Ann. Phys. (Berlin) **17**, No. 9 – 10, 691 – 704 (2008) / **DOI** 10.1002/andp.200810320

Maxwell's equations in Minkowski's world: their premetric generalization and the electromagnetic energy-momentum tensor

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Received 17 July 2008, revised 25 July 2008, accepted 25 July 2008 by U. Eckern
Published online 21 August 2008

Key words Classical electrodynamics, Maxwell's equations, Minkowski, premetric electrodynamics, energy-momentum tensor.

PACS 03.50.De, 06.30.Ka, 01.65.+g

In December 1907, Minkowski expressed the Maxwell equations in the very beautiful and compact 4-dimensional form: lor $f = -s$, lor $F^* = 0$. Here 'lor', an abbreviation of Lorentz, represents the 4-dimensional differential operator. We study Minkowski's derivation and show how these equations generalize to their modern premetric form in the framework of tensor and exterior calculus (valid also in general relativity). After mentioning some applications of premetric electrodynamics, we turn to Minkowski's discovery of the energy-momentum tensor of the electromagnetic field. We discuss how he arrived at it and how its premetric formulation looks like.

1 Introduction

In September 1908, Minkowski [40] presented a 4-dimensional geometrical version of special relativity theory; for the historical context, one may compare Walter [63]. A bit earlier [39], in December 1907, he gave a preliminary version, and he expressed Maxwell's theory of electrodynamics in a 4-dimensional form. For the Maxwell equations he found, as pointed out by Damour [7] in his historical analysis, the very compact representation

$$\boxed{\mathrm{lor}\, f = -s \qquad \text{and} \qquad \mathrm{lor}\, F^* = 0\,.} \tag{1}$$

Here $\mathrm{lor}_h := \partial/\partial x_h$ are the components of the 4-dimensional differential operator. The excitation f and the field strength F are represented in Maxwell's nomenclature by[1]

$$(f_{hk}) = \begin{pmatrix} 0 & H_z & -H_y & -iD_x \\ -H_z & 0 & H_x & -iD_y \\ H_y & -H_x & 0 & -iD_z \\ iD_x & iD_y & iD_z & 0 \end{pmatrix}, \quad (F_{hk}) = \begin{pmatrix} 0 & B_z & -B_y & -iE_x \\ -B_z & 0 & B_x & -iE_y \\ B_y & -B_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{pmatrix}, \tag{2}$$

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[1] Minkowski took $D = e$, $H = m$; $E = E$, $B = M$, that is, for the excitation $f \sim (e, m)$ and for the field strength $F \sim (E, M)$. Even though this seems logical and appropriate (e, E correspond to electric and m, M to magnetic), the Maxwellian notation is so deeply ingrained in our subconsciousness that any competing notation seems to have no chance.

with the imaginary unit $i := \sqrt{-1}$. The rows and columns of the matrices in (2) are numbered from 1 to 4. The 4-dimensional electric current is denoted by s. The star * used in $(1)_2$ is a duality operator, that is, $F^*_{hk} := \frac{1}{2}\hat{\epsilon}_{hklm}F_{lm}$, with $\hat{\epsilon}_{hklm} = \pm 1, 0$ as the totally antisymmetric Levi-Civita symbol and $\hat{\epsilon}_{1234} = +1$; moreover, the summation convention is applied. Minkowski demonstrated [39] that the equations in (1) are covariant under arbitrary Poincaré (inhomogeneous Lorentz) transformations.

For vacuum (in the "aether", as Minkowski said), excitation and field strength, in suitable units, are related by

$$f = F \qquad \text{or} \qquad (f_{hk}) = (F_{hk})\,. \tag{3}$$

Accordingly, Minkowski found the correct 4-dimensional representation of the Maxwell equations in the context of special relativity theory.

Later it became clear, perhaps for the first time to Einstein [11], that the Maxwell-Minkowski equations (1) can be generalized such that they become covariant under general coordinate transformations (diffeomorphisms) and metric independent. Then, only the generalization of the constitutive relation (3) remains metric dependent.

In the calculus of exterior forms, the 4-dimensional Maxwell equations for the excitation H and the field strength $\mathcal{F}$ eventually read[2]

$$\boxed{dH = J \qquad \text{and} \qquad d\mathcal{F} = 0\,.} \tag{4}$$

Here d is the exterior differential. The Minkowskian scheme (1) foreshadows in its compactness the exterior calculus version in (4). This point was stressed by Damour [7]. The equations in (4) are premetric[3] and, accordingly, also valid in this form in general relativity and in general relativistic field theories with torsion and curvature [48]. The 2-forms of the excitation $H = \frac{1}{2}\,H_{ij}dx^i \wedge dx^j$ and of the field strength $\mathcal{F} = \frac{1}{2}\,\mathcal{F}_{ij}dx^i \wedge dx^j$, with $i, j = 0, 1, 2, 3$, decompose in 1+3 according to [16]

$$H = -\mathcal{H} \wedge dt + \mathcal{D} \qquad \text{and} \qquad \mathcal{F} = E \wedge dt + B\,. \tag{5}$$

The signs in (5) are *not* conventional. They are rather dictated by the Lenz rule, see Itin et al. [25, 26]. If we put the latter two equations into matrix form in order to be able to compare them later with the results of Minkowski, we find

$$H = (H_{ij}) = \begin{pmatrix} 0 & \mathcal{H}_1 & \mathcal{H}_2 & \mathcal{H}_3 \\ -\mathcal{H}_1 & 0 & \mathcal{D}^3 & -\mathcal{D}^2 \\ -\mathcal{H}_2 & -\mathcal{D}^3 & 0 & \mathcal{D}^1 \\ -\mathcal{H}_3 & \mathcal{D}^2 & -\mathcal{D}^1 & 0 \end{pmatrix}, \quad \mathcal{F} = (\mathcal{F}_{ij}) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B^3 & -B^2 \\ E_2 & -B^3 & 0 & B^1 \\ E_3 & B^2 & -B^1 & 0 \end{pmatrix}. \tag{6}$$

Here $\mathcal{D}^a := \frac{1}{2}\epsilon^{abc}\mathcal{D}_{bc}$ and $B^a := \frac{1}{2}\epsilon^{abc}B_{bc}$, with the 3-dimensional Levi-Civita symbol $\epsilon^{123} = +1$, that is, $\mathcal{D}$ and B, as required by their definitions in physics, are 3-dimensional 2-forms, whereas E and $\mathcal{H}$ are 1-forms. Note that $\mathcal{D}^a$ and B^a are vector *densities*. We stress that the identifications in (6) are premetric, they are valid on any (well-behaved) differential manifold that can be split locally into time and space.

The excitation H is a 2-form *with* twist (sometimes called odd 2-form) and the field strength $\mathcal{F}$ a 2-form without twist (sometimes called even 2-form). These are consequences of the definitions of H and $\mathcal{F}$ via charge conservation and the Lorentz force density, respectively. For this reason, in exterior calculus, the constitutive law for vacuum ("the spacetime relation") reads

$$H = \lambda_0\,{}^\star\mathcal{F}\,. \tag{7}$$

[2] We take the notation of our book [16]. The only exception is that we denote the field strength by $\mathcal{F}$ in order not to mix it up with Minkowksi's F. For other presentations of Maxwell's equations in exterior calculus, see, e.g., Bamberg & Sternberg [5], Choquet-Bruhat et al. [6], Delphenich [8, 9], Frankel [14], Jancewicz [27], Lindell [34], Misner, Thorne & Wheeler [41], Russer [51], Thirring [60], and Trautman [61].

[3] We call a structure *premetric*, if it can be defined "before" a metric is available (or a metric exists, but it is not used).

Here $\lambda_0 = \sqrt{\varepsilon_0/\mu_0} \approx 1/377\,\Omega$ is the vacuum admittance; in the units Minkowski chose, $\lambda_0 = 1$. The metric-dependent Hodge star operator * in (7) maps a 2-form without twist into a form with twist, and vice versa.

In this paper we want to trace the way from Minkowski's system (1) and (2) with (3) to the premetric framework (4) and (6) with the metric-dependent spacetime relation (7). Subsequently, we analyze Minkowski's discovery of the energy-momentum tensor of the electromagnetic field and display its modern premetric version.

2 Minkowski's way to: $\mathbf{lor\, f = -s}$, $\mathbf{lor\, F^* = 0}$

Minkowski first introduced the fields f and F in Cartesian coordinates x, y, z and with an imaginary time coordinate it (he put $c = 1$), see (2). Afterwards he changed to the numbering of coordinates in the following way: $x_1 := x$, $x_2 := y$, $x_3 := z$, $x_4 := it$. His metric is Euclidean $ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 = g_{hk}dx_h dx_k$, with $g_{hk} = \mathrm{diag}(1, 1, 1, 1)$. Sometimes he also seems to use the signature $(-1, -1, -1, -1)$. In any case, as long as Minkowski sticks to Cartesian coordinates—and this is what he was doing—there is no need to distinguish contravariant (upper) from covariant (lower) indices. He displays the conventional Maxwell equations in component form. The inhomogeneous Maxwell equation reads

$$
\begin{aligned}
&& \frac{\partial f_{12}}{\partial x_2} &+ \frac{\partial f_{13}}{\partial x_3} + \frac{\partial f_{14}}{\partial x_4} = s_1\,,\\
\frac{\partial f_{21}}{\partial x_1} &+ && \frac{\partial f_{23}}{\partial x_3} + \frac{\partial f_{24}}{\partial x_4} = s_2\,,\\
\frac{\partial f_{31}}{\partial x_1} &+ \frac{\partial f_{32}}{\partial x_2} && + \frac{\partial f_{34}}{\partial x_4} = s_3\,,\\
\frac{\partial f_{41}}{\partial x_1} &+ \frac{\partial f_{42}}{\partial x_2} &+ \frac{\partial f_{43}}{\partial x_3} & \qquad = s_4\,,
\end{aligned}
\tag{8}
$$

and the homogeneous one

$$
\begin{aligned}
&& \frac{\partial F_{34}}{\partial x_2} &+ \frac{\partial F_{42}}{\partial x_3} + \frac{\partial F_{23}}{\partial x_4} = 0\,,\\
\frac{\partial F_{43}}{\partial x_1} &+ && \frac{\partial F_{14}}{\partial x_3} + \frac{\partial F_{31}}{\partial x_4} = 0\,,\\
\frac{\partial F_{24}}{\partial x_1} &+ \frac{\partial F_{41}}{\partial x_2} && + \frac{\partial F_{12}}{\partial x_4} = 0\,,\\
\frac{\partial F_{32}}{\partial x_1} &+ \frac{\partial F_{13}}{\partial x_2} &+ \frac{\partial F_{21}}{\partial x_3} & \qquad = 0\,.
\end{aligned}
\tag{9}
$$

Using modern notation, including the summation convention, see Schouten [52], we can rewrite (8) and (9) as

$$
\frac{\partial f_{hk}}{\partial x_k} = s_h \qquad \text{and} \qquad \frac{\partial F_{hk}}{\partial x_l} + \frac{\partial F_{kl}}{\partial x_h} + \frac{\partial F_{lh}}{\partial x_k} = 0\,. \tag{10}
$$

Here $h, k, ... = 1, 2, 3, 4$. Minkowski also introduced the dual of F_{hk} according to $F^*_{hk} := \frac{1}{2}\hat{\epsilon}_{hklm}F_{lm}$, with the Levi-Civita symbol $\hat{\epsilon}_{hklm} = \pm 1, 0$ and $\hat{\epsilon}_{1234} = +1$. Explicitly, we have

$$
F^* = (F^*_{hk}) = \begin{pmatrix} 0 & -iE_z & iE_y & B_x \\ iE_z & 0 & -iE_x & B_y \\ -iE_y & iE_x & 0 & B_z \\ -B_x & -B_y & -B_z & 0 \end{pmatrix}. \tag{11}
$$

Then we can put the homogeneous equation in a form reminiscent of the inhomogeneous one (apart from the sources):

$$\frac{\partial F^*_{hk}}{\partial x_k} = 0\,. \tag{12}$$

Subsequently Minkowski develops a type of Cartesian tensor calculus with a 4-dimensional differential operator called 'lor' (abbreviation of Lorentz). He introduces ordinary (co)vectors (space-time vectors of the 1st kind), like x_h and $\mathrm{lor}_h := \frac{\partial}{\partial x_h}$, and antisymmetric 2nd rank tensors (space-time vectors of the 2nd kind), like f_{hk} and F_{hk}. Then, he can represent $(10)_1$ and (12) symbolically as

$$\mathrm{lor}\, f = -s \qquad \text{and} \qquad \mathrm{lor}\, F^* = 0\,, \tag{13}$$

respectively, see (1). Using his Cartesian tensor calculus, Minkowski showed that Eqs. (13) are covariant under Poincaré transformations.

Einstein and Laub [13] thought that Minkowski's presentation of electrodynamics was very demanding from the point of view of the mathematics involved. Therefore, they essentially rederived Minkowski's results in a purely 3-dimensional formalism.

3 Premetric electrodynamics in tensor calculus

The next step in the development of the Maxwell equations occurred immediately after Einstein's fundamental 1915 paper on general relativity and even before his big survey paper on general relativity would appear. Now Einstein was in command of tensor calculus in arbitrary coordinate systems and observed [11] that Maxwell's equations in vacuum can be put in a generally covariant form by picking suitable field variables.

Einstein [11] used the metric $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ with signature $(1,-1,-1,-1)$, here $\mu,\nu,... = 0,1,2,3$, and he wrote Maxwell's equations as[4]

$$\frac{\partial F_{\rho\sigma}}{\partial x^\tau} + \frac{\partial F_{\sigma\tau}}{\partial x^\rho} + \frac{\partial F_{\tau\rho}}{\partial x^\sigma} = 0\,, \quad \mathfrak{F}^{\mu\nu} = \sqrt{-g}\,g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}\,, \quad \frac{\partial \mathfrak{F}^{\mu\nu}}{\partial x^\nu} = \mathcal{J}^\mu\,. \tag{15}$$

The field strength $F_{\rho\sigma}$ is a tensor, the excitation $\mathfrak{F}^{\mu\nu}$ a tensor density. In his "Meaning of Relativity" [12], in the part on general relativity, he even picked the letter ϕ for the field strength (instead of F) apparently in order to stress the different nature of ϕ and $\mathfrak{F}$.

The Maxwell equations $(15)_1$ and $(15)_3$ are generally covariant *and* metric independent. Since in general relativity the metric g is recognized as the gravitational potential, it is quite fitting that the fundamental field equations of electromagnetism do *not* contain the gravitational potential. The equations in (15) are specifically valid in Minkowskian spacetime. Then, of course, the metric occurring in $(15)_2$ is flat, i.e., its curvature vanishes. Hence by transforming Minkowski's $\mathrm{lor}\, f = -s$ and $\mathrm{lor}\, F^* = 0$ to curvilinear coordinates, one can equally well arrive at (15).

[4] Einstein used subscripts for denoting the coordinates x, i.e., x_τ etc. Moreover, we dropped twice the summation symbols Σ. Einstein's identifications, which were only worked out for vacuum, read

$$\mathfrak{F} = (\mathfrak{F}^{\mu\nu}) = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & H_z & -H_y \\ -E_y & -H_z & 0 & H_x \\ -E_z & H_y & -H_x & 0 \end{pmatrix}, \quad F = (F_{\rho\sigma}) = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & H_z & -H_y \\ E_y & -H_z & 0 & H_x \\ E_z & H_y & -H_x & 0 \end{pmatrix}. \tag{14}$$

Einstein discussed already earlier the general covariant form of Maxwell's equations, see [10]. However, the presentation was a bit different from (15) and also by far not as transparent as the one in [11]. Probably for this reason Einstein republished these ideas.

The gravitational potential only enters Eq. $(15)_2$. We call this equation the spacetime relation—it is the "constitutive law" of the vacuum. Needless to say that having understood that there exists a way to formulate Maxwell's equations in a generally covariant and metric-independent manner, going back to (10) would appear to be an anachronism. Einstein's argument in favor of his formalism, which is similar to the Minkowski version (10), and not following $(10)_1$ and (12), was that with his formalism one can derive the energy-momentum tensor of the electromagnetic field in a much more transparent way. This is certainly true. Still, it was Minkowski who discovered the energy-momentum expression [39] by performing products between the two space-time vectors of the 2nd kind f and F.

But the real decisive step was left to Kottler [29]. He derived the Maxwell equations from the conservation laws of electric charge and magnetic flux. The charges in a 3-dimensional volume and the flux lines piercing a 2-dimensional area are conserved. These two independent conservation laws are related to counting procedures and thus are independent of any length or time standard. It is for this reason that the two Maxwell equations $(15)_1$ and $(15)_3$ emerge as a priori independent structures and free of a metric. It is then left to the constitutive law to link the excitation $\mathfrak{F}^{\mu\nu}$ with the field strength $F_{\alpha\beta}$.

In (15) we displayed the Maxwellian system in the way Einstein wrote it. Changing now to modern notation, see Schouten [52] and Post [47], and using the conventions of [16], the premetric Maxwell equations read[5] $(i, j, ... = 0, 1, 2, 3)$

$$\partial_j \check{H}^{ij} = \check{J}^i \qquad \text{and} \qquad \partial_{[i}\mathcal{F}_{jk]} = 0\,. \tag{16}$$

The antisymmetrization bracket is defined according to $[ijk] := \frac{1}{6}(ijk - jik + jki - +...)$ i.e., a plus (minus) sign occurs before even (odd) permutations of the indices ijk. Here $\check{H}^{ij}$ are components of a tensor density, whereas $\mathcal{F}_{ij}$ are those of a tensor. For $\mathcal{F}_{ij}$ we have the identification $(6)_2$ and for $\check{H}^{ij} := \frac{1}{2}\epsilon^{ijkl}H_{kl}$, with H_{kl} identified according to $(6)_1$. Thus, explicitly,[6]

$$\check{H} = (\check{H}^{ij}) = \begin{pmatrix} 0 & \mathcal{D}^1 & \mathcal{D}^2 & \mathcal{D}^3 \\ -\mathcal{D}^1 & 0 & \mathcal{H}_3 & -\mathcal{H}_2 \\ -\mathcal{D}^2 & -\mathcal{H}_3 & 0 & \mathcal{H}_1 \\ -\mathcal{D}^3 & \mathcal{H}_2 & -\mathcal{H}_1 & 0 \end{pmatrix}. \tag{18}$$

If we substitute $\check{H}^{ij}$ into $(16)_1$ and use $J_{ijk} := \frac{1}{6}\hat{\epsilon}_{ijkl}\check{J}^l$, then (16) can be displayed in the more symmetric form of

$$\partial_{[i}H_{jk]} = J_{ijk} \qquad \text{and} \qquad \partial_{[i}\mathcal{F}_{jk]} = 0\,. \tag{19}$$

Today it is not only of academic interest to put Maxwell's equations in a general covariant form. On the surface of a neutron star, for example, where we might have strong magnetic fields of some 10^{10} tesla and a huge curvature of spacetime—we are, after all, not too far outside the Schwarzschild radius of the neutron star of some 3 km—the Maxwell equations keep their form (16). The metric of spacetime enters the vacuum relation, see $(15)_2$. There is no longer place for only a Poincaré covariant formulation of Maxwell's equations à la (10). A less extreme case is the Global Positioning System. Nevertheless, also the GPS rules out Poincaré covariant electrodynamics, see Ashby [4].

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5 Post [47] denotes $\check{H}^{ij}$ by $\mathfrak{G}^{\mu\nu}$. In Obukhov's investigations on the energy-momentum tensor [42], the $\check{H}^{ij}$ is simply named H^{ij}, that is, the ˇ is dropped.

6 If we abandon Minkowski's imaginary time coordinate and go over to x^i, with $i = 0, 1, 2, 3$, then excitation and field strength in Minkowski's framework read

$$\tilde{f} = (\tilde{f}_{hk}) = \begin{pmatrix} 0 & D_x & D_y & D_z \\ -D_x & 0 & H_z & -H_y \\ -D_y & -H_z & 0 & H_x \\ -D_z & H_y & -H_x & 0 \end{pmatrix}, \quad \tilde{F} = (\tilde{F}_{hk}) = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}. \tag{17}$$

3.1 Constitutive relation

The system of the $4+4$ Maxwell equations (16) for the $6+6$ independent components of the electromagnetic field $\check{H}^{ij}$ and $\mathcal{F}_{kl}$ is evidently underdetermined. To complete this system, a constitutive relation of the form

$$\check{H}^{ij} = \check{H}^{ij}(\mathcal{F}_{kl}) \tag{20}$$

has to be assumed. The constitutive relation (20) is independent of the Maxwell equations and its form can be determined by using experimental results. For general *magnetoelectric* media, including the vacuum as special case, we assume with Tamm [36, 57], v. Laue [32], and Post [47] the *local*[7] and *linear* premetric relation

$$\check{H}^{ij} = \frac{1}{2}\,\chi^{ijkl}\mathcal{F}_{kl}\,, \tag{21}$$

where χ^{ijkl} is a *constitutive tensor density* of rank 4 and weight $+1$, with the dimension $[\chi] = 1/\text{resistance}$. Since both $\check{H}^{ij}$ and $\mathcal{F}_{kl}$ are antisymmetric in their indices, we have $\chi^{ijkl} = -\chi^{jikl} = -\chi^{ijlk}$. An antisymmetric pair of indices corresponds, in four dimensions, to six independent components. Thus, the constitutive tensor can be considered as a 6×6 matrix with 36 independent components.

One can take the premetric Maxwell equations (16) together with the premetric ansatz (21) as an electrodynamic framework ("local and linear premetric electrodynamics"), see also Matagne [38]. Therein one can study the propagation of electromagnetic waves in the geometrical optics approximation. It turns out, see [16, 50], that the premetric, totally symmetric *Tamm-Rubilar tensor*[8]

$$\mathcal{G}^{ijkl}(\chi) := \frac{1}{4!}\,\hat{\epsilon}_{mnpq}\,\hat{\epsilon}_{rstu}\,\chi^{mnr(i}\,\chi^{j|ps|k}\,\chi^{l)qtu} \tag{22}$$

—a new structural element in theoretical physics—controls the light propagation via a generalized Fresnel equation $\mathcal{G}^{ijkl}q_iq_jq_kq_l = 0$. The q_i are the components of the wave vector of the light.[9] If birefringence is forbidden, then the light cone emerges [30], that is, the metric g_{ij} up to a (dilation) factor [19].

Let us come back to χ^{ijkl}. As a 6×6 matrix, it can be decomposed in its tracefree symmetric part (20 independent components), its antisymmetric part (15 components), and its trace (1 component). On the level of χ^{ijkl}, this *decomposition* is reflected in

$$\begin{aligned} \chi^{ijkl} &= {}^{(1)}\chi^{ijkl} + {}^{(2)}\chi^{ijkl} + {}^{(3)}\chi^{ijkl}\,. \\ 36 &= \quad 20 \quad\oplus\quad 15 \quad\oplus\quad 1\,. \end{aligned} \tag{23}$$

The third part, the *axion* part, is totally antisymmetric and as such proportional to the Levi-Civita symbol, ${}^{(3)}\chi^{ijkl} := \chi^{[ijkl]} = \alpha\,\epsilon^{ijkl}$. Note that α is a pseudoscalar since ϵ^{ijkl} has weight -1. Therefore, the weight of ϵ^{ijkl} is essential information. The second part, the *skewon* part, is defined according to ${}^{(2)}\chi^{ijkl} := \frac{1}{2}(\chi^{ijkl} - \chi^{klij})$. If the constitutive equation can be derived from a Lagrangian, which is the case as long as only reversible processes are considered, then ${}^{(2)}\chi^{ijkl} = 0$. The *principal* part ${}^{(1)}\chi^{ijkl}$ fulfills the symmetries ${}^{(1)}\chi^{ijkl} = {}^{(1)}\chi^{klij}$ and ${}^{(1)}\chi^{[ijkl]} = 0$. The premetric constitutive relation now reads

$$\check{H}^{ij} = \frac{1}{2}\left({}^{(1)}\chi^{ijkl} + {}^{(2)}\chi^{ijkl} + \alpha\,\epsilon^{ijkl}\right)\mathcal{F}_{kl}\,. \tag{24}$$

[7] A nonlocal generalization of this law is being investigated by Mashhoon, see his review [37].

[8] Parentheses around indices denote total symmetrization, that is, $(ijkl) := \frac{1}{24}\{ijkl + jikl + jkil + \cdots\}$. The covariant epsilon-system is marked by a hat: $\hat{\epsilon}_{ijkl} = \pm 1, 0;\ \hat{\epsilon}_{0123} = +1$.

[9] This scheme was generalized to Abelian gauge and string theory by Schuller, Wohlfahrt, et al., see the articles [49, 53] and the references given there. Also in these generalized schemes a Tamm-Rubilar type tensor plays a decisive role.

If we assume the existence of a metric g_{ij}, then *in vacuum* the constitutive tensor reduces to

$$\chi^{ijkl} = \lambda_0 \sqrt{-g} \left(g^{ik} g^{jl} - g^{il} g^{jk} \right) . \tag{25}$$

This formula is, up to a factor, determined uniquely by the symmetries of the principal part. On substitution of (25) into (21), we find the vacuum relation

$$\check{H}^{ij} = \lambda_0 \sqrt{-g} g^{ik} g^{jl} F_{kl} , \qquad \text{or} \qquad H_{ij} = \frac{\lambda_0}{2} \hat{\epsilon}_{ijmn} \sqrt{-g} g^{mk} g^{nl} F_{kl} , \tag{26}$$

compare $(15)_2$. Note that (25) belongs to the principal part ${}^{(1)}\chi^{ijkl}$.

In order to compare (24) with experiments, we have to split it into time and space parts. As shown in [16, 17] in detail, we can parameterize the *principal* part by the 6 permittivities $\varepsilon^{ab} = \varepsilon^{ba}$, the 6 permeabilities $\mu_{ab} = \mu_{ba}$, and the 8 magnetoelectric pieces $\gamma^a{}_b$ (the trace vanishes, $\gamma^c{}_c = 0$) and the *skewon* part by the 3 permittivities n_a, the 3 permeabilities m^a, and the 9 magnetoelectric pieces $s_a{}^b$. Then, the premetric constitutive relation (24) can be rewritten as

$$\begin{aligned} \mathcal{D}^a &= \left(\varepsilon^{ab} \; - \epsilon^{abc} \, n_c \right) E_b + (\quad \gamma^a{}_b + s_b{}^a - \delta^a_b s_c{}^c) \, B^b + \alpha \, B^a , \\ \mathcal{H}_a &= \left(\mu^{-1}_{ab} - \hat{\epsilon}_{abc} m^c \right) B^b + \left(-\gamma^b{}_a + s_a{}^b - \delta^b_a s_c{}^c \right) E_b - \alpha \, E_a . \end{aligned} \tag{27}$$

Here $\epsilon^{abc} = \hat{\epsilon}_{abc} = \pm 1, 0$ are the 3-dimensional Levi-Civita symbols, with $\epsilon^{123} = \hat{\epsilon}_{123} = +1$. As can be seen from our derivation, α is a 4-dimensional pseudo (or axial) scalar, whereas $s_c{}^c$ is only a 3-dimensional scalar. The cross-term $\gamma^a{}_b$ is related to the Fresnel-Fizeau effects. The skewon contributions m^c, n_c are responsible for electric and magnetic Faraday effects, respectively, whereas the skewon terms $s_a{}^b$ describe optical activity. Equivalent constitutive relations were formulated by Serdyukov et al. [54], p.86, and studied in quite some detail.

3.2 Some applications of premetric structures

The theory of a possible *skewon* part of (24) lies in its infancy, see [21, 44]. The *axion* part was widely neglected in the literature, if not even put to zero right away [47]. As we can see from (24) and (27), the axion piece obeys

$$^{(3)}\check{H}^{ij} = \frac{1}{2} \alpha \, \epsilon^{ijkl} \mathcal{F}_{kl} \qquad \text{or} \qquad \left\{ \begin{array}{l} \mathcal{D} = \quad \alpha B \\ \mathcal{H} = -\alpha E \end{array} \right\} . \tag{28}$$

• Tellegen [58, 59] introduced the *gyrator*, with $\alpha = const$, as a new device in the theory of two-port networks that "rotates" the field strength (B, E) into the excitation $(\mathcal{D}, \mathcal{H})$—that is, "voltages" into "currents"—and vice versa, see [17].

• In the *axion-electrodynamics* of Wilczek [65], the axion piece as a field is added to (26),

$$\check{H}^{ij} = \left(\lambda_0 \sqrt{-g} g^{ik} g^{jl} + \frac{1}{2} \alpha(x) \epsilon^{ijkl} \right) F_{kl} , \tag{29}$$

see also Itin [24]. Moreover, kinetic terms for the axion are added in order to make the axion a propagating field.

• Lindell & Sihvola [35, 56], again for $\alpha = const$, defined the *perfect electromagnetic conductor* (PEMC) via (28). Hopefully the PEMC can be realized by suitable metamaterials, see Sihvola [55]. Lindell & Sihvola arrived at the PEMC when studying electrodynamics in its 4-dimensional version. Then the minus sign in $\mathcal{H} = -\alpha E$ emerges naturally. Note, $\mathcal{D} = -sB$ and $\mathcal{H} = -sE$, with $s := s_c{}^c$, see (27), represent a (spatially) isotropic skewon that is only a 3-dimensional scalar. Thus, the relative sign in the second column of (28) is decisive.

• On the basis of experiments in solid state physics, we were able to show [20] that in Cr_2O_3 a phase can occur (below the Curie temperature) that obeys the relation (written in terms of exterior forms) ${}^{(3)}H = \alpha\mathcal{F}$, with $\alpha \approx 10^{-4}\lambda_0$. This proves that H is a form with twist and $\mathcal{F}$ one without twist thereby demonstrating the consistency of the system $dH = J,\ d\mathcal{F} = 0$ and ${}^{(3)}H = \alpha\mathcal{F}$.

4 Premetric electrodynamics in exterior calculus

One of the principal goals of exterior calculus is to put equations, in the case under consideration, Maxwell's equations and the constitutive law, into a manifestly coordinate independent form. In a way, in exterior calculus, the principle of coordinate covariance is built-in. If we handle the 2-form H, we handle a geometrical quantity which does not depend on coordinates at all. Only if we are interested in its components H_{ij}, with $H = \frac{1}{2} H_{ij} dx^i \wedge dx^j$, then these components are, of course, coordinate dependent. Thus, the Maxwell equations as such, namely $dH = J$ and $d\mathcal{F} = 0$ are just relations between forms, coordinates do not play a role. Naturally, such a calculus expresses the universality of Maxwell's equations in a particular impressive way.

Let us derive the Maxwell equations in modern language *ab ovo* in order to liberate the physics of Maxwell's equations from historical ballast, see [16] for details. Electrodynamics is based on two conservation laws. First, we have *electric charge conservation* (first axiom). The charge-current density 3-form J is closed $dJ = 0$ and, by de Rham's theorem, also exact,

$$dH = J, \tag{30}$$

with the excitation 2-form H as "charge-current potential" [62]. This is already the inhomogeneous Maxwell equation.

The current J is a 3-form *with* twist "since an electric charge has no screw-sense," cf. Schouten [52]. As a consequence, the excitation H also carries twist or, in the words of Perlick [46] (my translation), "...one *must* understand the excitation as a form with twist if one wants that the charge contained in a volume always has the same sign, independent of the orientation chosen."

We decompose the 4-dimensional excitation H into two pieces: one along the 1-dimensional time t and another one embedded in 3-dimensional space $(a, b = 1, 2, 3)$. We find (see [16])

$$H = -\mathcal{H} \wedge dt + \mathcal{D} = -\mathcal{H}_a dx^a \wedge dt + \frac{1}{2}\mathcal{D}_{ab} dx^a \wedge dx^b \tag{31}$$

which, if put in matrix form, yields the identification $(6)_1$. Substitution of (31) into (30) leads to the 3-dimensional inhomogeneous Maxwell equations,

$$\underline{d}\mathcal{D} = \rho\,, \qquad \underline{d}\mathcal{H} - \dot{\mathcal{D}} = j\,, \tag{32}$$

i.e., to the Coulomb-Gauss and the Oersted-Ampère-Maxwell laws, respectively. The underline denotes the 3-dimensional exterior differential and the dot differentiation with respect to time.

With charge conservation alone, we arrived at the inhomogeneous Maxwell equations (30). Now we need some more input for deriving the homogeneous Maxwell equations. The force on a charge density is encoded into the axiom of the *Lorentz force density* (second axiom)

$$f_\alpha = (e_\alpha \rfloor \mathcal{F}) \wedge J\,. \tag{33}$$

Here e_α is an arbitrary (or anholonomic) frame or tetrad, a basis of the tangent space, with $\alpha = 0, 1, 2, 3$, and $\rfloor$ denotes the interior product (contraction). In tensor language, we can express (33) as $\check{f}_i = \mathcal{F}_{ik}\,\check{J}^k$.

The axiom (33) should be read as an operational procedure for defining the electromagnetic field strength 2-form $\mathcal{F}$ in terms of the force density f_α, known from mechanics, and the current density J, known from charge conservation. According to (33), the field strength $\mathcal{F}$ turns out to be a 2-form without twist. Its 1+3 decomposition reads

$$\mathcal{F} = E \wedge d\sigma + B = E_a dx^a \wedge dt + \frac{1}{2} B_{ab} dx^a \wedge dx^b \,, \tag{34}$$

which yields the identification $(6)_2$.

Magnetic flux conservation is our third axiom. The field strength $\mathcal{F}$ is closed,

$$d\mathcal{F} = 0 \,. \tag{35}$$

This is the homogeneous Maxwell equation. Split into 1+3, we find

$$\underline{d}\, E + \dot{B} = 0 \,, \qquad \underline{d}\, B = 0 \,, \tag{36}$$

that is, Faraday's induction law and the sourcelessness of B. The laws (36) are also premetric. The Lenz rule, as the reason for the relative sign difference between the time derivatives in $(32)_2$ and $(36)_1$, is discussed in [25, 26].

We have then the Maxwellian set[10]

$$dH = J \qquad \text{and} \qquad d\mathcal{F} = 0 \,. \tag{37}$$

If decomposed into components, we recover the tensor calculus formulas in (19). This closes our considerations on the appropriate form of Maxwell's equations in 4-dimensional spacetime. For vacuum, the Hodge star maps the 2-form $\mathcal{F}$ without twist into the 2-form H with twist according to

$$H = \lambda_0 \, {}^\star\mathcal{F} \tag{38}$$

—this is equivalent to (26). The Hodge star, as applied to 2-forms in 4 dimensions, depends only on the conformally invariant part of the metric, see, e.g., Frankel [14] and Kastrup [28]. The constitutive relation for axion-electrodynamics simply reads

$$H = \lambda_0 \, {}^\star\mathcal{F} + \alpha\mathcal{F} \,, \tag{39}$$

that is, for the PEMC we have simply $H = \alpha\mathcal{F}$. For matter we have the relation

$$H = \kappa(\mathcal{F}) \tag{40}$$

as analog of (20). If the operator κ is local and linear, we find for its components

$$\kappa_{ij}{}^{kl} = \frac{1}{2} \hat{\epsilon}_{ijmn} \chi^{mnkl} \,. \tag{41}$$

[10] Damour [7] discussed Maxwell's equations in exterior calculus in his footnote 1. He found $\delta f = s$ and $\delta \star F = 0$. The codifferential is defined according to $\delta := \star d \star$. For a p-form ψ we have $\star\star\psi = (-1)^{p-1}\psi$. For the homogeneous equation we find $\star d \star\star F = -\star dF = 0$ or $dF = 0$, which coincides with $(37)_2$. Thus F can be identified with the field strength $\mathcal{F}$. We apply the star to Damour's inhomogeneous Maxwell equation and find $\star(\delta f) = \star\star d\star f = d \star f = S := \star s$ or $d \star f = S$. This equation only agrees with $(37)_1$, provided we put $H = \star f$. Consequently, in Damour's exterior calculus, f loses its operational interpretation as electromagnetic excitation. Of course, the codifferential δ is not required for the formulation of the Maxwell equations (37). The exterior differential d is sufficient.

5 Minkowski's discovery of the energy-momentum tensor

Minkowski's "greatest discovery was that at any point in the electromagnetic field *in vacuo* there exists a tensor of rank 2 of outstanding physical importance. ... *each component of the tensor* $E_p{}^q$ *has a physical interpretation,* which in every case had been discovered many years before Minkowski showed that these 16 components constitute a tensor of rank 2. The tensor $E_p{}^q$ is called the *energy tensor* of the electromagnetic field."

These words of Whittaker [64], Vol. 2, pp. 66 and 67, stress the central importance of this discovery. Einstein wouldn't have been able to derive the field equations of gravity of general relativity without being aware of this important quantity that figures as source of his (and Hilbert's) field equation of gravity. Einstein's reason for picking the scheme (15), and thus (16) with $(26)_1$, was that this allowed him to derive the energy-momentum tensor of the electromagnetic field in a more transparent way than Minkowski.

Minkowski starts with his "field vectors" f and F, as specified in (2), and multiplies them as matrices according to fF. The product matrix he decomposes, without giving any motivation, into a tracefree piece S (that is, with $\mathrm{tr}\, S = 0$), which he calls a spacetime matrix of the 2nd kind, and a trace piece $\mathcal{L}\,\mathbf{1}$, with the unit matrix $\mathbf{1}$ and with $\mathcal{L} := -\frac{1}{4}\mathrm{tr}(fF)$, which Minkowski identifies correctly as the Lagrangian density $\frac{1}{2}(H\cdot B - D\cdot E)$ of the electromagnetic field. He wrote, if we formulate it compactly in the notation of Cartesian tensor calculus,

$$S_{hk} = f_{hl}F_{lk} - \frac{1}{4}\delta_{hk} f_{lm}F_{ml}\,, \qquad \text{that is,} \qquad S_{hh} = 0\,. \tag{42}$$

In order to display the reality relations appropriately, he wrote his spacetime matrix as

$$S = (S_{hk}) = \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{pmatrix} = \begin{pmatrix} X_x & Y_x & Z_x & -iT_x \\ X_y & Y_y & Z_y & -iT_y \\ X_z & Y_z & Z_z & -iT_z \\ -iX_t & -iY_t & -iZ_t & T_t \end{pmatrix}. \tag{43}$$

With the help of (42), Minkowski identifies explicitly the Maxwell stress (momentum flux density)

$$\begin{aligned} X_x &= \frac{1}{2}\left(D_xE_x - D_yE_y - D_zE_z + H_xB_x - H_yB_y - H_zB_z\right)\,, \\ X_y &= D_yE_x + H_xB_y\,, \\ Y_x &= D_xE_y + H_yB_x \quad \text{etc.}\,, \end{aligned} \tag{44}$$

the Poynting vector (energy flux density)

$$T_x = H_zE_y - H_yE_z \quad \text{etc.}\,, \tag{45}$$

and the electromagnetic energy density

$$T_t = \frac{1}{2}\left(D_xE_x + D_yE_y + D_zE_z + H_xB_x + H_yB_y + H_zB_z\right)\,. \tag{46}$$

He did not name the electromagnetic momentum density X_t, Y_t, Z_t—which, by Lebedev [33], had already been discovered experimentally in 1901. Note that Minkowski's energy-momentum tensor is asymmetric in general and thus, because of its tracelessness, has 15 independent components (for the algebraic properties of energy-momentum currents, see Itin [23]).

We leave this historical track and turn first to *tensor calculus:* Starting with the Lorentz force density $\check{f}_i = \mathcal{F}_{ij}\check{J}^k$, substituting the inhomogeneous Maxwell equation $(16)_1$, and using $(16)_2$, we arrive at the

Minkowski energy-momentum tensor density[11]

$$\mathcal{T}_i{}^j = \frac{1}{4}\delta_i^j \mathcal{F}_{kl}\check{H}^{kl} - \mathcal{F}_{ik}\check{H}^{jk}\,, \qquad \mathcal{T}_k{}^k = 0\,, \tag{47}$$

together with $\check{f}_i = \partial_j \mathcal{T}_i{}^j + \mathcal{X}_i$ and the auxiliary force density $\mathcal{X}_i = \frac{1}{4}\left[\left(\partial_i \mathcal{F}_{jk}\right)\check{H}^{jk} - \mathcal{F}_{jk}\,\partial_i \check{H}^{jk}\right]$. Incidentally, already Minkowski proved [39] that $\mathcal{T}_i{}^j\,\mathcal{T}_j{}^k = \frac{1}{4}\,\delta_i^k\,\mathcal{T}^2$, with $\mathcal{T}^2 := \mathcal{T}_l{}^m \mathcal{T}_m{}^l$, a formula that was repeatedly rediscovered.

For *vacuum*, $\check{H}^{ij} = \lambda_0\,\sqrt{-g}g^{ik}g^{jl}\,F_{kl}$, and thus, in *Cartesian coordinates* $[g_{ij} = \mathrm{diag}(1,-1,-1,-1)]$, the auxiliary force density $\mathcal{X}_i = 0$. Then $\mathcal{T}_{ik} := g_{kj}\mathcal{T}_i{}^j = \mathcal{T}_{ki}$ is symmetric and conserved, see [16] for details—and we recover the result of Einstein [11]. At the same time, under those two conditions (vacuum and Cartesian coordinates), Minkowski's formula $K = \mathrm{lor}\ S$ re-emerges, where K is the 4-dimensional Lorentz force density and S Minkowski's energy-momentum matrix.

In *exterior calculus* we can apply an analogous procedure and find the energy-momentum 3-form as

$$\Sigma_\alpha := \frac{1}{2}\left[\mathcal{F}\wedge(e_\alpha\rfloor H) - H\wedge(e_\alpha\rfloor\mathcal{F})\right]\,, \qquad \vartheta^\alpha\wedge\Sigma_\alpha = 0\,, \tag{48}$$

where e_α denotes the frame (tetrad) and ϑ^α the coframe, with $e_\alpha\rfloor\vartheta^\beta = \delta_\alpha^\beta$. The expression $(48)_1$ translates into $(47)_1$ via $\Sigma_\alpha = \mathcal{T}_\alpha{}^\beta\,{}^\star\vartheta_\beta$. Accordingly, the formulas (47) and (48) incorporate Minkowski's energy-momentum of the electromagnetic field in a modern notation.

Apparently Minkowski's energy-momentum tensor is defined for matter, in particular also for *moving matter*. Abraham [1–3] introduced a symmetric tensor instead, since the symmetry was believed to be important from the point of view of angular momentum conservation, amongst other reasons. The dispute between the supporters of the Minkowski and the Abraham tensor is lingering on till today. Obukhov [42] reviewed this problem, partly following [16, 43, 45]. Obukhov [42] based his discussion on a general Lagrange-Noether framework. He recovers the Minkowski tensor as the *canonical* energy-momentum and shows that the balance equations of energy-momentum and angular momentum are always satisfied for an open electromagnetic system because of the asymmetry of the canonical tensor. Moreover, Obukhov is able to formulate an "abrahamization" prescription for symmetrizing a given asymmetric energy-momentum tensor, provided a timelike vector is available (a velocity of a medium, for example). Obukhov considers an ideal fluid with isotropic electric and magnetic properties interacting with the electromagnetic field. Thereby Obukhov resolves the perennial Abraham-Minkowski controversy and assigns the appropriate places for the Minkowski as well as for the Abraham tensor.

6 Discussion

Compare, in vacuum, Minkowski's system

$$\{\mathrm{lor} f = -s, \quad \mathrm{lor}\, F^* = 0; \quad f = F\}\,, \tag{49}$$

with the system in exterior calculus

$$\{dH = J, \qquad d\mathcal{F} = 0; \qquad H = {}^\star\mathcal{F}\}\,. \tag{50}$$

The first system is Poincaré covariant, the second one generally covariant.

Equation (49) is a special case of (50). Since in Minkowski's formalism $\mathrm{lor}\,F^* = 0$ in its version $(10)_2$ is already generally covariant, we rewrite it tentatively as $(\mathrm{lor}^*)\,F = 0$ and identify $\mathrm{lor}^* \to d$. Then, the inhomogeneous equation becomes $(\mathrm{lor}^*)\,f^* = s$ or $d\,f^* = s$, and we arrive at $\{d\,f^* = s,\ dF = 0;\ f = F\}$, or at (50), provided we identify $H \to f^*$ and $\mathcal{F} \to -F$. Roughly we could say that in the transition from

[11] If we use $\check{H}^{ij} = \frac{1}{2}\epsilon^{ijkl}H_{kl}$, then $\mathcal{T}_i{}^j = \frac{1}{4}\,\epsilon^{jklm}\left(H_{ik}\mathcal{F}_{lm} - \mathcal{F}_{ik}H_{lm}\right)$. It is remarkable that Ishiwara [22] in 1913, inspired by Minkowski's formalism, displayed the energy-momentum tensor as $\mathbf{T} = -\frac{1}{2}\left\{[[\mathbf{H}^*\mathbf{F}^*]] - [[\mathbf{F}\,\mathbf{H}]]\right\}$.

(49) to (50) the star migrated from the Maxwell equations to the constitutive relation for the vacuum. Of course, the system (49) is now history. The set (50), which was derived from a reasonable axiomatics of Maxwell's theory, has superseded it. Still, we find it remarkable how powerful Minkowski's formalism is and how closely (50) resembles (49).

And in exterior calculus Minkowski's energy-momentum matrix S_{hk} finds its particular compact expression as energy-momentum 3-form

$$\Sigma_\alpha := \frac{1}{2}\left[\mathcal{F} \wedge (e_\alpha \rfloor H) - H \wedge (e_\alpha \rfloor \mathcal{F})\right] . \tag{51}$$

Summing up: The Maxwell equations $dH = J$, $d\mathcal{F} = 0$, the Lorentz force density $f_\alpha = (e_\alpha \rfloor \mathcal{F}) \wedge J$, and Minkowski's energy momentum 'tensor' (51) represent the framework of premetric electrodynamics. As such, they are a modern generally covariant expression of Minkowski's 4-dimensional Poincaré covariant theory of electromagnetism.

Acknowledgements I am very grateful to Yuri Obukhov (Moscow) for thorough and detailed discussions on the subject of this article. Moreover, I would like to thank the following colleagues and friends for reading my article and for coming up with numerous suggestions and corrections: Gerhard Bruhn (Darmstadt), Hubert Goenner (Göttingen), Yakov Itin (Jerusalem), Bernard Jancewicz (Wrocław), Florian Loebbert (Golm), Bahram Mashhoon (Columbia, Missouri), Günter Nimtz (Köln), Dirk Pützfeld (Golm), Guillermo Rubilar (Concepción), Tilman Sauer (Pasadena), Gerhard Schäfer (Jena), Engelbert Schücking (New York), Ari Sihvola (Helsinki), and Yosef Verbin (Tel Aviv). At an earlier stage, this work was supported partly by the Deutsche Forschungsgemeinschaft (Bonn) with the grant HE 528/21-1.

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