

If  $f = \sum_n f_n x^n$  and  $g = \sum_n g_n x^n$  are power series, we write  $f \succeq g$  if  $f_n \geq g_n$  for all  $n$ . Thus if  $f_1 \succeq f_2 \succeq 0$  and  $g_1 \succeq g_2 \succeq 0$ , then  $f_1 g_1 \succeq f_2 g_2$ . If  $f \succeq g$  and  $h \succeq 0$ , then  $h(f) \geq h(g)$ . In particular, this is true when  $h = \exp$ .

*Lemma.* If  $k > 1$  and  $f \succeq 0$  satisfies  $f_m = O(m^{-k})$ , then  $\exp(f)_m = O(m^{-k})$ .

**Proof.** It suffices to prove the result when  $f_m = C m^{-k}$ . Suppose  $a(x) = \sum_{m=1}^{\infty} a_m x^m$  and  $b(x) = \sum_{m=1}^{\infty} b_m x^m$  are  $\succeq 0$  and  $c(x) = a(x)b(x) = \sum_m c_m x^m$ . If for all  $m$ ,  $a_m \leq A m^{-k}$  and  $b_m \leq B m^{-k}$ , then

$$\begin{aligned} c_m &= \sum_{i=1}^{\lfloor m/2 \rfloor} a_i b_{m-i} + \sum_{i=\lfloor m/2 \rfloor+1}^{m-1} a_i b_{m-i} \leq B(m/2)^{-k} \sum_{i=1}^{\infty} A i^{-k} + A(m/2)^{-k} \sum_{i=1}^{\infty} B i^{-k} \\ &= 2^{k+1} A B \zeta(k) m^{-k}. \end{aligned}$$

By induction on  $n$ ,

$$(f^n)_m \leq C^n 2^{(k+1)(n-1)} \zeta(k)^{n-1} m^{-k},$$

so for  $m > 0$ ,

$$\exp(f)_m \leq \frac{\exp(2^{k+1} C \zeta(k))}{2^{k+1} \zeta(k)} m^{-k}.$$

*Remark.* The theorem does not hold for  $k = 1$ , but one does have the following variant: if  $f_m = o(m^{-1})$  then  $\exp(f)_m = o(m^{\epsilon-1})$  for all  $\epsilon > 0$ .

Now we apply this lemma to the problem of partitions. Fix an integer  $n \geq 2$ . Let  $p_k$  denote the probability that a random permutation on  $k$  letters, drawn from a the uniform distribution on  $S_k$ , has an  $n^{\text{th}}$  root. The condition is equivalent to the statement that the number of  $r$ -cycles in the permutation is divisible by  $d_n(r)$ , where  $d = d_n(r)$  denotes the largest divisor of  $n$  such that  $r$  and  $\frac{n}{d}$  are relatively prime. By a standard generating function argument

$$p(x) = \sum_{k=0}^{\infty} p_k x^k = \prod_{r=1}^{\infty} \exp_{d_n(r)}(x^r/r),$$

where

$$\exp_k(x) := \sum_{i=0}^{\infty} \frac{x^{ik}}{(ik)!} \preceq \exp(x^k/k!).$$

Therefore,

$$p(x) \preceq q_1(x) q_2(x) := \exp\left(\sum_{(r,n)=1} \frac{x^r}{r}\right) \exp\left(\frac{\sum_{(r,n)>1} x^{rd_n(r)}}{r^2}\right).$$

We can break up the logarithm of the second multiplicand,  $\log(q_2(x))$ , into a finite sum over residue classes of  $n$ ; each such sum has  $O(m^{-2})$  coefficients, so  $\log(q_2)$  and hence  $q_2$  has  $O(m^{-2})$  coefficients. The  $q_1(x)$  term can be expressed by the Möbius inversion formula as

$$\prod_{d|n} (1 - x^d)^{-\mu(d)/d}$$

By Darboux's lemma [Kn-W], the coefficients of this power series are asymptotic to a multiple of  $m^{\frac{\phi(n)-n}{n}}$ .

Finally, we observe that if  $0 < \alpha < 1$ ,  $a(x) \succeq 0$  is any power series with coefficients  $a_m \sim Cm^{-\alpha}$ , and  $b(x) \succeq 0$  is a power series with coefficients  $b_m = O(m^{-2})$ , then the coefficients of  $c(x) = a(x)b(x)$  are asymptotic to  $(C \sum_i b_i)m^{-\alpha}$ . As the  $b_i$  are non-negative,  $a(x)b(x) \succeq a(x)(b_0 + b_1x + \dots + b_kx^k)$ . It follows that

$$\liminf_{m \rightarrow \infty} \frac{c_m}{m^\alpha} \geq \lim_{m \rightarrow \infty} C (b_0 + b_1(1 - 1/m)^{-\alpha} + \dots + b_k(1 - k/m)^{-\alpha}) = C \sum_{i=0}^k b_i,$$

and sending  $k \rightarrow \infty$

$$\liminf_{m \rightarrow \infty} \frac{c_m}{m^\alpha} \geq C \sum_{i=0}^{\infty} b_i.$$

On the other hand, fixing  $\beta \in (\alpha, 1)$ ,

$$c_n = \sum_{i=0}^{\lfloor n^\beta \rfloor} b_i a_{n-i} + \sum_{i=\lfloor n^\beta \rfloor+1}^n b_i a_{n-i} < \left( \left( \sum_{i=0}^{\infty} b_i \right) \sup_{j \in (n-n^\beta, n)} a_j \right) + \left( \sum_{i=\lfloor n^\beta \rfloor+1}^{\infty} b_i \right) \sup_j a_j.$$

As  $a_j$  is absolutely bounded,

$$\sum_{i=\lfloor n^\beta \rfloor+1}^{\infty} b_i = O(n^{-\beta}),$$

and  $a_m \sim Cm^{-\alpha}$ ,

$$\limsup_{n \rightarrow \infty} \frac{c_n}{n^\alpha} \leq C \sum_{i=0}^{\infty} b_i.$$

We conclude that  $p_m \sim Cm^{\frac{\phi(n)-n}{n}}$  for any fixed  $m$ .