QUASILINEAR ELLIPTIC EQUATIONS WITH SUB-NATURAL GROWTH AND NONLINEAR POTENTIALS

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ABSTRACT

In this thesis, we study quasilinear elliptic equations of the type

$$-\Delta_p u = \sigma u^q \quad \text{in } \mathbb{R}^n, \tag{0.1}$$

where $\Delta_p u = \nabla\cdot(\nabla u|\nabla u|^{p-2})$ is the p-Laplacian, $1 < p < \infty$, and $\sigma \geq 0$ is an arbitrary locally integrable function, or measure, in the *sub-natural growth* case $0 < q < p-1$.

Necessary and sufficient conditions on σ for the existence of *finite energy* and *weak* solutions to (0.1) are given. Sharp global pointwise estimates of solutions are obtained as well. We also discuss the uniqueness and regularity properties of solutions. As a consequence, characterization of solvability of the equation

$$-\Delta_p v = b\,\frac{|\nabla v|^p}{v} + \sigma \quad \text{in} \quad \mathbb{R}^n, \tag{0.2}$$

where $b > 0$, is deduced.

Our main tools are Wolff potential estimates, dyadic models, and related integral inequalities. Special nonlinear potentials of Wolff type associated with "sublinear" problems are constructed to obtain sharp bounds of solutions. We also treat equations with the fractional Laplacians $(-\Delta)^\alpha$. Our approach is applicable to more general quasilinear $\mathcal{A}$-Laplace operators $\mathrm{div}\mathcal{A}(x,\nabla\cdot)$ as well as the fully nonlinear k-Hessian operators.

Chapter 1

Introduction

This work is concerned with quasilinear problems of the following type:

$$\begin{cases} -\Delta_p u = \sigma u^q & \text{in } \mathbb{R}^n, \\ \liminf\limits_{|x|\to\infty} u(x) = 0, \ \ u > 0, \end{cases} \tag{1.1}$$

where $\Delta_p u = \nabla\cdot(\nabla u|\nabla u|^{p-2})$ is the p-Laplacian, $1 < p < \infty$, and $\sigma \geq 0$ is an arbitrary locally integrable function, or measure, in the case $0 < q < p-1$ (*sub-natural* growth rate).

We will give necessary and sufficient conditions for the existence of finite energy and weak solutions to (1.1). Sharp global pointwise bounds along with regularity properties of solutions are obtained as well. We identify crucial integral inequalities and introduce new nonlinear potentials of Wolff type adapted for these problems. We also study the fractional Laplacian equation

$$
(1.2) \qquad \begin{cases} (-\Delta)^{\alpha} u = \sigma u^q & \text{in } \mathbb{R}^n, \\ \liminf\limits_{|x|\to\infty} u(x) = 0, \quad u > 0, \end{cases}
$$

where $0 < \alpha < \frac{n}{2}$ and $0 < q < 1$.

In the classical case $p = 2$ and $0 < q < 1$, equation (1.1), or (1.2) with $\alpha = 1$, serves as a model *sublinear* elliptic problem. Such problems were studied by H. Brezis and S. Kamin in [BK92], where they gave necessary and sufficient conditions for the existence of *bounded* solutions. In particular, they proved that equation (1.1) has a bounded solution u if and only if $\mathbf{I}_2\sigma \in L^{\infty}(\mathbb{R}^n)$, where $\mathbf{I}_2\sigma$ is the Riesz potential of order 2 (Newtonian potential) defined by

$$
(1.3) \qquad \mathbf{I}_2\sigma(x) = \int_0^{\infty} \frac{\sigma(B(x,r))}{r^{n-2}} \frac{dr}{r} = c_n \int_{\mathbb{R}^n} \frac{d\sigma(y)}{|x-y|^{n-2}}, \quad x \in \mathbb{R}^n.
$$

Moreover, such a solution u is unique and there is a constant $c = c(n,q) > 0$ so that

$$
c^{-1} (\mathbf{I}_2\sigma(x))^{\frac{1}{1-q}} \leq u(x) \leq c\, \mathbf{I}_2\sigma(x), \quad \forall\, x \in \mathbb{R}^n.
$$

Both the lower and upper estimates are sharp in a sense, as was pointed out in [BK92]. However, there is a substantial gap between them. The difficulty here is to find matching lower and upper bounds of solutions and getting such a result is our motivation.

Analogous sublinear problems in bounded domains $\Omega \subset \mathbb{R}^n$ for various classes of σ have been extensively studied. In particular, Boccardo and Orsina [BO96], [BO12], and Abdel Hamid and Bidaut-Véron [AHBV10] gave sufficient conditions for the existence of solutions under the assumption $\sigma \in L^r(\Omega)$. Earlier results, under more restrictive assumptions on σ are due to Krasnoselskii [Kra64], and Brezis and Oswald [BrOs86] (see also the literature cited there).

We will consider two different types of solutions : *finite energy* and *weak* solutions. The first one is a solution which lies in $L^q_{\text{loc}}(\mathbb{R}^n, d\sigma)$ and belongs to the homogeneous Sobolev (or Dirichlet) space $L_0^{1,p}(\mathbb{R}^n)$ defined in Chapter 2. The second one only requires solutions in $L^q_{\text{loc}}(\mathbb{R}^n, d\sigma)$. Finding weak solutions is considered to be much more complicated than that of finite energy solutions. In both cases, we are able to obtain necessary and sufficient conditions for the existence of such solutions.

We employ powerful Wolff potential estimates due to Kilpeläinen and Malý, Trudinger and Wang, and Labutin in [KM94, La02, TW02b, KuMi14]. This makes it possible to replace the p-Laplacian Δ_p in the model problem (1.1) by a more general quasilinear operator $\text{div}\mathcal{A}(x, \nabla\cdot)$ with bounded measurable coefficients, under standard structural assumptions on $\mathcal{A}(x,\xi)$ which ensure that $\mathcal{A}(x,\xi)\cdot\xi \approx |\xi|^p$ [HKM06, MZ97], or a fully nonlinear operator of k-Hessian type [TW99, La02] (see also [PV09], [JV12]), and the fractional Laplacian $(-\Delta)^\alpha$ as well.

The Wolff potential $\mathbf{W}_{\alpha,p}\sigma$ of a nonnegative locally Borel measure σ on $\mathbb{R}^n$, is defined, for $1 < p < \infty$ and $0 < \alpha < \frac{n}{p}$, by ([HW83, AH96]):

$$\mathbf{W}_{\alpha,p}(x) = \int_0^\infty \left(\frac{\sigma(B(x,t))}{t^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t}. \tag{1.4}$$

Here $B(x,t) = \{y \in \mathbb{R}^n : \ |x-y| < t\}$ is a ball centered at $x \in \mathbb{R}^n$ of radius $t > 0$.

To study (1.1), we make use of the pointwise estimates of solutions to quasilinear equations in terms of Wolff potentials, which were discovered by Kilpeläinen and Malý in [KM94, Ki02]. One of their main theorems states that if $U \geq 0$ is a p-superharmonic (or locally renormalized, see Chapter 2) solution to the equation

$$
\begin{cases} -\Delta_p U = \sigma \quad \text{in} \quad \mathbb{R}^n, \\ \liminf_{|x|\to\infty} U(x) = 0, \end{cases} \tag{1.5}
$$

then there exists a constant $K > 0$ which depends only on p and n such that

$$
\frac{1}{K}\,\mathbf{W}_{1,p}\sigma(x) \leq U(x) \leq K\,\mathbf{W}_{1,p}\sigma(x), \quad x \in \mathbb{R}^n. \tag{1.6}
$$

Moreover, a solution U to (1.5) exists if and only if $1 < p < n$ and $\mathbf{W}_{1,p}\sigma \not\equiv +\infty$ (see [PV08]), or equivalently,

$$
\int_1^\infty \left(\frac{\sigma(B(0,t))}{t^{n-p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} < +\infty. \tag{1.7}
$$

We study the general framework of integral equations which are closely related to nonlinear elliptic PDE mentioned above,

$$
u = \mathbf{W}_{\alpha,p}(u^q d\sigma) \quad \text{in } \mathbb{R}^n, u \geq 0, \tag{1.8}
$$

where $1 < p < \infty, 0 < \alpha < \frac{n}{p}$. When $\alpha = 1$, it gives the corresponding p-Laplacian, and $\alpha = \frac{2k}{k+1}, p = k+1$ correspond to the k-Hessian operator. We notice that in the special case $p = 2$, (1.8) brings the fractional Laplace equation (1.2) into the equivalent integral form $u = \frac{1}{c(\alpha,n)}\,\mathbf{I}_{2\alpha}(u^q d\sigma)$, where $\mathbf{I}_{2\alpha}\mu$ is the Riesz potential of order 2α with $d\mu = u^q d\sigma$, defined by (2.13). For the sake of simplicity, the constant $c(\alpha, n)$ will be dropped.

We are also interested in the supersolution of (1.8), which is a solution of the following nonlinear integral inequality

$$u \geq \mathbf{W}_{\alpha,p}(u^q d\sigma) \quad \text{in } \mathbb{R}^n, u \geq 0. \tag{1.9}$$

Our first result states that if u is a nontrivial supersolution to either equation (1.1) or integral equation (1.8) with $\alpha = 1$, then u satisfies the following lower estimate

$$u(x) \geq C\, (\mathbf{W}_{1,p}\sigma(x))^{\frac{p-1}{p-1-q}}, \quad \forall\, x \in \mathbb{R}^n, \tag{1.10}$$

where $C > 0$ is a constant depending only on n, p and q. Consequently, a necessary condition for the existence of a nontrivial solution to (1.1) is that $\mathbf{W}_{1,p}\sigma \not\equiv +\infty$, i.e., (1.7) holds. Estimate (1.10) is one of our key tools to obtain the main results in this work.

Concerning the finite energy solutions, we make an observation due to Brezis and Browder, which says that, if there is a solution $u \in L_0^{1,p}(\mathbb{R}^n)$ to (1.1), then u belongs to $L^{1+q}(\mathbb{R}^n, d\sigma)$ globally. Combining this fact with (1.10), we arrive at a necessary condition

$$\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\sigma)^{\frac{(1+q)(p-1)}{p-1-q}}\, d\sigma < \infty. \tag{1.11}$$

As shown in Chapter 3, condition (1.11) holds if and only if there exists a constant $C > 0$ such that the following weighted norm inequality holds,

$$\Big(\int_{\mathbb{R}^n} |\varphi|^{1+q}\, d\sigma \Big)^{\frac{1}{1+q}} \leq C\, ||\nabla\varphi||_{L^p(\mathbb{R}^n)}, \quad \forall\, \varphi \in C_0^\infty(\mathbb{R}^n). \tag{1.12}$$

Some equivalent characterizations for (1.11) will be discussed in Chapter 3 as well.

One might ask whether (1.11) is sufficient for the existence of finite energy solutions to (1.1). An affirmative answer to this question is given in the same chapter. We remark that (1.11) yields $\mathbf{W}_{1,p}\sigma \not\equiv +\infty$, which is the necessary condition mentioned above. Moreover, we are able to show that such a finite energy solution is *unique*.

When studying weak solutions, it turns out that (1.1) is closely related to the following important integral inequality

$$\left(\int_{\mathbb{R}^n} |\varphi|^q \, d\sigma\right)^{\frac{1}{q}} \leq \varkappa \, ||\Delta_p \varphi||_{L^1(\mathbb{R}^n)}^{\frac{1}{p-1}}, \tag{1.13}$$

for all test functions φ such that $-\Delta_p \varphi \geq 0$, $\liminf_{x\to\infty} \varphi(x) = 0$. Here $\varkappa$ denotes the best constant in (1.13). Such an inequality (1.13) corresponds to the end-point case of the (L^p, L^q)-trace inequality for $p > 1$, and is less studied when comparing with the well-studied inequality (1.12).

Having (1.6) in hand, we see that (1.13) is equivalent to the inequality

$$||\mathbf{W}_{1,p}\nu||_{L^q(\mathbb{R}^n, d\sigma)} \leq \kappa (\nu(\mathbb{R}^n))^{\frac{1}{p-1}}, \quad \forall \nu \in M^+(\mathbb{R}^n), \tag{1.14}$$

here κ denotes the least constant in the above inequality. We will need a local version of the preceding inequality, where the measure $\sigma = \sigma_B$ is restricted to a ball B in $\mathbb{R}^n$:

$$||\mathbf{W}_{1,p}\nu||_{L^q(d\sigma_B)} \leq \kappa(B) (\nu(\mathbb{R}^n))^{\frac{1}{p-1}}, \quad \forall \nu \in M^+(\mathbb{R}^n), \tag{1.15}$$

where $\kappa(B)$ is the best constant in this inequality. These constants are used as building blocks in our key tool, a new nonlinear potential of Wolff type,

$$\mathbf{K}_{1,p,q}\sigma(x) = \int_0^\infty \left(\frac{\kappa(B(x,s))^{\frac{q(p-1)}{p-1-q}}}{s^{n-p}} \right)^{\frac{1}{p-1}} \frac{ds}{s}, \quad x \in \mathbb{R}^n. \tag{1.16}$$

This intrinsic nonlinear potential of Wolff type has never appeared before and will control the solutions to (1.1). Similarly to (1.10), we also have an estimate for a nontrivial supersolution u to (1.1) as follows

$$u(x) \geq C\,\mathbf{K}_{1,p,q}\sigma(x), \quad x \in \mathbb{R}^n, \tag{1.17}$$

where $C > 0$ is a constant depending only on n, p, and q. Consequently, another necessary condition is that $\mathbf{K}_{1,p,q}\sigma \not\equiv \infty$, or equivalently,

$$\int_1^\infty \left(\frac{\kappa(B(0,s))^{\frac{q(p-1)}{p-1-q}}}{s^{n-p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} < \infty. \tag{1.18}$$

The potential $\mathbf{K}_{1,p,q}\sigma$ together with the usual Wolff potential $\mathbf{W}_{1,p}\sigma$ provides sharp global pointwise estimates of the solutions to (1.1). Using both of them allow us to bridge the gap in the estimates of Brezis-Kamin. In particular, as we will show in Chapter 4, (1.7) and (1.18) are necessary and sufficient for the existence of a *miminal* weak solution u to (1.1). Moreover, such a solution u has matching lower and upper estimates as follows

$$c^{-1}\left(\mathbf{K}_{1,p,q}\sigma + (\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}}\right) \leq u \leq c\left(\mathbf{K}_{1,p,q}\sigma + (\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}}\right), \tag{1.19}$$

where $c > 0$ is a constant depending only on n, p, and q.

If one wishes to have a solution with $W^{1,p}_{\rm loc}$-regularity, then together with (1.7) and (1.18), a local version of (1.11) is needed, i.e.,

$$\int_B (\mathbf{W}_{1,p}\sigma_B)^{\frac{(1+q)(p-1)}{p-1-q}}\, d\sigma < \infty, \tag{1.20}$$

for all balls B in $\mathbb{R}^n$. We will also see that (1.20) is necessary in order to have a solution in $W^{1,p}_{\rm loc}(\mathbb{R}^n)$ to (1.1).

For the fractional Laplace equation (1.2), let $\mathfrak{K}(B)$ denote the least constant in the integral inequality

$$||\mathbf{I}_{2\alpha}\nu||_{L^q(d\sigma_B)} \le \mathfrak{K}(B)\nu(\mathbb{R}^n), \quad \forall \nu \in M^+(\mathbb{R}^n). \tag{1.21}$$

We define the corresponding nonlinear potential of Wolff type by

$$\mathbf{K}_{\alpha,q}\sigma(x) = \int_0^\infty \frac{\mathfrak{K}(B(x,s))^{\frac{q}{1-q}}}{s^{n-2\alpha}}\frac{ds}{s}, \quad x \in \mathbb{R}^n. \tag{1.22}$$

To ensure that both $\mathbf{K}_{\alpha,q}\sigma$ and $\mathbf{I}_{2\alpha}\sigma$ are not indentically infinite, the following condition should hold

$$\int_1^\infty \frac{\mathfrak{K}(B(0,s))^{\frac{q}{1-q}}}{s^{n-2\alpha}}\frac{ds}{s} + \int_1^\infty \frac{\sigma(B(0,s))}{s^{n-2\alpha}}\frac{ds}{s} < \infty. \tag{1.23}$$

As a consequence of our main results, (1.23) is necessary and sufficient for the existence of a solution u to the sublinear fractional Laplace equation (1.2), and there exists a minimal solution u which satisfies

$$c^{-1}\left(\mathbf{K}_{\alpha,q}\sigma + (\mathbf{I}_{2\alpha}\sigma)^{\frac{1}{1-q}}\right) \le u \le c\left(\mathbf{K}_{\alpha,q}\sigma + (\mathbf{I}_{2\alpha}\sigma)^{\frac{1}{1-q}}\right). \tag{1.24}$$

We also observe that if there is a nontrivial p-superharmonic supersolution to (1.1) or (1.8) with $\alpha = 1$, then the measure σ is absolutely continuous with respect to p-capacity $\text{cap}_p(\cdot)$, i.e., $\sigma(E) = 0$ if $\text{cap}_p(E) = 0$, for all compact sets E in $\mathbb{R}^n$, where

$$\text{cap}_p(E) = \inf\{||\nabla f||_{L^p}^p : f \geq 1 \text{ on } E,\ f \in C_0^\infty(\mathbb{R}^n)\}. \tag{1.25}$$

In Chapter 5, we study problem (1.1) under the assumption that

$$\sigma(E) \leq C(\sigma)\ \text{cap}_p(E) \text{ for all compact sets } E \subset \mathbb{R}^n, \tag{1.26}$$

where $C(\sigma)$ is a positive constant. Such a condition was considered in [JV10, JV12, Maz11]. Making use of the sub-supersolutions method, we obtain the main result in this direction which states that if both (1.7) and (1.26) hold, then there exists a distributional solution $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ to (1.1) and u satisfies

$$C^{-1}\Big(\mathbf{W}_{1,p}\sigma\Big)^{\frac{p-1}{p-1-q}} \leq u \leq C\,\Big(\mathbf{W}_{1,p}\sigma + (\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}}\Big), \tag{1.27}$$

where $C > 0$ depends only on n, p, q, and $C(\sigma)$.

Both estimates are sharp as in the Brezis-Kamin theorem. In the case $p = 2, 0 < q < 1$, we remark that if $\mathbf{I}_2\sigma \in L^\infty(\mathbb{R}^n)$ (considered by Brezis-Kamin), then by a well known result it follows that (1.26) holds with $p = 2$. Hence, by (1.27), there exists a bounded solution u to (1.1) with $p = 2$ and u satisfies

$$c^{-1}\,(\mathbf{I}_2\sigma)^{\frac{1}{1-q}} \leq u \leq c\,\mathbf{I}_2\sigma,$$

where $c = c(n, q)$. From this observation we see that our condition is weaker than that of Brezis-Kamin. Moreover, our results extend to possibly unbounded solutions as well.

Also, under conditions (1.7) and (1.26), we obtain the existence and pointwise estimates for a solution u to (1.1) with positive lower bound, i.e., equation of the type

$$\begin{cases} -\Delta_p u = \sigma u^q \quad \text{in} \quad \mathbb{R}^n, \\ \liminf\limits_{|x|\to\infty} u(x) = r, \end{cases} \tag{1.28}$$

where $r > 0$, and u has matching upper and lower bounds as follows

$$c^{-1}\Big(r + \mathbf{W}_{1,p}\sigma(x)\Big)^{\frac{p-1}{p-1-q}} \leq u(x) \leq c\Big(r + \mathbf{W}_{1,p}\sigma(x)\Big)^{\frac{p-1}{p-1-q}}, \quad x \in \mathbb{R}^n. \tag{1.29}$$

From (1.27), we can deduce, using (1.6) and (1.10),

$$\mathbf{W}_{1,p}\left((\mathbf{W}_{1,p}\sigma)^{\frac{(p-1)q}{p-1-q}}\, d\sigma\right) \leq \kappa\left(\mathbf{W}_{1,p}\sigma + (\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}}\right) < \infty \ \text{ a.e.}, \tag{1.30}$$

where $\kappa = \kappa(n, p, q)$ is a positive constant. It turns out that the preceding condition is enough to have a solution which satisfies (1.27). So (1.30) is not only necessary but also sufficient for the existence of solutions to (1.1) satisfying (1.27).

In the case $p = 2$ and σ is radial, we establish bilateral bounds of solution u to

(1.1) as follows.

$$
\begin{aligned}
(1.31)\quad & c^{-1}\left(\frac{1}{|x|^{n-2}}\left(\int_{|y|<|x|}\frac{d\sigma(y)}{|y|^{(n-2)q}}\right)^{\frac{1}{1-q}}+\left(\int_{|y|\geq|x|}\frac{d\sigma(y)}{|y|^{n-2}}\right)^{\frac{1}{1-q}}\right)\\
& \leq u(x)\leq c\left(\frac{1}{|x|^{n-2}}\left(\int_{|y|<|x|}\frac{d\sigma(y)}{|y|^{(n-2)q}}\right)^{\frac{1}{1-q}}+\left(\int_{|y|\geq|x|}\frac{d\sigma(y)}{|y|^{n-2}}\right)^{\frac{1}{1-q}}\right), \quad x\in\mathbb{R}^n.
\end{aligned}
$$

Let us make a remark that we will be referring to (1.1) with $1 < p < \infty$ and $0 < q < p-1$, as well as other nonlinear equations where analogous phenomena occur in a natural way, as *sublinear problems* in general. One of the main features that distinguishes them from the case $p \geq p-1$ is the absence of any smallness asumptions on σ.

Simultaneously with (1.1), we consider in Chapter 4 the equation with the singular natural growth in the gradient term, which is closely related to (1.1) :

$$
(1.32)\qquad \begin{cases} -\triangle_p v = b\,\frac{|\nabla v|^p}{v} + \sigma \quad \text{in} \quad \mathbb{R}^n, \\ \\ \liminf\limits_{x\to\infty} v = 0. \end{cases}
$$

where b is a positive constant defined by

$$
(1.33)\qquad b = \frac{q(p-1)}{p-1-q}, \quad 0 < q < p-1.
$$

Equation (1.32) with $p = 2$ in a bounded domain $\Omega \subset \mathbb{R}^n$ has been studied by D. Arcoya et al. in [ABLP10] and B. Abdellaoui et al. in [AGPW11], in which they gave sufficient conditions for the existence of solutions in certain Sobolev spaces under the assumption $\sigma \in L^s(\Omega)$ for some $s > 1$ (see also the literature given there).

It is well known that *formally* the substitution

$$v = \frac{p-1}{p-1-q}\, u^{\frac{p-1-q}{p-1}} \tag{1.34}$$

reduces (1.1) to (1.32), and vice versa. However, in general, this substitution fails for some solutions u and v since some certain singular measures can arise, as was first noticed by Ferone and Murat [FM00] who studied a similar phenomenon in the case $q = p-1$. Nevertheless, a careful justification enables us to give necessary and sufficient conditions for the existence of *weak* and *finite energy* solutions to (1.32), and obtain pointwise estimates of such solutions as well. As we will demonstrate in Chapter 4, if u is a solution to (1.1) then v is a solution to (1.32) via substitution (1.34); but in the opposite direction, if v is a weak solution to (1.32) then u is only a *supersolution* to (1.1), which is enough for our purposes.

The content of this thesis is as follows. In Chapter 2 we introduce basic definitions and notations, along with several useful results on quasilinear equations and Wolff potentials estimates that will be used throughout the text. In Chapter 3 we study finite energy solutions to (1.1) and prove the uniqueness property of such solutions. Chapter 4 is devoted to a study of weak solutions to equations (1.1) and (1.32). The intrinsic nonlinear potentials of Wolff type are introduced in this chapter as well. Chapter 5 is concerned with equation (1.1) under the assumption (1.26). We provide, in Appendix A, the proof of uniqueness of solutions to (1.5) in a bounded domain when σ is absolutely continuous with respect to $\mathrm{cap}_p(\cdot)$. Finally, the weak compactness property of $L_0^{1,p}(\mathbb{R}^n)$ is given in Appendix B.

The content of this dissertation is taken from the joint works with Professor Igor E. Verbitsky [CV14a, CV14b, CV15].

Chapter 2

Preliminaries

2.1 Notations, definitions

Given an open set $\Omega \subseteq \mathbb{R}^n$, we denote by $M^+(\Omega)$ the class of all nonnegative Borel measures in Ω which are finite on compact subsets of Ω. If $\sigma \in M^+(\Omega)$, the σ-measure of a measurable set $E \subset \Omega$ is denoted by $|E|_\sigma = \sigma(E) = \int_E d\sigma$. We write $A \approx B$ if there are two universal constants c_1 and c_2 such that $c_1 A \leq B \leq c_2 A$.

For $p > 0$ and $\sigma \in M^+(\Omega)$, we denote by $L^p(\Omega, d\sigma)$ ($L^p_{\mathrm{loc}}(\Omega, d\sigma)$, respectively) the space of all measurable functions f such that $|f|^p$ is integrable (locally integrable) with respect to σ. For $f \in L^p(\Omega, d\sigma)$, we set

$$||f||_{L^p(\Omega, d\sigma)} = \Big(\int_\Omega |f|^p \, d\sigma \Big)^{\frac{1}{p}}.$$

When $d\sigma = dx$, we write $L^p(\Omega)$ (respectively $L^p_{\mathrm{loc}}(\Omega)$), and denote Lebesgue measure of $E \subset \mathbb{R}^n$ by $|E|$.

The Sobolev space $W^{1,p}(\Omega)$ ($W^{1,p}_{\rm loc}(\Omega)$, respectively) is the space of all functions f such that $f \in L^p(\Omega)$ and $|\nabla f| \in L^p(\Omega)$ ($f \in L^p_{\rm loc}(\Omega)$ and $|\nabla f| \in L^p_{\rm loc}(\Omega)$, respectively). As usual, $W^{1,p}_0(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the Sobolev norm

$$|||f|||_{1,p} = ||f||_{L^p} + ||\nabla f||_{L^p}.$$

By $L^{1,p}_0(\Omega)$ we denote the homogeneous Sobolev space, i.e., the space of functions $f \in W^{1,p}_{\rm loc}(\Omega)$ such that $|\nabla f| \in L^p(\Omega)$, and $||\nabla f - \nabla \varphi_j||_{L^p(\Omega)} \to 0$ as $j \to \infty$ for a sequence $\varphi_j \in C_0^\infty(\Omega)$.

When $1 < p < n$ and $\Omega = \mathbb{R}^n$, we will identify $L^{1,p}_0(\mathbb{R}^n)$ with the space of all functions $f \in W^{1,p}_{\rm loc}(\mathbb{R}^n)$ such that $f \in L^{\frac{np}{n-p}}(\mathbb{R}^n)$ and $|\nabla f| \in L^p(\mathbb{R}^n)$. For $f \in L^{1,p}_0(\mathbb{R}^n)$, the norm $||f||_{1,p} = ||\nabla f||_{L^p}$ is equivalent to

$$||f||_{L^{\frac{np}{n-p}}(\mathbb{R}^n)} + ||\nabla f||_{L^p(\mathbb{R}^n)}.$$

It is easy to see that $C_0^\infty(\mathbb{R}^n)$ is dense in $L^{1,p}_0(\mathbb{R}^n)$ with respect to this norm (see, e.g., [MZ97], Sec. 1.3.4).

The dual Sobolev space $L^{-1,p'}(\Omega) = L^{1,p}_0(\Omega)^*$ is the space of distributions $\nu \in D'(\Omega)$ such that

$$||\nu||_{-1,p'} = \sup \frac{|\langle f, \nu \rangle|}{||f||_{1,p}} < +\infty,$$

where the supremum is taken over all $f \in L^{1,p}_0(\Omega)$, $f \neq 0$.

We write $\nu \in L^{-1,p'}_{\rm loc}(\Omega)$ if $\varphi\,\nu \in L^{-1,p'}(\Omega)$, for every $\varphi \in C_0^\infty(\Omega)$.

We denote by $W^{-1,p'}(\Omega)$ the dual space of $W^{1,p}_0(\Omega)$. We say $\nu \in W^{-1,p'}_{\rm loc}(\Omega)$ if $\varphi\,\nu \in W^{-1,p'}(\Omega)$, for every $\varphi \in C_0^\infty(\Omega)$.

We will need Wolff's inequality [HW83] (see also [AH96], Sec. 4.5) in the case

$\Omega = \mathbb{R}^n$ for $\nu \in M^+(\mathbb{R}^n)$:

(2.1) $$c^{-1}\,||\nu||^{p'}_{-1,p'} \le \int_{\mathbb{R}^n} \mathbf{W}_{1,p}\nu\, d\nu \le c||\nu||^{p'}_{-1,p'},$$

where $1 < p < n$, and c is a positive constant which depends only on n and p. There is a local version of Wolff's inequality (see [AH96], Theorem 4.5.5):

(2.2) $$\nu \in M^+(\mathbb{R}^n) \cap W^{-1,p'}_{\text{loc}}(\mathbb{R}^n) \Longleftrightarrow \int_B \mathbf{W}_{1,p}\nu_B\, d\nu_B < \infty, \text{ for all balls } B,$$

where $B = B(x, R)$, and $\nu_B = \nu|_B$.

The following theorem is due to Brezis and Browder [BB79] (see also [MZ97], Theorem 2.39).

Theorem 2.1. *Let $1 < p < n$. Suppose $u \in L^{1,p}_0(\mathbb{R}^n)$, and $\mu \in M^+(\mathbb{R}^n) \cap L^{-1,p'}(\mathbb{R}^n)$. Then $u \in L^1(\mathbb{R}^n, \mu)$ (for a quasicontinuous representative of u), and*

(2.3) $$\langle \mu, u\rangle = \int_{\mathbb{R}^n} u\, d\mu.$$

We observe that if, under the assumptions of this theorem, $-\Delta_p u = \mu$, then it follows (see [MZ97], Theorem 2.34)

(2.4) $$\langle \mu, u\rangle = \int_{\mathbb{R}^n} u\, d\mu = ||u||^p_{1,p} = ||\mu||^{p'}_{-1,p'}.$$

We next define the $\mathcal{A}$-Laplace operator as the mapping $\mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following structural assumptions:

$$x \to \mathcal{A}(x, \xi) \quad \text{is measurable for all } \xi \in \mathbb{R}^n,$$

$$\xi \to \mathcal{A}(x, \xi) \quad \text{is continuous for a.e. } x \in \mathbb{R}^n,$$

and there are constants $0 < \alpha \leq \beta < \infty$, such that for a.e. x in $\mathbb{R}^n$, and for all ξ in $\mathbb{R}^n$,

$$\mathcal{A}(x, \xi) \cdot \xi \geq \alpha |\xi|^p, \quad |\mathcal{A}(x, \xi)| \leq \beta |\xi|^{p-1},$$

$$(\mathcal{A}(x, \xi_1) - \mathcal{A}(x, \xi_2)) \cdot (\xi_1 - \xi_2) > 0 \quad \text{if } \xi_1 \neq \xi_2,$$

$$\mathcal{A}(x, \lambda\xi) = \lambda|\lambda|^{p-2} \mathcal{A}(x, \xi), \quad \text{if } \lambda \in \mathbb{R} \backslash \{0\}.$$

The p-Laplace operator corresponds to the choice $\mathcal{A}(x, \xi) = |\xi|^{p-2}\xi$.

For $u \in W^{1,p}_{\text{loc}}(\Omega)$, we define the p-Laplacian Δ_p $(1 < p < \infty)$ in the distributional sense, i.e., for every $\varphi \in C_0^\infty(\Omega)$,

$$\langle \Delta_p u, \varphi \rangle = \langle \operatorname{div}(|\nabla u|^{p-2} \nabla u, \varphi \rangle = - \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx. \tag{2.5}$$

We will extend the usual distributional definition of solutions u of $-\Delta_p u = \mu$, where $\mu \in W^{-1,p'}_{\text{loc}}(\Omega)$, to u not necessarily in $W^{1,p}_{\text{loc}}(\Omega)$. We will understand solutions in the following potential-theoretic sense using p-superharmonic functions, which is equivalent to the notion of locally renormalized solutions in terms of test functions (see [KKT09]).

A function $u \in W^{1,p}_{\text{loc}}(\Omega)$ is called p-harmonic if it satisfies the homogeneous equation $\Delta_p u = 0$. Every p-harmonic function has a continuous representative which coincides with u a.e. (see [HKM06]).

As usual, *p-superharmonic* functions are defined via a comparison principle. We say that $u : \Omega \to (-\infty, \infty]$ is p-superharmonic if u is lower semicontinuous, is not identically infinite in any component of Ω, and, whenever $D \Subset \Omega$ and $h \in C(\overline{D})$ is

p-harmonic in D with $h \le u$ on ∂D, then $h \le u$ in D.

A p-superharmonic function u does not necessarily belong to $\mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)$, but its truncations $T_k(u) = \min(k, \max(u, -k))$ do, for all $k > 0$. In addition, $T_k(u)$ are supersolutions, i.e., $-\operatorname{div}(|\nabla T_k(u)|^{p-2} \nabla T_k(u)) \ge 0$, in the distributional sense. The generalized gradient of a p-superharmonic function u defined by [HKM06]:

$$Du = \lim_{k \to \infty} \nabla(T_k(u)).$$

We note that every p-superharmonic function u has a quasicontinuous representative which coincides with u quasieverywhere (q.e.), i.e., everywhere except for a set of p-capacity zero (see [HKM06]). We will assume that u is always chosen to be quasicontinuous.

Let u be p-superharmonic, and let $1 \le r < \frac{n}{n-1}$. Then $|Du|^{p-1}$, and consequently $|Du|^{p-2}Du$, belongs to $L^r_{\mathrm{loc}}(\Omega)$ [KM92]. This allows us to define a nonnegative distribution $-\Delta_p u$ for each p-superharmonic function u by

$$-\langle \Delta_p u, \varphi \rangle = \int_\Omega |Du|^{p-2} Du \cdot \nabla \varphi \, dx, \tag{2.6}$$

for all $\varphi \in C^\infty_0(\Omega)$. Then by the Riesz representation theorem there exists a unique measure $\mu[u] \in M^+(\Omega)$ so that $-\Delta_p u = \mu[u]$, where $\mu[u]$ is called the Riesz measure of u.

Definition 2.2. For $\omega \in M^+(\Omega)$, u is said to be a (p-superharmonic) solution to the equation

$$-\Delta_p u = \omega \quad \text{in } \Omega \tag{2.7}$$

if u is p-superharmonic in Ω, and $\mu[u] = \omega$ (see [KM92], [KM94], [Ki02]).

Thus, if $\sigma \in M^+(\Omega)$, then $u \geq 0$ is a (*weak*) solution to the equation

$$-\Delta_p u = \sigma u^q \quad \text{in } \Omega \tag{2.8}$$

if u is p-superharmonic in Ω, $u \in L^q_{\text{loc}}(\Omega, d\sigma)$, and $d\mu[u] = u^q \, d\sigma$.

Alternatively, we will use the framework of *locally renormalized* solutions. This notion introduced by Bidaut-Véron [BiVe03], following the development of the theory of renormalized solutions in [MMOP99], is well suited for our purposes. As was shown recently in [KKT09], for $\omega \in M^+(\Omega)$ it coincides with the notion of a p-superharmonic solution in Definition 2.2.

In particular, a p-superharmonic function $u \geq 0$ satisfying (2.7) is a locally renormalized solution defined in terms of test functions (see [KKT09], Theorem 3.15). This means that, for all $\varphi \in C^\infty_0(\Omega)$ and $h \in W^{1,\infty}(\mathbb{R}_+)$ with h' having compact support, we have

$$\int_\Omega |Du|^p \, h'(u) \, \varphi \, dx + \int_\Omega |Du|^{p-2} Du \cdot \nabla\varphi \, h(u) \, dx = \int_\Omega h(u)\varphi \, d\,\omega. \tag{2.9}$$

The converse is also true, i.e., if u is a locally renormalized solution to (2.7) then there exists a p-superharmonic representative $\tilde{u} = u$ a.e.

We will call such solutions of (2.9) with $d\omega = u^q d\sigma$ (locally) renormalized, p-superharmonic, or simply solutions, of (2.8).

Definition 2.3. A function $u \geq 0$ is called a (renormalized) supersolution to (2.8) if

u is p-superharmonic in Ω, $u \in L^q_{\text{loc}}(\Omega, d\sigma)$, and

$$\int_\Omega |Du|^{p-2} Du \cdot \nabla \varphi \, dx \geq \int_\Omega u^q \varphi \, d\sigma, \quad \forall \, \varphi \in C_0^\infty(\Omega), \quad \varphi \geq 0. \tag{2.10}$$

As we will show below, supersolutions to (1.1) in the sense of Definition 2.3 are closely related to supersolutions associated with the integral equation (1.8), i.e., $u \in L^q_{\text{loc}}(\mathbb{R}^n, d\sigma)$ such that

$$u \geq \mathbf{W}_{\alpha,p}(u^q \, d\sigma) \quad d\sigma\text{-a.e.}, \tag{2.11}$$

in the case $\alpha = 1$.

We will employ some fundamental results of the potential theory of quasilinear elliptic equations. First, let us state a useful convergence result in [KM92], Theorem 1.17.

Theorem 2.4. *Suppose $\{u_j\}_j$ is a sequence of nonnegative p-superharmonic functions in an open set Ω. Then there is a subsequence $\{u_{j_k}\}_k$ of u_j which converges almost everywhere to a nonnegative function u which is either p-superharmonic or identically infinite in each component of Ω and $Du_{j_k} \to Du$ a.e. in $\{u < \infty\}$.*

The following important weak continuity result [TW02b] will be used to prove the existence of p-superharmonic solutions to quasilinear equations.

Theorem 2.5. *Suppose $\{u_j\}$ is a sequence of nonnegative p-superharmonic functions that converges a.e. to a p-superharmonic function u in an open set Ω. Then $\mu[u_j]$*

converges weakly to $\mu[u]$, i.e., for all $\varphi \in C_0^\infty(\Omega)$,

$$\lim_{j\to\infty} \int_\Omega \varphi \, d\mu[u_j] = \int_\Omega \varphi \, d\mu[u].$$

The next result [KM94] is concerned with global pointwise estimates of nonnegative p-superharmonic functions in terms of Wolff's potentials discussed in the Introduction.

Theorem 2.6 ([KM94])**.** *Let $1 < p < \infty$, and let u be a p-superharmonic function in $\mathbb{R}^n$ with $\liminf_{|x|\to\infty} u(x) = 0$.*

(i) If $p < n$ and $\omega = \mu[u]$, then

$$\frac{1}{K} \mathbf{W}_{1,p}\omega(x) \le u(x) \le K \, \mathbf{W}_{1,p}\omega(x), \quad x \in \mathbb{R}^n, \tag{2.12}$$

where K is a positive constant depending only on n and p.

(ii) In the case $p \ge n$, it follows that $u \equiv 0$.

For $0 < \alpha < n$ and $\sigma \in M^+(\mathbb{R}^n)$, the Riesz potential of order α is defined by

$$\mathbf{I}_\alpha \sigma(x) = \int_0^\infty \frac{\sigma(B(x,r))}{r^{n-\alpha}} \frac{dr}{r} = c_n \int_{\mathbb{R}^n} \frac{d\sigma(y)}{|x-y|^{n-\alpha}}, \quad x \in \mathbb{R}^n. \tag{2.13}$$

For $E \subset \mathbb{R}^n$, we define the Riesz capacity of E by

$$\mathrm{cap}_{\alpha,p}(E) = \inf\{ ||f||^p_{L^p(\mathbb{R}^n)} : f \in L^p(\mathbb{R}^n), f \ge 0, \mathbf{I}_\alpha f \ge 1 \text{ on } E \}. \tag{2.14}$$

We have, for all compact sets $E \subset \mathbb{R}^n$,

$$\frac{1}{c}\mathrm{cap_{1,p}(E)} \leq \mathrm{cap_p(E)} \leq \mathrm{c\,cap_{1,p}(E)}, \tag{2.15}$$

where $c = c(n,p)$ (see [MZ97]). We next define the truncated Wolff potential $\mathbf{W}^R_{\alpha,p}\sigma$, where $R > 0$, by

$$\mathbf{W}^R_{\alpha,p}\sigma(x) = \int_0^R \left(\frac{\sigma(B(x,t))}{t^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t}, \quad x \in \mathbb{R}^n.$$

In some instances, it is more convenient to work with the dyadic version of Wolff potential

$$\mathcal{W}_{\alpha,p}\sigma(x) = \sum_{Q\in\mathcal{D}} \left[\frac{\sigma(Q)}{|Q|^{1-\frac{\alpha p}{n}}}\right]^{\frac{1}{p-1}} \chi_Q(x), \tag{2.16}$$

where $\mathcal{D}$ is the set of dyadic cubes in $\mathbb{R}^n$. The shifted version of the Wolff's potential, for $\mu \in M^+(\mathbb{R}^n), t \in \mathbb{R}^n$, is defined by

$$\mathcal{W}^{d,t}_{\alpha,p}\mu(x) = \sum_{Q\in\mathcal{D}_t} \left[\frac{\mu(Q)}{|Q|^{1-\frac{\alpha p}{n}}}\right]^{\frac{1}{p-1}} \chi_Q(x),$$

where Q now denotes a shifted dyadic cube in the lattice $\mathcal{D}_t = \mathcal{D}+t = \{Q+t\}_{Q\in\mathcal{D}}$. Let us state a useful dyadic shifting lemma which goes back to the papers [FS71, GJ82].

Lemma 2.7. *Let $R > 0$, there exist constants $c, c_1 > 0$ depending only on n such that for all $x \in \mathbb{R}^n$:*

$$\mathbf{W}^R_{\alpha,p}\mu(x) \leq c_1\, R^{-n} \int_{|t|\leq cR} \mathcal{W}^{d,t}_{\alpha,p}\mu(x)dt. \tag{2.17}$$

See, for instance, [COV00]. We will need the following Wolff's inequality [HW83] (see also [AH96], Sec. 4.5) which gives precise estimates of the energy associated with the Wolff potential:

Theorem 2.8. *Suppose* $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, *and* $\sigma \in M^+(\mathbb{R}^n)$. *Then there exists a constant* $C > 0$ *depending only on* p, α, *and* n *such that*

$$\frac{1}{C} \int_{\mathbb{R}^n} (\mathbf{I}_\alpha \sigma)^{p'} \, dx \leq \int_{\mathbb{R}^n} \mathbf{W}_{\alpha,p}\sigma \, d\sigma \leq C \int_{\mathbb{R}^n} (\mathbf{I}_\alpha \sigma)^{p'} \, dx, \tag{2.18}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

2.2 Wolff potential estimates

We start with some useful estimates for Wolff potentials. Throughout this section we will assume that $\sigma \in M^+(\mathbb{R}^n)$ and $\sigma \neq 0$.

Lemma 2.9. *Suppose* $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, *and* $\sigma \in M^+(\mathbb{R}^n)$. *Let* $s = \min(1, p-1)$. *Then there exists a positive constant* c *which depends only on* n, p, *and* α *such that, for all* $x \in \mathbb{R}^n$ *and* $R > 0$,

$$\begin{aligned} & c^{-1} \int_R^\infty \left(\frac{\sigma(B(x,r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \\ & \leq \inf_{B(x,R)} \mathbf{W}_{\alpha,p}\sigma \leq \left(\frac{1}{|B(x,R)|} \int_{B(x,R)} \left[\mathbf{W}_{\alpha,p}\sigma(y)\right]^s \, dy \right)^{\frac{1}{s}} \\ & \leq c \int_R^\infty \left(\frac{\sigma(B(x,r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}. \end{aligned} \tag{2.19}$$

Proof. Without loss of generality we can assume that $x = 0$. We first prove the last estimate in (2.19). Clearly,

$$\frac{1}{|B(0,R)|}\int_{B(0,R)} [\mathbf{W}_{\alpha,p}\sigma(y)]^s \, dy \le I_1 + I_2,$$

where

$$I_1 = \frac{1}{|B(0,R)|}\int_{B(0,R)} \left(\int_0^R \left(\frac{\sigma(B(y,r))}{r^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dr}{r}\right)^s dy,$$

$$I_2 = \frac{1}{|B(0,R)|}\int_{B(0,R)} \left(\int_R^\infty \left(\frac{\sigma(B(y,r))}{r^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dr}{r}\right)^s dy.$$

To estimate I_2, notice that since $B(y,r) \subset B(0,2r)$ for $y \in B(0,R)$ and $r > R$, it follows

$$I_2 \le \left(\int_R^\infty \left(\frac{\sigma(B(0,2r))}{r^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dr}{r}\right)^s.$$

To estimate I_1, suppose first that $p \ge 2$ so that $s = 1$. Then using Fubini's theorem and Jensen's inequality we deduce

$$I_1 \le c\int_0^R \left(\frac{1}{|B(0,R)|}\int_{B(0,R)} \sigma(B(y,r))\, dy\right)^{\frac{1}{p-1}} \frac{dr}{r^{\frac{n-\alpha p}{p-1}+1}}.$$

Using Fubini's theorem again, we obtain

$$\int_{B(0,R)} \sigma(B(y,r))\, dy \le \int_{B(0,2R)} |B(y,r)|\, d\sigma = |B(0,1)|\, r^n\, \sigma(B(0,2R)).$$

Hence, there is a constant $c = c(n,p,\alpha)$ such that

$$\begin{aligned} I_1 &\le cR^{-\frac{n}{p-1}}\sigma(B(0,2R))^{\frac{1}{p-1}}\int_0^R r^{\frac{\alpha p}{p-1}-1}dr \\ &= cR^{\frac{\alpha p-n}{p-1}}\sigma(B(0,2R))^{\frac{1}{p-1}} \le c\int_R^\infty \left(\frac{\sigma(B(0,2r))}{r^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dr}{r}. \end{aligned}$$

Notice that this is the same estimate we have deduced above for I_2 with $s = 1$.

Let us now estimate I_1 for $1 < p < 2$ and $s = p - 1$. In this case, we will use the following elementary inequality: for every $R > 0$,

$$\left(\int_0^R \left(\frac{\phi(r)}{r^\gamma} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right)^{p-1} \leq c(p, \gamma) \int_0^{2R} \frac{\phi(r)}{r^\gamma} \frac{dr}{r},$$

where $\gamma > 0$, $1 < p < 2$, and ϕ is a non-decreasing function on $(0, \infty)$.

Applying the preceding inequality with $\phi(r) = \sigma(B(y, 2r))$ and $\gamma = n - \alpha p$, and estimating as in the case $p \geq 2$, using Fubini's theorem again, we obtain:

$$\begin{aligned} I_1 &\leq \frac{c}{|B(0,R)|} \int_{B(0,R)} \int_0^{2R} \frac{\sigma(B(y,r))}{r^{n-\alpha p}} \frac{dr}{r} dy \\ &\leq c\, R^{-n} \sigma(B(0, 2R)) \int_0^{2R} r^{\alpha p - 1} dr = c\, R^{-n+\alpha p} \sigma(B(0, 2R)) \\ &\leq c \left(\int_R^\infty \left(\frac{\sigma(B(0, 2r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right)^{p-1}, \end{aligned}$$

where c denotes different constants depending only on n, p, α. Combining the estimates for I_1 and I_2, we arrive at

$$\frac{1}{|B(0,R)|} \int_{B(0,R)} (\mathbf{W}_{\alpha,p}\sigma)^s \, dy \leq c \left(\int_R^\infty \left(\frac{\sigma(B(0, 2r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right)^s.$$

Making the substitution $\rho = 2r$ in the integral on the right-hand side completes the proof of the upper estimate in (2.19).

To prove the lower estimate, notice that

$$\mathbf{W}_{\alpha,p}\sigma(y) \geq \int_{2R}^\infty \left(\frac{\sigma(B(y,r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} = c \int_R^\infty \left(\frac{\sigma(B(y, 2\rho))}{\rho^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{d\rho}{\rho}.$$

Since $B(y, 2\rho) \supset B(0, \rho)$ for $y \in B(0, R)$ and $\rho > R$, there exists $c = c(n, p, \alpha) > 0$ such that

$$\inf_{B(0,R)} \mathbf{W}_{\alpha,p}\sigma \geq c \int_R^\infty \left(\frac{\sigma(B(0,\rho))}{\rho^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{d\rho}{\rho}.$$

$\square$

Corollary 2.10. *Suppose* $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, *and* $\sigma \in M^+(\mathbb{R}^n)$.

(i) $\mathbf{W}_{\alpha,p}\sigma \not\equiv +\infty$ *if and only if*

$$\int_1^\infty \left(\frac{\sigma(B(0,r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} < \infty. \tag{2.20}$$

(ii) Condition (2.20) *implies*

$$\int_t^\infty \left(\frac{\sigma(B(x,r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} < \infty, \quad \forall x \in \mathbb{R}^n, \, t > 0. \tag{2.21}$$

(iii) If (2.20) *holds, then* $\mathbf{W}_{\alpha,p}\sigma \in L^s_{\text{loc}}(dx)$, *where* $s = \min(1, p-1)$, *and*

$$\liminf_{|x| \to \infty} \mathbf{W}_{\alpha,p}\sigma(x) = 0. \tag{2.22}$$

Proof. We first verify statement (ii). Suppose (2.20) holds. We may assume $x \neq 0$, since for $x = 0$ (2.21) is obvious. Clearly, $B(x, r) \subset B(0, 2r)$ for $|x| < r$, and hence,

$$I_x := \int_{|x|}^\infty \left(\frac{\sigma(B(x,r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \leq \int_{|x|}^\infty \left(\frac{\sigma(B(0,2r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} < \infty.$$

It follows that (2.21) holds for $t \geq |x|$. If $t < |x|$, then

$$\int_t^\infty \left(\frac{\sigma(B(x,r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} = \int_t^{|x|} \left(\frac{\sigma(B(x,r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} + I_x < \infty,$$

since in the first integral $B(x,r) \subset B(0, 2|x|)$. Thus, (2.21) holds for all x and $t > 0$.

It remains to prove (2.22), since the other statements of Corollary 2.10 are immediate from (2.19) and (2.21). Suppose that (2.20) holds. For $R > 0$, let $A_R = \{\frac{R}{2} < |x| < R\}$. Then by the upper estimate of Lemma 2.9 (with $x = 0$),

$$\begin{aligned}\inf_{|x|>R/2} \mathbf{W}_{\alpha,p}\sigma(x) \le \inf_{A_R} \mathbf{W}_{\alpha,p}\sigma(x) &\le \left(\frac{1}{|A_R|} \int_{A_R} (\mathbf{W}_{\alpha,p}\sigma)^s \, dx \right)^{\frac{1}{s}} \\ &\le c \int_R^\infty \left(\frac{\sigma(B(0,r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r},\end{aligned}$$

where c does not depend on R. Since the right-hand side of the preceding inequality tends to zero as $R \to \infty$, we see that (2.22) holds. □

It is easy to see that if $\omega \in M^+(\mathbb{R}^n)$, and $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ is a weak solution to the equation $-\Delta_p u = \omega$, then $\omega \in W^{-1,p'}_{\mathrm{loc}}(\mathbb{R}^n)$. The converse statement is less obvious, and we were not able to find it in the literature. In the next lemma, for the sake of completeness, we give a proof in the case $\omega \ge 0$ using a series of Caccioppoli-type inequalities.

Lemma 2.11. *Suppose $1 < p < n$, and $\omega \in M^+(\mathbb{R}^n) \cap W^{-1,p'}_{\mathrm{loc}}(\mathbb{R}^n)$. If $u \ge 0$ is a p-superharmonic solution to the equation $-\Delta_p u = \omega$ in $\mathbb{R}^n$ such that $\liminf_{x\to\infty} u = 0$, then $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n, d\omega)$.*

Proof. Let us first show that $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n, d\omega)$ using Wolff's inequality [HW83]. Fix a ball $B = B(0, R)$, $R > 0$. By Theorem 2.6, u satisfies the Wolff potential estimate

(2.12). Hence,

$$\int_B u\,d\omega \le K\int_B \left(\int_0^R \frac{\omega(B(x,r))}{r^{n-p}}\right)^{\frac{1}{p-1}} \frac{dr}{r} d\omega(x)$$
$$+ K\int_B \int_R^\infty \left(\frac{\omega(B(x,r))}{r^{n-p}}\right)^{\frac{1}{p-1}} \frac{dr}{r} d\omega(x) := I + II.$$

Since $B(x,r) \subset 2B = B(0,2R)$ for $x \in B$ and $r < R$, we obtain by (2.2),

$$I \le K\int_B \int_0^R \left(\frac{\omega(B(x,r)\cap 2B)}{r^{n-p}}\right)^{\frac{1}{p-1}} \frac{dr}{r} d\omega(x)$$
$$\le K\int_{\mathbb{R}^n} \mathbf{W}_{1,p}\omega_{2B}\, d\omega_{2B} < \infty.$$

To estimate II, notice that $B(x,r) \subset B(0,2r)$, for $r > R$ and $x \in B$. Hence,

$$II \le K\omega(B) \int_R^\infty \left(\frac{\omega(B(0,2r))}{r^{n-p}}\right)^{\frac{1}{p-1}} \frac{dr}{r} < \infty$$

by Corollary 2.10.

We next show that $u \in L^s_{\text{loc}}(\mathbb{R}^n, dx)$ for $0 < s \le \frac{np}{n-p}$. Arguing as above, we use (2.12) and split the integral with respect to dr/r into two parts:

$$\int_B u^s dx \le c_s\, K^s \int_B \left(\int_0^R \left(\frac{\omega(B(x,r))}{r^{n-p}}\right)^{\frac{1}{p-1}} \frac{dr}{r}\right)^s dx$$
$$+ c_s\, K^s \int_B \left(\int_R^\infty \left(\frac{\omega(B(x,r))}{r^{n-p}}\right)^{\frac{1}{p-1}} \frac{dr}{r}\right)^s dx := III + IV,$$

where c is a constant depending only on s.

To estimate III, notice that by (2.1) $\omega_{2B} \in L^{-1,p'}(\mathbb{R}^n)$, and consequently there is a unique solution $u_{2B} \in L^{1,p}_0(\mathbb{R}^n)$ to the equation $-\Delta_p u_{2B} = \omega_{2B}$ in $\mathbb{R}^n$. Hence, by the

Sobolev inequality, $u_{2B} \in L^s_{\text{loc}}(\mathbb{R}^n)$ for $0 < s \le \frac{np}{n-p}$. Clearly, u_{2B} is p-superharmonic, and satisfies (2.12) with ω_{2B} in place of ω, i.e.,

$$\int_0^\infty \left(\frac{\omega(B(x,r) \cap 2B)}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \le K\, u_{2B}(x).$$

Since $B(x,r) \subset 2B$ for $x \in B$ and $r < R$, we estimate

$$III \le c \int_B \left(\int_0^R \left(\frac{\omega(B(x,r) \cap 2B)}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right)^s dx \le c \int_B u^s_{2B} dx < \infty.$$

The estimate of IV is similar to that of II:

$$IV \le c_s\, K\, |B| \left(\int_R^\infty \left(\frac{\omega(B(0,2r))}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right)^s < \infty$$

by Corollary 2.10. Thus, $u \in L^s_{\text{loc}}(\mathbb{R}^n, dx)$ for $s \le \frac{np}{n-p}$.

We next show that there exists $0 < \beta \le 1$ such that, for all balls B,

$$\int_B |Du|^p u^{\beta-1} dx < \infty. \tag{2.23}$$

Indeed, since u is p-superharmonic, it is a locally renormalized solution to $-\Delta_p u = \omega$ as discussed in Chapter 2. Let $u_k = \min(u, k)$, where $k > 0$. Note that u, and hence u_k, is locally bounded below. Using $h(u) = u_k^\beta$ $(0 < \beta \le 1)$ in (2.9), and a cut-off function $\varphi \in C_0^\infty(B)$ such that $0 \le \varphi \le 1$ and $\varphi = 1$ on $\frac{1}{2}B$, we obtain

$$\beta \int_{u \le k} |Du|^p u^{\beta-1} \varphi\, dx + \int_B |Du|^{p-2} Du \cdot \nabla\varphi\, u_k^\beta\, dx = \int_B u_k^\beta \varphi\, d\omega. \tag{2.24}$$

As was shown above, $u \in L^1_{\text{loc}}(\mathbb{R}^n,\, d\omega)$, and hence the right-hand side is bounded by

$$\int_B u^\beta \varphi \, d\omega \le \omega(B)^{1-\beta} \left(\int_B u \, d\omega \right)^\beta < \infty, \tag{2.25}$$

for $0 < \beta \le 1$.

Since u is p-superharmonic, we have $|Du| \in L^{r'(p-1)}$ for $r' < \frac{n}{n-1}$. By Hölder's inequality with exponents r' and $r > n$, we deduce from (2.24),

$$\begin{aligned} \beta \int_{u \le k} |Du|^p u^{\beta-1} \varphi \, dx &\le c \, ||Du||^{p-1}_{L^{r'(p-1)}(B,dx)} ||u||^\beta_{L^{\beta r}(B,dx)} \\ &\quad + \omega(B)^{1-\beta} \left(\int_B u \, d\omega \right)^\beta . \end{aligned} \tag{2.26}$$

If $\beta r = s \le \frac{np}{n-p}$, where $r > n$ and $\beta \le 1$, then the right-hand side of the preceding inequality is finite. Picking r so that $r > n$ and is arbitrarily close to n, and passing to the limit as $k \to \infty$, we obtain (2.23) for $\beta = \beta_0$, provided

$$0 < \beta_0 < \frac{p}{n-p}, \quad \beta_0 \le 1.$$

In the case $\frac{p}{n-p} > 1$, i.e., for $p > \frac{n}{2}$, we can set $\beta_0 = 1$, which shows that in fact $Du \in L^p(\frac{1}{2}B, dx)$, for all $B = B(0, R)$. Hence, $Du = \nabla u$ in the distributional sense, and consequently $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n)$.

For $1 < p \le \frac{n}{2}$, we fix s so that $p < s \le \frac{np}{n-p}$ which ensures that $u \in L^s_{\text{loc}}(\mathbb{R}^n, dx)$ as shown above. Applying Hölder's inequality with exponents p' and p, we obtain from (2.24) and (2.25),

$$\beta \int_{u\le k} |Du|^p u^{\beta-1}\varphi\, dx \le c\left(\int_B |Du|^p u^{\beta_0 -1} dx\right)^{\frac{1}{p'}} \left(\int_B u^{\beta p+(1-\beta_0)(p-1)} dx\right)^{\frac{1}{p}} + \omega(B)^{1-\beta}\left(\int_B u\, d\omega\right)^{\beta}.$$

Passing to the limit as $k\to\infty$, we deduce that (2.23) holds if $\beta \le 1$ and

$$\beta p + (1-\beta_0)(p-1) \le \beta p + p - 1 \le s.$$

In particular, (2.23) holds for $\beta = \beta_1 = \min\left(1, \frac{s-p+1}{p}\right)$.

If $\beta_1 = 1$, then $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ as above. In the case

$$\beta_1 = \frac{s-p+1}{p} < 1,$$

we set $\beta_j = \beta_1 + \frac{p-1}{p}\beta_{j-1}$, so that

$$\beta_j p + (1-\beta_{j-1})(p-1) = s, \quad j \ge 2.$$

In other words,

$$\beta_j = \frac{s-p+1}{p}\sum_{i=0}^{j-1}\left(\frac{p-1}{p}\right)^i, \quad j = 1, 2, \ldots .$$

Since

$$\lim_{j\to\infty}\beta_j = s-p+1 > 1,$$

we can choose $J \ge 2$ so that $\beta_1 \le \cdots \le \beta_{J-1} < 1$, but $\beta_J \ge 1$.

If $\beta_J > 1$, then we will replace β_J with $\beta_J = 1$. Clearly,

$$\beta_j p + (1-\beta_{j-1})(p-1) = s, \quad j = 2, 3, \ldots, J-1; \quad \beta_J\, p + (1-\beta_{J-1})(p-1) \leq s.$$

Arguing by induction, and using (2.23) with $\beta = \beta_j$, for $j = 2, 3, \ldots, J$, we estimate as above,

$$\begin{aligned}
\beta_j \int_{u \leq k} |Du|^p u^{\beta_j - 1} \varphi \, dx &\leq c \left(\int_B |Du|^p u^{\beta_{j-1}-1} dx \right)^{\frac{1}{p'}} \\
\times \left(\int_B u^{\beta_j p + (1-\beta_{j-1})(p-1)} dx \right)^{\frac{1}{p}} &+ \omega(B)^{1-\beta_j} \left(\int_B u \, d\omega \right)^{\beta_j} < \infty.
\end{aligned}$$

Since $\beta_J = 1$ at the last step, we arrive at the estimate

$$\int_{u \leq k} |Du|^p \varphi \, dx \leq C_B < \infty,$$

where C_B does not depend on k. Passing to the limit as $k \to \infty$, we conclude that $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n)$. $\square$

In the next theorem we obtain a lower bound for solutions of the integral inequality (2.11).

Theorem 2.12. *Let $1 < p < n$, $0 < q < p-1$, $0 < \alpha < \frac{n}{p}$, and $\sigma \in M^+(\mathbb{R}^n)$. Suppose $0 \leq u \in L^q_{\text{loc}}(\mathbb{R}^n, d\sigma)$ is a nontrivial solution to (2.11). Then there holds*

$$u(x) \geq C \left[\mathbf{W}_{\alpha,p} \sigma(x) \right]^{\frac{p-1}{p-1-q}}, \qquad \forall\, x \in \mathbb{R}^n, \tag{2.27}$$

where C is a positive constant depending only on n, p, q, and α.

The same lower bound holds for a nontrivial p-superharmonic supersolution to (1.1). *If $p \geq n$, there is only a trivial supersolution $u = 0$ on $\mathbb{R}^n$.*

Before proving Theorem 2.12, we need the following lemma.

Lemma 2.13. *Let $1 < p < \infty$ and $0 < \alpha < \frac{n}{p}$. Then, for every $r > 0$,*

$$\mathbf{W}_{\alpha,p}\left[(\mathbf{W}_{\alpha,p}\sigma)^r d\sigma\right] \geq \mathfrak{c}^{\frac{r}{p-1}} \left(\mathbf{W}_{\alpha,p}\sigma\right)^{\frac{r}{p-1}+1}, \tag{2.28}$$

where $\mathfrak{c} = \mathfrak{c}_{n,p,\alpha} > 0$ depends only on n, p, and α.

Proof. For $t > 0$, obviously,

$$\mathbf{W}_{\alpha,p}\sigma(y) = \int_0^t \left(\frac{\sigma(B(y,s))}{s^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{ds}{s} + \int_t^\infty \left(\frac{\sigma(B(y,s))}{s^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{ds}{s}.$$

For $y \in B(x,t)$, we have

$$\int_t^\infty \left(\frac{\sigma(B(y,s))}{s^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{ds}{s} = \int_{t/2}^\infty \left(\frac{\sigma(B(y,2r))}{(2r)^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dr}{r}$$

$$= \left(\frac{1}{2}\right)^{\frac{n-\alpha p}{p-1}} \int_{t/2}^\infty \left(\frac{\sigma(B(y,2s))}{s^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{ds}{s} \geq C_{n,p,\alpha} \int_t^\infty \left(\frac{\sigma(B(y,2s))}{s^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{ds}{s},$$

where $C_{n,p,\alpha} = \left(\frac{1}{2}\right)^{\frac{n-\alpha p}{p-1}}$. Since $s \geq t$ and $y \in B(x,t)$, then $B(y,2s) \supset B(x,s)$, which implies

$$\mathbf{W}_{\alpha,p}\sigma(y) \geq C_{n,p,\alpha} \int_t^\infty \left(\frac{\sigma(B(x,s))}{s^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{ds}{s}. \tag{2.29}$$

Notice that

$$\mathbf{W}_{\alpha,p}((\mathbf{W}_{\alpha,p}\sigma)^r d\sigma)(x) = \int_0^\infty \left(\frac{\int_{B(x,t)} [\mathbf{W}_{\alpha,p}\sigma(y)]^r d\sigma(y)}{t^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t}.$$

By (2.29), we obtain

$$\mathbf{W}_{\alpha,p}((\mathbf{W}_{\alpha,p}\sigma)^r d\sigma)(x) \geq$$

$$\geq \int_0^\infty \left(\frac{\int_{B(x,t)} \left[C_{n,p,\alpha} \int_t^\infty \left(\frac{\sigma(B(x,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^r d\sigma(y)}{t^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t}$$

$$\geq C_{n,p,\alpha}^{\frac{r}{p-1}} \int_0^\infty \left[\int_t^\infty \left(\frac{\sigma(B(x,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^{\frac{r}{p-1}} \left(\frac{\sigma(B(x,t))}{t^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t}.$$

Integrating by parts, we deduce

$$\mathbf{W}_{\alpha,p}((\mathbf{W}_{\alpha,p}\sigma)^r d\sigma)(x) \geq \frac{C_{n,p,\alpha}^{\frac{r}{p-1}}}{\frac{r}{p-1}+1} \left(\int_0^\infty \left(\frac{\sigma(B(x,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right)^{\frac{r}{p-1}+1}.$$

Clearly, $\frac{r}{p-1}+1 \leq e^{\frac{r}{p-1}}$, and hence, (2.28) follows with $\mathfrak{c} = e^{-1} C_{n,p,\alpha}$. This completes the proof of Lemma 2.13. $\square$

In particular, setting $\alpha = 1$ and $r = \frac{q(p-1)}{p-1-q}$ in Lemma 2.13, we deduce the following estimate used extensively below:

$$\mathbf{W}_{1,p} \left((\mathbf{W}_{1,p}\sigma)^{\frac{q(p-1)}{p-1-q}} d\sigma \right)(x) \geq \kappa \, (\mathbf{W}_{1,p}\sigma(x))^{\frac{p-1}{p-1-q}}, \tag{2.30}$$

where κ depends only on p, q, and n.

Proof of Theorem 2.12. Let $d\omega = u^q d\sigma$. Fix $x \in \mathbb{R}^n$ and pick $R > |x|$. Let $B =$

$B(0,R)$, and let $d\sigma_B = \chi_B \, d\sigma$. Iterating (2.11), we obtain

$$
\begin{aligned}
u(x) &\geq \mathbf{W}_{\alpha,p}\left[(\mathbf{W}_{\alpha,p}\omega)^q d\sigma_B\right](x) \\
&= \int_0^\infty \left(\frac{1}{t^{n-\alpha p}} \int_{B(x,t)\cap B} \mathbf{W}_{\alpha,p}\omega(z)^q d\sigma(z)\right)^{\frac{1}{p-1}} \frac{dt}{t}.
\end{aligned}
$$

We estimate,

$$
\mathbf{W}_{\alpha,p}\omega(z) = \int_0^\infty \left(\frac{\omega(B(z,s))}{s^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{ds}{s} \geq c \int_R^\infty \left(\frac{\omega(B(z,2s))}{s^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{ds}{s},
$$

where $c = c(n,p,\alpha) > 0$.

Notice that if $z \in B$ and $R \leq s$ then $B(z,2s) \supset B(0,s)$. Hence,

$$
\mathbf{W}_{\alpha,p}\omega(z) \geq c \int_R^\infty \left(\frac{\omega(B(0,s))}{s^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{ds}{s}.
$$

From this it follows,

$$
u(x) \geq [c\, M(R)]^{\frac{q}{p-1}} \, \mathbf{W}_{\alpha,p}\sigma_B(x),
$$

where

$$
M(R) = \int_R^\infty \left(\frac{\omega(B(0,s))}{s^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{ds}{s} > 0.
$$

Combining (2.11) with the preceding estimate, and using Lemma 2.13 with $r = q$ and σ_B in place of σ, we obtain

$$
\begin{aligned}
u(x) &\geq [c\, M(R)]^{(\frac{q}{p-1})^2} \, \mathbf{W}_{\alpha,p}\left[(\mathbf{W}_{\alpha,p}\sigma_B)^q d\sigma\right](x) \\
&\geq \mathfrak{c}^{\frac{q}{p-1}} \, [c\, M(R)]^{(\frac{q}{p-1})^2} \, \left[\mathbf{W}_{\alpha,p}\sigma_B(x)\right]^{1+\frac{q}{p-1}}.
\end{aligned}
$$

Iterating this procedure and using Lemma 2.13 with $r = q\sum_{k=0}^{j-1}(\frac{q}{p-1})^k$, we deduce

$$u(x) \geq \mathfrak{c}^{\sum_{k=1}^{j} k\left(\frac{q}{p-1}\right)^k} [c\, M(R)]^{(\frac{q}{p-1})^{j+1}} \; [\mathbf{W}_{\alpha,p}\sigma_B(x)]^{\sum_{k=0}^{j}(\frac{q}{p-1})^k},$$

for all $j = 2, 3, \ldots$. Since $0 < q < p-1$, obviously

$$\sum_{k=1}^{\infty} k\left(\frac{q}{p-1}\right)^k < \infty.$$

Letting $j \to \infty$ in the preceding estimate we obtain

$$u(x) \geq C\; [\mathbf{W}_{\alpha,p}\sigma_B(x)]^{\frac{p-1}{p-1-q}}, \quad B = B(0,R), \quad R > |x|,$$

where $C > 0$ depends only on n, p, q, and α. Letting $R \to \infty$ yields (2.27) for all $x \in \mathbb{R}^n$. □

The next lemma shows that if there exists a nontrivial solution to (2.11), then σ must be absolutely continuous with respect to the (α,p)-capacity defined by (2.14) (see [AH96], Sec. 2.2). As a consequence, if (1.1) has a nontrivial p-superharmonic supersolution, then σ is absolutely continuous with respect to the p-capacity defined by (1.25). Notice that $\mathrm{cap}_p(E) \approx \mathrm{cap}_{1,p}(E)$ for compact sets E.

Lemma 2.14. *Let $1 < p < \infty$, $0 < q < p-1$, $0 < \alpha < \frac{n}{p}$, and $\sigma \in M^+(\mathbb{R}^n)$. Suppose there is a nontrivial solution $u \in L^q_{\mathrm{loc}}(\mathbb{R}^n, d\sigma)$ to inequality (2.11). Then there exists a constant C depending only on n, p, q, α such that*

$$\sigma(E) \leq C\; \left[\mathrm{cap}_{\alpha,p}(E)\right]^{\frac{q}{p-1}} \left(\int_E u^q d\sigma\right)^{\frac{p-1-q}{p-1}}, \tag{2.31}$$

for all compact sets $E \subset \mathbb{R}^n$*.*

Proof. Let $d\omega = u^q d\sigma$. Then $u \geq \mathbf{W}_{\alpha,p}\omega$ $d\sigma$-a.e. By Theorem 1.11 in [V1],

$$\int_E \frac{d\omega}{(\mathbf{W}_{\alpha,p}\omega)^{p-1}} \leq C \,\mathrm{cap}_{\alpha,p}(E),$$

where C depends only on n, p, and α. Hence,

$$\int_E u^{q-p+1}\, d\sigma \leq \int_E \frac{d\omega}{(\mathbf{W}_{\alpha,p}\omega)^{p-1}} \leq C \,\mathrm{cap}_{\alpha,p}(E). \tag{2.32}$$

Note that $q - p + 1 < 0$. Using Hölder's inequality with exponents $r = \frac{p-1}{q}$ and $r' = \frac{p-1}{p-1-q}$, we have

$$\sigma(E) = \int_E u^{-\beta} u^{\beta} d\sigma \leq \left(\int_E u^{-\beta r} d\sigma\right)^{\frac{1}{r}} \left(\int_E u^{\beta r'}\, d\sigma\right)^{\frac{1}{r'}},$$

where $\beta = \frac{q(p-1-q)}{p-1} > 0$. Then $-\beta r = q-p+1$ and $\beta r' = q$, and since $u \in L^q_{\mathrm{loc}}(\mathbb{R}^n, d\sigma)$, the preceding estimate implies (2.31). $\square$

Chapter 3

Finite energy solutions

3.1 Main results

In this chapter, we study finite energy solutions to quasilinear elliptic equation (1.1). We are interested in solutions $u \in L_0^{1,p}(\mathbb{R}^n)$ to (1.1), and related integral inequalities. Here $L_0^{1,p}(\mathbb{R}^n)$ is the homogeneous Sobolev (or Dirichlet) space defined in Chapter 2 (see [HKM06], [MZ97], [Maz11]).

More precisely, u is called a *finite energy* solution to (1.1) if $u \in L_0^{1,p}(\mathbb{R}^n) \cap L^q_{\text{loc}}(\mathbb{R}^n, d\sigma)$, $u \geq 0$, and, for all $\varphi \in C_0^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx = \int_{\mathbb{R}^n} u^q \, \varphi \, d\sigma. \tag{3.1}$$

Finite energy solutions to (1.1) are critical points of the functional

$$H[\varphi] = \int_{\mathbb{R}^n} \frac{1}{p} \, |\nabla \varphi|^p \, dx - \int_{\mathbb{R}^n} \frac{1}{q+1} \, |\varphi|^{1+q} \, d\sigma.$$

We will give a *necessary and sufficient* condition for the existence of a nontrivial finite energy solution to (1.1), and prove its uniqueness.

Our main result is the following.

Theorem 3.1. *Let $0 < q < p-1$, $1 < p < n$, and let $\sigma \in M^+(\mathbb{R}^n)$. Then there exists a nontrivial solution $u \in L_0^{1,p}(\mathbb{R}^n) \cap L^q_{\text{loc}}(\mathbb{R}^n, d\sigma)$ to* (1.1) *if and only if*

$$\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\sigma)^{\frac{(1+q)(p-1)}{p-1-q}} \, d\sigma < \infty. \tag{3.2}$$

Furthermore, such a solution is unique. For $p \geq n$, (1.1) *has only a trivial solution $u = 0$.*

We observe that (3.2) yields $\sigma \in L^{-1,p'}_{\text{loc}}(\mathbb{R}^n)$, where $L^{-1,p'}(\mathbb{R}^n) = L_0^{1,p}(\mathbb{R}^n)^*$ is the dual Sobolev space (see definitions in Chapter 2). Consequently, σ is necessarily absolutely continuous with respect to the p-capacity $\text{cap}_p(\cdot)$.

Moreover, as was shown in [COV00], condition (3.2) holds if and only if there exists a constant C such that, for all $\varphi \in C_0^\infty(\mathbb{R}^n)$,

$$\left(\int_{\mathbb{R}^n} |\varphi|^{1+q} \, d\sigma \right)^{\frac{1}{1+q}} \leq C \, ||\nabla \varphi||_{L^p(\mathbb{R}^n)}. \tag{3.3}$$

An obvious sufficient condition which follows from Sobolev's inequality is $\sigma \in L^r(\mathbb{R}^n)$, $r = \frac{np}{n(p-1-q)+p(1+q)}$.

There is also an equivalent characterization of (3.3) in terms of capacities due to Maz'ya and Netrusov (see [Maz11], Sec. 11.6):

$$\int_0^{\sigma(\mathbb{R}^n)} \left(\frac{t}{\varkappa(\sigma, t)} \right)^{\frac{1+q}{p-1-q}} dt < +\infty, \tag{3.4}$$

where $\varkappa(\sigma, t) = \inf\{\ \text{cap}_p(E) :\ \sigma(E) \geq t\}$.

Thus, any one of the conditions (3.2), (3.3), and (3.4) is necessary and sufficient for the existence of a nontrivial finite energy solution to (1.1).

We now outline the contents of this chapter. In Sec. 3.2 we study the corresponding integral inequalities, deduce a necessary and sufficient condition for the existence of a finite energy solution, and construct a minimal solution. In Sec. 3.3 we prove the uniqueness property of finite energy solutions.

3.2 Existence and minimality of finite energy solutions

In this section, we deduce a necessary and sufficient condition for the existence of a finite energy solution, and construct a minimal solution to (1.1). We will assume that $1 < p < n$, since for $p \geq n$ there are only trivial nonnegative supersolutions on $\mathbb{R}^n$ (Theorem 2.12; see also [HKM06], Theorem 3.53). We start with the following lemma.

Lemma 3.2. *Suppose there exists a nontrivial supersolution $u \geq 0$, $u \in L_0^{1,p}(\mathbb{R}^n) \cap L^q_{\text{loc}}(\mathbb{R}^n, d\sigma)$ to* (1.1). *Then*

$$-\Delta_p u \in L^{-1,p'}(\mathbb{R}^n) \cap M^+(\mathbb{R}^n).$$

Moreover, $u \in L^{1+q}(\mathbb{R}^n, d\sigma)$ (for a quasicontinuous representative of u), and condition (3.2) *holds.*

Proof. Suppose $u \in L_0^{1,p}(\mathbb{R}^n) \cap L^q_{\text{loc}}(\mathbb{R}^n, d\sigma)$ is a supersolution to (1.1). By Hölder's inequality, we have, for every $\varphi \in C_0^\infty(\mathbb{R}^n)$,

$$|\langle \Delta_p u, \varphi\rangle| = \left| \int_{\mathbb{R}^n} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx \right| \leq ||\nabla u||^{p-1}_{L^p(\mathbb{R}^n)} ||\nabla \varphi||_{L^p(\mathbb{R}^n)}.$$

Hence, $\Delta_p u \in L^{-1,p'}(\mathbb{R}^n)$. If $\varphi \geq 0$, then

$$-\langle \Delta_p u, \varphi \rangle = \int_{\mathbb{R}^n} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx \geq \int_{R^n} \varphi \, u^q \, d\sigma \geq 0,$$

and consequently $-\Delta_p u \in M^+(\mathbb{R}^n)$.

It follows that $d\mu = u^q \, d\sigma \in M^+(\mathbb{R}^n) \cap L^{-1,p'}(\mathbb{R}^n)$. Let $\{\varphi_j\}$ be a sequence of nonnegative C_0^∞-functions such that $\varphi_j \to u$ in $L_0^{1,p}(\mathbb{R}^n)$. By definition,

$$\int_{\mathbb{R}^n} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi_j \, dx \geq \langle \mu, \varphi_j \rangle.$$

Hence,

$$\int_{\mathbb{R}^n} |\nabla u|^p \, dx = \lim_{j \to \infty} \int_{\mathbb{R}^n} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi_j \, dx \geq \lim_{j \to \infty} \langle \mu, \varphi_j \rangle = \langle \mu, u \rangle.$$

Let us assume as usual that u coincides with its quasicontinuous representative. Then, applying Theorem 2.1, we deduce

$$\langle \mu, u \rangle = \int_{\mathbb{R}^n} u \, d\mu = \int_{\mathbb{R}^n} u^{1+q} \, d\sigma < \infty.$$

By Theorem 2.12, it follows that if $u \not\equiv 0$, then $u \geq C \left(\mathbf{W}_{1,p}\sigma\right)^{\frac{p-1}{p-1-q}}$, and consequently (3.2) holds. $\square$

Let us define a nonlinear integral operator T by

$$T(f)(x) = \Big(\mathbf{W}_{\alpha,p}(|f|d\sigma)\Big)^{p-1}(x), \quad x \in \mathbb{R}^n. \tag{3.5}$$

Lemma 3.3. *Let $1 < p < \infty, 0 < q < p-1, 0 < \alpha < \frac{n}{p}$ and $\beta > 0$. Let $\sigma \in M^+(\mathbb{R}^n)$. Suppose*

$$\int_{\mathbb{R}^n} (\mathbf{W}_{\alpha,p}\sigma)^{\frac{(\beta+q)(p-1)}{p-1-q}} \, d\sigma < \infty. \tag{3.6}$$

Then T is a bounded operator from $L^{\frac{\beta+q}{q}}(\mathbb{R}^n, d\sigma)$ to $L^{\frac{\beta+q}{p-1}}(\mathbb{R}^n, d\sigma)$.

Proof. Let $f \in L^{\frac{\beta+q}{q}}(\mathbb{R}^n, d\sigma)$. Without lost of generality, we may assume that $f \geq 0$. Clearly,

$$||(\mathbf{W}_{\alpha,p}(fd\sigma))^{p-1}||_{L^{\frac{\beta+q}{p-1}}(d\sigma)} = \left(\int_{\mathbb{R}^n} \Big(\mathbf{W}_{\alpha,p}(fd\sigma)\Big)^{\beta+q} d\sigma\right)^{\frac{p-1}{\beta+q}}.$$

We have

$$\mathbf{W}_{\alpha,p}(fd\sigma)(x) \leq \int_0^\infty \left(\frac{\sigma(B(x,r))}{r^{n-\alpha p}}\right)^{p'-1} M_\sigma f(x)^{p'-1} \frac{dr}{r} = M_\sigma f(x)^{p'-1} \mathbf{W}_{\alpha,p}\sigma(x),$$

where the centered maximal operator M_σ is defined by

$$M_\sigma f(x) = \sup_{r>0} \frac{1}{\sigma(B(x,r))} \int_{B(x,r)} |f| \, d\sigma, \quad x \in \mathbb{R}^n.$$

It is well known that $M_\sigma : L^s(\mathbb{R}^n, d\sigma) \to L^s(\mathbb{R}^n, d\sigma)$ is a bounded operator for all $s > 1$. Let $s = \frac{\beta+q}{q}$. Then, using Hölder's inequality with the exponents $\frac{p-1}{q} > 1$ and

$\frac{p-1}{p-1-q}$, we estimate,

$$\int_{\mathbb{R}^n} \Big(\mathbf{W}_{\alpha,p}(f d\sigma)\Big)^{\beta+q} d\sigma \leq \int_{\mathbb{R}^n} (M_\sigma f)^{\frac{\beta+q}{p-1}} (\mathbf{W}_{\alpha,p}\sigma)^{\beta+q} d\sigma$$

$$\leq \left(\int_{\mathbb{R}^n} (M_\sigma f)^{\frac{\beta+q}{q}} d\sigma\right)^{\frac{q}{p-1}} \left(\int_{\mathbb{R}^n} (\mathbf{W}_{\alpha,p}\sigma)^{\frac{(\beta+q)(p-1)}{p-1-q}} d\sigma\right)^{\frac{p-1-q}{p-1}}$$

$$\leq C \left(\int_{\mathbb{R}^n} f^{\frac{\beta+q}{q}} d\sigma\right)^{\frac{q}{p-1}} \left(\int_{\mathbb{R}^n} (\mathbf{W}_{\alpha,p}\sigma)^{\frac{(\beta+q)(p-1)}{p-1-q}} d\sigma\right)^{\frac{p-1-q}{p-1}}.$$

Thus,

$$||\mathbf{W}_{\alpha,p}(f d\sigma)^{p-1}||_{L^{\frac{\beta+q}{p-1}}(d\sigma)} \leq c\, ||f||_{L^{\frac{\beta+q}{q}}(d\sigma)}.$$

$\square$

Remark 3.4. *It is not difficult to see that actually* (3.6) *is also necessary for the boundedness of the operator* $T : L^{\frac{\beta+q}{q}}(\mathbb{R}^n, d\sigma) \to L^{\frac{\beta+q}{p-1}}(\mathbb{R}^n, d\sigma)$ *(see, for example, [COV06]).*

Theorem 3.5. *Let* $1 < p < n, 0 < q < p-1, 0 < \alpha < \frac{n}{p}, \beta > 0$ *and* $\sigma \in M^+(\mathbb{R}^n)$. *Suppose that condition* (3.6) *holds. Then there exists a solution* $u \in L^{\beta+q}(\mathbb{R}^n, d\sigma)$ *to the integral equation* (1.8). *Conversely,* (3.6) *is also necessary for the existence of a solution* $u \in L^{\beta+q}(\mathbb{R}^n, d\sigma)$ *to* (1.8).

Proof. Suppose that (3.6) holds. By Lemma 3.3, we have, for all nonnegative $f \in L^{\frac{\beta+q}{q}}(\mathbb{R}^n, d\sigma)$,

$$\int_{\mathbb{R}^n} \Big(\mathbf{W}_{\alpha,p}(f d\sigma)\Big)^{\beta+q} d\sigma \leq C \left(\int_{\mathbb{R}^n} f^{\frac{\beta+q}{q}} d\sigma\right)^{\frac{q}{p-1}}. \tag{3.7}$$

Let $u_0 = c_0 \left(\mathbf{W}_{\alpha,p}\sigma\right)^{\frac{p-1}{p-1-q}}$, where $c_0 > 0$ is a small constant to be chosen later on. We construct a sequence of iterations u_j as follows:

$$u_{j+1} = \mathbf{W}_{\alpha,p}(u_j^q d\sigma), \quad j = 0, 1, 2, \ldots. \tag{3.8}$$

Applying Lemma 2.13, we have

$$u_1 = \mathbf{W}_{\alpha,p}(u_0^q d\sigma) = c_0^{\frac{q}{p-1}}\, \mathbf{W}_{\alpha,p}((\mathbf{W}_{\alpha,p}\sigma)^{\frac{q(p-1)}{p-1-q}}\, d\sigma) \geq c_0^{\frac{q}{p-1}}\, \mathfrak{c}^{\frac{q}{p-1-q}}\, (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}},$$

where $\mathfrak{c}$ is the constant in (2.28). Choosing c_0 so that $c_0^{\frac{q}{p-1}}\mathfrak{c}^{\frac{q}{p-1-q}} \geq c_0$, we obtain $u_1 \geq u_0$. By induction, we can show that the sequence $\{u_j\}_j$ is nondecreasing. Note that $u_0 \in L^{\beta+q}(\mathbb{R}^n, d\sigma)$ by assumption. Suppose that $u_0, \ldots, u_j \in L^{\beta+q}(\mathbb{R}^n, d\sigma)$ for some $j \geq 0$. Then

$$\int_{\mathbb{R}^n} u_{j+1}^{\beta+q}\, d\sigma = \int_{\mathbb{R}^n} \left(\mathbf{W}_{\alpha,p}(u_j^q\, d\sigma)\right)^{\beta+q} d\sigma.$$

Applying (3.7) with $f = u_j^q$, we obtain by induction,

$$\int_{\mathbb{R}^n} u_{j+1}^{\beta+q}\, d\sigma \leq C \left(\int_{\mathbb{R}^n} u_j^{\beta+q}\, d\sigma\right)^{\frac{q}{p-1}} < \infty. \tag{3.9}$$

Since $u_j \leq u_{j+1}$, the preceding inequality yields

$$\int_{\mathbb{R}^n} u_{j+1}^{\beta+q}\, d\sigma \leq C \left(\int_{\mathbb{R}^n} u_{j+1}^{\beta+q}\, d\sigma\right)^{\frac{q}{p-1}} < \infty.$$

Thus,

$$\left(\int_{\mathbb{R}^n} u_{j+1}^{\beta+q} d\sigma\right)^{\frac{p-1-q}{p-1}} \leq C < \infty.$$

Using the Monotone Convergence Theorem and passing to the limit as $j \to \infty$ in (3.8), we see that there exists $u = \lim_{j\to\infty} u_j$, such that $u \in L^{\beta+q}(\mathbb{R}^n, d\sigma)$ and u is a nontrivial solution to (1.8).

Conversely, suppose that there exists a solution $u \in L^{\beta+q}(\mathbb{R}^n, d\sigma)$ to (1.8). By Theorem 2.12, $u \geq C(\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}}$, and hence (3.6) follows. □

Lemma 3.6. *Let $u \in L^{1+q}(\mathbb{R}^n, d\sigma)$ be a nonnegative solution to the integral inequality* (2.11) *with $\alpha = 1$. Then*

$$u^q \, d\sigma \in L^{-1,p'}(\mathbb{R}^n). \tag{3.10}$$

Proof. Let $d\nu = u^q \, d\sigma$. We need to show that, for all $\varphi \in C_0^\infty(\mathbb{R}^n)$,

$$\left| \int_{\mathbb{R}^n} \varphi \, d\nu \right| \leq c \left(\int_{\mathbb{R}^n} |\nabla \varphi|^p \, dx \right)^{\frac{1}{p}}. \tag{3.11}$$

It is easy to see that the above inequality is equivalent to

$$\left| \int_{\mathbb{R}^n} \mathbf{I}_1 g \ d\nu \right| \leq c \left(\int_{\mathbb{R}^n} |g|^p \, dx \right)^{\frac{1}{p}}, \tag{3.12}$$

for all $g \in L^p(\mathbb{R}^n)$, where $\mathbf{I}_1 g$ is the Riesz potential of g of order 1. By duality, (3.12) is equivalent to

$$\int_{\mathbb{R}^n} (\mathbf{I}_1 \nu)^{p'} \, dx < \infty. \tag{3.13}$$

Using Wolff's inequality (2.18), we deduce that (3.13) holds if and only if

$$\int_{\mathbb{R}^n} \mathbf{W}_{1,p} \nu \, d\nu < \infty. \tag{3.14}$$

Notice that since $u \geq \mathbf{W}_{1,p}(u^q\, d\sigma)$ and $u \in L^{1+q}(\mathbb{R}^n, d\sigma)$ then

$$\int_{\mathbb{R}^n} \mathbf{W}_{1,p}\nu\, d\nu = \int_{\mathbb{R}^n} \mathbf{W}_{1,p}(u^q\, d\sigma)\, u^q\, d\sigma \leq \int_{\mathbb{R}^n} u^{1+q}\, d\sigma < \infty.$$

Thus, (3.13) holds. This completes the proof of the lemma. □

We will need a weak comparison principle which goes back to P. Tolksdorf's work on quasilinear equations. The following lemma is essentially known; we include a proof in the context of quasicontinuous solutions on the entire space for the convenience of the reader (see also [PV08], Lemma 6.9, for renormalized solutions in bounded domains).

Lemma 3.7. *Suppose $\mu, \omega \in M^+(\mathbb{R}^n) \cap L^{-1,p'}(\mathbb{R}^n)$. Suppose u and v are (quasicontinuous) solutions in $L_0^{1,p}(\mathbb{R}^n)$ of the equations $-\Delta_p u = \mu$ and $-\Delta_p v = \omega$, respectively. If $\mu \leq \omega$, then $u \leq v$ q.e.*

Proof. For every $\varphi \in L_0^{1,p}(\mathbb{R}^n)$, we have by Theorem 2.1,

$$\int_{\mathbb{R}^n} |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\, dx = \langle \mu, \varphi \rangle = \int_{\mathbb{R}^n} \varphi\, d\mu, \tag{3.15}$$

$$\int_{\mathbb{R}^n} |\nabla v|^{p-2}\nabla v \cdot \nabla\varphi\, dx = \langle \omega, \varphi \rangle = \int_{\mathbb{R}^n} \varphi\, d\omega. \tag{3.16}$$

Hence,

$$\int_{\mathbb{R}^n} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v) \cdot \nabla\varphi\, dx = \int_{\mathbb{R}^n} \varphi\, d\mu - \int_{\mathbb{R}^n} \varphi\, d\omega. \tag{3.17}$$

Since $\mu \le \omega$, it follows that, for every $\varphi \in L_0^{1,p}(\mathbb{R}^n)$, $\varphi \ge 0$, we have

$$\int_{\mathbb{R}^n} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v) \cdot \nabla\varphi \, dx \le 0. \tag{3.18}$$

Testing (3.18) with $\varphi = (u-v)^+ = \max\{u-v, 0\} \in L_0^{1,p}(\mathbb{R}^n)$, we obtain,

$$I = \int_{\mathbb{R}^n} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v) \cdot \nabla(u-v)^+ \, dx \le 0.$$

Let $A = \{x \in \mathbb{R}^n : u(x) > v(x)\}$, then

$$I = \int_A (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v) \cdot \nabla(u-v) \, dx \le 0.$$

Note that

$$(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v) \cdot \nabla(u-v) \ge 0.$$

Thus,

$$0 \le \int_A (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v) \cdot \nabla(u-v) \, dx = \int_A \varphi(d\mu - d\omega) \le 0.$$

It follows that $\nabla(u-v) = 0$ a.e. on A. By Lemma 2.22 in [MZ97], for every $a > 0$,

$$\text{cap}_p \{u - v > a\} \le \frac{1}{a^p} \int_A |\nabla(u-v)|^p \, dx = 0.$$

Consequently, $\text{cap}_p(A) = 0$, i.e., $u \le v$ q.e. □

We are now in a position to prove the main theorem of this section.

Theorem 3.8. *Let $1 < p < n$ and $0 < q < p-1$. Let $\sigma \in M^+(\mathbb{R}^n)$, $\sigma \ne 0$. Suppose* *that* (3.2) *holds. Then there exists a nontrivial solution* $w \in L_0^{1,p}(\mathbb{R}^n) \cap L^q_{\text{loc}}(\mathbb{R}^n, d\sigma)$ *to* (1.1)*. Moreover,* w *is a minimal solution, i.e.,* $w \le u$ $d\sigma$*-a.e. (q.e. for quasi-continuous representatives) for any nontrivial solution* $u \in L_0^{1,p}(\mathbb{R}^n) \cap L^q_{\text{loc}}(\mathbb{R}^n, d\sigma)$ *to* (1.1)*.*

Proof. We first show that there exists a solution $w \in L_0^{1,p}(\mathbb{R}^n) \cap L^q_{\text{loc}}(\mathbb{R}^n, d\sigma)$ to (1.1). Since (3.2) holds, applying Theorem 3.5 with $\alpha = 1$ and $\beta = 1$, we conclude that there exists a solution $v \in L^{1+q}(\mathbb{R}^n, d\sigma)$ to the integral equation (1.8) with $\alpha = 1$. By using a constant multiple $c\,v$ in place of v, we can assume that $v = K\mathbf{W}_{1,p}(v^q\,d\sigma)$, where K is the constant in (2.12). Then by Lemma 3.6 and Theorem 2.12,

$$v^q\,d\sigma \in L^{-1,p'}(\mathbb{R}^n), \quad \text{and} \quad v \ge C\,K^{\frac{p-1}{p-1-q}}\,(\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}}\,,$$

where C is the constant in (2.27).

We set $w_0 = c_0\,(\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}}$, $\quad d\omega_0 = w_0^q\,d\sigma$, where $c_0 > 0$ is a small constant to be determined later. We see that $w_0 \le v$ if $c_0 \le CK^{\frac{p-1}{p-1-q}}$. Hence,

$$w_0 \in L^{1+q}(\mathbb{R}^n, d\sigma), \quad \text{and} \quad \omega_0 \in L^{-1,p'}(\mathbb{R}^n).$$

Then there exists a unique nonnegative solution $w_1 \in L_0^{1,p}(\mathbb{R}^n)$ to the equation

$$-\Delta_p w_1 = \omega_0, \quad \text{and} \quad ||w_1||_{1,p}^{p-1} = ||\omega_0||_{-1,p'}.$$

(See (2.4)). Moreover, by Theorem 2.6,

$$0 \le w_1 \le K\,\mathbf{W}_{1,p}\omega_0 \le K\,\mathbf{W}_{1,p}(v^q d\sigma) = v.$$

Consequently, by Lemma 3.6,

$$w_1 \in L^{1+q}(\mathbb{R}^n, d\sigma), \quad \text{and} \quad w_1^q \, d\sigma \in L^{-1,p'}(\mathbb{R}^n).$$

We deduce using (2.30),

$$w_1 \geq \frac{1}{K} \mathbf{W}_{1,p}\omega_0 = \frac{c_0^{\frac{q}{p-1}}}{K} \, \mathbf{W}_{1,p}\Big(\, (\mathbf{W}_{1,p}\sigma)^{\frac{q(p-1)}{p-1-q}} \, d\sigma\Big)$$

$$\geq \frac{c_0^{\frac{q}{p-1}} \, \mathfrak{c}^{\frac{q}{p-1-q}}}{K} \, (\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}} = \frac{c_0^{\frac{q}{p-1}-1} \, \mathfrak{c}^{\frac{q}{p-1-q}}}{K} \, w_0.$$

Hence, for $c_0 \leq (K^{-1}\mathfrak{c}^{\frac{q}{p-1-q}})^{\frac{p-1}{p-1-q}}$, we have $v \geq w_1 \geq w_0$.

To prove the minimality of w, we will need $c_0 \leq C$, so we pick c_0 such that

$$0 < c_0 \leq \min \big\{ C \, K^{\frac{p-1}{p-1-q}}, \, (K^{-1}\mathfrak{c}^{\frac{q}{p-1-q}})^{\frac{p-1}{p-1-q}}, \, C \big\}. \tag{3.19}$$

Let us now construct by induction a sequence $\{w_j\}_{j\geq 1}$ so that

$$\begin{cases} -\Delta_p w_j = \sigma \, w_{j-1}^q \ \text{ in } \ \mathbb{R}^n, \quad w_j \in L_0^{1,p}(\mathbb{R}^n) \cap L^{1+q}(\mathbb{R}^n, d\sigma), \\ 0 \leq w_{j-1} \leq w_j \leq v, \ \text{q.e.}, \quad w_{j-1}^q \, d\sigma \in L^{-1,p'}(\mathbb{R}^n), \end{cases} \tag{3.20}$$

where $\sup_j \, ||w_j||_{1,p} < \infty$. We set $d\omega_j = w_j^q \, d\sigma$, so that

$$-\Delta_p w_j = \omega_{j-1}, \quad j = 1, 2, \ldots.$$

Suppose that $w_0, w_1, \ldots, w_{j-1}$ have been constructed. As in the case $j = 1$, we see that, since $\omega_{j-1} \in L^{-1,p'}(\mathbb{R}^n)$, there exists a unique $w_j \in L_0^{1,p}(\mathbb{R}^n)$ such that

$-\Delta_p w_j = \omega_{j-1}$, and by (2.4),

$$||w_j||_{1,p}^p = ||\omega_{j-1}||_{-1,p'}^{p'} = \int_{\mathbb{R}^n} w_j \, w_{j-1}^q \, d\sigma.$$

By Theorem 2.6, we get

$$w_j \leq K \, \mathbf{W}_{1,p} \omega_{j-1} = K \, \mathbf{W}_{1,p}(w_{j-1}^q d\sigma).$$

Using the inequality $w_{j-1} \leq v$, we see that

$$w_j \leq K \, \mathbf{W}_{1,p}(v^q \, d\sigma) = v.$$

Combining these estimates, we obtain

$$||w_j||_{1,p}^p = \int_{\mathbb{R}^n} w_j \, w_{j-1}^q \, d\sigma \leq \int_{\mathbb{R}^n} v^{1+q} \, d\sigma < \infty.$$

Consequently, $\{w_j\}$ is a bounded sequence in $L_0^{1,p}(\mathbb{R}^n)$. Notice that $w_{j-1} \leq w_j$ by the weak comparison principle (Lemma 3.7), since $\omega_{j-2} \leq \omega_{j-1}$, for $j \geq 2$.

Thus, the sequence (3.20) has been constructed. Letting $w = \lim_{j \to \infty} w_j$, and applying the weak continuity of the p-Laplace operator (Theorem 2.5), the Monotone Convergence Theorem, and Theorem B.1 (Lemma 1.33 in [HKM06]), we deduce the existence of a nontrivial solution $w \in L_0^{1,p}(\mathbb{R}^n)$ to (1.1).

We now prove the minimality of w. Suppose $u \in L_0^{1,p}(\mathbb{R}^n) \cap L_{\text{loc}}^q(\mathbb{R}^n, d\sigma)$ is any nontrivial solution to (1.1). Letting $d\mu = u^q \, d\sigma$, we have $u \in L^{1+q}(\mathbb{R}^n, d\sigma)$, and

$\mu \in L^{-1,p'}(\mathbb{R}^n)$ by Lemma 3.2. To show that $u \geq w$, notice that by Theorem 2.12,

$$u \geq C \left(\mathbf{W}_{1,p}\sigma\right)^{\frac{p-1}{p-1-q}},$$

where C is the constant in (2.27). By the choice of c_0 in (3.19), we have $w_0 \leq u$, so that $\omega_0 \leq \mu$. Therefore, by the weak comparison principle $w_1 \leq u$ q.e. Arguing by induction as above, we see that $w_{j-1} \leq w_j \leq u$ q.e. for $j \geq 1$. It follows that $\lim_{j\to\infty} w_j = w \leq u$ q.e., which proves that w is a minimal solution. □

Combining Lemma 3.2 and Theorem 3.8 we conclude the proof of the existence part of Theorem 3.1. In Sec. 3.3 below we will establish the uniqueness part using the existence of the minimal solution constructed in Theorem 3.8.

It is known that basic facts of potential theory stated in Chapter 2, including Wolff's potential estimates [KM94], and the weak continuity principle [TW02b], remain true for the $\mathcal{A}$-Laplacian. From the above results it follows that our methods work, with obvious modifications, not only for the p-Laplace operator, but for the general $\mathcal{A}$-Laplace operator $\operatorname{div} \mathcal{A}(x, \nabla u)$ as well. In particular, the following more general theorem holds.

Theorem 3.9. *Under the assumptions on $\mathcal{A}(x, \xi)$ stated in Chapter 2, together with the conditions of Theorem 3.1, the equation*

$$-\operatorname{div} \mathcal{A}(x, \nabla u) = \sigma\, u^q$$

has a solution $u \in L_0^{1,p}(\mathbb{R}^n) \cap L^q_{\mathrm{loc}}(\mathbb{R}^n, d\sigma)$ if and only if condition (3.2) *holds.*

3.3 Uniqueness

In this section, we prove the uniqueness of finite energy solutions to (1.1). We employ a convexity argument using some ideas of Kawohl [Ka00] (see also [BeKa02], [BF12]), together with the existence of the minimal solution established above.

Theorem 3.10. *Let $1 < p < \infty$ and let $0 < q < p-1$. Let $\sigma \in M^+(\mathbb{R}^n)$. Suppose that there exists a nontrivial solution $u \in L_0^{1,p}(\mathbb{R}^n) \cap L^q_{\text{loc}}(\mathbb{R}^n, d\sigma)$ to (1.1). Then such a solution is unique.*

Proof. Suppose $u, v \in L_0^{1,p}(\mathbb{R}^n) \cap L^q_{\text{loc}}(\mathbb{R}^n, \sigma)$ are both nontrivial solutions to (1.1). We first show that $u = v$ $d\sigma$-a.e. implies that $u = v$ as elements of $L_0^{1,p}(\mathbb{R}^n)$.

Indeed, suppose that $u = v$ $d\sigma$-a.e., and set $d\mu = u^q d\sigma = v^q d\sigma$, where $\mu \in M^+(\mathbb{R}^n)$, and

$$-\Delta_p u = -\Delta_p v = \mu. \tag{3.21}$$

As usual, we assume that u, v are quasicontinuous representatives (see, e.g., [HKM06], [MZ97]). Then by Lemma 3.2, u, $v \in L^{1+q}(\mathbb{R}^n, d\sigma)$, and

$$\int_{\mathbb{R}^n} \mathbf{W}_{1,p}\mu \, d\mu < +\infty.$$

By Wolff's inequality (2.18), this means that $\mu \in L^{-1,p'}(\mathbb{R}^n)$. It is well known ([MZ97], Sec. 2.1.5) that, for such μ, a finite energy solution to the equation $-\Delta_p u = \mu$ is unique. (See also Lemma 3.7 above.) Hence, from (3.21) we deduce $u = v$ q.e. and as elements of $L_0^{1,p}(\mathbb{R}^n)$.

We next show that if $u \geq v$ $d\sigma$-a.e. then $u = v$ $d\sigma$-a.e. By Theorem 2.12, it follows that $u(x) > 0$, $v(x) > 0$, for all $x \in \mathbb{R}^n$. Testing the equations

$$\int_{\mathbb{R}^n} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx = \int_{\mathbb{R}^n} u^q \varphi \, d\sigma, \quad \varphi \in L_0^{1,p}(\mathbb{R}^n) \cap L^q_{\text{loc}}(\mathbb{R}^n, d\sigma), \tag{3.22}$$

$$\int_{\mathbb{R}^n} |\nabla v|^{p-2} \nabla v \cdot \nabla \psi \, dx = \int_{\mathbb{R}^n} v^q \psi \, d\sigma, \quad \psi \in L_0^{1,p}(\mathbb{R}^n) \cap L^q_{\text{loc}}(\mathbb{R}^n, d\sigma), \tag{3.23}$$

with $\varphi = u, \psi = v$, respectively, we obtain

$$\int_{\mathbb{R}^n} |\nabla u|^p \, dx = \int_{\mathbb{R}^n} u^{1+q} \, d\sigma, \quad \int_{\mathbb{R}^n} |\nabla v|^p \, dx = \int_{\mathbb{R}^n} v^{1+q} \, d\sigma.$$

Let

$$\lambda_t(x) = \Big((1-t) v^p(x) + t u^p(x) \Big)^{\frac{1}{p}}.$$

Using convexity of the Dirichlet integral $\int_{\mathbb{R}^n} |\nabla u|^p \, dx$ in u^p ([Ka00]; see also the proof of Lemma 2.1 in [BF12]), we estimate, for all $t \in (0,1]$,

$$\int_{\mathbb{R}^n} |\nabla \lambda_t(x)|^p dx \le (1-t) \int_{\mathbb{R}^n} |\nabla v|^p \, dx + t \int_{\mathbb{R}^n} |\nabla u|^p \, dx$$

$$= t \left(\int_{\mathbb{R}^n} |\nabla u|^p \, dx - \int_{\mathbb{R}^n} |\nabla v|^p \, dx \right) + \int_{\mathbb{R}^n} |\nabla v|^p \, dx.$$

Thus,

$$\int_{\mathbb{R}^n} \frac{|\nabla \lambda_t(x)|^p - |\nabla \lambda_0(x)|^p}{t} \, dx \le \int_{\mathbb{R}^n} u^{1+q} \, d\sigma - \int_{\mathbb{R}^n} v^{1+q} \, d\sigma.$$

Using the inequality

$$|a|^p - |b|^p \ge p |b|^{p-2} b \cdot (a-b), \quad a, b \in \mathbb{R}^n, \tag{3.24}$$

we deduce

$$|\nabla \lambda_t|^p - |\nabla \lambda_0|^p \geq p|\nabla \lambda_0|^{p-2}\nabla \lambda_0 \cdot (\nabla \lambda_t - \nabla \lambda_0).$$

Notice that $\lambda_0 = v$, and consequently, for all $t \in (0, 1]$,

$$p\int_{\mathbb{R}^n} |\nabla v|^{p-2}\nabla v \cdot \frac{\nabla(\lambda_t - \lambda_0)}{t}\, dx \leq \int_{\mathbb{R}^n} u^{1+q} d\,\sigma - \int_{\mathbb{R}^n} v^{1+q} d\,\sigma. \tag{3.25}$$

Testing (3.23) with $\psi = \lambda_t - \lambda_0 \in L_0^{1,p}(\mathbb{R}^n)$, we obtain

$$\int_{\mathbb{R}^n} |\nabla v|^{p-2}\nabla v \cdot \nabla(\lambda_t - \lambda_0)\, dx = \int_{\mathbb{R}^n} v^q(\lambda_t - \lambda_0) d\sigma.$$

Hence, by (3.25), for all $t \in (0, 1]$,

$$p\int_{\mathbb{R}^n} v^q\, \frac{\lambda_t - \lambda_0}{t}\, d\sigma \leq \int_{\mathbb{R}^n} u^{1+q} d\,\sigma - \int_{\mathbb{R}^n} v^{1+q} d\,\sigma. \tag{3.26}$$

Clearly, $\lambda_t \geq \lambda_0$, since $u \geq v$. Applying Fatou's lemma, we obtain

$$\int_{\mathbb{R}^n} v^q\, \frac{u^p - v^p}{v^{p-1}}\, d\sigma \leq \liminf_{t\to 0} p \int_{\mathbb{R}^n} v^q\, \frac{\lambda_t - \lambda_0}{t}\, d\sigma.$$

Combining this and (3.26) yields

$$\int_{\mathbb{R}^n} (\frac{v^q u^p}{v^{p-1}} - v^{1+q})\, d\sigma \leq \int_{\mathbb{R}^n} u^{1+q} d\,\sigma - \int_{\mathbb{R}^n} v^{1+q} d\,\sigma.$$

Therefore, canceling the second terms on both sides, and taking into account that $u \geq v$ $d\sigma$-a.e., we arrive at

$$0 \geq \int_{\mathbb{R}^n} (\frac{v^q u^p}{v^{p-1}} - u^{1+q})\, d\sigma = \int_{\mathbb{R}^n} \frac{v^q u^p - u^{1+q} v^{p-1}}{v^{p-1}}\, d\sigma$$

$$= \int_{\mathbb{R}^n} \frac{v^q u^{1+q}(u^{p-1-q} - v^{p-1-q})}{v^{p-1}} \, d\sigma \geq 0.$$

Hence, $u = v \ d\sigma$-a.e.

We now complete the proof of the uniqueness property. Suppose that u and v are nontrivial finite energy solutions to (1.1). Then $\min(u, v) \geq w \ d\sigma$-a.e., where w is the nontrivial minimal solution constructed in Theorem 3.8. Therefore, as was shown above, $w = u = v \ d\sigma$-a.e., and also as elements of $L_0^{1,p}(\mathbb{R}^n)$. $\square$

Chapter 4

Weak solutions and intrinsic potentials of Wolff type

4.1 Main results

In this chapter, we study the equation

$$(4.1) \qquad \begin{cases} -\Delta_p u = \sigma u^q & \text{in} \quad \mathbb{R}^n, \\ \liminf\limits_{|x|\to\infty} u(x) = 0, & u > 0, \end{cases}$$

together with the closely related equation

$$(4.2) \qquad \begin{cases} -\triangle_p v = b\,\frac{|\nabla v|^p}{v} + \sigma & \text{in} \quad \mathbb{R}^n, \\ \liminf\limits_{|x|\to\infty} v(x) = 0, & v > 0, \end{cases}$$

where $1 < p < \infty, 0 < q < p-1, \sigma \in M^+(\mathbb{R}^n)$ and b is a positive constant defined by

$$b = \frac{q(p-1)}{p-1-q}. \tag{4.3}$$

We give *necessary and sufficient* conditions for the existence of *weak* solutions to (4.1). Sharp global *pointwise* estimates and regularity properties of solutions are provided as well. We also treat the fractional Laplacian equation of the type

$$\begin{cases} (-\Delta)^\alpha u = \sigma u^q \quad \text{in} \quad \mathbb{R}^n, \\ \liminf\limits_{|x|\to\infty} u(x) = 0, \quad u > 0, \end{cases} \tag{4.4}$$

for $0 < q < 1$ and $0 < \alpha < \frac{n}{2}$.

As a consequence of our results for (4.1), we will be able to give necessary and sufficient conditions for the existence of solutions to (4.2) and obtain bilateral pointwise bounds of such solutions as well.

As mentioned in the Introduction, equation (4.1) with $p = 2, 0 < q < 1$ is associated with the integral inequality

$$\left(\int_{\mathbb{R}^n} |\varphi|^q \, d\sigma \right)^{\frac{1}{q}} \leq \varkappa \, ||\Delta \varphi||_{L^1(\mathbb{R}^n)}, \tag{4.5}$$

for all test functions φ such that $-\Delta\varphi \geq 0$, $\liminf\limits_{x\to\infty} \varphi(x) = 0$.

This inequality represents the end-point case of the well-studied (L^p, L^q) weighted norm inequality for $p > 1$. The reader could consult [Maz11] for similar inequalities of this type.

We need a localized version of the preceding inequality, where the measure $\sigma = \sigma_B$

is restricted to a ball B in $\mathbb{R}^n$:

$$\left(\int_{\mathbb{R}^n} |\varphi|^q \, d\sigma_B \right)^{\frac{1}{q}} \leq \varkappa(B) \, ||\Delta \varphi||_{L^1(\mathbb{R}^n)}, \tag{4.6}$$

where $\varkappa(B)$ is the best constant in this inequality. We use these constants to build a special nonlinear potential of Wolff type,

$$\mathbf{K}\sigma(x) = \int_0^\infty \frac{\varkappa(B(x,s))^{\frac{q}{1-q}}}{s^{n-2}} \frac{ds}{s}, \quad x \in \mathbb{R}^n. \tag{4.7}$$

Together with the usually $\mathbf{I}_2\sigma$ potential, this new potential $\mathbf{K}\sigma$ provides sharp estimates for solutions of (1.1) with $p = 2$ and $0 < q < 1$.

Using both $\mathbf{K}\sigma$ and $\mathbf{I}_2\sigma$, we will able to bridge the gap in the estimates of Brezis-Kamin [BK92], as discussed in the Introduction, and extend these results to possibly singular (unbounded) solutions as well as more general nonlinear equations.

Simultaneously with (4.1), we will investigate equation (4.2) with singular gradient terms. We notice that equations (4.1) and (4.2) are *formally* related via the substitution

$$v = \frac{p-1}{p-1-q} \, u^{\frac{p-1-q}{p-1}}. \tag{4.8}$$

In general, the situation is more delicate since some certain singular measures can arise, as was first observed by Ferone and Murat [FM00] who studied a similar phenomenon in the case $q = p - 1$. We are, however, still able to give necessary and sufficient conditions for the existence of weak solutions to (4.2), and obtain pointwise estimates of such solutions as well.

Equations of the type (4.1) and (4.2) have been studied extensively, mostly in a bounded domain $\Omega \subset \mathbb{R}^n$ with $\sigma \in L^s(\Omega)$ for some $s > 1$, in [ABLP10, AGPW11, AHBV10, BO96, BO12, BrOs86, Kra64]. In these papers, the authors considered various existence results for solutions in some Sobolev spaces (see also futher references given there).

However, the precise conditions on σ which guarantee the existence of solutions are more subtle. In particular, σ could be an L^1_{loc}-function, or a measure singular with respect to Lebesgue measure. Notice that, as shown in Theorem 2.14, σ must be absolutely continuous with respect to p-capacity. Moreover, we will be able to offer a condition on σ which is not only necessary but also sufficient for the existence of solutions to (4.1).

For a ball B in $\mathbb{R}^n$, let denote by $\varkappa(B)$ the best constant in the inequality

$$\left(\int_B |\varphi|^q \, d\sigma\right)^{\frac{1}{q}} \leq \varkappa(B) \, ||\Delta_p \varphi||^{\frac{1}{p-1}}_{L^1(\mathbb{R}^n)}, \tag{4.9}$$

for all test functions φ such that $-\Delta_p \varphi \geq 0$, $\liminf\limits_{x\to\infty} \varphi(x) = 0$.

We define the intrinsic potential of Wolff type by

$$\mathbf{K}_{1,p,q}\sigma(x) = \int_0^\infty \left(\frac{\varkappa(B(x,s))^{\frac{q(p-1)}{p-1-q}}}{s^{n-p}}\right)^{\frac{1}{p-1}} \frac{ds}{s}, \quad x \in \mathbb{R}^n. \tag{4.10}$$

As we show below, $\mathbf{K}_{1,p,q}\sigma \not\equiv +\infty$ if and only if

$$\int_1^\infty \left(\frac{\varkappa(B(0,s))^{\frac{q(p-1)}{p-1-q}}}{s^{n-p}}\right)^{\frac{1}{p-1}} \frac{ds}{s} < \infty. \tag{4.11}$$

We notice that, by Theorem 2.12, a necessary condition for the existence of a nontrivial solution to (4.1) is that $\mathbf{W}_{1,p}\sigma \not\equiv +\infty$, or equivalently,

$$\int_1^\infty \left(\frac{\sigma(B(0,t))}{t^{n-p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} < +\infty. \tag{4.12}$$

We are ready to state our main theorem for equation (4.1).

Theorem 4.1. *Let $1 < p < n$, $0 < q < p-1$, and let $\sigma \in M^+(\mathbb{R}^n)$.*

(i) *If both* (4.11) *and* (4.12) *hold, then there exists a minimal renormalized (p-superharmonic) solution $u > 0$ to* (4.1) *such that*

$$c^{-1}\left[\mathbf{K}_{1,p,q}\sigma + (\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}}\right] \le u \le c\left[\mathbf{K}_{1,p,q}\sigma + (\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}}\right], \tag{4.13}$$

where $c > 0$ is a constant which depends only on p, q, and n.

(ii) *Conversely, if there exists a nontrivial renormalized supersolution u to* (4.1), *then both* (4.11) *and* (4.12) *hold, and u is bounded below by the minimal solution of statement* (i).

(iii) *In the case $p \ge n$ there are no nontrivial supersolutions on $\mathbb{R}^n$.*

In the next theorem, we characterize solutions with $W^{1,p}_{\text{loc}}$-regularity.

Theorem 4.2. *Let $0 < q < p-1, 1 < p < n$ and $\sigma \in M^+(\mathbb{R}^n)$. Then there exists a nontrivial solution $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ to* (4.1) *if and only if* (4.11) *and* (4.12) *hold, together with the local condition*

$$\int_B (\mathbf{W}_{1,p}\sigma_B)^{\frac{(1+q)(p-1)}{p-1-q}} \, d\sigma < \infty, \tag{4.14}$$

for all balls $B = B(0,r)$ in $\mathbb{R}^n$.

We remark that in Chapter 3, a global version of (4.14) is necessary and sufficient for the existence of finite energy solutions (i.e., $L_0^{1,p}(\mathbb{R}^n)$-solutions) to (4.1). We also proved the uniqueness property of such solutions.

We observe that (see Section 4.4) conditions (4.11), (4.12) and (4.14) are mutually independent.

We will study the general nonlinear integral equations which are closely related to all nonlinear PDE mentioned above,

$$u = \mathbf{W}_{\alpha,p}(u^q d\sigma), \tag{4.15}$$

where $1 < p < \infty$ and $0 < \alpha < \frac{n}{p}$. Here $\alpha = 1$ stands for the p-Laplacian, $p = 2$ and $0 < \alpha < \frac{n}{2}$ correspond to the fractional Laplacian $(-\Delta)^\alpha$ and $\alpha = \frac{2k}{k+1}, p = k+1$ represent the k-Hessian operator. We are also interested in the supersolution of (4.15), which is a nonnegative solution u to the integral inequality,

$$u \geq \mathbf{W}_{\alpha,p}(u^q d\sigma). \tag{4.16}$$

In Section 4.2, we will introduce a fractional version $\mathbf{K}_{\alpha,p,q}\sigma$ of the intrinsic Wolff potential (4.10) for all $p > 1, 0 < q < p-1, 0 < \alpha < \frac{n}{p}$ and deduce analogues of Theorem 4.1 and Theorem 4.2 for the $\mathcal{A}$-Laplacians and k-Hessians in Section 4.3.

Letting $p = 2$ in (4.15) brings the fractional Laplace equation (4.4) into the equivalent integral form $u = \mathbf{I}_{2\alpha}(u^q d\sigma)$, where $\mathbf{I}_{2\alpha}$ is the Riesz potential of order 2α. Equation (4.4) was studied by F. Punzo and G. Terrone [PT14], in which they considered bounded solutions and $0 < \alpha < \min\{1, \frac{n}{4}\}$. We will be able to extend their results to possibly unbounded solutions and $0 < \alpha < \frac{n}{2}$.

For the fractional Laplacian equation (4.4), let $\kappa(B)$ denote the least constant in the inequality

$$||\mathbf{I}_\alpha \nu||_{L^q(d\sigma_B)} \leq \kappa(B)\nu(\mathbb{R}^n), \quad \forall \nu \in M^+(\mathbb{R}^n), \tag{4.17}$$

where $0 < q < 1, 0 < \alpha < \frac{n}{2}$, and B is a ball in $\mathbb{R}^n$. We define the corresponding nonlinear potential of Wolff type by

$$\mathbf{K}_{\alpha,q,2}\sigma(x) = \int_0^\infty \frac{\kappa(B(x,s))^{\frac{q}{1-q}}}{s^{n-2\alpha}} \frac{ds}{s}, \quad x \in \mathbb{R}^n. \tag{4.18}$$

To ensure that $\mathbf{K}_{\alpha,q,2}\sigma$ and $\mathbf{I}_{2\alpha}\sigma$ are not indentically infinite, we impose the condition

$$\int_1^\infty \frac{\kappa(B(0,s))^{\frac{q}{1-q}}}{s^{n-2\alpha}} \frac{ds}{s} + \int_1^\infty \frac{\sigma(B(0,s))}{s^{n-2\alpha}} \frac{ds}{s} < \infty. \tag{4.19}$$

We state our main results for the fractional Laplacian equation (4.4).

Theorem 4.3. *Let $0 < \alpha < \frac{n}{2}$, $0 < q < 1$, and $\sigma \in M^+(\mathbb{R}^n)$.*

(i) *Suppose that* (4.19) *holds. Then there exists a minimal solution $u > 0$ to* (4.4) *such that $\liminf_{|x|\to\infty} u(x) = 0$, and*

$$c^{-1}\left[\mathbf{K}_{\alpha,q,2}\sigma + (\mathbf{I}_{2\alpha}\sigma)^{\frac{1}{1-q}}\right] \leq u \leq c\left[\mathbf{K}_{\alpha,q,2}\sigma + (\mathbf{I}_{2\alpha}\sigma)^{\frac{1}{1-q}}\right], \tag{4.20}$$

where $c > 0$ is a constant which depends only on α, q, and n.

(ii) *Conversely, if there exists a nontrivial supersolution u to* (4.4) *then* (4.19) *holds, and u satisfies the lower bound in* (4.20).

We are now in a position to discuss the relation between (4.1) and (4.2). We will show that (4.11) and (4.12) are necessary and sufficient for the solvability of (4.2) as well. In particular, if u is a solution to (4.1) then v is a solution to (4.2) via substitution (4.8); but in the opposite direction, if v is a weak solution to (4.2) then u is only a *supersolution* to (4.1), which is enough for our purposes. In order for u to be a genuine solution, one needs to impose extra assumptions on v described in the following theorem.

Theorem 4.4. *Let* $1 < p < \infty$ *and* $0 < q < p-1$. *Suppose* $b > 0$ *is defined by* (4.3), *and* $\sigma \in M^+(\mathbb{R}^n)$.

(i) If u *is a renormalized solution to* (4.1), *then* v *defined by* (4.8) *is a renormalized solution to* (4.2). *Consequently, if both* (4.11) *and* (4.12) *hold, then* (4.2) *has a renormalized solution* v *which satisfies both the lower estimate*

$$v \geq c^{-1} \left[\mathbf{W}_{1,p}\sigma + (\mathbf{K}_{1,p,q}\sigma)^{\frac{p-1-q}{p-1}} \right]. \tag{4.21}$$

and the matching upper estimate

$$v \leq c \left[\mathbf{W}_{1,p}\sigma + (\mathbf{K}_{1,p,q}\sigma)^{\frac{p-1-q}{p-1}} \right], \tag{4.22}$$

where $c > 0$ *depends only on* $n, p,$ *and* q.

(ii) Conversely, if (4.2) *has a solution* $v > 0$, *then for every ball* B *and* $d\omega_B = \frac{|Dv|^p}{v} \chi_B \, dx$, *we have*

$$||v||_{L^{\frac{q(p-1)}{p-1-q},\infty}(\omega_B)} < \infty, \tag{4.23}$$

Moreover, v satisfies the lower bound (4.21)*, and u defined by* (4.8) *is a supersolution to* (4.1)*; consequently both* (4.11) *and* (4.12) *hold.*

(iii) Furthermore, if v satisfies the strong-type version of (4.23)*, i.e.,*

$$||v||_{L^{\frac{q(p-1)}{p-1-q}}(\omega_B)} < \infty. \tag{4.24}$$

for every ball B, then u is actually a renormalized solution to (4.1)*.*

Next, we consider finite energy solutions to (4.2) and show that such solutions exist if and only if $b < 1$, i.e., $q < 1 - \frac{1}{p}$, and σ has finite energy: $\sigma \in L^{-1,p'}(\mathbb{R}^n) = L_0^{1,p}(\mathbb{R}^n)^*$. By (2.1), $\sigma \in L^{-1,p'}(\mathbb{R}^n)$ if and only if

$$\int_{\mathbb{R}^n} \mathbf{W}_{1,p}\sigma \, d\sigma < \infty. \tag{4.25}$$

Theorem 4.5. *Let $1 < p < \infty$ and $0 < q < p - 1$. Suppose $b > 0$ is defined by* (4.3)*, and $\sigma \in M^+(\mathbb{R}^n)$.*

(i) If $b < 1$, i.e., $q < 1 - \frac{1}{p}$, and σ has finite energy, then (4.1) *has a p-superharmonic solution u, and v defined by* (4.8) *is a finite energy solution to* (4.2)*.*

(ii) Conversely, if (4.2) *has a nontrivial finite energy solution $v \in L_0^{1,p}(\mathbb{R}^n)$, then $b < 1$ and σ has finite energy. Moreover, u defined by* (4.8) *is a p-superharmonic solution to* (4.1)*.*

The content of this chapter is as follows. In Section 4.2, we study the integral equation (4.15) and introduce the intrinsic Wolff potentials $\mathbf{K}_{\alpha,p,q}\sigma$. In Section 4.3 we prove our main theorem regarding equation (4.1) and briefly discuss more general quasilinear and fully nonlinear equations . In Section 4.4, we give a counterexample showing that the finiteness of the embedding constant $\kappa(B)$ is not enough for the existence of a global solution to (4.1) even if $\mathbf{W}_{1,p}\sigma < \infty$ a.e. The last Section (Sec. 4.5) is devoted to a study of equation (4.2) and its relations to (4.1).

4.2 Solutions of the integral equations

4.2.1 Weighted norm inequalities and intrinsic potentials $\mathbf{K}_{\alpha,p,q}\sigma$

Let $\sigma \in M^+(\mathbb{R}^n)$, $1 < p < \infty, 0 < q < p-1$ and $0 < \alpha < \frac{n}{p}$. We denote by κ the least constant in the weighted norm inequality

$$(4.26) \qquad ||\mathbf{W}_{\alpha,p}\nu||_{L^q(\mathbb{R}^n,d\sigma)} \leq \kappa(\nu(\mathbb{R}^n))^{\frac{1}{p-1}}, \quad \forall \nu \in M^+(\mathbb{R}^n).$$

We will also need a localized version of (4.26) for $\sigma_E = \sigma|_E$, where $E \subset \mathbb{R}^n$ is Borel, and $\kappa(E)$ is the least constant in

$$(4.27) \qquad ||\mathbf{W}_{\alpha,p}\nu||_{L^q(d\sigma_E)} \leq \kappa(E)(\nu(\mathbb{R}^n))^{\frac{1}{p-1}}, \quad \forall \nu \in M^+(\mathbb{R}^n).$$

We define the intrinsic potential of Wolff type in terms of $\kappa(B(x,s))$, the least constant in (4.27) with $E = B(x,s)$, by

$$(4.28) \qquad \mathbf{K}_{\alpha,p,q}\sigma(x) = \int_0^\infty \left(\frac{\kappa(B(x,s))^{\frac{q(p-1)}{p-1-q}}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s}, \quad x \in \mathbb{R}^n.$$

Remark 4.6. *We notice that, for $\alpha = 1$, in the definition of $\mathbf{K}_{1,p,q}\sigma$ we can use either the constant $\varkappa(B(x,s))$ in (4.9) or $\kappa(B(x,s))$ in (4.27) with $E = B(x,s)$, since by*

Theorem 2.6,

$$\frac{1}{K}\kappa(E) \le \varkappa(E) \le K\kappa(E), \tag{4.29}$$

where $K = K(n,p) > 0$ *is the constant in* (2.12).

We have the following key lemma whose proof is based on Vitali's covering lemma and weak-type maximal function inequalites.

Lemma 4.7. *Let* $1 < p < \infty, 0 < q < p-1, 0 < \alpha < \frac{n}{p}$ *and* $\sigma \in M^+(\mathbb{R}^n)$.

(i) Suppose $0 \le u \in L^q_{\text{loc}}(\mathbb{R}^n, d\sigma)$ *is a nontrivial solution of* (4.16). *Then* (4.27) *holds for all balls* B *with*

$$\kappa(B) \le c(\alpha,n,p,q)\left(\int_B u^q d\sigma\right)^{\frac{p-1-q}{(p-1)q}}. \tag{4.30}$$

(ii) In statement (i), if we have $u \in L^q(\mathbb{R}^n, d\sigma)$, *then* (4.26) *holds with*

$$\kappa \le c(\alpha,n,p,q)\left(\int_{\mathbb{R}^n} u^q d\sigma\right)^{\frac{p-1-q}{(p-1)q}}. \tag{4.31}$$

Proof. Let $d\omega = u^q d\sigma$. For $\nu \in M^+(\mathbb{R}^n)$, consider the maximal function

$$M^\nu_\omega(x) = \sup_{r>0}\frac{\nu(B(x,\frac{r}{5}))}{\omega(B(x,r))}, \quad x \in \mathbb{R}^n, \tag{4.32}$$

here we make the convention that $\frac{0}{0} = 0$. Let

$$E_t = \{x \in \mathbb{R}^n : M^\nu_\omega(x) > t\}, \quad t > 0.$$

Suppose $E_t \neq \emptyset$. For $x \in E_t$, there exists a ball $B(x, r_x)$ such that

$$\frac{\nu(B(x, \frac{r}{5}))}{\omega(B(x, r))} > t.$$

Thus $E_t \subset \cup_{x \in E_t} B(x, \frac{r_x}{5})$, and hence for any compact subset E of E_t, there exists a $k \in \mathbb{N}$ such that

$$E \subset \cup_{i=1}^{k} B(x_k, \frac{r_{x_k}}{5}).$$

By Vitali's covering lemma, we find disjoint balls $\{B(x_{j_l}, \frac{r_{x_{j_l}}}{5})\}_{l=1}^{m}$ such that

$$E \subset \cup_{l=1}^{m} B(x_{j_l}, r_{x_{j_l}}).$$

Thus,

$$\omega(E) \leq \sum_{l=1}^{m} \omega(B(x_{j_l}, r_{x_{j_l}})) \leq \frac{1}{t} \sum_{l=1}^{m} \nu(B(x_{j_l}, \frac{r_{x_{j_l}}}{5})) \leq \frac{1}{t} \nu(\mathbb{R}^n).$$

Therefore,

$$\sup_{t>0} t\omega(E_t) := ||M_\omega^\nu||_{L^{1,\infty}(d\omega)} \leq \nu(\mathbb{R}^n). \tag{4.33}$$

For $x \in \mathbb{R}^n$ such that $M_\omega^\nu(x) < \infty$, clearly, we have

$$\mathbf{W}_{\alpha,p}\nu(x) = \int_0^\infty \left(\frac{\nu(B(x,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} = 5^{\frac{n-\alpha p}{p-1}} \int_0^\infty \left(\frac{\nu(B(x, \frac{s}{5}))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s}$$

$$= 5^{\frac{n-\alpha p}{p-1}} \int_0^\infty \left(\frac{\nu(B(x, \frac{s}{5}))}{\omega(B(x,s))} \cdot \frac{\omega(B(x,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \leq 5^{\frac{n-\alpha p}{p-1}} \left(M_\omega^\nu(x)\right)^{\frac{1}{p-1}} \mathbf{W}_{\alpha,p}\omega(x)$$

$$\leq 5^{\frac{n-\alpha p}{p-1}} \left(M_\omega^\nu(x)\right)^{\frac{1}{p-1}} \mathbf{W}_{\alpha,p}(u^q d\sigma)(x) \leq 5^{\frac{n-\alpha p}{p-1}} \left(M_\omega^\nu(x)\right)^{\frac{1}{p-1}} u(x).$$

Note that if $\nu(B(x, \frac{s}{5})) > 0$ but $\omega(B(x,s)) = 0$ for some $s > 0$ then $M_\omega^\nu(x) = \infty$. By (4.33), it follows that the set of such $y \in B$ has ω-measure zero, and hence σ-measure zero, since by Lemma 2.9, we have $\inf_B u > 0$. Therefore,

$$||\mathbf{W}_{\alpha,p}\nu||^q_{L^q(d\sigma_B)} \leq c \int_B (M_\omega^\nu)^{\frac{q}{p-1}} u^q d\sigma \leq c \int_B (M_\omega^\nu(x))^{\frac{q}{p-1}} d\omega.$$

Let us recall the well-known inequality

$$||f||^r_{L^r(X,d\omega)} \leq c(r)\, \omega(X)^{1-r} ||f||^r_{L^{1,\infty}(X,d\omega)},$$

where $0 < r < 1$ and ω is a finite measure on X. Applying the preceding inequality with $f = M_\omega^\nu$ and $r = \frac{q}{p-1} < 1$, we obtain

$$||\mathbf{W}_{\alpha,p}\nu||^q_{L^q(d\sigma_B)} \leq c(\omega(B))^{1-\frac{q}{p-1}} ||M_\omega^\nu||^{\frac{q}{p-1}}_{L^{1,\infty}(d\omega)} \leq c(\omega(B))^{\frac{p-1-q}{p-1}} \nu(\mathbb{R}^n)^{\frac{q}{p-1}}, \tag{4.34}$$

where $c = c(\alpha, n, p, q)$. This proves statement (i) of Lemma 4.7.

If $u \in L^q(\mathbb{R}^n, d\sigma)$ then (ii) follows from (i) by letting $R \to \infty$ with $B = B(0,R)$.

□

We will also need a converse estimate to (4.27) for subsolution u_B of equation (4.15) with σ_B in place of σ, for a ball B.

Corollary 4.8. *Let $1 < p < \infty, 0 < q < p-1, 0 < \alpha < \frac{n}{p}$ and $\sigma \in M^+(\mathbb{R}^n)$. Suppose $0 \leq u_B \in L^q(\mathbb{R}^n, d\sigma_B)$ is a nontrivial solution to $u_B \leq \mathbf{W}_{\alpha,p}(u_B^q d\sigma_B)$ $d\sigma_B$-a.e. Then, for every ball B,*

$$\left(\int_B u^q d\sigma\right)^{\frac{p-1-q}{(p-1)q}} \leq \kappa(B). \tag{4.35}$$

Proof. Without loss of generality we may assume that $\kappa(B) < \infty$. Then using (4.27) with $d\nu = u_B^q d\sigma_B$, we have

$$\int_B u_B^q d\sigma \leq \int_B (\mathbf{W}_{\alpha,p}(u_B^q d\sigma_B))^q d\sigma \leq \kappa(B)^q \left(\int_B u_B^q d\sigma\right)^{\frac{q}{p-1}},$$

which yields (4.35). □

4.2.2 Solution in $L^q(\mathbb{R}^n, d\sigma)$

We next state our result concerning the existence of a global solution $u \in L^q(\mathbb{R}^n, d\sigma)$ to (4.15).

Theorem 4.9. *Let $\sigma \in M^+(\mathbb{R}^n)$. Then equation* (4.15) *has a solution $u \in L^q(\mathbb{R}^n, d\sigma)$ if and only if there exists a constant $\kappa > 0$ such that* (4.26) *holds.*

Proof. The necessity of (4.26) follows from Lemma 4.7 (ii). To prove its sufficiency, we first show that (4.26) implies

$$\int_{\mathbb{R}^n} (\mathbf{W}_{\alpha,p}\sigma)^{\frac{q(p-1)}{p-1-q}} d\sigma < \infty. \tag{4.36}$$

Indeed, fix a ball $B = B(0, R)$. Applying (4.26) with $d\nu = d\sigma_B$ we get

$$\int_{\mathbb{R}^n} (\mathbf{W}_{\alpha,p}\sigma_B)^q d\sigma \leq \kappa^q \sigma(B)^{\frac{q}{p-1}} < \infty.$$

Letting $v_0 = (\mathbf{W}_{\alpha,p}\sigma_B)^q$, where $v_0 \in L^1(\mathbb{R}^n, d\sigma)$, and using $d\nu = v_0 d\sigma$ in (4.26) we obtain

$$\int_{\mathbb{R}^n} (\mathbf{W}_{\alpha,p}(v_0 d\sigma))^q \, d\sigma \leq \kappa^q \left(\int_{\mathbb{R}^n} v_0 d\sigma\right)^{\frac{q}{p-1}} < \infty.$$

By Lemma 2.13 with $r = q$, we have

$$(\mathbf{W}_{\alpha,p}(v_0 d\sigma))^q = (\mathbf{W}_{\alpha,p}((\mathbf{W}_{\alpha,p}\sigma_B)^q d\sigma))^q \geq (\mathbf{W}_{\alpha,p}((\mathbf{W}_{\alpha,p}\sigma_B)^q d\sigma_B))^q$$

$$\geq \mathfrak{c}^{\frac{q^2}{p-1}} (\mathbf{W}_{\alpha,p}\sigma_B)^{q(\frac{q}{p-1}+1)}.$$

Let $v_1 = \mathfrak{c}^{\frac{q^2}{p-1}} (\mathbf{W}_{\alpha,p}\sigma_B)^{q(\frac{q}{p-1}+1)}$, then $v_1 \in L^1(\mathbb{R}^n, d\sigma)$, and

$$\int_{\mathbb{R}^n} v_1 d\sigma \leq \kappa^q \left(\int_{\mathbb{R}^n} v_0 d\sigma \right)^{\frac{q}{p-1}}.$$

Using (4.26) again with $d\nu = v_1 d\sigma$, we obtain

$$\int_{\mathbb{R}^n} (\mathbf{W}_{\alpha,p}(v_1 d\sigma))^q \, d\sigma \leq \kappa^q \left(\int_{\mathbb{R}^n} v_1 d\sigma \right)^{\frac{q}{p-1}} \leq \kappa^{q(1+\frac{q}{p-1})} \left(\int_{\mathbb{R}^n} v_0 d\sigma \right)^{\frac{q^2}{(p-1)^2}} < \infty.$$

By Lemma 2.13 with $r = q(\frac{q}{p-1} + 1)$, we have

$$(\mathbf{W}_{\alpha,p}(v_1 d\sigma))^q = \left(\mathbf{W}_{\alpha,p}(\mathfrak{c}^{\frac{q^2}{p-1}} (\mathbf{W}_{\alpha,p}\sigma_B)^{q(\frac{q}{p-1}+1)} d\sigma) \right)^q$$

$$\geq \mathfrak{c}^{\frac{q^2}{p-1}(1+2\frac{q}{p-1})} (\mathbf{W}_{\alpha,p}\sigma_B)^{q(\frac{q^2}{(p-1)^2}+\frac{q}{p-1}+1)}.$$

Letting

$$v_2 = \mathfrak{c}^{\frac{q^2}{p-1}(1+2\frac{q}{p-1})} (\mathbf{W}_{\alpha,p}\sigma_B)^{q(\frac{q^2}{(p-1)^2}+\frac{q}{p-1}+1)},$$

we obtain

$$\int_{\mathbb{R}^n} v_2 d\sigma \leq \kappa^{q(1+\frac{q}{p-1})} \left(\int_{\mathbb{R}^n} v_0 d\sigma \right)^{\frac{q^2}{(p-1)^2}} < \infty.$$

Letting

$$v_j = \mathfrak{c}^{\frac{q^2}{p-1}\sum_{k=1}^{j} k(\frac{q}{p-1})^{k-1}} (\mathbf{W}_{\alpha,p}\sigma_B)^{q\sum_{k=0}^{j}(\frac{q}{p-1})^k},$$

and arguing by induction, we see that

$$\int_{\mathbb{R}^n} v_j d\sigma \leq \kappa^{q\sum_{k=0}^{j-1}(\frac{q}{p-1})^k} \left(\int_{\mathbb{R}^n} v_0 d\sigma\right)^{(\frac{q}{p-1})^j} < \infty.$$

By Fatou's lemma,

$$\int_{\mathbb{R}^n} \liminf_{j\to\infty} v_j d\sigma \leq \liminf_{j\to\infty} \int_{\mathbb{R}^n} v_j d\sigma \leq \liminf_{j\to\infty} \kappa^{q\sum_{k=0}^{j-1}(\frac{q}{p-1})^k} \left(\int_{\mathbb{R}^n} v_0 d\sigma\right)^{(\frac{q}{p-1})^j}$$

$$= \kappa^{\frac{q(p-1)}{p-1-q}} < \infty.$$

Thus,

$$\mathfrak{c}^{\frac{q^2}{p-1}\sum_{k=1}^{\infty} k(\frac{q}{p-1})^{k-1}} \int_{\mathbb{R}^n} (\mathbf{W}_{\alpha,p}\sigma_B)^{\frac{q(p-1)}{p-1-q}} d\sigma \leq \kappa^{\frac{q(p-1)}{p-1-q}} < \infty. \tag{4.37}$$

Note that $B = B(0, R)$, letting $R \to \infty$ in (4.37) and using the Monotone Convergence Theorem yield (4.36).

Next, let $u_0 = c_0 (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}}$, where $c_0 > 0$ is a small constant to be chosen later on. We construct a sequence $\{u_j\}$ as follows:

$$u_{j+1} = \mathbf{W}_{\alpha,p}(u_j^q d\sigma), \quad j = 0, 1, 2, \ldots. \tag{4.38}$$

Applying Lemma 2.13, we estimate

$$u_1 = \mathbf{W}_{\alpha,p}(u_0^q d\sigma) = c_0^{\frac{q}{p-1}} \mathbf{W}_{\alpha,p}((\mathbf{W}_{\alpha,p}\sigma)^{\frac{q(p-1)}{p-1-q}} d\sigma) \geq c_0^{\frac{q}{p-1}} \mathfrak{c}^{\frac{q}{p-1-q}} (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}},$$

where $\mathfrak{c}$ is the constant in (2.28). Choosing c_0 so that $c_0^{\frac{q}{p-1}} \mathfrak{c}^{\frac{q}{p-1-q}} \geq c_0$, we obtain $u_1 \geq u_0$. By induction, we can show that the sequence $\{u_j\}$ is nondecreasing. Note that $u_0 \in L^q(\mathbb{R}^n, d\sigma)$ by (4.36). Suppose that $u_j \in L^q(\mathbb{R}^n, d\sigma)$ for some $j \geq 0$. Then, applying (4.26) with $d\nu = u_j^q d\sigma$, we obtain

$$\int_{\mathbb{R}^n} u_{j+1}^q \, d\sigma = \int_{\mathbb{R}^n} (\mathbf{W}_{\alpha,p}(u_j^q \, d\sigma))^q d\sigma \leq \kappa \left(\int_{\mathbb{R}^n} u_j^q \, d\sigma \right)^{\frac{q}{p-1}} < \infty.$$

Since $u_j \leq u_{j+1}$, the preceding inequality yields, for all $j = 0, 1, \ldots$,

$$\int_{\mathbb{R}^n} u_{j+1}^q \, d\sigma \leq \kappa^{\frac{p-1}{p-1-q}} < \infty.$$

Thus, using the Monotone Convergence Theorem and passing to the limit as $j \to \infty$ in (4.38), we see that $u = \lim_{j\to\infty} u_j$ is a nontrivial solution to (4.15) and $u \in L^q(\mathbb{R}^n, d\sigma)$. □

4.2.3 Solution in $L^q_{\mathrm{loc}}(\mathbb{R}^n, d\sigma)$

In this section, we prove our main theorem for the existence of a solution $u \in L^q_{\mathrm{loc}}(\mathbb{R}^n, d\sigma)$ to (4.15). We begin with the following lemma.

Lemma 4.10. *Suppose* $0 \leq u \in L^q_{\mathrm{loc}}(\mathbb{R}^n, d\sigma)$ *is a nontrivial solution to* (4.16). *Then, for all* $x \in \mathbb{R}^n$ *and* $t > 0$,

$$\sigma(B(x,t)) \left[\int_t^\infty \left(\frac{(\kappa(B(x,s)))^{\frac{q(p-1)}{p-1-q}}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^q \leq c \int_{B(x,t)} u^q d\sigma, \tag{4.39}$$

where c depends only on $\alpha, n, p,$ and q. Consequently,

$$\int_t^\infty \left(\frac{(\kappa(B(x,s)))^{\frac{q(p-1)}{p-1-q}}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} < \infty, \tag{4.40}$$

for all $x \in \mathbb{R}^n$ and $t > 0$.

Proof. Let $d\omega = u^q d\sigma$. We estimate

$$\int_{B(x,t)} u^q d\sigma \geq \int_{B(x,t)} (\mathbf{W}_{1,p}\omega)^q d\sigma = \int_{B(x,t)} \left[\int_t^\infty \left(\frac{\omega(B(y,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^q d\sigma(y).$$

Since $B(y,2s) \supset B(x,s)$ if $s \geq t$ and $y \in B(x,t)$, then

$$\begin{aligned}
&\int_{B(x,t)} \left[\int_t^\infty \left(\frac{\omega(B(y,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^q d\sigma(y) \\
&= c \int_{B(x,t)} \left[\int_t^\infty \left(\frac{\omega(B(y,2s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^q d\sigma(y) \\
&\geq c \int_{B(x,t)} \left[\int_t^\infty \left(\frac{\omega(B(x,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^q d\sigma(y) \\
&= c\,\sigma(B(x,t)) \left[\int_t^\infty \left(\frac{\omega(B(x,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^q \\
&\geq c\,\sigma(B(x,t)) \left[\int_t^\infty \left(\frac{(\kappa(B(x,s)))^{\frac{q(p-1)}{p-1-q}}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^q,
\end{aligned}$$

where $c = c(\alpha, n, p, q)$ and we have used (4.27) in the last line. Hence, (4.39) holds for all $x \in \mathbb{R}^n$ and $t > 0$.

It suffices to prove (4.40) for t large enough since by Lemma 4.7, $\kappa(B(x,s)) < \infty$ for all $x \in \mathbb{R}^n$ and $s > 0$. But (4.40) follows from (4.39) if we pick t large enough so that $\sigma(B(x,t)) > 0$. $\square$

Lemma 4.11. *Let $1 < p < \infty, 0 < q < p-1, 0 < \alpha < \frac{n}{p}$ and $\sigma \in M^+(\mathbb{R}^n)$. Suppose that* (4.40) *holds for $x = 0$ and $t = 1$, i.e.,*

$$\int_1^\infty \left(\frac{(\kappa(B(0,s)))^{\frac{q(p-1)}{p-1-q}}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} < \infty, \tag{4.41}$$

then (4.40) *holds for all $x \in \mathbb{R}^n, t > 0$, and $\mathbf{K}_{\alpha,p,q}\sigma \in L^q_{\mathrm{loc}}(\mathbb{R}^n, d\sigma)$.*

Proof. We first note that if (4.41) holds, then for every $t > 0$,

$$\int_t^\infty \left(\frac{(\kappa(B(0,s)))^{\frac{q(p-1)}{p-1-q}}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} < \infty.$$

Indeed, if $t > 1$ then it is trivial. If $t < 1$, then

$$\int_t^\infty \left(\frac{(\kappa(B(0,s)))^{\frac{q(p-1)}{p-1-q}}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} = \int_t^1 \left(\frac{(\kappa(B(0,s)))^{\frac{q(p-1)}{p-1-q}}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s}$$

$$+ \int_1^\infty \left(\frac{(\kappa(B(0,s)))^{\frac{q(p-1)}{p-1-q}}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s}$$

$$\leq \int_t^1 \left(\frac{(\kappa(B(0,1)))^{\frac{q(p-1)}{p-1-q}}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} + \int_1^\infty \left(\frac{(\kappa(B(0,s)))^{\frac{q(p-1)}{p-1-q}}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s}$$

$$= (\kappa(B(0,1)))^{\frac{q}{p-1-q}} \int_t^1 \left(\frac{1}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} + \int_1^\infty \left(\frac{(\kappa(B(0,s)))^{\frac{q(p-1)}{p-1-q}}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} < \infty.$$

Now, fix $x \in \mathbb{R}^n$, clearly, $B(x,s) \subset B(0, s+|x|)$, so

$$\int_t^\infty \left(\frac{(\kappa(B(0,s)))^{\frac{q(p-1)}{p-1-q}}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \leq \int_t^\infty \left(\frac{(\kappa(B(0,s+|x|)))^{\frac{q(p-1)}{p-1-q}}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s}$$

$$= \int_{t+|x|}^\infty \left(\frac{(\kappa(B(0,r)))^{\frac{q(p-1)}{p-1-q}}}{(r-|x|)^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r-|x|}.$$

If $t > |x|$, then $r - |x| > \frac{1}{2}r$ if $r \geq t + |x|$. Thus,

$$\int_{t+|x|}^\infty \left(\frac{(\kappa(B(0,r)))^{\frac{q(p-1)}{p-1-q}}}{(r-|x|)^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r-|x|} \leq 2^{\frac{n-\alpha p}{p-1}+1} \int_{t+|x|}^\infty \left(\frac{(\kappa(B(0,r)))^{\frac{q(p-1)}{p-1-q}}}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} < \infty.$$

If $t \leq |x|$ then

$$\int_{t+|x|}^\infty \left(\frac{(\kappa(B(0,r)))^{\frac{q(p-1)}{p-1-q}}}{(r-|x|)^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r-|x|} = \int_{t+|x|}^{2|x|} \left(\frac{(\kappa(B(0,r)))^{\frac{q(p-1)}{p-1-q}}}{(r-|x|)^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r-|x|}$$

$$+ \int_{2|x|}^\infty \left(\frac{(\kappa(B(0,r)))^{\frac{q(p-1)}{p-1-q}}}{(r-|x|)^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r-|x|}$$

$$\leq (\kappa(B(0,2|x|)))^{\frac{q}{p-1-q}} \int_{t+|x|}^{2|x|} \left(\frac{1}{(r-|x|)^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r-|x|}$$

$$+ 2^{\frac{n-\alpha p}{p-1}+1} \int_{2|x|}^\infty \left(\frac{(\kappa(B(0,r)))^{\frac{q(p-1)}{p-1-q}}}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} < \infty.$$

Let us now show $\mathbf{K}_{\alpha,p,q}\sigma \in L^q_{\text{loc}}(\mathbb{R}^n, d\sigma)$. Fix a ball $B(x,t)$ and let $B = B(x, 2t)$. We split $\mathbf{K}_{\alpha,p,q}\sigma$ into two parts and estimate

$$I := \int_{B(x,t)} \left[\int_0^t \left(\frac{(\kappa(B(y,s)))^{\frac{q(p-1)}{p-1-q}}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^q d\sigma(y),$$

$$II := \int_{B(x,t)} \left[\int_t^\infty \left(\frac{(\kappa(B(y,s)))^{\frac{q(p-1)}{p-1-q}}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^q d\sigma(y).$$

Notice that, in II, $B(y,s) \subset B(x,2s)$, hence, by (4.40),

$$II \leq \sigma(B(x,t)) \left[\int_t^\infty \left(\frac{(\kappa(B(x,2s)))^{\frac{q(p-1)}{p-1-q}}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^q < \infty.$$

In I, we note that $B(y,s) \subset B$ and thus,

$$I \leq \int_B (\mathbf{K}_{\alpha,p,q}\sigma_B)^q d\sigma.$$

Since $\kappa(B) < \infty$, by Theorem 4.9 with σ_B in place of σ, there exists a solution $u_B \in L^q(\mathbb{R}^n, d\sigma_B)$ to the equation $u_B = \mathbf{W}_{\alpha,p}(u_B^q d\sigma_B)$. Hence, by Lemma 4.7 with σ_B in place of σ,

$$(\kappa(B(y,s) \cap B))^{\frac{q(p-1)}{p-1-q}} \leq c \int_{B(y,s)} u_B^q d\sigma_B,$$

where $c = c(\alpha, n, p, q)$. This follows that

$$\int_B (\mathbf{K}_{\alpha,p,q}\sigma_B)^q d\sigma \leq c \int_B (\mathbf{W}_{\alpha,p}(u_B^q d\sigma_B))^q d\sigma = c \int_B u_B^q d\sigma < \infty.$$

Therefore, $\int_B (\mathbf{K}_{\alpha,p,q}\sigma)^q d\sigma < \infty$, and we complete the proof of the lemma. $\square$

Theorem 4.12. *Let* $1 < p < \infty, 0 < q < p-1, 0 < \alpha < \frac{n}{p}$ *and* $\sigma \in M^+(\mathbb{R}^n)$. *Suppose that* (2.20) *and* (4.41) *hold. Then there exists a solution* $u \in L^q_{\mathrm{loc}}(\mathbb{R}^n, d\sigma)$ *to* (4.15) *such that* $\liminf_{|x|\to\infty} u(x) = 0$ *and* u *satisfies*

$$C^{-1}\left(\mathbf{K}_{\alpha,p,q}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}}\right) \le u \le C\left(\mathbf{K}_{\alpha,p,q}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}}\right), \tag{4.42}$$

where $C > 0$ *depends only on* $\alpha, n, p,$ *and* q. *The lower bound in* (4.42) *holds for any nontrivial solution* $u \in L^q_{\text{loc}}(\mathbb{R}^n, d\sigma)$ *to the inequality* (4.16).

Proof. Let $u_0 = c_0\,(\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}}$, where c_0 is a constant which will be chosen later. We construct a sequence $\{u_j\}$ as follows

$$u_{j+1} = \mathbf{W}_{\alpha,p}(u_j^q d\sigma), \quad j = 0, 1, 2, \dots.$$

Choosing c_0 small enough and using Lemma 2.13, we can show that the sequence $\{u_j\}$ is nondecreasing. We need to check that u_j are well defined, i.e., $u_j \in L^q_{\text{loc}}(\mathbb{R}^n, d\sigma)$.

Let $d\omega_0 = u_0^q d\sigma$. We first show that, for all $x \in \mathbb{R}^n$ and $t > 0$,

(4.43)

$$\omega_0(B(x,t)) \le c[\kappa(B(x,2t))]^{\frac{(p-1)q}{p-1-q}} + c\,\sigma(B(x,t))\left(\int_t^\infty \left(\frac{\sigma(B(x,r))}{r^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dr}{r}\right)^{\frac{(p-1)q}{p-1-q}},$$

where $c = c(\alpha, n, p, q)$. We let $B = B(x, 2t)$ and denote by B^c the complement of B in $\mathbb{R}^n$. Clearly, for $y \in B(x,t)$,

$$\mathbf{W}_{\alpha,p}\sigma_{B^c}(y) = \int_0^\infty \left(\frac{\sigma(B^c \cap B(y,r))}{r^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dr}{r} = \int_t^\infty \left(\frac{\sigma(B^c \cap B(y,r))}{r^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dr}{r}.$$

If $r \ge t$, then $B(y,r) \subset B(x,2r)$, and consequently

$$\mathbf{W}_{\alpha,p}\sigma_{B^c}(y) \le \int_t^\infty \left(\frac{\sigma(B^c \cap B(x,2r))}{r^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dr}{r}$$

$$\leq \int_t^\infty \left(\frac{\sigma(B(x,2r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \leq c \int_t^\infty \left(\frac{\sigma(B(x,r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}.$$

Thus,

$$\int_{B(x,t)} (\mathbf{W}_{\alpha,p}\sigma)^{\frac{q(p-1)}{p-1-q}} \, d\sigma \leq c \int_{B(x,t)} (\mathbf{W}_{\alpha,p}\sigma_B)^{\frac{q(p-1)}{p-1-q}} \, d\sigma + c \int_{B(x,t)} (\mathbf{W}_{\alpha,p}\sigma_{B^c})^{\frac{q(p-1)}{p-1-q}} \, d\sigma$$

$$\leq c \int_{B(x,t)} (\mathbf{W}_{\alpha,p}\sigma_B)^{\frac{q(p-1)}{p-1-q}} \, d\sigma + c \, \sigma(B(x,t)) \left(\int_t^\infty \left(\frac{\sigma(B(x,r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right)^{\frac{q(p-1)}{p-1-q}}.$$

Note that $\kappa(B) < \infty$ by condition (4.41), then using Theorem 4.9 with σ_B in place of σ, there exists a solution $u_B \in L^q(\mathbb{R}^n, d\sigma_B)$ to the equation $u_B = \mathbf{W}_{\alpha,p}(u_B^q d\sigma_B)$. By Theorem 2.12, $u_B \geq C \, (\mathbf{W}_{\alpha,p}\sigma_B)^{\frac{p-1}{p-1-q}}$. We also have, by Corollary 4.8,

$$\int_B u_B^q d\sigma \leq (\kappa(B))^{\frac{(p-1)q}{p-1-q}}.$$

Therefore, combining the preceding estimates, we conclude that (4.43) holds and in particular $u_0 \in L^q_{\text{loc}}(\mathbb{R}^n, d\sigma)$. For $x \in \mathbb{R}^n$ and $t > 0$, we let

$$A_0(x,t) := \int_t^\infty \left(\frac{\omega_0(B(x,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s},$$

and

(4.44)
$$M(x,t) := \int_t^\infty \left(\frac{(\kappa(B(x,s)))^{\frac{q(p-1)}{p-1-q}}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} + \left(\int_t^\infty \left(\frac{\sigma(B(x,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right)^{\frac{p-1}{p-1-q}}.$$

By Corollary 2.10 and Lemma 4.11, $M(x,t) < \infty$ for all $x \in \mathbb{R}^n$ and $t > 0$. We now show that

$$A_0(x,t) \le c\, M(x,t), \quad \forall\, x \in \mathbb{R}^n, t > 0,$$

where $c = c(\alpha, n, p, q)$. Indeed, using (4.43), we have

$$A_0(x,t) \le c \int_t^\infty \left(\frac{(\kappa(B(x,2s)))^{\frac{q(p-1)}{p-1-q}}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s}$$

$$+c \int_t^\infty \left(\frac{\sigma(B(x,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \left(\int_s^\infty \left(\frac{\sigma(B(x,r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right)^{\frac{q}{p-1-q}} \frac{ds}{s}.$$

Making the substitution $\rho = 2s$ in the first term and estimating crudely the second term give us

$$A_0(x,t) \le c \int_t^\infty \left(\frac{(\kappa(B(x,\rho)))^{\frac{q(p-1)}{p-1-q}}}{\rho^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{d\rho}{\rho}$$

$$+c \left(\int_t^\infty \left(\frac{\sigma(B(x,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right)^{\frac{p-1}{p-1-q}} = c\, M(x,t).$$

Setting $d\omega_j = u_j^q d\sigma$, for $j = 1, 2, \ldots$, we claim that

(4.45)
$$\omega_j(B(x,t)) \le c[\kappa(B(x,t))]^q [\omega_j(B(x,2t))]^{\frac{q}{p-1}} + c\sigma(B(x,t)) \left[\int_t^\infty \left(\frac{\omega_j(B(x,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^q,$$

where $c = c(\alpha, n, p, q)$. First, we estimate

$$\omega_j(B(x,t)) = \int_{B(x,t)} (\mathbf{W}_{\alpha,p}\omega_{j-1})^q d\sigma = \int_{B(x,t)} \left[\int_0^\infty \left(\frac{\omega_{j-1}(B(y,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^q d\sigma(y)$$

$$\le c_q \int_{B(x,t)} \left[\int_0^t \left(\frac{\omega_{j-1}(B(y,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^q d\sigma(y)$$

$$+c_q \int_{B(x,t)} \left[\int_t^\infty \left(\frac{\omega_{j-1}(B(y,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^q d\sigma(y) := c_q(I + II).$$

We notice that if $y \in B(x,t)$ and $0 < s \leq t$ then $B(y,s) \subset B = B(x,2t)$. Hence, using (4.27) with $d\nu = \chi_B d\omega_{j-1}$, we obtain

$$I \leq \int_{B(x,t)} (\mathbf{W}_{\alpha,p}\nu)^q d\sigma \leq [\kappa(B(x,t))]^q [\omega_{j-1}(B(x,2t))]^{\frac{q}{p-1}}.$$

To estimate II, we see that $B(y,s) \subset B(x,2s)$ if $y \in B(x,t)$ and $s \geq t$. Thus,

$$II \leq \int_{B(x,t)} \left[\int_t^\infty \left(\frac{\omega_{j-1}(B(x,2s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^q d\sigma(y)$$

$$\leq c\sigma(B(x,t)) \left[\int_t^\infty \left(\frac{\omega_{j-1}(B(x,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^q.$$

Combining estimates I and II yield (4.45) for $j = 1, 2, \ldots$.

Now letting

$$A_j(x,t) = \int_t^\infty \left(\frac{\omega_j(B(x,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s}, \tag{4.46}$$

for $j = 1, 2, \ldots$, and using (4.45), we estimate

$$A_j(x,t) \leq c \int_t^\infty \left(\frac{[\kappa(B(x,s))]^q [\omega_{j-1}(B(x,2s))]^{\frac{q}{p-1}}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s}$$

$$+c \int_t^\infty \left(\frac{\sigma(B(x,s)) \left(\int_s^\infty \left(\frac{\omega_{j-1}(B(x,r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right)^q}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s}$$

$$\leq c \int_t^\infty \left(\frac{[\kappa(B(x,s))]^q [\omega_{j-1}(B(x,2s))]^{\frac{q}{p-1}}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s}$$

$$+c[A_{j-1}(x,t)]^{\frac{q}{p-1}} \int_t^\infty \left(\frac{\sigma(B(x,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s}.$$

Using Holder's inequality with exponents $\frac{p-1}{p-1-q}$ and $\frac{p-1}{q}$ in the first integral of the preceding expression, we get

$$A_j(x,t) \leq c[A_{j-1}(x,t)]^{\frac{q}{p-1}} \left(\int_t^\infty \left(\frac{(\kappa(B(x,s)))^{\frac{q(p-1)}{p-1-q}}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right)^{\frac{p-1-q}{p-1}}$$

$$+c\,[A_{j-1}(x,t)]^{\frac{q}{p-1}} \int_t^\infty \left(\frac{\sigma(B(x,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \leq c\,[A_{j-1}(x,t)]^{\frac{q}{p-1}} [M(x,t)]^{\frac{p-1-q}{p-1}},$$

where $c = c(\alpha, n, p, q)$. Arguing by induction, we obtain that $A_j(x,t) < \infty$ for all $x \in \mathbb{R}^n$ and $t > 0$. We see that $A_{j-1}(x,t) \leq A_j(x,t)$ since $\omega_{j-1} \leq \omega_j$. Hence, from the preceding estimate we deduce

$$A_j(x,t) \leq cM(x,t), \quad \forall\, x \in \mathbb{R}^n, t > 0, \tag{4.47}$$

where $c > 0$ depends only on α, n, p, and q. As a consequence of (4.47), we have

$$\omega_j(B(x,t)) \leq c\, t^{n-\alpha p} [M(x,t)]^{p-1}, \quad j = 1, 2, \ldots, \forall\, x \in \mathbb{R}^n, t > 0,$$

where $c = c(\alpha, n, p, q)$; hence, $u_j \in L^q_{\text{loc}}(\mathbb{R}^n, d\sigma)$ for all $j = 1, 2, \ldots$.

Using the Monotone Convergence Theorem and letting $u = \lim_{j\to\infty} u_j$, we see that

u is a solution to equation (4.15) and $u \in L^q_{\mathrm{loc}}(\mathbb{R}^n, d\sigma)$. By (4.47), we have

$$\int_t^\infty \left(\frac{\int_{B(x,s)} u^q d\sigma}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \leq c\, M(x,t) \leq c\, M(x,0), \tag{4.48}$$

where c depends only on α, n, p, q and

$$M(x,0) = \mathbf{K}_{\alpha,p,q}\sigma(x) + (\mathbf{W}_{\alpha,p}\sigma(x))^{\frac{p-1}{p-1-q}}.$$

Letting $t \to 0$ in (4.48), we obtain

$$u(x) = \mathbf{W}_{\alpha,p}(u^q d\sigma) \leq \int_0^\infty \left(\frac{\int_{B(x,s)} u^q d\sigma}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \leq c\, M(x,0).$$

On the other hand, by Lemma 4.7, we have $\int_{B(x,s)} u^q d\sigma \geq c\, [\kappa(B(x,s))]^{\frac{(p-1)q}{p-1-q}}$. Combining this with Theorem 2.12 yield the lower bound

$$u(x) \geq cM(x,0),$$

for any nontrivial solution $u \in L^q_{\mathrm{loc}}(\mathbb{R}^n, d\sigma)$ of the inequality $u \geq \mathbf{W}_{\alpha,p}(u^q d\sigma)$. The fact $\liminf_{|x|\to\infty} u(x) = 0$ follows from Corollary 2.10. Therefore, we complete the proof of Theorem 4.12. $\square$

4.2.4 Solution in $L^{1+q}_{\mathbf{loc}}(\mathbb{R}^n, d\sigma)$

In this section we will prove that the solution u to (4.15) obtained in Theorem 4.12 has the property $u \in L^{1+q}_{\text{loc}}(\mathbb{R}^n, d\sigma)$ under the additional assumption

$$\int_B (\mathbf{W}_{\alpha,p}\sigma_B)^{\frac{(1+q)(p-1)}{p-1-q}} \, d\sigma < \infty, \qquad \text{for all balls } B \text{ in } \mathbb{R}^n. \tag{4.49}$$

We will see later that (4.49) is also necessary for $u \in L^{1+q}_{\text{loc}}(\mathbb{R}^n, d\sigma)$.

Lemma 4.13. *Let* $1 < p < \infty, 0 < q < p-1, 0 < \alpha < \frac{n}{p}$ *and* $\sigma \in M^+(\mathbb{R}^n)$. *Suppose that* (2.20), (4.41) *and* (4.49) *hold. Then* $\mathbf{W}_{\alpha,p}\sigma \in L_{\text{loc}}^{\frac{(1+q)(p-1)}{p-1-q}}(\mathbb{R}^n, d\sigma)$ *and* $\mathbf{K}_{\alpha,p,q}\sigma \in L^{1+q}_{\text{loc}}(\mathbb{R}^n, d\sigma)$.

Proof. Let $x \in \mathbb{R}^n$ and $t > 0$. We need to verify

$$I := \int_{B(x,t)} (\mathbf{W}_{\alpha,p}\sigma)^{\frac{(1+q)(p-1)}{p-1-q}} d\sigma < \infty,$$

$$II := \int_{B(x,t)} (\mathbf{K}_{\alpha,p,q}\sigma)^{1+q} d\sigma < \infty.$$

First, we split I into two parts

$$I_a = \int_{B(x,t)} \left[\int_0^t \left(\frac{\sigma(B(y,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^{\frac{(1+q)(p-1)}{p-1-q}} d\sigma(y),$$

$$I_b = \int_{B(x,t)} \left[\int_t^\infty \left(\frac{\sigma(B(y,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^{\frac{(1+q)(p-1)}{p-1-q}} d\sigma(y).$$

To estimate I_b, we notice that $B(y,s) \subset B(x,2s)$ if $y \in B(x,t)$ and $t \leq s$, hence, making the substitution $r = 2s$, we obtain

$$I_b \leq c\sigma(B(x,t)) \left[\int_t^\infty \left(\frac{\sigma(B(x,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^{\frac{(1+q)(p-1)}{p-1-q}} < \infty,$$

by Corollary 2.10.

We now estimate I_a. Notice that $B(y,s) \subset B = B(x,2t)$ if $0 < s < t$ and $y \in B(x,t)$. Hence,

$$\int_0^t \left(\frac{\sigma(B(y,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \leq \mathbf{W}_{\alpha,p}\sigma_B(y),$$

which implies

$$I_a \leq \int_B [\mathbf{W}_{\alpha,p}\sigma_B(y)]^{\frac{(1+q)(p-1)}{p-1-q}} \, d\sigma(y) < \infty,$$

by (4.49), and thus $I < \infty$.

Similarly, we next estimate II by splitting $\mathbf{K}_{\alpha,p,q}\sigma$ into two parts,

$$II_a = \int_{B(x,t)} \left[\int_0^t \left(\frac{[\kappa(B(y,s))]^{\frac{q(p-1)}{p-1-q}}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^{1+q} d\sigma(y),$$

$$II_b = \int_{B(x,t)} \left[\int_t^\infty \left(\frac{[\kappa(B(y,s))]^{\frac{q(p-1)}{p-1-q}}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^{1+q} d\sigma(y).$$

To estimate II_b, notice that if $y \in B(x,t)$ and $t \leq s$, then $\kappa(B(y,s)) \leq \kappa(B(x,2s))$, hence, using Lemma 4.11 and making the substitution $r = 2s$, we get

$$II_b \leq c\sigma(B(x,t)) \left[\int_t^\infty \left(\frac{[\kappa(B(x,s))]^{\frac{q(p-1)}{p-1-q}}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^{1+q} < \infty.$$

To show $II_a < \infty$, we notice again that $B(y,s) \subset B = B(x,2t)$ if $0 < s < t$ and $y \in B(x,t)$. Then $\kappa(B(y,s)) = \kappa(B(y,s) \cap B)$, and so

$$II_a \le \int_{B(x,t)} \left[\int_0^\infty \left(\frac{[\kappa(B(y,s) \cap B)]^{\frac{q(p-1)}{p-1-q}}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^{1+q} d\sigma(y).$$

Since (4.49) holds, applying Theorem 3.5 in Chapter 3 with σ_B in place of σ, we see that there exists a global solution $u_B \in L^{1+q}(\mathbb{R}^n, d\sigma_B)$ to the equation $u_B = \mathbf{W}_{\alpha,p}(u_B^q d\sigma_B)$. Hence, using (4.42) with σ_B in place of σ, we get

$$\int_0^\infty \left(\frac{[\kappa(B(y,s) \cap B)]^{\frac{q(p-1)}{p-1-q}}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \le C\, u_B(y), \quad y \in \mathbb{R}^n.$$

Thus,

$$II_a \le C \int_B u_B^{1+q} d\sigma < \infty.$$

Therefore, $II < \infty$ and this completes the proof of Lemma 4.13. $\square$

Theorem 4.14. *Let* $1 < p < \infty, 0 < q < p-1, 0 < \alpha < \frac{n}{p}$ *and* $\sigma \in M^+(\mathbb{R}^n)$. *Suppose that* (2.20), (4.41) *and* (4.49) *hold. Then there exists a nontrivial solution* $u \in L^{1+q}_{\text{loc}}(\mathbb{R}^n, d\sigma)$ *to* (4.15). *Moreover, u satisfies* (4.42).

Conversely, if there exists a nontrivial solution $u \in L^{1+q}_{\text{loc}}(\mathbb{R}^n, d\sigma)$ *to the inequality* (4.16)*, then* (2.20)*,* (4.41) *and* (4.49) *hold.*

Proof. If (2.20) and (4.41) hold, then by Theorem 4.12, there exists a nontrivial solution $u \in L^q_{\text{loc}}(\mathbb{R}^n, d\sigma)$ to the equation $u = \mathbf{W}_{\alpha,p}(u^q d\sigma)$ and u satisfies (4.42). Using the upper estimate in (4.42) and Lemma 4.13 yield $u \in L^{1+q}_{\text{loc}}(\mathbb{R}^n, d\sigma)$.

Conditions (2.20) and (4.41) are also necessary for the existence of any nontrivial solution to (4.16), which follow from Theorem 4.12. By (2.27) and the fact $u \in$

$L^{1+q}_{\text{loc}}(\mathbb{R}^n, d\sigma)$, we conclude that (4.49) is necessary as well. $\square$

4.3 Proofs of Theorem 4.1, 4.2 and Theorem 4.3

We will need some versions of the comparison principle.

Lemma 4.15. *Let Ω be a bounded domain in $\mathbb{R}^n$. Suppose that $\mu, \nu \in L^{-1,p'}(\Omega)$, $0 \leq \mu \leq \nu$. Suppose that $u \in L^{1,p}_0(\Omega)$ and $v \in W^{1,p}(\Omega)$ are distributional solutions to the equations $-\Delta_p u = \mu$ and $-\Delta_p v = \nu$ in Ω, respectively. Then $u \leq v$ a.e. in Ω.*

Proof. First, we have

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla \varphi = \int_\Omega \varphi \, d\mu, \quad \forall\, \varphi \in C^\infty_0(\Omega), \tag{4.50}$$

$$\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla \phi = \int_\Omega \phi \, d\nu, \quad \forall\, \phi \in C^\infty_0(\Omega). \tag{4.51}$$

Since $\mu, \nu \in L^{-1,p'}(\Omega)$, by Theorem 2.1, (4.50) and (4.51) hold for any $\varphi, \phi \in L^{1,p}_0(\Omega)$. We notice that $u - \min\{u, v\} \in L^{1,p}_0(\Omega)$ since $0 \leq (u - \min\{u, v\}) \leq u \in L^{1,p}_0(\Omega)$. Plugging $\varphi = u - \min\{u, v\}$ and $\phi = u - \min\{u, v\}$ into (4.50) and (4.51) respectively, we obtain

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla(u - \min\{u, v\})dx = \int_\Omega (u - \min\{u, v\}) \, d\mu,$$

$$\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla(u - \min\{u, v\})dx = \int_\Omega (u - \min\{u, v\}) \, d\nu.$$

Subtracting above equations, we get

$$\int_{\Omega} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v) \cdot \nabla(u - \min\{u, v\}) dx$$

$$= \int_{\Omega} (u - \min\{u, v\})\, d\mu - \int_{\Omega} (u - \min\{u, v\})\, d\nu \leq 0.$$

On the other hand,

$$\int_{\Omega} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v) \cdot \nabla(u - \min\{u, v\}) dx$$

$$= \int_{\Omega} (|\nabla u|^{p-2}\nabla u - |\nabla \min\{u, v\}|^{p-2}\nabla \min\{u, v\}) \cdot \nabla(u - \min\{u, v\}) dx \geq 0.$$

Thus,

$$\int_{\Omega} (|\nabla u|^{p-2}\nabla u - |\nabla \min\{u, v\}|^{p-2}\nabla \min\{u, v\}) \cdot \nabla(u - \min\{u, v\}) dx = 0,$$

and this implies

$$\int_{\Omega} |\nabla(u - \min\{u, v\})|^p dx = 0.$$

Hence $u - \min\{u, v\} = 0$ a.e., so $u \leq v$ a.e. in Ω. $\square$

Lemma 4.16. *Let Ω be a bounded open set in $\mathbb{R}^n$. Suppose that μ, ν are nonnegative finite measures on Ω such that $\mu \leq \nu$, and μ is absolutely continuous with respect to p-capacity $\mathrm{cap}_p(\cdot)$. Suppose that u, v are nonnegative p-superharmonic functions in Ω whose Riesz measures are μ and ν, respectively, and $\min\{u, k\} \in L_0^{1,p}(\Omega)$ for all $k \in \mathbb{N}$. Then $u \leq v$ a.e. in Ω.*

Proof. Let $v_k = \min\{v, k\}$, then $v_k \in W^{1,p}(\Omega)$ is p-superharmonic and the Riesz measure $\nu_k = -\Delta_p v_k$ converges weakly to ν as $k \to \infty$ (see [HKM06], section 7).

Let $\mu_k = \chi_{\{v<k\}}\,\mu$, $k > 0$. Clearly, $\nu_k|_{\{v<k\}} = \nu|_{\{v<k\}}$, so $\mu_k \le \nu_k$.

For any $\varphi \in C_0^\infty(\Omega)$, we have

$$\left|\int_\Omega \varphi d\mu_k - \int_\Omega \varphi d\mu\right| = \left|\int_{v\ge k} \varphi d\mu\right| \le \int_{v\ge k} |\varphi| d\mu$$

$$\le \max_\Omega |\varphi| \mu(\{v \ge k\}) \to c\mu(\{v = \infty\}) \quad \text{ as } k \to \infty.$$

Since μ is absolutely continuous with respect to cap_p and v is p-superharmonic, this implies that $\mu(\{v = \infty\}) = 0$, and hence, $\mu_k \to \mu$ weakly.

We have $\mu_k \in L^{-1,p'}(\Omega)$ since $\mu_k \le \mu \in L^{-1,p'}(\Omega)$. So there is a unique solution $u_k \in L_0^{1,p}(\Omega)$ to the equation $-\Delta_p u_k = \mu_k$. By Lemma 4.15, we obtain $u_k \le v_k$ for all $k > 0$ and $u_j \le u_k$ if $j \le k$. Letting $\tilde{u} = \lim_{k\to\infty} u_k$, we see that $\tilde{u} \le v$. Since $\mu_k \to \mu$ weakly, we have $\tilde{u}$ is a p-superharmonic solution to the equation $-\Delta_p \tilde{u} = \mu$ and $\min\{\tilde{u}, k\} \in L_0^{1,p}(\Omega)$ for all $k > 0$. Since μ is absolutely continuous with respect to cap_p and $\min\{u, k\} \in L_0^{1,p}(\Omega)$ for all $k > 0$, by the uniqueness theorem (see [Kil99] or Appendix A), it follows that $u = \tilde{u}$ a.e., and consequently $u \le v$ a.e. □

Proof of the Theorem 4.1. Let $1 < p < n$. Suppose that (1.7) and (1.18) hold. Then by Theorem 4.12, there exists a nontrivial solution $v \in L^q_{\text{loc}}(\mathbb{R}^n, d\sigma)$ to the equation

$$v = K\mathbf{W}_{1,p}(v^q d\sigma). \tag{4.52}$$

where K is the constant in Theorem 2.6 and $\liminf_{|x|\to\infty} v(x) = 0$. By Theorem 2.12

(with $K^{p-1}\sigma$ in place of σ),

$$v \geq C\, K^{\frac{p-1}{p-1-q}} \, (\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}} \,,$$

where C is the constant in (2.27). We set

$$w_0 = c_0 \, (\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}} \,, \quad d\omega_0 = w_0^q \, d\sigma,$$

where $c_0 > 0$ is a small constant to be determined later. Since

$$w_0 \leq \frac{c_0}{CK^{\frac{p-1}{p-1-q}}} \, v,$$

it follows that, for $c_0 \leq CK^{\frac{p-1}{p-1-q}}$, $w_0 \leq v$.

Clearly, ω_0 is a locally finite Borel measure since $d\omega_0 \leq v^q d\sigma$ and $v \in L^q_{\mathrm{loc}}(d\sigma)$. By Lemma 2.14 with $\alpha = 1$, ω_0 is absolutely continuous with respect to $\mathrm{cap}_p(\cdot)$. Hence, by Theorem A.1, there exists a unique renormalized solution to the equation

$$-\Delta_p u_1^k = \omega_0 \chi_{B(0,2^k)} \text{ in } B(0,2^k), \quad u_1^k = 0 \text{ on } \partial B(0,2^k), \tag{4.53}$$

where $k = 0, 1, 2, \ldots$. Notice that $\{u_1^k\}_k$ is increasing by the comparison principle (Lemma 4.16). Moreover, by Theorem 2.6,

$$0 \leq u_1^k \leq K \, \mathbf{W}_{1,p}(\omega_0 \chi_{B(0,2^k)}) \leq K \, \mathbf{W}_{1,p}\omega_0 \leq K \, \mathbf{W}_{1,p}(v^q d\sigma) = v.$$

Letting $u_1 = \lim_{k\to\infty} u_1^k$ and using the weak continuity of the p-Laplace operator (Theorem 2.5) and the Monotone Convergence Theorem, we see that u_1 is a

p-superharmonic solution to the equation $-\Delta_p u_1 = \omega_0$ in $\mathbb{R}^n$. Since $u_1^k \le v$, so $u_1 \le v$, and hence $\liminf_{|x|\to\infty} u_1(x) = 0$. By Theorem 2.6,

$$0 \le u_1 \le K\,\mathbf{W}_{1,p}\omega_0 \le K\,\mathbf{W}_{1,p}(v^q d\sigma) = v.$$

We deduce, using (2.28),

$$u_1 \ge \frac{1}{K}\mathbf{W}_{1,p}\omega_0 = \frac{c_0^{\frac{q}{p-1}}}{K}\,\mathbf{W}_{1,p}\Big(\,(\mathbf{W}_{1,p}\sigma)^{\frac{q(p-1)}{p-1-q}}\,d\sigma\Big)$$

$$\ge \frac{c_0^{\frac{q}{p-1}}\,\mathfrak{c}^{\frac{q}{p-1-q}}}{K}\,\left(\mathbf{W}_{1,p}\sigma\right)^{\frac{p-1}{p-1-q}} = \frac{c_0^{\frac{q}{p-1}-1}\,\mathfrak{c}^{\frac{q}{p-1-q}}}{K}\,w_0.$$

Hence, for $c_0 \le \min\Big[\big(\mathfrak{c}^{\frac{q}{p-1-q}}K^{-1}\big)^{\frac{p-1}{p-1-q}}, CK^{\frac{p-1}{p-1-q}}\Big]$, we have $v \ge u_1 \ge u_0$.

Next, let us construct a sequence $\{u_j\}_{j\ge1}$ of functions which are p-superharmonic in $\mathbb{R}^n$, $u_j \in L^q_{\text{loc}}(d\sigma)$, so that

$$\text{(4.54)}\qquad \begin{cases} -\Delta_p u_j = \sigma u_{j-1}^q \text{ in } \mathbb{R}^n, \quad j = 2,3,\ldots, \\ c_j\,(\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}} \le u_j \le v, \\ 0 \le u_{j-1} \le u_j, \\ \liminf_{|x|\to\infty} u_j(x) = 0. \end{cases}$$

Here

$$c_j = \Big(\mathfrak{c}^{\frac{q}{p-1-q}}K^{-1}\Big)^{\sum_{l=0}^{j-1}\left(\frac{q}{p-1}\right)^l} c_0^{\left(\frac{q}{p-1}\right)^j}, \quad j = 1,2,3,\ldots.$$

Suppose that $u_0, u_1, ..., u_{j-1}$ have been constructed. Let $d\omega_{j-1} = u_{j-1}^q d\sigma$. Then $\omega_{j-1} \in M^+(\mathbb{R}^n)$, since $u_j \le v$, where $v \in L^q_{\text{loc}}(d\sigma)$ and ω_{j-1} is absolutely continuous with respect to $\text{cap}_p(\cdot)$. Using Theorem A.1 again, we see that there exists a unique renormalized solution u_j^k to the equation

$$-\Delta_p u_j^k = \omega_{j-1}\chi_{B(0,2^k)} \text{ in } B(0,2^k), \quad u_j^k = 0 \text{ on } \partial B(0,2^k).$$

By induction, let u_{j-1}^k be the unique solution of the equation

$$-\Delta_p u_{j-1}^k = \omega_{j-1}\chi_{B(0,2^k)} \text{ in } B(0,2^k), \quad u_{j-1}^k = 0 \text{ on } \partial B(0,2^k).$$

Since $u_{j-2} \leq u_{j-1}$, by Lemma 4.16, we deduce $u_j^k \geq u_{j-1}^k$.

Using Theorem 2.6, we have

$$0 \leq u_j^k \leq K\,\mathbf{W}_{1,p}(\omega_{j-1}\chi_{B(0,2^k)}) \leq K\,\mathbf{W}_{1,p}(v^q d\sigma) = v.$$

Letting $u_j = \lim_{k\to\infty} u_j^k$ and using Theorem 2.5 and the Monotone Convergence Theorem, we deduce that u_j is a p-superharmonic solution to the equation $-\Delta u_j = \sigma u_{j-1}^q$ in $\mathbb{R}^n$.

Moreover, $u_j \leq v$ since $u_j^k \leq v$ and hence $\liminf_{|x|\to\infty} u_j(x) = 0$. Furthermore, we have $u_{j-1} \leq u_j$ since $u_{j-1}^k \leq u_j^k$, for all $k \geq 1$. On the other hand, applying Theorem 2.6 and Lemma 2.13, and arguing by induction, we obtain

$$u_j \geq \frac{1}{K}\mathbf{W}_{1,p}(u_{j-1}^q d\sigma) \geq \frac{1}{K}\mathbf{W}_{1,p}\left(c_{j-1}^q\,(\mathbf{W}_{1,p}\sigma)^{\frac{q(p-1)}{p-1-q}}\,d\sigma\right)$$

$$\geq \mathfrak{c}^{\frac{q}{p-1-q}}K^{-1}c_{j-1}^{\frac{q}{p-1}}\,(\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}} = c_j\,(\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}}\,.$$

Again, letting $u = \lim_{j\to\infty} u_j$ and using Theorem 2.5 and the Monotone Convergence Theorem, we see that u is a p-superharmonic solution to the equation

$-\Delta_p u = \sigma u^q$ in $\mathbb{R}^n$. Thus, by Theorem 2.6, $u \geq \frac{1}{K}\mathbf{W}_{1,p}(u^q d\sigma)$. Using Theorem 4.12 with $\alpha = 1$, we obtain the lower bound in (4.13). Since $u \leq v$, we deduce the upper bound in (4.13) by applying Theorem 4.12. Clearly, $\liminf_{|x|\to\infty} u(x) = 0$ since $u \leq v$ and $\liminf_{|x|\to\infty} v(x) = 0$.

Let us prove now the minimality of u. Suppose $w \in L^q_{\text{loc}}(\mathbb{R}^n, d\sigma)$ is any nontrivial solution to (4.1). Let $d\nu = w^q d\sigma$. By Theorem 2.6, $w \geq \frac{1}{K}\mathbf{W}_{1,p}(w^q d\sigma)$. Hence, using Theorem 2.14 with $\alpha = 1$, we see that ν is absolutely continuous with respect to $\text{cap}_p(\cdot)$. By Theorem 2.12,

$$w \geq CK^{\frac{1-p}{p-1-q}}(\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}}.$$

We notice that, by the choice of c_0 above, $\omega_0 \leq \nu$. By Lemma 4.16, the function u_1^k obtained in (4.53) satisfies $u_1^k \leq w$ in $B(0, 2^k)$ for all $k > 0$, and hence $u_1 = \lim_{k\to\infty} u_1^k \leq w$. Repeating this argument by induction, we deduce that $u_j \leq w$, for $j = 2, 3, \ldots$. It follows that $\lim_{j\to\infty} u_j = u \leq w$, which proves the minimality of u. This proves the statement (i) of Theorem 4.1.

To prove statement (ii), suppose that u is a supersolution to (4.1). Then by Theorem 2.6, $u \geq \frac{1}{K}\mathbf{W}_{1,p}(u^q d\sigma)$. Hence, by Theorem 4.12 with $\alpha = 1$, both (4.11) and (4.12) hold.

Statement (iii) is a consequence of Theorem 2.6 (ii). $\square$

Proof of Theorem 4.2. If both (4.11) and (4.12) hold, by Theorem 4.1, there exists a p-superharmonic solution u to (4.1) such that (4.13) holds. Using Theorem 2.6, we have

$$u \geq \frac{1}{K} \mathbf{W}_{1,p}(u^q d\sigma). \tag{4.55}$$

If (4.49) holds, applying Theorem 4.14, we conclude that there exists a solution $v \in L^{1+q}_{\text{loc}}(\mathbb{R}^n, d\sigma)$ to the integral equation (4.15) such that v satisfies (4.42) with $\alpha = 1$. Hence, there exists $c > 0$ such that

$$c^{-1} v \leq u \leq cv \ d\sigma\text{-a.e.}$$

Consequently, $u \in L^{1+q}_{\text{loc}}(\mathbb{R}^n, d\sigma)$ and hence, by (4.55)

$$\int_B \mathbf{W}_{1,p}(u^q d\sigma_B) u^q d\sigma \leq c \int_B u^{1+q} d\sigma < \infty,$$

for every ball B. By the local Wolff's inequality (2.2), we see that $u^q d\sigma \in W^{-1,p'}_{\text{loc}}(\mathbb{R}^n)$. By Lemma 2.11, we deduce $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n)$.

Conversely, if there exists a solution $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n)$, then its quasi-continuous representative is p-superharmonic solution and $u^q d\sigma \in W^{-1,p'}_{\text{loc}}(\mathbb{R}^n)$. By Lemma 2.11, $u \in L^1_{\text{loc}}(\mathbb{R}^n, u^q d\sigma)$, and hence, for every ball B,

$$\int_B u^{1+q} d\sigma = \int_B u \cdot u^q d\sigma < \infty.$$

Hence, using Theorem 2.12, we conclude that (4.49) holds. By Theorem 4.12, we see that both (4.11) and (4.12) hold as well, which completes the proof of Theorem 4.2. $\square$

Proof of Theorem 4.3. We notice that (4.4) is understood in the sense

$$u = \mathbf{I}_{2\alpha}(u^q d\sigma) \ \text{ in } \mathbb{R}^n, \ u > 0.$$

Since $\mathbf{I}_{2\alpha}\sigma = \mathbf{W}_{\alpha,2}\sigma$, Theorem 4.3 is a special case of Theorem 4.1 with $p = 2$. $\square$

Remark 4.17. **(1)** Direct analogues of our main theorems hold for the more general quasilinear $\mathcal{A}$-Laplace operator $\operatorname{div} \mathcal{A}(x, \nabla u)$ in place of Δ_p:

$$-\operatorname{div} \mathcal{A}(x, \nabla u) = \sigma u^q \quad \text{in } \mathbb{R}^n, \qquad \liminf_{x \to \infty} u = 0, \tag{4.56}$$

under the standard monotonicity and boundedness assumptions on $\mathcal{A}$ which guarantee that the Wolff potential estimates (1.6) hold (see, e.g., [KM94], [KuMi14], [TW02b], [PV08]).

(2) Similar results hold for the fully nonlinear k-Hessian operator F_k $(k = 1, 2, \ldots, n)$ defined by

$$F_k[u] = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}, \tag{4.57}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the Hessian matrix $D^2 u$ on $\mathbb{R}^n$. In other words, $F_k[u]$ is the sum of the $k \times k$ principal minors of $D^2 u$, which coincides with the Laplacian $F_1[u] = \Delta u$ if $k = 1$.

Local Wolff potential estimates for the equation $F_k[u] = \mu$, where $\mu \in M^+(\mathbb{R}^n)$, in this case are due to Labutin [La02] (see also [TW02b]); global estimates analogous to (1.6) can be found in [PV08]. The corresponding "sublinear" equation can be written in the form

$$F_k[u] = \sigma\, |u|^q \quad \text{in } \mathbb{R}^n, \qquad \limsup_{x\to\infty} u = 0, \tag{4.58}$$

where $0 < q < k$, and $u \leq 0$ is a k-convex function.

Similar equations in the supercritical case $q > k$ were considered in [PV08], and in the critical case $q = k$, in [JV10]. Intrinsic nonlinear potentials of the type $\mathbf{K}_{\alpha,p,q}\sigma$ do not play a role there. However, the reduction of both (4.56) and (4.58) to (4.15) is carried over as in the case of the p-Laplacian treated above. See details in [PV08], [JV10], [JV12], [CV14a].

4.4 Example

Let $0 < q < 1$, and $0 < \alpha < \frac{n}{2}$. We will construct $\sigma \in M^+(\mathbb{R}^n)$ such that $\kappa(B(0,R)) < \infty$ for every $R > 0$, and the equation

$$\begin{cases} (-\Delta)^\alpha u \ = \sigma \quad \text{in } \ \mathbb{R}^n, \\ \liminf_{|x|\to\infty} u(x) = 0, \end{cases}$$

has a weak solution, but

$$\begin{cases} (-\Delta)^\alpha u \ = \sigma\, u^q \quad \text{in } \ \mathbb{R}^n, \\ \liminf_{|x|\to\infty} u(x) = 0, \end{cases}$$

has no weak solution. The condition $\kappa(B(0,R)) < \infty$ ensures that locally, for σ restricted to $B(0,R)$, the equation

$$\begin{cases} (-\Delta)^{\alpha} u = \chi_{B(0,R)} \sigma \, u^q \quad \text{in} \quad \mathbb{R}^n, \\ \inf_{\mathbb{R}^n} u = 0, \end{cases}$$

has a solution.

In other words, we need to construct a measure σ such that $\mathbf{I}_{2\alpha}\sigma < \infty$ a.e., that is,

$$\int_1^{\infty} \frac{\sigma(B(0,R))}{R^{n-2\alpha}} \frac{dR}{R} < \infty, \tag{4.59}$$

and $\kappa(B(0,R)) < \infty$ for every $R > 0$, but

$$\int_1^{\infty} \frac{[\kappa(B(0,R))]^{\frac{q}{1-q}}}{R^{n-2\alpha}} \frac{dR}{R} = \infty. \tag{4.60}$$

This requires $[\kappa(B(0,R))]^{\frac{q}{1-q}}$ to grow much faster than $\sigma(B(0,R))$ as $R \to \infty$.

Lemma 4.18. *Let $0 < q < 1, n \geq 2$ and $0 < \alpha < \frac{n}{2}$. If*

$$||\mathbf{I}_{2\alpha}\nu||_{L^q(d\sigma)} \leq \kappa(\sigma)\nu(\mathbb{R}^n), \quad \forall \nu \in M^+(\mathbb{R}^n), \tag{4.61}$$

then

$$\mathcal{K}(\sigma) := \sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{d\sigma(y)}{|x-y|^{(n-2\alpha)q}} \leq \kappa(\sigma)^q. \tag{4.62}$$

Proof. Let $\nu = \delta_x$ in (4.61), and take the supremum of the left-hand side over all

$x \in \mathbb{R}^n$. $\square$

We need the following lemma in the radially symmetric case which will be proved elsewhere.

Lemma 4.19. *Let $0 < q < 1$ and $0 < 2\alpha < n$. If $d\sigma = \sigma(|x|)dx$ is radially symmetric then condition (4.61) is equivalent to $\mathbf{I}_{2\alpha}\sigma \in L^{\frac{1}{1-q}, \frac{q}{1-q}}(d\sigma)$, and hence is not only necessary, but also sufficient for (4.61). Moreover, there exists $c = c(q, \alpha, n) > 0$ such that the least constant $\kappa(\sigma)$ in (4.61) satisfies*

$$\mathcal{K}(\sigma) \leq \kappa(\sigma)^q \leq c\mathcal{K}(\sigma). \tag{4.63}$$

Corollary 4.20. *Let $\sigma_{R,\gamma} = \chi_{B(0,R)}|x|^{-\gamma}$, where $0 \leq \gamma < n - q(n-2\alpha)$ and $R > 0$. Then*

$$\frac{\omega_n}{n - \gamma - q(n-2\alpha)} \leq \frac{\kappa(\sigma_{R,\gamma})^q}{R^{n-\gamma-q(n-2\alpha)}} \leq \frac{c}{n - \gamma - q(n-2\alpha)}, \tag{4.64}$$

where $c = c(q, \alpha, n)$, and $\omega_n = |S^{n-1}|$ is the surface area of the unit sphere.

Proof. Letting $x = 0$ in (4.62) we have

$$\mathcal{K}(\sigma_{R,\gamma}) = \int_{|y|<R} \frac{|y|^{-\gamma}}{|y|^{q(n-2\alpha)}} dy = \omega_n \int_0^R r^{-\gamma - q(n-2\alpha)+n-1} dr$$

$$= \frac{\omega_n}{n - \gamma - q(n-2\alpha)} R^{n-\gamma-q(n-2\alpha)}.$$

Hence (4.64) follows from the preceding estimate and Lemma 4.19. $\square$

Let

$$\sigma = \sum_{k=1}^{\infty} c_k \sigma_{k,\gamma_k}(x + x_k),$$

where $|x_k| = k$, $\gamma_k = n - q(n - 2\alpha) - \epsilon_k$, and c_k, ϵ_k are picked so that $\sum_{k=1}^{\infty} c_k < \infty$, and $\epsilon_k \to \infty$ fast enough; it is enough to set $c_k = 1/k^2$, $\epsilon_k = 1/k^{n+2}$.

Let $R > 0$. Clearly,

$$\sigma((B(0,R)) \leq \sum_{k=1}^{\infty} c_k \sigma_{k,\gamma_k}(B(x_k, R)) \leq \sum_{k=1}^{\infty} c_k \sigma_{k,\gamma_k}(B(0,R)).$$

Here

$$\sigma_{k,\gamma_k}(B(0,R)) = \omega_n \int_0^{\min(k,R)} r^{-\gamma_k + n - 1} dr$$

$$= \frac{\omega_n}{n - \gamma_k} \min(k,R)^{n-\gamma_k} \leq \frac{\omega_n}{q(n - 2\alpha)} \min(k,R)^{q(n-2\alpha)+\epsilon_k}.$$

Hence, for $R \geq 1$

$$\sigma(B(0,R)) \leq \frac{\omega_n}{q(n-2\alpha)} \sum_{k=1}^{N} c_k k^{q(n-2\alpha)+\epsilon_k} + \frac{\omega_n}{q(n-\alpha)} R^{q(n-2\alpha)+\epsilon_N} \sum_{k=N}^{\infty} c_k.$$

Picking N large enough so that $\epsilon_N < (1-q)(n-\alpha)$, we deduce (4.59).

Using Corollary 4.20, we will show that $\kappa(B(0,R)) < \infty$ for every $R > 0$. Indeed, since $\kappa(\sigma)$ is obviously invariant under translations,

$$\kappa(B(0,R))^q \leq \sum_{k=1}^{\infty} c_k [\kappa(\chi_{B(x_k,R)} \sigma_{k,\gamma_k})]^q. \tag{4.65}$$

If $k > 2R$, then $|x - x_k| < R$, $|x| < k$ and $|x_k| = k$ yields $k > |x| > \frac{k}{2}$.

Consequently, $\chi_{B(x_k,R)}\sigma_{k,\gamma_k}(x) \approx \frac{c}{k^{\gamma_k}}\chi_{B(x_k,R)}$. It follows that, for $\nu \in M^+(\mathbb{R}^n)$,

$$||\mathbf{I}_{2\alpha}\nu||^q_{L^q(\chi_{B(x_k,R)}d\sigma_{k,\gamma_k})} \leq \frac{c}{k^{\gamma_k}}||\mathbf{I}_{2\alpha}\nu||^q_{L^q(\chi_{B(x_k,R)}dx)} \leq \frac{c}{k^{\gamma_k}}[\kappa(\chi_{B(x_k,R)})]^q\nu(\mathbb{R}^n)^q.$$

Hence, by Corollary 4.20 with $\gamma = 0$ yields $[\kappa(\chi_{B(x_k,R)})]^q \approx R^{n-q(n-2\alpha)}$. Hence,

$$[\kappa(\chi_{B(x_k,R)}\sigma_{k,\gamma_k})]^q \leq \frac{c}{k^{\gamma_k}}R^{n-q(n-2\alpha)}.$$

From this and (4.65) we deduce

$$[\kappa(\sigma_{B(0,R)})]^q \leq \sum_{1\leq k\leq 2R} c_k\kappa(\sigma_{k,\gamma_k})^q + cR^{n-q(n-2\alpha)}\sum_{k>2R}^{\infty}\frac{c_k}{k^{\gamma_k}} < \infty.$$

Note that each term in the first sum is finite by Corollary 4.20 since $0 < \gamma_k < n - q(n-\alpha)$ is below the critical exponent.

Let us now show that (4.60) holds. By Lemma 4.19,

$$\begin{aligned}
[\kappa(B(0,R)]^q \geq \mathcal{K}(\sigma_{B(0,R)}) &= \sup_{x\in\mathbb{R}^n}\sum_{k=1}^{\infty} c_k\int_{|y|<R}\frac{\sigma_{k,\gamma_k}(y+x_k)}{|x-y|^{q(n-2\alpha)}}dy \\
\geq \sup_{k\geq 1} c_k\int_{|y|<R}\frac{\sigma_{k,\gamma_k}(y+x_k)}{|x_k+y|^{q(n-2\alpha)}}dy &= \sup_{k\geq 1} c_k\int_{|z-x_k|<R}\frac{\sigma_{k,\gamma_k}(z)}{|z|^{q(n-2\alpha)}}dz \\
= \sup_{k\geq 1} c_k\int_{|z-x_k|<R,\,|z|<k}\frac{dz}{|z|^{\gamma_k+q(n-2\alpha)}} &= \sup_{k\geq 1} c_k\int_{|z-x_k|<R,\,|z|<k}\frac{dz}{|z|^{n-\epsilon_k}}.
\end{aligned}$$

If $k \leq \frac{R}{2}$, then $B(0,k) \subset B(x_k,R)$. Hence, for $R > 2$,

$$[\kappa(B(0,R)]^q \geq \sup_{1\leq k\leq \frac{R}{2}} c_k\int_{|z|<k}\frac{dz}{|z|^{n-\epsilon_k}} \geq \omega_n \sup_{1\leq k\leq\frac{R}{2}}\frac{c_k}{\epsilon_k}k^{\epsilon_k}$$

$$\geq \omega_n \sup_{\frac{R}{4} \leq k \leq \frac{R}{2}} \frac{c_k}{\epsilon_k} \geq \omega_n 4^{-n} R^n.$$

Since $\frac{n}{1-q} > n - 2\alpha$, the preceding estimate gives (4.60), as desired.

We give some examples which show that conditions (4.11), (4.12) and (4.14) are mutually independent. Indeed, in the case $p = 2, 0 < q < 1, n \geq 3$, the previous example shows that (4.11) fails but (4.12) holds. If $\sigma(y) = \frac{1}{|y|^s}$, where $0 < s < 2$, we can see that (4.12) fails but both (4.11) and (4.14) hold. For $\sigma(y) = \frac{1}{|y|^s}\chi_{B(0,1)}$, where $\frac{n}{2}(1-q) + 1 + q < s < n$, we obtain (4.12) but not (4.14). Letting $\sigma(y) = \frac{1}{|y|^s}$, where $\frac{n}{2}(1-q) + 1 + q < s < n(1-q) + 2q$, gives (4.11) but not (4.14).

4.5 Equations with singular gradient terms

In this section, we will investigate the relationship between (4.1) and (4.2) and prove Theorem 4.4 and Theorem 4.5 using the framework of locally renormalized solutions. We will show that the substitution (4.8) will send a solution u of (4.1) to a solution v of (4.2), but in the opposite direction, a solution v of (4.2) just gives a supersolution u to (4.1). Notice that u is a genuine solution only under some extra restrictions on v as we can see from the following example.

Let $0 < q < 1, p = 2, n \geq 3$ and $\sigma = 0$, clearly, the function $v = c|x|^{-(1-q)(n-2)}$ is a solution to (4.2) for an appropriate $c > 0$, but the corresponding $u = c_1|x|^{-(n-2)}$ is only superharmonic, and not harmonic. It means that v satisfies (4.23), but not (4.24).

Proof of Theorem 4.4. (*i*) Suppose u is a p-superharmonic solution to equation (4.1).

Let $\gamma = \frac{p-1-q}{p-1}$, $v = \frac{1}{\gamma} u^\gamma$ and

$$u_k = \min\left(u,\, (\gamma k)^{\frac{1}{\gamma}}\right), \quad v_k = \min(v,\, k), \quad k = 1, 2, \ldots.$$

Note that v is p-superharmonic since x^γ is concave and increasing ([HKM06]). We have

$$\int_{\mathbb{R}^n} |Du|^{p-2} Du \cdot \nabla\phi \, dx = \int_{R^n} u^q \phi \, d\sigma, \quad \forall\, \phi \in C_0^\infty(\mathbb{R}^n). \tag{4.66}$$

By Theorem 3.15 in [KKT09], u is a (locally) renormalized solution to (4.1). Therefore,

$$\int_{\mathbb{R}^n} |Du|^{p-2} Du \cdot \nabla(h(u)\phi) \, dx = \int_{R^n} u^q h(u) \phi \, d\sigma, \tag{4.67}$$

for all $\phi \in C_0^\infty(\mathbb{R}^n)$ and $h \in W^{1,\infty}(\mathbb{R})$ with h' having compact support.

Let $\phi \in C_0^\infty(\mathbb{R}^n)$ and $h(u) = \frac{1}{u_k^q}$. Then

$$\int_{\mathbb{R}^n} |Du|^{p-2} Du \cdot \nabla(\frac{\phi}{u_k^q}) \, dx = \int_{R^n} u^q \frac{\phi}{u_k^q} \, d\sigma.$$

Consequently,

$$\int_{\mathbb{R}^n} |Du|^{p-2} Du \cdot \nabla\phi \, \frac{1}{u_k^q} \, dx = \int_{R^n} u^q \frac{\phi}{u_k^q} \, d\sigma + q \int_{\mathbb{R}^n} |Du|^{p-2} Du \cdot \nabla u_k \, \frac{\phi}{u_k^{1+q}} \, dx. \tag{4.68}$$

Notice that $Du = (\gamma v)^{\frac{1}{\gamma}-1} Dv$, so $|Du|^{p-1} = (\gamma v)^{\frac{q}{\gamma}} |Dv|^{p-1}$.

Since u is p-superharmonic,

$$(\gamma v)^{\frac{q}{\gamma}} |Dv|^{p-1} \in L^1_{\mathrm{loc}}(\mathbb{R}^n). \tag{4.69}$$

Hence,

$$\begin{aligned} \int_{\mathbb{R}^n} |Dv|^{p-2} Dv \cdot \nabla \phi \, \frac{(\gamma v)^{\frac{q}{\gamma}}}{(\gamma v_k)^{\frac{q}{\gamma}}} \, dx \\ = \int_{R^n} (\gamma v)^{\frac{q}{\gamma}} \frac{\phi}{(\gamma v_k)^{\frac{q}{\gamma}}} \, d\sigma + b \int_{\mathbb{R}^n} |Dv|^{p-2} Dv \cdot \nabla v_k \, \frac{(\gamma v)^{\frac{q}{\gamma}} \phi}{v_k (\gamma v_k)^{\frac{q}{\gamma}}} \, dx. \end{aligned} \tag{4.70}$$

Let $E = \text{support}(\phi)$; then $v_1 \geq \delta_E > 0$ a.e., and hence q.e. since v_1 is a positive superharmonic function. Notice that $\{v_k\}$ is increasing, so that $v_k \geq \delta_E > 0$ q.e. Consequently,

$$|Dv|^{p-2} Dv \cdot \nabla \phi \, \frac{(\gamma v)^{\frac{q}{\gamma}}}{(\gamma v_k)^{\frac{q}{\gamma}}} \leq \frac{||\nabla \phi||_{L^\infty(\mathbb{R}^n)}}{(\gamma \delta_E)^{\frac{q}{\gamma}}} \, |Dv|^{p-1} (\gamma v)^{\frac{q}{\gamma}} \text{ on } E.$$

Using (4.69) and the Dominated Convergence Theorem, we obtain

$$\int_{\mathbb{R}^n} |Dv|^{p-2} Dv \cdot \nabla \phi \, \frac{(\gamma v)^{\frac{q}{\gamma}}}{(\gamma v_k)^{\frac{q}{\gamma}}} \, dx \to \int_{\mathbb{R}^n} |Dv|^{p-2} Dv \cdot \nabla \phi \, dx, \tag{4.71}$$

as $k \to \infty$, where the right hand side is finite since v is p-superharmonic.

Assuming momentarily that $\phi \geq 0$, we deduce from (4.70),

$$0 \leq b \int_{\mathbb{R}^n} |Dv|^{p-2} Dv \cdot \nabla v_k \, \frac{(\gamma v)^{\frac{q}{\gamma}} \phi}{v_k (\gamma v_k)^{\frac{q}{\gamma}}} dx \leq \int_{\mathbb{R}^n} |Dv|^{p-2} Dv \cdot \nabla \phi \, \frac{(\gamma v)^{\frac{q}{\gamma}}}{(\gamma v_k)^{\frac{q}{\gamma}}} \, dx \leq C, \tag{4.72}$$

where $C > 0$ does not depend on k by (4.71). Obviously,

$$0 \le |Dv|^{p-2} Dv \cdot \nabla v_k \, \frac{(\gamma v)^{\frac{q}{\gamma}} \phi}{v_k (\gamma v_k)^{\frac{q}{\gamma}}} dx \le |Dv|^{p-2} Dv \cdot \nabla v_{k+1} \, \frac{(\gamma v)^{\frac{q}{\gamma}} \phi}{v_{k+1} (\gamma v_{k+1})^{\frac{q}{\gamma}}}.$$

Hence, using the Monotone Convergence Theorem and (4.72), we deduce

$$\int_{\mathbb{R}^n} |Dv|^{p-2} Dv \cdot \nabla v_k \, \frac{(\gamma v)^{\frac{q}{\gamma}} \phi}{v_k (\gamma v_k)^{\frac{q}{\gamma}}} \, dx \to \int_{\mathbb{R}^n} \frac{|Dv|^p \, \phi}{v} \, dx \le \frac{C}{b},$$

as $k \to \infty$. Hence,

$$\frac{|Dv|^p}{v} \in L^1_{\mathrm{loc}}(\mathbb{R}^n, dx). \tag{4.73}$$

Notice that, for all $\phi \in C_0^\infty(\mathbb{R}^n)$,

$$(\gamma v)^{\frac{q}{\gamma}} \frac{|\phi|}{(\gamma v_k)^{\frac{q}{\gamma}}} \le \frac{||\phi||_{L^\infty}}{(\delta_E)^{\frac{q}{\gamma}}} (\gamma v)^{\frac{q}{\gamma}} \ \text{ q.e. on } E = \mathrm{supp}(\phi).$$

Since σ is absolutely continuous with respect to p-capacity, it follows that the preceding estimate holds on $E\, d\sigma$-a.e. Using the Dominated Convergence Theorem and the fact that $(\gamma v)^{\frac{q}{\gamma}} = u^q \in L^1_{\mathrm{loc}}(\mathbb{R}^n, d\sigma)$, we obtain, for all $\phi \in C_0^\infty(\mathbb{R}^n)$,

$$\int_{R^n} (\gamma v)^{\frac{q}{\gamma}} \frac{\phi}{(\gamma v_k)^{\frac{q}{\gamma}}} \, d\sigma \to \int_{R^n} \phi \, d\sigma.$$

Clearly,

$$\left| |Dv|^{p-2} Dv \cdot \nabla v_k \, \frac{(\gamma v)^{\frac{q}{\gamma}} \phi}{v_k (\gamma v_k)^{\frac{q}{\gamma}}} \right| \le \frac{|Dv|^p |\phi|}{v}.$$

Using (4.73) and the Dominated Convergence Theorem again, we obtain

$$\int_{\mathbb{R}^n} |Dv|^{p-2} Dv \cdot \nabla v_k \, \frac{(\gamma v)^{\frac{q}{\gamma}} \phi}{v_k (\gamma v_k)^{\frac{q}{\gamma}}} \, dx \to \int_{\mathbb{R}^n} \frac{|Dv|^p \, \phi}{v} \, dx \quad \text{as } k \to \infty.$$

Therefore, letting $k \to \infty$ in (4.70), we deduce

$$\int_{\mathbb{R}^n} |Dv|^{p-2} Dv \cdot \nabla \phi \, dx = b \int_{\mathbb{R}^n} \frac{|Dv|^p \, \phi}{v} \, dx + \int_{R^n} \phi \, d\sigma, \quad \forall \, \phi \in C_0^\infty(\mathbb{R}^n).$$

Thus, v is a p-superharmonic (locally renormalized) solution to (4.2). Moreover, if both (4.11) and (4.12) hold, then by Theorem 4.1, the minimal solution u satisfies (4.13), and consequently v satisfies (4.22).

(ii) Conversely, suppose v is a p-superharmonic solution to (4.2). Let $\omega_k = -\Delta_p v_k$. Then $v_k \in W^{1,p}_{\text{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is p-superharmonic, and

$$-\Delta_p v_k = b \frac{|\nabla v_k|^p}{v_k} + \sigma \, \chi_{v<k} + \tilde{\omega}_k, \tag{4.74}$$

where $\tilde{\omega}_k$ is a nonnegative measure in $\mathbb{R}^n$ supported on $\{v = k\}$.

We have $u_k = (\gamma v_k)^{\frac{1}{\gamma}}$ and $u_k \in W^{1,p}_{\text{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ since $v_k \in W^{1,p}_{\text{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $\frac{1}{\gamma} = \frac{p-1}{p-1-q} > 1$. Let $\mu_k = -\Delta_p u_k$. Then it follows,

$$\mu_k = -\Delta_p u_k = -\Delta_p v_k \, (\gamma v_k)^{\frac{q}{\gamma}} - b \frac{|\nabla v_k|^p}{v_k} \, (\gamma v_k)^{\frac{q}{\gamma}} \geq 0. \tag{4.75}$$

Indeed, for any $\phi \in C_0^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \phi \, (\gamma v_k)^{\frac{q}{\gamma}} \, d\omega_k = \int_{\mathbb{R}^n} \nabla(\phi \, (\gamma v_k)^{\frac{q}{\gamma}}) \cdot \nabla v_k \, |\nabla v_k|^{p-2} \, dx$$

$$= \int_{\mathbb{R}^n} (\gamma v_k)^{\frac{q}{\gamma}} \nabla \phi \cdot \nabla v_k \, |\nabla v_k|^{p-2} \, dx + b \int_{\mathbb{R}^n} (\gamma v_k)^{\frac{q}{\gamma}} \phi \, \frac{|\nabla v_k|^p}{v_k} \, dx.$$

Hence,

$$\langle \phi, \mu_k \rangle = \int_{\mathbb{R}^n} \nabla \phi \cdot \nabla((\gamma v_k)^{\frac{1}{\gamma}}) \, |\nabla((\gamma v_k)^{\frac{1}{\gamma}})|^{p-2} \, dx = \int_{\mathbb{R}^n} \nabla \phi \cdot \nabla v_k \, |\nabla v_k|^{p-2} \, (\gamma v_k)^{\frac{q}{\gamma}} \, dx$$

$$= \int_{\mathbb{R}^n} \phi \, (\gamma v_k)^{\frac{q}{\gamma}} \, d\omega_k - b \int_{\mathbb{R}^n} \phi \, \frac{|\nabla v_k|^p}{v_k} \, (\gamma v_k)^{\frac{q}{\gamma}} \, dx = \int_{\mathbb{R}^n} \phi \, (\gamma v_k)^{\frac{q}{\gamma}} \, \chi_{v<k} \, d\sigma + \int_{\mathbb{R}^n} \phi \, (\gamma v_k)^{\frac{q}{\gamma}} \, d\tilde{\omega}_k,$$

where in the last expression we have used (4.74). From the preceding estimates it follows that $\langle \phi, \mu_k \rangle \geq 0$ if $\phi \geq 0$, and consequently u_k is p-superharmonic.

Clearly, $u = (\gamma v)^{\frac{1}{\gamma}} < +\infty$-a.e., and $u = \lim_{k \to +\infty} u_k$ is p-superharmonic in $\mathbb{R}^n$ as the limit of the increasing sequence of p-superharmonic functions u_k.

Since v is a p-superharmonic solution to (4.2), it follows that v is a locally renormalized solution ([KKT09]). Then for all $\phi \in C_0^\infty(\mathbb{R}^n)$ and $h \in W^{1,\infty}(\mathbb{R})$ with h' having compact support, we have

$$\int_{\mathbb{R}^n} |Dv|^{p-2} Dv \cdot \nabla(h(v)\phi) \, dx = b \int_{\mathbb{R}^n} \frac{|Dv|^p}{v} \, h(v)\phi \, dx + \int_{\mathbb{R}^n} h(v)\phi \, d\sigma. \tag{4.76}$$

Let $\phi \in C_0^\infty(\mathbb{R}^n), \phi \geq 0$. For $k > 0$, set $h(v) = (\gamma v_k)^{\frac{q}{\gamma}}$. Then

$$\int_{\mathbb{R}^n} |Dv|^{p-2} Dv \cdot \nabla((\gamma v_k)^{\frac{q}{\gamma}} \phi) \, dx = b \int_{\mathbb{R}^n} \frac{|Dv|^p}{v} \, (\gamma v_k)^{\frac{q}{\gamma}} \phi \, dx + \int_{\mathbb{R}^n} (\gamma v_k)^{\frac{q}{\gamma}} \phi \, d\sigma,$$

which implies

$$\int_{\mathbb{R}^n} |Dv|^{p-2} Dv \cdot \nabla \phi \, (\gamma v_k)^{\frac{q}{\gamma}} \, dx + b \int_{\mathbb{R}^n} |Dv|^{p-2} Dv \cdot \nabla v_k \, \frac{(\gamma v_k)^{\frac{q}{\gamma}}}{v_k} \, \phi \, dx$$

$$= b \int_{\mathbb{R}^n} \frac{|Dv|^p}{v} \, (\gamma v_k)^{\frac{q}{\gamma}} \phi \, dx + \int_{\mathbb{R}^n} (\gamma v_k)^{\frac{q}{\gamma}} \phi \, d\sigma.$$

Hence,

$$\int_{\mathbb{R}^n} |Dv|^{p-2} Dv \cdot \nabla\phi\, (\gamma v_k)^{\frac{q}{\gamma}}\, dx = b \int_{v>k} \frac{|Dv|^p}{v}\, (\gamma v_k)^{\frac{q}{\gamma}}\, \phi\, dx + \int_{\mathbb{R}^n} (\gamma v_k)^{\frac{q}{\gamma}} \phi\, d\,\sigma. \tag{4.77}$$

Consequently,

$$\int_{\mathbb{R}^n} |Dv|^{p-2} Dv \cdot \nabla\phi\, (\gamma v_k)^{\frac{q}{\gamma}}\, dx = b\gamma^{\frac{q}{\gamma}} k^{\frac{q(p-1)}{p-1-q}} \int_{v>k} \frac{|Dv|^p}{v}\, \phi\, dx + \int_{\mathbb{R}^n} (\gamma v_k)^{\frac{q}{\gamma}} \phi\, d\,\sigma. \tag{4.78}$$

Therefore,

$$\int_{\mathbb{R}^n} |Dv|^{p-2} Dv \cdot \nabla\phi\, (\gamma v_k)^{\frac{q}{\gamma}}\, dx \geq \int_{\mathbb{R}^n} (\gamma v_k)^{\frac{q}{\gamma}} \phi\, d\,\sigma. \tag{4.79}$$

Note that $Du = (\gamma v)^{\frac{q}{p-1-q}} Dv$, so that $|Du|^{p-1} = (\gamma v)^{\frac{q}{\gamma}} |Dv|^{p-1}$, and

$$\left|\, |Dv|^{p-2} Dv \cdot \nabla\phi\, (\gamma v_k)^{\frac{q}{\gamma}} \right| \leq |\nabla\phi|\, |Dv|^{p-1} (\gamma v)^{\frac{q}{\gamma}} \leq ||\nabla\phi||_{L^\infty(\mathbb{R}^n)} |Du|^{p-1}.$$

Notice that $|Du|^{p-1} \in L^1_{\text{loc}}(\mathbb{R}^n, dx)$. Using the Dominated Convergence Theorem, we obtain

$$\int_{\mathbb{R}^n} |Dv|^{p-2} Dv \cdot \nabla\phi\, (\gamma v_k)^{\frac{q}{\gamma}}\, dx \to \int_{\mathbb{R}^n} |Dv|^{p-2} Dv \cdot \nabla\phi\, (\gamma v)^{\frac{q}{\gamma}}\, dx = \int_{\mathbb{R}^n} |Du|^{p-2} Du \cdot \nabla\phi\, dx.$$

From this and (4.78), we deduce

$$b\gamma^{\frac{q}{\gamma}} \int_{v>k} \frac{|Dv|^p}{v}\, \phi\, dx \leq \frac{C(u,\phi)}{k^{\frac{q(p-1)}{p-1-q}}} < \infty.$$

Therefore,

$$||v||_{L^{\frac{q(p-1)}{p-1-q},\infty}(\phi\frac{|Dv|^p}{v}dx)} < \infty. \tag{4.80}$$

Using (4.79) and the Monotone Convergence Theorem, we deduce

$$\int_{\mathbb{R}^n} |Du|^{p-2}Du \cdot \nabla\phi \, dx \geq \int_{R^n} u^q \phi \, d\sigma, \quad \forall\, \phi \in C_0^\infty(\mathbb{R}^n),\, \phi \geq 0.$$

Notice that u is p-superharmonic in $\mathbb{R}^n$. This means that u is a supersolution to (4.1). By Theorem 4.1, u satisfies the lower bound in (4.13), and consequently v satisfies (4.21).

(iii) Suppose additionally that

$$\int_B v^{\frac{q(p-1)}{p-1-q}} \frac{|Dv|^p}{v} \, dx < \infty,$$

for every ball B in $\mathbb{R}^n$. Then

$$\int_{v>k} \frac{|Dv|^p}{v} (\gamma v_k)^{\frac{q}{\gamma}} \, \phi \, dx \to 0$$

by the Dominated Convergence Theorem. Letting $k \to \infty$ in (4.77), we deduce

$$\int_{\mathbb{R}^n} |Du|^{p-2}Du \cdot \nabla\phi \, dx = \int_{R^n} u^q \phi \, d\sigma, \quad \forall\, \phi \in C_0^\infty(\mathbb{R}^n).$$

Thus, u is a p-superharmonic, and hence a locally renormalized solution to (4.1). This completes the proof of Theorem 4.4. □

Proof of Theorem 4.5. (i) Suppose that (4.25) holds and $b < 1$, i.e, $q < 1 - \frac{1}{p}$. Let

$\gamma = \frac{p-1-q}{p-1}$ and $\beta = \frac{p-1-pq}{p-1}$; then $0 < \beta < 1$ and $\beta + q = \gamma$. Applying Theorem 3.5 with this β, there exists a nontrivial solution $w \in L^{\gamma}(\mathbb{R}^n, d\sigma) \cap L^q_{\rm loc}(\mathbb{R}^n, d\sigma)$ to the equation $w = K\mathbf{W}_{1,p}(w^q d\sigma)$, where K is the constant in (2.12) and $\liminf_{|x|\to\infty} w(x) = 0$. We have, by Theorem 2.12, $w \geq C\, K^{\frac{p-1}{p-1-q}} (\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}}$, where C is the constant in (2.27).

We set $w_0 = c_0\, (\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}}$, $d\omega_0 = w_0^q\, d\sigma$, where $c_0 > 0$ is a small constant to be chosen later. Clearly, $w_0 \leq w$ if $c_0 \leq CK^{\frac{p-1}{p-1-q}}$.

Notice that, by Theorem 2.14, ω_0 is absolutely continuous with respect to $\text{cap}_p(\cdot)$ since $d\omega_0 \leq w^q d\sigma, w \in L^q_{\rm loc}(\mathbb{R}^n, d\sigma)$ and $\sigma << \text{cap}_p$. Hence, for every $m \in \mathbb{N}$, there exists an increasing sequence of measures $\{\omega_0^{m,k}\}_k$ such that $\omega_0^{m,k} \in W^{-1,p'}(B(0, 2^m))$, $\omega_0^{m,k} \leq \omega_0$ and $\omega_0^{m,k}\chi_{B(0,2^m)} \to \omega_0\chi_{B(0,2^m)}$ weakly as $k \to \infty$ (see Corollary 6.5 in [Mik96], Appendix A). Thus, there exists a unique p-superharmonic function $u^k_{1,m} \in W^{1,p}_0(B(0, 2^m))$ such that $-\Delta_p u^k_{1,m} = \omega_0^{m,k}\chi_{B(0,2^m)}$ in $B(0, 2^m)$, i.e.,

$$\int_{B(0,2^m)} |\nabla u^k_{1,m}|^{p-2}\nabla u^k_{1,m} \cdot \nabla\varphi = \int_{B(0,2^m)} \varphi\, d\omega_0^{m,k}, \quad \forall\, \varphi \in C_0^\infty(B(0, 2^m)). \tag{4.81}$$

Moreover, by Theorem 2.6,

$$0 \leq u^k_{1,m} \leq K\, \mathbf{W}_{1,p}(\omega_0^{m,k}) \leq K\, \mathbf{W}_{1,p}\omega_0 \leq K\, \mathbf{W}_{1,p}(w^q d\sigma) = w.$$

For $\varepsilon > 0$, let $\varphi = (u^k_{1,m} + \varepsilon)^\beta - \varepsilon$; then $\varphi \in W^{1,p}_0(B(0, 2^m))$ and $\nabla\varphi = \frac{\beta\nabla u^k_{1,m}}{(u^k_{1,m}+\varepsilon)^{1-\beta}}$. Plugging φ into (4.81), we obtain

$$\beta \int_{B(0,2^m)} \frac{|\nabla u^k_{1,m}|^p}{(u^k_{1,m} + \varepsilon)^{1-\beta}} dx = \int_{B(0,2^m)} ((u^k_{1,m} + \varepsilon)^\beta - \varepsilon)\, d\omega_0^{m,k}. \tag{4.82}$$

Clearly, $(u_{1,m}^k+\varepsilon)^\beta-\varepsilon \le (u_{1,m}^k+\varepsilon)^\beta \le (u_{1,m}^k)^\beta+\varepsilon^\beta$, and

$$\int_{B(0,2^m)}((u_{1,m}^k)^\beta+\varepsilon^\beta)d\omega_0^{m,k} \le \int_{B(0,2^m)}((u_{1,m}^k)^\beta+\varepsilon^\beta)d\omega_0 \le \int_{B(0,2^m)} w^\beta w^q d\sigma$$

$$+\varepsilon^\beta \int_{B(0,2^m)} w^q d\sigma = \int_{B(0,2^m)} w^\gamma d\sigma + \varepsilon^\beta \int_{B(0,2^m)} w^q d\sigma < \infty.$$

Therefore, letting $\varepsilon \downarrow 0$ in (4.82) and using the Monotone and Dominated Convergence Theorems, we obtain

$$\beta \int_{B(0,2^m)} \frac{|\nabla u_{1,m}^k|^p}{(u_{1,m}^k)^{1-\beta}}dx = \int_{B(0,2^m)} (u_{1,m}^k)^\beta \, d\omega_0^{m,k}.$$

Consequently,

(4.83)
$$\beta \int_{B(0,2^m)} \frac{|\nabla u_{1,m}^k|^p}{(u_{1,m}^k)^{1-\beta}}dx \le \int_{B(0,2^m)} (u_{1,m}^k)^\beta \, d\omega_0 \le \int_{B(0,2^m)} w^\beta w^q d\sigma = \int_{B(0,2^m)} w^\gamma d\sigma.$$

Notice that the sequence $\{u_{1,m}^k\}_k$ is increasing by comparison principle. Hence, letting $u_{1,m}=\lim_{k\to\infty} u_{1,m}^k$ and using Theorem 2.5 together with the Monotone Convergence Theorem, we deduce that $u_{1,m}$ is a p-superharmonic solution to the equation $-\Delta_p u_{1,m}=\omega_0\chi_{B(0,2^m)}$ in $B(0,2^m)$ and $u_{1,m}\le w$. By Fatou's lemma, it follows from (4.83) that

$$\beta \int_{B(0,2^m)} \frac{|Du_{1,m}|^p}{(u_{1,m})^{1-\beta}}dx \le \int_{B(0,2^m)} w^\gamma d\sigma. \tag{4.84}$$

We also note that the sequence $\{u_{1,m}\}_{m=1}^\infty$ is increasing by the comparison principle (Lemma 4.16). Letting $u_1=\lim_{m\to\infty} u_{1,m}$ and using Theorem 2.5 and the Monotone Convergence Theorem, we see that u_1 is a p-superharmonic solution to the equation

$-\Delta_p u_1 = \omega_0$ in $\mathbb{R}^n$. Since $u_{1,m} \le w$, $u_1 \le w$, and hence $\liminf_{|x|\to\infty} u_1(x) = 0$.

Letting $m \to \infty$ in (4.84) and using Fatou's lemma and the Monotone Convergence Theorem, we obtain

$$\beta \int_{\mathbb{R}^n} \frac{|Du_1|^p}{(u_1)^{1-\beta}} dx \le \int_{\mathbb{R}^n} w^\gamma d\sigma. \tag{4.85}$$

We deduce, using (2.28),

$$u_1 \ge \frac{1}{K} \mathbf{W}_{1,p}\omega_0 = \frac{c_0^{\frac{q}{p-1}}}{K} \mathbf{W}_{1,p}\Big((\mathbf{W}_{1,p}\sigma)^{\frac{q(p-1)}{p-1-q}} \, d\sigma\Big)$$

$$\ge \frac{c_0^{\frac{q}{p-1}} \mathfrak{c}^{\frac{q}{p-1-q}}}{K} \, (\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}} = \frac{c_0^{\frac{q}{p-1}-1} \mathfrak{c}^{\frac{q}{p-1-q}}}{K} \, u_0.$$

Hence, for $c_0 \le (\mathfrak{c}^{\frac{q}{(p-1-q)}} K^{-1})^{\frac{p-1}{p-1-q}}$, we have $w \ge u_1 \ge u_0$.

Arguing by induction, we can construct a sequence $\{u_j\}_{j\ge1}$ of functions which are p-superharmonic in $\mathbb{R}^n$, $u_j \in L^q_{\text{loc}}(d\sigma)$ so that

$$\begin{cases} -\Delta_p u_j = \sigma u_{j-1}^q \text{ in } \quad \mathbb{R}^n \,, j = 2, 3, \dots, \\ c_j \, (\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}} \le u_j \le w, \\ 0 \le u_{j-1} \le u_j, \\ \liminf_{|x|\to\infty} u_j = 0, \end{cases} \tag{4.86}$$

and

$$\beta \int_{\mathbb{R}^n} \frac{|Du_j|^p}{(u_j)^{1-\beta}} \, dx \le \int_{\mathbb{R}^n} w^\gamma d\sigma, \tag{4.87}$$

where

$$c_j = \left(\frac{\mathfrak{c}^{\frac{q}{p-1-q}}}{K} \right)^{\sum_{l=0}^{j-1} \left(\frac{q}{p-1}\right)^l} c_0^{\left(\frac{q}{p-1}\right)^j}, \ j = 2, 3, \ldots .$$

Passing to the limit as $j \to \infty$ in (4.86) and using the weak continuity of the p-Laplacian and the Monotone Convergence Theorem, we see that $u = \lim_{j\to\infty} u_j$ is a p-superharmonic solution to the equation $-\Delta_p u = \sigma u^q$ in $\mathbb{R}^n$.

Letting $j \to \infty$ in (4.87) and using Fatou's lemma, we deduce

$$\beta \int_{\mathbb{R}^n} \frac{|Du|^p}{u^{1-\beta}} \, dx \leq \int_{\mathbb{R}^n} w^\gamma d\sigma < \infty. \tag{4.88}$$

Moreover, $\liminf_{|x|\to\infty} u(x) = 0$ since $u \leq w$ and $\liminf_{|x|\to\infty} w(x) = 0$.

Letting $v = \frac{1}{\gamma} u^\gamma$ and applying Theorem 4.4, we deduce v is a p-superharmonic solution to (4.2). Notice that, by (4.88)

$$\int_{\mathbb{R}^n} |Dv|^p \, dx = \int_{\mathbb{R}^n} \frac{|Du|^p}{u^{1-\beta}} \, dx < \infty.$$

Hence, $Dv = \nabla v$ and thus $v \in L_0^{1,p}(\mathbb{R}^n)$, i.e., v is a finite energy solution to (4.2).

(ii) Conversely, suppose $v \in L_0^{1,p}(\mathbb{R}^n)$, $v > 0$ is a p-superharmonic solution to (4.2). Then

$$\int_{\mathbb{R}^n} |\nabla v|^{p-2} \nabla v \cdot \nabla \phi \, dx = b \int_{\mathbb{R}^n} \frac{|\nabla v|^p}{v} \phi \, dx + \int_{\mathbb{R}^n} \phi \, d\sigma, \quad \forall \, \phi \in C_0^\infty(\mathbb{R}^n). \tag{4.89}$$

Let $\phi_k \in C_0^\infty(\mathbb{R}^n)$ such that $\phi_k \geq 0$ and $\nabla \phi_k \to \nabla v$ in $L^p(\mathbb{R}^n)$ and $\phi_k \to v$ q.e. in $\mathbb{R}^n$ as $k \to \infty$. Consequently, $\phi_k \to v \ d\sigma$-a.e. in $\mathbb{R}^n$ since σ is absolutely continuous with respect to $\mathrm{cap}_p(\cdot)$. Using ϕ_k in (4.89) in place of ϕ, we have

$$\int_{\mathbb{R}^n} |\nabla v|^{p-2}\nabla v \cdot \nabla \phi_k \, dx = b \int_{\mathbb{R}^n} \frac{|\nabla v|^p}{v} \phi_k \, dx + \int_{\mathbb{R}^n} \phi_k \, d\sigma.$$

By Fatou's lemma,

$$b \int_{\mathbb{R}^n} |\nabla v|^p \, dx + \int_{\mathbb{R}^n} v \, d\sigma \leq \liminf_{k\to\infty} \left(b \int_{\mathbb{R}^n} \frac{|\nabla v|^p}{v} \phi_k \, dx + \int_{\mathbb{R}^n} \phi_k \, d\sigma \right)$$

$$\leq \liminf_{k\to\infty} \int_{\mathbb{R}^n} |\nabla v|^{p-2}\nabla v \cdot \nabla \phi_k \, dx = \int_{\mathbb{R}^n} |\nabla v|^p dx < \infty.$$

Thus,

$$(1-b) \int_{\mathbb{R}^n} |\nabla v|^p dx \geq \int_{\mathbb{R}^n} v \, d\sigma > 0,$$

and consequently, $b < 1$ and $v \in L^1(\mathbb{R}^n, d\sigma)$. Notice that $v \geq c\mathbf{W}_{1,p}\sigma$, hence

$$\int_{\mathbb{R}^n} \mathbf{W}_{1,p}\sigma \, d\sigma \leq c \int_{\mathbb{R}^n} v \, d\sigma < \infty,$$

i.e., σ has finite energy.

Notice also that $v \geq \delta_B > 0$ on any ball $B \subset \mathbb{R}^n$, hence

$$\frac{v^{\frac{q(p-1)}{p-1-q}}}{v} = \frac{1}{v^{\frac{p-1-pq}{p-1-q}}} \leq \frac{1}{\delta_B^{\frac{p-1-pq}{p-1-q}}},$$

since $p - 1 - pq > 0$. Therefore,

$$\int_B \frac{|\nabla v|^p}{v} v^{\frac{q(p-1)}{p-1-q}} \, dx \leq \frac{1}{\delta_B^{\frac{p-1-pq}{p-1-q}}} \int_B |\nabla v|^p \, dx < \infty, \quad \text{for every ball } B.$$

Letting now $u = (\gamma v)^{\frac{1}{\gamma}}$ and applying Theorem 4.4 (ii), we deduce that u is a p-superharmonic solution to (4.1), which completes the proof of Theorem 4.5. $\square$

Chapter 5

Extension of the Brezis-Kamin theorem

5.1 Main results

In this chapter, we establish different approaches to study equations of the type

$$
\begin{cases} -\Delta_p u = \sigma\, u^q & \text{in } \mathbb{R}^n, \\ \\ \liminf\limits_{x\to\infty} u(x) = r, & u > 0, \end{cases} \tag{5.1}
$$

where $r \geq 0$, and other parameters are the same as before.

We will prove the existence of distributional solutions to (5.1) under certain conditions on σ. We also obtain sharp global pointwise bounds of solutions in terms of Wolff potentials. Necessary and sufficient conditions on σ for the existence of a certain class of solutions to (5.1) will be presented. When σ is radial, an explicit condition on σ for the existence of radial solutions and sharp bilateral bounds of solutions are given as well. We also treat equations with the fractional Laplacian $(-\Delta)^\alpha$, for $0 < \alpha < \frac{n}{2}$.

We study the following integral equations which are closely related to (5.1)

$$u = \mathbf{W}_{\alpha,p}(u^q d\sigma) + r, \quad u > 0, \tag{5.2}$$

where $r \geq 0$, and the integral inequality

$$u \geq \mathbf{W}_{\alpha,p}(u^q d\sigma). \tag{5.3}$$

Let us recall that a necessary condition for the existence of a solution to (5.2) with $\alpha = 1$ and (5.1) is that

$$\int_1^\infty \left[\frac{\sigma(B(0,t))}{t^{n-p}}\right]^{\frac{1}{p-1}} \frac{dt}{t} < +\infty. \tag{5.4}$$

We will see later that $c_0 \left(\mathbf{W}_{\alpha,p}\sigma\right)^{\frac{p-1}{p-1-q}}$ is a subsolution to (5.2) if c_0 is small enough. Therefore, in order to establish the existence of a solution to (5.2), if suffices to find a supersolution to (5.2). This is the main goal of this chapter.

As shown in Chapter 2, if there exists a solution to (5.1), then σ must be absolutely continuous with respect to the p-capacity $\mathrm{cap}_p(\cdot)$. In this chapter, we will assume that σ satisfies the following capacity condition

$$\sigma(E) \leq C(\sigma)\ \mathrm{cap}_p(E) \text{ for all compact sets } E \subset \mathbb{R}^n, \tag{5.5}$$

where $C(\sigma)$ is a positive constant. Such a condition was considered in [JV10, JV12, Maz11]. We will show that this condition is equivalent to

$$\int_E (\mathbf{W}_{1,p}\sigma_E)^s d\sigma \le c(p,n,s,C(\sigma))\,\sigma(E), \tag{5.6}$$

for every compact set E and for every $s > 0$. This implies that

$$\int_B (\mathbf{W}_{1,p}\sigma_B)^s d\sigma \le c(p,n,s,C(\sigma))\,\sigma(B), \tag{5.7}$$

for every ball B and for every $s > 0$.

Estimates (5.7) will be our key tool in finding a supersolution and obtaining global pointwise estimates of solutions to (5.2) with $\alpha = 1$.

As shown below, condition (5.5) along with (5.4) will ensure the existence of a solution u to (5.1) and such a u satisfies the following two-sided estimates when $r > 0$.

$$c^{-1}\Big(r + \mathbf{W}_{1,p}\sigma\Big)^{\frac{p-1}{p-1-q}} \le u \le c\Big(r + \mathbf{W}_{1,p}\sigma\Big)^{\frac{p-1}{p-1-q}}, \tag{5.8}$$

where $c = c(n,p,q,r,C(\sigma))$. If σ obeys (5.5) and (5.4) holds, it also guarantees the existence of a solution u to the ground state equation (5.1) with $r = 0$ and u satisfies

$$c^{-1}\left(\mathbf{W}_{1,p}\sigma\right)^{\frac{p-1}{p-1-q}} \le u \le c\left(\mathbf{W}_{1,p}\sigma + (\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}}\right), \tag{5.9}$$

where $c = c(n,p,q,r,C(\sigma)) > 0$. In the case $p = 2$, suppose that $\mathbf{I}_2\sigma \in L^\infty(\mathbb{R}^n)$, then by a well known result, (5.5) holds with $p = 2$. Therefore, by (5.9) and notice that $\mathbf{W}_{1,2}\sigma \equiv \mathbf{I}_2\sigma$, we deduce the existence of a solution u to (5.1) with $p = 2, r = 0$ such that

$$c^{-1}\left(\mathbf{I}_2\sigma\right)^{\frac{1}{1-q}} \le u \le c\left(\mathbf{I}_2\sigma + (\mathbf{I}_2\sigma)^{\frac{1}{1-q}}\right).$$

Since $\mathbf{I}_2\sigma \in L^\infty(\mathbb{R}^n)$, this implies

$$c^{-1}\left(\mathbf{I}_2\sigma\right)^{\frac{1}{1-q}} \le u \le c\,\mathbf{I}_2\sigma.$$

Thus, our estimates of solutions extend those of Brezis-Kamin. Moreover, it could extend to possibly singular (unbounded) solutions as well.

We can see that condition (5.5) does not depend on the "sub-critical" growth rate q at all. Initially we studied equation (5.1) in the *dyadic models*, so we propose another sufficient condition for the existence of solutions to the discrete model of (5.1) in terms of dyadic cubes which depends on q as follows.

$$\begin{aligned}(5.10)\quad \int_P \left(\sum_{Q\subset P}\left(\frac{\sigma(Q)}{|Q|^{1-\frac{p}{n}}}\right)^{\frac{1}{p-1}} \chi_Q(y)\right)^{\frac{(p-1)q}{p-1-q}} d\sigma(y)\\ \le c\,\sigma(P)\left(1+\sum_{R\supset P}\left(\frac{\sigma(R)}{|R|^{1-\frac{p}{n}}}\right)^{\frac{1}{p-1}}\right)^{\frac{(p-1)q}{p-1-q}},\end{aligned}$$

where P, Q, R are dyadic cubes in $\mathbb{R}^n$, and c is a positive constant. The analogue of (5.10) in the continuous case is

$$(5.11)\quad \int_{B(x,r)} (\mathbf{W}_{1,p}\sigma)^{\frac{(p-1)q}{p-1-q}}\, d\sigma \le c\,\sigma(B(x,r))\left(1+\int_r^\infty \left(\frac{\sigma(B(x,t))}{t^{n-p}}\right)^{\frac{1}{p-1}}\frac{dt}{t}\right)^{\frac{(p-1)q}{p-1-q}}.$$

From this and the additional assumption that $\mathbf{W}_{1,p}\sigma \not\equiv +\infty$, we can deduce

$$(5.12)\qquad \mathbf{W}_{1,p}\left((\mathbf{W}_{1,p}\sigma)^{\frac{(p-1)q}{p-1-q}}\, d\sigma\right) \le \kappa\left(\mathbf{W}_{1,p}\sigma + (\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}}\right) < \infty \ \text{ a.e.},$$

where $\kappa = \kappa(n,p,q)$ is a positive constant.

These conditions are actually weaker than the capacity condition (5.5). Indeed, as shown in Lemma 5.6 below, (5.5) implies (5.11), and thus (5.12). It is easy to find $\sigma \geq 0$ such that (5.11), (5.12) hold but (5.5) fails. Letting $\sigma(x) = \frac{1}{|x|^s}$ on the ball $B(0,1)$ and zero outside $B(0,1)$, where $2 < s < n$, gives such an example.

As we will show below, condition (5.12) is necessary and sufficient for the existence of a solution u to (5.1) with $r = 0$ so that

$$c^{-1}\Big(\mathbf{W}_{1,p}\sigma\Big)^{\frac{p-1}{p-1-q}} \leq u \leq c\left(\mathbf{W}_{1,p}\sigma + \Big(\mathbf{W}_{1,p}\sigma\Big)^{\frac{p-1}{p-1-q}}\right). \tag{5.13}$$

In Section 5.5, we provide an example which shows that (5.5) and (5.12) are sufficient but not necessary for the existence of a solution to (5.1).

We are now ready to state our first main result for equation (5.1) with positive lower bound of solutions.

Theorem 5.1. *Let $1 < p < n, 0 < q < p-1, r > 0$ and $\sigma \in M^+(\mathbb{R}^n)$. If (5.4) and (5.5) hold, then there exists a distributional solution $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ to (5.1) and u satisfies*

$$c^{-1}\Big(r + \mathbf{W}_{1,p}\sigma\Big)^{\frac{p-1}{p-1-q}} \leq u \leq c\Big(r + \mathbf{W}_{1,p}\sigma\Big)^{\frac{p-1}{p-1-q}},$$

where $c > 0$ is a constant depending only on $n, p, q, r,$ and $C(\sigma)$.

We see that the upper and lower estimates are matching. Our next theorem is concerned with the ground state problem (5.1).

Theorem 5.2. *Let $1 < p < n, 0 < q < p-1$ and $\sigma \in M^+(\mathbb{R}^n)$. If (5.4) and (5.5) hold, then there exists a mimimal p-superharmonic solution $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ to (5.1)* *with* $r = 0$ *and* u *satisfies*

$$c^{-1}\Big(\mathbf{W}_{1,p}\sigma\Big)^{\frac{p-1}{p-1-q}} \leq u \leq c\left(\mathbf{W}_{1,p}\sigma + \Big(\mathbf{W}_{1,p}\sigma\Big)^{\frac{p-1}{p-1-q}}\right), \tag{5.14}$$

where $c = c(n,p,q,C(\sigma)) > 0$. *In the case* $p \geq n$ *there are no nontrivial solutions on* $\mathbb{R}^n$.

We now turn to sublinear fractional Laplacian equation (4.4). Suppose that (5.5) holds with the fractional capacity $\mathrm{cap}_{\alpha,2}$ in place of p-capacity, i.e.,

$$\sigma(E) \leq c(\sigma)\mathrm{cap}_{\alpha,2}(E), \quad \text{for all compact sets } E \subset \mathbb{R}^n, \tag{5.15}$$

where $c(\sigma) > 0$ is a constant, and $\mathbf{I}_{2\alpha}\sigma \not\equiv +\infty$, or equivalently,

$$\int_1^{\infty} \frac{\sigma(B(0,t))}{t^{n-2\alpha}}\frac{dt}{t} < +\infty, \tag{5.16}$$

we deduce similar results for (4.4) as follows.

Theorem 5.3. *Let* $0 < q < 1, 0 < \alpha < \frac{n}{2}$ *and* $\sigma \in M^+(\mathbb{R}^n)$. *Suppose that* (5.15) *and* (5.16) *hold. Then there exists a mimimal solution* u *to* (4.4) *and* u *satisfies*

$$c^{-1}\Big(\mathbf{I}_{2\alpha}\sigma\Big)^{\frac{1}{1-q}} \leq u \leq c\left(\mathbf{I}_{2\alpha}\sigma + \Big(\mathbf{I}_{2\alpha}\sigma\Big)^{\frac{1}{1-q}}\right), \tag{5.17}$$

where $c > 0$ *is a constant depending only on* n, q, *and* $c(\sigma)$.

In the next theorem, we give necessary and sufficient conditions for the existence of solutions to (5.1) satisfying (5.13).

Theorem 5.4. *Let $1 < p < n, 0 < q < p-1$ and $\sigma \in M^+(\mathbb{R}^n)$. Then equation* (5.1) *with $r = 0$ has a p-superharmonic solution u which satisfies* (5.13) *if and only if* (5.12) *holds.*

We now outline the contents of this chapter. In Section 5.2, we study the equivalence between the capacity condition (5.5) and (5.6). In Section 5.3, we prove the existence and obtain pointwise estimates of solutions to (5.2). Section 5.4 is devoted to a proof of our main results regarding equation (5.1). In Section 5.5, we consider the case when σ is radial and $p = 2$. We give an explicit condition for the existence and provide sharp pointwise estimates of a radial solution. We also characterize condition (5.12) as well. We finally give a counter example showing that there exists a solution to (5.1) but condition (5.12) fails.

5.2 Capacity condition and Wolff potential estimates

Discrete Carleson measures appear in the following lemma

Lemma 5.5. *Suppose that $\sigma(E) \leq c\, cap_{\alpha,p}(E)$, then for all dyadic cubes $P \in \mathcal{D}$ and compact sets $E \subset \mathbb{R}^n$ and $t \in \mathbb{R}^n$,*

$$\sum_{Q \subset P,\, Q \in \mathcal{D}} \left(\frac{\sigma((Q+t) \cap E)}{|Q+t|^{1-\frac{\alpha p}{n}}} \right)^{\frac{1}{p-1}} \sigma((Q+t) \cap E) \leq c\, \sigma(P \cap E), \tag{5.18}$$

and

$$\sum_{Q \in \mathcal{D}} \left(\frac{\sigma((Q+t) \cap E)}{|Q+t|^{1-\frac{\alpha p}{n}}} \right)^{\frac{1}{p-1}} \sigma((Q+t) \cap E) \leq c\, \sigma(E). \tag{5.19}$$

This lemma was proved in [JV10]. We will need the following lemma.

Lemma 5.6. *Let $\sigma \in M^+(\mathbb{R}^n)$ and suppose that*

$$\sigma(E) \le C\, cap_{\alpha,p}(E), \quad \textit{for all compact sets } E \subset R^n. \tag{5.20}$$

Then the following inequality holds

$$\int_E (\mathbf{W}_{\alpha,p}\sigma_E)^s d\sigma \le c\,\sigma(E), \quad \textit{for all compact sets } E, \tag{5.21}$$

and for every $s > 0$. Conversely, if (5.21) *holds for a given $s > 0$, then* (5.20) *holds; and consequently,* (5.21) *holds for every $s > 0$.*

Proof. Suppose that (5.20) holds, then we have (see [V1])

$$\int_{\mathbb{R}^n} [\mathbf{I}_\alpha(\sigma_E)]^{p'} dx \le c\,\sigma(E), \tag{5.22}$$

for all compact sets $E \subset \mathbb{R}^n$. By Wolff's inequality, we obtain

$$\int_{\mathbb{R}^n} [\mathbf{I}_\alpha \sigma_E]^{p'} dx \ge c \int_{\mathbb{R}^n} \mathbf{W}_{\alpha,p}\sigma_E\, d\sigma_E = c \int_E \mathbf{W}_{\alpha,p}\sigma_E\, d\sigma. \tag{5.23}$$

Consequently,

$$\int_E \mathbf{W}_{\alpha,p}\sigma_E\, d\sigma_E \le c\,\sigma(E), \quad \text{for all compact sets } E.$$

If $0 < s < 1$, using preceding estimate and Hölder's inequality yield (5.21). If $s > 1$, we will show that (3.2) holds by using shifted dyadic lattice $\mathcal{D}_t$. Let E be a compact set in $\mathbb{R}^n$, by Lemma 2.7, we have

$$\mathbf{W}^r_{\alpha,p}\sigma_E(x) \le c_1\, r^{-n} \int_{|t|\le cr} \sum_{Q\in\mathcal{D}} \left[\frac{\sigma_E(Q+t)}{|Q+t|^{1-\frac{\alpha p}{n}}}\right]^{\frac{1}{p-1}} \chi_{Q+t}(x)dt.$$

Raising both sides to the power s, integrating over E with respect to $d\sigma_E$, and using Minkowski's inequality, we obtain

$$\int_E (\mathbf{W}^r_{\alpha,p}\sigma_E(x))^s\, d\sigma_E$$

$$\le c_1 \left(r^{-n} \int_{|t|\le cr} \left(\int_E \left[\sum_{Q\in\mathcal{D}} \left[\frac{\sigma_E(Q+t)}{|Q+t|^{1-\frac{\alpha p}{n}}}\right]^{\frac{1}{p-1}} \chi_{Q+t}(x) \right]^s d\sigma_E \right)^{\frac{1}{s}} dt \right)^s.$$

Applying Proposition 2.2 in [COV04], we have

$$\int_E \left[\sum_{Q\in\mathcal{D}} \left[\frac{\sigma_E(Q+t)}{|Q+t|^{1-\frac{\alpha p}{n}}}\right]^{\frac{1}{p-1}} \chi_{Q+t}(x) \right]^s d\sigma_E$$

$$\le c \sum_{Q\in\mathcal{D}} \left[\frac{\sigma_E(Q+t)}{|Q+t|^{1-\frac{\alpha p}{n}}}\right]^{\frac{1}{p-1}} \sigma_E(Q+t)$$

$$\cdot \left(\frac{1}{\sigma_E(Q+t)} \sum_{R\subset Q} \left[\frac{\sigma_E(R+t)}{|R+t|^{1-\frac{\alpha p}{n}}}\right]^{\frac{1}{p-1}} \sigma_E(R+t) \right)^{s-1}.$$

By (5.18), the latter term is uniformly bounded. Hence, using (5.19), we obtain

$$\int_E \left[\sum_{Q\in\mathcal{D}} \left[\frac{\sigma_E(Q+t)}{|Q+t|^{1-\frac{\alpha p}{n}}}\right]^{\frac{1}{p-1}} \chi_{Q+t}(x) \right]^s d\sigma_E \le c\,\sigma(E).$$

Thus,

$$\int_E (\mathbf{W}^r_{\alpha,p}\sigma_E)^s\, d\sigma \le c_1 \left(r^{-n} \int_{|t|\le cr} (c\,\sigma(E))^{\frac{1}{s}}\, dt \right)^s \le c\,\sigma(E).$$

Letting $r \to \infty$ and using the Monotone Convergence Theorem, we deduce

$$\int_E (\mathbf{W}_{\alpha,p}\sigma_E)^s \, d\sigma \le c\,\sigma(E),$$

where the constant $c > 0$ depends on α, p, n, s and C.

Conversely, suppose that (5.21) holds for a fixed $s > 0$. Let E be a compact set in $\mathbb{R}^n$ and suppose that $\sigma(E) > 0$. We write

$$\sigma(E) = \int_E (\mathbf{W}_{\alpha,p}\sigma_E)^{-\beta} (\mathbf{W}_{\alpha,p}\sigma_E)^{\beta} d\sigma,$$

where $\beta > 0$ will be chosen later. Using Hölder's inequality, we have

$$\sigma(E) \le \left(\int_E (\mathbf{W}_{\alpha,p}\sigma_E)^{-\beta r} d\sigma \right)^{\frac{1}{r}} \left(\int_E (\mathbf{W}_{\alpha,p}\sigma_E)^{\beta r'} d\sigma \right)^{\frac{1}{r'}},$$

where $r > 1$ and $r' = \frac{r}{r-1}$. Let us choose β and r such that $\beta r = p - 1$ and $\beta r' = s$; hence, $r = 1 + \frac{p-1}{s}$ and $\beta = \frac{s(p-1)}{s+p-1}$. Consequently,

$$\sigma(E) \le \left(\int_E \frac{d\sigma_E}{(\mathbf{W}_{\alpha,p}\sigma_E)^{p-1}} \right)^{\frac{1}{r}} \left(\int_E (\mathbf{W}_{\alpha,p}\sigma_E)^{s} d\sigma \right)^{\frac{1}{r'}}.$$

Applying Theorem 1.11 in [V1], we have

$$\int_E \frac{d\sigma_E}{(\mathbf{W}_{\alpha,p}\sigma_E)^{p-1}} \le c\,\mathrm{cap}_{\alpha,p}(E).$$

Combining this estimate and (5.21) yield

$$\sigma(E) \le c \left(\mathrm{cap}_{\alpha,p}(E) \right)^{\frac{1}{r}} (\sigma(E))^{\frac{1}{r'}}.$$

This gives us $\sigma(E) \leq c\,\mathrm{cap}_{\alpha,p}(E)$. Consequently, arguing as in the beginning of the proof, we also see that (5.21) holds for any $s > 0$. Hence, we complete the proof of Lemma 5.6. □

Remark 5.7. Suppose that (5.20) holds, from (5.21), we deduce

$$\int_B (\mathbf{W}_{\alpha,p}\sigma_B)^s d\sigma \leq c(\alpha, p, n, C, s)\, \sigma(B), \tag{5.24}$$

for every ball $B \subset \mathbb{R}^n$ and for every $s > 0$.

Remark 5.8. We observe that Lemma 5.6 for $s > \frac{1}{2}$ was proved by Nazarov, Treil and Volberg in [NTV03], where the authors used the Bellman function method.

5.3 Solutions of the nonlinear integral equations

5.3.1 Inhomogeneous problems

We state our result concerning equation (5.2).

Theorem 5.9. *Let $r > 0, 1 < p < n, 0 < q < p-1, 0 < \alpha < \frac{n}{p}$ and $\sigma \in M^+(\mathbb{R}^n)$. Suppose that (2.20) and (5.20) hold. Then there exists a solution u to (5.2) and u satisfies*

$$c^{-1}\left(r + (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}}\right) \leq u \leq c\left(r + (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}}\right), \tag{5.25}$$

where $c = c(n, p, q, \alpha, C, r)$. Moverer, $u \in L^s_{\mathrm{loc}}(\mathbb{R}^n, d\sigma)$, for every $s > 0$.

Proof. Let $w = c_0 \left(\mathbf{W}_{\alpha,p}\sigma\right)^{\frac{(p-1)}{p-1-q}}$, where $c_0 > 0$ is a small constant. Using Lemma 2.13, we can see that w is a subsolution to (5.2). Let

$$v = c\left(r + (\mathbf{W}_{\alpha,p}\sigma)^{\frac{(p-1)}{p-1-q}}\right),$$

where $c > 0$ is a large constant. We will show that v is a supersolution of (5.2). First, we estimate

$$\mathbf{W}_{\alpha,p}(v^q d\sigma)(x) = \int_0^\infty \left(\frac{\int_{B(x,t)} v^q d\sigma}{t^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t}$$

$$\leq c^{\frac{q}{p-1}} c_1 \left(r^{\frac{q}{p-1}}\, \mathbf{W}_{\alpha,p}\sigma + \int_0^\infty \left(\frac{\int_{B(x,t)} (\mathbf{W}_{\alpha,p}\sigma)^{\frac{(p-1)q}{p-1-q}} d\sigma}{t^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t}\right).$$

Let $\beta = \frac{(p-1)q}{p-1-q}$, we next estimate

$$\int_{B(x,t)} (\mathbf{W}_{\alpha,p}\sigma)^\beta d\sigma = \int_{B(x,t)} \left[\int_0^\infty \left(\frac{\sigma(B(y,s))}{s^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{ds}{s}\right]^\beta d\sigma(y)$$

$$\leq c_1 \int_{B(x,t)} \left[\int_0^t \left(\frac{\sigma(B(y,s))}{s^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{ds}{s}\right]^\beta d\sigma(y)$$

$$+ c_1 \int_{B(x,t)} \left[\int_t^\infty \left(\frac{\sigma(B(y,s))}{s^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{ds}{s}\right]^\beta d\sigma(y) = c_1\,(I + II).$$

For $y \in B(x,t)$ and $s \geq t$, we have $B(y,s) \subset B(x,2s)$, thus

$$II \leq \int_{B(x,t)} \left[\int_t^\infty \left(\frac{\sigma(B(x,2s))}{s^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{ds}{s}\right]^\beta d\sigma(y)$$

$$\leq \sigma(B(x,t))\left[\int_t^\infty \left(\frac{\sigma(B(x,2s))}{s^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{ds}{s}\right]^\beta \leq c_1\, \sigma(B(x,t))[\mathbf{W}_{\alpha,p}\sigma(x)]^\beta,$$

where $c_1 = c_1(\alpha, n, p, q)$. For $s \leq t$, we have $B(y,s) \subset B(x,2t)$, so

$$I = \int_{B(x,t)} \left[\int_0^t \left(\frac{\sigma(B(y,s)\cap B(x,2t))}{s^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{ds}{s}\right]^\beta d\sigma(y)$$

$$\leq \int_{B(x,2t)} \left[\int_0^t \left(\frac{\sigma(B(y,s)\cap B(x,2t))}{s^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{ds}{s}\right]^\beta d\sigma(y)$$

$$\leq \int_{B(x,2t)} \left[\mathbf{W}_{\alpha,p}\sigma_{B(x,2t)}\right]^\beta d\sigma(y).$$

Using (5.24), we obtain

$$\int_{B(x,2t)} \left[\mathbf{W}_{\alpha,p}\sigma_{B(x,2t)}\right]^\beta d\sigma(y) \leq c_2\, \sigma(B(x,2t),$$

where $c_2 = c_2(\alpha, n, p, q, C)$. Thus,

$$\int_{B(x,t)} (\mathbf{W}_{\alpha,p}\sigma)^\beta d\sigma \leq c_1\, (\mathbf{W}_{\alpha,p}\sigma(x))^\beta \sigma(B(x,t)) + c_2\, \sigma(B(x,2t)). \tag{5.26}$$

Consequently,

$$\left[\int_{B(x,t)} (\mathbf{W}_{\alpha,p}\sigma)^\beta d\sigma\right]^{\frac{1}{p-1}}$$

$$\leq c_1\, (\mathbf{W}_{\alpha,p}\sigma(x))^{\frac{q}{p-1-q}} [\sigma(B(x,t))]^{\frac{1}{p-1}} + c_2\, [\sigma(B(x,2t))]^{\frac{1}{p-1}}.$$

Hence,

$$\mathbf{W}_{\alpha,p}(v^q d\sigma) \leq c^{\frac{q}{p-1}} \tilde{c} \left(r^{\frac{q}{p-1}}\, \mathbf{W}_{\alpha,p}\sigma + c_2\, \mathbf{W}_{\alpha,p}\sigma + c_1\, (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}}\right).$$

$$\leq c^{\frac{q}{p-1}} c_3 \left(r + (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}} \right),$$

where $c_3 = c_3(n, p, q, \alpha, C, r)$. Therefore, picking c large enough yields $v \geq r + \mathbf{W}_{\alpha,p}(v^q d\sigma)$ and $v \geq w$.

Using the sub-supersolutions method, iterations and the Monotone Convergence Theorem, we deduce the existence of a nontrivial solution u to (5.2) and u satisfies

$$c^{-1} \left(r + (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}} \right) \leq u \leq c \left(r + (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}} \right),$$

where $c = c(n, p, q, \alpha, C, r)$. By Lemma 5.6, we deduce that $u \in L^s_{\text{loc}}(\mathbb{R}^n, d\sigma)$ for every $s > 0$. This concludes the proof of Theorem 5.9. □

5.3.2 Homogeneous problems

Having the same hypotheses as in Theorem 5.9, we obtain the following result for the homogeneous equation

$$u = \mathbf{W}_{\alpha,p}(u^q d\sigma). \tag{5.27}$$

Theorem 5.10. *Let* $1 < p < \infty, 0 < q < p-1, 0 < \alpha < \frac{n}{p}$ *and* $\sigma \in M^+(\mathbb{R}^n)$*. Suppose that* (2.20) *and* (5.20) *hold. Then there exists a solution* u *to* (5.27) *and* u *satisfies*

$$c^{-1}(\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}} \leq u \leq c\,(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}}), \tag{5.28}$$

where $c = c(n, p, q, \alpha, C)$*. Moverer,* $u \in L^s_{\text{loc}}(\mathbb{R}^n, d\sigma)$*, for every* $s > 0$*.*

Proof. Let $w = c_0\,(\mathbf{W}_{\alpha,p}\sigma(x))^{\frac{p-1}{p-1-q}}$ with small constant c_0. Then w is a subsolution to (5.27). Now, let

$$v = c_1 \left(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}} \right),$$

where c_1 is a large constant. Clearly, $w \leq v$. Arguing as in Theorem 5.9 and using (5.24), we can show that v is a supersolution to (5.27). Therefore, using again the sub-supersolutions method, iterations and the Monotone Convergence Theorem, we see that there exists a nontrivial solution u to (5.27) and u satisfies

$$c^{-1} (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}} \leq u \leq c \left(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}} \right), \tag{5.29}$$

where $c = c(n, p, q, \alpha, C) > 0$.

Given $s > 0$, then $u \in L^s_{\text{loc}}(\mathbb{R}^n, d\sigma)$ follows from (5.29) and Lemma 5.6. We also notice that $\liminf_{|x|\to\infty} u(x) = 0$ since $\liminf_{|x|\to\infty} \mathbf{W}_{\alpha,p}\sigma(x) = 0$ by Corollary 2.10. This completes the proof of Theorem 5.10. □

Instead of using capacity condition (5.20), suppose that

$$\mathbf{W}_{\alpha,p} \left((\mathbf{W}_{\alpha,p}\sigma)^{\frac{(p-1)q}{p-1-q}} \, d\sigma \right) \leq \kappa \left(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}} \right) < \infty \text{ a.e.}, \tag{5.30}$$

where κ is a positive constant, then we also obtain the existence of solutions to (5.27) as follows.

Theorem 5.11. *Let $1 < p < \infty, 0 < q < p-1, 0 < \alpha < \frac{n}{p}$ and $\sigma \in M^+(\mathbb{R}^n)$. Suppose that* (5.30) *holds, then there exists a solution u to* (5.27) *and u satisfies*

$$c^{-1}(\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}} \leq u \leq c \left(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}} \right), \tag{5.31}$$

where $c > 0$ is a constant depending only on α, n, p, q, κ. Conversely, suppose that there exists a nontrivial supersolution u to (5.27) *and u satisfies* (5.31), *then* (5.30) *holds with $\kappa = \kappa(p, q, c)$.*

Proof. Suppose that (5.30) holds. Let $w = c_0(\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}}$ with small constant c_0; then w is a subsolution to (5.27) as before. Let

$$v = c\left(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}}\right),$$

where $c > 0$ is a large constant. We estimate

$$\mathbf{W}_{\alpha,p}(v^q d\sigma) = c^{\frac{q}{p-1}}\mathbf{W}_{\alpha,p}\left(\left(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}}\right)^q d\sigma\right)$$

$$\leq a\, c^{\frac{q}{p-1}}\mathbf{W}_{\alpha,p}\left((\mathbf{W}_{\alpha,p}\sigma)^q d\sigma\right) + a\, c^{\frac{q}{p-1}}\mathbf{W}_{\alpha,p}\left((\mathbf{W}_{\alpha,p}\sigma)^{\frac{(p-1)q}{p-1-q}} d\sigma\right)$$

$$\leq a\, c^{\frac{q}{p-1}}\mathbf{W}_{\alpha,p}\left((\mathbf{W}_{\alpha,p}\sigma)^q d\sigma\right) + a\, c^{\frac{q}{p-1}}\kappa\left(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}}\right),$$

where $a = a(p, q)$. Next, we write

$$\mathbf{W}_{\alpha,p}\left((\mathbf{W}_{\alpha,p}\sigma)^q d\sigma\right)(x) = \int_0^\infty \left(\frac{\int_{B(x,t)}(\mathbf{W}_{\alpha,p}\sigma)^q d\sigma}{t^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t}.$$

Using Hölder and Young's inequalities, we obtain

$$\int_{B(x,t)}(\mathbf{W}_{\alpha,p}\sigma)^q d\sigma \leq \left(\int_{B(x,t)}(\mathbf{W}_{\alpha,p}\sigma)^{\frac{(p-1)q}{p-1-q}} d\sigma\right)^{\frac{p-1-q}{p-1}} \left[\sigma(B(x,t))\right]^{\frac{q}{p-1}}$$

$$\leq \tilde{b}\left(\int_{B(x,t)}(\mathbf{W}_{\alpha,p}\sigma)^{\frac{(p-1)q}{p-1-q}} d\sigma + \sigma(B(x,t))\right).$$

Thus,

$$\int_0^\infty \left(\frac{\int_{B(x,t)} (\mathbf{W}_{\alpha,p}\sigma)^q d\sigma}{t^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \leq b \int_0^\infty \left(\frac{\int_{B(x,t)} (\mathbf{W}_{\alpha,p}\sigma)^{\frac{(p-1)q}{p-1-q}} d\sigma}{t^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t}$$

$$+b \int_0^\infty \left(\frac{\sigma(B(x,t))}{t^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} = b\mathbf{W}_{\alpha,p} \left((\mathbf{W}_{\alpha,p}\sigma)^{\frac{(p-1)q}{p-1-q}} d\sigma \right)(x) + b\mathbf{W}_{\alpha,p}\sigma(x),$$

here $b = b(p,q)$. By (5.12), the last term is less than

$$b\kappa \left(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}} \right)(x) + b\mathbf{W}_{\alpha,p}\sigma(x)$$

$$= b(\kappa+1)\mathbf{W}_{\alpha,p}\sigma + b\kappa \left(\mathbf{W}_{\alpha,p}\sigma \right)^{\frac{p-1}{p-1-q}}.$$

Hence,

$$\mathbf{W}_{\alpha,p}(v^q d\sigma) \leq a\, c^{\frac{q}{p-1}} b(\kappa+1)\mathbf{W}_{\alpha,p}\sigma + a\, c^{\frac{q}{p-1}} b\kappa \left(\mathbf{W}_{\alpha,p}\sigma \right)^{\frac{p-1}{p-1-q}}$$

$$+a\, c^{\frac{q}{p-1}} \kappa \left(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}} \right)$$

$$\leq c^{\frac{q}{p-1}} a(b\kappa + b + \kappa)(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}}).$$

If c is chosen such that $c \geq c^{\frac{q}{p-1}} a(b\kappa + b + \kappa)$ and $c \geq c_0$ then we obtain $v \geq w$ and $v \geq \mathbf{W}_{\alpha,p}(v^q d\sigma)$. Using again the sub-supersolutions method, iterations and the Monotone Convergence Theorem, we deduce the existence of a solution u to the equation $u = \mathbf{W}_{\alpha,p}(u^q d\sigma)$ and

$$c^{-1} \left(\mathbf{W}_{\alpha,p}\sigma \right)^{\frac{p-1}{p-1-q}} \leq u \leq c \left(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}} \right).$$

Conversely, suppose there exists a nontrivial supersolution u to (5.27) and

$$c^{-1}\left(\mathbf{W}_{\alpha,p}\sigma\right)^{\frac{p-1}{p-1-q}} \le u \le c\left(\mathbf{W}_{\alpha,p}\sigma + \left(\mathbf{W}_{\alpha,p}\sigma\right)^{\frac{p-1}{p-1-q}}\right).$$

We see that

$$u \ge \mathbf{W}_{\alpha,p}(u^q d\sigma) \ge (c^{-1})^{\frac{q}{p-1}} \mathbf{W}_{\alpha,p}\left(\left(\mathbf{W}_{\alpha,p}\sigma\right)^{\frac{(p-1)q}{p-1-q}} d\sigma\right).$$

Therefore,

$$\mathbf{W}_{\alpha,p}\left(\left(\mathbf{W}_{\alpha,p}\sigma\right)^{\frac{(p-1)q}{p-1-q}} d\sigma\right) \le c\left(\mathbf{W}_{\alpha,p}\sigma + \left(\mathbf{W}_{\alpha,p}\sigma\right)^{\frac{p-1}{p-1-q}}\right) < \infty \text{ a.e.}$$

This completes the proof of Theorem 5.11. □

5.4 Proofs of Theorem 5.1, 5.2, 5.3 and Theorem 5.4

Proof of Theorem 5.1. Let $v \in L^{1+q}_{\mathrm{loc}}(\mathbb{R}^n, d\sigma)$ be a solution to the integral equation

$$v = K\mathbf{W}_{1,p}(v^q d\sigma) + r, \tag{5.32}$$

$\liminf_{|x|\to\infty} v(x) = r$, and v satisfies

$$c^{-1}\left(r + \left(\mathbf{W}_{1,p}\sigma\right)^{\frac{p-1}{p-1-q}}\right) \le v \le c\left(r + \left(\mathbf{W}_{1,p}\sigma\right)^{\frac{p-1}{p-1-q}}\right). \tag{5.33}$$

The existence of such a v follows from Theorem 5.9 with some modifications in the constants. We have

$$\int_B \mathbf{W}_{1,p}(v^q d\sigma_B)\, v^q\, d\sigma \leq c \int_B v^{1+q} d\sigma < \infty,$$

for every ball B. By (2.2), we see that $v^q d\sigma \in W^{-1,p'}_{\mathrm{loc}}(\mathbb{R}^n)$.

We let $u_0 = r$ and $B_k = B(0, 2^k)$, where $k = 0, 1, 2 \ldots$. We have $u_0^q d\sigma \in W^{-1,p'}(B_k)$ since $u_0 \leq v$ and $v^q d\sigma \in W^{-1,p'}_{\mathrm{loc}}(\mathbb{R}^n)$. Hence, there exists a unique p-superharmonic solution u_1^k to the equation

$$-\Delta_p u_1^k = \sigma u_0^q \text{ in } B_k\,, \quad u_1^k \geq r,\, u_1^k - r \in W_0^{1,p}(B_k). \tag{5.34}$$

(See, e.g., Theorem 21.6 in [HKM06]). By (2.12), we have

$$u_1^k - r \leq K\mathbf{W}_{1,p}(u_0^q d\sigma).$$

Since $u_0 \leq v$, we get

$$u_1^k \leq K\mathbf{W}_{1,p}(v^q d\sigma) + r = v.$$

We see that the sequence $\{u_1^k\}_k$ is increasing by Lemma 4.15. Letting $u_1 = \lim_{k\to\infty} u_1^k$ and using Theorem 2.5 and the Monotone Convergence Theorem, we deduce that u_1 is a p-superharmonic solution to the equation

$$-\Delta_p u_1 = \sigma u_0^q \text{ in } \mathbb{R}^n.$$

Moreover, $r \leq u_1 \leq v$ since $u_1^k \leq v$ and hence $\liminf_{|x|\to\infty} u_1(x) = r$. Clearly, we also have $u_0 \leq u_1$. We notice that $u_0^q d\sigma \in W^{-1,p'}_{\text{loc}}(\mathbb{R}^n)$ since $u_0 \leq v$ and $v^q d\sigma \in W^{-1,p'}_{\text{loc}}(\mathbb{R}^n)$. Therefore, applying Lemma 2.11, we conclude that $u_1 \in W^{1,p}_{\text{loc}}(\mathbb{R}^n)$.

Let us construct a sequence $\{u_j\}_j$ of p-superharmonic functions in $\mathbb{R}^n$, $u_j \in L^q_{\text{loc}}(\mathbb{R}^n, d\sigma)$, so that

$$\begin{cases} -\Delta_p u_j = \sigma u_{j-1}^q \text{ in } \quad \mathbb{R}^n, \quad j = 2, 3, \dots, \\ r \leq u_j \leq v, \; u_j \in W^{1,p}_{\text{loc}}(\mathbb{R}^n), \\ u_{j-1} \leq u_j, \\ \liminf_{|x|\to\infty} u_j(x) = r. \end{cases} \tag{5.35}$$

Suppose that $u_1, ..., u_{j-1}$ have been constructed. We see that $\sigma u_{j-1}^q \in W^{-1,p'}(B_k)$ since $u_{j-1} \leq v$ and $v^q d\sigma \in W^{-1,p'}_{\text{loc}}(\mathbb{R}^n)$. Thus, as before, there exists a unique p-superharmonic solution u_j^k to the equation

$$-\Delta_p u_j^k = \sigma u_{j-1}^q \text{ in } B_k, \quad u_j^k \geq r, \; u_j^k - r \in W_0^{1,p}(B_k). \tag{5.36}$$

Arguing by induction, let u_{j-1}^k be the unique solution of the equation

$$-\Delta_p u_{j-1}^k = \sigma u_{j-2}^q \text{ in } B_k, \quad u_{j-1}^k \geq r, \; u_{j-1}^k - r \in W_0^{1,p}(B_k). \tag{5.37}$$

Since $u_{j-2} \leq u_{j-1}$, by comparison principle, we deduce that

$$u_{j-1}^k \leq u_j^k, \quad \forall\, k \geq 1. \tag{5.38}$$

Using (2.12), we have

$$u_j^k - r \leq K \mathbf{W}_{1,p}(u_{j-1}^q d\sigma).$$

Since $u_{j-1} \leq v$, we obtain

$$u_j^k \leq K\mathbf{W}_{1,p}(v^q d\sigma) + r = v.$$

Again, by the comparison principle, we see that the sequence $\{u_j^k\}_k$ is increasing. Letting $u_j = \lim_{k\to\infty} u_j^k$ and using Theorem 2.5 and the Monotone Convergence Theorem, we see that u_j is a p-superharmonic solution to the equation

$$-\Delta_p u_j = \sigma u_{j-1}^q \text{ in } \mathbb{R}^n.$$

Moreover, $r \leq u_j \leq v$ since $u_j^k \leq v$ and hence $\liminf_{|x|\to\infty} u_j(x) = r$. We also have $u_{j-1} \leq u_j$ since $u_{j-1}^k \leq u_j^k$. We see that $u_{j-1}^q d\sigma \in W_{\text{loc}}^{-1,p'}(\mathbb{R}^n)$ since $u_{j-1} \leq v$ and $v^q d\sigma \in W_{\text{loc}}^{-1,p'}(\mathbb{R}^n)$. Therefore, by Lemma 2.11, it follows that $u_j \in W_{\text{loc}}^{1,p}(\mathbb{R}^n)$. By Theorem 2.5 and the Monotone Convergence Theorem, we deduce that u is a solution to the equation

$$-\Delta_p u = \sigma u^q \text{ in } \mathbb{R}^n.$$

Furthermore, $r \leq u \leq v$, and hence $\liminf_{|x|\to\infty} u(x) = r$. By (5.33), we get

$$u \leq c\Big(r + \mathbf{W}_{1,p}\sigma\Big)^{\frac{p-1}{p-1-q}}.$$

Using Lemma 2.11 again, we conclude that $u \in W_{\text{loc}}^{1,p}(\mathbb{R}^n)$ since $u^q d\sigma \in W_{\text{loc}}^{-1,p'}(\mathbb{R}^n)$. The lower estimate follows from (2.27) and the fact that $u \geq r$. This completes the proof of Theorem 5.1. $\square$

Proof of Theorem 5.2. Suppose that (5.4) and (5.5) hold, then by Theorem 5.10, there exists a nontrivial solution $v \in L^{1+q}_{\rm loc}(\mathbb{R}^n, d\sigma)$ to the equation

$$v = K\mathbf{W}_{1,p}(v^q d\sigma). \tag{5.39}$$

where K is the constant in (2.12) and $\liminf_{|x|\to\infty} v(x) = 0$. Moreover, there exists a constant $c = c(n, p, q, C(\sigma)) > 0$ such that

$$v \le c\left(\mathbf{W}_{1,p}\sigma + (\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}}\right). \tag{5.40}$$

Arguing as in the proof of Theorem 4.1, we deduce the existence of a minimal p-superharmonic solution u to the equation

$$-\Delta_p u = \sigma u^q \text{ in } \mathbb{R}^n.$$

Since $u \le v$, we see that $u \in L^{1+q}_{\rm loc}(\mathbb{R}^n, d\sigma)$. Hence,

$$\int_B \mathbf{W}_{1,p}(u^q d\sigma_B)\, u^q\, d\sigma \le c\int_B u^{1+q} d\sigma < \infty,$$

for every ball B. By the local Wolff's inequality (2.2), we see that $u^q d\sigma \in W^{-1,p'}_{\rm loc}(\mathbb{R}^n)$. Applying Lemma 2.11, we conclude that $u \in W^{1,p}_{\rm loc}(\mathbb{R}^n)$.

Moreover, by (2.27), $u \ge c\,(\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}}$, and consequently,

$$c^{-1}\,(\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}} \le u \le c\left(\mathbf{W}_{1,p}\sigma + (\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}}\right).$$

The conclusion in the case $p \ge n$ follows from Theorem 2.6 (ii). This completes the proof of Theorem 5.2. $\square$

Proof of Theorem 5.3. We notice that (4.4) is understood in the sense

$$u = \mathbf{I}_{2\alpha}(u^q d\sigma) \quad \text{in } \mathbb{R}^n, u \geq 0.$$

Since $\mathbf{I}_{2\alpha}(u^q d\sigma) = \mathbf{W}_{\alpha,2}(u^q d\sigma)$, Theorem 5.3 is just a special case of Theorem 5.10 with $p = 2$. $\square$

Proof of Theorem 5.4. Suppose there exists a p-superharmonic solution u to (5.1) with $r = 0$, and u satisfies (5.13). By Theorem 2.6, $u \geq \frac{1}{K}\mathbf{W}_{1,p}(u^q d\sigma)$. Consequently, by (2.27), $u \geq C(\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}}$; and hence,

$$u \geq c\mathbf{W}_{1,p}((\mathbf{W}_{1,p}\sigma)^{\frac{(p-1)q}{p-1-q}} d\sigma).$$

Therefore,

$$\mathbf{W}_{1,p}((\mathbf{W}_{1,p}\sigma)^{\frac{(p-1)q}{p-1-q}} d\sigma) \leq c\left(\mathbf{W}_{1,p}\sigma + (\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}}\right) < \infty \text{ a.e.}$$

Conversely, suppose that (5.12) holds, then applying Theorem 5.11 and arguing as in the proof of Theorem 4.1, we conclude the proof of Theorem 5.4. $\square$

Remark 5.12. Since we have used the powerfull Wolff potential estimates (see [KM92, KuMi14, La02, TW02b, PV08]), all of the results metioned above remain valid if one replaces the p-Laplacian Δ_p in the model problem (5.1) by a more general quasilinear operator $\mathrm{div}\mathcal{A}(x, \nabla\cdot)$, under standard structural assumptions on $\mathcal{A}(x,\xi)$ which ensure that $\mathcal{A}(x,\xi)\cdot\xi \approx |\xi|^p$ [HKM06, MZ97], or a fully nonlinear operator of k-Hessian type [TW02b, La02, PV08, JV12].

5.5 Radial case

In this section, let $\sigma \in M^+(\mathbb{R}^n)$ be radial. Let $p = 2, 0 < q < 1$ and $r = 0$, equation (5.1) becomes

$$(5.41) \qquad \begin{cases} -\Delta u = \sigma u^q \quad \text{in } \mathbb{R}^n, \\ \liminf_{|x|\to\infty} u(x) = 0. \end{cases}$$

We notice that a necessary condition for the existence of a solution to (5.41) is that σ must be absolutely continuous with respect to the 2-capacity $\text{cap}_2(\cdot)$, hence, σ has no atoms. Suppose that σ is radial and $\mathbf{I}_2\sigma \not\equiv +\infty$, then the Riesz potential $\mathbf{I}_2\sigma$ is given by (see Theorem 9.7 in [LL97])

$$\mathbf{I}_2\sigma(x) = c_n \left(\frac{\sigma(B(0,|x|)}{|x|^{n-2}} + \int_{B(0,|x|)^c} \frac{d\sigma(y)}{|y|^{n-2}} \right), \quad x \neq 0,$$

where $B(0,|x|)^c = \mathbb{R}^n \backslash B(0,|x|)$ and we drop the first term if $x = 0$. For the sake of convenience, the constant c_n will be dropped. We have the following proposition.

Proposition 5.13. *Let $0 < q < 1, n \geq 3$ and $\sigma \in M^+(\mathbb{R}^n)$ is radial. Suppose there exists a nontrivial solution u to (5.41), then*

$$(5.42) \qquad \int_{|y|<1} \frac{d\sigma(y)}{|y|^{(n-2)q}} < \infty \text{ and } \int_{|y|\geq 1} \frac{d\sigma(y)}{|y|^{n-2}} < \infty.$$

Moreover, for all $x \in \mathbb{R}^n$,

$$(5.43) \qquad u(x) \geq c^{-1} \left(\frac{1}{|x|^{n-2}} \left(\int_{B(0,|x|)} \frac{d\sigma(y)}{|y|^{(n-2)q}} \right)^{\frac{1}{1-q}} + \left(\int_{B(0,|x|)^c} \frac{d\sigma(y)}{|y|^{n-2}} \right)^{\frac{1}{1-q}} \right),$$

where $c = c(n,q) > 0$ and we drop the first term when $x = 0$. Conversely, suppose that (5.42) *holds, then there exists a radial solution u to* (5.41) *and u satisfies the upper estimate for all $x \in \mathbb{R}^n$,*

$$u(x) \leq c \left(\frac{1}{|x|^{n-2}} \left(\int_{B(0,|x|)} \frac{d\sigma(y)}{|y|^{(n-2)q}} \right)^{\frac{1}{1-q}} + \left(\int_{B(0,|x|)^c} \frac{d\sigma(y)}{|y|^{n-2}} \right)^{\frac{1}{1-q}} \right).$$

Proof of Proposition 5.13. Suppose that u is a solution of (5.41), then $u = c\,\mathbf{I}_2(u^q d\sigma)$. We first notice that $\mathbf{I}_2\sigma$ is radial. Therefore, the minimal solution to (5.41) constructed in Theorem 4.1 is radial. Thus, we may assume that u is radial. By (2.27),

$$u(x) \geq c\, (\mathbf{I}_2\sigma(x))^{\frac{1}{1-q}}, \quad x \in \mathbb{R}^n,$$

where $c = c(n,q) > 0$. Consequently,

$$u(x) \geq c \left(\int_{B(0,|x|)^c} \frac{d\sigma(y)}{|y|^{n-2}} \right)^{\frac{1}{1-q}}, \quad x \in \mathbb{R}^n.$$

We notice that

$$u(x) = c\,\mathbf{I}_2(u^q d\sigma)(x) = c\,\frac{\int_{B(0,|x|)} u^q d\sigma}{|x|^{n-2}} + c \int_{B(0,|x|)^c} \frac{u^q d\sigma(y)}{|y|^{n-2}}. \tag{5.44}$$

By (4.34), we have

$$||\mathbf{I}_2\nu||_{L^q(d\sigma_{B(0,|x|)})} \leq c \left(\int_{B(0,|x|)} u^q d\sigma \right)^{1-q} \nu(\mathbb{R}^n), \quad \forall \nu \in M^+(\mathbb{R}^n).$$

Let $\nu = \delta_0$, we get

$$\left(\int_{B(0,|x|)} \frac{d\sigma(y)}{|y|^{(n-2)q}}\right)^{\frac{1}{1-q}} \leq c \int_{B(0,|x|)} u^q d\sigma.$$

Therefore, we deduce from (5.44) that

$$u(x) \geq c\, \frac{1}{|x|^{n-2}} \left(\int_{B(0,|x|)} \frac{d\sigma(y)}{|y|^{(n-2)q}}\right)^{\frac{1}{1-q}}.$$

Thus, (5.43) holds, and hence, (5.42) follows since $u \not\equiv +\infty$.

Conversely, suppose that condition (5.42) holds. This implies that

$$\int_{B(0,|x|)} \frac{d\sigma(y)}{|y|^{(n-2)q}} < \infty \text{ and } \int_{|y|\geq|x|} \frac{d\sigma(y)}{|y|^{n-2}} < \infty, \forall x \neq 0.$$

We remark that equation (5.41) is equivalent to $u = \mathbf{I}_2(u^q d\sigma)$. Let

$u_0 = c_0\, (\mathbf{I}_2\sigma)^{\frac{1}{1-q}}$ with a small constant c_0, then $u_0 \leq \mathbf{I}_2(u_0^q d\sigma)$ as before.

$$\text{Let } v(x) = c\left(\frac{1}{|x|^{n-2}}\left(\int_{B(0,|x|)} \frac{d\sigma(y)}{|y|^{(n-2)q}}\right)^{\frac{1}{1-q}} + \left(\int_{B(0,|x|)^c} \frac{d\sigma(y)}{|y|^{n-2}}\right)^{\frac{1}{1-q}}\right).$$

We see that $v \geq \mathbf{I}_2(v^q d\sigma)$. Indeed, for $x \neq 0$, we have

$$\mathbf{I}_2(v^q d\sigma)(x) = \frac{1}{|x|^{n-2}} \int_{|y|<|x|} v^q d\sigma + \int_{|y|\geq|x|} \frac{v^q d\sigma}{|y|^{n-2}}$$

$$\leq c^q\, \frac{1}{|x|^{n-2}} \int_{|y|<|x|} \frac{1}{|y|^{(n-2)q}} \left(\int_{|z|<|y|} \frac{d\sigma(z)}{|z|^{(n-2)q}}\right)^{\frac{q}{1-q}} d\sigma(y)$$

$$+c^q\, \frac{1}{|x|^{n-2}} \int_{|y|<|x|} \left(\int_{|z|\geq|y|} \frac{d\sigma(z)}{|z|^{n-2}}\right)^{\frac{q}{1-q}} d\sigma(y)$$

$$+c^q \int_{|y|\geq|x|} \frac{1}{|y|^{n-2}} \frac{1}{|y|^{(n-2)q}} \left(\int_{|z|<|y|} \frac{d\sigma(z)}{|z|^{(n-2)q}}\right)^{\frac{q}{1-q}} d\sigma(y)$$

$$+c^q \int_{|y|\geq|x|} \frac{1}{|y|^{n-2}} \left(\int_{|z|\geq|y|} \frac{d\sigma(z)}{|z|^{n-2}} \right)^{\frac{q}{1-q}} d\sigma(y)$$

$$:= c^q(I + II + III + IV).$$

Clearly,

$$I \leq \frac{1}{|x|^{n-2}} \left(\int_{|y|<|x|} \frac{d\sigma(y)}{|y|^{(n-2)q}} \right)^{\frac{1}{1-q}}.$$

We write

$$II = \frac{1}{|x|^{n-2}} \int_{|y|<|x|} \left(\int_{|y|\leq|z|<|x|} \frac{d\sigma(z)}{|z|^{n-2}} + \int_{|z|\geq|x|} \frac{d\sigma(z)}{|z|^{n-2}} \right)^{\frac{q}{1-q}} d\sigma(y)$$

$$\leq c_1 \frac{1}{|x|^{n-2}} \int_{|y|<|x|} \left(\int_{|y|\leq|z|<|x|} \frac{d\sigma(z)}{|z|^{n-2}} \right)^{\frac{q}{1-q}} d\sigma(y)$$

$$+c_1 \frac{1}{|x|^{n-2}} \int_{|y|<|x|} d\sigma(y) \left(\int_{|z|\geq|x|} \frac{d\sigma(z)}{|z|^{n-2}} \right)^{\frac{q}{1-q}} = c_1(II_a + II_b).$$

We estimate

$$II_a = \frac{1}{|x|^{n-2}} \int_{|y|<|x|} \left(\int_{|y|\leq|z|<|x|} \frac{d\sigma(z)}{|z|^{(n-2)q}|z|^{(n-2)(1-q)}} \right)^{\frac{q}{1-q}} d\sigma(y)$$

$$\leq \frac{1}{|x|^{n-2}} \int_{|y|<|x|} \left(\int_{|y|\leq|z|<|x|} \frac{d\sigma(z)}{|z|^{(n-2)q}} \right)^{\frac{q}{1-q}} \frac{1}{|y|^{(n-2)q}} d\sigma(y)$$

$$\leq \frac{1}{|x|^{n-2}} \left(\int_{|y|<|x|} \frac{d\sigma(y)}{|y|^{(n-2)q}} \right)^{\frac{1}{1-q}}.$$

Using Young's inequality with exponents $\frac{1}{1-q}$ and $\frac{1}{q}$, we obtain

$$II_b \leq c_2 \left(\left(\frac{1}{|x|^{n-2}} \int_{|y|<|x|} d\sigma(y) \right)^{\frac{1}{1-q}} + \left(\int_{|z|\geq|x|} \frac{d\sigma(z)}{|z|^{n-2}} \right)^{\frac{1}{1-q}} \right).$$

We estimate

$$III \le c_1 \int_{|y|\ge|x|} \frac{1}{|y|^{n-2}|y|^{(n-2)q}} \left(\int_{|z|<|x|} \frac{d\sigma(z)}{|z|^{(n-2)q}}\right)^{\frac{q}{1-q}} d\sigma(y)$$

$$+c_1 \int_{|y|\ge|x|} \frac{1}{|y|^{n-2}|y|^{(n-2)q}} \left(\int_{|x|\le|z|<|y|} \frac{d\sigma(z)}{|z|^{(n-2)q}}\right)^{\frac{q}{1-q}} d\sigma(y)$$

$$\le c_1 \frac{1}{|x|^{(n-2)q}} \left(\int_{|z|<|x|} \frac{d\sigma(z)}{|z|^{(n-2)q}}\right)^{\frac{q}{1-q}} \int_{|y|\ge|x|} \frac{d\sigma(y)}{|y|^{n-2}}$$

$$+c_1 \int_{|y|\ge|x|} \frac{1}{|y|^{n-2}|y|^{(n-2)q}} \left(\int_{|x|\le|z|<|y|} \frac{|z|^{(n-2)(1-q)}d\sigma(z)}{|z|^{n-2}}\right)^{\frac{q}{1-q}} d\sigma(y)$$

$$\le c_1 \frac{1}{|x|^{(n-2)q}} \left(\int_{|z|<|x|} \frac{d\sigma(z)}{|z|^{(n-2)q}}\right)^{\frac{q}{1-q}} \int_{|y|\ge|x|} \frac{d\sigma(y)}{|y|^{n-2}}$$

$$+c_1 \left(\int_{|x|\le|z|} \frac{d\sigma(z)}{|z|^{n-2}}\right)^{\frac{q}{1-q}} \int_{|y|\ge|x|} \frac{d\sigma(y)}{|y|^{(n-2)}}.$$

Using Young's inequality again, we get

$$III \le c_1c_2 \frac{1}{|x|^{n-2}} \left(\int_{|z|<|x|} \frac{d\sigma(z)}{|z|^{(n-2)q}}\right)^{\frac{1}{1-q}} + (c_1c_2 + c_1) \left(\int_{|y|\ge|x|} \frac{d\sigma(y)}{|y|^{n-2}}\right)^{\frac{1}{1-q}}.$$

Clearly,

$$IV \le \left(\int_{|y|\ge|x|} \frac{d\sigma(y)}{|y|^{n-2}}\right)^{\frac{1}{1-q}}.$$

It is easy to show that $v(0) \ge \mathbf{I}_2(v^q d\sigma)(0)$, if c is chosen large enough. Therefore, we obtain that $\mathbf{I}_2(v^q d\sigma) \le v$. Using iterations, sub-supersolutions method and the Monotone Convergence Theorem, we deduce the existence of a radial solution u to the equation $u = \mathbf{I}_2(u^q d\sigma)$. Moreover,

$$u(x) \le c\left(\frac{1}{|x|^{n-2}}\left(\int_{|y|<|x|}\frac{d\sigma(y)}{|y|^{(n-2)q}}\right)^{\frac{1}{1-q}} + \left(\int_{|y|\ge|x|}\frac{d\sigma(y)}{|y|^{n-2}}\right)^{\frac{1}{1-q}}\right).$$

Therefore, u is a solution of (5.41) and we complete the proof of Proposition 5.13. $\square$

Next, we will characterize condition (5.12) when $p = 2$ and σ is radial. Suppose that $\mathbf{I}_2\sigma \not\equiv +\infty$, then for every $a > 0$,

$$\mathbf{I}_2(x) \le c(a,\sigma), \quad \text{for } |x| \ge a.$$

Indeed,

$$\mathbf{I}_2\sigma(x) = \frac{\sigma(B(0,|x|))}{|x|^{n-2}} + \int_{B(0,|x|)^c}\frac{d\sigma(y)}{|y|^{n-2}}$$

$$= \frac{\int_{|y|<a} d\sigma(y)}{|x|^{n-2}} + \frac{\int_{a\le|y|<|x|} d\sigma(y)}{|x|^{n-2}} + \int_{|y|\ge|x|}\frac{d\sigma(y)}{|y|^{n-2}}$$

$$\le \frac{\int_{|y|<a} d\sigma(y)}{|a|^{n-2}} + \int_{a\le|y|<|x|}\frac{d\sigma(y)}{|y|^{n-2}} + \int_{|y|\ge a}\frac{d\sigma(y)}{|y|^{n-2}} \le \frac{\int_{|y|<a} d\sigma(y)}{|a|^{n-2}} + 2\int_{|y|\ge a}\frac{d\sigma(y)}{|y|^{n-2}}.$$

We can also show that $\lim_{|x|\to\infty}\mathbf{I}_2\sigma(x) = 0$. If $\limsup_{|x|\to 0}\mathbf{I}_2\sigma(x) < +\infty$, then by the above observation, we have $\mathbf{I}_2\sigma \in L^\infty(\mathbb{R}^n)$. This implies that condition (5.12) holds with $p = 2$. Therefore, we need to focus on the case where $\limsup_{|x|\to 0}\mathbf{I}_2\sigma(x) = +\infty$. Let us denote

$$\mathbf{K}\sigma(x) = \frac{1}{|x|^{n-2}}\left(\int_{|y|<|x|}\frac{d\sigma(y)}{|y|^{(n-2)q}}\right)^{\frac{1}{1-q}}, x \ne 0.$$

Suppose that (5.12) holds with $p = 2$, by Theorem 5.2, there exists a solution u to (5.41) such that $u \le c(\mathbf{I}_2\sigma + (\mathbf{I}_2\sigma)^{\frac{1}{1-q}})$. On the other hand, by Proposition 5.13,

$u \geq c\,\mathbf{K}\sigma$. Therefore, condition (5.12) implies that

$$\mathbf{K}\sigma \leq c\left(\mathbf{I}_2\sigma + (\mathbf{I}_2\sigma)^{\frac{1}{1-q}}\right) < \infty \text{ a.e.} \tag{5.45}$$

Conversely, suppose that (5.45) holds, then by Proposition 5.13, there exists a solution u to (5.41) such that $u \leq c\left(\mathbf{K}\sigma + (\mathbf{I}_2\sigma)^{\frac{1}{1-q}}\right)$. Thus, $u \leq c\left(\mathbf{I}_2\sigma + (\mathbf{I}_2\sigma)^{\frac{1}{1-q}}\right)$. Hence, using (2.27) yields (5.12) with $p = 2$. Therefore, when $p = 2$, (5.12) holds if and only if (5.45) holds. We have the following proposition.

Proposition 5.14. *Let $\sigma \in M^+(\mathbb{R}^n)$ be radial, and $0 < q < 1$. Suppose that $\limsup_{|x|\to 0} \mathbf{I}_2\sigma(x) = +\infty$. There exists a constant $c > 0$ such that (5.45) holds if and only if $\int_{|y|\geq 1} \frac{d\sigma(y)}{|y|^{n-2}} < \infty$ and*

$$\limsup_{|x|\to 0} \frac{\frac{1}{|x|^{(n-2)(1-q)}} \int_{B(0,|x|)} \frac{d\sigma(y)}{|y|^{q(n-2)}}}{\int_{|y|\geq |x|} \frac{d\sigma(y)}{|y|^{n-2}}} < \infty. \tag{5.46}$$

Proof. Suppose that $\int_{|y|\geq 1} \frac{d\sigma(y)}{|y|^{n-2}} < \infty$ and (5.46) holds, then there exists δ, $0 < \delta < 1$, such that

$$\mathbf{K}\sigma(x) \leq c\,(\mathbf{I}_2\sigma(x))^{\frac{1}{1-q}}, \quad \forall\, 0 < |x| < \delta.$$

When $\delta \leq |x| \leq 1$, we have

$$\frac{1}{|x|^{n-2}}\left(\int_{B(0,|x|)} \frac{d\sigma(y)}{|y|^{(n-2)q}}\right)^{\frac{1}{1-q}} \leq \frac{1}{\delta^{n-2}}\left(\int_{B(0,1)} \frac{d\sigma(y)}{|y|^{(n-2)q}}\right)^{\frac{1}{1-q}}.$$

On the other hand,

$$\mathbf{I}_2\sigma(x) = \frac{1}{|x|^{n-2}}\int_{B(0,|x|)} d\sigma(y) + \int_{|y|\geq|x|} \frac{d\sigma(y)}{|y|^{n-2}} \geq \int_{B(0,\delta)} d\sigma(y) + \int_{|y|\geq 1} \frac{d\sigma(y)}{|y|^{n-2}}.$$

Therefore, there exists a constant $c = c(\delta, \sigma)$ such that

$$\mathbf{K}\sigma(x) \leq c\,\mathbf{I}_2\sigma(x), \quad \text{when } \delta \leq |x| \leq 1.$$

Suppose $|x| \geq 1$,

$$\int_{B(0,|x|)} \frac{d\sigma(y)}{|y|^{(n-2)q}} \leq \int_{0\leq|y|\leq 1} \frac{d\sigma(y)}{|y|^{(n-2)q}} + \int_{1\leq|y|<|x|} \frac{d\sigma(y)}{|y|^{(n-2)q}}.$$

By Hölder's inequality,

$$\int_{1\leq|y|<|x|} \frac{d\sigma(y)}{|y|^{(n-2)q}} \leq \left(\int_{1\leq|y|<|x|} d\sigma(y)\right)^{1-q} \left(\int_{1\leq|y|<|x|} \frac{d\sigma(y)}{|y|^{n-2}}\right)^{q}.$$

Thus,

$$\left(\int_{1\leq|y|<|x|} \frac{d\sigma(y)}{|y|^{(n-2)q}}\right)^{\frac{1}{1-q}} \leq \left(\int_{1\leq|y|<|x|} d\sigma(y)\right) \left(\int_{1\leq|y|<|x|} \frac{d\sigma(y)}{|y|^{n-2}}\right)^{\frac{q}{1-q}}$$

$$\leq \left(\int_{|y|<|x|} d\sigma(y)\right) \left(\int_{|y|\geq 1} \frac{d\sigma(y)}{|y|^{n-2}}\right)^{\frac{q}{1-q}}.$$

Hence, there exists a constant $c = c(\sigma, q)$ such that

$$\mathbf{K}\sigma(x) \leq c\,\mathbf{I}_2\sigma(x), \quad \text{for } |x| \geq 1.$$

Thus,

$$\mathbf{K}\sigma(x) \leq c\left(\mathbf{I}_2\sigma(x) + (\mathbf{I}_2\sigma(x))^{\frac{1}{1-q}}\right) < \infty, \quad \forall\, x \neq 0.$$

Conversely, suppose that (5.45) holds and $\limsup_{|x|\to 0} \mathbf{I}_2\sigma(x) = +\infty$, then for $|x|$

small enough we have

$$\mathbf{I}_2\sigma(x) \le (\mathbf{I}_2\sigma(x))^{\frac{1}{1-q}}.$$

Consequently,

$$\mathbf{K}\sigma(x) \le c\,(\mathbf{I}_2\sigma(x))^{\frac{1}{1-q}},$$

when $|x|$ is close to 0, i.e.,

$$\frac{1}{|x|^{(n-2)(1-q)}}\int_{|y|<|x|}\frac{d\sigma(y)}{|y|^{(n-2)q}} \le c\left(\frac{1}{|x|^{n-2}}\int_{B(0,|x|)} d\sigma(y) + \int_{|y|\ge|x|}\frac{d\sigma(y)}{|y|^{n-2}}\right). \tag{5.47}$$

We estimate

$$\frac{1}{|x|^{n-2}}\int_{B(0,|x|)} d\sigma(y) = \frac{1}{|x|^{(n-2)(1-q)}|x|^{(n-2)q}}\int_{0\le|y|\le\delta}\frac{|y|^{(n-2)q}\,d\sigma(y)}{|y|^{(n-2)q}}$$

$$+\frac{1}{|x|^{n-2}}\int_{\delta<|y|<|x|}\frac{|y|^{n-2}\,d\sigma(y)}{|y|^{n-2}} \le \frac{\delta^{(n-2)q}}{|x|^{(n-2)(1-q)}|x|^{(n-2)q}}\int_{0\le|y|\le\delta}\frac{d\sigma(y)}{|y|^{(n-2)q}}$$

$$+\int_{\delta<|y|<|x|}\frac{d\sigma(y)}{|y|^{n-2}}.$$

Let us pick $\delta = c_1\,|x|$ where c_1 is small enough, then we obtain

$$\frac{1}{|x|^{n-2}}\int_{B(0,|x|)} d\sigma(y) \le c_1^{(n-2)q}\,\frac{1}{|x|^{(n-2)(1-q)}}\int_{0\le|y|<|x|}\frac{d\sigma(y)}{|y|^{(n-2)q}}$$

$$+\int_{|y|>c_1\,|x|}\frac{d\sigma(y)}{|y|^{n-2}}.$$

Thus, by (5.47), we get

$$\frac{1}{|x|^{(n-2)(1-q)}}\int_{B(0,|x|)}\frac{d\sigma(y)}{|y|^{q(n-2)}} \le c\,c_1^{(n-2)q}\frac{1}{|x|^{(n-2)(1-q)}}\int_{B(0,|x|)}\frac{d\sigma(y)}{|y|^{q(n-2)}}$$

$$+c\int_{|y|\geq c_1\,|x|}\frac{d\sigma(y)}{|y|^{n-2}}+c\int_{|y|\geq|x|}\frac{d\sigma(y)}{|y|^{n-2}}.$$

Hence, if $c\,c_1^{(n-2)q}\leq\frac{1}{2}$ and $c_1<1$, then

$$\frac{1}{|x|^{(n-2)(1-q)}}\int_{B(0,|x|)}\frac{d\sigma(y)}{|y|^{q(n-2)}}\leq 4c\int_{|y|\geq c_1\,|x|}\frac{d\sigma(y)}{|y|^{n-2}}.$$

This implies that

$$\frac{1}{|x|^{(n-2)(1-q)}}\int_{B(0,c_1|x|)}\frac{d\sigma(y)}{|y|^{q(n-2)}}\leq 4c\int_{|y|\geq c_1\,|x|}\frac{d\sigma(y)}{|y|^{n-2}}.$$

Letting $\tilde{x}=c_1\,x$, we obtain

$$\frac{1}{|\tilde{x}|^{(n-2)(1-q)}}\int_{B(0,\,|\tilde{x}|)}\frac{d\sigma(y)}{|y|^{q(n-2)}}\leq\frac{4c}{c_1^{(n-2)(1-q)}}\int_{|y|\geq|\tilde{x}|}\frac{d\sigma(y)}{|y|^{n-2}}.$$

Therefore, (5.46) holds and we complete the proof of Proposition 5.14. □

Let us now give a counter example showing that there exists a measure $\sigma\geq 0$ such that $\mathbf{K}\sigma<\infty$, $\mathbf{I}_2\sigma<\infty$, i.e., there exists a solution u to (5.41) but condition (5.12) fails. Let

$$\sigma(y)=\begin{cases}\frac{1}{|y|^s(\log\frac{1}{|y|})^\beta}, & \text{if } |y|<1/2,\\ 0, & \text{if } |y|\geq 1/2,\end{cases}\tag{5.48}$$

where $2<s<n$, $s+(s-2)\frac{q}{1-q}=n$, and $\beta>1$.

For such a σ, it is easy to show that $\mathbf{K}\sigma<\infty$ and $\mathbf{I}_2\sigma<\infty$. Indeed, we just need to verify condition (5.42). Clearly, $\sigma\in L^1_{\mathrm{loc}}(dx)$ since $\int_{B(0,\frac{1}{2})}\sigma(y)dy<\infty$, so

$\sigma \in M^+(\mathbb{R}^n)$. Obviously,

$$\int_{|y|\geq 1} \frac{d\sigma(y)}{|y|^{n-2}} = 0 < \infty.$$

On the other hand, a direct calculation gives us

$$\int_{|y|<1} \frac{d\sigma(y)}{|y|^{(n-2)q}} = \int_{|y|<\frac{1}{2}} \frac{dy}{|y|^{(n-2)q}|y|^s(\log\frac{1}{|y|})^\beta} < \infty.$$

Thus, (5.42) holds, i.e, $\mathbf{K}\sigma < \infty$ and $\mathbf{I}_2\sigma < \infty$.

By the above proposition, it suffices to show that (5.46) fails. Indeed, for $|x|$ small, we can show that

$$\int_{B(0,|x|)} \frac{d\sigma(y)}{|y|^{(n-2)q}} = \int_0^{|x|} \frac{s^{n-1}d\sigma(s)}{s^{(n-2)q}} \approx \frac{|x|^{(n-2)(1-q)}(\log\frac{1}{|x|})^{1-q}}{|x|^{s-2}(\log\frac{1}{|x|})^\beta},$$

and

$$\int_{|y|\geq|x|} \frac{d\sigma(y)}{|y|^{n-2}} = \int_{|x|}^\infty s\,d\sigma(s) \approx \frac{1}{|x|^{s-2}(\log\frac{1}{|x|})^\beta}.$$

Therefore,

$$\limsup_{|x|\to 0} \frac{\frac{1}{|x|^{(n-2)(1-q)}}\int_{B(0,|x|)}\frac{d\sigma(y)}{|y|^{q(n-2)}}}{\int_{|y|\geq|x|}\frac{d\sigma(y)}{|y|^{n-2}}} \approx \limsup_{|x|\to 0}\left(\log\frac{1}{|x|}\right)^{1-q} = +\infty,$$

as claimed.

Appendix A

Uniqueness of solutions to quasilinear PDEs

In this appendix, we prove the uniqueness property of solutions to the equation

$$
\text{(A.1)} \qquad \begin{cases} -\Delta_p u = \mu, \text{ in } \Omega. \\ \min\{u, k\} \in W_0^{1,p}(\Omega), \quad \forall\, k \in \mathbb{N}, \end{cases}
$$

where Ω is an open bounded domain, $\mu \in M^+(\Omega), \mu(\Omega) < \infty$ and μ is absolutely continuous with respect to p-capacity. The proof follows from Theorem 6.6 in [Mik96], Theorem 3.3 in [Kil99] and Theorem 3.3 in [BGO96].

Theorem A.1. *Let Ω be a bounded open set in $\mathbb{R}^n$ and $1 < p < n$. Suppose that μ, ν are nonnegative finite Borel measures on Ω and μ, ν are absolutely continuous with respect to p-capacity. Then there exist u, v which are renormalized solutions to the equations $-\Delta_p u = \mu$ and $-\Delta_p v = \nu$ in Ω such that $T_k(u), T_k(v) \in W_0^{1,p}(\Omega)$, $\forall\, k \in \mathbb{N}$. Moreover, if $\mu \leq \nu$ then $u \leq v$ a.e. in Ω. Consequently, equation* (A.1) *has a unique* *solution.*

Before proving this theorem, let us state some useful lemmas. Let

$$T_k(s) = \max\{\min\{s,k\},-k\},\ T_k^+(s) = \max\{\min\{s,k\},0\},$$

and $T_k^-(s) = \min\{\max\{s,-k\},0\}$. Clearly,

$$T_k(s) = T_k^+(s) + T_k^-(s),\ \ T_k^+(s) = -T_k^-(-s),$$

and

$$T_k^+(s+t) \le T_k^+(s) + T_k^+(t),\ \ T_k^-(s+t) \ge T_k^-(s) + T_k^-(t).$$

Lemma A.2. *Suppose u is a p-superharmonic solution to* (A.1). *Then for each $M > 0$ and $k > 0$,*

$$\int_{k \le u \le k+M} |Du|^p \, dx \le c\, M \mu(\{u > k\}). \tag{A.2}$$

Proof. Testing (A.1) with $T_M(u - T_k u) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ yields (A.2).(See also [KX96, Mik96].) $\square$

Corollary A.3. *Suppose u is a p-superharmonic solution to* (A.1). *Then*

$$\lim_{k\to\infty} \int_{k \le u \le k+M} |Du|^p \, d\,x = 0. \tag{A.3}$$

Proof. We have $\mathrm{cap}_p(\{u = +\infty\}) = 0$ since u is p-superharmonic ([HKM06]). Hence $\mu(\{u = +\infty\}) = 0$ since $\mu << \mathrm{cap}_p$. Combining this with (A.2) yield (A.3). $\square$

Proof of Theorem A.1. Let us extend μ to $\mathbb{R}^n$ by setting $\mu(\mathbb{R}^n \backslash \Omega) = 0$. For some $R > 0$, we have

$$\mu(\{x \in \mathbb{R}^n : \mathbf{W}_{1,p}^R \,\mu(x) = +\infty\}) = 0,$$

since $\mu << \mathrm{cap}_p$ and $\mathrm{cap}_p(\{x \in \mathbb{R}^n : \mathbf{W}_{1,p}^R \,\mu(x) = +\infty\}) = \mathrm{cap}_p(\{u = +\infty\}) = 0$. Let $E_j = \{x \in \mathbb{R}^n : \mathbf{W}_{1,p}^R \,\mu(x) \leq j\}$. Let μ_j be the restriction of μ to E_j. Then $0 \leq \mu_j \leq \mu_{j+1} \leq \mu$. By the Dominated Convergence Theorem,

$$\lim_{j \to \infty} \int_{\mathbb{R}^n} \varphi \, d\,\mu_j = \int_{\mathbb{R}^n} \varphi \, d\,\mu,$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$, hence $\mu_j \to \mu$ weakly. We have

$$\int_{\mathbb{R}^n} \mathbf{W}_{1,p}^R \mu_j \, d\mu_j \leq \int_{\mathbb{R}^n} j \, d\mu_j \leq j \, \mu(\Omega) < \infty,$$

and so $\mu_j \in W^{-1,p'}(\Omega)$ by Wolff's inequality. By a well known result (see, e.g., Corollary 2.7 in [Mik96] or Theorem 21.6 in [HKM06]), there exists a unique nonnegative p-superharmonic function $u_j \in W_0^{1,p}(\Omega)$ such that $-\Delta_p u_j = \mu_j$.

Applying Theorem 2.4, there exists a subsequence of u_j which is labeled again by u_j such that $u_j \to u$ a.e. We have

$$\text{(A.4)} \qquad \int_\Omega |\nabla u_j|^{p-2} \nabla u_j \cdot \nabla \phi \, dx = \int_\Omega \phi \, d\,\mu_j, \quad \forall \phi \in W_0^{1,p}(\Omega).$$

Letting $\phi = \min\{u_j, k\} \in W_0^{1,p}(\Omega)$, we get

$$\text{(A.5)} \qquad \begin{aligned} \int_\Omega |\nabla(\min\{u_j, k\})|^p &\leq \int_\Omega |\nabla u_j|^{p-2} \nabla u_j \cdot \nabla(\min\{u_j, k\}) \, dx \\ &= \int_\Omega \min\{u_j, k\} \, d\,\mu_j \leq k\, \mu_j(\Omega) \leq k\, \mu(\Omega). \end{aligned}$$

By Poincaré inequality, there holds

$$\int_\Omega \min\{u_j, k\}\, dx \leq c\, k,$$

where c is independent of k and j. Thus $u = \lim_{j\to\infty} u_j < \infty,$ a.e.. Applying the weak continuity of the p-Laplacian, we obtain $-\Delta_p u = \mu$.

Moreover, for each $k \in \mathbb{N}, \min\{u, k\} \in W_0^{1,p}(\Omega)$ since $\min\{u_j, k\}$ is uniformly bounded in $W_0^{1,p}(\Omega)$ by (A.5).

Suppose $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, then $T_k^+(u-\varphi) \in W_0^{1,p}(\Omega)$ and

$$\int_\Omega |\nabla u_j|^{p-2}\nabla u_j \cdot \nabla T_k^+(u-\varphi) dx = \int_\Omega T_k^+(u-\varphi) d\,\mu_j.$$

Let $M = \sup|\varphi|$; then we have $u_j \leq u \leq k+M$ in $\{|u-\varphi| < k\}$ and $\nabla T_k^+(u-\varphi) = 0$ a.e. in $\{|u-\varphi| \geq k\}$. Consequently,

$$\int_\Omega |\nabla \min\{u_j, k+M\}|^{p-2}\nabla \min\{u_j, k+M\} \cdot \nabla T_k^+(u-\varphi) dx$$

$$= \int_\Omega |\nabla u_j|^{p-2}\nabla u_j \cdot \nabla T_k^+(u-\varphi) dx = \int_\Omega T_k^+(u-\varphi) d\mu_j.$$

By (A.5), $\{\nabla \min\{u_j, k+M\}\}_{j\in\mathbb{N}}$ is bounded in $L^p(\Omega)$ and $\nabla \min\{u_j, k+M\} \to \nabla \min\{u, k+M\}$ as $j \to \infty$. We deduce that $|\nabla \min\{u_j, k+M\}|^{p-2}\nabla \min\{u_j, k+M\}$ converges weakly to $|\nabla \min\{u, k+M\}|^{p-2}\nabla \min\{u, k+M\}$ in $L^{p'}(\Omega)$. In particular, since $\nabla T_k^+(u-\varphi) \in L^p(\Omega)$, we get

$$\int_\Omega |Du|^{p-2}Du \cdot \nabla T_k^+(u-\varphi) dx$$

$$= \int_\Omega |\nabla \min\{u, k+M\}|^{p-2}\nabla \ \min\{u, k+M\} \cdot \nabla T_k^+(u-\varphi)dx$$

$$= \lim_{j\to\infty} \int_\Omega |\nabla \min\{u_j, k+M\}|^{p-2}\nabla \ \min\{u_j, k+M\} \cdot \nabla T_k^+(u-\varphi)dx$$

$$= \lim_{j\to\infty} \int_\Omega T_k^+(u-\varphi)d\mu_j = \int_\Omega T_k^+(u-\varphi)d\mu,$$

where in the last term we have used the Dominated Convergence Theorem. Thus,

$$\int_\Omega |D\,u|^{p-2}D\,u\ \cdot \nabla T_k^+(u-\varphi)dx = \int_\Omega T_k^+(u-\varphi)d\,\mu. \tag{A.6}$$

Similarly, we obtain

$$\int_\Omega |D\,v|^{p-2}D\,v\ \cdot \nabla T_k^-(v-\psi)dx = \int_\Omega T_k^-(v-\psi)d\,\nu. \tag{A.7}$$

Plugging $\varphi = T_m(v)$ and $\psi = T_m(u)$ into (A.6) and (A.7) respectively, we have

$$\int_\Omega |D\,u|^{p-2}D\,u\ \cdot \nabla T_k^+(u-T_m(v))dx = \int_\Omega T_k^+(u-T_m(v))d\,\mu,$$

$$\int_\Omega |D\,v|^{p-2}D\,v\ \cdot \nabla T_k^-(v-T_m(u))dx = \int_\Omega T_k^-(v-T_m(u))d\,\nu.$$

Adding these equations up, we obtain

$$\begin{aligned} \int_\Omega |D\,u|^{p-2}D\,u\ \cdot \nabla T_k^+(u-T_m(v))dx + \int_\Omega |D\,v|^{p-2}D\,v\ \cdot \nabla T_k^-(v-T_m(u))\,dx \\ = \int_\Omega T_k^+(u-T_m(v))d\,\mu + \int_\Omega T_k^-(v-T_m(u))d\,\nu. \end{aligned} \tag{A.8}$$

Notice that $T_k^+(s) = -T_k^-(-s)$. Letting $m \to \infty$ and using the Dominated Convergence Theorem, the right hand side tends to

$$\int_\Omega T_k^+(u-v)d\,\mu + \int_\Omega T_k^-(v-u))d\,\nu = \int_\Omega T_k^+(u-v)d\,\mu - \int_\Omega T_k^+(u-v))d\,\nu \le 0.$$

Let us split Ω into five regions $\Omega = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5$, where

$$A_1 = \{|u-v| \le k,\, 0 \le u \le m,\, \le v \le m\},\, A_2 = \{|u-v| \le k,\, u > m,\, v > m\},$$

$$A_3 = \{|u-v| \le k,\, \le v < m,\, u > m\},\, A_4 = \{|u-v| \le k,\, 0 \le u < m,\, v > m\},$$

and $A_5 = \{|u-v| > k\}$.

In A_3, we have $T_m(v) = v$, so

$$\left|\int_{A_3} |D\,u|^{p-2} D\,u\ \cdot \nabla T_k^+(u - T_m(v))dx\right| = \left|\int_{A_3} |D\,u|^{p-2} D\,u\ \cdot \nabla T_k^+(u-v)\,dx\right|$$

$$= \left|\int_{A_3} |D\,u|^{p-2} D\,u\ \cdot (D\,u - D\,v)\,dx\right|.$$

Using Hölder's inequality and the fact $A_3 \subset \{m < u < m+k\}$ and $A_3 \subset \{m-k < v < m\}$, we get

$$= \left|\int_{A_3} |D\,u|^{p-2} D\,u\ \cdot (D\,u - D\,v)\,dx\right| \le \int_{A_3} |D\,u|^p\,dx + \left(\int_{A_3} |D\,u|^p dx\right)^{\frac{1}{p'}} \left(\int_{A_3} |D\,v|^p\,dx\right)^{\frac{1}{p}}$$

$$\le \int_{m<u<m+k} |D\,u|^p\,dx + \left(\int_{m<u<m+k} |D\,u|^p dx\right)^{\frac{1}{p'}} \left(\int_{m-k<v<m} |D\,v|^p\,dx\right)^{\frac{1}{p}}.$$

Notice that the last term converges to 0 as $m \to \infty$ by Corollary A.3. Next, we estimate

$$\left|\int_{A_3} |D\,v|^{p-2} D\,v\;\cdot \nabla T_k^-(v - T_m(u)) dx\right| = \left|\int_{A_3} |D\,v|^{p-2} D\,v\;\cdot \nabla T_k^-(v - m)\,dx\right|$$

$$= \int_{A_3} |D\,v|^p dx \le \int_{m-k<v<m} |D\,v|^p\,dx \to 0,$$

as $m \to \infty$ by Corollary A.3.

Similarly, we can show that the integrals over A_4 tend to zeros as $m \to \infty$. Now we estimate

$$I := \left|\int_{A_5} |D\,u|^{p-2} D\,u\;\cdot \nabla T_k^+(u - T_m(v)) dx\right|.$$

If $T_m(v) = v$ then $\nabla T_k^+(u - T_m(v)) = \nabla T_k^+(u - v) = 0$ since over A_5, $|u - v| > k$. So we just need to consider the case when $T_m(v) \neq v$, i.e., $T_m(v) = \pm\, m$. If $T_m(v) = m$, we also need to integrate over $|u - m| < k$, otherwise the integral is 0. We have $|u| \le |u - m| + m < k + m$ and $|u| \ge m - |u - m| > m - k$. Therefore,

$$I = \left|\int_{A_5} |D\,u|^{p-2} D\,u\;\cdot \nabla T_k^+(u - m)\,dx\right| = \int_{A_5} |D\,u|^p\,dx$$

$$\le \int_{m-k\le|u|\le m+k} |D\,u|^p\,dx \to 0 \quad \text{ as } m \to \infty.$$

In A_2, $T_m(v) = m$ and if we choose $u > 2m$ and $m > k$ then $u - T_m(v) = u - m > m > k$, thus $\nabla T_k^+(u - T_m(v)) = 0$. Similarly, $\nabla T_k^+(v - T_m(u)) = 0$ over A_2. Therefore, the integrals over A_2 are zero.

Notice that in A_1, if $u \le v$ then $T_k^+(u - T_m(v)) = T_k^+(u - v) = 0$. Thus, we just need to restrict A_1 in the set $\{|u-v|<k,\, 0\le u\le m, 0\le v\le m, u>v\}$. Therefore,

$$\lim_{m\to\infty}\int_{\{|u-v|<k,\,0\le u\le m,0\le v\le m,u\ge v\}}(|D\,u|^{p-2}D\,u-|D\,v|^{p-2}D\,v)\,\cdot(Du-Dv)\,dx$$

is finite and nonpositive. Since

$$(|D\,u|^{p-2}D\,u-|D\,v|^{p-2}D\,v)\,\cdot(Du-Dv)\ge c|Du-Dv|^p\ge 0,$$

applying Fatou's lemma, we obtain

$$\int_{\{|u-v|<k,u\ge v\}}(|D\,u|^{p-2}D\,u-|D\,v|^{p-2}D\,v)\,\cdot(Du-Dv)\,dx\le 0.$$

Thus $Du=Dv$ a.e. in $\{|u-v|<k, u>v\}$. Letting $k\to\infty$, we deduce $Du=Dv$ a.e. in $\{u>v\}$. We show now $|\{u>v\}|=0$. Indeed, if that is not the case, we have $Du=\nabla T_k(u)$ in $\{|u|<k\}$ and $Dv=\nabla T_k(v)$ in $\{|v|<k\}$, and consequently, $\nabla T_k(u)=\nabla T_k(v)$, in $\{|u|<k, |v|<k, u>v\}$. Notice that $T_k(u), T_k(v)\in W_0^{1,p}(\Omega)$. Hence, $T_k(u)=T_k(v)$ a.e. in $\{|u|<k, |v|<k, u>v\}$, i.e., $u=v$ in $\{|u|<k, |v|<k, u>v\}$, a contradiction. Hence, $u\le v$ a.e in Ω. The uniqueness of solutions to (A.1) immediately follows from the previous arguments. This completes the proof of Theorem A.1. $\square$

Appendix B

Weak compactness in $L_0^{1,p}(\mathbb{R}^n)$

We prove the following theorem concerning weak compactness in $L_0^{1,p}(\mathbb{R}^n)$ (see Theorem 1.33 in [HKM06]).

Theorem B.1. *Let $1 < p < n$. Suppose that $u_j \in L_0^{1,p}(\mathbb{R}^n)$ is a sequence converging to u a.e. If the sequence ∇u_j is bounded in $L^p(\mathbb{R}^n)$ then $u \in L_0^{1,p}(\mathbb{R}^n)$ and $\nabla u_j \to \nabla u$ weakly in $L^p(\mathbb{R}^n)$.*

To prove this theorem, we recall first the Mazur lemma.

Lemma B.2 (Mazur lemma)**.** *If X is a normed space and a sequence x_j converges weakly to x in X, then there exists a sequence $\tilde{x}_j$ of convex combinations of x_j,*

$$\tilde{x}_j = \sum_{k=1}^{j} \lambda_{j,k} x_k, \quad \lambda_{j,k} \geq 0, \quad \sum_{k=1}^{j} \lambda_{j,k} = 1,$$

such that $\tilde{x}_j \to x$ in the norm topology of X.

See [Yo80], page 120. We need the following theorem.

Theorem B.3 (Theorem 1.32 in [HKM06]). *Let Ω be an open set in $\mathbb{R}^n, 1 < p < n$. Suppose that u_j is a bounded sequence in $W^{1,p}(\Omega)$ such that $u_j \to u$ a.e. Then $u \in W^{1,p}(\Omega)$, $u_j \to u$ weakly in $L^p(\Omega)$ and $\nabla u_j \to \nabla u$ weakly in $L^p(\Omega)$. Furthermore, if $u_j \in W_0^{1,p}(\Omega)$ then $u \in W_0^{1,p}(\Omega)$.*

Proof of Theorem B.1. Let $D \subset\subset \mathbb{R}^n$. By hypotheses, $\sup_j ||u_j||_{L^{\frac{np}{n-p}}(\mathbb{R}^n)} < \infty$, and hence u_j is uniformly bounded in $W^{1,p}(D)$. By Theorem B.3 we have $u \in W^{1,p}(D)$ and

$$\nabla u_j \to \nabla u \text{ weakly in } L^p(D). \tag{B.1}$$

Note that ∇u_j is bounded in $L^p(\mathbb{R}^n)$, so by weak compactness in $L^p(\mathbb{R}^n)$, there exists a subsequence ∇u_{j_k} such that $\nabla u_{j_k} \to v$ weakly in $L^p(\mathbb{R}^n)$. Notice that

$$\nabla u_{j_k} \to \nabla u \text{ weakly in } L^p(D).$$

Thus $v = \nabla u$ a.e. in D for every $D \subset\subset \mathbb{R}^n$ and hence $v = \nabla u$ a.e. in $\mathbb{R}^n$. So

$$\nabla u_{j_k} \to \nabla u \text{ weakly in } L^p(\mathbb{R}^n).$$

Since the weak limit is independent of the choice of subsequence, it follows that

$$\nabla u_j \to \nabla u \text{ weakly in } L^p(\mathbb{R}^n).$$

To show $u \in L_0^{1,p}(\mathbb{R}^n)$, we now use the Mazur lemma. There exists a sequence $v_j \in L_0^{1,p}(\mathbb{R}^n)$ which is a convex combinations of u_j such that $\nabla v_j \to \nabla u$ in $L^p(\mathbb{R}^n)$.

Given $\varepsilon > 0$ we can find such a j and $\varphi \in C_0^\infty(\mathbb{R}^n)$ so that

$$\int_{\mathbb{R}^n} |\nabla v_j - \nabla u|^p dx < (\frac{\varepsilon}{2})^p, \quad \int_{\mathbb{R}^n} |\nabla \varphi - \nabla v_j|^p dx < (\frac{\varepsilon}{2})^p.$$

Thus,

$$\left(\int_{\mathbb{R}^n} |\nabla \varphi - \nabla u|^p dx \right)^{\frac{1}{p}} < \varepsilon.$$

Consequently, $u \in L_0^{1,p}(\mathbb{R}^n)$ by definition. $\square$

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