APPLICATIONS OF FOURIER ANALYSIS TO INTERSECTION BODIES

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The concept of an intersection body is central for the dual Brunn-Minkowski theory and has played an important role in the solution of the Busemann-Petty problem. A more general concept of $k$-intersection bodies is related to the generalization of the Busemann-Petty problem. We are interested in comparing classes of $k$-intersection bodies. In the first chapter we present the result that was published in J. Schlieper, *A note on $k$-intersection bodies*, Proc. Amer. Math. Soc., 135 (2007), 2081-2088. The result examines the conjecture that the classes of $k$-intersection bodies increase with $k$. In particular, the result constructs a 4-intersection body that is not a 2-intersection body. The second chapter is concerned with the geometry of spaces of Lorentz type. We define a 1-homogeneous functional based on Lorentz type norms. Consider the family of norms $||x||_{\pi(a)} = (a_i x_i^q + \cdots a_n x_n^q)^{1/q}$ where $a = (a_1, \ldots, a_n)$ with $a_1 \geq \cdots \geq a_n > 0$ and $\pi(a)$ is a permutation of the vector $a$. Define a 1-homogeneous functional based on this family of norms as follows: for $k \geq 1$,
\[ ||x||_k = \left( \sum_{\pi(a)} ||x||^{-k}_{\pi(a)} \right)^{-\frac{1}{k}}, \]
where the sum is taken over all permutations. We examine the geometric properties of the space $(\mathbb{R}^n, ||\cdot||_k)$. First, we determine the conditions when the star body $(\mathbb{R}^n, ||\cdot||_k)$ is a $k$-intersection body. Second, we find the extremal sections of the
star body \((\mathbb{R}^n, \|\cdot\|_k)\). Throughout this work we use the Fourier Analytic methods that were recently developed.
Chapter 1

Introduction

1.1 Preliminaries

We say that a closed bounded set \( K \in \mathbb{R}^n \) is a star body if for every \( x \in K \) each point of the interval \([0, x)\) is an interior point of \( K \), i.e. every straight line passing through the origin crosses the boundary of \( K \) at exactly two points and the origin is an interior point of \( K \).

The Minkowski functional of \( K \) is defined as

\[
\|x\|_K = \min\{a \geq 0 : x \in aK\}.
\]

The radial function of \( K \) is defined by

\[
\rho_K(x) = \|x\|_K^{-1}, \quad x \in \mathbb{R}^n.
\]

If \( x \in S^{n-1} \), then \( \rho_K(x) \) is the Euclidean distance from the origin to the boundary of \( K \) in the direction of \( x \). The radial metric on the set of all origin-symmetric star bodies in \( \mathbb{R}^n \) is defined by

\[
\rho(K, L) = \max_{x \in S^{n-1}} |\rho_K(x) - \rho_L(x)|.
\]

A star body is called \( k \)-smooth, \( k \in \mathbb{N} \cup 0 \), if the restriction of its Minkowski functional to the sphere \( S^{n-1} \) belongs to the space \( C^k(S^{n-1}) \) of \( k \)-times differentiable functions.
functions on the sphere $S^{n-1}$. We say that a body is infinitely smooth if it is $k$-smooth for every $k \in \mathbb{N}$.

Since we are going to consider powers of the Minkowski functional as distributions, we need the following fact, from [K3, p.14]

**Lemma 1.** Let $K$ be an origin-symmetric star body in $\mathbb{R}^n$. Then, for $0 < p < n$, the function $\|\cdot\|_K^{-p}$ is locally integrable on $\mathbb{R}^n$. Also if $f$ is a bounded integrable function of $\mathbb{R}^n$, then the function $\|\cdot\|_K^{-p}f(\cdot)$ is integrable on $\mathbb{R}^n$.

One can express the volume of $K$ in terms of the Minkowski functional by writing the volume of $K$ in polar coordinates:

$$\text{vol}_n(K) = \frac{1}{n} \int_{S^{n-1}} \|\theta\|_K^{-n} d\theta.$$ 

The Fourier transform of distributions is a main tool used throughout this work. Let $\phi$ be a function from the Schwartz space $S$ of rapidly decreasing differentiable functions on $\mathbb{R}^n$. We define the Fourier transform of $\phi$ by

$$\hat{\phi}(\xi) = \int_{\mathbb{R}^n} \phi(x) e^{-i(x, \xi)} dx, \quad \xi \in \mathbb{R}^n.$$ 

The Fourier transform of a distribution $f$ is defined by

$$\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle$$

for every test function $\phi$ from the space $S$.

We say that a distribution $f$ is **positive definite** if its Fourier transform is a positive distribution, i.e. $\langle \hat{f}, \phi \rangle \geq 0$ for every non-negative test function $\phi$ from the space $S$. 

1
Denote by $\gamma_q$ the Fourier transform of the function $z \mapsto \exp(-|z|^q)$, $z \in \mathbb{R}$.

The functions $\gamma_q$ can be calculated precisely for $q = 2$, where

$$
\gamma_2(t) = (e^{-z^2})\hat{\gamma}(t) = \sqrt{\pi} \exp \left( -\frac{t^2}{4} \right)
$$

and for $q = 1$, where

$$
\gamma_1(t) = \frac{2}{1 + t^2}.
$$

For other values of $q$, we have to study the properties of the functions $\gamma_q$ indirectly. These functions were studied by Polya [P2]. We will use the following properties of the functions $\gamma_q$ stated as the following lemmas.

**Lemma 2.** (Koldobsky, [K3, section 2.8]) For $0 < q \leq 2$, the function $\gamma_q$ is positive on $[0, \infty)$, and the function $\log(\gamma_q(\sqrt{x}))$ is convex on $[0, \infty)$. In other words, the function $\gamma_q'(t)/(t\gamma_q(t))$ is increasing on $(0, \infty)$.

**Lemma 3.** (Polya, [PS, Chapter 4, problem 154]) For any $q > 0$

$$
\lim_{t \to \infty} t^{1+q} \gamma_q(t) = 2\Gamma(q + 1) \sin \left( \frac{\pi q}{2} \right).
$$

The result also holds if $q$ is an even integer, when the limit is equal to zero. Thus, for even $q$, the function $\gamma_q$ decreases even faster than $t^{-1-q}$, the rate is exponential.

We also need the moments of the function $\gamma_q$, denoted by

$$
s_q(z) = \int_{\mathbb{R}} |t|^z \gamma_q(t) dt.
$$

The integral converges absolutely for $-1 < z < q$. The moments can be calculated precisely for all $q > 0$; see for example [K5].
Lemma 4. For any $q > 0$ and $-1 < z < q$, where $z$ is not an even integer,

$$s_q(z) = \frac{2^{z+2} \sqrt{\pi} \Gamma(-z/q) \Gamma((z+1)/2)}{q \Gamma(-z/2)}.$$  

In particular, if $q > 2$, the moments $s_q(z)$ are positive for $z \in (-1, 0)$ and $z \in (0, 2)$, and they are negative for $z \in (2, \min(q, 4))$.

For a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ denote by $x^* = (x_1^*, \ldots, x_n^*)$ the non-increasing permutation of the numbers $|x_1|, \ldots, |x_n|$. We shall consider the order statistics $x_k^*$ as functions of the variables $x_1, \ldots, x_n$. For $a_1 \geq \cdots \geq a_n \geq 0$ (not all zero) and $q > 0$, the expression $\| (x_1, \ldots, x_n) \|_\omega = (a_1 x_1^q + \cdots + a_n x_n^q)^{1/q}$ is the norm (or $q$-norm if $q < 1$) of an $n$-dimensional weighted Lorentz space which is denoted by $\ell_{\omega,q}^n$, where $\omega = (a_1, \ldots, a_n)$.

Consider the family of norms $\| x \|_{\pi(a)} = (a_1 x_1^q + \cdots + a_n x_n^q)^{1/q}$ where $a = (a_1, \ldots, a_n)$ with $a_1 \geq \cdots \geq a_n > 0$ and $\pi(a) = (a_i_1, \ldots, a_i_n)$ is a permutation of the vector $a$. We will define a 1-homogeneous functional based on this family of norms as follows: for $k \geq 1$,

$$\| x \|_k = \left( \sum_{\pi(a)} \| x \|_{\pi(a)}^{-k} \right)^{-\frac{1}{k}},$$  \hspace{1cm} (1.1)

where the sum is taken over all permutations.
Chapter 2

$k$-intersection bodies

2.1 Introduction

Let $D, L$ be origin symmetric star bodies in $\mathbb{R}^n$. Following Lutwak[LU], we say that $D$ is the intersection body of $L$ if the radius of $D$ in every direction is equal to the $(n - 1)$-dimensional volume of the section of $L$ by the central hyperplane orthogonal to this direction, i.e. for every $\xi \in S^{n-1}$,

$$\rho_D(\xi) = \|\xi\|_D^{-1} = Vol_{n-1}(L \cap \xi^\perp).$$  \hspace{1cm} (2.1)

where $\|x\|_D = \min\{a \geq 0 : x \in aD\}$ is the Minkowski functional of $D$ and $\xi^\perp$ is the hyperplane orthogonal to $\xi$. A more general class of intersection bodies can be defined as the closure in the radial metric of the class of intersection bodies of star bodies. The concept of an intersection body is important for the Busemann-Petty problem and its generalizations, (see [K3, Chap 5]). The Fourier analytic characterization states if $D$ is an intersection body of $L$ then for every $x \in S^{n-1}$ we have

$$\|x\|_D^{-1} = \frac{1}{\pi(n-1)}(\|\cdot\|_{L^{n+1}}^{-1})(x),$$  \hspace{1cm} (2.2)
where on the right-hand side we have the Fourier transform in the sense of distributions. We extend this equality to the whole \( \mathbb{R}^n \), as an equality of homogeneous functions of degree \(-1\). Since \( D \) is symmetric, the Fourier transform is self-invertible (up to a constant), so

\[
(\|x\|_{D}^{-1})^\wedge(\xi) = \frac{(2\pi)^n}{\pi(n-1)}\|\xi\|_L^{-n+1} > 0,
\]

(2.3)

which means that the locally integrable function \( \|x\|_{D}^{-1} \) represents a positive definite distribution. On the other hand, if \( \|x\|_{D}^{-1} \) is a positive definite distribution, whose Fourier transform is a continuous positive function on \( S^{n-1} \), we can define a symmetric star body \( L \) by

\[
\|\xi\|_L = \left( \frac{\pi(n-1)}{(2\pi)^n}\|\cdot\|_{D}^{-1}\wedge(\xi) \right)^{-1/(n-1)}, \quad \xi \in S^{n-1}
\]

(2.4)

We reverse the argument above and see that \( D \) is the intersection body of \( L \). We have outlined the proof of the following theorem that gives the Fourier analytic characterization of intersection bodies.

**Theorem 1.** (Koldobsky, [K7]) An origin-symmetric star body \( D \) n \( \mathbb{R}^n \) is an intersection body if and only if \( \|\cdot\|_{D}^{-1} \) represents a positive definite distribution on \( \mathbb{R}^n \).

The concept of embedding in \( L_p \) with \( p < 0 \) was introduced in [K2] as an extension of the same properties for \( p > 0 \):

**Definition 1.** Let \( 0 < p < n \). Let \( D \) be an origin symmetric star body in \( \mathbb{R}^n \). We say that the space \( (\mathbb{R}^n, \|\cdot\|_D) \) embeds in \( L_{-p} \) if there exists a finite Borel measure \( \mu \)
on $S^{n-1}$ so that for every even test function $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$
\int_{\mathbb{R}^n} \|x\|^{-k} \hat{\phi}(x) dx = \int_{S^{n-1}} d\mu(\xi) \int_0^\infty t^{k-1} \hat{\phi}(t\xi) dt.
$$

(2.5)

It was proved in [K2] that if $D$ is an origin symmetric star body in $\mathbb{R}^n$, then the space $(\mathbb{R}^n, \|\cdot\|_D)$ embeds in $L_p$, $p < -1$ if and only if $\|\cdot\|^p_D$ is a positive definite distribution. A direct connection between $k$-intersection bodies and embeddings in $L_{-k}$ was established in [K4] stated as the following theorem.

**Theorem 2.** Let $1 \leq k < n$. An origin symmetric star body $D$ in $\mathbb{R}^n$ is a $k$-intersection body if and only if the space $(\mathbb{R}^n, \|\cdot\|_D)$ embeds in $L_{-k}$. Both conditions are equivalent to $\|\cdot\|^{-k}_D$ being a positive definite distribution.

The advantage of this connection is that we can now try to extend known results on $L_p$ spaces to negative values of $p$. Every such extension gives new information about intersection bodies. For example, a well known fact is that for $0 < p < q \leq 2$, the space $L_q$ embeds isometrically in $L_p$, therefore $L_p$-spaces become larger when $p$ decreases from 2. This result was also extended to negative values of $p$, in [K2]. Namely, every finite dimensional subspace of $L_q$, with $0 < q \leq 2$, embeds in $L_{-p}$ for every $p \in (0, n)$. By Theorem 2, the unit ball of every finite dimensional subspace of $L_q$, with $0 < q \leq 2$, is a $k$-intersection body for every $k \in (0, n)$. One question still to answer is: For $-n < p < q < 0$, does every normed subspace that embeds in $L_q$ also embed in $L_p$? Theorem 2 allows us to restate the question in terms of intersection bodies. For integers $0 < q < p < n$, is a $q$-intersection body also a $p$-intersection body? If $q$ divides $p$ then every $q$-intersection body is a $p$-intersection body. Therefore, every 2-intersection body is a 4-intersection body (see [M]).
section 2.2.3, we construct a normed space that shows that there exist 4-intersection bodies that are not 2-intersection bodies.

Koldobsky [K3, Section 4.3] proved that the unit balls of the $\ell_q$ spaces with $q > 2$ are not $k$-intersection bodies for $k < n - 3$. This is part of a more general result that shows the same is true for any norm whose second directional derivative vanishes in a certain way. Refer to [K3, section 4.4] for the proof.

**Theorem 3.** Let $0 < p < n$ and let $X$ be an $n$-dimensional normed space with a normalized basis $e_1, \ldots, e_n$, $n \geq 3$, so that:

1. For every fixed $(x_2, \ldots, x_n) \in \mathbb{R}^{n-1}/0$, the function
   
   $$x_1 \mapsto \left\| x_1 e_1 + \sum_{i=2}^{n} x_i e_i \right\|$$

   has a continuous second derivative everywhere on $\mathbb{R}$, and

   $$\|x\|''_{x_1}(0, x_2, \ldots, x_n) = \|x\|''_{x_1}(0, x_2, \ldots, x_n) = 0,$$

   where $\|x\|'_x$ and $\|x\|''_x$ stand for the first and second partial derivatives by $x_1$ of the norm $\|x_1 e_1 + \ldots + x_n e_n\|$.

2. There exists a constant $C$ so that, for every $x_1 \in \mathbb{R}$ and every $(x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$ with $\|x_2 e_2 + \ldots + x_n e_n\| = 1$, one has

   $$\|x\|''_{x_1}(x_1, x_2, \ldots, x_n) \leq C.$$

3. Convergence in limit

   $$\lim_{x_1 \to 0} \|x\|''_{x_1}(x_1, x_2, \ldots, x_n) = 0$$

   is uniform with respect to $(x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$ with $\|x_2 e_2 + \ldots + x_n e_n\| = 1$. 

Then the function $\|\cdot\|^{-p}$ represents a positive definite distribution if and only if $p \in [n-3, n)$. In particular the convex set $x \in \mathbb{R}^n : \|\sum_{i=1}^n x_ie_i\| \leq 1$ is a $k$-intersection body if and only if $k$ is one of the numbers $n-3, n-2, n-1$, $k \neq 0$.

2.2 Strict inclusion of classes of $k$-intersection bodies

In this section, we present the results that appears in [SCHL].

We are interested in comparing classes of $k$-intersection bodies for different $k$. Koldobsky’s conjecture is that these classes increase with $k$. This conjecture comes from the connection of $k$-intersection bodies and embedding in $L_p$ with $p < 0$. The concept of embedding in $L_p$ with $p < 0$ was introduced in [K2] as an extension of the same properties for $p > 0$:

**Definition 2.** Let $0 < p < n$. Let $D$ be an origin symmetric star body in $\mathbb{R}^n$. We say that the space $(\mathbb{R}^n, \|\cdot\|_D)$ embeds in $L_{-p}$ if there exists a finite Borel measure $\mu$ on $S^{n-1}$ so that for every even test function $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \|x\|^{-k}D^{-k}\phi(x)dx = \int_{S^{n-1}} d\mu(\xi) \int_0^\infty t^{k-1}\hat{\phi}(t\xi)dt. \quad (2.6)$$

For $-1 < p < 0$, (2.6) is equivalent to the existence of a measure $\mu$ on $S^{n-1}$ so that for every $x$:

$$\|x\|^p = \int_{S^{n-1}} |(x, \xi)|^p d\mu(\xi).$$

This says the norm $\|x\|$ admits the Levy representation with exponent $p$. A special construction of a normed space with its Levy representation was used in [K1] to show there is a Banach space embedding isometrically into $L_{1/2}$ but not into
The construction also yielded an example of a Banach space embedding into $L_{1/4}$ but not into $L_{1/2}$. J. Borwein and the Center for Computational Mathematics at Simon Fraser University (unpublished) showed by computer methods that this algorithm yields examples of Banach spaces embedding into $L_{a/64}$ but not into $L_{(a+1)/64}$ for $a = 1, 2, \ldots, 63$. Recently, N. Kalton and A. Koldobsky ([KK]) showed that for every $0 < p < 1$ one can find a Banach space embedding isometrically into $L_p$ but not into any $L_q$ for $p < q \leq 1$. In section 2.2.2, we use the construction ideas of [K1] to construct a space that embeds isometrically in $L_{-1/3}$ while at the same time it does not embed in $L_{-1/6}$.

As mentioned in the chapter introduction, an open question is: For $-n < p < q < 0$, does every normed subspace that embeds in $L_q$ also embed in $L_p$? Theorem 2 allows us to restate the question in terms of intersection bodies. For integers $0 < q < p < n$, is a $q$-intersection body also a $p$-intersection body? If $q$ divides $p$ then every $q$-intersection body is a $p$-intersection body. Therefore, every 2-intersection body is a 4-intersection body (see [M]). In section 2.2.3, we construct a normed space that shows that there exist 4-intersection bodies that are not 2-intersection bodies. Recently, Yaskin [Y] extended the result to arbitrary integers $p$ and $q$ with $1 \leq q < p < n - 3$, i.e. there are $p$-intersection bodies that are not $q$-intersection bodies.
2.2.1 Main Idea

The idea of the construction originates from [SCHN], [K1]. Let \( f \) be an infinitely differentiable function on the unit sphere \( S^{n-1} \) in \( \mathbb{R}^n \). We perturb the Euclidean norm \( \|x\|_2 \) by means of the function \( f \). Given \( \lambda > 0 \), consider the function

\[
N_\lambda(x) = \|x\|_2 \left(1 + \lambda f \left( \frac{x}{\|x\|_2} \right) \right), \quad x \in \mathbb{R}^n.
\] (2.7)

One can choose \( \lambda \) small enough so that \( N_\lambda \) is a norm in \( \mathbb{R}^n \). This follows from considering the one dimensional case: if \( a, b \in \mathbb{R} \), \( g \) is a convex function on \( [a, b] \) with \( g'' > \delta > 0 \) on \( [a, b] \) and \( h \in C^2[a, b] \) then the function \( g + \lambda h \) has positive second derivatives on \( [a, b] \) for sufficiently small \( \lambda \)'s and, therefore, is convex on \( [a, b] \). Let \( \lambda_f = \sup \{ \lambda > 0 : N_t \text{ is a norm in } \mathbb{R}^n \text{ for every } t \leq \lambda \} \). For each \( \lambda \leq \lambda_f \), we denote by \( X_\lambda \) the \( n \)-dimensional normed space with the norm \( N_\lambda \). For \( q < 0 \), let \( \lambda_q = \sup \{ \lambda > 0 : X_t \text{ embeds in } L_q \text{ for every } t \leq \lambda \} \). Given \( -n < p < q < 0 \), we want to find a Banach subspace of \( L_p \) that is not isometric to a subspace of \( L_q \). If we find a function, \( f \), so that \( \lambda_q \) is strictly less than \( \lambda_p \) and \( \lambda_p \) is strictly less than \( \lambda_f \), then we get an example of a Banach space which embeds in \( L_p \) but does not embed in \( L_q \).

We will use the following characterization of finite dimensional spaces that embed in \( L_p, p > -1 \), see [K1] for the proof.

**Lemma 5.** Let \( p > -1 \), \( p \) not an even integer, let \( (X, \|\cdot\|) \) be an \( n \)-dimensional Banach space, and suppose there exists a continuous function \( b \) on the sphere \( S^{n-1} \) in \( \mathbb{R}^n \) such that, for every \( x \in \mathbb{R}^n \),

\[
\|x\|^p = \int_{S^{n-1}} |(x, \xi)|^p b(\xi) d\xi.
\]

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where \((x, \xi)\) stands for the scalar product in \(\mathbb{R}^n\). Then \(X\) embeds in \(L_p\) if and only if \(b\) is a non-negative (not identically zero) function.

We are going to choose special norms for which it is possible to calculate the function \(b\) and then check if \(b\) is non-negative. We will need the following lemma from [K1] that has been expanded to meet our needs here.

**Lemma 6.** For every \(x = (x_1, x_2, \ldots, x_n)\) from the unit sphere \(S^{n-1}\) in \(\mathbb{R}^n\) and every \(q > -1, q \neq 0, q \neq 2\), we have

\[
x_n^2 = \frac{\Gamma\left(\frac{n+q}{2}\right)}{2\pi^{n/2} \Gamma\left(\frac{q+1}{2}\right)} \int_{S^{n-1}} |(x, \xi)|^q \left(\frac{n+q}{q} \xi_n^2 - \frac{1}{q}\right) d\xi.
\]

Furthermore,

\[
x_n^4 = \frac{\Gamma\left(\frac{n+q}{2}\right)}{2\pi^{n/2} \Gamma\left(\frac{q+1}{2}\right)} \int_{S^{n-1}} |(x, \xi)|^q \left(\frac{(n+q+2)(n+q)(q+3)}{2q(q-2)} \xi_n^4 - \frac{6(n+q)}{q(q-2)} \xi_n^2 + \frac{3}{q(q-2)}\right) d\xi.
\]

**Proof.** It is a well known fact, [K3, p.56], that for every \(x \in \mathbb{R}^n\) and every \(k > 0\),

\[
(x_1^2 + \cdots + x_n^2)^k = \frac{\Gamma((n+2k)/2)}{2\pi^{(n-1)/2} \Gamma((2k+1)/2)} \int_{S^{n-1}} |(x, \xi)|^{2k} d\xi.
\]

Differentiate both sides of (2.10) by \(x_n\) twice, and then use that \(x \in S^{n-1}\) to get

\[
1 + (2k-2)x_n^2 = (2k-1) \frac{\Gamma((n+2k)/2)}{2\pi^{(n-1)/2} \Gamma((2k+1)/2)} \int_{S^{n-1}} |(x, \xi)|^{2k-2} \xi_n^2 d\xi,
\]

for \(2k-2 > -1\). Set \(q = 2k-2\) and use \(\Gamma(x+1) = x\Gamma(x)\) to get (2.8).

Now to get (2.9), differentiated both sides of (2.10) by \(x_n\) four times, remembering that \(x \in S^{n-1}\):

\[
3 + (2k-6)(2k-4)x_n^4 + 6(2k-4)x_n^2 =
\]

\[
(2k-1)(2k-3) \frac{\Gamma((n+2k)/2)}{2\pi^{(n-1)/2} \Gamma((2k+1)/2)} \int_{S^{n-1}} |(x, \xi)|^{2k-4} \xi_n^4 d\xi,
\]

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for $2k - 4 > -1$. Now using the properties of the $\Gamma$ function, setting $q = 2k - 4$ and using (2.8) we get (2.9).

### 2.2.2 First Example

For every $\lambda > 0$ define a function $N_\lambda$ on $\mathbb{R}^n$ by

$$N_\lambda(x) = (x_1^2 + \cdots + x_n^2)^{1/2} \left( 1 + \lambda \frac{x_1^2 + \cdots + x_n^2}{x_1^2 + \cdots + x_n^2} \right)^{-6}, \ x \in \mathbb{R}^n. \ (2.11)$$

**Lemma 7.** $N_\lambda$ is a convex function if and only if $\lambda \leq \frac{1}{11}$.

**Proof.** $N_\lambda$ is convex if and only if the following function in two-dimensions is convex.

$$f(x, y) = (x^2 + y^2)^{1/2} \left( 1 + \lambda \frac{x^2 + 2y^2}{x^2 + y^2} \right)^{-6}$$

The function $f$ is convex if and only if $a^2 \frac{d^2 f}{dx^2} + 2ab \frac{d^2 f}{dxdy} + b^2 \frac{d^2 f}{dy^2}$ is a non-negative function for any choice of $a, b$. Taking the second derivatives of the function $f$ and considering all choices of $a, b, x,$ and $y$, we have the following expression:

$$a^2 \frac{d^2 f}{dx^2} + 2ab \frac{d^2 f}{dxdy} + b^2 \frac{d^2 f}{dy^2} = \frac{(bx-ay)^2(x^2+y^2)^{(9/2)(x^4(1-10\lambda-11\lambda^2)+y^4(1+16\lambda+28\lambda^2)+2x^2y^2(1+3\lambda+80\lambda^2))}}{(x^4(1+\lambda)+y^4(1+2\lambda))^{9/2}}$$

Since we only consider positive values of $\lambda$, we choose $\lambda$ so that each of the following expressions is positive: $1-10\lambda-11\lambda^2$, $1+16\lambda+28\lambda^2$, and $1+3\lambda+80\lambda^2$. The latter two are always positive when $\lambda > 0$, and the first is positive if $\lambda \leq \frac{1}{11}$. Therefore, $f(x, y)$ is convex if and only if $\lambda \leq \frac{1}{11}$. \qed
Theorem 4. Let $n \geq 4$. If

$$\frac{1}{6n-8} < \lambda \leq \frac{63n - 105 - \sqrt{21}\sqrt{153n^2 - 390n + 428}}{36n^2 - 240n + 97} \leq \frac{1}{11},$$

then the Banach space $X_\lambda$ embeds in $L_{-1/3}$ and, at the same time $X_\lambda$ does not embed in $L_{-1/6}$.

Proof. First, we will prove show that $X_\lambda$ does not embed in $L_{-1/6}$ if $\lambda > \frac{1}{6n-8}$. For $x \in S^{n-1}$, we use (2.8) with $q = -1/6$ to obtain the following representation of the norm $N_\lambda$:

$$N_{\lambda}^{-1/6} = (1 + \lambda) + \lambda x_n^2
= \frac{\Gamma(\frac{n-1/6}{2})}{2\pi^{\frac{n-1}{2}}\Gamma(5/12)} \int_{S^{n-1}} |(x, \xi)|^{-1/6} b(\xi) d\xi,$$

where

$$b(\xi) = (1 + 7\lambda) - \xi_n^2 \lambda (6n - 1)$$

By Lemma 5, the space $X_\lambda$ embeds in $L_{-1/6}$ if and only if $b(\xi)$ is non-negative. The function $b(\xi)$ is linear in $\xi_n^2 \in [0, 1]$ with $b(0) > 0$ and is non-negative if $\lambda \leq \frac{1}{6n-8}$. Therefore, if $\lambda > \frac{1}{6n-8}$ then $X_\lambda$ does not embed in $L_{-1/6}$. This proves the left hand inequality. For the right hand side, we use Lemma 6 with $q = -1/3$ to get the following representation for the norm $N_\lambda$:

$$N_{\lambda}^{-1/3} = (1 + 2\lambda + \lambda^2) + (2\lambda + 2\lambda^2) x_n^2 + \lambda^2 x_n^4
= \frac{\Gamma(\frac{n-1/3}{2})}{2\pi^{\frac{n-1}{2}}\Gamma(1/3)} \int_{S^{n-1}} |(x, \xi)|^{-1/3} b(\xi) d\xi,$$

where

$$b(\xi) = (1 + 8\lambda + \frac{76}{7}\lambda^2) - \xi_n^2 (3n - 1)(2\lambda + \frac{32}{7}\lambda^2) + \xi_n^4 (3n - 1)(3n + 5)(\frac{4}{21}\lambda^2)$$
Lemma 5 says the space $X_\lambda$ embeds in $L_{-1/3}$ if and only if $b(\xi)$ is non-negative. The function $b(\xi)$ is quadratic in $\xi_n^2$ with positive leading coefficient. Since $b(0) > 0$, it is enough to make sure that the roots of $b(\xi)$ do not lie in the interval $[0, 1]$. If we choose $\lambda$ so that the roots occur for $\xi_n^2 \geq 1$, then $b(\xi)$ is non-negative for all $\xi \in S^{n-1}$. Remembering that $b(\xi)$ is quadratic in $\xi_n^2 \in [0, 1]$, we get the following inequality:

$$\frac{8}{21}(3n-1)(3n+5)\lambda^2 - (3n-1)(2\lambda + \frac{32}{7}\lambda^2) \leq \sqrt{(3n-1)^2(2\lambda + \frac{32}{7}\lambda^2)^2 - \frac{16}{21}(3n-1)(3n+5)\lambda^2(1 + 8\lambda + \frac{76}{7}\lambda^2)} \quad (2.13)$$

Solving for $\lambda$ we get

$$\lambda \leq \frac{63n - 105 - \sqrt{21\sqrt{153n^2 - 390n + 428})}}{36n^2 - 240n + 97}.$$

Thus, if we choose $\lambda$ this way, $b(\xi)$ will be nonnegative and the space $X_\lambda$ embeds isometrically in $L_{-1/3}$. To prove the theorem, it suffices to note that, for every $n \geq 4$,

$$\frac{63n - 105 - \sqrt{21\sqrt{153n^2 - 390n + 428})}}{36n^2 - 240n + 97} \leq \frac{1}{11} \quad \text{and the condition} \quad \frac{1}{6n-8} < \frac{63n - 105 - \sqrt{21\sqrt{153n^2 - 390n + 428})}}{36n^2 - 240n + 97}$$

holds if $n > \frac{1 + \sqrt{15}}{6}$. Since $n \geq 4 > \frac{1 + \sqrt{15}}{6}$, if we choose $\lambda$ so that (2.12) is satisfied, then $X_\lambda$ embeds isometrically in $L_{-1/3}$ but at the same time does not embed isometrically in $L_{-1/6}$.

Note that for $n \leq 3$, every normed space embeds in any $L_p$, $-1 < p < 0$ and this is why we only considered spaces with $n \geq 4$, see [K3, p.78].

2.2.3 Second Example

It is a well known fact due to L. Schwartz, (see [GV, page 152]), that a distribution is positive definite if and only if its Fourier transform is a positive distribution.
Using the above construction of \( \mathcal{N}_\lambda(x) \), we will take the Fourier transform in the sense of distributions of \((\mathcal{N}_\lambda(x))^p\) with \( p < -1 \) and check for which values of \( \lambda \) is \( (\mathcal{N}_\lambda(x))^p(\xi) \) positive everywhere on \( S^{n-1} \). Therefore by Theorem 2, for these values of \( \lambda \) the space \( X_\lambda \) embeds isometrically in \( L_p, p < -1 \). We will need the following lemma, which is based on the formula for the Fourier transform of the Euclidean norm, [GS, page 192]:

\[
\hat{\left( \|x\|_2^p \right)}(\xi) = \frac{2^{n+p} \pi^{n/2} \Gamma(\frac{n+p}{2})}{\Gamma(\frac{-p}{2})} \|\xi\|_2^{-n-p}.
\]

**Lemma 8.** Let \( x \in \mathbb{R}^n \) and \( \|x\|_2 \) be the Euclidean norm, then

\[
\left( \frac{x_n}{\|x\|_2^p} \right)(\xi) = \frac{2^{n+p} \pi^{n/2} \Gamma(\frac{n+p}{2})}{\Gamma(\frac{-p}{2})} (-i)^j \frac{\partial^j}{\partial \xi_n^j} \|\xi\|_2^{-n-p}.
\] (2.14)

Furthermore,

\[
\frac{\partial^2}{\partial \xi_n^2} \|\xi\|_2^{-n-p} = (n + p)(n + p + 2) \xi_n^2 \|\xi\|_2^{-n-p-2} - (n + p) \|\xi\|_2^{-n-p-1}
\] (2.15)

and

\[
\frac{\partial^4}{\partial \xi_n^4} \|\xi\|_2^{-n-p} = 3(n + p)(n + p + 2) \|\xi\|_2^{-n-p-2}
\]

\[-6(n + p)(n + p + 2)(n + p + 4) \xi_n^2 \|\xi\|_2^{-n-p-3}
\]

\[+(n + p)(n + p + 2)(n + p + 4)(n + p + 6) \xi_n^4 \|\xi\|_2^{-n-p-4}
\] (2.16)

**Proof.** We use the connection between the Fourier transform and differentiation:

\[
\hat{(x_n f(x))}(\xi) = (-i)^j \frac{\partial^j}{\partial \xi_n^j} \hat{f}(\xi). \]

Using this to compute \( \left( \frac{x_n}{\|x\|_2^p} \right)(\xi) \) we get (2.14). In order to obtain the rest of the results, we differentiate \( \|\xi\|_2^{-n-p} \) two and four times respectively. After collecting terms, we get the result. \( \square \)
For every $\lambda > 0$ define a function $N_\lambda$ on $\mathbb{R}^n$ by

$$N_\lambda(x) = (x_1^2 + \cdots + x_n^2)^{1/2} \left( 1 + \lambda \frac{x_1^2 + \cdots + x_{n-1}^2 + 5x_n^2}{x_1^2 + \cdots + x_n^2} \right)^{-1/2}, \quad x \in \mathbb{R}^n. \quad (2.17)$$

**Lemma 9.** $N_\lambda$ is a convex function if and only if $\lambda \leq \frac{1}{3}$.

**Proof.** $N_\lambda$ is convex if and only if the following function in two-dimensions is convex.

$$f(x, y) = (x^2 + y^2)^{1/2} \left( 1 + \lambda \frac{x^2 + 5y^2}{x^2 + y^2} \right)^{-1/2}$$

The function $f$ is convex if and only if $a^2 \frac{d^2 f}{dx^2} + 2ab \frac{d^2 f}{dxdy} + b^2 \frac{d^2 f}{dy^2}$ is a non-negative function for all choices of $a, b$. Taking the second derivatives of the function $f$ and considering all choices of $a, b, x,$ and $y$, we have the following expression:

$$a^2 \frac{d^2 f}{dx^2} + 2ab \frac{d^2 f}{dxdy} + b^2 \frac{d^2 f}{dy^2} = \frac{(bx - ay)^2(x^2(1 - 2\lambda - 3\lambda^2) + y^2(1 + 14\lambda + 45\lambda^2))}{(x^2(1 + \lambda) + y^2(1 + 5\lambda))^{5/2}}$$

Since we only consider positive values of $\lambda$, then choose $\lambda$ so that each of the following expressions are always positive: $1 - 2\lambda - 3\lambda^2$ and $1 + 14\lambda + 45\lambda^2$. The latter expression is always positive when $\lambda > 0$, and the first is positive if $\lambda \leq \frac{1}{3}$. Therefore, $f(x, y)$ is convex if and only if $\lambda \leq \frac{1}{3}$. \qed

**Theorem 5.** Let $n \geq 7$. If

$$\frac{1}{2n-7} < \lambda \leq \frac{3n - 18 - \sqrt{3} \sqrt{n^2 - 6n + 5}}{2n^2 - 30n + 103} \leq \frac{1}{3} \quad (2.18)$$

then the Banach space $X_\lambda$ embeds in $L_{-4}$ and, at the same time $X_\lambda$ does not embed in $L_{-2}$.  

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Proof. First, we will show that $X_\lambda$ is isometric to a subspace of $L_{-2}$ if and only if $\lambda \leq \frac{1}{2n-7}$. We need to find the values of $\lambda$ that will make $(N_\lambda)^{-2}$ positive definite for all $\xi \in S^{n-1}$. We will use Lemma 8 to compute the Fourier Transform of $(N_\lambda)^{-2}$.

$$(N_\lambda)^{-2} = (1 + \lambda)\|x\|_{2}^{-2} + (4\lambda)x_n^2\|x\|_2^{-4}$$

$$b(\xi) = \widehat{(N_\lambda)^{-2}}(\xi) = 2^{n-2}\pi^{n/2}\Gamma\left(\frac{n-2}{2}\right) \left((1 + 3\lambda) - \xi_n^2(n-2)(2\lambda)\right)$$

The function $b(\xi)$ is linear in $\xi_n^2 \in [0, 1]$ with $b(0) > 0$. If we choose $\lambda \leq \frac{1}{2n-7}$, then $b(\xi)$ is nonnegative for all $\xi \in S^{n-1}$. Therefore, by Lemma 5, if $\lambda > \frac{1}{2n-7}$ then $X_\lambda$ does not embed in $L_{-2}$. This proves the left hand inequality. For the right hand side, we again use Lemma 8 to compute the Fourier Transform of the norm $N_\lambda$ with exponent $q = -4$:

$$(N_\lambda)^{-4} = (1 + 2\lambda + \lambda^2)\|x\|_{2}^{-4} + (8\lambda + 8\lambda^2)x_n^2\|x\|_2^{-6} + 16\lambda^2x_n^4\|x\|_2^{-8}$$

$$b(\xi) = \widehat{(N_\lambda)^{-4}}(\xi)$$

$$= 2^{n-4}\pi^{n/2}\Gamma\left(\frac{n-4}{2}\right)[(1+4\lambda+5\lambda^2) - \xi_n^2(n-4)(2\lambda+6\lambda^2) + \xi_n^4(n-4)(n-2)(\frac{2}{3}\lambda^2)]$$

Lemma 5 says the space $X_\lambda$ embeds in $L_{-4}$ if and only if $b(\xi)$ is non-negative. The function $b(\xi)$ is quadratic in $\xi_n^2$ with positive leading coefficient. Since $b(0) > 0$, it is enough to make sure that the roots of $b(\xi)$ do not lie in the interval $[0, 1]$. If we choose $\lambda$ so that the roots occur for $\xi_n^2 \geq 1$, then $b(\xi)$ is non-negative for all $\xi \in S^{n-1}$. Remembering that $b(\xi)$ is quadratic in $\xi_n^2 \in [0, 1]$, we get the following inequality:

$$2\lambda(n-4)(n-2) - 3(n-4)(1 + 3\lambda) \geq$$

$$\sqrt{9(n-4)^2(1 + 3\lambda)^2 - 6(n-4)(n-2)(1 + 4\lambda + 5\lambda^2)} \quad (2.19)$$
Solving for \( \lambda \) we get

\[
\lambda \leq \frac{3n - 18 - \sqrt{3\sqrt{n^2 - 6n + 5}}}{2n^2 - 30n + 103}.
\]

Thus, if we choose \( \lambda \) this way, \( b(\xi) \) will be nonnegative and the space \( X_\lambda \) embeds isometrically in \( L_{-4} \). To prove the theorem, it suffices to note that, for every \( n \geq 7 \),

\[
\frac{3n - 18 - \sqrt{3\sqrt{n^2 - 6n + 5}}}{2n^2 - 30n + 103} \leq \frac{1}{3}
\]

and the condition

\[
\frac{1}{2n - 7} < \frac{3n - 18 - \sqrt{3\sqrt{n^2 - 6n + 5}}}{2n^2 - 30n + 103}
\]

holds if \( n > \frac{15 - \sqrt{19}}{2} \). Since \( n \geq 7 > \frac{15 - \sqrt{19}}{2} \) if we choose \( \lambda \) so that (2.12) is satisfied, then \( X_\lambda \) embeds isometrically in \( L_{-4} \) but at the same time does not embed isometrically in \( L_{-2} \).

By theorem 2, we have an example of a normed space whose unit ball is a 4-intersection body but is not a 2-intersection body. For \( 5 \leq n < 7 \), every normed space embeds in \( L_{-4} \), see [K3, p.78], so the example is for any space that doesn’t embed in \( L_{-2} \), for example \( c^n_q \), \( q > 2 \), see [K3, p.83].
Chapter 3
The geometry of spaces of Lorentz type

3.1 Introduction

Consider the family of norms \( \|x\|_{\pi(a)} = (a_1x_1^q + \cdots + a_nx_n^q)^{1/q} \) where \( a = (a_1, \ldots, a_n) \) with \( a_1 \geq \cdots \geq a_n > 0 \) and \( \pi(a) \) is a permutation of the vector \( a \). We will define a 1-homogeneous functional based on this family of norms as follows:

\[
\|x\|_k = \left( \sum_{\pi(a)} \|x\|_{\pi(a)}^{-k} \right)^{-\frac{1}{k}},
\]

where the sum is taken over all permutations. Our work examines some geometric properties of the star bodies \( (\mathbb{R}^n, \|x\|_k) \) for certain values of \( k \). First, we will use the recently developed Fourier transform criteria to determine when the star body \( (\mathbb{R}^n, \|x\|_k) \) is a \( k \)-intersection body. Second, we examine the extremal sections of the star body \( (\mathbb{R}^n, \|x\|_{-n+1}) \).

3.2 The space \( (\mathbb{R}^n, \|x\|_k) \)

The following theorem gives the conditions for the star body \( (\mathbb{R}^n, \|x\|_k) \) to be an intersection body.
Theorem 6. The star body $(\mathbb{R}^n, \|x\|_k)$ is an $k$-intersection body if $0 < q \leq 2$ and it is not a $k$-intersection body if $2 \leq q < \infty$ and $k < n - 3$.

Proof. By Theorem 2, we need to find the conditions when $\|x\|_k^{-k}$ is a positive definite distribution. Using the definition of the $\Gamma$-function we have:

$$\|x\|_k^{-k} = \sum_{\pi(a)} \|x\|_\pi^{-k} = \frac{q}{\Gamma(\frac{k}{q})} \sum_{\pi(a)} \int_0^\infty t^{k-1} \exp(-t^q \|x\|_\pi^q) dt.$$

Using the Fourier transform, we need to find the conditions when $(\|x\|_k^{-k})^\wedge(\xi)$ is positive.

$$(\|x\|_k^{-k})^\wedge(\xi) = \frac{q}{\Gamma(\frac{k}{q})} \sum_{\pi(a)} \int_0^\infty t^{k-1} \exp(-t^q \|x\|_\pi^q)^\wedge(\xi) dt$$

$$(\|x\|^{-k})^\wedge(\xi) = \frac{q}{\Gamma(\frac{k}{q})} \sum_{\pi(a)} \int \int_0^\infty t^{k-1} \exp(-t^q \|x\|_\pi^q) + it(x, \xi) dtdx$$

$$(\|x\|^{-k})^\wedge(\xi) = \frac{q}{\Gamma(\frac{k}{q})} \sum_{\pi(a)} \int_0^\infty t^{k-1} \left( \int \exp(-t^q a_1 \|x\|^q) \exp(i \xi_1) dx_1 \right) \cdots$$

$$\cdots \left( \int \exp(-t^q a_n \|x\|^q) \exp(i \xi_n) dx_n \right) dt$$

$$(\|x\|^{-k})^\wedge(\xi) = \frac{q}{\Gamma(\frac{k}{q})} \int_0^\infty (a_1 \cdots a_n)^{-\frac{1}{q}} t^{k-1-n} \sum_{\pi(a)} \gamma_q \left( \frac{\xi_1}{ta_n^{-1/q}} \right) \cdots \gamma_q \left( \frac{\xi_n}{ta_n^{-1/q}} \right) dt$$

(3.1)

where $\gamma_q(t)$ is the Fourier transform of the function $x \mapsto \exp(-|x|^q)$. First, the case $0 < q \leq 2$. By Lemma 2, $\gamma_q(\cdot)$ is positive everywhere on $\mathbb{R}$. Using this fact with 3.1,
\((\|x\|^{-k})(\xi)\) is positive and therefore the star body \((\mathbb{R}^n, \|x\|_k)\) is a \(k\)-intersection body.

For the case \(2 < q < \infty\), let’s consider the integral

\[
I(\alpha_1, \ldots, \alpha_{n-1}) = a_1^{1-\sum_{i=1}^{n-1} \alpha_i} \cdots a_n^{1-\sum_{i=1}^{n-1} \alpha_i} \int_{\mathbb{R}^{n-1}} |\xi_1|^{\alpha_1} \cdots |\xi_{n-1}|^{\alpha_{n-1}} (\|x\|^{-k})(\xi_1, \ldots, \xi_{n-1}, 1) d\xi_1 \cdots d\xi_{n-1}
\]

\[
= \frac{q}{\Gamma\left(\frac{\xi}{q}\right)} a_1^{1-\sum_{i=1}^{n-1} \alpha_i} \cdots a_n^{1-\sum_{i=1}^{n-1} \alpha_i} \int_{\mathbb{R}^{n-1}} |\xi_1|^{\alpha_1} \cdots |\xi_{n-1}|^{\alpha_{n-1}}
\int_0^\infty (a_1 \cdots a_n) \frac{1}{q} t^{k-1-n} \sum_{\pi(a)} \gamma_q \left(\frac{\xi_1}{ta_{i_1}^{1/q}}\right) \cdots \gamma_q \left(\frac{\xi_{n-1}}{ta_{i_{n-1}}^{1/q}}\right) \gamma_q \left(\frac{y}{ta_{i_n}^{1/q}}\right) dt d\xi_1 \cdots d\xi_{n-1}
\]

Changing variables, \(y = \frac{1}{t}\), we have

\[
= \frac{q}{\Gamma\left(\frac{\xi}{q}\right)} \sum_{\pi(a)} \int_{\mathbb{R}^{n-1}} \int_{-\infty}^\infty \frac{1}{a_{i_n}^{1/q}} \frac{y}{a_{i_1}^{1/q}} \frac{y}{a_{i_2}^{1/q}} \cdots \frac{y}{a_{i_{n-1}}^{1/q}} |\xi_1| \gamma_q \left(\frac{\xi_1}{ta_{i_1}^{1/q}}\right) \cdots \gamma_q \left(\frac{\xi_{n-1}}{ta_{i_{n-1}}^{1/q}}\right) \gamma_q \left(\frac{y}{ta_{i_n}^{1/q}}\right) dy d\xi_1 \cdots d\xi_{n-1}
\]

\[
= \frac{q}{2\Gamma\left(\frac{\xi}{q}\right)} \sum_{\pi(a)} \left( \int_{\mathbb{R}} \frac{y}{a_{i_1}^{1/q}} \gamma_q \left(\frac{\xi_1}{a_{i_1}^{1/q}}\right) d\xi_1 \right) \cdots \left( \int_{\mathbb{R}} \frac{y}{a_{i_{n-1}}^{1/q}} \gamma_q \left(\frac{\xi_{n-1}}{a_{i_{n-1}}^{1/q}}\right) d\xi_{n-1} \right)
\]

\[
\left( \int_{\mathbb{R}} \frac{y}{a_{i_n}^{1/q}} \gamma_q \left(\frac{\xi_n}{a_{i_n}^{1/q}}\right) dy \right)
\]

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\[
\frac{q(n-1)!}{2\Gamma\left(\frac{k}{q}\right)} s_q(\alpha_1) s_q(\alpha_2) \cdots s_q(\alpha_{n-1}) s_q(-k - \alpha_1 - \cdots - \alpha_{n-1}) \sum_{j=1}^{n} a_{ij}^k \quad (3.2)
\]

Thus, \( I(\alpha_1, \ldots, \alpha_{n-1}) \) converges if each of \( \alpha_1, \ldots, \alpha_{n-1}, \) and \(-k - \alpha_1 - \cdots - \alpha_{n-1}\) belong to the interval \((-1, q)\). Note that \( \sum_{j=1}^{n} a_{ij}^k > 0 \) and \( 0 < k \leq n \). Choose \( \alpha_1, \ldots, \alpha_{n-1} \in (-1, 0) \), then \( s_q(\alpha_i) > 0 \) by Lemma 4. Thus \(-k - \alpha_1 - \cdots - \alpha_{n-1} \in (-k, n - k - 1) \cap (-1, q)\). In order for \( I(\alpha_1, \ldots, \alpha_{n-1}) \) to be positive \(-k - \alpha_1 - \cdots - \alpha_{n-1} \) must be in the set \((-1, 0) \cup (0, 2)\). Since \( 2 < q < \infty \), then the interval \((-k, n - k - 1) \cap (-1, q)\) will contain a neighborhood of 2 if \( n - k - 1 > 2 \). If \( k < n - 3 \) then \( I(\alpha_1, \ldots, \alpha_{n-1}) \) changes sign. Thus the function \( \|x\|^{-k} \) is not positive definite. By Theorem 2, the star body \((\mathbb{R}^n, \|x\|)\) is not a \( k \)-intersection body for \( 2 < q < \infty \) if \( k < n - 3 \). \( \square \)

### 3.3 Extremal sections of the star body \((\mathbb{R}^n, \|x\|_k)\)

#### 3.3.1 Introduction

The Fourier analytic approach to sections of convex bodies is based on formulas expressing the volume in terms of the Fourier transform of the Minkowski functional of a body. Laplace [LA] is credited with the first formula of this kind. Laplace stated that the \((n-1)\)-dimensional volume of the central hyperplane section of the unit cube \( Q_n \) in \( \mathbb{R}^n \) perpendicular to the main diagonal is equal to

\[
Vol_{n-1}(Q_n \cap (\frac{1}{\sqrt{n}}, \cdots, \frac{1}{\sqrt{n}})^\perp) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left( \frac{\sin (r/\sqrt{n})}{r/\sqrt{n}} \right)^n dr. \quad (3.3)
\]

The latter integral appears in the central limit theorem for the sums of uniformly distributed random variables. Using Taylor Series, we can approximate \( \sin x/x \) as...
follows:
\[
\sin \frac{x}{x} \approx 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots.
\]

Thus, as \( n \to \infty \),
\[
Vol_{n-1}(Q_n \cap (\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}})_\perp) \approx \frac{1}{\pi} \int_{-\sqrt{n}}^{\sqrt{n}} \left(1 - \frac{r^2}{6n}\right)^n \, dr \approx \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-\frac{r^2}{6}} \, dr = \sqrt{\frac{6}{\pi}}.
\]

This result is surprising since the hyperplane section perpendicular to \((1, 1, 0, \ldots, 0)\) in every direction has volume \(\sqrt{2}\). The problem of finding the maximal central hyperplane section of the unit cube remained unsolved until the 1980's. Hensley [H] showed that the volumes of central hyperplane sections are bounded above by a constant not depending on the dimension. Ball [B] proved that the maximal volume is \(\sqrt{2}\). Both used a general formula for the volume of central hyperplane sections of the cube that first appeared in literature by Polya [P1]:
\[
Vol_{n-1}(Q_n \cap \xi_\perp) = \frac{1}{\pi} \int_{-\infty}^{\infty} \prod_{k=1}^{n} \frac{\sin r \xi_k}{r \xi_k} \, dr,
\]
where \(\xi \in S^{n-1}, \xi_\perp = x \in \mathbb{R}^n : (x, \xi) = 0\) is the central hyperplane orthogonal to the vector \(\xi\). We assume that \(\sin r \xi_k / r \xi_k = 1\) when \(\xi_k = 0\).

Meyer and Pajor [MP] found a similar formula for the central hyperplane sections of the balls \(B^q_n\), the unit balls of the spaces \(\ell^q_n\), \(1 \leq q \leq 2\), defined by
\[
B^q_n = \{x \in \mathbb{R}^n : ||x||_q = (|x_1|^q + \cdots + |x_1|^q)^{1/q} \leq 1\}
\]
Koldobsky [K6] used Fourier methods to expand the formula to all \(0 < q < \infty\).
\[
Vol_{n-1}(B^q_n \cap \xi_\perp) = \frac{q}{\pi(n-1)\Gamma(\frac{n-1}{q})} \int_{0}^{\infty} \prod_{k=1}^{n} \gamma_q(t \xi_k) \, dt,
\]
where \(\gamma_q(t) = (e^{-|t|^q})^{\cap}(t), t \in \mathbb{R}\). Both equations 3.4 and 3.5 are part of a more general formula based on Fourier transforms of powers of norms, [K3, Chapter 3].
Theorem 7. For any origin-symmetric star body $K$ in $\mathbb{R}^n$ and any $\xi \in S^{n-1}$,

$$\text{Vol}_{n-1}(K \cap \xi^\perp) = \frac{1}{\pi(n-1)}(\|\cdot\|_K^{n+1})^\vee(\xi)$$

where $\|\cdot\|_K$ is the Minkowski functional of $K$. The Fourier transform of $\|\cdot\|_K^{n+1}$ is considered in the sense of distributions and as such coincides with a homogeneous of degree $-1$ function on $\mathbb{R}^n$, which is continuous on $\mathbb{R}^n \setminus 0$.

3.3.2 The star body $(\mathbb{R}^n, \|x\|_{-n+1})$

As before, we define a 1-homogeneous functional based on the family of norms, $\|x\|_{\pi(a)}$, as follows:

$$\|x\|_k = \left( \sum_{\pi(a)} \|x\|_{\pi(a)}^{-k} \right)^{-\frac{1}{k}}. \quad (3.6)$$

Setting $k = n - 1$, defines the star body $(\mathbb{R}^n, \|x\|_{n-1})$. We will use the Fourier transform criteria to determine the volume of central hyperplane sections of the star body $(\mathbb{R}^n, \|x\|_{n-1})$.

Lemma 10. Let $B$ be the unit star body of the space $(\mathbb{R}^n, \|x\|_{n-1})$. Then

$$\text{Vol}_{n-1}(B \cap \xi^\perp) = \frac{q^{(a_1 \cdots a_n)} \pi(n-1)}{\Gamma(n-1)} \sum_{\pi(a)} \int_0^\infty t^{-2} \prod_{k=1}^n \gamma_q(t\xi_k) dt$$

Proof. Using the definition of the $\Gamma$-function we have:

$$\|x\|_{n-1}^{-n+1} = \sum_{\pi(a)} \|x\|_{n-1}^{-n+1} = \frac{q^{(a_1 \cdots a_n)} \pi(n-1)}{\Gamma(n-1)} \sum_{\pi(a)} \int_0^\infty t^{-2} \prod_{k=1}^n \gamma_q(t\xi_k) dt.$$

Taking the Fourier transform of both sides, we have

$$(\|x\|_{n-1}^{-n+1})^\vee(\xi) = \frac{q^{(a_1 \cdots a_n)} \pi(n-1)}{\Gamma(n-1)} \sum_{\pi(a)} \int_0^\infty t^{-2} \prod_{k=1}^n \gamma_q(t\xi_k) dt$$

$$= \frac{q^{(a_1 \cdots a_n)} \pi(n-1)}{\Gamma(n-1)} \sum_{\pi(a)} \int_0^\infty (a_1 \cdots a_n)^\frac{1}{q} \gamma_q \left( \frac{\xi_1}{ta_{i_1}^{1/q}} \right) \cdots \gamma_q \left( \frac{\xi_n}{ta_{i_n}^{1/q}} \right) dt$$

$\square$
The following theorem shows that the minimal section corresponds to the direction \((1/\sqrt{n}, \ldots, 1/\sqrt{n})\) and the maximal section to the direction \((1, 0, \ldots, 0)\).

Theorem 8. For every \(q \in (0, 2)\) and \(\xi \in S^{n-1}\),

\[
\frac{q}{\Gamma(\frac{n-1}{q})(a_1 \cdots a_n)^{1/q}} \int_0^\infty t^{-2} \sum_{\pi(a)} \gamma_q \left( \frac{1}{a_{1_1}^{1/q} t} \right) \cdots \gamma_q \left( \frac{1}{a_{1_n}^{1/q} t} \right) dt \\
\leq Vol_{n-1}(B_{\pi(a)} \cap \xi^\perp) \leq \frac{q}{\Gamma(\frac{n-1}{q})(a_1 \cdots a_n)^{1/q}} \int_0^\infty t^{-2} \sum_{\pi(a)} \gamma_q^{n-1}(0) \gamma_q \left( \frac{1}{a_{1_1}^{1/q} t} \right) dt
\]

(3.8)

Proof. By Lemma 10,

\[
Vol_{n-1}(B_{\pi(a)} \cap \xi^\perp) = \frac{q(a_1 \cdots a_n)^{-1}}{\pi(n-1)\Gamma(\frac{n-1}{q})} \sum_{\pi(a)} \int_0^\infty t^{-2} \prod_{k=1}^n \gamma_q(t\xi_k) dt
\]

Now we apply the properties of the function \(\gamma_q\) stated in Lemma 2. The function \(\ln(\gamma_q(\sqrt{\cdot}))\) is convex on \([0, \infty)\), i.e. for any \(0 < \xi_1 < \varepsilon_1 < \varepsilon_2 < \xi_2\) with \(\xi_1^2 + \xi_2^2 = \varepsilon_1^2 + \varepsilon_2^2 = 1\), one has \(\gamma_q(t\xi_1)\gamma_q(t\xi_2) \geq \gamma_q(t\varepsilon_1)\gamma_q(t\varepsilon_2)\) for all \(t > 0\). Therefore, the max\([\gamma_q(t\xi_1)\gamma_q(t\xi_2) \cdots \gamma_q(t\xi_n)]\) occurs when one coordinate is 1 and the rest are 0, while the min\([\gamma_q(t\xi_1)\gamma_q(t\xi_2) \cdots \gamma_q(t\xi_n)]\) occurs when all the coordinates are equal, namely \(\frac{1}{\sqrt{n}}\). The result of the theorem follows by applying this fact to the formula for the volume of sections of \(B\). \(\square\)
Bibliography


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