BOX APPROXIMATION AND RELATED TECHNIQUES IN SPECTRAL THEORY

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The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

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ABSTRACT

This dissertation is concerned with various aspects of the spectral theory of differential and pseudodifferential operators. It consists of two chapters.

The first chapter presents a study of a family of spectral shift functions ξ^r, each associated with a pair of self-adjoint Schrödinger operators on a finite interval $(0, r)$. Specifically, we investigate the limit behavior of the functions ξ^r when the parameter r approaches infinity. We prove that an ergodic limit of ξ^r coincides with the spectral shift function associated with the singular problem on the semi-infinite interval.

In the second chapter, we study the attractor of the dynamical system $r \mapsto \mathcal{A}_r$, where $\mathcal{A}_r$ is the truncated Wiener-Hopf operator surrounded by operators of multiplication by the function $e^{\frac{\alpha}{2}|\cdot|}$, $\alpha > 0$. We show that in the case when the symbol of the Wiener-Hopf operator is a rational function with two real zeros the dynamical system $r \mapsto \mathcal{A}_r$ possesses a nontrivial attractor of a limit-circle type.

In his dream the Box Man takes his box off.
Is this the dream he had before he began
living in a box or is it the dream of his life
after he left it ...

Kobo Abe

Introduction

In this dissertation we discuss various aspects of the spectral theory for differential and pseudodifferential operators on finite (i.e., regular problems) as well as on (semi)-infinite intervals (i.e., singular problems) with the emphasis on the relation between the two problems as the length of the interval gets large.

The dissertation consists of two chapters. Chapter 1 is devoted to the study of fine properties of the eigenvalue counting function of a one-dimensional Schrödinger operator on large intervals (with Dirichlet boundary conditions at the endpoints) when the length of the interval approaches infinity. Assuming that the corresponding problem on the half-line is in the limit-point case at infinity, we study the behavior of the counting function in the limit of large intervals. In addition to the standard requirements we assume that the potential V is a real-valued function integrable with a finite first moment at infinity. Our main result establishes a connection between the characteristics of the absolutely continuous spectrum of the half-line problem and the discrete spectrum of the finite-interval problem. More precisely, we prove that the ergodic limit of the difference of the eigenvalue distribution functions associated with the Schrödinger operator with potential V

and the free Schrödinger operator on the finite interval, coincides with the spectral shift function corresponding to the pair of half-line operators. We note that the idea of comparing spectral characteristics of the problems on finite and infinite intervals goes back to the classics of the spectral theory for ordinary differential operators. For a recent application of a similar approach, we refer to [19], where well-known results about eigenvalues were used to obtain new convexity properties for the phase shift.

In Chapter 2, we focus on a qualitative spectral analysis of the family

$$(0, \infty) \ni r \mapsto \mathcal{A}_r$$

of unbounded operators on the space $L^2(\mathbb{R})$. Here $\mathcal{A}_r = W_\alpha(I - P_r \mathcal{L} P_r) W_\alpha$, W_α is the operator of multiplication by the function $e^{\frac{\alpha}{2}|x|}$, $\alpha > 0$, $\mathcal{L}$ is a self-adjoint integral operator of convolution type with kernel L satisfying $e^{\beta|\cdot|}L(\cdot) \in L^\infty(\mathbb{R})$, $\beta > \alpha$, and P_r is a projection of $L^2(\mathbb{R})$ onto the subspace $L^2((-r, r))$. The main goal is to study the attractor of the dynamical system $r \mapsto \mathcal{A}_r$ in the sense of norm resolvent convergence in the case when the symbol $l = 1 - \widehat{L}$ of the integral operator $I - \mathcal{L}$ is a rational function with exactly two zeros, both real (our methods, however, can be easily extended to the case of finitely-many real zeros). We prove that the dynamical system $r \to \mathcal{A}_r$ has a limit cycle consisting of a special one-parameter family of self-adjoint extensions of the symmetric operator $\mathcal{A} = W_\alpha(I - \mathcal{L}) W_\alpha$.

We would like to mention that the study of integral equations of convolution type with meromorphic symbols is closely connected with the one of the quantum mechanical three-body problem with short-range forces, where the phenomenon known as the Efimov effect is known to arise [16] (see also [2], [30], [35], [36], and [38]). It is also related to the three-body problem with point-like interactions and the so-called "fall to the center" phenomenon (see [17], [18], and [28]). In addition we remark that the methods developed in Chapter 2 are useful in the spectral analysis of the Herbst hamiltonian introduced in [24] (especially in the non-semibounded case).

Notice that in both problems the transition from a truncated operator to the operator in the limit is not an easy task. In the first problem, the nature of the spectrum of the truncated operator is substantially different from that of the half-line operator, which makes the convergence of spectral characteristics possible only in some averaged sense. In the second problem, the formal limit object

$$\mathcal{A} = W_\alpha(I - \mathcal{L})W_\alpha$$

is a symmetric operator, neither essentially self-adjoint, nor semibounded from below, while all truncated operators are self-adjoint. This leads to a nontrivial asymptotic behavior of the dynamical system $r \to \mathcal{A}_r$ and we show that the family $\mathcal{A}_r$ has an attractor of limit circle type in the space of bounded operators.

Chapter 1

On the ergodic limit of the spectral shift function

1.1 Introduction

In this chapter, we study the spectral shift function associated with a Schrödinger operator on the interval $(0, r)$ (with Dirichlet boundary conditions at the endpoints) and the spectral shift function for the corresponding problem on $(0, \infty)$. The main goal is to establish a connection between the two problems when the cut-off parameter r is getting large.

The concept of a spectral shift function goes back to I. M. Lifshits, who introduced it in the 1950's in connection with some problems in solid state physics. In his work, there appeared what is now called the trace formula associated with two self-adjoint operators H and H_0,

$$\operatorname{tr}(f(H) - f(H_0)) = \int_{\mathbb{R}} \xi(\lambda) f'(\lambda) d\lambda, \tag{1.1.1}$$

valid for a wide class of functions f and a certain function ξ, that in general depends on H_0 and H, but not on f. [1]

[1] The trace formula can be considered a non-commutative analog of the Fundamental Theorem of Calculus.

Shortly after the new concept introduced by Lifshits became available to the specialists, M.G. Krein developed a formalism giving the trace formula a precise mathematical meaning ([26]). Initially, it was assumed that the difference $H - H_0$ belongs to the trace class. However, this condition is too restrictive. For instance, if H_0 is the free Schrödinger operator and $H = H_0 + V$, this condition never holds unless the potential V is identically zero. Therefore, for the concept to be useful for immediate applications – in quantum mechanics, for example – a further development of the theory was needed. In [27], Krein relaxed the trace class condition to the requirement that the difference of the resolvents of (abstract) self-adjoint operators H_0 and H be in the trace class, that is,

$$(H - z)^{-1} - (H_0 - z)^{-1} \in \mathbf{S}_1, \quad z \in \rho(H_0) \cap \rho(H), \tag{1.1.2}$$

where $\rho(H_0)$ and $\rho(H)$ are the resolvent sets of H_0 and H, respectively. Under condition (1.1.2), Krein proved that there exists a real function ξ, called the spectral shift function, satisfying

$$\int_{\mathbb{R}} |\xi(\lambda)| \, (\lambda^2 + 1)^{-1} d\lambda < \infty, \tag{1.1.3}$$

such that the trace formula (1.1.1) holds. Since (1.1.1) holds, in particular, for all Schwartz functions f, it defines ξ uniquely up to a constant term; under condition (1.1.2), however, one cannot determine the constant uniquely (see [37], where it is suggested how to determine ξ up to an integer-valued constant). In the case both H and H_0 are bounded from below, the standard way to fix the constant is to require that

$$\xi(\lambda) = 0 \quad \text{for} \quad \lambda < \inf\{\sigma(H_0) \cup \sigma(H)\}.$$

We will focus on the spectral shift function associated with a pair of Schrödinger operators on a half-line, $H_0 = -d^2/dx^2$ and $H = H_0 + V$, with Dirichlet boundary condition at the origin and restrict our attention to the case of the short-range potential. Namely, we assume

$$\int_0^\infty |V(x)|\,(1+|x|)dx < \infty.$$

It is well-known ([9]) that in this case the spectral shift function on the continuous spectrum can be represented as the scattering phase (see the full definition in the next section) up to a constant factor (cf. [5]). Therefore, it is natural to expect that the phase shift associated with the potential V can be evaluated as the pointwise limit of the phase shifts associated with the cut-off potential V^r, (where V^r coincides with V on a finite interval $(0, r)$ and is continued by zero outside that interval). Indeed, in [10] it was proved that this pointwise convergence takes place, which, translated into the language of the spectral shift functions, means that

$$\lim_{r\to\infty} \hat{\xi}(r,\lambda) = \xi(\lambda), \quad \lambda > 0, \tag{1.1.4}$$

Here $\hat{\xi}(r,\cdot)$ stands for the spectral shift function associated with the pair H_0 and $\widehat{H}^r$, where H_0 is the free operator and $\widehat{H}^r$ is the operator with the cut-off potential V^r, i.e., $\widehat{H}^r = H_0 + V^r$ (both acting in $L^2((0,\infty))$) and both $\xi(r,\cdot)$ and ξ are chosen to be continuous for $\lambda \geq 0$. In fact, in the short-range case the functions ξ and $\hat{\xi}$, initially defined almost everywhere, can be chosen to be continuous on the positive semi-axis and the pointwise convergence (1.1.4) takes place not only λ-almost everywhere, but everywhere.

Another way to look at the semi-axis (i.e., singular) problem is to consider the problem first on the finite interval $(0,r)$ (the box-approximation problem), followed by taking an appropriate thermodynamical limit as $r \to \infty$.

One can expect the spectral characteristics of the problem on the finite interval to approximate the corresponding characteristics of the singular problem in a suitable sense. It should be mentioned that in contrast to the case considered above (both H_0 and $\hat{H}^r$ live on $(0,\infty)$ and only the potential is cut off) the nature of the spectrum of the truncated problem is quite different. That is, the Schrödinger operator on a finite interval with Dirichlet boundary conditions has discrete spectrum only. Since the finite-interval spectral shift function $\xi(r,\cdot)$ coincides with the difference of the eigenvalue counting functions of the perturbed and unperturbed operators on the interval, one cannot expect pointwise convergence, like in (1.1.4), as the cut-off parameter gets larger. The reason is simple: a pointwise limit – if it exists – of a sequence of integer-valued functions is an integer-valued function, while $\xi(\cdot)$ is generically a non-constant continuous function. Therefore, in order for the suggested box-approximation method to work, one has to relax the type of convergence.

In the present work we prove that the average of spectral shift functions $\xi(r,\cdot)$ with respect to the cut-off variable r converges pointwise to the limit function $\xi(\cdot)$:

$$\lim_{R\to\infty} \frac{1}{R}\int_0^R \xi\,(r,\lambda)\,dr = \xi(\lambda), \quad \lambda \in \mathbb{R}, \tag{1.1.5}$$

provided that $\xi(\cdot)$ is chosen to be continuous from the right for $\lambda < 0$ and continuous for $\lambda \geq 0$ (see Theorem 1.3.2). The first results in this direction were obtained in [19], [23], and [34], where the weak convergence of spectral shift functions was established:

$$\lim_{r\to\infty}\int_{-\infty}^{\lambda}\xi\left(r,\lambda'\right)d\lambda'=\int_{-\infty}^{\lambda}\xi(\lambda')d\lambda',\quad \lambda\in\mathbb{R}. \tag{1.1.6}$$

Notice that (1.1.5) is not a direct consequence of (1.1.6).

1.2 Preliminaries

1.2.1 The Jost function and the phase shift

In this section, we recall the definition of the Jost function and discuss its properties. We will mostly follow [12] and [29] for the basic results.

Hypothesis 1.2.1. Assume that V is a real-valued function, integrable near zero and integrable with finite first moment in a neighborhood of infinity, that is,

$$\int_0^\infty |V(x)|\,(1+|x|)dx<\infty. \tag{1.2.1}$$

Denote by $f(k,\cdot)$ the Jost solution of the differential equation

$$-\psi''(k,x)+V(x)\psi(k,x)=k^2\psi(k,x),\quad \mathrm{Im}\,(k)\geq 0,\quad x\geq 0, \tag{1.2.2}$$

given by

$$f(k,x)=e^{ikx}-\int_x^\infty\frac{\sin(k(x-x'))}{k}V(x')f(k,x')dx', \tag{1.2.3}$$

$$\mathrm{Im}\,(k)\geq 0,\quad x\geq 0.$$

Notice that under condition (1.2.1), equation (1.2.3) has a unique solution the space $\{f\in L^2((0,\infty)) : f,f'\in AC([0,R]),\forall R>0\}$. Recall that the Jost solution can also be introduced as a solution of Schrödinger equation (1.2.2) possessing the following asymptotic behavior

$$f(k,x)\sim e^{ikx},\quad x\to\infty,\quad \mathrm{Im}\,(k)\geq 0.$$

Next, denote by $\mathcal{F}$ the Jost function

$$\mathcal{F}(k) = f(k, 0), \quad \mathrm{Im}\,(k) \geq 0. \tag{1.2.4}$$

Equations (1.2.3) and (1.2.4) imply the following representation

$$\mathcal{F}(k) = 1 + \int_0^\infty \frac{\sin(kx)}{k} V(x) f(k, x) dx.$$

It is well known (see [12], [29]) that under Hypothesis 1.2.1, the Jost function $\mathcal{F}$ is continuously differentiable for $\mathrm{Im}\,(k) \geq 0$ with the possible exception of $k = 0$; if one wants to include the point $k = 0$, one has to impose the additional requirement that the function V has a finite second moment near infinity,

$$\int_0^\infty |V(x)|\,(1 + x^2) dx < \infty.$$

It is easy to show ([12]) that the Jost function $\mathcal{F}$ satisfies the estimate

$$|\mathcal{F}(k) - 1| \leq C \int_0^\infty \frac{x\,|V(x)|}{1 + |k|\,x}\,dx,$$

and hence,

$$\lim_{k\to\infty} \mathcal{F}(k) = 1, \quad \mathrm{Im}\,k \geq 0. \tag{1.2.5}$$

Since $\mathcal{F}$ does not vanish on the positive real axis ([29]), one introduces the phase shift δ as a continuous function for $k \geq 0$ by

$$\mathcal{F}(k) = |\mathcal{F}(k)|\,e^{i\delta(k)},$$

with the following normalization at infinity (cf. (1.2.5))

$$\lim_{k\to+\infty} \delta(k) = 0.$$

We remark that since the Jost function $\mathcal{F}$ is continuously differentiable for $k > 0$, so is the phase shift δ.

1.2.2 Phase shifts for truncated potentials

In this section, we introduce the phase shift associated with the box approximation for the potential V and summarize some of its well-known properties that will be important for future considerations. In particular, we will be interested in certain estimates on partial derivatives of the phase shift.

First, some basic notation. Denote by V^r a compact-support approximation of the potential V,

$$V^r(x) = V(x)\chi_{[0,r]}(x), \quad x \geq 0,\ r > 0, \tag{1.2.6}$$

where $\chi_{[0,r]}$ is the characteristic function of the interval $[0, r]$. Denote by $\mathcal{F}^r$ and $\delta(r, \cdot)$ the Jost function and the phase shift associated with the truncated potential V^r, respectively. Recall that the phase shift δ is a solution of a nonlinear integral equation, called the variable phase equation ([10], [33]),

$$\delta(r, k) = k^{-1} \int_0^r V(r') \sin^2 (kr' - \delta(r', k))\, dr', \quad k > 0,\ r > 0. \tag{1.2.7}$$

The phase shift associated with a potential with compact support admits an estimate on its derivative that only depends on the range of the potential (not its strength). Such an estimate is uniform with respect to the energy variable if the corresponding Jost function does not vanish on the positive imaginary semi-axis, and it only holds at high energy otherwise. The precise statements are given in the following two results.

Lemma 1.2.2. *([12], [11]) Assume Hypothesis 1.2.1. Let the potential V^r be given by (1.2.6) and let $\mathcal{F}^r$ and $\delta(r, \cdot)$ be the Jost function and the phase shift associated with V^r, respectively. Assume, in addition, that $\mathcal{F}^r$ does not vanish on the positive* *imaginary semi-axis including zero, i.e.* $\mathcal{F}^r(k) \neq 0$, $k \in \mathrm{i}\mathbb{R}$, $\mathrm{Im}\,(k) \geq 0$. *Then* δ *is continuously differentiable for* $k \geq 0$ *and*

$$\frac{d\delta(r,k)}{dk} < r, \quad k \geq 0.$$

In particular, the function

$$rk - \delta(r,k)$$

is strictly increasing in k *for* $k \geq 0$.

Notice that the condition that the Jost function does not vanish on the positive imaginary semi-axis means that the corresponding Schrödinger operator does not have negative eigenvalues or a zero-energy resonance and so is rather restrictive. Without that condition, the following is true.

Theorem 1.2.3. *Assume Hypothesis 1.2.1. Let the potential* V^r *be given by (1.2.6) and let* $\delta(r,\cdot)$ *be the phase shift associated with* V^r. *Then, for every* $k_0 > 0$, *there exists* $R_0 > 0$, *such that for every* $r > R_0$, *the function*

$$rk - \delta(r,k)$$

is strictly increasing in k *for* $k \geq k_0$.

Proof. The proof is based on the fact that the partial derivative of the phase shift δ with respect to k is a solution of the following nonlinear integral equation

$$\begin{aligned}\frac{d}{dk}\,\delta(r,k) = k^{-2}\int_0^r V(r')\left(\sin^2\left(kr' - \delta(r',k)\right) - kr'\sin\left(2\left(kr' - \delta(r',k)\right)\right)\right)\\ \times \exp\left(k^{-1}\int_{r'}^r V(s)\sin\left(2\left(ks - \delta(s,k)\right)\right)ds\right)dr', \qquad (1.2.8)\end{aligned}$$

$$k > 0,\ r > 0,$$

which, in turn, is a direct consequence of the phase equation (1.2.7) (see [10], p. 42, Eq. (24)). Equation (1.2.8) obviously implies the inequality

$$\left|\frac{d}{dk}\,\delta(r,k)\right| \leq k^{-2}\int_0^r \left(\left|\sin^2\left(kr' - \delta(r',k)\right)\right| + kr'\left|\sin\left(2\left(kr' - \delta(r',k)\right)\right)\right|\right)$$
$$\times \exp\left(k^{-1}\int_{r'}^r |V(s)|\left|\sin\left(2\left(ks - \delta(s,k)\right)\right)\right| ds\right) |V(r')|\, dr'.$$

Replacing the upper limits of integration by ∞ and estimating $|\sin(\cdot)|$ from above by 1 and k from below by k_0, we get

$$\left|\frac{d}{dk}\,\delta(r,k)\right| \leq k_0^{-2}\int_0^\infty |V(r')|\,(1 + k_0 r')\, e^{k_0^{-1}\int_{r'}^\infty |V(s)|\,ds} dr'.$$

Observe that the RHS of the last inequality does not depend on r and hence for R_0 sufficiently large,

$$\left|\frac{d}{dk}\,\delta(r,k)\right| < R_0 = R_0(k_0).$$

Therefore,

$$\left|\frac{d}{dk}\,\delta(r,k)\right| < r$$

for every $r > R_0$, . Thus

$$r - \frac{d}{dk}\,\delta(r,k) > 0, \quad \text{for } r > R_0, \quad k \geq k_0,$$

completing the proof. □

Remark 1.2.4. Notice that one can choose R_0 to be, for instance,

$$R_0 = k_0^{-2}\int_0^\infty |V(r')|\,(1 + k_0 r')\, e^{k_0^{-1}\int_{r'}^\infty |V(s)|\,ds} dr'.$$

We also observe that under Hypotheses 1.2.1, the phase shifts $\delta(r,k)$ converge pointwise to $\delta(k)$ as $r \to \infty$. The precise statement is as follows.

Lemma 1.2.5. *([10], p. 11, Eq. (13); [8]) Assume Hypotheses 1.2.1. Let the potential V^r be given by (1.2.6). Let $\delta(k)$ and $\delta(r,k)$ be the phase shifts associated with the potential V and V^r, respectively. Then*

$$\lim_{r\to\infty} \delta(r,k) = \delta(k), \quad k > 0. \tag{1.2.9}$$

Proof. Since for every fixed $k > 0$ the Jost function $\mathcal{F}^r(k)$ is continuous in r and $\mathcal{F}^r(k) \neq 0$, the phase shift $\delta(r,k)$ is also continuous in r. Moreover,

$$\lim_{r\to\infty} \mathcal{F}^r(k) = \mathcal{F}(k), \quad k > 0,$$

and, as a result,

$$\lim_{r\to\infty} e^{\delta(r,k)} = e^{\delta(k)}, \quad k > 0,$$

which implies (1.2.9). □

1.2.3 Spectral shift functions

In this section, we introduce the spectral shift function, provide a rigorous meaning to (1.1.1), and recall some important results on spectral shift functions associated with Schrödinger operators.

We start with a general result which gives sufficient conditions for formula (1.1.1) to hold under the assumption that the difference of the resolvents belongs to the trace class. For details we refer to [37], [31]; see also [21].

Theorem 1.2.6. *(Theorem 8.7.1, [37]) Let H, H_0 be self-adjoint semi-bounded operators such that $(H-z)^{-1} - (H_0-z)^{-1}$ is trace class, $z \in \rho(H_0) \cap \rho(H)$. Then,* *for every function f on $(-\infty,\infty)$ such that f has two locally bounded derivatives, satisfying*

$$(\lambda^2 f'(\lambda))' = \mathcal{O}(|\lambda|^{-1-\varepsilon}), \quad |\lambda| \to \infty \tag{1.2.10}$$

and

$$\lim_{\lambda\to-\infty} f(\lambda) = \lim_{\lambda\to+\infty} f(\lambda), \qquad \lim_{\lambda\to-\infty} \lambda^2 f'(\lambda) = \lim_{\lambda\to+\infty} \lambda^2 f'(\lambda), \tag{1.2.11}$$

$f(H) - f(H_0)$ *is trace class. Moreover, there exists a real-valued measurable function ξ on $(-\infty,\infty)$, satisfying* (1.1.3)*, such that* (1.1.1) *holds for every such f.*

Spectral shift function for Schrödinger operators on the half-line

Let H_0 be the half-line free self-adjoint Schrödinger operator

$$H_0 = -\frac{d^2}{dx^2}, \tag{1.2.12}$$

defined on the domain

$$\begin{aligned} D(H_0) =&\{f \in L^2((0,\infty)) : \; f, f' \in AC([0,R]), \forall R > 0, \\ & f'' \in L^2((0,\infty)), \; f(0_+) = 0\}. \end{aligned}$$

Next, given V satisfying (1.2.1), let H be the perturbation of H_0,

$$H = -\frac{d^2}{dx^2} + V, \tag{1.2.13}$$

which is self-adjoint on the domain

$$\begin{aligned} D(H) =&\{f \in L^2((0,\infty)) : \; f, f' \in AC([0,R]), \forall R > 0, \\ & - f'' + Vf \in L^2((0,\infty)), \; f(0_+) = 0\}. \end{aligned}$$

We now check that the difference of resolvents of operators H_0 and H is trace class. This, combined with Theorem 1.2.6, will immediately imply that the spectral shift function associated with the pair (H, H_0) is well-defined.

Theorem 1.2.7. *Assume Hypothesis 1.2.1. Let H_0 and H be defined by (1.2.12) and (1.2.13), respectively, and let $z \in \rho(H_0) \cap \rho(H)$. Then*

$$(H - z)^{-1} - (H_0 - z)^{-1}$$

is a trace class operator.

Proof. For $x \in (0, \infty)$, we introduce the following factorization

$$V(x) = v(x)u(x), \quad v(x) = |V(x)|^{1/2}, \quad u(x) = |V(x)|^{1/2} \operatorname{sgn}(V(x)).$$

Using the resolvent identity (cf. [20]),

$$(H - z)^{-1} - (H_0 - z)^{-1} = -\overline{(H_0 - z)^{-1}u}(I + \overline{u(H_0 - z)^{-1}v})^{-1}v(H_0 - z)^{-1},$$

we get the following representation,

$$(H - z)^{-1} - (H_0 - z)^{-1} = X_1 Y X_2,$$

where

$$X_1 = -\overline{(H_0 - z)^{-1}u}, \quad Y = (I + \overline{v(H_0 - z)^{-1}u})^{-1}, \quad X_2 = v(H_0 - z)^{-1}.$$

Since $V \in L^1((0, \infty))$, both u and v belong to $L^2((0, \infty))$, as does the function $\frac{1}{x-z}$ for $z \notin \mathbb{R}$. Hence, both X_1 and X_2 are Hilbert-Schmidt operators (see [33], Theorem XI.20). Obviously, Y is bounded, and so $X_1 Y X_2$ is trace class, proving the theorem. $\square$

Combining Theorems 1.2.6 and 1.2.7, we immediately get

Corollary 1.2.8. *Assume Hypothesis 1.2.1. Let H_0 and H be defined by (1.2.12) and (1.2.13), respectively. Then there exists a unique real-valued measurable function ξ on $(-\infty, \infty)$ satisfying*

$$\xi(\lambda) = 0 \quad \textit{for} \quad \lambda < \inf \{\mathrm{spec}(H_0) \cup \mathrm{spec}(H)\}$$

and

$$\mathrm{tr}\,(f(H) - f(H_0)) = \int_{\mathbb{R}} \xi(\lambda) f'(\lambda) d\lambda$$

for every f with two locally bounded derivatives which satisfies conditions (1.2.10), (1.2.11).

The next important result describes the connection between the spectral shift function associated with two half-line Schrödinger operators and the phase shift.

Theorem 1.2.9. *([9], [5], [37], [6]) Assume Hypothesis 1.2.1. Let H_0 and H be defined by (1.2.12) and (1.2.13), respectively. Let δ be the phase shift associated with V and ξ the spectral shift function associated with the pair (H_0, H). Let $N(\lambda)$ be the number of negative eigenvalues of the operator H that are smaller than λ. Then*

$$\xi(\lambda) = \begin{cases} -\frac{1}{\pi}\delta(\sqrt{\lambda}), & \lambda \geq 0, \\ -N(\lambda), & \lambda < 0. \end{cases} \tag{1.2.14}$$

Spectral shift function for Schrödinger operators on a finite interval

Having dealt with the half-line case, we now discuss the properties of the spectral shift function for a pair Schrödinger operators on an interval $(0, r)$. Recall that the free Schrödinger operator

$$H_0^r = -\frac{d^2}{dx^2}$$

on

$$D(H_0^r) = \{f \in L^2((0,r)) : \ f, f' \in AC([0,r]),$$

$$f'' \in L^2((0,r)), \ f(0_+) = f(r_-) = 0\},$$

(with Dirichlet boundary conditions at $x = 0$ and $x = r$), has a simple discrete spectrum. Let $N_0^r = N_0^r(\lambda)$ be its counting function, i.e. the number of eigenvalues of H_0^r which are smaller than λ.

Next, assume Hypothesis 1.2.1 and consider the operator

$$H^r = -\frac{d^2}{dx^2} + V^r, \tag{1.2.15}$$

on the domain

$$D(H^r) = \{f \in L^2((0,r)) : \ f, f' \in AC([0,r]), \tag{1.2.16}$$

$$- f'' + V^r f \in L^2((0,r)), \ f(0_+) = f(r_-) = 0\},$$

again with Dirichlet boundary conditions at the endpoints. Here V^r is a restriction of V onto $(0,r)$; since the context is unambiguous, we use the same notation for this restriction as for the box-approximation of V defined by (1.2.6). Recall (see [25]) that H^r is self-adjoint. It is also well known ([13]) that operator H^r defined by (1.2.15), (1.2.16) has a simple spectrum. Let $N = N(\lambda)$ be its eigenvalue counting function.

One can prove that

$$(H^r - z)^{-1} - (H_0^r - z)^{-1}$$

is a trace class operator (the proof is identical to the one of Theorem 1.2.7). Therefore, the operators H^r and H_0^r satisfy the conditions of Theorem 1.2.6, and the spectral shift function $\xi(r,\cdot)$ associated with the pair (H_0^r, H^r) is well-defined. Moreover, one can easily check (see [26]) that

$$\xi(r,\lambda) = N_0^r(\lambda) - N^r(\lambda), \quad \lambda \in \mathbb{R}. \tag{1.2.17}$$

Weak convergence of spectral shift functions

As proved by R. Geisler, V. Kostrykin, and R. Schrader in [19], the weak convergence of spectral shift functions related to Schrödinger operators takes place. The proof is based on a Feynman-Kac formula and the precise result is as follows.

Theorem 1.2.10. *Assume Hypothesis 1.2.1. Let H_0 and H be defined by (1.2.12) and (1.2.13) respectively and let ξ be a spectral shift function associated with the pair (H_0, H). Next, let $\xi(r,\cdot)$ be a spectral shift function associated with (H_0^r, H^r). Then*

$$\lim_{r\to\infty} \int_{-\infty}^{a} \xi(r,\lambda)d\lambda = \int_{-\infty}^{a} \xi(\lambda)d\lambda, \quad a \in \mathbb{R}. \tag{1.2.18}$$

Remark 1.2.11. We would like to remark that Theorem 1.2.10 was proved in [19] for Schrödinger operators in arbitrary dimension under the condition that the potential V in [19] belongs to the Birman-Solomjak space $l^1(L^2)$,

$$V \in l^1(L^2) = \left\{ f \middle| \sum_{j\in\mathbf{Z}^n} \left[\int_{\Delta_j} |f(x)|^2 \, dx \right]^{1/2} = \|f\|_{1,2} \right\}, \tag{1.2.19}$$

where Δ_j are unit cubes with centers at $x = j$. However, in dimension one the assumption (1.2.1) is sufficient for the result to hold. We also would like to mention that in our case one can use the Feynman-Kac formula for the operator $e^{-tH} - e^{-tH_0}$

with H and H_0 being Schrödinger operators on the half-line (not the full line). In this case $e^{-tH} - e^{-tH_0}$ is an integral operator with the integral kernel given by

$$\left(e^{-tH} - e^{-tH_0}\right)(x, y) = \mathbb{E}_{0,x}^{t,y}\left\{e^{-\int_0^t V(b(s))ds} - 1; T_{b,\infty} > t\right\}, \ x, y \geq 0.$$

The corresponding formula on the finite domain is given by

$$\left(e^{-tH^r} - e^{-tH_0^r}\right)(x, y) = \mathbb{E}_{0,x}^{t,y}\left\{e^{-\int_0^t V(b(s))ds} - 1; T_{b,r} > t\right\}, \ x, y \geq 0.$$

Here $\mathbb{E}_{0,x}^{t,y}$ stands for the conditional expectation with respect to the probability measure of the Brownian motion starting from x at time 0 and conditioned to be at y at time t, and

$$T_{b,r} = \inf\{s > 0, \ b(s) \notin (0, r)\},$$

$$T_{b,\infty} = \inf\{s > 0, \ b(s) \leq 0\}.$$

In order to get the proof in our case one can follow the lines in [19]; we will omit the details.

1.3 The main result

In this section, we will prove the principal result of this chapter. The main ingredient of the proof is based on the fact that *i)* on the positive semi-axis the "thermodynamic" limit of finite-interval spectral shift functions $\xi(r, \cdot)$ exists and equals the phase shift $\delta(\cdot)$, and that *ii)* the negative eigenvalues of the truncated Schrödinger operator H^r converge to the corresponding eigenvalues of the half-line operator H (a result due to Bailey, Everitt, Weidmann, and Zettle, [4]). We first will state the result on the positive semi-axis.

Theorem 1.3.1. *Assume Hypotheses 1.2.1. Let δ be the phase shift associated with the potential V. Let $\xi(r,\cdot)$ be the spectral shift function associated with the pair of free and perturbed Schrödinger operators on $(0,r)$. Then*

$$\lim_{R\to\infty}\frac{1}{R}\int_0^R \xi(r,\lambda)dr = \pi^{-1}\delta\big(\sqrt{\lambda}\big), \quad \lambda > 0. \tag{1.3.1}$$

The proof of this theorem will be given in the following two subsections.

Now we are ready to prove our main theorem.

Theorem 1.3.2. *Assume Hypotheses 1.2.1. Let ξ be the left-continuous spectral shift function associated with the pair of free and perturbed Schrödinger operators on $\mathbb{R}_+$ and let $\xi(r,\cdot)$ be the left-continuous spectral shift function associated with the pair of free and perturbed Schrödinger operators on $(0,r)$. Then*

$$\lim_{R\to\infty}\frac{1}{R}\int_0^R \xi(r,\lambda)dr = \xi\big(\lambda\big), \quad \lambda \in \mathbb{R}. \tag{1.3.2}$$

Proof. First, recall that the negative eigenvalues of each truncated operator converge to the corresponding negative eigenvalues of the half-line operator as the cut-off parameter tends to infinity (cf. [4]). This implies a pointwise convergence of the corresponding counting functions, and, consequently, a pointwise convergence of the left-continuous spectral shift functions on the negative semi-axis,

$$\lim_{r\to\infty}\xi(r,\lambda) = \xi(\lambda), \quad \lambda < 0. \tag{1.3.3}$$

The convergence (1.3.3), equation (1.3.1), and formula (1.2.14), imply the statement of the theorem. $\square$

1.3.1 Proof of Theorem 1.3.1 in the case of positive potentials

We first provide a proof of Theorem 1.3.1 in the special case where the potential V is positive almost everywhere.

Introduce the left-continuous greatest integer function

$$\lfloor x \rfloor = \begin{cases} [x], & x \notin \mathbb{Z}, \\ x-1, & x \in \mathbb{Z}, \end{cases}$$

where $[x]$ is the greatest integer smaller or equal to x. In this notation, we obtain our first result, a formula for the distribution of the eigenvalues of the operator H^r in terms of the corresponding phase shift.

Lemma 1.3.3. *Let V^r be the cut-off potential defined by (1.2.6) and $\delta(r,\cdot)$ the corresponding phase shift. Assume, in addition, that $V^r \geq 0$ a.e. Then, for every fixed $k>0$,*

$$rk - \delta(r,k) > 0,$$

and the number of eigenvalues of the operator H^r in the interval $[0,\lambda_0)$ is given by

$$N^r(\lambda_0) = \lfloor h(r,\lambda_0) \rfloor, \tag{1.3.4}$$

with

$$h(r,\lambda) = \pi^{-1}\Big(r\sqrt{\lambda} - \delta\big(r,\sqrt{\lambda}\big)\Big). \tag{1.3.5}$$

Proof. Let φ^r be the solution of the Cauchy problem

$$-\varphi''(k,x) + V^r(x)\varphi(k,x) = k^2\varphi(k,x), \quad k>0,\ x \geq 0,$$

$$\varphi(k,0)=0, \quad \varphi'(k,0)=1.$$

Recall that φ^r satisfies the Volterra integral equation for $k > 0$, $x \geq 0$,

$$\varphi^r(k,x) = \frac{\sin(kx)}{k} + \int_0^x \frac{\sin\left(k\left(x-x'\right)\right)}{k} V^r\left(x'\right)\varphi^r\left(k,x'\right)dx',$$

and, therefore, admits the following representation for $x \geq r$,

$$\varphi^r(k,x) = \frac{1}{2i}\frac{|\mathcal{F}^r(k)|}{k}\left(e^{ikx}e^{-i\delta(x,k)} - e^{-ikx}e^{i\delta(x,k)}\right).$$

Notice that $\lambda > 0$ is an eigenvalue of the operator H^r if and only if

$$\varphi^r\left(\sqrt{\lambda}, r\right) = 0, \tag{1.3.6}$$

so that $h(r,\lambda)$ is an integer. Correspondingly, we see that the total number of eigenvalues of the operator H^r in the interval $[0,\lambda_0)$ is equal to the number of integer values that h attains in the same interval. To count the number of integer values of h, notice that by Lemma 1.2.2, h is strictly increasing in λ, $\lambda \geq 0$. Thus, the total number of $\lambda \in [0,\lambda_0)$ such that $h(r,\lambda)$ is an integer, is equal to $\lfloor h(r,\lambda_0)\rfloor$. Therefore, the number of eigenvalues of the operator H^r in the interval $[0,\lambda_0)$ is given by the same expression, proving the claim. $\square$

Corollary 1.3.4. *Under the hypotheses of Lemma 1.3.3, the spectral shift function associated with the pair (H_0^r, H^r) admits the representation:*

$$\xi(r,\lambda) = \left\lfloor \pi^{-1} r\sqrt{\lambda}\right\rfloor - \left\lfloor \pi^{-1}\left(r\sqrt{\lambda} - \delta(r,\sqrt{\lambda})\right)\right\rfloor, \quad \lambda \geq 0. \tag{1.3.7}$$

Proof. Since the spectrum of the operator H_0^r is given by

$$\sigma(H_0^r) = \left\{\left(\frac{\pi n}{r}\right)^2\right\}_{n=1}^{\infty},$$

one obviously has that

$$\begin{aligned} N_0^r(\lambda) &= \sharp\{\, \mu < \lambda \mid \mu \text{ is an eigenvalue of } H_0^r\} \\ &= \sharp\left\{ n \in \mathbb{N} \mid \left(\frac{\pi n}{r}\right)^2 < \lambda \right\} = \max_{n \in \mathbb{N}} \left\{ \left(\frac{\pi n}{r}\right)^2 < \lambda \right\} = \left\lfloor \pi^{-1} r \sqrt{\lambda} \right\rfloor . \end{aligned} \quad (1.3.8)$$

The assertion now follows from (1.2.17) together with (1.3.4) and (1.3.8). □

The following result is the final piece of the puzzle.

An ergodic lemma

Lemma 1.3.5. *Let $A \in \mathbb{R}$, then*

$$\lim_{R \to \infty} \frac{1}{R} \int_0^R \big(\lfloor x \rfloor - \lfloor x - A \rfloor \big)\, dx = A. \quad (1.3.9)$$

Proof. To evaluate the definite integral in (1.3.9), we remark that the region M bounded by the graphs of the functions $\lfloor x \rfloor$ and $\lfloor x - A \rfloor$ on the interval $[0, R]$, $R \geq A + 1$, can be represented as the union of two disjoint sets

$$M = M_1 \cup M_2,$$

where M_1 admits the representation

$$M_1 = \bigcup_{k=1}^{[R-A]} (k, k + A) \times (k - 1, k),$$

and the area of the set $M_2 = M \setminus M_1$ admits the estimate

$$|M_2| \leq (R - [R - A])^2 \leq (A + 1)^2. \quad (1.3.10)$$

Since the area of M_1 is obviously given by

$$|M_1| = \sum_{k=1}^{[R-A]} A = A[R - A], \quad (1.3.11)$$

combining (1.3.11) and (1.3.10) yields

$$\lim_{R\to\infty}\frac{1}{R}\int_0^R \big(\lfloor x\rfloor - \lfloor x-A\rfloor\big)\,dx = \lim_{R\to\infty}\frac{A[R-A]}{R} + \lim_{R\to\infty}\frac{|M_2|}{R} = A.$$

$\square$

Corollary 1.3.6. *Let a be a real-valued function on $[0,\infty)$. Assume that the limit*

$$A = \lim_{x\to\infty} a(x)$$

exists. Then

$$\lim_{R\to\infty}\frac{1}{R}\int_0^R \big(\lfloor x\rfloor - \lfloor x-a(x)\rfloor\big)\,dx = A. \tag{1.3.12}$$

Proof. Fix an arbitrary $\varepsilon > 0$ and let R_ε be such that

$$|a(x) - A| < \varepsilon, \quad x > R_\varepsilon.$$

Then

$$\lfloor x - a(x)\rfloor = \lfloor x - A\rfloor \quad \text{whenever} \quad |(x-A) - \lfloor x-A\rfloor| > \varepsilon.$$

Therefore,

$$\begin{aligned}\left|\frac{1}{R}\int_{R_\varepsilon}^R \big(\lfloor x\rfloor - \lfloor x-a(x)\rfloor\big)\,dx - \frac{1}{R}\int_{R_\varepsilon}^R \big(\lfloor x\rfloor - \lfloor x-A\rfloor\big)\,dx\right| \\ \le 2\varepsilon\frac{|R-R_\varepsilon|}{R} \le 2\varepsilon, \quad R \ge R_\varepsilon.\end{aligned} \tag{1.3.13}$$

Since

$$\lim_{R\to\infty}\frac{1}{R}\int_{R_\varepsilon}^R \big(\lfloor x\rfloor - \lfloor x-a(x)\rfloor\big)\,dx = \lim_{R\to\infty}\frac{1}{R}\int_{R_\varepsilon}^R \big(\lfloor x\rfloor - \lfloor x-A\rfloor\big)\,dx = 0,$$

inequality (1.3.13)) implies

$$\overline{\lim_{R\to\infty}}\left|\frac{1}{R}\int_0^R \big(\lfloor x\rfloor - \lfloor x-a(x)\rfloor\big)\,dx - \frac{1}{R}\int_0^R \big(\lfloor x\rfloor - \lfloor x-A\rfloor\big)\,dx\right| \le 2\varepsilon,$$

and thus,

$$\lim_{R\to\infty}\frac{1}{R}\int_0^R \left(\lfloor x\rfloor - \lfloor x-a(x)\rfloor\right) dx = \lim_{R\to\infty}\frac{1}{R}\int_0^R \left(\lfloor x\rfloor - \lfloor x-A\rfloor\right) dx, \qquad (1.3.14)$$

since ε can be chosen arbitrarily small. Now (1.3.12) follows from (1.3.14) by applying Lemma 1.3.5. $\square$

We are now in a position to give the proof of Theorem 1.3.1 in the special case of a positive potential.

Proof of Theorem 1.3.1 under the additional assumption $V \geq 0$. The condition $V \geq 0$ allows us to apply the result of Corollary 1.3.4, obtaining that

$$\lim_{R\to\infty}\frac{1}{R}\int_0^R \xi(r,\lambda)dr = \lim_{R\to\infty}\frac{1}{R}\int_0^R \left\lfloor \pi^{-1}r\sqrt{\lambda}\right\rfloor - \left\lfloor \pi^{-1}\left(r\sqrt{\lambda}-\delta(r,\sqrt{\lambda})\right)\right\rfloor dr.$$

Taking into account (1.2.9) and applying Corollary 1.3.6 completes the proof. $\square$

1.3.2 Proof of Theorem 1.3.1 in the general case

We are interested in obtaining a formula for the eigenvalue counting function N^r, similarly to (1.3.4). Theorem 1.2.3 is useful in evaluating the rank of the spectral projection $\mathbb{E}_{H^r}([\lambda_1,\lambda_2))$ for a finite interval $[\lambda_1,\lambda_2)$ on the positive semi-axis away from zero ($0<\lambda_1<\lambda_2$). More precisely, we have the following result:

Lemma 1.3.7. *Let V^r be the cut-off potential defined by* (1.2.6) *and $\delta(r,\cdot)$ the corresponding phase shift. Let h be defined by* (1.3.5)*. Then, for $0<\lambda_1<\lambda_2$,*

$$N^r(\lambda_2)-N^r(\lambda_1) = \lfloor h(r,\lambda_2)\rfloor - \lfloor h(r,\lambda_1)\rfloor,$$

for sufficiently large $r>0$.

Proof. Notice that $\lambda > 0$ is an eigenvalue of the operator H^r if and only if equation (1.3.6) holds, so that $h(r,\lambda)$ is an integer. Taking this into account, we see that the total number of eigenvalues of the operator H^r in the interval $[\lambda_1, \lambda_2)$ is equal to the number of integer values that the function $h(r,\cdot)$ attains in this interval. To count the number of the integer values of $h(r,\lambda)$, notice that by Theorem 1.2.3 the function h is strictly increasing in λ, $\lambda \geq \lambda_1$ for r large enough. Thus, the total number of $\lambda \in [\lambda_1, \lambda_2)$, such that $h(r,\lambda)$ is an integer, is equal to

$$\lfloor h(r,\lambda_2) \rfloor - \lfloor h(r,\lambda_1) \rfloor ,$$

for all r large enough. Therefore, the number of eigenvalues of the operator H^r in the interval $[\lambda_1, \lambda_2)$ is given by the same expression, proving the claim. $\square$

Remark 1.3.8. Notice that if the support of the potential V is compact, one can show that

$$N^r(\lambda) = \left\lfloor \pi^{-1}\Big(r\sqrt{\lambda} - \delta\big(\sqrt{\lambda}\big) \Big) \right\rfloor , \quad \lambda \geq 0,$$

where r is outside of the support of V and satisfies

$$r > \sup_{k\geq 0} \delta'(k). \tag{1.3.15}$$

Here δ is the phase shift associated with the problem on the infinite interval. Indeed, for r satisfying (1.3.15), the function $h(r,\lambda) = r\sqrt{\lambda} - \delta\big(\sqrt{\lambda}\big)$ is strictly increasing in λ for $\lambda \geq 0$ and for r outside of the support of V, (1.3.6) holds if and only if $h(r,\lambda)$ is an integer.

Proof of Theorem 1.3.1. We start with proving the existence of the limit

$$\lim_{R\to\infty} \frac{1}{R} \int_0^R \big(\xi(r,\lambda) - \xi(r,\mu)\big) dr, \quad \lambda, \mu > 0. \tag{1.3.16}$$

Using equation (1.2.17) and rearranging terms, we get

$$\begin{aligned}\xi(r,\lambda)-\xi(r,\mu)=&N_0^r(\lambda)-N^r(\lambda)-N_0^r(\mu)+N^r(\mu)\\=&N_0^r(\lambda)-N_0^r(\mu)-N^r(\lambda)+N^r(\mu),\quad \lambda,\mu\in\mathbb{R}.\end{aligned}$$

Next, Lemma 1.3.7 together with equation (1.3.8) implies

$$\begin{aligned}\xi(r,\lambda)-\xi(r,\mu)=&\left\lfloor\pi^{-1}r\sqrt{\lambda}\right\rfloor-\left\lfloor\pi^{-1}\Big(r\sqrt{\lambda}-\delta\big(r,\sqrt{\lambda}\big)\Big)\right\rfloor\\&+\left\lfloor\pi^{-1}\Big(r\sqrt{\mu}-\delta\big(r,\sqrt{\mu}\big)\Big)\right\rfloor-\left\lfloor\pi^{-1}r\sqrt{\mu}\right\rfloor,\ \ \lambda,\mu>0,\end{aligned}\tag{1.3.17}$$

provided r is large enough. Notice that $\lim_{R\to\infty}\frac{1}{R}\int_0^R f(r)dr$, if it exists, only depends on the behavior of the function f in a neighborhood of infinity. Therefore, using the representation (1.3.17), we obtain

$$\lim_{R\to\infty}\frac{1}{R}\int_0^R\big(\xi(r,\lambda)-\xi(r,\mu)\big)dr=I(\lambda)-I(\mu),\quad \lambda>0,\ \mu>0,\tag{1.3.18}$$

where

$$I(\nu)=\lim_{R\to\infty}\frac{1}{R}\int_0^R\left(\left\lfloor\pi^{-1}r\sqrt{\nu}\right\rfloor-\left\lfloor\pi^{-1}\Big(r\sqrt{\nu}-\delta\big(r,\sqrt{\nu}\big)\Big)\right\rfloor\right)dr,\quad \nu>0,$$

provided that $I(\nu)$ exists for each $\nu>0$. To calculate $I(\nu)$, recall that according to Lemma 1.2.5,

$$\lim_{r\to\infty}\pi^{-1}\delta\big(r,\sqrt{\nu}\big)=\pi^{-1}\delta\big(\sqrt{\nu}\big),\quad \sqrt{\nu}>0,$$

and hence one may apply Corollary 1.3.6 (after making the change of variable $x=\pi^{-1}\sqrt{\nu}r$) to conclude that

$$I(\nu)=\pi^{-1}\delta\big(\sqrt{\nu}\big).\tag{1.3.19}$$

Combining equations (1.3.18) and (1.3.19), we get

$$\lim_{R\to\infty} \frac{1}{R}\int_0^R \big(\xi(r,\lambda)-\xi(r,\mu)\big)dr = \pi^{-1}\left(\delta\big(\sqrt{\lambda}\big)-\delta\big(\sqrt{\mu}\big)\right), \quad \lambda,\mu>0. \tag{1.3.20}$$

For $a>0$, introduce the notation

$$\langle f\rangle = \int_a^{a+1} f(\mu)d\mu.$$

Integrating both sides of equation (1.3.20) with respect to μ over interval $(a,a+1)$ gives, for $\lambda>0$,

$$\left\langle \lim_{R\to\infty} \frac{1}{R}\int_0^R \big(\xi(r,\lambda)-\xi(r,\cdot)\big)dr \right\rangle = \pi^{-1}\left(\delta\big(\sqrt{\lambda}\big)-\left\langle\delta\big(\sqrt{\cdot}\big)\right\rangle\right). \tag{1.3.21}$$

Formally interchanging the integration and the limit procedures in the LHS of this equation (a rigorous explanation will be given below), we get

$$\lim_{R\to\infty} \frac{1}{R}\int_0^R \left(\xi(r,\lambda)-\langle\xi(r,\cdot)\rangle\right)dr = \pi^{-1}\delta\big(\sqrt{\lambda}\big)-\pi^{-1}\left\langle\delta\big(\sqrt{\cdot}\big)\right\rangle, \ \lambda>0. \tag{1.3.22}$$

By Theorem 1.2.10, we have the weak convergence of spectral shift functions, i.e.,

$$\lim_{r\to\infty}\langle\xi(r,\cdot)\rangle = \pi^{-1}\left\langle\delta\big(\sqrt{\cdot}\big)\right\rangle,$$

and therefore,

$$\lim_{R\to\infty} \frac{1}{R}\int_0^R \langle\xi(r,\cdot)\rangle dr = \pi^{-1}\left\langle\delta\big(\sqrt{\cdot}\big)\right\rangle. \tag{1.3.23}$$

Equation (1.3.23) together with (1.3.22) implies (1.3.1).

The conclusion of the theorem then depends on justifying the interchanging of the integration $\int_0^R$ and the limit (1.3.21). In other words, we need to show that the LHS in equation (1.3.21) is equal to the one in (1.3.22), namely

$$\begin{aligned}\int_a^{a+1}&\left[\lim_{R\to\infty} \frac{1}{R}\int_0^R \big(\xi(r,\lambda)-\xi(r,\mu)\big)dr\right]d\mu\\ &= \lim_{R\to\infty} \frac{1}{R}\int_0^R \left[\xi(r,\lambda)-\int_a^{a+1}\xi(r,\mu)d\mu\right]dr, \ \lambda>0.\end{aligned}$$

The first step consists in interchanging the limit with integration with respect to μ in the LHS of equation (1.3.21). For this purpose, it is sufficient to prove the uniform estimate

$$\left|\frac{1}{R}\int_0^R \big(\xi(r,\lambda)-\xi(r,\mu)\big)dr\right| \leq \left(\frac{1}{\sqrt{\lambda}}+\frac{1}{\sqrt{\mu}}\right)\int_0^\infty |V(r)|\,dr+2,\ \lambda,\mu>0, \tag{1.3.24}$$

and apply the dominated convergence theorem to the LHS of (1.3.21) to get

$$\begin{aligned}\int_a^{a+1}&\Big[\lim_{R\to\infty}\frac{1}{R}\int_0^R \big(\xi(r,\lambda)-\xi(r,\mu)\big)dr\Big]d\mu \\ &= \lim_{R\to\infty}\frac{1}{R}\int_a^{a+1}\int_0^R \big(\xi(r,\lambda)-\xi(r,\mu)\big)drd\mu,\ \lambda>0. \end{aligned}\tag{1.3.25}$$

It remains to check (1.3.24). Taking into consideration representation (1.3.17), one gets the following inequalities for $\lambda,\mu>0$,

$$\begin{aligned}\left|\frac{1}{R}\int_0^R \big(\xi(r,\lambda)-\xi(r,\mu)\big)dr\right| &\leq \sup_{r>0}|\xi(r,\lambda)-\xi(r,\mu)| \\ &\leq \sup_{r>0}\left|\left\lfloor\pi^{-1}\Big(r\sqrt{\lambda}-\delta\big(r,\sqrt{\lambda}\big)\Big)\right\rfloor-\left\lfloor\pi^{-1}r\sqrt{\lambda}\right\rfloor\right| \\ &+\sup_{r>0}\left|\left\lfloor\pi^{-1}\Big(r\sqrt{\mu}-\delta\big(r,\sqrt{\mu}\big)\Big)\right\rfloor-\left\lfloor\pi^{-1}r\sqrt{\mu}\right\rfloor\right|.\end{aligned}$$

Notice that the phase equation (1.2.7) implies the following estimate

$$\sup_{r>0}\left|\delta\left(r,\sqrt{\nu}\right)\right| \leq \frac{1}{\sqrt{\nu}}\int_0^\infty |V(r)|\,dr, \quad \nu>0, \tag{1.3.26}$$

which, together with the inequality

$$\sup_{r>0}\left|\left\lfloor\pi^{-1}\Big(r\sqrt{\nu}-\delta\big(r,\sqrt{\nu}\big)\Big)\right\rfloor-\left\lfloor\pi^{-1}r\sqrt{\nu}\right\rfloor\right| \leq \sup_{r>0}\left|\pi^{-1}\delta\big(r,\sqrt{\nu}\big)\right|+1,\ \nu>0,$$

gives (1.3.24).

Finally, to justify the change in the order of the integration in the RHS of (1.3.25), we remark that (1.3.17) together with (1.3.26) obviously implies

$$\sup_{\substack{0 \le r \le R \\ a \le \mu \le a+1}} |\xi(r,\lambda) - \xi(r,\mu)| < \infty, \quad \lambda > 0, \tag{1.3.27}$$

and, therefore, the function $\xi(r,\lambda) - \xi(r,\mu)$ is integrable over the rectangle $[0,R] \times [a, a+1]$. Thus one can apply Fubini's theorem to the repeated integral in the LHS of equation (1.3.25) to get (1.3.22). Thus, the proof is complete. $\square$

Chapter 2

The box approximation for a pseudo-differential operator with an unbounded symbol

2.1 Introduction

In this chapter, we study a family of unbounded operators $\mathcal{A}_r$ on the Hilbert space $L^2(\mathbb{R})$, parameterized by $r \in \mathbb{R}_+$, and defined by

$$\mathcal{A}_r = W_\alpha(I - P_r \mathcal{L} P_r) W_\alpha. \tag{2.1.1}$$

Here W_α, $\alpha > 0$, denotes the self-adjoint operator of multiplication by the function $e^{\frac{\alpha}{2}|x|}$,

$$(W_\alpha f)(x) = e^{\frac{\alpha}{2}|x|} f(x); \tag{2.1.2}$$

$\mathcal{L}$ is the self-adjoint integral operator of convolution type

$$(\mathcal{L}f)(x) = \int_{\mathbb{R}} L(x-y) f(y) dy, \quad f \in L^2(\mathbb{R}), \tag{2.1.3}$$

with symmetric kernel $L(x) = \overline{L(-x)}$; and P_r stands for the projection of $L^2(\mathbb{R})$ onto the subspace $L^2((-r, r))$.

Under mild assumptions on the kernel L, the operators $\mathcal{A}_r$ are well-defined self-adjoint operators on the (natural) domain

$$D(\mathcal{A}_r) = D(W_\alpha^2) = \left\{ f \in L^2(\mathbb{R}) : W_\alpha^2 f \in L^2(\mathbb{R}) \right\}.$$

The main goal of this chapter is to study the asymptotic behavior in the (norm) resolvent sense of the dynamical system $r \mapsto \mathcal{A}_r$ as r approaches infinity. It is natural to expect the answer to be expressed in terms of the "limiting" operator $\mathcal{A}$, given by the formal sandwiched operator,

$$\mathcal{A} = W_\alpha (I - \mathcal{L}) W_\alpha. \tag{2.1.4}$$

We remark that under the exponential fall-off assumption on the kernel, i.e., if $e^{\beta|x|}L(x)$ is a bounded function for some $\beta > \alpha$, the operator $\mathcal{A}$ is a well-defined symmetric operator on the domain $D(\mathcal{A}) = D(W_\alpha^2)$. Indeed, in this case the Fourier transform of L admits an analytic continuation to the strip $|\operatorname{Im} z| \leq \frac{\alpha}{2}$, which guarantees that $\mathcal{L}D(W_\alpha) \subset D(W_\alpha)$, and the symmetry of $\mathcal{A}$ on $D(W_\alpha^2)$ follows (cf. [1]).

However, one faces a possible difficulty: the symmetric operator $\mathcal{A}$ may not be essentially self-adjoint on the initial domain $D(\mathcal{A}) = D(W_\alpha^2)$ and the question arises which self-adjoint extension (or extensions) one should choose to describe the limiting behavior of the family $\mathcal{A}_r$.

In this chapter, we consider a "model" convolution operator $\mathcal{L}$ such that the symbol of $I - \mathcal{L}$ is a rational function with two real zeros. We prove that in this case the operator $\mathcal{A}$ has deficiency indices $(2, 2)$ and show that the attractor for the dynamical system $r \mapsto \mathcal{A}_r$ in the (norm) resolvent sense is a limit cycle consisting of a special one-parameter family of self-adjoint extensions of the operator

$\mathcal{A}$ (see Theorem 2.5.1), under the assumption that parameter r is increasing along sequences that avoid a thin "exceptional" set.

2.2 The adjoint operator and its properties

In this section we introduce and discuss some basic properties of the operator $\mathcal{A}$ of the form (2.1.4) and discuss its adjoint in details. It is convenient to summarize the necessary assumptions in the form of the following hypothesis.

Hypothesis 2.2.1. Assume that $\mathcal{A}$ is an operator of the form (2.1.4), where W_α is the operator of multiplication $(W_\alpha f)(x) = e^{\frac{\alpha}{2}|x|}f(x)$ with $\alpha < \frac{1}{3}$, and $\mathcal{L}$ is given by (2.1.3) with the integral kernel

$$L(x) = \omega e^{-|x|}, \quad \omega > 1/2, \quad x \in \mathbb{R}. \tag{2.2.1}$$

The following general result provides necessary conditions for operator $\mathcal{A}$ to be well-defined.

Lemma 2.2.2. *(cf.[1]) Let $\mathcal{L}$ be a convolution operator of the form* (2.1.3) *with kernel L satisfying $L(x) = \overline{L(-x)}$, and $e^{(3\alpha+\varepsilon)|\cdot|}L(\cdot) \in L^\infty(\mathbb{R})$ for some $\varepsilon > 0$. Assume, in addition, that the Fourier transform $\widehat{L}$ of L,*

$$\widehat{L}(x) = (\mathcal{F}L)(x) = \int_{\mathbb{R}} e^{-ixy}L(y)dy, \quad x \in \mathbb{R},$$

is bounded, that is, $\widehat{L} \in L^\infty(\mathbb{R})$. Let $\mathcal{A}$ be of the form (2.1.4)*, with W_α given by (2.1.2). Then $\mathcal{A}$ is a symmetric operator on $D(\mathcal{A}) = D(W_\alpha^2)$ with equal deficiency indices.*

Remark 2.2.3. Note that under Hypothesis 2.2.1, all assumptions of Lemma 2.2.2 hold and thus $\mathcal{A}$ is a symmetric operator with equal deficiency indices.

As discussed above, in order to describe the asymptotic behavior of the family $\mathcal{A}_r$, we need to be able to characterize self-adjoint extensions of the operator $\mathcal{A}$. The first step in this direction consists in describing the domain of the adjoint operator:

Theorem 2.2.4. *(cf.[1]) Suppose the assumptions of Lemma 2.2.2 hold. Assume in addition that the symbol l of the operator $I - \mathcal{L}$,*

$$l(s) = 1 - \widehat{L}(s), \quad s \in \mathbb{R},$$

satisfies the following conditions,

(i) the function l doesn't vanish on the boundary of the strip $|\operatorname{Im} z| \leq \frac{\alpha}{2}$,

$$l\left(s \pm i\frac{\alpha}{2}\right) \neq 0, \quad s \in \mathbb{R},$$

(ii) the function l has a finite number of zeros in the strip $|\operatorname{Im} z| \leq \frac{3}{2}\alpha$,

(iii) the function l^{-1} is bounded in a neighborhood of infinity in the strip $|\operatorname{Im} z| \leq \frac{3}{2}\alpha$.

Then

$$D(\mathcal{A}^*) = D(\mathcal{A}) \dotplus \operatorname{span}\left\{\mathcal{F}^{-1}U_{\alpha/2}q : q \in \mathcal{Q}\right\} \dotplus \operatorname{span}\left\{\mathcal{F}^{-1}U_{-\alpha/2}q : q \in \mathcal{Q}\right\},$$

where $\mathcal{Q}$ is the space of rational functions (vanishing at infinity) with poles only in the strip $|\operatorname{Im} z| < \alpha/2$, such that the function $l(\cdot)q(\cdot)$, $q \in \mathcal{Q}$, is analytic in the strip $|\operatorname{Im} z| < \alpha/2$, and $U_{\pm\alpha/2}$ denotes the operation of the complex shift,

$$(U_{\pm\alpha/2}g)(s) = g(s \pm i\alpha/2), \quad s \in \mathbb{R}.$$

Moreover, the operator $\mathcal{A}$ has deficiency indices (n, n), where $n = \dim \mathcal{Q}$.

Now we are ready to formulate the main result of this section.

Theorem 2.2.5. *Assume Hypothesis 2.2.1. Then the symbol l of the operator $I-\mathcal{L}$ has two real zeros, $\pm a$,*

$$a = \sqrt{2\omega - 1},$$

and the operator $\mathcal{A}$ is symmetric on $D(\mathcal{A}) = D(W_\alpha^2)$ with deficiency indices $(2,2)$ (and hence it admits a real four-parameter family of self-adjoint extensions). Moreover, the domain of the adjoint operator $\mathcal{A}^*$ is given by

$$D(\mathcal{A}^*) = D(\mathcal{A}) \dotplus \operatorname{span}\left\{f_a^-, f_{-a}^-, f_a^+, f_{-a}^+\right\}, \tag{2.2.2}$$

where

$$\begin{aligned} f_a^-(t) &= e^{iat} e^{\frac{\alpha}{2}t} \chi_{(-\infty,0)}(t), \quad & f_{-a}^-(t) &= e^{-iat} e^{\frac{\alpha}{2}t} \chi_{(-\infty,0)}(t), \\ f_a^+(t) &= e^{iat} e^{-\frac{\alpha}{2}t} \chi_{(0,\infty)}(t), \quad & f_{-a}^+(t) &= e^{-iat} e^{-\frac{\alpha}{2}t} \chi_{(0,\infty)}(t), \quad t \in \mathbb{R}. \end{aligned} \tag{2.2.3}$$

Proof. Clearly, the symbol l is given by the formula

$$l(x) = 1 - \omega \mathcal{F}^{-1}\left(e^{-|\cdot|}\right)(x) = \frac{x^2 - 2\omega + 1}{x^2 + 1} = \frac{x^2 - a^2}{x^2 + 1} \tag{2.2.4}$$

with

$$a = \sqrt{2\omega - 1}. \tag{2.2.5}$$

Since by Hypothesis 2.2.1, $\omega > 1/2$, l has two real zeros $\pm a$, and therefore, by Theorem 2.2.4, the deficiency indices of the operator $\mathcal{A}$ are $(2,2)$. Finally, to prove (2.2.2), observe that the space $\mathcal{Q}$ referred to in Theorem 2.2.4 is of the form

$$\mathcal{Q} = \operatorname{span}\left\{\frac{1}{x-a}, \frac{1}{x+a}\right\}.$$

By Theorem 2.2.4,

$$D(\mathcal{A}^*) = D(\mathcal{A}) \dotplus \operatorname{span}\left\{\mathcal{F}^{-1}U_{\alpha/2}q : q \in \mathcal{Q}\right\} \dotplus \operatorname{span}\left\{\mathcal{F}^{-1}U_{-\alpha/2}q : q \in \mathcal{Q}\right\}. \tag{2.2.6}$$

Now (2.2.2) follows from (2.2.6), since

$$\left(\mathcal{F}^{-1}U_{\pm\alpha/2}\left(\frac{1}{\cdot - a}\right)\right)(x) = \mp i f_a^{\pm},$$
$$\left(\mathcal{F}^{-1}U_{\pm\alpha/2}\left(\frac{1}{\cdot + a}\right)\right)(x) = \mp i f_{-a}^{\pm}.$$

This completes the proof. □

We will also need a result that describes the action of the adjoint operator $\mathcal{A}^*$ on the deficiency subspace (see Lemma 2.4.4).

Lemma 2.2.6. *Assume Hypothesis 2.2.1. Let the functions*

$$\{f_a^-, f_{-a}^-, f_a^+, f_{-a}^+\},$$

form a basis in the deficiency subspace $D(\mathcal{A}^*) \setminus D(\mathcal{A})$*, given by* (2.2.3)*. Then,*

$$(\mathcal{A}^* f_a^-)(t) = \frac{ia - \operatorname{sgn}(t)}{2} e^{-|t|(1-\alpha/2)}, \tag{2.2.7}$$
$$(\mathcal{A}^* f_a^+)(t) = \frac{-ia + \operatorname{sgn}(t)}{2} e^{-|t|(1-\alpha/2)}, \tag{2.2.8}$$
$$(\mathcal{A}^* f_{-a}^-)(t) = \frac{-ia - \operatorname{sgn}(t)}{2} e^{-|t|(1-\alpha/2)}, \tag{2.2.9}$$
$$(\mathcal{A}^* f_{-a}^+)(t) = \frac{ia + \operatorname{sgn}(t)}{2} e^{-|t|(1-\alpha/2)}. \tag{2.2.10}$$

In particular, $(\mathcal{A}^* f_{\pm a}^-)(t) = -(\mathcal{A}^* f_{\pm a}^+)(t)$, $t \in \mathbb{R}$.

Proof. We only prove equality (2.2.7), the rest is analogous.

By (2.1.4),

$$(\mathcal{A}^* f_a^-)(t) = e^{iat} e^{\frac{\alpha}{2}|t|} \chi_{(-\infty,0)}(t) - \omega e^{\frac{\alpha}{2}|t|} \int_{-\infty}^{0} e^{-|t-s|} e^{ias} ds. \tag{2.2.11}$$

For $t > 0$, $|t-s| = t - s$ and therefore,

$$\int_{-\infty}^{0} e^{-|t-s|} e^{ias} ds = e^{-t} \int_{-\infty}^{0} e^{(ia+1)s} ds = \frac{e^{-t}}{ia+1},$$

which together with (2.2.5), implies

$$(\mathcal{A}^* f_a^-)(t) = e^{\frac{\alpha}{2}|t|} e^{-t} \frac{-\omega}{ia+1} = e^{\frac{\alpha}{2}|t|} e^{-t} \frac{ia-1}{2}, \quad t > 0.$$

For $t < 0$, we have

$$\int_{-\infty}^{0} e^{-|t-s|} e^{ias} ds = \frac{e^t}{ia-1} + \frac{1}{\omega} e^{iat}. \tag{2.2.12}$$

Combining (2.2.12) with (2.2.11) and (2.2.5), we get

$$(\mathcal{A}^* f_a^-)(t) = e^{\frac{\alpha}{2}|t|} e^{t} \frac{-\omega}{ia-1} = e^{\frac{\alpha}{2}|t|} e^{t} \frac{ia+1}{2}, \quad t < 0,$$

completing the proof. □

2.3 The box approximation operators

In this section we study the invertibility properties of the truncated operators $\mathcal{A}_r$ given by (2.1.1) and provide an explicit representation for the inverse (when exists).

Recall that the truncated operator $\mathcal{A}_r$ admits the following representation

$$\mathcal{A}_r = W_\alpha^2 - W_\alpha P_r \mathcal{L} P_r W_\alpha, \quad r > 0.$$

Observe that the closure of $W_\alpha P_r \mathcal{L} P_r W_\alpha$ is a bounded operator on $L^2(\mathbb{R})$. Therefore, for every $r > 0$, $\mathcal{A}_r$ is well-defined and self-adjoint on $D(\mathcal{A}_r) = D(W_\alpha^2)$.

Introduce the "exceptional" set

$$\Xi = \bigcup_{k=-\infty}^{\infty} \left\{ \frac{1}{a} \left(\varphi + \frac{\pi k}{2} \right) \right\}, \tag{2.3.1}$$

where

$$\varphi = \arccos\left(\frac{\sqrt{2\omega - 1}}{\sqrt{2\omega}} \right) \in (0, \pi), \tag{2.3.2}$$

and a is given by (2.2.5), with ω being a parameter in the integral kernel L given by equation (2.2.1). Then we have the following result:

Theorem 2.3.1. *For every r outside the critical set, that is, $r \notin \Xi$, the operator $\mathcal{A}_r$ has a bounded inverse $\mathcal{A}_r^{-1}$ and for every $f \in L^2(\mathbb{R})$,*

$$\left(\mathcal{A}_r^{-1} f\right)(t) = e^{-\alpha|t|} f(t) + \chi_{[-r,r]} e^{-\frac{\alpha}{2}|t|} \int_{-r}^{r} \Gamma_{2\psi_r}(t,s) e^{-\frac{\alpha}{2}|s|} f(s) ds, \quad t > 0. \quad (2.3.3)$$

Here

$$\Gamma_\theta(t,s) = \frac{-2\omega}{a \sin(\theta)} \begin{cases} \cos\left(at - \frac{\theta}{2}\right) \cos\left(as + \frac{\theta}{2}\right), & s \leq t, \\ \cos\left(at + \frac{\theta}{2}\right) \cos\left(as - \frac{\theta}{2}\right), & t < s, \end{cases} \quad (2.3.4)$$

$$\psi_r = ar - \varphi, \quad (2.3.5)$$

with a and φ given by (2.2.5) and (2.3.2), respectively.

Proof. We remark that the operator $\mathcal{A}_r$ is block-diagonal with respect to the decomposition of $L^2(\mathbb{R})$ given by

$$L^2(\mathbb{R}) = L^2((-\infty, -r)) \oplus L^2((-r, r)) \oplus L^2((r, \infty)).$$

We therefore have

$$\mathcal{A}_r = W_\alpha^2 \oplus W_\alpha (I - \mathcal{L}_r) W_\alpha \oplus W_\alpha^2, \quad (2.3.6)$$

where $\mathcal{L}_r$ is the convolution operator on the Hilbert space $L^2((-r, r))$ (with semi-separable integral kernel):

$$(\mathcal{L}_r f)(x) = \omega \int_{-r}^{r} e^{-|x-y|} f(y) dy, \qquad x \in (-r, r). \quad (2.3.7)$$

Hence, $\mathcal{A}_r$ is invertible if and only if $I - \mathcal{L}_r$ is. Notice that $I - \mathcal{L}_r$ is a truncated Wiener-Hopf integral operator (cf. [7]) with the symbol l given by (2.2.4).

To prove the invertibility of the operator $I - \mathcal{L}_r$ under the condition $r \notin \Xi$, we consider the operator $\widetilde{\mathcal{L}}_r$ on $L^2((0, 2r))$, given by

$$(\widetilde{\mathcal{L}}_r f)(t) = \int_0^{2r} e^{-|t-s|} f(s) ds, \quad t \in (0, 2r).$$

Note that $I-\widetilde{\mathcal{L}}_r$ is unitarily equivalent to $I-\mathcal{L}_r$, since

$$I-\mathcal{L}_r = U\left(I-\widetilde{\mathcal{L}}_r\right)U^{-1}, \tag{2.3.8}$$

where $U: L^2((0,2r)) \to L^2((-r,r))$ is the shift operator

$$(Uf)(t) = f(t+r), \quad t \in [-r,r].$$

Observe that the function $\omega e^{-|t|}$ admits representation

$$\omega e^{-|t|} = \begin{cases} iCe^{-itA}(I-P)B, & 0 \le t \le 2r, \\ -iCe^{-itA}PB, & -2r \le t < 0, \end{cases}$$

with

$$C = \begin{pmatrix} i & -i \end{pmatrix}, \quad A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$
$$B = \begin{pmatrix} \omega \\ \omega \end{pmatrix}, \qquad P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, Theorem 2.6.1 implies that the operator $I-\widetilde{\mathcal{L}}_r$ is invertible if and only if $r \notin \Xi$ and

$$((I-\widetilde{\mathcal{L}}_r)^{-1}f)(t) = f(t) + \int_0^{2r} \widetilde{\Gamma}_{2\psi_r}(t,s)f(s)ds, \quad f \in L^2((0,2r)),\ t \in (0,2r), \tag{2.3.9}$$

with

$$\widetilde{\Gamma}_\theta(t,s) = \frac{-2\omega}{a\sin(\theta)} \begin{cases} \cos\left(a(t-r)-\frac{\theta}{2}\right)\cos\left(a(s-r)+\frac{\theta}{2}\right), & 0 \le s \le t \le 2r, \\ \cos\left(a(t-r)+\frac{\theta}{2}\right)\cos\left(a(s-r)-\frac{\theta}{2}\right), & 0 \le t < s \le 2r \end{cases} \tag{2.3.10}$$

and ψ_r given by (2.3.5). Now, relation (2.3.8) implies that the operator $I-\mathcal{L}_r$ is also invertible if and only if $r \notin \Xi$ with the inverse given by

$$((I-\mathcal{L}_r)^{-1}f)(t) = f(t) + \int_{-r}^{r} \Gamma_{2\psi_r}(t,s)f(s)ds, \quad f \in L^2((-r,r)),\ t \in (-r,r). \tag{2.3.11}$$

The statement of the lemma now follows from (2.3.6) and (2.3.11). $\square$

In the case where r belongs to the exceptional set Ξ (and, therefore, $\mathcal{A}_r$ is not invertible), the operator $\mathcal{A}_r$ has a nontrivial kernel, described in the following result:

Lemma 2.3.2. *Let $\mathcal{A}_r$ be given by (2.1.1) and $r \in \Xi$, the critical set given by* (2.3.1). *Then* $\mathrm{Ker}(\mathcal{A}_r) \neq 0$ *with*

$$e^{-\frac{\alpha}{2}|t|}\cos(at)\chi_{[-r,r]}(t) \in \mathrm{Ker}(\mathcal{A}_r), \quad \text{if } r = \frac{1}{a}\left(\varphi + \frac{\pi k}{2}\right) \text{ for } k \text{ even }, \tag{2.3.12}$$

$$e^{-\frac{\alpha}{2}|t|}\sin(at)\chi_{[-r,r]}(t) \in \mathrm{Ker}(\mathcal{A}_r), \quad \text{if } r = \frac{1}{a}\left(\varphi + \frac{\pi k}{2}\right) \text{ for } k \text{ odd }. \tag{2.3.13}$$

Proof. We will only prove (2.3.12); the proof of (2.3.13) is analogous. It suffices to show that the function $g \in L^2((-r,r))$,

$$g(t) = \cos(at), \quad t \in (-r,r),$$

is in the kernel of operator $I - \mathcal{L}_r$, with $\mathcal{L}_r$ given by (2.3.7). By the definition of $I - \mathcal{L}_r$,

$$((I - \mathcal{L}_r)g)(t) = \cos(at) - \omega \int_{-r}^{r} e^{-|t-s|}\cos(as)ds, \quad t \in (-r,r).$$

Let us rewrite

$$\int_{-r}^{r} e^{-|t-s|}\cos(as)ds = \int_{-r}^{t} e^{-(t-s)}\cos(as)ds + \int_{t}^{r} e^{t-s}\cos(as)ds \overset{def}{=} I_1 + I_2. \tag{2.3.14}$$

Using the elementary identity

$$\int e^{\alpha x}\cos(\beta x + d)dx = e^{\alpha x}\left(\frac{\beta}{\beta^2+\alpha^2}\sin(\beta x + d) + \frac{\alpha}{\beta^2+\alpha^2}\cos(\beta x + d)\right) + C, \tag{2.3.15}$$

$\alpha, \beta, d \in \mathbb{R}$, one gets

$$I_1 = \int_{-r}^{t} e^{-(t-s)} \cos(as)ds = e^{-t} \left[e^s \left(\frac{a}{a^2+1} \sin(as) + \frac{1}{a^2+1} \cos(as) \right) \right]_{-r}^{t}$$
$$= \frac{a}{a^2+1} \sin(at) + \frac{1}{a^2+1} \cos(at) - e^{-t-r} \left(\frac{a}{a^2+1} \sin(-ar) + \frac{1}{a^2+1} \cos(ar) \right).$$

Since $\cos(-ar) = \frac{a}{\sqrt{2\omega}}$ and $\sin(-ar) = -\frac{1}{\sqrt{2\omega}}$ for $r \in \Xi$, the second term in the last expression is zero and

$$I_1 = \frac{a}{a^2+1} \sin(at) + \frac{1}{a^2+1} \cos(at). \tag{2.3.16}$$

Similarly,

$$I_2 = \int_{t}^{r} e^{t-s} \cos(as)ds = e^{t} \int_{t}^{r} e^{-s} \cos(as)ds$$
$$= e^{t-r} \left(\frac{a}{a^2+1} \sin(ar) - \frac{1}{a^2+1} \cos(ar) \right) - \left(\frac{a}{a^2+1} \sin(at) + \frac{-1}{a^2+1} \cos(at) \right).$$

As before, the first term is zero, and so

$$I_2 = -\frac{a}{a^2+1} \sin(at) + \frac{1}{a^2+1} \cos(at). \tag{2.3.17}$$

Combining (2.3.16) and (2.3.17) with (2.3.14), and using that $a^2 + 1 = 2\omega$, we obtain

$$\int_{-r}^{r} e^{-|t-s|} \cos(as)ds = \frac{1}{\omega} \cos(at), \quad t \in (-r, r), \quad r \in \Xi,$$

proving the lemma. □

2.4 A family of self-adjoint extensions of the operator $\mathcal{A}$

In this section we introduce a special family of self-adjoint extensions $\mathcal{B}_\theta$, $\theta \in [0, 2\pi)$, of the operator $\mathcal{A}$ that describe the behavior of the operators $\mathcal{A}_r$ as r approaches infinity.

For $\theta \in [0, 2\pi)$, consider the family of pairs (F_θ, G_θ), $F_\theta, G_\theta \in D(\mathcal{A}^*)/D(\mathcal{A})$, given by

$$F_\theta = f_a^+ + f_a^- + e^{i\theta} f_{-a}^+ + e^{-i\theta} f_{-a}^-, \tag{2.4.1}$$

$$G_\theta = f_{-a}^+ + f_{-a}^- + e^{i\theta} f_a^- + e^{-i\theta} f_a^+. \tag{2.4.2}$$

It is easy to see that F_θ and G_θ are linearly independent if and only if θ is not an integer multiple of π. Indeed, consider the linear combination

$$aF_\theta + bG_\theta = (a + e^{-i\theta} b) f_a^+ + (a + e^{i\theta} b) f_a^- + (e^{i\theta} a + b) f_{-a}^+ + (e^{-i\theta} a + b) f_{-a}^-.$$

If $\sin(\theta) \neq 0$, each of the coefficients can vanish if and only if both a and b are zeros. Conversely, if $\sin(\theta) = 0$, the linear combination vanishes for every nonzero a and b such that $a = -b$.

Introduce the family $\mathcal{B}_\theta$ of extensions of $\mathcal{A}$,

$$\mathcal{A} \subset \mathcal{B}_\theta \subset \mathcal{A}^*,$$

with domain given by

$$D(\mathcal{B}_\theta) = D(\mathcal{A}) \dot{+} \mathcal{L}_\theta, \tag{2.4.3}$$

where

$$\mathcal{L}_\theta = \begin{cases} \operatorname{span}\{F_\theta, G_\theta\}, & \theta \in (0, \pi) \cup (\pi, 2\pi), \\ \operatorname{span}\left\{f_a^+ + f_{-a}^+, f_a^- + f_{-a}^-\right\}, & \theta = 0, \\ \operatorname{span}\left\{f_a^+ - f_{-a}^+, f_a^- - f_{-a}^-\right\}, & \theta = \pi. \end{cases}$$

One can verify that the family $\{\mathcal{L}_\theta\}_{\theta \in [0, 2\pi)}$ is a continuous family of two-dimensional planes in the four-dimensional space $D(\mathcal{A}^*)/D(\mathcal{A})$.

Next, we collect several results describing the (spectral) properties of the operators $\mathcal{B}_\theta$. Our first lemma shows that every extension $\mathcal{B}_\theta$ is a self-adjoint operator.

Lemma 2.4.1. *For every $\theta \in [0, 2\pi)$, $\mathcal{B}_\theta$ is self-adjoint.*

Proof. According to the general theory (cf. [3], [15], [14]), operator $\mathcal{B}_\theta$ is self-adjoint if and only if the corresponding linear manifold $\mathcal{L}_\theta$ is a Lagrangian plane in the deficiency subspace $D(\mathcal{A}^*)/D(\mathcal{A})$, i.e.,

$$[f, g] = (\mathcal{A}^* f, g) - (f, \mathcal{A}^* g) = 0, \quad f, g \in \mathcal{L}_\theta. \tag{2.4.4}$$

In turn, equation (2.4.4) holds if and only if $\mathcal{L}_\theta$ can be represented as the graph of a self-adjoint relation. Introduce the basis

$$\begin{aligned} b_1(t) &= p_a(t) = e^{iat} e^{-\frac{\alpha}{2}|t|}, & b_2(t) &= p_{-a}(t) = \frac{1}{i} e^{-iat} e^{-\frac{\alpha}{2}|t|}, \\ b_3(t) &= q_a(t) = \frac{1}{i} e^{iat} e^{-\frac{\alpha}{2}|t|} \operatorname{sgn}(t), & b_4(t) &= q_{-a}(t) = e^{-iat} e^{-\frac{\alpha}{2}|t|} \operatorname{sgn}(t). \end{aligned} \tag{2.4.5}$$

One can show (cf. [1]) that

$$[b_i, b_j] = \delta_{ij},$$

and therefore the basis $\mathbf{B} = \{b_j\}_{j=1}^4$ is canonical. In order to represent $\mathcal{L}_\theta$ in the new basis, we make the observation that

$$\begin{aligned} f_a^+ &= \frac{1}{2}(p_a + iq_a), & f_{-a}^+ &= \frac{1}{2}(ip_{-a} + q_{-a}), \\ f_a^- &= \frac{1}{2}(p_a - iq_a), & f_{-a}^- &= \frac{1}{2}(ip_{-a} - q_{-a}). \end{aligned} \tag{2.4.6}$$

We first consider the subspaces $\mathcal{L}_0$ and $\mathcal{L}_\pi$. Applying (2.4.6) to the generating vectors of those subspaces, we get

$$\mathcal{L}_0 = \operatorname{span}\left\{p_a + iq_a + ip_{-a} + q_{-a},\ p_a - iq_a + ip_{-a} - q_{-a}\right\}$$

and

$$\mathcal{L}_\pi = \operatorname{span}\left\{p_a + iq_a - ip_{-a} - q_{-a},\ p_a - iq_a - ip_{-a} + q_{-a}\right\}.$$

It is now clear that $\mathcal{L}_0$ and $\mathcal{L}_\pi$ can be thought of as the graphs of the operators M_0 and M_π, respectively, where

$$M_0 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad M_\pi = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

both acting from span $\{p_a,\ q_a\}$ to span $\{q_{-a},\ p_{-a}\}$; therefore, both are Lagrangian planes.

Next, we consider the subspace $\mathcal{L}_\theta$ for the remaining values of the parameter θ, namely those $\theta \in [0, 2\pi)$, with $\sin(\theta) \neq 0$. We start by finding new representations for the vectors F_θ and G_θ using equations (2.4.6):

$$\begin{aligned}
F_\theta &= f_a^+ + f_a^- + e^{i\theta} f_{-a}^+ + e^{-i\theta} f_{-a}^- \\
&= \frac{1}{2}(p_a + iq_a) + \frac{1}{2}(p_a - iq_a) + e^{i\theta}\frac{1}{2}(ip_{-a} + q_{-a}) + e^{-i\theta}\frac{1}{2}(ip_{-a} - q_{-a}) \\
&= p_a + i\cos(\theta)p_{-a} + i\sin(\theta)q_{-a}, \\
G_\theta &= f_{-a}^+ + f_{-a}^- + e^{i\theta} f_a^- + e^{-i\theta} f_a^+ \\
&= \frac{1}{2}(ip_{-a} + q_{-a}) + \frac{1}{2}(ip_{-a} - q_{-a}) + e^{i\theta}\frac{1}{2}(p_a - iq_a) + e^{-i\theta}\frac{1}{2}(p_a + iq_a) \\
&= ip_{-a} + \cos(\theta)p_a + \sin(\theta)q_a.
\end{aligned}$$

Therefore, the corresponding representation for $\mathcal{L}_\theta$ is given by

$$\mathcal{L}_\theta = \text{span}\left\{p_a + i\cos(\theta)p_{-a} + i\sin(\theta)q_{-a},\ \cos(\theta)p_a + ip_{-a} + \sin(\theta)q_a\right\}.$$

To complete the proof, observe that $\mathcal{L}_\theta$ is the graph of the self-adjoint operator

$$M_\theta = \frac{1}{\sin(\theta)} \begin{pmatrix} -\cos(\theta) & -i \\ i & -\cos(\theta) \end{pmatrix},$$

acting from span $\{p_a,\ p_{-a}\}$ to span $\{q_a,\ q_{-a}\}$. Indeed, an arbitrary element of $\mathcal{L}_\theta$ is of the form

$$(c_1 + c_2\cos(\theta))p_a + (c_1 i\cos(\theta) + c_2 i)p_{-a} + c_2\sin(\theta)q_a + c_1 i\sin(\theta)q_{-a},\ c_1, c_2 \in \mathbb{C},$$

where the coefficients obviously satisfy the following relation for all $c_1, c_2 \in \mathbb{C}$

$$\begin{pmatrix} c_2 \sin(\theta) \\ c_1 i \sin(\theta) \end{pmatrix} = M_\theta \begin{pmatrix} c_1 + c_2 \cos(\theta) \\ c_1 i \cos(\theta) + c_2 i \end{pmatrix}.$$

We conclude that $\mathcal{L}_\theta$ is a Lagrangian plane for every $\theta \in [0, 2\pi)$, proving the lemma. $\square$

The following results deal with two complementary situations: when zero belongs to the resolvent set of the operator $\mathcal{B}_\theta$ and when it is a part of the spectrum. In the former case, we provide a formula for the inverse $\mathcal{B}_\theta^{-1}$, and in the latter case we give a characterization of the corresponding eigenspace.

Theorem 2.4.2. *For every $\theta \in (0, \pi) \cup (\pi, 2\pi)$, the operator $\mathcal{B}_\theta$ has a bounded inverse given by $\mathcal{B}_\theta^{-1} = G_\theta$, where*

$$(G_\theta f)(t) = e^{-\alpha|t|} f(t) + e^{-\frac{\alpha}{2}|t|} \int_{-\infty}^{\infty} \Gamma_\theta(t, s) e^{-\frac{\alpha}{2}|s|} f(s) ds, \quad f \in L^2(\mathbb{R}), \ t \in \mathbb{R}. \quad (2.4.7)$$

Here $\Gamma_\theta(t, s)$ is defined by (2.3.4) with $-\infty < s, t < \infty$.

The proof of the theorem is based on the following result:

Lemma 2.4.3. *For every $\theta \in (0, \pi) \cup (\pi, 2\pi)$ one has $\operatorname{Ran}(G_\theta) \subset D(\mathcal{B}_\theta)$; equivalently, for every $f \in L^2(\mathbb{R})$,*

$$e^{-\alpha|\cdot|} f(\cdot) + e^{-\frac{\alpha}{2}|\cdot|} \int_{-\infty}^{\infty} \Gamma_\theta(\cdot, s) e^{-\frac{\alpha}{2}|s|} f(s) ds \in D(\mathcal{B}_\theta).$$

Proof. Recall that

$$D(\mathcal{B}_\theta) = D(\mathcal{A}) \dotplus \operatorname{span}\{F_\theta, G_\theta\},$$

where F_θ and G_θ are given by (2.4.1) and (2.4.2), respectively. Therefore, in order to prove the assertion, we have to find a decomposition of $(G_\theta f)(t)$ of the form

$$(G_\theta f)(t) = \widetilde{h}_0(t) + C_1 F_\theta(t) + C_2 G_\theta(t), \quad t \in \mathbb{R}, \tag{2.4.8}$$

where $\widetilde{h}_0 \in D(\mathcal{A})$ and $C_1, C_2 \in \mathbb{C}$.

First, observe that

$$e^{-\alpha|\cdot|} f(\cdot) \in D(\mathcal{A}).$$

Next, we will show that

$$e^{-\frac{\alpha}{2}|t|} \int_{-\infty}^{\infty} \Gamma_\theta(t,s) e^{-\frac{\alpha}{2}|s|} f(s) ds = h_0(t) + C_1 F_\theta(t) + C_2 G_\theta(t), \quad t \in \mathbb{R}.$$

Here $h_0 \in D(\mathcal{A})$ is independent of θ, and $C_1, C_2 \in \mathbb{C}$ are given by

$$C_1 = C\frac{1}{4}(f, p_{-a})_{L^2(\mathbb{R})}, \qquad C_2 = C\frac{1}{4}(f, p_a)_{L^2(\mathbb{R})}, \tag{2.4.9}$$

where

$$C = \frac{-2\omega}{a\sin(\theta)},$$

and p_a and p_{-a} are defined in (2.4.5). This will then complete the proof. To prove (2.4.8), let

$$K(s) = e^{-\frac{\alpha}{2}|s|} f(s), \quad s \in \mathbb{R}. \tag{2.4.10}$$

By the definition of Γ_θ (cf. (2.3.4)),

$$\begin{aligned} e^{-\frac{\alpha}{2}|t|} \int_{-\infty}^{\infty} \Gamma_\theta(t,s) K(s) ds = {} & Ce^{-\frac{\alpha}{2}|t|} \int_{-\infty}^{t} \cos(at - \theta/2)\cos(as + \theta/2) K(s) ds \\ & + Ce^{-\frac{\alpha}{2}|t|} \int_{t}^{\infty} \cos(at + \theta/2)\cos(as - \theta/2) K(s) ds, \ t \in \mathbb{R}. \end{aligned} \tag{2.4.11}$$

Applying the elementary equalities

$$4\cos(at-\theta/2)\cos(as+\theta/2) = e^{iat}e^{ias} + e^{iat}e^{-ias}e^{-i\theta} + e^{-iat}e^{ias}e^{i\theta} + e^{-iat}e^{-ias}$$

and

$$4\cos(at+\theta/2)\cos(as-\theta/2) = e^{iat}e^{ias} + e^{iat}e^{-ias}e^{i\theta} + e^{-iat}e^{ias}e^{-i\theta} + e^{-iat}e^{-ias}$$

to (2.4.11), we obtain, after some algebra,

$$e^{-\frac{\alpha}{2}|t|}\int_{-\infty}^{\infty}\Gamma_\theta(t,s)K(s)ds = C_1 e^{-\frac{\alpha}{2}|t|}e^{iat} + C_2 e^{-\frac{\alpha}{2}|t|}e^{-iat} + R(t), \quad t\in\mathbb{R}, \quad (2.4.12)$$

where

$$R(t) = \frac{C}{4}e^{-\frac{\alpha}{2}|t|}\Big[\int_{-\infty}^{t}(e^{iat}e^{-ias}e^{-i\theta} + e^{-iat}e^{ias}e^{i\theta})K(s)ds + \int_t^{\infty}(e^{iat}e^{-ias}e^{i\theta} + e^{-iat}e^{ias}e^{-i\theta})K(s)ds\Big], \quad t\in\mathbb{R}.$$

Next, we rewrite R as

$$R(t) = C_2 e^{-i\theta}e^{-\frac{\alpha}{2}|t|}e^{iat} + C_1 e^{i\theta}e^{-\frac{\alpha}{2}|t|}e^{-iat} + R_1(t), \quad t\in\mathbb{R}, \quad (2.4.13)$$

where

$$R_1(t) = \frac{C}{2}i\sin(\theta)e^{-\frac{\alpha}{2}|t|}\int_t^{\infty}(e^{iat}e^{-ias} - e^{-iat}e^{ias})K(s)ds, \quad t\in\mathbb{R}. \quad (2.4.14)$$

Combining (2.4.12), (2.4.13), and (2.4.14), and taking into account that

$$e^{-\frac{\alpha}{2}|t|}e^{iat} = f_a^+(t) + f_a^-(t) \quad \text{and} \quad e^{-\frac{\alpha}{2}|t|}e^{-iat} = f_{-a}^+(t) + f_{-a}^-(t), \quad t\in\mathbb{R},$$

we finally get

$$
\begin{aligned}
e^{-\frac{\alpha}{2}|t|}&\int_{-\infty}^{\infty}\Gamma_\theta(t,s)K(s)ds\\
&=C_1(f_a^+(t)+f_a^-(t))+C_2(f_{-a}^+(t)+f_{-a}^-(t))+C_2e^{-i\theta}(f_a^+(t)+f_a^-(t))\\
&\quad+C_1e^{i\theta}(f_{-a}^+(t)+f_{-a}^-(t))\\
&\quad+\frac{C}{2}i\sin(\theta)e^{-\frac{\alpha}{2}|t|}\int_t^{\infty}(e^{iat}e^{-ias}-e^{-iat}e^{ias})K(s)ds\\
&=C_1(f_a^+(t)+f_a^-(t)+e^{i\theta}f_{-a}^+(t)+e^{i\theta}f_{-a}^-(t))\\
&\quad+C_2(f_{-a}^+(t)+f_{-a}^-(t)+e^{-i\theta}f_a^+(t)+e^{-i\theta}f_a^-(t))\\
&\quad+\frac{C}{2}i\sin(\theta)e^{-\frac{\alpha}{2}|t|}\int_t^{\infty}(e^{iat}e^{-ias}-e^{-iat}e^{ias})K(s)ds\\
&=C_1F_\theta(t)+C_2G_\theta(t)+h_0(t),\quad t\in\mathbb{R},
\end{aligned}
$$

where

$$
\begin{aligned}
h_0(t)=&\frac{C}{2}i\sin(\theta)e^{-\frac{\alpha}{2}|t|}\int_t^{\infty}(e^{iat}e^{-ias}-e^{-iat}e^{ias})K(s)ds\\
&+C_1f_{-a}^-(t)(e^{i\theta}-e^{-i\theta})+C_2f_a^-(t)(e^{-i\theta}-e^{i\theta}),\quad t\in\mathbb{R}.
\end{aligned}
$$

It remains to show that $h_0\in D(\mathcal{A})$. Indeed,

$$
h_0(t)=2i\sin(\theta)e^{-\frac{\alpha}{2}|t|}\left[\frac{C}{4}\int_t^{\infty}(e^{iat}e^{-ias}-e^{-iat}e^{ias})K(s)ds+C_1f_{-a}^-(t)-C_2f_a^-(t)\right].
$$

Applying (2.2.3), we see that

$$
C_1f_{-a}^-(t)-C_2f_a^-(t)=-\left(C_2e^{iat}-C_1e^{-iat}\right)\chi_{(-\infty,0)}(t),\quad t\in\mathbb{R},
$$

which, together with (2.4.9), gives

$$
h_0(t)=\frac{2\omega}{a}e^{-\frac{\alpha}{2}|t|}\operatorname{sgn}(t)\begin{cases}\int_t^{\infty}\sin(a(t-s))K(s)ds, & t>0,\\ \int_{-\infty}^{t}\sin(a(t-s))K(s)ds, & t<0.\end{cases}\tag{2.4.15}
$$

Equation (2.4.15) together with (2.4.10) implies that $e^{\alpha|\cdot|}h_0(\cdot)\in L^2(\mathbb{R})$, completing the proof. □

We are now ready to prove Theorem 2.4.2.

Proof of Theorem 2.4.2. In order to prove that G_θ is an inverse of $\mathcal{B}_\theta$ one has to prove the following two assertions:

(i) for every $f \in L^2(\mathbb{R})$, $G_\theta f \in D(\mathcal{B}_\theta)$ and $\mathcal{B}_\theta G_\theta f = f$,

(ii) for every $h \in D(\mathcal{B}_\theta)$, $G_\theta \mathcal{B}_\theta h = h$.

This will imply that $\mathrm{Ran}(B_\theta) = L^2(\mathbb{R})$, $\mathrm{Ker}(B_\theta) = \{0\}$, $\mathcal{B}_\theta G_\theta = I$, and $G_\theta \mathcal{B}_\theta = I|_{D(\mathcal{B}_\theta)}$. Lemma 2.4.3 yields the first part of statement (i); the next step is to show that

$$\mathcal{B}_\theta G_\theta f = f, \quad \forall f \in L^2(\mathbb{R}). \tag{2.4.16}$$

Combining (2.4.7) with (2.1.4), we get

$$\begin{aligned}
(\mathcal{B}_\theta G_\theta f)(t) &= e^{\alpha|t|}(G_\theta f)(t) - \omega e^{\frac{\alpha}{2}|t|}\int_{-\infty}^{\infty} e^{-|t-s|}e^{\frac{\alpha}{2}|s|}(G_\theta f)(s)ds \\
&= f(t) + e^{\frac{\alpha}{2}|t|}\int_{-\infty}^{\infty}\Gamma_\theta(t,s)e^{-\frac{\alpha}{2}|s|}f(s)ds - \omega e^{\frac{\alpha}{2}|t|}\int_{-\infty}^{\infty} e^{-|t-s|}e^{-\frac{\alpha}{2}|s|}f(s)ds \\
&- \omega e^{\frac{\alpha}{2}|t|}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-|t-p|}\Gamma_\theta(p,s)e^{-\frac{\alpha}{2}|s|}f(s)dsdp, \quad t \in \mathbb{R},
\end{aligned}$$

implying that $\mathcal{B}_\theta G_\theta f = f$ for every $f \in L^2(\mathbb{R})$ if and only if

$$\Gamma_\theta(t,s) - \omega e^{-|t-s|} - \omega\int_{-\infty}^{\infty} e^{-|t-p|}\Gamma_\theta(p,s)dp \equiv 0. \tag{2.4.17}$$

In order to prove (2.4.17), assume, without loss of generality, that $t \geq s$. Then

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-|t-p|}\Gamma_\theta(p,s)dp &= \int_{-\infty}^{s} e^{-|t-p|}\Gamma_\theta(p,s)dp \\
&+ \int_{s}^{t} e^{-|t-p|}\Gamma_\theta(p,s)dp + \int_{t}^{\infty} e^{-|t-p|}\Gamma_\theta(p,s)dp \overset{def}{=} I_1 + I_2 + I_3.
\end{aligned} \tag{2.4.18}$$

Using (2.3.15), we get

$$\begin{aligned}
I_1 &= Ce^{-t}\cos(as-\theta/2)\int_{-\infty}^{s} e^p \cos(ap+\theta/2)dp \qquad (2.4.19)\\
&= Ce^{-t}\cos(as-\theta/2)\left[e^p\left(\frac{a}{a^2+1}\sin(ap+\theta/2)+\frac{1}{a^2+1}\cos(ap+\theta/2)\right)\right]_{-\infty}^{s}\\
&= Ce^{-t+s}\cos(as-\theta/2)\left(\frac{a}{a^2+1}\sin(as+\theta/2)+\frac{1}{a^2+1}\cos(as+\theta/2)\right),
\end{aligned}$$

$$\begin{aligned}
I_2 &= Ce^{-t}\cos(as+\theta/2)\int_{s}^{t} e^p \cos(ap-\theta/2)dp \qquad (2.4.20)\\
&= Ce^{-t}\cos(as+\theta/2)\left[e^p\left(\frac{a}{a^2+1}\sin(ap-\theta/2)+\frac{1}{a^2+1}\cos(ap-\theta/2)\right)\right]_{s}^{t}\\
&= C\cos(as+\theta/2)\left(\frac{a}{a^2+1}\sin(at-\theta/2)+\frac{1}{a^2+1}\cos(at-\theta/2)\right)\\
&\quad - Ce^{-t+s}\cos(as+\theta/2)\left(\frac{a}{a^2+1}\sin(as-\theta/2)+\frac{1}{a^2+1}\cos(as-\theta/2)\right),
\end{aligned}$$

and

$$\begin{aligned}
I_3 &= Ce^{t}\cos(as+\theta/2)\int_{t}^{\infty} e^{-p} \cos(ap-\theta/2)dp \qquad (2.4.21)\\
&= Ce^{t}\cos(as+\theta/2)\left[e^{-p}\left(\frac{a}{a^2+1}\sin(ap-\theta/2)-\frac{1}{a^2+1}\cos(ap-\theta/2)\right)\right]_{t}^{\infty}\\
&= -C\cos(as+\theta/2)\left(\frac{a}{a^2+1}\sin(at-\theta/2)-\frac{1}{a^2+1}\cos(at-\theta/2)\right).
\end{aligned}$$

Combining (2.4.19), (2.4.20), and (2.4.21), we immediately get

$$\begin{aligned}
I_1+I_2+I_3 &= C\frac{2}{a^2+1}\cos(as+\theta/2)\cos(at-\theta/2)\\
&\quad + C\frac{a}{a^2+1}e^{-t+s}\left[\cos(as-\theta/2)\sin(as+\theta/2)-\cos(as+\theta/2)\sin(as-\theta/2)\right]\\
&= \frac{1}{\omega}\Gamma_\theta(t,s)+e^{-|t-s|}C\frac{a}{a^2+1}\sin(\theta)=\frac{1}{\omega}\Gamma_\theta(t,s)-e^{-|t-s|},
\end{aligned}$$

which, together with (2.4.18), implies (2.4.17). This establishes (2.4.16).

To finish the proof, we need to show that

$$G_\theta \mathcal{B}_\theta f = f, \quad f \in D(\mathcal{B}_\theta). \qquad (2.4.22)$$

According to the structure of $D(\mathcal{B}_\theta)$ defined by (2.4.3), in order to prove (2.4.22), we need to check that the following three equations hold,

$$G_\theta \mathcal{B}_\theta f = f, \quad \text{for every } f \in D(\mathcal{A}),$$

$$G_\theta \mathcal{B}_\theta F_\theta = F_\theta, \tag{2.4.23}$$

$$G_\theta \mathcal{B}_\theta G_\theta = G_\theta. \tag{2.4.24}$$

If $f \in D(\mathcal{A})$, then

$$\begin{aligned}
(G_\theta \mathcal{B}_\theta f)(t) &= e^{-\alpha|t|}(\mathcal{A}f)(t) + e^{-\frac{\alpha}{2}|t|}\int_{-\infty}^{\infty} \Gamma_\theta(t,s) e^{-\frac{\alpha}{2}|s|}(\mathcal{A}f)(s)ds \\
&= f(t) - \omega e^{-\frac{\alpha}{2}|t|}\int_{-\infty}^{\infty} e^{-|t-s|}e^{\frac{\alpha}{2}|s|}f(s)ds + e^{-\frac{\alpha}{2}|t|}\int_{-\infty}^{\infty}\Gamma_\theta(t,s)e^{\frac{\alpha}{2}|s|}f(s)ds \\
&- \omega e^{-\frac{\alpha}{2}|t|}\int_{-\infty}^{\infty}\Gamma_\theta(t,p)\int_{-\infty}^{\infty} e^{-|p-s|}e^{\frac{\alpha}{2}|s|}f(s)dsdp, \quad t \in \mathbb{R},
\end{aligned}$$

implying that $G_\theta \mathcal{B}_\theta f = f$ for every $f \in D(\mathcal{A})$ if and only if

$$\Gamma_\theta(t,s) - \omega e^{-|t-s|} - \omega \int_{-\infty}^{\infty} e^{-|p-s|}\Gamma_\theta(t,p)dp \equiv 0. \tag{2.4.25}$$

But (2.4.25) follows immediately from (2.4.17), taking into account that

$$\Gamma_\theta(t,s) = \Gamma_\theta(s,t).$$

It remains to establish (2.4.23) and (2.4.24). We focus on (2.4.23), equation (2.4.24) being proven analogously. Using the fact that $\mathcal{B}_\theta$ is a restriction of the operator $\mathcal{A}^*$ and applying equations (2.2.7) – (2.2.10) together with (2.4.1), we get

$$\begin{aligned}
G_\theta(\mathcal{B}_\theta F_\theta) &= G_\theta(\mathcal{A}^*(f_a^+ + f_a^- + e^{i\theta}f_{-a}^+ + e^{-i\theta}f_{-a}^-)) = 2i\sin(\theta)G_\theta(\mathcal{A}^* f_{-a}^+) \\
&= 2i\sin(\theta)\left(e^{-\alpha|\cdot|}\mathcal{A}^* f_{-a}^+ + e^{-\frac{\alpha}{2}|\cdot|}\int_{-\infty}^{\infty}\Gamma_\theta(\cdot,s)e^{-\frac{\alpha}{2}|s|}(\mathcal{A}^* f_{-a}^+)(s)ds\right) \\
&= 2i\sin(\theta)e^{-\frac{\alpha}{2}|\cdot|}\left(\frac{ia+\operatorname{sgn}(\cdot)}{2}e^{-|\cdot|} + \int_{-\infty}^{\infty}\Gamma_\theta(\cdot,s)\frac{ia+\operatorname{sgn}(s)}{2}e^{-|s|}ds\right).
\end{aligned} \tag{2.4.26}$$

Now, write

$$\int_{-\infty}^{\infty} \Gamma_\theta(t,s)\frac{ia+\operatorname{sgn}(s)}{2}e^{-|s|}ds = \frac{ia-1}{2}\int_{-\infty}^{0}\Gamma_\theta(t,s)e^{s}ds \\ + \frac{ia+1}{2}\int_{0}^{\infty}\Gamma_\theta(t,s)e^{-s}ds, \quad t\in\mathbb{R}. \tag{2.4.27}$$

For the rest of the proof we only consider $t>0$; the case $t<0$ can be handled in a similar manner. Applying equation (2.3.4), we get

$$\begin{aligned}
&\int_{-\infty}^{0}\Gamma_\theta(t,s)e^{s}ds \\
&\quad = C\cos(at-\theta/2)\int_{-\infty}^{0}\cos(as+\theta/2)e^{s}ds \qquad (2.4.28)\\
&\quad = C\cos(at-\theta/2)\left[e^{s}\left(\frac{a}{a^2+1}\sin(as+\theta/2)+\frac{1}{a^2+1}\cos(as+\theta/2)\right)\right]_{-\infty}^{0}\\
&\quad = C\cos(at-\theta/2)\left(\frac{a}{a^2+1}\sin(\theta/2)+\frac{1}{a^2+1}\cos(\theta/2)\right)
\end{aligned}$$

and

$$\int_{0}^{\infty}\Gamma_\theta(t,s)e^{-s}ds = \int_{0}^{t}\Gamma_\theta(t,s)e^{-s}ds + \int_{t}^{\infty}\Gamma_\theta(t,s)e^{-s}ds \overset{def}{=} I_1+I_2, \tag{2.4.29}$$

Now,

$$\begin{aligned}
I_1 &= C\cos(at-\theta/2)\left[e^{-s}\left(\frac{a}{a^2+1}\sin(as+\theta/2)-\frac{1}{a^2+1}\cos(as+\theta/2)\right)\right]_{0}^{t}\\
&= C\cos(at-\theta/2)e^{-t}\left(\frac{a}{a^2+1}\sin(at+\theta/2)-\frac{1}{a^2+1}\cos(at+\theta/2)\right)\\
&\quad - C\cos(at-\theta/2)\left(\frac{a}{a^2+1}\sin(\theta/2)-\frac{1}{a^2+1}\cos(\theta/2)\right), \qquad (2.4.30)
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= C\cos(at+\theta/2)\left[e^{-s}\left(\frac{a}{a^2+1}\sin(as-\theta/2)-\frac{1}{a^2+1}\cos(as-\theta/2)\right)\right]_{t}^{\infty}\\
&= -C\cos(at+\theta/2)e^{-t}\left(\frac{a}{a^2+1}\sin(at-\theta/2)-\frac{1}{a^2+1}\cos(at-\theta/2)\right).
\end{aligned} \tag{2.4.31}$$

Combining (2.4.30) and (2.4.31) with (2.4.29), we obtain

$$\int_0^\infty \Gamma_\theta(t,s)e^{-s}ds = -e^{-t} - C\cos(at-\theta/2)\left(\frac{a}{a^2+1}\sin(\theta/2) - \frac{1}{a^2+1}\cos(\theta/2)\right). \tag{2.4.32}$$

Lastly, using (2.4.28), (2.4.32), and (2.4.27), we get

$$\begin{aligned}\int_{-\infty}^\infty &\Gamma_\theta(t,s)\frac{ia+\mathrm{sgn}(s)}{2}e^{-|s|}ds\\ &= -e^{-t}\frac{ia+1}{2} + \frac{\sin(\theta/2)}{\sin(\theta)}\cos(at-\theta/2) - i\frac{\cos(\theta/2)}{\sin(\theta)}\cos(at-\theta/2)\\ &= -e^{-t}\frac{ia+1}{2} + \frac{1}{2i\sin(\theta)}(e^{iat}+e^{-iat}e^{i\theta}), \quad t\in\mathbb{R},\end{aligned}$$

which, together with (2.4.26), gives

$$\begin{aligned}(G_\theta\mathcal{B}_\theta F_\theta)(t) &= 2i\sin(\theta)e^{-\frac{\alpha}{2}|t|}\left[\frac{ia+1}{2}e^{-t} - e^{-t}\frac{ia+1}{2} + \frac{1}{2i\sin(\theta)}\left(e^{iat}+e^{-iat}e^{i\theta}\right)\right]\\ &= e^{-\frac{\alpha}{2}|t|}\left(e^{iat}+e^{-iat}e^{i\theta}\right) = F_\theta(t), \quad t>0,\end{aligned}$$

completing the proof. □

We remark that in the case θ equals 0 or π, the corresponding operators $\mathcal{B}_\theta$ are not invertible. The following result describes their kernels.

Lemma 2.4.4. *The operators $\mathcal{B}_0$ and $\mathcal{B}_\pi$ are not invertible and*

$$f_a^+ + f_a^- + f_{-a}^+ + f_{-a}^- \in \mathrm{Ker}(\mathcal{B}_0)$$

and

$$f_a^+ + f_a^- - f_{-a}^+ - f_{-a}^- \in \mathrm{Ker}(\mathcal{B}_\pi).$$

Proof. Follows immediately from Lemma 2.2.6. □

2.5 The main result

In this section we will prove the main result of this chapter, which describes the behavior of the family of truncations $\mathcal{A}_r$ for large values of the parameter r. It turns out that $\mathcal{A}_r$ converges to the torus $\mathcal{B}_\theta$ (described in the previous section) in the norm resolvent sense, along sequences that avoid a particular "exceptional" set. More precisely, we have the following result:

Theorem 2.5.1. *Let $\mathcal{A}_r$ be given by (2.1.1) and $\mathcal{B}_\theta$ be the family of self-adjoint extensions of $\mathcal{A}$ with domain given by (2.4.3). Let Ξ be the exceptional set defined in* (2.3.1). *Then for every sequence $\{r_k\}_{k=1}^{\infty}$, $r_k = r_0 + \frac{1}{a}\pi k$, with $r_0 \notin \Xi$,*

$$\lim_{k\to\infty} \left\|(\mathcal{B}_{\theta_0} - z)^{-1} - (\mathcal{A}_{r_k} - z)^{-1}\right\| = 0 \quad \forall z \in \rho(\mathcal{B}_{\theta_0}) \cap \rho(\mathcal{A}_{r_k}) \text{ for all } k \in \mathbb{N}, \tag{2.5.1}$$

where

$$\theta_0 = 2\psi_{r_0} \ (\mathrm{mod}\ 2\pi),$$

and ψ_r is given by (2.3.5).

Proof. Using the identity,

$$(A - z)^{-1} - (B - z)^{-1} = (A - z)^{-1} A (A^{-1} - B^{-1}) B (B - z)^{-1},$$

which holds for arbitrary self-adjoint operators A and B such that $0 \in \rho(A) \cap \rho(B)$, we get

$$\left\|(A - z)^{-1} - (B - z)^{-1}\right\| \le C \left\|A^{-1} - B^{-1}\right\|, \tag{2.5.2}$$

with C independent of A and B. Therefore, it is sufficient to prove equation (2.5.1) for $z = 0$, namely,

$$\lim_{k\to\infty} \left\| \mathcal{B}_{\theta_0}^{-1} - \mathcal{A}_{r_k}^{-1} \right\| = 0,$$

for $\theta_0 = 2\psi_{r_0} \pmod{2\pi}$, and ψ_r given by (2.3.5). Notice that for $r_0 \notin \Xi$, the corresponding θ_0 satisfies $\sin(\theta_0) \neq 0$, and by Theorem 2.4.2, the operator $\mathcal{B}_{\theta_0}$ is invertible with the inverse given by (2.4.7). Let P_r be the projection of $L^2(\mathbb{R})$ onto the subspace $L^2((-r, r))$, and introduce

$$Q_r = I - P_r,$$

where I is the identity on $L^2(\mathbb{R})$. Recall that the inverse of the operator $\mathcal{B}_{\theta_0}$ is given by (2.4.7),

$$(\mathcal{B}_{\theta_0}^{-1} f)(t) = e^{-\alpha|t|} f(t) + e^{-\frac{\alpha}{2}|t|} \int_{-\infty}^{\infty} \Gamma_{\theta_0}(t, s) e^{-\frac{\alpha}{2}|s|} f(s) ds, \quad f \in L^2(\mathbb{R}),\ t \in \mathbb{R},$$

and the inverse of the operator $\mathcal{A}_{r_k}$, given by (2.3.3), is of the form

$$\left(\mathcal{A}_{r_k}^{-1} f\right)(t) = e^{-\alpha|t|} f(t) + \chi_{[-r,r]} e^{-\frac{\alpha}{2}|t|} \int_{-r}^{r} \Gamma_{\theta_0}(t, s) e^{-\frac{\alpha}{2}|s|} f(s) ds, \quad f \in L^2(\mathbb{R}),\ t \in \mathbb{R},$$

with

$$\Gamma_\theta(t, s) = \frac{C}{\sin(\theta)} \begin{cases} \cos\left(at - \frac{\theta}{2}\right) \cos\left(as + \frac{\theta}{2}\right), & s \leq t, \\ \cos\left(at + \frac{\theta}{2}\right) \cos\left(as - \frac{\theta}{2}\right), & t < s. \end{cases}$$

Denote

$$(Tf)(t) = e^{-\frac{\alpha}{2}|t|} \int_{-\infty}^{\infty} \Gamma_{\theta_0}(t, s) e^{-\frac{\alpha}{2}|s|} f(s) ds, \quad t \in \mathbb{R}.$$

In this notation one can represent the inverses $\mathcal{B}_{\theta_0}^{-1}$ and $\mathcal{A}_{r_k}^{-1}$ in the following way,

$$(\mathcal{B}_{\theta_0}^{-1} f)(t) = e^{-\alpha|t|} f(t) + (Tf)(t), \quad t \in \mathbb{R},$$

and

$$\left(\mathcal{A}_{r_k}^{-1} f\right)(t) = e^{-\alpha|t|} f(t) + (P_r T P_r f)(t), \quad t \in \mathbb{R}.$$

Clearly,

$$\left(\left(\mathcal{B}_{\theta_0}^{-1}-\mathcal{A}_{r_k}^{-1}\right)f\right)(t)=(Q_rTf)(t)+(P_rTQ_rf)(t),\quad t\in\mathbb{R},$$

and therefore,

$$\begin{aligned}\left\|\left(\mathcal{B}_{\theta_0}^{-1}-\mathcal{A}_{r_k}^{-1}\right)f\right\|_{L^2(\mathbb{R})}\leq&\left\|e^{-\frac{\alpha}{2}|\cdot|}\int_{-\infty}^{\infty}\Gamma_{\theta_0}(\cdot,s)e^{-\frac{\alpha}{2}|s|}f(s)ds\right\|_{Q_{r_k}L^2(\mathbb{R})}\\&+\left\|e^{-\frac{\alpha}{2}|\cdot|}\int_{-\infty}^{\infty}\Gamma_{\theta_0}(\cdot,s)e^{-\frac{\alpha}{2}|s|}(Q_{r_k}f)(s)ds\right\|_{P_{r_k}L^2(\mathbb{R})}.\end{aligned}$$

Next,

$$\left|\int_{-\infty}^{\infty}\Gamma_{\theta_0}(t,s)e^{-\frac{\alpha}{2}|s|}f(s)ds\right|\leq\frac{C_1}{\sin\theta_0}\|f\|_{L^2(\mathbb{R})},$$

and therefore,

$$\left\|e^{-\frac{\alpha}{2}|\cdot|}\int_{-\infty}^{\infty}\Gamma_{\theta_0}(\cdot,s)e^{-\frac{\alpha}{2}|s|}f(s)ds\right\|_{Q_{r_k}L^2(\mathbb{R})}\leq\frac{C_2}{\sin\theta_0}e^{-\frac{\alpha}{2}r_k}\|f\|_{L^2(\mathbb{R})}\longrightarrow 0,$$

as $k\to\infty$. Similarly,

$$\left|\int_{-\infty}^{\infty}\Gamma_{\theta_0}(t,s)e^{-\frac{\alpha}{2}|s|}(Q_{r_k}f)(s)ds\right|\leq\frac{C_3}{\sin\theta_0}e^{-\frac{\alpha}{2}r_k}\|f\|_{L^2(\mathbb{R})}$$

$$\left\|e^{-\frac{\alpha}{2}|\cdot|}\int_{-\infty}^{\infty}\Gamma_{\theta_0}(\cdot,s)e^{-\frac{\alpha}{2}|s|}(Q_{r_k}f)(s)ds\right\|_{P_{r_k}L^2(\mathbb{R})}\leq\frac{C_4}{\sin\theta_0}e^{-\frac{\alpha}{2}r_k}\|f\|_{L^2(\mathbb{R})}\longrightarrow 0,$$

as $k\to\infty$, which completes the proof of the theorem. $\square$

Remark 2.5.2. One can reformulate Theorem 2.5.1 in the following way. For $\kappa\geq 0$, define the neighborhood set Ξ_κ of the exceptional set Ξ as

$$\Xi_\kappa=\bigcup_{k=0}^{\infty}\left\{\left[\frac{1}{a}\left(\varphi+\frac{\pi k}{2}\right)-e^{-k\kappa},\frac{1}{a}\left(\varphi+\frac{\pi k}{2}\right)+e^{-k\kappa}\right]\right\}.\tag{2.5.3}$$

If $0\leq\kappa<\frac{\alpha}{2}$, then

$$\lim_{r\to\infty,r\notin\Xi_\kappa}\left\|(\mathcal{B}_{\theta_r}-z)^{-1}-(\mathcal{A}_r-z)^{-1}\right\|=0,\quad\forall z\in\mathbb{C}:\operatorname{Im}(z)\neq 0,\tag{2.5.4}$$

where $\theta_r = 2\psi_r \pmod{2\pi}$.

Equation (2.5.4) describes the asymptotic behavior of the system $r \mapsto \mathcal{A}_r$. It shows that the one-dimensional torus $\{\mathcal{B}_\theta\}_{\theta\in[0,2\pi)}$ attracts the family $\mathcal{A}_r$ as r approaches infinity away from the (exponentially diminishing) neighborhood of the critical set.

2.6 Appendix

In this section we will discuss the invertibility properties of convolution operators on a finite interval, that we used in the proof of Theorem 2.3.3. We refer to [22] for details.

Let K be a convolution operator on $L^2((0,r))$,

$$(K\varphi)(t) = \int_0^r k(t-s)\varphi(s)ds, \quad 0 \le t \le r, \tag{2.6.1}$$

with an integrable kernel function k. We assume that k can be written in the form

$$k(t) = \begin{cases} iCe^{-itA}(I-P)B, & 0 \le t \le r, \\ -iCe^{-itA}PB, & -r \le t < 0, \end{cases} \tag{2.6.2}$$

where A is a matrix of size $n \times n$ with no real eigenvalues, B and C are matrices of sizes $n \times 1$ and $1 \times n$, respectively, and P is the Riesz projection of A corresponding to the eigenvalues in the upper half plane. Then the following theorem holds.

Theorem 2.6.1. *(Theorem XIII.10.1, [22]) Let K be the integral operator on $L^2((0,r))$, defined by* (2.6.1)*, and assume that k admits the representation* (2.6.2)*. Let n be the order of A, and put $A^\times = A - BC$. Then $I - K$ is invertible if and only if the map*

$$S_r = Pe^{irA}e^{-irA^\times}P : \mathrm{Ran}(P) \to \mathrm{Ran}(P)$$

is invertible. In that case

$$((I-K)^{-1}f)(t) = f(t) + \int_0^r \gamma(t,s)f(s)ds, \quad 0 \le t \le r,$$

with

$$\gamma(t,s) = \begin{cases} iCe^{-itA^\times}\Pi_r e^{isA^\times}B, & 0 \le s \le t \le r, \\ -iCe^{-itA^\times}(I-\Pi_r)e^{isA^\times}B, & 0 \le t < s \le r. \end{cases}$$

Here Π_r *is the projection of* $\mathbb{C}^n$ *along* $\mathrm{Ran}(P)$ *defined by*

$$\Pi_r x = x - S_r^{-1}Pe^{irA}e^{-irA^\times}x, \quad x \in \mathbb{C}^n.$$

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