Frames and applications: Distribution of frame coefficients, integer frames and phase retrieval

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The undersigned, appointed by the Dean of the Graduate School, have examined the thesis entitled

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FRAMES AND APPLICATIONS: DISTRIBUTION OF FRAME COEFFICIENTS, INTEGER FRAMES AND PHASE RETRIEVAL

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ABSTRACT

The present dissertation is divided into two main areas: frame theoretic results and applications of frames. In particular, the beginning half develops the first detailed theory of the distribution of frame coefficients. Next, the first systematic study of integer frames is included. The latter half of the dissertation is concerned with the application of frames to the area of signal reconstruction. In particular, phase retrieval by subspaces components and norm retrieval are discussed.

Chapter 1

Introduction

Hilbert space frames were introduced by Duffin and Schaeffer in [26] while studying deep questions in non-harmonic Fourier series. Today they have broad application to problems in pure mathematics, applied mathematics, engineering, medicine and much more. Due to the redundancy, flexibility and stability of a frame, frame theory has proven to be a powerful area of research with applications to a wide array of fields, including signal processing, noise and erasure reduction, compressed sensing, sampling theory, data quantization, quantum measurements, coding, image processing, wireless communications, time-frequency analysis, speech recognition, bio-imaging, and much more. A fundamental problem for applications of frames is to classify and construct frames with the necessary properties for the application. This can often be very difficult if not impossible in practice.

Within this dissertation there will be two main focuses. The first half is dedicated to furthering the theory of frames so that others can potentially apply these new developments to their unique problems. In particular, we advance classification and construction results regarding frame coefficients by developing the first systematic study of the distribution of frame coefficients followed by a detailed study of integer frames. The latter half of this dissertation takes a more applied approach and considers a frame's impact on signal reconstruction. The main focus here is on the study of subspaces which allow phase retrieval and norm retrieval.

1.1 A brief review of Frame Theory

This section is devoted to reviewing necessary terms and theorems from finite frame theory which will be used throughout the remainder of the present dissertation. For a more in depth study of finite frame theory the reader is referred to [23].

Given $M \in \mathbb{N}$, we shall employ the notation $\mathcal{H}_M$ to represent a (real or complex) Hilbert space of (finite) dimension M. For the sake of brevity, if $J \subseteq \mathbb{N}$ and $I \subseteq J$, then we will use the notation $I^c := J \setminus I$.

Definition 1.1.1. A family of vectors $\{\varphi_i\}_{i=1}^N$ in $\mathcal{H}_M$ is a **frame** if there are constants $0 < A \leq B < \infty$ so that for all $x \in \mathcal{H}_M$,

$$A\|x\|^2 \leq \sum_{n=1}^{N} |\langle x, \varphi_i \rangle|^2 \leq B\|x\|^2,$$

where A and B are **lower** and **upper frame bounds**, respectively.

1. In the finite dimensional setting, a frame is simply a spanning set of vectors in the Hilbert space.

2. $\{\langle x, \varphi_i \rangle\}_{i=1}^N$ are called the **frame coefficients** of the vector $x \in \mathcal{H}_M$ with respect to frame $\{\varphi_i\}_{i=1}^N$.

3. The **optimal lower frame bound** and **optimal upper frame bound**, denoted A_{op} and B_{op}, are the largest lower frame bound and the smallest upper frame bound, respectively.

4. If $A = B$ is possible, then $\{\varphi_i\}_{i=1}^N$ is a **tight frame**. Moreover, if $A = B = 1$ is possible, then $\{\varphi_i\}_{i=1}^N$ is a **Parseval frame**.

5. If $\{\varphi_i\}_{i=1}^N$ is a unit norm tight frame (UNTF) then the frame bound will be $A = B = \frac{N}{M}$.

6. If there is a constant c so that $\|\varphi_i\| = c$ for all $i = 1, \ldots, N$ then $\{\varphi_i\}_{i=1}^N$ is an **equal norm frame**. Moreover, if $c = 1$ then $\{\varphi_n\}_{i=1}^N$ is a **unit norm frame**.

7. If there is a constant d so that $|\langle \varphi_i, \varphi_j \rangle| = d$ for all $1 \leq i \neq j \leq N$, then $\{\varphi_i\}_{i=1}^N$ is an **equiangular frame**.

Note, in the present dissertation we are only concerned with finite frames. Since frames in the finite dimensional setting are spanning sets, at times it is useful to look at subsets of a frame which are also spanning sets.

Definition 1.1.2. A frame $\{\varphi_i\}_{i=1}^N$ in $\mathcal{H}_M$ satisfies the **complement property** if for all subsets $S \subset \{1, \ldots, N\}$, either $\operatorname{span}\{\varphi_i\}_{i \in S} = \mathcal{H}_M$ or $\operatorname{span}\{\varphi_i\}_{i \in S^c} = \mathcal{H}_M$.

Definition 1.1.3. Given a frame $\Phi = \{\varphi_i\}_{i=1}^N$ in $\mathcal{H}_M$, the **spark** of Φ is defined as the cardinality of the smallest linearly dependent subset of Φ. When $\operatorname{spark}(\Phi) = M + 1$, every subset of size M is linearly independent, and Φ is said to be **full spark**.

Remark 1.1.4. Let $\Phi := \{\varphi_i\}_{i=1}^N$ be a frame in $\mathcal{H}_M$ such that $N \geq 2M - 1$.

1. If Φ is full spark then Φ has the complement property.

2. If Φ has the complement property and $M = 2N - 1$ then Φ is full spark.

The notion of spark is the measure of how resilient a frame is against erasures, so *full spark* is a desired property of a frame. In general, it is very difficult to check the spark of a frame. Moreover, it is shown in [2] that determining if a frame is full spark is NP-hard.

With every frame there are three main operators which are used throughout frame theory.

Definition 1.1.5. If $\{\varphi_i\}_{i=1}^N$ is a frame for $\mathcal{H}_M$, then the **analysis operator** of the frame is the operator $T : \mathcal{H}_M \to \ell_2(N)$ given by

$$T(x) = \{\langle x, \varphi_i \rangle\}_{i=1}^N.$$

The **synthesis operator** is the adjoint operator T^*, which satisfies

$$T^*\left(\{a_i\}_{i=1}^N\right) = \sum_{i=1}^N a_i \varphi_i.$$

It is necessary to note that for any frame $\{\varphi_i\}_{i=1}^N$ in $\mathcal{H}_M$ with analysis operator T, the matrix representation of its synthesis operator T^* with respect to some orthonormal basis $\{e_i\}_{i=1}^M$ of $\mathcal{H}_M$ is given by the following $M \times N$ matrix

$$\begin{bmatrix} | & | & \cdots & | \\ \varphi_1 & \varphi_2 & \cdots & \varphi_N \\ | & | & \cdots & | \end{bmatrix}$$

where the columns of T^* represent the coefficients of the frame vectors with respect to $\{e_i\}_{i=1}^M$. Due to this relationship between a frame and its matrix representation, we will not distinguish between a frame and its matrix and instead use the term frame interchangeably.

It is also important to note that since the columns of the synthesis matrix represent the coefficients of the frame vectors, then the square sum of each column represents the square norm of the frame vectors. Hence a frame is equal norm if all of the columns square sum to the same constant.

Composing the synthesis operator with the analysis operator turns out to be a widely used operator throughout frame theory and as such is dubbed the frame operator.

Definition 1.1.6. The **frame operator** is the positive, self-adjoint, invertible operator $S = T^*T$ on $\mathcal{H}_M$ which satisfies

$$S(x) = T^*T(x) = \sum_{i=1}^{N} \langle x, \varphi_i \rangle \varphi_i.$$

We say that a frame has a certain spectrum or certain eigenvalues if its frame operator S has this spectrum or respectively these eigenvalues. Note that the spectrum of a frame operator S is positive and real where the smallest and largest eigenvalues coincide with the optimal lower and upper frame bounds, respectively. For any frame with spectrum $\{\lambda_m\}_{m=1}^M$, the sum of its eigenvalues counting multiplicities equals the sum of the squares of the norms of its vectors:

$$\sum_{m=1}^{M} \lambda_m = \sum_{i=1}^{N} ||\varphi_i||^2.$$

This quantity will be exactly the number of vectors N when we work with unit norm frames.

Rephrasing the definition of a frame in terms of its frame operator yields the following equivalence; $\{\varphi_i\}_{i=1}^N$ is a frame if and only if there are constants $0 < A \leq B < \infty$ such that its frame operator S satisfies $AI \leq S \leq BI$, where I is the identity on $\mathcal{H}_M$.

In particular, the frame operator of a Parseval frame is the identity operator. This fact makes Parseval frames very helpful in applications because they possess the property of perfect reconstruction. That is, $\{\varphi_i\}_{i=1}^N$ is a Parseval frame for $\mathcal{H}_M$ if and only if for any $x \in \mathcal{H}_M$ we have

$$x = \sum_{i=1}^{N} \langle x, \varphi_i \rangle \varphi_i.$$

There is a direct method for constructing Parseval frames. For $N \geq M$, given an $N \times N$ unitary matrix, if we select any M rows from this matrix, then the column vectors from these rows form a Parseval frame for $\mathcal{H}_M$. Moreover, the leftover set of $N - M$ rows also have the property that its N columns form a Parseval frame for $\mathcal{H}_{N-M}$. The next theorem, known as Naimark's Theorem, says that this is the only way to obtain Parseval frames.

Theorem 1.1.7 (Naimark's Theorem; page 36 of [23]). Let $\Phi = \{\varphi_i\}_{i=1}^N$ be a frame for $\mathcal{H}_M$ with analysis operator T, let $\{e_i\}_{i=1}^N$ be the standard basis of $\ell_2(N)$, and let $P : \ell_2(N) \to \ell_2(N)$ be the orthogonal projection onto range(T). Then the following conditions are equivalent:

1. $\{\varphi_i\}_{i=1}^N$ is a Parseval frame for $\mathcal{H}_M$.
2. For all $i = 1, \ldots, N$, we have $Pe_i = T\varphi_i$.
3. There exist $\psi_1, \ldots, \psi_N \in \mathcal{H}_{N-M}$ such that $\{\varphi_i \oplus \psi_i\}_{i=1}^N$ is an orthonormal basis of $\mathcal{H}_N$.

Moreover, $\{\psi_i\}_{i=1}^N$ is a Parseval frame for $\mathcal{H}_{N-M}$.

Explicitly, we call $\{\psi_i\}_{i=1}^N$ the **Naimark complement** of Φ. If $\Phi = \{\varphi_i\}_{i=1}^N$ is a Parseval frame, then the analysis operator T of the frame is an isometry. So we can associate φ_i with $T\varphi_i = Pe_i$, and with a slight abuse of notation we have:

Theorem 1.1.8 (Naimark's Theorem). $\Phi = \{\varphi_i\}_{i=1}^N$ is a Parseval frame for $\mathcal{H}_M$ if and only if there is an N-dimensional Hilbert space $\mathcal{K}_N$ with an orthonormal basis

$\{e_i\}_{i=1}^N$ such that the orthogonal projection $P : \mathcal{K}_N \to \mathcal{H}_M$ satisfies $Pe_i = \varphi_i$ for all $i = 1, \ldots, N$. Moreover, the Naimark complement of Φ is $\{(I-P)e_i\}_{i=1}^N$.

Note that Naimark complements are only defined for Parseval frames. Furthermore, Naimark complements are only defined up to unitary equivalence. That is, if $\{\varphi_i\}_{i=1}^N \subseteq \mathcal{H}_M$ and $\{\psi_i\}_{i=1}^N \subseteq \mathcal{H}_{N-M}$ are Naimark complements, and U and V are unitary operators, then $\{U\varphi_i\}_{i=1}^N$ and $\{V\psi_i\}_{i=1}^N$ are also Naimark complements.

To clarify terminology, as mentioned in Naimark's Theorem and as will be throughout this dissertation, an *orthogonal projection* or simply a *projection* is a self-adjoint projection.

Although Parseval frames possess many nice properties, they are very structured and so we would like to generalize the idea of perfect reconstruction to an arbitrary frame. We can accomplish this through the use of *dual frames.*

Definition 1.1.9. Let $\{\varphi_i\}_{i=1}^N$ be a frame for $\mathcal{H}_M$. A frame $\{\psi_i\}_{i=1}^N$ is called a **dual frame** for $\{\varphi_i\}_{i=1}^N$, if

$$x = \sum_{i=1}^N \langle x, \varphi_i \rangle \psi_i \text{ for all } x \in \mathcal{H}_M.$$

In finite dimensions, we have seen that a frame is simply a spanning set of vectors. However, the decomposition of a signal with respect to a frame is not unique and sometimes this uniqueness is necessary. An orthogonal basis provides this uniqueness, but many times orthogonal bases are too restrictive and we need a more lacks condition. We will see that a *Riesz basis* provides this uniqueness and does not have as strong of a condition as orthogonality.

Definition 1.1.10. A family of vectors $\{\varphi_i\}_{i=1}^M$ in a Hilbert space $\mathcal{H}_M$ is called a **Riesz basis** with **lower** (respectively, **upper**) **Riesz bounds A** (respectively,

B), if, for all scalars $\{a_i\}_{i=1}^M$, we have

$$A\sum_{i=1}^{M}|a_i|^2 \leq \|\sum_{i=1}^{M} a_i\varphi_i\|^2 \leq B\sum_{i=1}^{M}|a_i|^2.$$

This concludes a brief introduction to finite frames and other terms which are necessary throughout the present dissertation.

Chapter 2

The Distribution of Frame Coefficients

2.1 Introduction to the distribution of frame coefficients

The most fundamental notion of a Hilbert space frame $\Phi = \{\varphi_i\}_{i=1}^N$ for $\mathcal{H}_M$ is the sequence of frame coefficients, $\{\langle x, \varphi_i \rangle\}_{i=1}^N$, for a vector $x \in \mathcal{H}_M$. Yet, we know little about the distribution of these coefficients, even for very specific frames. Frames are used in numerous areas of mathematics and as such a classification of the frame coefficients enlightens researchers to determine what types of frames can be useful for their problems. The first detailed study of the distribution of frame coefficients for general frames follows. For several special classes of frames, including unit norm tight frames and equiangular frames, we then use the concept of *majorization* to provide a strengthening of the more general results. We also study the distributions of products of frame coefficients for different vectors x, y. Lastly, we make a detailed study of the square sums of the distances from a vector x to the frame vectors and discover that in quite general cases, these sums are nearly equal for all vectors. The work in this chapter can be found in the paper *Distribution of frame coefficients* [12].

2.2 Estimating the number of non-zero frame coefficients

We start our analysis of the distribution of the frame coefficients by giving estimates of the minimal number of non-zero frame coefficients for various types of frames.

The next few results provide estimates on the number of indices for which the frame coefficients are non-zero. We will be looking at the minimal number of indices here since there is always a dense set of vectors which have non-zero inner products with all the frame vectors. For example, if $\{\varphi_i\}_{i=1}^N$ is a frame for $\mathcal{H}_M$ then $\varphi_i^\perp$ is a hyperplane for every $i \in [N]$. If we choose

$$x \notin \cup_{i=1}^N \varphi_i^\perp,$$

we have that

$$|\{i \in [N] : \langle x, \varphi_i \rangle \neq 0\}| = N.$$

In general, there may be few indices for a given vector x for which $\langle x, \varphi_i \rangle \neq 0$. For instance, given K copies of an orthonormal basis $\{e_i\}_{i=1}^M$ for $\mathcal{H}_M$, say $\{e_{ij}\}_{\substack{1\le i\le M\\ 1\le j\le K}}$, choosing $x = e_1$ gives

$$\langle x, e_{ij} \rangle = 0, \quad \forall\, i \in \{2, \dots, M\},\ j \in \{1, \dots, K\}. \tag{2.2.1}$$

That is, we have only K non-zero coefficients out of a total of KM. In other words, there are KM/M non-zero coefficients. This is actually minimal with respect to this property as we shall see in this section.

The above shows that the following estimate for the minimal number of non-zero frame coefficients is best possible in general.

Theorem 2.2.1. Let $\Phi = \{\varphi_i\}_{i=1}^N$ be a frame in $\mathcal{H}_M$ with frame bounds A, B and set $D := \max\{\|\varphi_i\|^2 : i \in [N]\}$. For any unit norm $x \in \mathcal{H}_M$,

$$|J_x| =: |\{i \in [N] : \langle x, \varphi_i \rangle \neq 0\}| \geq \frac{A}{D}. \tag{2.2.2}$$

In particular, if Φ is a unit norm frame, then $J_x \geq A$ and if it is a unit norm tight frame then $J_x \geq \frac{N}{M}$.

Moreover, if we have equality in (2.2.2) then the sub-collection of frame vectors $\{\varphi_i : i \in J_x\}$ spans a one-dimensional space.

Proof. Pick a unit norm $x \in \mathcal{H}_M$ and set $J_x := \{1 \leq i \leq N : \langle x, \varphi_i \rangle \neq 0\}$. Then

$$\begin{aligned} A = A\|x\|^2 &\leq \sum_{i=1}^N |\langle x, \varphi_i \rangle|^2 = \sum_{i \in J_x} |\langle x, \varphi_i \rangle|^2 \\ &\leq \sum_{i \in J_x} \|x\|^2 \|\varphi_i\|^2 = \sum_{i \in J_x} \|\varphi_i\|^2 \leq D|J_x|. \end{aligned} \tag{2.2.3}$$

Hence, $\frac{A}{D}$ is a lower bound for $|J_x|$ independent of each unit norm $x \in \mathcal{H}_M$. By definition of the infimum, it follows that (2.2.2) holds true.

Concerning the moreover part, if we have equality in (2.2.3), then

$$|\langle x, \varphi_i \rangle|^2 = \|\varphi_i\|^2, \text{ for all } i \in J_x.$$

It follows that $\varphi_i = c_i x$, for some $|c_i| = 1$, and all $i \in J_x$. □

The following theorem may be viewed as a generalization of Theorem 2.2.1.

Theorem 2.2.2. Let $C \in (0,1)$, let $\Phi = \{\varphi_i\}_{i=1}^N$ be a frame for $\mathcal{H}_M$ with frame bounds A, B and set $D := \max\{\|\varphi_i\|^2 : i \in [N]\}$. Given a unit norm $x \in \mathcal{H}_M$,

$$|K_x| =: |\{i \in [N] : |\langle x, \varphi_i\rangle|^2 > C\frac{A}{N}\}| \geq (1-C)\frac{A}{D}. \tag{2.2.4}$$

In particular, if Φ is a unit norm tight frame, then

$$|K_x| \geq (1-C)\frac{N}{M}. \tag{2.2.5}$$

Proof. For unit norm $x \in \mathcal{H}_M$, we have

$$\sum_{i \in K_x^c} |\langle x, \varphi_i\rangle|^2 \leq \frac{CA}{N}|K_x^c| \qquad \text{and} \qquad \sum_{i \in K_x} |\langle x, \varphi_i\rangle|^2 \leq D|K_x|. \tag{2.2.6}$$

Hence,

$$\begin{aligned} A = A\|x\|^2 &\leq \sum_{i=1}^N |\langle x, \varphi_i\rangle|^2 = \sum_{i \in K_x} |\langle x, \varphi_i\rangle|^2 + \sum_{i \in K_x^c} |\langle x, \varphi_i\rangle|^2 \\ &\leq D|K_x| + \frac{CA}{N}|K_x^c| = D|K_x| + \frac{CA}{N}(N - |K_x|) \\ &= \Big(D - \frac{CA}{N}\Big)|K_x| + CA. \end{aligned} \tag{2.2.7}$$

It follows that

$$A(1-C) \leq \Big(D - \frac{CA}{N}\Big)|K_x|. \tag{2.2.8}$$

By the definition of K_x and the Cauchy-Schwarz inequality, we may deduce that $(CA)/N < D$. Consequently

$$(1-C)\frac{A}{D} \leq (1-C)\frac{A}{D - (CA)/N} \leq |K_x|. \tag{2.2.9}$$

Since the lower bound in (2.2.9) is independent of the unit norm $x \in \mathcal{H}_M$, the validity of (2.2.4) follows by definition of the infimum. □

2.3 Distribution of the frame coefficients

In this section we will classify the distribution of the frame coefficients using *majorization.*

Definition 2.3.1.

1. A sequence of positive non-increasing numbers $a = \{a_i\}_{i=1}^N$ **majorizes** a different sequence of positive non-increasing numbers $b = \{b_i\}_{i=1}^N$, written $a \succ b$, if

$$\sum_{i=1}^{k} a_i \geq \sum_{i=1}^{k} b_i, \ \ \forall\, k \in \{1, 2, \ldots, N\} \quad \text{and} \quad \sum_{i=1}^{N} a_i = \sum_{i=1}^{N} b_i. \tag{2.3.1}$$

2. A sequence of positive non-increasing numbers $a = \{a_i\}_{i=1}^N$ **weakly majorizes** a different sequence of positive non-increasing numbers $b = \{b_i\}_{i=1}^N$, written $a \succ_W b$, if

$$\sum_{i=1}^{k} a_i \geq \sum_{i=1}^{k} b_i, \ \ \forall\, k \in \{1, 2, \ldots, N\}. \tag{2.3.2}$$

If the two sequences do not have the same number of terms; we agree to add zeroes to the end of the shorter sequence until they have the same lengths.

Furthermore, there is no harm in identifying the sequence $\{a_i\}_{i=1}^N$ as the vector $a := (a_1, \ldots, a_N) \in \mathbb{R}^N$.

It will be important for our work to understand what happens when partial sums of the majorization vectors are equal.

Proposition 2.3.2. Let $a := \{a_i\}_{i=1}^N \in \mathbb{R}^N$ be a non-negative, non-increasing sequence of numbers, let $b := (A, A, \ldots, A) \in \mathbb{R}^N$ where $A > 0$, and assume $a \succ_W b$. If there exists $m \in \{1, \ldots, N\}$ so that $\sum_{i=1}^{m} a_i = mA$, then $a_i = A$ for every $i \in \{m+1, \ldots, N\}$.

Proof. Assume there exists $m \in \{1, \ldots, N\}$ so that $\sum_{i=1}^{m} a_i = mA$. Then $a_m \leq A$ and hence, $a_i \leq A$ for every $i \in \{m+1, \ldots, N\}$. Assume for the moment that $a_i < A$ for some $i \in \{m+1, \ldots, N\}$, then

$$\begin{aligned} NA = \sum_{i=1}^{N} A \leq \sum_{i=1}^{N} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{N} a_i \\ < mA + (N-m)A = NA, \end{aligned} \tag{2.3.3}$$

which is a contradiction. Thus, $a_i = A$ for every $i \in \{m+1, \ldots, N\}$. $\square$

Next we recall the Schur-Horn Theorem because it shows that the results in this section are in general best possible, although in specific cases one can clearly create specialized frames which give better bounds.

Theorem 2.3.3. [23] Given two sequences of real numbers $a_1 \geq \cdots \geq a_N > 0$ and $\lambda_1 \geq \cdots \geq \lambda_M > 0$, the following are equivalent.

(i) $\{\lambda_i\}_{i=1}^{M} \succ \{a_i\}_{i=1}^{N}$.

(ii) There exists a frame $\{\varphi_i\}_{i=1}^{N}$ in $\mathcal{H}_M$ such that $\|\varphi_i\|^2 = a_i$ for each $i \in [N]$ having frame operator with eigenvalues $\{\lambda_i\}_{i=1}^{M}$.

In the next result, we create a vector that trivially majorizes the modulus of the frame coefficients in a unit norm tight frame.

Lemma 2.3.4. Let $\{\varphi_i\}_{i=1}^{KN}$ be a unit norm K-tight frame in $\mathcal{H}_N$ and assume $x \in$

$\mathcal{H}_N$ has unit norm. Define the vectors $a, b \in \mathbb{R}^{KN}$ by

$$b := (\underbrace{1, \dots, 1}_{K}, \underbrace{0, \dots, 0}_{KN-K}) \tag{2.3.4}$$

and $a := (a_i)_{i=1}^{KN}$ where a is $\{|\langle x, \varphi_i \rangle|^2\}_{i=1}^{KN}$ in non-increasing order. Then $b \succ a$.

Moreover, $a_i = 1$ for every $i \in \{1, \dots, K\}$ if and only if there is $I \subset [KN]$ so that $\varphi_i = c_i x$, with $|c_i| = 1$, for all $i \in I$.

Proof. This is clear since $\|x\| = 1 = \|\varphi_i\|$ and so $|\langle x, \varphi_i \rangle| \leq 1$ with equality if and only if $\varphi_i = c_i x$. □

The following proposition establishes a relationship between the largest eigenvalue of the frame operator and the number of frame coefficients of modulus one.

Proposition 2.3.5. Let $\{\varphi_i\}_{i=1}^N$ be a unit norm frame in $\mathcal{H}_M$ whose frame operator has eigenvalues $\lambda_1 \geq \cdots \geq \lambda_M$ and let $x \in \mathcal{H}_M$ be unit norm. Define the vector $a \in \mathbb{R}^N$ by $a := (a_i)_{i=1}^N$ where $a_i := |\langle x, \varphi_i \rangle|^2$ in non-increasing order. Then $a_i = 1$ for at most $i \leq \lfloor \lambda_1 \rfloor$ (where $\lfloor \cdot \rfloor$ represents the least integer function).

Proof. By definition, $\sum\limits_{i=1}^N |\langle x, \varphi_i \rangle|^2 \leq \lambda_1 \|x\|^2 = \lambda_1$. Suppose for the moment that $a_i = 1$ for every $i \in \{1, \dots, \lfloor \lambda_1 \rfloor + 1\}$, then

$$\begin{aligned}\lambda_1 \geq \sum_{i=1}^N |\langle x, \varphi_i \rangle|^2 &= \sum_{i=1}^{\lfloor \lambda_1 \rfloor + 1} |\langle x, \varphi_i \rangle|^2 + \sum_{i=\lfloor \lambda_1 \rfloor + 2}^{N} |\langle x, \varphi_i \rangle|^2 \\ &= (\lfloor \lambda_1 \rfloor + 1) + \sum_{i=\lfloor \lambda_1 \rfloor + 2}^{N} |\langle x, \varphi_i \rangle|^2 > \lambda_1, \end{aligned} \tag{2.3.5}$$

which is a contradiction. □

We are ready to give one of the main majorization results for frame coefficients of unit norm vectors.

Theorem 2.3.6. Let $\{\varphi_i\}_{i=1}^N$ be a frame for $\mathcal{H}_M$ with frame bounds $A \leq B$ and let

$$b := \left(\frac{A}{N}, \frac{A}{N}, \dots, \frac{A}{N}\right) \in \mathbb{R}^{\mathbb{N}}. \tag{2.3.6}$$

For any unit norm $x \in \mathcal{H}_M$, define $a := \{a_i\}_{i=1}^N \in \mathbb{R}^N$ to be $\{|\langle x, \varphi_i\rangle|^2\}_{i=1}^N$ arranged in non-increasing order. Then $a \succ_W b$. In particular, $a_1 \geq \frac{A}{N}$.

Moreover, if there exists an $m \in \{1, \dots N\}$ such that $\sum_{i=1}^m a_i = m\frac{A}{N}$ then $a_i = \frac{A}{N}$ for all $i \in \{m+1, \dots, N\}$.

Proof. We proceed by way of contradiction. If there is an $m \in \{1, \dots, N\}$ so that

$$\sum_{i=1}^m a_i < \sum_{i=1}^m \frac{A}{N} = m\frac{A}{N}, \tag{2.3.7}$$

then $a_m < \frac{A}{N}$. Furthermore, it follows that

$$a_i \leq a_m, \quad \forall\, i \in \{m, \dots, N\}, \tag{2.3.8}$$

and we may write for any unit vector $x \in \mathcal{H}_M$

$$\begin{aligned} A = A\|x\|^2 &\leq \sum_{i=1}^N a_i = \sum_{i=1}^m a_i + \sum_{i=m+1}^N a_i \\ &< m\frac{A}{N} + (N-m)a_m < m\frac{A}{N} + (N-m)\frac{A}{N} \\ &= A, \end{aligned} \tag{2.3.9}$$

a contradiction.

The moreover part follows from Proposition 2.3.2. □

Corollary 2.3.7. Let $\{\varphi_i\}_{i=1}^N$ be a unit norm tight frame in $\mathcal{H}_M$. For any unit norm $x \in \mathcal{H}_M$, let $a := \{a_i\}_{i=1}^N \in \mathbb{R}^N$ be $\{|\langle x, \varphi_i\rangle|^2\}_{i=1}^N$ arranged in non-increasing order and set

$$b := \Big(\frac{1}{M}, \frac{1}{M}, \dots, \frac{1}{M}\Big) \in \mathbb{R}^N. \tag{2.3.10}$$

Then $a \succ_W b$.

Given a frame $\{\varphi_i\}_{i=1}^N$ and a unit vector $x \in \mathcal{H}_M$, if x is orthogonal to a number of frame coefficients, we would expect the nonzero coefficients to start to grow, and they indeed do as we shall see in the following proposition.

Proposition 2.3.8. Let $\{\varphi_i\}_{i=1}^N$ be a frame in $\mathcal{H}_M$ with frame bounds A, B and let $I \subseteq \{1, \dots, N\}$ be such that $|I| = K \in \mathbb{N}$. For unit norm $x \in \mathcal{H}_M$ assume $x \perp \varphi_i$ for each $i \in I$, and let $a := \{a_i\}_{i=1}^N \in \mathbb{R}^N$ be $\{|\langle x, \varphi_i\rangle|^2\}_{i=1}^N$ in non-increasing order. Finally, for $A > 0$ set

$$b := \Big(\underbrace{\frac{A}{N-K}, \dots, \frac{A}{N-K}}_{N-K \text{ terms}}, \underbrace{0, \dots, 0}_{K \text{ terms}}\Big) \in \mathbb{R}^N. \tag{2.3.11}$$

Then $a \succ_W b$.

Proof. Without loss of generality, assume $I = \{N-K+1, \dots, N\}$ (if not, apply a permutation of the indices $\{1, \dots, N\}$ that makes this happen) and thus $I^c = \{1, \dots, N-K\}$. Furthermore, since $\{a_i\}_{i=1}^N$ is in non-increasing order and $x \perp \varphi_i$ for each $i \in I$, then $a_i = 0$ for all $i \in I$. Being that $a_i \geq 0$ for all $i \in \{1, \dots, N\}$, it suffices to show $\sum\limits_{i=1}^m a_i \geq \sum\limits_{i=1}^m \frac{A}{N-K}$ for all $m \in \{1, \dots, N-K\}$. We proceed by way of contradiction. If there exists $m \in \{1, \dots, N-K\}$ so that

$$\sum_{i=1}^m a_i < \sum_{i=1}^m \frac{A}{N-K} = m\frac{A}{N-K}, \tag{2.3.12}$$

then $a_m < \frac{A}{N-K}$. Moreover, it follows that

$$a_i \leq a_m, \ \ \forall\, i \in \{m, \dots, N-K\}. \tag{2.3.13}$$

Hence, for any unit vector $x \in \mathcal{H}_M$ we may write

$$\begin{aligned} A = A\|x\|^2 &\le \sum_{i=1}^{N} a_i = \sum_{i \in I^c} a_i + \sum_{i \in I} a_i = \sum_{i \in I^c} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{N-K} a_i \\ &< \sum_{i=1}^{m} \frac{A}{N-k} + \sum_{i=m+1}^{N-K} a_m = m\frac{A}{N-K} + (N-K-m)a_m \\ &< m\frac{A}{N-K} + (N-K-m)\frac{A}{N-K} = A, \end{aligned} \tag{2.3.14}$$

a contradiction. □

2.4 Bounding products of frame coefficients

In this section, we will consider the following problem.

Problem 2.4.1. Given an equiangular tight frame $\{\varphi_i\}_{i=1}^N$ in $\mathcal{H}_M$, when does there exist a $C \in (0, \infty)$ so that

$$\min_{\|x\|=1=\|y\|} \sum_{i=1}^{N} |\langle x, \varphi_i \rangle||\langle y, \varphi_i \rangle| \ge C? \tag{2.4.1}$$

Remark 2.4.2. If $M \le 2N - 2$ we can always choose x and y such that $x \perp \varphi_i$ for all $i \in I_x \subset \{1, \ldots, N\}$, $|I_x| \le N - 1$, and $y \perp \varphi_i$ for all $i \in I_y \subset \{1, \ldots, N\}$, $|I_y| \le N - 1$, so that $I_x \cap I_y = \emptyset$ making $C = 0$ in (2.4.1).

In light of Remark 2.4.2, the question in Problem 2.4.1 is really about finding the specific $C \in (0, \infty)$ for various M, N.

Remark 2.4.3. Consider the following Hilbert space $\mathcal{H}_{M_1} \oplus \mathcal{H}_{M_2}$ with vectors $\{\varphi_i\}_{i \in I}$ in $\mathcal{H}_{M_1}$ and $\{\varphi_i\}_{i \in J}$ in $\mathcal{H}_{M_2}$. Choosing unit norm $x \in \mathcal{H}_{M_1}$ and unit norm $y \in \mathcal{H}_{M_2}$ yields

$$\sum_{i \in I \cup J} |\langle x, \varphi_i \rangle||\langle y, \varphi_i \rangle| = 0. \tag{2.4.2}$$

Hence, the question of a non-zero specific lower bound does not make sense. One case where a non-zero ‘C’ exists is for equiangular frames.

To analyze Problem 2.4.1, we first determine the number of indices $i \in [N]$ for which $|\langle x, \varphi_i \rangle||\langle y, \varphi_i \rangle| \neq 0$.

Proposition 2.4.4. Let $\{\varphi_i\}_{i=1}^N$ be a frame for $\mathcal{H}_M$ with $N \geq 2M-1$. The following are equivalent.

(i) $\{\varphi_i\}_{i=1}^N$ has the complement property.

(ii) For every $x, y \in \mathcal{H}_M$ there holds $\sum_{i=1}^N |\langle x, \varphi_i \rangle||\langle y, \varphi_i \rangle| \neq 0$.

Moreover, if $\{\varphi_i\}_{i=1}^N$ is full spark, then

$$\big|\{i : |\langle x, \varphi_i \rangle||\langle y, \varphi_i \rangle| \neq 0\}\big| \geq N - (2M-2). \tag{2.4.3}$$

Proof.

(i)⇒(ii): We will prove the contrapositive.

Suppose there exist nonzero $x, y \in \mathcal{H}_M$ such that $\sum_{i=1}^N |\langle x, \varphi_i \rangle||\langle y, \varphi_i \rangle| = 0$ and define

$$\begin{aligned} I &:= \{1 \leq i \leq N : \langle x, \varphi_i \rangle = 0\}, \\ J &:= \{1 \leq i \leq N : \langle y, \varphi_i \rangle = 0\}. \end{aligned} \tag{2.4.4}$$

<u>Case 1:</u> If $I \cap J = \emptyset$, then $J = I^c$. Furthermore, $x \perp \varphi_i$ for all $i \in I$ and thus $\operatorname{span}(\{\varphi_i\}_{i \in I}) \neq \mathcal{H}_M$. Also, $y \perp \varphi_i$ for all $i \in J = I^c$ and thus $\operatorname{span}(\{\varphi_i\}_{i \in I^c}) \neq \mathcal{H}_M$. Therefore there exists a partition I, I^c of $\{\varphi_i\}_{i=1}^N$ for which neither set spans $\mathcal{H}_M$. Hence, $\{\varphi_i\}_{i=1}^N$ fails the complement property.

<u>Case 2:</u> If $I \cap J \neq \emptyset$, then $\big(J \setminus (I \cap J)\big) = I^c$. Furthermore, $x \perp \varphi_i$ for all $i \in I$ and thus $\operatorname{span}(\{\varphi_i\}_{i \in I}) \neq \mathcal{H}_M$. Also, $y \perp \varphi_i$ for all $i \in \big(J \setminus (I \cap J)\big) = I^c$ and thus $\operatorname{span}(\{\varphi_i\}_{i \in I^c}) \neq \mathcal{H}_M$. Therefore there exists a partition I, I^c of $\{\varphi_i\}_{i=1}^N$ for which neither set spans $\mathcal{H}_M$. Thus, $\{\varphi_i\}_{i=1}^N$ fails the complement property.

(ii)$\Longrightarrow$(i): (*Proof by contrapositive*)

Suppose $\{\varphi_i\}_{i=1}^N$ fails the complement property. This implies there exists a partition I, I^c of $\{1, \ldots, N\}$ such that $\text{span}(\{\varphi_i\}_{i\in I}) \neq \mathcal{H}_M$ and $\text{span}(\{\varphi_i\}_{i\in I^c}) \neq \mathcal{H}_M$. Thus, there exist $x, y \in \mathcal{H}_M$ with the property that $x \perp \{\varphi_i\}_{i\in I}$ and $y \perp \{\varphi_i\}_{i\in I^c}$. Therefore $\sum_{i=1}^N |\langle x, \varphi_i\rangle||\langle y, \varphi_i\rangle| = 0$, as desired.

For the moreover part, the full spark property yields

$$\big|\{i : |\langle x, \varphi_i\rangle| = 0\}\big| \leq M - 1 \quad \text{and} \quad \big|\{i : |\langle y, \varphi_i\rangle| = 0\}\big| \leq M - 1. \tag{2.4.5}$$

Hence,

$$\big|\{i : |\langle x, \varphi_i\rangle||\langle y, \varphi_i\rangle| \neq 0\}\big| \geq N - (2M - 2), \tag{2.4.6}$$

as desired. □

(As a side note, we would like to point out that (ii) of Proposition 2.4.4 is equivalent to a collection of vectors yielding phase retrieval, as seen in Theorem 4.2.1.)

Next, further analysis of Problem 2.4.1 is provided via an upper bound on the sum in Equation 2.4.1.

Lemma 2.4.5. Let $\{\varphi_i\}_{i=1}^N$ be a frame in $\mathcal{H}_M$ with frame bounds A, B. Then

$$\sup_{\|x\|=1=\|y\|} \Big\{ \sum_{i=1}^N |\langle x, \varphi_i\rangle||\langle y, \varphi_i\rangle| \Big\} \leq B. \tag{2.4.7}$$

Proof. Let $x, y \in \mathcal{H}_M$ be unit norm. Invoking Hölder's inequality gives

$$\begin{aligned} \sum_{i=1}^N |\langle x, \varphi_i\rangle||\langle y, \varphi_i\rangle| &\leq \Big(\sum_{i=1}^N |\langle x, \varphi_i\rangle|^2\Big)^{1/2}\Big(\sum_{i=1}^N |\langle y, \varphi_i\rangle|^2\Big)^{1/2} \\ &= B^{1/2}B^{1/2} = B. \end{aligned} \tag{2.4.8}$$

□

The following proposition concerns an upper bound estimate for the summation in (2.4.7). It should be noted that the proposition is neither a stronger nor weaker version of Lemma 2.4.5. This can be seen by making particular choices for N, M which make the bounds in the lemma better than the bounds in the proposition and vice versa.

Proposition 2.4.6. Let $N \geq 2M-1$ and let $\{\varphi_i\}_{i=1}^N$ be a frame for $\mathcal{H}_M$ with frame bounds A, B. Then

$$\sup_{\|x\|=1=\|y\|} \left\{ \sum_{i=1}^{N} |\langle x, \varphi_i\rangle||\langle y, \varphi_i\rangle| \right\} \leq (N-2M+2)\sqrt{\frac{B}{M}}. \tag{2.4.9}$$

Proof. Pick unit vectors $x, y \in \mathcal{H}_M$ so that $x \perp \varphi_i$ for all $i \in \{1, \ldots, M-1\}$ and $y \perp \varphi_i$ for all $i \in \{N-M+2, \ldots, N\}$. It follows that

$$B = B\|x\|^2 \geq \sum_{i=1}^{N} |\langle x, \varphi_i\rangle|^2 = \sum_{i=M}^{N} |\langle x, \varphi_i\rangle|^2. \tag{2.4.10}$$

Suppose momentarily that $|\langle x, \varphi_i\rangle|^2 > \frac{B}{M}$ for all $i \in \{M, \ldots, N\}$. Then

$$B \geq \sum_{i=M}^{N} |\langle x, \varphi_i\rangle|^2 > (N-M+1)\,\frac{B}{M} \geq M\frac{B}{M} = B, \tag{2.4.11}$$

which is a contradiction. Hence, there exists $i_1 \in \{M, \ldots, N\}$ such that $|\langle x, \varphi_{i_1}\rangle|^2 \leq \frac{B}{M}$. If the third inequality in (2.4.11) is strict, then using a similar proof by contradiction (as in (2.4.11)) at most $N-2M+1$ more times, we obtain there exist indices $i_2, \ldots, i_{N-2M+2}$ all not equal to i_1 so that $|\langle x, \varphi_{i_j}\rangle|^2 \leq \frac{B}{M}$ for every $j \in \{1, \ldots, N-2M+2\}$. Without loss of generality, we may assume $i_1 = M, i_2 =$

$M+1, \ldots, i_{N-2M+2} = N-M+1$. Then

$$\sum_{i=1}^{N} |\langle x, \varphi_i\rangle||\langle y, \varphi_i\rangle| = \sum_{i=M}^{N-M+1} |\langle x, \varphi_i\rangle||\langle y, \varphi_i\rangle| \leq \sum_{i=M}^{N-M+1} |\langle x, \varphi_i\rangle|$$
$$\leq (N-2M+2)\sqrt{\frac{B}{M}}. \tag{2.4.12}$$

Taking the supremum in (2.4.12) over all $x, y \in \mathcal{H}_M$ with $\|x\| = 1 = \|y\|$ yields the desired result. □

Remark 2.4.7. The bounds in Lemma 2.4.5 and Proposition 2.4.6 are different and hence a comparison is necessary to determine when each one is a potentially better bound. Let $N \geq 2M-1$ and let $\{\varphi_i\}_{i=1}^{N}$ be a unit norm tight frame for $\mathcal{H}_M$. If $N - \sqrt{N} + 2 \geq 2M$ then $(N-2M+2)\frac{\sqrt{N}}{M} \geq \frac{N}{M}$ and hence $\frac{N}{M}$ is a better upper bound for

$$\sup_{\|x\|=1=\|y\|} \Big\{ \sum_{i=1}^{N} |\langle x, \varphi_i\rangle||\langle y, \varphi_i\rangle| \Big\}. \tag{2.4.13}$$

Moreover, when $N - \sqrt{N} + 2 < 2M$ then $(N-2M+2)\frac{\sqrt{N}}{M} < \frac{N}{M}$ and hence $(N-2M+2)\frac{\sqrt{N}}{M}$ is a better upper bound for this supremum.

In relation to these upper bounds, we now consider lower bounds for the summation in (2.4.7). The next proposition says this estimate is at least the product of the tight frame bound and the inner product of the unit vectors.

Proposition 2.4.8. If $\{\varphi_i\}_{i=1}^{N}$ is a unit norm tight frame in $\mathcal{H}_M$, then for any unit norm $x, y \in \mathcal{H}_M$ we have

$$\frac{N}{M}|\langle x, y\rangle| \leq \sum_{i=1}^{N} |\langle x, \varphi_i\rangle||\langle y, \varphi_i\rangle|. \tag{2.4.14}$$

Proof. Let $x, y \in \mathcal{H}_M$ be unit norm and consider

$$\frac{N}{M}|\langle x, y\rangle| = \left|\left\langle x, \frac{N}{M}y\right\rangle\right| = \left|\left\langle x, \sum_{i=1}^{N}\langle y, \varphi_i\rangle\varphi_i\right\rangle\right| = \left|\sum_{i=1}^{N}\langle x, \varphi_i\rangle\overline{\langle y, \varphi_i\rangle}\right|$$
$$\leq \sum_{i=1}^{N}|\langle x, \varphi_i\rangle||\langle y, \varphi_i\rangle|. \tag{2.4.15}$$

□

Continuing our investigation for lower bounds in the summation of (2.4.7), we make a slight modification in the following lemma. To set the record straight, we do not allow both x and y to vary for this estimate, but instead we fix the vector $y \in \mathcal{H}_M$ to be one of the frame vectors and allow $x \in \mathcal{H}_M$ to vary. Before we proceed with the lemma we first must recall the Welch bound [23].

Remark 2.4.9. Let $\{\varphi_i\}_{i=1}^N$ be a unit norm, equiangular, tight frame for $\mathcal{H}_M$, then for all $i \neq j$ we have

$$|\langle \varphi_i, \varphi_j\rangle|^2 = \frac{N-M}{M(N-1)}. \tag{2.4.16}$$

Lemma 2.4.10. Let $\{\varphi_i\}_{i=1}^N$ be a unit norm, equiangular, tight frame in $\mathcal{H}_M$ and fix $j \in \{1, \ldots, N\}$. Then

$$\frac{N}{M}\sqrt{\frac{N-M}{M(N-1)}} \leq \inf\left\{\sum_{i=1}^{N}|\langle \varphi_j, \varphi_i\rangle||\langle x, \varphi_i\rangle| : x \in \mathcal{H}_M, \|x\| = 1\right\}. \tag{2.4.17}$$

In particular, when

(i) $N = 2M$, we get

$$\frac{2}{\sqrt{2M-1}} \leq \inf\left\{\sum_{i=1}^{N}|\langle \varphi_j, \varphi_i\rangle||\langle x, \varphi_i\rangle| : x \in \mathcal{H}_M, \|x\| = 1\right\}; \tag{2.4.18}$$

(ii) $N = \frac{M(M+1)}{2}$, we get

$$\frac{M+1}{2\sqrt{M+2}} \leq \inf\Big\{\sum_{i=1}^{N} |\langle \varphi_j, \varphi_i\rangle||\langle x, \varphi_i\rangle| : x \in \mathcal{H}_M, \|x\| = 1\Big\}. \tag{2.4.19}$$

Proof. Fix a unit norm $x \in \mathcal{H}_M$ and fix $j \in \{1, \ldots, N\}$. Next, define $I := \{1, \ldots, N\} \setminus \{j\}$. For simplicity, define $c := \sqrt{\frac{N-M}{M(N-1)}}$.

$$\begin{aligned}
\sum_{i=1}^{N} |\langle \varphi_j, \varphi_i\rangle||\langle x, \varphi_i\rangle| &= \sum_{i\in I} |\langle \varphi_j, \varphi_i\rangle||\langle x, \varphi_i\rangle| + |\langle \varphi_j, \varphi_j\rangle||\langle x, \varphi_j\rangle| \\
&= c\sum_{i\in I} |\langle x, \varphi_i\rangle| + |\langle x, \varphi_j\rangle| \\
&= c\sum_{i=1}^{N} |\langle x, \varphi_i\rangle| + (1-c)|\langle x, \varphi_j\rangle| \\
&\geq c\sum_{i=1}^{N} |\langle x, \varphi_i\rangle| \geq c\sum_{i=1}^{N} |\langle x, \varphi_i\rangle|^2 = c\,\frac{N}{M}. \qquad (2.4.20)
\end{aligned}$$

This is a lower bound for the set $\Big\{\sum_{i=1}^{N} |\langle \varphi_j, \varphi_i\rangle||\langle x, \varphi_i\rangle| : x \in \mathcal{H}_M, \|x\| = 1\Big\}$ which is independent of the unit norm $x \in \mathcal{H}_M$. □

2.5 Distance between vectors and frame vectors

In this section we give estimates for the squared sums of the distances between a vector and the frame vectors. We will discover some surprising uniformities for the equiangular case.

The following lemma establishes a relationship between the coefficients of a collection of vectors with the norms and inner products of these vectors. While interesting in its own right, this lemma serves as a tool in Proposition 2.5.2.

Lemma 2.5.1. Let $\{\varphi_i\}_{i=1}^N$ be a collection of vectors in $\mathbb{R}^M$ whose components are given as $\varphi_1 = (\varphi_{11}, \ldots, \varphi_{1M}), \ldots, \varphi_N = (\varphi_{N1}, \ldots, \varphi_{NM})$. Then

$$\sum_{j=1}^{M}\Big(\sum_{i=1}^{N}\varphi_{ij}\Big)^2 = \sum_{i=1}^{N}\sum_{j=1}^{M}\varphi_{ij}\Big(\sum_{k=1}^{N}\varphi_{kj}\Big)$$
$$= \sum_{i=1}^{N}\|\varphi_i\|^2 + 2\sum_{1\le i<k\le N}\langle\varphi_i,\varphi_k\rangle. \qquad (2.5.1)$$

Proof. Concerning the first equality, note that

$$\sum_{i=1}^{N}\sum_{j=1}^{M}\varphi_{ij}\Big(\sum_{k=1}^{N}\varphi_{kj}\Big) = \sum_{j=1}^{M}\sum_{i=1}^{N}\varphi_{ij}\Big(\sum_{k=1}^{N}\varphi_{kj}\Big) = \sum_{j=1}^{M}\Big(\sum_{k=1}^{N}\varphi_{kj}\Big)\sum_{i=1}^{N}\varphi_{ij}$$
$$= \sum_{j=1}^{M}\Big(\sum_{i=1}^{N}\varphi_{ij}\Big)^2. \qquad (2.5.2)$$

For the last equality in (2.5.1), consider

$$\sum_{j=1}^{M}\Big(\sum_{i=1}^{N}\varphi_{ij}\Big)^2 = \sum_{j=1}^{M}\Big(\sum_{i=1}^{N}\varphi_{ij}^2 + 2\sum_{1\le i<k\le N}\varphi_{ij}\varphi_{kj}\Big)$$
$$= \sum_{i=1}^{N}\sum_{j=1}^{M}\varphi_{ij}^2 + 2\sum_{1\le i<k\le N}\sum_{j=1}^{M}\varphi_{ij}\varphi_{kj}$$
$$= \sum_{i=1}^{N}\|\varphi_i\|^2 + 2\sum_{1\le i<k\le N}\langle\varphi_i,\varphi_k\rangle, \qquad (2.5.3)$$

as wanted. □

The proposition below establishes upper and lower bounds for the sum of the squares of the distances between any unit vector and the frame vectors of an equiangular tight frame in terms of the dimension and the angle between vectors. Notice that these bounds are surprisingly close to one another.

Proposition 2.5.2. Let $\{\varphi_i\}_{i=1}^N$ be a unit norm, equiangular, tight frame in $\mathbb{R}^M$.

Then for any unit norm $x \in \mathbb{R}^M$, we have

$$2\big(N - \sqrt{N[1+(N-1)c]}\big) \le \sum_{i=1}^{N} \|x - \varphi_i\|^2$$
$$\le 2\big(N + \sqrt{N[1+(N-1)c]}\big), \tag{2.5.4}$$

where $c := \sqrt{\frac{N-M}{M(N-1)}}$; i.e., $c = |\langle \varphi_i, \varphi_j \rangle|$, for all $i, j \in \{1, \ldots, n\}$ such that $i \neq j$.

Proof. Fix a unit norm $x \in \mathbb{R}^M$ and write the vectors $x, \varphi_1, \ldots, \varphi_N \in \mathbb{R}^M$ in terms of their components; that is, write

$$x = (x_1, \ldots, x_M), \quad \varphi_1 = (\varphi_{11}, \ldots, \varphi_{1M}),$$
$$\varphi_2 = (\varphi_{21}, \ldots, \varphi_{2M}), \ldots, \varphi_N = (\varphi_{N1}, \ldots, \varphi_{NM}). \tag{2.5.5}$$

Next, define the functions $f, g : \mathbb{R}^M \longrightarrow \mathbb{R}$ by

$$f(y) := \sum_{i=1}^{N} \|y - \varphi_i\|^2 \qquad \text{and} \qquad g(y) := \|y\|^2. \tag{2.5.6}$$

Inputting our unit norm $x \in \mathbb{R}^M$ into the functions f, g gives

$$f(x) = \sum_{i=1}^{N} \big(\|x\|^2 + \|\varphi_i\|^2 - 2\langle x, \varphi_i \rangle\big) = 2N - 2\sum_{i=1}^{N}\sum_{j=1}^{M} x_j \varphi_{ij};$$
$$g(x) = \sum_{i=1}^{M} x_i^2 = 1. \tag{2.5.7}$$

At this stage, the main idea is to use Lagrange multipliers on the function $f(x)$ subject to the constraint function $g(x)$ to identify any absolute extrema. To this end, we calculate $(\nabla f)(x)$ and $(\nabla g)(x)$ as

$$(\nabla f)(x) = -2\Big(\sum_{i=1}^{N} \varphi_{i1}, \sum_{i=1}^{N} \varphi_{i2}, \ldots, \sum_{i=1}^{N} \varphi_{iM}\Big);$$
$$(\nabla g)(x) = 2(x_1, x_2, \ldots, x_M). \tag{2.5.8}$$

According to the method of Lagrange, we need to solve the system of equations $(\nabla f)(x) = \lambda[(\nabla g)(x)]$, $g(x) = 1$ where $\lambda \in \mathbb{R}$. Following this recipe produces

$$x_1 = -\frac{1}{\lambda}\sum_{i=1}^{N} \varphi_{i1}, \quad x_2 = -\frac{1}{\lambda}\sum_{i=1}^{N} \varphi_{i2}, \quad \ldots, \quad x_M = -\frac{1}{\lambda}\sum_{i=1}^{N} \varphi_{iM}. \tag{2.5.9}$$

Inputting these coordinates into the function $g(x) = 1$ gives

$$1 = \frac{1}{\lambda^2}\sum_{j=1}^{M}\Big(\sum_{i=1}^{N} \varphi_{ij}\Big)^2. \tag{2.5.10}$$

Thanks to Lemma 2.5.1, we may write the above equation as

$$\lambda^2 = \sum_{i=1}^{N} \|\varphi_i\|^2 + 2\sum_{1\leq i<k\leq N} \langle \varphi_i, \varphi_k\rangle = N + 2\sum_{1\leq i<k\leq N} \langle \varphi_i, \varphi_k\rangle, \tag{2.5.11}$$

which implies

$$\lambda = \pm\Big(N + 2\sum_{1\leq i<k\leq N} \langle \varphi_i, \varphi_k\rangle\Big)^{1/2}. \tag{2.5.12}$$

Substituting the coordinates in (2.5.9) into the function f given in (2.5.7) and using Lemma 2.5.1 yields

$$\begin{aligned} f(x) &= 2N - 2\sum_{i=1}^{N}\sum_{j=1}^{M}\Big(-\frac{1}{\lambda}\sum_{k=1}^{N}\varphi_{kj}\Big)\varphi_{ij} \\ &= 2N + \frac{2}{\lambda}\Big[\sum_{i=1}^{N}\sum_{j=1}^{M}\varphi_{ij}\Big(\sum_{k=1}^{N}\varphi_{kj}\Big)\Big] \\ &= 2N + \frac{2}{\lambda}\Big[\sum_{i=1}^{N}\|\varphi_i\|^2 + 2\sum_{1\leq i<k\leq N}\langle \varphi_i, \varphi_k\rangle\Big] \\ &= 2N + \frac{2}{\lambda}[\lambda^2] = 2(N+\lambda). \end{aligned} \tag{2.5.13}$$

Thus, the global extrema for the function f is $2(N + \lambda)$. Since λ depends on $\langle \varphi_i, \varphi_k\rangle$ (and not on $|\langle \varphi_i, \varphi_k\rangle|$) for each $i, k \in \{1, \ldots, N\}$ such that $i \neq k$, then the largest and smallest values λ obtains are when $\langle \varphi_i, \varphi_k\rangle = c$ where $c > 0$. Since

$\{\varphi_i\}_{i=1}^N$ is an equiangular unit norm tight frame, then we know explicitly the value of c; i.e., $c = \sqrt{\frac{N-M}{M(N-1)}}$. Under these assumptions, λ in (2.5.12) becomes

$$\lambda = \pm\sqrt{N + 2\frac{N(N-1)}{2}c} = \pm\sqrt{N[1+(N-1)c]}. \tag{2.5.14}$$

With λ as such, the desired conclusion follows. □

Remark 2.5.3. In Proposition 2.5.2, the lower bound is always **positive**. Indeed, $2\big(N - \sqrt{N[1+(N-1)c]}\big) > 0$ if and only if $c < 1$, which is satisfied a priori.

We establish slightly weaker bounds for Proposition 2.5.2 in the following corollary. Although these bounds are weaker, they are more accessible.

Corollary 2.5.4. Let $\{\varphi_i\}_{i=1}^N$ be a unit norm, equiangular, tight frame in $\mathbb{R}^M$. Then for any unit norm $x \in \mathbb{R}^M$, we have

$$\begin{aligned} 2N(1-\sqrt{2c}) &< 2\big(N - \sqrt{N[1+(N-1)c]}\big) \\ &\le \sum_{i=1}^N \|x - \varphi_i\|^2 \\ &\le 2\big(N + \sqrt{N[1+(N-1)c]}\big) \\ &< 2N(1+\sqrt{2c}), \end{aligned} \tag{2.5.15}$$

where $c := \sqrt{\frac{N-M}{M(N-1)}}$; i.e., $c = |\langle \varphi_i, \varphi_j\rangle|$, $\forall i,j \in \{1,\ldots,n\}$ such that $i \neq j$.

Proof. First note

$$c = \sqrt{\frac{N-M}{M(N-1)}} \ge \sqrt{\frac{1}{M(N-1)}} \ge \sqrt{\frac{1}{N(N-1)}} > \frac{1}{N}. \tag{2.5.16}$$

For the right hand side of the inequality in (2.5.15), we have

$$\begin{aligned}
2\big(N + \sqrt{N[1+(N-1)c]}\big) &= 2\Big(N + N\sqrt{\frac{1}{N} + \frac{N-1}{N}c}\Big) \\
&< 2\Big(N + N\sqrt{\frac{1}{N} + c}\Big) \\
&< 2(N + N\sqrt{2c}) \\
&= 2N(1+\sqrt{2c}), \qquad (2.5.17)
\end{aligned}$$

as desired. For the left hand side of the inequality in (2.5.15), we have

$$\begin{aligned}
2\big(N - \sqrt{N[1+(N-1)c]}\big) &= 2\Big(N - N\sqrt{\frac{1}{N} + \frac{N-1}{N}c}\Big) \\
&> 2\Big(N - N\sqrt{\frac{1}{N} + c}\Big) \\
&> 2(N - N\sqrt{2c}) \\
&= 2N(1-\sqrt{2c}), \qquad (2.5.18)
\end{aligned}$$

as wanted. The last order of business is to check that $2N(1-\sqrt{2c}) > 0$ which happens if and only if $\frac{1}{2} > c$. We consider the following scenarios.

<u>Case 1:</u> $M \geq 4$. (Proof by contradiction)

Note that $c \geq \frac{1}{2}$ if and only if $N \geq \frac{NM}{4} + \frac{3M}{4}$. If $M \geq 4$ and $c \geq \frac{1}{2}$ then $N \geq \frac{NM}{4} + \frac{3M}{4} \geq N + 3$, a contradiction. Therefore $c < \frac{1}{2}$ when $M \geq 4$ (which forces $N \geq 4$).

<u>Case 2:</u> $M = 3$.

Note that $c < \frac{1}{2}$ if and only if $4N - 3M - MN < 0$. Thus, for $M = 3$, we have $4N - 3(3) - 3N < 0$ if and only if $N < 9$. Recall, in [23] it is shown in $\mathbb{R}^3$ that the maximum number of vectors in an equiangular unit norm tight frame is 6. Hence,

$c < \frac{1}{2}$ when $M = 3$ (and, hence, $3 \le N \le 6 < 9$).

<u>Case 3:</u> $M = 2$.

Recall, in [23] it is shown in $\mathbb{R}^2$ that the maximum number of vectors in an equiangular unit norm tight frame is 3. Also, from <u>Case 2:</u>, we know $c < \frac{1}{2}$ if and only if $4N - 3M - MN < 0$. For $M = 2$, this becomes $4N - 3(2) - 2N < 0$ which happens if and only if $N < 3$. Consequently, $c < 1/2$ when $M = 2$ and $N = 2$.

This finishes the proof of Corollary 2.5.4. □

Remark 2.5.5. For the special case $M = 2$ and $N = 3$, it follows that $c = 1/2$ (see (2.4.16)) and

$$\frac{1}{N} + c = \frac{1}{3} + \frac{1}{2} = \frac{5}{6} = \frac{5}{3} \cdot \frac{1}{2} = \frac{5}{3}c. \tag{2.5.19}$$

Thus, we actually get improved bounds for the case when $M = 2$ and $N = 3$; specifically, (see (2.5.17) and (2.5.18))

$$\begin{aligned} 2N\Big(1 - \sqrt{\frac{5}{3}c}\Big) &\le 2\big(N - \sqrt{N[1 + (N-1)c]}\big) \\ &\le \sum_{i=1}^{N} \|x - \varphi_i\|^2 \\ &\le 2\big(N + \sqrt{N[1 + (N-1)c]}\big) \\ &\le 2N\Big(1 + \sqrt{\frac{5}{3}c}\Big). \end{aligned} \tag{2.5.20}$$

The next corollary gives a surprising identity exhibited by simplex frames. Recall that up to multiplication by a unitary operator and switching (replacing a vector by its additive inverse) there is only one unit norm tight frame with $M + 1$ elements in $\mathbb{R}^M$. This frame can be obtained in the following manner. Let $\{e_i\}_{i=1}^{M+1}$ be the standard orthonormal basis for $\mathbb{R}^{M+1}$ and let P be the rank one orthogonal projection onto span $(\sum_{i=1}^{M+1} e_i)$. Then the vectors

$$\{\varphi_i\}_{i=1}^{M+1} := \left\{ \frac{(I-P)e_i}{\|(I-P)e_i\|} \right\}_{i=1}^{M+1}$$

form an equiangular, tight frame for $\mathbb{R}^M$, [23]. This frame is commonly referred to as the simplex frame in $\mathbb{R}^M$.

Definition 2.5.6. An M-element **simplex frame** is a set of $M+1$ equiangular, equal norm vectors in M dimensions.

Specifically, the sum of the square distances between any unit vector and the simplex frame vectors is constant.

Corollary 2.5.7. Let $\{\varphi_i\}_{i=1}^{M+1}$ be the simplex frame in $\mathbb{R}^M$. Then for any unit norm $x \in \mathbb{R}^M$, we have

$$\sum_{i=1}^{M+1} \|x - \varphi_i\|^2 = 2(M+1). \tag{2.5.21}$$

Proof. This follows immediately from Proposition 2.5.2 and the fact that $c = -\frac{1}{M} = -\frac{1}{N-1}$ for the simplex in $\mathbb{R}^M$. □

There is a significant generalization of the above corollary for the case when the frame vectors sum to zero.

Theorem 2.5.8. Let $\{\varphi_i\}_{i=1}^{N}$ be a unit norm tight frame in $\mathcal{H}_M$ and assume $\sum_{i=1}^{N} \varphi_i = \mathbf{0}$. Then $\sum_{i=1}^{N} \|x - \varphi_i\|^2 = 2N$ for any unit norm $x \in \mathcal{H}_M$.

Proof. For any unit norm $x \in \mathcal{H}_M$, we have

$$\begin{aligned}
\sum_{i=1}^{N} \|x - \varphi_i\|^2 &= \sum_{i=1}^{N} \|x\|^2 + \|\varphi_i\|^2 - \langle x, \varphi_i \rangle - \langle \varphi_i, x \rangle \\
&= \sum_{i=1}^{N} (2 - 2\text{Re}(\langle x, \varphi_i \rangle)) \\
&= 2N - 2\text{Re}\langle x, \sum_{i=1}^{N} \varphi_i \rangle \\
&= 2N - 2Re\langle x, 0 \rangle = 2N.
\end{aligned}$$

□

Finally, in furthering this discussion we determine very accurate approximations for the sums of squared products of distances from vectors to the frame vectors. The lower and upper bounds merely differ by $\frac{8N}{M}$, providing a close estimation for the sum.

Theorem 2.5.9. Let $\{\varphi_i\}_{i=1}^N$ be a unit norm tight frame in $\mathbb{R}^M$ and assume $\sum_{i=1}^{N} \varphi_i = \mathbf{0}$. Then for any unit norm $x \in \mathcal{H}_M$,

$$4N\Big(1 - \frac{1}{M}\Big) \leq \sum_{i=1}^{N} \|x - \varphi_i\|^2 \|y - \varphi_i\|^2 \leq 4N\Big(1 + \frac{1}{M}\Big). \tag{2.5.22}$$

Proof. For starters, note that for each $i \in \{1, \ldots, N\}$ and each unit norm $x \in \mathbb{R}^M$, we have

$$\|x - \varphi_i\|^2 = \|x\|^2 + \|\varphi_i\|^2 - 2\langle x, \varphi_i \rangle = 2\big(1 - \langle x, \varphi_i \rangle\big). \tag{2.5.23}$$

Using this, we may write

$$\begin{aligned}\sum_{i=1}^{N}\|x-\varphi_i\|^2\|y-\varphi_i\|^2 &= \sum_{i=1}^{N}2\big(1-\langle x,\varphi_i\rangle\big)2\big(1-\langle y,\varphi_i\rangle\big)\\ &= 4\sum_{i=1}^{N}\big(1-\langle x,\varphi_i\rangle-\langle y,\varphi_i\rangle+\langle x,\varphi_i\rangle\langle y,\varphi_i\rangle\big)\\ &= 4\Big(N-0-0+\langle x,\sum_{i=1}^{N}\langle y,\varphi_i\rangle\varphi_i\rangle\Big)\\ &= 4\Big(N+\langle x,\frac{N}{M}y\rangle\Big)=4N\Big(1+\frac{1}{M}\langle x,y\rangle\Big). \qquad (2.5.24)\end{aligned}$$

Using Cauchy-Schwarz and the fact that $x, y \in \mathcal{H}_M$ are unit norm, we may deduce $-1 \le \langle x, y\rangle \le 1$. Combining this with the above calculation gives the desired result. $\square$

Since frames are used in a variety of research areas, a comprehensive study of frames and in particular frame coefficients is necessary. Prior to this, there had been no research devoted to the distribution of frame coefficients. In this chapter we have not only studied this distribution but also the products of frame coefficients and have given numerous sequences and boundaries for them, thus furthering our understanding of frames. Since finite frames are spanning sets then it is important to determine how the frame vectors could be spaced within a Hilbert space as compared to any arbitrary vector. With certain frames we saw that this distance is almost uniform. In continuing our study of frames, and in particular frame coefficients, we will now investigate the properties of frames whose frame coefficients are integers when the frame is written against the eigenbasis of the frame operator.

Chapter 3

Integer Frames

3.1 Introduction to integer frames

Integer frames, which are frames whose vectors have all integer coordinates with respect to a fixed orthonormal basis for a Hilbert space, have the potential to mitigate quantization errors and transmission loses as well as speed up computation time. The first systematic study of this class of frames now follows with the focus on construction methods for equal norm, tight, and/or full spark integer frames. Our approach to classifying integer frames via construction techniques will be to find a matrix representation for the synthesis operator, which has all integer entries.

Note that any rank M, $M \times N$ matrix with all integer entries represents an N-element integer frame in $\mathcal{H}_M$, where the frame vectors are the columns of this matrix with respect to an orthonormal basis for $\mathcal{H}_M$. However, an arbitrary rank M, $M \times N$ integer matrix, in general, does not have enough "nice" properties to prove to be useful in applications. In applications, since integer frames can be implemented to speed up computation time, the frame operator and its associated eigenvalues should be readily available. The following theorem addresses this issue and defines the added properties needed when constructing “application ready" integer frames.

Theorem 3.1.1. [23] Let $T : \mathcal{H}_M \to \ell_2(N)$ be a linear operator, let $\{e_j\}_{j=1}^M$ be an orthonormal basis for $\mathcal{H}_M$, and let $\{\lambda_j\}_{j=1}^M$ be a sequence of positive numbers. Define A to be the $M \times N$ matrix representation of T^* with respect to $\{e_j\}_{j=1}^M$ and the standard basis $\{\hat{e}_i\}_{i=1}^N$ of $\ell_2(N)$. Then the following are equivalent.

1. $\{T^*\hat{e}_i\}_{i=1}^N$ forms a frame for $\mathcal{H}_M$ whose frame operator has eigenvectors $\{e_j\}_{j=1}^M$ and associated eigenvalues $\{\lambda_j\}_{j=1}^M$.

2. The rows of A are orthogonal and the j-th row square sums to λ_j.

3. The columns of A form a frame for $\ell_2(M)$ and

$$AA^* = \mathrm{diag}(\lambda_1, \ldots, \lambda_M).$$

As a result of Theorem 3.1.1, it is clear that if we impose the synthesis matrix of a frame to be represented against the eigenbasis of its frame operator, S, then the synthesis matrix will have orthogonal rows and the square sum of the rows will be the eigenvalues of the frame operator S. Moreover, a frame is tight if all eigenvalues of S are equal and hence if the square sum of all rows are equal. Because of this, all integer frames (unless stated otherwise) in the present chapter will be represented against the eigenbasis of their respective frame operator. Hence, this requires orthogonality between the rows of the frame matrix.

Note, when constructing integer frames it suffices to construct frames with rational coordinates because we can then multiply the frame by the greatest common denominator of the rationals in order to get an integer frame. Also, in finite dimensions since a frame is simply a spanning set, the zero vector could potentially be one or more elements of the frame. However, since we are concerned with using integer frames to mitigate quantization errors and to speed up computation time then the zero vector is not useful in our application since it provides no new information. Because of this, in the present chapter we will assume that no frames contain the zero vector.

Within this chapter we first start by giving methods for constructing larger frames from those with fewer vectors or those in lower dimensions. Next, we completely classify equal norm, tight, integer frames (ENTIF) in two dimensions and partially classify the case for three dimensions. The special case of frames having one more element than the dimension is then fully examined where we show that the existence of $M+1$-element frames in M dimensions is directly related to the existence of M-simplices having integer coordinates in M dimensions. Finally, it is shown that when dropping either the equal norm or tight assumptions, any number of vectors greater than the dimension can be obtained. The same is shown for equal norm integer frames which are *nearly tight*. The work in this chapter can be found in the paper *Integer frames* [24].

3.2 Combining frames

In this section, we will see how to subset and combine existing frames to obtain frames with more vectors. These results will be used throughout the remainder of this chapter and serve more as preliminary proofs.

The following result regarding sub-setting rows of the matrix representation of a frame is basic; but since it is used extensively throughout this chapter we state it formally here.

Proposition 3.2.1. If $A = (a_{ij})_{i=1, j=1}^{M, N}$ is an $M \times N$ frame matrix and $I \subset \{1, 2, \ldots, M\}$

then $B = (a_{ij})_{i \in I, j=1}^{N}$ is also a frame matrix.

The next few results provide ways of adjoining frames to construct larger frames. Although the following proposition is clear; we record it for future reference.

Proposition 3.2.2. Let A be an $M \times N_1$ matrix and B be an $M \times N_2$ matrix and suppose A and B both represent frames in $\mathcal{H}_M$ with N_1 and N_2 elements, respectively. Then the $M \times (N_1 + N_2)$ block matrix $[A, B]$ represents a frame with $N_1 + N_2$ elements in $\mathcal{H}_M$. Furthermore, if A and B are both tight frames then $[A, B]$ is also a tight frame. Lastly, if A and B are both of the same equal norm, then $[A, B]$ is also equal norm.

It is easy to see via induction that the preceding proposition also holds for any number of frames over the same Hilbert space. In addition, one can also adjoin frame matrices diagonally. This result is also clear so its proof is omitted.

Proposition 3.2.3. Suppose A and B are $M_1 \times N_1$ and $M_2 \times N_2$ matrices which represent frames in $\mathcal{H}_{M_1}$ and $\mathcal{H}_{M_2}$, respectively. Then the $(M_1 + M_2) \times (N_1 + N_2)$ block diagonal matrix

$$C = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{bmatrix}$$

represents an $N_1 + N_2$ element frame in $\mathcal{H}_{M_1+M_2}$. For C to be tight, A and B need to have the same tightness factor and for C to be equal norm, both A and B need to be equal norm with the same factor.

The last proposition of this section gives a method for constructing a new frame having twice the dimension and twice the number of elements.

Proposition 3.2.4. If A is an $M \times N$ matrix representing a frame in $\mathcal{H}_M$ and c is a nonzero scalar, then the $2M \times 2N$ matrix

$$B = \begin{bmatrix} cA & cA \\ cA & -cA \end{bmatrix}$$

represents a frame in $\mathcal{H}_{2M}$. The frame B is tight if A is tight and B is equal norm if A is equal norm.

To demonstrate the usefulness of Proposition 3.2.4, we consider building ENTIFs out of *Hadamard matrices.*

Definition 3.2.5. An $N \times N$ matrix A, having only ± 1 as its entries and satisfying $A^T A = N \cdot I_{N \times N}$ is called a **Hadamard matrix**.

We are interested in Hadamard matrices because if an $N \times N$ Hadamard matrix, A, exists then the $M \times N$ matrix formed by the first M rows of A is an N-element ENTIF in M dimensions. Also note that a Hadamard matrix itself represents an ENTIF and so Proposition 3.2.4, with $c = 1$, implies that if an $N \times N$ Hadamard matrix exists, then there is also a Hadamard matrix of size $2^k N \times 2^k N$ for all $k \geq 0$. Thus a frame with $2^k N$ elements can also be formed in M dimensions. This is summarized in the following theorem.

Theorem 3.2.6. Suppose an $N \times N$ Hadamard matrix exists for some $N \in \mathbb{N}$. Then an ENTIF with $2^k N$ elements in M dimensions exists for all $k \geq 0$ and $M \leq 2^k N$.

The preceding theorem is a generalization of a now standard construction of Sylvester, who showed that $2^K \times 2^K$ Hadamard matrices exist for all non negative integers K. Namely, let H_0 be the 1×1 matrix

$$H_0 = [1]$$

and iterate to obtain the $2^K \times 2^K$ matrix

$$H_K = \begin{bmatrix} H_{K-1} & H_{K-1} \\ H_{K-1} & -H_{K-1} \end{bmatrix}$$

for any positive integer K. Now form a new matrix by choosing the first $M \leq 2^K$ rows of H_K yielding an ENTIF where the square norms of the columns (frame vectors) equal M. It is worth noting that the ENTIF obtained in this way may not be full spark. For instance, forming a frame by keeping only the first half of the rows of H_K yields two copies of an orthonormal basis. In general, it is not known which subsets of the rows of a Hadamard matrix give a full spark frame.

It is a well-known result that $N \times N$ Hadamard matrices can only exist when $N = 1, 2, 4K$, where $K \geq 1$. However, the existence of a Hadamard matrix of size $4K$ is not yet known for all values of K and the formal statement that they do exist is called the Hadamard conjecture. This conjecture is over a century old and has proven itself to be one of the most difficult problems in mathematics.

A large number of Hadamard matrices are known to exist. The conjecture has been proven for all $4K \leq 664$ and there are only 13 cases that have not yet been shown for $4K \leq 2000$ [32]. Moreover, Theorem 3.2.6 gives large classes of ENTIFs, found from Hadamard matrices, for all of these dimensions. Hadamard matrices are a well studied topic of research which have yet to be classified and as a result Theorem 3.2.6 illustrates why classifying ENTIFs is similarly complicated. See [52] for an in-depth discussion on Hadamard matrices.

3.3 ENTIFs in two and three dimensions

This section addresses when (full spark) ENTIFs exist in two and three dimensions. The question of existence in two dimensions is completely classified, but only partial results are obtained in three dimensions.

In order to construct a full spark frame in the two dimensional case, the following result concerning the number of representations of an integer as the sum of two squares is needed.

Lemma 3.3.1. [8, Ch. XV] Let $n = 2^{a_0}p_1^{2a_1}\cdots p_r^{2a_r}q_1^{b_1}\cdots q_s^{b_s}$, where the p_i's are prime numbers of the form $4x-1$ for $i=\{1,\cdots,r\}$, the q_j's are prime numbers of the form $4x+1$ for $j=\{1,\cdots,s\}$, and $a_i, b_j \in \mathbb{Z}$, for $i=\{1,\cdots,r\}$ and for $j=\{1,\cdots,s\}$. If

$$B = (b_1+1)(b_2+1)\cdots(b_s+1),$$

then the number of distinct representations of n as the sum of two unequal squares, ignoring order, is given by

$$N_s(n) = \begin{cases} \frac{B}{2} & \text{if } B \text{ is even} \\ \frac{B-1}{2} & \text{if } B \text{ is odd} \end{cases}$$

As an application of this lemma, we will show that there exists a full spark, ENTIF in $\mathcal{H}_2$ with any even number of vectors.

Theorem 3.3.2. There exists a full spark, ENTIF in $\mathcal{H}_2$ with $2N$ elements for all positive integers N.

Proof. Taking $n = c^2 = 5^{2N}$ (and hence $q_1 = 5$ and $b_1 = 2N$) in Lemma 3.3.1 implies that c^2 has N distinct representations as the sum of two unequal squares, ignoring order. Hence, there exists $a_i, b_i \in \mathbb{Z}$ for $i = \{1, \cdots, N\}$ such that

$$c^2 = a_1^2 + b_1^2 = \cdots = a_N^2 + b_N^2.$$

If A is the $2 \times 2N$ matrix given by

$$A = \begin{bmatrix} a_1 & b_1 & \cdots & a_N & b_N \\ b_1 & -a_1 & \cdots & b_N & -a_N \end{bmatrix},$$

then A clearly represents an ENTIF and it is full spark since each representation of c^2 is unique. □

We can also diagonally adjoin the matrices created in Theorem 3.3.2 to obtain a $2MN$-element ENTIF in $\mathcal{H}_{2M}$ for any $M, N \in \mathbb{N}$. However when $N > 1$, since we are adjoining M copies of the same frame we lose the full spark property.

Corollary 3.3.3. There exists an ENTIF in $\mathcal{H}_{2M}$ with $2MN$ elements for any positive integers M and N.

Proof. Let A be a $2 \times 2N$ matrix representing an ENTIF frame in $\mathcal{H}_2$ (Theorem 3.3.2 guarantees one exists for all positive integers N). Let B be the $2M \times 2MN$ block diagonal matrix $B = \text{diag}(A, \ldots, A)$ obtained by adjoining M copies of A together as described in Proposition 3.2.3. Then B represents a $2MN$ element ENTIF frame in $\mathcal{H}_{2M}$. □

Remark 3.3.4. For $M > 1$, the frame B obtained in the proof of Corollary 3.3.3 is full spark only when $N = 1$, whence the frame is a basis.

We have seen that there exists a $2N$ element full spark, ENTIF in $\mathcal{H}_2$ for all positive integers N; however, this is not the case when the frame has $2N + 1$ elements for any positive integer N. In fact, there does not exist any ENTIFs in $\mathcal{H}_2$ with an odd number of elements. To prove this fact, we need to carefully examine the *parities* of two sets of integers which square sum to the same number.

Definition 3.3.5. Let $n, m \in \mathbb{N}$ and set $p = m + n$. A set of integers $(a_i)_{i=1}^p$ has **parity** $[m, n]$ if m integers in $(a_i)_{i=1}^p$ are even and n integers in $(a_i)_{i=1}^p$ are odd.

Proposition 3.3.6. Let $\{a_i\}_{i=1}^N$ and $\{b_j\}_{j=1}^M$ be integers satisfying

$$\sum_{i=1}^N a_i^2 = \sum_{j=1}^M b_j^2,$$

and let

$$I = \{1 \leq i \leq N : a_i \text{ is odd}\}, \text{ and } J = \{1 \leq j \leq M : b_j \text{ is odd}\}.$$

Then $|I| - |J|$ is divisible by 4.

Proof. First note that

$$\sum_{i \in I} a_i^2 + \sum_{i \in I^c} a_i^2 = \sum_{j \in J} b_j^2 + \sum_{j \in J^c} b_j^2.$$

and hence rearranging gives

$$\sum_{i \in I} a_i^2 - \sum_{j \in J} b_j^2 = \sum_{j \in J^c} b_j^2 - \sum_{i \in I^c} a_i^2.$$

Since all terms on the right hand side are squares of even integers, then $\sum_{j \in J^c} b_j^2 - \sum_{i \in I^c} a_i^2$ is divisible by 4. Next, since all terms on the left hand side are squares of odd integers, then $\sum_{i \in I} a_i^2 - \sum_{j \in J} b_j^2$ is divisible by 4 if and only if $|I| - |J|$ is divisible by 4. □

Corollary 3.3.7. Suppose that $A = \{a_i\}_{i=1}^M$ and $B = \{b_i\}_{i=1}^M$ satisfy

$$\sum_{i=1}^{M} a_i^2 = \sum_{i=1}^{M} b_i^2.$$

If the parity of A is $[m, M-m]$, then the parity of B is $[m+4k, M-m-4k]$ for some integer k.

Proof. The proof follows from Proposition 3.3.6. □

We are now equipped with the necessary facts to prove that ENTIFs with an odd number of elements do not exist in $\mathcal{H}_2$.

Theorem 3.3.8. An ENTIF with an odd number of elements does not exist in $\mathcal{H}_2$.

Proof. Suppose by way of contradiction that there exists an ENTIF, A, in $\mathcal{H}_2$ with $2N+1$ elements, for some $N \in \mathbb{N}$. Note that if A consisted of all even elements, then we could factor out the largest common factor of 2^k from each element of A, for some $k \in \mathbb{N}$, and we will be left with $2^k \hat{A}$, where $\hat{A}$ has at least one odd element. So without loss of generality, we may assume that A has at least one odd element. Furthermore, observe that if both rows contain all odd elements, then the inner product of the rows cannot be zero, which contradicts our assumption that the rows of any integer frame must be orthogonal. Hence, A must have at least one even element and at least one odd element.

Therefore, since the square sums of the columns must be equal then Corollary 3.3.7 implies that each column has parity $[1, 1]$. Hence the total number of odd elements in A is $2N+1$, an odd number. However, since the square sums of the rows must also be equal, then Corollary 3.3.7 also implies that if s_1 is the number of odd elements in the first row, then $s_1 - 4k$ is the number of odd elements in the second row for some integer k. Thus, the total number of odd elements in A

is $2(s_1 - 2k)$, which is an even number, hence a contradiction is met and such an A cannot exist. □

So far, we have fully classified (full spark) ENTIFs in $\mathcal{H}_2$ and we would similarly like to be able to fully classify ENTIFs in $\mathcal{H}_3$. However, the three dimensional case has further complications and hence only a partial classification is obtained. First, it is shown that ENTIFs having a number of vectors that is any multiple of three or any multiple of four exist in three dimensions.

Theorem 3.3.9. For any positive integer N, there exists an ENTIF in $\mathcal{H}_3$ with $3N$ elements and there exists an ENTIF in $\mathcal{H}_3$ with $4N$ elements.

Proof. Let A be any 3×3 integer matrix whose columns form an orthonormal basis for $\mathcal{H}_3$. Such matrices exist in abundance by first finding one with rational entries and then multiplying by the common denominator. However, one can simply choose A to be the 3×3 identity matrix. Then the matrix $[A, \cdots, A]$ obtained by adjoining N copies of A together, as in Proposition 3.2.2, is an ENTIF with $3N$ elements in $\mathcal{H}_3$. The $4N$ element case is obtained in a similar manner by adjoining N copies of the 4×4 Hadamard matrix, as described in Section 3.2. □

Corollary 3.3.10. For any positive integers M and N, there is an ENTIF in $\mathcal{H}_{3M}$ with $3MN$-elements and there is an ENTIF in $\mathcal{H}_{3M}$ with $4MN$-elements.

Proof. Redefine the matrix A in Corollary 3.3.3 to be a $3\times3N$ matrix or a $3\times4N$ matrix representing an ENTIF in $\mathcal{H}_3$, which is guaranteed by Theorem 3.3.9. Then the proof follows from the proof of Corollary 3.3.3 where B is now redefined to be a $3M \times 3MN$ block diagonal matrix, or a $3M \times 4MN$ block diagonal matrix, respectively. □

Remark 3.3.11. Unfortunately, for any $p, q \in \mathbb{N}$, we cannot adjoin p copies of a 3-element ENTIF with q copies of a 4-element ENTIF to get new ENTIFs in $\mathcal{H}_3$ because the square norms of their columns can never be the same.

Next, necessary conditions for when a matrix of size $3 \times (2N+1)$ represents an ENTIF is given, which will lead to proving that an ENTIF with five elements in three dimensions does not exist.

Theorem 3.3.12. If N is an integer with $N \geq 2$ such that $\gcd(2N+1, 3) = 1$ and A is a $3 \times (2N+1)$ matrix which represents an ENTIF in $\mathcal{H}_3$, then the parity of each column must be $[2, 1]$ and the number of odds in the i^{th} row is of the form $4m_i + k$ with $0 \leq k < 4$. Therefore,

$$4(m_1 + m_2 + m_3) + 3k = 2N + 1 \tag{3.3.1}$$

must hold. Furthermore, $4m_i + k, 4m_j + k \leq N$ for some $1 \leq i \neq j \leq 3$.

Proof. As in the proof of Theorem 3.3.8, it may be assumed without loss of generality that A has at least one even entry and at least one odd entry.

First, consider the case in which two rows, R_1 and R_2, of A have $0 \leq s_1 \leq N$ and $0 \leq s_2 \leq N$ even entries, respectively, and let R_3 represent the remaining row of A. At least one of s_1 or s_2 is nonzero since both rows having all odd elements would imply that the two rows are not orthogonal. Also, Corollary 3.3.7 implies that each column of A has parity $[1, 2]$ since at least one column has two odds by the assumption that $s_i \leq N$ for $i = \{1, 2\}$. That is, up to reordering the columns and/or rows, we are in the case where the frame matrix is of the form

$$\begin{bmatrix} e & \cdots & e & o & \cdots & o & o \cdots o \\ o & \cdots & o & e & \cdots & e & o \cdots o \\ o & \cdots & o & o & \cdots & o & e \cdots e \end{bmatrix}$$

where e symbolizes an even integer, o symbolizes an odd integer and there are s_1 even entries in row one (R_1), s_2 even entries in row two (R_2) and $2N+1-s_1-s_2$ even entries in row three (R_3).

Furthermore, since the elements of R_1 and R_2 both square sum to the same number due to A being a tight frame, then Corollary 3.3.7 also gives $s_2 = s_1 + 4k$ for some integer k. Hence, R_3 must have $s_1 + s_2 = 2s_1 + 4k$ odd entries and $2N - 2s_1 - 4k + 1 = 2\,(N - s_1 - 2k) + 1$ even entries due the parity restriction of the columns. Now, since R_3 has an odd number of even entries and A is tight, then by Corollary 3.3.7 we see that s_1 and s_2 must also be odd numbers because they possibly differ from the number of even elements in R_3 by a factor of four. However, taking the inner product of R_1 and R_2 gives the sum of $2(s_1 + 2k)$ even numbers and $2(N - s_1 - 2k) + 1$ odd numbers, which must be odd. That is, the inner product cannot be zero yielding a contradiction.

Next, consider the case that two rows R_1 and R_2 have $0 \le s_1 \le N$ and $0 \le s_2 \le N$ odd entries, respectively. Then the parity of each column must be $[2, 1]$ since at least one column has two even entries. Corollary 3.3.7 implies each row has $4m_i + k$ odds and equation (3.3.1) is obtained by summing the number of odds in all rows. □

Theorem 3.3.13. There does not exist a five element ENTIF in $\mathcal{H}_3$.

Proof. If such an ENTIF did exist, then from Theorem 3.3.12 there would exist integers $m \geq 0$ and $0 \leq k \leq 3$ satisfying $4m + 3k = 5$. However, by substituting in $k = 0, 1, 2, 3$, it is immediate that no such numbers exist and so a contradiction is met. □

Theorem 3.3.12 does not give a contradiction for any number of elements larger than five. For instance, there may exist an ENTIF represented by a 3×7 matrix with one odd element in each of the first two rows and five odd elements in the last row.

Problem 3.3.14. In $\mathcal{H}_3$, does there exist an ENTIF with N elements for $N = 7, 10, 11...$ for the cases not covered above? When does there exist full spark ENTIFs in $\mathcal{H}_3$?

We will see throughout the next few sections that it is very difficult, in general, to construct ENTIFs with an odd number of elements except in very special cases, such as the case when the dimension of the space is odd (and in this case, multiples of the dimension are obtained) or for some special classes of simplices.

Problem 3.3.15. Is there something fundamental about N being an odd integer that presents a block to producing ENTIFs or is it just our construction methods which are limited?

The last theorem presented in this section characterizes the number of odds in each row of a matrix representing an ENTIF in $\mathcal{H}_3$, based on the parity of the columns. The proof is similar to the proof for Theorem 3.3.12 and so it is omitted.

Theorem 3.3.16. Suppose N is an integer with $N \geq 2$ so that $\gcd(4N + 2, 3) = 1$ and define A to be a $3 \times (4N + 2)$ matrix representing an ENTIF. If the parity of each column is $[2, 1]$, then the number of odds in each row is of the form $4m_i + 2$ and $m = m_1 + m_2 + m_3 = N - 1$. If the parity of each column is $[1, 2]$, then the number of odds in each row is $4m_i$ and $m = m_1 + m_2 + m_3 = 2N - 1$.

3.4 ENTIFs with $M+1$ vectors in M dimensions

This section is dedicated to fully classifying when an ENTIF with $M+1$ vectors exists in M dimensions. We show that for such a frame to exist it must be an M-simplex, from which the result will follow from a previously known result.

Recall that unit norm tight frames with $M+1$ vectors in M-dimensions are all *unitarily equivalent* [22]. That is, there is a unitary operator on $\mathbb{R}^M$ which takes the $M+1$ elements of one unit norm, tight frame to the $M+1$ elements of another unit norm, tight frame.

Theorem 3.4.1. If A is an $M \times (M+1)$ matrix representing an ENTIF, then A is equiangular. Thus, the columns of A form an M-simplex with integer coordinates.

Proof. First append an additional row to A which is orthogonal to and has the same norm as all rows of A. Call this new $(M+1)\times(M+1)$ matrix A'. Since the rows of A' all have the same norm and the columns of A all have equal norm, the added row must be of the form $[\pm a, \pm a, \ldots, \pm a, \pm a]$ for some $a \neq 0$. By possibly multiplying columns by -1, which does not affect the orthogonality of the rows, it may be assumed that the last row of A' is $[a, a, \ldots, a, a]$.

Now, the norm squared of each row of A is $(M+1)a^2$ (tightness factor) since it must match the norm squared of the appended row. Therefore, the norm squared of each column of A is Ma^2 because of the relationship

$$\begin{aligned}(M+1)c &= \sum_{j=1}^{M+1} c = \sum_{j=1}^{M+1}\sum_{i=1}^{M} A_{ij}^2 \\ &= \sum_{i=1}^{M}\sum_{j=1}^{M+1} A_{ij}^2 = \sum_{i=1}^{M} d = Md,\end{aligned}$$

where A_{ij} is the entry of A in the i^{th} row and j^{th} column, c is the equal norm squared and d is the tightness factor squared. Furthermore, the columns of A' must be orthogonal since A' is a multiple of a unitary. Therefore, the inner product of any two columns of A is $-a^2$ and so A is equiangular. □

The full classification for when an M-simplex with integer coordinates exists was first proved by I.J. Schoenberg in [50] and was stated in a clearer fashion by I.G. Macdonald in [39] as follows.

Theorem 3.4.2. [39] There exists a regular M-simplex in $\mathbb{R}^M$ with vertices in $\mathbb{Z}^M$ if and only if $M+1$ is the sum of 1, 2, 4 or 8 odd squares.

Remark 3.4.3. Theorem 3.4.2 along with Theorem 3.4.1 imply that an $M+1$ element ENTIF in M dimensions does not exist for

$$M = 2, 4, 5, 10, 12, 13, 14, 16, 18, 20, 21, 22, 26, \ldots.$$

Next, an explicit construction of an ENTIF for the allowable values of M is given. Note that it is equivalent to constructing a regular M-simplex with vertices in $\mathbb{Q}^M$. The ideas presented are mostly due to R. Chapman [27].

Define $m = M+1$ and let $e_1, \ldots, e_m$ be the standard orthonormal basis of $\mathbb{Q}^m$. Put $v = e_1 + \cdots + e_m$. The main idea of the construction is to find a linear operator S on $\mathbb{Q}^m$ so that $S = T/\sqrt{m}$ and satisfies $Sv = e_m$, where T is an orthogonal matrix. Such an S preserves inner products and furthermore the set $\{Se_j\}_{j=1}^m$ forms another orthogonal set in which the m-th coordinate of Se_j is $1/m$ for all $1 \leq j \leq m$. Therefore, removing the last row of the matrix representation with respect to the standard orthonormal basis of S gives an $M+1$ element ENTIF in M dimensions.

To construct such an S, it is enough to find a linear operator $U : \mathbb{Q}^m \to \mathbb{Q}^m$ so that $U = Q/\sqrt{m}$ for some orthogonal operator Q and then compose U with the reflection R, the hyperplane with normal vector

$$\frac{Uv - e_m}{\|Uv - e_m\|}.$$

That is, $S = R \circ U$ and so $Sv = R(Uv) = e_m$ as required.

In the case that m is a perfect square, define $Ux = x/\sqrt{m}$. If m is the sum of $k = 2, 4$, or 8 odd squares, such as $m = a^2 + \cdots + h^2$, then let $Ux = A_k x/m$ where A_k is the block diagonal matrix having E_k down the diagonal m/k times and where the E_k are defined as

$$E_2 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$

$$E_4 = \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix},$$

$$E_8 = \begin{bmatrix} a & b & c & d & e & f & g & h \\ -b & a & -d & c & -f & e & -h & g \\ e & -f & g & -h & -a & b & -c & d \\ -f & -e & h & g & b & a & -d & -c \\ -d & -c & b & a & -h & -g & f & e \\ c & -d & -a & b & -g & h & e & -f \\ g & -h & -e & f & c & -d & -a & b \\ -h & -g & -f & -e & d & c & b & a \end{bmatrix}.$$

The operators given in each case are easily checked to have the described properties.

In the construction above, an $(M+1)\times(M+1)$ rational unitary matrix having a row with all entries being the same modulus was constructed. Any such matrix yields an ENTIF with $M+1$ elements in M dimensions by removing the constant modulus row. An identical proof technique as in the proof of Theorem 3.4.1 combined with Theorem 3.4.2 immediately implies these types of matrices exist if and only if $M+1$ is the sum of 1, 2, 4, or 8 odd squares. We summarize this discussion in the following theorem.

Theorem 3.4.4. An $(M+1)$-element ENTIF in $\mathcal{H}_M$ exists if and only if $M+1$ is the sum of $1, 2, 4,$ or 8 odd squares.

3.5 General equal norm, tight, integer frames

This section includes numerous results concerning ENTIFs in a general dimension. The main result in this section provides a way to adjoin two ENTIFs to obtain an ENTIF with N elements for all large enough N. In order to obtain this result, a basic number theoretic result is needed.

Lemma 3.5.1. [51] If $a, b \in \mathbb{N}$ such that $\gcd(a, b) = 1$, then for all integers $m \geq (a-1)(b-1)$, there is exactly one pair of non negative integers p and q such that $q < a$ and $m = pa + qb$.

Corollary 3.5.2. If $a, b \in \mathbb{Z}$ and g is defined to be $g := \gcd(a, b)$, then for every integer $m \geq (a/g-1)(b/g-1)$ there exist non negative integers p and q so that $gm = pa + qb$.

Proof. Note that $\gcd(a/g, b/g) = 1$, so Lemma 3.5.1 applies. Hence, there exist non negative integers p and q such that $m = p(a/g) + q(b/g)$. □

Combining Lemma 3.5.1 and Corollary 3.5.2 yields a fundamental result which states that if we can construct two ENTIFs in $\mathcal{H}_M$ such that the number of vectors in the two frames are relatively prime with the same equal norm constant, then we can construct ENTIFs with N-elements for all large N.

Theorem 3.5.3. Suppose A and B represent ENTIFs in $\mathcal{H}_M$ with N_1 and N_2 elements, respectively, such that A and B have the same equal norm constant. If $K = \gcd(N_1, N_2)$, then there is a KN-element ENTIF in $\mathcal{H}_M$ for all $N \geq (N_1/K - 1)(N_2/K - 1)$.

Proof. If $N \geq (N_1/K-1)(N_2/K-1)$, then Corollary 3.5.2 implies the existence of nonnegative c_N and d_N such that $KN = c_N \cdot N_1 + d_N \cdot N_2$. Therefore, Proposition 3.2.2 implies that the block matrix

$$[A, \ldots, A, B, \ldots, B],$$

where A appears c_N times and B appears d_N times is an ENTIF in $\mathcal{H}_M$ with KN elements. □

Theorem 3.5.3 leads to a number of corollaries implying the existence of ENTIFs.

Corollary 3.5.4. If $M \geq 3$ is an odd integer and K is the smallest integer such that $2^K \geq M^2$, then there is an ENTIF with N elements in $\mathcal{H}_{M^2}$ for all $N \geq (M^2 - 1)(2^K - 1)$.

Proof. The matrix $A = M \cdot I_{M^2 \times M^2}$ is an ENTIF with vectors having square norms M^2. Furthermore, an $M^2 \times 2^K$ frame matrix B which represents an ENTIF may be obtained by a $2^K \times 2^K$ Hadamard matrix (see Section 3.2) where the square norms of the columns of B are also M^2. Since $\gcd(M^2, 2^K) = 1$, Theorem 3.5.3 gives the desired result. □

Corollary 3.5.5. If P is an odd integer and $M = 2P$, and $K \geq 2$ is the smallest integer such that $2^K \geq M^2$, then there is an ENTIF with $4N$ elements in $\mathcal{H}_{M^2}$ dimensions for all $N \geq (P^2 - 1)(2^{K-2} - 1)$.

Proof. Choose A and B in exactly the same way as in the proof of Corollary 3.5.4. Since $\gcd(M^2, 2^K) = 4$, Theorem 3.5.3 gives the result. □

The next corollary is particularly interesting because it eliminates the necessity of knowing each Hadamard matrix before being able to construct certain ENTIF. It proves that if we have knowledge of two consecutive Hadamard matrices then we know when certain ENTIFs exist.

Corollary 3.5.6. If both $4N \times 4N$ and $4(N+1) \times 4(N+1)$ Hadamard matrices exist for $4N \geq M$, then for all $K \geq N(N-1)$ there is a $4K$-element ENTIF in $\mathcal{H}_M$.

Proof. Since $\gcd(4N, 4(N+1)) = 4$, Theorem 3.5.3 implies a $4K$-element ENTIF in $\mathcal{H}_M$ exists for $K \geq (4N/4 - 1)(4(N+1)/4 - 1) = N(N-1)$. □

The next example demonstrates the usefulness of Corollary 3.5.6.

Example 3.5.7. Since 8×8 and 12×12 Hadamard matrices exist, there are $4K$ element ENTIFs in $M \leq 8$ dimensions for all $K \geq 2$. Since only 13 Hadamard matrices are left to be shown to exist for all $4N \leq 2000$ (see Section 3.2), Corollary 3.5.6 gives a vast amount of ENTIFs in a large number of dimensions.

Next, we prove that there exists an ENTIF in $\mathcal{H}_5$ with an even number of elements for almost every positive even integer.

Corollary 3.5.8. For every $N \geq 12$, there is a $2N$ element ENTIF in $\mathcal{H}_5$.

Proof. Let a be any nonzero integer and let $b = 2a$. Then the 5×8 matrix

$$A = \begin{bmatrix} a & a & a & a & a & a & a & a \\ b & -b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & -b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b & -b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b & -b \end{bmatrix}$$

and the 5×10 matrix

$$B = \begin{bmatrix} a & -b & 0 & 0 & 0 & 0 & 0 & 0 & a & -b \\ b & a & a & -b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & a & a & -b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b & a & a & -b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b & a & b & a \end{bmatrix}$$

represent ENTIFs having the same equal norm squared, $a^2 + b^2$, so that Theorem 3.5.3 gives a $2N$-element ENTIF in $\mathcal{H}_5$ for all $N \geq 12$. □

Remark 3.5.9. By Theorem 3.4.2, a six element ENTIF does not exist in five dimensions. Due to Section 3.2 and Theorem 3.2.6, since the 8×8, 12×12, 16×16 and 20×20 Hadamard matrices exists, then there exists ENTIFs in $\mathcal{H}_5$ with 8, 12, 16 and 20 elements. Also, adjoining two copies of the 5×5 identity matrix, as in Proposition 3.2.2, yields a 10-element ENTIF in $\mathcal{H}_5$. Lastly, Corollary 3.5.8 proves the existence of ENTIFs with an even number of vectors, $2N$, in $\mathcal{H}_5$ for all $2N \geq 24$. Therefore, the only even element ENTIFs in $\mathcal{H}_5$ for which the existence is unknown are those with $N = 14, 18$ and 22 elements.

The last theorem of this section gives the existence of $4N^2$ and $8N^2$ element ENTIFs, from which Theorem 3.5.3 can be applied to obtain even more ENTIFs as illustrated in the subsequent example.

Theorem 3.5.10. If N is a positive integer, then there exists an ENTIF with

1. $4N^2$ vectors in $N^2 + 1$ dimensions

2. $4N^2$ vectors in $2N^2 + 1$ dimensions

3. $4N^2$ vectors in $3N^2 + 1$ dimensions

4. $8N^2$ vectors in $4N^2 + 1$ dimensions

5. $8N^2$ vectors in $4N^2 + 2$ dimensions

Proof. For (1)-(3), let b be a nonzero integer and $a = Nb$.

(1) For each $1 \leq j \leq N^2$, let B_j be the $(N^2+1) \times 4$ matrix having $[b, b, b, -b]$ as its first row, $[a, a, -a, a]$ as its $(j + 1)$ row, and all other rows having zero entries. If A is the $(N^2 + 1) \times 4N^2$ matrix given by $A = [B_1, \dots, B_{N^2}]$, then the choice of a and b ensure that A is the desired ENTIF. Note that the equal norm squared is $(N^2 + 1)b^2$.

(2) For each $1 \leq j \leq N^2$, let B_j be the $(2N^2 + 1) \times 4$ matrix having $[b, b, b, -b]$ as its first row, $[a, a, -a, a]$ as its $2j$ row, $[a, -a, a, a]$ as its $(2j + 1)$ row, and all other rows having zero entries. If A is the $(2N^2 + 1) \times 4N^2$ matrix given by $A = [B_1, \dots, B_{N^2}]$, then the choice of a and b ensures that A is the desired ENTIF. Note that the equal norm squared is $(2N^2 + 1)b^2$.

(3) For each $1 \leq j \leq N^2$, let B_j be the $(3n^2+1) \times 4$ matrix having $[b, b, b, -b]$ as its first row, $[a, a, -a, a]$ as its $(3j-1)$ row, $[a, -a, a, a]$ as its $3j$ row, $[-a, a, a, a]$ as its $(3j+1)$ row, and all other rows having zero entries. If A is the $(3N^2+1) \times 4N^2$ matrix given by $A = [B_1, \dots, B_{N^2}]$, then the choice of a and b ensures that A is the desired ENTIF. Note that the equal norm squared is $(3N^2 + 1)b^2$.

For (4) and (5), let b be a nonzero integer and $a = 2Nb$.

(4) For each $1 \leq j \leq N^2$, let B_j be the $(4N^2+1)\times 8$ matrix having $[b, b, b, b, b, b, b, b]$ as its first row, $[a, -a, 0, 0, 0, 0, 0, 0]$ as its $4j-2$ row, $[0, 0, a, -a, 0, 0, 0, 0]$ as its $4j-1$ row, $[0, 0, 0, 0, a, -a, 0, 0]$ as its $4j$ row, $[0, 0, 0, 0, 0, 0, a, -a]$ as its $4j+1$ row, and zero entries in all other rows. If A is the $(4N^2+1)\times 8N^2$ matrix given by $A = [B_1, \ldots, B_{N^2}]$, then the choice of a and b ensures that A is the desired ENTIF. Note that the equal norm squared is $(4N^2+1)b^2$.

(5) For each $1 \leq j \leq 2N^2$, let B_j be the $(4N^2+2)\times 4$ matrix having $[b, b, b, b]$ as its first row, $[b, -b, b, -b]$ as its second row, $[a, 0, -a, 0]$ as its $2j+1$ row, $[0, a, 0, -a]$ as its $2j+2$ row, and zero entries in all other rows. If A is the $(4N^2+2)\times 8N^2$ matrix given by $A = [B_1, \ldots, B_{2N^2}]$, then the choice of a and b ensures that A is the desired ENTIF. Note that the equal norm squared is $2b^2(1+2N^2)$. □

Example 3.5.11. Applying Theorem 3.5.10 proves that there exist ENTIFs with:

1. 4 vectors in $\mathcal{H}_2$, 16 in $\mathcal{H}_5$, 36 in $\mathcal{H}_{10}$, 64 in $\mathcal{H}_{17}$, ...
2. 4 vectors in $\mathcal{H}_3$, 16 in $\mathcal{H}_9$, 36 in $\mathcal{H}_{19}$, 64 in $\mathcal{H}_{33}$, ...
3. 4 vectors in $\mathcal{H}_4$, 16 in $\mathcal{H}_{13}$, 36 in $\mathcal{H}_{28}$, 64 in $\mathcal{H}_{49}$, ...
4. 8 vectors in $\mathcal{H}_5$, 32 in $\mathcal{H}_{17}$, 72 in $\mathcal{H}_{37}$, 128 in $\mathcal{H}_{65}$, ...
5. 8 vectors in $\mathcal{H}_6$, 32 in $\mathcal{H}_{18}$, 72 in $\mathcal{H}_{38}$, 128 in $\mathcal{H}_{66}$, ...

Furthermore, these ENTIFs can be adjoined to obtain multiplies of the given number of vectors.

Remark 3.5.12. Using Theorem 3.5.3, one can construct an $(N^2+1)\times 2(N^2+1)$ matrix in a similar fashion as matrix B in the proof of Corollary 3.5.8 and adjoin it to the matrix in Theorem 3.5.10(1) to obtain an ENTIF in N^2+1 dimensions with $4K$ elements for all $K \geq N^2(N^2-1)$. One can also do the same in $4N^2+1$ dimensions to obtain a $2K$ element ENTIF with $2K$ elements for all $K \geq 4N^2(4N^2-1)$.

3.6 Removing the equal norm or tightness assumption

Thus far we have only considered ENTIFs and, as we have seen, they can be quite difficult to construct. So in this section, we address the question of what can be obtained if one of the assumptions that the frame is equal norm or tight is removed. In either case, it will be shown that an integer frame of any size greater than the dimension can be obtained.

Theorem 3.6.1. If M and N are positive integers satisfying $N \geq M$, then there is an equal norm integer frame with N elements in $\mathcal{H}_M$.

Proof. For all $1 \leq i \leq M$, let A_i be the $M \times i$ matrix formed by the first i columns of the identity matrix $I_{M\times M}$. Write $N = cM + k$ for some integers $c \geq 0$ and $0 \leq k < M$. If $k = 0$, then the block matrix $C = [A_M \cdots A_M]$ where A_M is repeated c times is an equal norm (tight) integer frame with N elements. If $k > 0$, then the block matrix $C = [A_M \cdots A_M \; A_k]$ where A_M is repeated c times is a desired equal norm integer frame. □

Before proving that tight integer frames exist with any number of elements in any dimension, the following number theoretic result is needed.

Lemma 3.6.2. For every positive integer k, there exists a nonzero integer s such that s^2 can be written as a sum of i nonzero squares for all $1 \leq i \leq k$.

Proof. First recall the well-known Euclid's formula, which states that if m and n

are positive integers with $m > n$, then

$$a = m^2 - n^2, \quad b = 2mn, \quad c = m^2 + n^2$$

forms a Pythagorean triple, i.e., $a^2 + b^2 = c^2$. Suppose that m_0 and n_0 are odd integers with $m_0 > n_0$ and let (a_0, b_0, c_0) be the Pythagorean triple formed by m_0 and n_0 as given by Euclid's Formula. Since m_0 and n_0 are both odd, $c_0 = 2 \cdot m_1$ for some odd integer m_1. Letting $n_1 = 1$ gives another Pythagorean triple (a_1, b_1, c_1) generated by m_1 and n_1 in which $c_1^2 = a_1^2 + b_1^2$ and $b_1 = 2m_1 \cdot n_1 = c_0$. Thus

$$c_1^2 = a_1^2 + b_1^2 = a_1^2 + c_0^2 = a_1^2 + a_0^2 + b_0^2.$$

This process may be continued to find a number c_{i-2}, such that c_{i-2}^2 is the sum of $3 \leq i \leq k$ squares. This follows because in each step c_{i-3} is always of the form $2m_{i-2}$ for some odd integer m_{i-2} and so $b_{i-2} = c_{i-3}$ with $n_{i-2} = 1$. □

Theorem 3.6.3. If M and N are positive integers satisfying $N \geq M$, then there is a tight integer frame with N elements in $\mathcal{H}_M$.

Proof. Case 1: If M is even, let $k = (M-2)/2$. Let p be a nonzero integer such that p^2 can be written as a sum of i nonzero squares for all $1 \leq i \leq N - 2k - 1$, which exists by Lemma 3.6.2. Write

$$p^2 = a^2 + b^2 = a_1^2 + \cdots + a_{N-2k-1}^2$$

for some $a, b, a_i \in \mathbb{Z}$ and define

$$A_i = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$

for all $1 \leq i \leq k$. Define A to be the $M \times N$ matrix given by

$$A = \begin{bmatrix} A_1 & & & & & & \\ & \ddots & & & & & \\ & & A_k & & & & \\ & & & a_1 & \cdots & a_{N-2k-1} & \\ & & & & & & p \end{bmatrix}$$

where all of the empty entries in A are 0. Then A is a tight integer frame with N elements in M dimensions and tightness factor p.

Case 2: If M is odd, let $k = (M-1)/2$ and let p be a nonzero integer such that p^2 can be written as a sum of i nonzero squares for all $1 \leq i \leq N-2k$. Write

$$p^2 = a^2 + b^2 = a_1^2 + \cdots + a_{N-2k}^2$$

for some $a, b, a_i \in \mathbb{Z}$ and define A_i as above for all $1 \leq i \leq k$. Define A to be the $M \times N$ matrix given by

$$A = \begin{bmatrix} A_1 & & & & & \\ & \ddots & & & & \\ & & A_k & & & \\ & & & a_1 & \cdots & a_{N-2k} \end{bmatrix}$$

where all of the empty entries in A are 0. Then A is a tight integer frame with N elements in M dimensions and tightness factor p. □

Throughout this chapter, we have made numerous theorems and classifications for when ENTIFs, equal norm integer frames, and tight integer frames exist. This next result looks at frames which are almost ENTIFs and implies that there exists an equal norm frame in $\mathcal{H}_M$ with N elements for $N \geq M$ that is arbitrarily close to being tight.

First, the formal definition of a frame being arbitrarily close to tight is given and then the result is stated and proved.

Definition 3.6.4. A frame $\{\varphi_i\}_{i=1}^N$ is said to be (ε, A)**-tight** if there are constants $0 < \varepsilon < 1$ and $A > 0$ such that the lower and upper frame bounds are $(1-\varepsilon)A$ and $(1+\varepsilon)A$, respectively.

Theorem 3.6.5. Let M and N be positive integers such that $N \geq M$. For any $\varepsilon > 0$ and any orthonormal basis $\beta = \{e_i\}_{i=1}^M$ for $\mathcal{H}_M$, there exists a full spark, equal norm, integer frame $F = \{\varphi_i\}_{i=1}^N$ with respect to β for which F is $(\varepsilon, N/M)$-tight.

Proof. It is enough to show the existence of such a frame with rational coordinates. Begin by first picking a unit norm tight frame $\Psi = \{\psi_i\}_{i=1}^N$ in $\mathcal{H}_M$. Recall that the tight frame bound for Ψ is N/M. Note that Ψ may not have rational coordinates.

Let $0 < \varepsilon < 1$ be given and momentarily fix a $0 < \delta < 1$, which will be chosen later. Since vectors with rational coordinates are dense in S^{M-1}, the unit sphere in $\mathbb{R}^M$, vectors $F_1 = \{\varphi_i\}_{i=1}^N$ can be chosen to be linearly independent and satisfy

$$\|\varphi_i - \psi_i\| \leq \delta \cdot \sqrt{N}/M$$

for all $1 \leq i \leq M$.

Now let $\mathbb{H}_1$ be the collection of all hyperplanes in $\mathbb{R}^M$ generated by sets of $M-1$ vectors chosen from F_1. If

$$C_1 = \left(\bigcup_{H \in \mathbb{H}_1} H \right)^c,$$

then C_1 is also dense in S^{M-1} and so we can choose f_{M+1} in $S^{M-1} \cap C_1$ with rational coordinates so that

$$\|f_{M+1} - \psi_{M+1}\| \leq \delta \cdot \sqrt{N}/M.$$

Notice that by construction the set $F_2 = \{\varphi_i\}_{i=1}^{M+1}$ is full spark. Now choose the set of all hyperplanes $\mathbb{H}_2$ generated by F_2 and continue the same process until a frame $F = \{\varphi_i\}_{i=1}^{N}$, having all rational coordinate vectors, is obtained.

Now to prove that F is $(\epsilon, N/M)$-tight. The Cauchy-Schwarz inequality gives for any $x \in \mathbb{R}^M$,

$$\begin{aligned}
\left(\sum_{i=1}^{N} |\langle x, \varphi_i\rangle|^2\right)^{\frac{1}{2}} &\leq \left(\sum_{i=1}^{N} |\langle x, \psi_i\rangle|^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{N} |\langle x, \varphi_i - \psi_i\rangle|^2\right)^{\frac{1}{2}} \\
&= \sqrt{\frac{N}{M}}\|x\| + \left(\sum_{i=1}^{N} \|x\|^2 \|f - \psi_i\|^2\right)^{\frac{1}{2}} \\
&\leq \|x\| \left[\sqrt{\frac{N}{M}} + \left(\sum_{i=1}^{N} \frac{\delta^2 \cdot N}{M^2}\right)^{\frac{1}{2}}\right] \\
&= \|x\|(1+\delta)\sqrt{\frac{N}{M}}
\end{aligned}$$

proving that an upper frame bound for F is $(1+\delta)^2 N/M$. Similarly, a lower frame bound for F is $(1-\delta)^2 N/M$. Now choose δ so that

$$(1-\varepsilon) \cdot \frac{N}{M} \leq (1-\delta)^2 \cdot \frac{N}{M} \leq (1+\delta)^2 \cdot \frac{N}{M} \leq (1+\varepsilon) \cdot \frac{N}{M},$$

proving that F is $(\varepsilon, N/M)$-tight. □

Remark 3.6.6. The frame F constructed in Theorem 3.6.5 is not necessarily represented against the eigenbasis of its frame operator as in all previous cases in this chapter.

The proof of Theorem 3.6.5 relies heavily on the fact that the set of all rational coordinate points are dense in S^{M-1}. Unfortunately, the higher the dimension and closer the frame is to being tight forces the need to choose numbers in which the denominators are possibly massive. That is, using the proof technique above might lead to computationally inconvenient integer frames after clearing out the denominators. See [49] for more details concerning rational coordinate points on the sphere.

It is also worth noting that the technique used to prove Theorem 3.6.5 is a standard argument which shows that full spark, equal norm frames are *dense* in the space of all equal norm frames [23, Ch. 4].

Thus far, we have seen numerous results and characterizations regarding the construction and existence of ENTIFs, equal norm integer frames and tight integer frames as well as the distribution of frame coefficients as discussed in Chapter 2. Classifying frames and characteristics of different types of frames is essential because frames are used in a vast assortment of applied research problems. In particular, frames are used in an area known as signal reconstruction, which we will look at in the following chapter.

Chapter 4

Phase Retrieval

4.1 Introduction to phase retrieval

Signal reconstruction has been a longstanding topic of research in engineering and has applications to a wide array of problems. However, when a signal is received, usually there is a loss of information making the reconstruction of the desired signal a challenging task.

Traditionally and as described in [5], signal reconstruction consisted of three steps: first, the input signal is linearly transformed from its input domain (e.g. time, or space) into a transformed domain (e.g. time-frequency, time-scale, space-scale etc.); second, a (nonlinear) estimation operator is applied in this representation domain; third, a (left) inverse of the linear transformation at step one is applied to the signal obtained at step two in order to synthesize the estimated signal in the input domain.

Some linear transformations which have been used for signal reconstruction are the windowed Fourier transform, wavelet filterbanks, and local cosine basis as seen in [5, 44, 45, 53]. Likewise, many signal estimators have been proposed and studied in the literature, some of them statistically motivated, e.g. Wiener (MMSE) filter, Maximum A Posteriori (MAP), Maximum Likelihood (ML) etc., others having a rather ad-hoc motivation, e.g. spectral subtraction, psychoacoustically motivated audio and video estimators etc., as described in [5].

For years, knowledge of the phase of a signal, or an estimation thereof, was seen to be a necessary component when reconstructing a signal. However, often times, as a signal is passed through a linear system, the phase of the signal is lost and only the absolute values of linear measurement coefficients, called intensity measurements, are known. Many researchers believed that with enough information and redundancy, signal reconstruction should be possible without the use of phase. The process of reconstructing a signal from intensity measurements without the use of phase is known as *phase retrieval* and is currently a popular topic of research. Note that multiplying a signal by a global phase factor does not affect these measurements, so we seek signal recovery modulo a global phase factor.

Phase retrieval is a challenging problem and has a wide range of applications in numerous fields. As such it has been studied by Engineers, Mathematicians and Physicists alike. Important applications of phase retrieval occur in optics with applications to X-ray crystallography, electron microscopy and coherence theory [35, 37, 38, 40, 41, 42, 48, 54]. There are also applications in the areas of diffractive imaging [11, 13, 14], astronomical imagining [29, 36], X-ray tomography [30], and speech recognition technology [7, 34, 45, 46, 47, 53], just to name a few.

Moreover, this problem of phase retrieval is very similar to a problem in quantum theory known as state tomography. As described in [9], a pure quantum state is given by a rank-one projection on a finite-dimensional Hilbert space, or equivalently, by the vectors in the range of this projection. A state is experimentally accessible only through the magnitudes of its Hilbert-Schmidt inner products with other states. These inner products of projections can be interpreted as the squared magnitudes of the inner products of corresponding normalized vectors in the respective range of the projections. Thus, reconstructing a pure quantum state is the same as finding a vector, up to a unimodular constant, from the magnitudes of linear transform coefficients.

There are two main approaches to this problem of phase retrieval. One is to restrict the problem to a subclass of signals on which the intensity measurements become injective. The other is to use a larger family of measurements so that the intensity measurements map any signal injectively. The latter approach in phase retrieval first appears in [5] where the authors examine injectivity of intensity measurements for finite Hilbert spaces. The authors completely characterize measurement vectors in the real case which yield such injectivity, and they provide a surprisingly small upper bound on the minimal number of measurements required for the complex case. This sparked an incredible volume of current phase retrieval research [1, 4, 6, 15, 16, 17, 19, 20, 21, 28, 33] focused on algorithms and conditions guaranteeing injective and stable intensity measurements.

Given a signal x in a Hilbert space, intensity measurements may also be thought of as norms of x under rank one projections. Here the spans of measurement vectors serve as the one dimensional range of the projections. However in some scenarios, such as crystal twinning [31], the signal is projected onto higher dimensional subspaces and one has to recover this signal from the norms of these projections. In this scenario, there exists a similar phase retrieval problem: given subspaces $\{W_i\}_{i=1}^N$ of an M-dimensional Hilbert space $\mathcal{H}_M$ and orthogonal projections $P_i : \mathcal{H}_M \to W_i$, can we recover any $x \in \mathcal{H}_M$ (up to a global phase factor) from the measurements $\{\|P_i x\|\}_{i=1}^N$?

This problem was studied in [3] where the authors use semidefinite programming to develop a reconstruction algorithm for when $\{W_i\}_{i=1}^N$ are real equidimensional random subspaces. Most results using random intensity measurements require the cardinality of measurements to scale linearly with the dimension of the signal space along with an additional logarithmic factor [21]; but this logarithmic factor was recently removed in [20]. Similarly, signal reconstruction from the norms of equidimensional random subspace components is possible with the cardinality of measurements scaling linearly with the dimension [3].

In addition to these results concerning the phase retrieval problem using subspaces, and much like [5], we seek to better characterize the subspaces $\{W_i\}_{i=1}^N$ of $\mathcal{H}_M$ for which the measurements $\{\|P_i x\|\}_{i=1}^N$ are injective for all $x \in \mathcal{H}_M$.

To set notation, define the equivalence relation $\sim$ between two vectors $x, y \in \mathcal{H}_M$ to be such that $x \sim y$ if and only if there exists a unimodular constant c such that $x = cy$. Note that $c = \pm 1$ in $\mathbb{R}^M$ and for the complex case $\mathbb{C}^M$, $c \in \mathbb{T}^1$, where $\mathbb{T}^1$ is the complex unit circle. Next, given subspaces $\{W_i\}_{i=1}^N$ of $\mathcal{H}_M$ with orthogonal projections $P_i : \mathcal{H}_M \to W_i$, we consider the measurements $\mathcal{A} : \mathcal{H}_M / \sim \, \to \mathbb{R}^N$ given by

$$\mathcal{A}(x)(i) := \|P_i x\|^2. \tag{4.1.1}$$

The phrase "$\{W_i\}_{i=1}^N$ *allows phase retrieval*" will be synonymous with $\mathcal{A}$ being injective. For the well studied case of $\dim W_i = 1$ for all i, whether or not $\mathcal{A}$ is injective shall be referred to as the *classical phase retrieval* problem. Stated more explicitly,

Definition 4.1.1.

1. (Classical phase retrieval) A set of vectors $\{\varphi_i\}_{i=1}^N$ in $\mathbb{R}^M$ (or $\mathbb{C}^M$) allows

phase retrieval if for all $x, y \in \mathbb{R}^M$ (or $\mathbb{C}^M$) satisfying $|\langle x, \varphi_i \rangle| = |\langle y, \varphi_i \rangle|$ for all $i = 1, \ldots, N$, then $x = cy$ where $c = \pm 1$ in $\mathbb{R}^M$ (and for $\mathbb{C}^M$, $c \in \mathbb{T}^1$ where $\mathbb{T}^1$ is the complex unit circle).

2. (Phase retrieval by subspace components) Let $\{W_i\}_{i=1}^N$ be a collection of subspaces of $\mathcal{H}_M$ and let $\{P_i\}_{i=1}^N$ be the orthogonal projections onto these subspaces. We say that $\{W_i\}_{i=1}^N$ (or $\{P_i\}_{i=1}^N$) allows **phase retrieval** if for $x, y \in \mathcal{H}_M$ the equality $\|P_i x\| = \|P_i y\|$ for all $i = 1, \ldots, N$ implies $x = cy$ for some scalar c with $|c| = 1$.

Given a sequence of vectors $\{\varphi_i\}_{i=1}^N$ and an orthogonal projection P, we will frequently refer to $\{P\varphi_i\}_{i=1}^N$ as possessing certain properties, such as: full spark, complement property, phase retrieval, and so on. By this we mean that it has these properties in the range of P.

Much of the work in this chapter can be found in the papers *Phase retrieval by projections* [16] and *Phase retrieval and norm retrieval* [15]. For an up to date review of phase retrieval by vectors and projections please see [17, 25].

4.2 Phase retrieval with subspace components

In this section we provide several characterizations of subspaces $\{W_i\}_{i=1}^N$ which allow phase retrieval. We first discuss classical phase retrieval, connecting it to phase retrieval with subspace components. Next, we analyze subspaces and their Naimark complements which leads to us answering the following problem: If a Parseval frame allows phase retrieval, when does its Naimark complement allow phase retrieval? This culminates to a main result which provides an upper bound on the minimal number of subspaces required for phase retrieval (in $\mathbb{R}^M$ and

$\mathbb{C}^M$) by concretely constructing subspaces which allow phase retrieval. After this, we provide numerous classification results regarding subspaces which allow phase retrieval.

4.2.1 Phase retrieval by vectors and Naimark complements

This section is devoted to further analyzing classical phase retrieval and in particular classifies when a Parseval frame and its Naimark complement allow phase retrieval. We start by recalling the characterization of classical phase retrieval in the real case.

Theorem 4.2.1. [5] A frame $\{\varphi_i\}_{i=1}^N$ in $\mathbb{R}^M$ allows phase retrieval if and only if it has the complement property. In particular, a full spark frame with $2M-1$ vectors allows phase retrieval. Moreover, if $\{\varphi_i\}_{i=1}^N$ allows phase retrieval in $\mathbb{R}^M$, then $N \geq 2M-1$ and no set of $2M-2$ vectors allows phase retrieval.

Notice that if a family of vectors $\Phi = \{\varphi_i\}_{i=1}^N$ in $\mathbb{R}^M$ is full spark and $N \geq 2M-1$ then Φ necessarily has the complement property.

In general, it is not necessary for a frame to be full spark in order to allow phase retrieval. For example, as long as the frame contains a full spark subset of $2M-1$ vectors in $\mathbb{R}^M$, it will allow phase retrieval. However, if the frame contains exactly $2M-1$ vectors, then clearly it allows phase retrieval if and only if it is full spark.

We would like for the complement property or a similar property to be a necessary and sufficient condition for phase retrieval with subspaces. However, as we will see, this is not the case.

Proposition 4.2.2. If $\{W_i\}_{i=1}^N$ allows phase retrieval in $\mathcal{H}_M$, then it has the complement property.

Proof. Assume $\{W_i\}_{i=1}^N$ allows phase retrieval in $\mathcal{H}_M$ but fails to have the complement property. We will prove a contradiction exists. Since $\{W_i\}_{i=1}^N$ fails the complement property then there exists an $I \subset \{1, \ldots, N\}$ such that $\text{span}\{W_i\}_{i\in I} \neq \mathcal{H}_M$ and $\text{span}\{W_i\}_{i\in I^c} \neq \mathcal{H}_M$.

Thus there exists an $x, y \in \mathcal{H}_M$ such that $x \perp \text{span}\{W_i\}_{i\in I}$ and $y \perp \text{span}\{W_i\}_{i\in I}$. Note that since $\{W_i\}_{i=1}^N$ spans $\mathcal{H}_M$ then $x \neq cy$ for any scalar c.

Hence for all $i \in I$, we have

$$\|P_i(x+y)\| = \|P_iy\| = \|(-1)\|\|P_iy\| = \|P_i(x-y)\|.$$

Also, for all $i \in I^c$, we have

$$\|P_i(x+y)\| = \|P_ix\| = \|P_i(x-y)\|.$$

Therefore $\|P_i(x+y)\| = \|P_i(x-y)\|$ for all $i \in \{1, \ldots, N\}$. However, since $x \neq cy$ for any scalar c then $x+y \neq c(x-y)$ for any unimodular constant c, which contradicts the assumption that $\{W_i\}_{i=1}^N$ allows phase retrieval. □

Remark 4.2.3. The converse of Proposition 4.2.2 fails.

The following example illustrates Remark 4.2.3 nicely by providing a collection of subspaces which possess the complement property but fail phase retrieval.

Example 4.2.4. Let $\{\varphi_i\}_{i=1}^4$ be a set of full spark vectors in $\mathbb{R}^3$. Consider the subspaces

$$\begin{aligned} W_1 &= \text{span}\{\varphi_1, \varphi_2\} \\ W_2 &= \text{span}\{\varphi_1, \varphi_3\} \\ W_3 &= \text{span}\{\varphi_4\} \end{aligned}$$

By Example 4.3.7, we know that it takes $2M-1=5$ subspaces in $\mathbb{R}^3$ to do phase retrieval, hence $\{W_i\}_{i=1}^3$ fail phase retrieval. Suppose we partition these subspaces into two sets:

Case 1: If $J \supset \{1,2\}$, then $\text{span}\{W_i\}_{i\in J} = \mathbb{R}^3$.

Case 2: Suppose $1 \in J$ and $2 \in J^c$. If $3 \in J$, then $\text{span}\{W_i\}_{i\in J} = \mathbb{R}^3$. If $3 \in J^c$, then $\text{span}\{W_i\}_{i\in J^c} = \mathbb{R}^3$. Hence $\{W_i\}_{i=1}^3$ possess the complement property.

Since the complement property characterizes classical phase retrieval but is not sufficient in the subspace case, we need to further analyze various properties a frame could possess in order to find a classifying property for the higher dimensional case. We will now present results regarding frames which have the complement property, full spark property, and/or other similar properties. These results stand alone but they also help in analyzing further phase retrieval and norm retrieval results.

Next, we will compare phase retrieval for Parseval frames with their Naimark complements. For this we need a result from [10]:

Theorem 4.2.5. [10] Let P be a projection on $\mathcal{H}_N$ with orthonormal basis $\{e_i\}_{i=1}^N$ and let $\mathcal{I} \subset \{1,2,\ldots,N\}$. The following are equivalent:

(1) $\{Pe_i\}_{i\in\mathcal{I}}$ is linearly independent.

(2) $\{(I-P)e_i\}_{i\in\mathcal{I}^c}$ spans $(I-P)\mathcal{H}_N$.

First, we need to see that the full spark property passes from a frame to its Naimark complement.

Proposition 4.2.6. A Parseval frame is full spark if and only if its Naimark complement is full spark.

Proof. By Theorem 1.1.7 any Parseval frame can be written as $\{Pe_i\}_{i=1}^N$ where $\{e_i\}_{i=1}^N$ is an orthonormal basis for $\ell_2(N) \simeq \mathcal{H}_N$ and P is an orthogonal projection.

Furthermore, the Naimark complement of this Parseval frame is $\{(I-P)e_i\}_{i=1}^N$. We also have that $\{Pe_i\}_{i=1}^N$ is full spark if and only if for any subset $\mathcal{I} \subseteq \{1, ..., N\}$ such that $|\mathcal{I}| = M$, $\{Pe_i\}_{i\in\mathcal{I}}$ is linearly independent and spanning (in the image of P). So under this assumption, Theorem 4.2.5 implies that $\{(I-P)e_i\}_{i\in\mathcal{I}}$ is also linearly independent and spanning (in the image of $I-P$), hence $\{(I-P)e_i\}_{i=1}^N$ is also full spark. The other direction follows from the same argument by reversing the roles of P and $I-P$. □

In general, if a Parseval frame allows phase retrieval, its Naimark complement may not allow phase retrieval. This follows from the fact that there may not be enough vectors in the Naimark complement to satisfy the complement property, as shown in the next proposition.

Proposition 4.2.7. Let $\{\varphi_i\}_{i=1}^N$ be a Parseval frame for $\mathbb{R}^M$ which allows phase retrieval and such that its Naimark complement $\{\psi_i\}_{i=1}^N$ allows phase retrieval for $\mathbb{R}^{N-M}$. Then $2M-1 \leq N \leq 2M+1$.

Proof. Since $\{\varphi_i\}_{i=1}^N$ allows phase retrieval in $\mathbb{R}^M$ then by Theorem 4.2.1, $2M-1 \leq N$. Also since the Naimark complement $\{\psi_i\}_{i=1}^N$ allows phase retrieval in $\mathbb{R}^{N-M}$ then again by Theorem 4.2.1, $2(N-M)-1 \leq N$. That is, $N \leq 2M+1$. □

Unfortunately, even if we restrict the number of vectors in a Parseval frame to $2M-1 \leq N \leq 2M+1$, its Naimark complement still might not allow phase retrieval.

Example 4.2.8. Let $\{\varphi_i\}_{i=2}^{2M}$ be a set of full spark vectors in $\mathbb{R}^M$ with $M \geq 3$. Define $\varphi_1 = \varphi_2$ and let S be the frame operator for $\{\varphi_i\}_{i=1}^{2M}$. Note that $\{S^{-\frac{1}{2}}\varphi_i\}_{i=2}^{2M}$ is still a full spark set of vectors. Therefore $\{S^{-\frac{1}{2}}\varphi_i\}_{i=1}^{2M}$ allows phase retrieval. That is, for any partition $\mathcal{I}, \mathcal{I}^c \subset \{1, \ldots, 2M\}$, either $\mathcal{I}$ or $\mathcal{I}^c$ has at least M elements from the full spark family $\{S^{-\frac{1}{2}}\varphi_i\}_{i=2}^{2M}$ and hence spans $\mathbb{R}^M$.

Now we will see that the Naimark complement of $\{S^{-\frac{1}{2}}\varphi_i\}_{i=1}^{2M}$ fails phase retrieval. Partition $\{S^{-\frac{1}{2}}\varphi_i\}_{i=1}^{2M}$ into $\{S^{-\frac{1}{2}}\varphi_i\}_{i=1}^{2}$ and $\{S^{-\frac{1}{2}}\varphi_i\}_{i=3}^{2M}$. Observe that neither set is linearly independent since $\varphi_1 = \varphi_2$ and $M \geq 3$. By Theorem 4.2.5, the Naimark complement of each set does not span $\mathbb{R}^{2M-M} = \mathbb{R}^M$. Hence this is a partition of the Naimark complement of $\{S^{-\frac{1}{2}}\varphi_i\}_{i=1}^{2M}$ which fails complement property and therefore fails phase retrieval.

With the aid of full spark, we are able to pass phase retrieval to Naimark complements as long as we satisfy the restriction on the number of vectors.

Proposition 4.2.9. If $\Phi = \{\varphi_i\}_{i=1}^{N}$ is a full spark Parseval frame in $\mathbb{R}^M$ and $2M-1 \leq N \leq 2M+1$ then Φ allows phase retrieval in $\mathbb{R}^M$ and the Naimark complement of Φ allows phase retrieval in $\mathbb{R}^{N-M}$.

Proof. By Proposition 4.2.6, the Naimark complement of Φ is full spark in $\mathbb{R}^{N-M}$. Since $2M-1 \leq N$ and $2(N-M)-1 \leq N$, by Theorem 4.2.1 both Φ and its Naimark complement have the complement property in their respective spaces. □

Seeing that the full spark property is a necessity for classical phase retrieval with regards to Naimark complements, if we hope to extend this idea to phase retrieval by subspace components then a natural question to ask is when is a collection of vectors in the range of a projection full spark? The next few propositions help to clarify this.

Proposition 4.2.10. Let P be a projection of rank N on $\mathcal{H}_M$ with orthonormal basis $\{e_i\}_{i=1}^{M}$. The following are equivalent:

(1) $\{Pe_i\}_{i=1}^{M}$ is full spark.

(2) For all $I \subset \{1, 2, \ldots, M\}$ with $|I| = N$ we have:

$$\left[\text{span}\{(I-P)e_i\}_{i=1}^M\right] \cap \text{span}\{e_i\}_{i\in I} = \{0\}.$$

Proof. $\{Pe_i\}_{i=1}^M$ is not full spark if and only if there exists an $I \subset \{1, 2, \ldots, M\}$ with $|I| = N$ and $\{a_i\}_{i\in I}$ not all zero such that $\sum_{i\in I} a_i Pe_i = 0$ if and only if $\sum_{i\in I} a_i(I-P)e_i = \sum_{i\in I} a_i e_i$ if and only if $\left[span\{(I-P)e_i\}_{i=1}^M\right] \cap span\{e_i\}_{i\in I} \neq \{0\}$. □

Proposition 4.2.11. Let P be a projection of rank N on $\mathcal{H}_M$ with orthonormal basis $\{e_i\}_{i=1}^M$. The following are equivalent:

(1) $\{Pe_i\}_{i=1}^M$ is full spark.

(2) For every $I \subset \{1, 2, \ldots M\}$ with $|I| = M - N$ the vectors $\{(I-P)e_i\}_{i\in I}$ spans $(I-P)(\mathcal{H})$.

Proof. This follows immediately from Corollary 4.2.6. □

Instead of only applying projections to orthonormal bases, what if we slightly generalize these ideas and use Riesz bases?

Proposition 4.2.12. Let $\{\varphi_i\}_{i=1}^M$ be a Riesz basis with dual basis $\{\varphi_i^*\}_{i=1}^M$ for $\mathcal{H}_M$ and let P be an orthogonal projection on $\mathcal{H}_M$ of rank N. Let $I \subset \{1, \ldots, M\}$. The following are equivalent:

1. $\{P\varphi_i\}_{i\in I}$ spans $P\mathcal{H}_M$

2. $\{(I-P)\varphi_i^*\}_{i\in I^c}$ is independent.

Proof. (1) $\Rightarrow$ (2) (Proof by contrapositive.) Assume $\{(I-P)\varphi_i^*\}_{i\in I^c}$ is NOT independent. Choose $\{b_i\}_{i\in I^c}$ not all zero so that $\sum_{i\in I^c} b_i(I-P)\varphi_i^* = 0$. Then $x := \sum_{i\in I^c} b_i\varphi_i^* = \sum_{i\in I^c} b_i P\varphi_i^* \in P\mathcal{H}_M$. If $i_0 \in I$, then $\langle x, P\varphi_{i_0}\rangle = \langle Px, \varphi_{i_0}\rangle =$

$\langle x, \varphi_{i_0}\rangle = \sum_{i\in I^c} b_i\langle \varphi_i^*, \varphi_{i_0}\rangle = 0$ since $i \in I^c$ and $i_0 \in I$. So $x \perp \text{span}\{P\varphi_i\}_{i\in I}$ so $\{P\varphi_i\}_{i\in I}$ does not span $P\mathcal{H}_M$.

(2) $\Rightarrow$ (1) (Proof by contrapositive.) Assume $\text{span}\{P\varphi_i\}_{i\in I} \neq P\mathcal{H}_M$. Then there exists a non-zero $x \in P\mathcal{H}_M$ with $x \perp \text{span}\{P\varphi_i\}_{i\in I}$. Also $x = \sum_{i=1}^{M}\langle x, \varphi_i\rangle\varphi_i^*$. Now for $i \in I$, $0 = \langle x, P\varphi_i\rangle = \langle Px, \varphi_i\rangle = \langle x, \varphi_i\rangle$. Hence $\sum_{i\in I^c}\langle x, \varphi_i\rangle\varphi_i^* = x = Px = \sum_{i\in I^c}\langle x, \varphi_i\rangle P\varphi_i^*$. Thus, $\sum_{i\in I^c}\langle x, \varphi_i\rangle(I-P)\varphi_i^* = 0$. Therefore $\{(I - P)\varphi_i^*\}_{i\in I^c}$ is NOT independent. □

Proposition 4.2.13. Let $\{\varphi_i\}_{i=1}^M$ be a Riesz basis on $\mathcal{H}_M$ and P an orthogonal projection on $\mathcal{H}_M$ of rank N. The following are equivalent:

1. $\{P\varphi_i\}_{i=1}^M$ is full spark.

2. For all $I \subset \{1, 2, \ldots, M\}$ with $|I| = N$ we have:

$$\left[\text{span}\{(I-P)\varphi_i\}_{i=1}^M\right] \cap \text{span}\{\varphi_i\}_{i\in I} = \{0\}.$$

Proof. $\{P\varphi_i\}_{i=1}^M$ is not full spark if and only if there exists an $I \subset \{1, 2, \ldots, M\}$ with $|I| = N$ and $\{a_i\}_{i\in I}$ not all zero such that $\sum_{i\in I} a_i P\varphi_i = 0$ if and only if $\sum_{i\in I} a_i(I-P)\varphi_i = \sum_{i\in I} a_i\varphi_i$ if and only if $\left[\text{span}\{(I-P)\varphi_i\}_{i=1}^M\right] \cap \text{span}\{\varphi_i\}_{i\in I} \neq \{0\}$. □

A natural question to ask in light of Proposition 4.2.13 is if $\{\varphi_i\}_{i=1}^M$ is a Riesz basis for $\mathcal{H}_M$ and $\{P\varphi_i\}_{i=1}^M$ is full spark on its range for some rank-N projection P, then is $\{(I-P)\varphi_i\}_{i=1}^M$ full spark on its range? Unfortunately, the following example shows that the answer is no.

Example 4.2.14. Let $\{e_1, e_1 + e_2\}$ be a Riesz basis for $\mathbb{R}^2$, where $\{e_i\}_{i=1}^2$ is the standard orthonormal basis for $\mathbb{R}^2$. Let P be the rank-1 projection onto e_1. Then

$\{Pe_1, P(e_1+e_2)\} = \{e_1\}$ is full spark on its range. However, $\{(I-P)e_1, (I-P)(e_1+e_2)\} = \{0, e_2\}$ is not full spark on its range.

4.2.2 Phase retrieval with subspace components and classical phase retrieval

As the phase retrieval problem from norms of higher dimensional subspace components is a generalization of classical phase retrieval, several tools are pertinent to both problems. Our first approach to the phase retrieval problem from subspace components will be to reduce it to the classical case. Since a family of vectors allowing phase retrieval in $\mathcal{H}_M$ can contain a minimum of $2M-1$ vectors, at first glance one may be inclined to suggest that the minimal number of higher dimensional subspaces required to allow phase retrieval must be larger than $2M-1$. Specifically, for a one dimensional space W_i, the measurement $\mathcal{A}(x)(i) = \|P_i x\|$ can come from only $\pm P_i x$. For higher dimensional W_i, there is a continuum of $P_i x$ which give measurements $\mathcal{A}(x)(i) = \|P_i x\|$, and thus we appear to have less information in the subspace case. This intuition is flawed however as we only care about x as the pre-image of $\mathcal{A}$ and not $P_i x$ as the pre-image under the norm. We will in fact show $\mathcal{A}$ can be injective with $2M-1$ subspaces, and we begin with a few lemmas.

Lemma 4.2.15. Let $\{\varphi_i\}_{i=1}^N$ be full spark in an M-dimensional space. Let $\{\psi_m\}_{m=1}^M$ be an orthonormal basis for the M-dimensional space constructed as follows: Let ψ_1 be a random vector chosen according to a uniform probability distribution on the unit sphere S^{M-1}. Then ψ_2 is chosen from a uniform distribution on $S^{M-1} \cap [span(\psi_1)]^\perp$. Continue so that ψ_k is chosen from a uniform distribution on $S^{M-1} \cap [span(\{\psi_i\}_{i=1}^{k-1})]^\perp$. Then $\{\varphi_i\}_{i=1}^N \cup \{\psi_m\}_{m=1}^M$ is full spark with probability

1.

Proof. For $1 \leq k < M$, if $\{\varphi_i\}_{i=1}^N \cup \{\psi_m\}_{m=1}^k$ is full spark, and we desire $\{\varphi_i\}_{i=1}^N \cup \{\psi_m\}_{m=1}^{k+1}$ to be full spark, we must prove ψ_{k+1} does not lie in the span of any $M-1$ vectors from $\{\varphi_i\}_{i=1}^N \cup \{\psi_m\}_{m=1}^k$. Pick any such $M-1$ vectors, and denote this set by A. Let $W_k := [\text{span}(\{\psi_m\}_{m=1}^k)]^\perp$, and choose ψ_{k+1} as a random vector from $W_k \cap S^{M-1}$. Then $\{\varphi_i\}_{i=1}^N \cup \{\psi_m\}_{m=1}^{k+1}$ is full spark if and only if $\psi_{k+1} \notin \text{span}(A)$, and this will be true with probability 1 if and only if

$$\dim(\text{span}(A) \cap W_k) \leq (M-k) - 1. \tag{4.2.1}$$

This follows because $\text{span}(A) \cap W_k$ is a subset of the $(M-k)$-dimensional space W_k, and if this inequality holds, then this intersection has measure zero. Hence with probability 1, $\psi_{k+1} \notin (\text{span}(A) \cap W_k)$ but $\psi_{k+1} \in W_k$. We will prove (4.2.1) by induction.

Let $W_0 = \mathbb{R}^M$ and the first vector ψ_1 is chosen randomly from S^{M-1}. If A is any $M-1$ vectors in $\{\varphi_i\}_{i=1}^N$, we have

$$\dim(\text{span}(A) \cap W_0) = M - 1$$

so that $\{\varphi_i\}_{i=1}^N \cup \psi_1$ remains full spark with probability 1. Now assume $\{\varphi_i\}_{i=1}^N \cup \{\psi_m\}_{m=1}^k$ is full spark. Choose any $M-1$ vectors $A \subset \left(\{\varphi_i\}_{i=1}^N \cup \{\psi_m\}_{m=1}^k\right)$. We consider two cases.

Case 1: Suppose $\psi_k \notin A$. We may write

$$\text{span}(A) \cap W_k = (\text{span}(A) \cap W_{k-1}) \cap W_k.$$

By our induction hypothesis, both subspaces on the right hand side have dimension less than or equal to $M-k$, and thus (4.2.1) holds if we show the subspaces are not equal. Note $\psi_k \notin A$ is needed to apply the induction hypothesis here. Also note that $W_k \subseteq W_{k-1}$. For a contradiction, suppose $\text{span}(A) \cap W_{k-1} = W_k$. Switching to their orthogonal complements, since $\psi_k \in W_k^{\perp}$, we have $\psi_k \in [\text{span}(A) \cap W_{k-1}]^{\perp}$ which is a space of dimension k. Observing that $\psi_k \notin W_{k-1}^{\perp}$ which has dimension $k-1$ and

$$W_{k-1}^{\perp} \subset [\text{span}(A) \cap W_{k-1}]^{\perp},$$

It follows that ψ_k lies in a unique one-dimensional space determined by W_{k-1} and A. Since ψ_k was chosen randomly from $S^{M-1} \cap W_{k-1}$, this fails with probability 1, and we have proven (4.2.1).

Case 2: Suppose $\psi_k \in A$. Since $\dim(W_k) = M-k$, note $\dim(\text{span}(A) \cap W_k) \leq M-k$. For a contradiction, suppose

$$\dim(\text{span}(A) \cap W_k) = M - k. \tag{4.2.2}$$

Then

$$W_k \subset \text{span}(A). \tag{4.2.3}$$

Choose some $\varphi \in \{\varphi_i\}_{i=1}^{N}$ where $\varphi \notin A$. Then

$$\dim(\text{span}(A \setminus \psi_k) \cap W_k) \leq \dim(\text{span}(A \setminus \psi_k \cup \varphi) \cap W_k) \leq (M-k) - 1. \tag{4.2.4}$$

where the last inequality follows by applying case 1. Equations (4.2.2) and (4.2.4) imply

$$\dim(\text{span}(A \setminus \psi_k) \cap W_k) = (M-k) - 1.$$

However, since $\psi_k \perp W_k$ and $\psi_k \in A$, (4.2.2) and (4.2.3) imply

$$\dim(\operatorname{span}(A \setminus \psi_k) \cap W_k) = \dim(\operatorname{span}(A) \cap W_k) = M - k,$$

a contradiction. We conclude (4.2.1) must hold. $\square$

By successive applications of Lemma 4.2.15, we immediately obtain the following result.

Corollary 4.2.16. Any finite number of randomly constructed orthonormal bases as in Lemma 4.2.15 are full spark with probability 1.

Lemma 4.2.17. Let $M \geq 2$ be a natural number. Choose any natural numbers $M - 1 \geq I_1 \geq \cdots \geq I_M \geq 1$. There exists a real invertible $M \times M$ matrix which has precisely I_k ones in the k-th row and zeroes elsewhere.

Proof. We proceed by induction on the dimension. For $M = 2$ the result is clear. Assume the result holds for M, and consider $M + 1$ so that for some natural number $s \leq M + 1$ we have

$$M = I_1 = \cdots = I_s > I_{s+1} \geq \cdots \geq I_{M+1} \geq 1.$$

Applying the induction hypothesis to $I_1 - 1 = \cdots = I_s - 1 \geq I_{s+1} \geq \cdots \geq I_M$, we let $A = [a_{ij}]_{i,j=1}^{M}$ be an $M \times M$ invertible matrix with $I_k - 1 = M - 1$ ones in row k for $k = 1, \ldots, s$ and I_k ones in row k for $k = s + 1, \ldots, M$. We now create an

$(M+1) \times (M+1)$ matrix $B = [b_{ij}]_{i,j=1}^{M+1}$ by setting

$$b_{ij} = \begin{cases} a_{ij} & 1 \le i, j \le M \\ 1 & 1 \le i \le s, \ j = M+1 \\ 1 & i = M+1, \ 1 \le j \le I_{M+1} \\ 0 & \text{else.} \end{cases}$$

Note B has I_k ones in each row for $k = 1, \ldots, M+1$. Since $A = [a_{ij}]_{i,j=1}^{M} = [b_{ij}]_{i,j=1}^{M}$ is invertible, we may row reduce B to $\tilde{B} = [\tilde{b}_{ij}]_{i,j=1}^{M+1}$ where $[\tilde{b}_{ij}]_{i,j=1}^{M} = I_{M \times M}$ and row $M+1$ is left unchanged. Suppose $\tilde{B}$ were not invertible. Then row $M+1$ can be row reduced to all zeros, and examining the last entry in this row we must have

$$\sum_{i=1}^{I_{M+1}} \tilde{b}_{i,M+1} = 0. \tag{4.2.5}$$

Now consider $\tilde{B}_\ell$ to be the matrix identical to $\tilde{B}$ but switch $\tilde{b}_{M+1,M+1} = 0$ with $\tilde{b}_{M+1,\ell} = 1$ where $\ell \in \{1, \ldots, I_{M+1}\}$. If $\tilde{B}_\ell$ is also non-invertible, we again may row reduce the last row to all zeros, and similar to (4.2.5), we now have

$$\sum_{\substack{i=1 \\ i \neq \ell}}^{I_{M+1}} \tilde{b}_{i,M+1} = -1. \tag{4.2.6}$$

Notice (4.2.5) and (4.2.6) imply $\tilde{b}_{M+1,\ell} = 1$. However, since this holds for all $\ell \in \{1, \ldots, I_{M+1}\}$ this contradicts (4.2.5). It follows that either $\tilde{B}$ or $\tilde{B}_\ell$, for some $\ell \in \{1, \ldots, I_{M+1}\}$, must be invertible proving the result. □

Through the use of these previous lemmas as well as the classical phase retrieval result, we are now ready to prove when higher dimensional subspaces allow phase retrieval.

Theorem 4.2.18. Phase retrieval in $\mathbb{R}^M$ is possible using $2M-1$ subspaces each of any non-zero dimension less than $M-1$.

Proof. Let $\{\varphi_i\}_{i=1}^{2M-1}$ be a family of vectors in $\mathbb{R}^M$ with the complement property and the additional requirement that $\{\varphi_i\}_{i=1}^{M}$ and $\{\varphi_i\}_{i=M+1}^{2M-1}$ are orthonormal sets. Such a set exists by Corollary 4.2.16. Let $I_k \subseteq \{1,\dots,M\}$ for $k=1,\dots,M$, let $J_k \subseteq \{M+1,\dots,2M-1\}$ for $k=M+1,\dots,2M-1$, and let P_{I_k} and P_{J_k} denote the orthogonal projection onto $\text{span}(\{\varphi_i\}_{i\in I_k})$ and $\text{span}(\{\varphi_i\}_{i\in J_k})$ respectively. We consider the problem of phase retrieval from $\|P_{I_k}x\|$ and $\|P_{J_k}x\|$ for $x\in\mathbb{R}^M$ and for $k=1,\dots,2M-1$.

Let $A=[a_{kz}]_{k,z=1}^{M}$ be the $M\times M$ matrix whose rows correlate to I_k where $a_{kz}=1$ if $z\in I_k$ and zero otherwise. Define $B=[b_{kz}]_{k,z=1}^{M-1}$ similarly as the $(M-1)\times(M-1)$ matrix where $b_{kz}=1$ if $(z+M)\in J_k$ and zero otherwise. We first examine the subspaces $\text{span}(\{\varphi_i\}_{i\in I_k})$ for $k=1,\dots,M$. Notice for any $x\in\mathbb{R}^M$,

$$\|P_{I_k}x\|^2=\sum_{i\in I_k}|\langle x,\varphi_i\rangle|^2$$

so that we have the equation

$$\begin{bmatrix}\|P_{I_1}x\|^2\\ \vdots\\ \|P_{I_M}x\|^2\end{bmatrix}=A\begin{bmatrix}|\langle x,\varphi_1\rangle|^2\\ \vdots\\ |\langle x,\varphi_M\rangle|^2\end{bmatrix}. \tag{4.2.7}$$

Provided A is invertible, we can solve for $\{|\langle x,\varphi_i\rangle|\}_{i=1}^{M}$. We obtain a similar equation using B. So provided that A and B are both invertible, we can completely determine $\{|\langle x,\varphi_i\rangle|\}_{i=1}^{2M-1}$. Now we have reduced the problem to the one-dimensional case. Since $\{\varphi_i\}_{i=1}^{2M-1}$ were assumed to have the complement property, by Theorem 4.2.1 it follows that phase retrieval is possible using the subspaces $\text{span}(\{\varphi_i\}_{i\in I_k})$ and $\text{span}(\{\varphi_i\}_{i\in J_k})$ for $k=1,\dots,2M-1$.

All that remains is to pick $\{I_k\}_{k=1}^{M}$ and $\{J_k\}_{k=M+1}^{2M-1}$ so that A and B are invertible. We may choose any invertible matrix with I_k ones (J_k ones respectively) in each row. Note the number of ones in each row corresponds to the dimension of a subspace. Such invertible matrices exist by Lemma 4.2.17 for any $1 \leq I_k \leq M-1$ and $1 \leq J_k \leq M-2$. □

Notice we restrict the dimensions of the subspaces in this theorem to be less than $M-1$ since the matrix B is $(M-1) \times (M-1)$ and we need B to be invertible. However, we can obtain phase retrieval in $\mathbb{R}^M$ using subspaces of dimension less than M. To see this, suppose we are given a subspace W_i such that $\dim(W_i) = M-1$ and $W_i^{\perp} = \text{span}\{\varphi_i\}$. By considering the projection onto $W_i^{\perp}$, the intensity measurement $|\langle x, \varphi_i \rangle|$ should contain similar information as the measurement $\|P_i x\|$, where here P_i is the projection onto W_i, since

$$\|P_i x\|^2 = \|x\|^2 - |\langle x, \varphi_i \rangle|^2 . \tag{4.2.8}$$

Using notation from the proof of Theorem 4.2.18, the matrix A lets us solve for $|\langle x, \varphi_i \rangle|$ for $i = 1, \ldots, M$ giving us $\|x\|^2 = \sum_{i=1}^{M} |\langle x, \varphi_i \rangle|^2$. Now for the remaining subspaces corresponding to matrix B, we may indeed allow $M-1$ dimensional subspaces by considering instead orthogonal complements and using (4.2.8). This leads to the following corollary.

Corollary 4.2.19. Phase retrieval in $\mathbb{R}^M$ is possible using $2M-1$ subspaces each of any non-zero dimension less than M.

We mention similar arguments hold for the complex case. The authors of [28] show that $4M-4$ generic vectors allow phase retrieval in $\mathbb{C}^M$. As Corollary 4.2.16 holds for complex vector spaces, we may obtain $4M-4$ full spark vectors, say

$\{\varphi_i\}_{i=1}^{4M-4}$, which are the union of four orthogonal sets. We then may create four matrices of zeros and ones as in the real case and reduce the problem of phase retrieval to the classical case with measurement vectors $\{\varphi_i\}_{i=1}^{4M-4}$. Unfortunately phase retrieval with vector measurements in $\mathbb{C}^M$ is fundamentally different from $\mathbb{R}^M$, and there is no known necessary and sufficient condition for phase retrieval similar to the complement property. The orthogonality requirements here destroys the genericity of our $4M-4$ vectors and with it the guarantee that $\{\varphi_i\}_{i=1}^{4M-4}$ allows phase retrieval.

A result of [43] however, shows phase retrieval in $\mathbb{C}^M$ is possible with the rows of four generic $M \times M$ unitary matrices. Notice for any $x \in \mathbb{C}^M$, by measuring with the M rows of the first unitary matrix, we may determine $\|x\|$. At this point, measuring with any $M-1$ rows of another unitary determines that final measurement. Therefore, this result actually implies phase retrieval is possible in $\mathbb{C}^M$ with $4M-3$ vectors taken from 4 orthonormal sets. Taking these $4M-3$ vectors, the arguments above are now valid, and we have the following corollary for the complex case.

Corollary 4.2.20. Phase retrieval in $\mathbb{C}^M$ is possible using $4M-3$ subspaces each of any non-zero dimension less than M.

What we have done in this section is bound the minimal number of subspaces required for phase retrieval in $\mathbb{R}^M$ by $2M-1$ and in $\mathbb{C}^M$ by $4M-3$. In the case of real classical phase retrieval, $2M-1$ are also necessary; however, it is unclear whether or not these bounds are also necessary in the subspace case. In the next subsection we will characterize the subspaces which allow phase retrieval, and this characterization will highlight some of the difficulties in determining this answer.

4.2.3 Characterizing subspaces which allow phase retrieval

Much recent advancement for classical phase retrieval has come from lifting the problem into the space of self-adjoint operators. We may take a similar approach when using norms of projections as our measurements. Let $\mathcal{H}^{M\times M}$ be the $M(M+1)/2$ dimensional vector space of $M\times M$ self-adjoint real matrices. Given a family of subspaces $\{W_i\}_{i=1}^N$ of $\mathbb{R}^M$ with corresponding projections $P_i \in \mathcal{H}^{M\times M}$, define the operator $F : \mathcal{H}^{M\times M} \to \mathbb{R}^N$ as $FA(i) = \langle A, P_i\rangle_{HS}$. Here $\langle\ ,\ \rangle_{HS}$ is the Hilbert-Schmidt inner product. If we let $\{\varphi_{i,d}\}_{d=1}^{D_i}$ be an orthonormal basis for W_i, notice for any $x \in \mathbb{R}^M$,

$$F(xx^*)(i) = \langle xx^*, P_i\rangle_{HS} = \mathrm{Tr}(xx^*P_i) = \mathrm{Tr}(P_ix(P_ix)^*) = \|P_ix\|^2.$$

Therefore much like the classical phase retrieval problem [5, 6, 21], we may linearize the measurements by working in this higher dimensional space of self-adjoint operators. This identification allows a useful characterization for when subspaces allow phase retrieval. For classical phase retrieval, Lemma 9 in [6] provides this characterization.

Lemma 4.2.21. (Lemma 9 in [6]) Let $\Phi = \{\varphi_i\}_{i=1}^N$ be a family of vectors in $\mathbb{R}^M$. Then Φ allows phase retrieval if and only if the null space of $G : \mathcal{H}^{M\times M} \to \mathbb{R}^N$ given by $GA(i) = \langle A, \varphi_i\varphi_i^*\rangle_{HS}$ does not contain a matrix of rank 1 or 2.

If we generalize this result to projections of arbitrary ranks, it turns out that the characterization is identical. In fact, the same proof technique holds by replacing the operator G in Lemma 4.2.21 with its subspace component analog F. For this reason, we give the following as a corollary and omit the proof.

Corollary 4.2.22. Given subspaces $\{W_i\}_{i=1}^N$ in $\mathbb{R}^M$ with corresponding projections

P_i, $\{W_i\}_{i=1}^N$ allows phase retrieval if and only if there are no matrices of rank 1 or 2 in the null space of F.

Since we know $2M-1$ vectors are necessary for phase retrieval, one would hope the close similarities between the characterizations in Lemma 4.2.21 and Corollary 4.2.22 would provide insight into the minimal number of subspaces required for phase retrieval. Unfortunately it is difficult to draw any comparison between the two problems in this regard. The main issue here is that the space of rank 1 and rank 2 operators do not form a subspace in $\mathcal{H}^{M\times M}$, and null spaces of F and G may (or may not) intersect this space in fundamentally different ways. The minimal number $2M-1$ arises for phase retrieval with vector measurements since this is the fewest number of vectors which may have the complement property in $\mathbb{R}^M$.

We will now develop a characterization of subspaces $\{W_i\}_{i=1}^N$ in $\mathcal{H}_M$ which allow phase retrieval by relating the subspaces to the one dimensional case; but this also falls short to providing a minimal number of subspaces required. To accomplish this, we give a few preliminary results. Note that the following results hold in both the real and complex case.

Lemma 4.2.23. Let $\{W_i\}_{i=1}^N$ be subspaces of $\mathcal{H}_M$ allowing phase retrieval. For every orthonormal basis $\{\varphi_{i,j}\}_{j=1}^{J_i}$ of W_i, the set $\Phi = \{\varphi_{i,j}\}_{i=1,j=1}^{N,J_i}$ allows phase retrieval in $\mathcal{H}_M$.

Proof. Let P_i be the orthogonal projection onto W_i for each $i = \{1,\ldots,N\}$. For every $i = 1,2,\ldots,N$ and for any $x, y \in \mathcal{H}_M$ such that $|\langle x, \varphi_{i,j}\rangle| = |\langle y, \varphi_{i,j}\rangle|$ for all $j = \{1,\ldots,J_i\}$, we have

$$\|P_i x\|^2 = \sum_{j=1}^{J_i} |\langle x, \varphi_{i,j}\rangle|^2 = \sum_{j=1}^{J_i} |\langle y, \varphi_{i,j}\rangle|^2 = \|P_i y\|^2.$$

Since $\{W_i\}_{i=1}^N$ allows phase retrieval it follows that $x = cy$ for some $c \in \mathbb{C}$ with $|c| = 1$. Hence, Φ allows phase retrieval. $\square$

Lemma 4.2.24. Let P be a projection onto an N-dimensional subspace W, of $\mathcal{H}_M$. Given $x, y \in \mathcal{H}_M$, the following are equivalent:

1. $\|Px\| = \|Py\|$.

2. There exists an orthonormal basis $\{\varphi_i\}_{i=1}^N$ for W such that $|\langle x, \varphi_i\rangle| = |\langle y, \varphi_i\rangle|$, for all $i = 1, 2, \ldots, N$.

Proof. (1) $\Rightarrow$ (2): Consider the vectors $Px, Py \in W$ with $\|Px\| = \|Py\|$. We examine the following three cases.

Case 1: Assume $Px = cPy$ for some $|c| = 1$.

In this case, for any orthonormal basis $\{\varphi_i\}_{i=1}^N$ for W we have

$$|\langle x, \varphi_i\rangle| = |\langle x, P\varphi_i\rangle| = |\langle Px, \varphi_i\rangle| = |\langle cPy, \varphi_i\rangle| = |c||\langle y, P\varphi_i\rangle| = |\langle y, \varphi_i\rangle|,$$

for all $i = 1, 2, \ldots, N$ as desired.

Hence, for the next two cases, we can assume $Px \neq cPy$ for any $|c| = 1$.

Case 2: Assume $\langle Px, Py\rangle = 0$. Hence $\langle Py, Px\rangle = 0$.

In this case, let

$$\varphi_1 = \frac{Px + Py}{\|Px + Py\|}, \text{ and } \varphi_2 = \frac{Px - Py}{\|Px - Py\|}.$$

Note φ_1 and φ_2 are both unit norm. Letting $c = 1/(\|Px + Py\|\|Px - Py\|)$ we have

$$\langle \varphi_1, \varphi_2 \rangle = c\langle Px + Py, Px - Py \rangle = \|Px\|^2 - \|Py\|^2 + \langle Py, Px \rangle - \langle Px, Py \rangle = 0.$$

So $\{\varphi_1, \varphi_2\}$ is an orthonormal set. Also,

$$\begin{aligned} |\langle x, Px + Py \rangle| &= |\langle Px, Px \rangle + \langle Px, Py \rangle| = \|Px\|^2 \\ &= \|Py\|^2 = |\langle Py, Px \rangle + \langle Py, Py \rangle| = |\langle y, Px + Py \rangle|. \end{aligned}$$

Similarly, $|\langle x, Px-Py \rangle| = |\langle y, Px-Py \rangle|$. Hence, $|\langle x, \varphi_i \rangle| = |\langle y, \varphi_i \rangle|$ for $i = 1, 2$. Note that $Px, Py \in \operatorname{span}\{\varphi_1, \varphi_2\}$. Now, take $\{\varphi_i\}_{i=1}^N$ to be any orthonormal completion of $\{\varphi_1, \varphi_2\}$ to an orthonormal basis for W. Then, for all $i = 3, 4, \ldots, N$ we have

$$\langle x, \varphi_i \rangle = \langle x, P\varphi_i \rangle = \langle Px, \varphi_i \rangle = 0 = \langle Py, \varphi_i \rangle = \langle y, P\varphi_i \rangle = \langle y, \varphi_i \rangle.$$

Therefore, $|\langle x, \varphi_i \rangle| = |\langle y, \varphi_i \rangle|$ for all $i = \{1, \ldots, N\}$ where $\{\varphi_i\}_{i=1}^N$ is an orthonormal basis for W, as desired.

Case 3: $\langle Px, Py \rangle \neq 0$.

In this case, let $d = \langle Px, Py \rangle / |\langle Px, Py \rangle|$ so that $|d| = 1$, and let

$$\varphi_1 = \frac{Px + dPy}{\|Px + dPy\|}, \text{ and } \varphi_2 = \frac{Px - dPy}{\|Px - dPy\|}.$$

Note that φ_1 and φ_2 are both unit norm. Letting $c = 1/\|Px + dPy\|\|Px - dPy\|$

we have

$$\begin{aligned}
\langle\varphi_1,\varphi_2\rangle = & \quad c\langle Px+dPy,Px-dPy\rangle \\
= & \quad c(\|Px\|^2-\|dPy\|^2+\langle dPy,Px\rangle-\langle Px,dPy\rangle) \\
= & \quad c\Big(\|Px\|^2-|d|\|Py\|^2+d\langle Py,Px\rangle-\overline{d}\langle Px,Py\rangle\Big) \\
= & \quad c\Big((1-|d|)||Px||^2+\frac{\langle Px,Py\rangle\langle Py,Px\rangle}{|\langle Px,Py\rangle|}-\frac{\overline{\langle Px,Py\rangle}\langle Px,Py\rangle}{|\langle Px,Py\rangle|}\Big) \\
= & \quad c\Big(0+\frac{|\langle Px,Py\rangle|^2}{|\langle Px,Py\rangle|}-\frac{|\langle Px,Py\rangle|^2}{|\langle Px,Py\rangle|}\Big) \\
= & \quad 0.
\end{aligned}$$

Hence, $\{\varphi_1,\varphi_2\}$ is an orthonormal set. Now the proof follows as in Case 2.

(2) $\Rightarrow$ (1): This is immediate by:

$$\|Px\|^2=\sum_{i=1}^{N}|\langle x,\varphi_i\rangle|^2=\sum_{i=1}^{N}|\langle y,\varphi_i\rangle|^2=\|Py\|^2.$$

□

Combining Lemma 4.2.23 and Lemma 4.2.24, we arrive at a characterization for when $\{W_i\}_{i=1}^N$ allows phase retrieval in $\mathcal{H}_M$ in terms of orthonormal bases.

Theorem 4.2.25. Let $\{W_i\}_{i=1}^N$ be subspaces of $\mathcal{H}_M$. The following are equivalent:

1. $\{W_i\}_{i=1}^N$ allows phase retrieval in $\mathcal{H}_M$.
2. For every orthonormal basis $\{\varphi_{i,j}\}_{j=1}^{J_i}$ of W_i, the set $\{\varphi_{i,j}\}_{i=1,j=1}^{N,J_i}$ allows phase retrieval in $\mathcal{H}_M$.

Proof. (1) $\Rightarrow$ (2): This is Lemma 4.2.23.

(2) $\Rightarrow$ (1): Suppose we have $x,y\in\mathcal{H}_M$ with $\|P_ix\|=\|P_iy\|$, for all $i=1,2,\ldots,N$. By Lemma 4.2.24 we can choose orthonormal bases $\Phi=\{\varphi_{i,j}\}_{i=1,j=1}^{N,J_i}$

so that

$$|\langle x, \varphi_{i,j}\rangle| = |\langle y, \varphi_{i,j}\rangle|, \text{ for all } i, j.$$

By (2), Φ allows phase retrieval and so $x = cy$ for some $|c| = 1$. I.e. $\{W_i\}_{i=1}^N$ allows phase retrieval. □

In the real case, the complement property is a convenient property with which to work and this is why we present the above theorem in terms of orthonormal bases. However, the proof of this theorem doesn't require us to consider only vectors. Instead, the same general arguments hold if we take each W_i and split this subspace into orthogonal subspaces which span W_i.

Corollary 4.2.26. Let $\{W_i\}_{i=1}^N$ be subspaces of $\mathcal{H}_M$. The following are equivalent:

(a) $\{W_i\}_{i=1}^N$ allow phase retrieval for $\mathcal{H}_M$.

(b) For every choice of orthogonal subspaces $\{Z_{i,d}\}_{d=1}^{D_i}$ where $\bigoplus_{d=1}^{D_i} Z_{i,d} = W_i$, the subspaces $\{Z_{i,d}\}_{i=1,d=1}^{N,\ D_i}$ allow phase retrieval in $\mathcal{H}_M$.

4.2.4 Additional properties of subspaces regarding phase retrieval

At this point we have given several abstract characterizations of the subspaces $\{W_i\}_{i=1}^N$ which allow phase retrieval in $\mathcal{H}_M$. Note however the only subspaces which we have concretely shown to satisfy these characterizations are highly structured. That is, the only subspaces which we have shown to allow phase retrieval are those constructed in Theorem 4.2.18. For the special case when $\{W_i\}_{i=1}^N$ are hyperplanes, we will overcome this restriction of structure and produce highly non-structured subspaces which allow phase retrieval in Subsection 4.3.3. In general however, we believe any $2M - 1$ random subspaces should also allow phase retrieval much as $2M-1$ random vectors allow phase retrieval [4]. While we cannot prove this, we take an incremental step by showing the subspaces which allow phase retrieval are open in some sense. Specifically, we show when given subspaces $\{W_i\}_{i=1}^N$ which allow phase retrieval, there exist open balls $B_i(W_i, \epsilon)$ around each W_i such that for any $W_i' \in B_i(W_i, \epsilon)$, the subspaces $\{W_i'\}_{i=1}^N$ allow phase retrieval. We again require a few preliminary results to build to this end.

First we show that subspaces $\{W_i\}_{i=1}^N$ allow phase retrieval if and only if we cannot find nonzero orthogonal vectors $x, y \in \mathbb{R}^M$ such that $\mathcal{A}x = \mathcal{A}y$. This is useful in that to show $\{W_i\}_{i=1}^N$ allows phase retrieval, we need only show $\mathcal{A}(x) \neq \mathcal{A}(y)$ for all $x \perp y$ rather than all $x \neq \pm y$. We note that the argument used to prove this result may be extracted from the arguments needed for Lemma 4.2.21.

Lemma 4.2.27. Subspaces $\{W_i\}_{i=1}^N$ of $\mathbb{R}^M$ do not allow phase retrieval if and only if there exists nonzero $u, v \in \mathbb{R}^M$ with $u \perp v$ such that $\|P_i u\| = \|P_i v\|$ for all $i = 1, \ldots, N$.

Proof. The necessity direction is obvious, so for sufficiency, suppose $\{W_i\}_{i=1}^N$ do not allow phase retrieval. Then there exists nonzero $x, y \in \mathbb{R}^M$ with $x \neq \pm y$ such that $\|P_i x\| = \|P_i y\|$ for all $i = 1, \ldots, N$. In the operator space, this implies $F(xx^*) = F(yy^*)$, so that $xx^* - yy^*$ is in the null space of F, where F is the linear operator as defined in the beginning of Section 4.2.3. Note that $xx^* - yy^*$ is a rank two, symmetric operator. Thus by the Spectral Theorem, there exist orthogonal eigenvectors $u, v \in \mathbb{R}^M$ and nonzero scalars $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $xx^* - yy^* = \lambda_1 uu^* + \lambda_2 vv^*$. Then for all $i = 1, \ldots, N$ we have,

$$0 = F(\lambda_1 uu^* + \lambda_2 vv^*)(i) = \lambda_1 \|P_i u\|^2 + \lambda_2 \|P_i v\|^2.$$

Since u, v are nonzero, it follows that $\|P_i u\|^2 > 0$ and $\|P_i v\|^2 > 0$ for some i and λ_1, λ_2 must have opposite signs. This implies

$$|\lambda_1|\,\|P_i u\|^2 = |\lambda_2|\,\|P_i v\|^2.$$

Thus $u/\sqrt{|\lambda_1|}$, and $v/\sqrt{|\lambda_2|}$ are the orthogonal vectors we seek. □

Lemma 4.2.28. Suppose $\{W_i\}_{i=1}^N$ are subspaces allowing phase retrieval for $\mathbb{R}^M$. Then there exists a $\delta > 0$ such that for any $x \perp y$ where $1 = \|x\| \geq \|y\| > 0$, there exists an $1 \leq i \leq N$ such that

$$\big|\|P_i x\| - \|P_i y\|\big| > \delta.$$

Proof. We proceed by way of contradiction. Assume $\{W_i\}_{i=1}^N$ allow phase retrieval, and for every $j = 1, 2, \ldots$ there exist $x_j \perp y_j$ where $1 = \|x_j\| \geq \|y_j\| > 0$ so that

$$\big|\|P_i x_j\| - \|P_i y_j\|\big| \leq \frac{1}{j} \tag{4.2.9}$$

for every $i = 1, \ldots, N$. By switching to a subsequence, we may assume $x_j \to x$ with $\|x\| = 1$. Note there exists some i such that $P_i x \neq 0$ for otherwise $\{W_i\}_{i=1}^N$ would not allow phase retrieval. By (4.2.9), this also implies $P_i y_j$ does not converge to zero for some i and thus y_j cannot converge to zero. We therefore switch to a further subsequence such that $x_i \to x$, $y_i \to y$, $1 = \|x\| \geq \|y\| > 0$, and $x \perp y$. Moreover, equation (4.2.9) now implies $\|P_i x\| = \|P_i y\|$ for all $i = 1, \ldots, N$. We conclude $\{W_i\}_{i=1}^N$ does not allow phase retrieval - a contradiction. □

Combining these lemmas, we have the desired theorem.

Theorem 4.2.29. Suppose $\{W_i\}_{i=1}^N$ are subspaces allowing phase retrieval for $\mathbb{R}^M$.

Let $\{W_i'\}_{i=1}^N$ be subspaces with associated orthogonal projections Q_i. Then there exists an $\epsilon > 0$ such that when $\|P_i - Q_i\| < \epsilon$ for all $i = 1, \ldots, N$, then $\{W_i'\}_{i=1}^N$ allow phase retrieval.

Proof. By Lemma 4.2.27, it suffices to prove that for any nonzero $x \perp y$ there exists some i such that $\|Q_i x\| \neq \|Q_i y\|$. Take any nonzero $x \perp y$, and we may assume by scaling and switching x and y if necessary that $1 = \|x\| \geq \|y\| > 0$. By Lemma 4.2.28 there exists a $\delta > 0$ such that

$$\big|\|P_{i_0}x\| - \|P_{i_0}y\|\big| > \delta$$

for some $i_0 \in \{1, \ldots, N\}$. Then

$$\begin{aligned}
&\big|\|Q_{i_0}x\| - \|Q_{i_0}y\|\big| \\
&\qquad \geq \big|\|P_{i_0}x\| - \|P_{i_0}y\|\big| - \big|\|Q_{i_0}x\| - \|P_{i_0}x\|\big| - \big|\|P_{i_0}y\| - \|Q_{i_0}y\|\big| \\
&\qquad > \delta - \|P_{i_0}x - Q_{i_0}x\| - \|P_{i_0}y - Q_{i_0}y\| \\
&\qquad \geq \delta - 2\|P_{i_0} - Q_{i_0}\| \\
&\qquad > \delta - 2\epsilon \\
&\qquad > 0
\end{aligned}$$

when $\epsilon < \delta/2$. □

4.3 Norm retrieval

In this section we further analyze the subspaces which allow phase retrieval by showing that if a family of projections $\{W_i\}_{i=1}^N := \{P_i\}_{i=1}^N$ allows phase retrieval, it need not occur that $\{W_i^\perp\}_{i=1}^N := \{(I - P_i)\}_{i=1}^N$ allows phase retrieval. This leads to fully characterizing when subspaces and their orthogonal complements allow phase retrieval. The fundamental notion which connects phase retrieval to complements is something we call *norm retrieval.* After showing that norm retrieval is central to these questions, we make a detailed study of norm retrieval and its relationship to phase retrieval. In particular, we show that a collection of vectors $\{\varphi_i\}_{i=1}^N$ allows phase retrieval if and only if $\{T\varphi_i\}_{i=1}^N$ allows norm retrieval for every invertible operator T. Another fundamental idea which connects these results is the question of when the identity operator is in the span of $\{P_i\}_{i=1}^N$. Often times when a collection of vectors or projections allow phase retrieval then the identity is in their span. Moreover, we show that having the identity in the span of a family of projections allowing phase retrieval, will generally yield that the orthogonal complements allow phase retrieval. We also give a number of examples throughout showing that these results are best possible.

4.3.1 Norm retrieval and its impact on phase retrieval

A question which naturally arises when considering the problem of phase retrieval with subspace components is given subspaces $\{W_i\}_{i=1}^N$ of $\mathcal{H}_M$ which allow phase retrieval, do $\{W_i^\perp\}_{i=1}^N$ allow phase retrieval? We introduce a new fundamental property, norm retrieval, which is precisely what is needed to pass phase retrieval to orthogonal complements. Next, we make precise the definition of norm retrieval and then show its importance to phase retrieval. After this we will develop the basic properties of norm retrieval.

Definition 4.3.1. Let $\{W_i\}_{i=1}^N$ be a collection of subspaces in $\mathcal{H}_M$ and define $\{P_i\}_{i=1}^N$ to be the orthogonal projections onto each of these subspaces. We say that $\{W_i\}_{i=1}^N$ (or $\{P_i\}_{i=1}^N$) allows **norm retrieval** if for all $x, y \in \mathcal{H}_M$ satisfying $\|P_i x\| = \|P_i y\|$

for all $i = 1, \ldots, N$, then $\|x\| = \|y\|$.

Remark 4.3.2. Although trivial, it is important to point out that any collection of subspaces which allows phase retrieval necessarily allows norm retrieval. However, the converse need not hold since any orthonormal set of vectors does norm retrieval but has too few vectors to do phase retrieval.

We will start by showing that norm retrieval is precisely the condition needed to pass phase retrieval to orthogonal complements.

Proposition 4.3.3. Let $\{W_i\}_{i=1}^N$ be a collection of subspaces in $\mathcal{H}_M$ allowing phase retrieval and let $\{P_i\}_{i=1}^N$ be the projections onto these subspaces. The following are equivalent:

1. $\{I - P_i\}_{i=1}^N$ allows phase retrieval.

2. $\{I - P_i\}_{i=1}^N$ allows norm retrieval.

Proof. (1) $\Rightarrow$ (2) Since phase retrieval always implies norm retrieval then this is clear.

(2) $\Rightarrow$ (1) Let $x, y \in \mathcal{H}_M$ be such that $\|(I - P_i)x\|^2 = \|(I - P_i)y\|^2$ for all $i = 1, \ldots, N$. Since $\{I - P_i\}_{i=1}^N$ allows norm retrieval then this implies that $\|x\|^2 = \|y\|^2$. Since P and $(I - P)$ correspond to orthogonal subspaces then $\|x\|^2 = \|P_i x\|^2 + \|(I - P_i)x\|^2$ for all $i = 1, \ldots, N$. Thus, for all $i = 1, \ldots, N$ we have

$$\|P_i x\|^2 = \|x\|^2 - \|(I - P_i)x\|^2 = \|y\|^2 - \|(I - P_i)y\|^2 = \|P_i y\|^2.$$

Since $\{W_i\}_{i=1}^N$ allows phase retrieval then this implies that $x = cy$ for some scalar $|c| = 1$. Therefore $\{I - P_i\}_{i=1}^N$ allows phase retrieval. $\square$

We can think of norm retrieval as giving us one free measurement when trying to do phase retrieval. For example, let $\{e_i\}_{i=1}^3$ be an orthonormal basis for $\mathbb{R}^3$ and choose $\{\varphi_1, \varphi_2\}$ so that these 5 vectors are full spark. Hence, these vectors allow phase retrieval in $\mathbb{R}^3$. Now consider the family of vectors $\Phi := \{e_1, e_2, \varphi_1, \varphi_2\}$ in $\mathbb{R}^3$. Φ cannot do phase retrieval in general since it takes at least 5 vectors to do phase retrieval in $\mathbb{R}^3$. However, given unit norm $x, y \in \mathbb{R}^3$, Φ will do phase retrieval in this scenario since we have knowledge of the norms of the signals. To see this, assume

$$|\langle x, e_i\rangle| = |\langle y, e_i\rangle| \text{ and } |\langle x, \varphi_i\rangle| = |\langle y, \varphi_i\rangle| \text{ for } i = 1, 2.$$

Then

$$1 = \|x\|^2 = \sum_{i=1}^{3} |\langle x, e_i\rangle|^2,$$

and similarly for y. This implies

$$|\langle x, e_3\rangle|^2 = 1 - \sum_{i=1}^{2} |\langle x, e_i\rangle|^2 = 1 - \sum_{i=1}^{2} |\langle y, e_i\rangle|^2 = |\langle y, e_3\rangle|^2.$$

That is, x, y have the same modulus of inner products with all 5 vectors which allow phase retrieval and so $x = \pm y$.

A fundamental idea is to apply operators to vectors and subspaces which allow phase retrieval or norm retrieval. We now consider when operators preserve these concepts.

Proposition 4.3.4. If $\{\varphi_i\}_{i=1}^N$ is a frame in $\mathcal{H}_M$ which allows phase retrieval (respectively norm retrieval) then $\{P\varphi_i\}_{i=1}^N$ allows phase retrieval (respectively norm retrieval) for all orthogonal projections P on $\mathcal{H}_M$.

Proof. Let $x, y \in P(\mathcal{H}_M)$ such that $|\langle x, P\varphi_i\rangle|^2 = |\langle y, P\varphi_i\rangle|^2$ for all $i = \{1, \ldots, N\}$.

For all $i \in \{1, \ldots, N\}$, we have

$$|\langle x, \varphi_i \rangle|^2 = |\langle Px, \varphi_i \rangle|^2 = |\langle x, P\varphi_i \rangle|^2 = |\langle y, P\varphi_i \rangle|^2 = |\langle Py, \varphi_i \rangle|^2 = |\langle y, \varphi_i \rangle|^2.$$

Since $\{\varphi_i\}_{i=1}^N$ gives phase retrieval (respectively norm retrieval) then this implies $x = cy$ for some scalar $|c| = 1$ (respectively $\|x\| = \|y\|$). Therefore, $\{P\varphi_i\}_{i=1}^N$ allows phase retrieval (respectively norm retrieval). □

Although norm retrieval is preserved when applying any projection to the vectors, this does not hold when we apply an invertible operator to the vectors. The next theorem classifies when invertible operators maintain norm retrieval.

Theorem 4.3.5. Let $\{\varphi_i\}_{i=1}^N$ be vectors in $\mathcal{H}_M$. The following are equivalent:

1. $\{\varphi_i\}_{i=1}^N$ allows phase retrieval.
2. $\{T\varphi_i\}_{i=1}^N$ allows phase retrieval for all invertible operators T on $\mathcal{H}_M$.
3. $\{T\varphi_i\}_{i=1}^N$ allows norm retrieval for all invertible operators T on $\mathcal{H}_M$.

Proof. (1) ⇒ (2) Let T be any invertible operator on $\mathbb{R}^M$ and let $x, y \in \mathcal{H}_M$ be such that $|\langle x, T\varphi_i \rangle| = |\langle y, T\varphi_i \rangle|$ for all $i \in \{1, \ldots, N\}$. Then $|\langle T^*x, \varphi_i \rangle| = |\langle T^*y, \varphi_i \rangle|$ for all $i \in \{1, \ldots, N\}$. Since $\{\varphi_i\}_{i=1}^N$ allows phase retrieval then this implies $T^*x = cT^*y$ for some scalar $|c| = 1$. Since T is invertible and linear then $(T^*)^{-1}T^*x = (T^*)^{-1}cT^*y$ implies $x = cy$ and $|c| = 1$. Therefore, $\{T\varphi_i\}_{i=1}^N$ does phase retrieval.

(2) ⇒ (3) Since phase retrieval implies norm retrieval then this is clear.

(3) ⇒ (1) Choose nonzero $x, y \in \mathcal{H}_M$ such that $|\langle x, \varphi_i \rangle| = |\langle y, \varphi_i \rangle|$ for all $i \in \{1, \ldots, N\}$. By assumption, $\{T\varphi_i\}_{i=1}^N$ allows norm retrieval for all invertible operators T on $\mathcal{H}_M$.

Let T be any invertible operator on $\mathcal{H}_M$, then $(T^*)^{-1}$ is an invertible operator and hence $\{(T^*)^{-1}\varphi_i\}_{i=1}^N$ allows norm retrieval. For $Tx, Ty \in \mathcal{H}_M$, we have

$$\begin{aligned}
|\langle Tx, (T^*)^{-1}\varphi_i\rangle| &= |\langle T^{-1}Tx, \varphi_i\rangle| \\
&= |\langle x, \varphi_i\rangle| \\
&= |\langle y, \varphi_i\rangle| \\
&= |\langle T^{-1}Ty, \varphi_i\rangle| \\
&= |\langle Ty, (T^*)^{-1}\varphi_i\rangle|,
\end{aligned}$$

for every $i \in \{1, \ldots, N\}$. Hence, $||Tx|| = ||Ty||$ for any invertible operator T on $\mathcal{H}_M$.

Now we will be done if we can show that $\|Tx\| = \|Ty\|$ for any invertible T implies $y = cx$ for some scalar c with $|c| = 1$. First note that since the identity operator is invertible we have that $\|x\| = \|y\|$, so if $y = cx$ then it follows that $|c| = 1$. Now choose an orthonormal basis $\{e_j\}_{j=1}^M$ for $\mathcal{H}_M$ with $e_1 = \frac{x}{\|x\|}$ and suppose $y = \sum_{j=1}^M \alpha_j e_j$, so that $\|y\|^2 = \sum_{j=1}^M \alpha_j^2$. Define the operator T by $Te_1 = e_1$ and $Te_j = \frac{1}{2}e_j$ for $j = 2, ..., M$. Now we have that $\|x\|^2 = \|Tx\|^2 = \|Ty\|^2 = \alpha_1^2 + \sum_{j=2}^M \frac{1}{4}\alpha_j^2$, which implies that $\sum_{j=2}^M \alpha_j^2 = \frac{1}{4}\sum_{j=2}^M \alpha_j^2$, and so $\alpha_j = 0$ for $j = 2, ..., M$. Therefore, $y = \alpha_1 e_1 = \frac{\alpha_1}{\|x\|}x$ which competes the proof. □

At first glance one would think that retrieving the norm of a signal would be much easier than recovering the actual signal. However, Theorem 4.3.5 gives a new classification of phase retrieval in terms of norm retrieval and states that if every invertible operator applied to a frame allows norm retrieval then our original frame allows phase retrieval. This illustrates that recovering the norm of a signal may be more similar to recovering the actual signal than originally thought and hence may not be as easily achievable as anticipated.

Theorem 4.3.5 shows that if a frame does not allow phase retrieval, then we cannot apply an invertible operator to it in order to get a frame that does allow phase retrieval. In contrast, it is true that there exists at least one invertible operator which when applied to a frame allows norm retrieval.

Proposition 4.3.6. Given $\{\varphi_i\}_{i=1}^N$ spanning $\mathcal{H}_M$, there exists an invertible operator T on $\mathcal{H}_M$ so that the collection of orthogonal projections onto the vectors $\{T\varphi_i\}_{i=1}^N$ allows norm retrieval.

Proof. Without loss of generality, assume $\{\varphi_i\}_{i=1}^M$ are linearly independent. Choose an invertible operator T so that $T\varphi_i = e_i$ for all $i = 1, \ldots, M$, where $\{e_i\}_{i=1}^M$ is an orthonormal basis for $\mathcal{H}_M$. Thus, for any $x \in \mathcal{H}_M$, $||x||^2 = \sum_{i=1}^M |\langle x, e_i \rangle|^2 = \sum_{i=1}^M |\langle x, T\varphi_i \rangle|^2$. Hence, the collection of orthogonal projections onto the vectors $\{T\varphi_i\}_{i=1}^M$ allows norm retrieval. In particular, $\{T\varphi_i\}_{i=1}^N$ allows norm retrieval. □

4.3.2 Phase retrieval with orthogonal complements

Through the use of norm retrieval we can now further our analysis of phase retrieval for subspace components. In particular we will characterize the relationship between $\{W_i\}_{i=1}^N$ and $\{W_i^\perp\}_{i=1}^N$ in terms of phase retrieval.

First observe that there exist subspaces $\{W_i\}_{i=1}^N$ allowing phase retrieval such that $\{W_i^\perp\}_{i=1}^N$ do not.

Example 4.3.7. Let $\{\varphi_i\}_{i=1}^3$ and $\{\psi_i\}_{i=1}^3$ be orthonormal bases for $\mathbb{R}^3$ such that

$\{\varphi_i\}_{i=1}^3 \cup \{\psi_i\}_{i=1}^3$ is full spark. Consider the subspaces

$$\begin{aligned}
W_1 &= \text{span}(\{\varphi_1, \varphi_3\}) & W_1^\perp &= \text{span}(\{\varphi_2\}) \\
W_2 &= \text{span}(\{\varphi_2, \varphi_3\}) & W_2^\perp &= \text{span}(\{\varphi_1\}) \\
W_3 &= \text{span}(\{\varphi_3\}) & W_3^\perp &= \text{span}(\{\varphi_1, \varphi_2\}) \\
W_4 &= \text{span}(\{\psi_1\}) & W_4^\perp &= \text{span}(\{\psi_2, \psi_3\}) \\
W_5 &= \text{span}(\{\psi_2\}) & W_5^\perp &= \text{span}(\{\psi_1, \psi_3\}).
\end{aligned}$$

Then $\{W_i\}_{i=1}^5$ allow phase retrieval for $\mathbb{R}^3$ while the orthogonal complements $\{W_i^\perp\}_{i=1}^5$ do not.

To see this, notice the subspaces $\{W_i\}_{i=1}^5$ allow phase retrieval from a direct application of Theorem 4.2.18. Considering the orthogonal complements $\{W_i^\perp\}_{i=1}^5$ with associated orthogonal projections $\{Q_i\}_{i=1}^5$, notice $Q_1 + Q_2 = Q_3$. Thus the measurement $\|Q_3 x\|^2$ does not contribute any new information (or think of Q_3 as a linearly dependent operator which when removed does not change the null space associated with F, as in Corollary 4.2.22). Thus $\{W_i^\perp\}_{i=1}^5$ allows phase retrieval if and only if $\{W_i^\perp\}_{i\in\{1,2,4,5\}}$ allows phase retrieval. However, for the special case of $\mathbb{R}^3$, we can show 5 subspaces are necessary.

Indeed, suppose we have any subspaces $\{W_i\}_{i=1}^N$ in $\mathbb{R}^3$ and corresponding projections $\{P_i\}_{i=1}^N$ with $N \leq 4$. Since $\dim(\mathcal{H}^{3\times3}) = 6$, by the rank-nullity theorem, the dimension of the null space of F is greater than or equal to 2. Hence there exist two nonzero, linearly independent matrices $A, B \in \mathcal{H}^{3\times3}$ such that $F(A) = F(B) = 0$, where F is the operator as defined in the beginning of Subsection 4.2.3. If either matrix is rank 1 or 2, then $\{W_i\}_{i=1}^N$ in $\mathbb{R}^3$ do not allow phase retrieval by Corollary 4.2.22. So assume A and B are full rank and consider the continuous map

$$f : t \mapsto \det(A\cos t + B\sin t), \quad t \in [0, \pi].$$

Since $f(0) = \det(A) \neq 0$ and $f(\pi) = \det(-A) = (-1)^3\det(A) = -\det(A) \neq 0$, then by the intermediate value theorem there exists some $t_0 \in [0, \pi]$ such that $f(t_0) = \det(A\cos t_0 + B\sin t_0) = 0$. Therefore $C := A\cos t_0 + B\sin t_0$ is a rank 1 or 2 matrix, such that $F(C) = 0$. Note that $C \neq 0$ since A and B are nonzero, linearly independent matrices. Therefore C is a nonzero, rank 1 or 2 matrix in the null space of F and thus by Corollary 4.2.22, $\{W_i\}_{i=1}^N$ in $\mathbb{R}^3$ again fails phase retrieval.

There are also special cases when phase retrieval is always possible with orthogonal complements.

Theorem 4.3.8. Suppose $\{W_i\}_{i=1}^N$ are subspaces of $\mathcal{H}_M$ allowing phase retrieval with corresponding orthogonal projections $\{P_i\}_{i=1}^N$. If $I \in \text{span}(\{P_i\}_{i=1}^N)$ so that $I = \sum_{i=1}^N a_i P_i$ with $\sum_{i=1}^N a_i \neq 1$, then $\{W_i^\perp\}_{i=1}^N$ allow phase retrieval.

Proof. Note

$$\sum_{i=1}^N a_i(I - P_i) = \left(\sum_{i=1}^N a_i I\right) - I = \left(\left(\sum_{i=1}^N a_i\right) - 1\right) I,$$

so letting $b = \left(\sum_{i=1}^N a_i\right) - 1$, we have $I = \sum_{i=1}^N \frac{a_i}{b}(I - P_i)$. Thus the measurements $\|(I - P_i)x\|$ associated with $\{W_i^\perp\}_{i=1}^N$ allow one to determine

$$\begin{aligned}\|x\|^2 = \langle xx^*, I\rangle_{HS} = \langle xx^*, \sum_{i=1}^N \frac{a_i}{b}(I - P_i)\rangle_{HS} = \sum_{i=1}^N \frac{\bar{a}_i}{\bar{b}}\langle xx^*, I - P_i\rangle_{HS} \\ = \sum_{i=1}^N \frac{\bar{a}_i}{\bar{b}}\|(I - P_i)x\|^2.\end{aligned}$$

Since $\{W_i\}_{i=1}^N$ allow phase retrieval and $\|P_ix\|^2 = \|x\|^2 - \|(I-P_i)x\|^2$, it follows that $\{W_i^\perp\}_{i=1}^N$ allow phase retrieval. □

We now state one consequence of Theorem 4.3.8.

Theorem 4.3.9. Let $\{W_i\}_{i=1}^N$ be a collection of subspaces of $\mathcal{H}_M$ that allow phase retrieval and suppose further that $\dim(W_i) = K$ for every $i = 1, 2, ..., N$. Let P_i be the orthogonal projection onto W_i and suppose that $I \in \text{span}\{P_i\}_{i=1}^N$, then $\{W_i^\perp\}$ allows phase retrieval.

Proof. Since $I \in \text{span}\{P_i\}_{i=1}^N$ then there exists $\{a_i\}_{i=1}^N$ not all zero such that $I = \sum_{i=1}^N a_iP_i$. Then

$$M = \text{Tr}(I) = \text{Tr}(\sum_{i=1}^N a_iP_i) = \sum_{i=1}^N a_i\text{Tr}(P_i) = K\sum_{i=1}^N a_i$$

since $\text{Tr}(P_i) = K$ for every i. Therefore, $\sum_{i=1}^N a_i = \frac{M}{K} > 1$ since $K < M$ ($\{W_i\}_{i=1}^N$ are equidimensional subspaces which allow phase retrieval so $\dim(W_i) = K < M$), so the result follows from Theorem 4.3.8. □

Given a collection of orthogonal projections $\{P_i\}_{i=1}^N$, we cannot conclude that they allow phase retrieval just because $I \in \text{span}\{P_i\}_{i=1}^N$. For example, given any projection P, we have $I = P + (I-P)$ but certainly $\{P, I-P\}$ will not allow phase retrieval. However, we now show that this is enough to conclude that $\{P_i\}_{i=1}^N$ allows norm retrieval.

Proposition 4.3.10. Let $\{W_i\}_{i=1}^N$ be subspaces of $\mathcal{H}_M$, and let $\{P_i\}_{i=1}^N$ be the associated projections. If $I \in \text{span}\{P_i\}_{i=1}^N$, then $\{W_i\}_{i=1}^N$ gives norm retrieval.

Proof. Since $I \in \text{span}\{P_i\}_{i=1}^N$ then there exists $\{a_i\}_{i=1}^N$ not all zero such that

$I = \sum_{i=1}^{N} a_i P_i$. Notice for $x \in \mathcal{H}_M$ we have

$$\sum_{i=1}^{N} a_i \|P_i x\|^2 = \sum_{i=1}^{N} \langle a_i P_i x, x\rangle = \left\langle \sum_{i=1}^{N} a_i P_i x, x \right\rangle = \langle Ix, x\rangle = \|x\|^2$$

Let $x, y \in \mathcal{H}_M$ such that $\|P_i x\| = \|P_i y\|$ for all $i = 1, \ldots, N$. We have

$$\|x\|^2 = \sum_{i=1}^{N} a_i \|P_i x\|^2 = \sum_{i=1}^{N} a_i \|P_i y\|^2 = \|y\|^2.$$

Hence $\{W_i\}_{i=1}^{N}$ allows norm retrieval. □

The converse of Proposition 4.3.10 is far from true. One way to see this (at least for the real case) is as follows: For $2M \leq N \leq M(M+1)/2$ choose any full spark frame $\Phi = \{\varphi_i\}_{i=1}^{N}$ for $\mathbb{R}^M$ such that $\{\varphi_i \varphi_i^*\}_{i=1}^{N}$ is linearly independent (which happens generically, see e.g., [18]). Let S be the frame operator for Φ and define

$$\psi_i = \frac{S^{-1/2}\varphi_i}{\|S^{-1/2}\varphi_i\|}$$

with $P_i = \psi_i \psi_i^*$ (note that P_i is a rank one orthogonal projection). Since $\{S^{-1/2}\varphi_i\}_{i=1}^{N}$ is a Parseval frame it follows that

$$I = \sum_{i=1}^{N} \|S^{-1/2}\varphi_i\|^2 P_i. \tag{4.3.1}$$

Also, since $\{\varphi_i \varphi_i^*\}_{i=1}^{N}$ is linearly independent it follows that $\{P_i\}_{i=1}^{N}$ is linearly independent (and so (4.3.1) is the only way to write I as a linear combination of the P_i's). Also since $\{\varphi_i\}_{i=1}^{N}$ is full spark we know that $\|S^{-1/2}\varphi_i\| \neq 0$ for every $i = 1, 2, ..., N$. Therefore it follows that if $\mathcal{I} \subset \{1, 2, ..., N\}$ then $I \notin \text{span}\{P_i\}_{i \in \mathcal{I}}$. Furthermore, since $N \geq 2M$ and $\{\varphi_i\}_{i=1}^{N}$ is full spark it follows that $\{P_i\}_{i \in \mathcal{I}}$ allows phase retrieval (and hence norm retrieval) whenever $|\mathcal{I}| \geq 2N - 1$.

Although the above example proves that the converse of Proposition 4.3.10 is false, in the special case where $\sum_{i=1}^{N} \dim(W_i) = M$ the converse turns out to be true.

Proposition 4.3.11. A collection of unit norm vectors $\{\varphi_i\}_{i=1}^{N}$ in $\mathcal{H}_N$ allow norm retrieval if and only if $\{\varphi_i\}_{i=1}^{N}$ are orthogonal.

Proposition 4.3.11 is a consequence of the following more general theorem about subspaces.

Theorem 4.3.12. Let $\{W_i\}_{i=1}^{N}$ be a collection of subspaces of $\mathcal{H}_M$ with the property that $\sum_{i=1}^{N} \dim(W_i) = M$ and let P_i be the orthogonal projection onto subspace W_i for each $i = 1, \dots, N$. The following are equivalent:

1. $\{W_i\}_{i=1}^{N}$ allows norm retrieval

2. $\sum_{i=1}^{N} P_i = I$.

Proof. (2) $\Rightarrow$ (1) Follows from Proposition 4.3.10.

(1) $\Rightarrow$ (2) Pick some W_j and define V_j to be the span of the $N-1$ subspaces $\{W_i\}_{i\neq j}$, and let Q be the orthogonal projection onto V_j. Without loss of generality we may assume that W_j is not the zero subspace. Note that

$$\dim V_j \leq \sum_{i\neq j} \dim(W_i) = M - \dim(W_j).$$

Claim 1: $W_j \cap V_j = \{0\}$.

Proof of Claim: Assume to the contrary that $W_j \cap V_j$ is nontrivial. Then

$$\dim \operatorname{span}\{W_i\}_{i=1}^{N} < \dim V_j + \dim W_j \leq M.$$

This implies that there exists a nonzero $x_0 \in (\operatorname{span}\{W_i\}_{i=1}^{N})^{\perp}$ and hence $P_i x_0 = 0$ for all $i = 1, \dots, N$. However, since $\{W_i\}_{i=1}^{N}$ gives norm retrieval, we conclude that $x_0 = 0$, a contradiction. Thus $W_j \cap V = \{0\}$.

Claim 2: $P_j Q = Q P_j = 0$.

Proof of Claim 2: Assume toward a contradiction that $P_j Q \neq 0$, and thus $QP_j \neq 0$. Set $Y = \{x \in W_j : Qx = 0\}$. Since $QP_j \neq 0$ we see that $Y \neq W_j$.

Let Z be the orthogonal complement of Y in W_j. Since $V_j \cap W_j = \{0\}$ and $Z \subset W_j$, we conclude that $V_j \cap Z = \{0\}$.

Let $z \in Z \setminus \{0\}$. Since $z \notin V_j$ we have $Qz \neq z$, and since $z \notin Y$ we have $Qz \neq 0$. Set $x := Qz \neq 0$ (note $x \neq z$) and $y := (I - Q)z \neq 0$.

Note that

$$\begin{aligned}
\langle P_j x, y \rangle &= \langle P_j Q z, (I-Q)z \rangle = \langle P_j Q z, P_j (I-Q) z \rangle \\
&= \langle P_j Q z, P_j z - P_j Q z \rangle = \langle P_j Q z, z - P_j Q z \rangle \\
&= \langle P_j Q z, z \rangle - \|P_j Q z\|^2 = \langle Q z, P_j z \rangle - \|P_j Q z\|^2 \\
&= \langle Q z, z \rangle - \|P_j Q z\|^2 = \langle Q z, Q z \rangle - \|P_j Q z\|^2 \\
&= \|Qz\|^2 - \|P_j Q z\|^2
\end{aligned}$$

Subclaim: $\langle P_j x, y \rangle$ is nonzero and positive.

Proof of Subclaim: Notice that $\|Qz\|^2 = \|P_j Q z\|^2 + \|(I - P_j) Q z\|^2$ and hence $\|P_j Q z\|^2 \leq \|Qz\|^2$. Thus $\langle P_j x, y \rangle = \|Qz\|^2 - \|P_j Q z\|^2 \geq 0$. In particular, if $\langle P_j x, y \rangle = 0$ then this forces $P_j Q z = Qz$. Hence $Qz = x \in W_j$, contradicting the fact that $W_j \cap V_j = \{0\}$. Thus, $\langle P_j x, y \rangle$ is nonzero and positive.

Set $v_1 = x$ and $v_2 = x + \alpha y$ for some $\alpha \in \mathcal{H}$ which will be specified later. Since

$P_i(I-Q)=0$ for $i \neq j$, we have

$$\begin{aligned}\|P_i v_2\| &= \|P_i(x+\alpha y)\| = \|P_i(Qz+\alpha(I-Q)z)\| \\ &= \|P_iQz+\alpha P_i(I-Q)z\| = \|P_iQz\| = \|P_ix\| \\ &= \|P_i v_1\|\end{aligned}$$

for all $i \neq j$.

If $P_jy=0$ then we take α to be any nonzero scalar, and we have

$$\|P_jv_1\| = \|P_jx\| = \|P_jx+\alpha P_jy\| = \|P_jv_2\|.$$

If $P_jy \neq 0$ then we set $\alpha = -\frac{2\langle P_jx,y\rangle}{\|P_jy\|^2}$ and we have

$$\begin{aligned}\|P_jv_2\|^2 &= \|P_jx+\alpha P_jy\|^2 \\ &= \|P_jx\|^2 + \alpha\bar{\alpha}\|P_jy\|^2 + \bar{\alpha}\langle P_jx, P_jy\rangle + \alpha\langle P_jy, P_jx\rangle \\ &= \|P_jx\|^2 + \frac{4\langle P_jx,y\rangle\langle y,P_jx\rangle}{\|P_jy\|^4}\|P_jy\|^2 - \frac{2\langle y,P_jx\rangle\langle P_jx,y\rangle}{\|P_jy\|^2} - \frac{2\langle P_jx,y\rangle\langle y,P_jx\rangle}{\|P_jy\|^2} \\ &= \|P_jx\|^2 + \frac{4|\langle P_jx,y\rangle|^2}{\|P_jy\|^2} - \frac{2|\langle P_jx,y\rangle|^2}{\|P_jy\|^2} - \frac{2|\langle P_jx,y\rangle|^2}{\|P_jy\|^2} \\ &= \|P_jx\|^2 = \|P_jv_1\|^2.\end{aligned}$$

Thus $\|P_iv_1\| = \|P_iv_2\|$ for all $i=1,\ldots,N$. However, for any $\alpha \in \mathcal{H}\setminus\{0\}$ we have

$$\|v_2\|^2 = \|x\|^2 + |\alpha|^2\|y\|^2 > \|x\|^2 = \|v_1\|^2.$$

Hence $\{W_i\}_{i=1}^N$ does not allow norm retrieval, a contradiction to our assumption, and so $QP_j = P_jQ = 0$, which finishes the proof of Claim 2.

Therefore, we have that

$$W_j = \operatorname{im}(P_j) = \ker(Q) = V_j^{\perp}.$$

But if $i \neq j$ then $W_i \subseteq V_j$ and so $W_i \perp W_j$, from which (2) easily follows. □

4.3.3 Hyperplanes allowing phase retrieval

Hyperplanes are orthogonal complements of vectors and since classical phase retrieval is fully characterized in the real case then a natural question to ask is: Can we find examples of and classify when non-structured hyperplanes allow phase retrieval?

Using Theorem 4.3.8, we can answer this question positively and construct highly non-structured subspaces which allow phase retrieval. It is known that for any $N > M$, the full spark families of vectors $\{\varphi_i\}_{i=1}^N$ are a dense, open set of full measure within the families of vectors $\{\varphi_i\}_{i=1}^N$ such that $\sum_{i=1}^N \varphi_i\varphi_i^* = I$ [23]. From here it is easy to construct full spark vectors $\{\varphi_i\}_{i=1}^N$ in $\mathbb{R}^M$ with $\sum_{i=1}^N \varphi_i\varphi_i^* = I$ such that no two vectors are orthogonal. It follows that $\{\frac{\varphi_i}{\|\varphi_i\|}\}_{i=1}^N$ is full spark and thus allows phase retrieval. Letting P_i be the orthogonal projection onto span(φ_i), we have $P_i = \frac{\varphi_i\varphi_i^*}{\|\varphi_i\|^2}$ so

$$\sum_{i=1}^N \|\varphi_i\|^2 P_i = I,$$

and by Theorem 4.3.8, it follows that $\{W_i = (I - P_i)\mathbb{R}^M\}_{i=1}^N$ is a family of hyperplanes allowing phase retrieval. These are *unstructured* since structured subspaces would have the property that their orthogonal complements contain a large number of orthogonal vectors.

In light of these unstructured hyperplanes allowing phase retrieval, it is interesting to note that given subspaces $\{W_i\}_{i=1}^N$ which do not allow phase retrieval in

$\mathbb{R}^M$, it is always possible to find hyperplanes $\{W_i'\}_{i=1}^N$ such that $W_i \subseteq W_i'$ where $\{W_i'\}_{i=1}^N$ do not allow phase retrieval.

Proposition 4.3.13. If the subspaces $\{W_i\}_{i=1}^N$ do not allow phase retrieval in $\mathbb{R}^M$ then there exists $\{W_i'\}_{i=1}^N$ not allowing phase retrieval where $\dim W_i' = M-1$ and $W_i \subseteq W_i'$ for all $i = 1, \ldots, N$.

Proof. Since $\{W_i\}_{i=1}^N$ do not allow phase retrieval, there exists nonzero $x, y \in \mathbb{R}^M$ such that $x \neq \pm y$ and $\|P_i x\| = \|P_i y\|$ for all $i = 1, \ldots, N$. For i such that $\dim W_i = M-1$, let $W_i' = W_i$. For any other i, say $\dim W_i = D_i \leq M-2$, we construct W_i' as follows. Let $\{\varphi_1, \varphi_2\}$ be orthonormal vectors in $W_i^\perp$. Let $Z := \operatorname{span}(\{\varphi_1, \varphi_2\})$, with $P_Z : \mathbb{R}^M \to Z$ an orthogonal projection. Set

$$u := P_Z x \quad \text{and} \quad v := P_Z y,$$

and consider the function $f : Z \to \mathbb{R}$ given by

$$f(z) = |\langle u, z\rangle| - |\langle v, z\rangle| .$$

Let $z_1, z_2 \in Z$ be unit norm vectors such that $z_1 \perp u$ and $z_2 \perp v$. Then $f(z_1) \leq 0 \leq f(z_2)$, and by the intermediate value theorem, there exists a $z_0 \in Z$ where $f(z_0) = 0$ and hence $|\langle u, z_0\rangle| = |\langle v, z_0\rangle|$. We assume $z_0 \neq 0$. Note if $z_0 = 0$, then $z_1 = -z_2$, $f(z_1) = 0 = f(z_2)$, and we could instead choose $z_0 = z_1 \neq 0$. Letting

$W_i' = \text{span}(\{W_i, z_0\})$ with corresponding orthogonal projection P_i', we have

$$\begin{aligned}
\|P_i'x\|^2 &= \|P_ix\|^2 + \left|\left\langle x, \frac{z_0}{\|z_0\|}\right\rangle\right|^2 = \|P_ix\|^2 + \left|\left\langle x, P_Z\frac{z_0}{\|z_0\|}\right\rangle\right|^2 \\
&= \|P_ix\|^2 + \left|\left\langle u, \frac{z_0}{\|z_0\|}\right\rangle\right|^2 = \|P_iy\|^2 + \left|\left\langle v, \frac{z_0}{\|z_0\|}\right\rangle\right|^2 \\
&= \|P_iy\|^2 + \left|\left\langle y, P_Z\frac{z_0}{\|z_0\|}\right\rangle\right|^2 = \|P_i'y\|^2.
\end{aligned}$$

It follows that $\{W_i'\}_{i=1}^N$ do not allow phase retrieval. We now may iterate this argument until $\dim W_i' = M-1$ for all $i = 1, \dots N$. □

4.3.4 Using norm retrieval to characterize subspaces allowing phase retrieval

In this section we fully classify phase retrieval by subspaces components through the use of norm retrieval.

Since orthonormal bases are very restrictive, we would like to relax the conditions in Theorem 4.2.25 to see what properties the vectors within the subspaces have when the $\{W_i\}_{i=1}^N$ are assumed to allow phase retrieval. A natural next step would be to look at linearly independent vectors as opposed to orthogonal vectors. In particular, since unitary operators are the only linear operators which preserve orthogonality, by moving to linearly independent vectors we can lessen the conditions of our operators and look at invertible operators. This way we can see if invertible operators preserve phase retrieval. Hence we could look at linearly independent vectors as opposed to orthonormal vectors to see if any results fold out.

However, the following example shows that if $\{W_i\}_{i=1}^N$ allows phase retrieval in $\mathbb{R}^M$ and $\{\varphi_{ij}\}_{j=1}^{D_i}$ are linearly independent vectors within W_i for each $i \in$

$\{1, \ldots, N\}$ then it is not necessarily true that $\{\varphi_{ij}\}_{j=1,i=1}^{D_i,N}$ allows phase retrieval in $\mathbb{R}^M$.

Example 4.3.14. Let $\{e_i\}_{i=1}^3$ be an orthonormal basis for $\mathbb{R}^3$. Define the subspaces

$$W_1 = \text{span}\{e_1, e_2\}, W_2 = \text{span}\{e_2\}, W_3 = \text{span}\{e_3\},$$

$$W_4 = \text{span}\{\frac{e_1 + e_2}{2}\}, W_5 = \text{span}\{\frac{e_2 + e_3}{2}\}, W_6 = \text{span}\{\frac{e_1 + e_3}{2}\}.$$

Let $x \in \mathbb{R}^3$. Then $x = \sum_{i=1}^3 \alpha_i e_i$ where $\alpha_i = \langle x, e_i \rangle$ for $i = 1, 2, 3$. We have

$$||P_{W_i}x||^2 = \begin{cases} \alpha_1^2 + \alpha_2^2, & i = 1 \\ \alpha_2^2, & i = 2 \\ \alpha_3^2, & i = 3 \\ \frac{1}{2}\left(\alpha_1 + \alpha_2\right)^2, & i = 4 \\ \frac{1}{2}\left(\alpha_2 + \alpha_3\right)^2, & i = 5 \\ \frac{1}{2}\left(\alpha_1 + \alpha_3\right)^2, & i = 6 \end{cases}$$

First, we will show that we can recover $\pm x$ from $\{||P_{W_i}x||^2\}_{i=1}^6$.

We can recover the absolute values of the coefficients:

$$|\alpha_1| = \sqrt{||P_{W_1}x||^2 - ||P_{W_2}x||^2}, |\alpha_2| = ||P_{W_2}x||, |\alpha_3| = ||P_{W_3}x||.$$

Thus, if two of the coefficients are zero then we have $x = \pm|\alpha_i|e_i$ for some i. From now on we will assume at least two of the coefficients are nonzero.

Case 1: Assume $\alpha_1 = 0$. We may assume without loss of generality that $\alpha_2 > 0$, and thus $\alpha_2 = ||P_{W_2}x||$. Finally,

$$\alpha_3 = \frac{2||P_{W_5}x||^2 - ||P_{W_2}x||^2 - ||P_{W_3}x||^2}{2||P_{W_2}x||}.$$

Case 2: Assume $\alpha_1 \neq 0$. We may assume WLOG that $\alpha_1 > 0$, and thus

$$\alpha_1 = \sqrt{||P_{W_1}x||^2 - ||P_{W_2}x||^2}.$$

We have

$$\alpha_2 = \frac{2||P_{W_4}x||^2 - ||P_{W_1}x||^2}{2\sqrt{||P_{W_1}x||^2 - ||P_{W_2}x||^2}}$$

and

$$\alpha_3 = \frac{2||P_{W_6}x||^2 + ||P_{W_2}x||^2 - ||P_{W_1}x||^2 - ||P_{W_3}x||^2}{2\sqrt{||P_{W_1}x||^2 - ||P_{W_2}x||^2}}.$$

This shows that $\{W_i\}_{i=1}^6$ allows phase retrieval.

However, if we choose the linearly independent (and not orthonormal) basis $\{e_1+e_2, e_2\}$ for W_1 and the spanning element from the other subspaces, we get the set of vectors $\{e_1+e_2, e_2, e_2, e_3, e_1+e_2, e_2+e_3, e_1+e_3\} = \{e_1+e_2, e_2, e_3, e_2+e_3, e_1+e_3\}$. Notice that if we partition this set as follows $\{e_2+e_3, e_2, e_3\}, \{e_1+e_2, e_1+e_3\}$ then neither set spans $\mathbb{R}^3$. Hence this set does not have the complement property and therefore does not allow phase retrieval.

Remark 4.3.15. In Example 4.3.14, $\{W_i^\perp\}_{i=1}^6$ allows phase retrieval. To see this, we just need to see that this family allows norm retrieval. Let $\{Q_i\}_{i=1}^6$ be the orthogonal projections onto each of $\{W_i^\perp\}_{i=1}^6$. Then,

$$W_1^\perp = \text{span}\{e_3\}, \quad W_2^\perp = \text{span}\{e_1, e_3\} \quad W_3^\perp = \text{span}\{e_1, e_2\}.$$

So given a vector x, we have

$$\|Q_1 x\|^2 = |\langle x, e_3\rangle|^2 \text{ and } \|Q_2 x\|^2 = |\langle x, e_1\rangle|^2 + |\langle x, e_3\rangle|^2$$

$$\text{and } \|Q_3 x\|^2 = |\langle x, e_1\rangle|^2 + |\langle x, e_2\rangle|^2.$$

Hence we know $\{|\langle x, e_i\rangle|^2\}_{i=1}^3$ and so we know $\|x\|^2$.

In light of Example 4.3.14 we cannot replace "orthonormal bases" with "spanning sets" in Theorem 4.2.25 (2). The next theorem shows that the key property of orthonormal bases in Theorem 4.2.25 is not just that they span, but that they give norm retrieval.

Theorem 4.3.16. Let $\{W_i\}_{i=1}^N$ be subspaces of $\mathcal{H}_M$. The following are equivalent:

1. $\{W_i\}_{i=1}^N$ allows phase retrieval in $\mathcal{H}_M$.

2. For every sequence $\{\varphi_{i,j}\}_{j=1}^{J_i} \subset W_i$ which gives norm retrieval in W_i, the sequence $\{\varphi_{i,j}\}_{j=1,i=1}^{J_i,N}$ allows phase retrieval.

Proof. (1) $\Rightarrow$ (2) For each $i = 1, \ldots, N$ let $\{\varphi_{i,j}\}_{j=1}^{J_i}$ be a sequence in W_i which gives norm retrieval in W_i. Let $x, y \in \mathcal{H}_M$ such that $|\langle x, \varphi_{i,j}\rangle| = |\langle y, \varphi_{i,j}\rangle|$ for all $j = 1, \ldots, J_i$, $i = 1, \ldots, N$. For each $i = 1, \ldots, N$ let P_i be the projection onto W_i. We have

$$|\langle P_i x, \varphi_{i,j}\rangle| = |\langle x, P_i\varphi_{i,j}\rangle| = |\langle x, \varphi_{i,j}\rangle| = |\langle y, \varphi_{i,j}\rangle| = |\langle y, P_i\varphi_{i,j}\rangle| = |\langle P_i y, \varphi_{i,j}\rangle|,$$

and since $\{\varphi_{i,j}\}_{j=1}^{J_i}$ gives norm retrieval in W_i we have $\|P_i x\| = \|P_i y\|$ for all $i = 1, \ldots, N$. Since $\{W_i\}_{i=1}^N$ gives phase retrieval, we have $x = cy$ for some c with $|c| = 1$.

(2) $\Rightarrow$ (1) Since orthonormal bases give norm retrieval, (2) implies that each sequence $\{\varphi_{i,j}\}_{j=1,i=1}^{J_i,N}$ gives phase retrieval, where $\{\varphi_{i,j}\}_{j=1}^{J_i}$ is an orthonormal basis for W_i. Theorem 4.2.25 implies that the subspaces $\{W_i\}_{i=1}^N$ allow phase retrieval.

□

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