

General Relativistic Theory of Light Propagation
in the Field of Gravitational Multipoles

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The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled:

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in the Field of Gravitational Multipoles**

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Chapter 1

Introduction

1 Historical Background and Statement of the Problem.

It is well known that according to general relativity (GR) gravity alters propagation of electromagnetic waves. Among the classical tests of the theory are deflection of optical and radio waves from distant sources in gravitational field of the sun [1] and gravity-induced delay in time of propagation of radar signals sent towards a planet or a satellite so that they pass near the sun and returned to the Earth [2]. Recent measurements of these effects placed tight constraints on the post-Newtonian parameter γ which takes different values in different theories of gravity, showing that with the accuracy of 10^{-3} percent¹ this parameter should equal unity - the value it takes in GR [5, 6]. However, the classical experiments mentioned above deal only with propagation of electromagnetic waves in the static, spherically symmetric field of the sun. Among the experiments in time-dependent fields of the solar system bodies one can mention recent measurement of deflection of radio waves in the time-dependent field of moving Jupiter [7, 8] and proposed detection of spacetime perturbations due to solar oscillations by the

¹Doppler tracking measurements using Cassini spacecraft [3]. VLBI measurements of deflection of radio waves have reached .02 percent [4] (see also review [6]).

space-based detector of gravitational waves LISA [9, 10]. In these experiments electromagnetic waves propagate in the *near zone* of the source of gravitational field.

Astronomical measurements also seem to provide good opportunities for observing relativistic effects when electromagnetic waves propagate in gravitational wave fields. A priori one could hope to observe

- the effects due to a single localized source of gravitational waves, such as a binary star, when the line of sight to a distant source of electromagnetic radiation passes close to the source of gravitational waves;
- the effects due to stochastic background of gravitational waves produced by a large number of localized sources².

In the two situations pointed out above the potentially observable relativistic effects include variations in the apparent position and brightness of the light source due to bending (focusing) of light rays by time-dependent field of gravitational waves, variations in the time of arrival of pulsar impulses due to gravitational (Shapiro) time delay, time-dependent gravitational Doppler shift, and rotation of the polarization plane (Skrotskii effect). One may notice that in the case with localized sources of gravitational waves astronomic techniques seem to have an obvious advantage over the gravitational wave detectors [10, 12, 13]: electromagnetic signals from distant astronomical sources may propagate in the vicinity of the sources of gravitational radiation where the amplitude of the waves is significantly higher than near the Earth. Closer studies, however, have shown that the above-mentioned astrometric, photometric, and timing effects due to interaction of light and gravitational waves are smaller than one could a priori expect and unobservable with present techniques.

²Stochastic background of gravitational waves of various wavelengths also arises in cosmological models (see [11] and references therein).

In 1966 Zipoy [14] (see also [15]) considered propagation of light in time-dependent field of a single source of gravity, influence of a plane-gravitational-wave impulse on a beam of light, and various effects due to stochastic background of cosmological gravitational waves. He showed that in the case when a localized source of gravitational radiation is located close to the line of sight, the contributions to the effect of scintillation due to gravitational waves are negligible because of the transverse character of gravitational radiation and the largest contributions come from the near-zone Newtonian fields. The estimates for variations of intensity of the beam due to a binary close to the line of sight or due to a single pulse of plane gravitational waves from a distant source both yielded very small results³. For jitter in position of the source due to stochastic background of gravitational waves it was shown that the effect does not increase with the distance travelled within gravitational waves⁴. The estimated intensity fluctuations due to the stochastic background, under favorable model assumptions, were found to be of the order of a thousandth of a percent.

In 1978 Sazhin [17] first suggested that pulsar timing observations may be used to detect gravitational waves from a binary system close to the line of sight towards the pulsar. For the known pairs "binary system - pulsar" the effect was found to be several orders of magnitude below the observational threshold due to large impact parameters. In 1993 Sazhin and Saphonova [18] estimated the probability of observation of the effect when both the binary and the pulsar belong to the same globular cluster. Probability was found to be quite high, reaching .97 for one cluster. However, it seems that the effect calculated in these works is due to *near-zone, quasi-static* gravitational field. Since, as it was clearly shown in [14] and later works discussed below, the effect due to gravitational

³The parameters of the binary system were taken to be quite modest, however: masses of the components $m = M_{\odot} = 2 \times 10^{33}$ g., orbital period $T = 10$ y., impact parameter $d = 1$ l.y.

⁴The same result was soon obtained by Bertotti and Trevese [16] in the physical optics approximation. It was pointed out that the reason for "locality" of the effect is transversal character of the gravitational waves.

waves is negligibly small for this configuration. Thus, measurement of the effect discussed would not serve as evidence for existence of gravitational radiation.

In 1993-1994 there appeared several papers [19–22] which, contrary to the results of the early works [14, 15], erroneously claimed that there might exist significant photometric [19], or astrometric [20–22] effects due to gravitational waves from localized sources located close to the line of sight towards the source of light.

In 1998 Damour and Esposito-Farèse [23] considered light deflection and time delay effects due to complete time-dependent, retarded, quadrupolar field which is given as a sum of terms having different fall-off properties that correspond to near-zone (falling off as $1/r^3$ with the distance from the source), intermediate-zone ($1/r^2$) and wave ($1/r$) fields. The source of light and observer were formally assumed to be separated by infinite distance with the source of gravitational field located close to the line of sight. In agreement with the results of [14, 15] it was shown that in the considered configuration the wavelike $1/r$ and the intermediate-zone $1/r^2$ pieces of the field give no contribution to the effects. It was also shown that the leading time-dependent effects of time delay and light deflection are caused by the near-zone, quasi-static piece of the field and fall off with the impact parameter d as $1/d^2$ and $1/d^3$ respectively.

Recently Kopeikin [24, 25] developed an elegant formalism for integration of equations of light geodesics in the field of a localized source possessing constant mass and spin and time-dependent quadrupole moment. The formalism allows one to obtain analytical solutions to the equations for arbitrary relative positions of the source of light, observer, and the gravitating system. In [25] analytical expressions were obtained for light ray trajectory and observable effects of time delay and deflection of light. The observable effects were also analyzed in gravitational-lens approximation (small impact parameter case discussed above) and in the plane-gravitational-wave approximation. In the gravitational-

lens approximation the results of [25] were in complete agreement with those of [14, 15, 23].

It seemed that the formalism developed in [24, 25] could be successfully applied for solution of a more general problem of propagation of light in the field of a localized source of arbitrary multipolarity. To extend the results of [25] for localized sources possessing time-dependent multipole moments of arbitrary order was the motivation of this work. The exact statement of the problem is as follows. We consider propagation of electromagnetic signals through the time-dependent gravitational field of an isolated system emitting gravitational waves such as, for example, a binary star. Gravitational field is assumed to be weak along the light ray, and linear approximation of GR is used [26, 27]. However, we do not place any restriction on internal velocities of matter in the system which makes our formalism applicable to sources, such as supernovae, with fast internal motions. Our consideration is not limited to systems of specific type, but rather, following [25], we derive general formulae which can be used for arbitrary source characterized by its time-dependent multipole moments of arbitrary order. We assume the wavelength of light to be much smaller than the characteristic wavelength of the gravitational waves and work in the geometrical optics approximation. The relative positions of the source of light, the isolated system, and the observer are not restricted, which makes our formalism quite general and applicable for most practical situations. With the assumptions made we obtain an analytical solution to the equations of propagation of light (which in geometrical optics are the equations of null geodesics) giving linear in metric perturbations corrections to the Newtonian trajectory. We also consider the relativistic effects of time delay and deflection angle in case of arbitrary disposition of the source of light, gravitating system, and the observer. Making use of the formalism developed in [28] and [29], we also consider propagation of the polarization properties along the light ray and obtain analytical expressions for

the angle of gravity-induced rotation of the polarization plane - the effect first discussed in 1957 by G. V. Skrotskii [30, 31].

This work is essentially a generalization of [25] for the case when the isolated system possesses multipole moments of arbitrary order. We also generalize the results of [29] for Skrotskii effect for sources possessing higher order multipole moments (the results of [29] for isolated system emitting gravitational waves were obtained again in spin-dipole mass-quadrupole approximation). The results of the dissertation have been published in papers [32] and [33].

The structure of the thesis is as follows. In Section 2 of this chapter we introduce notations used throughout. In Chapter 2 the expressions for metric tensor perturbations are specified and the choice of coordinate system is discussed. It is argued that ADM-harmonic coordinates constructed in Section 3 of that chapter possess the properties that allow analytical integration of the equations of light propagation and simplify interpretation of observable effects. In Chapter 3 we provide a derivation of the laws of geometrical optics in curved space-time from the Maxwell's equations and give the equations for trajectory of photons and propagation of polarization properties of light in the form these equations will be used in the following chapters. Chapter 4 is devoted to the problem of integration of the equations. Analytical solutions are obtained for trajectory of photons and the angle of gravity-induced rotation of the polarization plane of light. In Chapter 5 we give the expressions for the observable relativistic effects of time delay and bending of light. The expressions for the effect of rotation of polarization plane are essentially given in Section 3 of Chapter 4. Chapter 6 considers two special cases of arrangement of light source, observer, and the source of gravity: the gravitational-lens and the plane-wave approximations. Asymptotic expressions for observable effects are derived and discussed in each case. Finally, in Chapter 7 we summarize the main results of this work. Appendix A gives the expressions for metric perturbations in terms of light-ray variables ξ and τ ,

introduced in Chapter 4. Appendix B contains expressions for gauge functions and gauge-dependent terms from the equations for trajectory of light rays and equation for rotation of the polarization plane.

2 Notations and Conventions

Metric tensor on the space-time manifold is denoted by $g_{\alpha\beta}$ and its perturbation $h_{\alpha\beta} = g_{\alpha\beta} - \eta_{\alpha\beta}$. The determinant of the metric tensor is negative and is denoted as $g = \det||g_{\alpha\beta}||$. The four-dimensional fully antisymmetric Levi-Civita symbol $\epsilon_{\alpha\beta\gamma\delta}$ is defined in accordance with the convention $\epsilon_{0123} = +1$. In the present paper we use a geometrodynamical system of units [26, 27] such that $c = G = 1$ where c is the fundamental speed and G is the universal gravitational constant. Space-time is assumed to be asymptotically-flat and covered by a single coordinate system $(x^0, x^1, x^2, x^3) = (t, x, y, z)$, where t and (x, y, z) are time and space coordinates respectively. This coordinate system is reduced at infinity to the Minkowskian coordinates. We shall also use the spherical coordinates (r, θ, ϕ) related to (x, y, z) by the standard transformation

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta. \quad (2.1)$$

Greek (spacetime) indices range from 0 to 3, and Latin (space) indices run from 1 to 3. If not specifically stated the opposite, the indices are raised and lowered by means of the Minkowski metric $\eta_{\alpha\beta} \equiv \text{diag}(-1, 1, 1, 1)$. Regarding this rule the following conventions for coordinates hold: $x^i = x_i$ and $x^0 = -x_0$. We also adopt notations, $\delta_{ij} \equiv \text{diag}(1, 1, 1)$, for the Kroneker symbol (a unit matrix), and, ϵ_{ijk} , for the fully antisymmetric 3-dimensional symbol of Levi-Civita with convention $\epsilon_{123} = +1$.

Repeated indices are summed over in accordance with the Einstein's rule [26, 27]. In the linearized approximation of general relativity used in this work there is no difference between spatial vectors and co-vectors as well as between

upper and lower space indices. Therefore, for a dot product of two space vectors we have

$$A^i B_i = A_i B_i \equiv A_1 B_1 + A_2 B_2 + A_3 B_3 . \quad (2.2)$$

In what follows, we shall commonly use spatial multi-index notations for Cartesian three-dimensional tensors [35], that is

$$\mathcal{I}_{A_l} \equiv \mathcal{I}_{a_1 \dots a_l} . \quad (2.3)$$

Tensor product of l identical spatial vectors k^i will be denoted as a three-dimensional tensor having l indices

$$k_{a_1 \dots a_l} \equiv k_{a_1} \dots k_{a_l} . \quad (2.4)$$

Full symmetrization with respect to a group of spatial indices of a Cartesian tensor will be distinguished with round brackets put around the indices

$$Q_{(a_1 \dots a_l)} \equiv \frac{1}{l!} \sum_{\sigma} Q_{\sigma(1) \dots \sigma(l)} , \quad (2.5)$$

where σ is the set of all permutations of $(1, 2, \dots, l)$ which makes $Q_{a_1 \dots a_l}$ fully symmetrical in $a_1 \dots a_l$.

It is convenient to introduce a special notation for symmetric trace-free (STF) Cartesian tensors by making use of angular brackets around STF indices. The explicit expression of the STF part of a tensor $Q_{a_1 \dots a_l}$ is [35]

$$Q_{\langle a_1 \dots a_l \rangle} \equiv \sum_{k=0}^{[l/2]} a_k^l \delta_{(a_1 a_2} \dots \delta_{a_{2k-1} a_{2k}} S_{a_{2k+1} \dots a_l) b_1 b_1 \dots b_k b_k} , \quad (2.6)$$

where $[l/2]$ is the integer part of $l/2$,

$$S_{a_1 \dots a_l} \equiv Q_{(a_1 \dots a_l)} \quad (2.7)$$

and numerical coefficients

$$a_k^l = \frac{(-1)^k}{(2k)!!} \frac{l!}{(2l-1)!!} \frac{(2l-2k-1)!!}{(l-2k)!} . \quad (2.8)$$

We also assume that for any integer $l \geq 0$

$$l! \equiv l(l-1) \dots 2 \cdot 1, \quad 0! \equiv 1, \quad (2.9)$$

and

$$l!! \equiv l(l-2)(l-4) \dots (2 \text{ or } 1), \quad 0!! \equiv 1. \quad (2.10)$$

One has, for example,

$$T_{\langle abc \rangle} \equiv T_{(abc)} - \frac{1}{5} \delta_{ab} T_{(cjj)} - \frac{1}{5} \delta_{bc} T_{(ajj)} - \frac{1}{5} \delta_{ac} T_{(bjj)}. \quad (2.11)$$

Cartesian tensors of the mass-type $\mathcal{I}_{\langle A_l \rangle}$ and spin-type multipoles $\mathcal{S}_{\langle A_l \rangle}$ entirely describing gravitational field outside of an isolated astronomical system are always STF objects that can be checked by inspection of the definition following from the multipolar decomposition of the metric tensor perturbation $h_{\alpha\beta}$ [35]. For this reason, to avoid appearance of too complicated index notations we shall omit in the following text the angular brackets around indices of these (and only these) tensors, that is we adopt: $\mathcal{I}_{A_l} \equiv \mathcal{I}_{\langle A_l \rangle}$ and $\mathcal{S}_{A_l} \equiv \mathcal{S}_{\langle A_l \rangle}$.

We shall also use transverse (T) and transverse-traceless (TT) Cartesian tensors in our calculations [27, 35]. These objects are defined by making use of the operator of projection P_{jk} onto the plane orthogonal to a unit vector k_j :

$$P_{jk} \equiv \delta_{jk} - k_{jk}. \quad (2.12)$$

Thus, one has [35]

$$Q_{a_1 \dots a_l}^T \equiv P_{a_1 b_1} P_{a_2 b_2} \dots P_{a_l b_l} Q_{b_1 \dots b_l}, \quad (2.13)$$

$$Q_{a_1 \dots a_l}^{\text{TT}} \equiv \sum_{k=0}^{[l/2]} b_k^l P_{(a_1 a_2} \dots P_{a_{2k-1} a_{2k}} W_{a_{2k+1} \dots a_l) b_1 b_1 \dots b_k b_k}, \quad (2.14)$$

where again $[l/2]$ is the integer part of $l/2$,

$$W_{a_1 \dots a_l} \equiv Q_{(a_1 \dots a_l)}^T \quad (2.15)$$

and numerical coefficients

$$b_k^l = \frac{(-1)^k l(l-k-1)!!}{4^k k!(l-2k)!} . \quad (2.16)$$

For instance,

$$Q_{ab}^{\text{TT}} \equiv \frac{1}{2} (P_{ai} P_{bj} Q_{ij} + P_{bi} P_{aj} Q_{ij}) - \frac{1}{2} P_{ab} (P_{jk} Q_{jk}) . \quad (2.17)$$

Polynomial coefficients will be used in some of our equations and they are defined by

$$C_l(p_1, \dots, p_n) \equiv \frac{l!}{p_1! \dots p_n!} , \quad (2.18)$$

where l and p_i are positive integers such that $\sum_{i=1}^n p_i = l$. We introduce a Heaviside unit step function $H(p - q)$ such that on the set of whole numbers

$$H(p - q) = \begin{cases} 0, & \text{if } p \leq q, \\ 1, & \text{if } p > q. \end{cases} \quad (2.19)$$

For any differentiable function $f = f(t, \mathbf{x})$ one uses notations: $f_{,0} = \partial f / \partial t$ and $f_{,i} = \partial f / \partial x^i$ for its partial derivatives. In general, comma standing after the function denotes a partial derivative with respect to a corresponding coordinate: $f_{,\alpha} \equiv \partial f(x) / \partial x^\alpha$. Overdot denotes a total derivative with respect to time $\dot{f} \equiv df/dt = \partial f / \partial t + \dot{x}^i \partial f / \partial x^i$ usually taken in this paper along the light ray trajectory $\mathbf{x}(t)$. Sometimes the partial derivatives with respect to space coordinate x^i will be also denoted as $\partial_i \equiv \partial / \partial x^i$, and the partial time derivative will be denoted as $\partial_t \equiv \partial / \partial t$. Covariant derivative with respect to the coordinate x^α will be denoted as ∇_α .

We shall use special notations for integrals with respect to time and for those taken along the light-ray trajectory. Specifically, the time integrals from a function $F(t, \mathbf{x})$, where \mathbf{x} is not specified, are denoted as

$$F^{(-1)}(t, \mathbf{x}) \equiv \int_{-\infty}^t F(\tau, \mathbf{x}) d\tau , \quad F^{(-2)}(t, \mathbf{x}) \equiv \int_{-\infty}^t F^{(-1)}(\tau, \mathbf{x}) d\tau . \quad (2.20)$$

Time integrals from the function $F(t, \mathbf{x})$ taken along the light ray so that spatial coordinate \mathbf{x} is a function of time $\mathbf{x} \equiv \mathbf{x}(t)$, are denoted as

$$F^{[-1]}(t, \mathbf{r}) \equiv \int_{-\infty}^t F(\tau, \mathbf{r}(\tau)) d\tau, \quad F^{[-2]}(t, \mathbf{r}) \equiv \int_{-\infty}^t F^{[-1]}(\tau, \mathbf{r}(\tau)) d\tau. \quad (2.21)$$

Integrals in equations (2.20) represent functions of time and space coordinates. Integrals in equations (2.21) are defined on the light-ray trajectory and are functions of time only.

Multiple time derivative from function $F(t, \mathbf{x})$ is denoted by

$$F^{(p)}(t, \mathbf{x}) = \frac{\partial^p F(t, \mathbf{x})}{\partial t^p}, \quad (2.22)$$

so that its action on the time integrals eliminates integration in the sense that

$$F^{(p)}(t, \mathbf{x}) = \frac{\partial^{p+1} F^{(-1)}(t, \mathbf{x})}{\partial t^{p+1}} = \frac{\partial^{p+2} F^{(-2)}(t, \mathbf{x})}{\partial t^{p+2}}, \quad (2.23)$$

and

$$F^{[p]}(t, \mathbf{x}) = \frac{d^{p+1} F^{[-1]}(t, \mathbf{x})}{dt^{p+1}} = \frac{d^{p+2} F^{[-2]}(t, \mathbf{x})}{dt^{p+2}}. \quad (2.24)$$

Spatial vectors will be denoted by bold italic letters, for instance $A^i \equiv \mathbf{A}$, $k^i \equiv \mathbf{k}$, etc. The Euclidean dot product between two spatial vectors \mathbf{a} and \mathbf{b} is denoted with dot between them: $a^i b_i = \mathbf{a} \cdot \mathbf{b}$. The Euclidean wedge (cross) product between two vectors is denoted with symbol \times , that is $\epsilon_{ijk} a^j b^k = (\mathbf{a} \times \mathbf{b})^i$. Other particular notations will be introduced as they appear in the text.

Chapter 2

Metric Tensor and Coordinate Systems

1 Metric Tensor

We assume that the gravitational field is weak along the light ray and work in the linear approximation of GR. Thus, the metric tensor can be expressed as the sum of the Minkowski metric and a small perturbation, $h_{\alpha\beta} \ll 1$, linear in the universal gravitational constant G :

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}. \tag{1.1}$$

We consider propagation of light in the space-time around an isolated gravitating system. The most general expressions for metric perturbations in this case were

given in [35] and read

$$h_{00}^{\text{can.}} = \frac{2\mathcal{M}}{r} + 2 \sum_{l=2}^{\infty} \frac{(-1)^l}{l!} \left[\frac{\mathcal{I}_{A_l}(t-r)}{r} \right]_{,A_l}, \quad (1.2)$$

$$h_{0i}^{\text{can.}} = -\frac{2\epsilon_{ipq}\mathcal{S}_p N_q}{r^2} - 4 \sum_{l=2}^{\infty} \frac{(-1)^l l}{(l+1)!} \left[\frac{\epsilon_{ipq}\mathcal{S}_{pA_{l-1}}(t-r)}{r} \right]_{,qA_{l-1}} \\ + 4 \sum_{l=2}^{\infty} \frac{(-1)^l}{l!} \left[\frac{\dot{\mathcal{I}}_{iA_{l-1}}(t-r)}{r} \right]_{,A_{l-1}}, \quad (1.3)$$

$$h_{ij}^{\text{can.}} = \delta_{ij} h_{00}^{\text{can.}} + q_{ij}^{\text{can.}}, \quad (1.4)$$

$$q_{ij}^{\text{can.}} = 4 \sum_{l=2}^{\infty} \frac{(-1)^l}{l!} \left[\frac{\ddot{\mathcal{I}}_{ijA_{l-2}}(t-r)}{r} \right]_{,A_{l-2}} - 8 \sum_{l=2}^{\infty} \frac{(-1)^l l}{(l+1)!} \left[\frac{\epsilon_{pq(i}\dot{\mathcal{S}}_j)pA_{l-2}(t-r)}{r} \right]_{,qA_{l-2}}. \quad (1.5)$$

Here \mathcal{M} and \mathcal{S}_i are the total mass and angular momentum of the system, \mathcal{I}_{A_l} and \mathcal{S}_{A_l} are two independent sets of mass-type and spin-type multipole moments, and $N^i = x^i/r$ is a unit vector directed from the origin of the coordinate system to the field point. Since the origin of the coordinate system has been chosen at the center of mass, the expansions (1.2) – (1.5) do not depend on the mass-type linear multipole moment \mathcal{I}_i .

The expressions for metric (1.2)-(1.5) are given in a special coordinate system [35] which belongs to the class of harmonic coordinates [27, 34]. In what follows we shall call this coordinate system *canonical harmonic*. Metric perturbations in a different coordinate system can be obtained by means of an infinitesimal coordinate transformation

$$x'^{\alpha} = x^{\alpha} - w^{\alpha} \quad (1.6)$$

from the *canonical harmonic* coordinates x^{α} to coordinates x'^{α} where w^{α} are gauge functions. Then in the new coordinate system x'^{α} one has [27]

$$h_{\alpha\beta} = h_{\alpha\beta}^{\text{can.}} + w_{\alpha,\beta} + w_{\beta,\alpha}. \quad (1.7)$$

We shall use the freedom in choosing the functions w^{α} to simplify the problem of integration of the equations of light propagation and analysis of observable

effects. It turns out that to this end two coordinate systems possess useful properties. These are the harmonic and the Arnowitt-Deser-Misner coordinates which will be considered in the following two sections. Here we would like to note that if gauge functions w^α satisfy the homogeneous wave equation

$$\square w^\alpha = 0, \quad (1.8)$$

the transformation (1.6) preserves the harmonic gauge conditions which can be formulated as $\square x^\alpha = 0$.

The most general solution of the equation (1.8) reads [35]

$$w^0 = \sum_{l=0}^{\infty} \left[\frac{\mathcal{W}_{A_l}(t-r)}{r} \right]_{,A_l}, \quad (1.9)$$

$$w^i = \sum_{l=0}^{\infty} \left[\frac{\mathcal{X}_{A_l}(t-r)}{r} \right]_{,iA_l} \quad (1.10)$$

$$+ \sum_{l=1}^{\infty} \left\{ \left[\frac{\mathcal{Y}_{iA_{l-1}}(t-r)}{r} \right]_{,A_{l-1}} + \left[\epsilon_{ipq} \frac{\mathcal{Z}_{qA_{l-1}}(t-r)}{r} \right]_{,pA_{l-1}} \right\},$$

where \mathcal{W}_{A_l} , \mathcal{X}_{A_l} , $\mathcal{Y}_{iA_{l-1}}$, and $\mathcal{Z}_{qA_{l-1}}$ are four sets of STF multipole moments depending on the retarded time.

2 The Harmonic Coordinate System

The harmonic coordinate system, analogous to the Lorentz gauge in electrodynamics, is defined in linear approximation by the conditions

$$2h^{\alpha\beta}_{,\beta} - h^{,\alpha} = 0 \quad (2.1)$$

where $h \equiv h^\alpha_\alpha$. The linearized Einstein field equations in harmonic coordinates reduce to wave equations for metric perturbations $h^{\alpha\beta}$.

The metric tensor perturbations in harmonic coordinates possess a property that can be used to simplify the problem of analytical integration of the equations

of light propagation. Specifically, the method of integration of equations of light geodesics, developed in [24] and [25] and used in this work (Ch. 4), requires that the metric perturbations be functions of the retarded time (compare Eqs. (1.2)-(1.5)). Not all coordinate systems possess this property. For instance, metric perturbations in the canonical ADM coordinate system, considered in the next section, contain terms that depend on instantaneous time, not allowing to use the method of integration mentioned above.

3 The Arnowitt-Deser-Misner Coordinate System

The Arnowitt-Deser-Misner (ADM) gauge conditions in the linear approximation read [36]

$$2h_{0i,i} - h_{ii,0} = 0, \quad 3h_{ij,j} - h_{jj,i} = 0. \quad (3.1)$$

For comparison, the harmonic gauge conditions (2.1) in the linear approximation have the form

$$2h_{0i,i} - h_{ii,0} = h_{00,0}, \quad 2h_{ij,j} - h_{jj,i} = -h_{00,i}. \quad (3.2)$$

The ADM coordinates have the property that test particles perturbed by gravitational waves stay at rest with respect to the coordinate grid. (Thus, ADM coordinate system is not inertial.) This property can be used to simplify analysis of observable effects. For example, when considering detection of a photon by an observer in the field of a gravitational wave one normally has to solve both the equations of motion of the photon and the observer. Using ADM coordinates allows to exclude the problem of motion of the observer.

The disadvantage of using the ADM coordinates for analysis of the effects in propagation of light through gravitational field of an isolated source is that, as it was mentioned above, the ADM metric perturbations may contain terms

that depend on "instantaneous" time [25]. This does not allow one to apply the method of integration of the equations of light propagation mentioned above (Ch. 4).

Fortunately, the classes of ADM and harmonic coordinates overlap. In [25] an *ADM-harmonic* coordinate system, satisfying both ADM and harmonic gauge conditions in linear approximation, was constructed for isolated system with constant mass and angular momentum and time-dependent quadrupole moment. We generalized this result for the case when the localized source possesses full multipole structure (1.2)-(1.5). The gauge functions

$$w^0 = \sum_{l=2}^{\infty} \frac{(-1)^l}{l!} \left[\frac{\mathcal{I}_{A_l}^{(-1)}(t-r)}{r} \right]_{,A_l}, \quad (3.3)$$

$$w^i = \sum_{l=2}^{\infty} \frac{(-1)^l}{l!} \left[\frac{\mathcal{I}_{A_l}^{(-2)}(t-r)}{r} \right]_{,iA_l} - 4 \sum_{l=2}^{\infty} \frac{(-1)^l}{l!} \left[\frac{\mathcal{I}_{iA_{l-1}}(t-r)}{r} \right]_{,A_{l-1}} \quad (3.4)$$

$$+ 4 \sum_{l=2}^{\infty} \frac{(-1)^l l}{(l+1)!} \left[\frac{\epsilon_{iba} \mathcal{S}_{bA_{l-1}}^{(-1)}(t-r)}{r} \right]_{,aA_{l-1}}$$

bring the metric perturbations $h_{\alpha\beta}$ to a remarkably simple form:

$$h_{00} = \frac{2\mathcal{M}}{r}, \quad (3.5)$$

$$h_{0i} = -\frac{2\epsilon_{ipq} \mathcal{S}_p N_q}{r^2}, \quad (3.6)$$

$$h_{ij} = \delta_{ij} h_{00} + h_{ij}^{TT}, \quad (3.7)$$

$$h_{ij}^{TT} = P_{ijkl} q_{kl}^{\text{can.}}. \quad (3.8)$$

Metric perturbations (3.5)-(3.8) satisfy both the harmonic and ADM gauge conditions in the first post-Minkovskian approximation. In (3.8) the TT-projection differential operator P_{ijkl} applied to symmetric tensors depending on both time

and spatial coordinates is given by

$$P_{ijkl} = (\delta_{ik} - \Delta^{-1}\partial_i\partial_k)(\delta_{jl} - \Delta^{-1}\partial_j\partial_l) - \frac{1}{2}(\delta_{ij} - \Delta^{-1}\partial_i\partial_j)(\delta_{kl} - \Delta^{-1}\partial_k\partial_l), \quad (3.9)$$

where Δ and Δ^{-1} denote the Laplacian and the inverse Laplacian respectively.

The *ADM-harmonic* coordinates combine the advantages of both coordinate systems and will be used in what follows for analysis of observable effects in propagation of electromagnetic signals through time-dependent gravitational field of a localized source.

Chapter 3

Equations for Propagation of Electromagnetic Signals

1 Geometrical Optics in Curved Spacetime

In this section, following [29] and [27] (§22.5), we provide a derivation of the main laws of geometrical optics in curved spacetime from the Maxwell's equations.

In absence of sources the Maxwell's equations for the electromagnetic field tensor $F_{\alpha\beta}$ in curved spacetime take the well known form [26, 27]:

$$\nabla_{\alpha}F_{\beta\gamma} + \nabla_{\beta}F_{\gamma\alpha} + \nabla_{\gamma}F_{\alpha\beta} = 0 , \quad (1.1)$$

$$\nabla_{\beta}F^{\alpha\beta} = 0 , \quad (1.2)$$

where ∇_{α} denotes a covariant derivative. The wave equation for $F_{\alpha\beta}$ can be derived from (1.1) and (1.2) and in vacuum ($R_{\alpha\beta} = 0$, where $R_{\alpha\beta}$ is the Ricci tensor) it assumes the form

$$\square_g F_{\alpha\beta} + R_{\alpha\beta\mu\nu}F^{\mu\nu} = 0 , \quad (1.3)$$

where $\square_g \equiv \nabla^{\alpha}\nabla_{\alpha}$ and $R_{\alpha\beta\gamma\delta}$ is the Riemann curvature tensor of the spacetime.

The solution to the Maxwell's equations corresponding to a high frequency wave is given by

$$F_{\alpha\beta} = \text{Re}\{A_{\alpha\beta} \exp(i\varphi)\} , \quad (1.4)$$

where $A_{\alpha\beta}$ is the complex amplitude and φ is the phase. The complex amplitude is a slowly varying function of position and time, while the phase rapidly changes with time and position.

The criterion for applicability of the geometrical optics approximation in curved spacetime is formulated in [27] as follows: let \mathcal{R} be the characteristic radius of curvature of the spacetime through which the electromagnetic wave propagates and \mathcal{L} – the characteristic length, over which the amplitude, wavelength, and polarization of the electromagnetic wave change significantly. Geometrical optics approximation can be applied whenever the wavelength of the electromagnetic radiation satisfies the following two conditions: 1) $\lambda \ll \mathcal{R}$; and 2) $\lambda \ll \mathcal{L}$. Then a small parameter $\varepsilon \equiv \lambda/\min\{\mathcal{L}, \mathcal{R}\}$ can be introduced, and the expansion of the electromagnetic field of a wave (1.4) in powers of ε is assumed to have the form

$$F_{\alpha\beta} = (a_{\alpha\beta} + \varepsilon b_{\alpha\beta} + \varepsilon^2 c_{\alpha\beta} + \dots) \exp\left(\frac{i\varphi}{\varepsilon}\right). \quad (1.5)$$

For critical discussion of this procedure see [38] and [40]. Substituting this expansion into (1.1), taking into account the definition of the electromagnetic wave vector $l_\alpha \equiv \partial\varphi/\partial x^\alpha$ and rearranging the terms with the same powers of ε lead to the chain of equations

$$l_\alpha a_{\beta\gamma} + l_\beta a_{\gamma\alpha} + l_\gamma a_{\alpha\beta} = 0, \quad (1.6)$$

$$\nabla_\alpha a_{\beta\gamma} + \nabla_\beta a_{\gamma\alpha} + \nabla_\gamma a_{\alpha\beta} = -i(l_\alpha b_{\beta\gamma} + l_\beta b_{\gamma\alpha} + l_\gamma b_{\alpha\beta}), \quad (1.7)$$

where the effects of curvature have been neglected. Another chain of equations is obtained by substituting (1.5) into (1.2):

$$l_\beta a^{\alpha\beta} = 0, \quad (1.8)$$

$$\nabla_\beta a^{\alpha\beta} + il_\beta b^{\alpha\beta} = 0. \quad (1.9)$$

The equation (1.8) shows that in the lowest-order approximation (terms of order $1/\varepsilon$) the amplitude of the electromagnetic field tensor is orthogonal to the wave

vector in the four-dimensional sense. Contracting (1.6) with l_α and accounting for (1.8) implies that l_α is a null vector

$$l_\alpha l^\alpha = 0. \quad (1.10)$$

Taking a covariant derivative from the equality (1.10) and using the fact that $\nabla_{[\alpha} l_{\beta]} = 0$ (since $l_\alpha \equiv \nabla_\alpha \varphi$) we obtain that l_α satisfies the geodesic equation

$$l^\beta \nabla_\beta l^\alpha = 0, \quad (1.11)$$

showing that the wave vector is parallel transported along itself. Equations (1.10) and (1.11) combined constitute an important result of the geometrical optics: in the lowest order approximation light rays are null geodesics.

Substituting (1.4) into (1.3) and considering terms of the order $1/\varepsilon$ one can obtain the law of propagation for the amplitude of the electromagnetic tensor

$$D_\lambda a_{\alpha\beta} + \vartheta a_{\alpha\beta} = 0, \quad (1.12)$$

where $D_\lambda \equiv l^\alpha \nabla_\alpha$ and $\vartheta \equiv (1/2)\nabla_\alpha l^\alpha$ is the expansion of the null congruence l^α .

2 Equations for Trajectory of a Photon

It was shown in the previous section that in the lowest order approximation of geometrical optics light rays, or trajectories of photons, are represented by null geodesics of the spacetime under consideration. In what follows we shall consider propagation of photons in a spacetime around an isolated gravitating system, with the metric of the spacetime defined by (1.7). We shall assume that the wavelength of the electromagnetic radiation is much smaller than the characteristic wavelength of the gravitational waves emitted by the gravitating system, so that the conditions for validity of geometrical optics approximation are satisfied.

Let us consider propagation of photons, subject to the initial-boundary conditions

$$\mathbf{x}(t_0) = \mathbf{x}_0, \quad \frac{d\mathbf{x}(-\infty)}{dt} = \mathbf{k}, \quad (2.1)$$

which specify the spacial velocity of photons at the past null infinity by a unit spacial vector \mathbf{k} and the positions of the photons at some initial instant of time t_0 .

Taking into account that $l^\alpha = dx^\alpha/d\lambda$, where λ is an affine parameter along the photon's trajectory, we can rewrite the equations of geodesics (1.11) in terms of coordinates of the photon

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma_{\beta\gamma}^\alpha \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda} = 0, \quad (2.2)$$

where

$$\Gamma_{\beta\gamma}^\alpha = \frac{1}{2}g^{\alpha\mu} (\partial_\gamma g_{\mu\beta} + \partial_\beta g_{\mu\gamma} - \partial_\mu g_{\beta\gamma}) \quad (2.3)$$

are the Christoffel symbols. Combining the four equations (2.2), we obtain the equations for spacial position of the photon as a function of the coordinate time:

$$\ddot{x}^i(t) = - (\Gamma_{\alpha\beta}^i - \Gamma_{\alpha\beta}^0 \dot{x}^i) \dot{x}^\alpha \dot{x}^\beta. \quad (2.4)$$

Substituting (2.3) into (2.4), taking into account that $\dot{x}^i = k^i + O(h)$, and keeping only linear in metric perturbations terms we can rewrite the equations (2.4) in the form

$$\begin{aligned} \ddot{x}^i(t) = & \frac{1}{2}h_{00,i} - h_{0i,0} - \frac{1}{2}h_{00,0}k^i - h_{ik,0}k^k - (h_{0i,k} - h_{0k,i})k^k \\ & - h_{00,k}k^k k^i - \left(h_{ik,j} - \frac{1}{2}h_{kj,i} \right) k^k k^j + \left(\frac{1}{2}h_{kj,0} - h_{0k,j} \right) k^k k^j k^i. \end{aligned} \quad (2.5)$$

In Chapter 4 we will integrate these equations analytically in the spacetime with metric given by Eq. (1.7), Ch. 2 around a localized system.

3 Equations for Propagation of Polarization Properties

In this section, following [28] and [29], we introduce the relativistic description of polarized electromagnetic radiation and give the equation (derived in [29]) for gravity-induced rotation of the polarization plane (Skrotskii effect).

To describe polarization properties of electromagnetic radiation one has to specify local reference frames along the light rays. At each point of spacetime we introduce a complex null tetrad $(l^\alpha, n^\alpha, m^\alpha, \bar{m}^\alpha)$ associated with a congruence of light rays [41, 42]. All the vectors of the tetrad are lightlike. The vectors l^α and n^α are real; vectors m^α and \bar{m}^α are complex and complex conjugate with respect to each other. The tetrad is normalized in such a way that the only nonvanishing products among the vectors are $l_\alpha n^\alpha = -1$ and $m_\alpha \bar{m}^\alpha = 1$.

It is obvious that such field of complex null tetrads is not uniquely determined by the congruence of null rays and the normalization conditions. The transformations

$$\begin{aligned}
 l'^\alpha &= A l^\alpha , & (3.1) \\
 n'^\alpha &= A^{-1} (n^\alpha + \bar{B} m^\alpha + B \bar{m}^\alpha + B \bar{B} l^\alpha) , \\
 m'^\alpha &= e^{-i\Theta} (m^\alpha + \bar{B} l^\alpha) , \\
 \bar{m}'^\alpha &= e^{i\Theta} (\bar{m}^\alpha + B l^\alpha) ,
 \end{aligned}$$

where A and Θ are real and B is a complex parameter, preserve the direction of the vector l^α and the normalization conditions [42]. The transformations (3.1) form a four-parameter subgroup of the Lorentz group. In addition to the complex null tetrad we introduce at each point in spacetime an orthonormal reference tetrad $e^\alpha_{(\beta)}$ defined as follows. Suppose at each point of spacetime there is an observer moving with four-velocity u^α . Let two of the vectors of the observer's

local reference frame be

$$e_{(0)}^\alpha = u^\alpha, \quad e_{(3)}^\alpha = (-l_\alpha u^\alpha)^{-1} [l^\alpha + (l_\beta u^\beta) u^\alpha]. \quad (3.2)$$

With such orientation of the reference frame the observer will see the electromagnetic wave propagating in the $+z$ direction. Spacelike unit vectors $e_{(1)}^\alpha$ and $e_{(2)}^\alpha$ are orthogonal to each other as well as to both $e_{(0)}^\alpha$ and $e_{(3)}^\alpha$ and thus are specified up to a spacial rotation.

The relationship between the vectors of the complex null tetrad and the frame $e_{(\beta)}^\alpha$ is given by

$$l^\alpha = -(l_\gamma u^\gamma) (e_{(0)}^\alpha + e_{(3)}^\alpha), \quad (3.3)$$

$$n^\alpha = -\frac{1}{2}(l_\gamma u^\gamma) (e_{(0)}^\alpha - e_{(3)}^\alpha), \quad (3.4)$$

$$m^\alpha = \frac{1}{\sqrt{2}} (e_{(1)}^\alpha + ie_{(2)}^\alpha), \quad (3.5)$$

$$\bar{m}^\alpha = \frac{1}{\sqrt{2}} (e_{(1)}^\alpha - ie_{(2)}^\alpha). \quad (3.6)$$

Vectors $e_{(1)}^\alpha$ and $e_{(2)}^\alpha$ and corresponding to them m^α and \bar{m}^α play an important role in description of polarization properties of electromagnetic radiation since they form a basis in the polarization plane.

The tensor of the electromagnetic field can be represented as

$$F_{\alpha\beta} = Re(\mathcal{F}_{\alpha\beta}), \quad (3.7)$$

where $\mathcal{F}_{\alpha\beta}$ is the complex field. The complex field can be expressed as

$$\mathcal{F}_{\alpha\beta} = \Phi l_{[\alpha} m_{\beta]} + \Psi l_{[\beta} \bar{m}_{\alpha]} \quad (3.8)$$

where Φ and Ψ are complex scalar functions.

The electric and magnetic fields in the rest frame of an observer moving with four-velocity u^α are defined as

$$E^\alpha = F^{\alpha\beta} u_\beta, \quad H^\alpha = (-1/2)\epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} u_\beta. \quad (3.9)$$

We also define the complex electric field as

$$\mathcal{E}^\alpha = \mathcal{F}^{\alpha\beta} u_\beta . \quad (3.10)$$

Polarization properties of light are completely characterized by the polarization tensor [26] (coherency matrix in [44])

$$J_{\alpha\beta} = \langle \mathcal{E}_\alpha \bar{\mathcal{E}}_\beta \rangle , \quad (3.11)$$

where the angular brackets notify an ensemble average equivalent to averaging over many periods of the wave.

The electromagnetic Stokes parameters are defined with respect to two of the observer's tetrad vectors $e_{(1)}^\alpha$ and $e_{(2)}^\alpha$ as follows [28]

$$S_0 = J_{\alpha\beta} \left[e_{(1)}^\alpha e_{(1)}^\beta + e_{(2)}^\alpha e_{(2)}^\beta \right] , \quad (3.12)$$

$$S_1 = J_{\alpha\beta} \left[e_{(1)}^\alpha e_{(1)}^\beta - e_{(2)}^\alpha e_{(2)}^\beta \right] , \quad (3.13)$$

$$S_2 = J_{\alpha\beta} \left[e_{(1)}^\alpha e_{(2)}^\beta + e_{(2)}^\alpha e_{(1)}^\beta \right] , \quad (3.14)$$

$$S_3 = i J_{\alpha\beta} \left[e_{(1)}^\alpha e_{(2)}^\beta - e_{(2)}^\alpha e_{(1)}^\beta \right] . \quad (3.15)$$

It is worth noting that both the Stokes parameters and the components of the polarization tensor of an electromagnetic wave can be determined from measurements of intensities of the wave after it passes through devices that transmit radiation of certain polarization only [44].

Using the definition of the polarization tensor (3.11) we can also rewrite the expressions for the Stokes parameters in terms of the electric field components

$$S_0 = \langle |\mathcal{E}_{(1)}|^2 + |\mathcal{E}_{(2)}|^2 \rangle , \quad (3.16)$$

$$S_1 = \langle |\mathcal{E}_{(1)}|^2 - |\mathcal{E}_{(2)}|^2 \rangle , \quad (3.17)$$

$$S_2 = \langle \mathcal{E}_{(1)} \bar{\mathcal{E}}_{(2)} + \bar{\mathcal{E}}_{(1)} \mathcal{E}_{(2)} \rangle , \quad (3.18)$$

$$S_3 = i \langle \mathcal{E}_{(1)} \bar{\mathcal{E}}_{(2)} - \bar{\mathcal{E}}_{(1)} \mathcal{E}_{(2)} \rangle , \quad (3.19)$$

where $\mathcal{E}_{(n)} = \mathcal{E}_\alpha e_{(n)}^\alpha$ for $n = 1, 2$. The Stokes parameters can be expressed in terms of the complex functions Φ and Ψ as well, since

$$\mathcal{E}_\alpha = -\frac{l^\alpha u_\alpha}{2} (\Phi m_\alpha + \Psi \bar{m}_\alpha) \quad (3.20)$$

and therefore

$$\mathcal{E}_{(1)} = -\frac{l^\alpha u_\alpha}{\sqrt{8}} (\Phi + \Psi) , \quad \mathcal{E}_{(2)} = -i\frac{l^\alpha u_\alpha}{\sqrt{8}} (\Phi + \Psi) . \quad (3.21)$$

To determine how the polarization properties of light vary along the ray we should specify the law of propagation for the tetrad vectors. The vectors of both the null tetrad and the tetrad $e_{(\beta)}^\alpha$ are postulated to be parallelly transported along the light rays. We specify the tetrads at the point of observation and thus specify them at every point along the ray. Thus, the equations for propagation of the tetrad vectors read

$$\frac{dm^\alpha}{d\lambda} + \Gamma_{\beta\gamma}^\alpha l^\beta m^\gamma = 0 , \quad (3.22)$$

$$\frac{d\bar{m}^\alpha}{d\lambda} + \Gamma_{\beta\gamma}^\alpha l^\beta \bar{m}^\gamma = 0 , \quad (3.23)$$

$$\frac{de_{(\mu)}^\alpha}{d\lambda} + \Gamma_{\beta\gamma}^\alpha l^\beta e_{(\mu)}^\gamma = 0 , \quad (3.24)$$

and the same laws hold for vectors l^α and n^α of the null tetrad.

It follows from (1.12), (3.10) and (3.24) that the complex amplitude of the electric field propagates along the ray according to the law

$$D_\lambda \mathcal{E}_\alpha + \vartheta \mathcal{E}_\alpha = 0 . \quad (3.25)$$

For the complex amplitudes Φ and Ψ one has

$$D_\lambda \Phi + \vartheta \Phi = 0 , \text{ and } D_\lambda \Psi + \vartheta \Psi = 0 . \quad (3.26)$$

Finally, from the equations (3.16) - (3.19) it follows that the Stokes parameters propagate according to the law

$$D_\lambda S_\alpha + 2\vartheta S_\alpha = 0 , \quad (3.27)$$

where $\alpha = 0, 1, 2, 3$. We would like to note that despite suggestive notation the Stokes parameters do not form a four-vector, because they do not behave as a vector under coordinate system transformations.

Any stationary or time-dependent axisymmetric gravitational field in general causes a relativistic effect of rotation of the polarization plane of an electromagnetic wave [29]. This effect was first discussed by Skrotskii [30] and afterwards by many researchers (see, for example, [39] and references therein). Recently the effect was studied in [29] where authors derived an expression for the angle of rotation of the polarization plane of an electromagnetic wave propagating through a weak gravitational field described by metric perturbations $h_{\alpha\beta}$. Here we give this expression without derivation which can be found in [29] or in [33]. If ϕ is the angle characterizing the orientation of the polarization ellipse, then one has

$$\frac{d\phi}{dt} = \frac{1}{2} k^\alpha k^j \epsilon_{j\hat{p}\hat{q}} \partial_q h_{\alpha\hat{p}} , \quad (3.28)$$

where the hat over the spatial indices denotes the projection onto the plane orthogonal to the propagation of light ray, for instance, $A^{\hat{i}} \equiv P^i_j A^j$. In the following chapter we shall integrate this equation analytically to obtain the angle of the rotation of the polarization plane for an electromagnetic signal propagating in gravitational field of an isolated system.

Chapter 4

Solution to the Equations of Propagation of Electromagnetic Signals

1 Method of Analytical Integration of the Equations of Propagation of Electromagnetic Signals

In this section we describe the method of analytical integration of equations of light geodesics in the field of a localized source developed by Kopeikin et al. in the series of publications [24], [25] and [33]. In subsequent sections this method will be applied to find solutions to the equation of geodesics and Skrotskii effect (Eqs. (2.5) and (3.28) of Chap. 3) in the spacetime with metric given by Eq. (1.7), Chap. 2.

We introduce new variables τ and ξ^i as follows:

$$\tau \equiv k_i x^i, \quad \xi^i \equiv P_j^i x^j, \quad (1.1)$$

where P_j^i is the operator of projection onto the plane perpendicular to k^i . Considering the unperturbed trajectory of the photon: $x_N^i = k^i(t - t_0) + x_0$, we see that the variable τ , characterizing the position of the photon, is proportional to

time

$$\tau \equiv k_i x_N^i = t - t^*, \quad (1.2)$$

where $t^* \equiv k_i x_0^i - t_0$ is the time of the closest approach of the electromagnetic signal to the origin of the coordinate system. Since t^* is a constant for a particular ray, one has $d\tau = dt$. This allows to change to the variable τ when calculating integrals along the unperturbed ray. Vector ξ^i is the vector from the origin of the coordinate system to the point of the closest approach. In terms of the new variables the unperturbed trajectory can be written as

$$x_N^i(\tau) = k^i \tau + \xi^i. \quad (1.3)$$

Since the vectors k^i and ξ^i are orthogonal to each other, the distance from the origin of the coordinate system to the point on unperturbed trajectory with coordinates τ and ξ^i can be expressed as

$$r_N = \sqrt{\tau^2 + d^2}, \quad (1.4)$$

where $d = |\boldsymbol{\xi}|$ is the impact parameter of the unperturbed light-ray trajectory with respect to the origin of the coordinate system.

We introduce the operators of differentiation with respect to τ and ξ^i

$$\hat{\partial}_\tau \equiv \frac{\partial}{\partial \tau}, \quad \hat{\partial}_i \equiv P_i^j \frac{\partial}{\partial \xi^j}. \quad (1.5)$$

Then for any smooth function $F(t, x^i)$ the following relationships are valid between the derivatives in the old and new variables taken on the unperturbed trajectory

$$\left[\left(\frac{\partial}{\partial x^i} + k_i \frac{\partial}{\partial t} \right) F(t, \mathbf{x}) \right]_{\mathbf{x}=\mathbf{x}_0+\mathbf{k}(t-t_0)} = \left(\frac{\partial}{\partial \xi^i} + k_i \frac{\partial}{\partial \tau} \right) F(t^* + \tau, \boldsymbol{\xi} + \mathbf{k}\tau), \quad (1.6)$$

$$\left[\frac{\partial}{\partial t} F(t, \mathbf{x}) \right]_{\mathbf{x}=\mathbf{x}_0+\mathbf{k}(t-t_0)} = \frac{\partial}{\partial t^*} F(t^* + \tau, \boldsymbol{\xi} + \mathbf{k}\tau). \quad (1.7)$$

In the left-hand sides of the equations (1.6) and (1.7) one has to first calculate the derivatives and only after that substitute the unperturbed trajectory $\mathbf{x} = \mathbf{x}_0 + \mathbf{k}(t-t_0)$, while in the right-hand sides one has to first substitute the unperturbed trajectory, parametrized by the variables τ and ξ^i , and then differentiate. Using the equation (1.7) in (1.6) one can obtain an expression for spacial derivative:

$$\left[\frac{\partial F(t, \mathbf{x})}{\partial x^i} \right]_{\mathbf{x}=\mathbf{x}_0+\mathbf{k}(t-t_0)} = \left(\frac{\partial}{\partial \xi^i} + k_i \frac{\partial}{\partial \tau} - k_i \frac{\partial}{\partial t^*} \right) F(t^* + \tau, \boldsymbol{\xi} + \mathbf{k}\tau). \quad (1.8)$$

Equations (1.7) and (1.8) can be used for changing over to the variables τ and ξ^i in the equations of light propagation. A useful property of the variables τ and ξ^i is that when one calculates integrals along the light rays the following formulae are valid

$$\int \frac{\partial}{\partial \tau} F(\tau, \boldsymbol{\xi}) d\tau = F(\tau, \boldsymbol{\xi}) + C(\boldsymbol{\xi}), \quad (1.9)$$

$$\int \frac{\partial}{\partial \xi^i} F(\tau, \boldsymbol{\xi}) d\tau = \frac{\partial}{\partial \xi^i} \int F(\tau, \boldsymbol{\xi}) d\tau, \quad (1.10)$$

where $C(\boldsymbol{\xi})$ is a function of $\boldsymbol{\xi}$. Equation (1.9) shows that the terms represented by partial derivatives with respect to τ can be immediately integrated. Equation (1.10) states that one can change the order of integration and differentiation which crucially simplifies the problem of integration of the equations as it is shown below.

Let us consider the geodesic equations (2.5), Ch. 3. All terms in the right-hand sides of the equations are proportional to the first-order partial derivatives of the metric perturbations $h_{\alpha\beta}$ with respect to time or spacial coordinates. The metric perturbations, given by the equations (1.7), Ch. 2, consist of the canonical and gauge-dependent parts. The canonical part (Eqns. (1.2)-(1.5) of Ch. 2) is given by a linear combination of different-order partial derivatives with respect to spacial variables of functions $[F(t-r)/r]$ (where $F(t-r)$ stands for a mass- or spin-type multipole moment). If one changes the variables to τ and

ξ^i in the geodesic equations (2.5), Ch. 3, integrates and then changes the order of differentiation and integration in the right-hand sides of the equations, the only two types of integrals that will appear in the solutions (not counting the gauge-dependent terms) will be the integrals of the type $[F(t-r)/r]^{[-1]}$ and $[F(t-r)/r]^{[-2]}$. The gauge-dependent terms, after changing the variables to τ and ξ^i , will appear in the equations of geodesics under the second-order partial derivative with respect to τ and thus can be immediately integrated (cf. Eq. (1.10)). A similar consideration can be made for the equation (3.28), Ch. 3, describing Skrotskii effect.

As it follows from the consideration above, the problem of integration of the equations of light propagation reduces to evaluation of integrals $[F(t-r)/r]^{[-1]}$ and $[F(t-r)/r]^{[-2]}$ where $F(t-r)$ denotes multipole moments of different type and order of the gravitating system.

Let us first consider the contributions to the relativistic effects in propagation of light due to mass-monopole and spin-dipole terms. We neglect the loss of energy and angular momentum due to gravitational radiation so that the total mass and angular momentum of the system are considered to be conserved. Then the contributions under consideration can be expressed in terms of integrals $[1/r]^{[-1]}$, $[\hat{\partial}_i(1/r)]^{[-1]}$, $[\hat{\partial}_{ia}(1/r)]^{[-1]}$, $[\hat{\partial}_i(1/r)]^{[-2]}$ and $[\hat{\partial}_{ia}(1/r)]^{[-2]}$ which can be evaluated.

The relativistic corrections to the light ray path and the relativistic effect of rotation of the polarization plane due to higher-order multipole moments (starting with the mass- and spin-quadrupole) are expressed in terms of integrals $[F(t-r)/r]^{[-1]}$ and $[F(t-r)/r]^{[-2]}$ along the unperturbed light ray, where $F(t-r)$ denotes a multipole moment. To evaluate these integrals we introduce the new variable [25]

$$y \equiv s - t^* = \tau - r(\tau) = \tau - \sqrt{d^2 + \tau^2}. \quad (1.11)$$

Then the following relationships hold:

$$\tau = \frac{y^2 - d^2}{2y}, \quad \sqrt{d^2 + \tau^2} = -\frac{1}{2} \frac{d^2 + y^2}{y}, \quad d\tau = \frac{1}{2} \frac{d^2 + y^2}{y^2} dy. \quad (1.12)$$

Making use of the new variable y and the equations (1.12) we can express the integrals under discussion as follows

$$\left[\frac{F(t-r)}{r} \right]^{[-1]} = - \int_{-\infty}^y \frac{F(t^* + \zeta)}{\zeta} d\zeta, \quad (1.13)$$

$$\left[\frac{F(t-r)}{r} \right]^{[-2]} = -\frac{1}{2} \int_{-\infty}^y \int_{-\infty}^{\eta} \frac{F(t^* + \zeta)}{\zeta} d\zeta d\eta - \frac{d^2}{2} \int_{-\infty}^y \frac{1}{\eta^2} \int_{-\infty}^{\eta} \frac{F(t^* + \zeta)}{\zeta} d\zeta d\eta, \quad (1.14)$$

where ζ and η are the new variables of integration and t^* is the time of the closest approach of the photon to the origin of the coordinate system. It is worth to note that t^* depends on the choice of coordinate system, which can be arbitrary in our formalism.

An important property of the integrals (1.13) and (1.14), expressed in terms of the variable y , is that they depend on τ and ξ^i only through the upper limits of integration (since $y \equiv \tau - \sqrt{d^2 + \tau^2}$) and the square of the impact parameter in the prefactor of the second integral in (1.14).

Differentiating (1.13) with respect to ξ^i and τ yields

$$\hat{\partial}_i \left\{ \left[\frac{F(t-r)}{r} \right]^{[-1]} \right\} = -\frac{F(t^* + y)}{y} \hat{\partial}_i y = -F(t^* + y) \hat{\partial}_i \ln(-y) = \frac{\xi^i}{yr} F(t-r), \quad (1.15)$$

$$\hat{\partial}_\tau \left\{ \left[\frac{F(t-r)}{r} \right]^{[-1]} \right\} = -\frac{F(t^* + y)}{y} \hat{\partial}_\tau y = -\frac{F(t^* + y)}{y} \left(1 - \frac{\tau}{r} \right) = \frac{F(t-r)}{r}. \quad (1.16)$$

The last result also follows from the equation (1.10). Thus, the derivatives of the integral $[F(t-r)/r]^{[-1]}$ with respect to either ξ^i or τ can be expressed in terms

of the integrand. Differentiating (1.14) with respect to ξ^i multiple times yields

$$\hat{\partial}_k \left\{ \left[\frac{F(t-r)}{r} \right]^{[-2]} \right\} = \xi^k \left\{ \frac{1}{y} \left[\frac{F(t-r)}{r} \right]^{[-1]} - \int_{-\infty}^y \frac{1}{\eta^2} \int_{-\infty}^{\eta} \frac{F(t^* + \zeta)}{\zeta} d\zeta d\eta \right\}, \quad (1.17)$$

$$\hat{\partial}_{jk} \left\{ \left[\frac{F(t-r)}{r} \right]^{[-2]} \right\} = \frac{\xi^k \xi^j}{y^2} \frac{F(t-r)}{r} + P^{jk} \left\{ \frac{1}{y} \left[\frac{F(t-r)}{r} \right]^{[-1]} - \int_{-\infty}^y \frac{1}{\eta^2} \int_{-\infty}^{\eta} \frac{F(t^* + \zeta)}{\zeta} d\zeta d\eta \right\}, \quad (1.18)$$

$$\hat{\partial}_{ijk} \left\{ \left[\frac{F(t-r)}{r} \right]^{[-2]} \right\} = \frac{1}{y} \left\{ \left(P^{ij} + \frac{\xi^{ij}}{yr} \right) \hat{\partial}_k + P^{jk} \hat{\partial}_i + \xi^j \hat{\partial}_{ik} \right\} \left[\frac{F(t-r)}{r} \right]^{[-1]}, \quad (1.19)$$

In the last expression all terms contain at least a first order derivative of the integral $[F(t-r)/r]^{[-1]}$ with respect to ξ^i . As it was shown above (Eq. (1.15)) integration is eliminated in this case. Thus, the third and higher-order derivatives of $[F(t-r)/r]^{[-2]}$ with respect to ξ^i can be expressed in terms of $F(t-r)$ and one does not have to perform integration. Explicitly evaluating the derivatives in (1.19) using (1.15) one gets

$$\begin{aligned} \hat{\partial}_{ijk} \left\{ \left[\frac{F(t-r)}{r} \right]^{[-2]} \right\} &= \frac{P^{ij} \xi^k + P^{jk} \xi^i + P^{ik} \xi^j}{y^2 r} F(t-r) \\ &+ \frac{\xi^i \xi^j \xi^k}{y^2 r^2} \left[\left(\frac{2}{y} - \frac{1}{r} \right) F(t-r) - \dot{F}(t-r) \right]. \end{aligned} \quad (1.20)$$

In the equations for trajectory of a photon and gravity-induced rotation of polarization plane the integrals $[F(t-r)/r]^{[-1]}$ and $[F(t-r)/r]^{[-2]}$ are differentiated with respect to τ and ξ^i and it can be shown that in all terms differentiation eliminates integration. In the following two sections the solutions are formally given in terms of integrals $[F(t-r)/r]^{[-1]}$ and $[F(t-r)/r]^{[-2]}$, but one should bear in mind that after performing differentiations using the formulae (1.9), (1.15) and (1.19) one can express the integrals in terms of corresponding integrands.

2 Solution to the Equations for Trajectory of a Photon

Using the formulae (1.7) and (1.8) and rewriting the equation of geodesics (2.5), Ch. 3, in terms of the variables τ and ξ^i one obtains

$$\frac{d^2 x^i(\tau)}{d\tau^2} = \frac{1}{2} k^\alpha k^\beta \hat{\partial}_i h_{\alpha\beta}^{\text{can.}} - \hat{\partial}_\tau \left(k^\alpha h_{i\alpha}^{\text{can.}} - \frac{1}{2} k^i k^j k^p q_{jp}^{\text{can.}} \right) - \hat{\partial}_{\tau\tau} (w^i - k^i w^0), \quad (2.1)$$

where in the right-hand side all functions are taken on the unperturbed light-ray trajectory (before performing differentiation). The gauge functions w^α have not yet been specified which means that equations (2.1) are gauge-invariant. Substituting into (2.1) the metric perturbations expressed in terms of τ and ξ^i (Eqns. (0.1)-(0.9) of Appendix A) and integrating once with respect to τ one obtains [33]

$$\dot{x}^i(\tau) = k^i + \dot{\Xi}^i(\tau, \boldsymbol{\xi}), \quad (2.2)$$

where

$$\dot{\Xi}^i(\tau, \boldsymbol{\xi}) = \dot{\Xi}_{(G)}^i(\tau, \boldsymbol{\xi}) + \dot{\Xi}_{(M)}^i(\tau, \boldsymbol{\xi}) + \dot{\Xi}_{(S)}^i(\tau, \boldsymbol{\xi}), \quad (2.3)$$

$$\dot{\Xi}_{(G)}^i(\tau, \boldsymbol{\xi}) = \hat{\partial}_\tau [(\varphi^i - k^i \varphi^0) + (w^i - k^i w^0)] , \quad (2.4)$$

$$\begin{aligned} \dot{\Xi}_{(M)}^i(\tau, \boldsymbol{\xi}) = & 2\mathcal{M} \left[\hat{\partial}_i \frac{1}{r} \right]^{[-1]} - k^i \frac{2\mathcal{M}}{r} + \quad (2.5) \\ & 2\hat{\partial}_i \sum_{l=2}^{\infty} \sum_{p=0}^l \sum_{q=0}^p \frac{(-1)^{l+p-q}}{l!} C_l(l-p, p-q, q) H(2-q) \times \\ & \left(1 - \frac{p-q}{l} \right) \left(1 - \frac{p-q}{l-1} \right) k_{\langle a_1 \dots a_p} \hat{\partial}_{a_{p+1} \dots a_l} \hat{\partial}_\tau^q \left[\frac{\mathcal{I}_{A_l}^{(p-q)}(t-r)}{r} \right]^{[-1]} - \\ & 2 \sum_{l=2}^{\infty} \sum_{p=0}^{l-1} \frac{(-1)^{l+p}}{l!} C_l(l-p, p) \left(1 - \frac{p}{l} \right) \times \\ & \left\{ \left(1 + \frac{p}{l-1} \right) k_{i \langle a_1 \dots a_p} \hat{\partial}_{a_{p+1} \dots a_l} \left[\frac{\mathcal{I}_{A_l}^{(p)}(t-r)}{r} \right] - \right. \\ & \left. \frac{2p}{l-1} k_{\langle a_1 \dots a_{p-1}} \hat{\partial}_{a_p \dots a_{l-1}} \left[\frac{\mathcal{I}_{i A_{l-1}}^{(p)}(t-r)}{r} \right] \right\} , \end{aligned}$$

$$\dot{\Xi}_{(S)}^i(\tau, \boldsymbol{\xi}) = 2k_j \epsilon_{jba} \mathcal{S}_b \left[\hat{\partial}_{ia} \frac{1}{r} \right]^{[-1]} - 2\hat{\partial}_a \frac{\epsilon_{iba} \mathcal{S}_b}{r} \quad (2.6)$$

$$\begin{aligned} & 4k_j \hat{\partial}_{ia} \sum_{l=2}^{\infty} \sum_{p=0}^{l-1} \sum_{q=0}^p \frac{(-1)^{l+p-q} l}{(l+1)!} C_{l-1}(l-p-1, p-q, q) H(2-q) \times \\ & \left(1 - \frac{p-q}{l-1} \right) k_{\langle a_1 \dots a_p} \hat{\partial}_{a_{p+1} \dots a_{l-1}} \hat{\partial}_\tau^q \left[\frac{\epsilon_{jba} \mathcal{S}_{b A_{l-1}}^{(p-q)}(t-r)}{r} \right]^{[-1]} + \\ & 4 \left(\hat{\partial}_a - k_a \hat{\partial}_{t^*} \right) \sum_{l=2}^{\infty} \sum_{p=0}^{l-1} \frac{(-1)^{l+p} l}{(l+1)!} C_{l-1}(l-p-1, p) \times \\ & \left(1 - \frac{p}{l-1} \right) k_{\langle a_1 \dots a_p} \hat{\partial}_{a_{p+1} \dots a_{l-1}} \left[\frac{\epsilon_{iba} \mathcal{S}_{b A_{l-1}}^{(p)}(t-r)}{r} \right] + \\ & 4k_j \sum_{l=2}^{\infty} \sum_{p=0}^{l-1} \frac{(-1)^{l+p} l}{(l+1)!} C_{l-1}(l-p-1, p) k_{\langle a_1 \dots a_p} \hat{\partial}_{a_{p+1} \dots a_{l-1}} \left[\frac{\epsilon_{jba_{l-1}} \mathcal{S}_{ib A_{l-2}}^{(p+1)}(t-r)}{r} \right] . \end{aligned}$$

Here $\dot{\Xi}_{(G)}^i(\tau, \boldsymbol{\xi})$ represents the gauge-dependent perturbations, $\dot{\Xi}_{(M)}^i(\tau, \boldsymbol{\xi})$ and $\dot{\Xi}_{(S)}^i(\tau, \boldsymbol{\xi})$ are the perturbations due to the mass and spin multipole moments, correspond-

ingly. The Heaviside function $H(p - q)$ is defined by the expression (2.19), Ch. 1 and $C_l(l - p, p - q, q)$ are the polynomial coefficients (2.18), Ch. 1. The gauge functions φ^α were introduced as follows: we collected in the equations (2.1) all terms with the second and higher-order derivatives with respect to τ and equated them to $k^i\varphi^0 - \varphi^i$. By introducing the functions φ^α we singled out the terms that can be immediately integrated. We do not separate $k^i\varphi^0 - \varphi^i$ into φ^0 and φ^i , since such separation is not unique while the functions $k^i\varphi^0 - \varphi^i$ are defined uniquely (Eqns. (0.4)-(0.6) in Appendix B).

The gauge functions w^α can be chosen arbitrarily. For the reasons discussed in Section 2 we choose w^α which make our coordinate system ADM-harmonic, that is satisfying both ADM and harmonic gauge conditions. These functions are given by the expressions (0.2) and (0.3) of Appendix B.

The second integration of (2.1) yields the expressions for trajectory of the photon

$$x^i(\tau) = x_N^i + \Delta_{(G)} \Xi^i + \Delta_{(M)} \Xi^i + \Delta_{(S)} \Xi^i, \quad (2.7)$$

where

$$\begin{aligned} \Delta_{(G)} \Xi^i &\equiv \Xi_{(G)}^i(\tau, \boldsymbol{\xi}) - \Xi_{(G)}^i(\tau_0, \boldsymbol{\xi}), \\ \Delta_{(M)} \Xi^i &\equiv \Xi_{(M)}^i(\tau, \boldsymbol{\xi}) - \Xi_{(M)}^i(\tau_0, \boldsymbol{\xi}), \\ \Delta_{(S)} \Xi^i &\equiv \Xi_{(S)}^i(\tau, \boldsymbol{\xi}) - \Xi_{(S)}^i(\tau_0, \boldsymbol{\xi}). \end{aligned} \quad (2.8)$$

The term

$$\Xi_{(G)}^i(\tau, \boldsymbol{\xi}) = (\varphi^i - k^i\varphi^0) + (w^i - k^iw^0), \quad (2.9)$$

represents the gauge-dependent part of the trajectory's perturbations, and the

gauge-independent terms due to the mass and spin multipoles are given by

$$\begin{aligned}
\Xi_{(M)}^i(\tau, \boldsymbol{\xi}) = & 2\mathcal{M} \left[\hat{\partial}_i \frac{1}{r} \right]^{[-2]} - 2\mathcal{M} k^i \left[\frac{1}{r} \right]^{[-1]} + \tag{2.10} \\
& 2\hat{\partial}_i \sum_{l=2}^{\infty} \sum_{p=0}^l \sum_{q=0}^p \frac{(-1)^{l+p-q}}{l!} C_l(l-p, p-q, q) H(2-q) \times \\
& \left(1 - \frac{p-q}{l} \right) \left(1 - \frac{p-q}{l-1} \right) k_{\langle a_1 \dots a_p} \hat{\partial}_{a_{p+1} \dots a_l} \hat{\partial}_\tau^q \left[\frac{\mathcal{I}_{A_l}^{(p-q)}(t-r)}{r} \right]^{[-2]} - \\
& 2 \sum_{l=2}^{\infty} \sum_{p=0}^{l-1} \frac{(-1)^{l+p}}{l!} C_l(l-p, p) \left(1 - \frac{p}{l} \right) \times \\
& \left\{ \left(1 + \frac{p}{l-1} \right) k_{i \langle a_1 \dots a_p} \hat{\partial}_{a_{p+1} \dots a_l} \left[\frac{\mathcal{I}_{A_l}^{(p)}(t-r)}{r} \right]^{[-1]} - \right. \\
& \left. \frac{2p}{l-1} k_{\langle a_1 \dots a_{p-1}} \hat{\partial}_{a_p \dots a_{l-1}} \left[\frac{\mathcal{I}_{iA_{l-1}}^{(p)}(t-r)}{r} \right]^{[-1]} \right\},
\end{aligned}$$

$$\begin{aligned}
\Xi_{(S)}^i(\tau, \xi) = & 2k_j \epsilon_{jba} \mathcal{S}_b \left[\hat{\partial}_{ia} \frac{1}{r} \right]^{[-2]} - 2\epsilon_{iba} \mathcal{S}_b \left[\hat{\partial}_a \frac{1}{r} \right]^{[-1]} - \quad (2.11) \\
& 4k_j \hat{\partial}_{ia} \sum_{l=2}^{\infty} \sum_{p=0}^{l-1} \sum_{q=0}^p \frac{(-1)^{l+p-q} l}{(l+1)!} C_{l-1}(l-p-1, p-q, q) H(2-q) \times \\
& \left(1 - \frac{p-q}{l-1} \right) k_{\langle a_1 \dots a_p} \hat{\partial}_{a_{p+1} \dots a_{l-1}} \hat{\partial}_\tau^q \left[\frac{\epsilon_{jba} \mathcal{S}_{bA_{l-1}}^{(p-q)}(t-r)}{r} \right]^{[-2]} + \\
& 4 \left(\hat{\partial}_a - k_a \hat{\partial}_{t^*} \right) \sum_{l=2}^{\infty} \sum_{p=0}^{l-1} \frac{(-1)^{l+p} l}{(l+1)!} C_{l-1}(l-p-1, p) \times \\
& \left(1 - \frac{p}{l-1} \right) k_{\langle a_1 \dots a_p} \hat{\partial}_{a_{p+1} \dots a_{l-1}} \left[\frac{\epsilon_{iba} \mathcal{S}_{bA_{l-1}}^{(p)}(t-r)}{r} \right]^{[-1]} + \\
& 4k_j \sum_{l=2}^{\infty} \sum_{p=0}^{l-1} \frac{(-1)^{l+p} l}{(l+1)!} C_{l-1}(l-p-1, p) \times \\
& k_{\langle a_1 \dots a_p} \hat{\partial}_{a_{p+1} \dots a_{l-1}} \left[\frac{\epsilon_{jba_{l-1}} \mathcal{S}_{ibA_{l-2}}^{(p+1)}(t-r)}{r} \right]^{[-1]} .
\end{aligned}$$

3 Solution to the Equations for Rotation of the Polarization Plane

We rewrite the equation (3.28) in terms of the variables τ and ξ^i using the relationship (1.8) and substitute the metric perturbations, expressed in terms of the variables τ and ξ^i (Eqns. (0.1)-(0.9) in Appendix A):

$$\frac{d\phi}{d\tau} = \frac{1}{2} k^\alpha k^j \epsilon_{j\hat{p}\hat{q}} \hat{\partial}_{\hat{q}} h_{\alpha\hat{p}}^{can.} + \frac{1}{2} k^j \epsilon_{j\hat{p}\hat{q}} \hat{\partial}_{q\tau} w^{\hat{p}} . \quad (3.1)$$

Integrating with respect to τ and changing the order of differentiation and integration yields

$$\phi = \phi_{(G)} + \phi_{(M)} + \phi_{(S)} + \phi_0 , \quad (3.2)$$

where ϕ_0 is the constant angle characterizing the initial orientation of the polarization ellipse in the plane formed by the vectors $\mathbf{e}_{(1)}$ and $\mathbf{e}_{(2)}$. The terms $\phi_{(G)}$,

$\phi_{(M)}$ and $\phi_{(S)}$ describe the contributions due to gauge-dependent terms and terms depending on mass- and spin-type moments, correspondingly.

The gauge-dependent part of the Skrotskii effect can be obtained immediately, since the gauge dependent terms appear in the equation (3.1) under the first-order derivative with respect to τ (cf. Eq. (1.9)). The integration yields

$$\phi_{(G)} = \frac{1}{2} k^j \epsilon_{j\hat{p}\hat{q}} \hat{\partial}_{\hat{q}} (w^{\hat{p}} + \chi^{\hat{p}}) , \quad (3.3)$$

where the gauge functions w^i and χ^i are given by the equations (3.4), Ch. 2 and (0.7), App. B. The gauge functions χ^i were introduced by collecting in the equation (3.1) all terms that can be eliminated by a gauge transformation.

The Skrotskii effect due to the mass and spin multipoles of the isolated system is given by

$$\begin{aligned} \phi_{(M)}(\tau) = & 2 \sum_{l=2}^{\infty} \sum_{p=0}^l \frac{(-1)^{l+p}}{l!} C_l(l-p, p) \frac{l-p}{l-1} \\ & \times k_{\langle a_1 \dots a_p} \hat{\partial}_{a_{p+1} \dots a_l \rangle j} \left[\frac{\epsilon_{j\hat{b}a_l} \mathcal{I}_{\hat{b}A_{l-1}}^{(p)}(t-r)}{r} \right]^{[-1]} , \end{aligned} \quad (3.4)$$

and

$$\begin{aligned} \phi_{(S)}(\tau) = & 2 \sum_{l=1}^{\infty} \sum_{p=0}^l \frac{(-1)^{l+p} l}{(l+1)!} C_l(l-p, p) \left(1 - \frac{p}{l}\right) \\ & \times \left[1 + H(l-1) \left(1 - \frac{2p}{l-1}\right) \right] k_{\langle a_1 \dots a_p} \hat{\partial}_{a_{p+1} \dots a_l \rangle} \left[\frac{\mathcal{S}_{A_l}^{(p+1)}(t-r)}{r} \right]^{[-1]} . \end{aligned} \quad (3.5)$$

Like in the equations for light-ray path (2.5)-(2.6) and (2.10)-(2.11) differentiation eliminates integrals in the right-hand sides of the equations (3.4)-(3.5).

Chapter 5

Observable Relativistic Effects

In this section we give the expressions for the observable relativistic effects of time delay and gravitational deflection of light. The expressions for another observable effect – rotation of the polarization plane were essentially given in Section 3 of the previous chapter.

1 Time Delay

In [25] the following expression was obtained for the time of propagation of the electromagnetic signals from emitter to the observer through the gravitational field of an isolated source:

$$t - t_0 = |\mathbf{x} - \mathbf{x}_0| + \Delta(\tau, \tau_0) , \quad (1.1)$$

$$\Delta(\tau, \tau_0) = \Delta_{(G)}(\tau, \tau_0) + \Delta_{(M)}(\tau, \tau_0) + \Delta_{(S)}(\tau, \tau_0) , \quad (1.2)$$

where $x_0^\alpha = (t_0, \mathbf{x}_0)$ and $x^\alpha = (t, \mathbf{x})$ are the coordinates of the points of emission and observation of the signal, $\Delta_{(G)}(\tau, \tau_0)$, $\Delta_{(M)}(\tau, \tau_0)$ and $\Delta_{(S)}(\tau, \tau_0)$ are functions describing the delay of the electromagnetic signal due to the gauge-dependent terms, mass and spin multipoles of the gravitational field of the isolated system correspondingly. These functions are expressed in terms of the perturbations of the signal's trajectory as

$$\Delta_{(G)}(\tau, \tau_0) = -k_i \left[\Xi_{(G)}^i(\tau, \boldsymbol{\xi}) - \Xi_{(G)}^i(\tau_0, \boldsymbol{\xi}) \right], \quad (1.3)$$

$$\Delta_{(M)}(\tau, \tau_0) = -k_i \left[\Xi_{(M)}^i(\tau, \boldsymbol{\xi}) - \Xi_{(M)}^i(\tau_0, \boldsymbol{\xi}) \right], \quad (1.4)$$

$$\Delta_{(S)}(\tau, \tau_0) = -k_i \left[\Xi_{(S)}^i(\tau, \boldsymbol{\xi}) - \Xi_{(S)}^i(\tau_0, \boldsymbol{\xi}) \right]. \quad (1.5)$$

We note that the expressions give the effect with respect to the coordinate time which has to be converted to the time measured by the observer. It is shown in [25] that if the observer's velocity is negligible with respect to the global ADM-harmonic coordinate system, the relationship between the coordinate time and proper time T of the observer is given by

$$T = \left(1 - \frac{\mathcal{M}}{r} \right) (t - t_i), \quad (1.6)$$

where t_i is the initial epoch of observation. In most cases the distance r is much greater than \mathcal{M} and coordinate time coincides with the proper time of the observer. If observer is moving with respect to the global coordinate system, an additional Lorentz transformation to the rest frame of the observer has to be performed. This case is considered in [33].

2 Deflection of Light

The expression for a unit observable vector s^i in the direction towards the source of light calculated at the point of observation was derived in [25] and reads

$$s^i(\tau, \boldsymbol{\xi}) = K^i + \alpha^i(\tau, \boldsymbol{\xi}) + \beta^i(\tau, \boldsymbol{\xi}) - \beta^i(\tau_0, \boldsymbol{\xi}) + \gamma^i(\tau, \boldsymbol{\xi}), \quad (2.1)$$

where

$$K^i = \frac{x^i - x_0^i}{|\mathbf{x} - \mathbf{x}_0|} \quad (2.2)$$

is the unit vector in the direction "observer – source of light", defined in the global coordinate system;

$$\alpha^i(\tau, \boldsymbol{\xi}) = -P_j^i \dot{\Xi}^j \quad (2.3)$$

is the vector describing the angle of light deflection;

$$\beta^i(\tau, \boldsymbol{\xi}) = \frac{P_j^i \left[\Xi_{(M)}^j(\tau, \boldsymbol{\xi}) + \Xi_{(S)}^j(\tau, \boldsymbol{\xi}) \right]}{|\mathbf{x} - \mathbf{x}_0|} \quad (2.4)$$

is a relativistic correction, introduced by the relationship

$$k^i = -K^i - \beta^i(\tau, \boldsymbol{\xi}) + \beta^i(\tau_0, \boldsymbol{\xi}). \quad (2.5)$$

The term

$$\gamma^i(\tau, \boldsymbol{\xi}) = -\frac{1}{2} P^{ij} k^q h_{jq}^{TT}(t, \mathbf{x}) \quad (2.6)$$

appears as a result of transformation from the the global ADM-harmonic system to the local frame of the observer and describes the perturbations of the observer's coordinate system due to the gravitational waves from the localized source. It was assumed that the observer is at rest with respect to the global ADM-harmonic system. In the case when the observer is moving one has to perform an additional Lorentz transformation to the rest frame of the observer.

Chapter 6

Discussion

In this chapter we give the approximate expressions for the observable relativistic effects of time delay, light deflection and gravitational rotation of the polarization plane of electromagnetic waves for two practically interesting cases of arrangement of the source of light, the observer and the gravitating system deflecting the light rays. The first case is the gravitational-lens approximation when both the source of light and the observer are located far away from the gravitating system and the light ray passes close to the system (Fig. 6.1). In the second case the source of light and observer are located in the wave zone of the gravitating system and are separated by a distance much smaller than the distance from either of these points to the system (Fig. 6.2). This case corresponds to the approximation of plane gravitational waves.

1 Gravitational-Lens Approximation

1.1 Some useful asymptotic expansions

In the Gravitational-Lens Approximation the impact parameter d of the light ray with respect to the barycenter of the astronomical system is much smaller than the distances from the system to either the observer or the source of light $d \ll \min[r, r_0]$. We assume that the system is located between the source of light

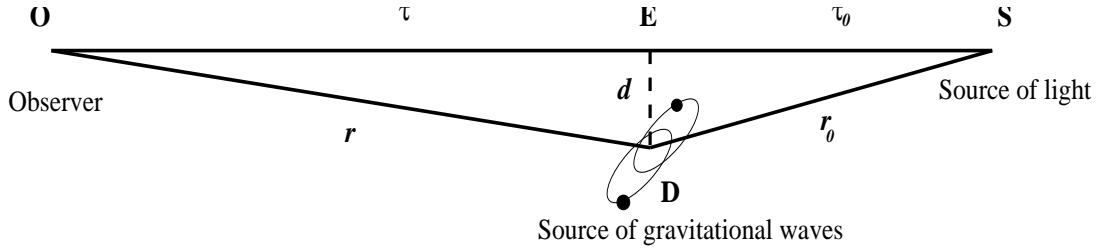


Figure 6.1: Relative configuration of observer (O), source of light (S), and a localized source of gravitational waves (D). The source of gravitational waves deflects light rays which are emitted at the moment t_0 at the point S and received at the moment t at the point O. The point E on the line OS corresponds to the moment of the closest approach of light ray to the deflector D. Distances are $OS = R$, $DO = r$, $DS = r_0$, the impact parameter $DE = d$, $OE = \tau > 0$, $ES = \tau_0 = \tau - R < 0$. The impact parameter d is small in comparison to all other distances.

and observer so that $\tau_0 < 0$ (see Fig. 6.1). Then two small parameters $\varepsilon \equiv d/r$ and $\varepsilon_0 \equiv d/r_0$ can be introduced. We also introduce the parameter $\varepsilon_\lambda \equiv d/\lambda$, where λ is the characteristic wavelength of the gravitational radiation emitted by the astronomical system. When light ray propagates through the near zone of the system ($d \ll \lambda$, see [27], [35]) this parameter is small. In the rest of this section we give some asymptotic expansions in powers of the parameters ε , ε_0 , and ε_λ which are useful for calculation of observable relativistic effects in the gravitational-lens approximation.

The variables y and y_0 can be expanded as follows:

$$y = \sqrt{r^2 - d^2} - r = -\varepsilon \frac{d}{2} + d \sum_{k=2}^{\infty} C_k \varepsilon^{2k-1} = -\varepsilon \frac{d}{2} + O(\varepsilon^3), \quad (1.1)$$

$$y_0 = -\sqrt{r_0^2 - d^2} - r_0 = -2r_0 + \varepsilon_0 \frac{d}{2} - d \sum_{k=2}^{\infty} C_k \varepsilon_0^{2k-1} = -2r_0 + \varepsilon_0 \frac{d}{2} + O(\varepsilon_0^3), \quad (1.2)$$

and

$$\frac{1}{yr} = -\frac{2}{d^2} - \frac{1}{d^2} \sum_{k=1}^{\infty} \mathcal{C}_k \varepsilon^{2k} = -\frac{2}{d^2} + O(\varepsilon^2), \quad (1.3)$$

$$\frac{1}{y_0 r_0} = \frac{1}{d^2} \sum_{k=1}^{\infty} \mathcal{C}_k \varepsilon_0^{2k} = O(\varepsilon_0^2), \quad (1.4)$$

where the numerical coefficient in the expansions is

$$\mathcal{C}_k = (-1)^k \frac{\frac{1}{2} \left(\frac{1}{2} - 1 \right) \cdot \dots \cdot \left(\frac{1}{2} - k + 1 \right)}{k!} = \begin{cases} -1/2, & \text{if } k = 1; \\ -\frac{(2k-3)!!}{(2k)!!}, & \text{if } k \geq 2. \end{cases} \quad (1.5)$$

For the variables t and t_0 one has

$$t = t^* + r - \varepsilon \frac{d}{2} + d \sum_{k=2}^{\infty} \mathcal{C}_k \varepsilon^{2k-1} = t^* + r - \varepsilon \frac{d}{2} + O(\varepsilon^3), \quad (1.6)$$

$$t_0 = t^* - r_0 + \varepsilon_0 \frac{d}{2} - d \sum_{k=2}^{\infty} \mathcal{C}_k \varepsilon_0^{2k-1} = t^* - r_0 + \varepsilon_0 \frac{d}{2} + O(\varepsilon_0^3), \quad (1.7)$$

where \mathcal{C}_k is given by (1.5) and t^* is the time of the closest approach of the light ray to the barycenter of the isolated system.

Using (1.6) and (1.7) we can write down the post-Newtonian expansions for functions of the retarded time $t - r$ as follows:

$$\begin{aligned} F(t - r) &= \sum_{n=0}^{\infty} \frac{(-1)^n d^n}{n!} \left(\frac{\varepsilon}{2} - \sum_{k=2}^{\infty} \mathcal{C}_k \varepsilon^{2k-1} \right)^n F^{(n)}(t^*) \\ &= F(t^*) - \varepsilon \frac{d}{2} \dot{F}(t^*) + O(\varepsilon^2 \varepsilon_\lambda^2), \end{aligned} \quad (1.8)$$

$$\begin{aligned} F(t_0 - r_0) &= \sum_{n=0}^{\infty} \frac{d^n}{n!} \left(\frac{\varepsilon_0}{2} - \sum_{k=2}^{\infty} \mathcal{C}_k \varepsilon_0^{2k-1} \right)^n F^{(n)}(t^* - 2r_0) \\ &= F(t^* - 2r_0) + \varepsilon_0 \frac{d}{2} \dot{F}(t^* - 2r_0) + O(\varepsilon_0^2 \varepsilon_\lambda^2). \end{aligned} \quad (1.9)$$

The asymptotic expansions of integrals of the stationary part of the metric have

the form:

$${}^{[-1]}\left[\frac{1}{r}\right] = -2 \ln d + \ln 2r + O(\varepsilon^2), \quad (1.10)$$

$${}^{[-1]}\left[\frac{1}{r_0}\right] = -\ln 2r_0 + O(\varepsilon_0^2), \quad (1.11)$$

$${}^{[-2]}\left[\frac{1}{r}\right] = -r - 2r \ln d + r \ln 2r - \varepsilon \frac{d}{2} \left[\frac{1}{2} - \ln \frac{d^2}{2r} \right] + O(\varepsilon^2), \quad (1.12)$$

$${}^{[-2]}\left[\frac{1}{r_0}\right] = -r_0 + r_0 \ln 2r_0 - \varepsilon_0 \frac{d}{2} \left[\frac{1}{2} + \ln 2r_0 \right] + O(\varepsilon_0^2). \quad (1.13)$$

The following estimates and expressions for derivatives of functions of retarded time are useful in analysis of observable effects in the gravitational-lens approximation:

$$\hat{\partial}_{\langle a_1 \dots a_l \rangle} \frac{F(t-r)}{r} = O\left(\varepsilon^m \varepsilon_\lambda^n \frac{F}{d^{l+1}}\right), \quad (1.14)$$

where $m \geq 2, n \geq 0$ for any l ;

$$\hat{\partial}_{\langle a_1 \dots a_l \rangle} \frac{F(t_0 - r_0)}{r_0} = O\left(\varepsilon^m \varepsilon_\lambda^n \frac{F}{d^{l+1}}\right), \quad (1.15)$$

where $m \geq 2, n \geq 0$ for any l ;

$$\hat{\partial}_\tau^l \frac{F(t-r)}{r} = O\left(\varepsilon^{l+1} \frac{F}{d^{l+1}}\right) + \sum_{n=1}^l \sum_{m=1}^n O\left(\varepsilon^{l-m+2j+1} \varepsilon_\lambda^m \frac{F}{d^{l+1}}\right), \quad (1.16)$$

where $j(n, m) = \min i_1$ with i_1 satisfying the two equations $m = \sum_{k=1}^l i_k$ and $n = \sum_{k=1}^l k i_k$ (all i_k are positive integers);

$$\hat{\partial}_\tau^l \frac{F(t_0 - r_0)}{r_0} = \sum_{m=0}^l O \left(\varepsilon^{l-m+1} \varepsilon_\lambda^m \frac{F}{d^{l+1}} \right); \quad (1.17)$$

$$\hat{\partial}_{\langle a_1 \dots a_l \rangle}^{[-1]} \left[\frac{F(t-r)}{r} \right] = -2F(t-r) \hat{\partial}_{\langle a_1 \dots a_l \rangle} \ln d + O \left(\varepsilon \varepsilon_\lambda \frac{F}{d^l} \right); \quad (1.18)$$

$$\hat{\partial}_{\langle a_1 \dots a_l \rangle}^{[-1]} \left[\frac{F(t_0 - r_0)}{r_0} \right] = O \left(\varepsilon^m \varepsilon_\lambda^n \frac{F}{d^l} \right), \quad (1.19)$$

where $m \geq 2, n \geq 0$ for any l ;

$$\hat{\partial}_{\langle a_1 \dots a_l \rangle}^{[-2]} \left[\frac{F(t-r)}{r} \right] = -2rF(t-r) \hat{\partial}_{\langle a_1 \dots a_l \rangle} \ln d + O \left(\varepsilon \varepsilon_\lambda \frac{F}{d^{l-1}} \right); \quad (1.20)$$

$$\hat{\partial}_{\langle a_1 \dots a_l \rangle}^{[-2]} \left[\frac{F(t_0 - r_0)}{r_0} \right] = O \left(\varepsilon^m \varepsilon_\lambda^n \frac{F}{d^{l-1}} \right), \quad (1.21)$$

where $m \geq 3$ and $n \geq 0$ for any l .

1.2 Asymptotic expressions for observable effects

In this section we give the approximate expressions for relativistic effects of time delay, light deflection and rotation of the polarization plane in the gravitational-lens approximation. In all expressions we write out explicitly only the terms of the zeroth order with respect to the small parameters ε and ε_0 .

For the relativistic time delay (Eqs. (1.1) - (1.5), Ch. 5) one has

$$\Delta(\tau, \tau_0) = \underset{(M)}{\Delta}(\tau, \tau_0) + \underset{(S)}{\Delta}(\tau, \tau_0), \quad (1.22)$$

where

$$\begin{aligned} \Delta_{(M)}(\tau, \tau_0) = & -4\mathcal{M} \ln d + 2\mathcal{M} \ln(4rr_0) - \\ & 4 \sum_{l=2}^{\infty} \sum_{p=0}^l \frac{(-1)^{l+p}}{l!} C_l(l-p, p) \left(1 - \frac{p}{l}\right) \left(1 - \frac{p}{l-1}\right) \times \\ & \hat{\partial}_{t^*}^p \mathcal{I}_{A_l}(t-r) k_{\langle a_1 \dots a_p} \hat{\partial}_{a_{p+1} \dots a_l \rangle} \ln d + O(\varepsilon \varepsilon_\lambda), \end{aligned} \quad (1.23)$$

$$\begin{aligned} \Delta_{(S)}(\tau, \tau_0) = & -4\epsilon_{iba} k_i \mathcal{S}_b \hat{\partial}_a \ln d + \\ & 8\epsilon_{iba} k_i \hat{\partial}_a \sum_{l=2}^{\infty} \sum_{p=0}^{l-1} \frac{(-1)^{l+p} l}{(l+1)!} C_{l-1}(l-p-1, p) \left(1 - \frac{p}{l-1}\right) \times \\ & \hat{\partial}_{t^*}^p \mathcal{S}_{bA_{l-1}}(t-r) k_{\langle a_1 \dots a_p} \hat{\partial}_{a_{p+1} \dots a_{l-1} \rangle} \ln d + O(\varepsilon \varepsilon_\lambda). \end{aligned} \quad (1.24)$$

The observable unit vector in the direction "observer - source of light" calculated at the point of observation is given by

$$s^i(\tau, \boldsymbol{\xi}) = K^i + \alpha^i(\tau, \boldsymbol{\xi}) + \beta^i(\tau, \boldsymbol{\xi}), \quad (1.25)$$

where we have neglected the quantities $\beta^i(\tau_0, \boldsymbol{\xi})$ and $\gamma^i(\tau, \boldsymbol{\xi})$, since the angle $\beta^i(\tau, \boldsymbol{\xi})$ (Eq. (2.4), Ch. 5) is negligibly small at the point of emission of light and $\gamma^i(\tau, \boldsymbol{\xi})$ is proportional to metric perturbations at the location of the observer (Eq. (2.6), Ch. 5) and thus is negligibly small as well.

The vector $\alpha^i(\tau, \boldsymbol{\xi})$, characterizing the deflection of light, and the relativistic correction $\beta^i(\tau, \boldsymbol{\xi})$ are given by the expressions

$$\alpha^i(\tau, \boldsymbol{\xi}) = \alpha_{(M)}^i(\tau, \boldsymbol{\xi}) + \alpha_{(S)}^i(\tau, \boldsymbol{\xi}), \quad (1.26)$$

where

$$\begin{aligned} \alpha_{(M)}^i(\tau, \boldsymbol{\xi}) = & 4\mathcal{M}\hat{\partial}_i \ln d \\ & + 4\hat{\partial}_i \sum_{l=2}^{\infty} \sum_{p=0}^l \frac{(-1)^{l+p}}{l!} C_l(l-p, p) \left(1 - \frac{p}{l}\right) \left(1 - \frac{p}{l-1}\right) \times \\ & \hat{\partial}_{t^*}^p \mathcal{I}_{A_l}(t-r) k_{\langle a_1 \dots a_p} \hat{\partial}_{a_{p+1} \dots a_l \rangle} \ln d + O(\varepsilon\varepsilon_\lambda), \end{aligned} \quad (1.27)$$

$$\begin{aligned} \alpha_{(S)}^i(\tau, \boldsymbol{\xi}) = & 4\varepsilon_{jba} k_j \mathcal{S}_b \hat{\partial}_{ia} \ln d - \\ & 8\varepsilon_{jba} k_j \hat{\partial}_{ia} \sum_{l=2}^{\infty} \sum_{p=0}^{l-1} \frac{(-1)^{l+p} l}{(l+1)!} C_{l-1}(l-p-1, p) \left(1 - \frac{p}{l-1}\right) \times \\ & \hat{\partial}_{t^*}^p \mathcal{S}_{bA_{l-1}}(t-r) k_{\langle a_1 \dots a_p} \hat{\partial}_{a_{p+1} \dots a_{l-1} \rangle} \ln d + O(\varepsilon\varepsilon_\lambda), \end{aligned} \quad (1.28)$$

and

$$\beta^i(\tau, \boldsymbol{\xi}) = \beta_{(M)}^i(\tau, \boldsymbol{\xi}) + \beta_{(S)}^i(\tau, \boldsymbol{\xi}), \quad (1.29)$$

where

$$\beta_{(M)}^i(\tau, \boldsymbol{\xi}) = -\frac{r}{R} \alpha_{(M)}^i + O(\varepsilon_R), \quad (1.30)$$

$$\beta_{(S)}^i(\tau, \boldsymbol{\xi}) = -\frac{r}{R} \alpha_{(S)}^i + O(\varepsilon_R). \quad (1.31)$$

We note that using the expressions (1.8) and (1.9), we can rewrite (1.23), (1.24), (1.27) and (1.28) expressing functions of retarded time $(t-r)$ in terms of their values at the moment of the closest approach of the photon and the gravitating system t^* . This is accomplished simply by substituting the value t^* instead of $(t-r)$ in the expressions (1.23), (1.24), (1.27) and (1.28) as the corrections will be of the first order with respect to the parameter ε (see Eq. (1.8)). Thus in the first approximation the relativistic effects depend only on the state of the light deflecting system at the moment of the closest approach of the photon to the barycenter of the system.

It is important to note also that in the expressions for time delay and deflection angle (1.23), (1.24), (1.27), and (1.28) the terms in which the order of

differentiation with respect to time p equals the order of the multipole moments l cancel out. Cancellation also takes place for the terms with $p = l - 1$. The terms with $p = l$ and $p = l - 1$ are likely to describe the effects due to the wave-zone $1/r$ and intermediate-zone $1/r^2$ fields (compare [23]), however we did not check this by a calculation. Thus, it is likely that these fields do not contribute to the leading-order terms¹ of the expressions for observable effects in gravitational-lens approximation. Due to the cancellation the leading terms in the equations for time delay and deflection angle fall off with impact parameter as $1/d^2$ and $1/d^3$ respectively for contributions due to multipole moments of all orders starting with quadrupole. Such strong dependence on impact parameter makes much more difficult any observation of time-dependent astrometric, timing or photometric effects in gravitational lensing (such as variations in apparent positions or brightness of light sources or variations in time of arrival of pulsar impulses) [23, 25, 29, 33]. After a series of erroneous papers [19–22], claiming that in the alignment under consideration there may be observable time-dependent effects due to interaction of electromagnetic and gravitational waves, suppression of the gravitational wave effects and the strong dependence on the impact parameter was clearly shown in [23] and, independently, in [25] and [29]. In [23] an estimate was also made for the fall-off properties of the terms generated by higher-order multipole moments. The authors state that the contributions due to higher-order moments will fall off with the impact parameter "at least as fast (and probably faster) than the quadrupolar one". This agrees with the results of our calculations stated above.

In the gravitational-lens approximation we can also reexpress our results for time delay and deflection angle in terms of the gravitational-lens potential ψ which is essentially the gravitational time delay in propagation of signals from

¹Terms of order ε^0 .

the source to the observer [23, 43]:

$$\Delta(\tau, \tau_0) = 2\mathcal{M} \ln(4rr_0) - 4\psi, \quad (1.32)$$

$$\alpha^i(\tau, \boldsymbol{\xi}) = 4\hat{\partial}_i\psi, \quad (1.33)$$

where

$$\begin{aligned} \psi = & \left\{ \mathcal{M} + \sum_{l=2}^{\infty} \sum_{p=0}^l \frac{(-1)^{l+p}}{l!} C_l(l-p, p) \left(1 - \frac{p}{l}\right) \left(1 - \frac{p}{l-1}\right) \times \right. \\ & \hat{\partial}_{t^*}^p \mathcal{I}_{A_l}(t^*) k_{\langle a_1 \dots a_p} \hat{\partial}_{a_{p+1} \dots a_l \rangle} + \\ & \epsilon_{jba} k_j \mathcal{S}_b \hat{\partial}_a - 2\epsilon_{jba} k_j \hat{\partial}_a \sum_{l=2}^{\infty} \sum_{p=0}^l \frac{(-1)^{l+p} l}{(l+1)!} C_{l-1}(l-p-1, p) \times \\ & \left. \left(1 - \frac{p}{l-1}\right) \hat{\partial}_{t^*}^p \mathcal{S}_{bA_{l-1}}(t^*) k_{\langle a_1 \dots a_p} \hat{\partial}_{a_{p+1} \dots a_{l-1} \rangle} \right\} \ln d + O(\epsilon\epsilon_\lambda). \end{aligned} \quad (1.34)$$

This expression takes into account multipole moments of all orders in the nonstationary case and presents a generalization of the corresponding results obtained in [24], [23] and [25].

The expressions for the angle of rotation of the polarization plane of light in the gravitational-lens approximation have the form

$$\begin{aligned} \phi_{(M)}(\tau) - \phi_{(M)}(\tau_0) = & -4 \sum_{l=2}^{\infty} \sum_{p=0}^l \frac{(-1)^{l+p}}{l!} C_l(l-p, p) \frac{l-p}{l-1} \times \\ & \hat{\partial}_{t^*}^p \epsilon_{j\hat{b}a_l} \mathcal{I}_{\hat{b}A_{l-1}}(t-r) k_{\langle a_1 \dots a_p} \hat{\partial}_{a_{p+1} \dots a_l \rangle j} \ln d + O(\epsilon\epsilon_\lambda), \end{aligned} \quad (1.35)$$

$$\begin{aligned} \phi_{(S)}(\tau) - \phi_{(S)}(\tau_0) = & -4 \sum_{l=1}^{\infty} \sum_{p=0}^l \frac{(-1)^{l+p} l}{(l+1)!} C_l(l-p, p) \left(1 - \frac{p}{l}\right) \times \\ & \left[1 + H(l-1) \left(1 - \frac{2p}{l-1}\right) \right] \times \\ & \hat{\partial}_{t^*}^{p+1} \mathcal{S}_{A_l}(t-r) k_{\langle a_1 \dots a_p} \hat{\partial}_{a_{p+1} \dots a_l \rangle} \ln d + O(\epsilon\epsilon_\lambda). \end{aligned} \quad (1.36)$$

2 Plane-Gravitational-Wave Approximation

2.1 Some useful asymptotic expansions

In the plane-gravitational-wave approximation we assume that both the source of light and the observer are located in the wave zone of the astronomical system emitting gravitational waves ($\min[r, r_0] \gg \lambda$, where λ is the characteristic wavelength of the gravitational waves emitted by the system [27, 35]). In addition we impose a restriction on the distance R between the source of light and the observer: $R \ll \min[r, r_0]$ (plane wave condition) (Fig. 6.2) and introduce the small parameters $\delta_\lambda = \lambda/r$, $\delta = R/r$ and $\delta_0 = R/r_0$.

The relationship among r , r_0 and R can be written as

$$r_0^2 = r^2 - 2rR \cos \theta + R^2 = r^2(1 - 2\delta \cos \theta + \delta^2), \quad (2.1)$$

where θ is the angle between the directions "observer – source of light" and "observer – gravitating system" (Fig 6.2). From the Eq. (2.1) it follows that

$$r_0 = r(1 - \delta \cos \theta + O(\delta^2)), \quad (2.2)$$

$$\frac{1}{r_0} = \frac{1}{r}(1 + \delta \cos \theta + O(\delta^2)). \quad (2.3)$$

For the variables τ and τ_0 we have

$$\tau = r \cos \theta, \quad \tau_0 = \tau - R = r \cos \theta - R. \quad (2.4)$$

The quantities d and y can be expressed as

$$d = r \sin \theta, \quad (2.5)$$

$$y = \tau - r = -r(1 - \cos \theta). \quad (2.6)$$

The instants of time $(t - r)$ and $(t_0 - r_0)$ are related by the equation

$$t_0 - r_0 = t - r - R(1 - \cos \theta). \quad (2.7)$$

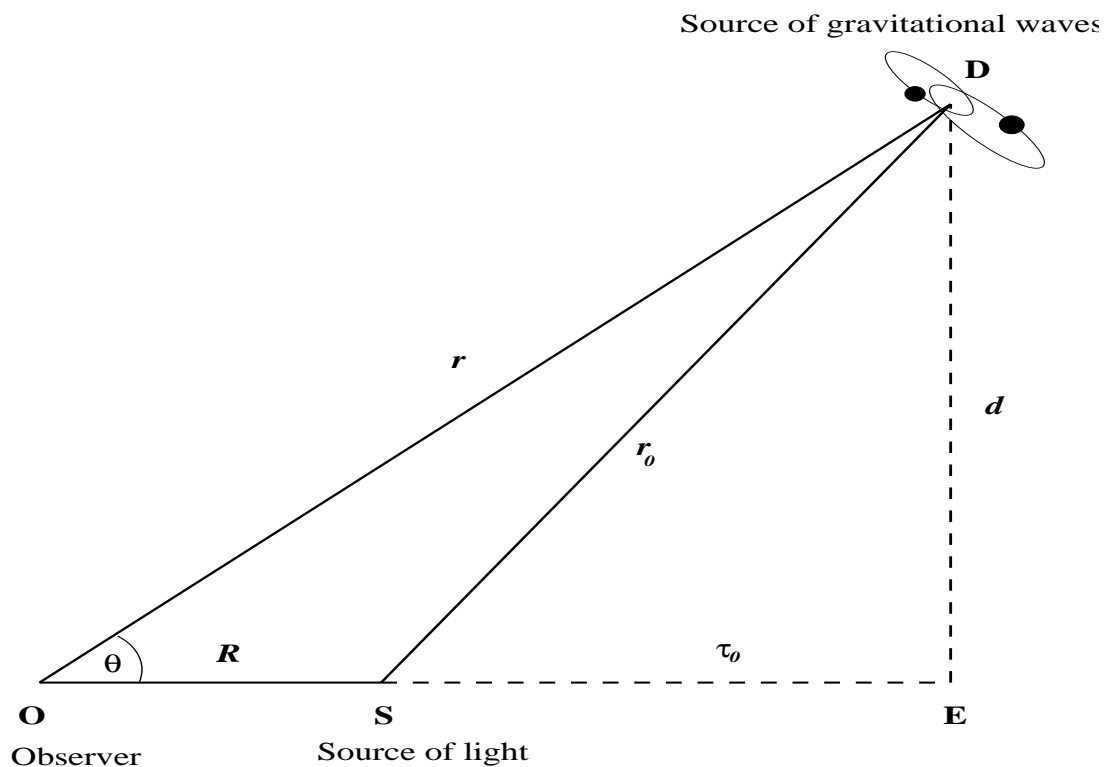


Figure 6.2: Relative configuration of observer (O), source of light (S), and a localized source of gravitational waves (D). The source of gravitational waves deflects light rays which are emitted at the moment t_0 at the point S and received at the moment t at the point O. The point E on the line OS corresponds to the moment of the closest approach of light ray to the deflector D. Distances are $OS = R$, $DO = r$, $DS = r_0$, the impact parameter $DE = d$, $OE = \tau = r \cos \theta$, $ES = \tau_0 = \tau - R$. The distance R is much smaller than both r and r_0 . The impact parameter d is, in general, not small in comparison to all other distances.

Using Eq. (2.7) we can write down the relationship between the values of functions of retarded time at the instants of emission and observation:

$$\hat{F}(t_0 - r_0) = F(t - r) - R(1 - \cos \theta) \dot{F}(t - r) + O((R/\lambda)^2). \quad (2.8)$$

Let us now give several useful asymptotic expressions for the derivatives of functions of retarded time

$$\hat{\partial}_\tau^n \frac{F(t - r)}{r} = (1 - \cos \theta)^n \hat{\partial}_{t^*}^n \left[\frac{F(t - r)}{r} \right] + O(\delta^2), \quad (2.9)$$

$$\hat{\partial}_{a_1 \dots a_n} \frac{F(t - r)}{r} = (-1)^n \frac{\xi_{a_1 \dots a_n}}{r^n} \hat{\partial}_{t^*}^n \left[\frac{F(t - r)}{r} \right] + O(\delta^2), \quad (2.10)$$

$$\hat{\partial}_\tau^{[-1]} \left[\frac{F(t - r)}{r} \right] = (1 - \cos \theta)^{n-1} \hat{\partial}_{t^*}^{n-1} \left[\frac{F(t - r)}{r} \right] + O(\delta^2), \quad (2.11)$$

$$\hat{\partial}_{a_1 \dots a_n}^{[-1]} \left[\frac{F(t - r)}{r} \right] = \frac{(-1)^n}{1 - \cos \theta} \frac{\xi_{a_1 \dots a_n}}{r^n} \hat{\partial}_{t^*}^{n-1} \left[\frac{F(t - r)}{r} \right] + O(\delta^2). \quad (2.12)$$

$$^{[-1]} \left\{ \left[\frac{F(t - r)}{r} \right]_{, < A_l >} \right\} = \sum_{p=0}^l \sum_{q=0}^p (-1)^{p-q} C_l(l - p, p - q, q) \times \quad (2.13)$$

$$\begin{aligned} & k_{< a_1 \dots a_p \hat{\partial}_{a_{p+1} \dots a_l >} \hat{\partial}_{t^*}^{p-q} \hat{\partial}_\tau^q \left[\frac{F(t - r)}{r} \right] = \\ & \frac{(-1)^l}{1 - \cos \theta} [N_{< A_l >} - k_{< A_l >}] \hat{\partial}_{t^*}^{l-1} \frac{F(t - r)}{r} + (-1)^l k_{< A_l >} \hat{\partial}_{t^*}^l \left[\frac{F(t - r)}{r} \right] + O(\delta_\lambda) = \\ & \frac{1}{1 - \cos \theta} \left[\frac{^{(-1)} F(t - r)}{r} \right]_{, < A_l >} - \frac{(-1)^l}{1 - \cos \theta} k_{< A_l >} \hat{\partial}_{t^*}^{l-1} \frac{F(t - r)}{r} + \\ & (-1)^l k_{< A_l >} \hat{\partial}_{t^*}^l \left[\frac{F(t - r)}{r} \right] + O(\delta_\lambda), \end{aligned}$$

where $N^i = x^i/r$. The last expression can be proved by induction. The expressions for functions taken at the instant of time $(t_0 - r_0)$ can be obtained by substitution t_0 , r_0 and θ_0 instead of t , r and θ respectively. Finally, the vector ξ^i can be represented as follows:

$$\xi^i = r (N^i - k^i \cos \theta). \quad (2.14)$$

2.2 Asymptotic expressions for observable effects.

In this section we give the expressions for the relativistic effects of time delay bending of light and rotation of the polarization plane in the approximation under consideration. We neglect all terms of order δ^2 , δ_0^2 and δ_λ^2 .

The expression for time delay can be written as

$$\Delta = \Delta_{(M)}(\tau, \tau_0) + \Delta_{(S)}(\tau, \tau_0), \quad (2.15)$$

where

$$\Delta_{(M)}(\tau, \tau_0) = 2\mathcal{M}\frac{R}{r} + \frac{2k_{ij}}{1 - \cos\theta} \sum_{l=2}^{\infty} \frac{(-1)^l}{l!} \left\{ \left[\frac{\dot{\mathcal{I}}_{ijA_{l-2}}(t-r)}{r} \right]_{,A_{l-2}}^{TT} - \left[\frac{\dot{\mathcal{I}}_{ijA_{l-2}}(t_0-r_0)}{r_0} \right]_{,A_{l-2}}^{TT} \right\}, \quad (2.16)$$

$$\Delta_{(S)}(\tau, \tau_0) = -2 \frac{\epsilon_{iba} k_i \xi_a \mathcal{S}_b}{1 - \cos\theta} \left(\frac{1}{r} - \frac{1}{r_0} \right) - \frac{4k_{ij}}{1 - \cos\theta} \sum_{l=2}^{\infty} \frac{(-1)^l l}{(l+1)!} \left\{ \left[\frac{\epsilon_{ba(i} \dot{\mathcal{S}}_{j)bA_{l-2}}(t-r)}{r} \right]_{,aA_{l-2}}^{TT} - \left[\frac{\epsilon_{ba(i} \dot{\mathcal{S}}_{j)bA_{l-2}}(t_0-r_0)}{r_0} \right]_{,aA_{l-2}}^{TT} \right\}. \quad (2.17)$$

Here the transverse-traceless part of the multipole moments tensors is taken with respect to the direction N^i . Taking into account the expressions (1.5), Ch. 2, for metric perturbations h_{ij} we can rewrite (2.15) – (2.17) as follows:

$$\Delta = \Delta_{(M)}(\tau, \tau_0) + \Delta_{(S)}(\tau, \tau_0) = 2\mathcal{M}\frac{R}{r} - 2 \frac{\epsilon_{iba} k_i \xi_a \mathcal{S}_b}{1 - \cos\theta} \left(\frac{1}{r} - \frac{1}{r_0} \right) - \frac{1}{2} \frac{k_{ij}}{1 - \cos\theta} \left({}^{(-1)}h_{ij}^{TT}(t-r) - {}^{(-1)}h_{ij}^{TT}(t_0-r_0) \right). \quad (2.18)$$

In a similar way we obtain the expressions for the observable direction towards the source of light from the point of observation:

$$s^i(\tau, \boldsymbol{\xi}) = K^i + \alpha^i(\tau, \boldsymbol{\xi}) + \gamma^i(\tau, \boldsymbol{\xi}), \quad (2.19)$$

where

$$\alpha^i(\tau, \boldsymbol{\xi}) = \frac{1}{2} \frac{k_{pq}^i}{1 - \cos \theta} [(\cos \theta - 2)k^i + N^i] h_{pq}^{TT}(t - r) + k^q h_{ip}^{TT}(t - r), \quad (2.20)$$

$$\gamma^i(\tau, \boldsymbol{\xi}) = -\frac{1}{2} P^{ij} k^q h_{jq}^{TT}(t, \mathbf{x}), \quad (2.21)$$

and we neglected the corrections $\beta^i(\tau, \xi)$ and $\beta^i(\tau_0, \xi_0)$.

The expressions (2.18) - (2.20) were obtained in [25] in the spin-dipole mass-quadrupole approximation. We have generalized the result of [25] for the case when the source possesses multipole moments of arbitrary order.

In the case when the distance between the observer and the source of light is much smaller than the wavelength of the gravitational radiation $R \ll \lambda$ the expression (2.18) deduces to a well known result

$$\frac{\Delta R}{R} = \frac{1}{2} k_{ij} h_{ij}^{TT}(t - r) + O(\delta^2) + O\left(\frac{R}{\lambda}\right), \quad (2.22)$$

where $\Delta R = c \Delta(\tau, \tau_0)$.

For the angle of rotation of the polarization plane, in agreement with [29], we obtain

$$\phi(\tau) - \phi(\tau_0) = \frac{1}{2} \frac{(\mathbf{k} \times \mathbf{N})^i k^j}{1 - \mathbf{k} \cdot \mathbf{N}} h_{ij}^{TT}(t - r) - \frac{1}{2} \frac{(\mathbf{k} \times \mathbf{N}_0)^i k^j}{1 - \mathbf{k} \cdot \mathbf{N}_0} h_{ij}^{TT}(t_0 - r_0). \quad (2.23)$$

The metric perturbations h_{ij} in this expression depend on multipole moments of all orders. Thus (2.23) generalizes the result of [29], where the effect is calculated in spin-dipole, mass-quadrupole approximation.

Chapter 7

Summary and Conclusions

We obtained analytical expressions for trajectory of light rays (Eqns. (2.7)-(2.11) of Ch. 4) in the time-dependent gravitational field of a localized source possessing full multipole structure and corresponding analytical expressions for relativistic effects of time delay, bending of light and rotation of the polarization plane (Ch. 4 and 5).

We showed that in the gravitational-lens approximation the leading time-dependent contributions to the effects of time delay and deflection of light fall off with the impact parameter as $1/d^2$ and $1/d^3$ respectively for contributions due to multipole moments of all orders. Such strong dependence on the impact parameter makes observation of time-dependent effects in gravitational lensing much more difficult. Also it is likely, but was not proven by a calculation, that the wave-like $1/r$ and the intermediate-zone $1/r^2$ fields do not contribute to the leading-order terms in the observable effects of time delay and deflection of light. This result would be in line with previous research, since in the considered configuration the effects due to gravitational radiation are expected to be negligible because of the transverse nature of gravitational waves [14, 23, 29].

We also obtained the expressions for observable effects of bending of light, time delay and rotation of the polarization plane in the field of a plane gravitational wave of arbitrary multipolarity.

Appendix A

The Metric Tensor in Terms of the Light-Ray Variables ξ and τ

The expressions for perturbations of the metric tensor (1.2) – (1.5), Ch. 2 in variables ξ and τ have the following form

$$h_{00}^{\text{can.}} = \frac{2\mathcal{M}}{r} + 2 \sum_{l=2}^{\infty} \sum_{p=0}^l \sum_{q=0}^p h_{(M)00}^{lpq}(t^*, \tau, \xi), \quad (0.1)$$

$$h_{0i}^{\text{can.}} = -\frac{2\epsilon_{iba}\mathcal{S}^b N^a}{r^2} + 4 \sum_{l=2}^{\infty} \sum_{p=0}^{l-1} \sum_{q=0}^p \left[h_{(S)0i}^{lpq}(t^*, \tau, \xi) + h_{(M)0i}^{lpq}(t^*, \tau, \xi) \right], \quad (0.2)$$

$$h_{ij}^{\text{can.}} = \delta_{ij} h_{00}^{\text{can.}} + q_{ij}^{\text{can.}}, \quad (0.3)$$

$$q_{ij}^{\text{can.}} = 4 \sum_{l=2}^{\infty} \sum_{p=0}^{l-2} \sum_{q=0}^p q_{(M)ij}^{lpq}(t^*, \tau, \xi) - 8 \sum_{l=2}^{\infty} \sum_{p=0}^{l-2} \sum_{q=0}^p q_{(S)ij}^{lpq}(t^*, \tau, \xi), \quad (0.4)$$

where

$$\begin{aligned} \mathbf{h}_{(M)00}^{lpq}(t^*, \tau, \boldsymbol{\xi}) &= \frac{(-1)^{l+p-q}}{l!} C_l(l-p, p-q, q) \\ & k_{\langle a_1 \dots a_p \hat{\partial}_{a_{p+1} \dots a_l} \rangle} \hat{\partial}_\tau^q \left[\frac{\mathcal{I}_{A_l}^{(p-q)}(t-r)}{r} \right], \end{aligned} \quad (0.5)$$

$$\begin{aligned} \mathbf{h}_{(M)0i}^{lpq}(t^*, \tau, \boldsymbol{\xi}) &= \frac{(-1)^{l+p-q}}{l!} C_{l-1}(l-p-1, p-q, q) \\ & k_{\langle a_1 \dots a_p \hat{\partial}_{a_{p+1} \dots a_{l-1}} \rangle} \hat{\partial}_\tau^q \left[\frac{\mathcal{I}_{iA_{l-1}}^{(p-q+1)}(t-r)}{r} \right], \end{aligned} \quad (0.6)$$

$$\begin{aligned} \mathbf{h}_{(S)0i}^{lpq}(t^*, \tau, \boldsymbol{\xi}) &= \frac{(-1)^{l+p-q} l}{(l+1)!} C_{l-1}(l-p-1, p-q, q) \\ & (\hat{\partial}_a + k_a \hat{\partial}_\tau - k_a \hat{\partial}_{t^*}) k_{\langle a_1 \dots a_p \hat{\partial}_{a_{p+1} \dots a_{l-1}} \rangle} \hat{\partial}_\tau^q \left[\frac{\epsilon_{iab} \mathcal{S}_{bA_{l-1}}^{(p-q)}(t-r)}{r} \right], \end{aligned} \quad (0.7)$$

$$\begin{aligned} \mathbf{q}_{(M)ij}^{lpq}(t^*, \tau, \boldsymbol{\xi}) &= \frac{(-1)^{l+p-q}}{l!} C_{l-2}(l-p-2, p-q, q) \\ & k_{\langle a_1 \dots a_p \hat{\partial}_{a_{p+1} \dots a_{l-2}} \rangle} \hat{\partial}_\tau^q \left[\frac{\mathcal{I}_{ijA_{l-2}}^{(p-q+2)}(t-r)}{r} \right], \end{aligned} \quad (0.8)$$

$$\begin{aligned} \mathbf{q}_{(S)ij}^{lpq}(t^*, \tau, \boldsymbol{\xi}) &= \frac{(-1)^{l+p-q} l}{(l+1)!} C_{l-2}(l-p-2, p-q, q) \\ & (\hat{\partial}_a + k_a \hat{\partial}_\tau - k_a \hat{\partial}_{t^*}) k_{\langle a_1 \dots a_p \hat{\partial}_{a_{p+1} \dots a_{l-2}} \rangle} \hat{\partial}_\tau^q \left[\frac{\epsilon_{ba(i} \mathcal{S}_{j) bA_{l-2}}^{(p-q+1)}(t-r)}{r} \right]. \end{aligned} \quad (0.9)$$

All quantities in the right side of expressions (0.5)–(0.9), which are explicitly shown as functions of x^i , $r = |\mathbf{x}|$ and t , must be understood as taken on the unperturbed light-ray trajectory and expressed in terms of ξ^i , $d = |\boldsymbol{\xi}|$, τ and t^* in accordance with the equations (1.2), (1.4), Ch. 4. For example, the ratio $\mathcal{I}_{A_l}^{(p-q)}(t-r)/r$ in equation (0.5) must be understood as

$$\frac{\mathcal{I}_{A_l}^{(p-q)}(t-r)}{r} \equiv \frac{\mathcal{I}_{A_l}^{(p-q)}(t^* + \tau - \sqrt{\tau^2 + d^2})}{\sqrt{\tau^2 + d^2}}, \quad (0.10)$$

and the same replacement rule applies to the other equations.

Appendix B

Gauge Functions

Gauge functions w^α generating the coordinate transformation from the canonical harmonic coordinate system to the ADM-harmonic are given by the equations (3.3), (3.4), Ch. 2. They transform the metric tensor as follows

$$h_{\alpha\beta}^{\text{can.}} = h_{\alpha\beta} - \partial_\alpha w_\beta - \partial_\beta w_\alpha, \quad (0.1)$$

where $h_{\alpha\beta}^{\text{can.}}$ is the canonical form of the metric tensor in harmonic coordinates given by equations (1.2)–(1.5), Ch. 2 and $h_{\alpha\beta}$ is the metric tensor given in the ADM-harmonic coordinates by equations (3.5)–(3.8), Ch. 2.

The gauge functions taken on the light-ray trajectory and expressed in terms of the variables ξ^i and τ can be written in the form

$$w^0 = \sum_{l=2}^{\infty} \sum_{p=0}^l \sum_{q=0}^p \int_{-\infty}^{\tau+t^*} du \, h_{(M)00}^{lpq}(u, \tau, \boldsymbol{\xi}), \quad (0.2)$$

$$w^i = (\hat{\partial}_i + k_i \hat{\partial}_\tau - k_i \hat{\partial}_{t^*}) \sum_{l=2}^{\infty} \sum_{p=0}^l \sum_{q=0}^p \int_{-\infty}^{\tau+t^*} dv \int_{-\infty}^v du \, h_{(M)00}^{lpq}(u, \tau, \boldsymbol{\xi}) \quad (0.3)$$

$$- 4 \sum_{l=2}^{\infty} \sum_{p=0}^{l-1} \sum_{q=0}^p \int_{-\infty}^{\tau+t^*} du \left[h_{(M)0i}^{lpq}(u, \tau, \boldsymbol{\xi}) + h_{(S)0i}^{lpq}(u, \tau, \boldsymbol{\xi}) \right],$$

where $h_{(M)00}^{lpq}(u, \tau, \boldsymbol{\xi})$, $h_{(M)0i}^{lpq}(u, \tau, \boldsymbol{\xi})$ and $h_{(S)0i}^{lpq}(u, \tau, \boldsymbol{\xi})$ are defined by the Eqns. (0.5), (0.6), App. A and (0.7) after making use of the substitution $t^* \rightarrow u$.

Linear combination $k^i \varphi^0 - \varphi^i$ of the gauge-dependent functions φ^α , introduced in equation (2.4), Ch. 4, is given by the expressions

$$k^i \varphi^0 - \varphi^i = (k^i \varphi_{(M)}^0 - \varphi_{(M)}^i) + (k^i \varphi_{(S)}^0 - \varphi_{(S)}^i), \quad (0.4)$$

$$\begin{aligned} k^i \varphi_{(M)}^0 - \varphi_{(M)}^i &= 2 \hat{\partial}_i \sum_{l=2}^{\infty} \sum_{p=2}^l \sum_{q=2}^p \frac{(-1)^{l+p-q}}{l!} C_l(l-p, p-q, q) \times \\ &\quad \left(1 - \frac{p-q}{l}\right) \left(1 - \frac{p-q}{l-1}\right) k_{\langle a_1 \dots a_p} \hat{\partial}_{a_{p+1} \dots a_l} \hat{\partial}_\tau^{q-2} \left[\frac{\mathcal{I}_{A_l}^{(p-q)}(t-r)}{r} \right] + \\ &\quad 2 \sum_{l=2}^{\infty} \sum_{p=1}^l \sum_{q=1}^p \frac{(-1)^{l+p-q}}{l!} C_l(l-p, p-q, q) \times \\ &\quad \left(1 - \frac{p-q}{l}\right) \left\{ \left(1 + \frac{p-q}{l-1}\right) k_{i \langle a_1 \dots a_p} \hat{\partial}_{a_{p+1} \dots a_l} \hat{\partial}_\tau^{q-1} \left[\frac{\mathcal{I}_{A_l}^{(p-q)}(t-r)}{r} \right] - \right. \\ &\quad \left. 2 \frac{p-q}{l-1} k_{\langle a_1 \dots a_{p-1}} \hat{\partial}_{a_p \dots a_{l-1}} \hat{\partial}_\tau^{q-1} \left[\frac{\mathcal{I}_{i A_{l-1}}^{(p-q)}(t-r)}{r} \right] \right\}, \end{aligned} \quad (0.5)$$

$$\begin{aligned}
k^i \varphi_{(S)}^0 - \varphi_{(S)}^i = & 2 \frac{\epsilon_{iab} k^a \mathcal{S}^b}{r} + \tag{0.6} \\
& 4k_j \hat{\partial}_{ia} \sum_{l=3}^{\infty} \sum_{p=2}^{l-1} \sum_{q=2}^p \frac{(-1)^{l+p-q} l}{(l+1)!} C_{l-1}(l-p-1, p-q, q) \times \\
& \left(1 - \frac{p-q}{l-1}\right) k_{\langle a_1 \dots a_p} \hat{\partial}_{a_{p+1} \dots a_{l-1} \rangle} \hat{\partial}_{\tau}^{q-2} \left[\frac{\epsilon_{jab} \mathcal{S}_{bA_{l-1}}^{(p-q)}(t-r)}{r} \right] - \\
& 4 \left(\hat{\partial}_a - k_a \hat{\partial}_{t^*} \right) \sum_{l=2}^{\infty} \sum_{p=1}^{l-1} \sum_{q=1}^p \frac{(-1)^{l+p-ql}}{(l+1)!} C_{l-1}(l-p-1, p-q, q) \times \\
& \left(1 - \frac{p-q}{l-1}\right) k_{\langle a_1 \dots a_p} \hat{\partial}_{a_{p+1} \dots a_{l-1} \rangle} \hat{\partial}_{\tau}^{q-1} \left[\frac{\epsilon_{iab} \mathcal{S}_{bA_{l-1}}^{(p-q)}(t-r)}{r} \right] - \\
& 4k_a \sum_{l=2}^{\infty} \sum_{p=0}^{l-1} \sum_{q=0}^p \frac{(-1)^{l+p-ql}}{(l+1)!} C_{l-1}(l-p-1, p-q, q) \times \\
& \left(1 - \frac{p-q}{l-1}\right) k_{\langle a_1 \dots a_p} \hat{\partial}_{a_{p+1} \dots a_{l-1} \rangle} \hat{\partial}_{\tau}^q \left[\frac{\epsilon_{iab} \mathcal{S}_{bA_{l-1}}^{(p-q)}(t-r)}{r} \right] + \\
& 4k_j \sum_{l=2}^{\infty} \sum_{p=1}^{l-1} \sum_{q=1}^p \frac{(-1)^{l+p-ql}}{(l+1)!} C_{l-1}(l-p-1, p-q, q) \times \\
& k_{\langle a_1 \dots a_p} \hat{\partial}_{a_{p+1} \dots a_{l-1} \rangle} \hat{\partial}_{\tau}^{q-1} \left[\frac{\epsilon_{jba_{l-1}} \mathcal{S}_{ibA_{l-2}}^{(p-q+1)}(t-r)}{r} \right]
\end{aligned}$$

Gauge-dependent term generated by equation (3.3), Ch. 4, for the rotation of the plane of polarization of electromagnetic wave is a pure spatial vector χ^i that can be decomposed in two linear parts corresponding to the mass and spin multipoles:

$$\chi^i = \chi_{(M)}^i + \chi_{(S)}^i, \tag{0.7}$$

where

$$\begin{aligned} \chi_{(M)}^i = & 4 \sum_{l=2}^{\infty} \sum_{p=1}^{l-1} \sum_{q=1}^p \frac{(-1)^{l+p-q}}{l!} C_{l-1}(l-p-1, p-q, q) \left(1 - \frac{p-q}{l-1}\right) \\ & \times k_{\langle a_1 \dots a_p \hat{\partial}_{a_{p+1}} \dots a_{l-1} \rangle} \hat{\partial}_{\tau}^{q-1} \left[\frac{\mathcal{I}_{iA_{l-1}}^{(p-q+1)}(t-r)}{r} \right], \end{aligned} \quad (0.8)$$

$$\begin{aligned} \chi_{(S)}^i = & -4 \sum_{l=1}^{\infty} \sum_{p=0}^{l-1} \sum_{q=0}^p \frac{(-1)^{l+p-ql}}{(l+1)!} C_{l-1}(l-p-1, p-q, q) \\ & \times \left[1 - \frac{p-q}{l-1} H(l-1) \right] \left[H(q)(\hat{\partial}_a - k_a \hat{\partial}_{t^*}) + k_a \hat{\partial}_{\tau} \right] \\ & \times k_{\langle a_1 \dots a_p \hat{\partial}_{a_{p+1}} \dots a_{l-1} \rangle} \hat{\partial}_{\tau}^{q-1} \left[\frac{\epsilon_{iba} \mathcal{S}_{bA_{l-1}}^{(p-q)}(t-r)}{r} \right] \\ & + 4 \sum_{l=3}^{\infty} \sum_{p=1}^{l-1} \sum_{q=1}^p \frac{(-1)^{l+p-ql}}{(l+1)!} C_{l-1}(l-p-1, p-q, q) \left(1 - \frac{p}{l}\right) \left(1 - \frac{q}{p}\right) \\ & \times k_{\langle a_1 \dots a_p \hat{\partial}_{a_{p+1}} \dots a_{l-1} \rangle_a} \hat{\partial}_{\tau}^{q-1} \left[\frac{\epsilon_{baa_{l-1}} \mathcal{S}_{ibA_{l-2}}^{(p-q)}(t-r)}{r} \right]. \end{aligned} \quad (0.9)$$

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