

SOME UPPER BOUNDS ON THE NUMBER OF RESONANCES FOR MANIFOLDS WITH INFINITE CYLINDRICAL ENDS

T. CHRISTIANSEN

ABSTRACT. We prove some sharp upper bounds on the number of resonances associated with the Laplacian, or Laplacian plus potential, on a manifold with infinite cylindrical ends.

The purpose of this note is to bound the number of resonances, poles of the meromorphic continuation of the resolvent, associated to a manifold with infinite cylindrical ends. These manifolds have an infinity which is in some sense one-dimensional, even though the manifold is n -dimensional. The bounds we obtain on the resonances reflect this dichotomy- we obtain one bound like that for one-dimensional scattering, and other bounds of the type expected for n -dimensional manifolds. As part of our study of resonances, we relate poles of the resolvent to L^2 eigenvalues and to poles of appropriately defined “scattering matrices.”

1. INTRODUCTION

A smooth Riemannian manifold X is said to be a manifold with cylindrical ends if it can be decomposed as

$$X = X_{\text{comp}} \sqcup_{i=1}^{m_0} X_i,$$

where $X_i = [a_i, \infty)_t \times Y_i$, (Y_i, g_i) is a compact Riemannian manifold, X_{comp} is a compact manifold with boundary $\sqcup_1^{m_0} Y_i$, and the metric on X_i is $(dt)^2 + g_i$. We may also allow X itself to be a manifold with boundary, and then the ends may take the form $[a_i, \infty)_t \times Y_i$, with Y_i a smooth, compact manifold with boundary. In this case, we require that the boundary of X_c be compact and smooth except for a finite number of corners corresponding to $a_i \times \partial Y_i$. An example of such a manifold is a waveguide, a domain with smooth boundary in the plane, with one or more infinite straight ends. If we allow X to have a boundary, we will consider the Laplacian with Dirichlet or Neumann boundary conditions.

Let Δ_{Y_i} be the Laplacian on (Y_i, g_i) , and let Y be the disjoint (and, if $m_0 \geq 1$, disconnected) union of the Y_i . Let Δ_Y be the Laplacian on Y ; that is, if $f \in C^\infty(Y)$, then $(\Delta_Y f)|_{Y_i} = \Delta_{Y_i} f|_{Y_i}$. Let $\{\sigma_j^2\}$, $\sigma_1^2 \leq \sigma_2^2 \leq \sigma_3^2 \leq \dots$ be the set of all eigenvalues of Δ_Y , repeated according to their multiplicity, and let $\nu_1^2 < \nu_2^2 < \nu_3^2 < \dots$ be the *distinct* eigenvalues of Δ_Y . Then the resolvent of the Laplacian Δ on X , or

Partially supported by an NSF grant.
e-mail address: tjc@math.missouri.edu.

of $\Delta + V$, for $V \in L_{\text{comp}}^\infty(X)$ real-valued, has a meromorphic continuation to the Riemann surface \hat{Z} on which $(z - \nu_j^2)^{1/2}$ is a single-valued function for all j ([11, 16]). Thus, the resonances, poles of the meromorphic continuation of the resolvent, are associated to points in \hat{Z} . The complicated nature of this Riemann surface makes more difficult the question of bounding the number of resonances, and makes necessary the restrictions we place on $\{\nu_j^2\}$.

Part of our study of resonances is to understand the relationship between poles of the resolvent and their multiplicities and poles of a “scattering matrix.” For a manifold with cylindrical ends, there are several reasonable objects to call the scattering matrix. One is an infinite dimensional matrix. In Section 3 we define this matrix and associated finite-dimensional matrices which contain the information about poles which we desire. We show that if $z_0 \in \hat{Z}$ is not a ramification point of \hat{Z} and is a pole of the resolvent, then there is an associated L^2 eigenfunction with an appropriate expansion or there is a pole of a “scattering matrix” at z_0 . We make this precise and address the issue of multiplicities in Theorem 3.1 and Proposition 3.3. In [2] such a relationship is noted, though to our knowledge no proof appears in the literature, so we include it here for completeness.

We will often require that there exists an $\alpha > 0$ such that

$$(H1) \quad \nu_m^2 - \nu_{m-1}^2 \geq \alpha \nu_m$$

for all sufficiently large m . Examples of cross-sectional manifolds that satisfy such requirements are spheres, an interval with Dirichlet or Neumann boundary conditions, and projective space. We can easily construct a manifold that satisfies (H1) with as many ends as we wish by taking all the cross sections Y_i to be the same manifold from the previous list. Other situations are possible as well, of course.

Let P be the operator Δ , the Laplacian, or $\Delta + V$, for real-valued $V \in L_{\text{comp}}^\infty(X)$. Then, for $z \in \mathbb{C} \setminus [\nu_1^2, \infty)$, $R(z) = (P - z)^{-1}$ is bounded on $L^2(X)$ except, perhaps, for a finite number of z . It has a meromorphic continuation to the Riemann surface \hat{Z} described earlier. We bound the number of resonances in certain regions of \hat{Z} . In doing so, we take the view that resonances near the physical sheet, the sheet of \hat{Z} on which the resolvent is bounded, are more interesting, as they have greater physical relevance. Let $r_j(z) = (z - \nu_j^2)^{1/2}$.

Assuming the hypothesis (H1), we simplify the study of the resonances somewhat and are then able to better bound them.

Theorem 1.1. *Assume X satisfies the hypothesis (H1) and let $\beta < 1$. Then, in the connected components of $\{z' \in \hat{Z} : |r_m(z')| < \beta\sqrt{\alpha\nu_m}\}$ that meet the physical sheet, there are at most $\mathcal{O}(m^{n-1})$ resonances.*

We remark that we count all poles with their multiplicities and that the poles of the resolvent include eigenvalues. In [5, 18] there is an example of a family of manifolds that has $\liminf_{\lambda \rightarrow \infty} N(\lambda)\lambda^{-n} > 0$, where $N(\lambda)$ is the number of eigenvalues of the Laplacian with norm less than λ^2 . Since the cross-sectional manifolds in the example can be taken to be $n - 1$ -dimensional unit spheres, this

shows that the order appearing in this theorem is optimal. This type of bound is indicative of the n -dimensional nature of the manifold.

In a simpler case, we can say a bit more.

Theorem 1.2. *Let $X = \mathbb{R} \times Y$ and suppose X satisfies (H1), and let $\rho > 0$ be fixed. Consider the operator $\Delta + V$, for real-valued $V \in L_{\text{comp}}^\infty(X)$. Then, on the connected components of $\{z \in \hat{Z} : |r_m(z)| < \rho\}$ that meet the physical sheet, the number of poles is bounded by $C(1 + m^{n-2})$.*

In case $V = V(s)$, $s \in \mathbb{R}$, this theorem is easy to prove, and this example shows that the bound is of optimal order.

Our proofs of these two theorems involve an adaptation of techniques developed by Melrose ([15]), Zworski ([24, 25]), and Vodev ([21]) to bound the number of poles. (See [12] and [22] for further references and results.) It involves constructing an approximation to the resolvent to find a holomorphic function whose zeros include the poles of the resolvent. Then we bound the function and apply Jensen's theorem. This requires some knowledge of the function at a "base point." For us, that will mean a lower bound. Since we will be changing the base point, we need some kind of uniform lower bound and that is different from these other applications of this technique. In order to do this, we will construct approximations of the resolvent especially well-suited to the regions where we work.

The following theorem does not require the hypothesis (H1), and its proof uses a different technique.

Theorem 1.3. *Fix a sheet of \hat{Z} , and let $\{z_k\}$ be the resonances of P on this sheet. Then*

$$\sum \frac{|\text{Im } r_1(z_k)|}{|r_1(z_k)|^2} < \infty.$$

This theorem is an analogue of what one finds in one-dimensional scattering theory (see [9, 23]), where the natural variable to consider is $\lambda = z^{1/2}$.

The problem of obtaining upper bounds on the resonance-counting function has been widely studied for Euclidean (e.g. [9, 15, 21, 23, 25]) and hyperbolic scattering (e.g. [12, 13, 17, 19]). For a survey and further references, see [22] or [26].

In this paper we use results of [16], which studied the Laplacian on compact manifolds with boundary and exact b -metrics. Under a change of variable, a special case of such manifolds is the class of manifolds considered here (see also [11]). The papers [5, 18] independently obtained that the number of eigenvalues less than λ^2 of the Laplacian grows at most like λ^n . The existence of eigenvalues or complex resonances has been studied in, for example, [1, 2, 3, 5, 6, 7, 18] and references.

In finishing the paper, the author received a copy of [8]. There Edward obtains a result similar to our Theorem 1.1 for the Laplacian on waveguides, that is, domains in the plane which outside of a compact set coincide with $(-\infty, \infty) \times \pi$, and thus fall in the category of manifolds which we consider. The waveguides satisfy hypothesis (H1) and their ν_m^2 are very regularly distributed for either the

Dirichlet or Neumann Laplacian. Use the metric induced on \hat{Z} by the pull-back of the metric on \mathbb{C} to define distance on \hat{Z} . Then summing over the $\nu_m^2 \leq r$ we can obtain from our Theorem 1.1 an upper bound

$$\#\{z_j : z_j \text{ is a resonance of } \Delta,\$$

$$\text{dist}(z_j, \text{physical sheet}) < c_1 \sqrt{|\pi(z_j)|}, |\pi(z_j)| < r\} = \mathcal{O}(r)$$

for some $c_1 > 0$, where $\pi : \hat{Z} \rightarrow Z$ is projection. This may be compared to Theorem 2 of [8], where for $c_1 = 1/2$, Edward obtains a bound $\mathcal{O}(r^{3+\epsilon})$, any $\epsilon > 0$.

Remark. We note that the “black-box” formalism of [20] can be adapted to this situation. Thus we could replace P by \tilde{P} , a more general self-adjoint, compactly supported perturbation of the Laplacian than the ones considered here. If \tilde{P} satisfies the assumptions of [20], properly interpreted for this setting, and is bounded below, then Theorem 1.3 will hold for \tilde{P} . Let $\tilde{P}^\# = \tilde{P}|_{\{t < \max(a_i)\}}$, with Dirichlet boundary conditions at $\{t = \max(a_i)\}$. Let

$$N_{\tilde{P}^\#}(\lambda) = \#\{\lambda_j^2 : \lambda_j^2 \text{ is an eigenvalue of } \tilde{P}^\#, \lambda_j^2 \leq \lambda^2\}.$$

If, in addition to the other assumptions,

$$N_{\tilde{P}^\#}(\lambda + a) - N_{\tilde{P}^\#}(\lambda - a) = \mathcal{O}_a(\lambda^q),$$

$q \geq n - 1$, for any constant a , then an analogue of Theorem 4.1 holds, with $\mathcal{O}_\beta(m^{n-1})$ replaced by $\mathcal{O}_\beta(m^q)$.

Acknowledgments. It is a pleasure to thank Maciej Zworski for suggesting the problem of counting resonances in this setting. I am grateful to him and to Dan Edidin for helpful discussions. Thanks to Julian Edward for helpful comments.

2. PRELIMINARIES

Let $r_j(z) = (z - \nu_j^2)^{1/2}$ and identify the physical sheet of \hat{Z} as being the part of \hat{Z} on which $\text{Im } r_j(z) > 0$ for all j and all z and on which $R(z)$ is bounded. Other sheets will be identified, when necessary, by indicating for which values of j $\text{Im } r_j(z) < 0$. Each sheet can be identified with $\mathbb{C} \setminus [\nu_1^2, \infty)$. With this language, there are points in \hat{Z} which belong to no sheet but which belong to the boundary of the closure of two sheets, and the ramification points, which correspond to $\{\nu_j^2\}$ and belong to the closure of four sheets (except for ramification points corresponding to ν_1^2). We note that sheets that meet the physical sheet are characterized by the existence of a $J \in \mathbb{N}$ such that

$$\text{Im } r_j(z) < 0 \text{ for all } z \text{ on that sheet if and only if } j \leq J.$$

Let $\{\phi_j\}$ be an orthonormal set of eigenfunctions of Δ_Y associated with $\{\sigma_j^2\}$.

We shall often abuse notation and use y for a coordinate in either Y or Y_i . On an end, we use the coordinates (t, y) , with $t \in (a_i, \infty)$ and $y \in Y_i$.

We define an operator on $[0, \infty)_t \times Y_y$. Using the same notation for an operator and its Schwartz kernel, let

$$R_{ej}(z) = \frac{i}{2(z - \sigma_j^2)^{1/2}} (e^{i|t-t'|(z-\sigma_j^2)^{1/2}} - e^{i|t+t'|(z-\sigma_j^2)^{1/2}}) \phi_j(y) \overline{\phi_j(y')},$$

where we take $\text{Im}(z - \sigma_j^2)^{1/2} > 0$ for $z \in \mathbb{C} \setminus [\sigma_1^2, \infty)$ corresponding to the physical sheet of \hat{Z} . Then, for $z \in \mathbb{C} \setminus [\sigma_1^2, \infty)$,

$$(1) \quad R_Y(z) = \sum_j R_{ej}(z)$$

is the resolvent of the Dirichlet Laplacian on $[0, \infty) \times Y$. As an operator from $L^2_{\text{comp}}([0, \infty) \times Y)$ to $H^2_{\text{loc}}([0, \infty) \times Y)$ it has a holomorphic continuation to the Riemann surface \hat{Z} and we will see that it is this that determines the surface to which the resolvent of P has a meromorphic continuation.

In general, we shall use z to stand for a point in \hat{Z} and $\pi(z)$ to represent its projection to \mathbb{C} . For $w \in \mathbb{R}^m$, $\langle w \rangle = (1 + |w|^2)^{1/2}$.

3. THE SCATTERING MATRIX AND POLES OF THE RESOLVENT

There are several reasonable definitions of the scattering matrix in this setting. We recall some results of [4, 16].

Recall that $r_k(z) = (z - \nu_k^2)^{1/2}$, and let $\tilde{r}_j(z) = (z - \sigma_j^2)^{1/2}$, with $\text{Im} \tilde{r}_j > 0$ when z is in the physical sheet and $\tilde{r}_j(z) = r_k(z)$ if $\sigma_j^2 = \nu_k^2$. In addition, fix a coordinate t on the ends, so that $t = 0$ lies on the ends. Then, for all but a finite number of z in the physical space, there are functions $\Phi_j(z, p)$ with

$$(2) \quad (P - \pi(z))\Phi_j(z, p) = 0$$

and, on the ends,

$$(3) \quad \Phi_j(z, t, y) = e^{-i\tilde{r}_j(z)t} \phi_j(y) + \sum S_{mj}(z) e^{i\tilde{r}_m(z)t} \phi_m(y).$$

The Φ_j have a meromorphic continuation to all of \hat{Z} and thus, so do the S_{mj} . The $S_{mj}(z)$ depend on the choice of the coordinate t in a fairly straight-forward way (see [5]). This dependence is not important here as it does not change the location of the scattering poles, so we ignore it but we do consider the coordinate t to be fixed throughout, and chosen so that $\{p \in X : t = 0\} \subset \sqcup X_i$.

There are several reasonable choices of objects to call the scattering matrix. One possibility is the infinite matrix of the $S_{ij}(z)$, as in [16]. Another, which is well-defined for z on the boundary of the physical sheet, is a normalized, finite-dimensional matrix of the $S_{ij}(z)$, where the dimension changes as z crosses a ν_j^2 . This is used in [4] and has the advantage of being unitary (though we note that the variable used in [4] is $\lambda = z^{1/2}$). Here, however, this is unnecessarily complicated as it requires the introduction of $(z - \nu_i^2)^{1/4}$.

We shall work with finite-dimensional matrices, of the form

$$(S_{ij}(z))_{i,j \in \mathcal{E}}$$

for some set $\mathcal{E} \subset \mathbb{N}$, where \mathcal{E} is chosen, depending on z , to be most helpful for our purposes.

3.1. Definitions and Some Properties. In order to describe the poles and their multiplicities of the resolvent and matrices of the $S_{ij}(z)$, we introduce some notation.

Definition 3.1. *We define the multiplicity of a pole of the resolvent $R(z)$ at z_0 to be*

$$m_{z_0}(R) = \dim \text{Image } \Xi_{z_0}(R),$$

where $\Xi_{z_0}(R)$ is the singular part of R at the point z_0 .

It follows as in [13, Lemma 2.4] that, if z_0 is not a ramification point of \hat{Z} , then $m_{z_0}(R)$ is also the rank of the residue of R at z_0 .

In order to define the multiplicity of the pole of a matrix, we shall use the following lemma. Though it may be well known, we include it and a proof for the convenience of the reader.

Lemma 3.1. *Suppose $A(z)$ is a $d \times d$ -dimensional meromorphic matrix, invertible for some value of z . Then, near z_0 , it can be put into the form*

$$A(z) = E(z) \left(\sum_{j=1}^p (z - z_0)^{-k_j} P_j + \sum_{j=p+1}^{p'} (z - z_0)^{l_j} P_j + P_0 \right) F(z)$$

where $E(z)$, $F(z)$, and their inverses are holomorphic near z_0 , and $P_i P_j = \delta_{ij} P_i$, $\text{tr } P_0 = d - p'$, $\text{tr } P_i = 1$, $i = 1, \dots, p'$. The k_j and l_j are, up to rearrangement, uniquely determined.

Proof. We outline a proof. Without loss of generality we may assume that $z_0 = 0$.

First we show the existence of such a decomposition. Choose k such that $z^k A(z) = B(z)$ is holomorphic at 0. Now the proof of the existence of such a decomposition follows much like a proof from [10, Section VI.2.4]. Let $B(z) = (b_{ij}(z))$. Choose an element $b_{ij}(z)$ that vanishes to the lowest order at 0, and by permuting rows and columns make this element $b_{11}(z)$. By subtracting from the k th row the first row multiplied by $b_{k1}(z)/b_{11}(z)$ (which is holomorphic near $z = 0$), we can make all the entries in the first row, other than the first one, zero. Similar column operations reduce $B(z)$ to the form

$$\begin{pmatrix} b_{11}(z) & 0 & \cdots & 0 \\ 0 & c_{22}(z) & \cdots & c_{2d}(z) \\ \vdots & \vdots & \cdots & \vdots \\ 0 & c_{d2}(z) & \cdots & c_{dd}(z) \end{pmatrix}.$$

Repeating the procedure on the $(d-1) \times (d-1)$ dimensional matrix of the c_{ij} , we obtain a matrix of the form

$$\begin{pmatrix} b_{11}(z) & 0 & 0 & \cdots & 0 \\ 0 & c_{22}(z) & 0 & \cdots & 0 \\ 0 & 0 & \alpha_{33}(z) & \cdots & \alpha_{3d}(z) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \alpha_{d3}(z) & \cdots & \alpha_{dd}(z) \end{pmatrix}.$$

Repeating this procedure a finite number of times, we obtain a matrix with only the diagonal entries, $h_{ii}(z)$, nonzero. Since $h_{ii}(z) = z^{m_i} g_i(z)$, with $g_i(0) \neq 0$ and $g_i(z)$ holomorphic near 0, by multiplying by a diagonal matrix with nonzero entries $1/g_i(z)$ we obtain a diagonal matrix with diagonal entries of the form z^{m_i} . Finally, multiplying the whole thing by $z^{-k}I$ we obtain a matrix equivalent to $A(z)$ near $z = 0$. We note that the construction ensures that E and F are holomorphic near 0. Moreover, since each of E and F is the product of elementary matrices and one diagonal matrix (holomorphic near 0, with nonzero determinant), E and F are both invertible near 0.

To see the uniqueness of the k_i, l_i , we prove it for $B(z) = z^k A(z)$, where B is holomorphic at 0, and use the straight-forward relationship between A and B . Suppose there are two such decompositions: $B(z) = E_1(z)D_1(z)F_1(z) = E_2(z)D_2(z)F_2(z)$, where $E_i(z)$ and $F_i(z)$ are as in the statements of the lemma and the D_i are diagonal matrices with nonzero entries $d_{mm,i} = z^{l_{m,i}}$. By row and column operations we can make $0 \leq l_{1,i} \leq l_{2,i} \leq \dots \leq l_{d,i}$. We have

$$(4) \quad E(z)D_1(z) = D_2(z)F(z)$$

for new matrices $E(z), F(z)$, holomorphic and invertible near 0. Thus it is easy to see that

$$\text{rank}(D_1(0)) = \text{rank}(D_2(0)) \equiv r_0.$$

Using (4) and the fact that $l_{j,i} = 0$ if and only if $j \leq r_0$, we obtain that, if $E(z) = (e_{ij}(z)), F(z) = (f_{ij}(z))$,

$$(5) \quad e_{ij}(0) = 0 = f_{ji}(0) \text{ if } j \leq r_0 \text{ and } i > r_0.$$

We finish the proof of the uniqueness by induction. Let $r_{j,i} = \text{rank}(D_i^{(j)}(0))$, and notice it suffices to prove that $r_{j,1} = r_{j,2}$ for all j , and that we have $\sum_j r_{j,i} = d$. Suppose we have shown that $r_{q,1} = r_{q,2} \equiv r_q$ for $q \leq N$, and that, if $R_N = r_0 + r_1 + \dots + r_N$,

$$(6) \quad e_{ij}(0) = 0 = f_{ji}(0) \text{ for } j \leq R_N, i > R_N.$$

If $v = (v_1, v_2, \dots, v_d)^t$, let $\Pi_s v = (0, \dots, 0, v_{s+1}, \dots, v_d)^t$ for $s \in \mathbb{N}, s \leq d$. Then

$$\text{rank}(\Pi_{R_N} E(0) \Pi_{R_N}) = d - R_N = \text{rank}(\Pi_{R_N} F(0) \Pi_{R_N}),$$

using (6) and the fact that $E(0)$ and $F(0)$ are invertible. Since, for $j \leq N$, $D_1^{(j)}(0)\Pi_{R_N} = 0$ and $\Pi_{R_N}D_2^{(j)}(0) = 0$, we get

$$\begin{aligned} \text{rank} \left(\frac{d^{N+1}}{dz^{N+1}} (\Pi_{R_N} E(z) D_1(z) \Pi_{R_N})|_{z=0} \right) &= \text{rank} \left(\Pi_{R_N} E(0) D_1^{(N+1)}(0) \Pi_{R_N} \right) \\ &= \text{rank} \left(D_1^{(N+1)}(0) \right) = \text{rank} \left(\frac{d^{N+1}}{dz^{N+1}} (\Pi_{R_N} D_2(z) F(z) \Pi_{R_N})|_{z=0} \right) \\ &= \text{rank} \left(D_2^{(N+1)}(0) \right) \end{aligned}$$

and thus $r_{1,N+1} = r_{2,N+1} \equiv r_{N+1}$. If $R_N + r_{N+1} < d$, from (4) we obtain

$$e_{ij}(0) = 0 = f_{ji}(0) \text{ for } j \leq R_N + r_{N+1}, i > R_N + r_{N+1}.$$

The induction need only continue until $R_{N+1} = d$, finishing the proof. \square

Definition 3.2. Let $A(z)$ be a meromorphic matrix, invertible for some values of z , and let $k_1, k_2, \dots, k_p, l_{p+1}, l_{p+2}, \dots, l_{p'}$ be as in Lemma 3.1. Set

$$\mu_{m_{z_0}}(A) = \sum_{j=1}^p k_j,$$

the “maximal multiplicity” of the pole of A at z_0 . Set

$$\mu_{d_{z_0}}(A) = \sum_{j=1}^p k_j - \sum_{j=p+1}^{p'} l_j,$$

the “determinantal multiplicity” of the pole of A at z_0 .

We note that

$$\mu_{d_{z_0}}(A) = \min\{j \in \mathbb{Z} : (z - z_0)^j \det A(z) \text{ is regular at } z_0\}.$$

Each of these measures of multiplicity will be useful.

Suppose $\mathcal{E} \subset \mathbb{N}$ is a finite subset. Let

$$\tilde{\mathcal{E}} = \{j \in \mathbb{N} : \sigma_j^2 = \nu_l^2 \text{ for some } l \in \mathcal{E}\}.$$

Define the matrix

$$S_{\mathcal{E}}(z) = (S_{ij}(z))_{i,j \in \tilde{\mathcal{E}}}.$$

For $z \in \hat{Z}$, let

$$\mathcal{E}_z = \{j \in \mathbb{N} : \text{Im } r_j(z) \leq 0\}$$

and

$$\mathcal{J}_z = \{j \in \mathbb{N} : j \leq \max \mathcal{E}_z\}.$$

If $\mathcal{E} \subset \mathbb{N}$ is a finite set, define $w_{\mathcal{E}} : \hat{Z} \rightarrow \hat{Z}$ as follows. To z we may associate the set of square roots $\{r_j(z)\}$. Then $w_{\mathcal{E}}(z)$ may be determined by saying it is the element of \hat{Z} associated to the set $\{r_j(w_{\mathcal{E}}(z))\}$, with

$$r_j(w_{\mathcal{E}}(z)) = \begin{cases} -r_j(z), & \text{if } j \in \mathcal{E} \\ r_j(z), & \text{if } j \notin \mathcal{E}. \end{cases}$$

For $J \in \mathbb{N}$, define a similar operator $w_J : \hat{Z} \rightarrow \hat{Z}$, where

$$r_j(w_J(z)) = \begin{cases} -r_j(z), & \text{if } j \leq J \\ r_j(z), & \text{if } j > J. \end{cases}$$

This operator appears in the relations (14) and (15).

The following Proposition generalizes similar results of [16] and [4].

Proposition 3.1. *Let $\mathcal{E} \subset \mathbb{N}$ be a finite set. For $j \in \tilde{\mathcal{E}}$,*

$$\Phi_j(w_{\mathcal{E}}(z)) = \sum_{k \in \tilde{\mathcal{E}}} S_{kj}(w_{\mathcal{E}}(z)) \Phi_k(z)$$

and

$$(S_{\mathcal{E}}(z))^{-1} = S_{\mathcal{E}}(w_{\mathcal{E}}(z)).$$

Proof. For $j \in \tilde{\mathcal{E}}$, consider

$$\Psi_j(z) = \Phi_j(w_{\mathcal{E}}(z)) - \sum_{k \in \tilde{\mathcal{E}}} S_{kj}(w_{\mathcal{E}}(z)) \Phi_k(z).$$

Then

$$(P - \pi(z)) \Psi_j(z) = 0$$

and, on the ends, Ψ_j has an expansion

$$(7) \quad \Psi_j(z, t, y) = e^{i\tilde{r}_j(z)t} \phi_j(y) - \sum_{k, l \in \tilde{\mathcal{E}}} S_{kj}(w_{\mathcal{E}}(z)) S_{lk}(z) e^{i\tilde{r}_l(z)t} \phi_l(y) \\ + \sum_{l \notin \tilde{\mathcal{E}}} \left(S_{lj}(w_{\mathcal{E}}(z)) - \sum_{k \in \tilde{\mathcal{E}}} S_{kj}(w_{\mathcal{E}}(z)) S_{lk}(z) \right) e^{i\tilde{r}_l(z)t} \phi_l(y).$$

Since, for z on the physical sheet, $\text{Im } \tilde{r}_l(z) > 0$ for all l , $\Psi_j(z) \in L^2(X)$ there and thus $\Psi_j(z) \equiv 0$ for all z in the physical space. By analytic continuation, $\Psi_j(z) \equiv 0$ for all $z \in \hat{Z}$. This proves the first part of the Proposition. It also shows that, for $j, l \in \tilde{\mathcal{E}}$,

$$\sum_{k \in \tilde{\mathcal{E}}} S_{kj}(w_{\mathcal{E}}(z)) S_{lk}(z) = \delta_{jl};$$

that is, $(S_{\mathcal{E}}(z))^{-1} = S_{\mathcal{E}}(w_{\mathcal{E}}(z))$. \square

3.2. Relation between Poles of the Resolvent and Poles of $S_{\mathcal{J}_{z_0}}(z)$. First, we recall some results of [16, Sections 6.7, 6.8] on the nature of poles of the resolvent on the boundary of the physical space.

Proposition 3.2. *Suppose z lies on the boundary of the physical space. Then, if z is not a ramification point of \hat{Z} , the multiplicity of z as a pole of the resolvent is equal to the dimension of the L^2 null space of $P - \pi(z)$. If z is a ramification point, then the resolvent has a double pole at z , with coefficient the projection onto*

the L^2 null space of $P - \pi(z)$, if there is any. The residue has rank equal to the dimension of

$$\{f : (P - \pi(z))f = 0, f \notin L^2(X), \frac{\partial}{\partial t}f \in L^2(X)\}.$$

Since

$$(8) \quad \Phi_j(z, p) = \chi(t)e^{-i\tilde{r}_j(z)t}\phi_j - (P - z)^{-1}(P - \pi(z))\chi(t)e^{-i\tilde{r}_j(z)t}\phi_j,$$

where $\chi(t) \in C^\infty(\mathbb{R})$ is supported in $t > \max(a_i)$ and is one in a neighborhood of infinity, it is clear by their definition that the $S_{ij}(z)$ cannot have a pole unless $R(z)$ has a pole. More can be said, and we begin our study of the relationship between the poles of the resolvent and poles of the scattering matrix with the following

Theorem 3.1. *Suppose $z_0 \in \hat{Z}$, and z_0 is not a ramification point. Then*

$$\begin{aligned} m_{z_0}(R) \\ = \mu m_{z_0}(S_{\mathcal{J}_{z_0}}) + \dim\{f \in L^2(X) : (P - \pi(z))f = 0, f \sim \sum_{j \notin \tilde{\mathcal{J}}_{z_0}} c_j e^{ir_j(z)t}\phi_j\} \end{aligned}$$

where the expansion must be valid on any end.

In proving this theorem, we will use some techniques from [13, Section 2].

We will call

$$(9) \quad \{f : (P - \pi(z))f = 0, f \sim \sum_{j \notin \tilde{\mathcal{J}}_{z_0}} c_j e^{ir_j(z)t}\phi_j\}$$

the set of eigenfunctions of type I. This depends on z_0 of course, and we will note the dependence in cases of possible confusion.

We first show that

Lemma 3.2. *If z_0 is not a ramification point of \hat{Z} , then*

$$\begin{aligned} \mu m_{z_0}(S_{\mathcal{J}_{z_0}}) \\ \leq m_{z_0}(R) - \dim\{f \in L^2(X) : (P - \pi(z))f = 0, f \sim \sum_{j \notin \tilde{\mathcal{J}}_{z_0}} c_j e^{ir_j(z)t}\phi_j\}. \end{aligned}$$

Proof. Just as in Lemma 2.4 of [13], whose notation we use, one can prove that if $R(z)$ has a pole at z_0 , for z_0 not a ramification point of \hat{Z} , then near z_0

$$(10) \quad R(z) = \sum_{k=1}^p \frac{A_k(z_0)}{(z - z_0)^k} + H(z_0, z)$$

where

$$A_k(z_0) = \sum_{l,m=1}^q a_k^{lm}(z_0)\varphi_l \otimes \varphi_m$$

and

$$(\varphi_l \otimes \varphi_m)f(p) = \varphi_l(p) \int_{p' \in X} \varphi_m(p')f(p').$$

As in [13], we have

$$(P - \pi(z_0))A_k(z_0) = A_{k+1}(z_0) = A_k(z_0)(P - \pi(z_0)).$$

If $a_k(z_0) = (a_k^{lm}(z_0))_{1 \leq l, m \leq q}$, then $a_1(z_0)$ is symmetric with rank q , $d(z_0) = a_1(z_0)^{-1}a_2(z_0)$ is nilpotent, and $a_k(z_0) = a_1(z_0)d^{k-1}(z_0)$, $k > 1$. Moreover, φ_l has an expansion on the ends of the form

$$(11) \quad \varphi_l(t, y)|_{t > \max a_i} = \sum_j \sum_{m \leq m_l} c_{ljm} t^m e^{i\tilde{r}_j(z_0)t} \phi_j(y)$$

([16]) and the φ_l are linearly independent. Let $B(z)$ be the matrix $B(z) = (b^{lm}(z))_{l, m \leq p}$ with

$$b^{lm}(z) = \sum_{k=1}^p \frac{a_k^{lm}(z_0)}{(z - z_0)^k}.$$

Then $B(z)$ can be written, as in Proposition 2.11 of [13], as

$$(12) \quad B(z) = E^\#(z) \left[\sum_{j=1}^{p'} (z - z_0)^{-k_j} P_j + P_0 \right] F^\#(z)$$

where $E^\#(z)$, $F^\#(z)$ and their inverses are holomorphic near z_0 , $P_i P_j = \delta_{ij} P_i$, $\text{tr } P_i = 1$, $i \neq 0$, $\text{tr } P_0 = q - p'$, and $k_1 + k_2 + \dots + k_{p'} = q$.

First assume that there are no type I eigenfunctions of P with eigenvalue $\pi(z_0)$. We recall that we can construct the generalized eigenfunctions Φ_j , $j \in \tilde{\mathcal{J}}_{z_0}$, as indicated in (8). We then obtain the entries of $S_{\mathcal{J}_{z_0}}(z)$ by first restricting Φ_j to $t = 0$ and then extracting the coefficients of ϕ_k .

This, taken with the representation (12), shows that the singular part of $S_{\mathcal{J}_{z_0}}(z)$ near z_0 will be given by

$$E^b(z) \left[\sum (z - z_0)^{-k_j} P_j + P_0 \right] F^b(z)$$

where $E^b(z)$, $F^b(z)$ are holomorphic matrices near z_0 which may not be invertible, or even square, and this proves the lemma in this special case.

To handle the case where $\pi(z_0)$ is an eigenvalue of P with

$$(13) \quad n_e(z_0) = \dim\{f \in L^2(X) : (P - \pi(z_0))f = 0, f \sim \sum_{j \notin \tilde{\mathcal{J}}_{z_0}} c_j e^{i\tilde{r}_j(z)t} \phi_j\} > 0,$$

we need to be a bit more careful. We first remark that for a general $z \in \hat{Z}$, $R(z)\psi$ is well-defined if

$$\psi \in e^{-\langle t \rangle \max(0, -\text{Im } r_j(z))} L^2(X).$$

This can be seen from the construction of the analytic continuation of the resolvent in [16]. Now suppose ψ is an L^2 eigenfunction of P . Then it is exponentially decreasing, and, for $J \in \mathbb{N}$, if z and $w_J(z)$ both lie on the boundary of the physical space,

$$(14) \quad (R(z) - R(w_J(z)))\psi = 0.$$

By analytic continuation, if $z, w_J(z)$ are such that

$$\psi \in e^{-(t) \max(0, -\operatorname{Im} r_j(z), -\operatorname{Im} r_j(w_J(z)))} L^2(X),$$

then (14) holds. By repeatedly applying (14) and using our knowledge of the structure of the resolvent on the closure of the physical space, we obtain that if

$$\psi \in \{f \in L^2(X) : (P - \pi(z_0))f = 0, f \sim \sum_{j \notin \tilde{\mathcal{J}}_{z_0}} c_j e^{ir_j(z)t} \phi_j\},$$

then there is a neighborhood of z_0 so that in this neighborhood

$$R(z) = \frac{1}{(\pi(z_0) - \pi(z)) \|\psi\|^2} \psi \otimes \bar{\psi} + B_1(z, z_0, \psi)$$

with $B_1(z, z_0, \psi)\psi = 0$. That is, each L^2 eigenfunction with an expansion at infinity of this type contributes to the singularities of $R(z)$ in the expected way. Using the fact that if $j \in \mathcal{J}_{z_0}$

$$\int_X \bar{\psi} (P - \pi(z_0)) \chi(t) e^{-i\tilde{r}_j(z_0)t} \phi_j(y) = 0,$$

we see that the singularities of $R(z)$ at z_0 corresponding to L^2 eigenfunctions of type I do not contribute to the singularities of $S_{\mathcal{J}_{z_0}}(z)$ at z_0 . After making this observation, the proof for the case $n_e(z_0) > 0$ follows just as in the case $n_e(z_0) = 0$. \square

The proof of Theorem 3.1 will be completed by

Lemma 3.3. *If z_0 is not a ramification point of \hat{Z} , then*

$$\begin{aligned} \mu m_{z_0}(S_{\mathcal{J}_{z_0}}) + \dim\{f \in L^2(X) : (P - \pi(z))f = 0, f \sim \sum_{j \notin \tilde{\mathcal{J}}_{z_0}} c_j e^{ir_j(z)t} \phi_j\} \\ \geq m_{z_0}(R). \end{aligned}$$

Proof. We use, for $J \in \mathbb{N}$,

$$(15) \quad R(z) - R(w_J(z)) = \frac{i}{2} \sum_{\sigma_j^2 \leq \nu_j^2} \Phi_j(z) \otimes \Phi_j(w_J(z)) \frac{1}{\tilde{r}_j(z)}.$$

Equation (15) holds, by Stone's formula, [16, Section 6.8], and [4, Section 2.2], for z on the boundary of the physical space, with $\nu_j^2 < \pi(z) < \nu_{j+1}^2$, and then holds on the rest of \hat{Z} by analytic continuation. Using Proposition 3.1, we may write (15) as

$$(16) \quad R(z) - R(w_J(z)) = \frac{i}{2} \sum_{\sigma_j^2 \leq \nu_j^2} \sum_{\sigma_m^2 \leq \nu_j^2} S_{mj}(z) \Phi_m(w_J(z)) \otimes \Phi_j(w_J(z)) \frac{1}{\tilde{r}_j(z)}.$$

Recalling equations (10) and (11), we see that if ψ is in the image of the singular part of $R(z)$ at z_0 , then ψ is a linear combination of eigenfunctions of type I (see (9)) and

$$(17) \quad \{f : (P - \pi(z_0))^k = 0 \text{ for some } k \in \mathbb{N}; f \sim \sum_j \sum_{m \leq m_l} b_{jm} t^m e^{i\tilde{r}_j(z_0)t} \phi_j(y), \\ b_{jm} \neq 0 \text{ for some } j \in \tilde{J}_{z_0}, \text{ some } m\}.$$

We call functions of the form (17) type II. Now take $J = \max \mathcal{E}_{z_0}$. If g is in the image of the singular parts of $R(z)$ at both z_0 and at $w_J(z_0)$, then it must be of type I.

It is the appearance of the type II functions which we must understand. Suppose g is in the image of the residue of $R(z)$ at z_0 and is of type II. Then, since it is not in the image of the residue of $R(z)$ at $w_J(z_0)$, it must be in the image of the residue of

$$(18) \quad \sum_{\sigma_j^2 \leq \nu_j^2} \sum_{\sigma_m^2 \leq \nu_j^2} S_{mj}(z) \Phi_m(w_J(z)) \otimes \Phi_j(w_J(z)) \frac{1}{\tilde{r}_j(z)}$$

at $z = z_0$. First assume that $\Phi_j(w_J(z))$, $j \in \tilde{J}_{z_0}$ has no poles at z_0 . Recalling that we may write $S_{\mathcal{J}_{z_0}}(z)$ near z_0 as in Lemma 3.1, we can write (18) as

$$(19) \quad \sum_{j=1}^p (z - z_0)^{-k_j} \Psi_j^1(z) \otimes \Psi_j^2(z) + H(z, z_0)$$

where $k_1 + \dots + k_p = \mu m_{z_0}(S_{\mathcal{J}_{z_0}}(z))$, and $\Psi_j^1(z)$, $\Psi_j^2(z)$, and $H(z, z_0)$ are holomorphic near z_0 . Then it is easy to see that the rank of the image of the residue of (19) cannot exceed $\mu m_{z_0}(S_{\mathcal{J}_{z_0}}(z))$, and the total rank of the residue of $R(z)$ at z_0 cannot exceed

$$\mu m_{z_0}(S_{\mathcal{J}_{z_0}}) + \dim\{f \in L^2(X) : (P - \pi(z))f = 0, f \sim \sum_{j \notin \tilde{J}_{z_0}} c_j e^{i\tilde{r}_j(z)t} \phi_j\}.$$

To finish the proof, we need only understand what happens if $\Phi_j(w_J(z))$ has a pole at z_0 for some $j \in \tilde{J}_{z_0}$. However, a pole of $\Phi_j(w_J(z))$ cannot contribute to the singularity of $R(z)$ at z_0 because the expansion on the ends of the singular part is of the wrong form. Therefore, the proof of this case follows much as the proof of the previous one. \square

The following proposition will also be useful.

Proposition 3.3. *Suppose $z_0 \in \hat{Z}$ is such that $\pi(z_0)$ is not in the spectrum of P . Then*

$$m_{z_0}(R) = \mu d_{z_0}(S_{\mathcal{E}_{z_0}}).$$

Proof. We sketch the proof of this proposition, as it is very similar to the proof of Theorem 3.1.

Just as in Lemma 3.2, one can show that $\mu m_{z_0}(S_{\mathcal{E}_{z_0}}) \leq m_{z_0}(R)$. Here, of course,

$$\dim\{f \in L^2(X) : (P - \pi(z))f = 0, f \sim \sum_{j \notin \tilde{\mathcal{J}}_{z_0}} c_j e^{i\tilde{r}_j(z)t} \phi_j\} = 0,$$

by our assumption on $\pi(z_0)$. Moreover, $S_{\mathcal{E}_{z_0}}(z)$ has no zeros at z_0 as a zero would imply the existence of an L^2 eigenfunction, and thus

$$\mu m_{z_0}(S_{\mathcal{E}_{z_0}}) = \mu d_{z_0}(S_{\mathcal{E}_{z_0}}).$$

Finishing the proof thus requires showing that

$$m_{z_0}(R) \leq \mu m_{z_0}(S_{\mathcal{E}_{z_0}}).$$

This can be done as in Lemma 3.3, first noting that, if $\psi_l \neq 0$ is in the image of the singular part of R at z_0 , then on the ends

$$(20) \quad \psi_l(t, y) = \sum_j \sum_{m \leq m_j} b_{ljm} t^m e^{i\tilde{r}_j(z_0)t} \phi_j(y)$$

where $b_{ljm} \neq 0$ for some $j \in \mathcal{E}_{z_0}$ and some m . In fact, because $\pi(z_0)$ is not an eigenvalue of P , the linear independence of $\psi_1, \psi_2, \dots, \psi_q$ in the image of the singular part of $R(z)$ at z_0 is equivalent to the linear independence of

$$\sum_{j \in \tilde{\mathcal{E}}_{z_0}} \sum_{m \leq m_j} b_{ljm} t^m e^{i\tilde{r}_j(z_0)t} \phi_j(y),$$

$l = 1, \dots, q$. Using this and the fact that $\Xi_{z_0}(R)$ is symmetric, where Ξ_{z_0} is the singular part at z_0 , we obtain that

$$\begin{aligned} & \dim \text{Image } \Xi_{z_0}(R) \\ & \leq \dim \text{Image } \Xi_{z_0} \left(\sum_{j, m \in \tilde{\mathcal{E}}_{z_0}} S_{mj}(z) \Phi_m(w_J(z)) \otimes \Phi_j(w_J(z)) \frac{1}{\tilde{r}_j(z)} \right). \end{aligned}$$

Then, if $\Phi_j(w_J(z))$ is regular at z_0 for all $j \in \tilde{\mathcal{J}}_{z_0}$, it is easy to see that

$$m_{z_0}(R) \leq \mu m_{z_0}(S_{\mathcal{E}_{z_0}}).$$

Again, if $\Phi_j(w_J(z))$ has a pole at z_0 , it does not contribute to the singularities of $R(z)$ at z_0 but corresponds instead to a singularity of $R(z)$ at $w_J(z_0)$. \square

4. PROOF OF THEOREM 4.1

In this section we bound the number of poles of the resolvent in neighborhoods of the ramification points on the boundary of the physical sheet, with the size of the neighborhoods increasing. We will use the fact that if $\chi = 1$ for $t < \max(a_i, 0)$, then $\dim \text{Image } \Xi_{z_0}(R) = \dim \text{Image } \Xi_{z_0}(R\chi)$, where $\Xi_{z_0}(T)$ is the singular part of T at z_0 .

We recall Theorem 1.1

Theorem. *Assume X satisfies the hypothesis (H1) and let $\beta < 1$. Then, in the connected components of $\{z' \in \hat{Z} : |r_m(z')| < \beta\sqrt{\alpha\nu_m}\}$ that meet the physical sheet, there are at most $\mathcal{O}(m^{n-1})$ resonances.*

A corollary to this is

Theorem 4.1. *Assume X satisfies the hypothesis (H1) and let $\beta < 1$. Then, on the closure of the sheet with $\text{Im } r_j(z) < 0$ if and only if $j \leq m$, there are at most $\mathcal{O}_\beta(m^{n-1})$ poles of the resolvent of P with $|r_m(z)| < \beta\sqrt{\alpha\nu_m}$.*

We shall use the Fredholm determinant method used in, for example, [12, 15, 21, 24, 25]. We will find a trace class operator $K_m^\#(z)$ so that, in the desired region, the poles of the resolvent are contained in the zeros of $I + K_m^\#(z)$, and thus in the zeros of $\det(I + K_m^\#(z))$. In addition, $K_m^\#(z_0) = 0$, where $z_0 \in \hat{Z}$ is a “base point” which depends on m . To do this, we first construct an approximation of the resolvent adapted to this problem and valid in this region, obtaining a compact operator $K_m(z)$ so that the poles of the resolvent in this region are contained in the zeros of $I + K_m(z)$. Further manipulations simplify $I + K_m(z)$.

For $m > 1$, there are two ramification points on the boundary of the physical sheet of \hat{Z} which correspond to ν_m^2 , and thus, for large m , two connected components of $\{z' \in \hat{Z} : |r_m(z')| < \beta\sqrt{\alpha\nu_m}\}$. We shall work near the ramification point which is reached by taking a limit as $\text{Im } \pi(z) \downarrow 0$, $\text{Re } \pi(z) \rightarrow \nu_m^2$. We will designate this ramification point by $(\nu_m^2)_+$. A similar analysis works near the other point, $(\nu_m^2)_-$, obtained by taking a limit for z in the physical space, $\text{Im } \pi(z) \uparrow 0$, $\text{Re } \pi(z) \rightarrow \nu_m^2$.

We assume $m > 1$.

For $c > \max(a_i, 0)$, let

$$X_c = X_{\text{comp}} \sqcup_{i=1}^m (X_i \cap (a_i, c] \times Y_i).$$

For $\zeta \in \mathbb{C}$, let $R_c(\zeta) = (P|_{X_c} - \zeta)^{-1}$ be the resolvent of P on X_c with Dirichlet boundary conditions. (If X has a boundary and we are considering Neumann boundary conditions on X , we use Neumann boundary conditions on X_c here.) Recall that $R_Y(z) = (D_t^2 + \Delta_Y - z)^{-1}$ is defined by (1).

For $i = 1, 2, 3$, choose $\chi_i \in C_c^\infty(X)$ so that $\chi_i \chi_{i+1} = \chi_i$, $i = 1, 2$, $\chi_1 \equiv 1$ on $X_{\max(a_i, 0)}$, $\nabla \chi_i$ only depends on t , and $|\nabla \chi_i| < \gamma$, $|\Delta \chi_i| < \gamma$. It suffices to take $\gamma \leq \alpha(1 - \beta)^2 (1000(1 + \alpha))^{-1}$. Choose c_0 so that the support of χ_3 is properly contained in X_{c_0} .

Choose z_0 in the physical plane so that $\pi(z_0) = \nu_m^2 + \frac{1}{4}\alpha(1 - \beta)^2 \nu_m i$ and let

$$E_m(z) = \chi_3 R_{c_0}(\pi(z)) \Pi_m \chi_2 + \chi_3 R_{c_0}(\pi(z_0)) (1 - \Pi_m) \chi_2 + (1 - \chi_1) R_Y(z) (1 - \chi_2).$$

Here Π_M projects off of the eigenfunctions of $P|_{X_{c_0}}$ with eigenvalues in $(\nu_m^2 - 5\alpha\nu_m, \nu_m^2 + 5\alpha\nu_m)$. This of course depends on c_0 , but we omit this dependence in our notation. Then

$$(21) \quad (P - \pi(z))E_m(z) = I + \tilde{K}_m(z),$$

where

$$\begin{aligned} \tilde{K}_m(z) &= [P, \chi_3] (R_{c_0}(\pi(z))\Pi_m + R_{c_0}(\pi(z_0))(1 - \Pi_m)) \chi_2 \\ &\quad - \chi_3(\pi(z) - \pi(z_0))R_{c_0}(\pi(z_0))(1 - \Pi_m)\chi_2 - [P, \chi_1]R_Y(z)(1 - \chi_2), \end{aligned}$$

and, if the domain is restricted to $L_{\text{comp}}^2(X)$, $\tilde{K}_m(z)$ is meromorphic on \hat{Z} , with the poles corresponding to poles of $R_{c_0}(\pi(z))\Pi_m$. Let $\chi \in C_c^\infty(X)$ be 1 on the support of χ_3 . Then

$$(22) \quad (P - \pi(z))E_m(z)\chi = \chi(I + \tilde{K}_m(z)\chi) = \chi(I + K_m(z))$$

and $K_m(z)$ is compact.

Our choice of z_0 and γ guarantee that $I + K_m(z_0)$ is invertible, with norm bounded by 2. Then, by analytic Fredholm theory, $I + K_m(z)$ is invertible at all but a discrete set of points of \hat{Z} . The points where it is not invertible correspond to zeros of $I + K_m(z)$, and the poles of the cut-off resolvent $(P - z)^{-1}\chi$ are included in the union of the zeros of $I + K_m(z)$ and the poles of $E_m(z)$. However, we will restrict our attention to a region that does not include any poles of $E_m(z)$.

Using the fact that $I + K_m(z_0)$ is invertible, with norm bounded by 2, we obtain that in the region in question, the poles of the resolvent are contained in the zeros of

$$(23) \quad I + (I + K_m(z_0))^{-1}(K_m(z) - K_m(z_0)).$$

Now we restrict our attention to a region where $|r_m(z) - r_m(z_0)| < \rho\sqrt{\alpha\nu_m}$, where

$$\rho = (\beta^2/4 + 3/4)^{1/2}$$

and z lies on one of the four sheets that meet the ramification point $(\nu_m^2)_+$, with m large. In this region, we have

$$\|(I + K_m(z_0))^{-1}[P, \chi_3] (R_{c_0}(\pi(z)) - R_{c_0}(\pi(z_0))) \Pi_m \chi_2\| \leq \frac{1}{3},$$

and

$$(24) \quad \|((I + K_m(z_0))^{-1}[P, \chi_1] \sum_{\sigma_j^2 > \nu_{m+1}^2} (R_{e_j}(z) - R_{e_j}(z_0)) (\chi - \chi_2))\| \leq \frac{1}{3}.$$

For (24), we are using the fact that $[P, \chi_1]$ depends only on t . Therefore, in this region the poles of the resolvent are contained in the zeros of

$$(25) \quad I + K_m^\#(z),$$

where

$$K_m^\#(z) = L_m(z) (K_{1m}(z) + K_{2m}(z))$$

with

$$(26) \quad K_{1m}(z) = -\chi_3(\pi(z) - \pi(z_0))R_{c_0}(\pi(z_0))(1 - \Pi_m)\chi_2,$$

$$(27) \quad K_{2m}(z) = -[P, \chi_1] \sum_{\sigma_j^2 \leq \nu_{m+1}^2} (R_{ej}(z) - R_{ej}(z_0)) (\chi - \chi_2),$$

and the norm of $L_m(z)$ is bounded, independent of m and z , as long as we stay in the region described.

Now $K_m^\#(z)$ is trace class. We consider the function

$$h(z) = \det(I + K_m^\#(z)),$$

and note that $h(z_0) = 1$ and that h is holomorphic on \hat{Z} when $|r_m(z) - r_m(z_0)| < 2\sqrt{\alpha\nu_m}$. We will apply Jensen's theorem to h to obtain an upper bound on the number of zeros of the resolvent in this region, and to do this we need the following lemma.

Lemma 4.1. *If $|r_m(z) - r_m(z_0)| \leq \rho\sqrt{\alpha\nu_m}$ and z lies in the connected component of $\{z' \in \hat{Z} : |r_m(z') - r_m(z_0)| \leq \rho\sqrt{\alpha\nu_m}\}$ containing $(\nu_m^2)_+$, then $|h(z)| \leq Ce^{C\langle m \rangle^{n-1}}$.*

Proof. We remark that because of Weyl's law and the hypothesis (H1), for large m ν_m is bounded above and below by constant multiples of m .

We use the property that

$$(28) \quad |\det(I + A + B)| \leq \det(I + |A|)^2 \det(I + |B|)^2$$

(e.g. [12, Lemma 6.1]). Since $(I - \Pi_m)\chi_2$ has rank bounded by Cm^{n-1} and $\|R_{c_0}(\pi(z_0))\| \leq \frac{C}{\langle \nu_m \rangle}$, we obtain, in the region with $|r_j(z) - r_j(z_0)| \leq \rho\sqrt{\alpha\nu_m}$, and thus $|\pi(z) - \pi(z_0)| \leq C\langle \nu_m \rangle$,

$$|\det(1 + |L_m(z)K_{1m}(z)|)| \leq Ce^{C\langle m \rangle^{n-1}}.$$

Consider next

$$\begin{aligned} L_m(z)K_{2m}(z) &= L_m(z)[P, \chi_1] \sum_{\sigma_j^2 \leq \nu_{m+1}^2} (R_{ej}(z) - R_{ej}(z_0)) (\chi - \chi_2) \\ &= L_m(z)[P, \chi_1] \sum_{j=1}^m \left(e^{i|t-t'|r_j(z)} - e^{i|t+t'|r_j(z)} - e^{i|t-t'|r_j(z_0)} + e^{i|t+t'|r_j(z_0)} \right) \\ &\quad \times \sum_{\sigma_l^2 = \nu_j^2} \phi_l(y) \overline{\phi_l(y')} (\chi - \chi_2). \end{aligned}$$

Since $[P, \chi_1]$ and $\chi - \chi_2$ have disjoint supports,

$$[P, \chi_1] \left(e^{i|t-t'|r_j(z)} - e^{i|t+t'|r_j(z)} - e^{i|t-t'|r_j(z_0)} + e^{i|t+t'|r_j(z_0)} \right) \phi_l(y) \overline{\phi_l(y')} (\chi - \chi_2)$$

is a rank four operator with norm bounded by $Ce^{C|\operatorname{Im} r_j(z)|}$. We need, therefore, to bound

$$(29) \quad \sum_1^{m+1} |\operatorname{Im} r_j(z)| \#\{\sigma_l^2 : \sigma_l^2 = \nu_j^2\}.$$

We have

$$\#\{\sigma_l^2 : \sigma_l^2 = \nu_j^2\} \leq Cj^{n-2}.$$

On the region of interest, $|\operatorname{Im} r_j(z)|$ can be bounded by $2\alpha\nu_m(|\nu_m^2 - \nu_j^2|)^{-1/2}$, if $j < m$, and $\sqrt{\alpha\nu_m}$, if $j = m$. Therefore,

$$(30) \quad \sum_1^{m+1} |\operatorname{Im} r_j(z)| \#\{\sigma_l^2 : \sigma_l^2 = \nu_j^2\} = \sum_1^{m-2} |\operatorname{Im} r_j(z)| \#\{\sigma_l^2 : \sigma_l^2 = \nu_j^2\} + \mathcal{O}(m^{n-1}).$$

To bound (29), then, it suffices to bound

$$(31) \quad 2\alpha\nu_m \int_0^{\nu_m^2} (\nu_m^2 - l^2)^{-1/2} l^{n-2} dl \\ = 2\alpha\nu_m^{n-1} \int_0^{\nu_m^{-1}/\nu_m} (1 - s^2)^{-1/2} s^{n-2} ds \leq \alpha C\nu_m^{n-1}.$$

□

Proof of Theorem 1.1. The poles of the resolvent in the region in question are contained in the zeros of $h(z)$. Working near the point $(\nu_m^2)_+$, we use $r_m(z)$ as a coordinate, which we may do as long as we keep away from other ramification points or regions where $r_m(z)$ fails to be one-to-one. We apply Jensen's Theorem to h , using a circle centered at z_0 and having radius $\rho\sqrt{\alpha\nu_m}$. Then the theorem follows because the disk $|r_m(z)| \leq \beta\sqrt{\alpha\nu_m}$ is properly contained in the disk $|r_m(z) - r_m(z_0)| \leq \rho\sqrt{\alpha\nu_m}$, with the ratio of the distance between the two boundaries and the radius of the larger disk bounded from below, independent of m . □

5. PROOF OF THEOREM 1.2

We recall the statement of Theorem 1.2:

Theorem. *Let $X = \mathbb{R} \times Y$ and suppose X satisfies (H1), and let $\rho > 0$ be fixed. Consider the operator $\Delta + V$, for real-valued $V \in L_{\text{comp}}^\infty(X)$. Then, on the connected components of $\{z \in \hat{Z} : |r_m(z)| < \rho\}$ that meet the physical sheet, the number of poles is bounded by $C(1 + m^{n-2})$.*

We prove this by the Fredholm determinant method as in the previous theorem. We assume that $\rho > 1$.

Let $R_0(z) = (\Delta - z)^{-1}$; its Schwartz kernel is given by

$$(32) \quad R_0(z) = \sum_{j=1}^{\infty} \frac{i}{2r_j(z)} e^{i|t-t'|r_j(z)} \sum_{\sigma_l^2 = \nu_j^2} \phi_l(y) \bar{\phi}_l(y'),$$

and let $R_V(z) = (\Delta + V - z)^{-1}$. We have

$$(33) \quad (\Delta + V - \pi(z))R_0(z) = I + VR_0(z).$$

Let $\chi(t) \in C_c^\infty(X)$ be one on the support of V , with $|\chi| \leq 1$. Then, if $z_0 \in \hat{Z}$ is not a ramification point and $R_V(z)\chi$ has a pole at z_0 , then $I + VR_0(z_0)\chi$ has nontrivial null space.

For $r > 0$, $m \in \mathbb{N}$, set $B_{m,r}$ to be the connected components of $\{z \in \hat{Z} : |r_m(z)| < r\}$ that meet the physical sheet.

Let $\tilde{\rho} = \max(\rho, 2\|V\|_\infty^{1/2})$, and restrict z to $B_{m,4\tilde{\rho}}$. Take m sufficiently large that the only ramification points in $B_{m,4\tilde{\rho}}$ correspond to ν_m^2 . Then

$$I + VR_0(z)\chi = (I + K_1(z))(I + (I + K_1(z))^{-1}K_2(z))$$

where $K_1(z)$ has Schwartz kernel

$$K_1(z) = V(t, y) \sum_{j \neq m} \frac{i}{2r_j(z)} e^{i|t-t'|r_j(z)} \sum_{\sigma_i^2 = \nu_j^2} \phi_i(y) \bar{\phi}_i(y') \chi(t')$$

and $K_2(z)$

$$K_2(z) = V(t, y) \frac{i}{2r_m(z)} e^{i|t-t'|r_m(z)} \sum_{\sigma_i^2 = \nu_m^2} \phi_i(y) \bar{\phi}_i(y') \chi(t').$$

By choosing m sufficiently large, so that

$$\frac{A^{1/2}}{|\alpha\nu_m - 16\tilde{\rho}^2|^{1/2}} \exp\left(A \frac{16\tilde{\rho}^2}{\sqrt{2\alpha\nu_m}}\right) < \frac{1}{2\|V\|},$$

where $A = \max_{t, t' \in \text{supp } \chi} |t - t'|$, we have $\|(I + K_1(z))^{-1}\| \leq 2$ on $B_{m,4\tilde{\rho}}$. The poles of $R_V(z)$ in $B_{m,4\tilde{\rho}}$, other than the ramification point, correspond to values of z for which $I + (I + K_1(z))^{-1}K_2(z)$ has non-trivial null space. We remark that $R_0(z)$ has a pole of rank $M_Y(\nu_m^2)$ at the ramification point, where $M_Y(\nu_m^2)$ is the multiplicity of ν_m^2 as an eigenvalue of Δ_Y , and this can contribute a pole of up to the same multiplicity to $R_V(z)$, even if $I + (I + K_1(z))^{-1}K_2(z)$ is invertible there.

Let

$$K_3(z) = (I + K_1(z))^{-1}K_2(z).$$

The poles of $R_V(z)$ in $B_{m,4\tilde{\rho}}$ are contained in the zeros of $I + K_3(z)$ in the same region, except, possibly, at the ramification points, as discussed above. Now we shall work near $(\nu_m^2)_+$ as in the proof of the previous theorem, as a similar analysis will work for the other connected component of $B_{m,4\tilde{\rho}}$. Choose z_0 in the physical space with $\pi(z_0) = \nu_m^2 + 4i\|V\|_\infty$. Then $\|K_2(z_0)\| \leq 1/4$, and

$$\left\| (I + (I + K_1(z_0))^{-1}K_2(z_0))^{-1} \right\| = \|(I + K_3(z_0))^{-1}\| \leq 2.$$

Let

$$h(z) = \left(\frac{r_m(z)}{r_m(z_0)} \right)^{2M_Y(\nu_m^2)} \det(I + (I + K_3(z_0))^{-1}(K_3(z) - K_3(z_0))).$$

Then $h(z_0) = 1$, and, except, possibly, for some at the ramification point, the poles of $R_V(z)$ in the connected component of $B_{m,4\tilde{\rho}}$ that includes $(\nu_m^2)_+$, are contained in the zeros of $h(z)$ in the same region. This misses at most $M_Y(\nu_m^2) = \mathcal{O}(\nu^{n-2})$

poles of R_V at the ramification point. An application of Jensen's theorem on a circle centered at z_0 and with $|r_m(z) - r_m(z_0)| \leq 3\tilde{\rho}$ will then finish the proof, after we have proved the following lemma.

Lemma 5.1. *On the connected component of $B_{m,4\tilde{\rho}}$ that contains $(\nu_m^2)_+$,*

$$|h(z)| \leq \exp(C(1 + |m|^{n-2})).$$

Proof. Let $R_{ems}(z)$ be the operator with Schwartz kernel

$$\frac{i}{2r_m(z)} \chi(t) (e^{i|t-t'|r_m(z)} - 1)$$

and let $R_{em0}(z_0)$ be the operator with Schwartz kernel

$$\frac{i}{2r_m(z_0)} \chi(t) e^{i|t-t'|r_m(z_0)}.$$

Let

$$A_1(z) = (I + K_1(z))^{-1} V \frac{i}{2r_m(z)} \sum_{\sigma_i^2 = \nu_m^2} \phi_l \otimes \bar{\phi}_l \chi.$$

Then

$$\begin{aligned} K_3(z) - K_3(z_0) &= ((I + K_1(z))^{-1} V R_{ems}(z) - (I + K_1(z_0))^{-1} V R_{em0}(z_0)) \\ &\quad \times \sum_{\sigma_i^2 = \nu_m^2} \phi_l \otimes \bar{\phi}_l \chi + A_1. \end{aligned}$$

Now we shall use (28) and

$$|\det(I + |BT|)| \leq \det(I + \|B\| |T|)$$

(e.g. [12, Lemma 6.1]). Then

$$\begin{aligned} &|\det(I + (I + K_3(z_0))^{-1} (K_3(z) - K_3(z_0)))| \\ &\leq \det(I + \|(I + K_3(z_0))^{-1} (I + K_1(z))^{-1} V\| \|R_{ems}(z)\| \sum_{\sigma_i^2 = \nu_m^2} |\phi_l \otimes \bar{\phi}_l \chi|)^4 \\ &\quad \times \det(I + \|(I + K_3(z_0))^{-1} (I + K_1(z_0))^{-1} V\| \|R_{em0}(z_0)\| \sum_{\sigma_i^2 = \nu_m^2} |\phi_l \otimes \bar{\phi}_l \chi|)^4 \\ &\quad \times \det(I + |(I + K_3(z_0))^{-1} A_1|)^2. \end{aligned}$$

On $B_{m,4\tilde{\rho}}$, $\|(I + K_3(z_0))^{-1} (I + K_1(z))^{-1}\| \leq C$. For a compact operator A , let $\mu_1(A) \geq \mu_2(A) \geq \mu_3(A) \geq \dots$ be the characteristic values of A ; that is, the eigenvalues of $|A^*A|^{1/2}$. Then, if A is trace class,

$$\det(I + |A|) = \prod (I + \mu_j(A)).$$

For $p = 0, 1, 2, \dots$,

$$\mu_{pM_Y(\nu_m^2)+j}(R_{ems}(z)) \sum_{\sigma_i^2 = \nu_m^2} \phi_l \otimes \bar{\phi}_l \chi = \tilde{\mu}_p(r_m(z)), \quad j = 1, \dots, M_Y(\nu_m^2).$$

Here we use the fact that $R_{ems}(z)$ depends on z only through $r_m(z)$. The $\tilde{\mu}_p(w)$ are independent of m . Therefore,

$$\begin{aligned} |\det(I + \|(I + K_3(z_0))^{-1}(I + K_1(z))^{-1}V\|R_{ems}(z) \sum_{\sigma_i^2 = \nu_m^2} \phi_l \otimes \overline{\phi_l \chi})| \\ \leq C^{M_Y(\nu_m^2)} \leq C^{(m)^{n-2}}. \end{aligned}$$

The same argument gives

$$|\det(I + \|(I + K_3(z_0))^{-1}(I + K_1(z_0))^{-1}V\|R_{em0}(z_0) \sum_{\sigma_i^2 = \nu_m^2} \phi_l \otimes \overline{\phi_l \chi})| \leq C^{(m)^{n-2}}.$$

A similar argument bounds

$$r_m(z)^{2M_Y(\nu_m^2)} \det(I + |(I + K_3(z_0))^{-1}A_1|)^2.$$

The rank of $(I + K_3(z_0))^{-1}A_1$ is $M_Y(\nu_m^2)$, and $\|(I + K_3(z_0))^{-1}A_1\|$ is bounded for $4\tilde{\rho} \geq |r_m(z)| > c > 0$. Since $r_m(z)(I + K_3(z_0))^{-1}A_1$ is bounded on $B_{m,4\tilde{\rho}}$,

$$r_m^{M_Y(\nu_m^2)} \det(I + |(I + K_3(z_0))^{-1}A_1|) \leq C^{M_Y(\nu_m^2)} \leq C^{(m)^{n-2}}$$

on $B_{m,4\tilde{\rho}}$. □

A consequence of Theorem 1.2 is

Corollary 5.1. *Let $X = \mathbb{R} \times Y$ and suppose X satisfies (H1). Let $V \in L_{\text{comp}}^\infty(X)$ be real-valued, and let*

$$N(\lambda) = \#\{\lambda_j^2 \leq \lambda^2 : \lambda_j^2 \text{ is an eigenvalue of } \Delta + V\}.$$

Then

$$N(\lambda) = \mathcal{O}(\lambda^{n-1}).$$

Proof. Suppose $\tau \in \mathbb{R}_+$. Then, using (32) and (33), we see that if

$$\frac{\|V\|(\text{length supp } V + 1)}{\min_j |r_j(\tau)|} \leq \frac{1}{2}$$

then τ cannot be an eigenvalue of $\Delta + V$. Here

$$\text{length supp } V = \max_{a,b \in \text{supp } V} |a - b|.$$

This means that any eigenvalue must lie within a fixed distance of some ν_j^2 . Theorem 1.2 provides a bound on the number of eigenvalues within such a ball; summing over the ν_j^2 we obtain the corollary. □

6. PROOF OF THEOREM 1.3

We recall Theorem 1.3:

Theorem. *Fix a sheet of \hat{Z} , and let $\{z_k\}$ be the resonances of P on this sheet. Then*

$$\sum \frac{|\operatorname{Im}(r_1(z_k))|}{|r_1(z_k)|^2} < \infty.$$

In the proof of this theorem we shall use Proposition 3.3 and Carleman's Theorem, which we recall (e.g. [14, Section V.i]).

Theorem (Carleman). *If $F(\zeta)$ is a holomorphic function in the region $\operatorname{Im} \zeta \geq 0$, $F(0) = 1$, and if $a_k = r_k e^{i\theta_k}$ ($k = 1, 2, \dots$) are its zeros in this region, then*

$$\begin{aligned} \sum_{r_k \leq R} \left(\frac{1}{r_k} - \frac{r_k}{R^2} \right) \sin \theta_k &= \frac{1}{\pi R} \int_0^\pi \ln |F(Re^{i\theta})| \sin \theta d\theta \\ &+ \frac{1}{2\pi} \int_0^R \left(\frac{1}{x^2} - \frac{1}{R^2} \right) \ln |F(x)F(-x)| dx + \frac{1}{2} \operatorname{Im} F'(0). \end{aligned}$$

We note that Carleman's Theorem also holds for a function $F(\zeta)$ which is holomorphic in $\operatorname{Im} \zeta > 0$ and continuous in $\operatorname{Im} \zeta \geq 0$. In order to see this, apply Carleman's Theorem to $F_\epsilon(\zeta) = F(\zeta + i\epsilon)/F(i\epsilon)$, $\epsilon > 0$. Then, since both sides of the equation are continuous in ϵ for small $\epsilon \geq 0$, the theorem holds in this case as well.

Proof of Theorem 1.3. Fix a sheet of \hat{Z} , and let

$$\mathcal{E} = \{j \in \mathbb{N} : \operatorname{Im} r_j(z) < 0 \text{ on this sheet}\}.$$

By Proposition 3.3, the poles of the resolvent on this sheet (but not on its boundary) correspond, with multiplicity, to the poles of $\det S_\mathcal{E}(z)$ on this sheet. We have, by Proposition 3.1,

$$S_\mathcal{E}^{-1}(z) = S_\mathcal{E}(w_\mathcal{E}(z)),$$

and, if z lies on the sheet with

$$\operatorname{Im} r_j(z) < 0 \text{ if and only if } j \in \mathcal{E},$$

then $w_\mathcal{E}(z)$ lies on the physical sheet. Therefore, we reduce the problem to a question about the zeros of $\det S_\mathcal{E}(z)$ for z on the physical sheet.

It is helpful to identify the physical sheet with the upper half plane using the variable $\zeta = r_1(z)$. Let $\Psi_{ij}(\zeta) = S_{ij}(z(\zeta))$ and $\Psi(\zeta) = S_\mathcal{E}(z(\zeta))$. Using the fact that $S_\mathcal{E}(z)$ is meromorphic on \hat{Z} we can extend Ψ to the closed upper half plane by continuity, except, perhaps, for a finite number of points corresponding to poles of $S_\mathcal{E}(z)$. We shall also call these points poles of $\Psi(\zeta)$. The matrix $\Psi(\zeta)$ has at most a finite number of poles in the closed upper half plane.

To prove the theorem, we shall use Carleman's theorem applied to a multiple of $\det \Psi(\zeta)$, chosen so that the product is holomorphic in the upper half plane and continuous on its closure. In order to do this, we need bounds on $\det \Psi$. Let

$J = \max\{j \in \mathcal{E}\}$. If $\zeta \in \mathbb{R}$, $\zeta^2 + \sigma_1^2 > \nu_j^2$, then $|\Psi_{ij}| \leq C$ ([4]). To bound $|\Psi_{ij}(\zeta)|$ away from the real axis, we need to bound $|S_{ij}(z)|$, where S_{ij} is determined by the expansion on the ends of Φ_j , see (2) and (3). We recall that

$$(34) \quad \Phi_j(z(\zeta)) = \chi(t)e^{-i\tilde{r}_j(\zeta^2 + \sigma_1^2)t} \phi_j - (P - \zeta^2 - \sigma_1^2)^{-1}(P - \zeta^2 - \sigma_1^2)\chi(t)e^{-i\tilde{r}_j(\zeta^2 + \sigma_1^2)t} \phi_j$$

where $\chi(t) \in C^\infty(\mathbb{R})$ is supported in $t > \max(a_i)$ and is one in a neighborhood of infinity.

We note that for $j \in \tilde{\mathcal{E}}$,

$$(35) \quad \begin{aligned} \|(P - \zeta^2 - \sigma_1^2)\chi(t)e^{-i\tilde{r}_j(\zeta^2 + \sigma_1^2)t} \phi_j\| &\leq C\langle \zeta^2 + \sigma_1^2 - \sigma_j^2 \rangle^{1/2} e^{C|\operatorname{Im} \tilde{r}_j(\zeta^2 + \sigma_1^2)|} \\ &\leq C\langle \zeta^2 + \sigma_1^2 - \sigma_j^2 \rangle^{1/2} e^{C(\operatorname{Im} \zeta)} \end{aligned}$$

and

$$(36) \quad \left\| \left(e^{i\tilde{r}_l(\zeta^2 + \sigma_1^2)t} \phi_l \right)_{|t>a} \right\| = (2|\operatorname{Im} \tilde{r}_l(\zeta^2 + \sigma_1^2)|)^{-1/2} e^{-a \operatorname{Im} \tilde{r}_l(\zeta^2 + \sigma_1^2)}.$$

Since

$$\|(P - \zeta^2 - \sigma_1^2)^{-1}\| \leq (\operatorname{dist}(\zeta^2 + \sigma_1^2, \sigma(P)))^{-1},$$

we have, using (3), (34), (35) and (36),

$$|\Psi_{lj}(\zeta)| \leq \frac{C\langle \zeta^2 + \sigma_1^2 - \sigma_l^2 \rangle^{1/2}}{\operatorname{dist}(\zeta^2 + \sigma_1^2, \sigma(P))} e^{C(\operatorname{Im} \zeta)}$$

when $|\zeta|$ is large. This proves

$$|\det \Psi(\zeta)| \leq C e^{C(\operatorname{Im} \zeta)}$$

if $\operatorname{Im} \zeta > \epsilon > 0$ and $|\zeta|$ large. To obtain a bound in the closure of the upper half-plane, apply the Phragmen-Lindelöf theorem to

$$h(\zeta) = \prod_1^{k_0} \frac{\zeta - \zeta_j}{\zeta - \zeta_j + Mi} \det \Psi(\zeta)$$

where $\zeta_1, \zeta_2, \dots, \zeta_{k_0}$ are the poles of $\det \Psi(\zeta)$ in the closed upper half plane and M is chosen sufficiently large that $M > \operatorname{Im} \zeta_j$, $j = 1, \dots, k_0$. Then $h(\zeta)$ is holomorphic in the upper half plane, continuous in the closed upper half plane, and

$$|h(\zeta)| \leq C e^{C(\operatorname{Im} \zeta)}$$

in the closed upper half plane. An application of Carleman's Theorem to $h(\zeta)$ finishes the proof. \square

REFERENCES

- [1] A. Aslanyan and E.B. Davies, *Separation of variables in perturbed cylinders*, preprint.
- [2] A. Aslanyan, L. Parnovski, and D. Vassiliev, *Complex resonances in acoustic waveguides*, Quart. J. Mech. Appl. Math. **53** (2000), no. 3, 429–447.
- [3] W. Bulla, F. Gesztesy, W. Renger, B. Simon, *Weakly coupled bound states in quantum waveguides*, Proc. Amer. Math. Soc. **125** (1997), no. 5, 1487–1495.
- [4] T. Christiansen, *Scattering theory for manifolds with asymptotically cylindrical ends*. Journal of Functional Analysis **131**, 2, (1995), 499–530.
- [5] T. Christiansen and M. Zworski, *Spectral asymptotics for manifolds with cylindrical ends*, Ann. Inst. Fourier **45**, 1 (1995), 251–263.
- [6] E.B. Davies and L. Parnovski, *Trapped modes in acoustic waveguides*, Quart. J. Mech. Appl. Math. **51** (1998), no. 3, 477–492.
- [7] P. Duclos, P. Exner, B. Meller, *Exponential bounds on curvature-induced resonances in a two-dimensional Dirichlet tube*, Helv. Phys. Acta **71** (1998), no. 2, 133–162.
- [8] J. Edward, *On the resonances of the Laplacian on waveguides*, preprint.
- [9] R. Froese, *Asymptotic distribution of resonances in one dimension*, J. Differential Equations **137** (1997), no. 2, 251–272.
- [10] F.R. Gantmacher, *The Theory of Matrices*, Volume I. Chelsea Publishing Company, New York, 1959.
- [11] L. Guillopé, *Théorie spectrale de quelques variétés à bouts*, Ann. Scient. Ec. Norm. Sup. **22**, 4, (1989), 137–160.
- [12] L. Guillopé and M. Zworski, *Upper bounds on the number of resonances for non-compact Riemann surfaces*, J. Functional Analysis **129**, No. 2 (1995), 364–389.
- [13] L. Guillopé and M. Zworski, *Scattering asymptotics for Riemann surfaces*, Annals of Mathematics **145** (1997), 597–660.
- [14] B. Ja. Levin, *Distribution of zeros of entire functions*, American Mathematical Society, Providence, R.I. 1964 viii+493 pp.
- [15] R.B. Melrose, ‘*Polynomial bounds on the distribution of poles in scattering by an obstacle*’, Journées “Équations aux Dérivées partielles” Saint-Jean-de-Monts, 1984.
- [16] R.B. Melrose, *The Atiyah-Patodi-Singer Index Theorem*, A.K. Peters, Wellesley, MA 1993.
- [17] W. Müller, *Spectral geometry and scattering theory for certain complete surfaces of finite volume*, Invent. Math. **109** (1992), 265–305.
- [18] L. Parnovski, *Spectral asymptotics of the Laplace operator on manifolds with cylindrical ends*, Int. J. Math. **6** (1995), 911–920.
- [19] A. Selberg, *Göttingen lectures in: Collected Works*, Vol. I, 626–674, Springer-Verlag, Berlin, 1989.
- [20] J. Sjöstrand and M. Zworski, *Complex scaling and the distribution of scattering poles*, J. of the AMS, **4**, No. 4 (1991), 729–769.
- [21] G. Vodev, *Sharp bounds on the number of scattering poles for perturbations of the Laplacian*, Comm. Math. Phys. **146** (1992), 205–216.
- [22] G. Vodev, *Resonances in the Euclidean scattering*, Cubo Matemática Educacional **3** no. 1 (2001), 317–360.
- [23] M. Zworski, *Distribution of poles for scattering on the real line*, J. Funct. Anal. **73** (2) (1987), 277–296.
- [24] M. Zworski, *Sharp polynomial bounds on the number of scattering poles of radial potentials*, J. Funct. Anal. **82** (1989), 370–403.
- [25] M. Zworski, *Sharp polynomial bounds on the number of scattering poles*, Duke Math. Jour. **59** (2) (1989), 311–323.
- [26] M. Zworski, *Counting scattering poles*. In: Spectral and scattering theory (Sanda, 1992), 301–331, Lecture Notes in Pure and Appl. Math., 161, Dekker, New York, 1994.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI COLUMBIA, MISSOURI 65211