ERDŐS DISTANCE PROBLEM IN THE HYPERBOLIC HALF-PLANE

A Thesis presented to the Faculty of the Graduate School at the University of Missouri-Columbia

---

In Partial Fulfillment of the Requirements for the Degree

Master of Arts

---

by STEVEN SENGER

Dr. Alex Iosevich, Thesis Supervisor

DECEMBER 2009

The undersigned, appointed by the Dean of the Graduate School, have examined the thesis entitled

ERDŐS DISTANCE PROBLEM IN THE HYPERBOLIC HALF-PLANE

Presented by Steven Senger,

a candidate for the degree of master of mathematics,

and hereby certify that, in their opinion, it is worthy of acceptance.

Alex Iosevich

William Banks

Sergei Kopeikin

ACKNOWLEDGEMENTS

I would like to thank Alex Iosevich for his continued support, advice, and friendship throughout the duration of this, and many research projects.

ABSTRACT

The Erdős distance problem asks for the minimum number of distinct distances determined by large finite point sets in the plane. The aim of this work is to investigate how the classical techniques employed in the study of the Erdős distance problem carry over to the hyperbolic half-plane.

Contents

1. Introduction

The Erdős Distance Problem says: given a set of n points in the plane, what is the cardinality of the distance set, or how many distinct distances are determined? The obvious upper bound is $O(n^2)$. The lower bound is much less trivial. There have been numerous improvements in lower bound estimations in the Euclidean setting since the problem was first posed. In his original paper, Erdős [1] demonstrated the lower bound of $\Omega(n^{\frac{1}{2}})$. Leo Moser [2] used Erdős' result for his improvement to $\Omega(n^{\frac{2}{3}})$, but with a slightly different approach to the earlier bound. More complicated arguments have led to better results in recent years. Although the conjecture of $\Omega(n^{1-\epsilon})$ has yet to be demonstrated in the Euclidean setting, there has been some work on analogs of the problem in other settings. Here we explore the problem in the hyperbolic half-plane. First, a few basic facts about the hyperbolic half plane will be outlined. This will be followed by series of arguments explaining how each bound is attained in the hyperbolic half-plane, and how each argument is modified to work in the new setting. For the sake of brevity, throughout the rest of the paper, assume circles, bisectors and distances are hyperbolic unless specified otherwise. Also, c shall denote an unspecified, positive constant.

2. Hyperbolic Half-Plane

The hyperbolic half-plane is fairly well-behaved, but there are a few details that one should be familiar with prior to plunging into the arguments ahead. The hyperbolic, or Poincairé half-plane is characterized as being a surface with constant negative Gaussian curvature. This has several obvious consequences that will become apparent when dealing with distance sets. The metric will behave differently, so our notion of straight will change, as will the idea of a bisector.

The hyperbolic metric can be thought of as the standard Euclidean metric divided by y value. It is generally computed by minimizing a particular line integral over all paths connecting two points. There is also a formula using logarithms and some trigonometric functions. Since we are only counting how many distinct distances there are, we are not concerned with the actual values of the various distances, just that they are distinct. As will become apparent soon enough, it is sufficient to consider a set circles of different radii to get a handle on the number of distinct distances. So rather than bogging ourselves down with calculations involving the hyperbolic metric directly, we will be better served by learning about the hyperbolic notion of a circle. Fortunately, the locus of points that form a circle look exactly the same in the hyperbolic half-plane as the Euclidean plane. The difference is in determining the center and radius. Given a point in the hyperbolic half-plane, (H, K), and a hyperbolic radius, R, one need only draw a Euclidean circle centered at (h, k), with radius r, where $h = H$,

$k = K\cosh(R)$, and $r = K\sinh(R)$. Even though there are some hyperbolic trigonometric functions involved, we won't need their explicit values in the following arguments. What's important is that hyperbolic circles inherit many properties from Euclidean circles, such as intersecting with one another at most twice. Also, two points on the same circle centered at a given point are both the same distance from the center.

Euclidean straight lines are easy to envision, but what does *straight* mean in the hyperbolic sense? Since the hyperbolic half-plane has curvature everywhere, the hyperbolic notion of *straight* might not look very straight at all. It is important to keep in mind what straight means. Let us define a straight line, in general, as the shortest distance between two points. To find the shortest distance between two points, we need to consult the aforementioned line integral. Luckily for us, it turns out that the shortest path between two points can be found as follows. Draw the unique Euclidean circle centered at the x-axis that passes through both points. The shortest hyperbolic path between the two is along the arc connecting the two points. Of course, if the two points share the same x value, the shortest distance will be a vertical line. So hyperbolic straight lines, or as we will call them henceforth, *geodesics*, are not too difficult to deal with.

The last hyperbolic object that we'll need to have a basic understanding of is a bisector. Given two points, let us define a bisector as the locus of points equidistant from the pair. In other words, the bisector of the points p and p' is the set $\{q : d(p, q) = d(p', q)\}$, where

$d(p,q)$ is the hyperbolic distance from p to q. In the Euclidean setting, these are quite familiar straight lines. However, given a slightly more complicated way to measure distance, the bisectors could turn out to be a problem. As the following propositions will show, hyperbolic bisectors are rather well behaved.

Proposition 2.1. *Bisectors in the hyperbolic half-plane are geodesics.*

Proof. Consider two points in the hyperbolic half-plane, p and q. Give them the Cartesian coordinates (x_1, y_1) and (x_2, y_2), respectively.

Suppose that $y_1 = y_2$. The bisector of p and q in this case is clearly a straight vertical line, halfway between the points. One way to visualize this is by drawing hyperbolic circles with hyperbolic centers at each point, and hyperbolic radii equal to half of the diastance between p and q. Repetitively dilating each circle by equal amounts and marking the intersections will result in a locus of points that form our straight vertical line. One could also note that this case is similar to the Euclidean case, as any distortion in the y-direction will not affect either a point or an infinitely long straight vertical line.

In the case that $y_1 \neq y_2$, one need only rotate hyperbolically about the point p until q', the point q mapped under hyperbolic rotation, has a Cartesian y-coordinate equal to y_1. Draw the straight vertical line halfway between p and q'. Recall that this line is the bisector of p and q', as in the previous case. Upon hyperbolically rotating back an equal amount, such that q' is mapped back to q, notice that the straight line has been mapped to a semicircle with its center on the x-axis. This is because straight lines and semicircles centered at the x-axis are geodesics in the hyperbolic half-plane, and geodesics are mapped onto one another by hyperbolic rigid motions, or distance preserving maps. $\square$

Proposition 2.2. *Given two points, p and q, on a hyperbolic circle, C their bisector, l passes through the center c of the circle C. Furthermore, the hyperbolic angle $\angle pcq$, is bisected by l.*

Proof. This proof is similar to that of Proposition 2.1. Rotate the hyperbolic half-plane hyperbolically about c, until p and q are equidistant from the vertical line passing through c. Call this line l'. Call the image of p under the hyperbolic rotation p'. Define q' similarly. Notice that the bisector of p' and q' is l', and that it passes through c. Also notice that l' forms equal hyperbolic angles with the segments $\overline{cp'}$ and $\overline{cq'}$. Call l the hyperbolically rotated image of l'. Notice that l, the bisector of p and q will still go through c. Hyperbolically rotating the half-plane back will preserve angles, so l will bisect $\angle pcq$. $\square$

The symbol $A \gtrsim B$ means that $A \geq cB$, for some constant c as A and B grow large. Also, $A \approx B$ means that both $A \gtrsim B$ and $B \gtrsim A$.

3. The $n^{\frac{1}{2}}$ Argument

There is very little diferent, in the argument for $n^{\frac{1}{2}}$, between the Euclidean and hyperbolic planes. In fact, the only real difference is the location of the centers of the circles of the respective settings. The key point of this argument is that circles can intersect no more than twice. This is true in both the hyperbolic and the Euclidean settings. The hyperbolic argument follows, but the proof for the Euclidean setting is nearly identical.

Theorem 3.1. *Given a set, E, of n points in the hyperbolic half-plane, there exists a point which determines $n^{\frac{1}{2}}$ distinct hyperbolic distances.*

Proof. In order to count hyperbolic distances from a given point, we can count the number of hyperbolic circles centered at that point that cover the rest of the points in the set. The number of such hyperbolic circles would then determine the number of distinct hyerbolic distances from that point, since any two points the same hyperbolic distance away would lie on the same hyperbolic circle.

Consider two points, p and q, in the hyperbolic half-plane. Centered around each, draw enough hyperbolic circles to cover the rest of the points in the set E. Let s and t be the number of hyperbolic circles around p and q respectively. Note that all of the points in E, except for p and q, lie on the intersections of the s and t hyperbolic circles. There are at most $2st$ intersections, because each hyperbolic circle centered at p can intersect each hyperbolic circle centered at q at most twice. So $2st \geq (n-2)$. This means that either $s \gtrsim n^{\frac{1}{2}}$ or $t \gtrsim n^{\frac{1}{2}}$. So in either case, at least one of p or q will have $\gtrsim n^{\frac{1}{2}}$ different hyperbolic circles centered at it. This will mean that that point has at least $n^{\frac{1}{2}}$ distinct hyperbolic distances which can be measured from it. □

4. The $n^{\frac{2}{3}}$ Argument

Classically, Moser achieved this bound with a beautiful method [2] using a logical contiunation to the ideas in the prior argument. He partitioned the points into subsets that determined mutually exclusive distance sets. Unfortunately, extending these ideas would involve unwieldy computations with the hyperbolic metric. It can probably be done, but for the hyperbolic setting, graph theory proves to be a more useful technique. A graph theoretic result from Székely's $n^{\frac{4}{5}}$ argument [3] will be more instructive for the following proofs. The following theorem, which will be key in the following arguments, was first proved by Leighton [9], and later, independently by Ajtai, Chvátal, Newborn, and Szemerédi [10].

Theorem 4.1. *Given a topological multigraph, G, with e edges, v vertices, and a maximum edge multiplicity of m, either $e < 5vm$ or $Cr(G) \gtrsim \frac{e^3}{v^2 m}$.*

In this, and the proofs to follow, topological multigraphs will be used. For all topological multigraphs in this paper, we will assume that the edges are piecewise continuously differentiable curves. This will prevent two edges from crossing each other too many times. Once again, the proof is very similar to the Euclidean case. The crucial difference here is the behavior of the bisectors. Proposition 2.1 will be the key to extending this result from its Euclidean counterpart.

Theorem 4.2. *Given a set, E, of n points in the hyperbolic half-plane, there exists a point which determines $n^{\frac{2}{3}}$ distinct hyperbolic distances.*

Proof. As in the $n^{\frac{1}{2}}$ argument, we will count the number of hyperbolic distances a single point determines by counting the distinct number of hyperbolic circles centered at that point.

We will now draw circles with centers at each point in E. Only draw circles that intersect with at least one other point in E, but draw enough circles so that every other point is covered by circles centered at that point. Let t be the largest number of circles centered at any point. In other words, let $t = \max_{p \in E} |\{d(p,q) : q \in E\}|$, where $d(p,q)$ denotes the hyperbolic distance from p to q. This means that no more than t distances can be measured from any point in the set E. From here, we seek to show that $t \gtrsim n^{\frac{2}{3}}$. If $t \gtrsim n^{\frac{2}{3}}$, there are more than $n^{\frac{2}{3}}$ distinct distances from some point in E. Therefore, we may assume $t \lesssim n^{\frac{2}{3}}$.

Define the topological multigraph, G, by using each point in E as a vertex, and every arc of every circle as an edge. We are guaranteed that these edges will satisfy our constraint of piecewise continuously differentiability, because the arcs of hyperbolic circles will look just like the arcs of Euclidean circles. Delete edges from circles that contain only one or two points. Recall that we have assumed for now that $t \lesssim n^{\frac{2}{3}}$. Since every one of the n points has at most t circles centered at it, there could be no more than t circles deleted for each point. Since there are n points, there can be at most $nt \lesssim n^{\frac{5}{3}}$ circles deleted, each with at most two points. So in this way we lose only on the order of $n^{\frac{5}{3}}$ edges. This will keep us on the order of n^2 edges left in G. Let e be the number of edges left in G after our deletion.

Consider the crossing number of G, $Cr(G)$. The crossing number is defined as the fewest number of crossings of edges over any redrawing of the same topological multigraph. Clearly $Cr(G) \leq 2n^2t^2$, since circles intersect no more than twice and there are no more than nt circles.

Let m be the maximum edge multiplicity, or the largest number of edges between any two vertices of G. If $m \geq n^{\frac{2}{3}}$, then there are two vertices which have more than $n^{\frac{2}{3}}$ edges connecting them. This means that there are $n^{\frac{2}{3}}$ circles through the two points in E represented by the vertices in G. If we consider the bisector of these two points, we would discover that it held more than $n^{\frac{2}{3}}$ points. Proposition 2.1 shows that bisectors in the hyperbolic half-plane are geodesics. If there are s points on l, a given geodesic, we will need to draw at least $\frac{s}{2}$ circles centered at any point on l to cover the rest of the points on l. This is because circles can intersect geodesics, which are either arcs of circles or straight lines, no more than twice. Therefore, in this case, we would have drawn at least $\frac{n^{\frac{2}{3}}}{2}$ circles centered at any point on the bisector. This would make $t \gtrsim n^{\frac{2}{3}}$. So it is apparent that in hyperbolic space, the familiar interactions between geodesics and circles allow for a fair amount of analogous argument.

By the argument in the last paragraph, either we are done, or we are guaranteed that $m \leq n^{\frac{2}{3}}$. In the latter case, we can apply Theorem 4.1 with $m = n^{\frac{2}{3}}$. This gives us that the $Cr(G) \geq \frac{e^3}{n^2m}$.

$$\frac{n^6}{n^2m} \leq \frac{e^3}{n^2m} \leq Cr(G) \leq n^2t^2$$

Recall that $m \leq n^{\frac{2}{3}}$ and solve for t to obtain the desired result. $\square$

5. The $n^{\frac{4}{5}}$ Argument

The next proof will use the same idea as before, with the key results being Theorem 4.1 and Proposition 2.1 once again, but there is an added inclusion/exclusion argument to make the edge deletion more efficient. Also, the celebrated Szemerédi-Trotter incidence theorem [4] will be leaned upon heavily for the rest of the results in this area.

Theorem 5.1. *(Szemerédi-Trotter [4])*
(a) Given n distinct points in the plane, then number, L_m, of lines incident to at least $m > 2$ points is

$$L_m \lesssim \frac{n^2}{m^3} + \frac{n}{m},$$

where the first term is contributed by lines incident to less than $\sqrt{n}$ points, and the latter term is due to lines incident to more than $\sqrt{n}$ points.
(b) Given a set of n points and l lines in the Euclidean plane, the number of incidences of points and lines is at most $c[(nl)^{\frac{2}{3}} + n + l]$.

Of course, Theorem 5.1 seems to be intended for applications in the Euclidean plane. However, Székely [5] gives a short proof of Theorem 5.1, by way of Theorem 4.1. In this proof, the difference between the hyperbolic and Euclidean notions of *straight* does not change the outcome. So Theorem 5.1 can be applied in the hyperbolic setting, by using hyperbolic geodesics in place of Euclidean straight lines.

Theorem 5.2. *Given a set, E, of n points in the hyperbolic half-plane, there exists a point which determines $n^{\frac{4}{5}}$ distinct hyperbolic distances.*

Proof. This proof is very similar to that of Theorem 4. Define G and t as before. Once again, it is obvious that $Cr(G) \leq 2n^2t^2$. The difference here is that we choose the maximum edge multiplicity $m \lesssim t^{\frac{1}{2}}$, and construct a new topological multigraph, G', from G, by deleting edges with multiplicity greater than m. In order for this to be a productive endeavor, G' needs to still have cn^2 edges. This is shown in the following proposition.

Proposition 5.3. *The number of pairs, (l, a), where a is an arc of G', and l is a bisector of the vertices of a and incident to at least m vertices, is at most $c\frac{tn^2}{m^2} + ctn \log n$.*

Proof. First, notice that the number of geodesics incident to 2^i points is at most $c\frac{n^2}{2^{3i}}$, provided $2^i \leq \sqrt{n}$. This is because if there were more, the total number of such geodesics would exceed the bound from Theorem 5.1, part (a). Let the number of points on a curve l be denoted $|l|$. Notice that for each geodesic, the number of bisected edges of G' is at most $2t|l|$. To see this, recall that there are fewer than t circles centered at any point by assumption, and for a geodesic to bisect any edge, the two points that determine the edge must lie on a circle centered at a point on the geodesic. So the number of pairs, (l, a), with $m \leq |l| \leq 4\sqrt{n}$ is at most

$$\sum_{i:m\leq 2^i\leq\sqrt{n}} \frac{ctn^2 2^i}{2^{3i}} \leq \frac{ctn^2}{m^2}.$$

This takes care of the geodesics incident to fewer than $\sqrt{n}$ points. If a given geodesic is incident to more than $\sqrt{n}$ points, the Szemerédi-Trotter theorem will no longer help. This case is even easier though, in light of a simple inclusion-exclusion argument [5]. Since geodesics can intersect each other at most only once, by definition, we're guaranteed that there can only be so many geodesics incident to a relatively large number of points. After recognizing this, there are merely a few simple things to count and we are done.

Call each geodesic incident to more than $l \geq \sqrt{2n}$ points A_i, and let $|A_i|$ be the number of points incident to that geodesic. Suppose N_l is the number of geodesics with between l and $2l$ points, where $l \geq 4\sqrt{n}$. For the proposition to hold, we need $N_l \leq \frac{4n}{l}$. So given $l \geq \sqrt{n}$, let us suppose that $N_l \geq \frac{2n}{l}$, and arrive at a contradiction.

$$n = |E| \geq \left| \bigcup_{l \leq |A_i| \leq 2l} |A_i| \right| \geq \sum_{i=1}^{N_l} \left| A_i \setminus \left(\bigcup_{j=1}^{i-1} A_j \right) \right|,$$

upon possibly reordering the A_i's to put those considered in the union first. This sum is clearly greater than or equal to

$$\sum_{i=1}^{N_l} \max(0, m-i) \geq \sum_{i=1}^{\frac{4n}{l}} \max(0, m-i) \geq \sum_{i=1}^{\sqrt{n}} \max(0, 4\sqrt{n} - i) \geq$$

$$\geq \sqrt{n}(4\sqrt{n} - \sqrt{n}) \geq 3n.$$

So we have a contradiction, implying that $N_l \leq \frac{4n}{l}$.

Now, to get the total number of bisected edges contributed by the geodesics A_i, we'll sum over all of them.

$$\sum_{i:|A_i| \geq |\sqrt{n}} 2t|A_i| \leq \sum_{i:2^j \leq i \leq 2^{j+1}} 2^{j+1} 2t N_{2^j} \leq \sum_{i:2^j \leq i \leq 2^{j+1}} 2^{j+1} 2t \frac{4n}{2^j} \leq$$

$$\leq 4tn \sum_{\sqrt{n} \leq 2^i \leq n} 1 \leq 4tn \log n.$$

This completes the proof of the Proposition 5.3. $\square$

So, we'll remove edges of multiplicity higher than some m, so that we still have a positive proportion of edges in G', but gain in the lower bound for $Cr(G)$. Specifically, we need to optimize the following inequality:

$$c\frac{tn^2}{m^2} + ctn\log n \lesssim n^2.$$

The log term doesn't bother us, so we need only concern ourselves with the other term, which yields the bound $m \gtrsim \sqrt{t}$. Since we want m as low as possible, we will pick $m \approx \sqrt{t}$. When we plug this new value of m into our lower bound for $Cr(G)$, we get this:

$$n^2t^2 \gtrsim Cr(G) \gtrsim \frac{e^3}{n^2m} \approx \frac{n^6}{n^2\sqrt{t}}.$$

Solving this inequality yields the desired result. $\square$

6. The $n^{\frac{6}{7}}$ Argument

When investigating possible improvements to the preceeding argument, the most natural thing to consider is the what keeps the argument from doing better. One of the obstacles that prevents a sharper result is the difficulty in controlling high edge multiplicities in the multigraph. High edge multiplicities occur when a pair of points have a bisector that crosses through a large number of points. These points will generate circles going through the bisected pair, which will in turn yield a large number of edges. In this argument, we will investigate incidences of such points and bisectors. This technique was initially a success for Solymosi and Tóth [7]. Similar techniques have been employed by the current record holders, Katz and Tardos [8], who got $n^{\frac{19}{22}}$. So this style of argument is currently state of the art. The following theorem by Beck [6], will be used.

Theorem 6.1. *Given n points in the plane, either there is a line incident to at least $\frac{n}{100}$ points, or there are on the order of n^2 lines incident to at least two points.*

The theorem is stated in the Euclidean setting, but Beck admits that it could be used much more generally. It is merely an exercise to check that each step in the proof holds for hyperbolic geodesics in place of the Euclidean lines. This helps us get rid of some of the cases that trivially satisfy our claimed distance bound, but won't work with the bisector argument to follow. With all of the previous theorems in tow, we proceed to the main result.

Theorem 6.2. *Given a set, E, of n points in the hyperbolic half-plane, there exists a point which determines $n^{\frac{6}{7}}$ distinct hyperbolic distances.*

Proof. Centered at each of the n points, draw circles so that every other point is on a circle. Remove each circle that contains fewer than three points. This will eliminate some degenerate cases for the graph theoretic portion of the argument. We won't lose too many circles doing this, because if a positive proportion of our circles had fewer than three points, we'd have $t \approx n$. Call the maximum number of circles about any point t. This means that there is a point that determines t distinct distances. We know, from the last Theorem 5.2, that $t \gtrsim n^{\frac{4}{5}}$. By Theorem 6.1, applied to hyperbolic geodesics instead of Euclidean lines, we have n^2 gedesics incident to two points. Otherwise, we would trivially have cn distinct distances. Therefore, we can be assured that there are roughly cn points in E that have cn geodesics running through them and another point in E. Call the set of these points B.

Fix a point $a \in B$. Let $E_a \subset E \backslash \{a\}$ be a maximal set such that for each point $q \in E_a$, the geodesic $\overline{aq}$ contains no other point of E_a. Now consider C_a, the set of all circles centered at $a \in E$ that are incident to at least three points of E_a. Let E'_a denote the set of all elements of E_a which belong to a circle in C_a. Clearly, $|E'_a| \approx n$, since $|E_a| \approx n$ and $t \lesssim n$ by assumption. After deleting at most one point from each circle in C_a, partition the remaining points into pairwise disjoint consecutive triples, (q_1, q_2, q_3). Clearly, the number of such triples over all circles around a is cn.

A bisector b is called *rich* if it is incident to at least k points of E, where k will be chosen later. A triple is *good* if the bisector of one of the pairs in the triple is not rich. Otherwise it is called *bad*. If half of the triples associated with a given point, $p \in B$ are good, call that point *good*. Call g the number of good points in B.

We will eventually compare incidences of bad points and rich bisectors. However, to stack the deck in our favor, we will first get a value for k that will balance a sufficiently large number of rich bisectors to a sufficiently large number of bad points. The method here uses the same ideas employed in [7].

Define a topological multigraph, G, with the point set E as its vertices, and the arcs between pairs of points in a good triple without a rich bisector as the edges. This way, we'll have exactly one edge for each good triple. Clearly, the number of vertices of G, $v_G \approx n$. The number of edges, $e_G \approx gn$, as there are at least $\frac{cn}{2}$ edges incident to each good point. The multiplicity of each edge must be below k, as the bisector of each associated point pair must not be rich, by assumption.

Now we apply Theorem 4.1 to G with $k = \frac{n^2}{t^2}$. Either $e_G \leq 5v_G k$ or $Cr(G) \gtrsim \frac{e_G^3}{kv_G^2}$. In the former case,

$$gn \approx e_G < 5v_G k \approx \frac{n^3}{t^2} \leq n^{\frac{3}{2}}.$$

This gives us that $g \lesssim n^{\frac{1}{2}} < n$. In the latter case,

$$Cr(G) \gtrsim \frac{e_G^3}{kv_G^2} = \frac{g^3 t^2}{n}.$$

Clearly, $Cr(G) \leq 2n^2t^2$, by counting the maximal number of circles about any point, which can only intersect twice.

Comparing these two bounds for the crossing number of G also gives us that $g \lesssim n$. This means that we have cn bad points in B.

At this time, we need to use a lemma from the original Solymosi-Tóth paper.

Lemma 6.3. *Let T be a set of N triples, (a_i, b_i, c_i), of distinct real numbers such that $a_i < b_i < c_i$ for $i = 1, ..., N$, and assume that $c_i < a_{i+1}$ for all but at most $t-1$ indices i. Let $W = \{a_i + b_i, a_i + c_i, b_i + c_i | i = 1, ..., N\}$. Then*

$$|W| \gtrsim \frac{N}{t^{\frac{2}{3}}}.$$

This bound cannot be improved.

Apply Lemma 6.3 to the system of $\gtrsim n$ disjoint bad triples along the circles centered at a fixed point $a \in B$. Map each point, u, that is, in such a bad triple to the hyperbolic angle the hyperbolic ray $\overline{au}$ forms with the geodesic segment between a and a point directly below a. By construction, this mapping is an injection into the real numbers. There are at most t triples mapped into a range containing 0, and we remove them. The remaining triples form a set, W, of $N \gtrsim n$ triples satisfying the requirements of Lemma 6.3. Notice that there are at most two orientations in W that correspond to the same rich geodesic through a. So Lemma 6.3 implies that for each point a, the number of rich geodesics through a is $\gtrsim \frac{n}{t^{\frac{2}{3}}}$. This gives us that the number of incidences of rich geodesics and bad points is

$$I \gtrsim \frac{n^2}{t^{\frac{2}{3}}}.$$

So by Theorem 5.1, part (a), we have a bound on the number of rich geodesics, $L_k \lesssim \frac{n^2}{k^3} \approx \frac{t^6}{n^4}$. Then, by part (b) of the same theorem, the number of incidences of rich lines and bad points, I, satisfies the following upper bound:

$$I \lesssim n^{\frac{2}{3}} L_m^{\frac{2}{3}} + n + L_m \lesssim \frac{t^4}{n^2} + \frac{t^6}{n^4} + n \lesssim \frac{t^4}{n^2}.$$

Comparing the upper and lower bounds on I gives us the desired result of $t \gtrsim n^{\frac{6}{7}}$.

□

References

[1] P. Erdős, *On sets of distances of n points*, Amer. Math. Monthly **53** (1946), 248-250.

[2] L. Moser, *On the different distances determined by n points*, Amer. Math. Monthly **59** (1952), 85-91.

[3] L. A. Székely, *Crossing numbers and hard Erdős problems in discrete geometry*, Combin. Probab. Comput. **6** (1997), no. 3, 353-358.

[4] Szemerédi, E. and Trotter, W. T. (1983) Extremal problems in discrete geometry *Combinatorica* **3** 381-392.

[5] Székely, L. A. (1987) *Inclusion-exclusion formulae without higher terms*, Ars Combinatoria **23B** 7-20.

[6] J. Beck, *On the lattice property of the plane and some problems of Dirac, Motzkin, and Erdős in combinatorial geometry*, Combinatorica **3** (1983), no. 3-4, 281-297.

[7] J. Solymosi and C. Tóth, Distinct distances in the plane, Discrete Comput. Geom. **25** (2001), no. 4, 629-634.

[8] N. H. Katz and G. Tardos, *A new entropy inequality for the Erdős distance problem*, Towards a Theory of Geometric Graphs, (ed. J. Pach) Contemporary Mathematics **342** (2004), 119-120.

[9] T. Leighton, *Complexity Issues in VLSI, Foundations of Computer Series*, MIT Press, Cambridge, MA, 1983.

[10] M. Ajtai, V. Chvátal, M. Newborn, and E. Szemerédi, *Crossing-free subgraphs*, Ann. Discrete Mathematics **12** (1982), 9-12.