If $f = \sum_n f_n x^n$ and $g = \sum_n g_n x^n$ are power series, we write $f \geq g$ if $f_n \geq g_n$ for all $n$. Thus if $f_1 \geq f_2 \geq 0$ and $g_1 \geq g_2 \geq 0$, then $f_1 g_1 \geq f_2 g_2$. If $f \geq g$ and $h \geq 0$, then $h(f) \geq h(g)$. In particular, this is true when $h = \exp$.

Lemma. If $k > 1$ and $f \geq 0$ satisfies $f_m = O(m^{-k})$, then $\exp(f)_m = O(m^{-k})$.

Proof. It suffices to prove the result when $f_m = Cm^{-k}$. Suppose $a(x) = \sum_{m=1}^\infty a_m x^m$ and $b(x) = \sum_{m=1}^\infty b_m x^m$ are $\geq 0$ and $c(x) = a(x)b(x) = \sum_m c_m x^m$. If for all $m$, $a_m \leq Am^{-k}$ and $b_m \leq Bm^{-k}$, then

$$c_m = \sum_{i=1}^{[m/2]} a_i b_{m-i} + \sum_{i=[m/2]+1}^{m-1} a_i b_{m-i} \leq B(m/2)^{-k} \sum_{i=1}^\infty A_i^{-k} + A(m/2)^{-k} \sum_{i=1}^\infty B_i^{-k} = 2^{k+1}AB\zeta(k)m^{-k}.$$ 

By induction on $n$,

$$(f^n)_m \leq C^m 2^{(k+1)(n-1)}\zeta(k)^{n-1}m^{-k},$$

so for $m > 0$,

$$\exp(f)_m \leq \frac{\exp(2^{k+1}C\zeta(k))}{2^{k+1}\zeta(k)}m^{-k}.$$ 

Remark. The theorem does not hold for $k = 1$, but one does have the following variant: if $f_m = o(m^{-1})$ then $\exp(f)_m = o(m^{\epsilon-1})$ for all $\epsilon > 0$.

Now we apply this lemma to the problem of partitions. Fix an integer $n \geq 2$. Let $p_k$ denote the probability that a random permutation on $k$ letters, drawn from a the uniform distribution on $S_k$, has an $n^{th}$ root. The condition is equivalent to the statement that the number of $r$-cycles in the permutation is divisible by $d_n(r)$, where $d = d_n(r)$ denotes the largest divisor of $n$ such that $r$ and $\frac{n}{r}$ are relatively prime. By a standard generating function argument

$$p(x) = \sum_{k=0}^\infty p_k x^k = \prod_{r=1}^\infty \exp_{d_n(r)}(x^r/r),$$

where

$$\exp_k(x) = \sum_{i=0}^\infty \frac{x^{ik}}{(ik)!} \leq \exp(x^k/k!).$$

Therefore,

$$p(x) \preceq q_1(x)q_2(x) := \exp\left(\sum_{(r,n) = 1}^{\frac{n}{r}} \frac{x^r}{r} \exp\left(\frac{\sum_{(r,n) > 1} x^r d_n(r)}{r^2}\right)\right).$$

We can break up the logarithm of the second multiplicand, $\log(q_2(x))$, into a finite sum over residue classes of $n$; each such sum has $O(m^{-2})$ coefficients, so $\log(q_2)$ and hence $q_2$ has $O(m^{-2})$ coefficients. The $q_1(x)$ term can be expressed by the M"obius inversion formula as

$$\prod_{d \mid n} (1 - x^d)^{-\mu(d)/d}$$
By Darboux’s lemma [Kn-W], the coefficients of this power series are asymptotic to a multiple of \( m^{\phi(n) - n/n} \).

Finally, we observe that if \( 0 < \alpha < 1 \), \( a(x) \geq 0 \) is any power series with coefficients \( a_m \sim C m^{-\alpha} \), and \( b(x) \geq 0 \) is a power series with coefficients \( b_m = O(m^{-2}) \), then the coefficients of \( c(x) = a(x)b(x) \) are asymptotic to \((C \sum b_i)m^{-\alpha}\). As the \( b_i \) are non-negative, \( a(x)b(x) \geq a(x)(b_0 + b_1 x + \cdots + b_k x^k) \). It follows that

\[
\liminf_{m \to \infty} \frac{c_m}{m^\alpha} \geq \lim_{m \to \infty} C \left( b_0 + b_1 (1 - 1/m)^{-\alpha} + \cdots + b_k (1 - k/m)^{-\alpha} \right) = C \sum_{i=0}^{k} b_i,
\]

and sending \( k \to \infty \)

\[
\liminf_{m \to \infty} \frac{c_m}{m^\alpha} \geq C \sum_{i=0}^{\infty} b_i.
\]

On the other hand, fixing \( \beta \in (\alpha, 1) \),

\[
c_n = \left[ n^\beta \right] \sum_{i=0}^{[n^\beta]} b_i a_{n-i} + \sum_{i=[n^\beta]+1}^{n} b_i a_{n-i} < \left( \sum_{i=0}^{\infty} b_i \sup_{j \in (n-n^\beta, n]} a_j \right) + \left( \sum_{i=[n^\beta]+1}^{\infty} b_i \right) \sup_j a_j.
\]

As \( a_j \) is absolutely bounded,

\[
\sum_{i=[n^\beta]+1}^{\infty} b_i = O(n^{-\beta}),
\]

and \( a_m \sim C m^{-\alpha} \),

\[
\limsup_{n \to \infty} \frac{c_n}{n^\alpha} \leq C \sum_{i=0}^{\infty} b_i.
\]

We conclude that \( p_m \sim C m^{\phi(n) - n/n} \) for any fixed \( m \).