TWO PATHS TO ADVANCED PLACEMENT CALCULUS: AN EXAMINATION OF SECONDARY STUDENTS' MATHEMATICAL UNDERSTANDING EMERGING FROM INTEGRATED AND SINGLE-SUBJECT CURRICULA

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ABSTRACT

This study examined high school students' mathematical understanding of calculus readiness concepts after studying four years of college preparatory (integrated or single-subject) mathematics. Data were collected from 505 students, 201 experienced an integrated curriculum (IC) and 304 experienced a single-subject curriculum (SSC). Additional data were collected on a subset of 199 of these students who enrolled in AP Calculus, 59 studied from the integrated curriculum (APIC) and 140 studied from the single-subject curriculum (APSSC). All 505 students took the *Precalculus Concept Assessment (PCA)* a 25 item multiple choice assessment focused on functions. In addition, the AP Calculus students completed two open-ended tasks related to functions.

After adjusting for prior achievement with the Iowa Algebra Aptitude Test (IAAT), SSC students performed statistically higher ($F = 9.39$, $p = .002$) than IC students on the *PCA*; whereas, APSSC and APIC students performed comparably ($F = 3.54$, $p = .063$). The item analysis of the *PCA* suggests that students are completing four years of college preparatory mathematics (integrated and single-subject) with a procedural understanding of functions. For instance, most students could evaluate and solve functions correctly; however, they exhibited multiple misconceptions about rates of change and function inverse.

AP Calculus students tended to use similar solution strategies when solving the two open-ended tasks, including using a graph, an equation, a table, or using multiple methods. Although students used similar strategies, their ability to use these strategies effectively differed.

AP Calculus students from each curricula pathway demonstrated errors related to rates of change. Students tended to use words such as slope, rate of change, and steepness interchangeably. Student errors on the two open-ended tasks and the *PCA* revealed a lack of understanding of the interpretation or meaning of rate of change. Students successfully calculated the rate of change of linear functions; however, when the rate of change was not linear, students struggled to calculate it, represent it correctly on the graph, or correctly interpret the rate in a real world context.

CHAPTER 1: INTRODUCTION

> No reform is a panacea, but in order to institute reform those who are in favor make all sorts of claims for it, and they raise expectations. When those expectations are not met, the public – always skeptical of reform in mathematics – starts to perceive that the reform, just like the mathematics they took in schools, is not successful. So the public perception is of failure even if the overall situation is for the better. (Usiskin, 1999, p. 7)

History is a great teacher. We must learn from the past and use that information wisely to make changes for the future. However in education, the public perception of what and how students learn is always a key ingredient in the acceptance or rejection of change. Mathematics curriculum is not exempt from public scrutiny. Over the past 50 years, major revisions in mathematics textbooks have occurred. Each attempt provided information and insight to future curriculum developers, yet the public viewed the reform textbooks developed during these eras as failures (Usiskin, 1999).

The struggle for direction on mathematics curriculum and teaching in K-12 schools in the United States began in the early 1950s, and a crisis-reform-reaction cycle continues today. Fey and Graeber (2003) describe the situation as follows:

> A prominent social, political, or professional group calls attention to serious problems in students' performance and recommends action, only to find that reform initiatives ultimately run up against resistance from opposing views and the deeply conservative nature of educational reform rhetoric often settles down to a quieter pattern of business as usual, at best moderately perturbed by the energetic calls for change in standard practices. (p. 521)

Currently, mathematics education is in the reaction stage of this cycle as many debate the effectiveness of new mathematics curriculum for K-12 students.

In 1992, in response to the National Council of Teachers of Mathematics (NCTM) release of *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics, 1989) the National Science Foundation (NSF) initiated a wave of major reform by funding five different secondary mathematics curriculum projects. Each project was developed by independent groups of writers to provide multiple ways in which mathematics curricula might reflect the vision of the *Standards*. Although each curriculum series is different, they include one common feature, namely the integration of mathematical topics that students experience to promote connections. In contrast to the traditional sequence of secondary mathematics courses Algebra, Geometry, Algebra II, and Precalculus, integrated mathematics curricula allow students to study algebra, geometry, and data each year of high school. In addition these curricula provide opportunities for students to engage in higher order thinking, experience contextual problems, and use technology. Integrated curricula "represent a truly radical approach to secondary mathematics education" (St. John, Fuller, Houghton, Tambe, & Evans, 2004, p. 94). More specifically these curricula represent a "blend of mathematical concepts chosen from a wide variety of mathematical fields and emphasiz[e] relations among the concepts within mathematics, as well as between mathematics and other disciplines" (Lott, 2001, p. 1). Integrated curricula balance the development between concepts and procedures, use of technology, and various types of representation (Stein, Remillard, & Smith, 2007).

Integrated curricula contain a common thread that is interwoven throughout each course, creating a vertical model of incorporating concepts from different strands. Burkhardt (2001) suggests the "main advantage of integrated curricula are that they build on essential connections, help make mathematics more usable, avoid long gaps in learning, allow a balanced curriculum, and support equity" (p. 2). Although integrated curricula may be fairly new within the United States, teachers and students from many countries other than the U.S. have used them for many years (Jun, 2005; Nagaoka, 2005; Pang, 2005; Reys, B. J., Reys, & Chavez, 2004; Usiskin, 2003; Yee, 2005).

In contrast to integrated programs, single-subject curricula have long dominated classrooms in the U.S. The course sequence of Algebra 1, Geometry, Algebra II, and Precalculus has existed in the United States since the turn of the 20th century. In 1894, the release of the *Report of the Committee of Ten* (National Education Association) greatly influenced the school system organization with their recommendations for single-subject course sequencing. Schools were organized in this manner to accommodate students who only attended high school for one or two years. The committee believed students should learn a combination of algebra and geometry rather than an in depth study of algebra. In the single-subject sequence, students spend an entire year learning a specific mathematical strand in depth, such as geometry. The connections within mathematics are made less often because students are immersed in one specific strand each year (Burkhardt, 2001). Although school conditions have changed significantly during the last 100 years, this single-subject structure is viewed by many as sacred.

Comparison between Integrated and Single-subject Curricula

The integrated and single-subject curricula are significantly different in their organizational structure. The expectations and roles for students and teachers are also different. The integrated curriculum was designed so teachers can be classroom facilitators for students who are actively engaged in mathematical learning (St. John et al., 2004). After students have developed a conceptual knowledge of the topic, “teacher[s] step in to apply definitions, standard labels, and (sometimes) standard procedural techniques related to the concept” (Stein et al., 2007, p. 331). The NSF integrated curriculum espouses the philosophy that students learn by constructing new knowledge connected to their prior knowledge. Students are given mathematical problems based on real world situations to understand the importance of mathematics and make connections to the concepts being taught. Integrated curricula were developed to "keep post-high school education and career options open for all students" (Senk & Thompson, 2003, p. 311).

On the other hand, single-subject curricula were designed to provide "distinct programs for college-bound and non-college bound students" (Senk & Thompson, 2003, p. 300). However, these books are often organized as discrete two-page lessons taught independent of one another. In fact, researchers from the Third International Mathematics and Science Study (TIMSS) found U.S. teachers taught using "frequent reviews of relatively unchallenging, procedurally oriented mathematics during lessons that are unnecessarily fragmented" (Hiebert, Stigler, Jacobs, Givvin, Garnier, Smith et al., 2005, p. 116). Hiebert and Grouws (2007) agreed, suggesting U.S. mathematics teachers design lessons to reinforce lower-level mathematics and do not challenge students' thinking. Stein et al. (2007) state, "students typically are expected to master definitions and standard algorithmic procedures *before* they are exposed to opportunities to apply their knowledge" (p. 331). Most single-subject curricula contain some real-world contextual problems, but the focus is students learning algorithms and then practicing them (Stein et al., 2007).

Some mathematics educators have conducted research on integrated and single-subject curricula; however, it has not provided evidence sufficient for selecting one type of curriculum over another (National Research Council, 2004). The lack of convincing research on integrated curricula has raised the question: Is integrated curriculum beneficial for students' mathematical understanding?

Critics of the integrated curricula sometimes refer to it as "*fuzzy math*" or "*new new math*" (Senk & Thompson, 2003). Mathematicians, mathematics educators, teachers, and parents continue to debate and ask if students in integrated courses understand mathematics *better* than before? Would students learn *better* from a different organizational structure of the mathematics curriculum? What are students gaining or missing by learning from integrated mathematics curricula? These questions are difficult to answer because factors other than the textbook cannot be controlled, yet they impact student learning. Hiebert (1999) states, "if researchers cannot prove that one course of action is *the* best one, it follows that researchers cannot prescribe a curriculum and a pedagogical approach for all students and for all time" (p. 7). However, researchers can document the current status of reform efforts such as what mathematics curriculum is being used and how students learn from it. Researchers can also document the effectiveness of curricula such as "what students *can* learn under what kinds of conditions" (Hiebert, 1999, p. 9).

Statement of the Problem

The first published versions of NSF-funded integrated high school mathematics curricula appeared in 1997. In the past 10 years, researchers have begun to document student learning from these curricula; however, only recently have high school students had an opportunity to complete the entire four years of the integrated sequence. Over the past decade, many secondary teachers have made adaptations to their pedagogical methods and have become more comfortable teaching from integrated curriculum. For these teachers it meant becoming familiar with mathematics content in ways far different than how they learned. It also required teachers to change their classroom from teacher centered to student centered.

School districts have been reluctant to adopt integrated mathematics textbooks because of the many opponents who criticize them and claim students are not learning mathematics. However, some secondary schools offer parallel paths in their mathematics program, which allow students to take either integrated or single-subject courses. Students in these districts complete four years of college preparatory mathematics courses that will prepare them for either Advanced Placement (AP) Calculus or AP Statistics, depending on course offerings available at the school.

The difference in organizational structure and expectations raises the question: What are the similarities and differences in how students understand mathematics after completing either an integrated or single-subject curriculum path? Figure 1.1 displays two options for students as they enter high school mathematics courses and shows that students learn mathematics in both paths. However, what mathematics is learned and how it is learned are influenced by the implemented curriculum. A key question is: what mathematical knowledge will students possess when they emerge from these two different curriculum paths? Figure 1.1 shows that students' from these two curricular paths will merge into a common AP Calculus classroom where they will continue to learn mathematics. An important practical question is: Will students in one of these groups be disadvantaged or advantaged? Students who have completed four years of mathematics in different curricular paths may display similarities and differences in their strategies for solving problems, types of mathematical errors, misconceptions, ability to solve questions requiring different levels of cognitive demand and items focused on specific representations (e.g., graphic, numeric, and symbolic). As curriculum designers revise and develop new curricula, they need to understand what students are learning or not learning from existing curricula. Stein et al. (2007) points out "few studies have connected the curriculum (or tasks) *as enacted* with student learning or achievement" (p. 358); however, even fewer studies have connected high school mathematics curricula with student learning (Harwell, Post, Maeda, Davis, Cutler, Andersen et al., 2007). Research at the high school level needs to examine what students who have experienced integrated or single-subject curricula have learned and what strategies they have developed.

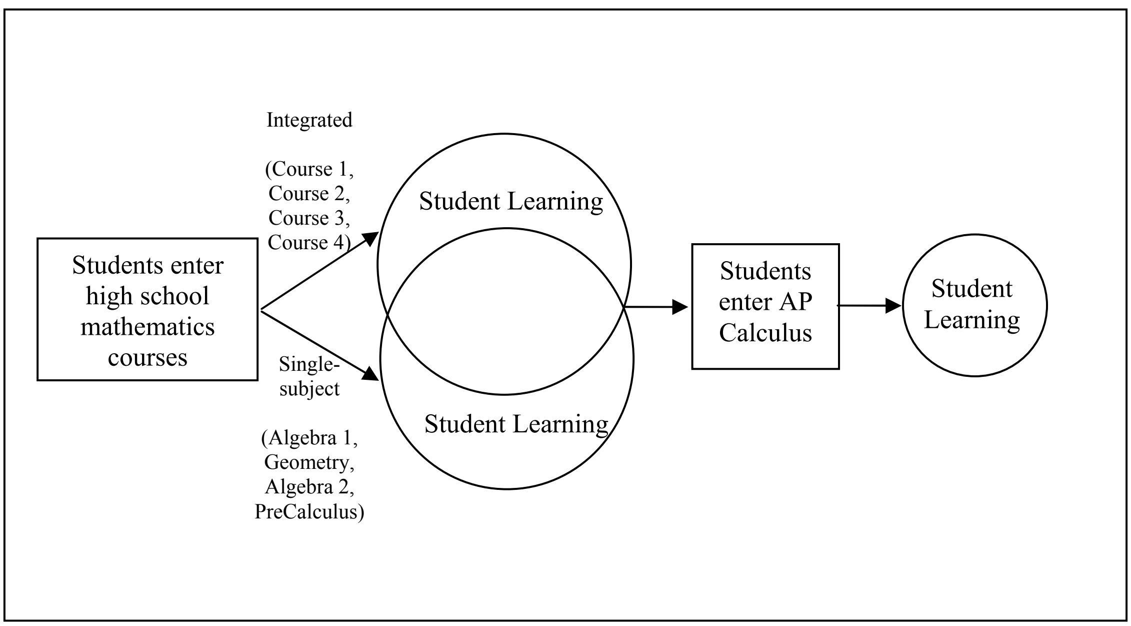


Figure 1.1. Students' mathematical course paths

Purpose of the Study

The purpose of this study was to investigate similarities and differences in mathematical understanding of high school students who completed four years of college preparatory (integrated or single-subject) mathematics curricula. More specifically, this study investigated students' ability to solve tasks requiring different levels of cognitive demand, items focused on specific representations (e.g., graphic, numeric, and symbolic), and common misconceptions they had about functions before entering AP Calculus after completing four years of college preparatory (integrated or single-subject) curricula. Strategies students used to solve open-ended problems and errors they made were analyzed to understand how their previous mathematics curriculum impacted what they did and how they did it.

Research Questions

This study sought to answer the following research questions:

1. How do high school students who studied four years of college preparatory mathematics curriculum (integrated or single-subject) perform and compare on calculus readiness concepts in terms of:
    a. overall performance;
    b. responses to items by levels of cognitive demand;
    c. responses to items focused on specific representations (e.g., graphic, numeric, symbolic); and
    d. common misconceptions about functions?
2. How do high school students who studied four years of college preparatory mathematics curriculum (integrated or single-subject) and enrolled in AP Calculus perform and compare on college readiness concepts in terms of:
    a. overall performance;
    b. responses to items by levels of cognitive demand;

c. responses to items focused on specific representations (e.g., graphic, numeric, symbolic); and

d. common misconceptions about functions?

3. How do high school students who studied four years of college preparatory mathematics curriculum (integrated or single-subject) and enrolled in AP Calculus perform and compare in solution strategies and errors on open-ended mathematical tasks that focus on functions during the first quarter of AP Calculus?

Conceptual Framework

Professional organizations, NCTM and the National Research Council (NRC), encouraged changes in instructional practices that emphasize problem solving, reasoning, communicating mathematically, and conceptual understanding (National Council of Teachers of Mathematics, 1989, 2000; National Research Council, 1989). However, if teachers changed their instructional practices they must also change their assessments to understand what and how students think. Most large-scale assessments measure student knowledge with multiple-choice questions. These items have some advantages, including a large data sample can be collected quickly, more items can be administered in a short period of time, procedural knowledge can be measured easily, and student responses can be scored quickly and reliably (Cai, 1997). However, multiple-choice items only require students to produce an answer with no indication of how they arrived at the answer, which makes it difficult for researchers and teachers to analyze and understand student thinking (Cai, Lane, & Jakabcsin, 1996). Robitaille and Travers (1992) argue that "information about how students approach the solution of a given problem is … more important than whether or not they are able to recognize the correct solution" (p. 708). Resnick (1988) advocates examining characteristics of students' solutions, including strategies, representations, and errors. More recently, Lesh and Zawojewski (2007) counsel that research should "focus on students' interpretations, representations, and reflections …" (p. 765).

If researchers and teachers are to understand what students know about mathematics then open-ended tasks must be administered and analyzed for understanding. Open-ended tasks "require students to synthesize information, justify their solutions processes, and evaluate their answers…" (Lane, 1993, p. 16). These tasks allow researchers and teachers to *learn* what students think and understand about mathematics because they must not only produce an answer, but also "show their solution processes and give justification for their answers" (Cai, Lane et al., 1996, p. 138). Cai and Silver (1995) corroborated that Chinese students performed better on procedural questions related to division than U.S. students; however, utilizing open-ended tasks they discovered Chinese students performed no different than U.S. students on questions requiring mathematical understanding or mathematical problem solving. Open-ended tasks allow researchers and teachers to assess mathematical understanding and complex problem solving that cannot be done with typical multiple choice items (Cai, 1997; Cai & Silver, 1995).

Generally, assessments are scored for correctness and total scores are reported as a percentage, scale score, or raw score. To analyze students' solutions on open-ended tasks a qualitative analysis must be conducted. Messick (1989) described three methods for analyzing student processes on a performance task. First, *protocol analysis*, students are asked to think aloud as they solve a problem. Second, *analysis of reasons*, students provides written rationale for their responses. Finally, *analysis of errors*, inferences are drawn about processes from the incorrect procedures, concepts, or representations used to solve the problems. Although *protocol analysis* is a powerful method for gaining insight into what students understand and know about mathematics, it is difficult to conduct on a large scale. The process of collecting, coding and analyzing interview data is labor intensive. However, *analysis of reasons* and *analysis of errors* can reveal students' mathematical thinking on written open-ended tasks.

The QUASAR (Quantitative Understanding: Amplifying Students Achievement and Reasoning) project (Cai, Magone, Wang, & Lane, 1996; Stein & Lane, 1996; Stein, Smith, Henningsen, & Silver, 2000) designed a qualitative analytic framework to analyze middle school student responses on multiple open-ended tasks (see Figure 1.2) by analyzing solution processes, "including mathematical communication, solutions strategies, and mathematical errors" (Cai, Magone et al., 1996, p. 830).

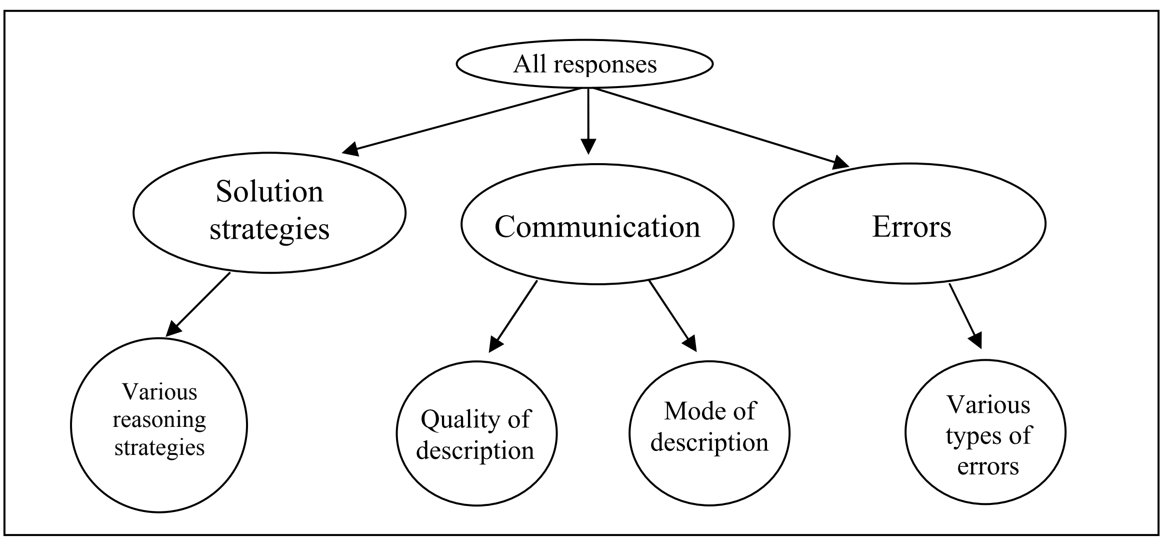


Figure 1.2. Generic qualitative analytic framework for analyzing student work on the QUASER Cognitive Assessment Instrument (Cai, Magone et al., 1996)

Although this framework was used to analyze middle school students' written work, it provides a foundation to begin analyses of high school students' work.
An adaptation of Figure 1.2 was used to analyze similarities and differences in AP Calculus students' strategies for solving open-ended tasks and errors or misconceptions (see Figure 1.3). Each question was first separated by correct and incorrect answers. Each correct answer was analyzed for solution strategies and each incorrect answer was analyzed for solution strategies and errors.

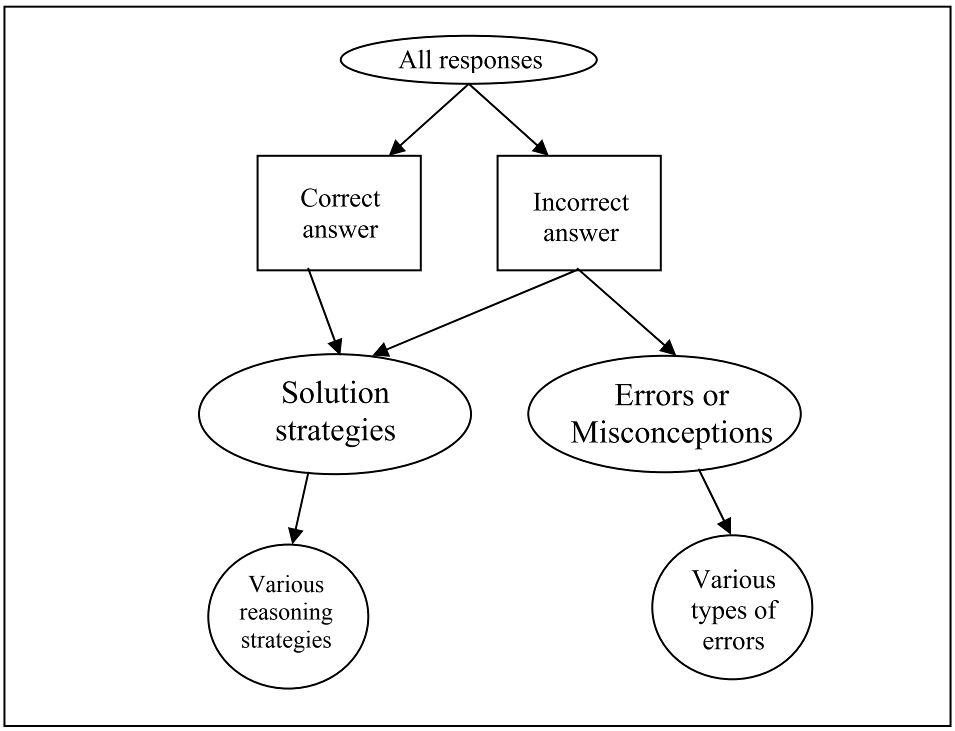


Figure 1.3. Adapted framework for analyzing high school students' mathematical understanding

Definition of Terms

This section provides definitions of different terms used in this study. Although different definitions exist for many commonly used terms in mathematics and mathematics education (Usiskin, 2008), I define the terms for how they were used within this study.

The field of mathematics education defines integrated curriculum many ways. Usiskin (2003) describes different kinds and levels of integration within mathematics curriculum: unifying concepts or merging different areas of mathematics (strands or interdisciplinary). He suggests that some people believe single-subject curriculum is integrated because it contains different strands (algebra, geometry, and data); however, these strands are not typically related to each other and are usually taught separately.

Integrated curriculum

For this study, an integrated curriculum textbook series contains strands (algebra, geometry, and data) in which connections are continuously made among these strands. The specific integrated curriculum used in this study was the *Contemporary Mathematics in Context: A Unified Approach* (Coxford, Fey, Hirsch, Schoen, Burrill, Hart et al., 2003) series, one of the NSF high school curricula developed around NCTM's *Standards*.

Single-subject curriculum

This study referred to the single-subject curriculum as any secondary mathematics curriculum that was designed for use in a single-subject course: Algebra 1, Geometry, Algebra II, and Precalculus. The specific single subject curricula used for this study was a combination of three University of Chicago School Mathematics Project (UCSMP) textbooks, *Algebra, Geometry*, and *Advanced Algebra* (McConnell, Brown, Usiskin, Senk, Widerski, Anderson et al., 1996; Senk, Thompson, Viktora, Usiskin, Ahbel, Levin et al., 1996; Usiskin, Hirschhorn, Coxford Jr., Highstone, Lewellen, Oppong et al., 1997) and a Prentice Hall Precalculus textbook, *Precalculus Enhanced with Graphing Utilities* (Sullivan & Sullivan III, 2006).

Precalculus Concept Assessment

The *Precalculus Concept Assessment* (*PCA*) (Engelke, Oehrtman, & Carlson, 2005) is an instrument used to assess students' understanding of functions. The *PCA* has also been used to determine calculus readiness for undergraduate college students. The *PCA* contains 25 multiple choice questions and was administered in 50 minutes. The *PCA* was developed using distracters that were common misconceptions students have about functions.

Strategies

For this study, strategies were defined as the general plan or direction students select to solve an open-ended task. More specifically, this was the process or steps students wrote on their paper.

Errors

For this study, errors were defined as mistakes students made when solving open-ended problems or misconceptions students demonstrate about the mathematics being assessed.

Significance of the Study

Although integrated mathematics curricula were published and in use since 1997, limited research exists on what students learn from the integrated curriculum. Harwell et al. (2007) stated, "it is time now to devote research energy to the more important question of what mathematics *Standards*-based [integrated] students are learning that traditional [single-subject] are not and vice versa, …" (p. 96). Schoen and Hirsch (2003b) call for more research to study the effect of the latest version of integrated textbooks on student achievement in settings where students are accustomed to the expectations of the curriculum. This study addresses both of these recommendations by analyzing students' mathematical knowledge after successfully completing four years of mathematics in each path.

Summary

A driving force behind most curriculum reform is *what* should be taught and *how*. The NSF-funded mathematics curriculum was a major effort to initiate change to help all students learn. Although these curricula have been available for some time now, the research base is limited because few students have used them for an extended period of time.

This study reports on students from two high schools that completed four years of college preparatory mathematics courses (integrated or single-subject). More specifically, the first focus was on performance of students who completed four years of college preparatory (integrated or single-subject) mathematics curriculum on the *PCA*. The second focus was on AP Calculus students' strategies used to solve open-ended problems and the mathematical errors they made while solving these tasks. In the next chapter the relevant research on integrated and single-subject high school mathematics curricula and the impact of these curricula on student learning are reviewed.

CHAPTER 2: LITERATURE REVIEW

The primary focus of this study was to describe similarities and differences in the mathematical understanding of students who completed four years of integrated mathematics curricula or single-subject coursework. More specifically, this study examined students' performance on functions, students' solution strategies when solving open-ended tasks, mathematical errors they made when solving open-ended problems, and how successful they are in solving items that require different levels of cognitive demand. Researching how students use curriculum and its impact on achievement is not new; however, research on the use of newer curricula supported with funds from the National Science Foundation (NSF) is limited (National Research Council, 2004; U.S. Department of Education, 2002). This chapter focuses on reported research and relevant documents that inform the work of this study. More specifically this chapter includes: (a) a brief historical view of the mathematics curriculum evolutionary process, (b) research documenting the effects of integrated curriculum, *Core-Plus*, on student achievement, (c) research on calculus students, both college and high school students, (d) research on student solution strategies, (e) research on mathematical errors, and (g) frameworks for analyses of levels of cognitive demand.

This chapter is divided into six sections. First, a brief historical perspective of how and why mathematics curriculum has changed is provided. This discussion included a backdrop for understanding some of the current mathematics curriculum issues being experienced in the United States. The second section describes research on how the *Core-Plus* curriculum has impacted high school students' achievement. This section leads into the research on students' understanding of functions, which provides background information on the difficulties students have with some foundational topics in calculus. The next two sections describe research on students' solution strategies and errors when solving open-ended tasks. The concluding section describes different frameworks used to analyze levels of cognitive demand for each question. Knowledge from these sections will inform the methods and procedures utilized in this study.

Historical Perspective

Decisions about mathematics curricula follow a crisis-reform-reaction cycle that continues today as the field struggles to answer *what* mathematics should be taught and *how*. As mathematics educators and others propose or implement curriculum change the amount and intensity of discussion increases. Debates about mathematics curricula have occurred for decades and continue today in the United States.

A major cycle of mathematics curriculum change began in 1951. University of Illinois engineering faculty believed high school students were entering their classrooms without the mathematics background necessary to succeed in college. They joined the faculty in education and mathematicians to develop the pamphlet, *Mathematical Needs of Prospective Students in the College of Engineering of the University of Illinois*, to guide high school students in their course taking choices (Jones & Coxford Jr., 2003). Max Berbman, with members from the colleges of Education, Engineering, and Liberal Arts and Sciences, organized and directed the *University of Illinois Committee on School Mathematics* (UICSM) to investigate the mathematics content and teaching of high school courses. (Jones & Coxford Jr., 2003).

The UICSM developed a first-of-its-kind model four year high school mathematics curriculum for college bound students that included a completely new set of courses that reflected formalism and symbolism never before seen in high school mathematics in the United States. Furthermore, in addition to curriculum development, the UICSM field tested their curriculum, and required teachers to be trained extensively before using their program.

During the same decade, the College Entrance Exam Board (CEEB) appointed the Commission on Mathematics (College Entrance Exam Board, 1959) to examine all secondary mathematics curriculum and recommend how to modernize, modify, and improve it (Jones & Coxford Jr., 2003). The CEEB wanted to make sure that what was taught matched what was assessed and that students were actually prepared for college. Recommendations included a balance between concepts and skills, deductive reasoning, the use of mathematical properties, equalities and inequalities, unifying mathematics ideas, and suggestions for courses in geometry, trigonometry and a senior level mathematics course (Jones & Coxford Jr., 2003). Although the CEEB report was not published until 1959, the UICSM textbooks modeled many of these recommendations and influenced future curricula, such as the *School Mathematics Study Group* (*SMSG*).

Outside forces also influence curriculum changes. The launching of Sputnik in 1957 caused the nation to re-evaluate what mathematics was required, but more importantly how schools could facilitate the development of engineers and scientists that could keep pace with those in other countries (Jones & Coxford Jr., 2003). The CEEB's recommendations and the development of the UICSM curriculum provided an avenue to help students develop understanding of mathematics and prepare them to enter college mathematics courses. The "new math" era began with textbooks that contained new and more advanced content that would prepare students for college. Teachers required more training and professional development because they were not prepared to teach the mathematics content in these new curricula.

The “new math” curricula changed the emphasis on the type of knowledge students were expected to learn, such as a reduction on rote memorization and an increased emphasis on concepts and proof (Davis, 1974). These curricula also impacted teacher pedagogy. For example, Gray (1974) said “The major change has been more in how math is taught, rather than what math is taught” (p. 39). Students were asked to understand mathematics conceptually rather than apply formulas with no meaning. Students and parents had difficulty understanding this new approach to mathematics. Many “new math” curricula “were designed for mathematically capable students heading for further academic study of mathematics” (Fey, 1978, p. 342). These curricula were characterized as “emphasizing pure mathematics and neglecting the traditional diet of ‘life survival’ applications that were prominent ….” (p. 342) in curricula that was replaced.

The “new math” curricula developed during the 1950s and 1960s aroused unprecedented public discussion about mathematics curriculum and led to new research and interest in student learning (Mueller, 1966). During the 1960s the government instituted significant funding for curriculum development in mathematics and science. For the first time government agencies responded to the need for new mathematics textbooks and became stakeholders in the education system. The funding allowed more people opportunities to work on mathematics curriculum projects and helped promote dissemination of the curricula to a larger group of teachers and students across the U.S.

At the same time, curriculum developers began field testing textbooks, in-servicing teachers, and collaborating with mathematicians. These reform efforts are still used today. For the first time curriculum developers worked with classroom teachers to produce high quality textbooks that were not created in a short period of time. These curricula took years to conceptualize, develop, field-test, revise and produce a final edition. Before the UICSM curriculum emerged, teachers were not given special professional development to facilitate their use of specific curriculum materials. However, this changed because it was clear that traditional training of mathematics teachers had not prepared them to implement mathematics curricula that required additional depth of mathematical knowledge and a different orientation of the teacher's role.

The "new math" era marked the largest national mathematics curriculum reform in the United States. It focused on preparing materials for secondary school mathematics "which expressed the modern views and role of mathematics" (Jones & Coxford Jr., 2003, p. 72). For the first time, educators seriously addressed how they could prepare mathematics students for college and not just teach the basic skills.

1970s

Some prominent mathematicians were critical of the "new math" curricula and how it was implemented. Their concerns together with concerns and confusion expressed by many parents stalled further mathematics curriculum development in the early 1970s. In fact, initiatives were made to refocus the mathematics curriculum and return to the basics. Public and professional discussions centered on two questions:

1) “Have the reformers chartered the right direction for the content goals of school curricula?
2) Do the programs that implement their ideas work as intended?” (Fey & Graeber, 2003, p. 531)

Morris Kline, a mathematician, was a prominent critic of the “new math” textbooks because “algebra, geometry, and trigonometry were replaced by set theory, symbolic logic, matrices, and Booliean algebra” (Jones & Coxford Jr., 2003, p. 82). He thought these subjects were inappropriate for high school students to study.

Concerns were expressed about how mathematics was taught and what was being learned. A report in 1977 from the College Board showed that over the last ten years student scores on the Scholastic Aptitude Test (SAT) declined. The government reduced its funding for curriculum development during the seventies because of political resistance and societal fears that school districts were losing power to make local decisions about curricula (Fey, 1978).

The public called for curricula to be designed for less able students. In addition parental concern for students’ perceived lack of computational skills pushed the field to ask what basic mathematics skills do K-12 students need to know and how do teachers help students obtain these skills. In 1975, the National Institute of Education conducted a meeting to discuss two questions:

- “What are basic mathematical skills and learning?
- What are the major problems related to children’s acquisition of basic mathematical skills and learning, and what role should the National Institute

> of Education play in addressing these problem?" (Conference on Basic Mathematical Skills, 1976, p. 2)

Although the Conference on Basic Mathematical Skills recommended ten basic skills essential for K-12 students little attention was initially given to these skills. However, one year later the National Council of Supervisors of Mathematics (NCSM) built on the work done at the Conference on Basic Mathematical Skills and released *The Position Paper on Basic Mathematical Skills* (1977), which recommended the following skills: (1) problem solving, (2) applying mathematics to everyday situations, (3) alertness to the reasonableness of results, (4) estimation and approximation; (5) appropriate computational skills, (6) geometry, (7) measurement, (8) reading, interpreting, and constructing tables, charts and graphs, (9) using mathematics to predict, and (10) computer literacy. Their position paper was a response to societal calls to return to the basics, but more importantly it also provided a unified definition for basic skills that included more than computation.

The 1970s was a decade of "groping for a clearer focus and sense of direction" (Hill, S., 1983). The field had to respond to opponents and public dissatisfaction about the "new math" curriculum. There was also "widespread concern for specifying standards of minimal competency in mathematics to be attained by all students…" (Fey, 1978, p. 342). During this decade, one curriculum reform ended and a new one began.

Mathematics educators spent the 1970s reacting to what happened during the "new math" era, but were also unhappy with the "back to basics" movement. New curricula needed to be designed for all students; however, mathematics educators and mathematicians needed to come to consensus on what students should learn and how it should be taught. The National Science Foundation sponsored conferences to discuss "questions about the nature of mathematics, the way it is learned, and the way it should be taught…" (Fey & Graeber, 2003, p. 549) and paved the way for the rise of professional organizations and national recommendations that began during the 1970s.

1980s

A crisis level was reached during the 1970s and forced professional organizations to work for consensus and understanding. The crisis level also caused many to call for reform and initiated new ways of viewing mathematics education. The first major report released by the National Council of Teachers of Mathematics (NCTM), *An Agenda for Action* (1980) "was a response to particular demands and pressing concerns" (Hill, S., 1983, p. 1). The *Agenda for Action* was developed after many discussions and recommendations from conferences and reports during the seventies. "The *Agenda* [*for Action*] expressed a vision that was endorsed by numerous groups, given relatively wide circulation, and was used to guide NCTM publications and actions from 1980 until the 1989 release of the first *Standards* document…" (Fey & Graeber, 2003, p. 553). The report called for all K-12 students to experience important topics in mathematics: problem solving, basic skills, and technology. Consequently new curricula developed during the 1980s focused on problem solving. The *Agenda for Action* called for increased emphasis with collecting, organizing, and analyzing data as well as the use of maps and diagrams to help understand problem situations (Fey & Graeber, 2003). The definition of basic skills was adopted from the NCSM position paper on *Basic Mathematical Skills* requiring students to do more than simple computations. Calculators were introduced in the late seventies; however, NCTM was the first professional organization to endorse judicious calculator use for all students in grades K-12. The *Agenda for Action* called for a decreased emphasis in paper-pencil computations and the use of logarithmic tables because of calculators. The *Agenda for Action also* recommended that students should be required to learn more mathematics. These recommendations “helped turn the tide away from the minimal competency, drill-and-skill mentality of the ‘back to basics’ 1970s toward a more balanced view of what was appropriate for school mathematics curricula” (Coxford, 2003, p. 611).

Soon after the release of *An Agenda for Action*, the National Commission on Excellence in Education (NCEE) released *A Nation at Risk* (1983) that brought to America’s attention the necessity of change in mathematics education. NCEE recommended that all students who graduate from high school complete a minimum of 3 years of mathematics course work. Specific changes in high school content were recommended:

> high school [mathematics] should equip graduates to: (a) understand geometric and algebraic concepts; (b) understand elementary probability and statistics; (c) apply mathematics in everyday situations; and (d) estimate, approximate, measure, and test the accuracy of their calculations. In addition to the traditional sequence of studies available for college-bound students, new, equally demanding mathematics curricula need to be developed for those who do not plan to continue their formal education immediately. (National Commission on Excellence in Education, 1983, p. 26)

This document declared a new vision for high school mathematics content and organization. Mathematics educators now had a foundation for new curricula development that could assist all students.

In response to the call for revitalization of mathematics education the National Research Council (NRC), with assistance from the Mathematical Sciences Education Board, the Board on Mathematical Sciences, and the Committee on the Mathematical

Sciences in the Year 2000, conducted a study of U.S. mathematics classrooms from kindergarten through graduate school to identify strengths and weaknesses of the educational system. The result was *Everybody Counts* (1989), a national call to change mathematics education so all students could receive high-quality education. *Everybody Counts* presented a model of collaboration between groups concerned with mathematics education that imparted guidance and direction for the field. The report not only recommended needed changes, but also provided a new vision for mathematics education. Specifically, the U.S. has an "underachieving curriculum that follows a spiral of almost constant radius, reviewing each year so much of the past that little new learning takes place" (p. 45). The NRC recommended secondary school mathematics textbooks should focus on "the transition from concrete to conceptual mathematics" and that all students should be exposed to core concepts of useful mathematics so they can lead "intelligent lives as productive citizens" (p. 49). Although *Everybody Counts* established a need for change and provided a vision for the field, more importantly it forecasted the significance of the NCTM *Standards* that provided a vision of mathematical expectations for all students.

Shirley Hill, in her 1980 presidential address, recommended NCTM impact policymakers and this could happen if the organization spoke as a united voice. NCTM's *Agenda for Action* was a first attempt at creating a policy document for reforming mathematics curricula in the 1980s. However, after an NCTM committee tried to evaluate the impact of the *Agenda for Action* on textbooks and classrooms it was noted that the document was "only a 'loose framework' rather than a complete curriculum guide, and more 'specificity' was needed" (McLeod, 2003, p. 762). This realization prompted

NCTM to focus on curriculum and evaluation standards. Specifically a committee was given the charge to:

> 1) Create a coherent vision of what it means to be mathematically literate both in a world that relies on calculators and computers to carry out mathematical procedures and in a world where mathematics is rapidly growing and is extensively being applied in diverse fields.
> 2) Create a set of standards to guide the revision of the school mathematics curriculum and its associated evaluation toward this vision. (p. 1)

The eventual result was a document entitled *Curriculum and Evaluation Standards for School Mathematics* (1989). This document built on the principles and guidelines of the *Agenda for Action,* but also provided "a broad framework to guide reform in school mathematics in the next decade" (p. v). More specifically, it provided "a set of standards for mathematics curricula in North American schools (K-12) and for evaluating the quality of both the curriculum and student achievement" (p. v). The *Standards* provided educators with a vision of what mathematics curricula should include. People interested in the quality of K-12 mathematics were encouraged to use the S*tandards* to improve teaching and learning of mathematics. Many earlier reports laid the groundwork for the release of a document that would initiate change in K-12 mathematics. The impact of the *Standards* was huge as states, teachers, and even the general public began to discuss the need to have a mathematically literate generation of students. Thus NCTM became a unified voice on *what* students should be taught and *how* mathematics should be taught.

The *Standards* presented a vision of what it meant to be mathematically competent, but also provided society with new goals and direction for what was valued in mathematics education. "New social goals for education include (1) mathematically literate workers, (2) lifelong learning, (3) opportunity for all, and (4) an informed electorate" (National Council of Teachers of Mathematics, 1989, p. 3). Students were to learn more than facts, procedures, and symbols. They were to learn to value mathematics, gain confidence in their ability to do mathematics, become problem solvers, and communicate and reason mathematically (National Council of Teachers of Mathematics, 1989). Learning was to become an active process rather than static understanding that relied on bits and pieces of information that had no meaning (Schoenfeld, 2004).

Schoenfeld (2004) argued that the *Standards* document was both a radical and conservative document. It was radical because it challenged the “assumptions underlying the traditional curricula” (p. 267). The traditional curricula of this time were designed for students who intended to pursue higher education, yet 50% of students dropped out of mathematics courses after the ninth grade. The *Standards* included a table of topics that should receive either increased or decreased attention. The topics recommended to receive deceased emphasis were the “meat and potatoes of the traditional curriculum” (p. 268). On the conservative side, the *Standards* document was written for mathematics teachers everywhere. Although the *Standards* were vague, “this was part of their genius…” (p. 268) because people had to interpret and reevaluate what was important for their students and society.

These four documents, *Agenda for Action, A Nation at Risk, Everybody Counts,* and *Curriculum and Evaluation Standards for School Mathematics,* offered a conclusive recommendation that “teaching of mathematics in American schools needed to change at all grade levels in order to provide the best possible learning opportunities for all students and that the need for mathematically literate citizenry had never been greater” (St. John et al., 2004, p. 1). Each document reported different approaches for change in the K-12 mathematics system: change mathematical content and organization, increase mathematics course requirement, provide a vision for the future of mathematics education and mathematics curricula, and become a voice in the fight for reform and helping students learn mathematics. Each organization shared a similar view that mathematics instruction should emphasize engaging and challenging students to solve complex problems that contain multiple mathematical ideas (Reys, B. J., Robinson, Sconiers, & Mark, 1999; St. John et al., 2004).

These documents laid a foundation for a vision not reflected in existing mathematics curriculum. Commercial publishers have been reluctant to commit significant financial resources to develop new products that deviate significantly from the programs that have significant market share (Reys, B. J. & Reys, 2006). Furthermore the demand to conceptualize a new secondary mathematics program, and then develop, field-test and revise it prior to selling it has never been done by a commercial publisher. These circumstances produced the setting for the NSF to solicit proposals during the 1990s for the development of new instructional materials consistent with the vision of mathematics curriculum endorsed by the *Standards* (Reys, B. J. et al., 1999; Robinson & Robinson, 1999; St. John et al., 2004).

> These curricula were to provide teachers with tools that could help them move beyond traditional practice, imbuing their teaching with new methods and topics that would make mathematics education more relevant and meaningful to young people of the 21st century. (St. John et al., 2004, p. 2)

The NSF awarded five independent groups funds to develop new secondary mathematics curricula based on recommendations for reform voiced during the 1980s. These groups worked independently to develop integrated mathematics curricula thereby providing multiple programs that used a different organization structure than what had been typical in the United States.

Integrated Curricula and its Effect on Student Achievement

"IN EDUCATION, *integration* means the simultaneous consideration of different aspects of knowledge" (Usiskin, 2003, p. 13). Two main ways knowledge can be integrated are through unifying concepts or merging different areas of mathematics. An integrated curriculum based on a unifying concept uses a topic that "cuts across most, if not all, branches of mathematics" (p. 16), such as functions. The *School Mathematics Study Group (SMSG)* used "unifying concepts as sets, mathematical systems, and logic throughout their materials" (p. 16). Another example, the *Agenda for Action* proposed developing mathematics curriculum around problem solving.

Types of Integration

Most early 1900s curricula focused on one branch or topic in mathematics. For example, high school students who took an algebra class learned very little geometry. Similarly geometry students did not learn about coordinate geometry or make connections to algebra. However, *SMSG* broke this tendency by merging solid and plane geometry in one course. Integrated curricula can be developed by merging different areas of mathematics. Three ways of integrating mathematics topics are: (1) removing distinctions between areas of mathematics, (2) integrating by strand, and (3) interdisciplinary (Usiskin, 2003).

By removing distinctions between different areas of mathematics topics such as, probability and statistics, geometry, algebra, and functions, students are better able to make connections between areas, but they are also exposed to multiple areas each year. The NSF-funded mathematics curricula are an example of this type of integration. Five high school projects, *Contemporary Mathematics in Context (Core-Plus), Interactive*

Mathematics Program (IMP), MATH connections: A Secondary Mathematics Core Curriculum, Mathematics: Modeling our World (MMOW), and *SIMMS Integrated Mathematics,* were developed using real-world applications or themes, and connections among algebra, functions, geometry, trigonometry, probability, statistics, and discrete mathematics were made as appropriate within each theme.

The most common integration is by strands. Most textbooks provide multiple strands, algebra, geometry, and data that students are expected to learn each year; however, these strands are typically divided by chapter and connections are frequently lost. Students most often struggle to make connections between different strands because they are taught individually. For example, in Russia and China algebra and geometry are taught during the same year, but at different times. On certain days of the week students learn algebra and the other days are spent learning geometry; however, students are not using one textbook, but different books for each strand (Usiskin, 2003).

Finally, integration by multiple disciplines is one of the oldest and well known types of integration. Typically mathematics and science are usually grouped together because they compliment each other, and *SIMMS* integrated mathematics was developed with this goal. However, it is "difficult to give enough attention to the different concepts in different subjects simultaneously, yet demonstrate the differences in their importance, and also link them in some coherent way" (Usiskin, 2003, p. 22).

Size of Integration

Usiskin (2003) further contends that curricula can be considered integrated based on the grain size. He describes six curriculum grain sizes from smallest to largest:

1) The questions and problems asked during teaching

2) A lesson

3) A unit or chapter in a textbook

4) An entire course, typically completed in a semester or year

5) The mathematics curriculum (what students complete during their K-12 education)

6) The school curriculum (includes multiple areas of schooling)

The smallest integration grain size exists in the questions and problems teachers ask students during class. A problem students are given to solve may integrate multiple mathematics concepts or may integrate different strands such as geometry and algebra. A lesson is typically completed during one class period. The integration may be in connecting student learning from the previous day to the new concepts being taught in the new lesson. A unit or chapter in a textbook is one of the more popular types of integration. For example, a typical chapter in a geometry book might be on coordinate geometry. This chapter will integrate geometry concepts: parallel, perpendicular, circles, and lines with algebra concepts: find the distance between two lines, writing equations of lines, deciding whether two lines are parallel or perpendicular, and writing equations for circles. The fourth type of integration is a course, which would be similar to single-subject curricula. Students make connections between chapters. Topics taught at the beginning of the year are necessary to build knowledge and understanding of new topics later in the year. Many mathematics textbooks could be considered integrated at the course level. However, an integrated school mathematics curriculum is different, because what is taught at each grade is important and imperative for students to understand. The connections are not made solely across the chapters, but also across grade levels. What students learn in grade 10 is connected to what they learned in grades K-9. The sixth type of integration is much larger and it is across the entire school curriculum, meaning multiple disciplines are involved. For example, English, mathematics, science, and history might be integrated into one class. This type of comprehensive integration across disciplines is rare.

In a school mathematics curriculum students are not expected to learn everything in a single year, yet topics in subsequent years build on their prior understanding. Typically upon completion of Algebra I students first complete a course in Geometry and then a course in Algebra II; however, it may be the case that a student decides to take Algebra II first and then Geometry. "If these subjects can be interchanged, neither is necessary for the other" (Usiskin, 2003, p. 26).

The five high school NSF-funded curricula mentioned previously are integrated at the school mathematics curriculum level. Students need to take each course to progress, build their mathematical knowledge, and gain a full understanding of the mathematics required at the high school level. Each program was designed to replace the Algebra 1, Geometry, Algebra II, and Precalculus sequence and prepare students to enter either AP mathematics courses, collegiate mathematics courses, or the work force. Each program has its own format, style, and organization with emphasis being given to various mathematical strands, but students should finish with a similar understanding and knowledge of mathematics.

Research on Integrated Curricula Conducted Internally

Today high school students in the United States use either integrated or single-subject curricula. Researchers have compared students who use these two types of curricula with standardized achievement tests and college entrance exams, such as the Iowa Test of Basic Skills (ITBS); Iowa Test of Educational Development (ITED); Preliminary Scholastic Aptitude Tests (PSAT); Stanford Achievement Test, Ninth edition (SAT-9); Scholastic Aptitude Tests (SAT); American College Testing (ACT) examination; or the New Standards Reference Examination (NSRE) (Abeille & Hurley, 2001; Harwell et al., 2007; Huntley, Rassmussen, Villarubi, Sangtong, & Fey, 2000; Lott, Hirstein, Allinger, Walen, Burke, Lundin et al., 2003; Schoen, Fey, Hirsch, & Coxford, 1999; Schoen & Hirsch, 2003a; Webb, 2003). Results reported are mixed; with one group of students performing statistically higher than the other on some assessments, and when other assessments are utilized no statistical difference was found between the two groups of students (integrated and single-subject).

Researchers have reported that high school students who used an NSF-funded high school curricula either outperform or perform equally on standardized achievement assessments when compared to peers in a similar single-subject course (Abeille & Hurley, 2001; Lott et al., 2003; Schoen & Hirsch, 2003a; Webb, 2003). Abeille and Hurley (2001) found high school students who used *Mathematics Modeling our World* scored significantly higher than the national average on the SAT-9. Schoen and Hirsch (2003a) found high school students who used *Core-Plus* course 1, when matched on previous achievement and aptitude, significantly outperformed students enrolled in either Pre-Algebra or Algebra courses and performed equally with students enrolled in an Accelerated Geometry course. Whereas, Webb (2003) found no difference between students who used the *Interactive Mathematics Program* and students who used single-subject curricula on ITBS scores. Lott et al. (2003) found similar results for students who used *Systematic Initiative for Montana Mathematics and Science*. *SIMMS* students did not perform differently than students who had used single-subject curricula on the Preliminary Scholastic Aptitude Test (PSAT) mathematics portion. These mixed results contribute to the necessity for more research on how students who use the different NSF-funded curricula perform on standardized achievement exams. These mixed results also raise questions about the alignment between standardized assessments and these five NSF-funded curricula.

Researchers also found students who used an NSF-funded curricula did not perform differently on college entrance exams when compared to peers who used single-subject textbooks (Schoen & Hirsch, 2003a; Webb, 2003) Schoen and Hirsch (2003a) found no difference in student performance on the SAT and ACT mathematics portion for those who used either *Core-Plus* or single-subject curricula during high school. Webb (2003) also found students who completed three years of the *Interactive Mathematics Program* performed no different on the SAT mathematics portion than students who completed three years of single-subject curricula. Overall, students who used these two NSF-funded curricula have not performed differently than students who use single-subject curricula on measures of college aptitude.

All of the research reported above contains at least one curriculum designer or personnel internal to the projects. The NRC called for involvement of researchers that are independent of the curriculum developers. The NRC (2004) also reported in *On Evaluating Curricular Effectiveness* that approximately 60% of comparative studies were considered “minimally methodologically adequate”. The weaknesses of these studies were based on “inadequate attention to experimental design, insufficient evidence of the independence of evaluators in some studies, …” (p. 164).

Research on Core-Plus External to the Project

As students continue to use the NSF-funded curricula the research base will continue to grow, but more importantly the National Research Council (2004) recommended that studies should be conducted external to the project. In the last 5-8 years researchers not associated with the development of the NSF-funded curricula have begun publishing their results (Harwell et al., 2007; Hill, R. O. & Parker, 2006; Huntley et al., 2000; Milgram, unpublished work; Smith, J. P., III & Star, 2000; Smith, J. P., III & Star, 2007). Although some researchers are finding similar results, others are reporting differing outcomes. Huntley et al. (2000) found similar results to the project evaluators that students who use *Core-Plus* tend to perform better on contextual open-ended questions, problem-solving situations, and problems related to statistics and probability. However, Huntley et al. found *Core-Plus* students did poorer on questions of symbolic manipulation when a calculator was unavailable, which was in opposition to what the developers, Schoen and Hirsch (2003a) found.

Hill and Parker (2006) compared students who had completed the *Core-Plus* curricula and those who had not completed *Core-Plus* in their freshman mathematics course at the University of Michigan and their grade they received in that course. A total of 3200 students’ data were collected from forty-five high schools. Six high schools used *Core-Plus* for their high school mathematics curricula. Hill and Parker found significantly more students who completed *Core-Plus* curricula were taking remedial courses and getting lower grades.

The curriculum developers responded to the Hill and Parker (2006) study by noting three serious flaws: (1) the sample of students described used a draft field-test version of *Core-Plus*; (2) two schools categorized as "only *Core-Plus*" actually had multiple mathematics textbooks that teachers used during the school years studied; and (3) two schools categorized as "supplemented *Core-Plus*" actually used *Core-Plus* textbooks for all mathematics classes (Fey, Hart, Hirsch, Schoen, & Watkins, 2007). The last two problems affect the data analysis that was conducted because they included schools that should not be and excluded schools that should be included. Although Hill and Parker were informed of these problems they did not make any changes in their analysis.

Milgram (unpublished work) used data collected by George Bachelis to analyze how students from two high schools, Lahser and Andover, performed at the university level. Both high schools are located in the same school district and had similar populations; however, students at Andover used *Core-Plus* as their mathematics textbooks and students at Andover used single-subject textbooks. Data were collected through student surveys. The results indicated that ACT scores for students at Andover, who completed four years of *Core-Plus,* dropped significantly during the five year implementation of the curricula when compared with students at Lahser high school. Milgram also found more students (70%) who had completed the *Core-Plus* curricula were taking remedial mathematics courses in college compared to students (44%) who completed the single-subject curricula. It was noted that fewer Andover graduates (3%) took Calculus the first year in college than Lasher graduates (27%). Both the Hill and

Parker and Milgram studies support critics who are concerned about students' mathematics preparation for college.

The curriculum developers, however, refuted the results of Milgram's study. They state, "the survey is invalid due to serious design flaws and the report draws incorrect conclusions" (Hirsch & Schoen, 2007, p. 1). The flaws found in this study are: (1) the data are self-reported by self-selected students, which lead to unreliable data; (2) invalid generalizations were made based on a pilot version of the curriculum; and (3) incorrect conclusions were made about *Core-Plus* students not being prepared for college mathematics because data collected from the University of Michigan revealed opposite results. Although the curriculum developers encourage researchers to continue studying how students who use *Core-Plus* perform in college, they "urge … for more valid research based on published versions of curricula" (Fey et al., 2007, p. 1).

Another line of research impacting the field is how students adjust to changes from one type of curriculum to another. Researchers involved in the current *Mathematical Transitions Project* (Smith, J. P., III & Star, 2000; Smith, J. P., III & Star, 2007) located at Michigan State University, have followed high school and college students as they move from traditional (single-subject) curriculum to standards-based (integrated) curriculum or vice versa. The researchers are studying transitional periods in the students' lives. Transitional periods consist of: moving from junior high to high school, high school to college, or from one state to another. At these transition points, students are forced to move from one curricula type to another because of availability or offerings that exist at the next level (high school or college). The focus of Smith and

Star's research is on how these transitions affect students' learning of mathematics when forced to switch between these two curricula types.

Preliminary results from *the Mathematical Transitions Project* (Star, Berk, Burdell, Hoffmann, Lazarovici, Lewis et al., 2001) indicate "different students react to curricular shifts differently in each school context" (p. 118). Researchers focused on "(1) student achievement, (2) significant differences students notice and report, (3) changes in disposition towards mathematics, and (4) changes in learning approaches" (p. 118) as changes occurred within curricula type. They determined that students experiencing a significant change in two or more of the four categories were experiencing a mathematical transition.

Smith and Star (2007) recently reported that approximately four-fifths of the students who took integrated or single-subject mathematics courses and then transition from high school mathematics to university mathematics experienced a significant drop in their achievement level. However, they also found no difference in student achievement for those who moved into or out of a reform program. This is significant because it implies that "students found it increasingly difficult to achieve at the same levels in mathematics as they progressed from … high school to university" (p. 23) and it does not seem to be dependent on the curricula type.

In another large scale study, Harwell et al. (2007) found students who used one of three standards-based (integrated) curricula, *Core-Plus, Interactive Mathematics Program,* or *Mathematics Modeling our Way,* preformed at or above the national average on traditional mathematics topics as measured by national standardized exams, SAT-9 and the New Standards Reference Exam in Mathematics. Harwell et al. (2007) states,

> District teachers and administrators (and parents) need to recognize that there is emerging evidence that *Standards*-based curricula do not impede the mathematical performance and development of high achieving students, and in some cases appear to have contributed to better student performance. (p. 95)

Researchers continue to find mixed results on how students perform on standardized achievement tests and at the college level; however, the majority of results reveal no difference between students who use integrated or single-subject curricula. Because only limited results supporting integrated curricula have been published many opponents continue to mandate a return to what is conventional (Klein, 2000). However, Hiebert (2003) asserts that traditional approaches to learning mathematics have not produced desired results in student achievement for the past 30 years and the field must change.

Summary. Researchers need to continue studying how students who have experienced integrated or single-subject curricula perform on assessments. However, Harwell et al. (2007) advises the field to switch research focus from achievement to understanding what students are learning from these two curricula types. “It will be prudent in future comparative studies to probe the depth of student understanding, their flexibility in accessing various mathematical processes and skills while engaged in problem solving …” (p. 95). The NRC (2004) recommends comparative studies should “include a variety of measures … question type (open ended, multiple choice), type of test (international, national, local), …” (p. 165). This proposed study will execute these recommendations by analyzing student work on the two open-ended tasks and the *PCA*.

Research on Calculus Students’ Understanding

Calculus was first recommended for secondary students in 1923 as an elective course (Ferrini-Mundy & Gaudard, 1992). However, during the 1950s students were encouraged to learn higher level mathematics and this lead to the development of Advanced Placement (AP) Calculus courses. Students had the opportunity to receive college credit while still in high school if they completed a calculus course and passed the AP exam. Since the first exam in 1955, the AP Calculus program has grown at a rate of 7-8% per year (Bressoud, 2004). There are two courses and examinations within the AP calculus program. The AB course covers the content that is typically taught in first semester college calculus, while the BC course covers content typically taught in two semesters of college calculus. In 2004, over 225,000 students took an AP calculus exam with over 50,000 of these students taking the BC exam (Bressoud, 2004). The AP calculus program continues to grow and this has raised concerns about whether high school students are experiencing a similar calculus course that college students are receiving.

Both Dickey (1986) and Ferrini-Mundy and Gaudard (1992) compared high school students with college calculus students and found similar results. Dickey (1986) found AP calculus BC students achieved as well as or better than college calculus students completing similar course content on a multiple choice test. Ferrini-Mundy and Gaudard (1992) found first semester college calculus students who had completed a full year of high school calculus (AP or not) were more successful on two instruments that assessed algebra, trigonometry, and spatial visualization rotation than those students who had either completed a partial year of calculus or had no experience before entering college calculus. A difference in procedural proficiency was found between students who had a partial year of calculus and those that had no experience with calculus. These results indicate students who complete AP Calculus perform as well or better than college calculus students; however, the emphasis may be on gaining procedural knowledge.

Although "the value of calculus lies in its potential to reduce complex problems to simple rules and procedures" (Aspinwall & Miller, 2001, p. 89) many students are more apt at the procedural process without grasping the conceptual understanding necessary to apply the mathematics. Students who only gain procedural knowledge of calculus struggle with conceptual understanding (Baker, Cooley, & Trigueros, 2000; Berry & Nyman, 2003; Ferrini-Mundy & Gaudard, 1992; Habre & Abboud, 2006; Szydlik, 2000; White & Mitchelmore, 1996; Williams, 1991).

During the 1980s more students were entering college and enrolling in calculus, yet more students were failing calculus than ever before. Lynn Steen, president of the Mathematical Association of America at the time, stated "in large universities, fewer than half of the students who begin calculus finish the term with a passing grade" (as cited in Friedler, 2004, p. 5). Two conferences, *Toward a Lean and Lively Calculus* and *Calculus for a New Century: A Pump not a Filter*, began the discussion for change in the content and pedagogy of calculus in the United States. Some reasons for change were: (1) new technology; (2) current methods were causing many students to fail; (3) other disciplines wanted input; and (4) rethinking of textbooks. In 1987, NSF announced its desire to help in the calculus reform. They encouraged the development of new textbooks that focused on conceptual understanding and problem solving with a reduction in the procedural skills. Eight different calculus curricula programs were developed and are currently used across U.S. college campuses. The programs implemented new and different types of problems. For instance, the Harvard Calculus Consortium (Hughes-Hallet, Gleason, Flath, Lock, Gordon, Lomen et al., 2002) introduced the “Rule of 4”, which was that every concept was presented graphically, verbally, numerically, and algebraically. Ganter (2001) summarized the research on the reform effort in *Changing Calculus, A Report on Evolution Efforts and National Impact from 1988-1998* and concluded:

- Most faculty believed there was a need to change the teaching of calculus; however, they were not sure it was going in the right direction.
- The success or failure of the reform effort was not dependent on what was implemented, but how, by whom, and in what setting.
- The reform effort motivated many conversations about how calculus was taught.
- Students were concerned about their lack of computational skills.
- Reform students are enrolling in more non-required mathematics courses beyond calculus, implying that reform calculus generated more interest in mathematics.
- Eighty-eight percent of all the studies conducted (111) found reform efforts were positive for at least one measure of student improvement.
- Computers can help students be more successful in calculus.

Although the calculus reform provided change, many mathematicians did not agree with the new textbooks because they removed or decreased attention to topics such as proof and mathematical language (Klein & Rosen, 1997; Wu, 1997). Yet others argue that these new calculus textbooks “are a middle ground” (Tall, 1997, p. 289) between the intuitive learning of calculus and the formal theory of mathematical analysis, which provides more students the chance to learn and be successful with calculus.

Typically calculus students learn topics in three broad areas: limits and continuity, derivatives, and integrals; however, their foundation of knowledge should be grounded in understanding functions. Researchers found students have

> various conceptions of a function, that it is given by a formula, that if *y* was a function of *x*, it must include *x* in the formula, that its graph was expected to have a recognizable shape…, and that it was to have certain ‘continuous’ properties. (Tall, 1997, p. 299)

The concept of functions is fundamental to learning calculus; however, students shallow understanding of function becomes a "way of working with other calculus concepts such as limit" (Ferrini-Mundy & Graham, 1991, p. 630).

Functions

Researchers have documented students incomplete understanding of functions as they enter calculus (Ferrini-Mundy & Graham, 1994; Markovits, Eylon, & Bruckheimer, 1988; Tall, 1997). Students' understanding of functions tends to be either an algebraic equation/formula or a continuous graph. Ferrini-Mundy and Graham (1994) found a student's judgment of whether a graph was a function or not was whether they could write an equation to fit the graph. Markovits et al. (1988) found students struggled identifying functions if the graph was disconnected or a piecewise function was given. Tall (1997) found similar results when working with students that "the graph of a function should have other properties, such as being regular, persistent, reasonable increasing…" (p. 10). Students' understanding of function is critical for understanding high level mathematics, especially calculus concepts: limits, derivatives, and integrals.

Breidenback, Dubinsky, Hawks, and Nichols (1992) and Dubinsky and Harel (1992) described two ways students think about functions: *action,* and *process*. *Action* oriented is described as "the ability to plug numbers into an algebraic expression and calculate" (Dubinsky & Harel, 1992, p. 85). These students tend to think about the function one step at a time as they input a value into the function and calculate the correct output value for a specific function. However, if these students are not given an algebraic equation they usually struggle to interpret the function. Whereas, *process* oriented is described as having the ability to use "a dynamic transformation of quantities according to some repeatable means that, given the same original quantity, will always produce the same transformed quantity" (Dubinsky & Harel, 1992, p. 85).

Carlson (1998) built on both, Breidenback et al. (1992) and Dubinsky and Harel (1992), work studying high-performing undergraduate students' concept of function. Results revealed that college algebra students and second semester calculus students have a range of misconceptions about functions. Some of these misconceptions were similar to earlier results that a function is "defined by a single algebraic formula and all functions must be continuous" (Carlson, 1998, p. 141). However, other misconceptions were new: difficulty interpreting rate of change, difficulty translating between an algebraic and graphical representation, and understanding the role of independent and dependent variables to list a few. The concept of function is difficult for students and it takes a number of years to "understand and orchestrate individual function components" (Carlson, 1998, p. 143).

Carlson and Oehrtman (2005) extended previous work with regard to *action* and *process* orientations arguing that students ability to answer function focused tasks was related to two types of reasoning.

> First, students must develop an understanding of functions as general processes that accept input and produce output. Second, they must be able to attend to the changing value of output and rate of change as the independent variable is varied through an interval in the domain. (p. 5)

Those students who held an *action* view of functions had misconceptions about piecewise functions being several functions, could not reason over an entire interval only at a specific point, and could not relate the domain and range to the inputs and outputs of a function. Without an understanding of functions accepting inputs and producing outputs students will struggle to reverse (inverse functions) the process. Most students are able to algebraically or geometrically find the inverse of a function; however, their answer "has little or no real meaning" (p. 7). Students who hold an *action* view are able to work with functions procedurally, but have little conceptual understanding.

Carlson and Oehrtman (2005) identified students who held an *action* view to use either an algebra or geometry approach to finding inverse functions. Students who used an algebra approach simply switched the variables (x and y) and solved for the y variable. Students who used a geometry approach reflected the given function over the line y = x to find the inverse function. However, most of their students did not understand why these procedures worked or how to use them in a different context. Students who demonstrated a *process* view of inverse functions were able to use a reversal process that defined a mapping of output values to input values.

Carlson and Oehrtman (2005) also identified students who held an *action* view of composition of functions as those who substituted a formula or expression in for x. Students who held a *process* view were able to coordinate two input-output processes. These students understood that the input in one function would result in an output that would be used as the input in a second function.

Students who held a *process* view saw the entire function and how the input values affected the output values. This was described as covariational reasoning, the ability to vary inputs and outputs at the same time and interpret or understand their influence on the rate of change. Students who hold a *process* view were better able to understand composition of functions and inverses. A *process* view of function is essential for understanding calculus concepts such as the limit, derivative, and integral.

Summary. Most of these studies confirm that students' conceptual understanding of function is weak. Few studies investigated high school students so questions exist as to whether students who completed four years of college preparatory mathematics (integrated or single-subject) courses have these same misconceptions and weak conceptual knowledge. If similar studies were conducted with AP Calculus students would the results change? Are there differences in students' conceptual knowledge that have learned from two different types of curricula? These questions will be addressed in this study.

Analyses of Student Strategies

Many students believe one correct way exists to solve a problem because mathematics is a list of rules and formulas (Cai, Magone et al., 1996; Santos-Trigo, 1996). Teachers try to overcome this belief by instructing students in different problem solving techniques, including pattern recognition, guessing and checking, working backwards, solving a simpler problem, and drawing a picture. Begle (1979) stated "problem-solving strategies are both problem- and student-specific" (p. 145). If this is true, then students may use different strategies to solve the same task because they prefer one strategy over another or they were taught to use specific strategies by their teacher based on their textbook.

Researchers have investigated how students of different age groups solve open-ended tasks. Studies have focused on different aspects of solving items: using different types of representations (algebraic, graphical, verbal) (Gagatsis & Shiakalli, 2004; Hegarty & Kozhevnikov, 1999); using informal or formal strategies (Koedinger & Nathan, 2004); describing the process students used to find an answer (Hall, Kibler, Wenger, & Truxaw, 1989; Sebrechts, Enright, Bennett, & Martin, 1996); and comparing how students from two different backgrounds solve problems (Cai, 1995, 1997, 2000; Cai & Silver, 1995; Silver, Leung, & Cai, 1995).

Hegarty and Kozhevnikov (1999) and Gagatsis and Shiakalli (2004) both focused on how students use different types of representations to solve open-ended problems. Although the researchers studied different age students, sixth grade boys and university students, they found similar results. Hegarty and Kozhevnikov (1999) focused on how students use either schematic and pictorial representations. Schematic representations were defined as “representing the spatial relationships between objects and imagining spatial transformations” (p. 685); whereas, pictorial representations were defined as “constructing vivid and detailed visual images” (p. 685). Sixth grade boys were given 15 problem solving questions to answer and then each problem was coded in four ways. First, whether the answer was correct or incorrect. Second, whether the student used a visual-spatial representation to solve the problem. Third, whether the students’ visual-spatial representation was schematic. Fourth, whether the students’ visual-spatial representation was pictorial. Student strategies were coded pictorial if a student used an image, rather than a relationship between the objects. A major finding was students who used schematic representations were more successful with mathematics problem solving; whereas, students who used pictorial representations were less successful with mathematics problem solving.

Gagatsis and Shiakalli (2004) found that university students who were able to move between different representations, graphical, algebraic, or verbal, had better problem solving abilities. The researchers developed two tests to investigate student success with moving from one type of representation to another: algebraic, graphical, and verbal. The first test required students to move from verbal representations of functions to graphical and algebraic representations. The second test required students to move from graphical representations of functions to verbal and algebraic representations. The authors found a correlation between students' ability to move between representations and problem solving ability. They also found students perceive verbal and graphical representations "as two different tasks and not as different means of representing the same idea" (p. 652). More importantly they report students' struggle moving from graphical representations to verbal and algebraic representations.

Koedinger and Nathan (2004) focused on the informal and formal strategies high school algebra students use to solve algebraic story problems and equations. They defined formal strategies to be an algebraic equation; whereas, informal strategies were either guess and test or unwind which was defined as "the student reverses the process described in the problem… addressing the last operation first and inverts each operation to work backward to obtain the start value" (Koedinger & Nathan, 2004, p. 146). Students used the unwind strategy most often (50%) when solving story problems. However, when students were given equations to solve they left it blank 32% of the time. This suggests that students' understanding of formal algebraic strategies is weak and also difficult for them to understand.

Another group of researchers have described how students solve specific tasks to create their framework for analyzing written work (Hall et al., 1989; Sebrechts et al., 1996). Both Hall et al. (1989) and Sebrechts et al. (1996) analyzed undergraduate students' strategies and developed their frameworks based on actual student responses.

Hall et al. (1989) had undergraduates solve algebra story problems and then divided students' work into episodes. These episodes were defined as breaks when the student changed direction or started a new process of thinking. Each episode was analyzed to classify students' strategies in three major categories, *strategic*, *tactical*, and *conceptual* coherence. Strategic coherence was the process or goal students used to find an answer to the problem, which included three sub-categories: comprehension, solution attempt, and verification. Comprehension was how the student represented the different aspects of the problem before solving the problem. Solution attempt was the steps students completed that led them to an answer. Finally, verification was how the student checked their answer. Tactical coherence was the method the student used to achieve their answer. This was coded into six categories: annotation, algebra, model-based reasoning, ratio, unit, and procedure. Annotation was the step of writing the given information on their paper, choosing appropriate formulas needed to solve the problem, or drawing a picture to represent the given information. Algebra was coded when a student used one or more equations to solve for a variable. Model-based reasoning was defined as executing a model of the problem situation: simulations, created a simpler problem, or made jumps in intervals because they saw a pattern. Ratio was coded when students made a relation between two quantities: whole-part, part-whole, part-part, proportions, or scaling. Unit was coded when students use the units of measurements to solve the problem. Finally, procedure was defined as executing a sequence of steps other than algebraic or arithmetic manipulations such as using the formula for average. Conceptual coherence was how students conceptualized the problem situation, including how the student used the constraints to solve the problem.

Sebrechts et al. (1996) used a similar technique with undergraduates who completed 20 released items from the Graduate Record Examination (GRE). Student solution strategies were analyzed by creating an initial taxonomy of actual solutions. The taxonomy was then adjusted and expanded by additional researchers who reanalyzed students' work. The final taxonomy separated strategies into *primary* and *collateral*. Primary strategies were those used in 82% of all solutions. Collateral strategies were not the primary means of getting the final answer, but they did provide help in getting to the solution. The four primary strategies were equation formulation, ratio setup, simulation, and other. Equation formulation was coded when students used an equation to solve a problem. Ratio setup was coded when students used a specific equation in the form $\frac{a}{b} = \frac{c}{d}$. Simulations were described as situations when students modeled the problem by using different steps. For instance one of the problems required students to find how much money was invested at 8% interest given the total amount of money invested at two different interest rates and the income from the investment. Some students would start with different amounts of money in each investment and found the result was too high and made adjustments in the increments until they found the correct answer. Finally, other was a category for strategies that were not easily classified.

Another group of researchers have compared how students from different backgrounds solve open-ended tasks (Cai, 1995, 1997, 2000; Cai & Silver, 1995; Silver et al., 1995). Silver, Leung, and Cai (1995) described how Japanese and American fourth graders solve problems. Students were given a picture of marbles arranged in a diamond shape (see Figure 2.1) and asked to decide how many marbles there were in as many ways as they could.

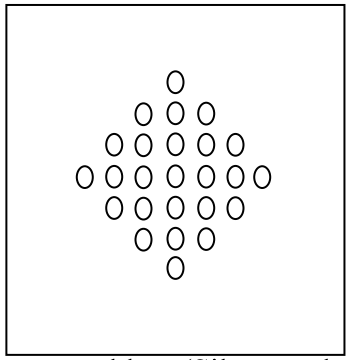


Figure 2.1. Marble arrangement problem (Silver et al., 1995, p. 37)

Items were examined both individually and collectivity for correctness, strategy used to solve the problem, and the type of explanation used to justify answers. American and Japanese fourth grade students both used enumeration (counting) and grouping (systematic organization) strategies to solve this problem. However, more surprising was U.S. students used visual explanations four times more often than Japanese students. On the other hand, Japanese students used a mixture of verbal and symbolic explanations twice that of U.S. students.

Cai and Silver (1995) compared the success of Chinese and U.S. fifth and sixth grade students on division-with-remainder problems, both computational and story problems. Differences were found between students in the two countries. Chinese students were more successful executing the division algorithm than U.S. students on computational problems. However, Chinese students were less successful in providing a correct answer for the story problem. In fact, only 24% of the Chinese students provided correct answers, which was about the same performance level as their U.S. counterparts.

Cai (1995) further investigated similarities and differences in Chinese and U.S. sixth grade students solution strategies, errors, and mode of representation on five open-ended tasks. He found "almost every solution strategy used by U.S. students was also used by Chinese students, and vice versa. Moreover, the solution strategies used most frequently by U.S. students …was also the strategy used most frequently by Chinese students" (Cai, 1995, p. 102). Although the solution strategies were similar across the two countries the "Chinese students' solutions strategies tended to be more elegant than those of U.S. students" (Cai, 1997, p. 7). Some other differences in students' solution strategies were: U.S. students used trial and error more frequently than Chinese students when it was an appropriate strategy; and U.S. students used visual representations more often than Chinese students.

Cai (2000) more recently compared U.S. and Chinese sixth grade students on differences in solving process-constrained and process-open problems. Process-constrained problems were items that could be solved with a standard algorithm or an application of a known algorithm. A process-open problem did not lend itself to a formal algorithm. A specific qualitative coding was developed for the 12 items to examine the solution strategies, mathematical errors, and mathematics representations. A random sample of students' work from both countries was used to determine the different types of strategies used. Cai found Chinese students use algorithms and symbolic representations more often than U.S. students. On the other hand, U.S. students prefer using concrete visual representations more often than Chinese students.

Summary. All of these studies provide evidence that students use multiple strategies to solve open-ended tasks. Another interesting result is when students from different countries are compared their solution strategies are similar; however, their preferred process is different. It may be the case that students are influenced by their teacher, the type of curriculum they use, or the country in which they reside as to the choice of strategies that they are more efficient with and use more often.

Students' use of strategies can be influenced by the curriculum they use during their course work. Although there may not be one correct strategy when solving a particular problem, there may be more efficient and less efficient strategies for solving it. Students who use integrated and single-subject curricula may have learned and tend to use different strategies to solve similar problems. Students may have multiple strategies to choose from, but their choice to use a particular strategy and how efficient they are with their chosen strategy is important to understand. It is essential for students to chose an appropriate strategy and use it efficiently if they are to be successful in mathematical problem solving (Cai, Magone et al., 1996). This study will examine the different strategies students who have completed the two different curricula use to solve problems and how efficient they are with these strategies.

Student Error Analyses

Since the early 1900s, researchers have conducted error analysis. This line of research is beneficial for helping teachers understand how students learn and improve their own teaching (Radatz, 1979). Radatz (1980) classified error analysis studies into five categories:

> (1) listing all potential error techniques; (2) determining the frequency distribution of these error techniques across age groups; (3) analyzing special difficulties, particularly encountered when doing written division, and when operation with zero; (4) determining the persistence of individual error techniques; and (5) attempting to classify and group errors. (p. 19)

Most of these studies analyzed elementary students' arithmetic errors with the focus on students' procedural misconceptions. Researchers changed their focus to systematic errors defined as "errorful rules that produced a pattern of incorrect responses" (Confrey, 1990, p. 33), which provided a connection between their errors and mathematical understanding. Knowledge of common misconceptions and errors are a powerful tool for helping students to understand mathematics. The research on error analysis has continued to grow with a focus on conceptual and procedural errors at the high school and college level (Hall et al., 1989; Koedinger & MacLaren, 1997; Koedinger & Nathan, 2004; Movshovitz-Hadar, Zaslavsky, & Inbar, 1987; Orton, 1983a, 1983b; Porter & Masingila, 2000; Pugalee, 2004). Each researcher identified distinct categories of errors for student work.

High School Student Error Analyses

Two different ways of analyzing high school mathematics students' errors emerged in the literature. First, Movshovitz-Hadar, Zaslavsky, and Inbar (1987) developed a detailed model of errors in high school mathematics. They classified errors by documenting "performance without appealing to processes in the students' minds that might or might not have yielded the errors committed and without faulting what the student did not do" (p. 4). Error analysis was conducted on solutions to eighteen open-ended questions from Israel's 11th grade graduation exam. Six error categories were found that describe what students did wrong, not what they were supposed to do.

1. *Misused data* – "errors related to some discrepancy between the data as given in the item and how the examinee treated them" (p. 9).
2. *Misinterpreted language* – "mathematical errors that deal with an incorrect translation of mathematical facts described in one (possibly symbolic) language to another (possibly symbolic)" (p. 10).

3. *Logically invalid inference* – “errors that deal with fallacious reasoning and not with specific content; … new information invalidly drawn from a given piece of information or from a previously inferred one” (p. 10).
4. *Distorted theorem or definition* – “errors that deal with a distortion of a specific and identifiable principle, rule, theorem, or definition” (p. 11).
5. *Unverified solution* – “each step taken by the examinee was correct in itself, but the final result as presented is not a solution to the stated problem” (p. 12).
6. *Technical error* – “computational errors …, errors in extracting data from tables, errors in manipulating elementary algebraic symbols …, and other mistakes in executing algorithms usually mastered in elementary or junior high school mathematics” (p. 12).

Movshovitz-Hadar, Zaslavsky, and Inbar argue “classification of errors by their operational nature seems more promising than a classification by their causes …” (p. 13) as Radatz (1979) did in his work. This model can be used to identify tendencies of students to make certain types of errors across different mathematics topics and different cognitive level questions.

A second method for analyzing high school mathematics students’ errors was much broader (Koedinger & MacLaren, 1997; Koedinger & Nathan, 2004; Pugalee, 2004). Each of these researchers had students solve open-ended tasks and then sorted the errors into two categories: *procedural* and *conceptual*. Procedural errors were defined as mistakes in performing arithmetic and incorrect mathematical operations. Conceptual errors were defined as forgetting to change the sign, confusing order of operations, selecting an inappropriate model or representation for the given information, or any error that was not procedural.

Undergraduate Student Error Analyses

Research conducted at the college level on mathematics students' errors tends to provide a smaller grain size in how and what errors were analyzed. Donaldson (1963) describes three types of errors: *structural*, *arbitrary*, and *executive*. *Structural* errors were defined as those where students did not understand a principle or concept that was essential for solving the problem. *Arbitrary* errors were defined as those where the students failed to use the given information to solve the problem. *Executive* errors were defined as those where students made arithmetic mistakes, but understood the concept correctly. These three categories were designed for analysis of elementary students' arithmetic errors; however, Orton (1983a; 1983b) used these three categories to analyze high school and college students' errors on calculus, differentiation and integration, tasks. Orton found with older students it was difficult to distinguish between *arbitrary* errors and the other two types. Generally he found students committed many more *structural* errors than *arbitrary* and *executive*. Older students' errors were difficult to code into one of the three types of errors. For example, students were given a cubic function and asked to find the coordinates of the point or points on the curve where there is a turning point or stationary point (see Figure 2.2). Students were able to get the equation $3x^2 - 6x = 0$, but then some students were unable to solve this equation. Some incorrectly canceled x by dividing through by x; therefore losing the solution x=0; whereas, other students incorrectly factored the equation to $3x(x-6) = 0$. Therefore the errors for this item were coded as both *structural* and *executive*.

Find the coordinates of the point or points on the curve

$$y = x^3 - 3x^2 + 4$$

at which there is a turning point or stationary point. Determine also what kind of point you have found.

Figure 2.2. Task that had multiple error codes (Orton, 1983a, p. 250)

Both Hall et al. (1989) and Porter and Masingila (2000) divided errors into two broad categories: *conceptual* and *manipulative (procedural)* errors. Although both groups of researchers analyzed undergraduate work, each focused on different mathematics content. Hall et al. (1989) had undergraduates solve algebra story problems and then categorized the errors into two layers. The first layer was rather broad with conceptual and manipulative errors. Conceptual errors were then broken down into two groups. One occurred when students included constraints that were not given in the problem (errors of commission) and the other excluded given constraints required for the problem (errors of omission). Manipulation errors were separated into three categories: algebraic, variable, and arithmetic. Algebraic errors were those that involved an algebraic procedure. Variable errors were those where students switched the meaning of variables in the middle of the problem or students used variables to label quantities. Arithmetic errors were those that involved computations.

Porter and Masingila (2000) also conducted analyses of undergraduates mathematics errors on three class exams and a final exam for a calculus course. They built on Movshovitz-Hadar et al. (1987) error analysis framework. However, they only classified errors into two categories of *procedural* and *conceptual*. Procedural errors were those where students committed syntax errors and errors in carrying out a procedure.

Conceptual errors were defined as the selection of inappropriate procedures, misunderstanding of mathematical terms and errors in logic.

Sebrechts el al. (1996) provide a different method for analyzing undergraduates students' mathematics errors. As previously mentioned, college students were given 20 released items from the GRE and then six categories of errors were created based on the student errors. The six categories were: *situation comprehension, strategy formulation, plan construction, general plan development, specific plan development,* and *procedural implementation*. Situation comprehension was defined as the student's inability to represent the problem situation correctly. For example, students solved for something not requested, they did not use the givens, or they made additional assumptions that were not given in the problem. Strategy formulation was defined as the student's inability to develop a solution strategy. For example, students would only get part of the solution or they would guess without knowing how to proceed. The third category, plan construction was defined as the student's inability to create an appropriate plan to solve the problem. Most of these errors occurred with setting up the correct equations, ratios or simulations to solve the problem. General plan development was defined as the student's errors in choosing a plan that was required for many of the problems. For example, converting between different units was particularly difficult for students across multiple problems. The next category was similar to the last; specific plan development was defined as the student's error in choosing a plan because only a few specific types of problems existed. For example, distinguishing clock and elapsed time was considered a specific error, yet students only found this on less than 3 problems out of 20. The final group of errors was procedural implementation defined as computational and transcriptional errors.

Summary. Radatz (1979) argued that classifying errors is not only about student difficulties, but also a function of the teacher, curriculum, environment, or any interaction between these three. He warned

> It is often quite difficult to make a sharp separation among the possible causes of a given error because there is such a close interaction among causes. The same problem can give rise to errors from different sources, and the same error can arise from different problem-solving processes. (p. 171)

Although difficulties arise in error analyses, teachers need to understand what and how their students are thinking if they are to improve their own instruction.

Current research on mathematical error analyses conducted with high school or undergraduate students provide some similarities and differences that will inform this study. Two broad categories of mathematical errors have emerged: conceptual and procedural. However, researchers are divided between creating smaller sub-groups and leaving it in two broad groups. To gain greater understanding of students' mathematical errors, smaller grain size categories will be used in this study to identify different levels of conceptual and procedural errors.

Frameworks for Analyses of Cognitive Demand

Research on how mathematical tasks influence what students learn in mathematics began in the late 1970s and early 1980s. Doyle (1983) reviewed research in cognitive psychology on how tasks influence how and what students learn. "Tasks influence learners by directing their attention to particular aspects of content and by specifying ways of processing information" (p. 161). He identified four academic tasks: memory tasks, procedural or routine tasks, comprehension or understanding tasks, and opinion tasks. Each task requires different levels of cognitive processes to complete the problem.

For example, a procedural task requires students to understand *how* to complete a specific problem; whereas, a comprehension task requires students to understand *why* and *when* to use a specific procedure to complete a problem.

Doyle suggests tasks should be described by cognitive levels or cognitive demand, which he defined as “the cognitive processes students are required to use in accomplishing it” (Doyle, 1988, p. 170). He found junior high school teachers give students two types of tasks, familiar and novel, but mostly they rely on familiar tasks. Students solve familiar tasks by repeating what they have seen and relying on their memory or a known algorithm. Students solve these lower level tasks quickly and without assistance from other students or the teacher. However, when students assemble information and procedures from many sources in a different way than was taught, they are solving novel tasks. Students need more time and guidance from classmates and the teacher to solve these higher level tasks because multiple cognitive processes are required.

Marx and Walsh (1988) built on Doyle’s work arguing that classroom work is constrained by “the setting in which tasks are initiated, and the manner in which tasks are delivered to students” (p. 208). This provided a new descriptive theory of classroom tasks that would influence the connection between teaching and learning. Hiebert and Wearne (1993) confirmed Marx and Walsh’s theory, providing empirical evidence of how tasks are presented to students influenced their learning. Hiebert and Wearne (1993) investigated student learning of place value and multi-digit addition and subtraction in six second-grade classrooms at one elementary school for 12 weeks. Two classroom implemented alternative instruction that followed the same topics being taught in the other four classrooms, but it was different than what was typically found in the other classrooms. The alternative instruction focused on students gaining conceptual understanding of place value and computational strategies. Students were given fewer problems to work, but worked on them longer and discussed them with other students and the teacher. Students were also asked to explain their thinking and how they had solved the problems. They found students in the alternative classrooms had a higher performance level than students in the other four classrooms that did not receive the alternative instruction.

This research on classroom tasks provided the foundation for the Mathematical Tasks Framework developed by the QUASAR project (Henningsen & Stein, 1997; Stein & Lane, 1996; Stein et al., 2000). The Mathematical Tasks Framework represents student learning as a function of task implementation. Three crucial phases of task implementation are (1) how tasks are represented in textbooks or curricula material, (2) how teachers set up tasks for students, and (3) how students complete tasks in the classroom. Although the implementation of tasks is important, the level of thinking, cognitive demand, students must use to solve problems is also influential in how and what is learned. Stein et al. (2000) define cognitive demand as "the kind and level of thinking required of students in order to successfully engage with and solve the task" (p. 11). The framework provides four levels of cognitive demand for tasks: *Memorization, Procedures without Connections, Procedures with Connections, and Doing Mathematics*. Stein et al. (1996) emphasize the importance of examining levels of cognitive demand of tasks because it influences student learning.

> The mathematical tasks with which students become engaged determine not only what substance they learn but also how they come to think about, develop, use,

> and make sense of mathematics. Indeed, an important distinction that permeates research on academic tasks is the difference between tasks that engage students at a surface level and tasks that engage students at a deeper level by demanding interpretation, flexibility, the shepherding of resources, and the construction of meaning. (p. 459)

QUASAR researchers found "student learning gains were greatest in classrooms in which instructional tasks consistently encouraged high-level student thinking and reasoning and least in classrooms in which instructional task were consistently procedural in nature" (Stein et al., 2000, p. 4).

Smith (2000) used the QUASAR Mathematical Task Framework to analyze the level of cognitive demand of tasks. She focused on the last two aspects of the Mathematical Task Framework, how teachers set up tasks and then how students complete tasks in the classroom. She made comparisons across countries and found that U.S. lessons contain fewer *making connections* than the other countries. She also found that U.S. lessons containing *making connections* tasks are not completed by students *using connections*, but rather by *using procedures* or answers only.

How students perform on tasks that require different levels of cognitive demand is so important that the National Assessment of Educational Progress (NAEP) and Trends in International Mathematics and Science Study (TIMSS) developed frameworks to incorporate cognitive demand dimensions. The NAEP framework categorizes cognitive demand by three levels: *low, medium,* and *high*. These levels are similar to those of the Mathematical Task framework; however, they combine *Procedures without Connections* and *Procedures with Connections* into the medium level. The TIMSS framework also categorizes levels of cognitive demand with three skills: (1) *knowing facts, procedures, and concepts,* (2) *applying knowledge and conceptual understanding,* and (3) *reasoning*.

Although both the NAEP and TIMSS frameworks could be used to analyze the cognitive demand of tasks, the elimination of a middle category forces some questions to be coded at a medium level when they actually require low levels of cognitive demand. By lumping all tasks that require students to use procedures with or without connections it does not provide a small enough grain size to pick up differences between these types of questions. Although both types of questions require students to use procedures differences exist in questions that have connections such as requiring students to move between different representations.

Summary. For the purpose of this study, the four levels of cognitive demand, *Memorization, Procedures without Connections, Procedures with Connections, and Doing Mathematics* presented in the Mathematical Task Framework will be used to analyze the Precalculus Concept Assessment. Each question will be coded at one level of cognitive demand. Questions of differing levels of cognitive demand will be used to analyze what students have learned and how the type of curricula influenced learning. Integrated curricula contain more contextual problems, more problems that can be solved using multiple strategies, and fewer procedural problems (Senk & Thompson, 2003). Single-subject curricula typically contain more procedural problems and few strategies are explored when solving problems (O'Brien, 1999). It is hypothesized that students who used integrated curriculum experience more higher level questions than students who used single-subject curriculum.

The four levels of cognitive demand provide a model for describing and categorizing tasks. However, the focus of research has been on how tasks follow the three phases of implementation: how tasks are represented in textbooks or curricula material, how teachers set up tasks for students, and how students complete tasks in the classroom. This study will focus on the first phase of the Mathematical Task Framework, namely how tasks are addressed by students taking the Precalculus Concept Assessment.

Summary

In this chapter a brief historical view of how mathematics curriculum has changed during the crisis-reform-reaction cycle was provided. Today the field is attending to many critics of the integrated mathematics curricula by trying to identify the benefits for students who experience this type of organizational structure at the secondary level. More research is needed by people not associated with the NSF projects. A review of the current research on the impact of integrated curriculum on high school students was reported. The results provide an uncertain a picture of *how* students are performing on standardized assessments; however, more research is needed to identify *what* students are learning from the integrated and single-subject curricula. Researchers are also concerned with the lack of conceptual knowledge college calculus students are gaining, yet the mathematics for collegiate work is established in high school. Therefore, researchers need to document the current status of what mathematics curriculum is being used and what students are learning from the different curricula types, but also document the effectiveness of these curricula on student learning at the high school level.

This chapter described how the research on students' solution strategies, errors, and mathematical tasks has informed this study. Specifically, the research on students' solution strategies and errors has provided multiple frameworks to analyze student work. The frameworks that will be used for data analysis will be discussed in the next chapter. The framework for levels of cognitive demand will be used to categorize each test item on the *PCA*. This literature review has provided procedures and methods that have informed this study and the body of research have confirmed a need for this study.

CHAPTER 3: METHODOLOGY

This study examined students' mathematical understanding after completing four years of college preparatory (integrated or single-subject) mathematics courses in two high schools in the Midwest. First, an analysis of mathematical understanding for all students who completed four years of college preparatory mathematics courses (integrated or single-subject) was conducted. Second, an analysis of mathematical understanding of students who completed four years of college preparatory (integrated or single-subject) mathematics and enrolled in AP Calculus during the 2007-08 school year was completed. Finally, a qualitative analysis of AP Calculus students' solution strategies and mathematical errors was conducted.

Students' mathematical understanding was measured with three instruments, the Precalculus Concept Assessment (*PCA*) and two open-ended tasks focused on functions. Responses on the *PCA* were analyzed by cognitive demand, specific representations (e.g., graphic, numeric, symbolic), and overall score. Item analysis was conducted to investigate students' performance on different concepts of functions taught prior to entering AP Calculus. The open-ended tasks were analyzed for students' solution strategies and their mathematical errors.

This chapter includes a description of the methods for selecting students and the instruments used for data collection. Next, a detailed description of the procedures for data collection is discussed. The chapter concludes with a description of the data coding and analyses used to answer the research questions stated in Chapter 1.

Sample Selection

A mid-western school district, with approximately 17,000 students, was the site of the study. The district has two high schools that enroll students in grades 10-12, and data were collected from students in each school. The state department of education reported that 80-90% of students in the district extend their education past high school and 70% of the students attend a four-year college or university.

In 1998 the district adopted the textbook series, *Contemporary Mathematics in Context: A Unified Approach* (*Core-Plus*) (Coxford et al., 2003) and began offering both an integrated and single-subject high school mathematics course sequence. That is, students in the district have the option of an integrated course sequence (Integrated 1, 2, 3, and 4) using *Core-Plus* or a sequence of single-subject mathematics courses (Algebra I, Geometry, Algebra II, PreCalculus) using three textbooks from the *University of Chicago School Mathematics Project*, *Algebra* (McConnell et al., 1996), *Geometry* (Usiskin et al., 1997), *Advanced Algebra* (Senk et al., 1996) and then *Precalculus Enhanced with Graphing Utilities* (Sullivan & Sullivan III, 2006).

The study sample included all students who completed a four year college preparatory (integrated or single-subject) mathematics sequence. Table 3.1 displays the number of students who completed either integrated (IC) or single-subject (SSC) mathematics courses at the two schools.

Table 3.1

Enrollment of high school students completing a four-year college preparatory mathematics sequence by curriculum path for 2006-2007 school year

	# of IC students	# of SSC students	Total students
Number of Students	201	304	505

A sub-sample of all students who entered AP Calculus after completing four years of college preparatory (integrated or single-subject) curricula was also identified. A total of 207 students enrolled in AP Calculus AB or BC at the two high schools during the 2007-2008 year; however, 8 of these students were excluded from the study because they transferred into the district, had not taken the *PCA* exam, or had switched curricular paths. Table 3.2 displays the number of students in the sub-sample who enrolled in AP Calculus AB or BC. Although all students who took calculus used the same textbook, *Calculus: Graphical, Numerical, Algebraic* (Finney, Demana, Waits, & Kennedy, 2003), the courses are different. The AP Calculus AB course mirrors a first semester college calculus course. Topics studied include limits, derivatives, and an introduction to integrals. The AP Calculus BC course mirrors a two semester sequence of calculus. Topics include those studied in AB; however, more depth is given to integration and students are introduced to sequences and series as well as parametric and polar functions.

Table 3.2

Enrollment in AP Calculus by curriculum path for 2007-2008 school year

	Course	# of IC students	# of SSC students	Total students by course
	AP Calculus AB	47	85	132
	AP Calculus BC	12	55	67
Total students		59	140	199
% of all students who enrolled in AP Calculus		29%	46%	39%

Participants

For the purpose of this study, only student data for those who completed four years of college preparatory (integrated or single-subject) mathematics courses were included in the analysis. Thus, students who switched curriculum paths were excluded from the study. Approval by the teachers, schools, school district administration, and the University of Missouri Institutional Review Board was gained prior to the collection of any data.

Instruments

This study utilized one instrument as the baseline measure of students' prior achievement and three other instruments were used to measure student knowledge after completing a four year college preparatory (integrated or single-subject) mathematics sequence. The baseline measure, the *Iowa Algebra Aptitude Test* (*IAAT*), was selected for two reasons. First, the district has students take this assessment at the end of the seventh grade, which is typically before students enter one of the two curricular paths. The second reason was that the *IAAT* was available data for students in the study. Therefore, the *IAAT* was used to determine if there were differences in student achievement for the two groups (integrated or single-subject) before entering a curriculum path.

Iowa Algebra Aptitude Test (IAAT)

The fourth edition of the *Iowa Algebra Aptitude Test* (*IAAT*) was taken by all seventh grade students in the district and served as the baseline measure of student achievement (Schoen & Ansley, 1993). The *IAAT* was developed to indicate mathematics aptitude and assist in placing students in Algebra 1. The exam contains 63 multiple choice items with four answer choices for each item. The content of the exam includes interpreting mathematical information graphically and symbolically, translating between algebraic expressions and verbal statements, finding relationships between two sets of data, and using symbols to determine any algebraic misconceptions. Students who had missing baseline data were excluded from the comparison analyses.

Reliability and validity of the IAAT. The *IAAT* is a reliable measure for the four constructs that are tested. The Kuder-Richardson Formula 20 reliability coefficient for the composite score for 7^{th} grade students was 0.93 (Schoen & Ansley, 1993, p. 15). This refers to a high internal consistency among the four parts of the test. The *IAAT* was also found to be a valid measure when compared with students' Algebra 1 grades and exams. The correlations ranged from 0.74 to 0.84.

Precalculus Concept Assessment

Students who completed their fourth year of either integrated or single-subject mathematics courses in 2006-2007 took the *Precalculus Concept Assessment* (*PCA*) (Carlson, Oehrtman, & Engelke, in review) a 25 item multiple choice instrument that assesses understanding of functions. Each item includes five answer choices. The *PCA* was developed by faculty from the Department of Mathematics at Arizona State University and was designed to reflect core content and common misconceptions students have about functions (Carlson et al., in review). Therefore, the answers students chose provided information about common misconceptions they had after completing four years of college preparatory mathematics.

Reliability and validity of the PCA. Reliability provides a gauge of the consistency of measures that are obtained. The *PCA* is a relatively new instrument used to characterize students reasoning and understanding of functions, which are prerequisites for understanding concepts of calculus. Carlson et al. (in review) administered the *PCA* to 902 college precalculus students and found the “Cronbach alpha of 0.73 indicating some degree of overall coherence” (p. 23). Although the *PCA* has not been validated nationally, Carlson and colleagues conducted clinical interviews with more than 150 college students to validate the *PCA* items (in review). Each question was tested and validated to guarantee that students who selected a specific distracter (i.e., answer choice) consistently provided similar justifications during interviews (Engelke et al., 2005).

Open-ended tasks

Students who completed four years of college preparatory (integrated or single-subject) mathematics curriculum and enrolled in AP Calculus for the 2007-2008 school year completed two open-ended tasks, *Piecewise Function* and *Filling a Tank,* (See Appendix A). The *Piecewise Function* task was adapted from a released 2003 AP Calculus free response item by an AP Calculus teacher for precalculus students. The *Filling a Tank* task was taken from examples developed by Peter Taylor (1992). The three AP Calculus teachers were originally given four tasks directly related to the concept of function and asked to select two tasks they thought to be most beneficial for students learning calculus. The *Piecewise Function* task and the *Filling a Tank* task were chosen by all three teachers. In the *Piecewise Function* task, students are given a graph of a function and asked to find rates of change as well as write equations for the piecewise function. In the *Filling a Tank* task, students compare average flow (rate of change) with instantaneous flow. Both tasks require students to demonstrate their understanding of functions; however, one task is based in a contextual setting and the other is not. Each task has multiple parts where students show their work and explain their thinking. Students worked on the two tasks individually during a scheduled class period during the first week of class.

Reliability and validity of scoring guides for open-ended tasks. Scoring guides for each open-ended task were developed by the researcher based on pilot student work. The scoring guides were developed by assigning points for different portions of the answer on each question. This allowed students to receive partial credit, and provided a finer grain analysis of their performance. Each guide was shared and discussed with other researchers, including both mathematicians and mathematics educators. Revisions in the scoring guides were made based on feedback gathered from the other researchers. Then each scoring guide was piloted by scoring a sub-set of pilot student work on the two open-ended tasks. Additional revisions were made to scoring guides to assure the points awarded were discriminatory of students understanding of the concept.

To obtain reliability, two doctoral students were trained using the scoring guides and then asked to score 10 pilot students' work. Percent of agreement was checked was checked with the researcher on each task. The *Piecewise Function* task required scorers to make 19 scoring decisions and high reliability was achieved with an average agreement of 91% on the task. The *Filling a Tank* task required scorers to make 17 scoring decisions and an average reliability of 97% was achieved. The high rates of agreement on the pilot student work suggest the likelihood of high reliability of the resulting scores on the two open-ended tasks for the AP Calculus student work in this study.

Data Collection

The district mathematics specialist provided the necessary baseline data (*IAAT* scores) for all students who completed four years of college preparatory (integrated or single-subject) mathematics curriculum during the 2006-2007 school year. Students' mathematics course taking pattern during high school was also provided, including a course title for each student during their 7^{th}-12^{th} grade schooling.

The *PCA* was administered during a two-week window at the end of April 2007 and these data provided the basis for the analysis of students' understanding before entering AP Calculus. Students' names were kept confidential by using identification numbers; in addition, exams were kept in a locked file drawer in the researcher's office. Student responses on the *PCA* were used to analyze misconceptions about functions, ability to solve questions of different levels of cognitive demand, ability to solve items focused on specific representations (e.g., graphic, numeric, symbolic) related to functions, and overall performance.

The open-ended tasks were administered to all AP Calculus students during August 2007, the first week of the new school year, which was agreed upon by all three AP Calculus teachers. AP Calculus student work on the two open-ended tasks provided the basis for the analysis of solution strategies and mathematical errors at the beginning of AP Calculus. Again, all data were kept confidential and locked in a file drawer in the researcher's office. Each student's work was analyzed for solution strategies and mathematical errors.

Data Coding

1. <u>Data scoring for multiple choice questions</u>

The *PCA* exam was scored by awarding 1 point for each correct answer. All incorrect and blank answers were scored 0 points. These points were added for an overall score on the *PCA*. An item analysis was conducted on all questions. Questions left blank were also analyzed to examine what type of problems (graphic, numeric, or symbolic) students did not answer.

2. Data scoring and coding for open-ended tasks

All students' open-ended tasks were scored according to two analyses: a scoring guide (a quantitative analysis) and a cognitive analysis of how AP Calculus students solved the open-ended tasks (a qualitative analysis) (Lane, 1993; Magone, Cai, Silver, & Wang, 1994). The quantitative analysis was based on a scoring rubric (see Appendix B) for each open-ended task. The qualitative analysis was based on a cognitive framework (Figure 1.4) of students' solution strategies and mathematical errors.

Quantitative Analysis

The quantitative analysis was based on the scoring rubric created for the two open-ended tasks. Both scoring rubrics were developed and then validated with pilot student data that were collected on the same tasks. The scoring rubrics were then revised to capture the work students had written. Each question was worth different point values because of the nature of the tasks and students could receive partial credit depending on the work shown. Point values ranged from 3-7 on the *Piecewise Function* task and 1-6 on the *Filling a Tank* task. Students could also lose points if they did not address certain parts of the question (see Appendix B for a detailed description of the scoring rubric that was applied to each task). Descriptive statistics were reported for both groups of students on each question for both tasks. An ANCOVA was conducted to determine if there was any statistical difference between the two groups of students on the two tasks.

Qualitative Analysis

The qualitative analysis for each open-ended task focused on two aspects: *solution strategies* and *mathematical errors*. A specific coding scheme was developed from the existing data. Codes were determined by the different strategies and errors that students displayed on these open-ended tasks.

The qualitative analysis helps distinguish between different levels of student responses. Different students may use different strategies to solve a problem and receive the same numerical score. Similarly, students may make different mathematical errors and receive the same numeric score on a problem. The qualitative analysis provided a more detailed description of what students understand as well as what strategies are used by students who receive a higher numeric score and what errors are made by students who receive a lower numeric score.

3. <u>Data coding for multiple choice</u>

All questions from the *PCA* (multiple choice) were coded by cognitive demand level and specific representations (e.g., graphic, numeric, symbolic). Cognitive demand was coded using the QUASAR framework (Henningsen & Stein, 1997; Silver & Stein, 1996; Stein et al., 1996; Stein & Lane, 1996; Stein et al., 2000). Each question was coded as one of four levels: *doing mathematics, procedures with connections, procedures without connections, or memorization*. Figure 3.1 lists the characteristics of each level of cognitive demand.

Levels of Demands

Lower-level demands (memorization):

- Involve either reproducing previously learned facts, rules, formulas, or definitions or committing facts, rules, formulas or definitions to memory
- Cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure
- Are not ambiguous. Such tasks involve the exact reproduction of previously seen material, and what is to be reproduced is clearly and directly stated.
- Have no connection to the concepts or meaning that underlie the facts, rules, formulas, or definitions being learned or reproduced.

Lower-level demands (procedures without connections):

- Are algorithmic. Use of the procedure either is specifically called for or is evident from prior instruction, experience, or placement of the task.
- Require limited cognitive demand for successful completion. Little ambiguity exists about what needs to be done and how to do it.
- Have no connection to the concepts or meaning that underlie the procedure being used
- Are focused on producing correct answers instead of on developing mathematical understanding
- Require no explanations or explanations that focus solely on describing the procedure that was used

Higher-level demands (procedures with connections):

- Focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas.
- Suggest explicitly or implicitly pathways to follow that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts
- Usually are represented in multiple ways, such as visual diagrams, manipulatives, symbols, and problem situations. Making connections among multiple representations helps develop meaning.
- Require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students need to engage with conceptual ideas that underlie the procedures to complete the task successfully and that develop understanding.

Higher-level demands (doing mathematics):

- Require complex and non-algorithmic thinking—a predictable, well-rehearsed approach or pathway is not explicitly suggested by the task, task instructions, or a worked-out example.
- Require students to explore and understand the nature of mathematical concepts, processes, or relationships
- Demand self-monitoring or self-regulation of one's own cognitive processes
- Require students to access relevant knowledge and experiences and make appropriate use of them in working through the task
- Require students to analyze the task and actively examine task constraints that may limit possible solution strategies and solutions
- Require considerable cognitive effort and may involve some level of anxiety for the student because of the unpredictable nature of the solution process required

Figure 3.1. Characteristics of Levels of Cognitive Demand (Smith, M. S. & Stein, 1998, p. 348)

Representation Analysis

Representation analysis was conducted using the different representations students were expected to know before entering AP Calculus, more specifically functions related to graphs, tables, and equations. Therefore, the representation analysis for the *PCA* was coded into three groups of analyzing functions numerically, graphically, and symbolically. All the questions on the *PCA* assessed students understanding of functions and used different representations. Questions with multiple representations were coded as such.

4. Reliability checks for all data coding

Reliability was checked at various stages throughout the study. Two doctoral students in mathematics education participated in reliability checks on scoring the open-ended tasks and coding items focused on specific representations (e.g., graphic, numeric, symbolic). Two high school teachers participated in reliability checks on coding levels of cognitive demand. First, doctoral students/teachers were trained on either the scoring rubric or the coding framework used. Then after practicing coding and answering any questions, each doctoral student/teacher scored a subset of student exams or coded exam questions by themselves (see Table 3.3). The two doctoral students'/teachers' results were used to compare with the researcher's scoring and an inter-rater reliability was calculated to report consistency in scoring open-ended tasks, coding for levels of cognitive demand and representation analysis.

Table 3.3

Reliability checks

Reliability tasks	Number and type of questions	Reliability with Researcher
Scoring of open-ended tasks	40 open-ended tasks from student responses (20 of each task)	*Piecewise Function* task: 99% *Filling a Tank* task: 95%
Coding Levels of Cognitive Demand on the *PCA*	25 multiple choice questions	84%
Coding three representations on the *PCA*	25 multiple choice questions	100%

Data Analysis

Different data analyses were conducted to answer the research questions stated in Chapter 1. This section provides the reader with each research question and follows with a description of each analysis based on students' performance on the *PCA* and/or the open-ended tasks. The alpha level used for all statistical tests was conducted at the 0.05 level unless otherwise noted. Before any analyses were conducted an ANOVA was used to determine if the two groups of students (integrated and single-subject) were similar when they entered the two curriculum paths.

Specifically, was there a statistically significant difference between student mean scores on the 7th grade *Iowa Algebra Aptitude Test* (*IAAT*) for the two curricula paths? If the two groups were not statistically different, then it was assumed that students entered the two mathematics paths at equal levels of achievement. If the two groups were statistically different, then the students' *IAAT* score was used as a covariate to control for differences in prior achievement.

Research Questions 1 and 2 and an overview of analyses

1. How do high school students who studied four years of college preparatory mathematics curriculum (integrated or single-subject) perform and compare on calculus readiness concepts in terms of:
    a. overall performance;
    b. responses to items by levels of cognitive demand;
    c. responses to items focused on specific representations (e.g., graphic, numeric, symbolic); and
    d. common misconceptions about functions?
2. How do high school students who studied four years of college preparatory mathematics curriculum (integrated or single-subject) and enrolled in AP Calculus perform and compare on college readiness concepts in terms of:
    a. overall performance;
    b. responses to items by levels of cognitive demand;
    c. responses to items focused on specific representations (e.g., graphic, numeric, symbolic); and
    d. common misconceptions about functions?

Descriptive results on the *PCA* were compiled for students in both curriculum paths. The mean and standard deviation were reported for students overall score on the *PCA*, the four levels of cognitive demand, and the three specific representations (e.g., graphic, numeric, symbolic). An item analysis was conducted for each question to report common misconceptions of students in the two curriculum paths.

A comparison was conducted between students in the two curriculum paths. Four comparisons were conducted: (1) overall scores on the *PCA*; (2) sub-scores on *PCA* questions by the four levels of cognitive demand; (3) sub-scores on the *PCA* questions by the three representations; and (4) misconceptions about functions. These scores were used to analyze differences between students who completed four years of college preparatory (integrated or single-subject) mathematics course work. Depending on whether a statistically significant difference between the groups (IC and SSC or APIC and APSSC) on achievement prior to entering a curricula path was found determined the statistical analysis. If no prior achievement differences existed, an analysis of variance (ANOVA) was conducted to determine if there was statistically significant difference between student scores on the *PCA* for the two curricula paths. If there was a statistically significant difference in prior achievement then an analysis of covariance (ANCOVA) using the *IAAT* score as a covariate was used to determine if there was a statistically significant difference between student scores on the *PCA* for the two curricula paths.

The *PCA* exam had four sub-scores for the four levels of cognitive demand. Because of the four levels of cognitive demand, essentially four dependent variables, a different statistical test was utilized. A one-way multivariate analysis of variance (MANOVA) is a more conservative statistical test than an ANOVA and can help reduce type 1 errors (Vincent, 2005). The four sub-scores were used to analyze differences between students who completed four years of integrated or single-subject mathematics course work. Again, depending on the results of statistical significance on prior achievement there were two different possible statistical analyses. If no difference in prior achievement was found then a MANOVA was conducted to determine if there is a statistically significant difference between students' scores on questions of the four levels of cognitive demand for the two types of curricula. However, if a difference in prior achievement existed, then a one-way multivariate analysis with a covariate (MANCOVA) using the *IAAT* scores as the covariate was applied.

The *PCA* had three sub-scores for each representation (tables, graphs, and equations). These scores were used to analyze differences between students who completed a four year college preparatory (integrated or single-subject) mathematics course work. Again if no difference in prior achievement was found then a MANOVA was conducted to determine if there is a statistically significant difference between students' scores on questions from the three types of representations for the two types of curricula. However, if a difference in prior achievement was found, then a MANCOVA using the *IAAT* scores as the covariate was used.

The comparison of mathematical misconceptions was both a quantitative and qualitative analysis. A binomial logistic regression was conducted to estimate the odds that one group of students (integrated or single-subject) were better able to correctly answer each *PCA* items. The logistic regression model was used because the dependent variable (correct/incorrect) was dichotomous and the independent variables (*IAAT* and curriculum type) were continuous and categorical.

The qualitative analysis focused on the commonly chosen wrong answers. The answer choices for each question on the *PCA* were developed to assess different misconceptions students have about functions. Therefore, it was determined that if more than 20% of students in a particular curriculum pathway chose a wrong answer it would be considered a misconception. These results were descriptive of the different misconceptions students had about functions. Student errors were also counted to report their frequency on each question by curricula type.

Research Question 3 and an overview of the analysis

3. How do high school students who studied four years of college preparatory mathematics curriculum (integrated or single-subject) and enrolled in AP Calculus perform and compare in solution strategies and errors on open-ended mathematical tasks that focus on functions during the first quarter of AP Calculus?

Descriptive results were compiled for students in both curriculum paths on the two open-ended tasks. Means and standard deviations were reported for students' overall score and on each question of the open-ended tasks. An analysis was conducted for each question to report whether students in the two curriculum paths attempted each question or left it blank.

A quantitative comparison of overall score on each task was conducted between students who completed a four year college preparatory (integrated or single-subject) mathematics curriculum. Whether there was a difference between the groups to begin the study or not determined which statistical analysis was conducted. If no prior achievement differences existed, an ANOVA was conducted to determine if there was statistically significant difference between student scores on each open-ended task for the two curricula paths. If there was a statistically significant difference in prior achievement then an ANCOVA using the *IAAT* score as a covariate was used to determine if there was a statistically significant difference between student scores on the two open-ended tasks for the two curricula paths.

Another comparison examined the qualitative analysis of the students' strategies and mathematical errors when solving open-ended tasks. Questions on the open-ended tasks were coded for solution strategies and errors. These results were descriptive of the different strategies and errors students made and used when solving the two open-ended tasks. Student errors were reported by frequency for each task by curricula type.

Summary

In this chapter, the methodology for selecting the sample of students and the data collecting instruments were summarized. Student performance on exams was examined through an analysis of questions requiring different levels of cognitive demand, items focused on specific representations (e.g., graphic, numeric, symbolic), and overall score. The analysis of students' performance on the *PCA* provided a profile of the similarities and differences in mathematical understanding of functions for those who had completed a four year college preparatory (integrated or single-subject) mathematics courses prior to entering AP Calculus. Using both quantitative and qualitative analysis of AP Calculus students' performance on two open-ended tasks provided a more complete picture of the similarities and differences in their mathematical understanding for those who had completed four years of integrated or single-subject mathematics courses.

CHAPTER 4: DATA ANAYLSIS AND RESULTS

This study examined students' mathematical understanding after completing four years of college preparatory (integrated or single-subject) mathematics courses. Students' mathematical understanding was measured with three instruments, the Precalculus Concept Assessment (*PCA*) and two open-ended tasks that focused on functions. Responses on the *PCA* were analyzed by overall score, levels of cognitive demand, and types of representations (e.g., graphic, numeric, symbolic). An item analysis was conducted to investigate how students in the two groups (integrated or single-subject) performed on different aspects of functions taught prior to entering AP Calculus and to identify misconceptions of students across curriculum types. The open-ended tasks were analyzed for students' solution strategies and mathematical errors.

The following three research questions guided the analysis:

1. How do high school students who studied four years of college preparatory mathematics curriculum (integrated or single-subject) perform and compare on calculus readiness concepts in terms of:
    a. overall performance;
    b. responses to items by levels of cognitive demand;
    c. responses to items focused on specific representations (e.g., graphic, numeric, symbolic); and
    d. common misconceptions about functions?
2. How do high school students who studied four years of college preparatory mathematics curriculum (integrated or single-subject) and enrolled in AP Calculus perform and compare on college readiness concepts in terms of:

a. overall performance;

b. responses to items by levels of cognitive demand;

c. responses to items focused on specific representations (e.g., graphic, numeric, symbolic); and

d. common misconceptions about functions?

3. How do high school students who studied four years of college preparatory mathematics curriculum (integrated or single-subject) and enrolled in AP Calculus perform and compare in solution strategies and errors on open-ended mathematical tasks that focus on functions during the first quarter of AP Calculus?

This chapter is divided into three major sections. First, quantitative results of the *PCA* are reported by curriculum type (integrated and single-subject) and sample (all students and AP Calculus students). Second, the *PCA* item analysis and common misconceptions are reported. Finally, the quantitative and qualitative results of the open-ended tasks are reported for AP Calculus students.

Precalculus Concept Assessment (*PCA*) Results by Curriculum Type

The *PCA* was administered in the spring of 2007 to 520 high school students who were completing their fourth year of college preparatory mathematics curriculum (integrated or single-subject). Fifteen students had switched between the two curricular paths during high school; therefore, they were not included in the analyses. Of the 505 students studied, 201 students completed four years of integrated mathematics curriculum (IC) and 304 students completed four years of single-subject mathematics curriculum (SSC). Of the 505 students who took the *PCA*, 199 enrolled in AP Calculus for the 2007-

2008 school year. Fifty-nine AP Calculus students completed four years of integrated mathematics curriculum (APIC) and 140 AP Calculus students completed four years of single-subject mathematics curriculum (APSSC).

Overall Performance

The *PCA* contains 25 questions and students were awarded one point for each correct answer and 0 points for an incorrect or blank answer. The descriptive statistics, not adjusted for prior achievement, of total score on the *PCA* for all students who completed four years of college preparatory mathematics (integrated or single-subject) are shown in Table 4.1.

Table 4.1

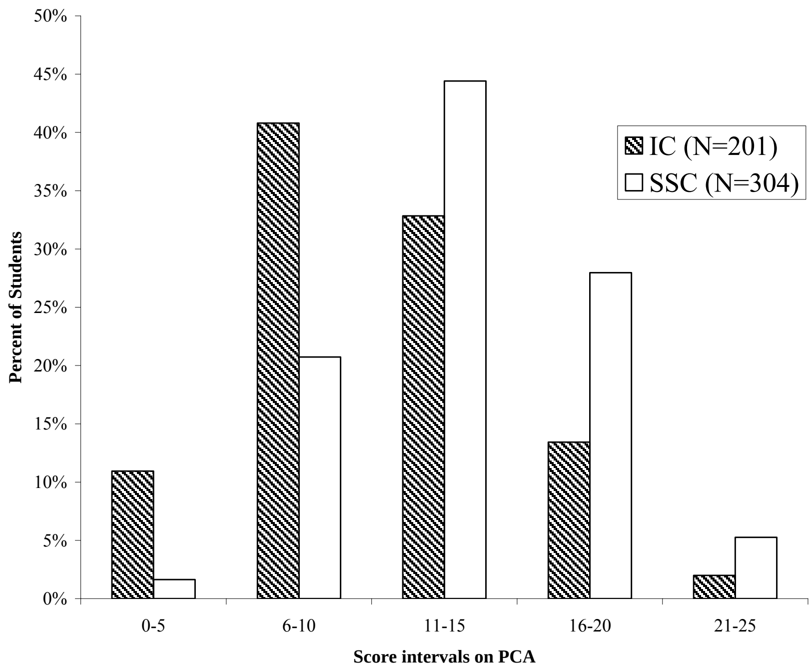
Descriptive statistics of PCA total scores for high school students who studied four years of college preparatory mathematics (integrated or single-subject)

Curriculum Type	*N*	Mean*	SD	Range
IC	201	10.73	4.34	2-22
SSC	304	13.65	4.21	4-25

*Not adjusted for prior achievement

All students. The top graph in Figure 4.1 displays the total score distributions for students on the *PCA*. SSC students scored higher on the *PCA* than IC students. Seventy-two percent of SSC students scored between 11 and 20 points while 46% of IC students scored in the same range on the *PCA*.

All Students



AP Calculus Students

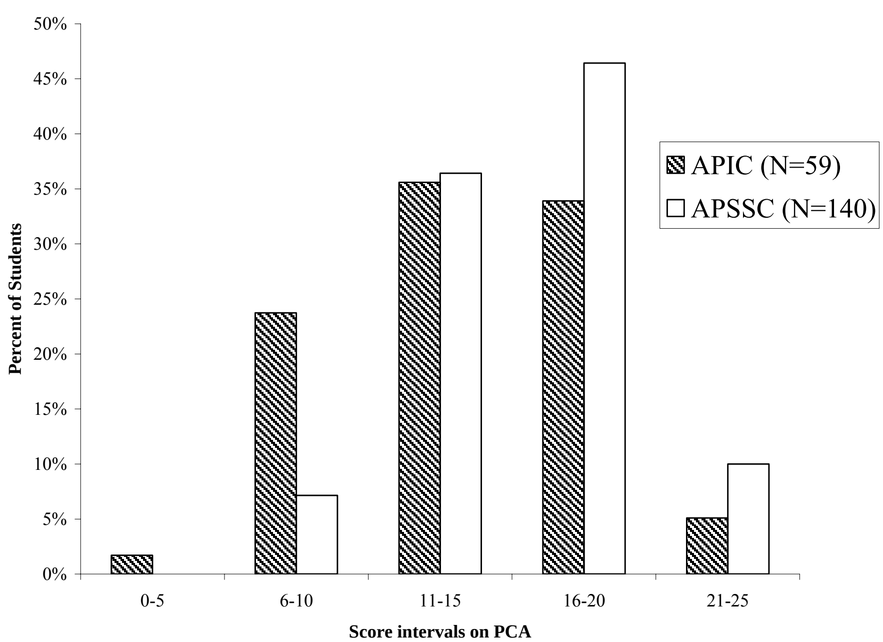


Figure 4.1. Score distributions on the *PCA* for all high school students who studied four years of college preparatory mathematics (integrated or single-subject) and those who enrolled in AP Calculus.

AP Calculus students. The descriptive statistics, not adjusted for prior achievement, of total scores on the *PCA* for the sample of AP Calculus students are shown in Table 4.2. The bottom graph in Figure 4.1 displays the total score distributions for AP Calculus students on the *PCA*. This graph reveals that AP Calculus student scores are approximately normally distributed for both groups. More than half (56%) of APSSC students scored between 16 and 25 points on the *PCA* compared with 39% of APIC students who scored above a 16 on the *PCA*. Two APSSC students received perfect scores on the *PCA*; whereas, the highest score for APIC students was a 22.

Table 4.2

Descriptive statistics of PCA total scores for AP Calculus students who studied four years of college preparatory mathematics (integrated or single-subject)

Curriculum Type	*N*	Mean*	SD	Range
APIC	59	13.90	4.17	5-22
APSSC	140	15.99	3.68	6-25

*Not adjusted for prior achievement

Prior Achievement Analysis

To identify if the two groups of students (integrated and single-subject) were statistically different on prior achievement before entering a specific curriculum path a t-test was conducted on the 316 students for which the district had Iowa Algebra Aptitude Test (*IAAT*) scores that were collected in the 7^{th} grade. Not all students in the study had an *IAAT* score for various reasons (e.g. transferred into the district after 7^{th} grade or absent on exam day).

All students. The descriptive statistics on the *IAAT* for all students are presented in Table 4.3. SSC students had a statistically significant higher mean on the *IAAT* than IC students ($t = 5.71$, $p = .000$).

Table 4.3

Descriptive statistics of IAAT scores for high school students who studied four years of college preparatory mathematics (integrated or single-subject)

Curriculum Type	*N*	Mean	Std. Deviation	Std. Error	*t*	*p*
IC	142	58.47	20.77	1.74	5.71*	.000
SSC	174	71.00	17.54	1.33		

*SSC students *IAAT* mean was significantly higher than IC students

AP Calculus students. The descriptive statistics on the *IAAT* for AP Calculus students are presented in Table 4.4. APSSC students also had a statistically significant higher mean on the *IAAT* than APIC students ($t = 3.04$, $p = .003$). Therefore, all remaining statistically analyses for both samples used students' *IAAT* scores as the covariate to account for differences in prior achievement.

Table 4.4

Descriptive statistics of IAAT scores for AP Calculus students who studied four years of college preparatory mathematics (integrated or single-subject)

Curriculum Type	*N*	Mean	Std. Deviation	Std. Error	*t*	*p*
APIC	40	74.03	16.09	2.54	3.04*	.003
APSSC	76	82.92	14.34	1.64		

*APSSC students *IAAT* mean was significantly higher than APIC students

Comparison on PCA Score

All students. An analysis of covariance (ANCOVA) was conducted to determine the statistical difference between the mean overall score on the *PCA* for the two groups (integrated and single-subject) of students in both samples adjusted for *IAAT* scores. The adjusted means for overall scores on the *PCA* by curriculum type for all students is presented in Table 4.5. SSC students performed significantly higher on the *PCA* (*F* =

9.39, p = .002) than IC students after adjusting for differences in prior achievement (*IAAT*).

Table 4.5

Adjusted descriptive statistics of PCA scores for high school students who studied four years of college preparatory mathematics (integrated or single-subject) using IAAT scores

Curriculum path	*N*	Mean[a]	Std. Error	*F*	*p*
IC	142	11.62	0.306	9.39*	.002
SSC	174	12.91	0.275		

[a]Covariates appearing in the model were evaluated at the following values: *IAAT* = 65.37
*SSC students adjusted mean *PCA* score was significantly higher than IC students.

AP Calculus students. The adjusted means for overall scores on the *PCA* by curriculum type for AP Calculus students is presented in Table 4.6. No statistically significant difference between the mean overall scores on the *PCA* for APSSC and APIC students was found after adjusting for differences in prior achievement (F = 3.54, p = .063).

Table 4.6

Adjusted descriptive statistics of PCA scores for AP Calculus students who studied four years of college preparatory mathematics (integrated or single-subject) using IAAT scores

Curriculum path	*N*	Mean[a]	Std. Error	*F*	*p*
APIC	40	14.85	0.550	3.54	.063
APSSC	76	16.14	0.394		

[a]Covariates appearing in the model were evaluated at the following values: *IAAT* = 79.85

Summary PCA Overall Performance

SSC students had a higher overall mean score on the *PCA* and performed significantly higher than IC students. APSSC students had a higher overall mean score on the *PCA* than APIC students; however, there was no statistically difference on the overall mean score between these two groups of students who were in AP Calculus.

Responses to PCA Items by Levels of Cognitive Demand

Each *PCA* item was coded for the level of cognitive demand (*memorization, procedures without connections, procedures with connections,* and *doing mathematics*) required by students (see Appendix C). No *PCA* items were coded as *memorization*. Ten items were coded as *procedures without connections*. Twelve items were coded as *procedures with connections*. The remaining three items were coded as *doing mathematics*. Figures 4.2, 4.3, and 4.4 display the score distributions for both samples of students (integrated and single-subject) on *PCA* items coded for the three levels of cognitive demand (*procedures without connections, procedures with connections,* and *doing mathematics*) without adjusting for prior achievement.

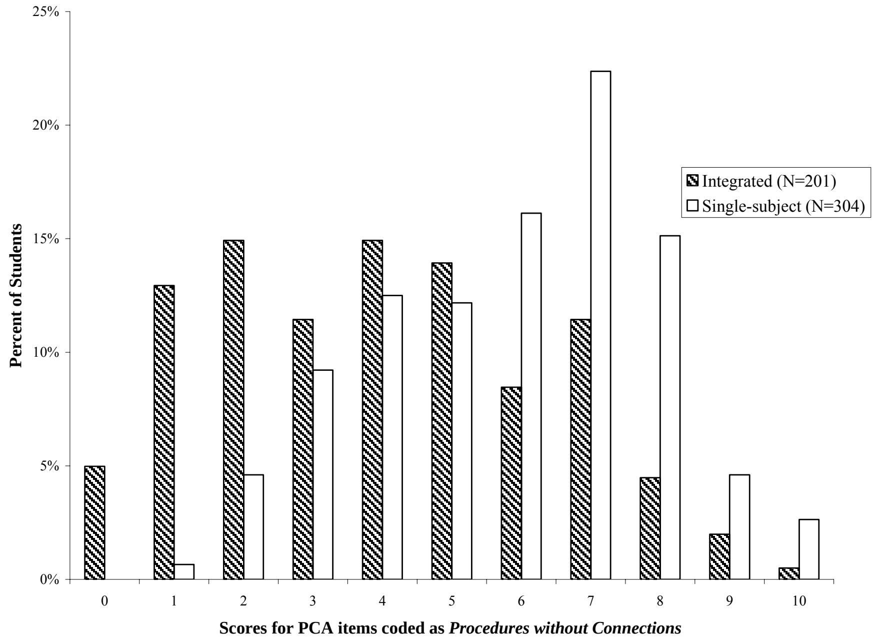
Procedures without Connections

Ten items were coded as *procedures without connections*, requiring a lower-level of cognitive demand. On these items students generally used a procedure they were familiar with or were taught in class and no explanation was required.

All students. Approximately three-fourths (73%) of the SSC students answered at least half of the items coded as *procedures without connections* correctly, compared to just over two-fifths (41%) of IC students (Figure 4.2, top). The top graph displays IC students' scores on items coded as *procedures without connections* skewed to the right and SSC students' scores skewed to the left. This is interpreted as SSC students correctly answered more items that require a procedure they were taught.

AP Calculus students. APSSC students also answered more *procedures without connections* items correctly than APIC students (Figure 4.2, bottom). The majority (84%) of AP Calculus students, APSSC (90%) and APIC (73%), answered at least half of the items correctly. Nearly one-fifth (19%) of APIC students answered 3 or fewer items correctly, while 4% of APSSC students answered 3 or fewer questions correctly.

All Students



AP Calculus Students

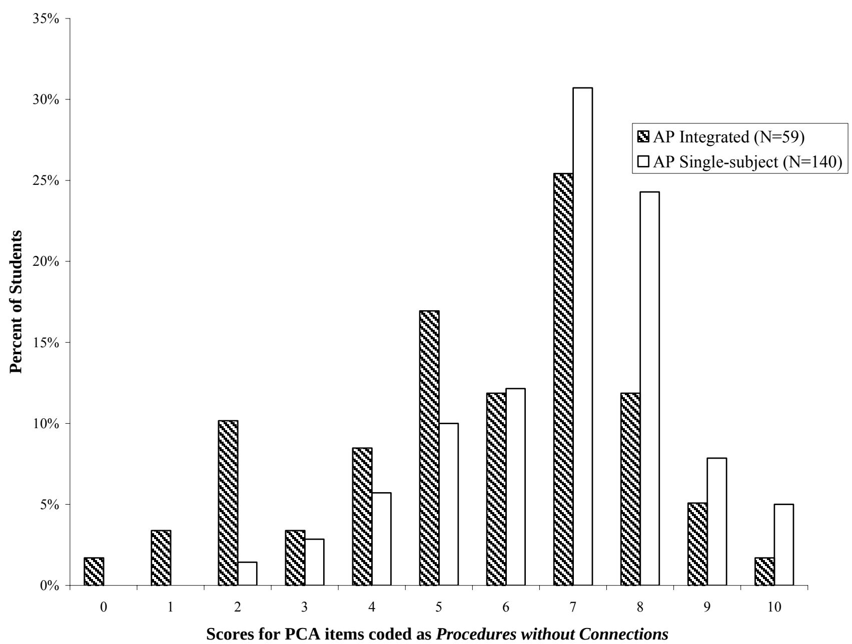


Figure 4.2. Score distributions on *PCA* items coded as *procedures without connections* for high school students who studied four years of college preparatory mathematics (integrated or single-subject) and those who enrolled in AP Calculus.

Procedures with Connections

Twelve items were coded as *procedures with connections*, requiring a high-level of cognitive demand. Students interpreted situations, used multiple representations, or used a procedure that required a deeper understanding of a mathematics concept.

All students. Students who studied from either curriculum type tended to answer *procedures with connections* similarly (Figure 4.3). The graph on the top displays the majority (68%) of all students, SSC (74%) and IC (59%), answered at least six or more of the items correctly.

AP Calculus students. The graph on the bottom of Figure 4.3 is skewed more to the left, meaning overall AP Calculus students answered more *procedures with connections* items correctly than the other sample (all students). The majority (87%) of AP Calculus students who studied either curriculum (integrated or single-subject) correctly answered six of more of the items.

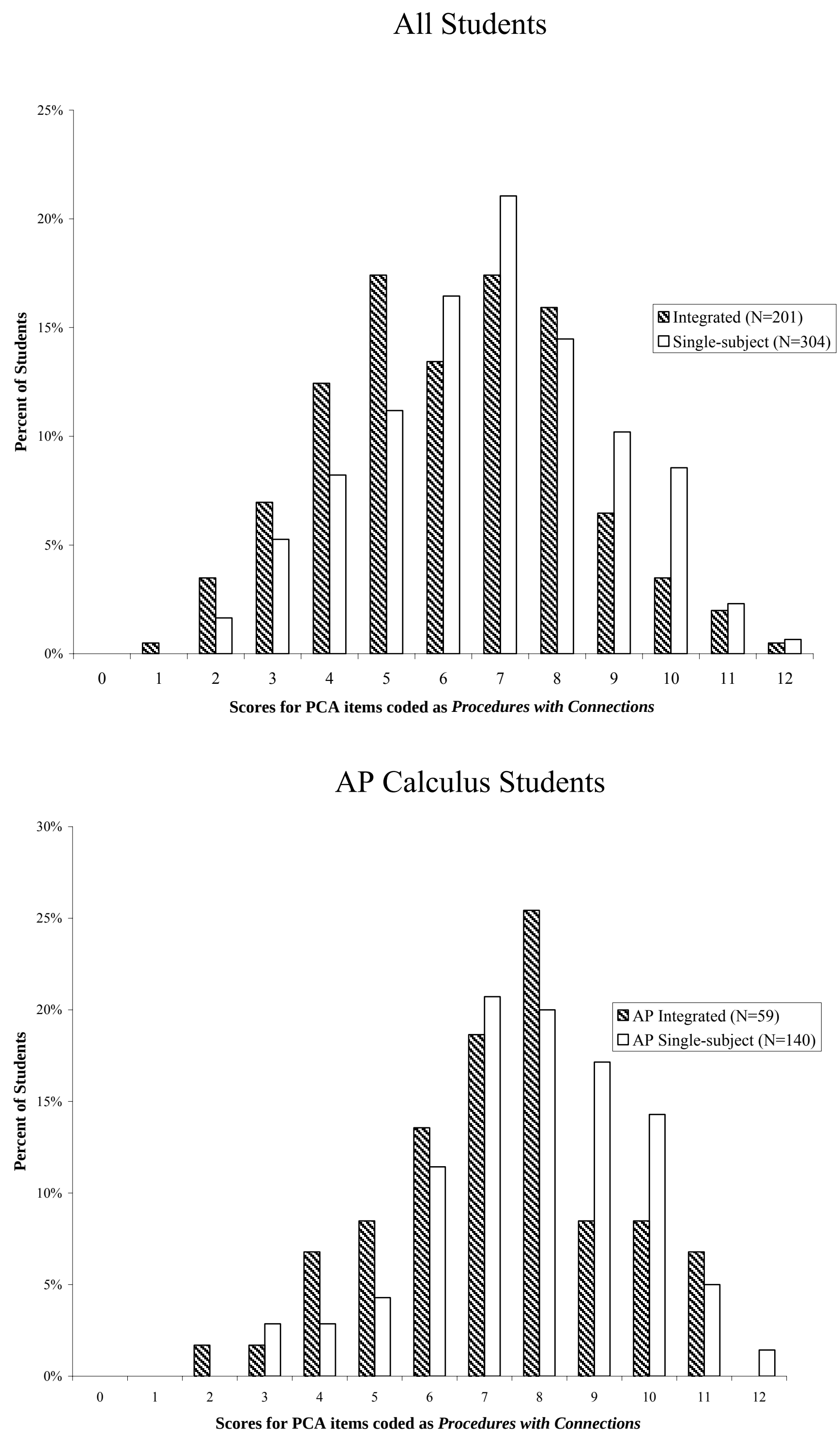


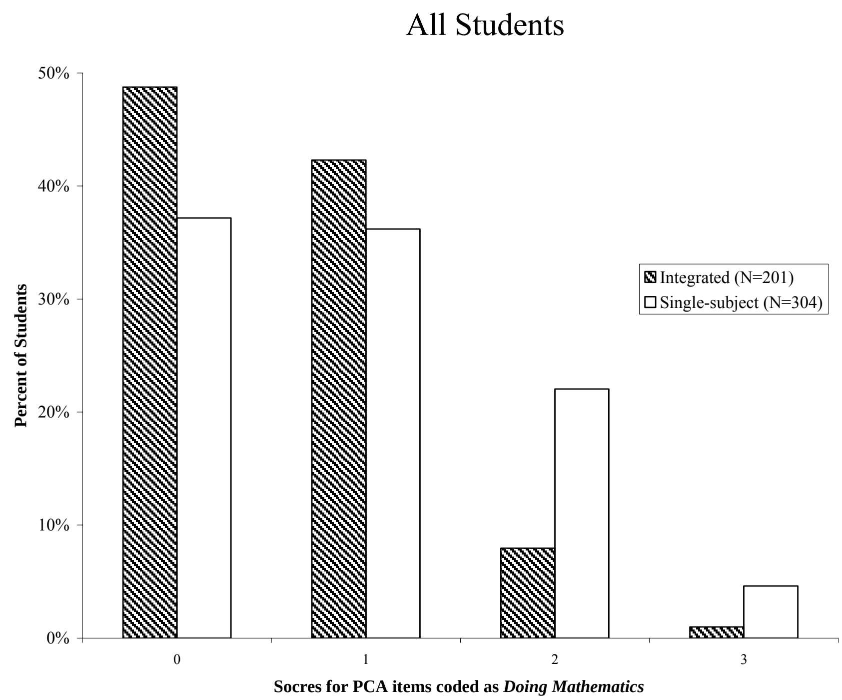
Figure 4.3. Score distributions on *PCA* items coded as *procedures with connections* for high school students who studied four years of college preparatory mathematics (integrated or single-subject) and those who enrolled in AP Calculus.

Doing mathematics

Three items were coded as *doing mathematics*, requiring a high-level of cognitive demand. Students needed to use complex and non-algorithmic thinking to solve these unfamiliar problems.

All students. Few students from either curriculum path answered all three items correctly. The majority (55%) of students from either curriculum pathway answered 2 or fewer items correctly. Thirty-seven percent of SSC students did not answer any item correctly; while 49% of IC students did not answer any *doing mathematics* items correctly (Figure 4.4, top).

AP Calculus students. The results were different for the sample of AP Calculus students (Figure 4.4, bottom) with more students able to correctly answer at least one item. A higher percentage (9%) of APSSC students answered all three items correctly compared to the APIC students (2%). The majority (70%) of students from either curriculum (integrated or single-subject) answered 2 or fewer items correctly. Nineteen percent of APSSC students and 34% of APIC students were unable to answer any *PCA* items coded as *doing mathematics*.



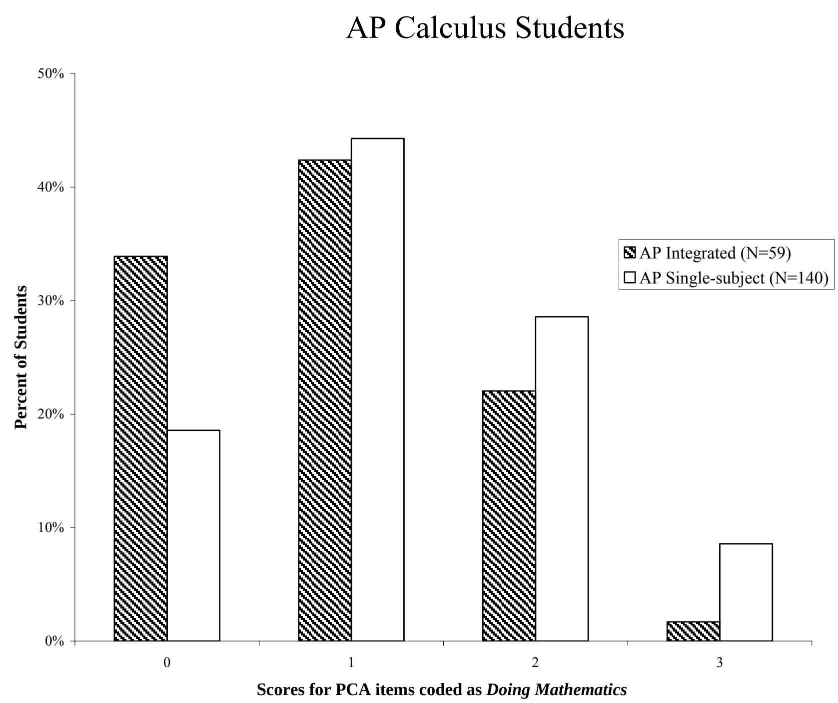


Figure 4.4. Score distributions on *PCA* items coded as doing mathematics for high school students who studied four years of college preparatory mathematics (integrated or single-subject) and those who enrolled in AP Calculus.

Comparison on Levels of Cognitive Demand Analysis

To determine if there was a statistically significant difference between students' mean scores for the three levels of cognitive demand a one-way multivariate analysis with a covariate (MANCOVA) was conducted for both samples using the *IAAT* scores as the covariate.

All students. The Wilks' Lambda (Pedhazur, 1997) was significant (F = 10.305, p = .000) among the cognitive levels. Therefore, three ANOVAs were conducted to determine which cognitive demand level was significantly different for the two curricula paths. The descriptive statistics adjusted for *IAAT* scores on *PCA* items for the three levels of cognitive demand by curricula are displayed in Table 4.7. The F for *procedures without connections* was found to be significant in favor of SSC students (F = 25.136, p = .000), but no significance was found for either *procedures with connections* (F = .026, p = .872) or *doing mathematics* (F = 1.562, p = .212). SSC students performed statistically higher on *PCA* items that required *procedures without connections* than IC students after adjusting for differences in prior achievement (*IAAT*).

AP Calculus students. The Wilks' Lambda was not significant (F = 1.468 p = .227) for AP Calculus students. Therefore, no further analysis was conducted since students in the two curricula paths did not differ significantly on *PCA* items requiring different levels of cognitive demand.

Table 4.7

Adjusted descriptive statistics of PCA scores by levels of cognitive demand for high school students who studied four years of college preparatory mathematics (integrated or single-subject) using IAAT scores

Level of Cognitive Demand	Curriculum Path	Points Possible	Mean[a]	Std. Error	*F*	*p*
Doing Mathematics	IC	3	0.72	0.06	1.562	.212
	SSC		0.83	0.06		
Procedures with Connections	IC	12	6.54	0.16	.026	.872
	SSC		6.58	0.14		
Procedures without Connections	IC	10	4.35	0.17	25.136*	.000
	SSC		5.50	0.15		

[a]Covariates appearing in the model were evaluated at the following values: *IAAT* = 65.37.
*SSC students mean score was significantly higher than IC students

Summary for PCA items by Levels of Cognitive Demand

PCA items were coded and analyzed by four levels of cognitive demand. No *PCA* items were coded in the lowest category, *memorization*. SSC and IC students did not perform statistically different on items coded as *procedures with connections* or *doing mathematics*; however, SSC students performed statistically higher than IC students on *PCA* items coded as *procedures without connections*.

APSSC and APIC students did not perform statistically different on items coded for the three levels of cognitive demand (*procedures without connections*, *procedures with connections*, or *doing mathematics*).

Responses to PCA items by Representation Type

Each *PCA* item was also coded for the type of mathematical representation (graphic, numeric, symbolic) students utilized to solve the question (see Appendix D). Nine items were coded as a *graphical representation* because students used or evaluated a graph. Two items were coded as a *numerical representation* because students used or evaluated a table. Sixteen items were coded as *symbolic representation* because students evaluated or wrote an equation. Two of the 25 items were coded in two categories, graphic and symbolic, because students used both representations to solve or evaluate the questions. Figures 4.5, 4.6, and 4.7 display the score distributions for both samples of students (integrated and single-subject) on *PCA* items coded for the three types of representations (graphic, numeric, and symbolic) without adjusting for prior achievement.

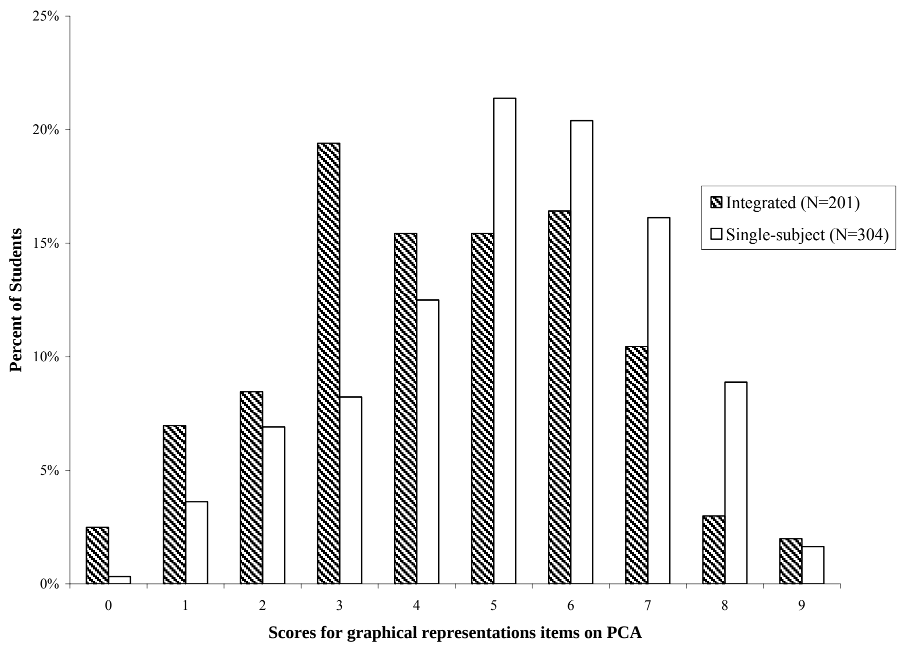
Graphic

Nine items were coded as *graphical representations*. Students were required to interpret, evaluate, or describe graphical features of different types of functions.

All students. Students who studied from either curriculum pathway answered graphical questions similarly (Figure 4.5). The graph on the top displays the majority of SSC students (68%) and just under half (47%) of IC students correctly answered the majority of graphical items.

AP Calculus students. The graph on the bottom of Figure 4.5 is skewed more to the left, meaning overall AP Calculus students answered more graphical representation items correctly. The majority (84%) of AP Calculus students, APSSC (87%) and APIC (78%), correctly answered the majority of graphical items

All Students



AP Calculus Students

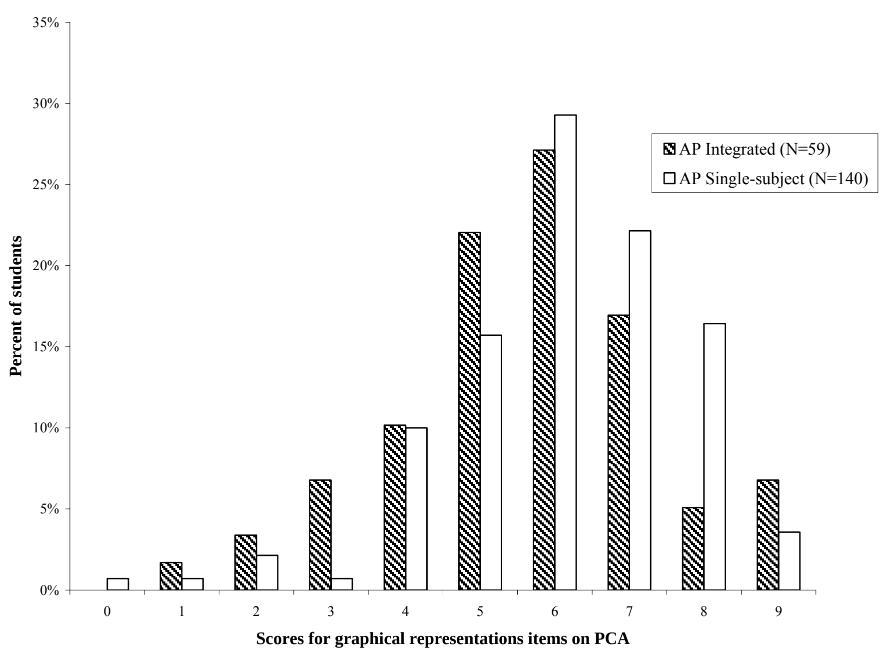


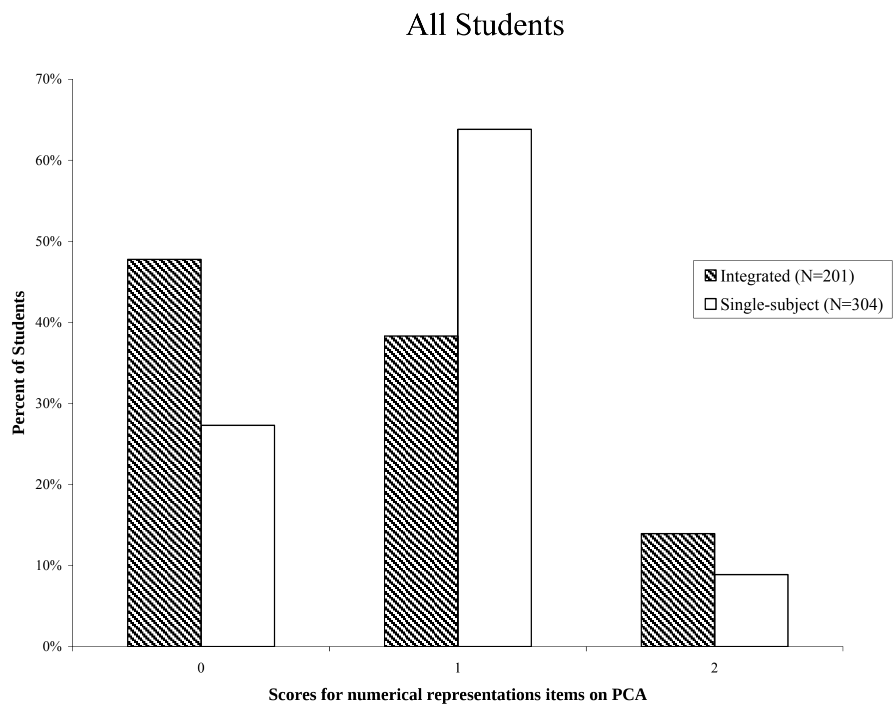
Figure 4.5. Score distributions on *PCA* items requiring a *graphical representation* for high school students who studied four years of college preparatory mathematics (integrated or single-subject) and those who enrolled in AP Calculus.

Numeric

Two items were coded as *numerical representations*. Students were required to analyze or interpret a table of values.

All students. Fourteen percent of IC students answered both items correctly compared to 9% of SSC students. However, 27% of SSC students and 48% percent of IC students did not answer either numerical item correctly (Figure 4.6). Most (83%) students from both curricula answered at least one item correctly.

AP Calculus students. The results were different for the sample of AP Calculus students (Figure 4.6, bottom). Twice the percentage of APIC students (29%) than APSSC students (13%) answered both items correctly. Approximately, the same percentages of students from both curricula, APIC (19%) and APSSC (17%), were unable to answer either numerical item. As found with all students, the majority of AP Calculus students (82%) from both curricula answered at least one item correctly.



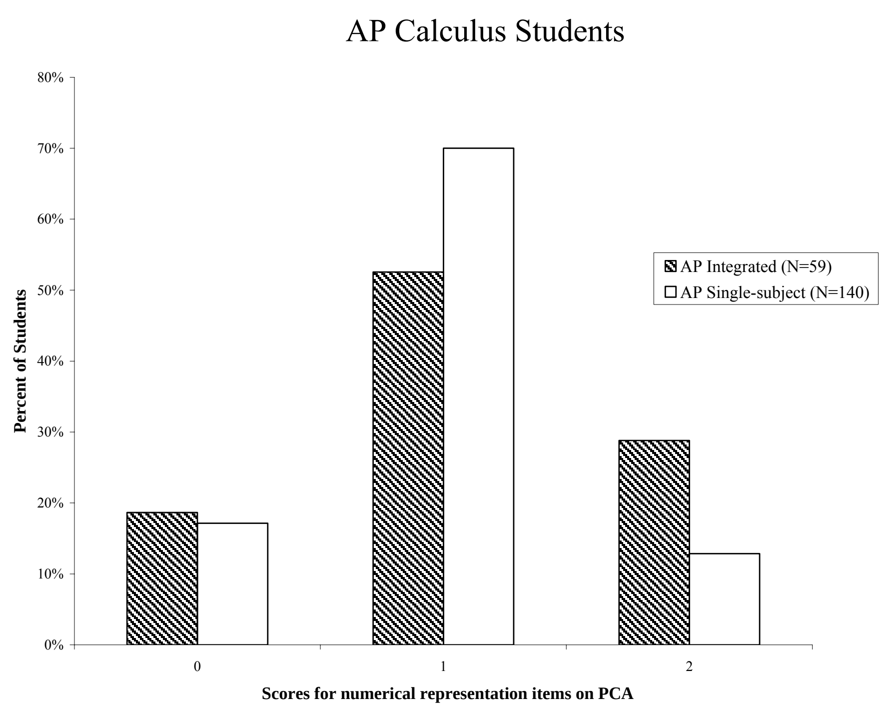


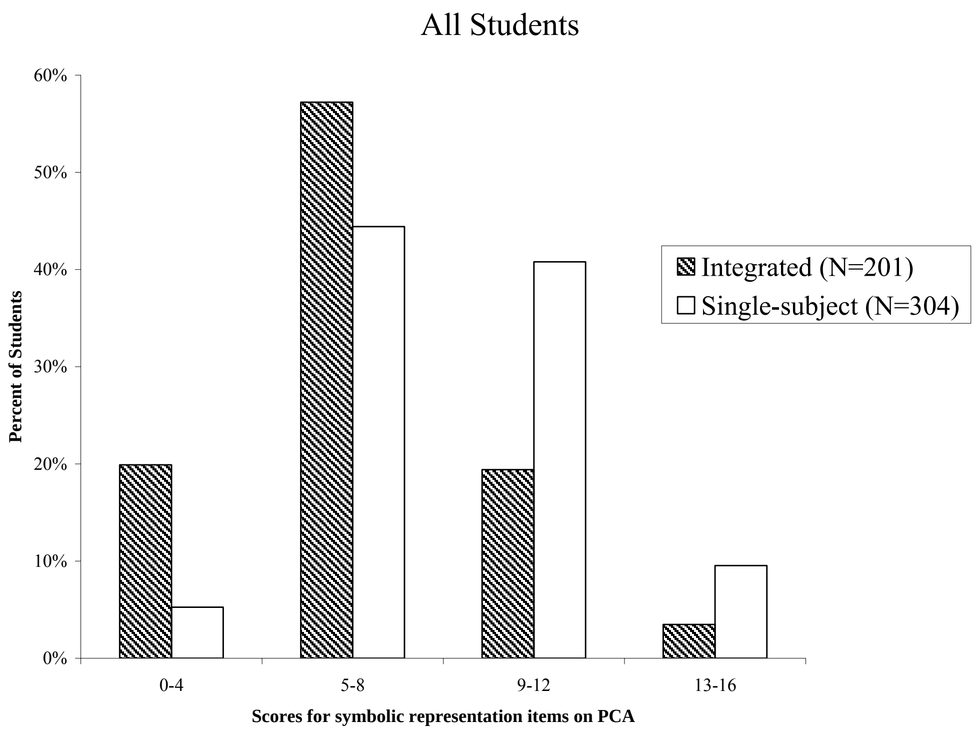
Figure 4.6. Score distributions on *PCA* items requiring a *numerical representation* for high school students who studied four years of college preparatory mathematics (integrated or single-subject) and those who enrolled in AP Calculus.

Symbolic

Sixteen items were coded as symbolic representations. These items required students to evaluate, solve, interpret, analyze and write different types of functions.

All students. SSC students answered more symbolic questions correctly than IC students. Just over twice the percentage of SSC students (51%) as IC students (22%) answered at least 7 items correctly (Figure 4.7, top). Five percent of SSC students scored a 4 or lower on symbolic items; whereas, 20% of IC students scored a 4 or lower on these *PCA* items.

AP Calculus students. APSSC students answered more symbolic representation items correctly than APIC students. Seventy-three percent of APSSC students answered at least 9 symbolic items correctly compared with 44% of APIC students. Four percent of APSSC students answered at least 15 items correctly, while no APIC students answered this many questions correctly. No APSSC students got 4 or fewer questions correct, while 7% of APIC students answered 4 or fewer items.



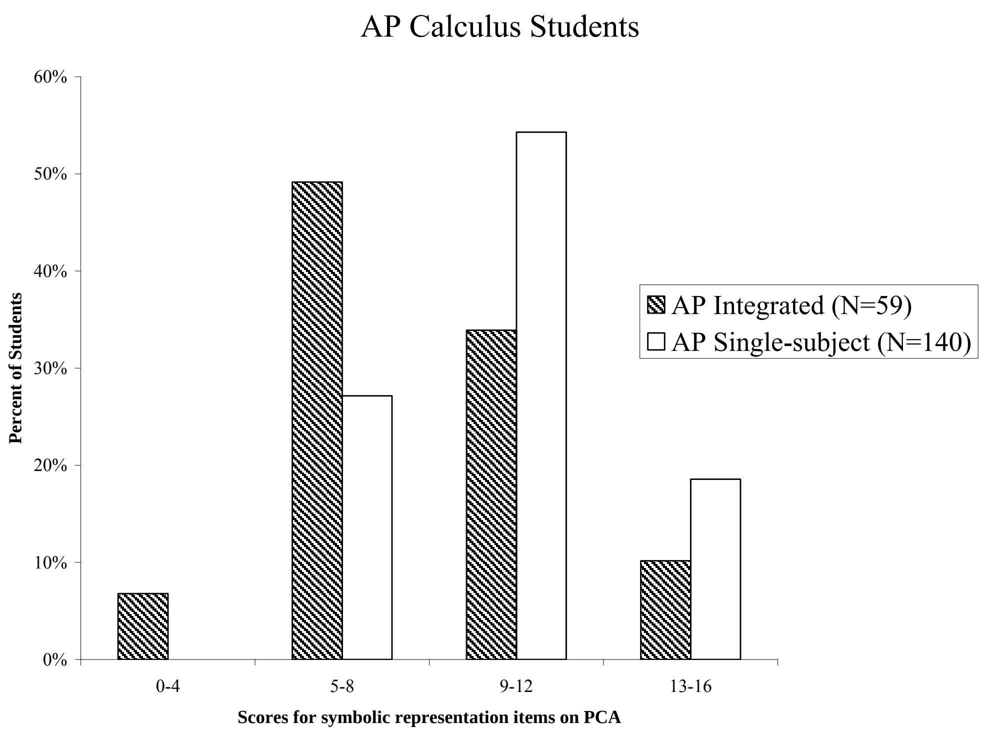


Figure 4.7. Score distributions on *PCA* items requiring a *symbolic representation* for high school students who studied four years of college preparatory mathematics (integrated or single-subject) and those who enrolled in AP Calculus.

Comparison on Representations

To determine if there was a statistical difference between students mean scores for the three representation types a one-way multivariate analysis with a covariate (MANCOVA) was conducted for both samples using the *IAAT* as the covariate.

All students. The Wilks' Lambda was significant (F = 6.358, p = .000) among the representation types for students who studied four years of college preparatory mathematics (integrated or single-subject). Therefore, three ANOVAs were conducted to determine which representation types were significantly different for the two curricula paths. The resulting analysis adjusted for *IAAT* scores on *PCA* items by the three types of representation are shown in Table 4.8. The F for symbolic representations was found to be significant in favor of SSC students (F = 14.651, p = .000), but no significance was found for graphical (F = .531, p = .467) or numerical representations (F = 0.55, p = .815). SSC students performed statistically higher on *PCA* items that required symbolic representations than IC students.

Table 4.8

Adjusted descriptive statistics on PCA scores by three types of representations for high school students who studied four years of college preparatory mathematics (integrated or single-subject) using IAAT scores

Representation Type	Curriculum Path	Points Possible	Mean[a]	Std. Error	F	p
Graphic	IC	9	4.69	0.145	.531	.467
	SSC		4.84	0.130		
Numeric	IC	2	0.76	0.054	.055	.815
	SSC		0.78	0.048		
Symbolic	IC	16	7.21	0.201	14.651*	.000
	SSC		8.27	0.181		

[a]Covariates appearing in the model were evaluated at the following values: *IAAT* = 65.37.
*SSC students mean score was significantly higher than IC students

AP Calculus students. The Wilks' Lambda was significant (F = 6.634, p = .001) among the representation types for AP Calculus students. Therefore, three ANOVAs were conducted to determine which representation types were significantly different for the two curricula paths and the resulting analysis adjusted for *IAAT* scores on *PCA* items by the AP Calculus students are shown in Table 4.9. The F for numerical was found to be significant, (F = 4.971, p = .028) and the F for symbolic representation was found to be significant, (F = 6.996, p = .009). The F for graphical representation was not found to be significant, (F = 1.055, p = .306). APIC students performed statistically higher on *PCA* items that required numerical representations than APSSC students, while APSSC students performed statistically higher on *PCA* items that require symbolic representations than APIC students.

Table 4.9

Adjusted descriptive statistics on PCA scores by three types of representations for AP Calculus students who studied four years of college preparatory mathematics (integrated or single-subject) using IAAT scores

Representation Type	Curriculum Path	Points Possible	Mean[a]	Std. Error	F	p
Graphic	APIC	9	5.92	0.231	1.055	.306
	APSSC		6.21	0.165		
Numeric	APIC	2	1.27	0.094	4.971†	.028
	APSSC		1.01	0.067		
Symbolic	APIC	16	8.89	0.376	6.996*	.009
	APSSC		10.14	0.270		

[a]Covariates appearing in the model were evaluated at the following values: *IAAT* = 79.85.
*APSSC students mean score was significantly higher than APIC students
†APIC students mean score was significantly higher than APSSC students

Summary of PCA Items by Representation Type

PCA items were coded and analyzed by three representation types (graphic, numeric, and symbolic). IC and SSC students did not perform statistically different on

PCA items coded as graphical and numerical; however, SSC students performed statistically higher on items coded as symbolic. Although there was no significance found, IC students had a higher mean on numerical items than SSC students.

APIC and APSSC students did not perform statistically different on *PCA* items coded as graphical. APIC students performed statistically higher on *PCA* items coded as numerical, while APSSC students performed statistically higher on *PCA* items coded as symbolic.

Summary

Table 4.10 summarizes the results for all students (IC, SSC, APIC, and APSSC) on overall *PCA* performance, items of different levels of cognitive demand, and items of different representation types after adjusting for prior achievement (*IAAT*). SSC students performed significantly higher than IC students on the *PCA*, items coded as *procedures without connections*, and items coded as symbolic. APIC and APSSC students were not found to be significantly different on overall performance of the *PCA* or levels of cognitive demand; however, APIC students scored significantly higher on *PCA* items coded as numeric and APSSC students scored significantly higher on *PCA* items coded as symbolic.

Table 4.10

Summary of results for all high school students who studied four years of college preparatory mathematics (integrated or single-subject) on overall performance on the PCA, performance on items of different levels of cognitive demand, and performance on item of different representation types after adjusting for prior achievement

Curriculum Path	Overall Performance	Levels of Cognitive Demand	Representation Type
IC			
SSC	Significantly higher	Significantly higher on *procedures without connections* items	Significantly higher on symbolic items
APIC	No difference	No difference	Significantly higher on numeric items
APSSC			Significantly higher on symbolic items

Item Analysis and Common Misconceptions about Functions

As described previously, the *PCA* contains 25 multiple choice questions. The *PCA* was developed with five answer choices that represent a correct answer and four common misconceptions students have about functions. Student exams were coded by the answer choice selected on each item.

PCA Item Analysis

The 25 *PCA* items were categorized in two ways: (1) difficulty level for all students regardless of their curricular path, and (2) the function concepts assessed. Difficulty level was determined according to how students in this study performed. An item was coded *easy* if 80% or more of the students in both samples answered it correctly. An item was coded *difficult* if 50% or less of the students in *either* sample answered it correctly. The remaining items were coded *moderate*. Table 4.11 displays the item numbers coded in each category.

Table 4.11

Summary of difficulty level for PCA items regardless of curricula paths

	Easy	Moderate	Difficult
PCA Item Numbers	1, 16, 18, 19	3, 5, 6, 7, 9, 12, 24, 25	2, 4, 8, 10, 11, 13, 14, 15, 17, 20, 21, 22, 23
Total # of items	4 items	8 items	13 items

The four easy items can be described by two topics, evaluating functions and describing the end behavior of a function. The eight moderate items were distributed into three topics, evaluating and solving equations using function notation, describing the end behavior of a function, and interpreting rates of change. The thirteen difficult items were consolidated into three topics, interpreting rate of change, interpreting the meaning of inverse functions, and writing or interpreting composition of functions.

Carlson and colleagues (in review) coded *PCA* items by four understandings: (1) function evaluation; (2) rate of change; (3) function composition; and (4) function inverse. In this current study, a fifth understanding emerged; solving functions and was added to the four that were previously reported by Carlson. Table 4.12 displays the five function concepts (function evaluation, solving functions, rates of change, function composition, and function inverse) that were assessed by multiple questions from the *PCA*. It is important to note that multiple questions may have assessed more than one function concept and therefore, were coded in more than one area. While a few *PCA* items did not fit in any of the five function concepts, for example, using proportional reasoning (item #3).

Table 4.12

Summary of function concepts assessed by most PCA items

	Function Evaluation	Solving Functions	Rates of Change	Function Composition	Function Inverse
PCA Item Numbers	1, 5, 6, 11, 12, 16, 20	2, 4, 9	8, 10, 11, 15, 19, 22	4, 5, 12, 16, 17, 20, 23	2, 4, 9, 10, 13, 14, 23
Total # of items	7 items	3 items	6 items	7 items	7 items

PCA items were grouped by function concept to analyze how students from each curriculum path performed on the five concepts. Table 4.13 displays the average percent of correct answers, without adjusting for prior achievement, for the five concepts assessed on the *PCA* by curricular paths. SSC students tended to answer function evaluation, solving functions, and function composition items correctly; however, IC students did not score higher than 50% on any of the five concept groups. AP Calculus students from both curricular paths tended to answer questions from each of the concepts similarly with function evaluation, solving functions, and function composition being areas that were not as difficult as rates of change and function inverse items.

Table 4.13

Mean percent of correct answers for function concepts assessed on PCA for high school students who studied four years of college preparatory mathematics (integrated or single-subject)

Curriculum	# of *PCA* Items	7	3	6	7	7
	N	Function Evaluation	Solving Functions	Rates of Change	Function Composition	Inverse Function
IC	201	45.20	47.10	38.31	34.83	30.49
SSC	304	65.18	62.39	41.17	56.00	37.10
APIC	59	63.44	64.97	45.48	52.06	41.40
APSSC	140	74.80	74.52	49.17	66.73	49.90

*Percents are not adjusted for prior achievement

Comparison on PCA Function Concepts

To determine if there was a statistical difference between the mean percent of correct answers for the five function concepts assessed on the *PCA* a one-way multivariate analysis with a covariate (MANCOVA) was conducted for both samples using the *IAAT* as the covariate.

All students. The Wilks' Lambda was significant ($F = 9.681$, $p = .000$) among the function concepts for students who studied four years of college preparatory mathematics (integrated or single-subject). Therefore, five ANOVAs were conducted to determine which function concepts were significantly different for the two curricula paths. The resulting analysis, adjusted for *IAAT* scores, on *PCA* items by the five function concepts is displayed in Table 4.14. The F for function evaluation ($F = 23.636$, $p = .000$) and function composition ($F = 29.974$, $p = .000$) were found to be significant in favor of SSC students. No significance was found for the other three function concepts: solving functions ($F = 2.482$, $p = .116$), rates of change ($F = 1.537$, $p = .216$), or function inverse ($F = 0.028$, $p = .867$). SSC students performed statistically higher than IC students on *PCA* items that required function evaluation and function composition.

Table 4.14

Adjusted mean percents of correct answers for function concepts assessed on PCA for high school students who studied four years of college preparatory mathematics (integrated or single-subject) using IAAT scores

Function Concept	# of *PCA* items	Mean[a] (S.E.) IC (N=142)	Mean[a] (S.E.) SSC (N=174)	F	p
Function Evaluation	7	48.43 (1.65)	60.73 (1.84)	23.636*	.000
Solving Functions	3	52.26 (2.35)	57.93 (2.61)	2.482	.116
Rates of Change	6	41.37 (1.48)	38.56 (1.83)	1.537	.216
Function Composition	7	38.93 (1.64)	52.71 (1.83)	29.974*	.000
Function Inverse	7	34.18 (1.56)	34.58 (1.73)	0.028	.867

[a]Covariates appearing in the model were evaluated at the following values: *IAAT* = 65.37.
*SSC students mean score was significantly higher than IC students

AP Calculus students. The Wilks' Lambda was significant (F = 2.391, p = .042) among the function concepts for AP Calculus students. Therefore, five ANOVAs were conducted to determine which concepts were significantly different for the two curricula paths. The resulting analysis, adjusted for *IAAT* scores, on *PCA* items for the AP Calculus students is shown in Table 4.15. The F for function evaluation (F = 5.256, p = .024) and function composition (F = 7.132, p = .009) were found to be significant in favor of APSSC students. No significance was found for the other three content areas: solving functions (F = 1.544, p = .217), rates of change (F = 0.261, p = .610), or function inverse (F = 0.047, p = .828). APSSC students performed statistically higher than APIC students on *PCA* items that required function evaluation and function composition.

Table 4.15

Adjusted mean percents of correct answers for function concepts assessed on PCA for AP Calculus students who studied four years of college preparatory mathematics (integrated or single-subject) using IAAT scores

Function Concept	# of *PCA* items	Mean[a] (S.E.) APIC (*N*=40)	Mean[a] (S.E.) APSSC (*N*=76)	*F*	*p*
Function Evaluation	7	65.98 (3.16)	75.05 (2.27)	5.256*	.024
Solving Functions	3	69.77 (4.57)	76.88 (3.27)	1.544	.217
Rates of Change	6	50.35 (3.25)	48.28 (2.33)	0.261	.610
Function Composition	7	57.18 (3.25)	68.03 (2.33)	7.132*	.009
Function Inverse	7	46.79 (3.40)	45.86 (2.44)	0.047	.828

[a]Covariates appearing in the model were evaluated at the following values: *IAAT* = 79.85.
*APSSC students mean score was significantly higher than APIC students

Summary

PCA items were coded and analyzed by difficulty level (easy, moderate, and difficult) and function concepts (evaluation, solving, rates of change, function composition, function inverse). After adjusting for prior achievement, IC and SSC students did not perform statistically different on *PCA* items that assessed solving functions, computing or interpreting rates of change, and interpreting inverse functions; however, SSC students performed statistically higher on items that assessed evaluating functions and working with composite functions. IC students had a higher mean than SSC students on items assessing rates of change.

After adjusting for prior achievement, APIC and APSSC students did not perform statistically different on *PCA* items that assessed solving functions, computing or interpreting rates of change, and interpreting inverse functions; however, APSSC students performed statistically higher on items that assessed evaluating functions and working with composite functions. APIC students had a higher mean than APSSC students on items related to rates of change and inverse functions.

After adjusting for prior achievement, all four groups of students (IC, SSC, APIC, and APSSC) tended to incorrectly answer *PCA* items that assessed rates of change and inverse functions. IC and APIC students also tended to incorrectly answer items related to composition of functions.

Misconceptions by Curricula

To determine if students had misconceptions on a *PCA* item, this general rule was used: if more than 20% of the students in a specific curricula group chose a specific incorrect answer, then it would be considered a misconception. This guideline was used because it was expected with five answer choices that at least 20% of students could guess any answer choice. Therefore, greater than 20% would suggest the answer choice was not guessed, but thought to be true because of some misconception students held about the question. Readers are reminded that students were found to be statistically different in mathematical achievement levels prior to entering their specific curricular paths. Therefore, the results on misconceptions need to be interpreted with caution. It may be the case that SSC or APSSC students did not manifest misconceptions because they began the study with a higher level of mathematics achievement and understanding. Therefore, it can be anticipated that IC and APIC students may display more misconceptions because they scored significantly lower on prior achievement.

All students. Table 4.16 displays each *PCA* item with the level of difficulty and the percent of students (IC, SSC, APSSC, and APIC) who answered the questions correctly. SSC students scored higher on 22 of the 25 *PCA* items, while IC students scored higher on the remaining 3 items. Nine items (7, 8, 10, 11, 13, 15, 18, 19, 23) had less than 10% difference between the percent of students in each curriculum pathway who answered correctly. Six items (1, 2, 9, 14, 21, 25) had between 10% to 19% difference. Finally, ten items (3, 4, 5, 6, 12, 16, 17, 20, 22, 24) ranged between 20% to 30% difference.

Table 4.16

Percentage of all students and AP Calculus students who answered each PCA item correct by curricula

Item #	Difficulty Level	% of ALL students		% of AP Calculus students	
		IC	SSC	APIC	APSSC
1	Easy	70	87	83	91
2	Hard	37	51	51	63
3	Moderate	54	74	66	89
4	Hard	37	59	56	73
5	Moderate	40	63	68	80
6	Moderate	44	70	70	85
7	Moderate	71	62	76	78
8	Hard	15	16	17	21
9	Moderate	67	77	89	88
10	Hard	33	29	32	35
11	Hard	15	15	19	21
12	Moderate	46	71	76	83
13	Hard	20	11	34	13
14	Hard	7	17	14	24
15	Hard	39	44	54	56
16	Easy	70	98	90	99
17	Hard	8	34	20	50
18	Easy	82	90	88	93
19	Easy	79	81	93	91
20	Hard	31	51	39	64
21	Hard	16	35	22	44
22	Hard	24	45	36	59
23	Hard	12	16	15	19
24	Moderate	48	70	66	81
25	Moderate	50	64	61	74

*Percents are not adjusted for prior achievement

AP Calculus students. APSSC students scored higher on 22 of the 25 *PCA* items, while APIC students scored higher on the remaining 3 items. Item 13 was answered similarly for both samples (IC/SSC and APIC/APSSC) with IC and APIC students having a higher percentage of correct answers. However, results were mixed for the two samples on items 9 and 19. SSC students performed higher than IC students on these two items, but APIC students performed higher than APSSC students on these same two questions.

Performance on twelve items (1, 7, 8, 9, 10, 11, 12, 15, 16, 18, 19, 23) differed by less than 10% across students in each curriculum pathway. Six items (3, 13, 17, 20, 21, 22) were found to have a difference range between 21% to 29%.

Misconception Analysis

As stated previously, if more than 20% of students in a specific curricula group chose an incorrect answer, then the response was considered to be a misconception. Students exhibited misconceptions on different function content on 18 of the 25 *PCA* items.

All students. A summary of the difficulty level and the function content assessed for each *PCA* item is summarized in Table 4.17. In addition, the group of students (integrated or single-subject) who displayed a misconception (20% or more of students) on the item without adjusting for prior achievement is noted. After analyzing all the answer choices, it was found that SSC students demonstrated at least one misconception on 40% of the 25 items. IC students exhibited at least one misconception on 72% of the 25 items.

PCA items were sorted into three groups according to misconceptions: (1) items with no misconceptions for either group of students, (2) items missed by at least 20% of IC students, (3) items missed by at least 20% of both IC and SSC students. Students from neither curriculum pathway displayed misconceptions on seven items (1, 7, 9, 16, 18, 19, 25). Students from both curricula pathways seem to understand the behavior of functions whether given an equation or graph. Both groups of students were successful interpreting and using function notation to evaluate a given function both symbolically and graphically. Furthermore, students interpreted and described the rate of change of an exponential function.

Table 4.17

Summary of difficulty level and group who displayed misconceptions on PCA items for all students

Item #	Difficulty Level	Function Concept Assessed	Group*
1	Easy	Evaluation (symbolic)	-
2	Hard	Solve/Function inverse (graphic)	IC
3	Moderate	Proportions	IC
4	Hard	Solve/Inverse/Function composition (symbolic)	IC
5	Moderate	Evaluation/ Function composition (graphic)	IC
6	Moderate	Evaluation (graphic)	IC
7	Moderate	Exponential function (symbolic)	-
8	Hard	Rate of change (graphic)	IC/SSC
9	Moderate	Solve/Function inverse (graphic)	-
10	Hard	Rate of change/ Function inverse (graphic)	IC/SSC
11	Hard	Evaluation/Rate of change (symbolic)	IC/SSC
12	Moderate	Evaluation/Function composition (numeric)	IC
13	Hard	Function inverse (numeric)	IC/SSC
14	Hard	Function inverse (symbolic)	IC/SSC
15	Hard	Rate of change	IC/SSC
16	Easy	Evaluation/ Function composition (symbolic)	-
17	Hard	Function composition (symbolic)	IC/SSC
18	Easy	Rational function (symbolic)	-
19	Easy	Rate of change (graphic)	-
20	Hard	Evaluation/Function composition	IC
21	Hard	Domain (symbolic)	IC/SSC
22	Hard	Rate of change (symbolic)	IC/SSC
23	Hard	Composition/Function inverse	IC/SSC
24	Moderate	End behavior (graphic)	IC
25	Moderate	End behavior (symbolic)	-

*Not adjusted for prior achievement

Only IC students displayed misconceptions on the following eight items: 2, 3, 4, 5, 6, 12, 20, 24. These eight items are characterized by four misconceptions. The first misconception appeared for 20% to 24% of IC students. Students were not able to interpret or evaluate symbolic function notation properly, which may be the reason why they translated to a graph or words incorrectly. For example, students were given $h(x) = 6$ and asked to solve for x. Approximately 42% of IC students used the 6 as the x coordinate and found the y-coordinate or the point (x, y) as the answer.

A second misconception appeared for more than 25% of IC students. They struggled to interpret and work with composite functions across different representations (symbolic, graphic, and numeric). IC students did not seem to recognize that a composition of functions required them to evaluate two functions at different points. For example, students were given $h(r(-3))$ and asked to answer using a given graph. IC students evaluated the first function correctly, but then moved vertically to the h function reporting this new point as their answer instead of using the output value as the new input for the h function.

The third misconception arose for 33% of IC students. Students were to use multiplicative reasoning on a proportional reasoning item; however, the answer the IC students incorrectly selected used additive reasoning.

The fourth misconception appeared for 29% of IC students. Students were asked to describe the behavior of a rational function and the incorrect answer selected only reflected what happened when x increased and ignored what happen when x approached 0.

Students from both curricula had misconceptions on the remaining ten items (8, 10, 11, 13, 14, 15, 17, 21, 22, 23). These ten items are characterized by four misconceptions. The first misconception appeared for 20% to 63% of IC students and 24% to 74% of SSC students. Students worked with inverse functions incorrectly. The misconception that both groups of students demonstrated was confusing function inverse and multiplicative inverse or defining an inverse to be 1 over the function. For example, students were given a table of values and asked to evaluate $h^{-1}(5)$. Students from both integrated (38%) and single-subject (66%) incorrectly chose the answer to be $\frac{1}{h(5)}$.

A second misconception appeared for 20% to 60% of IC students and 27% to 66% of SSC students. Students interpreted rates of change incorrectly whether given a graph or an equation. The misconceptions that both groups of students demonstrated were iconic translation between a picture and a graph, relating average velocity to rate of change, and the meaning of a multiplicative statement. For example, students were to describe the relationship between two cars given a graph of both cars speed. Fifty-nine percent of IC students and 66% of SSC students either interpreted the graph as a position graph, recognized that the graph was speed, but could not overcome the iconic translation, or they figured the average speed between the start and end was equivalent for both cars concluding that the cars should be at the same position at the end of the drive.

The third misconception appeared for 45% of SSC students and 46% of IC students. Students had to write a composite function for the area of a circle as a function of time. Both IC and SSC students did not include the speed as part of the radius, thus not squaring the speed. However, SSC students displayed an additional misconception, continually adding the speed to the radius or forgetting to square the time when completing the composition of functions. As described above, students worked with one function correctly, but not with two functions.

The fourth misconception appeared for 46% of IC students and 60% of SSC students. Students had to find the domain of a composite function that was both rational and contained a radical. These students were able to identify that the domain needed to be restricted so the denominator would not equal 0; however, they did not recognize that the numerator also needed a restriction because it was a square root.

In summary, on five *PCA* items (8, 14, 15, 21, 23) students from both curricula had the same misconception(s). The remaining six items both groups had a similar misconception, but one group of students (integrated or single-subject) had an additional misconception. For example, on one item students were required to interpret the meaning of the slope in a contextual situation. Twenty-seven percent of SSC students and 34% of IC students demonstrated a misunderstanding of the multiplicative meaning of five times greater. However, another 20% of IC students did not recognize equivalent statements in describing the slope. Although students from both curricula had a common misconception on this item, there was an additional misconception for IC students. Another example of this was on an item that students were required to use a function to find the average velocity over a specific time period. Forty-six percent of the SSC students and 49% of IC students found the average velocity between four integer values. Although students from both curricula had a common misconception, 23% of the SSC students exhibited an additional misconception, namely these students found the average between the start and end values.

AP Calculus students. A summary of the *PCA* items, the difficulty level, the function concept being assessed, and the group of AP Calculus students (integrated or single-subject) who exhibited misconceptions without adjusting for prior achievement is found in Table 4.18. After analyzing all the answer choices, it was found that APSSC students had misconceptions on 36% of the 25 items. The APIC students had misconceptions on 64% of the 25 items.

Table 4.18

Summary of difficulty level and group who displayed misconceptions on PCA items for AP Calculus students

Item #	Difficulty Level	Mathematical Content	Groups*
1	Easy	Evaluation (symbolic)	-
2	Hard	Solve/Function inverse (graphic)	-
3	Moderate	Proportions	APIC
4	Hard	Solve/Inverse/Function composition (symbolic)	APIC
5	Moderate	Evaluation/ Function composition (graphic)	APIC
6	Moderate	Evaluation (graphic)	APIC
7	Moderate	Exponential function (symbolic)	-
8	Hard	Rate of change (graphic)	APIC/APSSC
9	Moderate	Solve/Function inverse (graphic)	-
10	Hard	Rate of change/ Function inverse (graphic)	APIC/APSSC
11	Hard	Evaluation/Rate of change (symbolic)	APIC/APSSC
12	Moderate	Evaluation/Function composition (numeric)	-
13	Hard	Function inverse (numeric)	APIC/APSSC
14	Hard	Function inverse (symbolic)	APIC/APSSC
15	Hard	Rate of change	APIC/APSSC
16	Easy	Evaluation/ Function composition (symbolic)	-
17	Hard	Function composition (symbolic)	APIC/APSSC
18	Easy	Rational function (symbolic)	-
19	Easy	Rate of change (graphic)	-
20	Hard	Evaluation/Function composition	APIC
21	Hard	Domain (symbolic)	APIC/APSSC
22	Hard	Rate of change (symbolic)	APIC
23	Hard	Composition/Function inverse	APIC/APSSC
24	Moderate	End behavior (graphic)	APIC
25	Moderate	End behavior (symbolic)	-

*Not adjusted for prior achievement

PCA items were sorted into three groups according to misconceptions: (1) items with no misconceptions for either group of students, (2) items missed by at least 20% of APIC students, and (3) items missed by at least 20% of both APIC and APSSC students. Many of the misconceptions discussed in the previous section for all students were found for AP Calculus students as well. However, two items (2, 12) that IC students demonstrated misconceptions on did not appear for APIC students and one item (22) that both IC and SSC students displayed misconceptions with did not appear as a misconception for APSSC students. Although the results presented for AP Calculus students are by curriculum, the focus is on those items that were different or revealed new misconceptions.

AP Calculus students from both curricula did not demonstrate misconceptions (20% or more of students) on nine items. On seven items (1, 7, 9, 16, 18, 19, 25) AP Calculus students displayed the same misconceptions as all students; however, AP Calculus students did not display misconceptions on items 2 and 12. The sample of IC students displayed misconceptions on these two items, yet AP Calculus students did not display these misconceptions. The sample of AP Calculus students in both curricula seemed to understand the behavior of functions whether given an equation or graph. They were successful interpreting and using function notation to evaluate functions shown both symbolically and graphically. Students were also able to interpret and describe the rate of change of an exponential function. In addition, AP Calculus students evaluated a composite function using a table correctly.

Only APIC students exhibited misconceptions on the following seven items: 3, 4, 5, 6, 20, 22, 24. The four misconceptions were described previously. The first misconception, not interpreting symbolic function notation properly, may have lead students to translate to a graph or words incorrectly, and appeared for 22% to 24% of APIC students.

The second misconception, interpreting and working with composite functions across different representations (symbolic, graphic, and numeric) appeared for 22% to 24% of APIC students. These students did not seem to recognize that a composition of functions required them to evaluate two functions at different points.

The third misconception, using additive instead of multiplicative reasoning, arose for 25% of APIC students. Students did not recognize that the proportionality problem required a multiplicative relationship.

The fourth misconception, only acknowledging what happened with a rational function as x increased and ignoring what happens when x approached 0, appeared for 22% of APIC students.

Item 22 displayed misconceptions for all students in both curricula in the earlier analysis. However, when analyzing the sample of AP Calculus students, only APIC students struggled with this item. A quarter of APIC students displayed the misconception of not understanding the multiplicative meaning of five times greater when analyzing rate of change of a linear function. APSSC students did not display this misconception.

Students from both curricula had misconceptions on the remaining nine items (8, 10, 11, 13, 14, 15, 17, 21, 23). These items were described by four misconceptions. The first misconception, confusing function inverse and multiplicative inverse or defining an inverse to be 1 over the function, appeared for 27% to 71% of APIC students and 52% to 73% of APSSC students.

The second misconception involved an iconic translation between a picture and a graph. Students did not relate average velocity to rate of change, or did not understand the meaning of a multiplicative statement. This misconception appeared for 20% to 56% of APIC students and 29% to 61% of APSSC students.

The third misconception arose on item 17, yet each group of students (APIC and APSSC) displayed different misconceptions from each other as well as from the sample of all students. On the contextual problem students were to write a composite function with area of a circle as a function of time. APIC students (39%) did not include the speed as part of the radius, thus not squaring the speed and 20% of APIC students did not think the problem could be solved. On the other hand, APSSC students (25%) continually added the speed to the radius or forgot to square the time when completing the composition of functions.

The fourth misconception appeared when students recognized that the domain needed to be restricted so the denominator would not equal 0; however, they did not recognize that the numerator also needed a restriction because it was a square root. This misconception occurred with great frequency, as it was reflected by 50% of APSSC students and 58% of APIC students.

In summary, AP Calculus students from both curricula exhibited the same misconception on six items (10, 11, 13, 14, 15, 21). Both groups of students had similar misconceptions on two items (8, 23), which were described previously, but one group (integrated or single-subject) had an additional misconception. Finally, as discussed before, AP Calculus students in both groups had different misconceptions on one item (17).

Comparison on each PCA Item

A binomial logistic regression was conducted to estimate the odds that one group of students (integrated or single-subject) were better able to correctly answer each *PCA* item. The logistic regression model was used because the dependent variable (correct/incorrect) was dichotomous and the independent variables (*IAAT* and curriculum type) were continuous and categorical (Pedhazur, 1997).

The first step in the logistic regression model is to compute a Chi-square. If it was significant, then this indicated that either one or both of the independent variables (*IAAT* and curriculum type) were significant factors in the logistic regression model. This is interpreted as the probability that the correct answer for the item can be predicted based on the students *IAAT* score and/or the curriculum type.

The second step in the logistic regression model is to investigate which of the independent values (*IAAT* and/or curriculum type) were factors that influenced the model. If the Wald statistic (Pedhazur, 1997) was statistically significant, this implied that both independent variables (*IAAT* and curriculum type) were significant factors in the logistic regression model. The odds ratio (*Exp(B)*) was interpreted as a ratio above 1 predicted that IC students had better odds than SSC students of answering the question correctly, while a ratio below 1 predicted that SSC students had better odds than IC students of answering the question correctly. The closer the odds ratio is to 1, the less the curriculum type (integrated and single-subject) affected the dependent variable (answering the question correctly/incorrectly). For example, the odds ratio for item 7 on the *PCA* for all students was 1.861, this was interpreted as the odds of correctly answering this item were 1.861 times as large for IC students as for SSC students.

All students. Table 4.19 shows the model Chi-square test values for all students. Twenty-two of the 25 *PCA* items had a significant Chi-squared value. This was interpreted as either one or both of the independent variables were factors that could contribute to the model. Table 4.20 displays each *PCA* items with the Wald statistic and the estimated odds ratio (*Exp(B)*) for the independent variable (curriculum type) for all students. Six items (3, 6, 12, 16, 17, 20) had Wald statistics that were significant $(p \leq .05)$ and the odds were greater for SSC students than for IC students to answer these questions correctly. Two other items (7, 13) had Wald statistics that were significant $(p \leq .02)$ and the odds were greater for IC students than for SSC students to answer these questions correctly.

AP Calculus students. Table 4.21 provides the model Chi-square test values for AP Calculus students. Eleven of the 25 *PCA* items had a significant Chi-square value. This was interpreted as either one or both of the independent variables were factors that could contribute to the model. Table 4.22 displays each *PCA* items with the Wald statistic and the estimated odds ratio (*Exp(B)*) for the independent variable (curriculum type) for AP Calculus students. Four items (6, 16, 17, 20) had Wald statistics that were significant $(p \leq .05)$ and the odds were greater for APSSC students than for APIC students to answer these questions correctly. One item (13) had a Wald statistic that was significant $(p \leq .002)$ and the odds were greater for APIC students than for APSSC students to answer this question correctly.

Table 4.19

Model Chi-square test for coefficients (IAAT and curriculum type) by item for all students

Item Number	Chi-square (χ^2)	*p*
1	15.81*	0.000
2	37.49*	0.000
3	72.22*	0.000
4	48.02*	0.000
5	56.89*	0.000
6	55.86*	0.000
7	14.95*	0.000
8	9.97*	0.000
9	32.58*	0.000
10	3.58	0.167
11	3.01	0.222
12	56.07*	0.000
13	12.75*	0.002
14	9.70*	0.008
15	34.02*	0.000
16	85.39*	0.000
17	70.67*	0.000
18	27.05*	0.000
19	39.76*	0.000
20	26.77*	0.000
21	23.52*	0.000
22	12.73*	0.002
23	3.26	0.196
24	32.56*	0.000
25	17.69*	0.000

* Significantly different at $p < 0.05$

Table 4.20

Logistic Regression for each item by curriculum type for all students

Item #[a]	B	S.E.	Wald	df	*p*	*Exp*(*B*)
1	-0.492	0.291	2.871	1	0.090	0.611
2	-0.228	0.250	0.829	1	0.363	0.796
3	-0.558	0.268	4.326*	1	0.038	0.573
4	-0.447	0.250	3.199	1	0.074	0.640
5	-0.447	0.253	3.125	1	0.077	0.640
6	-0.653	0.254	6.624*	1	0.010	0.521
7	0.621	0.266	5.464‡	1	0.019	1.861
8	0.209	0.332	0.396	1	0.529	1.232
9	-0.102	0.276	0.136	1	0.713	0.903
10	0.433	0.262	2.734	1	0.098	1.542
11	0.259	0.340	0.583	1	0.445	1.296
12	-0.706	0.247	7.552*	1	0.006	0.494
13	0.988	0.330	8.982‡	1	0.003	2.686
14	-0.657	0.375	3.076	1	0.079	0.518
15	0.211	0.255	0.688	1	0.407	1.235
16	-3.134	0.615	25.953*	1	0.000	0.044
17	-1.492	0.369	16.382*	1	0.000	0.225
18	-0.488	0.371	1.725	1	0.189	0.614
19	0.412	0.343	1.441	1	0.230	1.510
20	-0.575	0.247	5.405*	1	0.020	0.563
21	-0.518	0.283	3.343	1	0.067	0.596
22	-0.312	0.243	1.647	1	0.199	0.732
23	-0.096	0.335	0.083	1	0.773	0.908
24	-0.436	0.386	1.279	1	0.258	0.647
25	-0.358	0.244	2.156	1	0.142	0.699

[a] Variable(s) entered: *IAAT* and Curriculum type (0 = single-subject and 1 = integrated).
* Odds of answering question correctly were significantly higher for SSC students at $p \leq 0.05$
‡ Odds of answering question correctly were significantly higher for IC students at $p \leq 0.02$

Table 4.21

Model Chi-square test for coefficients (IAAT and curriculum type) by item for AP Calculus students

Item Number	Chi-square (χ^2)	p
1	0.0983	0.612
2	2.97	0.227
3	22.90*	0.000
4	9.00*	0.011
5	2.48	0.290
6	4.35	0.114
7	4.46	0.108
8	4.25	0.119
9	1.42	0.492
10	4.92	0.085
11	0.92	0.631
12	2.05	0.359
13	10.68*	0.005
14	1.10	0.577
15	6.31*	0.043
16	6.56*	0.038
17	26.35*	0.000
18	10.77*	0.005
19	6.95*	0.031
20	10.08*	0.006
21	5.26	0.072
22	13.79*	0.001
23	4.01	0.135
24	7.31*	0.026
25	2.22	0.330

* Significantly different at $p < 0.05$

Table 4.22

Logistic Regression for each item by curriculum type for AP Calculus students

Item #[a]	B	S.E.	Wald	df	*p*	*Exp*(*B*)
1	-0.130	0.593	0.048	1	0.826	0.878
2	-0.525	0.415	1.601	1	0.206	0.591
3	-1.035	0.546	3.599	1	0.058	0.355
4	-0.323	0.485	0.500	1	0.480	0.724
5	-0.614	0.462	1.768	1	0.184	0.541
6	-1.069	0.524	4.164*	1	0.041	0.343
7	-0.288	0.485	0.352	1	0.553	0.750
8	0.054	0.505	0.011	1	0.915	1.056
9	-0.243	0.606	0.161	1	0.688	0.784
10	0.189	0.442	0.183	1	0.669	1.208
11	.0236	0.516	0.210	1	0.647	1.267
12	-0.137	0.563	0.059	1	0.808	0.872
13	1.520	0.490	9.642‡	1	0.002	4.573
14	-0.518	0.513	1.017	1	0.313	0.596
15	0.280	0.422	0.441	1	0.506	1.323
16	-2.286	1.140	4.024*	1	0.045	0.102
17	-1.266	0.483	6.880*	1	0.009	0.282
18	-1.467	0.874	2.816	1	0.093	0.231
19	1.917	1.205	2.532	1	0.112	6.801
20	-1.056	0.419	6.344*	1	0.012	0.348
21	-0.388	0.433	0.801	1	0.371	0.679
22	-0.552	0.446	1.536	1	0.215	0.576
23	0.306	0.520	0.347	1	0.556	1.358
24	-18.388	4555.082	0.000	1	0.997	0.000
25	-0.421	0.435	0.939	1	0.333	0.656

[a] Variable(s) entered: *IAAT* and Curriculum type (0 = single-subject and 1 = integrated).
* Odds of answering item correctly were significantly higher for APSSC students at $p \leq 0.05$
‡ Odds of answering item correctly are significantly higher for APIC students at $p = 0.002$

Summary of PCA Item Analysis and Misconceptions about Functions

The difficulty level of the *PCA* items ranges from easy (4 questions) to difficult (16 questions) with the remaining items (8 questions) coded as moderate. SSC students scored higher than IC students on 22 of the 25 *PCA* items. The three *PCA* items that IC students scored higher on were either difficult or moderate in difficulty. APSSC students answered a higher percentage of *PCA* items correctly than APIC students.

After adjusting for prior achievement, SSC and APSSC students performed statistically higher on *PCA* items that assessed evaluating functions and working with composite functions. Students from all four groups (IC, SSC, APIC, APSSC) tended to incorrectly answer questions related to interpreting rates of change and inverse functions.

SSC students had fewer misconceptions because they outperformed IC students; however, on 10 of the 25 items both groups of SSC and IC students displayed common misconceptions. These common misconceptions were categorized by four mathematical concepts. First, students struggled with inverse functions confusing it with multiplicative inverse. Second, students struggled with interpreting the meaning of rate of change on a graph. Third, students struggled with using or creating composite functions symbolically. Fourth, students struggled to find the domain of a composite function.

IC students had three additional misconceptions that immerged from 8 *PCA* items. These misconceptions were: evaluating function notation symbolically, interpreting a composite function, and interpreting rates of change. Although the APSSC and APIC students had fewer misconceptions on *PCA* items than the SSC and IC students, the misconceptions that did appear were similar. These results must be interpreted with caution because prior achievement was higher for SSC and APSSC students.

After adjusting for prior achievement, SSC students had significantly higher odds than IC students of correctly answering six *PCA* items (3, 6, 12, 16, 17, 20). IC students had significantly higher odds than SSC students of correctly answering two *PCA* items (7, 13). APSSC students had significantly higher probability than APIC students of correctly answering four *PCA* items (6, 16, 17, 20), while APIC students had significantly higher probability than APSSC students of correctly answering one *PCA* item (13).

Open-ended Task Results by Curriculum Type

Two open-ended tasks were administered during the first week of class in August 2007 to the 207 high school students who entered AP Calculus AB or BC. Eight of these students did not take the *PCA* in the spring; therefore, they were excluded from the study. It was reported in earlier that 199 students had enrolled in AP Calculus for the 2007 school year; however, six of these students were absent the day the open-ended tasks were administered. Therefore data were collected on the 193 remaining students, 57 students completed four years of integrated mathematics curriculum (APIC) and 136 students completed four years of single-subject mathematics curriculum (APSSC).

Each open-ended task is discussed separately with both quantitative and qualitative results provided for students from each curriculum pathway. First, a quantitative analysis of performance on the open-ended tasks is presented. Next, a qualitative analysis describes the solution strategies students used and errors committed by curriculum pathway.

Overall Performance on Piecewise Function Task

The *Piecewise Function* task (Appendix A) contained 4 questions. Students were given a graph of a function and asked to determine the rate of change, write the piecewise function, and solve the function given a y-value. All questions were not worth the same number of points and the total number of points possible on the task was 24 (see Appendix B scoring rubric). The descriptive statistics for the total scores on the

Piecewise Function task for AP Calculus students are shown in Table 4.23. Both groups of AP students scored across the range of 0 to 24. One (2%) APIC student and 19 (14%) APSSC students received perfect scores on the *Piecewise Function* task. On the other hand, two (4%) APIC and one (1%) APSSC students scored 0 on this task.

Table 4.23

Descriptive statistics of total scores on the Piecewise Function task for AP Calculus students who studied four years of college preparatory mathematics (integrated or single-subject)

Curriculum Type	*N*	Mean*	SD	Range
APIC	57	14.18	6.56	0-24
APSSC	134	18.65	5.18	0-24

* Not adjusted for prior achievement

Figure 4.8 displays the total score distributions for AP Calculus students who studied four years of college preparatory mathematics (integrated or single-subject) on the *Piecewise Function* task. As noted APIC student scores are spread evenly across the graph while APSSC student scores are negatively skewed (to the left). Sixty-three percent of APSSC students scored between 19 and 24 points compared with 30% of APIC students who scored in the same range on the *Piecewise Function* task.

Performance on Individual Questions

As stated previously, each question was worth different points depending on what the students were required to do. Questions 1 and 2 required students to identify interval(s) on which the function had a specific rate of change and the greatest rate of change. These two questions were worth 7 points each. Question 3 required students to write the equation for the piecewise function given in the graph. This question was worth 6 points. Finally, question 4 required students to solve the function given a y-value. This was worth four points because three different x-values were required and students had to explain how they got their answer. The descriptive statistics for each question on the *Piecewise Function* task for AP Calculus students are shown in Table 4.24.

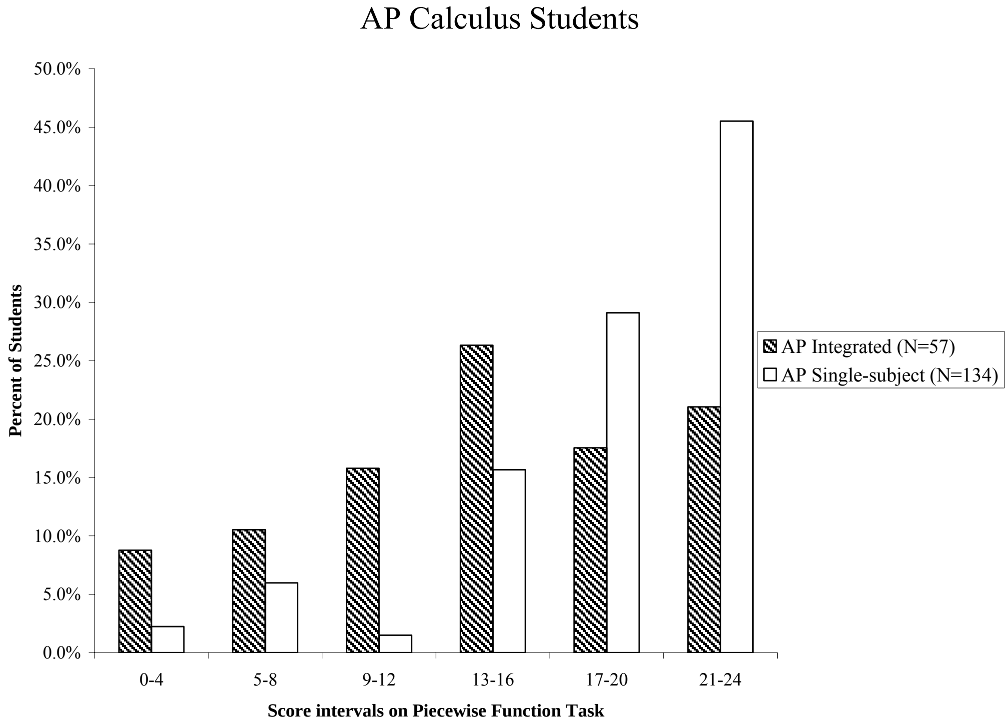


Figure 4.8. Score distributions on the *Piecewise Function* task for AP Calculus students who studied four years of college preparatory mathematics (integrated or single-subject).

Table 4.24

Descriptive statistics by question for the Piecewise Function task for AP Calculus students who studied four years of college preparatory mathematics (integrated or single-subject)

Question #	Points Possible	Mean* (SD) APIC (*N*=57)	Mean* (SD) APSSC (*N*=134)	Range
Question 1	7	5.37 (2.16)	6.33 (1.42)	0-7
Question 2	7	4.19 (2.70)	5.44 (2.16)	0-7
Question 3	6	2.32 (2.38)	4.01 (2.19)	0-6
Question 4	4	2.30 (1.79)	2.87 (1.58)	0-4

*Not adjusted for prior achievement

Figure 4.9 displays the score distributions on the four questions of the piecewise function task. At least 50% of APSSC students answered questions 1, 2, and 4 completely and correctly; whereas, less than 50% of APIC students answered each question completely and correctly. Both groups (integrated and single-subject) of students had difficultly with question 3 which required them to write equations for the piecewise function. More APIC students (67%) than APSSC students (41%) received a score of 3 or lower on question three.

A larger percentage of APIC students received zeros on all four questions than APSSC students. The largest percentage of APIC students (42%) who received a zero was on item 3. While the largest percentage of APSSC students (19%) who received a zero was on item 4.

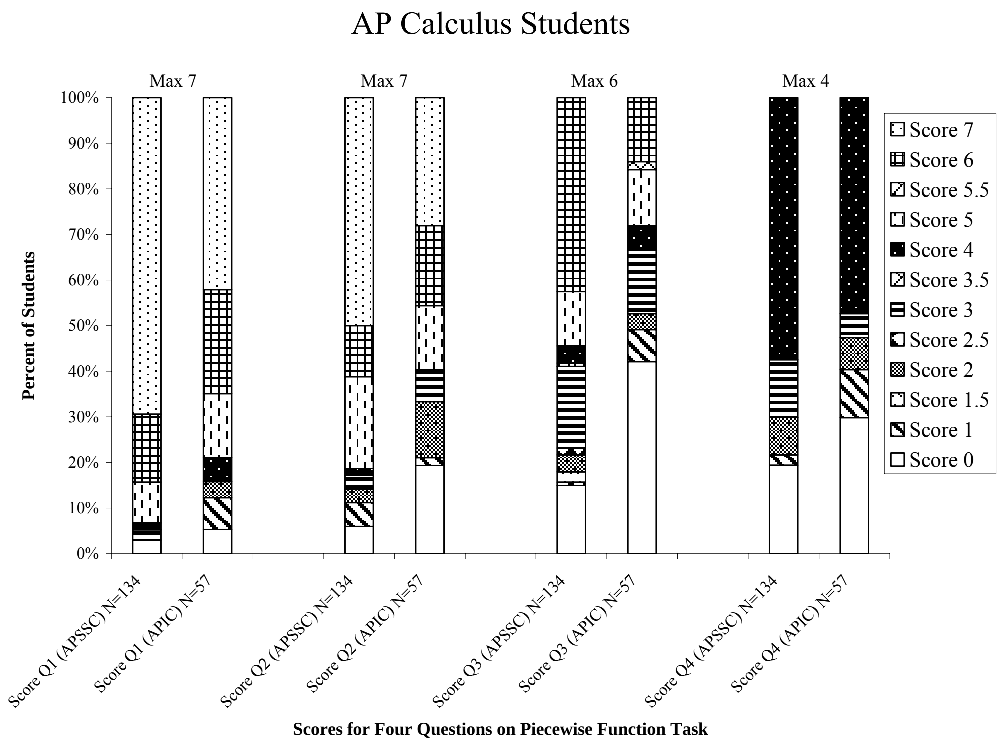


Figure 4.9. Score distributions on the four questions of the *Piecewise Function* task for AP Calculus students who studied four years of college preparatory mathematics (integrated or single-subject).

In an attempt to describe whether a score of zero meant a student had a misconception, meaning the question was attempted, or the student left the item blank, students' responses were coded as either attempted or blank. The criterion used was if the student offered any work, then the response was coded as attempted; whereas, if the student did not offer any work, then the response was coded as blank. Figure 4.10 displays the percentage of students from each curriculum who attempted or left the items blank. As seen from the graph APSSC students had a higher percentage of attempts on each of the four questions in the *Piecewise Function* task; whereas, APIC students had a higher percentage of blanks with the largest percentage on question 3.

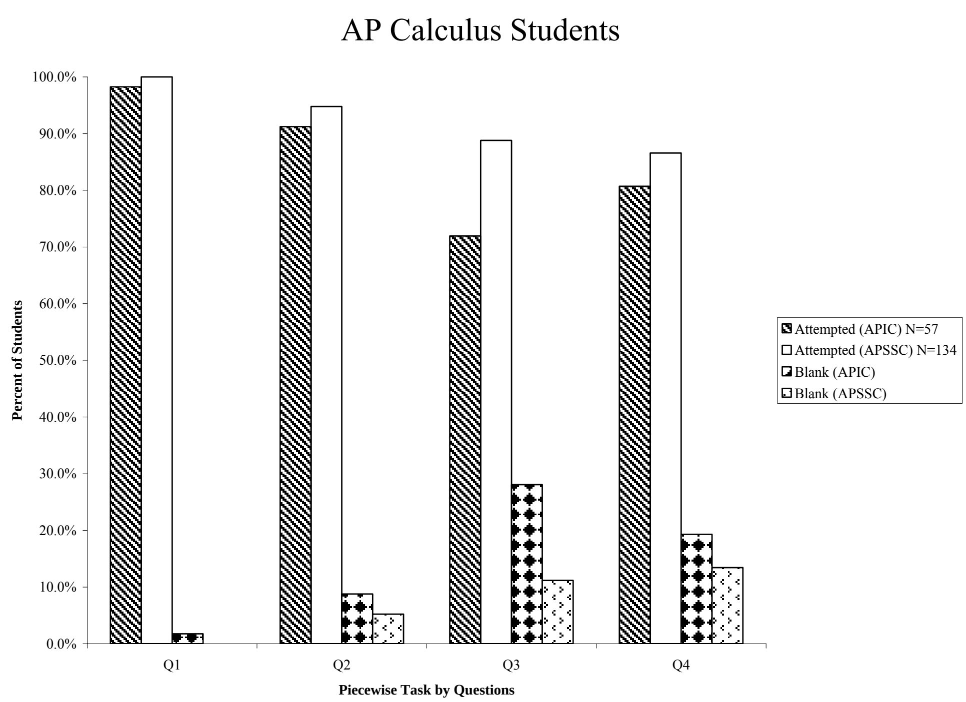


Figure 4.10. Distribution of attempted versus blank items on the *Piecewise Function* task for AP Calculus students who studied four years of college preparatory mathematics (integrated or single-subject).

Comparison on Piecewise Function Task

An analysis of covariance (ANCOVA) was conducted to determine the statistical difference between the mean overall score on the *Piecewise Function* task for the two groups (integrated and single-subject) of students adjusted for *IAAT* scores. The adjusted means for overall scores on this task by curriculum type for AP Calculus students are presented in Table 4.25. A statistically significant difference between the mean overall score on the *Piecewise Function* task for AP Calculus students who studied four years of college preparatory mathematics was found (F = 10.320, p = .002). APSSC students performed significantly higher overall on the *Piecewise Function* task than APIC students after adjusting for differences in prior achievement (*IAAT*).

Table 4.25

Adjusted descriptive statistics of total scores on Piecewise Function task for AP Calculus students who studied four years of college preparatory mathematics (integrated or single-subject) using IAAT scores

Curriculum path	*N*	Mean[a]	Std. Error	*F*	*p*
APIC	38	15.18	0.899	10.320*	.002
APSSC	74	18.77	0.636		

[a]Covariates appearing in the model were evaluated at the following values: *IAAT* = 80.21
*APSSC adjusted mean on *Piecewise Function* task score was significantly higher than APIC students.

Solution Strategies on Piecewise Function Task

After all *Piecewise Function* tasks were scored numerically, a qualitative coding was applied. This purpose was to determine what strategies students used to solve each of the four questions on the *Piecewise Function* task. This section provides an analysis of the different solution strategies students employed on the four questions.

Question 1. The first question required students to determine what interval(s) of x the function had a rate of change of 2 and then explain how they determined the interval(s). Four categories immerged from student work: *formula, graph, multiple*, and *cannot tell*. The *formula* category was used when students used the slope formula to justify their answer (see Figure 4.11).

1. For what interval(s) of *x* does *f(x)* have a rate of change of 2? Explain how you got your answer.

(2,4) $m = \frac{4-0}{2-0} = \frac{4}{2} = 2$ $m=2$
(0,0)

Interval x: [0,2]
To obtain my answer, I visually found the line which had a slope of 2 and solved to confirm my visual observation.

Figure 4.11. Example of student #1 work on the *Piecewise Function* task question 1 coded as formula.

The *graph* category was used when students referred to the graph as the line increased by 2 units on the y and 1 unit on the x (see Figure 4.12).

1. For what interval(s) of *x* does *f(x)* have a rate of change of 2? Explain how you got your answer.

$0 \le x \le 2$

When you look @ the graph, you can see that for each interval of x between 0 and 2, the graph goes up by 2. After x=2 however, a negative rate of change is seen.

Figure 4.12. Example of student #29 work on the *Piecewise Function* task question 1 coded as graph.

The *multiple* category was used when students used a combination of two or more strategies (see Figure 4.13).

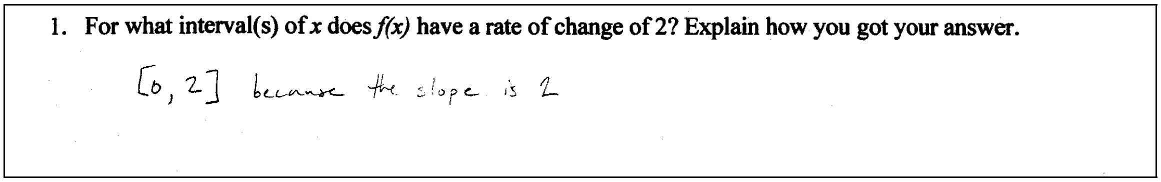
1. For what interval(s) of *x* does *f(x)* have a rate of change of 2? Explain how you got your answer.

rate change = 2 from (0,0) to (2,4) and (2,4) to (0,4)

by looking at the graph & determining the slope of each segment

Figure 4.13. Example of student #65 work on the *Piecewise Function* task question 1 coded as multiple strategies.

The final two categories, *cannot tell* and *blank* were used when student work was unable to be coded into one of the other three categories or it was left blank. Generally, students explained that the slope was 2, but it was not clear how the students found the rate of change (see Figure 4.14).


1. **For what interval(s) of *x* does *f(x)* have a rate of change of 2? Explain how you got your answer.**

[0, 2] because the slope is 2

Figure 4.14. Example of student #4 work on the *Piecewise Function* task question 1 coded as cannot tell.

Figure 4.15 displays the percent of students who used each strategy by curriculum pathway for question 1. Overall, AP Calculus students from both curricula (integrated and single-subject) used similar strategies to solve question 1. Almost 40% of students, APIC (36%) and APSSC (37%) used the slope formula to explain their reasoning. Another fourth of students, APIC (23%) and APSSC (28%), used the graph to explain their reasoning. Another third of students, APIC (37%) and APSSC (33%), did not provide enough information to code their strategies.

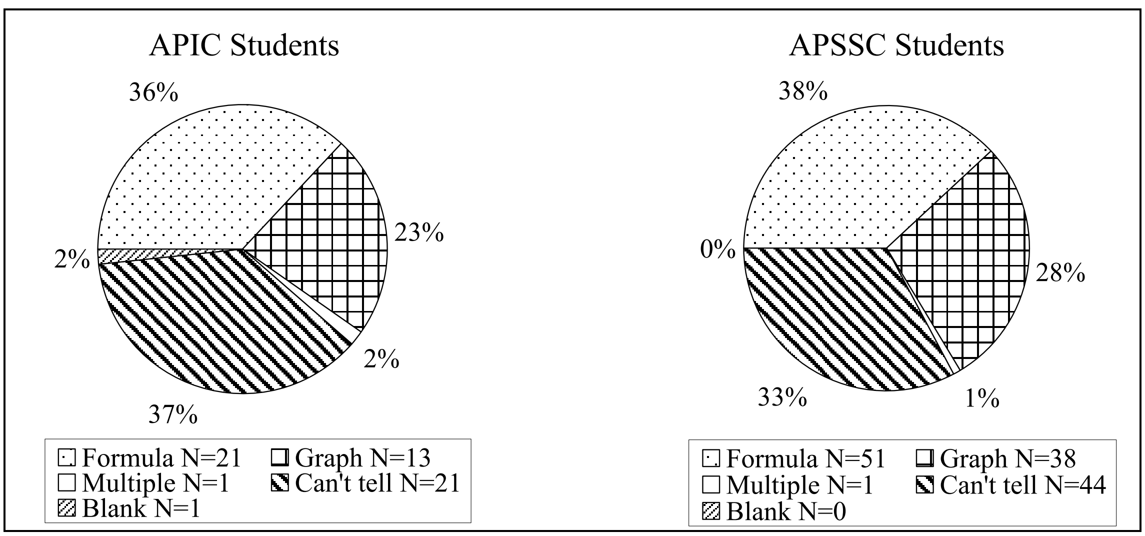


Figure 4.15. Solution strategies on question 1 of the *Piecewise Function* task for AP Calculus students who studied four years of college preparatory mathematics (integrated or single-subject).

Figure 4.16 displays the scores for students in each curriculum by their solution strategies for question 1. The graph does not include the *multiple* category because only one APSSC student and one APIC student used this method. Overall AP Calculus students were efficient using both the slope formula and the graph because no students received less than 4 points when using either of these strategies. Although AP Calculus students used similar strategies, students' success rate on question 1 using the three strategies was different. The APSSC students seem to use these strategies more efficiently than APIC students. The majority of APSSC students who used the slope formula or the graph strategy received perfect scores compared to a smaller percentage of APIC students who used these same strategies and received the same number of points.

AP Calculus Students

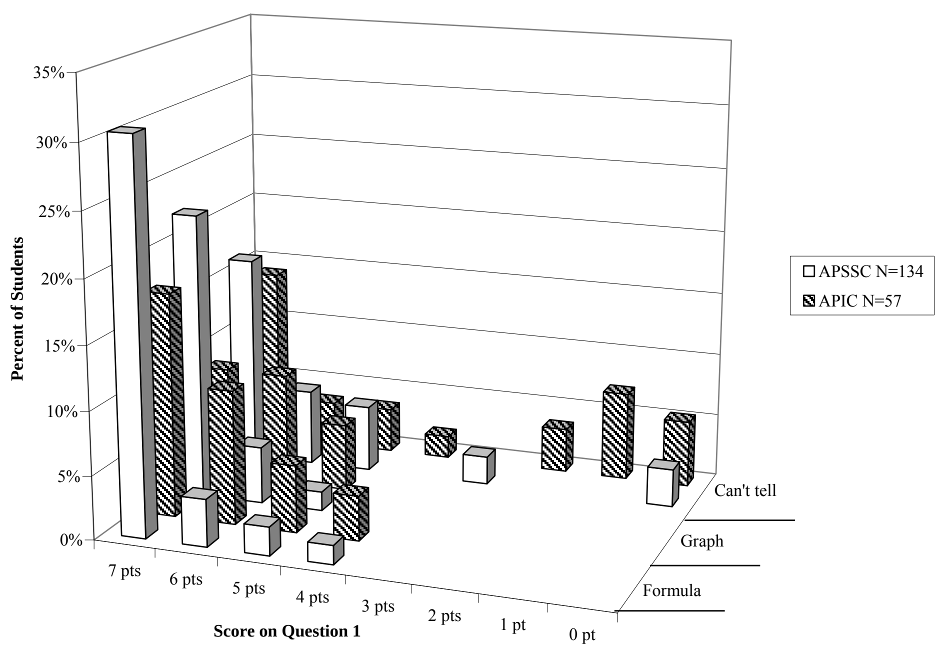
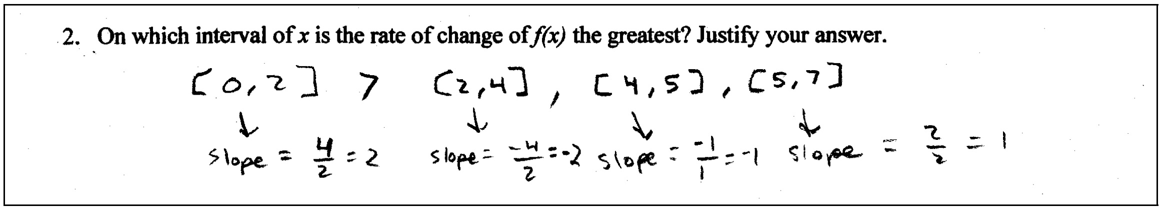


Figure 4.16. Solution strategies and scores on question 1 of the *Piecewise Function* task for AP Calculus students who studied four years of college preparatory mathematics (integrated or single-subject).

Question 2. The second question required students to determine what interval of x the function had the *greatest* rate of change and justify their answer. Five categories immerged from student work: *formula, graph, multiple*, *cannot tell,* and *blank*. The *formula* category was coded when students used the slope formula to justify their answer by showing that the only interval that had a slope of 2 was [0,2] (see Figure 4.17).

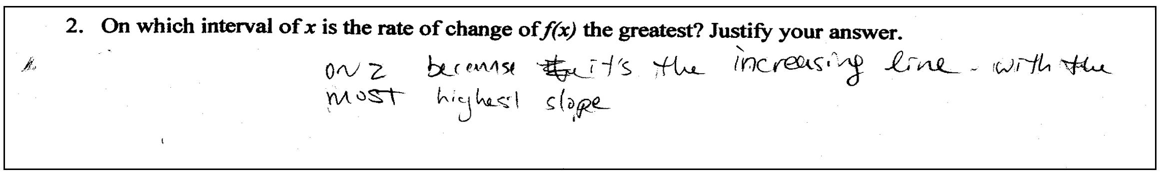

2. On which interval of x is the rate of change of $f(x)$ the greatest? Justify your answer.

[0,2] > [2,4], [4,5], [5,7]

slope = $\frac{4}{2} = 2$ slope = $\frac{-4}{2} = -2$ slope = $\frac{-1}{1} = -1$ slope = $\frac{2}{2} = 1$

Figure 4.17. Example of student #110 work on the *Piecewise Function* task question 2 coded as formula.

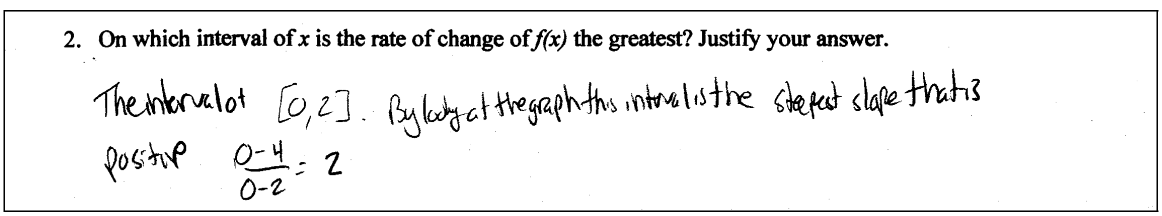
The *graph* category was used when students referred to the largest slope and/or an increasing line (see Figure 4.18).


2. On which interval of x is the rate of change of $f(x)$ the greatest? Justify your answer.

on 2 because ~~it~~ it's the increasing line - with the most highest slope

Figure 4.18. Example of student #71 work on the *Piecewise Function* task question 2 coded as graph.

The *multiple* category was coded when students used a combination of two or more strategies (see Figure 4.19).


2. On which interval of x is the rate of change of $f(x)$ the greatest? Justify your answer.

The interval of [0,2]. By looking at the graph this interval is the steepest slope that is postive $\frac{0-4}{0-2} = 2$

Figure 4.19. Example of student #134 work on the *Piecewise Function* task question 2 coded as multiple strategies.

The final two categories, *cannot tell* and *blank* were used when student work was unable to be coded into one of the other three categories or it was blank. Generally, students referred to the rate of change, but it was not clear whether they used the slope formula, the graph, or another method to find the rate of changes (see Figure 4.20).

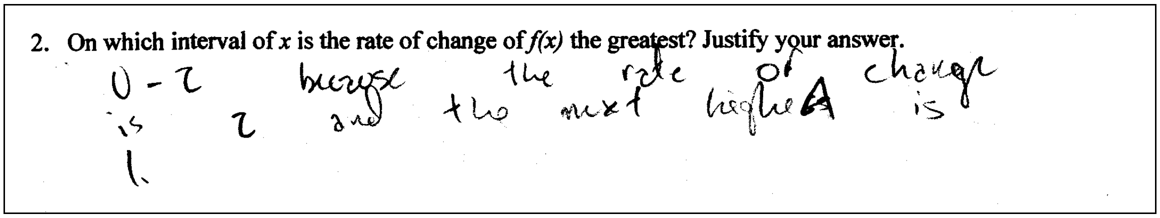


Figure 4.20. Example of student #31 work on the *Piecewise Function* task question 2 coded as cannot tell.

Figure 4.21 displays the percentage of students who used each strategy by curriculum pathway. Overall, AP Calculus students used similar strategies to solve question 2; however, the majority (79%) of students in either curricula did not write enough work to determine their strategy. Nine percent of APIC students and 12% of APSSC students used the slope formula to justify their answer. Four percent of both APIC and APSSC students used the graph to justify their answer.

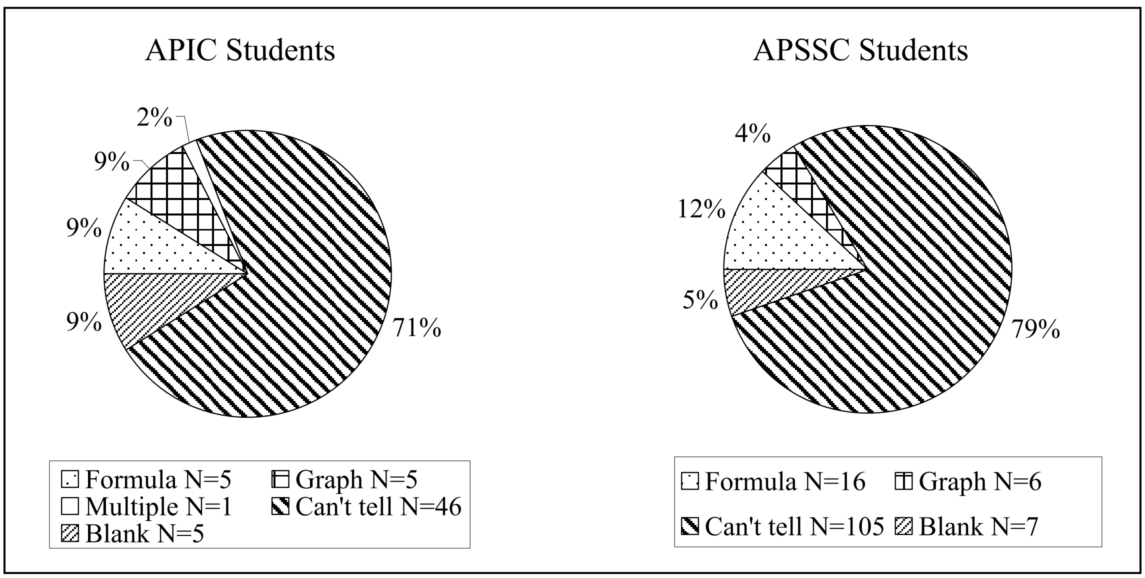


Figure 4.21. Solution strategies on question 2 of the *Piecewise Function* task for AP Calculus students who studied four years of college preparatory mathematics (integrated or single-subject).

Figure 4.22 displays the scores for students in each curriculum by their solution strategies for question 2. The graph does not include the *multiple* category because only one APIC student used this method. Although a small percentage of students used an identifiable strategy, their success rate on question 2 using these three strategies was different. Students in both groups who used the slope formula tended to use this strategy correctly. However, APSSC students who used the graph tended to score more points on question 2 than APIC students who used the same strategy. Finally, although the strategy was not identifiable, generally students were able to correctly answer the question. It may be the case that students used their answer and work from question 1 to help in answering question 2 and this may be why a solution strategy was not clearly identified.

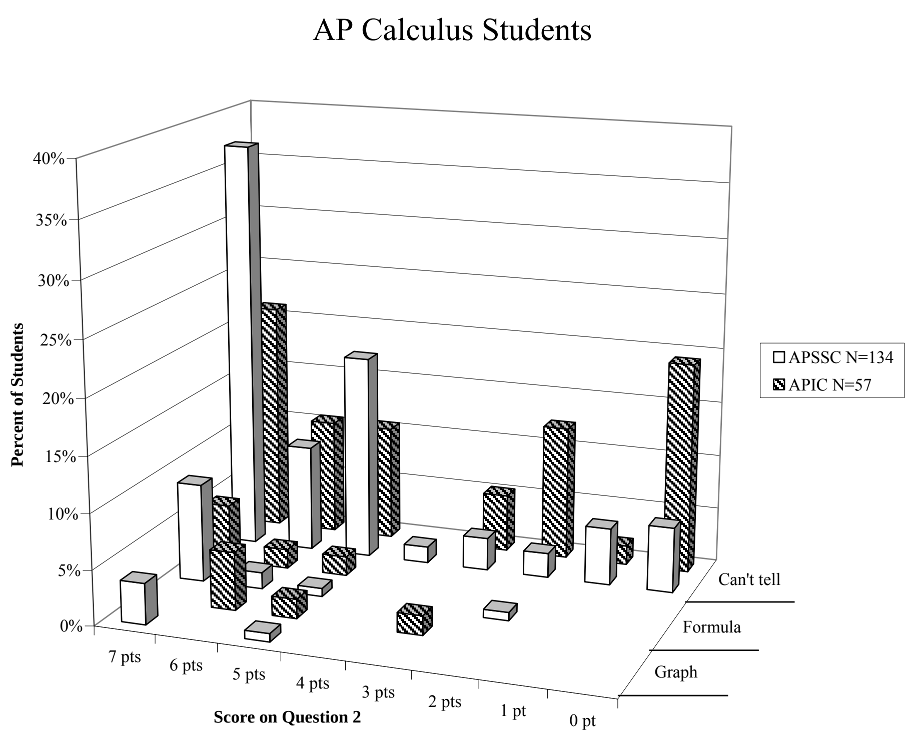
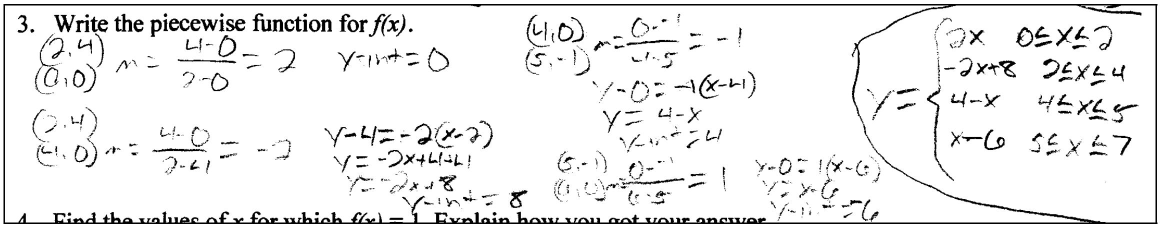


Figure 4.22. Solution strategies and scores on question 2 of the *Piecewise Function* task for AP Calculus students who studied four years of college preparatory mathematics (integrated or single-subject).

Question 3. The third question required students to write the piecewise function given in the graph. Five categories immerged from student work: *slope-intercept equation*, *slopes but no intercepts, transformation along the y-axis, cannot tell,* and *blank*. The *slope-intercept equation* category was coded when students wrote their equations using the slope-intercept form of a line (see Figure 4.23).



3. Write the piecewise function for $f(x)$.

$(2,4)$ $(0,0)$ $m = \frac{4-0}{2-0} = 2$ $y\text{-int} = 0$

$(2,4)$ $(4,0)$ $m = \frac{4-0}{2-4} = -2$ $y-4 = -2(x-2)$ $y = -2x+4+4$ $y = -2x+8$ $y\text{-int} = 8$

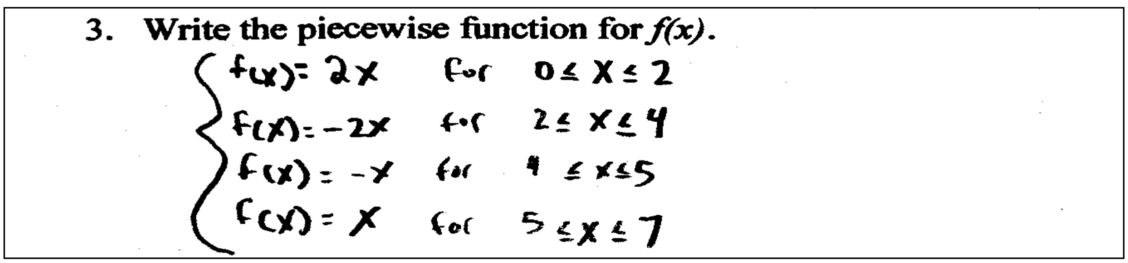
$(4,0)$ $(5,-1)$ $m = \frac{0--1}{4-5} = -1$ $y-0 = -1(x-4)$ $y = 4-x$ $y\text{-int} = 4$

$(5,-1)$ $(6,0)$ $m = \frac{0--1}{6-5} = 1$ $y-0 = 1(x-6)$ $y = x-6$ $y\text{-int} = -6$

$$y = \begin{cases} 2x & 0 \le x \le 2 \\ -2x+8 & 2 \le x \le 4 \\ 4-x & 4 \le x \le 5 \\ x-6 & 5 \le x \le 7 \end{cases}$$

Figure 4.23. Example of student #1 work on the *Piecewise Function* task question 3 coded as slope-intercept equation.

The *slopes but no intercepts* category was used when students found the slopes for each line and wrote an equation using the slopes, but did not take into consideration the y-intercept. (see Figure 4.24).

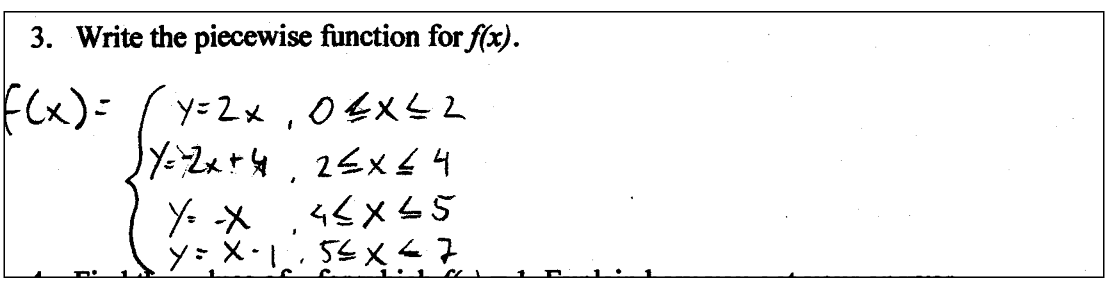


3. Write the piecewise function for $f(x)$.

$$\begin{cases} f(x) = 2x & \text{for } 0 \le x \le 2 \\ f(x) = -2x & \text{for } 2 \le x \le 4 \\ f(x) = -x & \text{for } 4 \le x \le 5 \\ f(x) = x & \text{for } 5 \le x \le 7 \end{cases}$$

Figure 4.24. Example of student #7 work on the *Piecewise Function* task question 3 coded as slopes but no intercepts.

The *transformation along the y-axis* category was used when students found the correct slope, but then used the y-value where the line started as the y-intercept for their equation (see Figure 4.25).

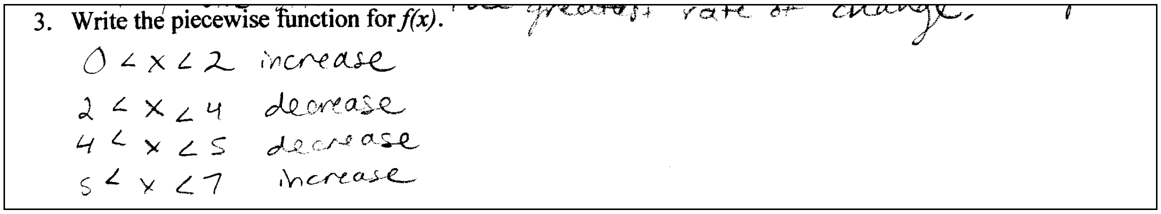


3. Write the piecewise function for $f(x)$.

$$f(x) = \begin{cases} y = 2x, & 0 \le x \le 2 \\ y = -2x+4, & 2 \le x \le 4 \\ y = -x, & 4 \le x \le 5 \\ y = x-1, & 5 \le x \le 7 \end{cases}$$

Figure 4.25. Example of student #167 work on the *Piecewise Function* task question 3 coded as transformation along the y-axis.

The final two categories, *cannot tell* and *blank* were used when student work was unable to be coded into one of the other three categories or it was blank. Generally, students left the question blank or completed only the restriction part of the question (see Figure 4.26).


3. Write the piecewise function for *f(x)*. greatest rate of change,

$0 < x < 2$ increase
$2 < x < 4$ decrease
$4 < x < 5$ decrease
$5 < x < 7$ increase

Figure 4.26. Example of student #29 work on the *Piecewise Function* task question 3 coded as cannot tell.

AP Calculus students used multiple strategies to solve question 3. Figure 4.27 displays the percentage of students who used each strategy by curricula. Sixty-two percent of APSSC students used the slope-intercept equation to write the equations for the piecewise function compared to 37% of APIC students. About the same percentage of students from both curricula pathways, APIC (18%) and APSSC (19%), found the correct slopes and used those for the equation; however, they did not include a y-intercept in their equations. More than twice the percentage of APIC students (12%) as APSSC students (5%) did not provide enough information to code a strategy. Similarly, more than twice the percentage of APIC students (28%) as APSSC students (11%) left the item blank.

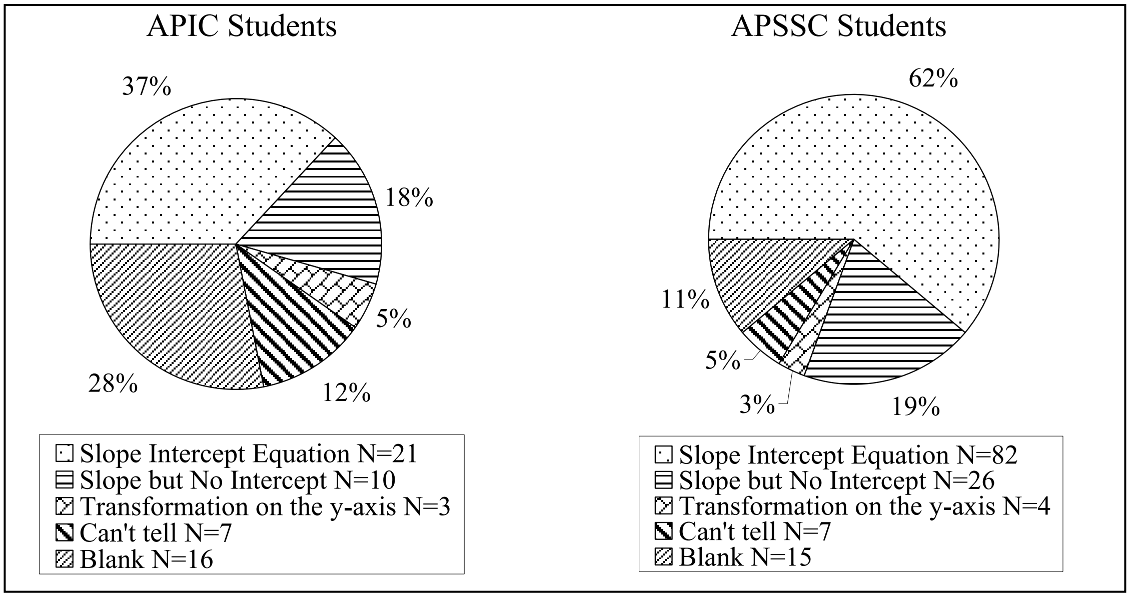


Figure 4.27. Solution strategies on question 3 of the *Piecewise Function* task for AP Calculus students who studied four years of college preparatory mathematics (integrated or single-subject).

Figure 4.28 displays the scores for students in each curriculum pathway by their solution strategies for question 3. The graph does not include the *transformation along the y-axis* category because only four APSSC students and three APIC students used this method. As might be expected, students from both curricula pathways were more successful when they used the slope-intercept equation. However, APSSC students tended to use this strategy more efficiently than APIC students as 43% of APSSC students received a perfect score on question 3 compared with 14% of APIC students.

AP Calculus Students

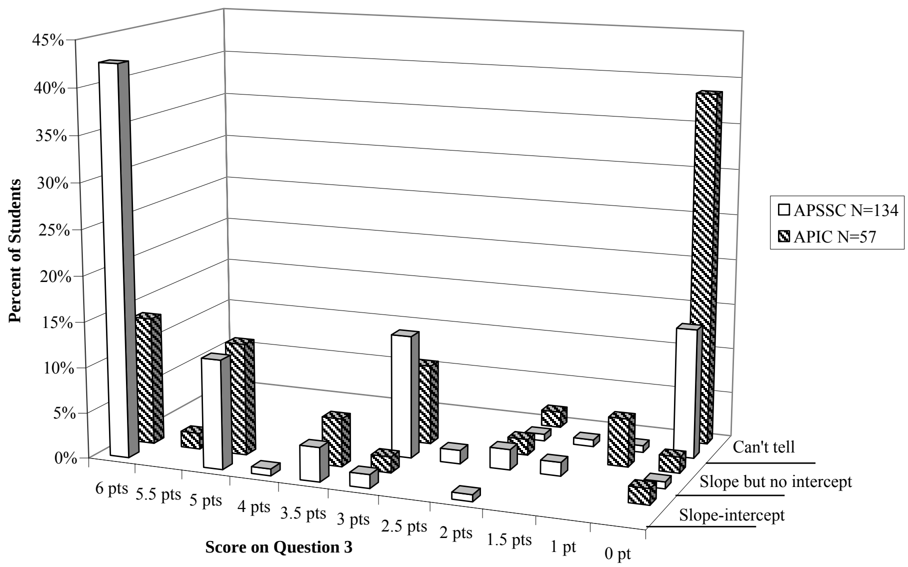
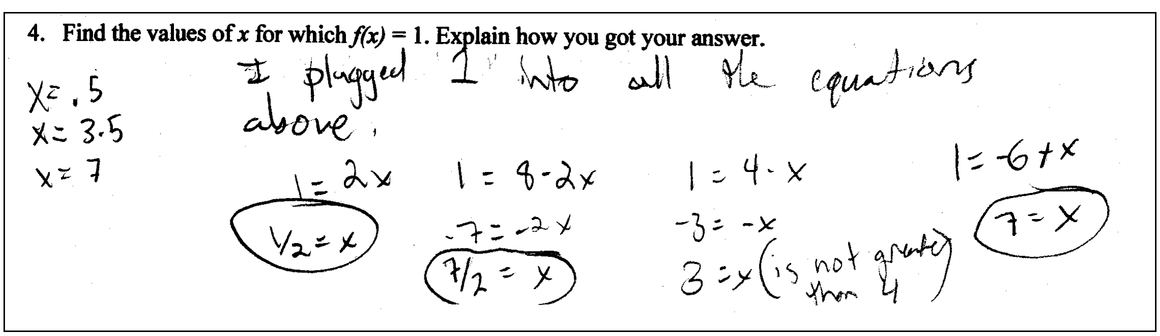


Figure 4.28. Solution strategies and scores on question 3 of the *Piecewise Function* task for AP Calculus students who studied four years of college preparatory mathematics (integrated or single-subject).

Question 4. The fourth question required students to solve the function for y = 1 and then explain how they got their answer. Six categories emerged from student work: *equation*, *graph, table, multiple, cannot tell* and *blank*. The *equation* category was used when students set their equations from question 3 equal to 1 and solved each for x (see Figure 4.29).


4. Find the values of *x* for which *f(x)* = 1. Explain how you got your answer.

$x = .5$
$x = 3.5$
$x = 7$

I plugged 1 into all the equations above.

$1 = 2x$ $\quad$ $1/2 = x$

$1 = 8 - 2x$ $\quad$ $-7 = -2x$ $\quad$ $7/2 = x$

$1 = 4 - x$ $\quad$ $-3 = -x$ $\quad$ $3 = x$ (is not greater than 4)

$1 = -6 + x$ $\quad$ $7 = x$

Figure 4.29. Example of student #58 work on the *Piecewise Function* task question 4 coded as using equations.

The *graph* category was coded when students used the graph of $f(x)$ by either drawing a line at y = 1 or referring to a line crossing the function at y = 1 (see Figure 4.30).

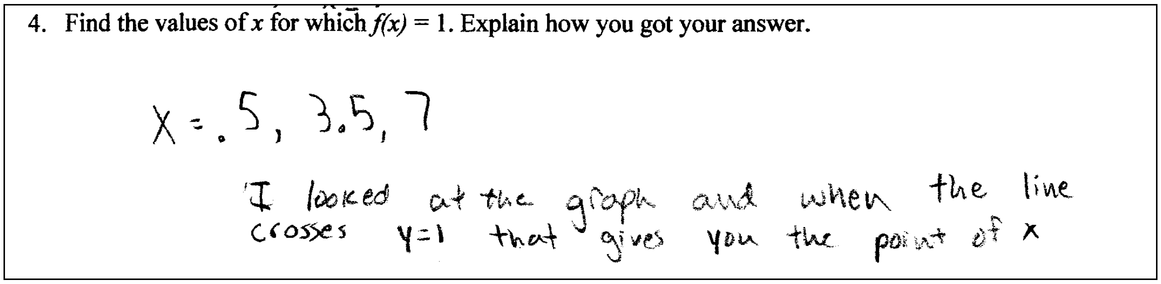


Figure 4.30. Example of student #32 work on the *Piecewise Function* task question 4 coded as using the graph.

The *table* category was used when students created a table of values identifying x-values that had a y-value equal to 1 (see Figure 4.31).

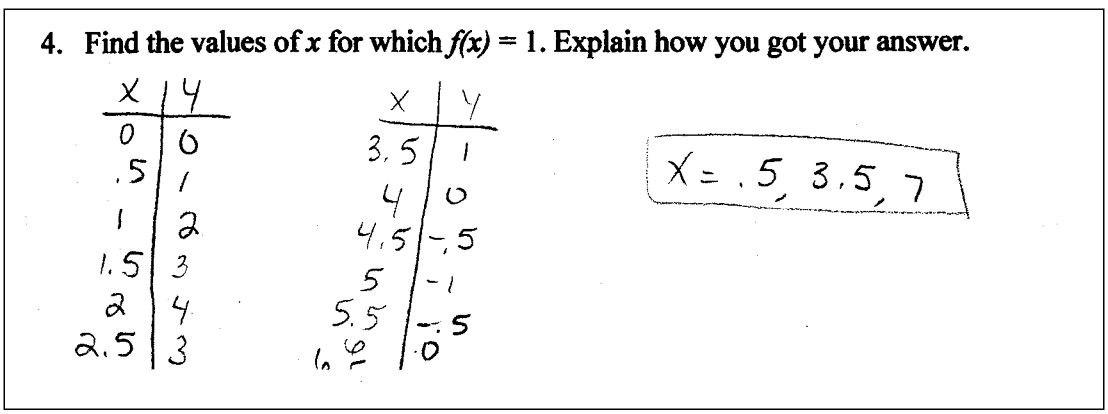
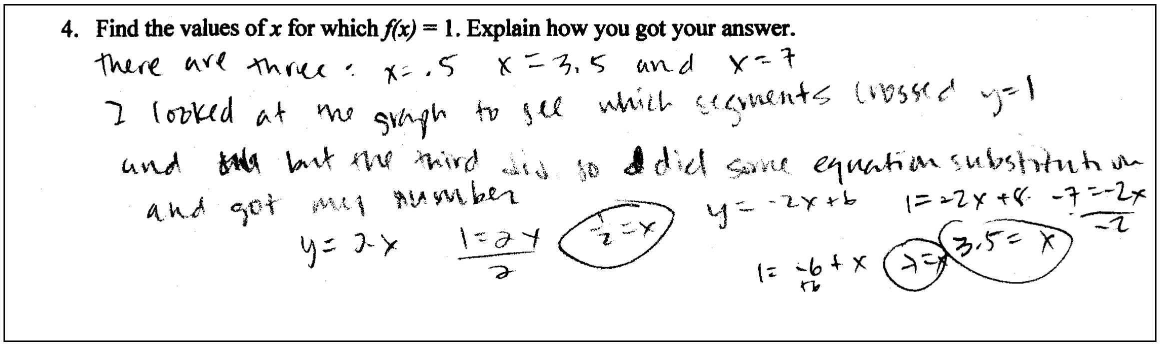


Figure 4.31. Example of student #149 work on the *Piecewise Function* task question 4 coded as using a table.

The *multiple* category was used when students combined more than one strategy to answer the question. The majority (81%) of students in both curricula who utilized multiple strategies used the equations they had found in question 3 and the graph to solve the function (see Figure 4.32).

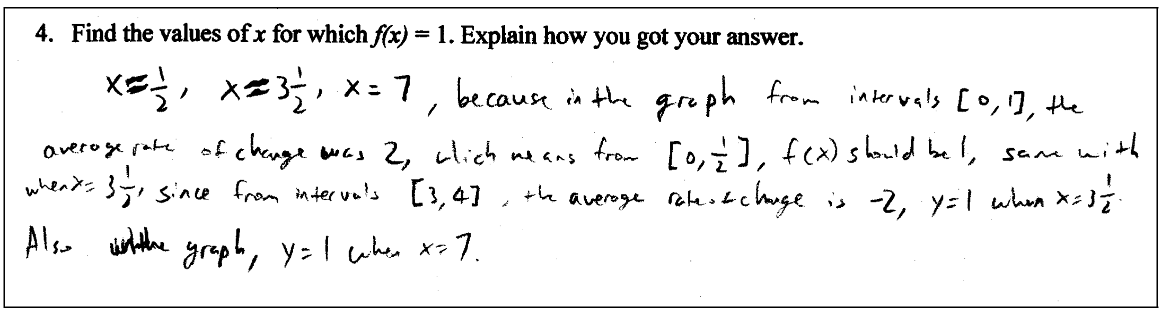


4. Find the values of *x* for which *f(x)* = 1. Explain how you got your answer.

there are three: $x=.5$ $x=3.5$ and $x=7$
I looked at the graph to see which segments crossed $y=1$
and but the third did so I did some equation substitution and got my number
$y=2x$ $\frac{1=2x}{2}$ $\frac{1}{2}=x$ $y=-2x+b$ $1=-2x+8$ $\frac{-7=-2x}{-2}$
$1=-6+x$ $7=x$ $3.5=x$

Figure 4.32. Example of student #85 work on the *Piecewise Function* task question 4 coded as using both equations and the graph.

Another 10% of all AP Calculus students who utilized multiple strategies used a combination of the graph and rates of change to solve the function (see Figure 4.33).

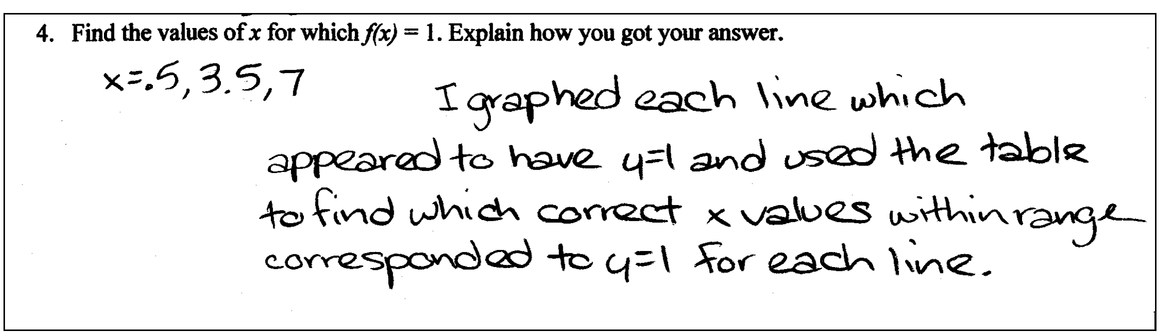


4. Find the values of *x* for which *f(x)* = 1. Explain how you got your answer.

$x=\frac{1}{2}$, $x=3\frac{1}{2}$, $x=7$, because in the graph from intervals [0, 1], the average rate of change was 2, which means from $[0, \frac{1}{2}]$, $f(x)$ should be 1, same with when $x=3\frac{1}{2}$, since from intervals [3, 4], the average rate of change is -2, $y=1$ when $x=3\frac{1}{2}$. Also with the graph, $y=1$ when $x=7$.

Figure 4.33. Example of student #148 work on the *Piecewise Function* task question 4 coded as using the graph and the rates of change.

Finally, 7% of students from both curricula who utilized multiple strategies used the equations they had found in question 3 and a table to solve the function (see Figure 4.34).



4. Find the values of *x* for which *f(x)* = 1. Explain how you got your answer.

$x=.5, 3.5, 7$

I graphed each line which appeared to have $y=1$ and used the table to find which correct x values within range corresponded to $y=1$ for each line.

Figure 4.34. Example of student #46 work on the *Piecewise Function* task question 4 coded as using both equations and a table.

The final two categories, *cannot tell* and *blank* were used when student work was unable to be coded into one of the other four categories or it was blank.

Figure 4.35 displays the percent of students by curriculum pathway who used each strategy on question 4. Overall AP Calculus students used similar strategies to solve question 4. However, twice the percentage of APSSC students (28%) as APIC students (14%) used the equation method. More than 20% of students from each curriculum pathway, APIC (23%) and APSSC (30%), used the graph to solve the function. About the same percentage of students, APIC (21%) and APSSC (22%), used multiple strategies to answer the question. Just over twice the percentage of APIC students (19%) as APSSC students (6%) did not give enough information to code a strategy. Finally, about the same percentage of students from each curriculum pathway, APIC (19%) and APSSC (13%) left this question blank.

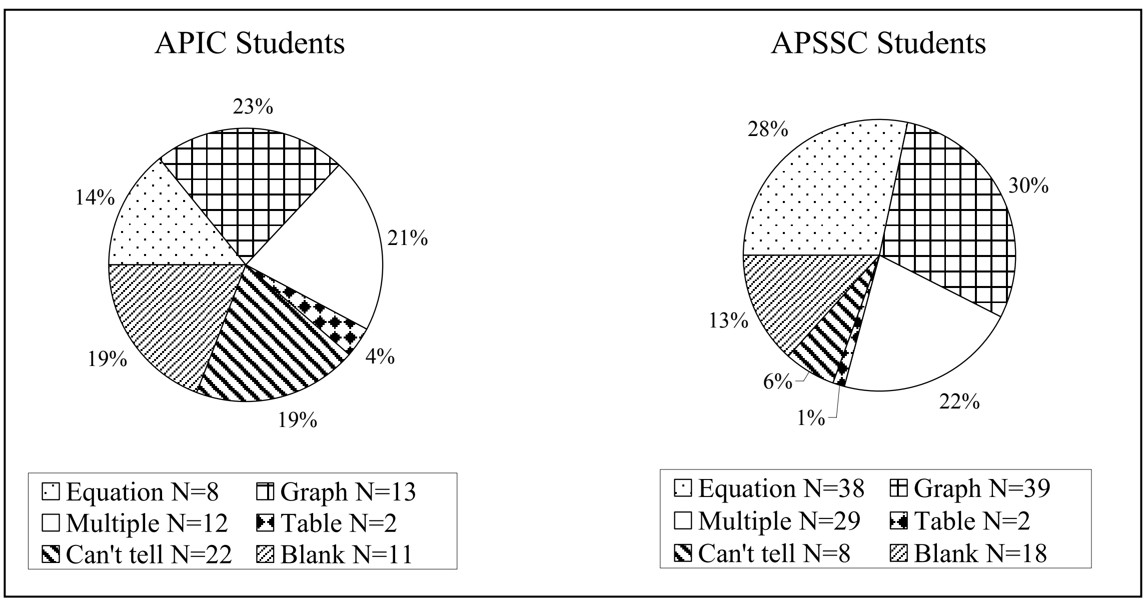


Figure 4.35. Solution strategies on question 4 of the *Piecewise Function* task for AP Calculus students who studied four years of college preparatory mathematics (integrated or single-subject).

Figure 4.36 displays the scores for students in each curriculum pathway by their solution strategies for question 4. The graph does not include the *table* category because only two AP Calculus students used this method. Again, AP Calculus students used similar strategies; however, their success rate on question 4 using the four strategies was different. APIC students demonstrated efficiency with different strategies on this question because no student received a zero when using the equations, graph, or multiple strategies to solve the function. On the other hand, APSSC students were proficient with either the graph or multiple strategies because no student received a zero when using these two strategies. The most successful method for APSSC students was the graph method, with approximately one-fourth of students receiving full credit. APIC students were equally successful when they used the graph or multiple strategies, with 18% of students receiving full credit using either of two strategies.

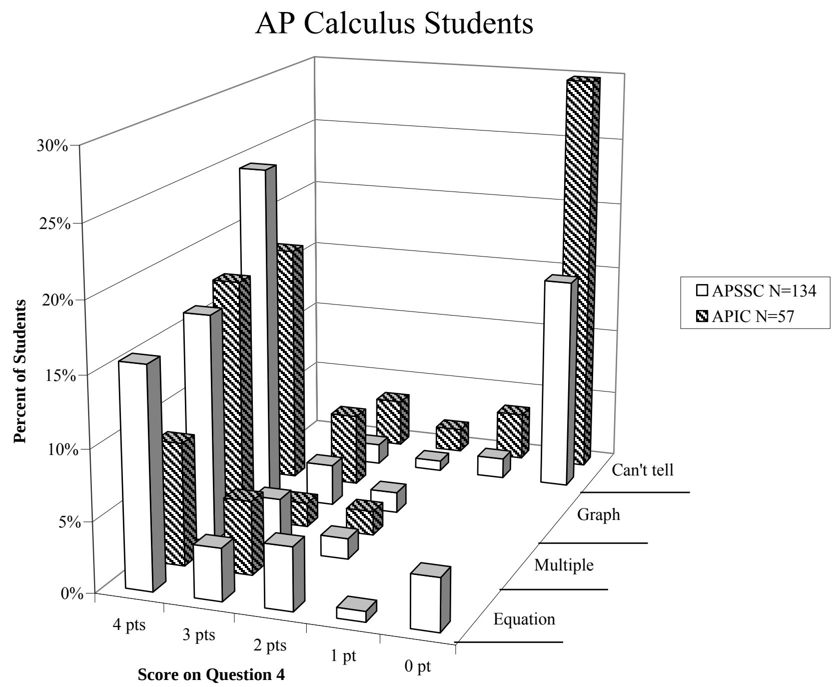
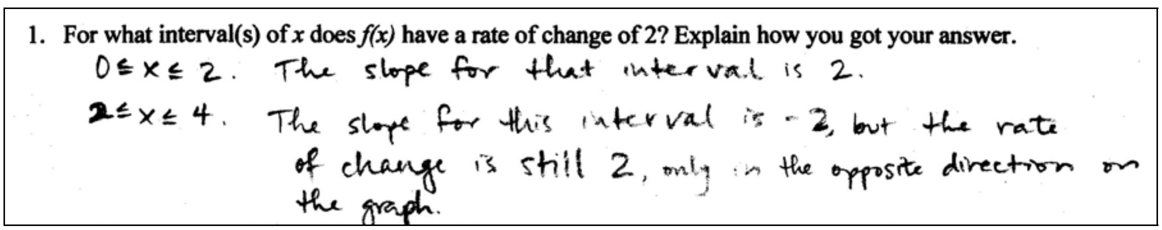


Figure 4.36. Solution strategies and scores on question 4 of the *Piecewise Function* task for AP Calculus students who studied four years of college preparatory mathematics (integrated or single-subject).

Errors on Piecewise Function Task

A second coding was completed to analyze student errors. This section provides an analysis of the different mathematical errors students committed on the four questions of the *Piecewise Function* task. Student errors were grouped by curriculum pathway and compared across curricula. Although different percentages of students made errors on different aspects of each question, only those errors that were common to at least 20% or more of the students are reported. Twenty percent was utilized as the cutoff based on conversations with the three AP Calculus teachers involved in this study. They agreed that they became concerned if 20% or more of the students in their class made common errors.

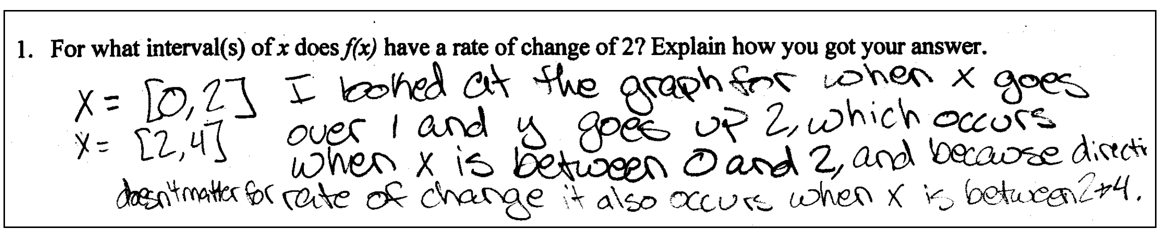
Question 1. The first question required students to determine what interval(s) of x the function had a rate of change of 2 and then explain how they determined the interval(s). Almost half (47%) of APIC students and 24% of APSSC students identified an incorrect interval as having a rate of change of 2. Students either chose the intervals [0, 2] and [2, 4] or the interval [0, 4]. Students who chose the intervals [0, 2] and [2, 4] explained that the slopes were different for these two segments (2 and -2), but the rate of change was the same (see Figure 4.37). While other students argued that the rate of change was the same for the two lines even though the direction of the segments was opposite (see Figure 4.38).


1. **For what interval(s) of *x* does *f(x)* have a rate of change of 2? Explain how you got your answer.**

$0 \le x \le 2$. The slope for that interval is 2.

$2 \le x \le 4$. The slope for this interval is -2, but the rate of change is still 2, only in the opposite direction on the graph.

Figure 4.37. Example of student #2 error on the rate of change for the *Piecewise Function* task question 1.

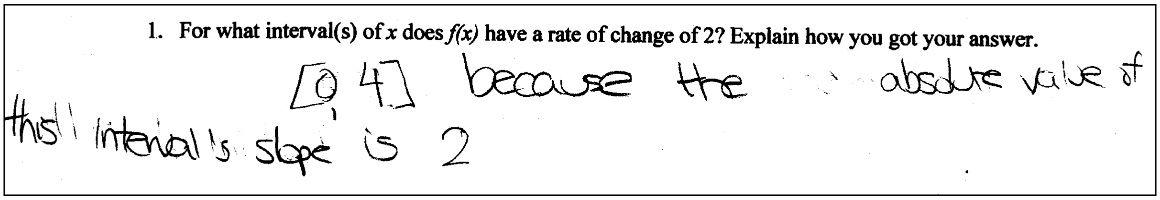

1. **For what interval(s) of *x* does *f(x)* have a rate of change of 2? Explain how you got your answer.**

x = [0,2]
x = [2,4]

I looked at the graph for when x goes over 1 and y goes up 2, which occurs when x is between 0 and 2, and because directi doesn't matter for rate of change it also occurs when x is between 2+4.

Figure 4.38. Example of student #184 error on the rate of change for the *Piecewise Function* task question 1.

Other students identified the interval [0, 4] as having a rate of change of 2. Although these students had similar explanations as the students who chose the intervals [0, 2] and [2, 4], these students also described that the rate of change was the absolute value of the slope (see Figure 4.39). These students did not recognize that at the point x = 2 there is no rate of change because the limit from the left and the right are different.


1. For what interval(s) of *x* does *f(x)* have a rate of change of 2? Explain how you got your answer.

[0, 4] because the absolute value of this interval's slope is 2

Figure 4.39. Example of student #137 error on the rate of change for the *Piecewise Function* task question 1.

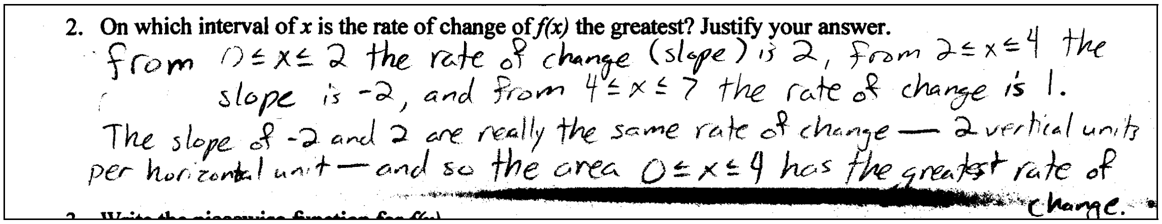
Question 2. The second question required students to determine what interval of x the function had the *greatest* rate of change and then justify their answer. Table 4.26 lists the errors students made across the two curricula pathways. Similar errors and results were found as in question 1 with twice the percentage of APIC students (30%) as APSSC students (15%) identified the intervals [0, 2] and [2, 4] as having a rate of change of 2 and therefore it was the *greatest*. The students' arguments were similar to question 1, as they took the absolute value of slopes to find the rate of change. It is interesting to note that 5 APIC students and 6 APSSC students who responded with the interval [0, 2] and [2, 4] or [0, 4] in question 1, got question 2 correct. This may be due to the use of the word *greatest* in the question.

Table 4.26

Errors on question 2of the Piecewise Function task for AP Calculus students who studied four years of college preparatory mathematics (integrated or single-subject)

Errors	% APIC students (*N*=57)	% APSSC students (*N*=134)
Students included the interval [2, 4]	30	15
Students chose the interval [0, 4]	16	24

Twenty-four percent of APSSC students identified the interval [0, 4] compared to 16% of APIC students. This error is interesting because a larger percentage (24%) of APSSC students gave this answer on question 2 than on question 1 (9%). Some of the students argued that the slopes of the graph were different, but the rate of changes were equal (see Figure 4.40). Again these students did not recognize that on the continuous function at the point x = 2 the rate of change is undefined.


2. **On which interval of *x* is the rate of change of *f(x)* the greatest? Justify your answer.**

from $0 \le x \le 2$ the rate of change (slope) is 2, from $2 \le x \le 4$ the slope is -2, and from $4 \le x \le 7$ the rate of change is 1. The slope of -2 and 2 are really the same rate of change — 2 vertical units per horizontal unit — and so the area $0 \le x \le 4$ has the greatest rate of change.

Figure 4.40. Example of student #102 error on the rate of change for the *Piecewise Function* task question 2.

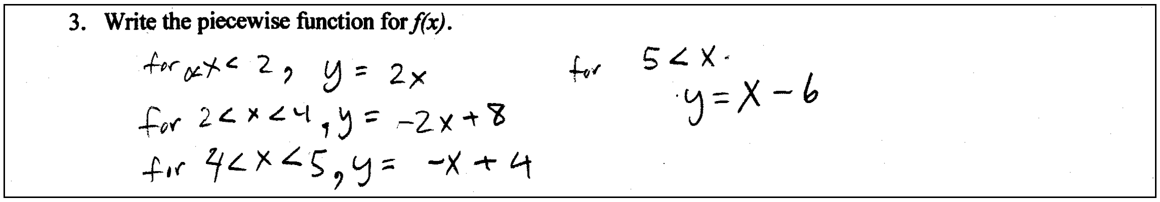
Question 3. The third question required students to write an equation for the piecewise function. Table 4.27 lists the errors students made across the two curricula. As was mentioned previously, question 3 was the most difficult for students in both curricula pathways.

Table 4.27

Errors on question 3 of the Piecewise Function task for AP Calculus students who studied four years of college preparatory mathematics (integrated or single-subject)

Errors	% APIC students (*N*=57)	% APSSC students (*N*=134)
At least two incorrect restrictions	53	25
At least two incorrect equations	40	43

APIC students had a difficult time writing the restriction for the piecewise function with more than 50% of students writing at least 2 restrictions incorrectly. On the other hand, one-quarter of APSSC students wrote at least two of the four restrictions incorrectly. Errors with restrictions made only by APSSC students were: (1) they forgot to include the endpoint in one of the two restrictions (see Figure 4.41); and (2) they wrote the restrictions based on their y-values and not the x-values (see Figure 4.42).


3. Write the piecewise function for $f(x)$.

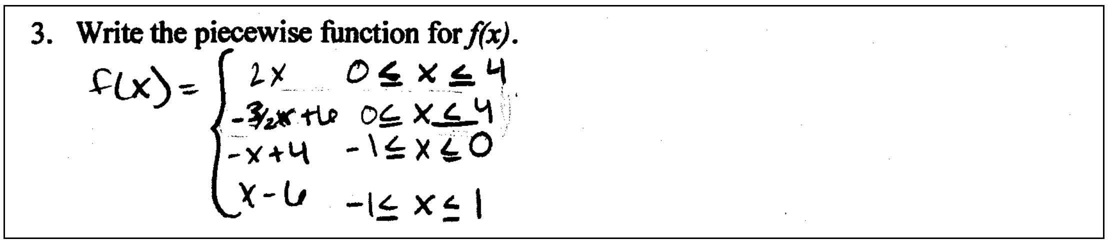
for $0 < x < 2$, $y = 2x$

for $2 < x < 4$, $y = -2x + 8$

for $4 < x < 5$, $y = -x + 4$

for $5 < x$. $y = x - 6$

Figure 4.41. Example of student #166 error on the restrictions for the *Piecewise Function* task question 3.


3. Write the piecewise function for $f(x)$.

$$f(x) = \begin{cases} 2x & 0 \le x \le 4 \\ -\frac{3}{2}x + 6 & 0 \le x \le 4 \\ -x + 4 & -1 \le x < 0 \\ x - 6 & -1 \le x \le 1 \end{cases}$$

Figure 4.42. Example of student #93 error on the restrictions using y-values for the *Piecewise Function* task question 3.

One-half (51%) of AP Calculus students from each curriculum pathway did not correctly write more than two of the four equations for the piecewise function. All four lines were linear functions, yet 43% of APSSC students and 40% of APIC students were unable to write correct equations. Students demonstrated two errors when writing the equation and both dealt with the y-intercept. In the first error students did not include a y-intercept. While in the second error students used incorrect y-intercepts such as the end point of each line.

Question 4. The fourth question required students to find the x values that would satisfy $f(x) = 1$ and explain how they got their answer. This question was easy for most students and no errors were found.

Summary of Performance on the Piecewise Function Task

Overall, APSSC students performed statistically higher on the *Piecewise Function* task than APIC students. APSSC students also had higher means on all four questions of the open-ended task than APIC students. A higher percentage of APSSC students attempted to answer each of the four questions than APIC students.

Strategies AP Calculus students used were similar across the two curricula; however, a larger percentage of APSSC students scored higher than APIC students when using the graph or slope-intercept equation. A larger percentage of APIC students scored higher than APSSC students when they used multiple strategies, such as a graph, formula and/or table.

Both APIC and APSSC students struggled with the concept of rate of change and what it meant in relationship to a given graph. APIC students did not consider the magnitude of the line as important when answering questions about the rate of change. APIC students also had difficulty with writing linear functions. APSSC students also did not consider the magnitude of the line as important when discussing rate of change and they did not recognize that there was no slope at a certain place on the graph because the limits were not equal.

Overall Performance on Filling a Tank Task

The *Filling a Tank* task (Appendix A) contained six parts. Students were given information about a hose that was filling an empty tank, but the flow rate changed because the value was closed and then re-opened. Students had to interpret the verbal meaning to draw a graph of the situation. Multiple times throughout the task students were asked to explain their thinking and what they had done. In the second half of the task a new function was introduced that represented water being pumped out of the tank at the same time that the hose was still filling the tank. These questions led to a final question about the instantaneous flow rate of the water from the hose. The questions previous to the last question about instantaneous flow rate would be prerequisite skills for calculus students. However, the instantaneous rate of change question was a new concept that most AP Calculus students would not have learned prior to entering the course; however, it is an important concept students learn during the AP Calculus course.

All questions were not worth the same number of points and the total number of points possible on the task was 20 (see Appendix B for scoring guide). The descriptive statistics for the total scores on the *Filling a Tank* task for AP Calculus students is shown in Table 4.28. Neither group of students (integrated and single-subject) scored below a 2 on the task with the range being 2 to 20. One (1.8%) APIC student and 2 (1.5%) APSSC students received perfect scores on this task.

Table 4.28

Descriptive statistics of total scores on the Filling a Tank task for AP Calculus students who studied four years of college preparatory mathematics (integrated or single-subject)

Curriculum Type	*N*	Mean*	SD	Range
APIC	57	11.81	3.88	2-20
APSSC	135	12.16	3.81	2-20

*Not adjusted for prior achievement

The graph in Figure 4.43 displays the total score distributions for students who studied four years of college preparatory mathematics (integrated or single-subject) on the *Filling a Tank* task. The graph shows both groups of students scores are negatively skewed (to the left). The majority of students in both curricula, APSSC (57%) and APIC (53%), scored between 10 and 15 points on this task.

AP Calculus Students

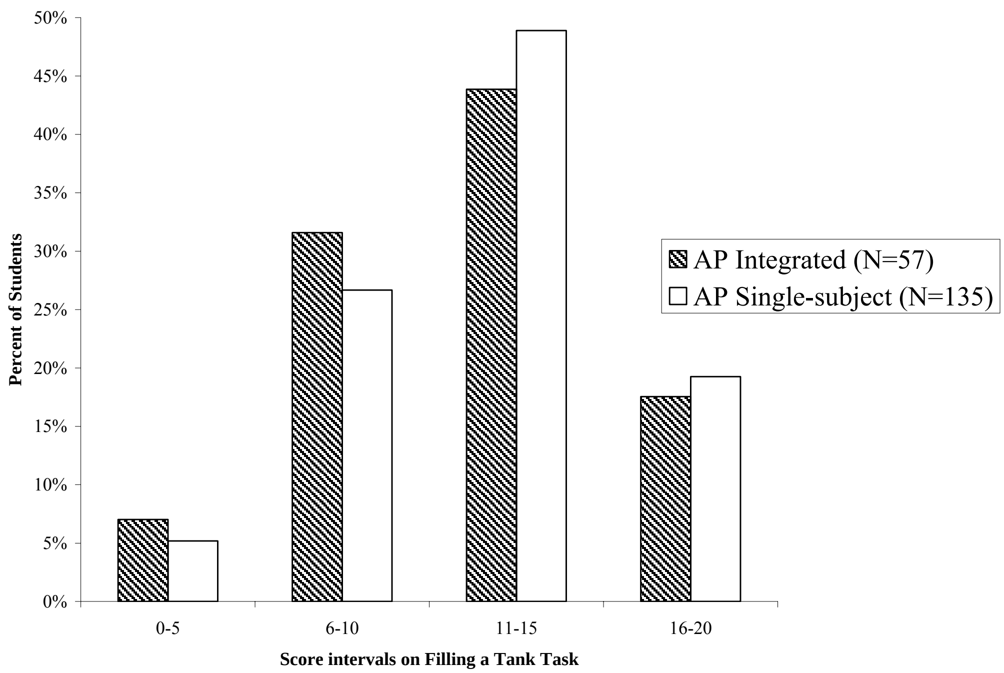


Figure 4.43. Score distributions on the *Filling a Tank* task for AP Calculus students who studied four years of college preparatory mathematics (integrated or single-subject).

Performance on Individual Questions

As stated previously, each question was worth different points depending on what students were required to do. Question A required students to draw a graph to model the volume of water in the tank at any time and explain how they decided to draw the graph. Students had to interpret three different parts on the graph and give their explanation. This question was worth 6 points. Question B required students to find the average rate of change over two different intervals that were given. This question was worth 2 points for each rate of change. Question C required students to explain how the rate of changes found in question B could be interpreted on their graph. This question was worth 3 points. Question D required students to draw another graph, which had their original graph from question A, but also included the new function to represent water being pumped out of the tank and how would they interpret the total volume in the tank at any time. This question was worth 4 points. Question E required students to explain and find where the volume would be at a maximum point. This question was worth 4 points. Finally, question F required students to find the instantaneous flow rate from the hose into the tank at the maximum point. This question was worth one point. The descriptive statistics for each question on the *Filling a Tank* task for the AP Calculus students is shown in Table 4.29.

Table 4.29

Descriptive statistics by question for the Filling a Tank task for AP Calculus students who studied four years of college preparatory mathematics (integrated or single-subject)

Question #	Points Possible	Mean* (SD) APIC (*N*=57)	Mean* (SD) APSSC (*N*=135)	Range
Question A	6	4.39 (1.41)	4.87 (1.30)	1-6
Question B	2	1.44 (0.71)	1.59 (0.64)	0-2
Question C	3	2.02 (1.14)	2.04 (1.18)	0-3
Question D	4	2.28 (1.22)	2.02 (1.37)	0-4
Question E	4	1.53 (1.39)	1.45 (1.43)	0-4
Question F	1	0.16 (0.37)	0.18 (0.38)	0-1

* Not adjusted for prior achievement

The score distributions for AP Calculus students on the six questions of the *Filling a Tank* task are displayed in Figure 4.44. Fifty percent of students from both curricula (integrated and single-subject) were able to answer questions B and C completely and correctly. Both groups (integrated and single-subject) of students had difficultly with questions E and F, which required them to find the maximum volume and then find the instantaneous rate of flow at the maximum point. The majority of students, APIC (67%) and APSSC (69%), received a 2 or lower on question E. A smaller percentage of APIC students (19%) received zeros on all six questions than APSSC students (26%). The most difficult question for both groups was question F with 82% of APIC students and 84% of APSSC students receiving a zero.

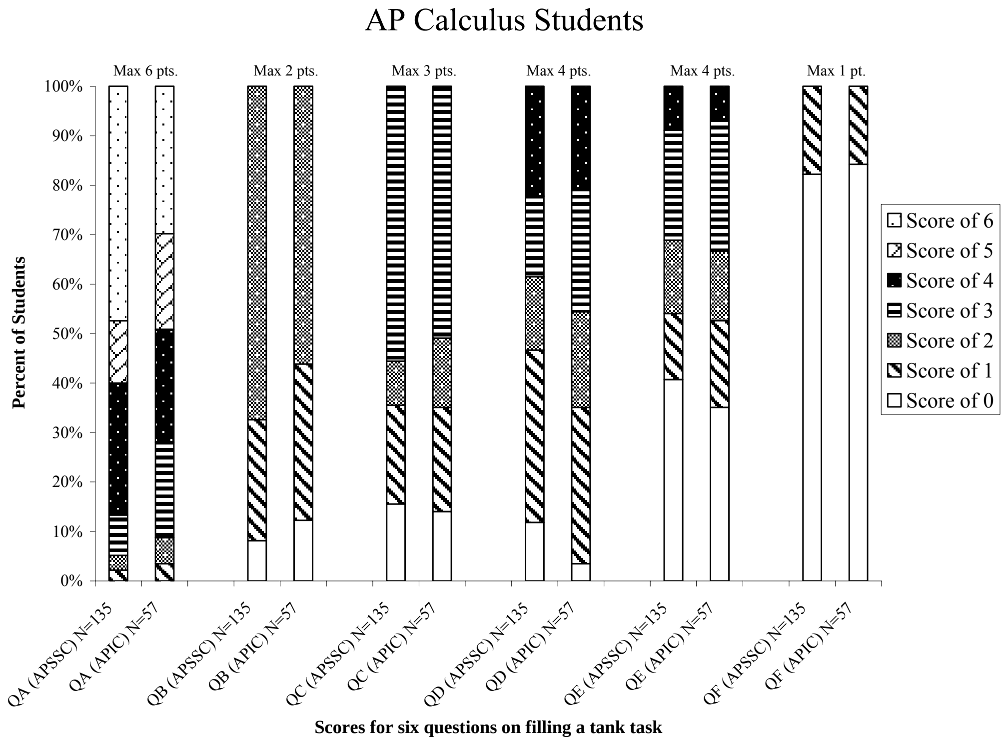


Figure 4.44. Score distributions on the six questions of the *Filling a Tank* task for AP Calculus students who studied four years of college preparatory mathematics (integrated or single-subject).

In an attempt to understand whether a score of zero meant a student had a misconception, meaning the question was attempted, or the student did not understand the concept, the item was left blank; students' responses were coded as either attempted or blank. The same criterion described previously was used. That is, if the student wrote anything it was coded as attempted; whereas, if the student did not write anything it was coded as blank. Figure 4.45 displays the percentage of students from each curriculum pathway who attempted or left the items blank. As is seen from the graph APIC students had a higher percentage of attempts in five of the six questions in the *Filling the Tank* task; while, APSSC students had a higher percentage of blanks.

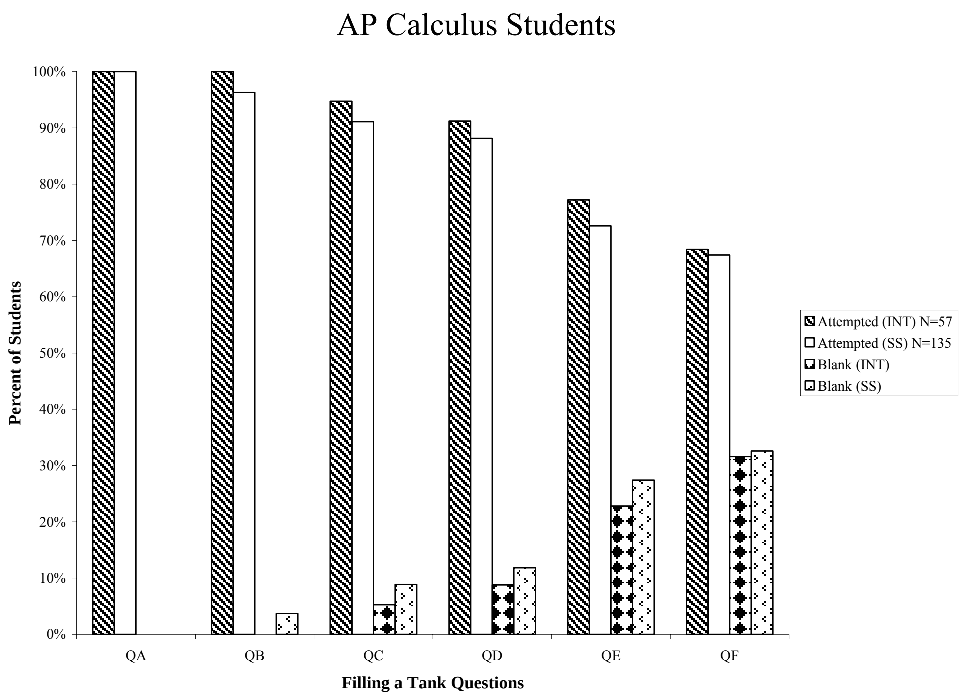


Figure 4.45. Distribution of attempted versus blank items on the *Filling a Tank* task for AP Calculus students who studied four years of college preparatory mathematics (integrated or single-subject).

Comparison on Filling a Tank Task

An analysis of covariance (ANCOVA) was conducted to determine the statistical difference between the mean overall score on the *Filling a Tank* task for the two groups (integrated and single-subject) of students adjusted for *IAAT* scores. The adjusted means for overall scores on the *Filling a Tank* task by curriculum type for AP Calculus students are presented in Table 4.30. No statistically significant difference between the mean overall score on the *Filling a Tank* task for AP Calculus students was found ($F = 2.591$, $p = .110$) after adjusting for prior achievement with *IAAT* scores.

Table 4.30

Adjusted descriptive statistics of total scores on Filling a Tank task for AP Calculus students who studied four years of college preparatory mathematics (integrated or single-subject) using IAAT scores

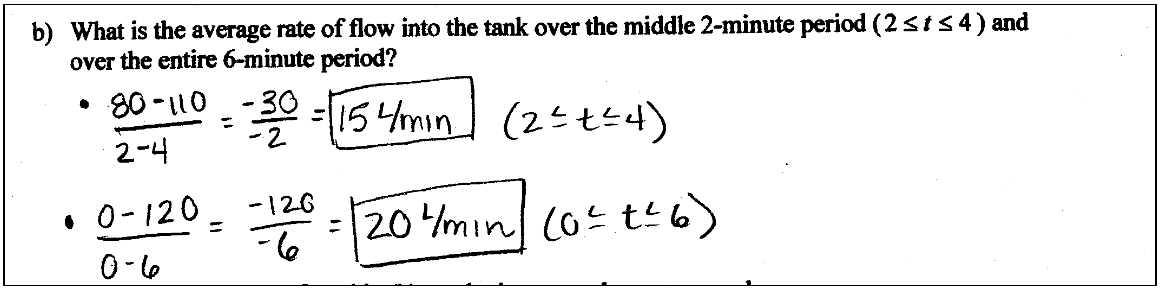
Curriculum path	*N*	Mean[a]	Std. Error	*F*	*p*
APIC	38	11.65	0.573	2.591	.110
APSSC	75	12.79	0.403		

[a]Covariates appearing in the model are evaluated at the following values: *IAAT* = 80.04

Solution Strategies on Filling a Tank Task

After all of the *Filling a Tank* tasks were scored numerically, a qualitative coding was applied. A first coding was used to determine what type of strategies students used to solve two of the six questions (B and E) on the *Filling a Tank* task. Four of the questions were not analyzed for strategies because students either had to explain their thinking or draw a graph that did not provide any strategies that could be analyzed. Therefore, this section provides an analysis of the different solution strategies students employed on questions B and E only.

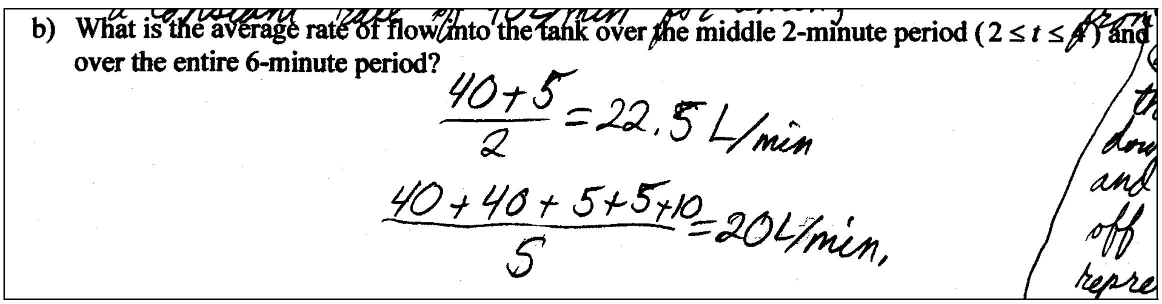
Question B. This item required students to find the average rate of flow during the middle 2 minute period and over the entire 6 minute period. Six categories immerged from student work: *slope-formula, average formula, table, multiple, cannot tell,* and *blank*. The *slope-formula* category was coded when students used the slope formula with the two sets of points (see Figure 4.46).


b) **What is the average rate of flow into the tank over the middle 2-minute period ($2 \le t \le 4$) and over the entire 6-minute period?**

- $\frac{80-110}{2-4} = \frac{-30}{-2} = \boxed{15\ L/min}$ $(2 \le t \le 4)$
- $\frac{0-120}{0-6} = \frac{-120}{-6} = \boxed{20\ L/min}$ $(0 \le t \le 6)$

Figure 4.46. Example of student #9 work on the *Filling a Tank* task question B coded in the slope-formula category.

The *average formula* category was coded when students used the average rates of flow given in the problem to calculate the average by adding the rates and dividing by the total number of rates (see Figure 4.47).

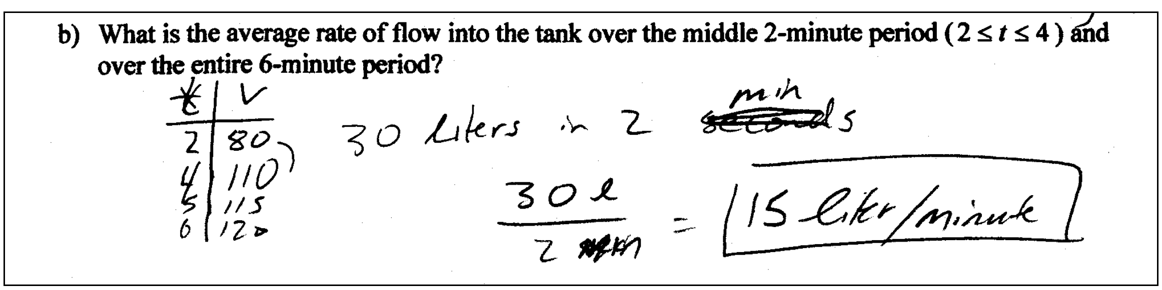

b) **What is the average rate of flow into the tank over the middle 2-minute period ($2 \le t \le 4$) and over the entire 6-minute period?**

$\frac{40+5}{2} = 22.5\ L/min$

$\frac{40+40+5+5+10}{5} = 20\ L/min.$

Figure 4.47. Example of student #183 work on the *Filling a Tank* task question B coded in the average formula category.

The *table* category was coded when students used a table to determine the average rate of change (see Figure 4.48).


b) **What is the average rate of flow into the tank over the middle 2-minute period ($2 \le t \le 4$) and over the entire 6-minute period?**

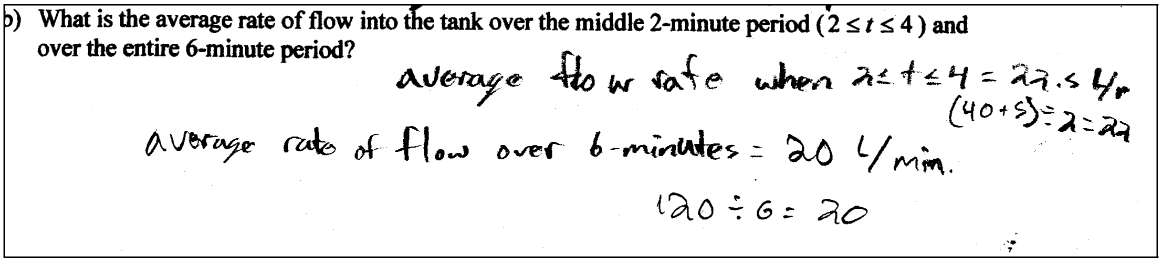
t	V
2	80
4	110
5	115
6	120

30 Liters in 2 ~~seconds~~ min

$\frac{30\ l}{2\ min} = \boxed{15\ liter/minute}$

Figure 4.48. Example of student #20 work on the *Filling a Tank* task question B coded in the table category.

The *multiple* category was coded when students used a combination of two or more strategies (see Figure 4.49).


b) **What is the average rate of flow into the tank over the middle 2-minute period ($2 \le t \le 4$) and over the entire 6-minute period?**

average flow rate when $2 \le t \le 4 = 22.5$ L/r

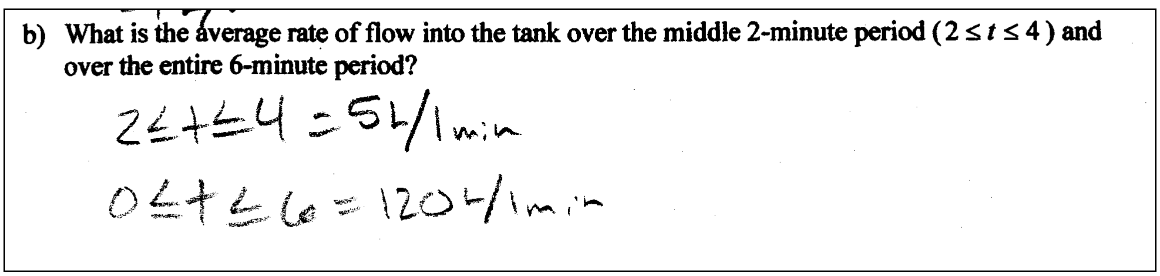
$(40+5) \div 2 = 22$

average rate of flow over 6-minutes = 20 L/min.

$120 \div 6 = 20$

Figure 4.49. Example of student #140 work on the *Filling a Tank* task question B coded in the multiple category.

The final two categories, *cannot tell* and *blank* were used when student work was unable to be coded into one of the other four categories or it was left blank. Generally, students stated the rate of change for both time periods, but it was not clear how they calculated those rates (see Figure 4.50).


b) **What is the average rate of flow into the tank over the middle 2-minute period ($2 \le t \le 4$) and over the entire 6-minute period?**

$2 \le t \le 4 = 5$L/1min

$0 \le t \le 6 = 120$L/1min

Figure 4.50. Example of student #111 work on the *Filling a Tank* task question B coded in the cannot tell category.

Figure 4.51 displays the percentage of students who used each strategy by curriculum pathway for question B. Overall, AP Calculus students used similar strategies to solve question B. Seventy-five percent of APIC students and 84% of APSSC students used the slope formula to find the average rates of change. Three times the percent of APIC students (9%) as APSSC students (3%) used the average formula to find the average rates of change. The same percentage of APIC and APSSC students (11%) did not provide enough information to code their strategies.

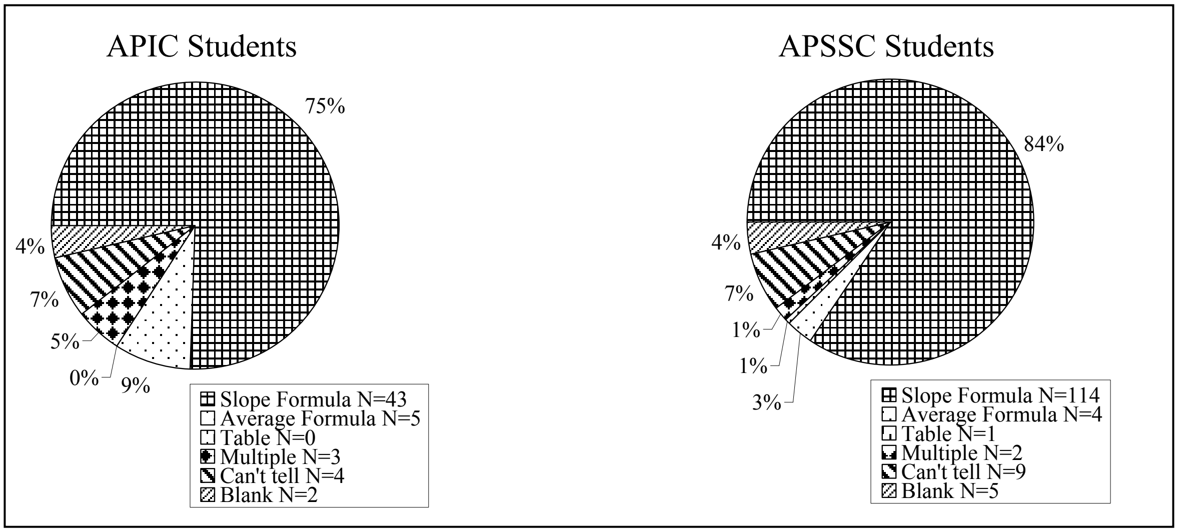


Figure 4.51. Solution strategies on question B of the *Filling a Tank* task for AP Calculus students who studied four years of college preparatory mathematics (integrated or single-subject).

Figure 4.52 displays the scores for students in each curriculum pathway by their solution strategies for question B. The graph does not include the two categories *table* and *multiple* because only five students used these strategies. Overall AP Calculus students were successful using the slope formula with the majority of students, APIC (51%) and APSSC (64%) receiving full credit on question B. On the other hand, no student from either curriculum pathway received a score of 2 when they used the average formula strategy. In fact, most students who used the average formula received a zero on this question.

AP Calculus Students

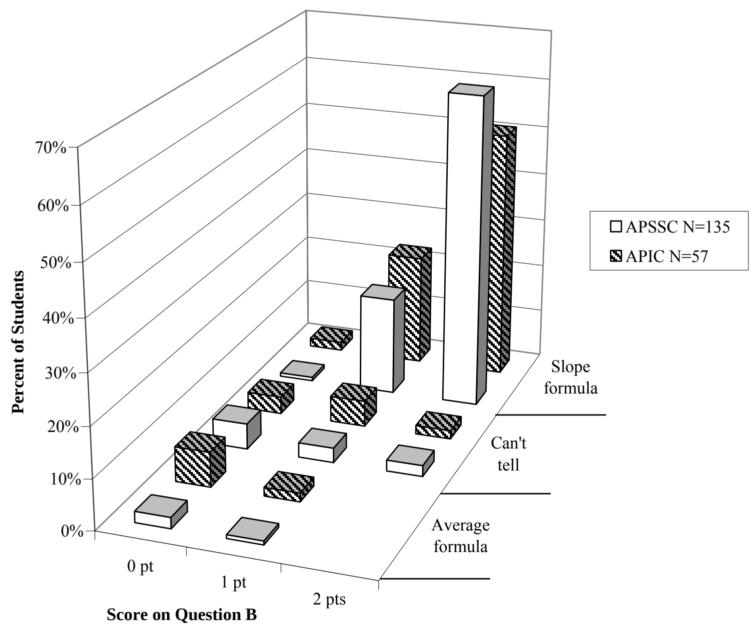
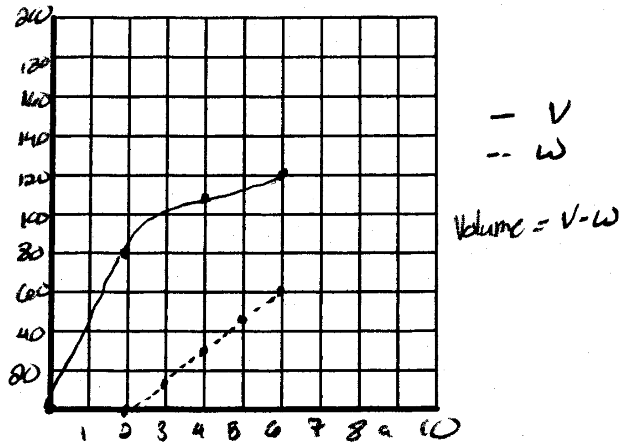


Figure 4.52. Solution strategies and scores on question B of the *Filling a Tank* task for AP Calculus students who studied four years of college preparatory mathematics (integrated or single-subject).

Question E. This question required students to find the maximum volume of water in the tank and explain how they determined the maximum point. Eight categories immerged from student work: *formula, graph, table, compare rates of change, subtract y-values*, *multiple, cannot tell,* and *blank*. The *formula* category was used when students either created an equation or formula to calculate the maximum volume (see Figure 4.53).



e) **Show or explain how to find on your graph of (d) the point at which the water level in the tank is at a maximum.** The W pump shuts off a T=6, so to find when the value is 180, simply take V(6) - W(6) + V(x) = 120 which cancels out to V(x) = 60. Since the rate is 5 L/min, The tank will fill at 18 minutes. (original 6 minutes + 12 minutes)

Figure 4.53. Example of student #158 work on the *Filling a Tank* task question E coded in the formula category.

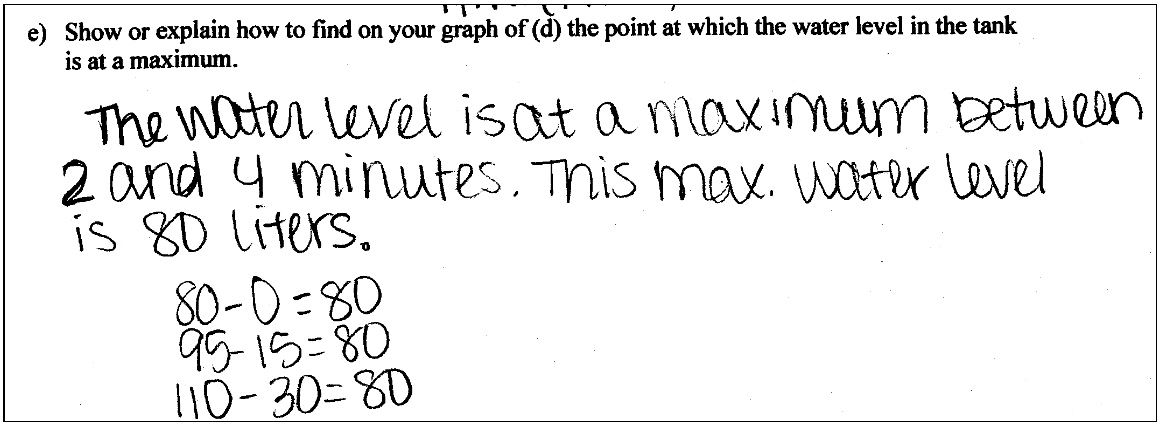
The *graph* category was used when students referred to the highest point on the graph or a specific point where two functions intersect (see Figure 4.54).

e) **Show or explain how to find on your graph of (d) the point at which the water level in the tank is at a maximum.**

The water level is @ it's Maximum at the graph's highest point, or 90 L (4 min)

Figure 4.54. Example of student #106 work on the *Filling a Tank* task question E coded in the graph category.

The *table* category was coded when students created a table by using the two functions that were pumping water in and out of the tank to find the total amount of water left in the tank at any time (see Figure 4.55).



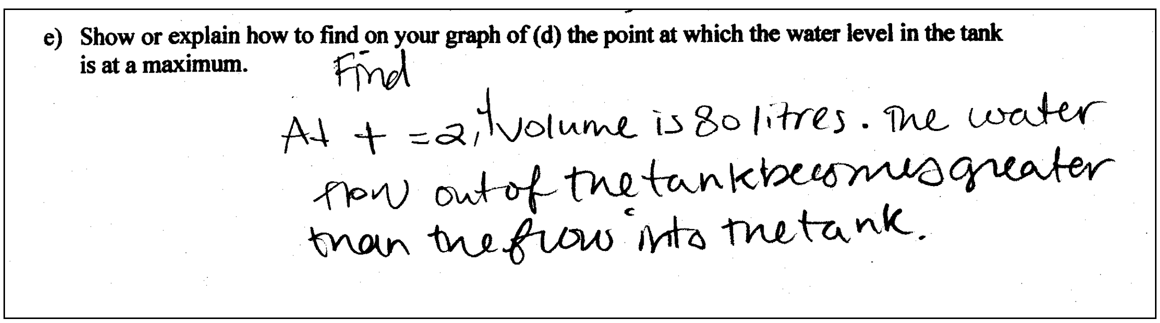
e) **Show or explain how to find on your graph of (d) the point at which the water level in the tank is at a maximum.**

The water level is at a maximum between 2 and 4 minutes. This max. water level is 80 liters.

80-0=80
95-15=80
110-30=80

Figure 4.55. Example of student #154 work on the *Filling a Tank* task question E coded in the table category.

The *compare rates of change* category was used when students compared the rates of change of the two functions to determine the point where more water was being pumped out of the tank than being pumped into the tank (see Figure 4.56).



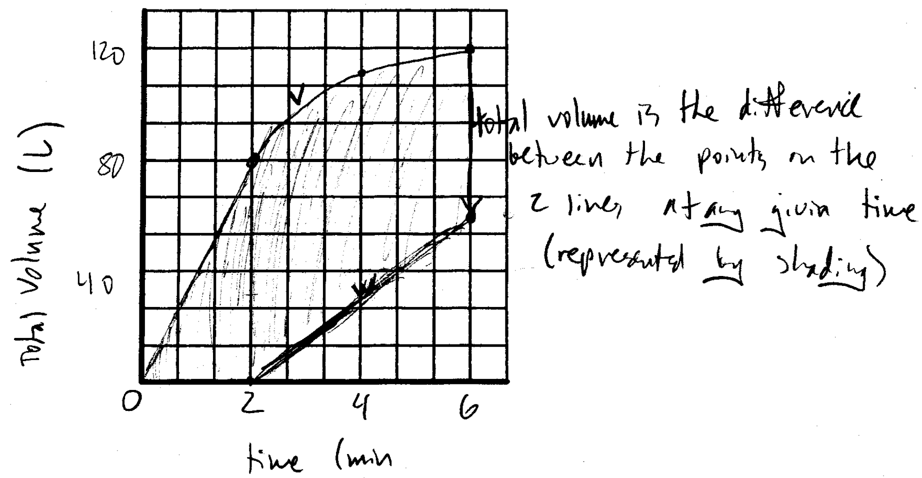
e) **Show or explain how to find on your graph of (d) the point at which the water level in the tank is at a maximum.**

Find

At t =2, volume is 80 litres. The water flow out of the tank becomes greater than the flow into the tank.

Figure 4.56. Example of student #5 work on the *Filling a Tank* task question E coded in the compare rates of change category.

The *subtract y-values* category was used when students discussed either subtracting the y-values of the two functions or that the greatest distance between the two functions is when the volume is at its maximum point (see Figure 4.57).



e) Show or explain how to find on your graph of (d) the point at which the water level in the tank is at a maximum.

The maximum is where the difference between the two volumes at any given time is the greatest (at approximately 2⅔ minutes)

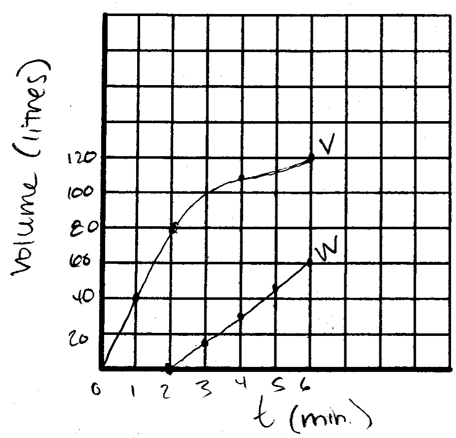
Figure 4.57. Example of student #62 work on the *Filling a Tank* task question E coded in the subtract y-values category.

The *multiple* category was coded when students used a combination of two or more strategies. Students used different combinations of strategies to find the maximum volume: using the graph and comparing rates of change (see Figure 4.58), using the graph and subtracting y-values (see Figure 4.59), and comparing rates of change for both functions and subtracting the y-values of each function (see Figure 4.60).

e) Show or explain how to find on your graph of (d) the point at which the water level in the tank is at a maximum.

The max point will be at the utmost point or coordinate on the graph. from 2 – 4 min, the ROC is 15 $\frac{l}{m}$. so you will be stuck @ 80. from 4 – 6 min, 5 $\frac{l}{m}$. So you will lose 20 ml in the last 2 min. So 80 is the max

Figure 4.58. Example of student #52 work on the *Filling a Tank* task question E coded in the multiple category.



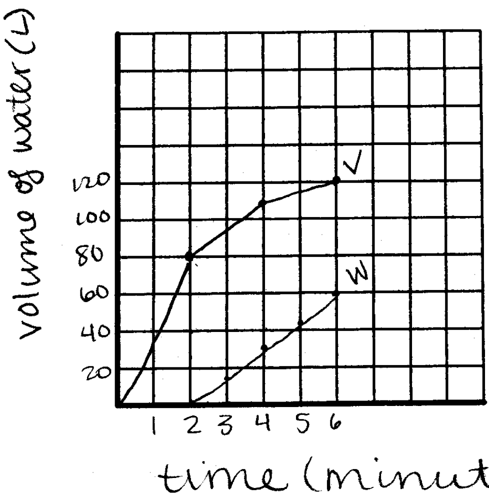
e) Show or explain how to find on your graph of (d) the point at which the water level in the tank is at a maximum.

At each time t, subtract the W coordinate from the V coordinate to find the amount of water in the tank at each time.

0 min.	0 - 0 = 0	4 min	110 - 30 = 80
1 min	40 - 0 = 40	5 min	115 - 45 = 70
2 min	80 - 0 = 80	6 min	120 - 60 = 60
3 min	100 - 15 = 85		

max level: 3 min, 85 litres

Figure 4.59. Example of student #180 work on the *Filling a Tank* task question E coded in the multiple category.



The volume of water (L) is the difference of y coordinates at time, t.
For example at t = 6
Water in = 120L
Water out = 60L
120L − 60L = 60L at 6 minutes

e) **Show or explain how to find on your graph of (d) the point at which the water level in the tank is at a maximum.**

The point at which there is the greatest space between V and W is the maximum volume. This is at x values [2, 4], when the pump is first started and the flow out of the tank is equal to the flow into the tank.

Figure 4.60. Example of student #164 work on the *Filling a Tank* task question E coded in the multiple category.

The final two categories, *cannot tell* and *blank* were used when student work was unable to be coded into one of the other six categories or the student left the question blank. Generally, students gave a maximum point, but did not explain how they had found this point (see Figure 4.61).

e) **Show or explain how to find on your graph of (d) the point at which the water level in the tank is at a maximum.**

The maximum amount of water in the tank is at t=2, with 80L in the tank

Figure 4.61. Example of student #60 work on the *Filling a Tank* task question E coded in the cannot tell category.

Figure 4.62 displays the percentage of students who used each strategy by curriculum pathway on question E. AP Calculus students from both curricula (integrated and single-subject) used some similar strategies to solve question E; however, only APSSC students wrote a formula or equation to find the maximum volume. A higher percentage of APIC students (25%) than APSSC students (18%) used the comparing rates of change category. All other categories were used by about the same percentage of students.

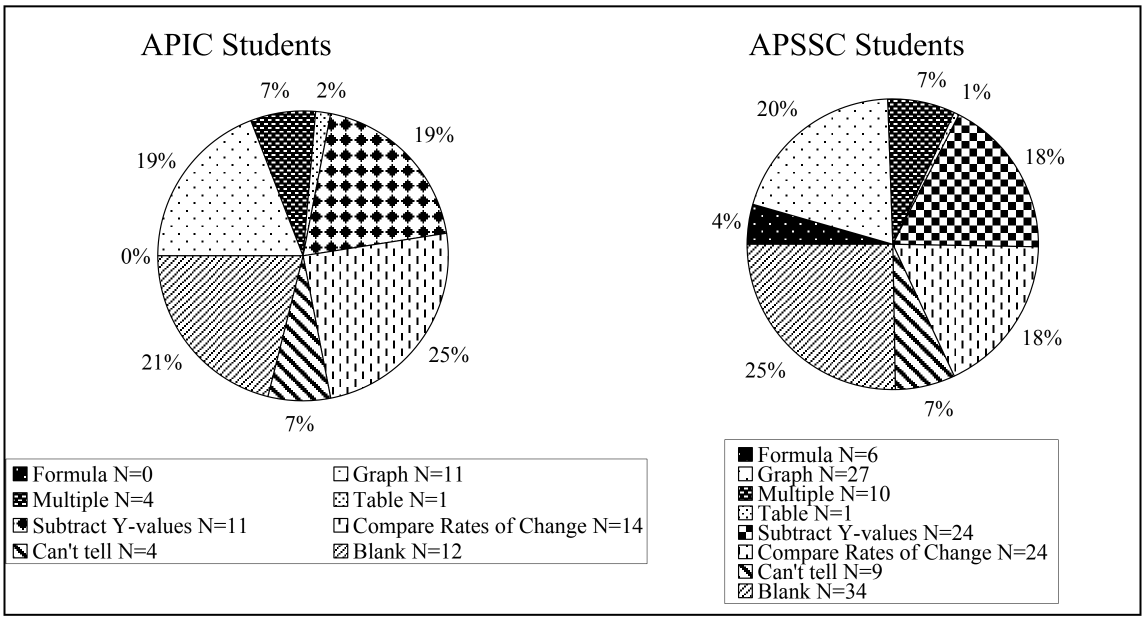


Figure 4.62. Solution strategies on question E of the *Filling a Tank* task for AP Calculus students who studied four years of college preparatory mathematics (integrated or single-subject).

Table 4.31 displays the scores for students in each curriculum pathway by their solution strategies for question E. The table does not include the *table* category because only two AP Calculus students, one from each curriculum pathway, used this method. Although students from both curricula (integrated and single-subject) used some similar strategies, students' success rates on question E using the seven strategies were different. Question E was a difficult item because only 8% of all AP Calculus students received a perfect score; however, almost half of students in each curriculum pathway, APSSC (46%) and APIC (47%) received a 2 or higher on this item. The 4% of APSSC students who used the formula strategy did not receive full credit on this question. Of the students who used the graph method about a quarter of APSSC students and a little more than a quarter (27%) of APIC students received a 3 or higher when using this method. Almost twice as many APIC students (75%) as APSSC students (40%) who used the multiple strategies scored a 3 or higher on this item. Ten percent of APSSC students scored a 4 when using multiple strategies. Both groups of students were successful when comparing the rates of change for the two flow rates because about the same percentage of students, APSSC (63%) and APIC (64%), received a 3 or higher. The most successful strategy for students in both curricula was subtracting the y-values to find the largest difference. Eighteen percent of APIC and 21% of APSSC students used the method flawlessly as they received perfect scores when subtracting y-values of the two functions.

Table 4.31

Solution strategies and scores on question E of the Filling a Tank task for AP Calculus students who studied four years of college preparatory mathematics (integrated or single-subject).

Solution Strategies APSSC (*N*=135) APIC (*N*=57)	Total Percent using strategies	Percent of Students 0 pt.	1 pt.	2 pts.	3 pts.	4 pts.
Formula						
APSSC	4.4	0.7	1.5	1.5	0.7	0
APIC	0	0	0	0	0	0
Graph						
APSSC	20.1	6.7	6.7	1.5	3	2.2
APIC	19.4	8.8	5.3	0	3.5	1.8
Multiple						
APSSC	7.3	2.2	1.5	0.7	2.2	0.7
APIC	7.1	0	1.8	0	5.3	0
Compare rates of change						
APSSC	17.7	2.2	0.7	3.7	8.9	2.2
APIC	24.6	0	3.5	5.3	14	1.8
Subtract y-values						
APSSC	17.8	0	0	7.4	6.7	3.7
APIC	19.4	1.8	1.8	8.8	3.5	3.5
Can't tell						
APSSC	31.9	28.9	3	0	0	0
APIC	28.1	24.6	3.5	0	0	0

Errors on Filling a Tank Task

A second coding was completed to analyze student errors. In this section the errors that AP Calculus students made on the *Filling a Tank* task are reported by the six questions. Student errors were grouped by curriculum pathway and compared across curricula. Errors committed by 20% or more of students in either curricula were analyzed to determine similarities and differences across the groups.

Question A. The first question required students to graph the volume of water in a tank at any time and then explain how they drew the graph. Table 4.32 lists the errors students made across the two curricula. Just over a third of AP Calculus students, APIC (39%) and APSSC (33%), drew the middle section of the graph incorrectly. Students generally drew a straight line, interpreting the rate of change as constant during the time period two to four minutes (see Figure 4.63). However, the problem states that at 2 minutes the tap is gradually closed until at t = 4 when the flow rate equals 5 L/min. Other students interpreted the rate of 5 L/min as the slope of the line during the time period [2, 4] (see Figure 4.64).

Table 4.32

Errors on question A of Filling a Tank task for AP Calculus students who studied four years of college preparatory mathematics (integrated or single-subject)

Errors	% of APIC students (*N*=57)	% of APSSC students (*N*=135)
Students drew an incorrect middle section of the graph.	39	33
Students' explanation was about the scale and which variable should go on the x and y axes.	21	13
Students' explanation discussed only two of the rates or they "plotted given points".	28	29

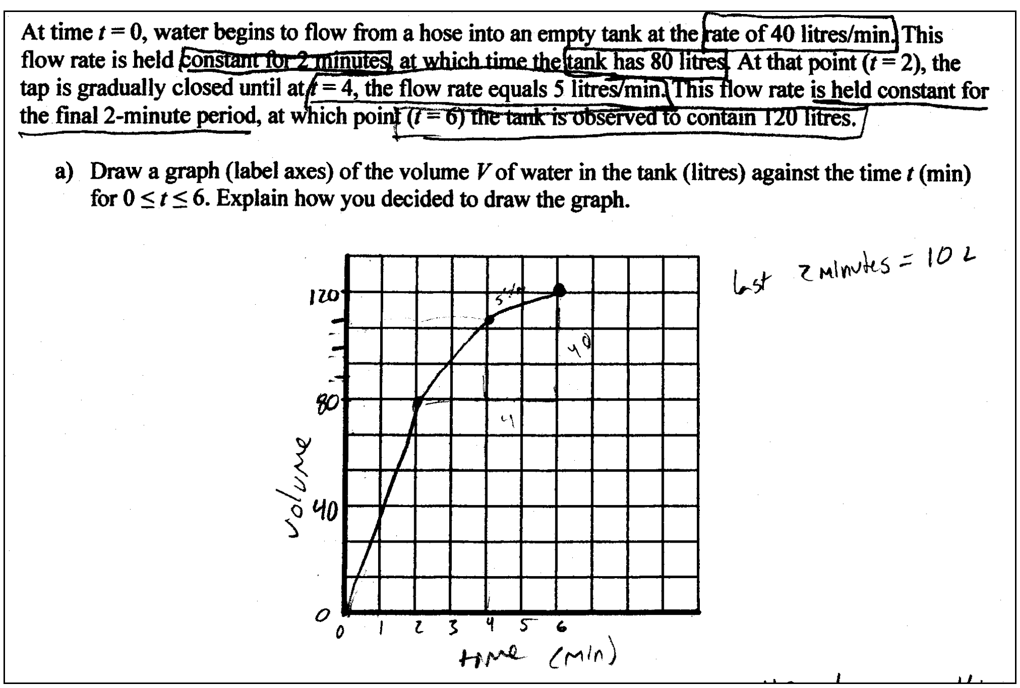


Figure 4.63. Example of student #124 error on graphing second segment of the *Filling a Tank* task question A.

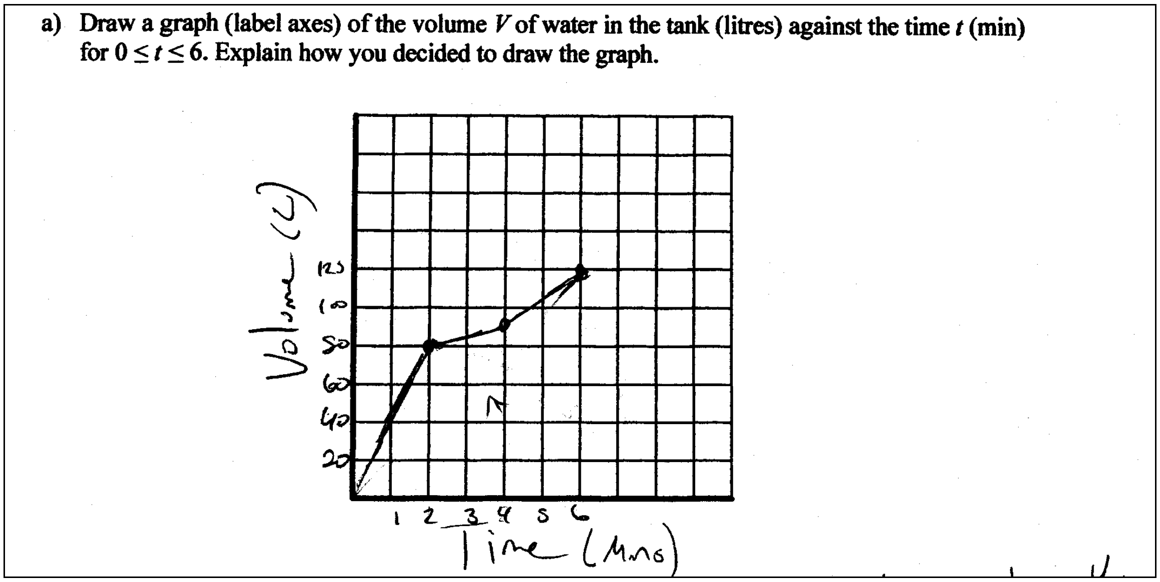
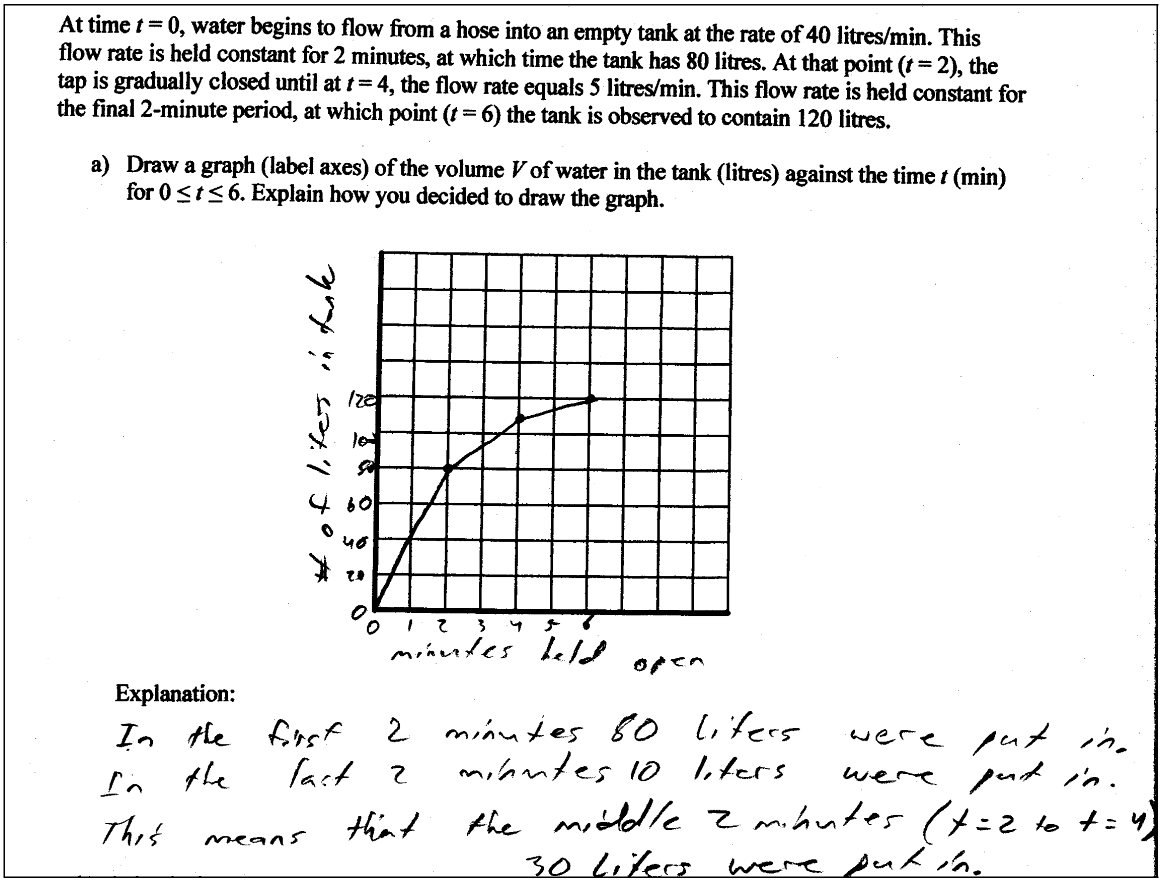


Figure 4.64. Example of student #10 error on graphing second segment of the *Filling a Tank* task question A.

The final two errors dealt with students' explanations not being complete. Approximately the same percentage of APIC (21%) and APSSC (26%) students who drew a correct graph did not receive full credit on their explanation. A larger percentage of APIC students (21%) than APSSC students (13%) explained how they created their graph (e.g., what scale to use on each axis, what variable to put on each axis) rather than how they drew the function. Approximately the same percentage of students in each curricula, APIC (28%) and APSSC (29%), explained how they used the two rates to draw the two segments, but then the third segment was either guessed or students connected the dots (see Figure 4.65).


At time $t = 0$, water begins to flow from a hose into an empty tank at the rate of 40 litres/min. This flow rate is held constant for 2 minutes, at which time the tank has 80 litres. At that point ($t = 2$), the tap is gradually closed until at $t = 4$, the flow rate equals 5 litres/min. This flow rate is held constant for the final 2-minute period, at which point ($t = 6$) the tank is observed to contain 120 litres.

a) Draw a graph (label axes) of the volume V of water in the tank (litres) against the time t (min) for $0 \le t \le 6$. Explain how you decided to draw the graph.

Explanation:

In the first 2 minutes 80 liters were put in. In the last 2 minutes 10 liters were put in. This means that the middle 2 minutes ($t=2$ to $t=4$) 30 liters were put in.

Figure 4.65. Example of student #55 error on explanation of the *Filling a Tank* task question A.

Question B. The second question required students to find the average rate of flow during two different time periods. One-fourth of APIC students and less than one-fifth of APSSC students (17%) found an incorrect rate of change for the interval [2, 4]. Some of the reasons this error appeared were students had an incorrect graph, students incorrectly used the rates of change to find the slope, and students misread their graph selecting wrong coordinate points.

Question C. The third question required students to explain how the rates they had found in question B could be interpreted on the graph in question A. Nearly one-fourth of students, APIC (26%) and APSSC (22%), were unable to explain the meaning of the rates of change on their graph. Students referred to the rates as describing if the function was increasing or decreasing or the flow rate of the water into the tank being faster or slower. Although these explanations are correct, students did not interpret the meaning of the rates of change on their graph (see Figure 4.66).

c) Explain how the rates found in (b) can be interpreted on your graph.

In the two minute period, the graph is going up by approximately 5 litres for each minute passed. Overall, the rate of flow changes throughout the graph but 20 is the average flow rate overall.

Figure 4.66. Example of student #125 error on explanation relating rates of change to graph on the *Filling a Tank* task question C.

Question D. The fourth question required students to draw their original volume function from question A on a graph, draw a new function (W) to represent water pumped out of the tank, and explain how to find the volume of water that remained in the tank at any time. Table 4.33 lists the errors students made across the two curricula.

Table 4.33

Errors on question D of Filling a Tank task for AP Calculus students who studied four years of college preparatory mathematics (integrated or single-subject)

Errors	% of APIC students (N=57)	% of APSSC students (N=135)
Incorrect W function, correct explanation.	25	16
Students drew a linear W function, but it was incorrect for various reasons (i.e., wrong starting point, decreasing, or increasing).	23	21

Approximately one-fifth of AP Calculus students, APIC (25%) and APSSC (16%), drew an incorrect W function, but their explanation of how to find the total volume in the tank was correct. Although the students' function for water being pumped out (W) was incorrect, they were able to reason and use their understanding of the situation to explain how to find the total volume of the water left in the tank.

Just over one-fifth of students from both curricula pathways, APIC (23%) and APSSC (21%), drew an incorrect W function for various reasons. Some students drew an increasing linear function for W; however, they started the graph at (0, 0), (2, 10), or (2, 15). Other students drew W as a decreasing linear function starting at the point (2, 80) and ending at the point (6, 20) (see Figure 4.67). Students had a difficult time overcoming the iconic translation that if water is being pumped out it should be represented with a graph that is decreasing.

Now suppose that, in addition to the above sequence of events, a pump is started up at $t = 2$, and , for the next 4 minutes, pumps water out of the tank at the constant rate of 15 litres/min.

d) What we have called V above, now represents the total amount of water that has flowed into the tank at time t; let W represent the total amount of water pumped out of the tank at time t. Plot V and W against t on the same set of axes. Show on your graph how to interpret the volume of water in the tank at any time.

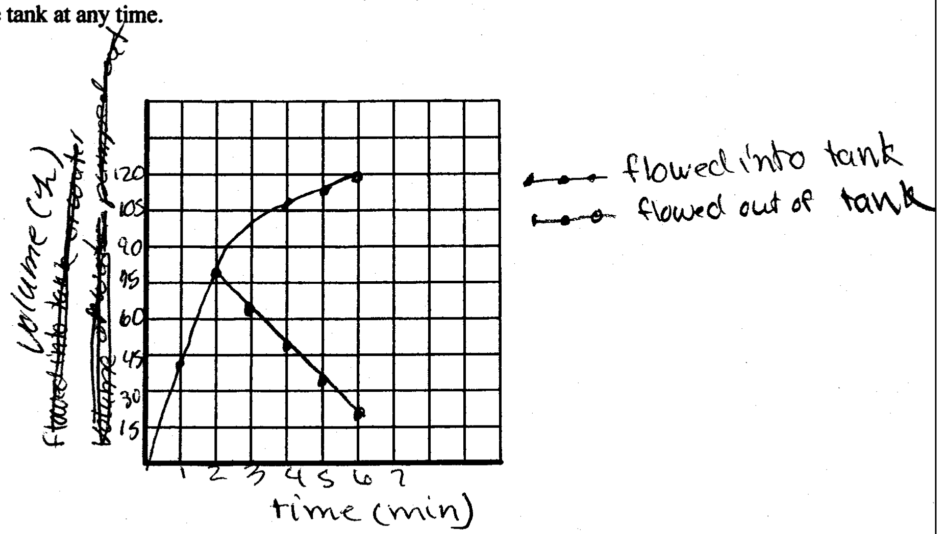


Figure 4.67. Example of student #48 error on graphing the W function on the *Filling a Tank* task question D.

Question E. The fifth question required students to show or explain how to find on their graph from question D the point that represented the maximum water level in the tank. Table 4.34 lists the errors students made across the two curricula.

Table 4.34

Errors on question E of Filling a Tank task for AP Calculus students who studied four years of college preparatory mathematics (integrated or single-subject)

Errors	% of APIC students (*N*=57)	% of APSSC students (*N*=135)
Students estimated the maximum water level to be (2, 80).	44	35
Students did not give a maximum point.	26	23

More than one-third of students, APIC (44%) and APSSC (35%), estimated the maximum water level to be at (2, 80). Students made this error for two reasons. First, the majority of students (63%) had at least one incorrect graph, which lead to an incorrect maximum value. Second, another group of students explained that the point (2, 80) was where water began to be pumped out of the tank; therefore, the water level could not rise above 80 L. What students did not take into account was the rate at which water was being pumped into the tank was greater than the rate of water being pumped out of the tank.

Approximately one-fourth of students, APIC (26%) and APSSC (23%), did not provide a maximum point. It is impossible to determine if students did not read the question completely or if students did not understand that they needed to provide the maximum point.

Question F. The sixth question required students to find the instantaneous flow rate at the maximum water level. Almost one-third of students, APIC (32%) and APSSC (30%), found an incorrect instantaneous rate of change. These students were split among three instantaneous rates of change (0 L/min, 40 L/min, and -10 L/min). Students who found the instantaneous rate of 0 L/min explained that the volume of water pumped in was equal to the volume of water pumped out at the maximum point (see Figure 4.68).

f) What is the instantaneous flow rate from the hose into the tank at this point?

$\frac{0L}{min}$ because the volume of water being pumped in equals the volume of water being pumped out.

Figure 4.68. Example of student #28 error on finding the instantaneous rate on the *Filling a Tank* task question F.

Another group of students found the instantaneous flow rate to be 40 L/min. Students found the slope of the line using the point (2, 80) and another point very close but prior to (2, 80). For instance one student used the points (1.999, 79.96) and (2, 80) to find the rate of change of 40 L/min (see Figure 4.69).

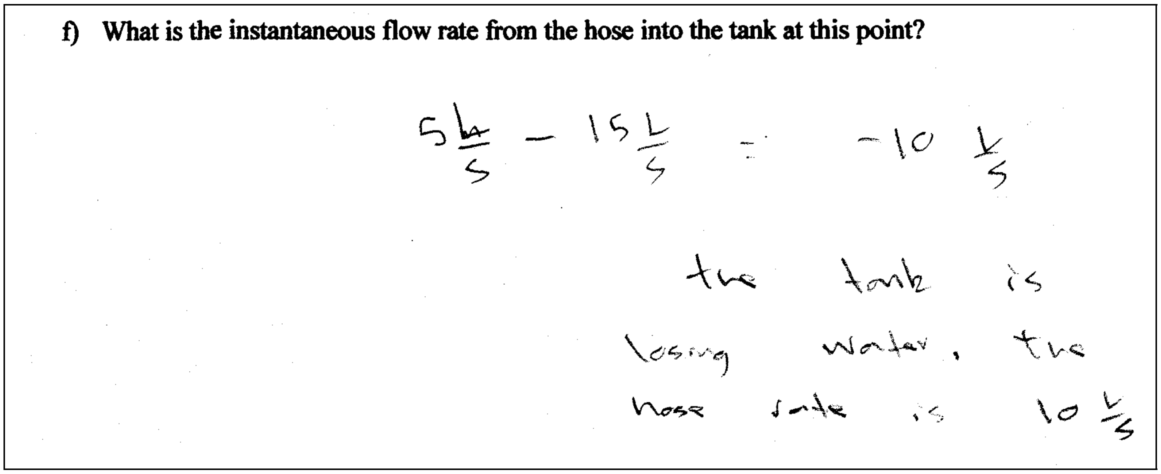
f) What is the instantaneous flow rate from the hose into the tank at this point?

(1.999, 79.96)(2, 80)

$$\frac{80-79.96}{2-1.999} = 40 \text{ L/min}$$

Figure 4.69. Example of student #188 error on finding the instantaneous rate on the *Filling a Tank* task question F.

Another popular answer for the instantaneous flow rate was -10 L/min. Students used the rates of change for the different functions and subtracted them. Students found the rate of change for the volume function at 2 minutes was 5 L/min and the rate of change for the W function at 2 minutes was 15 L/min. After subtracting these two values students reported the rate of change of -10 L/min (see Figure 4.70).



f) What is the instantaneous flow rate from the hose into the tank at this point?

$5\frac{L}{s} - 15\frac{L}{s} = -10\frac{L}{s}$

the tank is losing water, the hose rate is $10\frac{L}{s}$

Figure 4.70. Example of student #126 error on finding the instantaneous rate on the *Filling a Tank* task question F.

Summary of Performance on the Filling a Tank Task

APIC and APSSC students performed quite similarly on the *Filling a Tank* task. No statistical significance between mean overall scores was found. APIC students scored slightly higher on questions D and E; whereas, APSSC students scored slightly higher on questions A, B, C, and F. APIC students had a smaller percentage of students score 0 on individual questions. APIC students also had a higher percentage of students who attempted five of the six questions.

Solution strategies used by AP Calculus students from either curriculum pathway were similar. The majority of students used the slope-formula and was successful using it. Students were also fairly successful subtracting y-values of two functions to find a maximum point. APSSC students tended to use a formula strategy, but were not successful when it was applied.

The errors committed by both groups of AP Calculus students were similar across each question. Students had difficulty, multiple times, with understanding what was meant by a rate of change that was continually changing. If the function was linear, most students were able to get the correct rate of change and interpret it; however, once the function changed because of a varying rate of change students began to apply linear function procedures incorrectly. Students in both groups had difficulty interpreting the meaning of the rate of change in this contextual task. A few errors were specific to either group of students in one curriculum or the other. APSSC students tended to use the wrong points when finding a rate of change; whereas, APIC students used averages to find rate of change instead of specific points.

Summary

In this chapter, the results of the analysis of high school students' mathematical understanding after completing four years of college preparatory mathematics (integrated or single-subject) were described. More specifically, student performance on the *Precalculus Concept Assessment* (*PCA*) was summarized along with an item analysis of misconceptions related to functions. In addition an analysis of AP Calculus students' solution strategies and errors on the *Piecewise Function* task and the *Filling a Tank* task was provided.

Overall, high school students from each curriculum pathway demonstrated a limited understanding of functions based on the three instruments used in this study. SSC students performed statistically higher on the *PCA* than IC students; whereas, APSSC and APIC students performed comparably. On average, after adjusting for prior achievement, APSSC, APIC, and SSC students scored at or above 12.91 on the *PCA*. Carlson et al. (in review) predicted 13 as a cutoff in predicting success in college Calculus. On the other hand, the mean performance of IC students was 11.62 on the *PCA*.

The item analysis of the *PCA* supports the fact that students seem to be completing four years of college preparatory mathematics (integrated and single-subject) with a procedural understanding of functions. For instance students tend to evaluate and solve functions correctly; however, they exhibited multiple misconceptions about rate of change and inverse functions. These topics are essential for students as they continue into college calculus or AP Calculus at the high school.

AP Calculus students tended to use similar solution strategies when solving the two open-ended tasks. The solution strategies employed by students included using a graph, an equation, a table, or using multiple methods to solve the questions on the two tasks. Although students used similar strategies, their ability to use these strategies effectively differed. APSSC students tended to score more points on questions because they used strategies effectively.

AP Calculus students from each curricula pathway demonstrated errors related to rates of change. Students tended to use words such as slope, rate of change, and steepness interchangeably. Student errors on the two open-ended tasks and the *PCA* revealed a lack of understanding of the interpretation or meaning of rate of change. Students successfully calculated the rate of change of linear functions; however, when the rate of change was not linear, students struggled to calculate it, represent it correctly on the graph, or interpret the meaning in relation to a real world context.

In the next chapter, an overview of the entire study is provided. The findings of this study are discussed using relevant research to support the conclusions and implications of this study are examined. Finally, the limitations of this research study are summarized and recommendations for future research regarding students' mathematical understanding after completing four years of college preparatory mathematics (integrated and single-subject) are discussed.

CHAPTER 5: SUMMARY, DISCUSSION, AND RECOMMENDATIONS

This study examined similarities and differences in students' mathematical understanding after studying four years of college preparatory (integrated or single-subject) mathematics. More specifically, this study investigated students' performance and common misconceptions on tasks requiring different levels of cognitive demand and focused on specific representations (e.g., graphic, numeric, and symbolic). Strategies that AP Calculus students used to solve open-ended tasks, and the errors they made were analyzed to understand how their previous mathematics courses impacted what they did and how they did it.

This chapter is divided into five sections. The first section presents a brief overview of the study. The second section includes the key findings of the study. The third section outlines the implications of the study. The fourth section presents a discussion of the limitations of the study. Finally, the chapter concludes with recommendations for future research.

Summary of the Study

Many school districts are reluctant to adopt an integrated mathematics course pathway because opponents claim that integrated curricula materials do not prepare students for collegiate-level mathematics. Other schools have adopted an integrated mathematics pathway and reported their students are prepared for collegiate level mathematics. Still other secondary schools offer parallel course pathways, which allow students to take a series of either integrated or single-subject courses. Students in these districts complete four years of college preparatory mathematics courses that prepare them for either Advanced Placement (AP) Calculus or AP Statistics, depending on course offerings available at the school.

The difference in organizational structure and expectations raises the question: What are the similarities and differences in how students understand mathematics after successfully completing either an integrated or single-subject curriculum path? Students in either path are learning mathematics; however, what and how the mathematics is learned is influenced by the curriculum and how it is implemented. A key question is: what mathematical knowledge will students from these two different curriculum paths possess? For students continuing on into AP Calculus an important practical question is: Will students from either curricula pathway be disadvantaged or advantaged?

Students who have completed four years of mathematics in different curricular paths may display similarities and differences in their strategies for solving problems, types of mathematical errors, misconceptions, ability to solve questions requiring different levels of cognitive demand and focused on specific representations (e.g., graphic, numeric, and symbolic).

As curriculum designers revise and develop new curricula, they should understand what students are learning or not learning from existing curricula. Stein et al. (2007) points out “few studies have connected the curriculum (or tasks) *as enacted* with student learning or achievement” (p. 358). Even fewer studies have connected high school mathematics curricula with student learning (Harwell et al., 2007). Research needs to examine what mathematics students who have experienced high school integrated or single-subject mathematics curricula have learned and what strategies they have developed.

Purpose of the Study

This study documented similarities and differences in mathematical understanding of high school students who completed four years of college preparatory (integrated or single-subject) mathematics curricula. This study sought to answer the following research questions:

1. How do high school students who studied four years of college preparatory mathematics curriculum (integrated or single-subject) perform and compare on calculus readiness concepts in terms of:
    a. overall performance;
    b. responses to items by levels of cognitive demand;
    c. responses to items focused on specific representations (e.g., graphic, numeric, symbolic); and
    d. common misconceptions about functions?
2. How do high school students who studied four years of college preparatory mathematics curriculum (integrated or single-subject) and enrolled in AP Calculus perform and compare on college readiness concepts in terms of:
    a. overall performance;
    b. responses to items by levels of cognitive demand;
    c. responses to items focused on specific representations (e.g., graphic, numeric, symbolic); and
    d. common misconceptions about functions?
3. How do high school students who studied four years of college preparatory mathematics curriculum (integrated or single-subject) and enrolled in AP

Calculus perform and compare in solution strategies and errors on open-ended mathematical tasks that focus on functions during the first quarter of AP Calculus?

Methodology

In order to assess students' understanding of functions the *Precalculus Concept Assessment* (*PCA*) was administered to 505 high school students who studied four years of college preparatory mathematics curriculum (integrated or single-subject). Of the 505 students, 201 studied four years of the integrated mathematics curriculum (IC) and 304 studied four years of the single-subject mathematics curriculum (SSC). Of the 505 students who took the *PCA*, 199 enrolled in AP Calculus the following year, including 59 students who studied four years of the integrated mathematics curriculum (APIC) and 140 students who studied four years of the single-subject mathematics curriculum (APSSC).

In addition, AP Calculus students completed two open-ended tasks, *Filling a Tank* and *Piecewise Function,* (See Appendix A). Both tasks required students to demonstrate their understanding of functions; however, one task was contextually based and the other was not. Each task included multiple components in which students were asked to show their work and explain their thinking. Students worked on the two tasks individually during their regular class period during the first week of their AP Calculus class in August 2007.

The *PCA* questions were coded for levels of cognitive demand (Stein et al., 2000) and types of representation. Student responses were used to analyze performance and misconceptions about functions. A baseline measure, the *Iowa Algebra Aptitude Test*

(*IAAT*), was used to equate the two groups (integrated and single-subject). SSC and APSSC students' scores were statistical higher than IC and APIC students, indicating that students entered the two curricula paths with different prior achievement. An analysis of covariance (ANCOVA) was used to compare students on overall performance. A multivariate analysis of covariance (MANCOVA) was used to compare students' performance on questions requiring different levels of cognitive demand and on items reflecting specific representations (e.g., graphic, numeric, symbolic) related to functions.

Scoring guides (see Appendix B) for the two open-ended tasks were developed. Validity and reliability of the scoring guides and coding of student responses was established. All student work was scored quantitatively and then coded qualitatively for solution strategies and mathematical errors. An ANCOVA was used to compare students' overall performance on the two open-ended tasks using the *IAAT* as the covariate.

Results and Discussion of Key Findings

Performance of high school students on the PCA

Data from this study support findings for the three research questions. This section reports the results by research question. It also includes a discussion of the relevance of the findings in relationship to other research studies.

1. How do high school students who studied four years of college preparatory mathematics curriculum (integrated or single-subject) perform and compare on calculus readiness concepts in terms of:
    a. overall performance;
    b. responses to items by levels of cognitive demand;

c. responses to items focused on specific representations (e.g., graphic, numeric, symbolic); and
d. common misconceptions about functions?

Overall performance. Results from this study indicate that, after adjusting for prior achievement, SSC students perform significantly higher ($F = 9.39$, $p = .002$) than IC students (12.91 to 11.62) on the *PCA*. Two SSC students received a perfect score of 25; whereas, the highest score obtained by IC students was 22. Other researchers (Harwell et al., 2007; Schoen et al., 1999; Schoen & Hirsch, 2003a) have reported no difference between student achievement (integrated and single-subject) on standardized tests. However, the results of this study are consistent with results reported by other researchers who found SSC students performed statistically higher than IC students on instruments measuring algebra skills and performance in college mathematics courses (Hill, R. O. & Parker, 2006; Huntley et al., 2000).

Although the *PCA* was not designed as a calculus readiness exam it has served as a decent predictor. Carlson, Oehrtman & Engelke (in review) found college students enrolled in a college algebra course who took the *PCA* and received a score of 13 or above continued on in calculus and earned a grade of C or better. The results of the current study indicate 60% of SSC students scored a 13 or above on the *PCA*; whereas, approximately one-third of IC students scored in this same range. These results may indicate that SSC students are better prepared to enter calculus than IC students after completing four years of a mathematics curriculum.

Responses to items by levels of cognitive demand. All *PCA* items were coded by the four levels of cognitive demand (*memorization, procedures without connections,* *procedures with connections, and doing mathematics)*; however, only three of the four codes were used in this study since no *PCA* item was coded as *memorization*. After adjusting for prior achievement, SSC students performed statistically higher (F = 25.136, p = .000) than IC students (5.50 to 4.35) on *PCA* items coded as *procedures without connections*. This indicates that SSC students were more proficient than IC students in using procedures they were taught. These results were expected, as others (Huntley et al., 2000) have found similar results with SSC students scoring higher on procedural items. This may be due to the majority of single-subject curricula designed to have students learn procedures and then practice extensively (O'Brien, 1999); whereas, integrated curricula were designed for students to work with contextual problems and use multiple strategies with fewer procedural problems (Senk & Thompson, 2003).

PCA items coded as *procedures with connections* or *doing mathematics* require a higher level of cognitive demand. It was hypothesized that integrated students would perform better on higher level questions than single-subject students because integrated curricula were designed to have students solve unknown or higher level questions (Senk & Thompson, 2003).

In this study, IC and SSC students answered these items similarly. Students did not perform statistically different on *PCA* items coded as *procedures with connections* (F = .026, p = .872) or *doing mathematics* (F = 1.562, p = .212). These results were unexpected because others (Huntley et al., 2000; Schoen & Hirsch, 2003a) have found IC students were better able to answer higher cognitive demand questions because the curricula were designed for students to interpret situations; use multiple representations;

use procedures that require a deeper understanding of a mathematics concept; and use complex and non-algorithmic thinking to solve unfamiliar problems.

Responses to items focused on specific representations (e.g., graphic, numeric, symbolic). All *PCA* items were coded for three representation types (graphic, numeric, and symbolic). Graphical items were the easiest for all students regardless of curriculum type. After adjusting for prior achievement, IC and SSC students did not perform statistically different on graphic ($F = .531$, $p = .467$) or numeric items ($F = .055$, $p = .815$). However, SSC students performed statistically higher ($F = 14.651$, $p = .000$) than IC students on *PCA* items that required symbolic representations. As Huntley et al. (2000) found, SSC students tend to do better with symbolic notation because it is given more attention in their curriculum.

Common misconceptions about function. The *PCA* was designed with distracters that are common misconceptions students have about functions. Therefore, if 20% or more of a specific curricula group chose a particular incorrect answer then it was considered a misconception because it was selected at a rate higher than random. IC students displayed misconceptions on items related to evaluating and solving functions; interpreting rates of change; function composition; and function inverse. SSC students displayed similar misconceptions on items related to interpreting rates of change and function inverse.

On function composition items IC students displayed misconceptions with regard to completing two input-output processes. Although the function composition items required students to use different representations to solve questions, when students were given the symbolic representations they seemed to be more successful. When given the function in other representations (numeric or graphic) students seemed to complete one input-output process, but were confused on what or how to work with the second input-output process.

Both IC and SSC student misconceptions with regard to rates of change were consistent with not interpreting the entire graph, but instead basing their answer on specific values or points (Carlson & Oehrtman, 2005). For example, given a graph (speed vs. time) with two functions on the same axes, students either interpreted it as a position graph, or they calculated the average speed, which was identical for the two cars, and subsequently interpreted this as the two cars were at the same position after one hour. Students did not consider all intervals of the graph and therefore made an incorrect conclusion about the position of the two cars.

On multiple function inverse items most IC and SSC students exhibited a misconception between the meaning of multiplicative inverse and function inverse. These results indicate that students in this study have a relatively weak procedural understanding of inverse functions, and most often were not able to apply their procedures in different contexts. Students in both curricula also demonstrated confusion with taking the reciprocal of the function.

The misconceptions that IC students displayed with evaluating, solving, interpreting rates of change, function composition, and function inverse raises some important questions. As some critics have argued, students who use integrated curricula lack important procedural understandings for higher level mathematics and the results of this item analysis reveal that this may be true. However, the results also indicate that students who use single-subject curricula have similar misconceptions with interpreting rates of change and function inverse. All of these concepts are important as students finish a college preparatory mathematics course. Moreover, as students continue on in either Calculus at the college level or enter AP Calculus in high school these misconceptions may hinder their understanding of the concepts of limits, derivatives, and integrals all essential to the study of calculus.

Performance of AP Calculus students on the PCA

2) How do high school students who studied four years of college preparatory mathematics curriculum (integrated or single-subject) and enrolled in AP Calculus perform and compare on college readiness concepts in terms of:
   a. overall performance;
   b. responses to items by levels of cognitive demand;
   c. responses to items focused on specific representations (e.g., graphic, numeric, symbolic); and
   d. common misconceptions about functions?

Overall Performance. No statistically significant difference ($F = 3.54$, $p = .063$) between the mean overall scores on the *PCA* for APSSC and APIC students (16.14 to 14.85) was found after adjusting for differences in prior achievement. Critics of the integrated curricula have suggested that high achieving students would be "held back" if they used the integrated curricula. The results from this study are contrary to this belief and reveal there is no statistical difference between integrated and single-subject AP Calculus students in their achievement level on the *PCA*.

The majority of AP students from both curricula, APSSC (85%) and APIC (63%), scored a 13 or above on the *PCA*. These results indicate that the high school students who completed either curricula path were performing at or above the level reported to receive a passing grade in college calculus (Carlson et al., in review).

Responses to items by levels of cognitive demand. After adjusting for prior achievement, APIC and APSSC students did not perform statistically different (F = 1.468, p = .227) from each other on items coded for the three levels of cognitive demand (*procedures without connections, procedures with connections,* and *doing mathematics)*. Overall AP Calculus students answered more higher level cognitive demand questions correctly than the sample of all students completing four years of college preparatory mathematics.

Responses to items focused on specific representations (e.g., graphic, numeric, symbolic). After adjusting for prior achievement, APIC students performed statistically higher (F = 4.971, p = .028) than APSSC students on *PCA* items that required numerical representations. On the other hand, APSSC students performed statistically higher (F = 6.996, p = .009) than APIC students on *PCA* items that required symbolic representations. Finally, APIC and APSSC students did not perform statistically different (F = 1.055, p = .306) on graphical items.

The numerical representation results were surprising although, it may be the case that students in the integrated curriculum tend to work more with tables than students in the single-subject curriculum. Although the curricula were not analyzed, the integrated textbooks have been described as focusing on various types of representation (Stein et al., 2007). This may account for why APIC students scored significantly higher on these items because they experienced numeric representations within their curriculum. The symbolic representation results are to be expected as other researchers (e.g., Huntley et al., 2000) have found similar results with single-subject students scoring higher than integrated students on symbolic manipulations items.

Common misconceptions about function. As described previously, the *PCA* was designed with distracters that are common misconceptions students have about functions. Therefore, if 20% or more of a specific curricula group chose a particular incorrect answer then it was considered a misconception. APIC students exhibited misconceptions with interpreting rates of change and working with composite and inverse functions. APSSC students had similar misconceptions related to interpreting rates of change and working with inverse functions.

Both groups of AP Calculus students displayed *action-oriented* views of functions because the majority (70%) of students correctly answered the evaluating and solving items (Dubinsky & Harel, 1992). This provides evidence that AP Calculus students who have completed either curriculum were able to obtain the procedural skills necessary for evaluating and solving functions regardless of the curricula.

A majority (64%) of the APSSC students correctly answered questions involving function composition compared to about half (48%) of the APIC students. However, an analysis of the function composition questions revealed that 6 of the 8 items required students to use a symbolic representation of the function, and this may explain why APIC students were less successful in answering these questions given their background with symbolic manipulation compared to APSSC students. However, as Dubinsky and Harel (1992) describe, it may be the case that APSSC students are moving towards a *process-oriented* view of functions; whereas, the APIC students still have an *action- oriented* view of functions.

As was reported above for all students, the AP Calculus students also struggled with interpreting rates of change and function inverse. Less than half the AP students in either curriculum were able to correctly interpret the rate of change or find or evaluate an inverse function. The majority of AP Calculus students seem to have a strong *action-orientation* of function as Carlson and Orhrtman (2005) describe. For example, students were given a graph of volume as a function of time as had to write a linear function to represent time as a function of volume. This may seem like a simple task; however, most students did not recognize or understand that the graph was the inverse of the function that was needed. Fifty-two percent of APSSC and 42% of APIC students incorrectly wrote $v(t) = 5t$, which was a correct function for the given graph, but not a correct equation for time as a function of volume. Another 20% of APIC students wrote $t(v) = 5v$ showing some understanding that they needed to have time as a function of volume; however they did not seem to recognize that the inputs and outputs were not related correctly for the given graph.

These results on interpreting rates of change are discouraging because this is a critical AP Calculus topic. If students do not have a solid understanding of rates of change, how can they be expected to develop an understanding of limits and derivatives? Even more discouraging is the fact that it does not matter which curriculum path students came through, all exhibited similar misconceptions about functions that are crucial for learning higher level mathematics, including calculus concepts.

Performance of AP Calculus students on Open-ended Tasks

3) How do high school students who studied four years of college preparatory mathematics curriculum (integrated or single-subject) and enrolled in AP

Calculus perform and compare in solution strategies and errors on open-ended mathematical tasks that focus on functions during the first quarter of AP Calculus?

Performance on the Piecewise Function task

The open-ended *Piecewise Function* task provided students the opportunity to use different strategies to answer the four questions embedded within the task. Overall, after adjusting for prior achievement, the APSSC students performed statistically higher than APIC students on this task (F = 10.320, p = .002). APSSC students also had higher averages on individual questions than APIC students. A higher percentage of APSSC students also attempted more of the questions than APIC students.

The results of APSSC students scoring higher on the *Piecewise Function* task were not surprising. Based on descriptions of single-subject curricula by others, (Senk & Thompson, 2003; Stein et al., 2007) students master definitions and standard algorithmic procedures through repetition. The *Piecewise Function* task was a non-contextual procedural task that required students to know how to compute rates of change and write and solve linear functions. Students studying the single-subject curricula should be proficient at completing this task after completing four years of college preparatory mathematics. However, the integrated curricula focuses attention on solving problems based on real world situations. It may be the case that the integrated students were at a disadvantage on this item because they had no context in which to relate the mathematics and make sense of the results. Another possible reason for the performance is that piecewise functions are not introduced in the integrated curricula or teachers chose not to teach this concept. Although a content analysis of the curricula was not part of this study, an examination of the index of the *Core-Plus* curriculum found only two piecewise function problems in the four textbooks with no section or unit devoted to piecewise functions.

On the other hand, it was surprising that APSSC students attempted more questions than APIC students. The integrated curriculum was designed so teachers can be classroom facilitators for students who are actively engaged in mathematical learning (St. John et al., 2004). If students are learning in an actively engaged classroom, it follows that they would be more willing to attempt a problem than to leave it blank.

Solution strategies students used on Piecewise Function task. Students in both curricula used similar strategies to solve the four questions on the piecewise function task. Approximately the same percentage of students used the graph and slope formula on the first two questions. Students from both curricula pathways used the slope-intercept form of a line to write their equations on question three. On the final question students used the graph to solve their equations most often. The results of this study indicate students in both curricula tended to use appropriate solution strategies.

Although students used similar strategies, they were not always successful with the strategies they chose to use. For example, a higher percentage of APSSC than APIC students used the formula strategy to find the rate of change; however, both groups of students were more successful with the graphing strategy. When students were required to write equations for the piecewise function, APSSC students were more successful using the slope-intercept formula than APIC students.

Errors students made on Piecewise Function task. Although student solution strategies were similar across the two curricula, errors were distinguished by the curricula. For example, 47% of APIC students gave an incorrect interval for the rate of change as compared to 24% of APSSC students. Although both groups of students were able to use the slope formula to find the rate of change, APIC students did not attend to the direction of the rate of change. Finding the rate of change takes into account both magnitude and direction; however, APIC students only attended to the magnitude in their explanations and justifications. Although some APSSC students made this same error, the percentage was almost half that of the APIC students.

Another error specific to the APIC students was writing the domain (x-value) restrictions on a piecewise function. The majority (53%) of APIC students denoted incorrect restrictions in the answer, whereas, a quarter of APSSC students made this same error. Errors made only by APSSC students tended to be either forgetting to include the endpoint in their restrictions or they used the y-values of the coordinate point to write their restrictions. On the other hand, APIC students either left the question blank or wrote equations with no restrictions. These errors demonstrate a different understanding of piecewise functions for the two groups of students. Although APSSC students had errors, their understanding seemed more complete because they realized that the equations needed restrictions. However, APIC students who wrote equations without restrictions may not have realized a need for the restrictions. Without interviewing students, an answer for why students made these errors is not possible.

Writing equations for linear functions was another error made by approximately the same percentage of students in both groups on the *Piecewise Function* task. More specifically, about 40% of students were not able to write linear functions for the piecewise function regardless of the curricula. This error is rather surprising because all of these students have completed four years of college preparatory mathematics. Linear functions are the first function introduced to high school students and typically reviewed each year. Yet, 40% of AP Calculus students were not able to write these equations. Frequent errors made writing the equations were: (1) students found a correct rate of change, but excluded the y-intercepts; (2) students wrote a correct rate of change, but then did a transformation on the y-axis rather than on the x-axis for the y-intercepts. Both of these errors were committed by approximately the same percentage of students in each curriculum.

Performance on Filling a Tank task. After adjusting for prior achievement, there was no statistically significant difference ($F = 2.591$, $p = .110$) between the APSSC and APIC students on the open-ended *Filling a Tank* task. APSSC students had higher averages than APIC students on four of the questions; whereas, APIC students had higher averages than APSSC students on the other two questions. A higher percentage of APIC students attempted more of the questions than APSSC students.

The absence of difference in performance between the two groups of students on the *Filling a Tank* task was not surprising because based on other studies (Harwell et al., 2007; Huntley et al., 2000; Schoen & Hirsch, 2003a) integrated students usually perform equal to or better than single-subject students on problem solving tasks. The *Filling a Tank* task was contextual and required students to understand the procedural steps for finding rate of change; but more importantly, they had to interpret the meaning of a real world situation into mathematics on a graph. The task also required students to understand rates of change that were not all linear. Given their four years of high school mathematics, both groups of students should have had the necessary mathematical knowledge and skills to complete the task.

The fact that APIC students attempted more questions on this task than APSSC students was not surprising. The integrated curriculum was designed for students to be actively engaged in mathematical learning (St. John et al., 2004). This task provided APIC students with a familiar situation where they could become actively engage in the mathematics using a contextual situation.

Solution strategies students used on Filling a Tank task. Students in each curriculum used similar strategies to solve the two questions on the *Filling a Tank* task that were analyzed. The majority of students, APSSC (85%) and APIC (75%), used the slope formula on the first question. There were differences in the most popular strategy used by each group for the second question. APSSC students used the graph while APIC students compared rates of change to determine the maximum water level.

These results suggest that students in each curriculum developed similar strategies during their mathematics preparation. Although the organizational structure of the mathematics curricula may be different, students from each curriculum tended to choose strategies appropriate for solving the task.

Although students used similar strategies, they were not always successful with the strategies they chose to use. For example, 20% of APSSC students used the graph strategy to find the maximum water level in the tank; however, only one-fourth of these students were successful finding the maximum water level in the tank. Although a smaller percentage of APSSC students (18%) used the comparing rates of change method, these students were more successful with 11% finding the maximum water level.

The majority of APSSC (64%) and APIC (51%) students were successful using the slope formula for finding rates of change for two different time periods.

Errors students made on Filling a Tank task. Student solution strategies were similar across the two curricula as were the errors. On three of the six questions students from each curriculum demonstrated errors with rates of change. For example, one-third of students from each curriculum drew an incorrect middle section of the graph. The middle section of the graph related to a varying rate of change. Although a large percentage of students in each curriculum successfully found rates of change with the slope formula on the second question, the majority of these students were unable to explain the meaning or draw a correct graph with a varying rate of change. These concepts are foundational in the learning of calculus concepts.

Another error across curricula was drawing an incorrect graph of water being pumped out of the tank. Students did not interpret the contextual situation into a graphical representation correctly. Although the rate of change given was constant, 16% of APSSC and 25% of APIC students did not draw a correct function. Another 21% of APSSC students and 23% of APIC students recognized the function was linear; however, either they did not start at a correct point or they drew a decreasing function to represent water being pumped out of the tank.

Summary

Since the information from this segment of the study is based on student written responses not one-on-one interviews, the reasons why students used particular solution strategies or made the errors is impossible to determine.

A reoccurring theme across responses to all three instruments for both groups of students was misconceptions and errors related to rate of change, a foundational concept in the study of calculus. Regardless of their curricular path, AP Calculus students seemed to have procedural knowledge of how to calculate the rate of change, but seem to lack the conceptual knowledge that is needed to learn calculus topics.

Implications of the Study

This section provides a discussion of the implications of findings for various stakeholders. The discussion begins with an examination of the implications for curriculum developers, then moves to suggestions for researchers and concludes with implications for teachers and society in general.

Implications for curriculum developers

As research is conducted and reported, curriculum developers gain information to inform future changes to improve textbooks to help students better understand and learn mathematics. While approaches to curriculum development vary greatly, ideally, each new edition of a curriculum should be based on how the curriculum has worked in classrooms, including feedback from teachers and students who have used the piloted or latest versions of the curriculum.

Results of this study indicate that curriculum developers, both single-subject and integrated, need to provide additional attention to the topics of rates of change, function composition, and function inverse. These three concepts are crucial for higher level mathematics, yet results of this study reveal low performance for students regardless of their curriculum path. Students were well versed on the procedural skills of the slope formula; however, they lacked the ability to interpret rates of change in different contexts such as on a graph or in a word problem.

The vocabulary used within textbooks for rate of change is important and may, in fact, have contributed to student errors and misconceptions. Students struggled to find a rate of change of 2 on a piecewise function graph. Student explanations point to a misunderstanding of the meaning of slope, rate of change, and steepness. A content analysis of the vocabulary terms in these curricula was not done. However, a cursory search in the textbooks for rate of change and slope found instances in both curricula where the words slope and steepness were related as were rate of change and steepness. Although these words are used interchangeably, the question is: are they all defined to represent the same concept? Steepness relates to the magnitude of a line; however, rate of change and slope relate to the magnitude and direction of a line.

Curriculum developers of the integrated mathematics curriculum should be concerned that students who completed four years of college preparatory mathematics and performed above-average on a nationally normed assessment *(IAAT)* struggled with evaluating and solving functions. This study was based on students using the first edition of their textbook. Changes in the second edition of this curriculum have focused on the symbolic manipulation and connecting algebraic forms of different types of representations such as numeric and graphic (*Core-Plus* Mathematics Project, 2007). Therefore, it might be the case that some of the misconceptions found in the study will not be valid for students studying from the newer edition.

Implications for researchers

Studies conducted by external researchers (Harwell et al., 2007; Hill, R. O. & Parker, 2006; Huntley et al., 2000; Smith III & Star, 2007) have documented mixed results on what high school students who use an integrated mathematics curricula learn. Researchers need to continue to delve deeper into *what* high school students using the two curricula are learning or not learning in each curricula (Harwell et al., 2007).

Part of the results of this study concur with Harwell et al. (2007) "that *Standards*-based [integrated] curricula do not impede the mathematical performance and development of high achieving students" (p. 95). There were no statistically significant differences on overall scores on the *PCA* or the two open-ended tasks between AP Calculus students who studied either of the two curricula. The majority of the AP Calculus students scored above a 13 on the *PCA*, which should qualify them to be successful in calculus (Carlson et al., in review).

Other researchers (Cai, Lane et al., 1996; Lesh & Zawojewski, 2007; Resnick, 1988; Robitaille & Travers, 1992) have argued that *how* students solve problems may be more important information than whether students get correct answers, particularly to understand the benefits and limitations of particular curricula alternatives. The mathematics education field would benefit by delving deeper into how students approach problems; what types of representations they choose to use; and how they interpret answers.

This study provides a glimpse into AP Calculus students' solution strategies and errors on open-ended tasks related to functions. Results revealed that students across the two curricula are using similar strategies. Although students used two different curricula, similar strategies are being learned by all students. Noticeably, some of the errors students made are different across the curricula, and other errors, such as interpreting rates of change, are common across the curricula. This leads to more questions about *what* students are learning or not learning from the two curricula.

The research agenda for the field should move from deciding which group performs better, to *how* students are solving, representing, and justifying mathematical problem situations based on whatever curriculum they have experienced. Additional research on performance on different mathematical topics within the geometry and statistics and probability strands that have been found to be difficult for U.S. students should be pursued. This line of research will provide curriculum developers with valuable information to help revise and strengthen their mathematics curriculum.

Implications for teachers

Teachers are an integral part of the school system because they help students learn mathematics. Teachers who are not aware of the two curricula pathways and what each has to offer, may not be able to provide guidance and direction to students. New students enter teachers' classrooms every year with prior knowledge of mathematics. The teacher's job is to build on this prior knowledge so students continue learning and building new knowledge of mathematics. If teachers are not aware of what students learned from their curriculum, how can they begin to build on prior knowledge?

Results from this study indicate that although students' solution strategies are similar across the two curricula, single-subject students tend to be more effective in using whatever strategies they chose. This is important for teachers, but especially for AP Calculus teachers because as both groups of students enter the same class some integrated students may struggle. Not because they do not understand the mathematics, but because their skills have not been practiced enough and they may make more errors.

This study also indicates that students from both curricula struggle with the concept of rate of change. This is a central idea of calculus, yet the expected level of knowledge is not being achieved. AP Calculus teachers must understand that students entering their classroom have misconceptions about rate of change. Many students are capable of using the slope formula to find the rate of change; however, they are unable to interpret the meaning of rate of change in a contextual situation or with a given graph.

AP Calculus students also showed misconceptions about inverse functions. The majority of students seemed to not understand the difference between multiplicative inverse and function inverse. If teachers are not aware of these misconceptions, teaching topics such as limits and derivatives will likely be difficult. However, if teachers are aware of these common misconceptions, they can work through them at the start of the course so all students are on the same playing field as they begin their study of calculus.

Implications for society

Critics (e.g., Hill, R. O. & Parker, 2006; e.g., Wu, 1997) of the integrated curricula have documented that students are not learning the mathematics that they should. However, the results of this study indicate that students successfully completed four years of either the integrated or single-subject curricula and entered AP Calculus with similar mathematical understandings. Although this study represents one school district in the Midwest, it provides a glimpse into what high school students are learning in the two curricula. As Figure 1.1 depicts, there is an overlapping of the mathematical topics that these students learn during their high school mathematic courses. The results of the study indicate that the intersection of the two circles may be greater than previously thought.

Society needs to be familiar with both curricula paths and be able to identify the benefits of each curriculum for the learning of mathematics (Reys, R. E. & Reys, in press). Although the curricula are organized differently, the fundamental mathematical concepts are addressed in both curricula. There may not be a “right” curriculum for every student as Hiebert (1999) stated, “if researchers cannot prove that one course of action is *the* best one, it follows that researchers cannot prescribe a curriculum and a pedagogical approach for all students and for all time” (p. 7). However, researchers can document the current status of reform efforts such as what mathematics curriculum is being used and how students learn from it. Researchers can also document the effectiveness of curricula such as “what students *can* learn under what kinds of conditions” (Hiebert, 1999, p. 9).

Limitations of the Study

This study compared two groups of students who earned four units of mathematics credit in an integrated course sequence or in a single-subject sequence. All students earned four units of mathematics, yet in addition to following two different curricula paths these students had different mathematics teachers and this represents an important variable that was not controlled. Teachers are an integral part of what happens in the classroom. They influence what students have the opportunity to learn as well as how they are taught the mathematics. The NRC report (2004) argues strongly for the need to collect implementation data so researchers know students had an opportunity to experience the curriculum being investigated. However, this study relied on data gathered at the end of students four years of mathematics course work. The students in each curricula path experienced different teachers and there is no guarantee that teachers taught the curriculum as intended by the developers.

Another limitation of this study is the number of students in the study and how they were assigned to the two groups. The sample was 505 students in one district from two high schools that were enrolled in Integrated 4, Honors Integrated 4, Precalculus, and Honors Precalculus and had taken the previous mathematics courses in their respective sequencing. However, the sample was further reduced to those students for whom an *IAAT* score was available. Students were also not randomly assigned to the two curricula paths, which limits the inferences about causation.

The study was also limited to the use of two open-ended tasks that were not piloted with other students to gain insight into problems with the wording of questions on the two tasks. The *Piecewise Function* task was taken from an AP Calculus teacher who had simplified a past AP Calculus free-response question from the 2003 exam. The three AP Calculus teachers involved in the study helped choose the two open-ended tasks for their students, yet none of the teachers or the researcher realized that two of the questions would prove to be ambiguous for students because of wording and/or the graph that was provided. For example, on question 1 of the *Piecewise Function* task the question asked students "for what interval(s) of x does f(x) have a rate of change of 2". A better question (and more mathematically correct) would have been to add the word *average* before rate of change.

Another limitation of this study was that five NSF high school integrated curricula are used in U.S. classrooms; however, this study focused solely on the integrated curriculum developed by the *Core-Plus Mathematics Project*. The results of this study are based only on students who completed the first edition of the *Contemporary Mathematics in Context: A Unified Approach* series. Recently a second edition of the *Core-Plus* textbooks was developed and changes were made to reflect years of research with students and teachers to develop skills and concepts that were deemed weak in the previous edition. Three of these changes were:

1) An accelerated introduction of symbol-based reasoning – including symbol manipulation and proof.
2) More explicit development of symbol sense – connect algebraic forms to numeric, graphic, and context interpretations and implications.
3) Earlier introduction of inverse functions and logarithms. (*Core-Plus* Mathematics Project, 2007)

The students in this study used the first edition of the *Core-Plus* curricula; therefore, the results are a reflection of an older version of the curriculum the students in this study had used or experienced.

A final limitation of this study was the absence of a content-analysis of the curricula used by the students. Understanding what students have had the opportunity to learn within their textbook sequence and what teachers chose to implement is critical in interpreting results based on an assessment that may or may not be aligned with the curricula. Understanding how mathematics topics and concepts are introduced and built upon would also help to determine why students used specific strategies or why they committed certain errors. A content-analysis and teacher implementation data could also help in establishing what vocabulary words are used within the two curricula and how students might be interpreting specific words such as rate of change, slope, and steepness.

Recommendations for Future Research

The "math wars" have pushed the mathematics education field to supply answers about the different mathematics curricula that are available for use in schools. Limited research has been conducted at the high school level in regard to integrated mathematics curricula. Although this study provides a snapshot of high school students mathematical understanding after studying from either integrated or single-subject curricula, additional research needs to be conducted on *what* students are learning from these two curricula. This study focused on students understanding of functions and algebra, other studies should investigate other content areas. More research needs to be conducted at different points through the high school years. It may be beneficial for data to be collected after each year of learning from these curricula. Although the curricula are organized differently, it may be useful to know when students are learning specific topics to examine the impact of the mathematical sequence.

An extension of this study would be to follow students through the end of AP Calculus. Research should be conducted to analyze how the initial misconceptions students have about rate of change affect them during their AP Calculus course and what influence it has on their ability to be successful in the class. Research is needed to understand how students perceive these curricula by researching what students believe to be the benefits of the different curricula. Another area of needed research is what happens when students switch between mathematics curricula pathways. Many students switch between the two curricula pathways, whether it is in the same school or because they move to a new area. Understanding what students are learning or not learning from being in each curriculum path can help in revisions of current curricula.

Another area of needed research would be to follow all students who have completed the four years of either curriculum as they transition into college. Although some research has been conducted in this area (e.g., Smith, J. P., III & Star, 2007), the focus has been on student attitudes and ability to move into a different curricula type (from integrated to single-subject). A different dimension of research would focus on identifying misconceptions of functions prior to entering college, followed by investigating how those misconceptions affect their learning in college mathematics.

Another area of research concerns the content-analysis of the integrated and single-subject curricula. Research should explore what mathematics is available for students to learn in the two curricula. If students are to have a similar mathematical understanding after studying four years of mathematics in these two curricula, then it will be important to investigate what students have an opportunity to learn from textbooks or their teachers. Research should also focus on the amount of time that is spent on specific topics. Although each curriculum develops the concept of rate of change, it is important to understand not only how much attention is given to this topic, but also how the mathematics is introduced and built upon throughout the curriculum. Finally, research should investigate the different mathematical vocabulary used within the curricula. As was found in this study, students seem to be using the words such as rate of change, steepness, and slope interchangeably; however, steepness relates to the magnitude of the line and not direction. It is important that curricula use appropriate language.

Finally, research is needed to better understand how teachers implement these two curricula and the influence of the implementation on students' learning of the mathematics. Studying how teachers who have different mathematical knowledge background teach the mathematics from these two curricula can provide new information to support teacher preparation. Research is also needed to determine how teachers interpret these two curricula. Teachers who are teaching both curricula may combine what they believe to be robust from both curricula to create a “new hybrid curriculum”. If so, what effect will this have on student learning? Are students hurt by this decision or are they helped to have a clearer understanding of the mathematics being taught?

Summary

This study documented high school students’ mathematical knowledge after studying four years of college preparatory mathematics courses (integrated or single-subject). One sample of 505 students and a sub-sample of 199 students who enrolled in AP Calculus participated in the study. Data were collected from four instruments, one multiple choice assessment [*Iowa Algebra Aptitude Test (IAAT)*] to document prior achievement before entering either curriculum path, another multiple choice assessment [Precalculus Concept Assessment (*PCA*)] and two open-ended tasks (*Filling a Tank* and *Piecewise Function*) to document students understanding of functions. Students’ solution strategies, errors, and misconceptions with regards to functions were analyzed. A qualitative and quantitative analysis was conducted to describe how students in each curriculum performed on the three different instruments. In addition, the analysis was used to describe the solution strategies students employed, as well as common errors students made across the two curricula.

The findings confirm that AP Calculus students in both curricula are using similar strategies as they solve open-ended tasks related to functions. Data also documents that students who have studied from either curricula have similar misconceptions about rate of change and function inverse. Students understanding of rate of change is procedural oriented and they struggle to interpret rates of change or varying rates of change. Student misconceptions with regard to function inverse seem to be based on confusion between multiplicative and function inverses. This study indicates that statistically significant differences exist between the two groups of students on items that require either symbolic or numeric representations after controlling for prior achievement. Overall, after adjusting for prior achievement, the single-subject students outperformed the integrated students on the *PCA*. However, students from both curricular paths who were enrolled in AP Calculus performed at the same level on the *PCA* after adjusting for prior achievement.

Helping high school students learn higher level mathematics is imperative for their continued success in college and keeping career options open. Societal views of the integrated and single-subject curricula have raised questions and some schools have been reluctant to adopt an integrated curriculum. Opponents who criticize the integrated curricula claim that students are not learning mathematics. This study provided findings that can add to the research already conducted on what students who study from the integrated and single-subject learn or do not learn. This study also provided mixed results and is a reminder of the complexity of researching student mathematical learning in the educational system. Although one study will not answer the question, this study contributes to the mathematics education field by providing documentation of the effectiveness of curricula by examining "what students *can* learn under what kinds of conditions" (Hiebert, 1999, p. 9).

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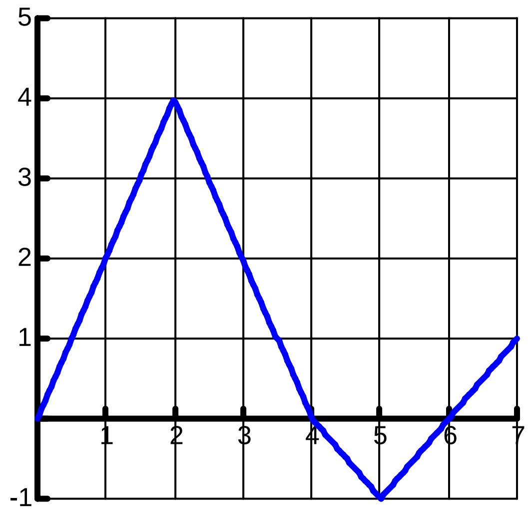
APPENDIX A

Two Open-ended Tasks

ID#:______________________ NAME:_____________________

Piecewise Functions

***Show all work on the paper provided. Explain your thinking process for each question.**



Graph of f(x)

Let *f(x)* be a function defined on the closed interval [0, 7]. The graph of *f(x)*, consisting of four line segments, is shown above.

1. For what interval(s) of *x* does *f(x)* have a rate of change of 2? Explain how you got your answer.

2. On which interval of *x* is the rate of change of *f(x)* the greatest? Justify your answer.

3. Write the piecewise function for *f(x)*.

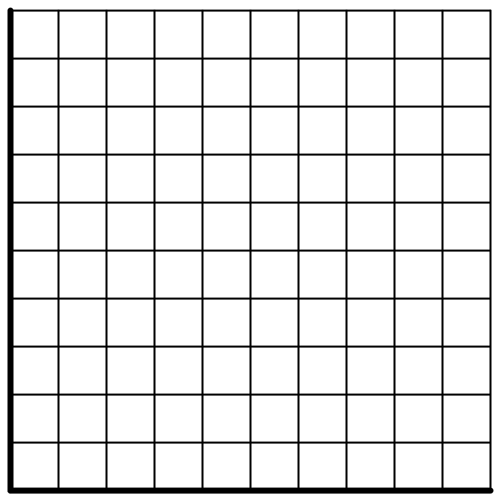
4. Find the values of *x* for which *f(x)* = 1. Explain how you got your answer.

ID#:______________________ NAME:_____________________

Filling a Tank (Taylor, 1992)

At time $t = 0$, water begins to flow from a hose into an empty tank at the rate of 40 litres/min. This flow rate is held constant for 2 minutes, at which time the tank has 80 litres. At that point ($t = 2$), the tap is gradually closed until at $t = 4$, the flow rate equals 5 litres/min. This flow rate is held constant for the final 2-minute period, at which point ($t = 6$) the tank is observed to contain 120 litres.

a) Draw a graph (label axes) of the volume V of water in the tank (litres) against the time t (min) for $0 \leq t \leq 6$. Explain how you decided to draw the graph.



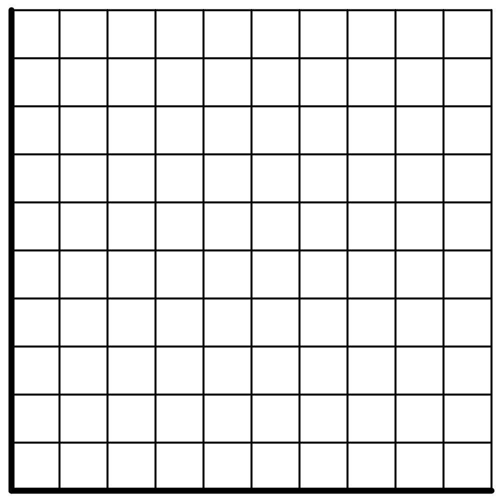
Explanation:

b) What is the average rate of flow into the tank over the middle 2-minute period ($2 \leq t \leq 4$) and over the entire 6-minute period?

c) Explain how the rates found in (b) can be interpreted on your graph.

Now suppose that, in addition to the above sequence of events, a pump is started up at t = 2, and , for the next 4 minutes, pumps water out of the tank at the constant rate of 15 litres/min.

d) What we have called V above, now represents the total amount of water that has flowed into the tank at time t; let W represent the total amount of water pumped out of the tank at time t. Plot V and W against t on the **same set of axes**. Show on your graph how to interpret the volume of water in the tank at any time.



e) Show or explain how to find on your graph of (d) the point at which the water level in the tank is at a maximum.

f) What is the instantaneous flow rate from the hose into the tank at this point?

APPENDIX B

Scoring Guides for Open-ended Tasks

Scoring Rubric: Piecewise Function Task 1

1.	Student gives the correct interval [0,2] *Student gives the interval [0,2] and [2,4] 2 points* *Student gives the interval [0,4] 1 point*	3 points
	Student provides correct explanation for an interval (either [0,2], [2,4], or [0,4] by: Implicitly or explicitly refers to OR finds the magnitude of the slope to be 2. BUT NOT I found the slope or used the slope OR Uses the graph by creating a right triangle or shows the change in y over the change in x is equal to 2. BUT NOT I used the graph or the graph shows you OR Equivalent argument	4 points

Total Points	**7 points**

Strategy Employed:

A – Formula B – Graph C – Table D – Multiple E – None or can't tell

2.	Student gives the correct interval [0,2] *Student gives the interval [0,2] and [2,4]* *2 points* *Student gives the interval [0,4]* *1 point*	3 points
	Student provides a valid justification for the correct interval(s) by: Using the slope of each line to provide justification for their answer OR Uses the graph to explain that the interval(s) chosen have the greatest slope or rate of change. OR Equivalent argument	4 points

Total Points	**7 points**

Strategy Employed:

A – Formula B – Graph C – Table D – Multiple E – None or can't tell

3.	Student provides y = 2x or equivalent equation	1 point
	Student provides the correct restrictions $0 \le x \le 2$, $0 \le x < 2$ or $x \le 2$	½ point
	Student provides y = -2x + 8 or equivalent equation	1 point
	Student provides the correct restrictions $2 \le x \le 4$ or $2 \le x < 4$	½ point
***	*Student provides the equation* $y = -2\|x-2\|+4$ *instead of the two equations* y = 2x and y = -2x + 8 *(2 points)*	
	Student provides the correct restrictions $0 \le x \le 4$ *or* $x \le 4$ *instead of* $0 \le x \le 2$ *and* $2 \le x \le 4$ *(1 point)*	
	Student provides y = -x + 4 or equivalent equation	1 point
	Student provides the correct restrictions $4 \le x \le 5$ or $4 \le x < 5$	½ point
	Student provides y = x – 6 or equivalent equation	1 point
	Student provides the correct restrictions $5 \le x \le 7$ or $x \ge 5$	½ point
***	*Student provides the equation* $y = \|x-5\|-1$ *for instead of the two equations* y = -x + 4 and y = x – 6 *(2 points)*	
	Student provides the correct restrictions $4 \le x \le 7$ *or* $x \ge 4$ *instead of* $4 \le x \le 5$ *and* $5 \le x \le 7$ *(1 point)*	

Total Points	**6 points**

*** Special cases

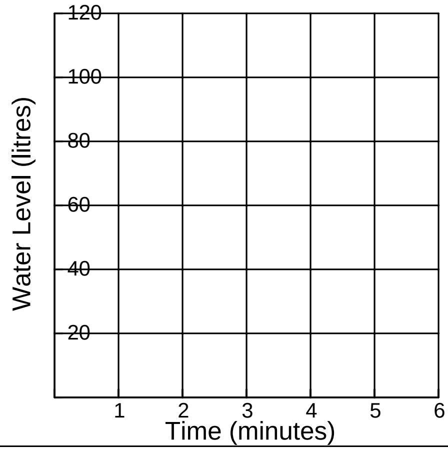
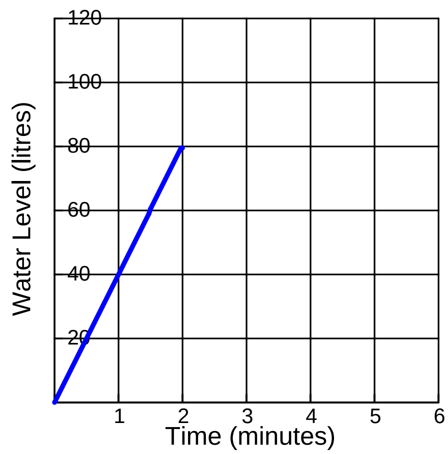
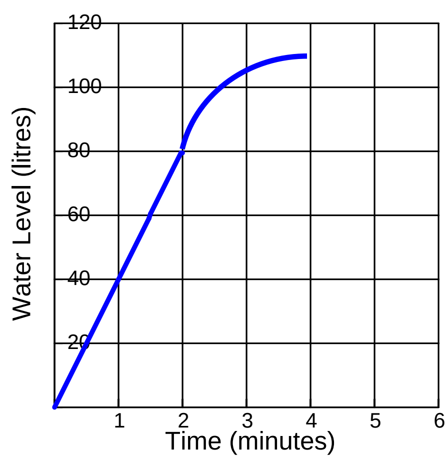
4.	Student provides x = $\frac{1}{2}$	1 point
	Student provides x = $\frac{7}{2}$	1 point
	Student provides x = 7	1 point
	Student provides correct explanation for the correct x-values by: Uses the equations by setting y = 1 and solving for x in each equation OR Uses the graph implicitly (describes what they did) or explicitly (drawing a line at y = 1) and then finds the three points where the function crosses y = 1. BUT NOT I used the graph or the graph shows you OR Equivalent argument	1 point
	Student sets all four equation found in question 3 equal to 1 and solves for x resulting in 4 x-values and does not recognize that one answer does not exist. *OR* *Student uses the equation y = -x + 4 solving for x and getting 3* *(-1 point)*	

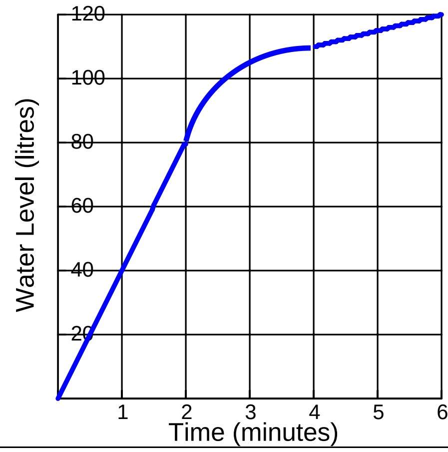
Total Points	**4 points**

Strategy Employed:

A – Equation B – Graph C – Table D – Multiple E – None or can't tell

Scoring Rubric: Filling a Tank Task 2

a.	Student labeled both axes with numbers 	1 point
	Student draws one correct segment 	1 point
	Student draws two correct segments 	1 point

<table>
<tr><td></td><td>Student draws a complete and correct graph
</td><td>1 point</td></tr>
<tr><td></td><td>Student ‘s explanation refers to:

The slope or flow rate of the water at the three different time periods (constant or slows down)

OR

Equivalent argument

Partial Credit (1 point)
Student refers to one or two specific time period but not all three

OR

Student only explains that they plotted given points and connecting the dots

OR

It started fast and then gradually slowed down
BUT NOT
I followed the directions</td><td>2 points</td></tr>
</table>

Total Points	**6 points**

b.	Student gives the correct flow rate for t = [2,4] as 15 litres/min	1 point
	Student gives the correct flow rate for t = [0,6] as 20 litres/min	1 point
	Student does not have correct units or leaves off the units (-1 point)	

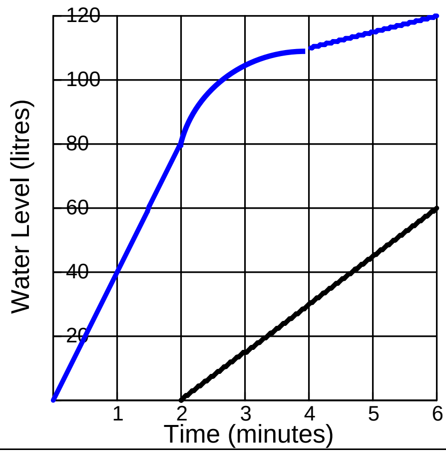
Total Points	**2 points**

Strategy Employed:

A – Formula B – Graph C – Table D – Multiple E – None or can't tell

c.	Student provides an explanation that relates to the graph: By drawing a straight line (secant) between the initial and endpoints on the graph OR Refers to the slope as the rate of change OR Relates the rate to the steepness (concavity) of the line OR Equivalent argument	1 point
	Student mentions [0,6] either explicitly or implicitly [i.e., overall, from the beginning to the end, chosen intervals]	1 point
	Student mentions [2,4], either explicitly or implicitly [i.e., in the middle]	1 point

Total Points	**3 points**

<table>
<tr><td>d.</td><td>Student labeled both axes with numbers
</td><td>1 point</td></tr>
<tr><td></td><td>Student draws a correct graph of W</td><td>1 point</td></tr>
<tr><td></td><td>Students interpret the volume of water in the tank by:

Labeling on the graph the area between the two functions (W and V)

OR

Student draws a new function to show the amount of water in the tank by subtracting the two values and finding a new point.

OR

Equivalent method</td><td>2 points</td></tr>
</table>

Total Points	**4 points**

e.	Student finds the maximum water level at approximately 2.8 minutes [accept answers between 2.5 and 3.2 minutes] or at approximately 87 litres [accept answers between 84-90 litres] *Partial Credit (1 point)* *Student finds the water level to be between 2 and 4 minutes or between 80-90 litres*	2 points
	Student explains using the graph that they have drawn (whether correct or not) to find the maximum water level by: Subtracting the two functions (V-W) to find the largest difference in volume OR Discusses the relationship between the V and W graph and how by translating the W function up to match the V function this will be the point of the maximum water level. (Discusses the maximum will happen when the rate at which the flow in is equal to the flow out) OR Uses the new function on the graph that accounts for the difference between the V and W functions (whether they are correct or not) and has labeled or given the maximum point. OR Equivalent argument *Partial Credit (1 point)* *Student refers to a new graph by subtracting values, but does not label the maximum point.*	2 points

Total Points	**4 points**

Strategy Employed:				
A – Formula	B – Graph	C – Table	D – Multiple	E – None or can't tell

f.	Student find the instantaneous flow rate at 15 litres/min	1 point

Total Points	**1 points**

APPENDIX C

Levels of Cognitive Demand Codes assigned *PCA* items

	Memorization	*Procedures without Connections*	*Procedures with Connections*	*Doing Mathematics*
PCA Item Numbers		1, 3, 4, 5, 6, 12, 14, 16, 21, 23	2, 7, 9, 10, 11, 13, 18, 19, 20, 22, 24, 25	8, 15, 17
Percent of Items (N=25)	0%	40%	48%	12%

APPENDIX D

Codes assigned *PCA* items for Representation Type

	Graphic	*Numeric*	*Symbolic*
PCA Item Numbers	2, 5, 6, 8, **9**, **10**, 15, 19, 24	12, 13	1, 3, 4, 7, **9**, **10**, 11, 14, 16, 17, 18, 20, 21, 22, 23, 25

Bolded items were coded in two different categories because both types of representation were used to complete the item.

VITA

Dawn Teuscher was born on September 22, 1970 in San Diego, California. She is the daughter of Lynn and Cheryl Teuscher and sister to six brothers and one sister. She attended public school in Encinitas, California and graduated from San Dieguito High School class of 1988. She has earned the following degrees: BA in Mathematics Education from Brigham Young University, Provo, Utah (1994); MS in Exercise and Sport Science from University of Utah, Salt Lake City, Utah (2000); PhD in Mathematics Education from University of Missouri, Columbia, Missouri (2008). In between degrees she taught high school mathematics for 10 years. Dawn is currently a member of the School of Educational Innovation and Teacher Preparation at Arizona State University Polytechnic Campus located in Mesa, Arizona.