of mathematics used to support knowledge of learners. The following episode is an additional example of where Amanda used her knowledge of mathematics to support a statement regarding her knowledge of learners. In this episode, Amanda discussed how two of her students incorrectly determined that the number of tiles required for the 20th step of the cross problem (see appendix) was 37. Prior to this mistake, the students completed the first seven steps of the table shown in figure 4.9. At this point, the students were asked to determine the number of tiles for the 20th step. The students wrote 20 in the last row of the table (3 rows below the seventh step they had just completed) and proceeded to add three 4's to 25 (the total so far for the seventh step) to obtain a total of 37 tiles for the 20th step (see figure 4.9). The mistake made by the student was that he placed 20 in the 10th row of the table and then applied a recursive rule to the total so far column to calculate the 20th step—they had actually calculated the value for the 10th step. More specifically, in table 4.9 the student noticed that the terms in the "total so far" column increased by 4 (recursive reasoning). He continued this pattern from the 25 in the 7th row of the "totals so far" column, adding three 4's to the 25 to obtain the value 37 for the 10th row.

Step Number	Step Size	Total So Far
1	1	1
2	4	5
3	4	9
4	4	13
5	4	17
5	4	21
7	4	25
20		37

Figure 4.9: Table used by students to solve Cross Problem

L

Amanda: At the end (of the table) there is a blank spot and they wrote 20 (for the 20th step of the pattern) and they were incorrect in how they got to 20.

L

Amanda: They were stuck because of the table. They had added 12 (four 3's) to 25 (the value of the seventh step) and they went, instead they went 8, 9, 10 (adding 4 tiles for steps 8, 9, and 10). It was basically step 10, but they viewed it as the 20th step. That was that one group and then I think someone did it on their homework but I do not remember what mistake they made.

M

L

(Amanda SR3, lines 31-46)

In this example, Amanda described a misunderstanding that a group of students had while using a table to calculate the 20th step of the Cross Problem. Amanda began by stating the mistake (They wrote 20 in the l0th spot and were incorrect in how they obtained it) and describing what the students had done (they added 12 to 25). Her knowledge of mathematics was then used to explain the mistake. (A recursive rule was used to calculate the 20^{th} step, but the 20 was located in the wrong place on the table) She quickly returned to comment on student thinking, adding that someone in the class made this mistake on his homework. This excerpt is an example of a situation where Amanda used her knowledge of mathematics (M) to clarify or make sense of student thinking (knowledge of learners). In general, these episodes involved time discussing student thinking with short excursions into mathematics to provide an explanation of the student thinking.

Example 1: Knowledge of learners prompted an exploration of mathematics. The following example illustrates how student thinking (knowledge of learners) prompted Amanda to explore her knowledge of mathematics. In this case, Amanda discussed the difficulties her students had in developing a rule for the cross problem (see appendix H). This prompted Amanda to shift the conversation to her difficulty in developing a rule for the straw problem (see appendix I) and a subsequent exploration of the mathematics.

L *Amanda*: They were having such difficulties coming up with a rule.

Researcher: Do you think the rule from the tower of 3 pattern was easier?

Amanda: I think it was obvious.

Researcher: Here it is a lot more …

L *Amanda*: They want it to be clear, I think even I want it to be. They want to be able to say, "I see what it is doing, there has to be something I can tell you really quick." Which is funny, Emily and I talked about the straw problem for a second--I was so frustrated, "How does this stupid pattern work?" It was frustrating me that we couldn't come up with a formula, because it has to fit, I was trying to make these connections, because there is a one and somehow they are connected. M

Researcher: I see you pointing to the one that is causing you problems.

Amanda: Well we kept saying, "Do you include in the first one (pointing to the extra straw in the straw problem) to make a rule about having four or do you make a rule about the one or do you make that a separate part of your formula? M

In this example, Amanda commented that, "they (her students) were having such difficulties coming up with a rule." This comment regarding how her students struggled to develop a rule appeared to be an impetus for Amanda's mathematical exploration of a rule for the straw problem. Although Amanda never developed a mathematical rule for the straw problem, her focus shifted from talking about her students (knowledge of learners) to talking about the mathematics of the situation. This was a common pattern among the episodes where mathematics became the focus of attention; some element of student thinking led to a need, or desire, to explore a mathematical idea.

Example 2: Knowledge of learners prompted the investigation of mathematics. The following episode is an additional example where Amanda's knowledge of learners led her to investigate a mathematical idea, in this case to discover a new mathematical technique. Amanda talked about the effects of the *Investigations in Number, Data, and Space* Curriculum (Tierney, Nemiovsky, Noble, & Clements, 1998) on her students. She described how her students' thought processes were different from hers and that she often learned mathematics from her students. In this example, Amanda described a problem done earlier in the semester where her students were multiplying a number by 98.

L { *Amanda:* The next day I came in and the kid goes times 100 minus two groups. You know, I was like, "Man, why didn't I see that?" } why didn't I see that it was really close to the landmark 100. I have taught that, but my mind is so one tracked towards, I learned a new way to do it through them. } M

Instead of using the traditional algorithm for multiplication, Amanda's student multiplied the number by 100 and subtracted twice the original number from the result (Instead of 98*X, the student took 100*X – 2X). This was not what Amanda expected, as she completed the problem in her mind using the standard algorithm. Amanda commented that her traditional background created roadblocks to how she thought and that she learned a considerable amount from listening to her students. In this example, her examination of student thinking (L) led to new personal insights about mathematics (M).

In summary, Amanda primarily used her knowledge of mathematics to support, clarify, and justify her comments about student thinking—this occurred in 14 of the 18 episodes coded to include Amanda's knowledge of mathematics and her knowledge of learners. At times, her knowledge of learners prompted an investigation of mathematics, but this was less frequent, occurring in only three episodes. The next section investigates how Emily used her knowledge of mathematics and her knowledge of learners simultaneously.

Emily

Table 4.10 lists the 14 episodes where Emily used her knowledge of mathematics in combination with her knowledge of learners. A description of how these knowledge types were integrated is included.

Episode (line #s)	Type of Interaction
Emily Initial (184-189)	Discussion of M leads to implications for L
Emily SR1 (57-62)	Knowledge of M used to support statement about L
Emily SR1 (122-135)	Discussion of M leads to implications for L
Emily SR1 (152-169)	Knowledge of L prompts exploration of M
Emily SR1 (307-317)	Discussion of M leads to implications for L
Emily SR1 (532-536)	Discussion of M leads to implications for L
Emily SR2 (15-16)	Knowledge of L prompts exploration of M
Emily SR2 (18-57)	Discussion of M leads to implications for L
Emily SR2 (116-129)	Knowledge of L prompts exploration of M
Emily SR2 (131-138)	Knowledge of M used to support statement about L
Emily SR3 (444-472)	Knowledge of M used to support statement about L
Emily SR3 (681-696)	Knowledge of M used to support statement about L
Emily SR3 (699-765)	Knowledge of L prompts exploration of M
Emily SR3 (820-851)	Knowledge of L prompts exploration of M

Table 4.10: Emily's use of knowledge of mathematics and knowledge of learners

In 5 of the 14 episodes, Emily explored or discovered a mathematical idea (knowledge of mathematics) that was prompted by a discussion of student thinking (knowledge of learners). For 5 of the episodes, Emily began by discussing mathematical ideas (M), which were followed by a description of the implications these mathematical ideas had for students (L); 3 of these 5 cases were the result of direct prompts from me asking about the mathematics of the lesson. In 4 of the 14 episodes, Emily used her knowledge of mathematics to clarify, support, or explain a statement she made about her students.

Example 1: Knowledge of learners prompted an exploration of mathematics. The following example illustrates how Emily's knowledge of student thinking (L) prompted her to think about, and explore, her knowledge of mathematics (M). In this case, Emily discussed a strategy that one of her students used while attempting to find the 245th step of the straw problem (see appendix I). The student wanted to find the number of straws required to build a chain of 500 squares by dividing this value by two and adjusting to find the number of straws required to build a chain of 245 squares. Prior to this episode, the class looked at groups of 5 squares and noticed that the first 5 squares required 16 straws, but each subsequent group of 5 squares only required 15 straws. Figure 4.11 illustrates this idea.

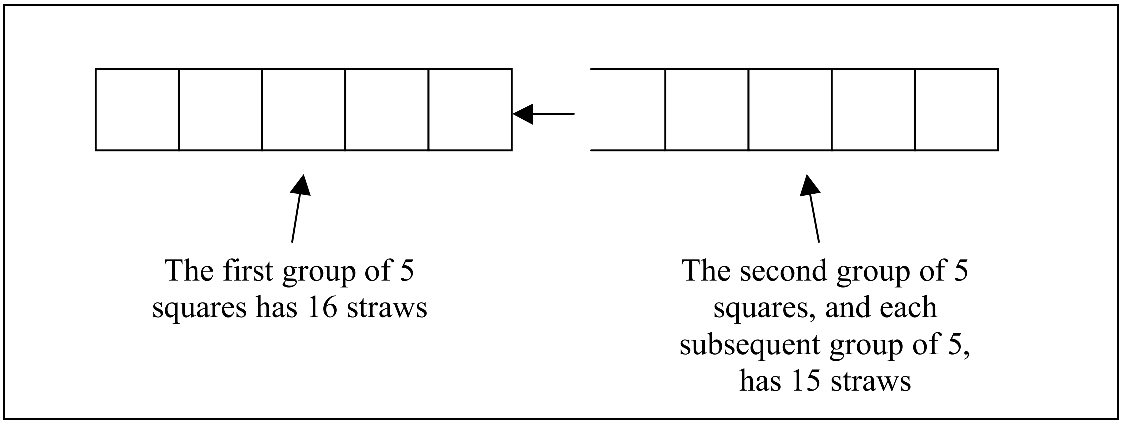


Figure 4.11: Groups of 5 Strategy

Using this knowledge the student calculated the number of straws required for 500 squares by taking 1 times 16 (the first group of 5 squares) and adding 99 times 15 (the next 99 groups of 5 squares). The student knew that 99 groups of 5 squares and 1 group of 5 squares would result in the desired 500 squares. Emily did not initially understand how the student attempted to calculate the 245th step. However, she acknowledged and explained the steps of his strategy (knowledge of learners) and then attempted to uncover the mathematics behind his strategy.

Researcher: What were you hoping he would come up with?

L { *Emily:* I don't know. I did not know what he was thinking, but I was interested to see what he would try to do. I didn't know. } Because I didn't even know what it would look like or how it compared to what the others students did …

L { *Emily:* He took 15 straws and multiplied it by 100, okay wait; I have go to think about that for a minute because I don't know what he is saying. }

L *Emily*: 15 times 99, plus 16, but that wouldn't work would it? Oh, if he was trying to find 500 squares it would. M

Researcher: So is he going for 500?

L *Emily:* Yeah, because he is still stuck on that. (The student attempted this strategy earlier in the lesson)

Researcher: Did he say 15 times 99 and 1 times 16?

Emily: Yes, which would work. But what happens when he divides though? Can he just divide? I don't know.

Emily: What he needed to say was that 500 squares is equal to 1600 straws. So he needed to say that 16 times 100 is equal to 500 squares. 1600 straws is equal to 500 squares. M

(Emily, Stimulated Recall 3, 699-753)

In this example, Emily described a strategy that a student used to find the 245th case of the straw problem, "15 times 99, plus 16." However, she did not understand the student's logic and questioned whether the strategy could be applied to this situation, which led her to investigate the mathematics behind the rule. Thus, it appears that Emily's lack of understanding regarding the strategy of the student (L) was the impetus for her to investigate the mathematics of the situation (M). Although Emily incorrectly concluded that, "What he needed to say was that 500 squares is equal to 1600" the episode does illustrate that student thinking led her to think about, and explore, the mathematics of the situation. It appears that Emily may have multiplied 100 by 16 to arrive at 1600, making the incorrect assumption that each group of 5 has 16 straws. The next example illustrates how Emily's use of her knowledge of mathematics often led to implications concerning her knowledge of learners.

Example 1: Discussion of M leads to implications for L. The following episode illustrates how Emily's use of her knowledge of mathematics led to an implication concerning learners. In this episode, Emily was asked by me to predict the mathematical ideas that will be brought out by the upcoming lesson on algebraic generalization. Using her knowledge of mathematics, Emily responded that the ideas of multiplication and arrays were connected to the idea of patterns that her students would be investigating (knowledge of mathematics). Because Emily understood the connection between arrays, multiplication, and linear patterns and she knew that her students had previously completed a unit on multiplication, which included the idea of arrays, she predicted that her students would revert back to these ideas and make connections between the patterns and multiplication.

> *Researcher:* What mathematical ideas do you think will be brought out by going through this?
>
> [M] *Emily:* I think obviously multiplication. I think arrays, like we talked about earlier, which is multiplication. [L] I think this seeing a pattern, I think a lot of them will revert back to those younger grades seeing the blocks, when they see those blocks, "Oh, I know what is going to happen."
>
> (Emily, Initial Interview, 181-189)

In this example, Emily used her mathematical knowledge to state that multiplication and arrays were connected to the idea of pattern. She combined mathematical knowledge with knowledge of learners, noting that students had previously discussed these topics. She predicted that they would apply the ideas from these topics during the process of investigating patterns (knowledge of learners).

Example 2: Discussion of M leads to implications for L. The following episode is an additional example where Emily's knowledge of mathematics led to a statement concerning student learning. In this excerpt, Emily discussed the doubling strategy, stating when the strategy would and would not work. After making a statement concerning the mathematics of the doubling strategy, she related this information to the mathematics her students had done in the past, multiplication, suggesting a potential reason for the continued misuse of the doubling strategy in her class.

M { *Emily:* I should have brought that up though … will doubling strategy work every time in every pattern, it won't so they have got to know, it has got to be a steady increase, because if it is not a steady increase then you can't just double it. }

L { They have learned that because it (the doubling strategy) }

M { always works with multiplication and division, if you know 50 goes into 100 twice, then you know 500 will go into 1000. }

(Emily, Stimulated Recall 1, 532-536)

Using her knowledge of mathematics, Emily stated that the doubling strategy did not apply to every pattern and stressed that it was a fact that her students "have got to know."

She continued and explained, "it has got to be a steady increase" for the strategy to work, a fact that is not true--there are many linear increasing situations where the doubling strategy does not work. Following a discussion of the mathematics, she turned to her knowledge of learners to reference a topic in her students' prior mathematical experience, multiplication, where they used the doubling strategy successfully. She then explained that for multiplication the doubling strategy would always work and provided an example (knowledge of mathematics). The reference to her students' prior use of the doubling strategy in multiplication appeared to suggest a potential cause for the continued misuse of the doubling strategy in her class. Emily's students might think that if the doubling strategy always worked for multiplication then it should also work in the new situation involving patterns. This argument implied the need for the doubling misconception to be addressed.

Example 1: Discussion of mathematics used to support knowledge of learners. The following episode is an example where Emily used her knowledge of mathematics to support a statement regarding student thinking. This is distinct from the previous category where a conversation that was focused on mathematics led to an implication about learners. In this episode, the focus of conversation is on student thinking and the knowledge of mathematics is used to explain or justify student thinking.

In this episode, Emily described a strategy that her student, Eddie, used to solve the straw problem (see Appendix I). In this episode Eddie attempted to find the number of straws required to build a chain of 500 squares. He noticed that the first 5 squares of the straw problem required 16 straws and incorrectly assumed that every subsequent set of 5 squares would also require 16 straws. In this episode, Emily used her knowledge of mathematics to discuss the thinking involved in a student's strategy and to provide an explanation as to why she chose to focus on this particular student strategy.

Researcher: I must have missed this, how did this conversation arise?

L { *Emily:* When Eddie said, "and then I went from five, I took that and multiplied that by a hundred and then I multiplied 16 by 100, too." And then John said that not every five squares has 16 straws. }

Researcher: So John brought that up?

L { *Emily:* Yes, John said that not every five squares has 16 straws, it's only the first five squares. }

Researcher: So why did you choose to focus on that strategy?

L { *Emily:* The main reason is because I think it was a very obvious thing for them to see } because it was one less like it was, because of that one that is different, because of that first step that is different, it is not normal because of the first straw. } M

(Emily, Stimulated Recall 3, 444-472)

Amanda restated Eddie's strategy; "I took 5 and multiplied it by 100 and then multiplied 16 by 100." At this point Eddie used proportional reasoning to conclude that since 5 squares required 16 straws then 500 squares, which is 5 times 100, must require 16 times 100 straws. Amanda continued by describing the adjustment that John proposed to Eddie's strategy, "not every five squares has 16 straws, it is only the first five squares." When Emily was asked why she chose to focus on this strategy, she used her knowledge of mathematics to support her decision. She stated that the context of the problem allowed her students to see and understand why the first set of 5 squares contained one more straw than the other groups of 5 squares. Thus, she chose to address the strategy because she understood that the mathematical representation of the problem lent itself to understanding the strategy proposed by Eddie and John.

In summary, Emily used her knowledge of mathematics in combination with her knowledge of learners in three different ways. First, Emily's knowledge of learners prompted her to think about, and explore, her knowledge of mathematics. Second, Emily's discussion of mathematics led to an implication concerning her students. Finally, Emily used her knowledge of mathematics to support statements regarding her knowledge of learners. Emily used these different combinations of her knowledge of learners and knowledge of curriculum/instructional strategies in roughly the same proportion. This illustrates Emily's flexibility in how she drew upon her knowledge in the act of teaching.

Summary of Knowledge of Mathematics and Knowledge of Learners

Amanda and Emily used the integration of their knowledge of mathematics and knowledge of learners in different ways. Amanda primarily used her knowledge of mathematics to support her knowledge of learners, this occurred in 14 of 18 episodes. For instance, Amanda often discussed student thinking using her knowledge of learners and then used her knowledge of mathematics to clarify what students were saying or to illuminate the understanding, or misunderstanding, that a student had. This was significantly higher than Emily, who only used mathematics to support her knowledge of learners in 4 of 14 episodes. Amanda's focus on learners, in relation to Emily, was consistent with the data in Table 4.4, which revealed that Amanda used her knowledge of learners 52.9 percent of the time compared to 38.8 percent for Emily.

In contrast, Emily seemed more willing to grapple with mathematical issues. For Emily, the most common context for integrating her knowledge of mathematics with her knowledge of learners was in her attempts to understand and explore the mathematics of student thinking; this occurred in 5 of 14 episodes. This differed from how Amanda used her knowledge, Amanda used the knowledge of mathematics to clarify student thinking, Emily often investigated mathematics to understand the mathematics. In addition, during five episodes Emily initiated conversations about mathematics, which were then followed up by a discussion of the implications for her students. This never occurred in Amanda's conversations, even thought both were asked about the mathematics involved in the lesson during each interview. Table 4.12 summarizes how Amanda and Emily combined their knowledge of mathematics and their knowledge of learners.

Type of Interaction	Amanda	Emily
Statement about L is supported by Knowledge of M	14	4
Knowledge of L prompts exploration of M	4	5
Knowledge of M leads to implications for L	0	5

Table 4.12: Amanda and Emily's use of M and L combined

In summary, Amanda focused on her knowledge of learners and used mathematics mainly in support. Emily used her knowledge of mathematics in support of her knowledge of learners, but also used her knowledge of learners to stimulate her knowledge of mathematics. Emily utilized the combination of her knowledge of mathematics and her knowledge of learners in different ways and appeared more flexible in how this integration took place. The next section explores the ways that Amanda and Emily integrated their knowledge of learners in combination with their knowledge of curriculum and instructional strategies.

Knowledge of Learners and the Knowledge of Curriculum/Instructional Strategies

Eighty episodes occurred where Amanda and Emily used their knowledge of learners in connection with their knowledge of curriculum and instructional strategies. Of these 80 episodes, 42 came from Amanda and 38 were from Emily. Two distinct types of interactions were identified: (a). Forty-two episodes were focused on curriculum and instructional strategies. In these episodes, Amanda and Emily discussed an instructional issue and used their knowledge of learners either to explain student reactions, or anticipated reactions, to the instruction being discussed (15 episodes) or to justify or support an instructional decision or claim that was being made (27 episodes); and (b) in 40 episodes student thinking (knowledge of learners) was the focus of Amanda and Emily's conversations. For these episodes, Amanda and Emily discussed student thinking that occurred in, or was related to, their classroom and used their knowledge of curriculum and instructional strategies in support of this discussion. They used their knowledge of curriculum and instructional strategies in two different ways. In 19 episodes, the knowledge of instruction was used to provide an explanation of the student thinking being discussed (i.e., the student was thinking in this fashion because I did this). In addition, in 21 episodes the discussion of student thinking led to an implication about curriculum and instruction (i.e., Since the students were thinking in this fashion I should have done, or need to do, this). The next two sections demonstrate the differences in how

Amanda and Emily individually used their knowledge of learners in connection with their knowledge of curriculum and instructional strategies.

It must be noted that since thee knowledge combinations in this section do not directly include the knowledge of mathematics they have a tendency to be general in nature. Marks (1990) noted that the boundary between PCK and the knowledge of content and pedagogy was often indistinguishable. It is in this section that the boundary is most likely to get blurred.

Amanda

Table 4.13 lists the episodes where Amanda used her knowledge of learners in connection with her knowledge of curriculum/instructional strategies. A description of how these two knowledge forms were integrated is included.

Episode (line #s)	Type of Interaction
Amanda Initial (218-227)	Focus on L – Implication for CI
Amanda Initial (245-256)	Focus on CI – L used to describe student reaction
Amanda Initial (334-345)	Focus on CI – L used to support CI decision/claim
Amanda Initial (349-368)	Focus on CI – L used to support CI decision/claim
Amanda Initial (384-400)	Focus on CI – L used to describe student reaction
Amanda SR1 (107-116)	Focus on L – Implication for CI
Amanda SR1 (200-212)	Focus on CI – L used to support CI decision/claim
Amanda SR1 (308-319)	Focus on CI – L used to support CI decision/claim
Amanda SR1 (442-461)	Focus on L – CI used to explain/situate student thinking
Amanda SR1 (466-468)	Focus on CI – L used to support CI decision/claim
Amanda SR2 (81-111)	Focus on CI – L used to support CI decision/claim
Amanda SR2 (123-141)	Focus on CI – L used to describe student reaction
Amanda SR2 (163-194)	Focus on CI – L used to explain student reaction and to support CI decision
Amanda SR2 (213-251)	Focus on L – Implications for CI
Amanda SR2 (343-370)	Focus on CI – L used to support CI decision/claim
Amanda SR2 (372-391)	Focus on CI – L used to support CI decision/claim
Amanda SR2 (448-461)	Focus on L – CI used to explain/situate student thinking
Amanda SR2 (486-497)	Focus on L – Implications for CI
Amanda SR2 (601-606)	Focus on L – Implication for CI
Amanda SR2 (679-704)	Focus on L – CI used to explain/situate student thinking

Amanda SR2 (705-719)	Focus on L – Implications for CI
Amanda SR2 (920-932)	Focus on L – Implications for CI
Amanda SR2 (947-961)	Focus on L – Implications for CI
Amanda SR2 (965-974)	Focus on CI – L used to support CI decision/claim
Amanda SR3 (48-65)	Focus on CI – L used to support CI decision/claim
Amanda SR3 (107-118)	No direct connection (Focus on L – CI side note)
Amanda SR3 (120-148)	Focus on CI – L used to support CI decision/claim
Amanda SR3 (207-215)	Focus on L – CI used to explain/situate student thinking
Amanda SR3 (244-249)	Focus on L – CI used to explain/situate student thinking
Amanda SR3 (269-299)	Focus on L – Implications for CI
Amanda SR3 (319-325)	Focus on L – CI used to explain/situate student thinking
Amanda SR3 (431-439)	Focus on CI – L used to describe student reaction
Amanda SR3 (457-503)	Focus on L – CI used to explain/situate student thinking
Amanda SR3 (552-562)	Focus on L – Implications for CI
Amanda SR3 (776-781)	Focus on CI – L used to support CI decision/claim
Amanda SR3 (799-808)	Focus on CI – L used to support CI decision/claim
Amanda Final (13-20)	Focus on L – CI used to explain/situate student thinking
Amanda Final (62-86)	Focus on L – CI used to explain/situate student thinking
Amanda Final (397-429)	Focus on L – Implications for CI
Amanda Final (459-481)	Focus on L – Implications for CI
Amanda Final (588-597)	Focus on CI – L used to support CI decision/claim
Amanda Final (157-182)	Focus on L – CI used to explain/situate student thinking

Table 4.13: Amanda's knowledge of learners and curriculum/instructional strategies

Forty-two of Amanda's episodes included the integration of knowledge of learners and knowledge of curriculum and instructional strategies. Of these episodes, 22 focused on knowledge of learners and used the knowledge of curriculum and instructional strategies to support ideas about learners. Two distinct ways were identified for how the knowledge of curriculum and instructional strategies was used in support: (a) 12 instances involved Amanda describing and discussed student thinking that occurred in her classroom (knowledge of learners) and stating an implication for instruction (e.g., My students were thinking like this, therefore I need to do this) and (b) 10 episodes occurred where Amanda discussed the student thinking that occurred in her classroom and then used knowledge of instruction to explain why this thinking occurred (e.g.., the student was thinking in this fashion because I did this). During both types of interaction Amanda focused on her knowledge of learners, but used her knowledge of curriculum and instructional strategies in support.

In 19 episodes Amanda focused on her knowledge of curriculum and instructional strategies, using her knowledge of learners to support ideas related to curriculum and instructional strategies. This occurred in two distinct ways: (a) in 5 episodes Amanda discussed an instructional issue (CI) and used her knowledge of learners (L) to describe student reactions, or anticipated reactions, to the instruction being discussed; and (b) for 15 episodes Amanda described an instructional decision or claim (CI) and used her knowledge of learners to justify that claim or decision. Both types of interactions focused on Amanda's knowledge of curriculum and instructional strategies, but used her knowledge of learners in different ways in support.

Commonalities existed among the episodes where Amanda's knowledge of learners was the focal point and those where her knowledge of curriculum and instructional strategies was the focal point. For both groups of episodes, interaction occurred where one form of knowledge was used to justify a claim involving the other (i.e., L used to justify claim regarding CI; CI used to justify claim about L). In addition, another type of interaction was identified in which one form of knowledge was used to describe an implication that resulted form a discussion of the other. (i.e., CI leads to a description of L; L leads to an implication for CI). The first case involved an argument in support of the knowledge of focus; the second described a result that occurred because of the knowledge of focus.

During the coding process, one episode was not coded into one of the four categories described above. In this episode, Amanda used both her knowledge of learners and her knowledge of curriculum and instructional strategies, but did not articulate a relationship between the two aspects of knowledge. In addition, a single episode was coded into two categories. In this episode, Amanda focused on curriculum and instructional strategies and used her knowledge of learners to support her instructional decision and to describe how the students reacted to the lesson. The following episodes illustrate the four different ways in which Amanda integrated her knowledge of learners with her knowledge of curriculum and instructional strategies.

Example 1: Knowledge of learners led to an implication about instruction. The following episode illustrates how Amanda's knowledge of learners led to an implication regarding curriculum and instructional strategies. In this episode, Amanda described a situation that occurred in her classroom. The class had investigated the "Tower of 3" problem (see Appendix J) and translated the pattern into a table, similar to the table provided in Figure 4.9. After completing the table, Amanda asked students to look for patterns within the table so they would create generalizations. She stated earlier that she expected them to see the recursive pattern within the "total so far" column that increased by 4 each step. After the lesson, Amanda stated her surprise with the variety of connections that students identified. This observation led to an implication about her instruction.

L

Amanda: I did think it was interesting that some of them were making connections in this way versus across, they were all making such different connections and I think that just showed me how different as a teacher they can learn, their minds are going on a different track, and it just reinforced to me that to go up there and for me to show them my strategy for looking at the table I have blocked half my class from figuring it out and understanding it because they are not thinking the way I think, so I have only caught the kids that think the way I do. That is what I thought and that is what I continue to think is so neat about TERC is that you have the opportunity through me questioning them to bring out all the different aspects and talk about it together.

CI

(Amanda, Stimulated Recall 1, 107-116)

In this episode, Amanda used her knowledge of learners to describe the variety of strategies her students used to find patterns within the table. She stated, "they were making connections in this way versus across, they were all making such different connections." She recognized that her students were not only looking across the columns of the table for patterns; they were also looking down the columns. This observation reinforced the importance of giving students time to investigate and discover mathematics (Instructional implication). Amanda declared that if she had not given them time to investigate, and instead shown them her strategy, it would have been detrimental to their understanding. She stated that it would have, "blocked half the class from figuring it out and understanding it because they are not thinking the way I think." This example, illustrates how a discussion focused on student thinking (L) led to an implication about instruction (CI). The next example provides an additional example of this type of interaction.

Example 2: Knowledge of learners led to an implication about instruction. The following episode is an additional example where Amanda's knowledge of learners led to an implication about instruction. In this episode, Amanda described an occurrence in class where two students completed a table. One student mistakenly placed the 'total so far' values in the 'step size' column. The mistake was identified and corrected by his partner. On reflecting on this episode, Amanda made a statement regarding the benefits of working in groups.

L

> *Amanda:* One of them started off counting the wrong amount, they had the wrong amount in the table. Oh, I remember what it was. The thing that threw it off was the partner who was filling it in, he had it right, but the partner filling it in was writing the total so far into the step size column. So that did come up again, I remember that now, it was quickly remedied by the partner seeing it, which reinforces to me the whole idea that working with partners helps because they get somebody to say, hopefully, wait that is not right as where I feel like that child could go on forever if they were working individually before I got there. What if it takes awhile before I get around to that person, you are doing the wrong thing and it is already so set in their mind.

CI

(Amanda, Stimulated Recall 3, 552-562)

In this episode, Amanda used her knowledge of learners to describe a situation where two students filled in a table. As the students worked, a miscommunication occurred between the pair. As one student gave numbers to be placed into the table, the second recorded those numbers in the wrong column. The first student recognized this mistake. Amanda used her knowledge of curriculum and instructional strategies to make an implication from this observation. She stated, “this reinforces to me the whole idea that working with partners helps because they get somebody to say, hopefully, wait this is not right.” This example illustrates how a discussion focused on student thinking (L) led to a statement about instruction (CI).

Example 3: Discussion of learners explained by knowledge of curriculum/instructional strategies. The following episode illustrates how Amanda’s knowledge of learners was explained, or situated, by her knowledge of curriculum and instructional strategies. In this episode, Amanda commented on a student who participated more than usual during the lesson that morning. When asked why this occurred, Amanda stated that it was due to the situated nature of the problem within a context and that this student was given concrete materials to model the situation. Amanda commented on the role manipulatives play in providing opportunities for her students to have success.

L *Amanda:* I was surprised that Kate had raised her hand and was giving answers. She usually does not participate. It was like a reversal.

Researcher: Why do you think she was participating?

CI *Amanda*: I think it is because we are using a context, we are using blocks, we are showing it, we are showing it in two different formats so we are including two totally different types of thinkers. When I was looking at how I would teach it I am looking at the blocks in the book. I did not need the actual blocks, but I needed to look, I did draw the picture and see it for myself. I think that is a problem, a lot of times we have eliminated most of the manipulatives, most of the time we don't spend time looking at that stuff and that is where some of these kids still are.

(Amanda Stimulated Recall 1, lines 442-458)

In this episode, Amanda used her knowledge of curriculum and instructional strategies to explain why a student participated in class. Initially, Amanda used her knowledge of learners to describe what occurred during class, "I was surprised that Kate raised her hand and was giving answers, she usually does not participate." Amanda used her knowledge of curriculum and instructional strategies to provide an explanation for this increase in participation. She stated, "I think it was because we are using context, we are using blocks, we are showing it in two different formats so we are including two totally different types of thinkers." This episode illustrates how Amanda used her knowledge of curriculum and instructional strategies to explain the participation of a student.

Example 4: Discussion of learners explained by knowledge of curriculum and instructional strategies. The following episode is an additional example of how Amanda used her knowledge of curriculum and instructional strategies to justify student thinking (knowledge of learners). In this episode, Amanda described a student, George, who developed an incorrect rule for the double staircase problem (see Appendix G). George thought that if he added across the columns in the table he would obtain the correct number of steps (i.e., step number plus step size equals total so far). In a prior lesson, the class worked the "Tower of 3" problem (see Appendix J) and developed a "multiply by 3" rule in which they multiplied the step number by the step size (always 3) to determine the "total so far." In the case of the "Tower of 3" problem this strategy was appropriate. Amanda believed this prior strategy impacted George's thinking.

L { *Amanda*: I do think that when George says that and he is looking at the step sizes, or the number of steps, and he adds across, } that obviously came from yesterday. I think it came from the day before where you could multiply across. } CI

(Amanda, Stimulated Recall 2, 448-450)

Amanda used her knowledge of learners to describe how George used the table to find a rule for the double staircase problem. She stated, "he is looking at the step sizes, or the number of steps, and he adds across" referring to his strategy of adding the step number plus the step size to obtain the total so far. Next, she used her knowledge of curriculum and instructional strategies to provide an explanation for this strategy. Amanda attributed George's incorrect strategy to the previous lesson where students developed a correct rule by looking across the columns of the table. Amanda was able to recognize an attribute of the instruction from the prior lesson that may have impacted George's thinking. In this episode, her knowledge of curriculum and instructional strategies helped her explain why George employed this strategy.

Example 5: Discussion of curriculum and instructional strategy, knowledge of learners used to describe student reaction. The following episode illustrates how Amanda used her knowledge of learners to describe student reactions to a curriculum and instructional decision. In this example, she felt pressure to increase the pace of her lesson so that students would not get bored (this was her 3rd lesson on algebraic generalization). However, upon reflecting on the lesson she acknowledged that this increased pace might have led to student misunderstandings.

CI — *Amanda:* I think on this one, I just wanted to move on … because I was feeling like, "we have got to get through this, we are still on lesson 1" and I want to get to where they are doing this on their own before they get bored out of their minds, but I have got to show them graphing, so I didn't spend the time to question and involve them all and so I think they left with a lot of misunderstandings, not misunderstandings completely, but not that full understanding that Jeff has and we didn't explore it because I felt like I didn't have time and I think that is a huge difference from how I taught the first day to how I am teaching in this lesson. — L — CI

(Amanda, Stimulated Recall 3, 431-439)

Amanda described an instructional decision to increase the pace of instruction. She stated, " I just want to move on … we have got to get through this before they get bored out of their minds." In order to increase the pace of instruction Amanda stated that she "didn't spend time to question and involve them all." This instructional decision impacted students, leading her to conclude that, "I think they left with a lot of misunderstandings." This example illustrates how Amanda used her knowledge of curriculum/instructional strategies to discuss an instructional decision and then used her knowledge of learners to describe the impact this decision had on her students. The next episode provides an additional example of this type of interaction.

Example 6: Discussion of curriculum/instructional strategy, knowledge of learners used to describe student reaction. The following episode is an additional example of how Amanda used her knowledge of learners to describe how students reacted to an instructional decision. In this example, Amanda discussed how she addressed a misconception from the day before. In that lesson, the students worked the double stair case problem (see Appendix G) and had difficulties distinguishing between a pattern that increased by two each step and one whose difference increased by two each step. The lesson ended with Brian clearly articulating the difference between the two ideas. At the beginning of this lesson, Amanda chose not to go on and graph the pattern as she had originally planned, but instead returned to building the double staircase with blocks and disucssed Brian's argument one more time.

CI

> *Amanda:* I don't think if I would have switched and started with a graph it would have helped. I think when Brian made that point the day before, and I came back to it when I drew it up there again, it allowed them to see it again and it was like a fresh start and so I do think that influenced their understanding, you know what I am saying, it helped them to see it better. I do think they saw it much more clearly today and there was very little confusion.

L

(Amanda, Stimulated Recall 2, 132-137)

Amanda described an instructional decision to not move forward with graphing the pattern, but instead returned to building the pattern with blocks and went over an argument presented by Brian the previous day. Amanda used her knowledge of learners to describe her students' reactions to this instructional decision. She stated, "It helped them see it better, I think they saw it much more clearly today and there was very little confusion.” This episode illustrates how Amanda used knowledge of learners to describe how students reacted to an instructional decision.

Example 7: Decision or claim regarding curriculum and instructional strategies justified by knowledge of learners. The following episode illustrates how Amanda used knowledge of learners (L) to justify an instructional decision (CI). During the second lesson, Amanda gave her students time to draw their own patterns and to create tables that represented these patterns. Several questions arose during the lesson: What does it mean to be a pattern? Can I change the pattern? Does it always have to increase? Amanda believed it was important to provide her students the opportunity to explore these ideas and used this time to work individually with students. Amanda justified her instructional decision by providing anecdotal evidence that her students were thinking about these ideas on their own.

CI

Amanda: On this part I thought that I would just try to get them to draw it, I wasn't going to tell them if they were right or wrong, I might ask them questions to try to get them there, but I wasn't going to bring everybody together and say, "how did you do this, okay, you are right, you are wrong." I wasn't even going to point that out as a whole class and I knew that and so I kind of looked at it as more of an experience and I looked at it as an opportunity to let these kids go further, lets really think about what you are doing, most of the kids were thinking a lot about it on their own, that was happening and, so I felt like it wasn't a time where I had to make it solid for these kids who weren't getting it. It was a time for them to just have an experience, I don't even mean because of this situation, I mean in this curriculum it is a point where they just get an experience with it. I am trying it out for the first time and I don't think it would have helped for me to push them much further.

L

CI

(Amanda, Stimulated Recall 2, 343-356)

In this episode, Amanda used her knowledge of curriculum and instructional strategies to describe her decision to provide students time to build and explore patterns. She stated, "I kind of looked at it as more of an experience and I looked at it as an opportunity for these kids to go further." She used knowledge of learners to defend her decision noting the amount of thinking the activity stimulated, explaining "most of the kids were thinking a lot about it on their own, that was happening … it was a time for them to just have an experience." This episode illustrates how Amanda used knowledge of learners to justify an instructional decision. The next episode provides an additional example of this type of interaction.

Example 8: Decision or claim regarding curriculum/instructional strategies was justified by knowledge of learners. The following episode is an additional example of how Amanda used knowledge of learners to support an instructional idea. In this example, Amanda wanted see how her students would tackle a more difficult task. To justify this goal, she provided evidence that her students were capable of high-level thinking. She described an event where two students debated the validity of their rules for the straw problem (see Appendix I). One student developed the rule $3n+2$ and the other developed the rule $3n+1$. Both students took 3 times n because they realized that each additional square required 3 additional straws. The debate was whether 1 or 2 extra straws were needed to close the pattern, see Figure 4.14 for an illustration of the $3n+1$ rule.

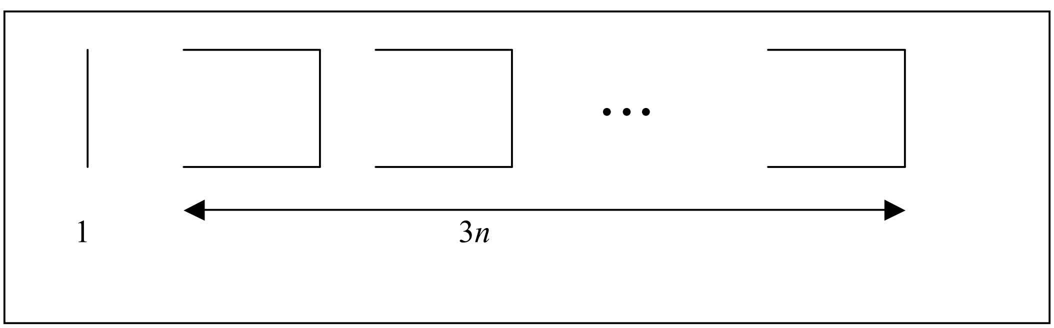


Figure 4.14: Representation of $3n+1$ rule

In the end, the student who developed the $3n+1$ ruled convinced the other student he was correct. Amanda then provided some instructional strategies that would bring out higher order thinking.

CI *Amanda:* I would love to see how their minds would take it to a harder task. I know I sent it home for homework and it was interesting because two of the kids came up to me and said, with the straw pattern, one said I proved the other one wrong because he was adding the last straw on again and it was already added on, I can't remember exactly what he was doing, but he was off, he had too many and then the other kid proved it to him. L

CI So I would really like to see them work those problems and be able to ask them questions like, "how what made you decide to do this?" see if they would use a table, can they draw it and see how they could generalize it to other problems besides building blocks.

(Amanda, Stimulated Recall 3, 799-808)

In this episode, Amanda used her knowledge of curriculum and instructional strategies to state a goal, "I would love to see how their mind would take to a harder task." To justify this goal Amanda used knowledge of learners to provide evidence that students were capable of high-level thinking involving patterns. She described a situation where two students shared their rules and were able to reconcile their differences. This episode illustrates how Amanda used knowledge of learners to support an instructional goal. The next section explores how Emily used her knowledge of learners and her knowledge of curriculum and instructional strategies simultaneously.

Emily

Table 4.15 lists the episodes where Emily used her knowledge of learners in combination with her knowledge of curriculum/instructional strategies. A description of how these knowledge types were integrated is included.

Episode (line #s)	Type of Interaction
Emily Initial (104-114)	Focus on CI – L used to describe student reaction
Emily Initial (192-208)	Focus on CI – L used to support CI decision/claim
Emily Initial (237-252)	Focus on CI – L used to describe student reaction
Emily Initial (283-297)	Focus on L – Implication for CI
Emily SR1 (5-53)	Focus on L – Implication for CI
Emily SR1 (92-95)	Focus on CI – L used to describe student reaction
Emily SR1 (185-212)	Focus on CI – L used to describe student reaction
Emily SR1 (256-283)	Focus on L – CI used to explain/situate student thinking
Emily SR1 (491-511)	Focus on L – CI used to explain/situate student thinking
Emily SR1 (519-525)	Focus on L – Implication for CI – L used to support CI decision/claim
Emily SR1 (560-572)	Focus on L – Implication for CI
Emily SR1 (576-589)	Focus on CI – L used to describe student reaction
Emily SR1 (591-639)	Focus on L – Implications for CI
Emily SR1 (678-696)	None
Emily SR2 (144-166)	Focus on CI – L used to support CI decision/claim
Emily SR2 (183-196)	Focus on L – CI used to explain/situate student thinking
Emily SR2 (214-228)	Focus on CI – L used to support CI decision/claim – L used to describe student reaction
Emily SR2 (229-251)	Focus on CI – L used to support CI decision/claim
Emily SR2 (274-285)	Focus on CI – L used to describe student reaction – Implications for CI
Emily SR2 (296-323)	Focus on CI – L used to support CI decision/claim
Emily SR2 (416-431)	Focus on L – Implications for CI
Emily SR2 (438-480)	Focus on CI – L used to support CI decision/claim
Emily SR2 (489-508)	Focus on L – Implications for CI
Emily SR2 (549-594)	Focus on CI – L used to support CI decision/claim
Emily SR2 (608-625)	Focus on L – CI used to explain/situate student thinking
Emily SR2 (627-635)	Focus on CI – L used to support CI decision/claim
Emily SR2 (663-667)	Focus on L – CI used to explain/situate student thinking
Emily SR2 (681-704)	Focus on CI – L used to describe student reaction
Emily SR2 (720-748)	Focus on CI – L used to describe student reaction
Emily SR2 (824-863)	Focus on L – Implications for CI
Emily SR3 (62-88)	Focus on L – CI used to explain/situate student thinking

Emily SR3 (151-170)	Focus on L – CI used to explain/situate student thinking
Emily SR3 (225-229)	Focus on L – Implications for CI
Emily SR3 (334-384)	Focus on L – CI used to explain/situate student thinking
Emily SR3 (386-410)	Focus on CI – L used to support CI decision/claim
Emily SR3 (528-552)	Focus on CI – L used to describe student reaction and to support CI decision/claim
Emily SR3 (1014-1066)	Focus on CI – L used to support CI decision/claim
Emily SR3 (1155-1182)	Focus on L – CI used to explain/situate student thinking

Table 4.15: Emily's knowledge of learners and curriculum/instructional strategies

In 38 episodes Emily combined her knowledge of learners with her knowledge of curriculum and instructional strategies. Of these 38 episodes, 18 focused on her knowledge of learners, with the knowledge of curriculum and instructional strategies used to support her knowledge of learners. Two distinct ways occurred in which Emily's knowledge of curriculum and instructional strategies supported her knowledge of learners: (a) Nine episodes involved a situation where she described student thinking that occurred in her classroom (knowledge of learners) and then stated an implication this had for instruction; and (b) Nine other episodes involved Emily describing student thinking that occurred in her classroom and using her knowledge of curriculum and instructional strategies to provide an explanation for the student thinking that was observed.

In addition, 22 episodes involved conversations where Emily focused on her knowledge of curriculum and instructional strategies, with her knowledge of learners used to support her knowledge of curriculum and instructional strategies. This occurred in two distinct ways: (a) Ten episodes involved a situation where Emily described an instructional issue (CI) and used her knowledge of learners (L) to explain student reactions, or anticipated reactions, to the instruction being discussed; and (b) Twelve episodes that involved Emily using her knowledge of learners (L) to justify an instructional claim or decision (CI).

One episode was not coded into any of the four categories. This episode included both the knowledge of learners and the knowledge of curriculum and instructional strategies, but did not contain a clear connection between these two components of PCK. In addition, four episodes included multiple categories. For example, one case described a curriculum and instructional decision (CI), which led to a description of student reaction to the instruction (L), which in turn led to implications about future instruction (CI). The following episodes provide a sample of the four categories described above.

Example 1: Knowledge of learners led to an implication about curriculum and instructional strategies. The following episode illustrates how Emily's use of her knowledge of learners led to an implication for curriculum and instruction. In this example, Emily described two students who had generated a formula for the double staircase problem (see Appendix G). The students developed the rule step number times steps size minus the total so far from the previous step (Step Number * Step Size – Previous Total so Far). Figure 4.16 depicts the table used for this problem. For example, in the second step of the double staircase problem the step number was 2, the step size was 4, and the total so far was 2. Plugging these values into the student's rule results in 2 * 4 – 2 = 6, which is the correct total so far for the second step.

Step Number	Step Size	Total so Far
1	2	(2)
(2)	(4)	**6**
3	6	12
4	8	20
5	10	30
6	12	42
7	14	56

Figure 4.16: Table for Double Staircase Problem

This formula seemed to surprise Emily and made her reconsider asking her students about the 20th step so quickly and to question whether she wanted to get into a discussion about formulas at this point in the lesson.

L

Emily: Okay, First, what is interesting is I went over to talk to Dan (another 5th grade teacher at Pinewood Hills elementary school) as soon as I was done because my mind was racing over that last, the double staircase problem, and the boys who had come up with a formula. I mean they had come up with multiply the …

Researcher: Step number times new tiles minus total so far.

Emily: Yes, Yes, What the heck is that? And I thought, Okay, is this where we are going to go with this (a discussion of formulas) and then I thought, I think my question wasn't a good one, to ask about the 20th step, I think I went to 20 too fast and so what I am thinking is that tomorrow when I go in I am going to totally back track and I am going to go back to that table and look at it again, it is going to kill me not to teach this tomorrow, you have no idea. CI

(Emily, Stimulated Recall 1, 8-20)

Emily used her knowledge of learners to describe the rule presented by the two boys. The formula appeared to surprise Emily and caused her to question whether her class was ready for such a discussion. She used her knowledge of curriculum and instructional strategies to critique her instruction, "I think my question wasn't a good one, to ask about the 20th step, I think I went to 20 too fast." Emily then stated an implication this had for future instruction, "I think tomorrow I am going to back track … go back to that table and look at it again." This episode illustrates how Emily's use of her knowledge of learners led to implications concerning curriculum and instructional strategies. The next episode provides an additional example of this type of interaction.

Example 2: Knowledge of learners led to an implication about curriculum/instructional strategies. The following episode is an additional example where Emily's use of her knowledge of learners led to an implication about instruction. In this episode, Emily commented on a student who had worked the Tower of Three

Problem (see Appendix J), developed a table for the pattern, and inserted values for the first four steps. An illustration of this table is provided in Figure 4.17.

Step Number	Step Size	Total so Far
1	3	3
2	3	6
3	3	9
4	3	12
5		
6		
7		

Figure 4.17: Table for Tower of Three Problem

When asked to calculate the 20th step, the student took the value for the 4th step (12) and multiplied it by 20, obtaining the value 240. The student appeared to treat the first four steps as a single unit, but misapplied how this assumption could be used to obtain the answer. In order for this strategy to be applied correctly the student must recognize the 5 groups of 4 in 20. Hence, to calculate the value for the 20th step, the student should have multiplied the value of the 4th step (12) by 5. Although this strategy can be applied correctly to this pattern, it cannot be directly applied to all linear situations. When Emily reflected on this strategy, she stated an adjustment that should have been made to her instruction.

L *Emily*: What she was thinking was pretty smart, because she was trying to make it an easier problem, she was going to take what she already knew, there are 12 tiles for the 4th step, but she got the total mixed up with the …

Researcher: So she multiplied by 20, and she shouldn't have, she should have multiplied by something else.

L *Emily*: Right, 5, which is what she came up with because she was looking at the first 4 steps as one step and it wasn't and I should have said that, I should have said that it wasn't one step. CI

(Emily, Stimulated Recall 1, lines 629-639)

Emily used her knowledge of learners to analyze and describe a student's solution to the Tower of Three Problem. She stated the misunderstanding the student made, "she was looking at that (first 4 steps) as one step and it wasn't" and correctly identified the adjustment; the student should have multiplied the value of the fourth step by 5, instead of 20. Once Emily understood the student's mistake, she used her knowledge of curriculum/instructional strategies to critique her instruction, "I should have said that it wasn't one step." This comment on instruction was an implication of Emily's understanding of the student's misconception. This example illustrates how Emily's use of her knowledge of learners led to an implication for instruction.

Example 1: Statement about learners explained using knowledge of curriculum/instructional strategies. The following episode illustrates how Emily's knowledge of learners was explained, or situated, by her knowledge of curriculum and instructional strategies. In this episode, Emily's students worked the Tower of 3 Problem and created a table, see Figure 4.17. Using the table, the class generated the rule (step number) X (step size) = (Total so Far). For example, in the fourth step, the step number is 4, the step size is 3, and the total so far is 12, which is 4 times 3. In addition, the students discovered a recursive pattern within the total so far column, that the total increased by 3 each step. At this point, I asked Emily if her students understood the connection between multiplication (step number X 3) and repeated addition (the recursive rule of plus 3). Emily described the student thinking related to this issue and made a hypothesis as to why her students were thinking in this manner.

Researcher: Are they seeing that connection?

L { *Emily:* Repeated addition, they should, just because of everything we have done up to this point. They understand that multiplication is repeated addition and I would think 8 times 3 was a known fact. }

Researcher: The person who found that 8 times 3 is 24, what cued her to do that?

L { *Emily:* Well, I think that they picked up the pattern, that is was 1 times 3, 2 times 3, } and I think even the way that it was drawn on, with me numbering those rows, you could see 1 times 3. } CI

(Emily, Stimulated Recall 1, 489-505)

In response to my question, Emily used her knowledge of learners to describe the students' strategies and stated, "they understand that multiplication is repeated addition." However, it is unclear whether she believed her students had made a connection between the two rules that were generated. Following the discussion of student strategies, Emily used her knowledge of curriculum/instructional strategies to give a hypothesis as to where the explicit rule came from, "I think the way that it was drawn …. you could see 1 times 3." Emily believed that the organization of the table--having the columns step number, step size, and total so far--led to the explicit rule. This example illustrates how Emily used her knowledge of curriculum/instructional strategies to explain a rule generated by a student.

Example 2: Discussion of learners explained by knowledge of curriculum/instructional strategies. The following episode is an additional example of how Emily used her knowledge of curriculum and instructional strategies to explain student thinking (knowledge of learners). In this episode, Emily discussed the transition students made from representing patterns with tables to representing them with graphs. She expected students to struggle with this transition and was surprised when this did not occur.

L { *Emily:* I did not expect them to get the graphs that easy, as easy as they did. I thought that would be a huge thing, I thought that would be very difficult, } and I don't know if it was because of the work we did with coordinate graphs, or coordinate grids, I don't know, I didn't see it going that quick. } CI

L { I was really kind of surprised about it. I though it would be more of a problem. }

(Emily, Stimulated Recall 2, 663-667)

Emily used her knowledge of learners to describe her students' reaction to the lesson on graphs, "I did not expect them to get the graphs that easy." She then used her knowledge of curriculum/instructional strategies to suggest an explanation for this unexpected result. Emily stated, "I don't know if it was because of the work we did with coordinate graphs, or coordinate grids, I don't know, I didn't see it going that quick." This example illustrates how Emily used her knowledge of curriculum/instructional strategies to provide a potential explanation for student thinking.

Example 1: Discussion of curriculum/instructional strategy, knowledge of learners used to describe student reaction. The following example illustrates how Emily used her knowledge of learners to predict how her students would react to instruction. In this example, Emily described what she deemed to be an important goal for a unit on patterns. She wanted her students to move from solving patterns recursively (adding on to a previous value) to developing an explicit formula.

> *Researcher*: Preparing for this chapter on patterns what ideas do you think are important for 5^{th} graders to take out of this unit?

CI *Emily*: I think the most important thing is that … what I want to see them do is to not just keep adding on, for example in the first lesson, find a pattern and then begin to think about a formula and how to plug numbers in and see how that works. I think that probably more than half of my class will do that. I think I will have 5, 4 or 5, that are just; they are going to keep adding on. If I can move them just a little bit, that would be my goal for them. Whereas with those higher-level kids trying to push them even further, but I think everybody is going to be at a different place. L CI

(Emily, Initial Interview, lines 104-114)

Emily used her knowledge of curriculum/instructional strategies to describe her goal of encouraging students to generalize and develop a formula (explicit rule). Following the statement of her goal, she provided a prediction that "more than half of my class will do that" with only 4 or 5 continuing to think recursively. By including a statement concerning student thinking, that most of her students would be able to develop a formula, Emily provided credence to her instructional goal. Although her statements are focused on instruction, Emily used knowledge of learners to support her belief that the goal would be met. The next episode provides an additional example of this type of interaction.

Example 2: Discussion of curriculum/instructional strategy, knowledge of learners used to describe student reaction. The following episode is an additional example where Emily used her knowledge of learners to describe her students' reaction to instruction. In this example, Emily talked about a discussion she had with Amanda regarding the difficulties students were having with the double staircase problem (see Appendix G.) From this conversation, Emily anticipated that her students would struggle with the task as well; the following episode revealed her surprise when they did not.

CI { *Emily:* I felt like it moved a lot faster than I thought it was going to, because after talking to Amanda I thought this was going to take awhile.

And they were totally like, Nope, we are done with that, lets move on to the next thing and I think that was my biggest shock, that it was that quick. } L

(Emily, Stimulated Recall 1, 92-95)

In this example, Emily described the implementation of the double staircase problem. She stated, "I felt like it moved a lot faster that I thought it was going to" referring to her expectation that her students would struggle. Following this, Emily used her knowledge of learners to describe the students reaction, "They were like, Nope, we are done with that, lets move on to the next thing," which surprised Emily. This example illustrates how Emily used her knowledge of learners to adjust her instruction due to the impact on student thinking.

Example 1: Decision or claim regarding curriculum/instructional strategies justified by knowledge of learners. The following episode illustrates how Emily used her knowledge of learners (L) to justify an instructional decision (CI). In this example, Emily discussed her use of manipulatives. She stated that manipulatives provided opportunities for her lower-end students to succeed, but they also created the instructional dilemma of challenging the high-end students who quickly saw patterns while using the manipulatives. Emily supported her instructional claim by providing examples of student thinking.

Researcher: Do you think the visual representation (manipulatives) brought out different types of thinking?

CI *Emily:* Absolutely, it is huge, especially for those that are lower on that continuum. For some of my kids, I could draw out this table and they would see it, I would probably have 5 or 6 in here and you will be able to pick them out, who will just go, "Oh well, you are multiplying, you are multiplying the step number times the new tiles to get it every single L time." That is what I am worried about with them today, I am worried about those high kids shutting down because they figure stuff out so quickly and then they kind of go mellow, "we get this, lets move on." And so every day I worry about that and a lot of times I send those kids off to do something else, they will go back to their table, that is that CI differentiating, which I think is just as important as meeting those lower kids and I think so many times those higher kids get left, we just go, "well you get it," so sit there and pick your nose and let us teach these other kids and that is not fair; and you can do it with minimal planning because usually, not always, but usually those higher level kids are pretty independent workers as well so you can put them back at a table with higher, some higher stuff and let them work. I won't do that today, just because I am not really sure, maybe I am wrong about what they are going to do because they may totally take with it. } L } CI

(Emily, Initial Interview, 192-208)

Emily described the benefits of using manipulatives. She stated, "It is huge for those that are lower on that continuum. For some of those kids, I could draw out this table and they would see it." Emily discussed the challenges that using manipulatives created, she used her knowledge of learners to describe how some of her high-level students might react to the blocks, "I would probably have 5 or 6 in here ... who will just go, oh well, you are multiplying the step number times the new tiles to get it every single time … we get this, lets move on." The fact that several of her high-end students would catch on quickly caused an instructional dilemma, which required Emily to return to her knowledge of curriculum/instructional strategies. Emily stated, "That is what I am worried about with them … I think so many times those higher kids get left out, we just go, well you get it, so sit there and pick your nose and let us teach the other kids and that is not fair." Emily then provided a solution; "you can put them at a table with some higher stuff and let them work." She supported the feasibility of her solution by using her knowledge of learners to state the characteristics of that group of students, "those higher level kids are pretty independent." This excerpt illustrates how Emily used her knowledge of learners to justify the claim that manipulatives are beneficial.

Example 2: Decision/claim regarding curriculum/instructional strategy supported by knowledge of learners. The following episode is an additional example where Emily used her knowledge of learners to support an instructional decision. In this episode, Emily decided to postpone the introduction of graphs until after her students had more experience generating patterns and tables.

CI

> *Emily:* I did not plan that, it was like this is where we need to go because I didn't know if they would get it (graphs), I wanted them to see the (L) connections between the graph and the table, but I thought if I tried to talk about this now they weren't going to be able to understand what I was trying to say. (L) So I thought it was important for them to have a couple of different experiences before I asked them to graph it so I was trying to get them to think about what it was going to look like. A lot of them were doing that. (L)

(Emily, Stimulated Recall 2, 559-566)

Emily used her knowledge of curriculum/instructional strategies to make a decision regarding graphs, "I thought it was important for them to have a couple of different experiences before I asked them to graph it." Emily used her knowledge of learners to support this statement. She was concerned that if she introduced graphs at that moment her students would not understand. Emily stated, "I didn't know if they would get it," and "If I tried to talk about this (graphs) now they weren't going to be able to understand." Thus, Emily made an instructional decision based upon an anticipated response from her students.

Comparing the Knowledge of Learners and Knowledge of C & I for Amanda and Emily

Amanda and Emily integrated their knowledge of learners and their knowledge of curriculum/instructional strategies in similar ways. Table 4.18 lists the types of interactions that occurred and how often Amanda and Emily used them.

Type of Interaction	Amanda	Emily	Total
Focus on L – Implications for CI	12	9	21
Focus on L – CI used to explain/situate student thinking	10	9	19
Total episodes focused on L	**22**	**18**	**40**
Focus on CI – L used to describe student reaction	5	10	15
Focus on CI – L used to support CI decision/claim	15	12	27
Total episodes focused on CI	**20**	**22**	**42**

Table 4.18: Amanda and Emily's use of L and CI combined

Amanda had an approximately equal number of episodes that focused on learners (22) as episodes that focused on curriculum and instruction (20). This is in distinct contrast to the data found for how she combined her knowledge of mathematics with her knowledge of learners. All 18 of Amanda's episodes focused on learners, none were focused on mathematics. Thus it appeared that Amanda's knowledge of learners was her primary focus when used in combination with her knowledge of mathematics. In contrast, when Amanda used her knowledge of learners in connection with her knowledge of curriculum and instructional strategies re she had discussions that were focused on both learners and instruction. Together, this portrays a picture of how Amanda used her knowledge, she made a statement and discussed issues concerning learners and curriculum/instructional strategies and then supported these statements with a combination of the other forms of knowledge.

The data add further insight into Table 4.4, which revealed that Amanda used her knowledge of learners 52.9 percent of the time and her knowledge of curriculum/instructional strategies only 35.9 percent of the time. The data in Table 4.4 implied that Amanda had a preference for using her knowledge of learners. It now appears that this difference is partly a result of their being a larger number of episodes that combine her knowledge of mathematics and learners than mathematics and curriculum/nstructional strategies, thus skewing the data. When Amanda used her knowledge of learners in connection with her knowledge of curriculum and instruction, each was the focus of conversation at about the same rate.

Emily appeared extremely flexible in how she utilized her knowledge of mathematics and her knowledge of learners, which is consistent with the flexibility she demonstrated when she combined her knowledge of mathematics and her knowledge of learners. Emily had almost equal numbers of episodes in all four categories of knowledge interaction identified. Similarly, Amanda also appeared flexible in how she connected these two forms of knowledge, although this flexibility was not evident when Amanda combined her knowledge of mathematics with her knowledge of learners. The fact that all four ways of combining these two forms of knowledge were prevalent for both Amanda and Emily, with such regularity, suggests that the integration of their knowledge of mathematics with their knowledge of curriculum and instructional strategies was important and useful to both teachers.

One difference did exist between how Amanda and Emily integrated their knowledge of learners and their knowledge of curriculum/instructional strategies; Amanda did not use her knowledge of learners to describe student reactions to an instructional decision as often as she used the other three types of interactions.

Knowledge of Mathematics and the Knowledge of Curriculum/Instructional Strategies

Amanda and Emily used their knowledge of mathematics in combination with their knowledge of curriculum and instructional strategies during 10 episodes. Of these 10 episodes, 3 excerpts were from Amanda and 7 from Emily. The most common type of interaction (8 of the 10 total episodes) was the use of their knowledge of mathematics to support and clarify a statement involving curriculum and instruction. The next two sections provide further detail as to how Amanda and Emily used their knowledge of mathematics in connection with their knowledge of curriculum/instructional strategies.

Amanda

Table 4.19 lists the episodes where Amanda used her knowledge of mathematics in connection with her knowledge of curriculum and instructional strategies. A description of how these knowledge types were integrated is included.

Episode (line #s)	Type of Interaction
Amanda Initial (131-137)	M used to support/clarify CI
Amanda SR1 (576-581)	M used to support/clarify CI
Amanda SR3 (150-152)	M used to support/clarify CI

Table 4.19: Amanda's knowledge of mathematics and curriculum/instructional strategies

Amanda combined her knowledge of mathematics with her knowledge of curriculum/instructional strategies during three episodes. In all three episodes Amanda used her knowledge of mathematics to clarify and expand on the instructional ideas discussed. The following example illustrates how Amanda integrated these two aspects of her PCK.

Example 1: Knowledge of mathematics used to support/clarify instruction. The following episode is an example where Amanda used her knowledge of mathematics to clarify and a support a discussion about curriculum and instruction. In this example, a group of students had developed two rules for a pattern and were debating which rule was correct. In reaction to this, Amanda suggested the possibility that both were correct.

CI

> *Amanda:* I don't remember saying that, but hearing myself say that, I think part of it was the whole idea of seeing if they both work, are we going to get to the same answer, because I want to get them out of my strategy works your strategy doesn't work and go more towards they both work, which one is more efficient, rather than mine works yours doesn't. This idea of efficiency is important.

M

(Amanda, Stimulated Recall 1, 576-581)

Amanda discussed a philosophy she had about group work--that developing understanding is more important than who is right. She stated, "I want to get them out of my strategy works your strategy doesn't work" and to investigate whether both rules could be correct. Amanda used her knowledge of mathematics to state that encouraging the exploration of multiple solution paths could lead to a discussion of efficiency, which she deemed an important aspect of mathematics. The mathematical idea of efficiency was used to support her instructional philosophy.

Emily

Table 4.20 lists the episodes where Emily used her knowledge of mathematics in connection with her knowledge of curriculum and instructional strategies. A description of how these knowledge types were integrated is included.

Episode (line #s)	Type of Interaction
Emily Initial (90-99)	Discussion of M has implications for CI
Emily Initial (165-179)	M used to support/clarify CI
Emily Initial (214-232)	M used to support/clarify CI
Emily SR2 (59-80)	Discussion of M has implications for CI
Emily SR2 (510-544)	M used to support/clarify CI
Emily SR2 (865-899)	M used to support/clarify CI
Emily SR3 (419-439)	M used to support/clarify CI

Table 4.20: Emily's knowledge of mathematics and curriculum/instructional strategies

Emily combined her knowledge of mathematics with her knowledge of curriculum/instructional strategies during seven episodes. Of these episodes, five involved Emily using her knowledge of mathematics to clarify and support a statement about curriculum and instruction. The two remaining episodes included a discussion of a mathematical issue that resulted in an instructional implication. The following episodes provide examples of the two categories described above.

Example 1: Knowledge of mathematics used to support/clarify Instruction. The following episode is an example where Emily used her knowledge of mathematics to clarify and support a discussion about curriculum and instruction. In this example, Emily described her instructional goal of pushing students to understand the meaning of pattern.

During the explanation of this goal, Emily brought out a mathematical issue to further refine her goal.

CI

Emily: I think the goal is for them to see a pattern, to understand how it builds and what is happening. I think that is what I want them to be able to do, that you can create something and do it in such a manner that you can figure out what is going to happen next no matter haw many steps you go forward, predicting later steps in the patterns. I don't think we will get to creating their own patterns that change in a regular way; I don't know that we will get that far. I think that is Thursday; we will be looking more at that. Really, I think it is even more than predicting you are implying that it is up for interpretation, like it could or couldn't happen, I think in math, especially this, you should be able to know this is what is going to happen at the 20th step or the 100th step. So that is what I am thinking.

M

(Emily, Initial Interview, 168-179)

Emily described a goal for a unit on algebraic generalization, "I think the goal is for them to see a pattern … that you can create something and do it in such a manner that you can figure out what is going to happen next." She predicted how far her students would progress the first day, "I don't think we will get to creating their own patterns." Next, Emily used her knowledge of mathematics to expand on the goal for the lesson, "I think it is even more than predicting, you are implying that it is up for interpretation." In this comment, Emily suggested an additional mathematical complication of describing patterns, that the description is based upon your interpretation of how the pattern continues. In other words, some patterns do not provide enough information to be able to determine what comes next. During class, Emily stressed the importance of being able to describe a pattern so that everyone had a collective understanding of what came next.

Example 2: Knowledge of mathematics used to support/clarify Instruction. The following episode is an example where Emily used her knowledge of mathematics to support an instructional decision. In this example, Emily's class was discussing the straw problem (see Appendix X) when one of her students stated that the total number of straws was always four times the number of squares. In response to this, Emily directed the class to the table (see Figure 4.21) and asked if the total so far would ever be a multiple of four after the first step. She immediately stated, "that was not a good question." The following excerpt is Emily's reaction to this question

Step Number	Step Size	Total so Far
1	4	4
2	3	7
3	3	10
4	3	13
5	3	16
6	3	20
7	3	233

Figure 4.21: Table for the Straw Problem

CI *Emily:* Well because I was going to ask if that (the total so far) was ever going to be a multiple of 4. Is that total ever going to be a multiple of 4, but it will be eventually so I was like, I couldn't say that because that is not right. M

(Emily, Stimulate Recall 3, 428-430)

In this example, Emily used her knowledge of curriculum and instructional strategies to discuss a question she asked to her class, "Is that (the total so far) ever going to be a multiple of 4?" She immediately determined that this was not a good question and used her knowledge of mathematics to support this claim. She stated, "but it will be (a multiple of 4) eventually." This example illustrates how Emily used her knowledge of mathematics to support an instructional claim.

Example 3:Discussion of mathematics led to an implication about instruction. The following episode is an example where Emily's knowledge of mathematics led to an implication involving instruction. In this example, Emily gave her students the opportunity to create their own patterns. During this time, one group asked whether a pattern could have a step size of zero. The group struggled with this concept, they felt that you should be able to have step size of zero, but could not represent such a step using their blocks. This discussion led Emily to think about, and question, the concept of pattern and the way the curriculum presented the material.

M *Emily:* But it doesn't change so is it a pattern? If you do it that way it is just one block.

Researcher: If you do it this way you never see the step because you can't draw a step of zero?

Emily: Right, you can't, and I wonder if they (Investigations) were purposeful about that or if they overlooked it. CI

(Emily, Stimulated Recall 2, 59-71)

In reaction to her students' comments, Emily used her knowledge of mathematics to question her conception of pattern, "but it doesn't change, so is it a pattern?" She referred to the physical representation, "if you do it this way it is just one block" to reinforce the idea that you can't see the step in the blocks because you can't represent a change of zero. Emily then wondered if the *Investigations* curriculum anticipated the confusion a step size of zero created. She stated, "I wonder if Investigations was purposeful about that or if they overlooked it." Emily's question led her to ponder the design of the curriculum.

Comparing Amanda and Emily's Knowledge of M and Knowledge of C & I

Amanda and Emily rarely used their knowledge of mathematics in connection with their knowledge of curriculum and instructional strategies. This pairing occurred only 10 times during the duration of the study, as compared to 80 episodes that featured the combination of their knowledge of learners and their knowledge of curriculum and instructional strategies and 32 episodes which combined their knowledge of mathematics with their knowledge of learners. Table 4.22 depicts the interactions that occurred when Amanda and Emily connected their knowledge of mathematics with their knowledge of curriculum and instructional strategies.

Type of Interaction	Amanda	Emily	Total
M used to support/clarify CI	3	5	8
Episodes focused on CI	**3**	**5**	**8**
Discussion of M has implications for CI	0	2	2
Episodes focused on M	**0**	**2**	**2**

Table 4.22: Amanda and Emily's use of M and CI

For eight of the 10 episodes, their knowledge of curriculum and instructional strategies was the primary focus of conversation. In these cases, their knowledge of mathematics was used to clarify or support an instructional decision or claim. These data suggest the relative roles their knowledge of mathematics and their knowledge of curriculum and instructional strategies played in their thinking. Amanda and Emily often devoted time to discuss curriculum or instructional issues that at times required their knowledge of mathematics to clarify or support their claims, but rarely had mathematics as their focus.

In particular, mathematics was never a primary focus for Amanda; it was always used to support her knowledge of curriculum and instructional strategies (see Table 4.22) or knowledge of learners (see table 4.12). This contrast with Emily's knowledge use, she occasionally focused on the mathematics of algebraic generalization, using her knowledge of learners (5 episodes) or knowledge of curriculum/instructional strategies (2 episodes) in support.

Similar to the previous knowledge pairings, Emily appeared more flexible in how she combined her knowledge of mathematics with her knowledge of curriculum and instructional strategies. Emily used her knowledge of mathematics in combination with her knowledge of curriculum/instructional strategies in two different ways. More specifically, Emily used her knowledge of mathematics to clarify a mathematical issue that occurred during a discussion of curriculum and instruction, as did Amanda. However, Emily also discussed mathematical issues that resulted in curricular implications.

In summary, Amanda focused on her knowledge of curriculum and instructional strategies and used her knowledge of mathematics mainly in support. Emily also used her knowledge of mathematics in support of her knowledge of curriculum and instructional strategies, but also investigated the mathematical ideas, which led to implications for instruction. Emily appeared more flexible in how she utilized her knowledge of mathematics and her knowledge of curriculum/instructional strategies. The next section explores the ways in which Amanda and Emily integrated all three aspects of pedagogical content knowledge.

Summary

Amanda and Emily connected their knowledge of mathematics, their knowledge of learners, and their knowledge of curriculum and instructional strategies in a variety of ways. In the prior three sections, I analyzed how they combined M & L, L & CI, and M & CI and described the types of interactions that occurred. A summary of the different interaction types is provided in Table 4.23.

Description of Knowledge Interactions			
	Primary Focus	Support/ Implication	Description
M & L	M	L	Discussion of M leads to implication for L
	L	M	Knowledge of L leads to exploration of M
	L	M	M used to support/explain statement about L
CI & L	CI	L	L used to describe student reaction to CI
	CI	L	L used to support CI decision/claim
	L	CI	Discussion of L leads to implications for CI
	L	CI	CI used to explain/situate student thinking
M & CI	M	CI	Discussion of M leads to implications for CI
	CI	M	M used to support/clarify statement about CI

Table 4.23: Description of knowledge interactions for M&L, L&CI, and M&CI

One pattern that existed in these interactions was that one aspect of PCK was the focus of conversation, while a second knowledge form played a supportive role. The secondary aspect supported the primary knowledge form in two distinct ways: (a) the second form of knowledge was used to support/clarify/explain what was being discussed with the primary knowledge form, and (b) a discussion that used the primary form of knowledge led to an implication or investigation using the second.

For example, Amanda often used her knowledge of learners to discuss student thinking that occurred during a lesson. At times she used her knowledge of curriculum and instructional strategies to explain why the student was thinking in the fashion being described. In this example, the secondary form of knowledge (CI) was used to support/clarify/explain the primary form of knowledge (L). In this case, the focus of conversation remained with the primary knowledge form, all statements using the second form of knowledge were connected directly back to the primary knowledge type. Figure 4.24 illustrates this notion.

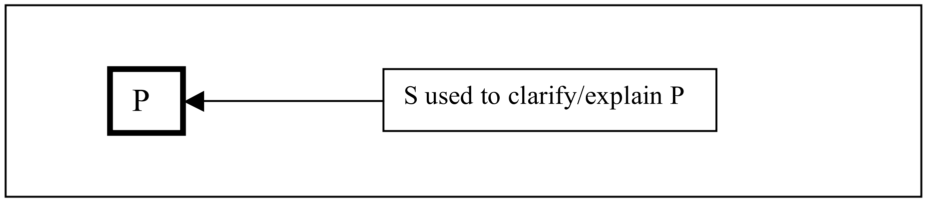


Figure 4.24: Secondary knowledge used to explain primary

In this illustration the bold box represents the primary knowledge form used in the episode. The box to the right describes a comment that used a second form of knowledge in order to support/clarify/explain the statement made using P. The arrow pointing from S towards P signifies that this piece of knowledge was used to support P and was connected directly back to the discussion about P.

In contrast, a discussion of student thinking could also lead to an implication for future instruction. (i.e., my students were thinking like this, thus I need to do this in the future.) In this case, the primary form of knowledge (L) led to an implication or investigation of a secondary form of knowledge (CI). The use of the secondary form of knowledge was different in this case. The focus of conversation momentarily shifted to the secondary form of knowledge and was not connected directly back to the primary form of knowledge. This is illustrated in Figure 4.25.

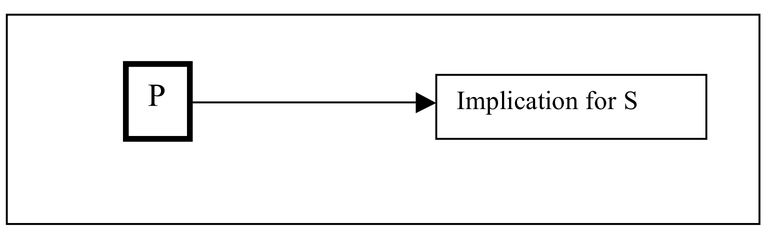


Figure 4.25: Primary leads to implication for secondary

In this illustration, the bold box with a P represents the initial, or primary, form of knowledge used in the episode. The box on the right represents a comment made using a second aspect of PCK that was initiated by the discussion of P. This comment was not used to support a claim about P, but instead was an implication, or tangent, that came about because of the initial conversation. The arrow pointing from P to S represents this shift in focus.

Given the two distinct roles of the secondary form of knowledge, I expanded Table 4.23 to include all possible combinations of the knowledge of mathematics, learners, and curriculum/instructional strategies; this is presented in Table 4.26. The arrows depict the two different roles of the secondary form of knowledge.

Use of Knowledge Combinations				
	Primary Focus		Support/ Implication	
M & L	M	→	L	Discussion of M leads to implication for L
	M	←	L	L used to support/explain statement about M
	L	→	M	Discussion of L leads to exploration of M
	L	←	M	M used to support/explain statement about L
L & CI	CI	→	L	L used to describe student reaction to CI
	CI	←	L	L used to support/explain CI decision/claim
	L	→	CI	Discussion of L leads to implications for CI
	L	←	CI	CI used to explain/situate student thinking
M & CI	M	→	CI	Discussion of M leads to implications for CI
	M	←	CI	CI used to support/explain statement about M
	CI	→	M	Discussion of CI leads to exploration of M
	CI	←	M	M used to support/clarify statement about CI

Table 4.26: Use of Knowledge Combinations

Table 4.27 summarizes how Amanda and Emily connected the three aspects of PCK. This summary used the expanded characterization provided in Table 4.26.

Summary of Amanda and Emily's Knowledge Use					
	Primary Focus	Support/ Implication	Amanda	Emily	Total
M & L	M →	L	0	5	5
	M ←	L	0	0	0
	L →	M	4	5	9
	L ←	M	14	4	18
CI & L	CI →	L	5	10	15
	CI ←	L	15	12	27
	L →	CI	12	9	21
	L ←	CI	10	9	19
M & L	M →	CI	0	2	2
	M ←	CI	0	0	0
	CI →	M	0	0	0
	CI ←	M	3	5	8

Table 4.27: Summary of Amanda and Emily's Knowledge Use

Isolating the interactions that were used to support a claim versus those that stated an implication brought out another distinction between Amanda and Emily. Table 4.28 provides the number of interactions of each type.

Interaction Type	Amanda	Emily
P ← S S used to clarify/support P	42 (67%)	30 (49%)
P → S P leads to implication for S	21 (33%)	31 (51%)

Table 4.28: Type of Interaction

Amanda was more likely to combine two forms of knowledge for the purpose of support and clarification; she was less likely to combine knowledge to state an implication. Emily, in contrast, used both types of knowledge interactions equally. The next section investigates how Amanda and Emily connected their knowledge of mathematics, learners, and curriculum/instructional strategies.

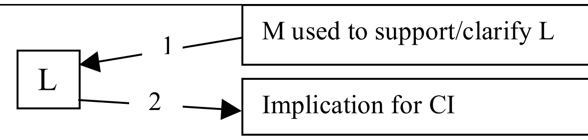
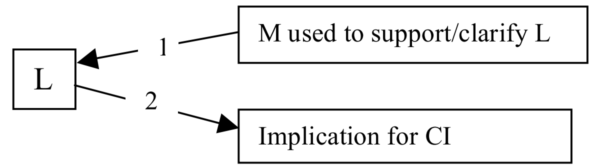
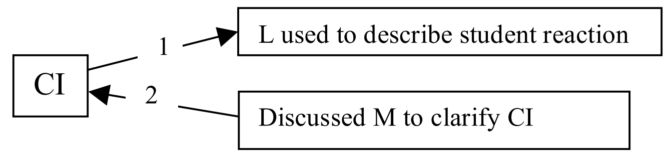
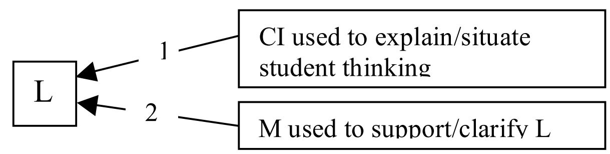
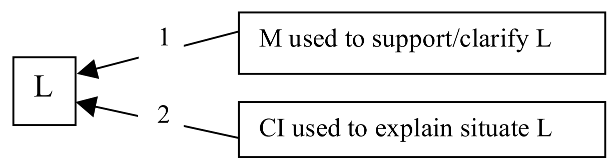
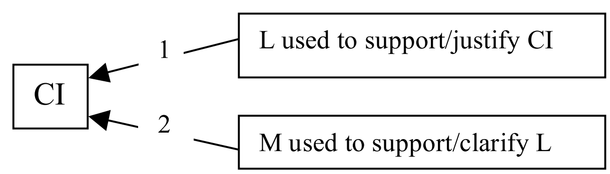
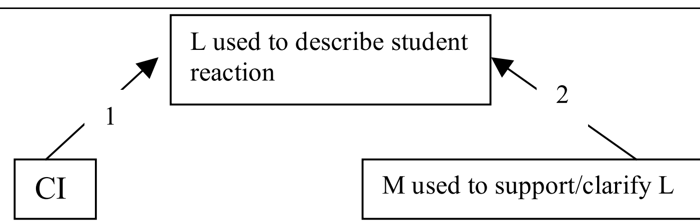
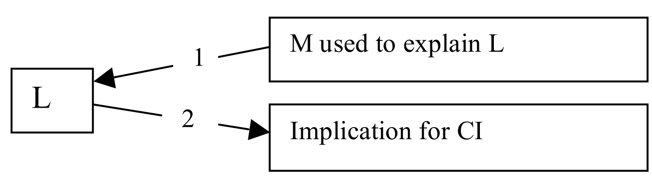
Knowledge of Mathematics, Learners, and Curriculum/Instructional Strategies

This section presents an analysis of how Amanda and Emily connected their knowledge of mathematics, their knowledge of learners, and their knowledge of curriculum and instructional strategies. The analysis for this section drew upon the findings from the prior three sections. For each episode, the connections between knowledge forms were mapped out using the interaction types identified in Table 4.27. Each episode mapping has a primary knowledge form, demarked by a bold square, and subsequent secondary knowledge forms that were used in support. Arrows were used to depict whether the secondary knowledge form clarified or explained the primary knowledge aspect, or whether it was an implication. In addition, numbers were included on each arrow to depict the order in which the connections were made.

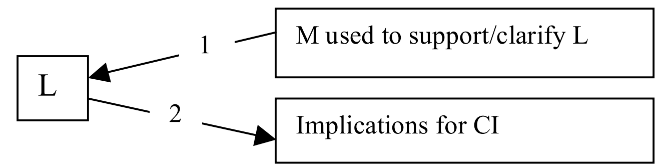
Amanda had 30 episodes where she connected her knowledge of mathematics, learners, and curriculum/instructional strategies. Emily had 10 episodes where she connected her knowledge of mathematics, learners, and curriculum/instructional strategies. The following sections are an analysis of these episodes.

Amanda.

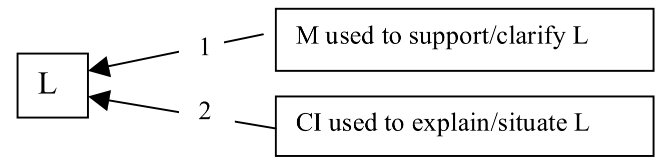
Table 4.29 lists the episodes where Amanda connected her knowledge of mathematics, knowledge of learners, and her knowledge of curriculum/instructional strategies. A mapping of how these knowledge types were integrated is included.

Episode (line #s)	Interaction
Amanda Initial (144-177)	
Amanda Initial (228-243)	
Amanda Initial (262-281)	
Amanda SR1 (33-56)	
Amanda SR1 (414-435)	
Amanda SR1 (508-526)	
Amanda SR1 (532-558)	
Amanda SR1 (633-640)	

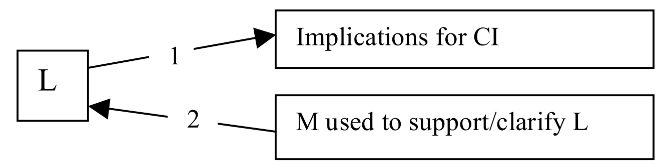
Amanda SR1 (678-702)



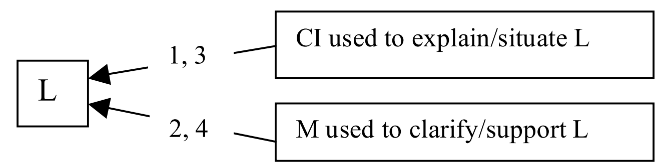
Amanda SR1 (816-841)



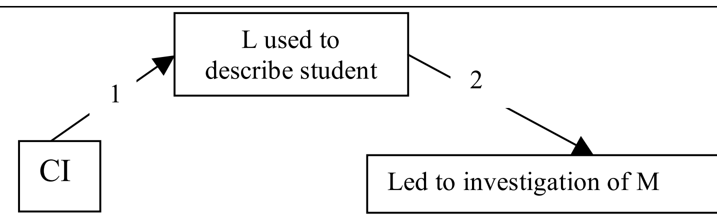
Amanda SR2 (536-540)



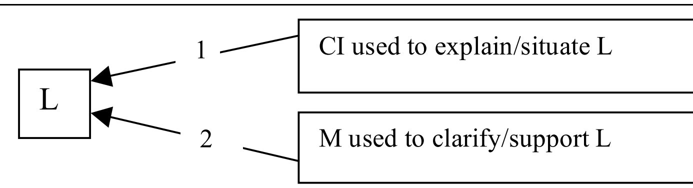
Amanda SR2 (633-678)



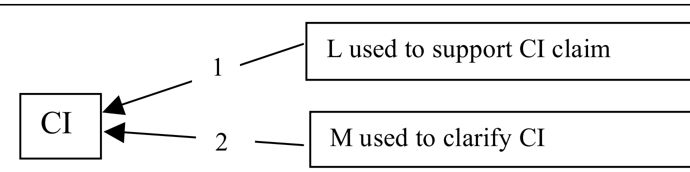
Amanda SR2 (727-776)



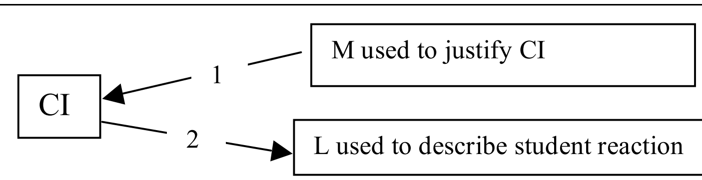
Amanda SR2 (933-946)

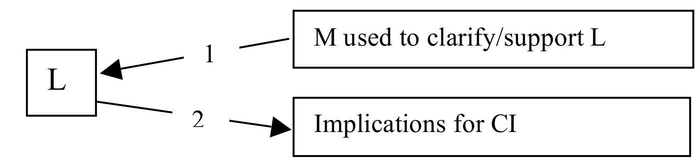
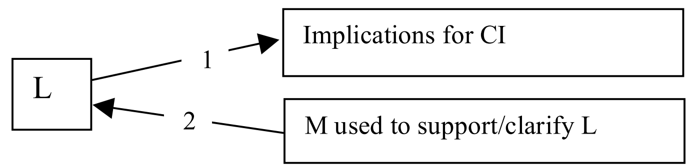
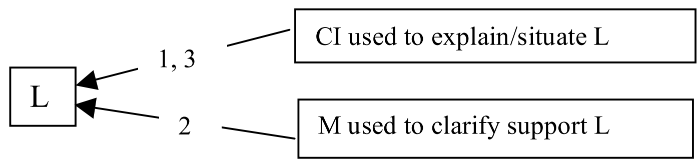
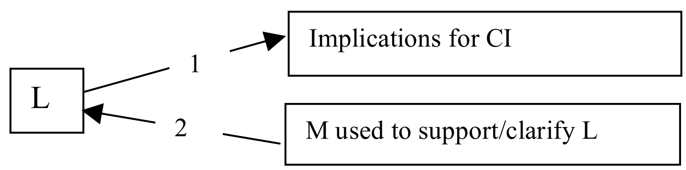
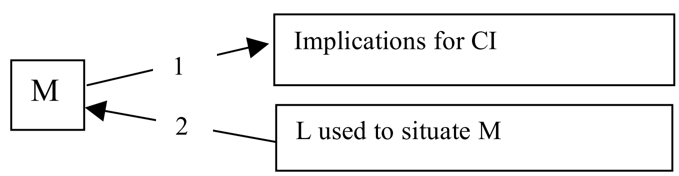
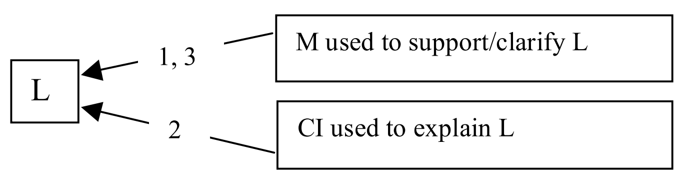
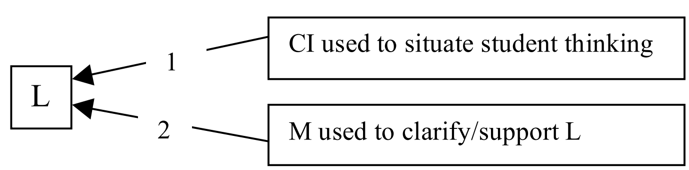
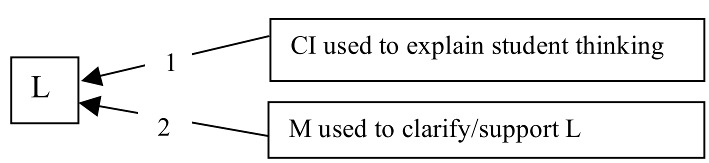


Amanda SR3 (69-99)



Amanda SR3 (101-108)



Amanda SR3 (159-164)	
Amanda SR3 (171-206)	
Amanda SR3 (219-243)	
Amanda SR3 (369-389)	
Amanda SR3 (393-398)	
Amanda Final (26-60)	
Amanda Final (101-123)	
Amanda Final (248-252)	

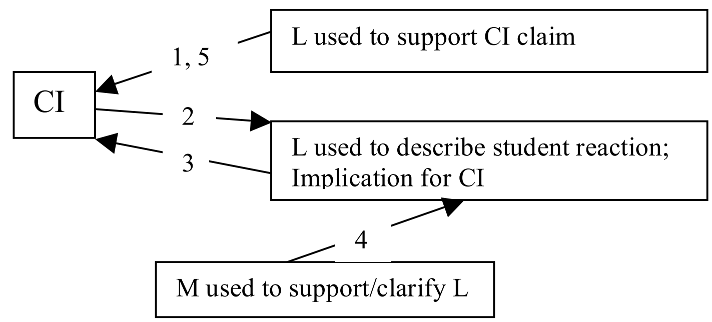
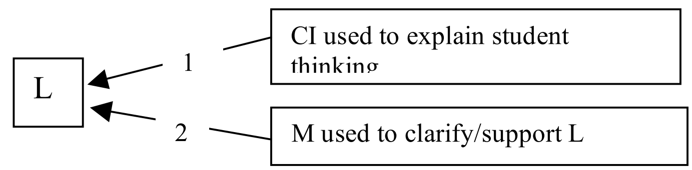
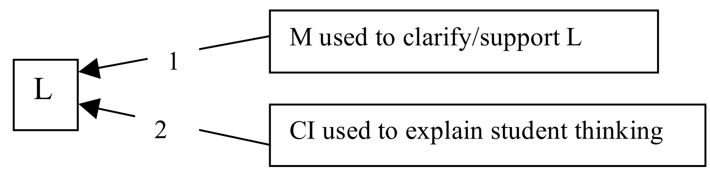
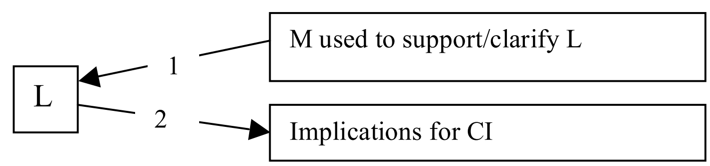
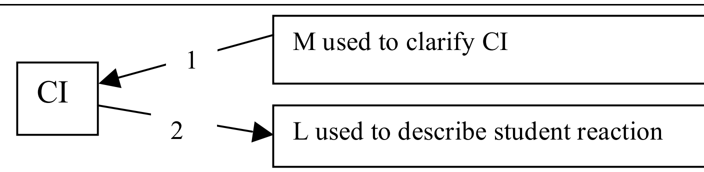
Amanda Final (325-363)	
Amanda Final (431-446)	
Amanda Final (483-496)	
Amanda Final (506-542)	
Amanda Final (690-717)	

Table 4.29: Amanda's knowledge of M, L, and CI

Example: The following example illustrates how Amanda integrated her knowledge of mathematics, knowledge of learners, and knowledge of curriculum/instructional strategies. In this excerpt Amanda used her knowledge of learners to discuss an episode of student thinking that she found interesting. She used her knowledge of mathematics and her knowledge of curriculum and instruction to support this discussion.

In this example, two students worked the Cross Problem (see Appendix H.) After they created the pattern with tiles and generated a table for the first four steps they asked if they could change the pattern for the 5th step. Figure 4.30 represents the first four steps of the Cross Problem; the different shadings represent the different steps of the pattern. Table 4.31 represents the table generated by the students.

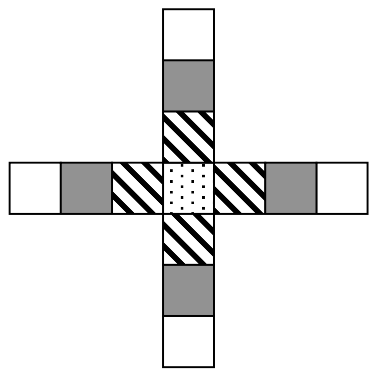


Figure 4.30: First 4 steps of the Cross Problem

Step Number	Step Size	Total so Far
1	1	1
2	4	5
3	4	9
4	4	13
5		

Table 4.31: Table for the Cross Problem

Amanda encouraged the students to investigate what would happen if they changed the pattern. The students added a fifth step that is illustrated in Figure 4.32.

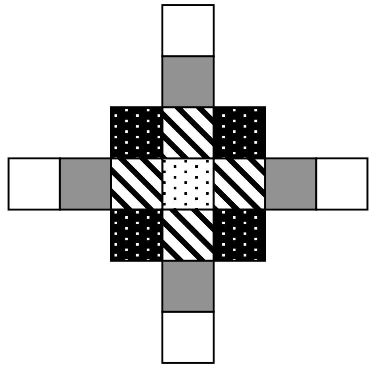


Figure 4.32: Altered 5th step of the Cross Problem

An interesting conversation developed as the students realized that even though they had changed how they created the pattern, the table remained the same. This realization caused both the students and Amanda to revise their conception of pattern.

Amanda: That was really interesting to me, the two girls sitting over
there drew this pattern (the cross problem) and they drew the table and
L they put a 1 and then they put a 4 and they knew they kept adding 4
and she said, "can I change the pattern when I get to step 5, can I
decide to change the pattern." I said, "What happens if you change the (1)
L pattern?" and she said it will change the table and so I thought, "she CI
(2) gets that she is completely changing the pattern" and so the pattern
will only work until there and then she is developing a new pattern, but M
I thought why not let her try to develop a new pattern so I let her go on CI
L that and when I came back through, she had continued to add 4, but
she added them here. So then I asked her, "How would I know that (3)
you put them there" and eventually she said, "Oh, you wouldn't, I CI
would have to tell them that I switched where I was putting them." So
that was interesting to me because my train of thinking was that she
(4) L realized she was going to change the whole table, but she thought that
changing where they went changed the table and it didn't change the M
table and she realized that. So I thought it was kind of cool that I didn't
know that because I left her to do it and she realized, "Oh, it didn't
change the table."

(Amanda SR2, lines 637-653)

In this example, Amanda used her knowledge of learners to describe an interesting excerpt of student thinking. She stated, "That was really interesting to me, the two girls sitting over there drew this pattern (the Cross Problem) and drew the table" and then asked, "Can I change the pattern when I get to step 5?" Amanda used her knowledge of curriculum and instructional strategies to react to the students' question, "What happens if you change the pattern?" By asking this question Amanda provided a setting to situate the student's response, "it will change the table." At this point, the students believed that if you change the pattern, the table would also change. In this portion of the episode Amanda is focused on student thinking and used her knowledge of curriculum and instructional strategies to clarify and situate the students' responses. Figure 4.33 illustrates this interaction, which is labeled with a 1 on the transcript.

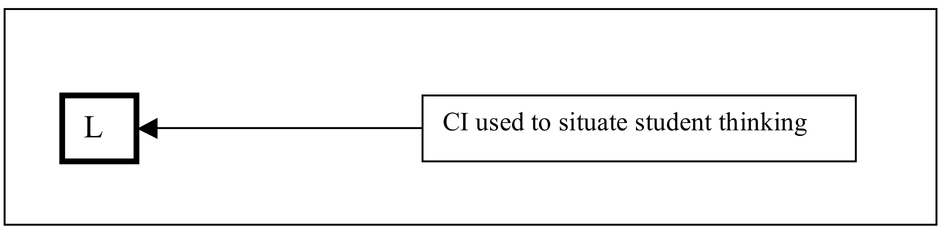


Figure 4.33: First Interaction type used in Amanda's episode

Amanda then stated the mathematics involved in the situation, "the pattern will only work until there (the 4th step) and then she is developing a new pattern. This statement of mathematics was used to clarify her conversation about student thinking and is illustrated by Figure 4.34 and labeled with a 2 on the transcript.

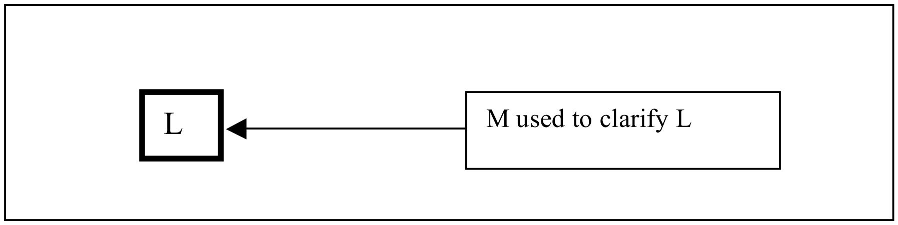


Figure 4.34: Second interaction type used in Amanda's episode

Amanda then returned to her knowledge of curriculum and instruction to further explain and situate student thinking. Amanda stated what she did next, "I thought, why not let her try to develop a new pattern" and then described how the students added the 5th step, illustrated in Figure 4.32. In response to their placement of the tiles, Amanda asked, "How would you know to place the tiles there" to which one student responded, "Oh, you wouldn't, I would have to tell them that I switched where I was putting them." In this portion of the transcript Amanda used her knowledge of curriculum and instruction to situate, and tell the story of, student thinking. Figure 4.35 illustrates this interaction, which is labeled with a 3 on the transcript.

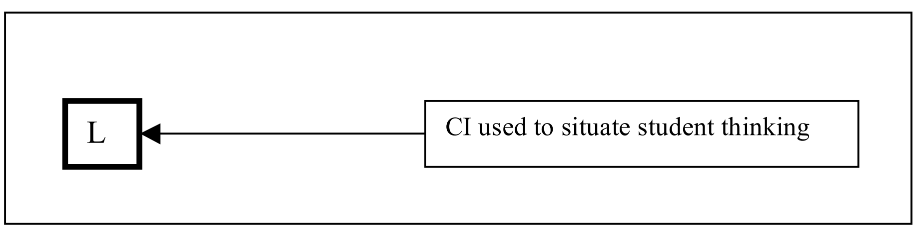


Figure 4.35: Third interaction type used in Amanda's episode

The description of student thinking led to a mathematical comment regarding how changing the pattern does not always change the table. Amanda used her knowledge of mathematics to state, "She thought where they (the tiles) went (in building the pattern) changed the table and it didn't." In this instance, mathematics was used to support the ongoing conversation about student thinking by stating the correct mathematical interpretation of the situation. Figure 4.36 illustrates this interaction, which is labeled with a 4 on the transcript.

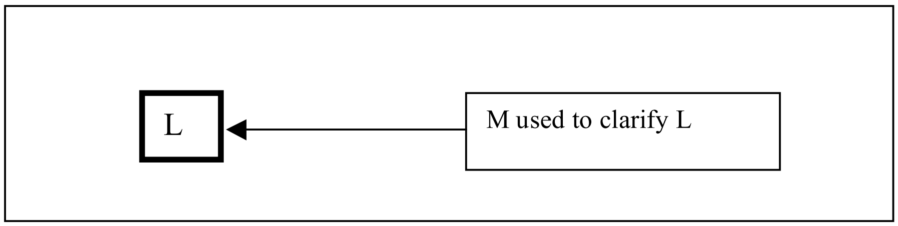


Figure 4.36: Fourth interaction type used in Amanda's episode

This example illustrates the complex way in which Amanda integrated her knowledge of mathematics, knowledge of learners, and knowledge of curriculum/instructional strategies. During this episode, Amanda focused on student thinking (knowledge of learners) and used her knowledge of mathematics and her knowledge of curriculum and instruction to support this conversation. Figure 4.37 presents the complete mapping of knowledge interaction for this episode.

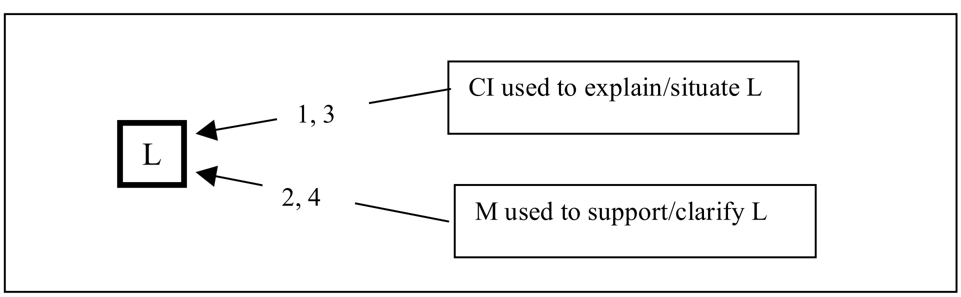
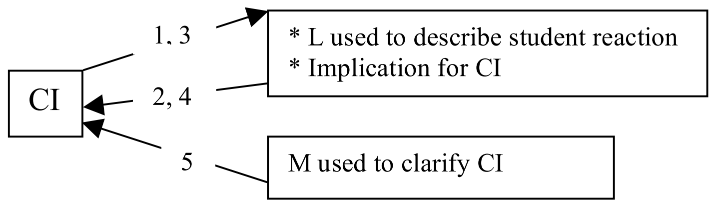
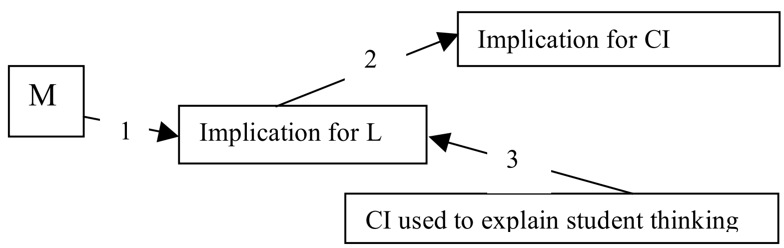
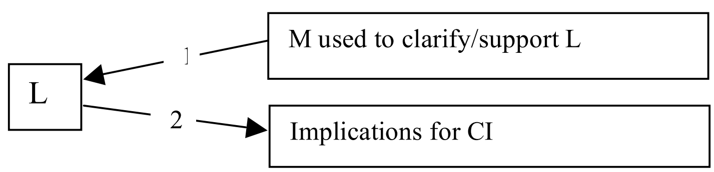
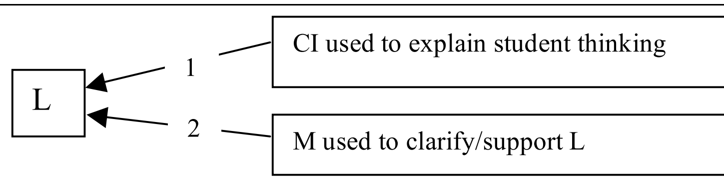
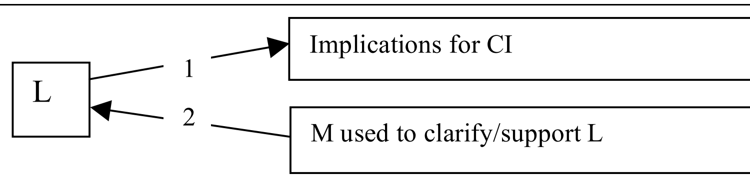
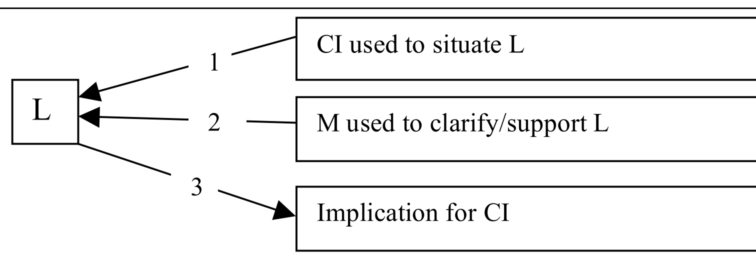


Figure 4.37: Mapping of knowledge interaction for episode

Emily

Table 4.38 lists the episodes where Emily used her knowledge of mathematics, knowledge of learners, and her knowledge of curriculum/instructional strategies simultaneously. A description of how these knowledge types were integrated is included.

Episode (line #s)	Type of Interaction
Emily Initial (254-277)	
Emily Initial (317-326)	
Emily SR1 (66-81)	
Emily SR1 (344-382)	
Emily SR2 (82-114)	
Emily SR2 (758-811)	

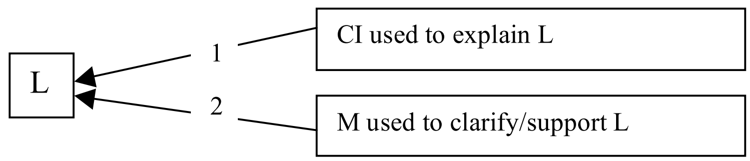
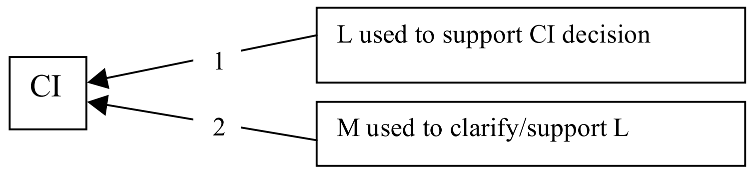
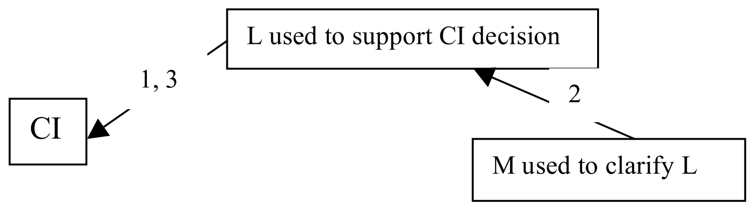
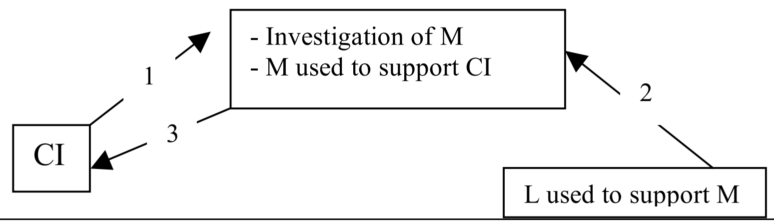
Emily SR3 (178-198)	
Emily SR3 (292-309)	
Emily SR3 (493-526)	
Emily SR3 (771-799)	

Table 4.38 Emily's knowledge of M, L, and CI

Example. The following example illustrates how Emily integrated her knowledge of mathematics, knowledge of learners, and knowledge of curriculum/instructional strategies. In this episode Emily discussed the Cross Problem (see Appendix H). She had difficulties working the problem the night before, which led her to reflect on the mathematics of the problem. The discussion led Emily to make several implications using other forms of knowledge.

Emily: I think that if we get to the Cross Problem, that was a struggle for
M   me to figure out, because I naturally look at patterns from left to right   (1)
and this pattern expands in all directions. I am concerned that they are   L
(2)   not going to see the pattern. The colors designate the steps, when the
CI   colors change so do the steps, so I have got to make sure I am very clear
about the colors and why those colors are there; I really think some of   L   (3)
my students will have an issue with left to right simply because every
CI   pattern we have done up to this point wasn't like that.

In this example, Emily used her knowledge of mathematics to analyze her struggles with the Cross Problem. She stated, "that was a struggle for me to figure out because I naturally look at patterns from left to right and this pattern expands in all directions." This mathematical feature of the problem led Emily to believe that her students would also struggle with the problem. Emily stated, "I am concerned that they are not going to see the pattern." In this portion of the episode a conversation about the mathematics of the Cross Problem led Emily to make an implication regarding her students. Figure 4.39 represents this interaction, which is labeled with a 1 on the transcript.

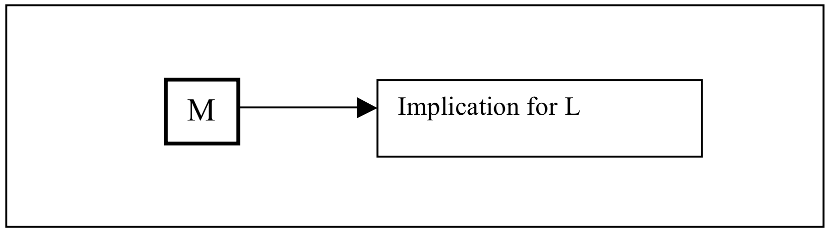


Figure 4.39: First interaction type used in Emily's episode

Emily was concerned that her students were not going to understand what constituted a step in the Cross Problem. She used her knowledge of curriculum to state that different colors were used to designate steps and stated, "I have got to make sure that I am very clear about the colors and why those colors are there." If Emily planned to emphasize the role that the colors played to alleviate some of the issues the students would have with how the pattern expands. In this portion of the episode her focus has shifted to the learner and their potential struggles with the Cross Problem and has stated an implication that may help resolve this potential problem. Figure 4.40 illustrates this interaction, which is labeled with a 2 on the transcript.

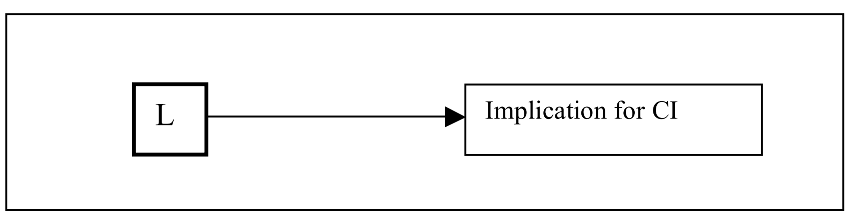


Figure 4.40: Second interaction type used in Emily's episode

Amanda returned to her concerns about students, "I really think some of my students will have an issue with left to right." She supported this claim by stating that all of the patterns they had previously worked had grown from left to right, or at least in a single direction. For example, in prior lessons they had worked the Tower of Three and Double Staircase Problem. Figure 4.41 illustrates how the Tower of Three pattern grows in a single direction, in contrast to the Cross Problem (Figure 4.30). Different shadings designate the different steps.

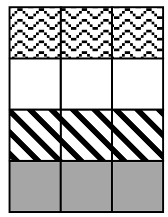


Figure 4.41: Tower of Three Pattern

Emily used the curricular fact that her students had never seen a pattern like the Cross Problem to support her statement that her students would struggle. Figure 4.42 illustrates this interaction, which is labeled with a 3 on the transcript.

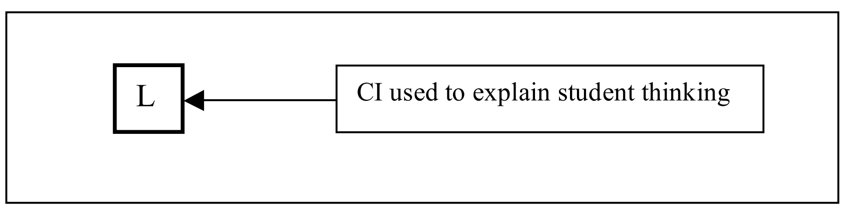


Figure 4.42: Third interaction type used in Emily's episode

This example illustrates the complex way in which Emily integrated her knowledge of mathematics, knowledge of learners, and knowledge of curriculum/instructional strategies. During this episode, Emily started with a focus on mathematics, which led to an implication regarding learners. She shifted her attention to focus on this implication about learners and used her knowledge of curriculum and instruction being used to support this statement and to provide an implication. Figure 4.43 presents the complete mapping of knowledge interaction for this example.

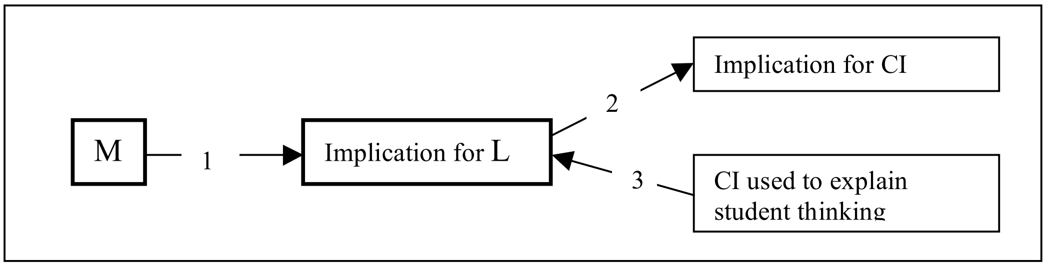


Figure 4.43: Mapping of knowledge interaction for Emily's episode

Discussion

Amanda and Emily connected their knowledge of mathematics, knowledge of learners, and knowledge of curriculum and instructional strategies in a variety of ways. Table 4.48 summarizes how Amanda and Emily connected the three aspects of PCK. The data in this table adds the data from the M, L, & CI combination to Table 4.27.

Summary of Amanda and Emily's Knowledge Use						
	Primary Focus		Support/ Implication	Amanda	Emily	Total
M & L	M	→	L	0	6	6
	M	←	L	1	1	2
	L	→	M	5	5	10
	L	←	M	38	10	48
CI & L	CI	→	L	11	12	23
	CI	←	L	19	15	34
	L	→	CI	22	15	37
	L	←	CI	23	13	46
M & L	M	→	CI	1	2	3
	M	←	CI	0	0	0
	CI	→	M	0	1	1
	CI	←	M	8	8	6

Table 4.48: Summary of Knowledge Use for all knowledge Combinations

Isolating the interactions that were used to support a claim versus those that stated an implication brought out another distinction between Amanda and Emily. Table 4.49 provides the number of interactions of each type and includes the data from the M, L, and CI combination.

Interaction Type	Amanda	Emily
P ⟵ S S used to clarify/support P	89 (70%)	47 (53%)
P ⟶ S P leads to implication for S	39 (30%)	41 (47%)

Table 4.49: Type of Interaction for all combinations

Amanda was more likely to combine two forms of knowledge for the purpose of support and clarification; she was less likely to combine knowledge to state an implication. Emily, in contrast, used both types of knowledge interactions equally. Table 4.50 depicts the primary focus of Amanda and Emily's knowledge interactions.

Primary Focus	Amanda	Emily
M	2	9
L	88	43
CI	38	36

Table 4.50: Primary Focus for all combinations

Table 4.50 supports the assertion that Amanda was very focused on her knowledge of learners. Eighty-eight of Amanda's interactions had learners as the primary focus. Emily, on the other hand, was more balanced between her knowledge of learners and knowledge of curriculum and instructional strategies being the focus of an interaction.

Summary

In summary, Amanda and Emily used their knowledge of mathematics, knowledge of learners, and knowledge of curriculum and instructional strategies in an integrated fashion. In addition, their knowledge of learners played a crucial role in helping them connect their knowledge of mathematics with their knowledge of curriculum and instructional strategies. They integrated their knowledge in two distinct ways: (a) to use one form of knowledge to support/justify a claim made using a second form of knowledge and (b) to state an implication that one form of knowledge had for another. The main purpose of this integration was to support either instructional decisions or the analysis of student thinking, it was rarely used to support a mathematical claim.

CHAPTER 5: SUMMARY, DISCUSSION, AND CONCLUSIONS

In this study, I investigated how two fifth grade teachers drew upon their pedagogical content knowledge (PCK) when providing instruction in algebraic generalization. In particular, I studied the use, and interaction, of three subcomponents of PCK: knowledge of mathematics, knowledge of learners, and knowledge of curriculum and instructional strategies. The purpose of this work was to answer the following research questions:

9) To what extent do two fifth grade teachers draw upon their knowledge of mathematics, their knowledge of learners, and their knowledge of curriculum and instructional strategies when designing, implementing, and reflecting on instruction related to algebraic generalization?
10) How do two fifth grade teachers when designing, implementing, and reflecting on instruction related to algebraic generalization, use these three aspects of pedagogical content knowledge simultaneously?

This chapter is organized into six sections related to the research questions listed above. These six sections include: (a) situating this chapter by providing a summary of the findings presented in chapter 4, (b) discussing the findings from this study in relation to the current research in mathematics education, (c) sharing implications of my work for practice and research, (d) elaborating on my suggestions for future research, (e) discussing the limitations of the study, with suggestions for improvement, and (f) concluding with remarks regarding the importance of this work to the field of mathematics education.

Summary of Findings

The following section presents a brief overview of the findings presented in chapter 4 for the purpose of situating the discussion that follows. The section is divided into two parts that address each of the research questions. Assertions generated from the findings are intermixed throughout this section.

Research Question 1

The first research question focused on the extent to which Emily and Amanda used their knowledge of mathematics, their knowledge of learners, and their knowledge of curriculum and instructional strategies in the act of teaching algebraic generalization. Table 5.1 summarizes the percentage of Amanda and Emily's conversations that were coded as one of the three aspects of PCK.

	K of M	K of L	K of CI
Amanda	16.1%	52.9%	35.9%
Emily	13.8%	38.8%	31.8%
Total	**14.8%**	**47.7%**	**35.8%**

Table 5.1: Percentage of Interviews coded as M, L, or CI

From this table, it is evident that Amanda and Emily used their knowledge of learners extensively during their conversations about algebraic generalization (47.7 percent of the time). They also used their knowledge of curriculum and instructional strategies frequently (35.8 percents) and to a lesser extent their knowledge of mathematics (14.8 percent). One difference that existed between these two teachers was that Amanda used her knowledge of learners 52.9 percent of the time as compared to Emily who only used her knowledge of learners 38.8 percent of the time. Hence, Amanda utilized her knowledge of learners more frequently while discussing her teaching. In contrast, Emily used her knowledge of learners (38.8 percent) at approximately the same rate as she used her knowledge of curriculum and instructional strategies (31.8 percent).

However, the data in Table 5.1 do not provide detailed information regarding the integrated nature of Amanda and Emily's use of these three aspects of pedagogical content knowledge. To gauge the extent of integration, each transcript was segmented into episodes--portions of the transcripts where the teacher discussed a single thought. Each episode was coded for the types of knowledge utilized and a tally was made of the various combinations of knowledge forms (M, L, CI, M&L, M&CI, L&CI). Table 5.2 displays the percentage of episodes that used these different knowledge combinations.

	M	L	CI	M&L	M&CI	L&CI	M,L,&CI
Amanda	2.5%	13.2%	8.3%	14.0%	2.5%	32.2%	27.3%
Emily	1.8%	12.5%	8.0%	13.4%	8.0%	41.1%	11.6%
Total	**2.2%**	**12.9%**	**8.2%**	**13.7%**	**5.2%**	**36.5%**	**19.7%**

Table 5.2: Amanda and Emily's integrated use of knowledge

The data (see Table 5.3) confirm that most episodes of Amanda and Emily involved combinations of knowledge bases within PCK, with the combination of knowledge of learners and knowledge of curriculum and instructional strategies most prevalent (32.2 and 41.1 percent for Amanda and Emily respectively). Amanda appeared more likely to use all three knowledge forms. Of note, the combination involving the use of knowledge of mathematics and knowledge of curriculum and instructional strategies occurred the least, representing only 5.2 percent of the episodes.

	Single Aspect of PCK	Multiple Aspects of PCK
Amanda	24.0%	76.0%
Emily	22.3%	74.1%
Total	**21.5%**	**74.7%**

Table 5.3: Amanda and Emily's use of PCK

The data in Table 5.3 demonstrate that Amanda and Emily had a smaller percentage of episodes that used a single knowledge type rather than multiple knowledge types--both teachers used multiple knowledge forms approximately 75 percent of the time. These results indicate the importance of the use of integrated knowledge forms during the instructional decision-making process. Although these data inform us that Amanda and Emily used their knowledge in an integrated fashion, we need further information on the nature of this knowledge. This is the focus of the next research question.

Assertion 1: Knowledge of learners can played a crucial role in helping teachers connect their knowledge of mathematics and their knowledge of curriculum and instructional strategies.

Assertion 2: Teachers draw upon multiple knowledge bases when making instructional decisions.

Research Question 2

The second research question focused on *how* Amanda and Emily used their knowledge of mathematics, knowledge of learners, and knowledge of curriculum and instructional strategies in the act of teaching. For this question, each knowledge combination (M&L, L&CI, M&CI) was analyzed using elements of grounded theory to uncover how the teachers connected the various aspects of pedagogical content knowledge. In the interactions involving various knowledge bases, one aspect of PCK was the focus of the conversation, while a second knowledge type played a supportive role. The supporting knowledge was either used to clarify or explain what was being discussed by the primary knowledge base, or was used to describe an implication that was drawn from the primary form of knowledge. Table 5.4 provides a list of models for the ways in which the different knowledge combinations were used.

Use of Knowledge Combinations				
	Primary Focus		Support/ Implication	Knowledge Interaction
M & L	M	→	L	Discussion of M leads to implication for L
	M	←	L	L used to support/explain statement about M
	L	→	M	Discussion of L leads to exploration of M
	L	←	M	M used to support/explain statement about L
L & CI	CI	→	L	L used to describe student reaction to CI
	CI	←	L	L used to support/explain CI decision/claim
	L	→	CI	Discussion of L leads to implications for CI
	L	←	CI	CI used to explain/situate student thinking
M & CI	M	→	CI	Discussion of M leads to implications for CI
	M	←	CI	CI used to support/explain statement about M
	CI	→	M	Discussion of CI leads to exploration of M
	CI	←	M	M used to support/clarify statement about CI

Table 5.4: Description of Knowledge Combinations

Amanda and Emily used these knowledge combinations in different proportions. Table 5.5 summarizes how they connected the three aspects of PCK during their interviews. The quantities represent the total number of interactions in which each knowledge combination was used.

Summary of Amanda and Emily's Knowledge Use					
	Primary Focus	Support/ Implication	Amanda	Emily	Total
M & L	M →	L	0	6	6
	M ←	L	1	1	2
	L →	M	5	5	10
	L ←	M	38	10	48
CI & L	CI →	L	11	12	23
	CI ←	L	19	15	34
	L →	CI	22	15	37
	L ←	CI	23	13	46
M & L	M →	CI	1	2	3
	M ←	CI	0	0	0
	CI →	M	0	1	1
	CI ←	M	8	8	16

Table 5.5: Summary of Knowledge Use for all knowledge Combinations

Amanda and Emily employed their knowledge differently. For example, Amanda often placed more emphasis on learners than did Emily. Table 5.6 provides the number of interactions where each of the three knowledge forms was the primary focus of conversation.

Primary Focus	Amanda	Emily	Total
M	2 (1.6%)	9 (10.2%)	11 (5.1%)
L	88 (68.8%)	43 (48.9%)	131 (60.6%)
CI	38 (29.7%)	36 (40.9%)	74 (34.3%)
Total	128	88	216

Table 5.6: Primary Focus of Conversation

Emily primarily drew on her knowledge of learners (48.9 percent) and her knowledge of curriculum and instructional strategies (40.9 percent). This difference may be indicative of the experience gap between Amanda and Emily. Emily's experience may have afforded her more opportunities to connect knowledge in different ways. In contrast, Amanda and Emily rarely focused on their knowledge of mathematics (1.6 percent for Amanda and 10.2 percent for Emily).

Finally, differences emerged in the ways that Amanda and Emily drew on their knowledge. Table 5.7 isolates the knowledge interactions that were used to support a claim versus those that were used to state an implication of a claim.

Interaction Type	Amanda	Emily
P ⟵ S S used to clarify/support P	89 (70%)	47 (53%)
P ⟶ S P leads to implication for S	39 (30%)	41 (47%)

Table 5.7: Type of Interaction for all combinations

Amanda was more likely than Emily to combine two forms of knowledge for the purpose of support and clarification (70 percent of her knowledge interactions); she was less likely to combine knowledge in order to state an implication (30 percent of knowledge interactions). Emily, in contrast, used both types of knowledge interactions in approximately equal amounts (53 percent of the time in support, 47 percent as an implication).

Assertion 3: Teachers draw upon combinations of PCK in two distinct ways: (a) to use one form of knowledge to support/justify a claim made using a second form of knowledge and (b) to state an implication that one form of knowledge had for another.

Assertion 4:Pedagogical content knowledge involves more than just factual knowledge; it is a process of drawing upon various knowledge forms in order to transform mathematics into a form that best meets the instructional needs of students.

The models provided in Table 5.4 were also used to map out the interactions that occurred when all three forms of knowledge were used in a single episode. An example mapping is provided in Figure 5.8. These mappings illustrate the complex process in which Amanda and Emily justified instructional claims.

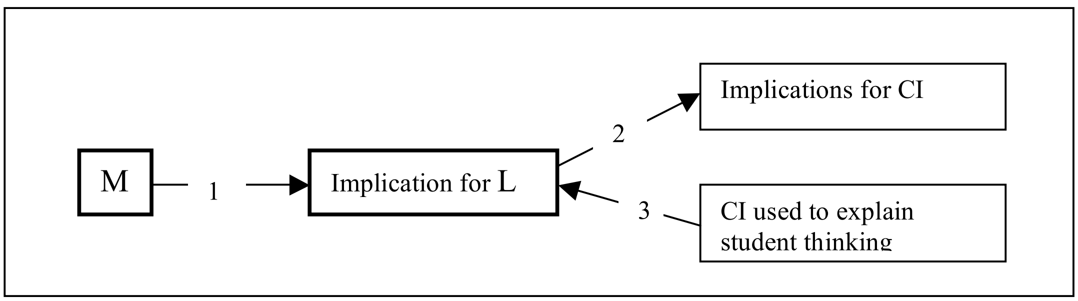


Figure 5.8: Mapping of knowledge interaction for one of Emily's episodes

In summary, Amanda and Emily often used their knowledge of mathematics, knowledge of learners, and knowledge of curriculum and instructional strategies in an integrated fashion. In particular, approximately 75 percent of the episodes utilized multiple forms of knowledge. However, Amanda and Emily did not employ their knowledge bases in the same proportions, or in the same manner. Emily appeared to have a more diverse repertoire of knowledge use, which may be indicative of her years of experience. Knowledge of learners was the most common knowledge form used, followed closely by their knowledge of curriculum and instructional strategies. Most of

Amanda and Emily's conversations focused on learners and curriculum/instructional strategies, with all three forms of knowledge used in support.

Discussion

This study examined how two fifth grade teachers drew upon their knowledge of mathematics, their knowledge of learners, and their knowledge of curriculum and instructional strategies to provide instruction in algebraic generalization. In this section I discuss the findings of this study in relation to current literature on PCK, and use these findings to expand the construct of PCK.

Revisiting Pedagogical Content Knowledge

The purpose of this study was to understand how two teachers used their PCK in the act of teaching. I first discuss the integrated nature of PCK and its place among the other forms of teacher knowledge. I follow this with a discussion of the role that the various aspects of pedagogical content knowledge play in the enactment of instruction.

The nature of PCK. Shulman (1986) saw the fallacy of focusing on isolated components of teacher knowledge and recommended a model that combined multiple aspects of knowledge. He criticized teacher evaluation programs that focused primarily on pedagogical knowledge for teaching, ignoring the connection to content. He believed that focusing solely on content, or solely on pedagogy, did not transfer directly into quality teaching. With this in mind, he developed the theory related to pedagogical content knowledge to better explain the knowledge of teaching. Shulman concluded:

> Pedagogical content knowledge is of special interest because it identifies the distinctive bodies of knowledge for teaching. It represents the blending of content and pedagogy into an understanding of how particular topics, problems, or issues

> are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction. Pedagogical content knowledge is the category most likely to distinguish the understanding of the content specialist from that of a pedagogue. (p. 8)

In this statement, Shulman brings out the notion that PCK is an integrated form of knowledge. Shulman and Grossman (1988) refined their notion of PCK to include this integrated feature and identified five subcomponents: knowledge of alternative content frameworks, knowledge of student understanding and misconceptions of a subject, knowledge of curriculum, knowledge of particular content for the purpose of teaching, and knowledge of specific pedagogical strategies.

The data presented in this study support and expand on Shulman's notion of PCK. For instance, the data show that Amanda and Emily drew on multiple knowledge bases to make instructional decisions a majority of the time (74.7 percent). In addition, Amanda and Emily used combinations of their knowledge of mathematics, their knowledge of learners, and their knowledge of curriculum and instructional strategies. These findings are in line with Shulmnan's (1986) statement, "It (PCK) represents the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction" (p. 8). In addition, the larger proportion of episodes that utilized multiple forms of knowledge hint at the importance of PCK in the process of instruction and supports Shulman's statements that "pedagogical content knowledge is of special interest because it identifies the distinctive bodies of knowledge for teaching."

However, disagreement exists on how pedagogical content knowledge relates to other forms of teacher knowledge. Shulman argued that it is a critical piece, but identified it as one of several different forms of knowledge that should be considered. Neagoy (1995), on the other hand, saw pedagogical content knowledge as *the* knowledge for teaching and argued that the important aspects of teacher knowledge are subsets of pedagogical content knowledge.

My study took Neagoy's construct as an assumption and applied it to the definition of PCK, defining it as the integration of other forms of knowledge for the purpose of instruction. Given this assumed definition, PCK was the predominant knowledge form used in the episodes of both Amanda and Emily. However, some episodes that were classified as PCK under this refined definition would be questionable under Shulman's original definition. The following example illustrates the discrepancy between the two definitions. In this episode, Amanda described an occurrence in class where two students were filling in a table. One of the students mistakenly placed the 'total so far' values in the 'step size' column. The mistake was caught and corrected by his partner. On reflecting on this episode, Amanda made a statement regarding the benefits of working in groups.

L *Amanda:* One of them started off counting the wrong amount, they had the wrong amount in the table. Oh, I remember what it was. The thing that threw it off was the partner who was filling it in, he had it right, but the partner filling it in was writing the total so far into the step size column. So that did come up again, I remember that now, it was quickly remedied by the partner seeing it, which reinforces to me the whole idea that working with partners helps because they get somebody to say, hopefully, wait that is not right as where I feel like that child could go on forever if they were working individually before I got there. What if it takes awhile before I get around to that person, you are doing the wrong thing and it is already so set in their mind. CI

(Amanda, Stimulated Recall 3, 552-562)

In this episode, Amanda made a clear connection between her knowledge of learners and her knowledge of curriculum and instructional strategies as it pertains to the implementation of her lesson on algebraic generalization. However, Shulman (1987) stated that a teacher's PCK was a transformation of "content knowledge he or she possesses into forms that are pedagogically powerful" (p. 15). This episode lacks a clear connection as to how the teacher transformed content knowledge into a form that is useful for instruction and is more general in nature. This illustrates that these two definitions of PCK are not the same and alludes to the difficulty in defining PCK. Marks (1990) stated, "a precise demarcation of pedagogical content knowledge from subject matter knowledge and general pedagogical knowledge is somewhat arbitrary" (p. 8). This suggests that certain episodes will clearly be PCK, while others will fall on the fringe of definition.

In addition, Shulman talked about pedagogical content knowledge as the "blending of content and pedagogy" (1987, p. 8); this is not necessarily the same idea as knowledge integration. For instance, a teacher may have a piece of PCK that has an element of mathematical knowledge and an element of pedagogical knowledge, yet is a single fact. The integration of knowledge projects the image that you are incorporating two different aspects of knowledge together for the purpose of instruction. I believe the ideas of PCK and knowledge integration are closely related, but not synonymous.

Despite the controversy regarding how PCK relates to other forms of knowledge, and regardless of the exact definition; it is the critical piece of teacher knowledge. PCK is the type of knowledge that is distinct to teachers because it integrates elements of other forms of knowledge for the purpose of instruction. The definition itself distinguishes it from other forms of knowledge; Shulman's notion that developing PCK requires transforming content knowledge into teachable representations suggests that PCK is more than just factual knowledge.

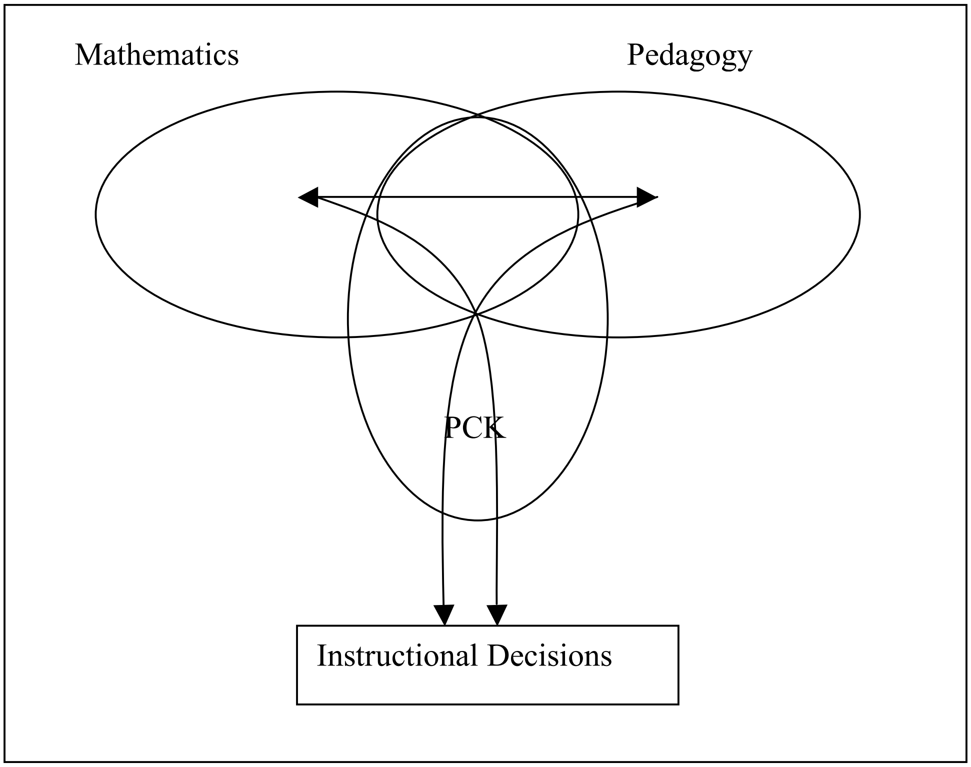
Through examining PCK, I characterize the knowledge used by Amanda and Emily as they engaged in the act of teaching algebraic concepts. To understand the knowledge for teaching we must unravel the complex nature of PCK. The next section extends the prior work on PCK to elaborate on the interaction and role these different subcomponents of PCK play in the act of teacher decision-making.

The role of knowledge of learners. An analysis of the role and function of the subcomponents of PCK used by Amanda and Emily points toward the important role that the knowledge of learners played in teacher decision-making. This form of knowledge was predominant, accounting for 47.7 percent of Amanda and Emily's conversations. Moreover, 60.6 percent of the knowledge interactions had the knowledge of learners as their focus, with the knowledge of mathematics and the knowledge of curriculum and instructional strategies used in support.

The crucial role that the knowledge of learners played for Amanda and Emily needs to become a more integral part of current practices within the mathematics education community. A glance at teacher education programs across the country reveals that teacher preparation involves a combination of mathematics, methods, and general education courses, with the knowledge of learners interspersed within this preexisting framework. There is work that is bringing knowledge of learners to the forefront, as the work of Cognitively Guided Instruction (Carpenter, Thomas, Fennema, Franke, Levi, & Epson, 1999) and the Integrating Mathematics and Pedagogy (Philipp, Sowder, Ambrose, Chauvot, Clement, & Schappelle, 2005) projects illustrate. Both projects used student thinking as a catalyst to connect teachers' knowledge of mathematics and pedagogy. Integrating student thinking into courses for teachers is important as Borko, Livingston, McCaleb, and Mauro (1988) found that a lack of understanding of student thinking limited the development of PCK.

Given that Amanda and Emily used their knowledge of learners prominently in their conversations, the first question we need to ask is, "What role does the knowledge of learners play in the process of transforming content into pedagogy?" Not only is the knowledge of learners the form of knowledge used most often by Amanda (52.9 percent) and Emily (38.8 percent) in their discussions of instruction, it also seems to play a critical role in connecting their knowledge of curriculum and instructional strategies to their knowledge of mathematics. Looking at the episodes discussed by Amanda and Emily, 86.7 percent of Amanda's episodes and 78.6 percent of Emily's episodes involved the knowledge of learners. In addition, their knowledge of mathematics and their knowledge of curriculum and instructional strategies were used in combination, without the knowledge of learners, only 5.2 percent of the time. Hence, Amanda and Emily's knowledge of learners played a critical role in how they used their knowledge of mathematics and their knowledge of curriculum and instructional strategies.

Shulman (1986) characterized pedagogical content knowledge as the missing link that connected content knowledge and the knowledge of pedagogy. He believed that PCK was the knowledge that allowed a teacher to transform their knowledge of mathematics and their knowledge of pedagogy into useful representations for the classroom. The data support a view of pedagogical content knowledge as a bridge that encourages teachers to connect their knowledge of mathematics to their knowledge of pedagogy. This connection makes the knowledge of pedagogy and mathematics useful for classroom instruction. Figure 5.9 depicts this interpretation.



5.9: My interpretation of Shulman's view of PCK

Ironically, only 5.2 percent of Amanda and Emily's episodes directly connect their knowledge of mathematics with their knowledge of curriculum and instructional strategies, in the absence of their knowledge of learners. Furthermore, looking at table 5.5, of the 226 total interactions among the three knowledge forms, only 20 interactions (8.8 percent) were made between the knowledge of mathematics and the knowledge of curriculum and instructional strategies. Given that Amanda and Emily's knowledge of mathematics was rarely used in direct contact with their knowledge of curriculum, in the absence of their knowledge of learners, further clarification of how PCK brings the knowledge of mathematics and pedagogy together for instruction is warranted. How does PCK act as this missing link that Shulman suggests?

Given that pedagogical content knowledge has various subcomponents that include the knowledge of mathematics, the knowledge of learners, and the knowledge of curriculum and instructional strategies, one could conclude from the data of this study that the knowledge of learners played an important role in encouraging Amanda and Emily to connect their knowledge of mathematics with their knowledge of curriculum and instructional strategies. This can be illustrated by the common trend among the mappings of table 4.29 and 4.38 of the existence of the M →L→CI pattern. The episode provided below is an illustration of this use of the knowledge of learners. In this example, Emily discussed her goal of encouraging students to develop explicit formulas.

CI { *Emily*: Well, this is what I am going to do, I want to go back and I want them to start building this (the double staircase), I am just going to do the double staircase for a while just so they, I want to see what they end up coming up with because my thinking, James (another 5th grade teacher at Amanda and Emily's school) was like, "they are not going to be able to come up with that" and I said, "I don't know," and he said, "maybe a few of them, but I don't think all of them will be able to come up with it." } L

Researcher: Come up with what?

M { *Emily*: Like a formula basically, like how do you plug in numbers so that if I give you the 100th step you are going to be able to know, I mean because you are looking for the pattern. I know that I have found some, but I know there have got to be other ones that, because I certainly didn't think of what Brett and Josh thought of. (Referring back to the boys mentioned at the very beginning) L

Researcher: Lets talk about that, what do you think of their formula?

M *Emily*: Well the only thing, my issue with that, what I kept running into was that they didn't know the total so far to subtract. L

CI In hindsight I wish I would have not even said, I think I gave them too much, I wish I would have said, "OK, lets do that, lets plug that in" and we did that to a degree, but I think I was thinking so much myself that I didn't let them grapple with it.

In this example, Emily discussed her strategy for encouraging students to develop generalizations (CI). This led to a conversation regarding the students' ability to develop explicit rules (L), which was followed by a mathematical elaboration of what she expected the students to be able to do (M). She then discussed a rule generated by two of her students to support her instructional claim that her students could develop an explicit rule (L). Next, she stated a misconception given by a student (M/L); this included a comment on the mathematics of the misconception. This reflection on student thinking ultimately caused her to suggest a change in her instruction (CI). This example illustrates the movement in her knowledge use from CI to L to M to L to CI. In this example, Emily's knowledge of learners was used as a bridge to connect her knowledge of mathematics with her knowledge of curriculum and instructional strategies. With this in mind, I have modified my interpretation (see Figure 5.10) to take the role of the knowledge of learners into account.

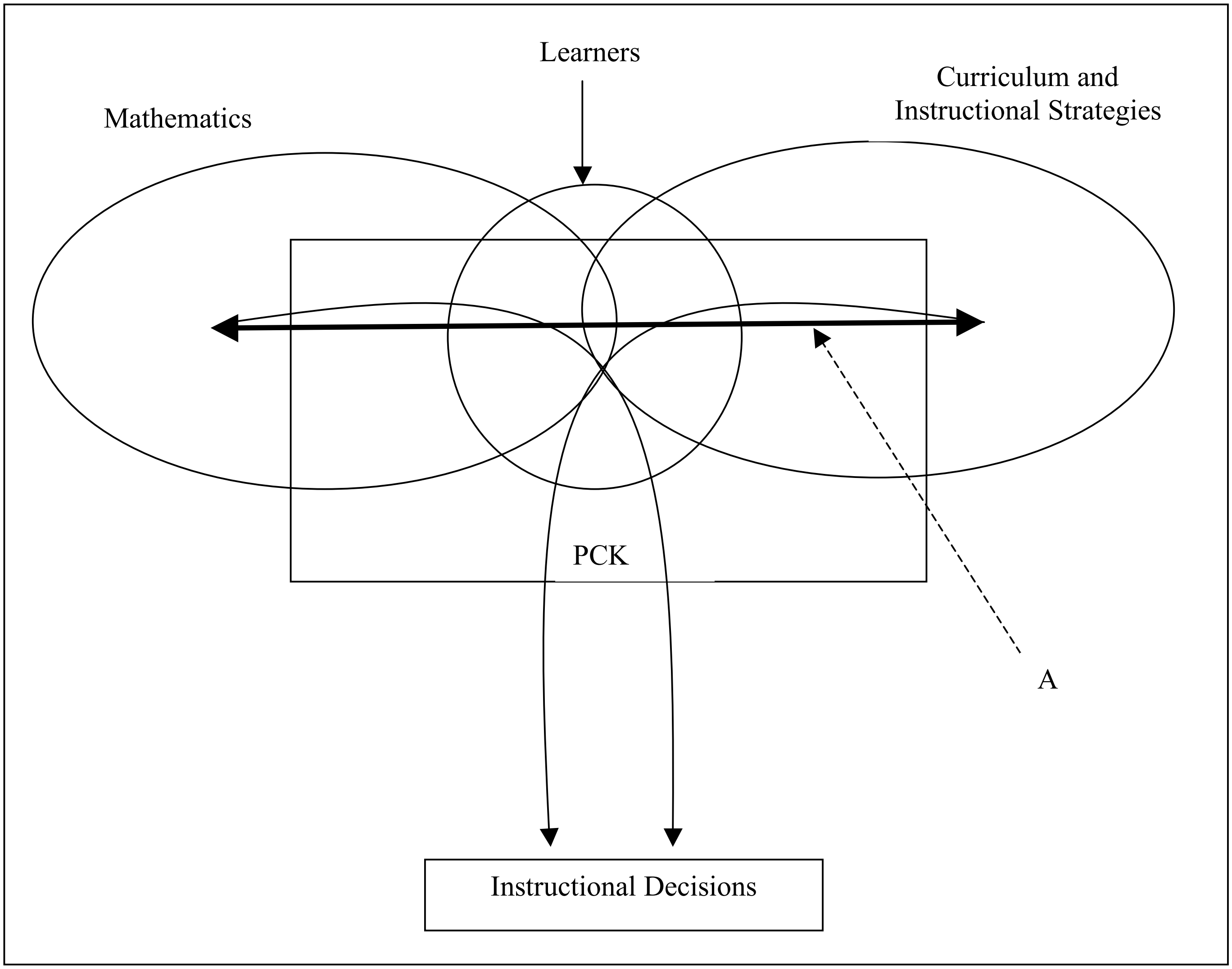


Figure 5.10: Relationship of the components of PCK- The role of L

In this new interpretation, the knowledge of learners was utilized to help the teacher make a connection between her knowledge of mathematics and her knowledge of curriculum and instructional strategies, as demonstrated by the CI-L-M-L-CI pattern of knowledge used by the teachers in the example above. The double arrow A shows the flow of knowledge between the knowledge of instruction and the knowledge of mathematics.

The role of knowledge of mathematics and knowledge of learners. This model can be further delineated if you consider how Amanda and Emily used the various knowledge interactions in their discussion of teaching. Table 5.11 depicts Amanda and Emily's use of their knowledge of mathematics in combination with their knowledge of learners.

Summary of Amanda and Emily's Knowledge Use					
	Primary Focus	Support/ Implication	Amanda	Emily	Total
M & L	M →	L	0	6	6
	M ←	L	1	1	2
	L →	M	5	5	10
	L ←	M	38	10	48

Table 5.11: Amanda and Emily's use of M&L

From the data, we can see that 58 interactions involved the knowledge of learners as the primary focus of conversation, 48 of which used the knowledge of mathematics in support of a claim about learners. There were only 8 total episodes where mathematics was the focus. Hence, when these two forms of knowledge were used in connection the knowledge of mathematics was mainly used to support a claim about learners. Figure 5.12 represents the addition of this information to the evolving model.

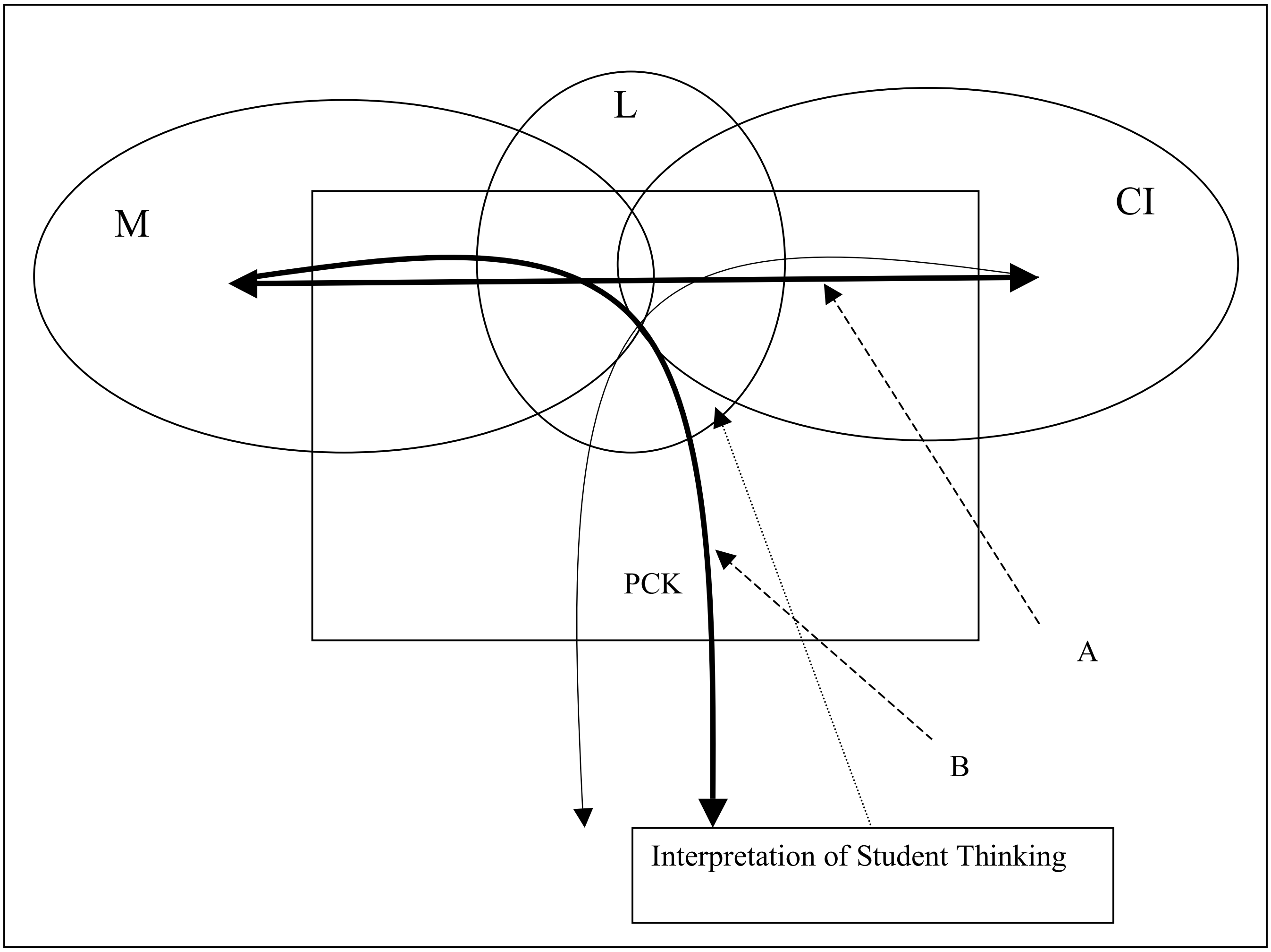


Figure 5.12: Relationship of the components of PCK- The role of M&L

Arrow B represents the interaction of Amanda and Emily's knowledge of mathematics with their knowledge of learners; this interaction often resulted in an interpretation of student thinking. If this interpretation was a new insight about student thinking it was then added to their existing knowledge of learners. The dotted arrow starting at the Interpretation of Student Thinking and returning to the knowledge of learners represents this.

The role of knowledge of learners and knowledge of curriculum and instructional strategies. Further additions to the model can be made if we consider how Amanda and Emily combined their knowledge of learners with their knowledge of curriculum and instructional strategies. Table 5.13 depicts Amanda and Emily's use of their knowledge of learners and their knowledge of curriculum and instruction.

Summary of Amanda and Emily's Knowledge Use						
	Primary Focus		Support/ Implication	Amanda	Emily	Total
CI & L	CI	→	L	11	12	23
	CI	←	L	19	15	34
	L	→	CI	22	15	37
	L	←	CI	23	13	46

Table 5.13: Amanda and Emily's use of CI&L

The data show that numerous interactions occurred where curriculum and instructional strategies (57 interactions) and learners (83 interactions) were the focus of conversation. In addition, both forms of knowledge were supported by, and were implications of, other forms of knowledge. In Figure 5.14, arrow C represents the interaction between the knowledge of curriculum/instructional strategies and the knowledge of learners that often resulted in either an implication or a justification of one of the forms of knowledge.

This new model represents the overall use of Amanda and Emily's knowledge of mathematics, learners, and curriculum and instructional strategies. It describes the interactions that occurred among the three subcomponents of PCK and the purpose for their collective use. Most of the outcomes, or focus of conversations, were on learners or curriculum and instructional strategies. In addition, the model portrays whether the knowledge interaction was used in support of an instructional claim or as an implication. Finally, the central role that the knowledge of learners played in connecting the knowledge of mathematics and the knowledge of curriculum and instructional strategies is also incorporated. There is one significant interaction that is not yet illustrated in the model, the fact there were 16 interactions in which the knowledge of mathematics was used to support an instructional claim--without the aid of the knowledge of learners. I have added arrow D to illustrate this final connection. Although other interactions did exist, I chose not to include them due to the infrequency of their use.

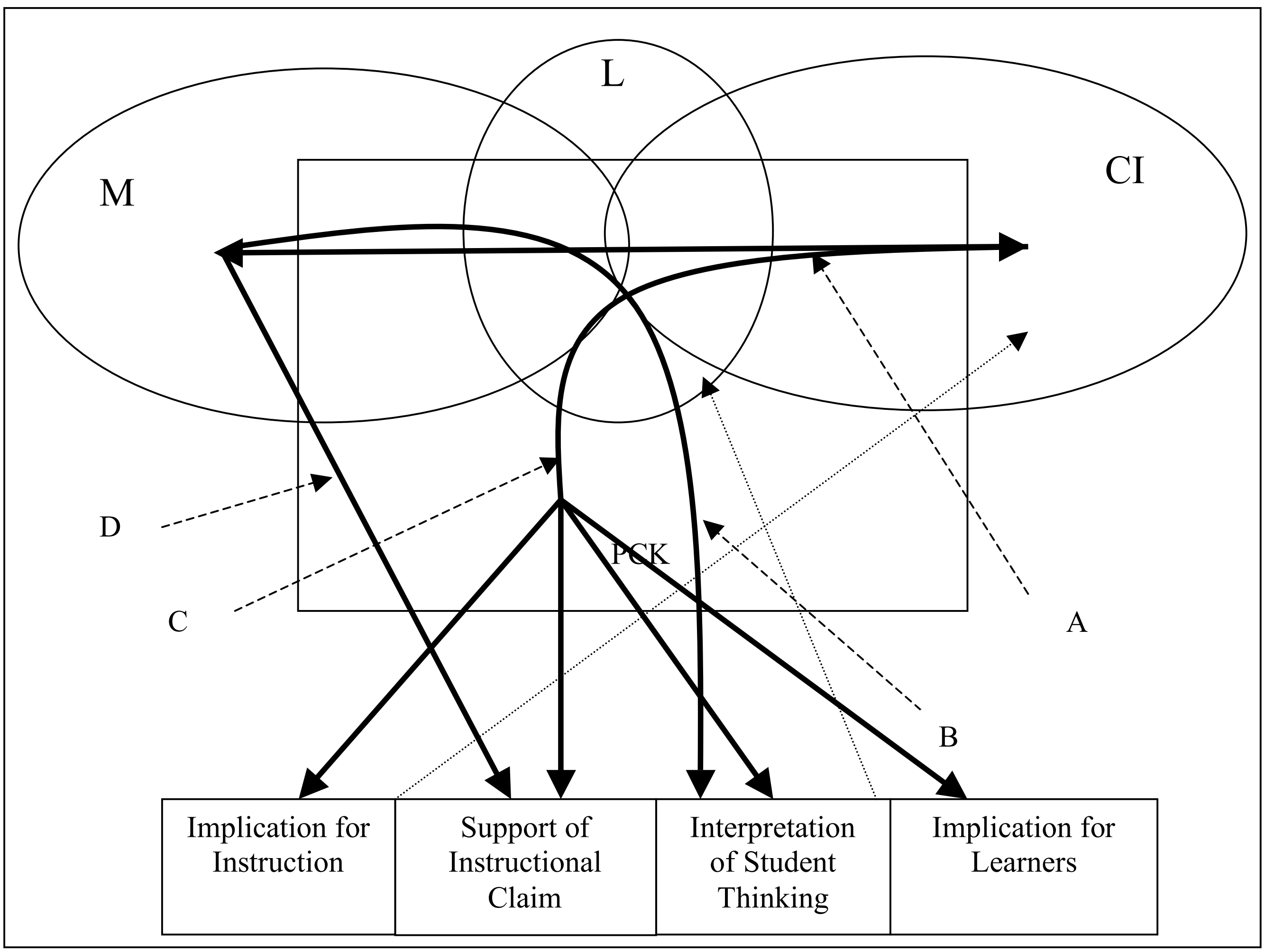


Figure 5.14: Relationships among the components of PCK

Shulman (1986) described pedagogical content knowledge as the missing link that connected a teacher's knowledge of pedagogy with their knowledge of mathematics. Shulman and Grossman (1988) later stated that PCK is made up of multiple forms of knowledge and identified five subcomponents. The model provided in Figure 5.14 describes the interactions that occurred among three aspects of PCK for Amanda and Emily.

The model of Amanda and Emily's knowledge use reveals some important characteristics that might otherwise have gone unnoticed. For instance, one characteristic was the critical role that the knowledge of learners played in how Amanda and Emily used their pedagogical content knowledge to make instructional decisions. With this in mind, we must return to the questions, "What knowledge do teachers need in order to teach effectively" and "Where does student thinking fit into our teacher education programs?" If we want to help teachers better utilize their pedagogical content knowledge, and to get the most out of their knowledge of mathematics and pedagogy, we must have a better understanding of how these knowledge sets are used collectively for instruction.

The role of knowledge of mathematics. This study of pedagogical content knowledge revealed characteristics of how the subcomponents of PCK interacted. For instance, the knowledge of learners appeared to play a critical role in how Amanda and Emily connected their knowledge of mathematics with their knowledge of curriculum and instructional strategies. In addition, the data revealed the role that mathematics played in the decision making process of teachers. The knowledge of mathematics was primarily used in a supportive role, usually to help support or justify a claim about learners or curriculum and instructional strategies. This may partially explain the fact that subject matter alone does not ensure effective teaching performance (Kahan, Cooper, and Bethea, 2003) and suggests that how mathematical knowledge is used in conjunction with other forms of teacher knowledge is crucial. Table 5.15 depicts the ways in which Amanda and Emily used their knowledge of mathematics.

Summary of Amanda and Emily's Knowledge Use					
	Primary Focus	Support/ Implication	Amanda	Emily	Total
M & L	M →	L	0	6	6
	M ←	L	1	1	2
	L →	M	5	5	10
	L ←	M	38	10	48
M & L	M →	CI	1	2	3
	M ←	CI	0	0	0
	CI →	M	0	1	1
	CI ←	M	8	8	16

Table 5.15: The role of Mathematics

Of the 86 interactions that involved mathematics, only 11 had mathematics as the focus of conversation; in all other interactions mathematics played a supportive role. The main use of mathematics was to either justify/explain a claim about learners (48 interactions) or to justify/explain a claim about curriculum and instructional strategies (16 interactions). The use of mathematics by Emily and Amanda suggests the need for a reconceptualization of our view of how teachers use their knowledge of mathematics and a subsequent adjustment in the content of the mathematics component of our teacher preparation programs to account for this change in vision. If teachers use their knowledge of mathematics to support their interpretation of student thinking, then content that is connected to student strategies and misconceptions needs to be included in the mathematics curriculum for teachers.

Variability. The fact that pedagogical content knowledge is the blending of several forms of knowledge for the purpose of making instructional decisions creates variability in how it is used by teachers. In addition, the different forms of teacher knowledge interact and influence each other (Voss et al., 1986; Alexander et al., 1989; Walker, 1987). This variability was evident in the use of PCK by Amanda and Emily. For instance, Amanda focused primarily on her knowledge of learners and used knowledge combinations for the purpose of justifying claims about learners. Emily, on the other hand, was more balanced in her use of the various knowledge bases of PCK. Emily had roughly the same number of episodes that were focused on learners as she had focused on curriculum and instructional strategies. Furthermore, she also demonstrated more variety in how she combined her knowledge. Emily was the more experienced teacher of the two, which raises the questions of how experience impacts the integration and use of knowledge.

Figure 5.14 represents a summary of how Amanda and Emily together used their knowledge of PCK. If a model were generated to describe the data from just Amanda, the schematic would need to be adjusted to portray her focus on learners and her use of knowledge to justify. The basic components of the model would remain the same, but there would be a different emphasis placed on certain relationships. Moreover, with a sample of only two teachers, other connections may become evident that are important to the implementation of PCK that were not used by Amanda or Emily. However, despite this limitation, Figure 5.14 does appear to be a useful tool in mapping out a teachers' use of her pedagogical content knowledge. The next section focuses on the implications this study has for professional development and research.

Implications

In the following section, I provide the implications that my study of pedagogical content knowledge has for the professional development of teachers and for future research methodologies.

Professional Development

The fact that Amanda and Emily used knowledge in an integrated fashion has implications for the development of future and current teachers. If teachers use knowledge in an integrated way, then it seems reasonable that they should experience the integrated nature of this knowledge while they are learning to become teachers, or continuing their education. However, research has shown that preservice teachers have underdeveloped pedagogical content knowledge (Borko et al., 1992; Borko & Putnam, 1986) and hence lack experience integrating various knowledge forms in making instructional decisions. Unfortunately, there has historically been a compartmentalization of these different forms of knowledge in the courses required of teachers. Ball (2000) stated,

> Subject matter knowledge and pedagogy have been peculiarly and persistently divided in the conceptualization and curriculum of teacher education and learning to teach. This fragmentation of practice leaves teachers on their own with the challenge of integrating subject matter knowledge and pedagogy in the contexts of their work. Yet, being able to do this is fundamental to engaging in the core tasks of teaching, and it is critical to being able to teach all students well. (p. 241)

Hence, a need exists to develop experiences for teachers that better reflect the integrated nature of the knowledge used in teacher practice. There are several examples of professional development projects that integrate content and pedagogy (Ball, 1988; Simon & Blume, Year). In these experiences the learning of content knowledge is done in connection with pedagogy. More research needs to be done to understand the effect of these types of experiences on teacher knowledge and practice.

In addition to developing courses that integrate content and methods, or other forms of knowledge, we must also soften the boundaries that exist between these terms. Neagoy (1995) commented that the education community breaks knowledge up into distinct parts for discussion that are indistinguishable in practice. Marks (1990) stated that,

> Another important finding was that the three primary types of knowledge—subject matter, pedagogical content, and pedagogy—are more overlapping and integrated than discrete. In light of this, university faculties in education and in the academic disciplines would be well advised to soften the traditional distinctions between content and methods. (p. 10)

Since knowledge is used in an integrated fashion in the classroom, and these separate knowledge forms are hard to distinguish, I recommend that courses with the goal of preparing future, or current, teachers have a blending of knowledge types. These courses should emphasis the connections that exist among the different forms of knowledge, make explicit the use of various knowledge bases, and illustrate how they are used collectively. For example, if you are discussing a mathematical topic, expand on how this knowledge would influence how you would design a lesson or the student thinking that in would evoke.

The Importance of Context. The fact that Amanda and Emily used their knowledge in an integrated fashion while they taught illuminates the importance of context to the use of teacher knowledge. I previously discussed the need for teacher educators to view teacher knowledge as integrated; the next important question to answer is "how do teachers acquire integrated forms of knowledge?" Ball (2000), referring to learning content knowledge in an integrated fashion, stated that "one promising possibility is to design and explore opportunities to learn content that are situated in contexts in which subject matter is used, a core activity of teaching" (p. 246). Hence, acquiring integrated teacher knowledge may be linked to experiencing this knowledge within the context of teaching. This study did not address the question of how teacher knowledge is acquired, but the findings suggest that context is an avenue worth further exploration.

Teacher development programs should provide opportunities for teachers to discuss and learn knowledge in the context of practice. The context of the classroom provides teachers with knowledge that is incorporated into the decision-making process; this is illustrated by Amanda and Emily's use of their knowledge of learners. Eraut (1994) commented that, "The key to unlocking these problems (of teacher education) was my realization that learning knowledge and using knowledge are not separate processes but the same process" (p. 25). If teacher knowledge and teacher practice are indeed the same process, then connections between knowledge and practice need to be incorporated into every aspect of professional development. Inservice teachers need to be encouraged to implement and explore new ideas and knowledge in the context of teaching, and to reflect on these experiences. Preservice teachers need to be given opportunities to observe and participate in classroom instruction that connects to the ideas they learn in their course and fieldwork. We must develop means to simulate classroom experiences—peer teaching, lesson study—that allow students to benefit from the context of teaching. I discussed earlier that education courses should be an integration of knowledge forms. In addition, we must also situate these courses in the context of teacher practice.

The role of student thinking. One implication of the findings relates to the question, "What knowledge do teachers need?" A glance at teacher education programs across the country reveals that teachers take a combination of mathematics, methods, and general education courses to become qualified teachers. The crucial role that the knowledge of learners played in how Amanda and Emily used their PCK suggests that the use of student thinking can serve as a springboard for professional development. Research has shown that knowledge of student thinking is important to the development of pedagogical content knowledge (Borko, Livingston, McCaleb, & Mauro, 1988) and that preservice teachers often lack such knowledge of students and are unaware of the informal knowledge that students bring to the classroom (Borko & Putnam, 1996; Grouws & Schultz, 1996).

Furthermore, if the knowledge of learners can play a role in connecting the knowledge of mathematics to the knowledge of curriculum and instructional strategies, then we should use student thinking as a vehicle for the integration and learning of the knowledge of mathematics and pedagogy. Developing courses that are designed around the mathematical and pedagogical issues of student thinking appears to be one way to help preservice teachers interconnect their knowledge as they will when teaching. We need to use student thinking as an explicit springboard into explaining and justifying instructional decision-making.

The role of mathematics. Emily and Amanda used their knowledge of mathematics primarily to justify/explain a claim about learners or to justify/explain a claim about curriculum and instructional strategies. This use of mathematics begs the question, “What mathematics do teachers need to understand to be effective teachers?” The data suggest that the mathematics of student thinking and the mathematics of curriculum should be included. Unfortunately, the mathematics of student thinking and curriculum are not always given this necessary attention in content courses for teachers.

This raises the question of, “How do we determine the mathematical makeup of teacher coursework?” Historically, this determination was done based on opinion or a mathematical analysis of school curriculum. Recently, this identification of knowledge has become more informed by research on student learning. However, more needs to be done to identify the critical components of mathematical knowledge required for teaching. This knowledge should reflect how teachers, such as Amanda and Emily, use their knowledge of mathematics in the classroom.

The complexity and difficulty of this issue derives from the fact that teaching is a complex endeavor that occurs in an ever-changing environment, which makes it difficult to research in its entirety. Furthermore, the teacher knowledge useful to teachers may differ depending on a variety of contextual factors such as curriculum, teacher philosophy, and student characteristics. Different instructional settings place different demands on a teachers' repertoire of teacher knowledge (e.g. Schifter, 1998; Stein, Silver, & Smith, 1998). How then should this research be conducted?

Simply understanding the mathematics presented in a textbook is not sufficient to address all the needs of teachers, other perspectives must be considered. From a mathematical viewpoint, teaching a particular topic well requires "deep" understanding of the underlying mathematical concepts. From a student thinking perspective, we realize the challenges created by divergent student strategies, each with mathematical underpinnings that need to be understood by the teacher. "One possible reason for which the problem of what teachers need to know remains unsolved is that people have been looking at [teacher knowledge] from different angles, having available only limited research that concentrated on connecting the knowledge produced by different perspectives" (Stylianides & Ball, 2005, p. 1). However, it still remains one of the most critical questions that we as mathematics educators must answer, and to this point considerable work needs to be done to better understand this phenomena.

Several approaches are possible for answering the question, "What mathematics do teachers need?" Although this wasn't the focus of my study, the perspective used for my study, looked at the knowledge used in classroom practice, could be utilized to answer this question as well. What mathematical knowledge did these teachers draw upon when making instructional decisions in the classroom? In addition, the findings of this study suggest that investigating the mathematics of student thinking may be a crucial avenue for identifying content knowledge for teaching.

Stylianides and Ball (2005) discussed six strategies for answering this question, two of which are the study of mathematical practice and the study of student thinking.

We return to Eraut (1994) to support this perspective as he states, “professional knowledge cannot be characterized in a manner that is independent of how it is learned and how it is used” (p. 19). His argument is that if you want to understand professional knowledge you must understand how it is used in practice. Stylianides and Ball (2005) suggests that this approach “begins with the study of records of mathematical practice (e.g. videotapes of classroom lessons, transcripts, student work, teacher journal) to uncover and describe in fine mathematical detail the work in which teachers engage in order to understand the mathematical knowledge needed for effective functioning in the course of this work” (p. 27). Unfortunately, this approach has not been widely explored as Stylianides and Ball state, “Although a growing number of studies have provided detailed analyses of school mathematics practice with relative emphasis on elementary students’ activity in reasoning and proving (e.g. Ball & Bass, 2003; Carpenter et al., 2003; Lampert, 1990, 1992; Maher & Martino, 1996; Reid, 2002; Zack, 1997), only few have presented this work as part of a principal interest to uncover less exposed mathematical entailments of the work for teachers, seeking useful answer to questions about what teachers need to know and be able to do in order to teach” (Ball & Bass, 2000, p. 198).

When different perspectives are simultaneously considered, a broader and more complete answer to our ultimate question begins to appear. Stylianides and Ball (2005) suggest that connecting the knowledge produced from different perspectives may ultimately aid in addressing the question of mathematical content knowledge for teaching. The data from this study suggest that the perspective of classroom practice and student thinking may be important aspects for this determination.

Research Methodologies

The fact that Amanda and Emily used their knowledge in an integrated fashion during instruction has several implications for the mathematics education research community. First, if teacher knowledge is used in an integrated fashion in the classroom, it behooves the educational community to study this phenomenon in its complexity. Unfortunately, this has not yet been fully realized. Fennema and Franke (1992) concluded that,

> Common sense suggests that knowledge is not monolithic. It is a large integrated, functioning system with each part difficult to isolate. While many have speculated about the components of teacher knowledge, few have received major attention from researchers: content knowledge, knowledge of learning, knowledge of mathematical representations, and pedagogical knowledge. Some have studied knowledge as integrated, but most have not (p. 148).

Investigating a complex phenomenon such as teacher knowledge presents difficulties for educational researchers. According to Eraut (1994), "in practice different types of knowledge and modes of cognition are integrated into professional performance in ways that are difficult to unravel, either conceptually or empirically" (p. 19). This difficulty requires new research methodologies to be developed and refined to address the complexity of teacher knowledge.

An unintended consequence of the isolated study of teacher knowledge is the isolated presentation of that knowledge to teachers. Learning knowledge in isolation may lead to knowledge that is not pertinent or useful to the act of teaching. Ball (2000) stated, "usable content knowledge is not something that teacher education, in the main, provides effectively" (p. 242). In order to determine the elements of teacher knowledge that are useful in the act of teaching, we must start by integrating teacher knowledge in practice and in connection with other forms of knowledge. For instance, if we want to understand what mathematical content knowledge is useful to teachers, we must look at the mathematical components of PCK that are connected to the act of teaching. "Our understanding of the content knowledge needed in teaching must start with practice. We must understand better the work that teachers do and analyze the role played by content knowledge in that work" (Ball, 2000, p. 244). Hence, in addition to conducting research that addresses the complexity of teacher knowledge, we must also study this knowledge within the context of practice.

The overview of the history of research on teacher knowledge provided in Chapter 2 brought additional methodological considerations. How do we determine what knowledge is useful in promoting student learning and understanding? For many years researchers used multiple-choice tests, or other easily obtainable measures, to acquire an understanding of teacher knowledge. Eraut (1994) argued that:

> Most examinations guaranteed only that knowledge they were able to test; and this seldom extended to practical competence. Hence one of the main consequences of their introduction was the transformation of large areas of the professional knowledge base into codified forms, which suited the textbooks needed to prepare students for what were from the outset very traditional examinations. (p. 7).

Eraut (1992) continued his argument stating that a result of this codifying process was that much of what is taught in teacher preparation programs does not include the practical aspects of teacher knowledge that are necessary for successful introduction into the teaching community, but rather more formal knowledge. He stated that,

> The norms of higher education tend to favour scientific knowledge rather than professional knowledge, and to encourage research priorities from those likely to be espoused by the professions, thus helping to widen the gap between professional educators and their erstwhile professional colleagues. (p. 9).

Renick (1987) also suggested that we close the gap between the theory of the university and the practice of the classroom. This phenomenon in higher education may in fact have its roots from research on teacher education that is divorced from the practice of teaching. Leinhardt et al. (1991) supported this statement when they stated, "the knowledge that a teacher needs to have or uses in the course of teaching a particular school-level curriculum in mathematics is different from the knowledge of advanced topics that a mathematician might have" (p. 188).

Eraut's arguments encouraged a more integrated view of teacher knowledge as he stated:

> Such segmentation and packing of knowledge for credit-based systems seems inappropriate preparation for professional work which involves using several different types of knowledge in an integrated way; and the pedagogic approaches needed for linking book knowledge with practical experience are almost impossible to implement where there is little continuity in the membership of the student group. (p. 10).

Ball (1988) has begun to address this issue by developing and using mathematics instruments that probed teachers' knowledge of mathematics in the context of everyday situations that occur in the course of teaching.

In summary, we need to develop and apply research methodologies that investigate teacher knowledge in the act of practice. We must gain an understanding of the knowledge used by teachers in the classroom if we are going to further the concept of teacher knowledge.

Future Research

Professional Development

One of the interesting side notes to this study was that both teachers claimed to have learned a considerable amount from participating in this study. This should not be a surprise given the previous discussion of the benefits of integrating the learning and usage of knowledge. By requiring Amanda and Emily to constantly reflect on their teaching, I forced them to intertwine their learning of knowledge with its application to the act of teaching. Figure 5.2 illustrates the process of using the different aspects of their knowledge to create conclusions about instruction and student thinking. In essence, the teachers were using their prior knowledge in an integrated manner to create new knowledge. Hence, the act of reflecting on their teaching was a means of learning. Is this process of reflecting on knowledge in the context of practice is a viable way to develop integrated teacher knowledge? I recommend that we continue to develop and study models of teacher learning that take into account the integrated manner in which knowledge is used and created.

To investigate the connection between using and learning knowledge I examined the length of the comments provided by the teachers. When Amanda and Emily applied their knowledge, comments were, at times, relatively short. However, when they struggled with a concept, their discussions were significantly longer. For instance, a particular explanation of a single thought at times took multiple paragraphs. Such lengthy expositions increased the likelihood of that thought containing multiple forms of knowledge; one might argue that it would almost necessitate multiple forms of knowledge. In essence, some decisions made by the teachers evoked considerable discussion and involved multiple forms of knowledge, while others provoked only a passing comment.

The fact that Amanda and Emily's episodes varied in length, and they used their knowledge differently, may be indicative of another characteristic of teacher knowledge--that a particular teacher's knowledge is in constant development, with some areas more developed and constant, with other areas still evolving. Some decisions that a teacher encounters in the classroom have been addressed hundreds of times and have become tacit in nature, not requiring an analysis of various forms of knowledge in order to make a decision. However, other decisions are relatively new and require contemplation. These new decisions are more likely to produce longer explanations that need multiple forms of knowledge. How do we get teachers to reflect on important aspects of teaching so that they can develop integrated knowledge?

I discussed in this chapter that teachers need to acquire knowledge in an integrated fashion, and the above arguments suggest that using knowledge in an integrated fashion may perpetuate the learning of integrated knowledge. More research needs to be conducted on how teachers develop and learn to use their knowledge in an integrated manner in the classroom. What can we do as teacher educators to promote this learning? Can opportunities be provided to them in their teacher preparation programs to develop this integrated teacher knowledge? If not, how do we adjust our programs to accommodate this? Do we develop programs that extend beyond the tradition four years of schooling to extend into the first few years of experience so that learn through the act of practice? It is not enough to know what knowledge is necessary to teach; we must also know how to develop that knowledge. We need to develop models for how this learning takes place, and research the effectiveness of different professional development models that address integrated teacher knowledge.

The impact of contextual factors on teacher knowledge

In selecting the teachers for this study, I chose two individuals who used a reform-oriented curriculum and who held beliefs that were consistent with *Standards* based instruction. This raises the question, would I have observed the same results if I had chosen participants who were teaching out of a traditional text, or who held different beliefs about teaching and learning? How do contextual factors affect the use of knowledge in the classroom? Would teachers who are using a different curriculum, with high fidelity, use these three types of knowledge in different combinations when making decisions in the classroom? Do student characteristics change the way in which teachers use their knowledge? My initial reaction is that a more traditional curriculum would require less knowledge in general, less integrated forms of knowledge, and less reliance on student thinking. It seems reasonable to assume that a teacher's personal philosophy of learning and teaching would impact their use of knowledge in the classroom. Finally, I think it is important to investigate the spread of knowledge used by teachers at different grade levels, with different courses, and with students of varying abilities. Again, my hypothesis is that teachers at higher grade levels, and with more advanced classes and students, would use a higher percentage of mathematical knowledge when making instructional decisions. More research needs to be conducted to understand the difference in the way knowledge is used among different groups of teachers and under different school conditions.

3-Dimensional Knowledge Packages

This study of pedagogical content knowledge focused on Amanda and Emily's pedagogical content knowledge in a general sense. I did not analyze the data to the level of isolating the content of their statements. For instance, Emily's understanding that the doubling strategy would not work for the straw problem led her to recognize the misconception that student' believed that it would work, which then led her to spend instructional time having the students explore the issue. Here we have a connection that Emily made between three different aspects of PCK. Ma (1999) developed knowledge packages that described the connections that American and Chinese teachers made between different aspects of their mathematical knowledge of division of fractions. The data in this study could be further analyzed to create knowledge packages that outline the connections between Amanda and Emily's knowledge of mathematics, knowledge of learners, and knowledge of curriculum and instructional strategies, as it relates to algebraic generalization.

Teacher Decision Making

My goal in this study was to understand how teachers drew upon their knowledge of mathematics, their knowledge of learners, and their knowledge of curriculum and instructional strategies when making instructional decisions in the classroom. However, my study only describes the cognitive part of the decision-making process. In order for teachers to make decisions they must also gather information from their students, a social process. Figure 5.16 represents this process.

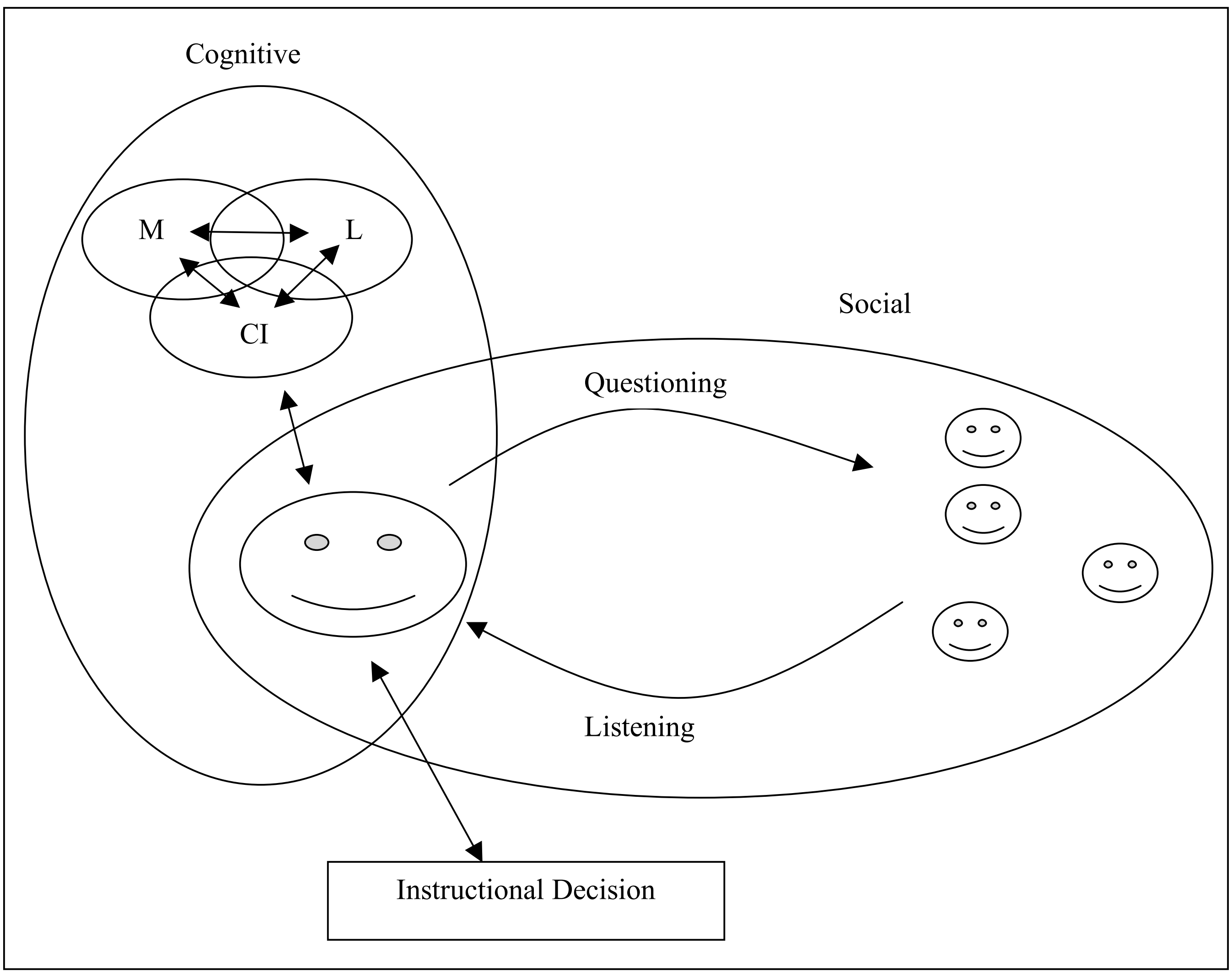


Figure 5.16: Teacher Decision Making Framework

In this framework, the teacher constantly gathers information from her students through a questioning/listening cycle. At the same time she processes this information using her integrated forms of teacher knowledge. Through this process she combines the information she gathers from her students with her teacher knowledge to generate and to support instructional claims. My research focused on the cognitive aspect of this process, but research needs to be conducted that connects these two aspects in order to gain a more complete understanding of the teacher decision-making process.

Limitations

There were several limitations that need to be considered when interpreting the findings of this study. First, the teachers selected for this study, and the curriculum they used, were not representative of the fifth grade teacher population. The knowledge used by teachers is related to the philosophy they hold about teaching and learning. Given that both Amanda and Emily used the *Investigations in Number, Data, and Space* curriculum (Tierney, Nemirovsky, Noble, & Clements, 1998), which promote and build upon student thinking, and both held teaching beliefs consistent with the curriculum, it seems reasonable that their comments and arguments focused on their knowledge of learners. This may have resulted in the high percentage of knowledge of learners found in the results.

Furthermore, the *Investigations in Number, Data, and Space* curriculum (Tierney, Nemirovsky, Noble, & Clements, 1998) unit *Patterns of Change* often commented on how students will react, offered suggestions on how to teach, and presented the mathematics behind the lesson, all of which could impact the conversations of the teachers.

My questioning inevitably affected the responses given by the teachers. I stated at the beginning of each interview that I wanted them to comment on their knowledge of mathematics, student thinking, and curriculum, but a bias towards one particular type of knowledge may have developed unintentionally during the interview.

Another difficulty that arose during data analysis was the coding of comments into the three forms of knowledge. Many comments had components of more than one form of knowledge that became almost indistinguishable. Should other aspects of PCK have been included? Would this additional form of knowledge led to a better understanding of how teachers used their knowledge of PCK?

In addition, my perspective had an influence on what I saw, the questions I asked, and the responses provided by the teachers. I have a keen interest in student thinking that may have resulted in a focus on the knowledge of learners aspect of the framework and resulted in a skewing of the data. In addition, Amanda stated that she had a focus on student thinking as well.

Furthermore, data involving Amanda and Emily's background regarding each aspect of the model was not collected. It was observed that both teachers used their knowledge of mathematics the least. My conclusion was that this occurred because their knowledge of mathematics was primarily being used in support, it was not the focal point of their conversations. An alternative explanation could be that there background in mathematics was not as strong as the other two areas and so it was utilized less.

Finally, my conclusions are based on data from two teachers. Thus, my conclusions are meant to be a description of what occurred with Amanda and Emily under the conditions that existed in their classrooms at the time of data collection. The results may provide insight into the structure of PCK that warrants further investigation, but one should be careful to the extent that these findings are generalized without further research.

Final Thoughts

The data in my study allowed for an examination of how the various forms of knowledge were used in collaboration to make instructional decisions. As was stated previously, most decisions or comments made by Amanda and Emily involved either a curriculum and instruction decision or an examination of student thinking. In either case, a conclusion was usually made and the other forms of knowledge were used to provide support for that particular conclusion. For instance, Emily may have commented on a particular decision concerning what a student was thinking and then provided support for that decision that included elements of her knowledge of mathematics, learners, and curriculum/instructional strategies.

In essence, this support can be seen as the teacher's justification of her decision. The more avenues of support a teacher can provide, the stronger her belief is that she made the correct decision. Justifying decisions from different perspectives (i.e., using different knowledge bases) serves as a method to self justify an instructional decision. Furthermore, the knowledge of learners played an integral role in the process of knowledge integration. This understanding of how teachers use knowledge provides further insight into how we can stimulate the use of these more complex forms of knowledge. Not only should we mix content and methods and provide teachers with opportunities to interact with the context of teaching, we need to push our teachers to justify these contextual decisions and conclusions, which is often done through their knowledge of student thinking. In a sense, we are not just providing them with knowledge, we are developing their ability to create and justify knowledge about their own classroom. Teacher development programs will never be able to provide all the knowledge that a teacher needs, the best we can do is to provide them the tools to obtain this information in an efficient and effective manner.

Pedagogical content knowledge is critical to the success of teachers and has been the focus of much thought, debate, and research since Shulman coined the term. Despite this attention, it remains one of the most misunderstood constructs in education. Much attention has been given to defining PCK as it has gone through many revisions through the years, but more attention needs to be spent on understanding how teachers use this form of knowledge in the classroom. I, too, struggled with defining PCK and came to the conclusion that PCK is more than factual knowledge, or the integration of various forms of knowledge; a critical aspect of PCK includes combining and connecting knowledge for the purpose of instruction. The importance of my study to the larger mathematics education community is that it continues the investigation of this elusive construct PCK and investigated how teachers integrated three of the subcomponents of PCK in order to make instructional decisions. I studied more than just factual knowledge, but how and why they integrated their knowledge. In essence, I have gone beyond just looking at PCK as factual knowledge, and began to look as the process of integration.

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Appendix A

Letter to potential subjects

Dear Teacher,

My name is David Barker and I am a doctoral student from the University of Missouri. I am trying to locate three 5th grade teachers that would be interested in participating in my dissertation research. I am writing to you because John Lannin recommended you for inclusion in this project.

I am interested in how teachers use their knowledge to design and implement lessons related to patterns and hence I am very interested in the Patterns of Change Unit in Investigations. My goal is to better understand the knowledge that teachers use in the classroom so that we can better inform the content, design, and implementation of teacher development programs. To help understand this knowledge, I am inviting three 5th grade teachers to be participants in my dissertation study. My study will involve: 1) an initial interview with the teacher (~30 minutes); 2) Videotaped observations of the teacher teaching approximately three math lessons, including follow-up interviews in which the teacher would watch the video-taped lesson and react to and provide insight into how they used their knowledge in making instructional decisions (~ three 60 minute interviews); and 3) A follow-up interview (~45 minutes) occurring two weeks after the last observation.

All reporting of data will be confidential and copies of any published finding involving the data will be sent to the participants at their request. Data collection will take no more than 4-6 hours outside of a teacher's regular school responsibilities for which the teacher will receive a $XXX stipend. In addition, a workshop on algebraic reasoning in the early grades and/or a session on the findings of the study will be provided for the participating schools. The study in itself would be a professional development opportunity, as we would be working together to better understand the knowledge that goes into teaching. If you are interested in participating in this study, or have any questions, please contact me (ddb21d@mizzou.edu), or my dissertation advisor Dr. John Lannin (lanninj@missouri.edu).

Thank you for considering this request!

Best regards,
David Barker

Appendix B

Overview provided to research participants

Contact:

David Barker: email: ddb21d@mizzou.edu
Phone: 573-882-0382
John Lannin: email LanninJ@missouri.edu
Phone: 573-882-6774

Purpose:

The purpose of my research is to: 1) understand the nature of teachers' knowledge of algebraic generalization and 2) To understand how teachers' use their knowledge in designing and implementing instruction. Once we have a better understanding of the knowledge that teachers use in practice we can utilize that information to improve the content, design, and implementation of teacher development programs.

Incentives:

- $XXX Stipend for each participating teacher.
- Workshop, approved by Linda Coutts, dealing with the algebraic generalization strategies that students use and the factors that influence student use of a particular strategy.

Participant Involvement:

- Initial Interview – completed week prior to observations (45 minutes)
- Video-taped observation (approx. 3 class periods)
- Stimulate Recall Interview – Teachers will be asked to watch the video of their teaching and reflect on the knowledge they used in making instructional decisions. (60 minutes each)
- Follow-up Interview – completed two weeks after observations (45 minutes) (4-6 total hours of out of class time)

Participant Requirements:

- Experience using Investigations in Number, Data, and Space sections
- Recommended by Principal, Coordinator, or Mathematics Education Faculty as being an exceptional teacher.

Time Frame: Data to be collected by the end of the school year.

Appendix C

Initial Survey and Interview Protocol

Description of study to be read to the participants:

The purpose of this project is to better understand your thinking during mathematics instruction. This initial interview is intended to get an understanding of your background in providing instruction in Algebraic Generalization and your preparation for teaching this unit of Patterns of Change.

Part I: Background

1. How many years have you taught (any grade or subject) prior to this school year?

2. What grade levels have you taught and how many years did you teach each grade level?

3. What degrees do you hold?

4. What teacher certifications do you hold?

5. Describe the nature and content of your professional development in mathematics education during the past three years?

6. What curriculum have you used in teaching mathematics?

Part II: Algebraic Generalization.

1: Have you ever taught lessons that involve the investigation of patterns? Have you ever taught lessons that involve making generalizations? How many years have you taught such lessons?

What types of tasks have you used related to patterns or generalization? (Prompt for specific examples.)

What mathematical ideas are important for 5th grade students to understand related to patterns? Generalizations?

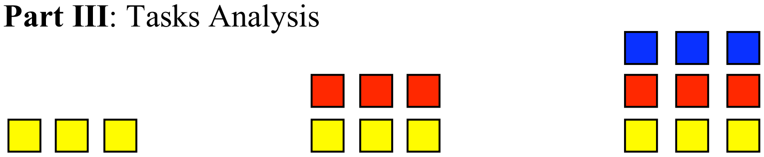
2: Describe a typical day of mathematics instruction in your class.

3: What is the overall goal of this series of lessons in *Patterns of Change*? What do you want your students to get out of these lessons?

What ideas do you think your students will struggle with?

4: What is the main goal of the lesson today?

Part III: Tasks Analysis



Step 1 Step 2 Step 3

What do you think my next step will be? How many tiles will I need to add?

Step Number	New Tiles (step size)	Total So Far
1	3	3
2	3	6
3	3	9
4		
5		

Questions:

1. What is the goal of this task?
2. What mathematical ideas are brought out by this task?

   What features of the task help bring out these ideas?
3. How do you expect your students to respond to the questions given in the task? How will these responses vary?
4. For the last question, "If I tell you the step number, what calculations would you do to get the total?" What strategies do you think students will use to find the fifth, seventh? Tenth? Nth? Will these strategies change?

   What is influencing the students to use these strategies?

What mathematics do you see coming out of the variety of strategies, or thinking, that students are using in this problem?

5. What misconceptions might result from this problem? What aspects of the task might lead to these misconceptions?
6. What will the students struggle with?
7. What are the strengths of the task? What are the weaknesses? What changes would you make?

Appendix D

Stimulated Recall Interview Protocol

Description of Study to be read to the participants:

The purpose of this research study is to better understand your thinking during instruction and your thinking about tasks. During this interview we are particularly interested in the types of knowledge and thinking that went into decisions made during the observed lessons.

Interview – Part I: The participant will be given the opportunity to watch the selected portions of video in their entirety. The teacher will be given time to reflect and provide further information regarding the thinking behind his/her instructional decisions.

Interview – Part II: The participant will then be asked to answer specific questions concerning specific portions of the videotape. Questions during this section will be specific to the episode in question and may include some of the following:

- Why did you ask that question?
- What do you think the student was thinking when they gave that response?
- Was the student correct? What misconceptions regarding the mathematics do they have?
- Why did you choose to follow-up on that student's response?
- What misunderstandings do you think your students are having at this point? What did you do to address them?
- What do you think your students understand well at this point in the lesson? What evidence do you have to support this?
- How do you think your students our understanding the concept of _______ at this point in the lesson? What adjustments did you make?
- How has the task influenced the way your students are thinking about the mathematics?
- What will you do tomorrow?

Appendix E

Final Interview Protocol

Interview – Part I: Patterns of Change

1. How do you think students perceive the idea of pattern? How did this change as a result of working through this unit? How do you think students view the generality of their rules?

2. Looking at this unit as a whole, answer the following questions:

    a. What are the goals of these tasks?

    b. What mathematical ideas are brought out as students work through these tasks?

    c. What features of the tasks, suggested questions, or representations help bring out these ideas?

    d. What misconceptions might result from these lessons? What aspects of the lesson might lead to these misconceptions?

    e. What did the students struggle with?

    f. What are the strengths of this lesson? What are the weaknesses? What changes would you make?

    g. What is your overall opinion of this unit? Why do you think so?

3. Looking at the four patterns given, answer the following questions:

    a. How are these patterns the same and how are they different?

    b. Are there certain tasks that were easier for your students and some that were more difficult?

    c. What difficulties did they have with any of these patterns?

    d. If you were revising this curriculum how would you change the selection of these tasks?

4. Reflecting on the suggested questions, and the questions that you asked, answer the following:

    a. Which questions do you feel are critical to the success of the lesson? Are there questions you would not ask?

    b. Are there other issues, that you as a teacher need to bring out, that are not addressed by these questions?

5. The following questions deal with the representations used in the first investigation:

    a. What types of representations do the student use in this investigation? What other representations do you have your students use?

    b. What mathematical ideas do these representations help to develop? How do they help to develop these ideas?

    c. Do the students struggle with connecting representations? What connections do you expect students to make among the representations? What connections do you expect the students not to make? What do you do to help build these connections?

    d. What patterns do you expect the students to see when analyzing the tables?

6. The following questions concern student strategy use:

    a. For the last question, "If I tell you the step number, what calculations would you do to get the total?" What strategies do you think students will use to find the fifth? Seventh? Tenth? Nth? Will these strategies change?

    b. What influences students to use these strategies?

    c. What mathematics do you see coming out of the variety of strategies, or thinking, that students are using in this problem?

Interview - Part II: Critique of Student Work

In this portion of the interview we will be focusing specifically on Pattern #3. I will be giving you 4 excerpts of student work and then ask you some questions about the student responses.

7. Ask the following questions for the all four vignettes:

a. How is the student attempting to solve the problem? Is the student correct? Why or why not?

   If the student is not correct, how would you address the strategy? When will the strategy the student is using work and when will it not work?

b. What mathematics is involved in the solution? What understanding or misunderstanding does this student have?

c. What aspects of the problem do you think the student was focusing on? How did the task influence her to address the problem in this particular fashion?

d. How is this student's strategy the same or different from the prior strategies? (Ask for vignette 2, 3, and 4)

Strategy 1

I noticed that after the first step there was 1 total tile, for the second step there are 5 total tiles, and for the third step there are 9 tiles. There is an increase of four each time so to get the number of stickers for a rod of length 4 you simply need to add 9+4 to get 13.

Strategy 2

I counted the total number of tiles for step 4 and found 13. To calculate the total number of tiles for step 7 we need to add 12 more stickers because there are three more groups of 4. Hence, the 7^{th} step is 13+12 or 25.

Strategy 3

The total number of tiles at step 5 is 17. If I wanted to know how many total tiles there are for step 10 I would simply double the 17 to get 34. If I wanted the total number of tiles for step 40, I would take 17 times 4 to get 68.

Strategy 4

I noticed that each leg of the cross pattern as one less tile than the step number. My rule would be 4 times the step number minus 1 plus one additional one for the tile in the middle.

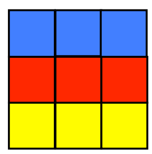
Appendix F

Combined Observation
Amanda – Lesson #1
April 14, 2005

Observation Transcript
Stimulated Recall Interview

Amanda: Be thinking, what did I do to our pattern? Think to yourself right now. I just want you to keep thinking, because I am going to show you another one, making sure everybody is watching.

The teacher places three blue tiles directly above the tiles already on the overhead.



A: There with my teaching, if you want me to point this out, I am trying to get them to think about it before I let anybody say their answer, not take the first answer, which is hard for even sometimes.

D: I noticed that you really stressed, "Stop, think"

A: Stop and think.

[2:45]
Amanda: Now that you have seen that, what do you think my next would be? What do you think I would do next if I were going to add to this pattern? ______, what do you think?

Student: Maybe 3 more in a different row

Amanda: 3 more in a different row. Do you think I add 3 more on this side?

A: And I don't think that that is something that I am really good at, I think I need to do it even more often, but once again that issue of timing, I felt rushed.

VITA

David D. Barker was born on July 6, 1973 in Mansfield, Missouri. He graduated from Mountain Grove High School in Mountain Grove, Missouri in 1991. In addition, he has earned the following degrees: B.S. in Mathematics Education from the University of Missouri-Columbia in Columbia, Missouri (1997); M. A. in Mathematics from the University of Missouri-Columbia in Columbia, Missouri (2002); Ph. D. in Mathematics Education from the University of Missouri-Columbia in Columbia, Missouri (2007). He is currently an assistant professor at Illinois State University in Normal, Illinois.