BOUNDARY VALUE PROBLEMS FOR THE STOKES SYSTEM IN ARBITRARY LIPSCHITZ DOMAINS

A Dissertation presented to

the Faculty of the Graduate School

at the University of Missouri

In Partial Fulfillment of

the Requirements for the Degree of

Doctor of Philosophy

---

by

MATTHEW WRIGHT

Dr. Marius Mitrea, Dissertation Supervisor

May 2008

The undersigned, appointed by the dean of the Graduate School, have examined the dissertation entitled

BOUNDARY VALUE PROBLEMS FOR THE STOKES SYSTEM
IN ARBITRARY LIPSCHITZ DOMAINS

presented by Matthew Wright,

a candidate for the degree of Doctor of Philosophy,

and hereby certify that, in their opinion, it is worthy of acceptance.

Professor Marius Mitrea

Professor David Retzloff

Professor Steve Hofmann

Professor Yuri Latushkin

Professor Dorina Mitrea

ACKNOWLEDGEMENTS

First of all, I would like to thank my graduate advisor, Marius Mitrea, for his continued support and encouragement throughout the process of completing this dissertation. Without his help, none of this research would have been possible. I thank him for sharing his passion for mathematics with me and for fostering my interest in harmonic analysis and its applications to partial differential equations. He is an excellent teacher inside and outside of the classroom, and I especially appreciate his willingness and ability to make complex concepts seem simple by taking the time to clearly explain every detail.

I would also like to thank the other members of my graduate committee, David Retzloff, Steve Hofmann, Yuri Latushkin, and Dorina Mitrea, who have also been very supportive throughout the completion of this work. I am greatful to all the faculty and staff of the mathematics department at the University of Missouri who have contributed to my development as a mathematician, as a teacher, and as a person over the past years.

Lastly, I would like to thank my lovely wife Kelly for her daily encouragement and love. She is my inspiration in all that I do, and without her support, none of my accomplishments would be possible.

Contents

BOUNDARY VALUE PROBLEMS FOR THE STOKES SYSTEM

IN ARBITRARY LIPSCHITZ DOMAINS

Matthew Wright

Dr. Marius Mitrea, Dissertation Supervisor

Abstract

The goal of this work is to treat the main boundary value problems for the Stokes system, i.e.,

(i) the Dirichlet problem with L^p-data and nontangential maximal function estimates,

(ii) the Neumann problem with L^p-data and nontangential maximal function estimates,

(iii) the Regularity problem with L_1^p-data and nontangential maximal function estimates,

(iv) the transmission problem with L^p-data and nontangential maximal function estimates,

(v) the Poisson problem with Dirichlet condition in Besov-Triebel-Lizorkin spaces,

(vi) the Poisson problem with Neumann condition in Besov-Triebel-Lizorkin spaces,

in Lipschitz domains of arbitrary topology in $\mathbb{R}^n$, for each $n \geq 2$. Our approach relies on boundary integral methods and yields constructive solutions to the aforementioned problems.

1 Introduction

1.1 Description of main well-posedness results

Informally speaking, the goal of the present work is to prove optimal well-posedness results for (homogeneous and inhomogeneous) boundary-value problems for the Stokes system in Lipschitz domains with arbitrary topology, in all space dimensions and for all major types of boundary conditions (Dirichlet, Neumann, transmission). The boundary data is selected from Lebesgue, Sobolev, Hardy, Besov and Triebel-Lizorkin spaces and the smoothness of the solutions is measured accordingly.

At the core of our analysis is the transmission problem for the Stokes system, on which we wish to elaborate first. Let Ω be a Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, and define $\Omega_+ := \Omega$ and $\Omega_- = \mathbb{R}^n \setminus \bar{\Omega}$. The transmission boundary value problems for the Stokes system studied here is of the type

$$(T_\mu) \quad \begin{cases} \Delta \vec{u}_\pm = \nabla \pi_\pm \ \text{ in } \ \Omega_\pm, \\ \operatorname{div} \vec{u}_\pm = 0 \ \text{ in } \ \Omega_\pm, \\ \vec{u}_+|_{\partial\Omega} - \vec{u}_-|_{\partial\Omega} = \vec{g} \in L^p_1(\partial\Omega), \\ \partial^\lambda_\nu(\vec{u}_+, \pi_+) - \mu\, \partial^\lambda_\nu(\vec{u}_-, \pi_-) = \vec{f} \in L^p(\partial\Omega), \\ M(\nabla\vec{u}_\pm), M(\pi_\pm) \in L^p(\partial\Omega). \end{cases} \tag{1.1}$$

Here, Δ is the Laplacian, $\mu \in [0, 1)$ is a fixed parameter, $\nu := \nu_+$ is the outward unit normal to Ω_+. For $1 < p < \infty$, $L^p_1(\partial\Omega)$ is the classical L^p-based Sobolev spaces of order one on $\partial\Omega$, M denotes the non-tangential maximal operator (cf. (2.5)), and

$$\partial^\lambda_\nu(\vec{u}_\pm, \pi_\pm) := (\nabla\vec{u}_\pm{}^\top + \lambda\nabla\vec{u}_\pm)\vec{\nu} - \pi_\pm\vec{\nu} \tag{1.2}$$

is a family of co-normal derivatives, indexed by a parameter $\lambda \in \mathbb{R}$ (more detailed definitions are given in subsequent chapters). In this way, we can simultaneously treat various types of Neumann boundary conditions. For example, when $\lambda = 0$, (1.2) corresponds to the co-normal derivative treated in [34], whereas when $\lambda = 1$, (1.2) corresponds to the "slip condition" considered in [23].

Two closely related boundary value problems are the Neumann problem and the Dirichlet problem with (maximally) regular data:

$$(N)\ \begin{cases} \Delta\vec{u} = \nabla\pi \ \text{ in } \ \Omega, \\ \operatorname{div}\vec{u} = 0 \ \text{ in } \ \Omega, \\ \partial_\nu^\lambda(\vec{u},\pi) = \vec{f} \in L^p(\partial\Omega), \\ M(\nabla\vec{u}),\, M(\pi) \in L^p(\partial\Omega) \end{cases} \qquad (R)\ \begin{cases} \Delta\vec{u} = \nabla\pi \ \text{ in } \ \Omega, \\ \operatorname{div}\vec{u} = 0 \text{ in } \Omega, \\ \vec{u}|_{\partial\Omega} = \vec{g} \in L_1^p(\partial\Omega), \\ M(\nabla\vec{u}),\, M(\pi) \in L^p(\partial\Omega). \end{cases} \tag{1.3}$$

From this point forth, we will refer to (R) as the Regularity problem. Fabes, Kenig, and Verchota proved in [34] that (N) and (R) are well-posed if $2-\varepsilon < p < 2+\varepsilon$, where $\varepsilon = \varepsilon(\partial\Omega) > 0$. Building on the work in [21], [75], Z. Shen has established in [82] a weak maximum principle for the Dirichlet problem for the Stokes system in Lipschitz domains in $\mathbb{R}^3$. Interpolating this L^∞ bound with the L^p-estimates from [34], with p near 2, shows that the Dirichlet problem for the Stokes system in three-dimensional Lipschitz domains with data in L^p, is solvable whenever $2-\varepsilon < p < \infty$. However, as pointed out by P. Deuring on p. 16 of [29], "*this leaves open the question of whether these solutions may be constructed by means of the boundary layer method, and how to deal with exterior problems and slip boundary conditions.*"

With these aims in mind, let us briefly discuss the relevance of the transmission problem itself. From a physical point of view, the transmission problem

$$(T)\ \begin{cases} \mu_\pm\Delta\vec{u}_\pm = \nabla\pi_\pm \ \text{ in } \ \Omega_\pm, \\ \operatorname{div}\vec{u}_\pm = 0 \ \text{ in } \ \Omega_\pm, \\ \vec{u}_+|_{\partial\Omega} - \vec{u}_-|_{\partial\Omega} = \vec{g}, \\ \sigma^\lambda\vec{u}_+ - \sigma^\lambda\vec{u}_- = \vec{f}, \end{cases} \tag{1.4}$$

where

$$\sigma^\lambda\vec{u}_\pm := \mu_\pm(\nabla\vec{u}_\pm{}^\top + \lambda\nabla\vec{u}_\pm)\vec{\nu} - \pi_\pm\vec{\nu}, \tag{1.5}$$

describes the flow of a viscous incompressible fluid within and around a stationary particle occupying the domain Ω_+ which is further embedded into a second porous medium Ω_-. In this context, $\vec{u}_+$ and π_+ are the volume-averaged fluid velocity and pressure fields of the inner flow, whereas $\vec{u}_-$ and π_- have analogous roles for the outer flow. In the specific case when $\lambda = 1$, this is a standard problem that arises when studying the low Reynolds number deformation of a viscous drop immersed in another fluid (see [78]; [76], Sec. 7.2). Here, μ_+ denotes the viscosity of the drop, while μ_- denotes the viscosity of the surrounding fluid. The case when $\vec{g} = 0$ is often of particular interest, since this introduces the physically relevant restriction that the velocities $\vec{u}_+$ and $\vec{u}_-$ must match on the boundary. The reader is referred to M. Kohr and I. Pop's monograph [55] for a more detailed discussion in this regard and for ample references to the engineering literature dealing with transmission problems for the Stokes system.

If we re-denote the term $\mu_\pm \vec{u}_\pm$ in (1.4) as simply $\vec{u}_\pm$ and let $\mu := \mu_-/\mu_+$ denote the ratio of the viscosities of the two fluids, we can rewrite the transmission problem in the form

$$
(T^1_\mu) \quad \begin{cases} \Delta \vec{u}_\pm = \nabla \pi_\pm \ \text{ in } \ \Omega_\pm, \\ \operatorname{div} \vec{u}_\pm = 0 \ \text{ in } \ \Omega_\pm, \\ \mu \vec{u}_+|_{\partial\Omega} - \vec{u}_-|_{\partial\Omega} = \vec{g}, \\ \partial^\lambda_\nu(\vec{u}_+, \pi_+) - \partial^\lambda_\nu(\vec{u}_-, \pi_-) = \vec{f}. \end{cases} \tag{1.6}
$$

Above, we have also re-denoted the term $\mu_- \vec{g}$ as simply $\vec{g}$, but since we will be interested in considering these problems for general values of $\vec{f}$ and $\vec{g}$, this is of little consequence. Going one step further, if we replace $\pi_\pm$ with $\mu_\pm \pi_\pm$ and $\vec{f}$ with $\mu_+ \vec{f}$ in (1.4), we can write a third form of the transmission problem,

$$
(T_\mu^2) \quad \begin{cases} \Delta \vec{u}_\pm = \nabla \pi_\pm \quad \text{in} \quad \Omega_\pm, \\ \operatorname{div} \vec{u}_\pm = 0 \quad \text{in} \quad \Omega_\pm, \\ \vec{u}_+|_{\partial\Omega} - \vec{u}_-|_{\partial\Omega} = \vec{g}, \\ \partial_\nu^\lambda(\vec{u}_+, \pi_+) - \mu \partial_\nu^\lambda(\vec{u}_-, \pi_-) = \vec{f}. \end{cases} \tag{1.7}
$$

Since the viscosities μ_+ and μ_- are positive numbers, these changes have no effect on the solvability of these problems and so, throughout our work, we will consider the form of the transmission problem that is most convenient for the particular goals we have in mind. One advantage of these last two descriptions comes from analyzing the limiting cases. For example, if we consider the case when $\mu_- << \mu_+$, studying (T_μ^1) for $\mu = 0$ yields information about the Regularity problem (R) in Ω_-, and studying (T_μ^2) for $\mu = 0$ yields information about the Neumann problem (N) in Ω_+. Similarly, if $\mu_+ << \mu_-$, analyzing (T_μ^1) and (T_μ^2) will lead to results for the Regularity problem (R) in Ω_+ and for the Neumann problem (N) in Ω_-. Our main results are as follows (the reader is referred to the subsequent chapters for the relevant notation employed below):

Theorem 1.1 *Assume that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded Lipschitz domain and set $\Omega_+ := \Omega$, $\Omega_- := \mathbb{R}^n \setminus \bar{\Omega}$. Also, fix $\mu \in (0,1)$ and $\lambda \in (-1,1]$. Then there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ such that for each*

$$
\tfrac{2(n-1)}{n+1} - \varepsilon < p < 2 + \varepsilon \tag{1.8}
$$

the transmission boundary value problem, concerned with finding two pairs of functions $(\vec{u}_\pm, \pi_\pm)$ in $\Omega_\pm$ satisfying

$$
\begin{cases} \Delta \vec{u}_\pm = \nabla \pi_\pm, \ \operatorname{div} \vec{u}_\pm = 0 \quad \text{in} \quad \Omega_\pm, \\ M(\nabla \vec{u}_\pm), \ M(\pi_\pm) \in L^p(\partial\Omega), \\ \vec{u}_+\Big|_{\partial\Omega} - \vec{u}_-\Big|_{\partial\Omega} = \vec{g} \in h_1^p(\partial\Omega), \\ \partial_\nu^\lambda(\vec{u}_+, \pi_+) - \mu\, \partial_\nu^\lambda(\vec{u}_-, \pi_-) = \vec{f} \in h^p(\partial\Omega), \end{cases} \tag{1.9}
$$

and the decay conditions

$$\vec{u}_-(x) = \begin{cases} O(|x|^{2-n}) \quad \textit{as} \quad |x| \to \infty, \quad \textit{if} \quad n \geq 3, \\ -\frac{1}{\mu} E(x) \Big(\int_{\partial\Omega} \vec{f}\, d\sigma \Big) + O(|x|^{-1}) \quad \textit{as} \quad |x| \to \infty, \quad \textit{if} \ n = 2, \end{cases} \tag{1.10}$$

$$\partial_j \vec{u}_-(x) = -\tfrac{1}{\mu} (\partial_j E)(x) \Big(\int_{\partial\Omega} \vec{f}\, d\sigma \Big) + O(|x|^{-n}) \quad \textit{as} \quad |x| \to \infty, \ 1 \leq j \leq n, \tag{1.11}$$

$$\pi_-(x) = \begin{cases} O(|x|^{1-n}) \quad \textit{as} \quad |x| \to \infty, \quad \textit{if} \quad n \geq 3, \\ \frac{1}{\mu} \Big\langle (\nabla E_\Delta)(x)\, , \int_{\partial\Omega} \vec{f}\, d\sigma \Big\rangle + O(|x|^{-2}) \quad \textit{as} \quad |x| \to \infty, \quad \textit{if} \ n = 2, \end{cases} \tag{1.12}$$

has a unique solution. In addition, there exists $C > 0$ such that

$$\|M(\nabla \vec{u}_\pm)\|_{L^p(\partial\Omega)} + \|M(\pi_\pm)\|_{L^p(\partial\Omega)} \leq C \|\vec{g}\|_{h^p_1(\partial\Omega)} + C \|\vec{f}\|_{h^p(\partial\Omega)}. \tag{1.13}$$

In the previous theorem as well as in the following results, the Hardy space $h^p(\partial\Omega)$, and its regular version $h^p_1(\partial\Omega)$, are as defined in (2.97).

Theorem 1.2 *Assume that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded Lipschitz domain. Then there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ such that for each*

$$2 - \varepsilon < p < \infty \quad \textit{if} \ \ n = 2, 3 \tag{1.14}$$

$$2 - \varepsilon < p < \tfrac{2(n-1)}{n-3} + \varepsilon \quad \textit{if} \ \ n \geq 4, \tag{1.15}$$

the interior Dirichlet boundary value problem

$$\begin{cases} \Delta \vec{u} = \nabla \pi, \ \ \operatorname{div} \vec{u} = 0 \quad \textit{in} \ \ \Omega, \\ M(\vec{u}) \in L^p(\partial\Omega), \\ \vec{u}\Big|_{\partial\Omega} = \vec{f} \in L^p_{\nu_+}(\partial\Omega), \end{cases} \tag{1.16}$$

has a solution, which is unique modulo adding functions which are locally constant in Ω to the pressure term. In addition, there exists a finite constant $C > 0$ such that

$$\|M(\vec{u})\|_{L^p(\partial\Omega)} \leq C\|\vec{f}\|_{L^p(\partial\Omega)}. \tag{1.17}$$

Similar results are valid for the exterior Dirichlet problem, formulated much as (1.16) with the additional decay conditions

$$\vec{u}(x) = \begin{cases} O(|x|^{2-n}) \ \textit{as}\ |x| \to \infty, & \textit{if} \ \ n \geq 3, \\ E(x)\vec{A} + O(1) \ \textit{as}\ |x| \to \infty, & \textit{if}\ n = 2, \end{cases} \tag{1.18}$$

$$\partial_j\vec{u}(x) = \begin{cases} O(|x|^{1-n}) \ \textit{as}\ |x| \to \infty, & \textit{if} \ \ n \geq 3, \\ \partial_j E(x)\vec{A} + O(|x|^{-2}) \ \textit{as}\ |x| \to \infty, & \textit{if}\ n = 2, \end{cases} \tag{1.19}$$

$$\pi(x) = \begin{cases} O(|x|^{1-n}) \ \textit{as}\ |x| \to \infty, & \textit{if} \ \ n \geq 3, \\ \langle \nabla E_\Delta(x), \vec{A}\rangle + O(|x|^{-2}) \ \textit{as}\ |x| \to \infty, & \textit{if}\ n = 2, \end{cases} \tag{1.20}$$

for some a priori given constant $\vec{A} \in \mathbb{R}^2$. Also, the standard nontangential maximal operator in (1.17) should be replaced by its truncated version.

Here we wish to mention that, while this work was in its final stages of preparation, we have learned that the case of the interior Dirichlet problem in which the Lipschitz domain $\Omega \subset \mathbb{R}^n$ has a connected boundary and $n \geq 4$ has also been treated by J. Kilty in [54], using a different approach. The limiting case $p = \infty$ has been dealt with by Z. Shen in [82], for Lipschitz domains in $\mathbb{R}^3$. In [82], Shen also establishes the well-posedness of the Dirichlet problem in three-dimensional Lipschitz domains with connected boundary for data in the Hölder space $C^\alpha(\partial\Omega)$, with $0 < \alpha < \alpha_o$. Here we give another proof of this result, via integral operators. In addition, we also treat the case of the Dirichlet problem for the Stokes system in the case in which the data is from BMO and the solution satisfies Carleson measure estimates. See Theorem 9.16 and Theorem 9.17 for details.

Our next result concerns the so-called Regularity problem, and is a version of the Dirichlet problem (1.16) corresponding to the case when the boundary data is maximally regular (i.e., belonging to boundary Hardy and Sobolev spaces of order one).

Theorem 1.3 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain. Then there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ such that for each p as in (1.8), the interior Regularity boundary value problem*

$$\begin{cases} \Delta\vec{u} = \nabla\pi, \ \ \mathrm{div}\,\vec{u} = 0 \quad in \ \ \Omega, \\ M(\nabla\vec{u}), \, M(\pi) \in L^p(\partial\Omega), \\ \vec{u}\Big|_{\partial\Omega} = \vec{f} \in h^p_{1,\nu_+}(\partial\Omega), \end{cases} \tag{1.21}$$

has a solution, which is unique modulo adding functions which are locally constant in Ω to the pressure.

In addition, there exists a finite constant $C > 0$ such that

$$\|M(\nabla\vec{u})\|_{L^p(\partial\Omega)} + \|M(\pi)\|_{L^p(\partial\Omega)} \leq C\|\vec{f}\|_{h^p_1(\partial\Omega)}. \tag{1.22}$$

Similar results are valid for the exterior Regularity problem, formulated much as (1.21) with the additional decay conditions (1.18)-(1.20).

Theorem 1.4 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain and fix $\lambda \in (-1, 1]$. Then there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ such that for each p as in (1.8) the interior Neumann boundary value problem*

$$\begin{cases} \Delta\vec{u} = \nabla\pi, \ \ \mathrm{div}\,\vec{u} = 0 \quad in \ \ \Omega, \\ M(\nabla\vec{u}), \, M(\pi) \in L^p(\partial\Omega), \\ \partial^\lambda_\nu(\vec{u}, \pi) = \vec{f} \in h^p(\partial\Omega), \end{cases} \tag{1.23}$$

has a solution if and only if

$$\vec{f} \in \mathrm{Im}\left(-\tfrac{1}{2}I + K^*_\lambda : h^p_{\Psi^\lambda_+}(\partial\Omega) \to h^p_{\Psi^\lambda_+}(\partial\Omega)\right). \tag{1.24}$$

Moreover, this solution is unique modulo adding to the velocity field functions from $\Psi^\lambda(\Omega)$. In addition, there exists a finite constant $C > 0$ such that

$$\|M(\nabla\vec{u})\|_{L^p(\partial\Omega)} + \|M(\pi)\|_{L^p(\partial\Omega)} \leq C\|\vec{f}\|_{h^p(\partial\Omega)}. \tag{1.25}$$

Finally, a similar result holds for the exterior domain $\mathbb{R}^n \setminus \bar{\Omega}$ after including the decay conditions

$$\vec{u}(x) = \begin{cases} O(|x|^{2-n}) \quad as \quad |x| \to \infty, \quad if \quad n \geq 3, \\ E(x)\Big(\int_{\partial\Omega} \vec{f}\, d\sigma\Big) + O(|x|^{-1}) \quad as \quad |x| \to \infty, \quad if \ n = 2, \end{cases} \tag{1.26}$$

$$\partial_j\vec{u}(x) = (\partial_j E)(x)\Big(\int_{\partial\Omega} \vec{f}\, d\sigma\Big) + O(|x|^{-n}) \quad as \quad |x| \to \infty, \ 1 \leq j \leq n, \tag{1.27}$$

$$\pi(x) = \begin{cases} O(|x|^{1-n}) \quad as \quad |x| \to \infty, \quad if \quad n \geq 3, \\ \Big\langle(-\nabla E_\Delta)(x)\,,\, \int_{\partial\Omega} \vec{f}\, d\sigma\Big\rangle + O(|x|^{-2}) \quad as \quad |x| \to \infty, \quad if \ n = 2. \end{cases} \tag{1.28}$$

More precisely, a solution to the exterior problem satisfying the above decay conditions exists if and only if

$$\vec{f} \in \mathrm{Im}\Big(\tfrac{1}{2}I + K_\lambda^* : h^p_{\Psi_-^\lambda}(\partial\Omega) \to h^p_{\Psi_-^\lambda}(\partial\Omega)\Big), \tag{1.29}$$

and solutions are unique modulo adding to the velocity field functions from $\Psi^\lambda(\mathbb{R}^n \setminus \bar{\Omega})$.

Our approach is based on boundary integral methods, and for each of the problems listed in Theorems 1.1-1.4, we are able to represent the solution in terms of hydrostatic layer potentials. In this strategy, one is led to study the invertibility properties of certain principal-value singular integral operators on Lipschitz surfaces. These operators are of Calderón-Zygmund type, so their boundedness on Lebesgue and Hardy type spaces follows from known results. The key ingredient in proving the invertibility of these operators is obtaining bounds from below. We accomplish this by devising some new Rellich type identities for the Stokes system.

The most physically relevant Neumann-type boundary condition is the so-called "slip condition", corresponding to (1.2) with $\lambda = 1$. Interestingly, it is precisely this boundary condition which is most challenging from the point of view of our analytical treatment.

This is because the usefulness of the Rellich type identities alluded to above is substantially diminished when $\lambda = 1$, due to the fact that the quadratic energy form associated with (1.2) when $\lambda = 1$ is only semi-positive definite (as opposed to being strictly positive definite when $|\lambda| < 1$). This difficulty was first encountered by Dahlberg, Fabes, Kenig and Verchota in their work on the L^2 Dirichlet and Neumann problems for the Stokes and Lamé systems in [23], [34]. As a remedy, these authors have developed some auxiliary estimates, which they termed boundary Korn inequalities, which were specifically designed to compensate for the lack of coerciveness of the Rellich estimates.

In the case of the transmission boundary value problem for the Stokes system considered here, these Korn inequalities fail to be as useful as they have been in the aforementioned works. This has to do with the very nature of the transmission problem, in which two (pairs) of solutions $(\vec{u}_+, \pi_+)$ and $(\vec{u}_-, \pi_-)$, which interact across the Lipschitz interface, are considered simultaneously. In this scenario, deriving Korn inequalities for each of them separately is of little value since, in turn, these inequalities cannot be further combined algebraically in order to relate them to the transmission boundary data, i.e.,

$$\vec{u}_+|_{\partial\Omega} - \vec{u}_-|_{\partial\Omega} \quad \text{and} \quad \partial_\nu^\lambda(\vec{u}_+, \pi_+) - \mu\partial_\nu^\lambda(\vec{u}_-, \pi_-). \tag{1.30}$$

The technical innovation we develop in order to address this significant issue is to produce some more elaborate Rellich type identities which, by design, have Korn-like identities built directly into them. The upshot of this is that working with identities in place of estimates is amenable to algebraic manipulations which can then fully take advantage of the transmission-like interaction between $(\vec{u}_+, \pi_+)$ and $(\vec{u}_-, \pi_-)$.

All the above considerations are relevant in the treatment of boundary value problems with L^2 data. As already suggested above, the central role in our treatment is played by the transmission problem. Subsequently, we explain how the Dirichlet/Regularity and Neumann problems can be viewed as limiting cases of this. To obtain well-posedness results for L^p-data with $p \neq 2$, following the seminal work of Dahlberg-Kenig [20], [21], we rely on atomic estimates in dimensions $n = 2, 3$, and on a recent remarkable advance of Z. Shen [83] in dimensions $n \geq 4$. Shen's original scheme is to start with the L^2 theory, then prove L^p results for $p > 2$ (the critical p corresponding to the Sobolev exponent in the embedding $L^2_1(\partial\Omega) \hookrightarrow L^p(\partial\Omega)$) using certain reverse Hölder estimates, and finally interpolate. This cannot be directly applied in our setting since the natural range of p's for which the L^p-transmission problem is solvable is a subset of $(1, 2]$. We overcome this difficulty by introducing and solving a suitable dual transmission problem.

As is well-known, in the case of the Dirichlet boundary problem for the Stokes system, i.e. for

$$\Delta\vec{u} = \nabla\pi, \;\; \operatorname{div}\vec{u} = 0 \;\text{ in }\; \Omega, \quad \vec{u}\Big|_{\partial\Omega} = \vec{f}, \tag{1.31}$$

the boundary datum $\vec{f}$ satisfies the necessary compatibility condition

$$\int_{\partial\Omega} \langle \nu, \vec{f}\rangle \, d\sigma = 0 \tag{1.32}$$

whenever $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain. This creates the following technical difficulty when addressing the issue of well-posedness of (1.31) for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ when the boundary datum $\vec{f}$ belongs to the (regular) Hardy space $h^{1,p}_{at}(\partial\Omega)$, $\frac{n-1}{n} < p \leq 1$. The latter is the ℓ^p-span of certain building blocks (satisfying suitable support, size and smoothness conditions), called regular atoms. Hence, it is natural to seek a solution for (1.31) when $\vec{f} = \sum_j \lambda_j a_j$ with $(\lambda_j)_j \in \ell^p$ and the a_j's regular atoms, as $\vec{u} = \sum_j \lambda_j \vec{u}_j$ where $\vec{u}_j$ solves (1.31) for the boundary datum a_j. However, even though the original datum $\vec{f}$ satisfies the necessary compatibility condition (1.32), there is no guarantee that each individual atom a_j does. We overcome this issue by first addressing the solvability of (1.31) in the case when $\Omega \subset \mathbb{R}^n$ is the *unbounded* domain lying above the graph of a (real-valued) Lipschitz function. In this setting, condition (1.32) no longer plays a role. We then develop appropriate localization techniques (carried out at the level of singular integral operators) in order to eventually handle the case of *bounded* Lipschitz domains. This idea influences our overall strategy in dealing with all types of boundary conditions for the Stokes system treated in our work.

Having developed a satisfactory theory for the Stokes system with L^p (and atomic) data and nontangential maximal function estimates, we next consider the inhomogeneous Stokes problem on Besov-Triebel-Lizorkin spaces in Lipschitz domains. The key idea is to view the former results as limiting/critical cases of the latter, and use interpolation. There are, nonetheless, significant difficulties in carrying out this program, a fact frequently noted in the literature. For example, discussing the status of the Poisson problem for the Stokes system in Lipschitz domains, P. Deuring writes on p. 3 of [30]: "*We see that for solutions of the Poisson problem [for the Dirichlet Laplacian] on Lipschitz domains, a rather complete L^p-theory is available, whereas for the Stokes system, only a L^2-theory could be developed. This, admittedly, was difficult enough, but this still raises the question what to expect if* $p \neq 2$."

A related open problem, posed on p. 195 of [28], asks whether for an arbitrary bounded Lipschitz domain Ω there holds

$$\left.\begin{array}{l} \Delta\vec{u} - \nabla\pi = \vec{f} \in L^2(\Omega) \\[2ex] \operatorname{div}\vec{u} = 0 \quad \text{in} \quad \Omega \\[2ex] \vec{u} \in W_0^{1,2}(\Omega),\ \ \pi \in L^2(\Omega) \end{array}\right\} \Longrightarrow \vec{u} \in W^{3/2,2}(\Omega). \tag{1.33}$$

A similar issue is raised in the case of Neumann boundary conditions. In the same setting, Deuring also asks if

$$\left.\begin{array}{l} \Delta\vec{u} = \nabla\pi \quad \text{in} \quad \Omega \\[2ex] \operatorname{div}\vec{u} = 0 \quad \text{in} \quad \Omega \\[2ex] M(\vec{u}) \in L^2(\partial\Omega) \end{array}\right\} \Longrightarrow \vec{u} \in W^{1/2,2}(\Omega). \tag{1.34}$$

Here we provide answers to the above questions and extend previous work in the literature by proving Theorem 1.5 and Theorem 1.6 below. In order to facilitate stating them, we introduce some notation. Let $B^{p,q}_\alpha(\mathbb{R}^n)$ and $F^{p,q}_\alpha(\mathbb{R}^n)$ denote the standard Besov and Triebel-Lizorkin scales of spaces in $\mathbb{R}^n$ (cf. § 11.1 for more details). Given $\Omega \subset \mathbb{R}^n$ Lipschitz and $0 < p, q \leq \infty$, $\alpha \in \mathbb{R}$, we set

$$\begin{aligned} B^{p,q}_\alpha(\Omega) &:= \{u \in \mathcal{D}'(\Omega) :\ \exists\, v \in B^{p,q}_\alpha(\mathbb{R}^n) \text{ with } v|_\Omega = u\}, \\ B^{p,q}_{\alpha,0}(\Omega) &:= \{u \in B^{p,q}_\alpha(\mathbb{R}^n) :\ \operatorname{supp} u \subseteq \overline{\Omega}\}, \end{aligned} \tag{1.35}$$

with similar definitions for $F_\alpha^{p,q}(\Omega)$ and $F_{\alpha,0}^{p,q}(\Omega)$. Also, $B_s^{p,q}(\partial\Omega)$ stands for the Besov class on the Lipschitz manifold $\partial\Omega$, obtained by transporting (via a partition of unity and pull-back) the standard scale $B_s^{p,q}(\mathbb{R}^{n-1})$. (In general, we make no notational distinction between these smoothness spaces of scalar-valued functions and their natural counterparts for vector-valued functions.) Finally, for $\varepsilon > 0$ and $n \geq 2$ let us introduce a two dimensional region $\mathcal{R}_{n,\varepsilon}$ in the $(s, 1/p)$-plane, which depends on the dimension as follows:

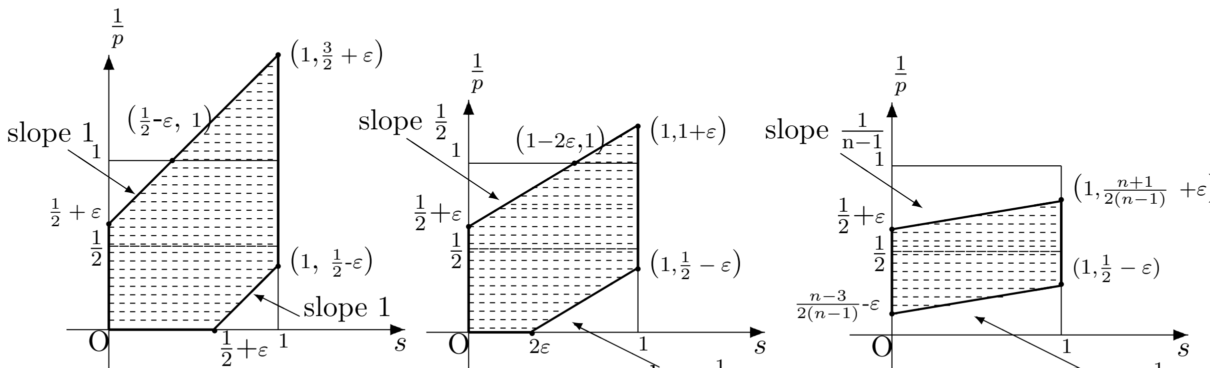

Figure 1: $\mathcal{R}_{n,\varepsilon}$ for $n = 2$

Figure 2: $\mathcal{R}_{n,\varepsilon}$ for $n = 3$

Figure 3: $\mathcal{R}_{n,\varepsilon}$ for $n \geq 4$

The theorem below deals with the case of Dirichlet boundary conditions.

Theorem 1.5 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, and assume that $\frac{n-1}{n} < p \leq \infty$, $0 < q \leq \infty$, $(n-1)(\frac{1}{p}-1)_+ < s < 1$. Consider the following boundary value problem*

$$\begin{aligned} &\Delta\vec{u} - \nabla\pi = \vec{f} \in B^{p,q}_{s+\frac{1}{p}-2}(\Omega), \quad \operatorname{div}\vec{u} = g \in B^{p,q}_{s+\frac{1}{p}-1}(\Omega), \\ &\vec{u} \in B^{p,q}_{s+\frac{1}{p}}(\Omega), \quad \pi \in B^{p,q}_{s+\frac{1}{p}-1}(\Omega), \quad \operatorname{Tr}\vec{u} = \vec{h} \in B_s^{p,q}(\partial\Omega), \end{aligned} \tag{1.36}$$

subject to the (necessary) compatibility condition

$$\int_{\partial\mathcal{O}} \langle \nu, \vec{h}\rangle \, d\sigma = \int_{\mathcal{O}} g(x)\, dx, \tag{1.37}$$

for every component $\mathcal{O}$ of Ω.

Then there exists $\varepsilon = \varepsilon(\Omega) \in (0,1]$ such that (1.36) is well-posed (with uniqueness modulo locally constant functions in Ω for the pressure), if the pair (s,p) belongs to the region $\mathcal{R}_{n,\varepsilon}$, depicted above.

Furthermore, the solution has an integral representation formula in terms of hydrostatic layer potential operators and satisfies natural estimates. Concretely, there exists a finite, positive constant $C = C(\Omega, p, q, s, n)$ such that

$$\|\vec{u}\|_{B^{p,q}_{s+\frac{1}{p}}(\Omega)} + \|\pi\|_{B^{p,q}_{s+\frac{1}{p}-1}(\Omega)/\mathbb{R}_{\Omega_+}} \leq C\|\vec{f}\|_{B^{p,q}_{s+\frac{1}{p}-2}(\Omega)} + C\|g\|_{B^{p,q}_{s+\frac{1}{p}-1}(\Omega)} + C\|\vec{h}\|_{B^{p,q}_{s}(\partial\Omega)}. \tag{1.38}$$

Moreover, analogous well-posedness results hold on the Triebel-Lizorkin scale, i.e., for the problem

$$\begin{aligned} &\Delta\vec{u} - \nabla\pi = \vec{f} \in F^{p,q}_{s+\frac{1}{p}-2}(\Omega), \quad \operatorname{div}\vec{u} = g \in F^{p,q}_{s+\frac{1}{p}-1}(\Omega), \\ &\vec{u} \in F^{p,q}_{s+\frac{1}{p}}(\Omega), \quad \pi \in F^{p,q}_{s+\frac{1}{p}-1}(\Omega), \quad \operatorname{Tr}\vec{u} = \vec{g} \in B^{p,p}_{s}(\partial\Omega), \end{aligned} \tag{1.39}$$

where the data is, once again, made subject to (1.37). This time, in addition to the previous conditions imposed on the indices p, q, it is also assumed that $p, q < \infty$.

In the class of Lipschitz domains we conjecture that this result is sharp. When $\partial\Omega \in C^1$, one may take $\varepsilon = 1$. This follows by combining the results in [32] with those of the current work. Theorem 1.5 refines a long list of results in the literature. When $\partial\Omega$ is sufficiently smooth, various cases (typically corresponding to Sobolev spaces with an integer amount of smoothness) have been dealt with by L. Cattabriga [14], R. Temam [88], Y. Giga [39], W. Varnhorn [92], R. Dautray and J.-L. Lions [25], among others, when $\partial\Omega$ is (at least of) class C^2. This has been subsequently extended by C. Amrouche and V. Girault [4] to the case when $\partial\Omega \in C^{1,1}$ and, further, by G.P. Galdi, C.G. Simader and H. Sohr [37] when $\partial\Omega$ is Lipschitz, with a small Lipschitz constant.

There is also a wealth of results related to Theorem 1.5 in the case when Ω is a polygonal domain in $\mathbb{R}^2$, or a polyhedral domain in $\mathbb{R}^3$. A extended account of this field of research can be found in V.A. Kozlov, V.G. Maz'ya and J. Rossmann's monograph [59], which also contains pertinent references to earlier work. Here we also wish to mention the recent work by V. Maz'ya and J. Rossmann [65]. Comparison between the regularity results obtained in [59], [65] and our Theorem 1.5 shows that the latter is optimal, at least if $n = 2, 3$.

In the case of the inhomogeneous Neumann problem we shall prove the following.

Theorem 1.6 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, with connected complement, and fix $\frac{n-1}{n} < p \leq \infty$, $0 < q \leq \infty$, and $(n-1)(\frac{1}{p}-1)_+ < s < 1$. Then there exists $\varepsilon = \varepsilon(\Omega) \in (0,1]$ such that the Poisson problem for the Stokes system with Neumann boundary condition*

$$\begin{aligned} &\Delta \vec{u} - \nabla \pi = \vec{f}\Big|_{\Omega}, \quad \vec{f} \in B^{p,q}_{s+\frac{1}{p}-2,0}(\Omega), \quad \operatorname{div} \vec{u} = 0 \ \text{in } \Omega, \\ &\vec{u} \in B^{p,q}_{s+\frac{1}{p}}(\Omega), \quad \pi \in B^{p,q}_{s+\frac{1}{p}-1}(\Omega), \quad \partial^{\lambda}_{\nu}(\vec{u},\pi)_{\vec{f}} = \vec{h} \in B^{p,q}_{s-1}(\partial\Omega), \end{aligned} \tag{1.40}$$

has a unique solution (modulo adding to the velocity functions from $\Psi^{\lambda}(\Omega)$) if the pair s, p belongs to the region $\mathcal{R}_{n,\varepsilon}$ described before, and the data $(\vec{f}, \vec{h})$ satisfy the necessary compatibility condition

$$\int_{\Omega} \langle \vec{f}, \psi \rangle \, dx = \int_{\partial\Omega} \langle \vec{h}, \psi \rangle \, d\sigma, \qquad \forall\, \psi \in \Psi^{\lambda}(\Omega). \tag{1.41}$$

In addition, the solution (normalized so that $\int_{\Omega} \langle \vec{u}(x), \psi(x) \rangle \, dx = 0$ for every $\psi \in \Psi^{\lambda}(\Omega)$) satisfies the estimate

$$\|\vec{u}\|_{B^{p,q}_{s+\frac{1}{p}}(\Omega)} + \|\pi\|_{B^{p,q}_{s+\frac{1}{p}-1}(\Omega)} \leq C\|\vec{f}\|_{B^{p,q}_{s+\frac{1}{p}-2,0}(\Omega)} + C\|\vec{h}\|_{B^{p,q}_{s-1}(\partial\Omega)}. \tag{1.42}$$

Moreover, an analogous well-posedness result holds for the problem

$$\begin{aligned} &\Delta \vec{u} - \nabla \pi = \vec{f}\Big|_{\Omega}, \quad \vec{f} \in F^{p,q}_{s+\frac{1}{p}-2,0}(\Omega), \quad \operatorname{div} \vec{u} = 0 \ \text{in } \Omega, \\ &\vec{u} \in F^{p,q}_{s+\frac{1}{p}}(\Omega), \quad \pi \in F^{p,q}_{s+\frac{1}{p}-1}(\Omega), \quad \partial^{\lambda}_{\nu}(\vec{u},\pi)_{\vec{f}} = \vec{h} \in B^{p,p}_{s-1}(\partial\Omega), \end{aligned} \tag{1.43}$$

assuming that $p, q < \infty$.

Finally, if the condition that the complement of Ω is connected is dropped (i.e., $\Omega \subset \mathbb{R}^n$ is an arbitrary Lipschitz domains), then problems (1.40), (1.43) have solutions for data $(\vec{f}, \vec{h})$ belonging to a finite co-dimensional subspace of $B^{p,q}_{s+1/p-2,0}(\Omega) \oplus B^{p,q}_{s-1}(\partial\Omega)$ and

$F^{p,q}_{s+1/p-2,0}(\Omega) \oplus B^{p,p}_{s-1}(\partial\Omega)$, *respectively, and uniqueness holds up to a finite dimensional space.*

Above, $\partial^{\lambda}_{\nu}(\vec{u},\pi)_{\vec{f}}$ should be thought of as a re-normalization of the conormal derivative (1.2) relative to $\vec{f}$. See Theorem 10.16 and the discussion preceding it for a more precise formulation. Here we only wish to point out that when $\partial\Omega \in C^1$ and $\lambda = 1$, corresponding to the so-called slip boundary condition, one can take $\varepsilon = 1$.

Theorems 1.5-1.6 are proved by interpolating the end-point cases addressed in Theorems 1.2-1.4. This is done at the level of boundary layer potentials and solutions for the problems described in Theorems 1.5-1.6 are produced in a constructive manner, via integral representation formulas.

1.2 Consequences of the solvability of the inhomogeneous problem

Here we record several relevant consequences of the well-posedness results from Theorems 1.5-1.6.

Denote by $\mathbf{G}_D$ the Green operator for the inhomogeneous problem for the incompressible Stokes system with Dirichlet boundary conditions. That is, formally, if $(\vec{u},\pi)$ solve

$$\Delta\vec{u} - \nabla\pi = \vec{f} \ \text{ in } \ \Omega, \quad \operatorname{div}\vec{u} = 0 \ \text{ in } \ \Omega, \quad \operatorname{Tr}\vec{u} = 0 \ \text{ on } \ \partial\Omega, \tag{1.44}$$

then

$$\mathbf{G}_D\vec{f} := \vec{u}. \tag{1.45}$$

Corollary 1.7 *If Ω is a bounded, Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, then there exists some small number $\varepsilon = \varepsilon(\Omega) > 0$ such the operators*

$$\mathbf{G}_D : B^{p,q}_{\alpha}(\Omega) \longrightarrow B^{p,q}_{\alpha+2}(\Omega), \tag{1.46}$$

$$\mathbf{G}_D : F^{p,q}_{\alpha}(\Omega) \longrightarrow F^{p,q}_{\alpha+2}(\Omega), \tag{1.47}$$

are well-defined and bounded whenever $0 < q \leq \infty$ and the point with coordinates $(\alpha - 1/p + 2, 1/p)$ belongs to the region $\mathcal{R}_{n,\varepsilon}$ in Figures 1-3.

The two-dimensional region of points with coordinates $(\alpha, 1/p)$ for which $(\alpha - 1/p + 2, 1/p) \in \mathcal{R}_{3,\varepsilon}$ is depicted below:

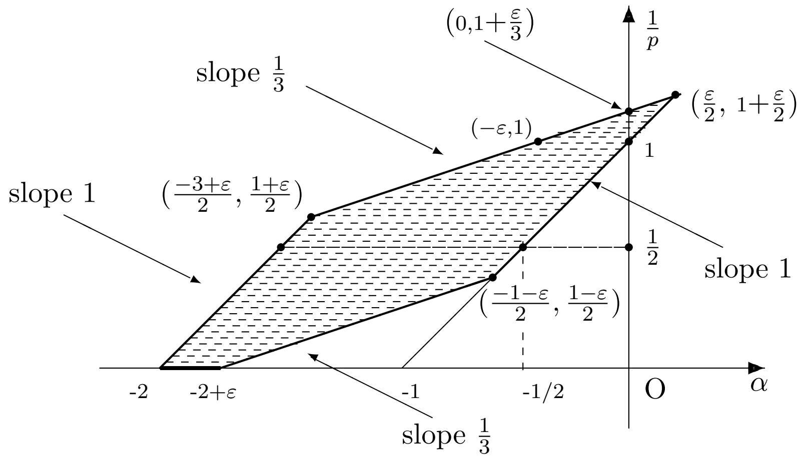

Figure 4

Thus, in the setting of a bounded Lipschitz domains $\Omega \subset \mathbb{R}^3$, the operators

$$\nabla^2 \mathbf{G}_D : B^{p,q}_{\alpha}(\Omega) \longrightarrow B^{p,q}_{\alpha}(\Omega), \tag{1.48}$$

$$\nabla^2 \mathbf{G}_D : F^{p,q}_{\alpha}(\Omega) \longrightarrow F^{p,q}_{\alpha}(\Omega), \tag{1.49}$$

are bounded whenever $0 < q \leq \infty$ and the point with coordinates $(\alpha, 1/p)$ belongs to the pentagonal region from Figure 4.

It is interesting to specialize this result to the Triebel-Lizorkin scale with $q = 2$ and $\alpha = 0$, in which case one obtains that

$$\begin{aligned} &\nabla^2 \mathbf{G}_D : h^p(\Omega) \longrightarrow h^p(\Omega) \quad \text{boundedly,} \\ &\text{if } \Omega \subset \mathbb{R}^3 \text{ is a bounded Lipschitz domain} \\ &\text{and } 1 - \varepsilon < p < 1 \text{ for some } \varepsilon = \varepsilon(\Omega) > 0. \end{aligned} \tag{1.50}$$

Corresponding to the two-dimensional case we have

$$
\begin{aligned}
&\nabla^2 \mathbf{G}_D : h^p(\Omega) \longrightarrow h^p(\Omega) \quad \text{boundedly,} \\
&\text{if } \Omega \subset \mathbb{R}^2 \text{ is a bounded Lipschitz domain} \\
&\text{and } \tfrac{2}{3} - \varepsilon < p < 1 \text{ for some } \varepsilon = \varepsilon(\Omega) > 0.
\end{aligned}
\tag{1.51}
$$

For the Laplace operator, similar results (valid in all space dimensions) have been established in [63], [64]. This answered in the affirmative a conjecture made by D.-C. Chang, S. Krantz and E. Stein (cf. [15], [16]) regarding the regularity of the harmonic Green potentials on Hardy spaces in Lipschitz domains. Here we prove the analogue of the Chang-Krantz-Stein conjecture for the Stokes system for arbitrary Lipschitz domains in the three dimensional setting. Analogous results are valid for $\mathbf{G}_N$, the Green operator associated with the inhomogeneous Stokes problem with Neumann boundary conditions.

When specialized to the case $\alpha = -1$ and $q = 2$, the operator (1.47) becomes

$$
\begin{aligned}
&\mathbf{G}_D : W^{-1,p}(\Omega) \longrightarrow W^{1,p}(\Omega) \quad \text{boundedly,} \\
&\text{if } \tfrac{2n}{n+1} - \varepsilon < p < \tfrac{2n}{n-1} + \varepsilon \text{ for some } \varepsilon = \varepsilon(\Omega) > 0,
\end{aligned}
\tag{1.52}
$$

where $W^{s,p}(\Omega)$ stands for the usual L^p-based Sobolev space of smoothness s in Ω. This follows from a brief inspection of the region in Figures 1-3. As a corollary, for every bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ there exists $p = p(\Omega) > 3$ such that the operator in (1.52) is well-defined and bounded. A similar result is valid for $\mathbf{G}_N$. In the case of $\mathbf{G}_D$, a result of this type has first been obtained by R. Brown and Z. Shen in [10] (at least if $\partial\Omega$ is connected and for Dirichlet boundary conditions). When $\Omega \subset \mathbb{R}^2$ is a bounded Lipschitz domain, the same type of conclusion holds for some $p = p(\Omega) > 4$. Let us also single out the following low-dimensional result:

Corollary 1.8 *Assume that Ω is either a convex polygon in $\mathbb{R}^2$, or a convex polyhedron in $\mathbb{R}^3$. Then*

$$
\mathbf{G}_D : L^p(\Omega) \longrightarrow W^{2,p}(\Omega) \quad \textit{boundedly, whenever } 1 < p \leq 2. \tag{1.53}
$$

Indeed, this follows by interpolating between the case $\frac{2}{3} - \varepsilon < p < 1$, contained in (1.51), and the case $p = 2$, which has been dealt with by R.B. Kellogg and J.E. Osborn in [52], when $\Omega \subset \mathbb{R}^2$ is a convex polygon, and by M. Dauge in [24] and by V.A. Kozlov, V.G. Maz'ya and C. Schwab in [60] when $\Omega \subset \mathbb{R}^3$ is a convex polyhedron. Theorem 1.8 should be compared with the result implied by the work of V. Kozlov and V. Maz'ya in [56], to the effect that

$$\begin{gathered} \nabla \mathbf{G}_D : L^q(\Omega) \longrightarrow L^\infty(\Omega) \quad \text{boundedly, } \forall\, q > 2, \\ \text{provided } \Omega \subset \mathbb{R}^2 \text{ is a bounded convex domain.} \end{gathered} \tag{1.54}$$

This portion of our work can be regarded as the natural analogue of the treatment of D. Jerison and C. Kenig of the inhomogeneous Dirichlet problem for the Laplacian in Sobolev-Besov spaces in Lipschitz domains from [46]. Here, we are able to extend this to the case of the Stokes system in a Lipschitz domain Ω, remove the assumption that $\partial\Omega$ is connected, handle boundary conditions of Neumann type, and work of more general scales of spaces (including non locally convex Besov and Triebel-Lizorkin spaces).

We continue by recording the following significant consequence of Theorem 1.5. Related versions for smooth domains have been proved by C. Amrouche and V. Girault in [4], [5], and by V. Girault and P.-A. Raviart in [40]. To state it, introduce $F^{p,q}_{\alpha,z}(\Omega) := \{u|_\Omega : u \in F^{p,q}_\alpha(\mathbb{R}^n)\ \mathrm{supp}\, u \subseteq \overline{\Omega}\}$, plus a similar definition for $B^{p,q}_{\alpha,z}(\Omega)$.

Corollary 1.9 *For every bounded, Lipschitz domain Ω in $\mathbb{R}^n$, $n \geq 2$, there exists some small number $\varepsilon = \varepsilon(\Omega) > 0$ such that*

$$F^{p,q}_{\alpha,z}(\Omega;\mathbb{R}^n) = \{\vec{v} \in F^{p,q}_{\alpha,z}(\Omega;\mathbb{R}^n) : \mathrm{div}\,\vec{v} = 0\} \oplus \{\vec{u} \in F^{p,q}_{\alpha,z}(\Omega;\mathbb{R}^n) : \Delta\vec{u} \in \nabla F^{p,q}_{\alpha-1}(\Omega)\}, \tag{1.55}$$

$$B^{p,q}_{\alpha,z}(\Omega;\mathbb{R}^n) = \{\vec{v} \in B^{p,q}_{\alpha,z}(\Omega;\mathbb{R}^n) : \mathrm{div}\,\vec{v} = 0\} \oplus \{\vec{u} \in B^{p,q}_{\alpha,z}(\Omega;\mathbb{R}^n) : \Delta\vec{u} \in \nabla B^{p,q}_{\alpha-1}(\Omega)\}, \tag{1.56}$$

where the direct sums are topological, whenever the point with coordinates $(\alpha - 1/p + 2, 1/p)$ belongs to the region $\mathcal{R}_{n,\varepsilon}$ in Figures 1-3 and $0 < q \leq \infty$. In particular, corresponding to the case when $\alpha = 1$ in (1.55),

$$W^{1,p}_0(\Omega;\mathbb{R}^n) = \{\vec{v} \in W^{1,p}_0(\Omega;\mathbb{R}^n) : \mathrm{div}\,\vec{v} = 0\} \oplus \{\vec{u} \in W^{1,p}_0(\Omega;\mathbb{R}^n) : \Delta\vec{u} \in \nabla L^p(\Omega)\}, \tag{1.57}$$

provided $\frac{2n}{n+1} - \varepsilon < p < \frac{2n}{n-1} + \varepsilon$.

Indeed, if $\vec{w} \in F^{p,q}_{\alpha,z}(\Omega;\mathbb{R}^n)$ is arbitrary and the pair $(\vec{u},\pi) \in F^{p,q}_{\alpha,z}(\Omega;\mathbb{R}^n) \times F^{p,q}_{\alpha-1}(\Omega)$ solves (1.39) for $\vec{f} := \Delta\vec{w} \in F^{p,q}_{\alpha-2}(\Omega;\mathbb{R}^n)$, $\vec{g} := 0$, and $\vec{h} := 0$, then $\vec{w} = \vec{u} + (\vec{w} - \vec{u})$ is the desired decomposition. That sum in the right-hand side of (1.55) is direct is immediate from the uniqueness statement for (1.39). This proves (1.55), and the argument for (1.56) is similar. Finally, (1.57) is a direct consequence of (1.55).

We next discuss the analogue of the off-diagonal estimates for the Green operator associated with the Dirichlet Laplacian in Lipschitz domains, established by B.E.J. Dahlberg in [19].

Corollary 1.10 *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Then there exists $\varepsilon = \varepsilon(\Omega) > 0$ with the property that if*

$$1 < p < \tfrac{3}{2} + \varepsilon \quad \textit{and} \quad \tfrac{1}{q} = \tfrac{1}{p} - \tfrac{1}{3} \tag{1.58}$$

then the operator

$$\nabla \mathbf{G}_D : L^p(\Omega) \longrightarrow W^q_1(\Omega) \tag{1.59}$$

is well-defined and bounded.

A similar result holds in the case when Ω is a bounded Lipschitz domain in $\mathbb{R}^2$, granted that (1.58) is replaced by $1 < p < \frac{4}{3} + \varepsilon$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$.

To justify this, consider an arbitrary vector field $\vec{f} \in L^p(\Omega)$ and, by taking the convolution of $\vec{f}$ (extended by zero to $\mathbb{R}^3$) with the fundamental solution for the Stokes system in the free space, construct two functions $\vec{w} \in W^p_2(\Omega)$ and $\rho \in W^p_1(\Omega)$ such that $\Delta\vec{w} - \nabla\rho = \vec{f}$, $\operatorname{div}\vec{w} = 0$ in Ω, and $\|\vec{w}\|_{W^p_2(\Omega)} + \|\rho\|_{W^p_1(\Omega)} \leq C\|\vec{f}\|_{L^p(\Omega)}$. Then $\mathbf{G}_D\vec{f} = \vec{w} - \vec{u}$, where the pair $(\vec{u},\pi)$ solves $\Delta\vec{u} - \nabla\pi = 0$, $\operatorname{div}\vec{u} = 0$ in Ω, and $\operatorname{Tr}\vec{u} = \operatorname{Tr}\vec{w}$ on $\partial\Omega$. Note that the compatibility condition (1.37) is automatically satisfied in this case. Also, $\vec{w} \in W^p_2(\Omega) \hookrightarrow W^q_1(\Omega)$ if $1/q = 1/p - 1/3$ and, accordingly, $\operatorname{Tr}\vec{w} \in B^{q,q}_{1-1/q}(\partial\Omega)$. Then Theorem 1.5 implies that $\vec{u} \in W^q_1(\Omega)$, $\pi \in L^q(\Omega)$, granted that the point with coordinates $(1 - 1/q, q)$ belongs to the pentagonal region $\mathcal{R}_{3,\varepsilon}$ described in Figure 2. A simple analysis shows that this is always the case whenever $\frac{3}{2+\varepsilon} < q < \frac{3}{1-\varepsilon}$, for some $\varepsilon = \varepsilon(\Omega) > 0$. The bottom line is that

$$\vec{f} \in L^p(\Omega) \Longrightarrow \mathbf{G}_D \vec{f} \in W_1^q(\Omega) \quad \text{if} \quad \tfrac{3}{2+\varepsilon} < q < \tfrac{3}{1-\varepsilon}, \ \tfrac{1}{q} = \tfrac{1}{p} - \tfrac{1}{3}. \tag{1.60}$$

Next, (1.47) with $\alpha = 0$, $q = 2$, and classical embeddings give

$$\nabla \mathbf{G}_D : F_0^{p,2}(\Omega) \longrightarrow F_1^{p^*,2}(\Omega) \quad \text{if} \quad \tfrac{3}{3+\varepsilon} < p < 1, \ \tfrac{1}{p^*} = \tfrac{1}{p} - \tfrac{1}{3}. \tag{1.61}$$

Interpolating by the complex method between (1.60) and (1.61) then yields (1.59) in full, as long as $\frac{1}{q} = \frac{1}{p} - \frac{1}{3}$ and $1 < q < \frac{3}{1-\varepsilon}$, a condition implied by (1.58). Finally, the reasoning for the two-dimensional case is similar.

We conclude with a discussion pertaining to the regularity properties of solutions of elliptic systems in domains with conical singularities. Consider the inhomogeneous Dirichlet problem

$$L(D)u = f \ \text{ in } \ \Omega, \quad \text{with zero boundary conditions,} \tag{1.62}$$

where $L(D)$ is a homogeneous, strongly elliptic, constant coefficient, formally self-adjoint system of order $2m$, $m \in \mathbb{N}$, and $\Omega \subset \mathbb{R}^n$ is a domain with a conical point at the origin $O \in \mathbb{R}^n$. Assume that f vanishes near O and u is the variational solution of (1.62). As is well-known, u admits a power-logarithmic asymptotic expansion near O. Somewhat more precisely, near the origin u behaves like a linear combination of terms of the form

$$|x|^{\lambda_j} \sum_{0 \le \ell \le l_j} \frac{(\log |x|)^{l_j - \ell}}{(l_j - \ell)!} \, w_{\ell,j}\Big(\tfrac{x}{|x|}\Big), \tag{1.63}$$

where the exponents $\lambda_j \in \mathbb{C}$ are the eigenvalues of a certain polynomial operator pencil (on a domain that is cut out of the unit sphere by the cone with vertex at O which is tangent to the boundary of Ω), and the functions $w_{\ell,j}$ are generalized eigenvectors corresponding to λ_j. The operator pencil arises when taking the Mellin transform of $L(D)$ and of the operators intervening in the boundary conditions along this tangent cone.

Specific information about the nature of the eigenvalues λ_j yields, in turn, regularity properties for the solution u. For example,

$$p < \min_j \left\{ \frac{n}{k - \operatorname{Re}\lambda_j} \right\} \Longrightarrow u \in W_k^p \text{ near } O. \tag{1.64}$$

In [57], V. Kozlov and V. Maz'ya have shown that, in the above setting,

$$\operatorname{Re}\lambda_j > m - (n-1)/2. \tag{1.65}$$

As a consequence of (1.64)-(1.65), we then have

$$u \in W_k^p \text{ near } O, \text{ whenever } p < \frac{n}{k - m + (n-1)/2} + \varepsilon, \tag{1.66}$$

where $\varepsilon = \varepsilon(\Omega) > 0$. Moreover, in [58], V. Kozlov and V. Maz'ya have also shown that (1.65) and, hence, (1.66), is sharp in the case when $2m \geq n$.

When $m = 1$, i.e., when $L(D)$ is a second order operator, the above analysis gives that

$$u \in W_1^p \text{ near } O, \text{ whenever } p < \frac{2n}{n-1} + \varepsilon. \tag{1.67}$$

While, strictly speaking, the Stokes system does not fit into this general narrative since it is not elliptic in the sense of I.G. Petrovskii, the same circle of ideas can be adapted to this somewhat nonstandard case. See, e.g., the work of V.A. Kozlov, V.G. Maz'ya and C. Schwab in [60] as well as the monograph [59] for the lower dimensional case ($n = 2, 3$).

The relevance of the above observation is that $\frac{2n}{n-1}$ is also the critical integrability exponent we have identified in (1.52). Thus, our results are consistent with the predictions of the regularity theory for domains with conical singularities, and are sharp when $n = 2, 3$. While it is not entirely clear whether that is also true when $n \geq 4$, we conjecture that this is indeed the case.

Acknowledgments. We are indebted to R. Brown, M. Costabel, V. Maz'ya, S. Nicaise and Z. Shen for several stimulating discussions and for their interest in this work. Partial support from the NSF grants DMS-0653180 and DMS/FRG-0456306 is also gratefully acknowledged.

2 Smoothness spaces and Lipschitz domains

For a brief review of the Besov and Triebel-Lizorkin scales in the entire Euclidean space $\mathbb{R}^n$, the reader is referred to § 11.1.

2.1 Graph Lipschitz Domains

We start with a few basic definitions. A *graph Lipschitz* domain $\Omega \subset \mathbb{R}^n$ is simply the domain lying above the graph of a real-valued Lipschitz function. That is,

$$\begin{aligned} &\Omega := \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \, x_n > \varphi(x')\}, \text{ where } x' = (x_1, ..., x_{n-1}), \\ &\varphi : \mathbb{R}^{n-1} \to \mathbb{R} \text{ is Lipschitz, i.e., } \nabla\varphi \text{ exists and belongs to } L^\infty(\mathbb{R}^{n-1}). \end{aligned} \tag{2.1}$$

We denote by $d\sigma$ the surface measure on $\partial\Omega$, and by ν the outward unit normal defined a.e. (with respect to $d\sigma$) on $\partial\Omega$. Hereafter, we will define $\Omega_\pm$ by

$$\Omega_+ := \Omega \quad \text{ and } \quad \Omega_- := \mathbb{R}^n \setminus \bar{\Omega}. \tag{2.2}$$

Next, we define the cones

$$\Gamma_\kappa^\pm := \{y = (y', y_n) \in \mathbb{R}^n_+ : \, |y'| < \pm\kappa y_n\}, \tag{2.3}$$

and for any $x \in \mathbb{R}^n$, define

$$\Gamma_\kappa^\pm(x) := x + \Gamma_\kappa^\pm. \tag{2.4}$$

In order to introduce the classical non-tangential maximal operator M, fix some $\kappa = \kappa(\partial\Omega)$ with $\kappa^{-1} > \|\nabla\varphi\|_{L^\infty}$. Then it can be shown that $\Gamma_\kappa^\pm(x) \subseteq \Omega_\pm$ for all $x \in \partial\Omega$. When the value of κ is understood, we will often drop it from the notation and write $\Gamma_\kappa^\pm(x) = \Gamma^\pm(x)$. Now, for an arbitrary $u : \Omega_\pm \to \mathbb{R}$, we set

$$M(u)(x) := \sup \{|u(y)| : \, y \in \Gamma^\pm(x)\}, \qquad x \in \partial\Omega. \tag{2.5}$$

These conical regions also play a fundamental role in defining non-tangential restrictions to the boundary. Again for u defined in $\Omega_{\pm}$, set

$$u\Big|_{\partial\Omega}(x) := \lim_{\substack{y\to x\\ y\in\Gamma^{\pm}(x)}} u(y), \qquad \text{for a.e. } x\in\partial\Omega. \tag{2.6}$$

Similarly, if $\langle\cdot,\cdot\rangle$ denotes the canonical inner product in $\mathbb{R}^n$ (although, later, the same symbol is going to be occasionally used for the pairing between a space and its dual), we set

$$\partial_\nu u(x) := \Big\langle \nu(x)\,,\ \lim_{\substack{y\to x\\ y\in\Gamma^{\pm}(x)}} (\nabla u)(y)\Big\rangle, \qquad \text{for a.e. } x\in\partial\Omega. \tag{2.7}$$

By $L^p(\partial\Omega)$ we denote the Lebesgue space of measurable, p-th power integrable functions on $\partial\Omega$, with respect to the surface measure $d\sigma$. Next, consider the first-order tangential derivative operators $\partial_{\tau_{jk}}$, acting on a compactly supported function ψ of class C^1 in a neighborhood of $\partial\Omega$ by

$$\partial_{\tau_{jk}}\psi := \nu_j(\partial_k\psi)\Big|_{\partial\Omega} - \nu_k(\partial_j\psi)\Big|_{\partial\Omega}, \qquad j,k=1,\dots,n. \tag{2.8}$$

For every $f\in L^1_{loc}(\partial\Omega)$ define the functional $\partial_{\tau_{kj}}f$ by setting

$$\partial_{\tau_{kj}}f : C^1_0(\mathbb{R}^n)\ni\psi\mapsto \int_{\partial\Omega} f\,(\partial_{\tau_{jk}}\psi)\,d\sigma. \tag{2.9}$$

Thus, if $f\in L^1_{loc}(\partial\Omega)$ has $\partial_{\tau_{kj}}f\in L^1_{loc}(\partial\Omega)$, the following integration by parts formula holds:

$$\int_{\partial\Omega} f\,(\partial_{\tau_{jk}}\psi)\,d\sigma = \int_{\partial\Omega} (\partial_{\tau_{kj}}f)\,\psi\,d\sigma, \qquad \forall\,\psi\in C^1_0(\mathbb{R}^n). \tag{2.10}$$

For each $p\in(1,\infty)$ we can then define the Sobolev type space

$$L^p_1(\partial\Omega) = \Big\{ f\in L^p(\partial\Omega) : \partial_{\tau_{jk}}f\in L^p(\partial\Omega),\quad j,k=1,\dots,n\Big\}. \tag{2.11}$$

For each $1<p<\infty$ this becomes a Banach space when equipped with the natural norm

$$\|f\|_{L_1^p(\partial\Omega)} := \|f\|_{L^p(\partial\Omega)} + \sum_{j,k=1}^{n} \|\partial_{\tau_{jk}} f\|_{L^p(\partial\Omega)}. \tag{2.12}$$

If we set

$$\nabla_{tan} f := \Big(\nu_k \partial_{\tau_{kj}} f\Big)_{1\le j\le n}, \qquad \forall\, f \in L_1^p(\partial\Omega), \tag{2.13}$$

then for each function $f \in L_1^p(\partial\Omega)$

$$\partial_{\tau_{jk}} f = \nu_j (\nabla_{tan} f)_k - \nu_k (\nabla_{tan} f)_j, \qquad j,k = 1,...,n, \tag{2.14}$$

σ-a.e. on $\partial\Omega$. In particular,

$$\|\nabla_{tan} f\|_{L^p(\partial\Omega)} \approx \sum_{j,k=1}^{n} \|\partial_{\tau_{jk}} f\|_{L^p(\partial\Omega)} \approx \sum_{j=1}^{n-1} \|\partial_{\tau_{jn}} f\|_{L^p(\partial\Omega)}, \qquad \forall\, f \in L_1^p(\partial\Omega). \tag{2.15}$$

Furthermore, if $1 < p, p' < \infty$ are such that $1/p + 1/p' = 1$ then

$$\int\limits_{\partial\Omega} (\partial_{\tau_{jk}} f)\, g \, d\sigma = \int\limits_{\partial\Omega} f\, (\partial_{\tau_{kj}} g)\, d\sigma \tag{2.16}$$

for every $f \in L_1^p(\partial\Omega)$, $g \in L_1^{p'}(\partial\Omega)$. In general, we shall call a first-order differential operator *tangential* if it can be written as a (variable coefficient) linear combination of the operators $\partial_{\tau_{jk}}$.

If $\Omega \subset \mathbb{R}^n$ is the domain lying above the graph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ then, for each $p \in (1,\infty)$,

$$f \in L_1^p(\partial\Omega) \iff f(\cdot, \varphi(\cdot)) \in L_1^p(\mathbb{R}^{n-1}), \tag{2.17}$$

with equivalence of norms. As a corollary, we obtain from this that for any bounded Lipschitz domain Ω in $\mathbb{R}^n$,

$$\mathrm{Lip}(\partial\Omega) \hookrightarrow L_1^p(\partial\Omega) \quad \text{and} \quad C^\infty(\mathbb{R}^n)\Big|_{\partial\Omega} \hookrightarrow L_1^p(\partial\Omega) \quad \text{densely} \tag{2.18}$$

whenever $1 < p < \infty$.

For each $1 < p < \infty$, $L_1^p(\partial\Omega)$ is a Banach space, densely embedded into $L^p(\partial\Omega)$. Furthermore, since the mapping

$$J : L_1^p(\partial\Omega) \longrightarrow \Big[L^p(\partial\Omega)\Big]^{1+\frac{(n-1)n}{2}}, \qquad Jf := \Big(f, (\partial_{\tau_{jk}} f)_{1\leq j,k\leq n}\Big), \tag{2.19}$$

is bounded both from above and below, its image is closed. Now, $L_1^p(\partial\Omega)$ is isomorphic to the latter space and, hence, is reflexive. Thus, if for each $1 < p < \infty$, we set

$$L_{-1}^p(\partial\Omega) := \Big(L_1^{p'}(\partial\Omega)\Big)^*, \qquad 1/p + 1/p' = 1, \tag{2.20}$$

it follows that

$$\Big(L_{-1}^p(\partial\Omega)\Big)^* = L_1^{p'}(\partial\Omega), \qquad 1/p + 1/p' = 1. \tag{2.21}$$

We can now prove the following result.

Corollary 2.1 *Let Ω be a Lipschitz domain in $\mathbb{R}^n$, $1 < p < \infty$ and fix $j, k \in \{1, ..., n\}$. Then the operator*

$$\partial_{\tau_{jk}} : L_1^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \tag{2.22}$$

extends in a (unique) compatible fashion to a bounded, linear mapping

$$\partial_{\tau_{jk}} : L^p(\partial\Omega) \longrightarrow L_{-1}^p(\partial\Omega). \tag{2.23}$$

Proof. For every $f \in L^p(\partial\Omega)$, set

$$\langle \partial_{\tau_{jk}} f, g \rangle := \int_{\partial\Omega} f \partial_{\tau_{kj}} g \, d\sigma, \qquad \forall\, g \in L_1^{p'}(\partial\Omega), \tag{2.24}$$

where $1/p + 1/p' = 1$. Then the desired conclusion follows from the boundary integration by parts formula (2.16). $\square$

Corollary 2.2 *Assume that Ω is a Lipschitz domain in $\mathbb{R}^n$ and that $1 < p < \infty$. Then for every $f \in L_{-1}^p(\partial\Omega)$ there exist $g_0, g_{jk} \in L^p(\partial\Omega)$, $1 \leq j, k \leq n$ (not necessarily unique) with the property that*

$$f = g_0 + \sum_{j,k=1}^{n} \partial_{\tau_{jk}} g_{jk} \quad \textit{in} \;\; L_{-1}^p(\partial\Omega). \tag{2.25}$$

Furthermore,

$$\|f\|_{L_{-1}^p(\partial\Omega)} \approx \inf \Big[\|g_0\|_{L^p(\partial\Omega)} + \sum_{j,k=1}^{n} \|g_{jk}\|_{L^p(\partial\Omega)} \Big], \tag{2.26}$$

where the infimum is taken over all representations of f as in (2.25).

Proof. Let $p' \in (1, \infty)$ be such that $1/p + 1/p' = 1$. If $f \in L_{-1}^p(\partial\Omega)$ is regarded as a functional $f : L_1^{p'}(\partial\Omega) \to \mathbb{R}$, then $f \circ J^{-1} : \mathrm{Im}\, J \to \mathbb{R}$ is well-defined, linear and bounded (where J is as in (2.19) with p' in place of p). At this stage, the Hahn-Banach Theorem in conjunction with Riesz's Representation Theorem ensure the existence of $g_0, g_{jk} \in L^p(\partial\Omega)$ such that (2.25)-(2.26) hold. $\square$

Let us also note that, as a simple application of the one of the standard consequences of the Hahn-Banach theorem,

$$L^p(\partial\Omega) \hookrightarrow L_{-1}^p 1(\partial\Omega) \quad \text{densely, for every} \;\; p \in (1, \infty). \tag{2.27}$$

For an unbounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, the *homogeneous* L^p-Sobolev space of order one is defined as

$$\dot{L}^p_1(\partial\Omega) := \{f \in L^p_{loc}(\partial\Omega) : \partial_{\tau_{jk}} f \in L^p(\partial\Omega),\ 1 \leq j,k \leq n\}. \tag{2.28}$$

Clearly, for each $p \in (1,\infty)$, $\dot{L}^p_1(\partial\Omega)$ becomes a Banach space modulo constants when equipped with the homogeneous norm $\|f\|_{\dot{L}^p_1(\partial\Omega)} := \|\nabla_{tan} f\|_{L^p(\partial\Omega)}$.

2.2 Hardy spaces on graph Lipschitz surfaces

Throughout this section, we shall assume that Ω is as in (2.1), i.e., the *unbounded domain in* $\mathbb{R}^n$ *lying above the graph of the Lipschitz function* $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$. A surface ball $S_r(x)$ is any set of the form $B_r(x) \cap \partial\Omega$, with $x \in \partial\Omega$ and $0 < r < \infty$. When the center is already specified or of no particular importance, we simplify the notation by writing S_r.

For $\frac{n-1}{n} < p \leq 1$, the *homogeneous* Hardy space is then defined by

$$H^p_{at}(\partial\Omega) := \Big\{f = \sum_j \lambda_j a_j : a_j\ (p,p_o)\text{-atom},\ (\lambda_j)_j \in \ell^p\Big\}, \tag{2.29}$$

where the series converges in $\mathrm{Lip}_c(\partial\Omega)'$, the dual of $\mathrm{Lip}_c(\partial\Omega)$, and equipped with the usual infimum norm. Here, $1 < p_o \leq \infty$ is a fixed parameter and a measurable function $a : \partial\Omega \to \mathbb{R}$ is called a (p,p_o)-*atom* if there exists a surface ball $S_r \subset \partial\Omega$ such that

$$\operatorname{supp} a \subseteq S_r, \quad \|a\|_{L^{p_o}(\partial\Omega)} \leq r^{(n-1)\left(\frac{1}{p_o}-\frac{1}{p}\right)} \quad \text{and} \quad \int_{\partial\Omega} a\, d\sigma = 0. \tag{2.30}$$

Given the atomic characterization of Hardy spaces in the Euclidean setting, we have

$$f \in H^p_{at}(\partial\Omega) \Longleftrightarrow f(\cdot,\varphi(\cdot))\sqrt{1+|\nabla\varphi(\cdot)|^2} \in H^p_{at}(\mathbb{R}^{n-1}). \tag{2.31}$$

In particular, this shows that different choices of the parameter p_o in (2.30) yield the same vector space and topology on $H^p_{at}(\partial\Omega)$. Let us also recall here the the well-known fact that

$$H^p_{at}(\mathbb{R}^{n-1}) = \dot{F}^{p,2}_0(\mathbb{R}^{n-1}) \qquad \text{if } \frac{n-1}{n} < p \leq 1, \tag{2.32}$$

where $\dot{F}^{p,q}_s(\mathbb{R}^{n-1})$ stands for the homogeneous Triebel-Lizorkin space in $\mathbb{R}^{n-1}$. See the discussion on p. 42 in [36]. For a precise definition, as well as basic properties of the latter scale see, e.g., [35], [90]. Here we only wish to point out that, as remarked on p. 44 in [36],

$$\|g\|_{\dot{F}^{p,q}_s(\mathbb{R}^{n-1})} \approx \sum_{j=1}^{n-1} \|\partial_j g\|_{\dot{F}^{p,q}_{s-1}(\mathbb{R}^{n-1})} \tag{2.33}$$

whenever $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$.

Recall that, for $\frac{n-1}{n} < p \leq 1$ and $\varepsilon > 0$, a (p,ε)-*molecule* adapted to a surface ball $S_r \subset \partial\Omega$ is a function $m \in L^1(\partial\Omega) \cap L^2(\partial\Omega)$ satisfying

$$\begin{aligned}
&(i) \quad \int_{\partial\Omega} m(x)\, d\sigma(x) = 0, \\
&(ii) \quad \Big(\int_{S_{16r}} |m(x)|^2\, d\sigma(x)\Big)^{1/2} \leq r^{(n-1)\left(\frac{1}{2}-\frac{1}{p}\right)}, \\
&(iii) \quad \Big(\int_{S_{2^{k+1}r}\setminus S_{2^k r}} |m(x)|^2\, d\sigma(x)\Big)^{1/2} \leq 2^{-\varepsilon k}\Big(2^k r\Big)^{(n-1)\left(\frac{1}{2}-\frac{1}{p}\right)}, \qquad \forall\, k \geq 4.
\end{aligned} \tag{2.34}$$

It is well-known that there exists a finite constant $\kappa = \kappa(\partial\Omega, p, \varepsilon) > 0$ such that

$$m \text{ is a } (p,\varepsilon)\text{-molecule} \Longrightarrow m \in H^p_{at}(\partial\Omega) \quad \text{and} \quad \|m\|_{H^p_{at}(\partial\Omega)} \leq \kappa. \tag{2.35}$$

For uniformity of notation, we find it convenient to define

$$H^p(\partial\Omega) := \begin{cases} H^p_{at}(\partial\Omega) & \text{for } \frac{n-1}{n} < p \leq 1, \\ L^p(\partial\Omega) & \text{for } p > 1. \end{cases} \tag{2.36}$$

Corresponding to one unit more on the smoothness scale we have the 'regular' Hardy space $H^{1,p}_{at}(\partial\Omega)$, defined for $\frac{n-1}{n} < p \leq 1$ as the ℓ^p-span of 'regular' atoms. More specifically, if $[f]$ denotes the class of f modulo constants, define

$$\begin{aligned}
H^{1,p}_{at}(\partial\Omega) \;\; := \;\; & \Big\{[f] :\, f \in L^1_{loc}(\partial\Omega) \text{ and there exist } (\lambda_i)_i \in \ell^p \text{ and } a_i \text{ regular } (p,p_o)\text{-atoms} \\
& \text{with } \partial_{\tau_{jn}} f = \sum_{i=1}^{\infty} \lambda_i \partial_{\tau_{jn}} a_i \text{ whenever } 1 \leq j \leq n-1\Big\},
\end{aligned} \tag{2.37}$$

where the series converges in $\mathrm{Lip}\,(\partial\Omega)'$. Also, set $\|f\|_{H^{1,p}_{at}(\partial\Omega)} := \inf\,[\sum |\lambda_i|^p]^{1/p}$, where the infimum is taken over all possible representations. Here, if $(n-1)/n < p \leq 1 < p_o \leq \infty$, a function $a \in L^{p_o}_1(\partial\Omega)$ is called a *regular* (p,p_o)-*atom* if there exists a surface ball S_r so that

$$\operatorname{supp} a \subseteq S_r, \qquad \|\nabla_{tan} a\|_{L^{p_o}(\partial\Omega)} \leq r^{(n-1)\left(\frac{1}{p_o}-\frac{1}{p}\right)}. \tag{2.38}$$

In analogy with (2.31), it can be shown that

$$[f] \in H^{1,p}_{at}(\partial\Omega) \Longleftrightarrow [f(\cdot, \varphi(\cdot))] \in \dot{F}^{p,2}_1(\mathbb{R}^{n-1}). \tag{2.39}$$

Much as before, this shows that different choices of the parameter p_o in (2.38) yield the same vector space and topology on $H^{1,p}_{at}(\partial\Omega)$. We also set

$$H^p_1(\partial\Omega) := \begin{cases} H^{1,p}_{at}(\partial\Omega) \ \text{ for } \ \frac{n-1}{n} < p \leq 1, \\ \\ \dot{L}^p_1(\partial\Omega) \ \text{ for } \ p > 1. \end{cases} \tag{2.40}$$

An alternative characterization of the quasi-norm in the space $H^p_1(\partial\Omega)$ is as follows.

Lemma 2.3 *Let Ω be as in (2.1) and assume that $\frac{n-1}{n} < p < \infty$. Then for each $j,k \in \{1, ..., n\}$*

$$\partial_{\tau_{jk}} : H^p_1(\partial\Omega) \longrightarrow H^p(\partial\Omega) \tag{2.41}$$

is a bounded operator. Furthermore,

$$H^{1,p}_{at}(\partial\Omega) = \Big\{[f] : \ f \in L^1_{loc}(\partial\Omega) \text{ and } \partial_{\tau_{jn}} f \in H^p_{at}(\partial\Omega) \ 1 \leq j \leq n-1\Big\}, \tag{2.42}$$

and, in fact,

$$\|[f]\|_{H^p_1(\partial\Omega)} \approx \sum_{j=1}^{n-1} \|\partial_{\tau_{jn}} f\|_{H^p(\partial\Omega)}. \tag{2.43}$$

Proof. The claim about (2.41) follows straight from definitions when $1 < p < \infty$, and by analyzing the action of this operator on atoms when $\frac{n-1}{n} < p \leq 1$. This also yields the right-pointing inequality in (2.43). Now, the opposite inequality is trivial for $1 < p < \infty$, so there remains to justify it when $\frac{n-1}{n} < p \leq 1$. In this scenario, we note that for every $j \in \{1, ..., n-1\}$ we have

$$\partial_{\tau_{jn}} f \in H^p_{at}(\partial\Omega) \quad \Leftrightarrow \quad \sqrt{1+|\nabla\varphi(x')|^2}(\partial_{\tau_{jn}} f)(x', \varphi(x')) \in H^p_{at}(\mathbb{R}^{n-1}) \tag{2.44}$$

$$\Leftrightarrow \quad \partial_j[f(x', \varphi(x'))] \in H^p_{at}(\mathbb{R}^{n-1}) \Leftrightarrow \partial_j[f(x', \varphi(x'))] \in \dot{F}^{p,2}_0(\mathbb{R}^{n-1}),$$

by (2.32). In concert with (2.33), this ensures that

$$\partial_{\tau_{jn}} f \in H^p_{at}(\partial\Omega) \ \text{ for every } \ j \in \{1, ..., n-1\} \Longrightarrow f(x', \varphi(x')) \in \dot{F}^{p,2}_1(\mathbb{R}^{n-1}). \tag{2.45}$$

If we now recall that, as proved in Proposition 3.4 in [66],

$$H^{1,p}_{at}(\mathbb{R}^{n-1}) = \dot{F}^{p,2}_1(\mathbb{R}^{n-1}) \qquad \text{for } \ \tfrac{n-1}{n} < p \leq 1, \tag{2.46}$$

it follows that

$$\partial_{\tau_{jn}} f \in H^p_{at}(\partial\Omega) \ \text{ for every } \ j \in \{1, ..., n-1\} \Longrightarrow f \in H^{1,p}_{at}(\partial\Omega). \tag{2.47}$$

This membership statement is accompanied by natural estimates and this finishes the proof of (2.43). Now, (2.42) follows from this equivalence. □

The space $H^{1,p}_{at}(\partial\Omega)$ in (2.37) is defined modulo constants. A "realization" of this as a space of genuine functions is as follows. If $\frac{n-1}{n} < p \leq 1$ and $p^* \in (1, \infty)$ is such that

$$\tfrac{1}{p^*} = \tfrac{1}{p} - \tfrac{1}{n-1} \tag{2.48}$$

we set

$$\widetilde{H}^{1,p}_{at}(\partial\Omega) := \Big\{ f \in L^{p^*}(\partial\Omega) : \ f = \sum_{j=1}^{\infty} \lambda_j a_j \text{ in } L^{p^*}(\partial\Omega), \ (\lambda_j)_j \in \ell^p, \ a_j \text{ regular } (p, p_o)\text{-atom} \Big\}, \tag{2.49}$$

and equip it with the natural infimum quasi-norm. We then have:

Proposition 2.4 *If $\frac{n-1}{n} < p \leq 1$, then the application*

$$\widetilde{H}^{1,p}_{at}(\partial\Omega) \ni f \mapsto [f] := f + \mathbb{R} \in H^{1,p}_{at}(\partial\Omega) \tag{2.50}$$

is an isomorphism.

Proof. The mapping (2.50) is clearly one-to-one. The fact that this is also onto follows from the lemma below. □

Lemma 2.5 *Let u be a tempered distribution in $\mathbb{R}^n$ with the property that $\partial_j u \in H^p(\mathbb{R}^n)$, $j = 1, ..., n$, for some $p \in (\frac{n}{n+1}, n)$. Then there exists $c \in \mathbb{R}$ such that $u - c \in L^{p^*}(\mathbb{R}^n)$, where $p^* := \frac{np}{n-p}$.*

Proof. For each $1 \leq j \leq n$, consider T_j to be the convolution integral operator in $\mathbb{R}^n$ with the kernel $(\partial_j E_\Delta)(x)$, where E_Δ denotes the fundamental solution for the Laplacian in $\mathbb{R}^n$. Classical Calderón-Zygmund theory implies that

$$\partial_k T_j = T_j \partial_k : H^p(\mathbb{R}^n) \longrightarrow H^p(\mathbb{R}^n), \qquad 1 \leq j, k \leq n, \ \frac{n}{n+1} < p < \infty, \tag{2.51}$$

are bounded operators. Furthermore, if $\frac{n}{n+1} < p < \infty$, we have

$$\partial_j T_j = I, \quad \text{the identity operator on } H^p(\mathbb{R}^n), \tag{2.52}$$

where repeated indices indicate summation, and if

$$\frac{n}{n+1} < p < n, \quad \frac{1}{p^*} := \frac{1}{p} - \frac{1}{n}, \quad 1 < p^* < \infty, \tag{2.53}$$

then

$$T_j : H^p(\mathbb{R}^n) \longrightarrow L^{p^*}(\mathbb{R}^n) \tag{2.54}$$

boundedly, by the Fractional Integration Theorem.

Next, let u be a tempered distribution in $\mathbb{R}^n$ with the property that there exists $p \in (\frac{n}{n+1}, n)$ such that $\partial_j u \in H^p(\mathbb{R}^n)$ for each $j = 1, ..., n$. Set

$$f_j := \partial_j u \in H^p(\mathbb{R}^n), \qquad j = 1, ..., n, \tag{2.55}$$

and note that, in the sense of distributions,

$$\partial_k f_j = \partial_j f_k, \qquad j, k = 1, ..., n. \tag{2.56}$$

We claim that, in the sense of distributions,

$$\partial_k (u - T_j f_j) = 0, \qquad k = 1, ..., n. \tag{2.57}$$

Once (2.57) has been established, it follows that the tempered distribution $u - T_j f_j$ must be a constant c which, in turn, implies that

$$u - c = T_j f_j \in L^{p^*}(\mathbb{R}^n). \tag{2.58}$$

which is what we wanted to prove. Therefore, it remains to justify (2.57). Using notational conventions introduced earlier, we can re-write this in the equivalent form

$$f_k = \partial_k (T_j f_j), \qquad k = 1, ..., n. \tag{2.59}$$

To prove (2.59), based on (2.52) and (2.56), for each k we write

$$\partial_k (T_j f_j) = T_j (\partial_k f_j) = T_j (\partial_j f_k) = \partial_j (T_j f_k) = f_k, \tag{2.60}$$

as desired. $\square$

As a corollary of Proposition 2.4, we obtain that the definition of $\widetilde{H}^{1,p}_{at}(\partial\Omega)$ is independent of the particular choice of $p_o \in (1, \infty]$. Let us also point out here that, when used in concert with (2.43), the fact that (2.50) is an isomorphism further entails

$$\|f\|_{\widetilde{H}^{1,p}_{at}(\partial\Omega)} \approx \|[f]\|_{H^{1,p}_{at}(\partial\Omega)} \approx \sum_{j=1}^{n-1} \|\partial_{\tau_{jn}} f\|_{H^p_{at}(\partial\Omega)}, \quad \text{uniformly for } f \in \widetilde{H}^{1,p}_{at}(\partial\Omega). \tag{2.61}$$

A distinctive feature of $\widetilde{H}^{1,p}_{at}(\partial\Omega)$ is that this space is local. This can be justified by analyzing the action of multiplication by $\psi \in \mathrm{Lip}_c(\partial\Omega)$ on regular atoms. To this end, it is trivial to check that, if $\frac{n-1}{n} < p \leq 1 < p_o \leq \infty$, then for each $\eta > 0$ there exists $C = C(\partial\Omega, \psi, \eta, p, p_o) > 0$ such that

$$\begin{array}{c} A \text{ regular } (p,p_o)\text{-atom supported in a surface ball of radius } \leq \eta \\ \Longrightarrow C^{-1}\psi\, A \text{ is a regular } (p,p_o)\text{-atom on } \partial\Omega. \end{array} \tag{2.62}$$

A more refined version of this result, allowing for atoms supported in surface balls of arbitrary radii, is as follows.

Lemma 2.6 *Let Ω be Lipschitz domain in $\mathbb{R}^n$ and assume that $\frac{n-1}{n} < p \leq 1$ and $p^* \leq p_o \leq q \leq \infty$, where p^* is as in (2.48). If $\psi \in \mathrm{Lip}_c(\partial\Omega)$ then $\psi\, A$ is, up to a fixed multiplicative constant, a regular (p,p_o)-atom on $\partial\Omega$ whenever A is a regular (p,q)-atom on $\partial\Omega$.*

Proof. To fix ideas, let us assume that $\mathrm{supp}\,\psi \subseteq S_1$, a surface ball of radius 1, and that $\|\psi\|_{L^\infty(\partial\Omega)} + \|\nabla_{tan}\psi\|_{L^\infty(\partial\Omega)} \leq 1$. Fix a regular (p,q)-atom A on $\partial\Omega$, i.e. a function $A \in L^q_1(\partial\Omega)$ satisfying $\mathrm{supp}\, A \subseteq S_r$, for some $r > 0$, and $\|\nabla_{tan} A\|_{L^q(\partial\Omega)} \leq r^{(n-1)\left(\frac{1}{q}-\frac{1}{p}\right)}$. In particular, Poincaré's inequality gives $\|A\|_{L^q(\partial\Omega)} \leq Cr\|\nabla_{tan} A\|_{L^q(\partial\Omega)} \leq C\, r^{1+(n-1)\left(\frac{1}{q}-\frac{1}{p}\right)}$. Next, introduce $\tilde{r} := \min\{r, 1\} > 0$ and note that $\mathrm{supp}\,(\psi\, A) \subseteq S_{\tilde{r}}$. Going further, write $\nabla_{tan}(\psi\, A) = \psi\, \nabla_{tan} A + (\nabla_{tan}\psi)A =: I + II$, and use Hölder's inequality in order to estimate

$$\begin{aligned} \|I\|_{L^{p_o}(\partial\Omega)} &\leq \|\psi\|_{L^\infty(\partial\Omega)} \|\nabla_{tan} A\|_{L^{p_o}(S_{\tilde{r}})} \leq C\tilde{r}^{(n-1)\left(\frac{1}{p_o}-\frac{1}{q}\right)} \|\nabla_{tan} A\|_{L^q(\partial\Omega)} \\ &\leq C\tilde{r}^{(n-1)\left(\frac{1}{p_o}-\frac{1}{q}\right)} r^{(n-1)\left(\frac{1}{q}-\frac{1}{p}\right)} \leq C\tilde{r}^{(n-1)\left(\frac{1}{p_o}-\frac{1}{p}\right)} \end{aligned} \tag{2.63}$$

and

$$\begin{aligned} \|II\|_{L^{p_o}(\partial\Omega)} &\leq \|\nabla_{tan}\psi\|_{L^\infty(\partial\Omega)} \|A\|_{L^{p_o}(S_{\tilde{r}})} \leq C\tilde{r}^{(n-1)\left(\frac{1}{p_o}-\frac{1}{q}\right)} \|A\|_{L^q(\partial\Omega)} \\ &\leq C\tilde{r}^{(n-1)\left(\frac{1}{p_o}-\frac{1}{q}\right)} r^{1+(n-1)\left(\frac{1}{q}-\frac{1}{p}\right)} \leq C\tilde{r}^{(n-1)\left(\frac{1}{p_o}-\frac{1}{p}\right)}. \end{aligned} \tag{2.64}$$

It is only in the last step above that $p_o \geq p^*$ is needed (when r is large). Altogether, the estimates (2.63)-(2.64) give $\|\nabla_{tan}(\psi\, A)\|_{L^{p_o}(\partial\Omega)} \leq C\tilde{r}^{(n-1)\left(\frac{1}{p_o}-\frac{1}{p}\right)}$, so $C^{-1}\psi\, A$ is a regular (p, p_o)-atom. $\square$

We can now formally state the following.

Lemma 2.7 *Let Ω be as before, and assume that ψ is a Lipschitz function, compactly supported on $\partial\Omega$. Then for every $p \in (\frac{n-1}{n}, 1]$*

$$f \in \widetilde{H}^{1,p}_{at}(\partial\Omega) \Longrightarrow \psi f \in \widetilde{H}^{1,p}_{at}(\partial\Omega), \tag{2.65}$$

plus a naturally accompanying estimate.

Proof. This is a direct consequence of Lemma 2.6. $\square$

The spaces $H^p_{at}(\partial\Omega)$ and $H^{1,p}_{at}(\partial\Omega)$ have inhomogeneous counterparts, denoted by $h^p_{at}(\partial\Omega)$ and $h^{1,p}_{at}(\partial\Omega)$, respectively. To be precise, fix a graph Lipschitz domain $\Omega \subset \mathbb{R}^n$ as in (2.1) and assume that $\frac{n-1}{n} < p \leq 1 < p_o \leq \infty$. Also, fix a threshold $\eta > 0$. Call a function $a \in L^1_{loc}(\partial\Omega)$ an *inhomogeneous* (p, p_o)-*atom* if for some surface ball $S_r \subseteq \partial\Omega$

$$\begin{gathered} \operatorname{supp} a \subseteq S_r, \quad \|a\|_{L^{p_o}(\partial\Omega)} \leq r^{(n-1)\left(\frac{1}{p_o}-\frac{1}{p}\right)}, \text{ and} \\ \text{either } r = \eta, \quad \text{or } r < \eta \text{ and } \int_{\partial\Omega} a\, d\sigma = 0. \end{gathered} \tag{2.66}$$

We then define $h^p_{at}(\partial\Omega)$ as the ℓ^p-span of inhomogeneous (p, p_o)-atoms and equip it with the natural infimum-type quasi-norm. One can check that this is a "local" quasi-Banach space, in the sense that

$$h^p_{at}(\partial\Omega) \text{ is a module over } C^\alpha(\partial\Omega) \text{ for any } \alpha > (n-1)\Big(\tfrac{1}{p} - 1\Big). \tag{2.67}$$

Different choices of the parameters p_o, η lead to equivalent quasi-norms and

$$\Big(h^p_{at}(\partial\Omega)\Big)^* = C^{(n-1)\left(\frac{1}{p}-1\right)}(\partial\Omega). \tag{2.68}$$

It is also useful to note that

$$L^q_{comp}(\partial\Omega) \subset h^p_{at}(\partial\Omega), \quad \text{whenever } \tfrac{n-1}{n} < p \leq 1, \ \ q > 1. \tag{2.69}$$

Furthermore, for each $p \in (\frac{n-1}{n}, 1]$,

$$f \in h^p_{at}(\partial\Omega) \Longleftrightarrow f(\cdot, \varphi(\cdot))\sqrt{1 + |\nabla\varphi(\cdot)|^2} \in h^p_{at}(\mathbb{R}^{n-1}) = F^{p,2}_0(\mathbb{R}^{n-1}), \tag{2.70}$$

in analogy with the case of homogeneous Hardy spaces. This characterizations shows that as far as the space $h^p_{at}(\partial\Omega)$ is concerned, the particular values of the parameters p_o and η (used in the normalization and support size of atoms) are immaterial.

Lemma 2.8 *If Ω is as in (2.1), then*

$$H^p_{at}(\partial\Omega) \hookrightarrow h^p_{at}(\partial\Omega), \qquad \forall\, p \in (\tfrac{n-1}{n}, 1]. \tag{2.71}$$

Proof. Of course, in the definitions of the various types of atoms discussed above, we could have replaced "surface balls" with "surface cubes" (i.e., subsets of $\partial\Omega$ which, in graph coordinates, project onto genuine $(n-1)$-dimensional cubes whose sides are parallel to the coordinate axes in $\mathbb{R}^{n-1}$).

It suffices to show that there exists a finite constant $C > 0$ with the property that each (p, ∞)-atom $a : \partial\Omega \to \mathbb{R}$ supported in a surface cube Q of side-length $r \geq \eta$ has $\|a\|_{h^p_{at}(\partial\Omega)} \leq C$. To see this, pick $N \in \mathbb{N}$ such that $\eta 2^{N-1} < r \leq \eta 2^N$ and cover Q with $2^{N(n-1)}$ surface cubes Q_j of side-length comparable with η. Then

$$a = \sum_{j=1}^{2^{N(n-1)}} \lambda_j b_j, \qquad \text{where } \lambda_j := \left(\tfrac{r}{\eta}\right)^{-\frac{n-1}{p}} \quad \text{and} \quad b_j := \left(\tfrac{r}{\eta}\right)^{\frac{n-1}{p}} a\chi_{Q_j}. \tag{2.72}$$

Then $\operatorname{supp} b_j \subseteq Q_j$, $\|b_j\|_{L^\infty(\partial\Omega)} \leq \eta^{-\frac{n-1}{p}}$, and $\sum_{j=1}^{2^{N(n-1)}} |\lambda_j|^p \leq 2^{N(n-1)}(r/\eta)^{-(n-1)} \leq 2^{n-1}$. The desired conclusion follows. $\square$

With Ω, p, p_o as before and $\eta > 0$ arbitrary, we next define

$$h^{1,p}_{at}(\partial\Omega) := \Big\{f \in \mathrm{Lip}_c(\partial\Omega)' : f = \sum_j \lambda_j a_j,\ (\lambda_j)_j \in \ell^p \text{ and } a_j \text{ regular } (p,p_o)\text{-atom}$$

$$\text{supported in a surface ball of radius } \leq \eta \text{ for every } j\Big\}, \tag{2.73}$$

where the series converges in $\mathrm{Lip}_c(\partial\Omega)'$, and equip it with the natural infimum quasi-norm. Next, if p^* is as in (2.48) then, by Poincaré's inequality,

$$a \text{ regular } (p,p_o)\text{-atom} \Longrightarrow \|a\|_{L^{p^*}(\partial\Omega)} \leq C(\partial\Omega,p,p_o), \tag{2.74}$$

$$\left.\begin{array}{l} a \text{ regular } (p,p_o)\text{-atom supported} \\ \text{in a surface ball of radius } \leq \eta \end{array}\right\} \Longrightarrow \|a\|_{L^{p}(\partial\Omega)} \leq C(\partial\Omega,\eta,p,p_o). \tag{2.75}$$

Thus, if $f = \sum_{j=1}^\infty \lambda_j a_j$ is an atomic decomposition of $f \in h^{1,p}_{at}(\partial\Omega)$, it follows that the series $\sum_{j=1}^\infty \lambda_j a_j$ converges both in $L^{p^*}(\partial\Omega)$ and $L^p(\partial\Omega)$. As a consequence,

$$h^{1,p}_{at}(\partial\Omega) \hookrightarrow L^p(\partial\Omega) \cap L^{p^*}(\partial\Omega) \tag{2.76}$$

and, hence,

$$h^{1,p}_{at}(\partial\Omega) \hookrightarrow \widetilde{H}^{1,p}_{at}(\partial\Omega) \hookrightarrow L^{p^*}(\partial\Omega) \tag{2.77}$$

boundedly, for each $p \in (\frac{n-1}{n},1]$. In particular,

$$\|f\|_{L^{p^*}(\partial\Omega)} \leq C\|f\|_{\widetilde{H}^{1,p}_{at}(\partial\Omega)}, \qquad \text{uniformly for } f \in \widetilde{H}^{1,p}_{at}(\partial\Omega). \tag{2.78}$$

Let us also record here the fact that, if $\frac{n-1}{n} < p \leq 1$, we have

$$f \in h^{1,p}_{at}(\partial\Omega) \Longleftrightarrow f(\cdot,\varphi(\cdot)) \in F^{p,2}_1(\mathbb{R}^{n-1}). \tag{2.79}$$

In particular, various choices of the parameters p_o, η in (2.73) yield the same vector space and topology on $h^{1,p}_{at}(\partial\Omega)$. The equivalence (2.79) also shows that the space $h^{1,p}_{at}(\partial\Omega)$, $p \in (\frac{n-1}{n},1]$, is local, in the sense that for every function $\psi \in \mathrm{Lip}_c(\partial\Omega)$, we have

$$f \in h^{1,p}_{at}(\partial\Omega) \Longrightarrow \psi f \in h^{1,p}_{at}(\partial\Omega), \tag{2.80}$$

plus a natural estimate.

The fact that $F^{p,2}_1(\mathbb{R}^{n-1}) = \{f \in L^p(\mathbb{R}^{n-1}) \cap \mathcal{S}'(\mathbb{R}^{n-1}) : [f] \in \dot{F}^{p,2}_1(\mathbb{R}^{n-1})\}$ for $\frac{n-1}{n} < p \leq 1$ yields another alternative characterization of $h^{1,p}_{at}(\partial\Omega)$, namely

$$h^{1,p}_{at}(\partial\Omega) = \Big\{ f \in L^1_{loc}(\partial\Omega) : \ f \in L^p(\partial\Omega) \text{ and } \partial_{\tau_{jn}} f \in H^p_{at}(\partial\Omega), \ 1 \leq j \leq n-1 \Big\}, \tag{2.81}$$

and moreover,

$$\|f\|_{h^{1,p}_{at}(\partial\Omega)} \approx \|f\|_{L^p(\partial\Omega)} + \sum_{j=1}^{n-1} \|\partial_{\tau_{jn}} f\|_{H^p_{at}(\partial\Omega)}. \tag{2.82}$$

Let us also note here that if Ω is as in (2.1) and $\frac{n-1}{n} < p \leq 1$, then for each $j \in \{1, ..., n-1\}$,

$$\partial_{\tau_{jn}} \ : \ h^{1,p}_{at}(\partial\Omega) \longrightarrow H^p_{at}(\partial\Omega) \quad \text{ boundedly.} \tag{2.83}$$

Indeed, this is implicit in (2.81)-(2.82).

We conclude this section by recording an elementary yet useful result.

Lemma 2.9 *Let Σ be the graph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ with $\varphi(0) = 0$ and fix two functions $\xi \in C^\infty_0(B(0,1))$, $\zeta \in C^\infty_0(B(0,4))$, with $\zeta \equiv 1$ on $B(0,2)$. Also, assume that $k : \Sigma \times \Sigma \setminus \text{diag} \to \mathbb{R}$ is such that*

$$|k(x,y)| \leq \kappa |x-y|^{-(n-1)}, \quad |\nabla_x k(x,y)| \leq \kappa |x-y|^{-n}, \quad \forall\, (x,y) \in \Sigma \times \Sigma \setminus \text{diag}, \tag{2.84}$$

and set

$$\mathcal{T} f(x) := \int_\Sigma (1 - \zeta(x)) k(x,y) \xi(y) f(y)\, d\sigma(y), \qquad x \in \Sigma. \tag{2.85}$$

Then for every $j, k \in \{1, ..., n\}$, $p \in (\frac{n-1}{n}, 1]$ and $q \in (1, \infty)$, the operator

$$\partial_{\tau_{jk}}\mathcal{T} : L^q(\partial\Omega) \longrightarrow H^p_{at}(\Sigma) \tag{2.86}$$

is well-defined, linear and bounded.

Proof. Let $\psi \in C_0^\infty(B(0,3/2))$ be such that $0 \leq \psi \leq 1$ and $\psi \equiv 1$ on $B(0,1)$. Set $\psi_0(x) := \psi(x)$, $\psi_1(x) := \psi(x/2) - \psi(x)$ and $\psi_i(x) := \psi_1(2^{-i+1}x)$ for $i = 2, 3....$ Then ψ_i is supported in the annulus $\Delta_i := \{x \in \mathbb{R}^n : 2^{i-1} \leq |x| \leq 2^{i+1}\}$ and $\sum_{i=0}^N \psi_i(x) = \psi(2^{-N}x)$ for $N = 0, 1, ...$ In particular, $\sum_{i=0}^\infty \psi_i(x) = 1$. Next, note that if $\|f\|_{L^q(\Sigma)} \leq 1$ then $|\mathcal{T}f(x)| \leq C2^{-i(n-1)}$ and $\partial_{\tau_{jk}}|\mathcal{T}f(x)| \leq C2^{-in}$ on $\Delta_i \cap \Sigma$. For $i = 0, 1, ...$, we now set $a_i := 2^{(i+1)[n-(n-1)/p]}\partial_{\tau_{jk}}[\psi_i\mathcal{T}f]$, $\lambda_i := 2^{-(i+1)[n-(n-1)/p]}$. Then $\operatorname{supp} a_i \subset B(0, 2^{i+1}) \cap \Sigma$, $\|a_i\|_{L^\infty(\Sigma)} \leq C \cdot 2^{-(i+1)(n-1)/p}$ and $\int_\Sigma a_i \, d\sigma = 0$. Consequently, each a_i is a fixed multiple of a (p, ∞)-atom on Σ. Furthermore, $\sum_{i=0}^\infty \lambda_i^p < \infty$ by our assumptions on p. Since $\partial_{\tau_{jk}}[\mathcal{T}f] = \sum_{i=0}^\infty \lambda_i a_i$, it follows that $\partial_{\tau_{jk}}[\mathcal{T}f] \in H^p_{at}(\Sigma)$ and $\|\partial_{\tau_{jk}}[\mathcal{T}f]\|_{H^p_{at}(\Sigma)} \leq C$. This finishes the proof of the lemma. $\square$

2.3 Bounded Lipschitz domains

Call an open set $\Omega \subset \mathbb{R}^n$ a *bounded Lipschitz domain* if there exist $M > 0$ and a family of hyper-planes Π_i, $i = 1, ..., m$, a choice of the unit normal N_i to Π_i, and a function $\varphi_i : \Pi_i \to \mathbb{R}$ with $|\varphi_i(x) - \varphi_i(y)| \leq M|x - y|$ for all $x, y \in \Pi_i$, which also satisfy the following additional properties. First, for each i, in the system of coordinates induced by (Π_i, N_i) in $\mathbb{R}^n$, there exists an open, upright, doubly truncated, circular cylinder Z_i such that $\{Z_i\}_{i=1}^m$ covers $\partial\Omega$. Second, if Ω_i is the domain lying above the graph of φ_i, once again considered in the system of coordinates induced by (Π_i, N_i) in $\mathbb{R}^n$, and if tZ_i denotes the concentric dilation of Z_i by factor $t > 0$ then for each i,

$$\begin{aligned} \Omega \cap 2(M+1)Z_i &= \Omega_i \cap 2(M+1)Z_i, \\ \partial\Omega \cap 2(M+1)Z_i &= \partial\Omega_i \cap 2(M+1)Z_i. \end{aligned} \tag{2.87}$$

In the sequel, we shall call (Z_i, φ_i) a *coordinate chart* for Ω and refer to $\partial\Omega_i$ as the graph of φ_i in the system of coordinates induced by Z_i. Also, a constant is said to depend on the *Lipschitz character of* Ω if its size is controlled in terms of m, the number of cylinders $\{Z_i\}_i$, the size of these cylinders and the constant M.

Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, set $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^n \setminus \bar{\Omega}$. The nontangential approach regions $\Gamma^{\pm}_{\kappa}(x)$, $x \in \partial\Omega$, are defined as $\Gamma^{\pm}_{\kappa}(x) := \{y \in \Omega_{\pm} : |x-y| < (1+\kappa)\,\mathrm{dist}\,(y, \partial\Omega)\}$, where $\kappa > 0$ is a fixed parameter, while at every boundary point the nontangential maximal function is given by $M(u)(x) := \sup\{|u(y)| : y \in \Gamma^{\pm}_{\kappa}(x)\}$ (with the choice of sign depending on whether u is defined in Ω_+ or Ω_-).

For a bounded Lipschitz domain, the spaces $L^p(\partial\Omega)$ and $L^p_1(\partial\Omega)$ when $1 < p < \infty$, as well as $H^p_{at}(\partial\Omega)$, $\widetilde{H}^{1,p}_{at}(\partial\Omega)$, $h^p_{at}(\partial\Omega)$ and $h^{1,p}_{at}(\partial\Omega)$ when $p \in (\frac{n-1}{n}, 1]$, can be defined as before. As a consequence, when $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and $\frac{n-1}{n} < p \leq 1$, we have:

$$\begin{aligned}
&h^p_{at}(\partial\Omega) = H^p_{at}(\partial\Omega) + \mathbb{R} = H^p_{at}(\partial\Omega) + L^q(\partial\Omega) \ \text{ for each } \ q > 1, \\
&h^p_{at}(\partial\Omega) \hookrightarrow L^{p^*}_{-1}(\partial\Omega), \qquad \text{where } p^* \text{ is as in (2.48)}, \\
&L^q_1(\partial\Omega) \hookrightarrow h^{1,p}_{at}(\partial\Omega) = \widetilde{H}^{1,p}_{at}(\partial\Omega) \hookrightarrow L^{p^*}(\partial\Omega), \quad \text{for each } \ q > 1, \\
&h^p_{at}(\partial\Omega), \quad h^{1,p}_{at}(\partial\Omega) \ \text{ are modules over } \ \mathrm{Lip}\,(\partial\Omega).
\end{aligned} \tag{2.88}$$

Next, we record a couple of technical results which will not enter the discussion until later on.

Lemma 2.10 *Assume that $\frac{n-1}{n} < p \leq 1$ and that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain. Also, fix a coordinate cylinder (Z, φ) and denote by Σ the graph of φ in the coordinate system induced by Z. Finally, let $\xi \in C^\infty_0(Z)$. Then there exists $C > 0$ such that*

$$\|\widetilde{\xi f}\|_{h^{1,p}_{at}(\partial\Omega)} \leq C\|f\|_{\widetilde{H}^{1,p}_{at}(\Sigma)}, \tag{2.89}$$

$$\|\widetilde{\xi f}\|_{h^{1,p}_{at}(\partial\Omega)} \leq C\|f\|_{h^{1,p}_{at}(\Sigma)}, \tag{2.90}$$

$$\|\widetilde{\xi f}\|_{\widetilde{H}^{1,p}_{at}(\Sigma)} \leq C\|\widetilde{\xi f}\|_{h^{1,p}_{at}(\Sigma)} \leq C\|f\|_{h^{1,p}_{at}(\partial\Omega)}, \tag{2.91}$$

where tilde denotes the extension by zero outside the support (naturally interpreted in each case).

Proof. Indeed, (2.89) is implied by Lemma 2.6, whereas (2.90) is a direct consequence of (2.62), and (2.91) follows from (2.77) and (2.62). □

In turn, the estimates (2.89)-(2.91) permit one to prove that many of the properties established for the scale $h^{1,p}_{at}(\partial\Omega)$ when Ω is a graph Lipschitz domain have natural counterparts in the setting of bounded Lipschitz domains. We continue by recording the analogue of (2.81) in the case when $\Omega \subset \mathbb{R}^n$ is a *bounded* Lipschitz domain.

Proposition 2.11 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, and assume that $\frac{n-1}{n} < p \leq 1$ and p^* is as in (2.48). Also, assume that $1 < q \leq p^*$. Then*

$$\begin{aligned} h^{1,p}_{at}(\partial\Omega) &= \Big\{ f \in L^{p^*}(\partial\Omega) : \partial_{\tau_{jk}} f \in H^p_{at}(\partial\Omega),\ 1 \leq j,k \leq n \Big\} \\ &= \Big\{ f \in L^{q}(\partial\Omega) : \partial_{\tau_{jk}} f \in h^p_{at}(\partial\Omega),\ 1 \leq j,k \leq n \Big\}, \end{aligned} \tag{2.92}$$

and in addition,

$$\|f\|_{h^{1,p}_{at}(\partial\Omega)} \approx \|f\|_{L^{p^*}(\partial\Omega)} + \sum_{j,k=1}^{n} \|\partial_{\tau_{jk}} f\|_{H^p_{at}(\partial\Omega)} \approx \|f\|_{L^q(\partial\Omega)} + \sum_{j,k=1}^{n} \|\partial_{\tau_{jk}} f\|_{h^p_{at}(\partial\Omega)}. \tag{2.93}$$

Proof. To get started, we claim that for each $j,k \in \{1,...,n\}$, the tangential derivative operator

$$\partial_{\tau_{jk}} : h^{1,p}_{at}(\partial\Omega) \longrightarrow H^p_{at}(\partial\Omega) \tag{2.94}$$

is well-defined, linear, and bounded. To prove this, fix $1 < p_o \leq \infty$ and observe that $\partial_{\tau_{jk}} a$ is a (p,p_o)-atom whenever a is a regular (p,p_o)-atom. It is therefore natural to try to define the operator (2.94) as

$$\partial_{\tau_{jk}} f := \sum_i \lambda_i \partial_{\tau_{jk}} a_i \quad \text{whenever} \quad f = \sum_i \lambda_i a_i \quad \text{in} \quad h^{1,p}_{at}(\partial\Omega). \tag{2.95}$$

Nonetheless, due to the redundancy in the atomic representations of functions in $h^{1,p}_{at}(\partial\Omega)$ the above observation alone does not guarantee that this operator is well-defined. See, e.g., the discussion in [7]. In order to overcome this difficulty, it suffices to show that if $\{\lambda_j\}_j \in \ell^p$ and a_j, $j \in \mathbb{N}$ are (p, p_o)-regular atoms, then

$$\sum_i \lambda_i a_i = 0 \quad \text{in} \quad h^{1,p}_{at}(\partial\Omega) \quad \Longrightarrow \quad \sum_i \lambda_i \partial_{\tau_{jk}} a_i = 0 \quad \text{in} \quad h^p_{at}(\partial\Omega). \tag{2.96}$$

This, however, is a consequence of (2.76), the second line in (2.88), and (2.23). Hence, the operator (2.94) is well-defined and bounded.

Turning to (2.92), let us note that, thanks to (2.88) and (2.94), the three spaces are listed in increasing order. Hence, it suffices to show that if $f \in L^q(\partial\Omega)$ has $\partial_{\tau_{jk}} f \in h^p_{at}(\partial\Omega)$ for $1 \leq j, k \leq n$, then $f \in h^{1,p}_{at}(\partial\Omega)$. Note that all spaces involved are modules over $\mathrm{Lip}\,(\partial\Omega)$. Hence, using a smooth partition of unity, matters can be reduced to the case when $\partial\Omega$ is replaced by $\Sigma \subset \mathbb{R}^n$, the graph of a real-valued Lipschitz function defined in $\mathbb{R}^{n-1}$, and f is compactly supported on Σ. By further flattening Σ to $\mathbb{R}^{n-1}$ using a bi-Lipschitz change of variables, we arrive at the following question. Prove that if $f \in L^q_{comp}(\mathbb{R}^{n-1}) \hookrightarrow h^p_{at}(\mathbb{R}^{n-1})$ has $\partial_j f \in h^p_{at}(\mathbb{R}^{n-1})$ for every $j = 1, ..., n-1$, then $f \in F^{p,2}_1(\mathbb{R}^{n-1})$. However, since $h^p_{at}(\mathbb{R}^{n-1}) = F^{p,2}_0(\mathbb{R}^{n-1})$ for $\frac{n-1}{n} < p \leq 1$, this latter claim follows from well-known lifting results for Triebel-Lizorkin spaces (cf., e.g., Proposition 2 on p. 19 in [79]). Finally, the equivalences in (2.93) are implicit in the above reasoning. $\square$

In keeping with notation introduced in (2.36) and (2.40), if $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, we set

$$h^p(\partial\Omega) := \begin{cases} h^p_{at}(\partial\Omega) & \text{for } \frac{n-1}{n} < p \leq 1, \\ L^p(\partial\Omega) & \text{for } p > 1, \end{cases} \qquad h^p_1(\partial\Omega) := \begin{cases} h^{1,p}_{at}(\partial\Omega) & \text{for } \frac{n-1}{n} < p \leq 1, \\ L^p_1(\partial\Omega) & \text{for } p > 1. \end{cases} \tag{2.97}$$

Let us also point out that all these spaces have natural vector-valued versions, although we shall make no notational distinction between the scalar and the vector-valued case; each time, this should be clear from the context.

2.4 Besov and Triebel-Lizorkin spaces in Lipschitz domains

Given an arbitrary open subset Ω of $\mathbb{R}^n$, we denote by $f|_\Omega$ the restriction of a distribution f in $\mathbb{R}^n$ to Ω. For $0 < p, q \leq \infty$ and $s \in \mathbb{R}$ we then set

$$\begin{aligned} &B_s^{p,q}(\Omega) := \{f \text{ distribution in } \Omega : \exists\, g \in B_s^{p,q}(\mathbb{R}^n) \text{ such that } g|_\Omega = f\}, \\ &\|f\|_{B_s^{p,q}(\Omega)} := \inf\{\|g\|_{B_s^{p,q}(\mathbb{R}^n)} : g \in B_s^{p,q}(\mathbb{R}^n),\ g|_\Omega = f\}, \qquad f \in B_s^{p,q}(\Omega). \end{aligned} \tag{2.98}$$

A similar definition is given for $F_s^{p,q}(\Omega)$ in the case when $p < \infty$. From the corresponding density result in $\mathbb{R}^n$, it follows that for any bounded Lipschitz domain Ω and any $0 < p, q < \infty$, $s \in \mathbb{R}$,

$$C^\infty(\overline{\Omega}) \hookrightarrow B_s^{p,q}(\Omega) \cap F_s^{p,q}(\Omega) \quad \text{densely.} \tag{2.99}$$

The existence of a universal extension operator for Besov and Triebel-Lizorkin spaces in an arbitrary Lipschitz domain $\Omega \subset \mathbb{R}^n$ has been established by V. Rychkov in **[80]**. To state this result, let $\mathcal{R}_\Omega$ denote the operator of restriction to Ω, which maps distributions from $\mathbb{R}^n$ into distributions in Ω,

$$\mathcal{R}_\Omega(u) := u\Big|_\Omega, \qquad u \text{ distribution in } \mathbb{R}^n. \tag{2.100}$$

Theorem 2.12 ([80]) *Let $\Omega \subset \mathbb{R}^n$ be either a bounded Lipschitz domain, the exterior of a bounded Lipschitz domain, or an unbounded Lipschitz domain. Then there exists a linear, continuous operator E_Ω, mapping distributions in Ω into tempered distributions in $\mathbb{R}^n$, such that whenever $0 < p, q \leq +\infty$, $s \in \mathbb{R}^n$,*

$$E_\Omega : A_s^{p,q}(\Omega) \longrightarrow A_s^{p,q}(\mathbb{R}^n) \text{ boundedly, satisfying } \mathcal{R}_\Omega(E_\Omega f) = f,\ \forall\, f \in A_s^{p,q}(\Omega), \tag{2.101}$$

for $A = B$ or $A = F$, in the latter case assuming $p < \infty$.

This and standard properties of retractions allow one to establish interpolation results for Besov and Triebel-Lizorkin spaces in Lipschitz domains. More specifically, we have the following analogue of Theorems 11.1-11.2.

Theorem 2.13 *Suppose Ω is a bounded Lipschitz domain in $\mathbb{R}^n$. Let $\alpha_0, \alpha_1 \in \mathbb{R}$, $\alpha_0 \neq \alpha_1$, $0 < q_0, q_1, q \leq \infty$, $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$. Then*

$$
\begin{aligned}
(F^{p,q_0}_{\alpha_0}(\Omega), F^{p,q_1}_{\alpha_1}(\Omega))_{\theta,q} &= B^{p,q}_{\alpha}(\Omega), \quad 0 < p < \infty, & (2.102)\\
(B^{p,q_0}_{\alpha_0}(\Omega), B^{p,q_1}_{\alpha_1}(\Omega))_{\theta,q} &= B^{p,q}_{\alpha}(\Omega), \quad 0 < p \leq \infty. & (2.103)
\end{aligned}
$$

Furthermore, if $\alpha_0, \alpha_1 \in \mathbb{R}$, $0 < p_0, p_1 < \infty$ and $0 < q_0, q_1 \leq \infty$ satisfy $\min\{q_0, q_1\} < \infty$, then

$$
[F^{p_0,q_0}_{\alpha_0}(\Omega), F^{p_1,q_1}_{\alpha_1}(\Omega)]_{\theta} = F^{p,q}_{\alpha}(\Omega), \tag{2.104}
$$

where $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

If $\alpha_0, \alpha_1 \in \mathbb{R}$, $0 < p_0, p_1, q_0, q_1 \leq \infty$ and $\min\{q_0, q_1\} < \infty$, then also

$$
[B^{p_0,q_0}_{\alpha_0}(\Omega), B^{p_1,q_1}_{\alpha_1}(\Omega)]_{\theta} = B^{p,q}_{\alpha}(\Omega), \tag{2.105}
$$

where $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

Finally, the same interpolation results remain valid if the spaces $B^{p,q}_s(\Omega)$, $F^{p,q}_s(\Omega)$ are replaced by $B^{p,q}_{s,0}(\Omega)$ and $F^{p,q}_{s,0}(\Omega)$, respectively.

Recall now the standard L^p-based Sobolev spaces in a Lipschitz domain Ω:

$$
W^p_k(\Omega) := \Big\{ f \in L^p(\Omega); \quad \partial^\gamma f \in L^p(\Omega), \ \forall\, \gamma : |\gamma| \leq k \Big\}, \quad 1 < p < \infty, \ k \in \mathbb{N}_0, \tag{2.106}
$$

equipped with the norm

$$
\|f\|_{W^p_k(\Omega)} := \sum_{|\gamma| \leq k} \|\partial^\gamma f\|_{L^p(\Omega)}. \tag{2.107}
$$

In view of Theorem 2.12, for any Lipschitz domain Ω, we have

$$
W^p_k(\Omega) = F^{p,2}_k(\Omega), \quad 1 < p < \infty, \quad k \in \mathbb{N}_0. \tag{2.108}
$$

For $0 < p, q \leq \infty$, $s \in \mathbb{R}$, we set

$$
\begin{aligned}
&A^{p,q}_{s,0}(\Omega) := \{f \in A^{p,q}_s(\mathbb{R}^n) : \operatorname{supp} f \subseteq \overline{\Omega}\}, \\
&\|f\|_{A^{p,q}_{s,0}(\Omega)} := \|f\|_{A^{p,q}_s(\mathbb{R}^n)}, \quad f \in A^{p,q}_{s,0}(\Omega),
\end{aligned}
\tag{2.109}
$$

where, as usual, either $A = F$ and $p < \infty$ or $A = B$. Thus, $B^{p,q}_{s,0}(\Omega)$, $F^{p,q}_{s,0}(\Omega)$ are closed subspaces of $B^{p,q}_{s,0}(\mathbb{R}^n)$ and $F^{p,q}_{s,0}(\mathbb{R}^n)$, respectively. In the same vein, we also define

$$
L^p_{s,0}(\Omega) := \{f \in L^p_s(\mathbb{R}^n) : \operatorname{supp} f \subseteq \overline{\Omega}\}, \quad 1 < p < \infty, \ s \in \mathbb{R}, \tag{2.110}
$$

with the norms inherited from $L^p_{s,0}(\mathbb{R}^n)$.

For $0 < p, q \leq \infty$ and $s \in \mathbb{R}$, we also introduce

$$
\begin{aligned}
&A^{p,q}_{s,z}(\Omega) := \{f \text{ distribution in } \Omega : \exists\, g \in A^{p,q}_{s,0}(\Omega) \text{ with } g|_\Omega = f\}, \\
&\|f\|_{A^{p,q}_{s,z}(\Omega)} := \inf\{\|g\|_{A^{p,q}_s(\mathbb{R}^n)} : g \in A^{p,q}_{s,0}(\Omega),\ g|_\Omega = f\}, \qquad f \in A^{p,q}_{s,z}(\Omega),
\end{aligned}
\tag{2.111}
$$

(where, as before, $A = F$ and $p < \infty$ or $A = B$) and, in keeping with earlier conventions,

$$
L^p_{s,z}(\Omega) := F^{p,2}_{s,z}(\Omega) = \{f \text{ distribution in } \Omega : \exists\, g \in L^p_{s,0}(\Omega) \text{ with } g|_\Omega = f\}, \tag{2.112}
$$

if $1 < p < \infty$, $s \in \mathbb{R}$. For further use, let us also make the simple yet important observation that the operator of restriction to Ω induced linear, bounded mappings in the following settings

$$
\mathcal{R}_\Omega : A^{p,q}_s(\mathbb{R}^n) \longrightarrow A^{p,q}_s(\Omega) \text{ and } \mathcal{R}_\Omega : A^{p,q}_{s,0}(\mathbb{R}^n) \longrightarrow A^{p,q}_{s,z}(\Omega) \tag{2.113}
$$

for $0 < p, q \leq \infty$, $s \in \mathbb{R}$.

It follows that if Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ and $0 < p, q < \infty$, $s \in \mathbb{R}$, then

$$
\widetilde{C^\infty_0(\Omega)} \hookrightarrow A^{p,q}_{s,0}(\Omega) \text{ densely}, \tag{2.114}
$$

$$
C^\infty(\overline{\Omega}) \hookrightarrow A^{p,q}_s(\Omega) \text{ densely}, \tag{2.115}
$$

$$
C^\infty_0(\Omega) \hookrightarrow A^{p,q}_{s,z}(\Omega) \text{ densely}, \tag{2.116}
$$

where, as before, tilde denotes the extension by zero outside Ω and A stands for either B or F. Indeed, the same proof as in the Remark 2.7 on p. 170 of [46] gives (2.114) and a minor variation of it justifies (2.114) as well. Finally, (2.116) is a consequence of (2.114) and the fact that $\mathcal{R}_\Omega$ maps $A^{p,q}_{s,0}(\Omega)$ continuously onto $A^{p,q}_{s,z}(\Omega)$.

Proposition 2.14 ([91]) *Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, and suppose that $0 < p, q \leq \infty$ and $s > \max\Big(1/p - 1, n(1/p - 1)\Big)$. Then extension by zero defined as*

$$\tilde{f}(x) := \begin{cases} f(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega, \end{cases} \tag{2.117}$$

induces a linear and bounded operator from $B^{p,q}_{s,z}(\Omega)$ to $B^{p,q}_{s,0}(\Omega)$ and, if $p < \infty$, from $F^{p,q}_{s,z}(\Omega)$ to $F^{p,q}_{s,0}(\Omega)$. Furthermore, if $\max\Big(1/p - 1, n(1/p - 1)\Big) < s < 1/p$ and $0 < p, q < \infty$, this operator also maps $B^{p,q}_{s}(\Omega)$ to $B^{p,q}_{s,0}(\Omega)$ and, if $\min\{p, 1\} \leq q$, $F^{p,q}_{s}(\Omega)$ to $F^{p,q}_{s,0}(\Omega)$.

If $1 < p, q < \infty$ and $1/p + 1/p' = 1/q + 1/q' = 1$, then

$$\Big(A^{p,q}_{s,z}(\Omega)\Big)^* = A^{p',q'}_{-s}(\Omega) \quad \text{if } \ s > -1 + \tfrac{1}{p}, \tag{2.118}$$

$$\Big(A^{p,q}_{s}(\Omega)\Big)^* = A^{p',q'}_{-s,z}(\Omega) \quad \text{if } \ s < \tfrac{1}{p}. \tag{2.119}$$

Furthermore, for each $s \in \mathbb{R}$ and $1 < p, q < \infty$, the spaces $A^{p,q}_{s}(\Omega)$ and $A^{p,q}_{s,0}(\Omega)$ are reflexive. As a consequence of (2.118)-(2.119) let us also note the following useful result:

Proposition 2.15 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, and assume that $1 < p, q < \infty$, $1/p + 1/p' = 1/q + 1/q' = 1$. Then*

$$\Big(B^{p,q}_{s}(\Omega)\Big)^* = B^{p',q'}_{-s}(\Omega), \qquad \Big(F^{p,q}_{s}(\Omega)\Big)^* = F^{p',q'}_{-s}(\Omega), \tag{2.120}$$

provided $-1 + 1/p < s < 1/p$.

There is yet another type of smoothness space which will play a significant role for us. Specifically, for $\Omega \subset \mathbb{R}^n$ Lipschitz domain, we set

$$\overset{\circ}{A}{}^{p,q}_{s}(\Omega) := \text{ the closure of } C^\infty_0(\Omega) \text{ in } A^{p,q}_{s}(\Omega), \qquad 0 < p, q \leq \infty, \ s \in \mathbb{R}, \tag{2.121}$$

where, as usual, $A = F$ or $A = B$. For every $0 < p, q < \infty$ and $s \in \mathbb{R}$, we then have

$$A^{p,q}_{s,z}(\Omega) \hookrightarrow \overset{\circ}{A}{}^{p,q}_{s}(\Omega) \hookrightarrow A^{p,q}_{s}(\Omega), \qquad \text{continuously.} \tag{2.122}$$

The second inclusion is trivial from (2.121), whereas the first can be justified as follows. If $f \in A^{p,q}_{s,z}(\Omega)$, then there exists $u \in A^{p,q}_{s,0}(\Omega)$ such that $\mathcal{R}_\Omega(u) = f$. By (2.114), there exists a sequence $u_j \in C^\infty_0(\Omega)$ such that $\tilde{u}_j \to u$ in $A^{p,q}_s(\mathbb{R}^n)$, which then implies $u_j = \mathcal{R}_\Omega(\tilde{u}_j) \to \mathcal{R}_\Omega(u) = f$ in $A^{p,q}_s(\Omega)$. This proves that $f \in \overset{\circ}{A}{}^{p,q}_{s}(\Omega)$ and the desired conclusion follows easily from this.

Going further, Proposition 3.1 in **[91]** ensures that

$$\overset{\circ}{A}{}^{p,q}_{s}(\Omega) = A^{p,q}_{s}(\Omega) = A^{p,q}_{s,z}(\Omega), \qquad A \in \{F, B\}, \tag{2.123}$$

whenever $0 < p, q < \infty$, $\max\Big(1/p - 1, n(1/p-1)\Big) < s < 1/p$, and $\min\{p, 1\} \le q < \infty$ in the case $A = F$. Other cases of interest have been considered in [64], from which we quote the following result.

Proposition 2.16 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then*

$$\overset{\circ}{F}{}^{p,q}_{s}(\Omega) = F^{p,q}_{s,z}(\Omega) \tag{2.124}$$

provided

$$\begin{aligned} &0 < p < \infty, \quad \min\{1, p\} \le q < \infty, \quad \textit{and} \\ &\exists\, k \in \mathbb{N}_0 \ \ \textit{so that} \ \ \max\Big(\tfrac{1}{p} - 1, n(\tfrac{1}{p} - 1)\Big) < s - k < \tfrac{1}{p}. \end{aligned} \tag{2.125}$$

Furthermore,

$$\overset{\circ}{B}{}^{p,q}_{s}(\Omega) = B^{p,q}_{s,z}(\Omega) \tag{2.126}$$

whenever

$$0 < p, q < \infty \ \ \textit{and} \ \exists\, k \in \mathbb{N}_0 \ \ \textit{so that} \ \ \max\Big(\tfrac{1}{p} - 1, n(\tfrac{1}{p} - 1)\Big) < s - k < \tfrac{1}{p}. \tag{2.127}$$

2.5 Smoothness spaces on Lipschitz boundaries

For $a \in \mathbb{R}$ set $(a)_+ := \max\{a, 0\}$. Consider three parameters p, q, s subject to

$$0 < p, q \leq \infty, \qquad (n-1)\Big(\tfrac{1}{p} - 1\Big)_+ < s < 1, \tag{2.128}$$

and assume that $\Omega \subset \mathbb{R}^n$ is the upper-graph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$. We then define $B_s^{p,q}(\partial\Omega)$ as the space of locally integrable functions f on $\partial\Omega$ for which the assignment $\mathbb{R}^{n-1} \ni x \mapsto f(x, \varphi(x))$ belongs to $B_s^{p,q}(\mathbb{R}^{n-1})$ (cf. § 11.1). We then define

$$\|f\|_{B_s^{p,q}(\partial\Omega)} := \|f(\cdot, \varphi(\cdot))\|_{B_s^{p,q}(\mathbb{R}^{n-1})}. \tag{2.129}$$

As far as Besov spaces with a negative amount of smoothness are concerned, in the same context as above we set

$$f \in B_{s-1}^{p,q}(\partial\Omega) \Longleftrightarrow f(\cdot, \varphi(\cdot))\sqrt{1 + |\nabla\varphi(\cdot)|^2} \in B_{s-1}^{p,q}(\mathbb{R}^{n-1}), \tag{2.130}$$

$$\|f\|_{B_{s-1}^{p,q}(\partial\Omega)} := \|f(\cdot, \varphi(\cdot))\sqrt{1 + |\nabla\varphi(\cdot)|^2}\|_{B_{s-1}^{p,q}(\mathbb{R}^{n-1})}. \tag{2.131}$$

As is well-known, the case when $p = q = \infty$ corresponds to the usual (non-homogeneous) Hölder spaces $C^s(\partial\Omega)$, defined by the requirement that

$$\|f\|_{C^s(\partial\Omega)} := \|f\|_{L^\infty(\partial\Omega)} + \sup_{\substack{x \neq y \\ x,y \in \partial\Omega}} \frac{|f(x) - f(y)|}{|x - y|^s} < +\infty. \tag{2.132}$$

All the above definitions then readily extend to the case of (bounded) Lipschitz domains in $\mathbb{R}^n$ via a standard partition of unity argument.

We now recall several properties of the Besov scales just introduced above which are going to be of importance for us later on.

Proposition 2.17 *For $(n-1)/n < p < \infty$ and $(n-1)(1/p - 1)_+ < s < 1$,*

$$\|f\|_{B_s^{p,p}(\partial\Omega)} \approx \|f\|_{L^p(\partial\Omega)} + \left(\int_{\partial\Omega}\int_{\partial\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^{n-1+sp}}\, d\sigma(x)\, d\sigma(y)\right)^{1/p}. \tag{2.133}$$

See [64] for a proof of the equivalence (2.133).

Theorem 2.18 ([64]) *Let Ω be a Lipschitz domain in $\mathbb{R}^n$ and assume that the indices p and s satisfy $\frac{n-1}{n} < p \le \infty$ and $(n-1)(\frac{1}{p}-1)_+ < s < 1$. Then the following hold:*

(i) The restriction to the boundary extends to a linear, bounded operator

$$\mathrm{Tr} : B^{p,q}_{s+\frac{1}{p}}(\Omega) \longrightarrow B^{p,q}_s(\partial\Omega) \quad \text{for} \quad 0 < q \le \infty. \tag{2.134}$$

Moreover, for this range of indices, Tr *is onto and has a bounded right inverse*

$$\mathrm{Ex} : B^{p,q}_s(\partial\Omega) \longrightarrow B^{p,q}_{s+\frac{1}{p}}(\Omega). \tag{2.135}$$

(ii) If $p \neq \infty$, then similar considerations hold for

$$\mathrm{Tr} : F^{p,q}_{s+\frac{1}{p}}(\Omega) \longrightarrow B^{p,p}_s(\partial\Omega). \tag{2.136}$$

In particular, the operator (2.136) has a linear, bounded right inverse

$$\mathrm{Ex} : B^{p,p}_s(\partial\Omega) \longrightarrow F^{p,q}_{s+\frac{1}{p}}(\Omega). \tag{2.137}$$

Theorem 2.19 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and assume that $\frac{n-1}{n} < p < \infty$, $(n-1)(1/p-1)_+ < s < 1$ and $\min\{1,p\} \le q < \infty$. Then*

$$F^{p,q}_{s+1/p,z}(\Omega) = \{f \in F^{p,q}_{s+1/p}(\Omega) : \mathrm{Tr}\, f = 0\} \tag{2.138}$$

and

$$C^\infty_c(\Omega) \hookrightarrow F^{p,q}_{s+1/p,z}(\Omega) \quad \text{densely.} \tag{2.139}$$

Furthermore, a similar result is valid for the scale of Besov spaces. More specifically, if $\frac{n-1}{n} < p < \infty$, $(n-1)(1/p-1)_+ < s < 1$ and $0 < q < \infty$, then

$$B^{p,q}_{s+1/p,z}(\Omega) = \{f \in B^{p,q}_{s+1/p}(\Omega) : \mathrm{Tr}\, f = 0\} \tag{2.140}$$

and

$$C_c^\infty(\Omega) \hookrightarrow B^{p,q}_{s+1/p,z}(\Omega) \quad \textit{densely.} \tag{2.141}$$

Proposition 2.20 *Suppose that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$. Furthermore, assume that $0 < p, q, q_0, q_1 \le \infty$ and that*

$$\begin{aligned} &\textit{either} \;\; (n-1)\Big(\tfrac{1}{p}-1\Big)_+ < s_0 \neq s_1 < 1, \\ &\textit{or} \;\; -1+(n-1)\Big(\tfrac{1}{p}-1\Big)_+ < s_0 \neq s_1 < 0. \end{aligned} \tag{2.142}$$

Then, with $0 < \theta < 1$, $s = (1-\theta)s_0 + \theta s_1$,

$$(B^{p,q_0}_{s_0}(\partial\Omega), B^{p,q_1}_{s_1}(\partial\Omega))_{\theta,q} = B^{p,q}_s(\partial\Omega). \tag{2.143}$$

Furthermore, if $0 < p_i, q_i \le \infty$ are such that $\min\{q_0, q_1\} < \infty$ and either one of the following two conditions

$$\begin{aligned} &\textit{either} \;\; (n-1)\Big(\tfrac{1}{p_i}-1\Big)_+ < s_i < 1, \;\; i = 0,1, \\ &\textit{or} \;\; -1+(n-1)\Big(\tfrac{1}{p_i}-1\Big)_+ < s_i < 0, \;\; i = 0,1, \end{aligned} \tag{2.144}$$

is satisfied then

$$[B^{p_0,q_0}_{s_0}(\partial\Omega), B^{p_1,q_1}_{s_1}(\partial\Omega)]_\theta = B^{p,q}_s(\partial\Omega), \tag{2.145}$$

where

$$0 < \theta < 1, \quad s := (1-\theta)s_0 + \theta s_1, \quad \tfrac{1}{p} := \tfrac{1-\theta}{p_0} + \tfrac{\theta}{p_1} \; \textit{and} \; \tfrac{1}{q} := \tfrac{1-\theta}{q_0} + \tfrac{\theta}{q_1}. \tag{2.146}$$

Proposition 2.21 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and fix $(n-1)/n < p < \infty$, $0 < q \le \infty$, and $(n-1)(\frac{1}{p}-1)_+ < s < 1$. Then, for each $j, k \in \{1, ..., n\}$, the tangential derivative operator*

$$\partial_{\tau_{jk}} : B^{p,q}_s(\partial\Omega) \longrightarrow B^{p,q}_{s-1}(\partial\Omega) \tag{2.147}$$

is well-defined, linear, and bounded.

Next, we discuss an atomic decomposition result for the space $B^{p,p}_{s-1}(\partial\Omega)$ when $(n-1)/n < p < \infty$ and $(n-1)(\frac{1}{p}-1)_+ < s < 1$. For a given, fixed parameter $\eta = \eta(\partial\Omega) > 0$, call $a_S \in L^\infty(\partial\Omega)$ an atom for $B^{p,p}_{s-1}(\partial\Omega)$ if

$$(1)\ \exists\, S = S_r, \text{ surface ball, such that } \operatorname{supp}(a_S) \subseteq S, \tag{2.148}$$

$$(2)\ \|a_S\|_{L^\infty(\partial\Omega)} \leq r^{s-1-\frac{n-1}{p}}, \tag{2.149}$$

$$(3)\ \int_{\partial\Omega} a_S(x)\, d\sigma(x) = 0 \ \text{ when } r < \eta. \tag{2.150}$$

We have:

Proposition 2.22 *For any bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ there exists $\eta = \eta(\partial\Omega) > 0$ such that the following is true. If $(n-1)/n < p \leq 1$ and $(n-1)(\frac{1}{p}-1) < s < 1$ then*

$$\begin{aligned} \|f\|_{B^{p,p}_{s-1}(\partial\Omega)} \ \approx\ & \inf\Big\{\Big(\sum_S |\lambda_S|^p\Big)^{1/p} : \\ & f = \sum_S \lambda_S a_S,\ a_S \text{ are } B^{p,p}_{s-1}(\partial\Omega) \text{ atoms},\ \{\lambda_S\}_S \in \ell^p\Big\}, \end{aligned} \tag{2.151}$$

uniformly for $f \in B^{p,p}_{s-1}(\partial\Omega)$.

Lemma 2.23 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and assume that $k : \partial\Omega \times \partial\Omega \setminus \mathrm{diag} \to \mathbb{R}$ is such that*

$$|k(x,y)| \leq \kappa|x-y|^{-(n-1)}, \quad |\nabla_y k(x,y)| \leq \kappa|x-y|^{-n}, \quad \forall\,(x,y) \in \partial\Omega \times \partial\Omega \setminus \mathrm{diag}. \tag{2.152}$$

For a fixed function $\xi \in C^\infty_0(\mathbb{R}^n)$ set $\tilde{k}(x,y) := [\xi(x) - \xi(y)]k(x,y)$ and introduce

$$\mathcal{C}f(x) := \int_{\partial\Omega} \tilde{k}(x,y) f(y)\, d\sigma(y), \qquad x \in \partial\Omega. \tag{2.153}$$

Then for every $s \in (0,1)$ and $q \in (1,\infty)$, the operator

$$\mathcal{C} : B^{q,q}_{-s}(\partial\Omega) \longrightarrow L^q(\partial\Omega) \tag{2.154}$$

is well-defined, linear, and bounded.

Proof. Consider first the case of (2.154) when $q = 1$. Our goal is to show that there exists $C > 0$ such that

$$\|\mathcal{C}a\|_{L^1(\partial\Omega)} \leq C \tag{2.155}$$

for every $B^{1,1}_{-s}(\partial\Omega)$-atom a. Recall the parameter η from Proposition 2.22 and note that if a is an atom supported in a surface ball of radius $\geq \eta$ then $\|a\|_{L^1(\partial\Omega)} \leq C(\eta, \partial\Omega) < \infty$. Thus, (2.155) holds in this case since $\mathcal{C}$ maps $L^1(\partial\Omega)$ boundedly into itself. When a is a $B^{1,1}_{-s}(\partial\Omega)$-atom supported in a surface ball $S_r(x_o)$ with $x_o \in \partial\Omega$ and $0 < r \leq \eta$, it is elementary to establish that

$$\int_{S_{2r}(x_o)} |\mathcal{C}a(x)|\, d\sigma(x) \leq Cr^{1-s} \leq C \quad \text{and} \quad \int_{\partial\Omega\setminus S_{2r}(x_o)} |\mathcal{C}a(x)|\, d\sigma(x) \leq Cr^{1-s} \ln r \leq C \tag{2.156}$$

for some finite $C = C(\partial\Omega, \eta, \kappa) > 0$. From this, (2.155) follows. Hence, (2.154) holds when $q = 1$. Since, by Schur's lemma, $\mathcal{C}$ maps $L^p(\partial\Omega)$ boundedly into itself whenever $1 < p < \infty$, the claim about (2.154) follows in its full generality from what we have just proved and interpolation. $\square$

We shall now briefly discuss the Triebel-Lizorkin spaces on the boundary of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, denoted in the sequel by $F^{p,q}_s(\partial\Omega)$. Compared with the Besov scale, the most important novel aspect here is the possibility of allowing the endpoint case $s = 1$ as part of the general discussion if $q = 2$. To discuss this in more detail, assume that

$$\begin{aligned} &\text{either} \;\; 0 < p < \infty, \quad 0 < q \leq \infty, \quad (n-1)\Big(\tfrac{1}{\min\{p,q\}} - 1\Big)_+ < s < 1, \\ &\text{or} \quad \tfrac{n-1}{n} < p < \infty, \quad q = 2 \quad \text{and} \quad s = 1. \end{aligned} \tag{2.157}$$

The starting point in introducing Triebel-Lizorkin spaces on $\partial\Omega$ is the case when Ω is the domain lying above the graph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$. In this case, if (p, q, s) are as in (2.157), we define $F^{p,q}_s(\partial\Omega)$ as the collection of all locally integrable functions on $\partial\Omega$ such that

$$\|f\|_{F^{p,q}_s(\partial\Omega)} := \|f(\cdot, \varphi(\cdot))\|_{F^{p,q}_s(\mathbb{R}^{n-1})} < +\infty, \tag{2.158}$$

and $F^{p,q}_{s-1}(\partial\Omega)$ is defined as the collection of all functionals $f \in (\mathrm{Lip}_c(\partial\Omega))'$ such that

$$\|f\|_{F^{p,q}_{s-1}(\partial\Omega)} := \|f(\cdot,\varphi(\cdot))\sqrt{1+|\nabla\varphi(\cdot)|^2}\|_{F^{p,q}_{s-1}(\mathbb{R}^{n-1})} < +\infty. \tag{2.159}$$

When (p,q,s) are as in (2.157), the Triebel-Lizorkin scale in $\mathbb{R}^{n-1}$ is invariant under pointwise multiplication by Lipschitz maps as well as composition by Lipschitz diffeomorphisms. In turn, this can be used to define $F^{p,q}_s(\partial\Omega)$ and $F^{p,q}_{s-1}(\partial\Omega)$ when Ω is a bounded Lipschitz domain, via a standard partition of unity argument.

Some basic properties of the spaces just introduced are as follows. First,

$$F^{p,2}_0(\partial\Omega) = h^p(\partial\Omega), \qquad F^{p,2}_1(\partial\Omega) = h^p_1(\partial\Omega), \qquad \tfrac{n-1}{n} < p < \infty, \tag{2.160}$$

where $h^p(\partial\Omega)$, $h^p_1(\partial\Omega)$ have been introduced in (2.97). Second,

Proposition 2.24 *Let Ω be an arbitrary bounded Lipschitz domain in $\mathbb{R}^n$. Assume that the indices $s, s_0, s_1, p, p_0, q_0, q, p_1, q_1, \theta$ are as in (2.146) and each of the two triplets (p_0, q_0, s_0) and (p_1, q_1, s_1) satisfies (2.157). If also $\min\{q_0, q_1\} < \infty$ then*

$$[F^{p_0,q_0}_{s_0}(\partial\Omega), F^{p_1,q_1}_{s_1}(\partial\Omega)]_\theta = F^{p,q}_s(\partial\Omega), \quad [F^{p_0,q_0}_{s_0-1}(\partial\Omega), F^{p_1,q_1}_{s_1-1}(\partial\Omega)]_\theta = F^{p,q}_{s-1}(\partial\Omega). \tag{2.161}$$

Finally, assume that each of the two triplets (p, q_0, s_0) and (p, q_1, s_1) satisfies (2.157) then

$$(F^{p,q_0}_{s_0}(\partial\Omega), F^{p,q_1}_{s_1}(\partial\Omega))_{\theta,q} = B^{p,q}_s(\partial\Omega), \quad (F^{p,q_0}_{s_0-1}(\partial\Omega), F^{p,q_1}_{s_1-1}(\partial\Omega))_{\theta,q} = B^{p,q}_{s-1}(\partial\Omega) \tag{2.162}$$

if $s_0 \neq s_1$, $0 < \theta < 1$, $s = (1-\theta)s_0 + \theta s_1$ and $0 < q \leq \infty$.

3 Rellich identities for divergence form, second-order systems

3.1 Green formulas

Let Ω be a domain in $\mathbb{R}^n$ and denote by $C^\infty(\bar{\Omega})$ the class of smooth, complex-valued functions defined in a neighborhood of $\bar{\Omega}$. Also, for two fixed nonnegative integers N, M, set $\mathcal{E} := [C^\infty(\bar{\Omega})]^N$, $\mathcal{F} := [C^\infty(\bar{\Omega})]^M$. In the sequel, we let $\langle u, v\rangle := \sum_{\beta=1}^N u_\beta v_\beta$ denote the pointwise inner product in $\mathcal{E}, \mathcal{F}$, etc. Note that this pairing does not involve any complex conjugation (i.e., is bilinear). Next let $D : \mathcal{E} \longrightarrow \mathcal{F}$ be the linear mapping given by

$$Du(x) = \Big(\sum_{|\gamma|\leq m} a_\gamma^{\alpha\beta}(x)\partial^\gamma u_\beta(x)\Big)_\alpha, \qquad u \in \mathcal{E},\, x \in \bar{\Omega}, \tag{3.1}$$

i.e. a differential operator of order m in Ω, with smooth, complex-valued coefficients in $\bar{\Omega}$, acting on vector-valued functions. Its formal transpose is then given by

$$D^\top : \mathcal{F} \longrightarrow \mathcal{E}, \quad D^\top v(x) := \Big(\sum_{|\gamma|\leq m} (-1)^{|\gamma|}\partial^\gamma[a_\gamma^{\alpha\beta}(x)v_\alpha(x)]\Big)_\beta, \qquad v \in \mathcal{F},\, x \in \bar{\Omega}. \tag{3.2}$$

If the superscript c denotes complex conjugation then D^*, the adjoint of D is

$$D^* : \mathcal{F} \longrightarrow \mathcal{E}, \quad D^* u := \Big[D^\top(u^c)\Big]^c. \tag{3.3}$$

In fact, if we set $D^c u := (Du^c)^c$ (i.e. conjugate the coefficients of D), then

$$D^* = (D^\top)^c = (D^c)^\top, \quad D^\top = (D^*)^c = (D^c)^*, \tag{3.4}$$

and adjunction, transposition and conjugation are all involutions.

Going further, recall that the principal symbol of (3.1) is the mapping

$$\sigma(D;\xi)u := \Big(i^m \sum_{|\gamma|=m} a_\gamma^{\alpha\beta}\xi^\gamma u_\beta\Big)_\alpha, \qquad \xi \in \mathbb{R}^n, \quad u \in \mathcal{E}, \tag{3.5}$$

where, throughout this section, $i := \sqrt{-1}$. It follows that, for each $\xi \in \mathbb{R}^n$ and each differential operator D of order m,

$$\begin{gathered}\sigma(D^c;\xi) = (-1)^m\sigma(D;\xi)^c, \quad \sigma(D^\top;\xi) = (-1)^m\sigma(D;\xi)^\top, \\ \text{and} \quad \sigma(D;\xi)^* = \sigma(D^*;\xi).\end{gathered} \tag{3.6}$$

Also, for any two differential operators D_1, D_2,

$$\sigma(D_1D_2;\xi) = \sigma(D_1;\xi)\sigma(D_2;\xi), \qquad \xi \in \mathbb{R}^n, \tag{3.7}$$

whenever the composition is meaningful.

Recall next that for a *first-order* differential operator $D : \mathcal{E} \to \mathcal{F}$, the following integration by parts formulas are valid:

$$\int_\Omega \langle Du, v^c\rangle\, dx = \int_\Omega \langle u, (D^* v)^c\rangle\, dx - \int_{\partial\Omega} \langle i\sigma(D;\nu)u, v^c\rangle\, d\sigma, \tag{3.8}$$

$$\int_\Omega \langle Du, v\rangle\, dx = \int_\Omega \langle u, D^t v\rangle\, dx - \int_{\partial\Omega} \langle i\sigma(D;\nu)u, v\rangle\, d\sigma, \tag{3.9}$$

where $d\sigma$ is the surface measure on $\partial\Omega$ (assumed to be reasonably smooth), ν is the outward unit normal to Ω, and the functions $u \in \mathcal{E}$, $v \in \mathcal{F}$, are sufficiently well-behaved near $\partial\Omega$.

We continue to assume that $D : \mathcal{E} \to \mathcal{F}$ is a first-order differential operator and consider $A : \bar{\Omega} \to \mathbb{C}^{M\times M}$ a smooth, matrix-valued function (also occasionally identified with a *zero-order* differential operator mapping $\mathcal{F}$ into $\mathcal{F}$). With D and A as above, introduce the *second-order* differential operator

$$L := -D^* A D, \qquad L : \mathcal{E} \longrightarrow \mathcal{F}, \tag{3.10}$$

and the associated *conormal derivative*

$$\partial_\nu^A := i\sigma(D^*;\nu)AD, \qquad \partial_\nu^A : \mathcal{E} \longrightarrow \mathcal{F}|_{\partial\Omega}. \tag{3.11}$$

For further reference, let us note here that

$$\sigma(\partial_\nu^A;\xi) = i\sigma(D^*;\nu)A\sigma(D;\xi), \tag{3.12}$$

so that in particular,

$$\sigma(\partial_\nu^A;\nu) = -i\sigma(L;\nu). \tag{3.13}$$

Also,

$$A = A^* \Longrightarrow L = L^* \Longrightarrow \sigma(L;\xi)^* = \sigma(L;\xi),\ \forall\, \xi \in \mathbb{R}^n. \tag{3.14}$$

It follows from (3.8) that

$$\begin{aligned}\int_\Omega \langle Lu, v^c\rangle\,dx &= -\int_\Omega \langle D^* ADu, v^c\rangle\,dx\\ &= -\int_\Omega \langle ADu, (Dv)^c\rangle\,dx + \int_{\partial\Omega} \langle \partial_\nu^A u, v^c\rangle\,d\sigma. \end{aligned} \tag{3.15}$$

Taking the complex conjugates of both sides and interchanging u and v also yields

$$\int_\Omega \langle u, (Lv)^c\rangle\,dx = -\int_\Omega \langle A^* Du, (Dv)^c\rangle\,dx + \int_{\partial\Omega} \langle u, (\partial_\nu^A v)^c\rangle\,d\sigma. \tag{3.16}$$

In particular,

$$A = A^* \Longrightarrow \int_\Omega \langle Lu, v^c\rangle - \langle u, (Lv)^c\rangle\,dx = \int_{\partial\Omega} \langle \partial_\nu^A u, v^c\rangle - \langle u, (\partial_\nu^A v)^c\rangle\,d\sigma, \tag{3.17}$$

i.e. the complex Green formula. Going further, note that replacing v by v^c in (3.17) yields the real Green formula

$$\int_\Omega \langle Lu, v\rangle\,dx = \int_\Omega \langle u, Lv\rangle\,dx + \int_{\partial\Omega} \langle \partial_\nu^A u, v\rangle\,d\sigma - \int_{\partial\Omega} \langle u, \partial_\nu^A v\rangle\,d\sigma \tag{3.18}$$

if $A^c = A$, $D^c = D$ (i.e., A and D have real coefficients) and $A = A^\top$.

3.2 The mother of all Rellich identities for second order systems

We continue to employ notation introduced in the previous section. Throughout this section, we shall assume that

$$Du(x) = \Big(\sum_{j=1}^n a_j^{\alpha\beta}(x)\partial_j u_\beta(x)\Big)_{1\le\alpha\le M}, \qquad u \in [C^\infty(\bar\Omega)]^N,\ x \in \bar\Omega, \tag{3.19}$$

is a first-order differential operator with C^1 coefficients and that the matrix A is self-adjoint, i.e.

$$A^* = A. \tag{3.20}$$

Then L, defined as in (3.10), becomes a self-adjoint, second-order partial differential operator. In order to continue, we need one more piece of notation. Specifically, if $\vec h = (h_j)_j : \bar\Omega \to \mathbb{R}^n$ is a smooth vector field, we set

$$\nabla_h^{\mathcal{E}} u := (\nabla_h u_j)_\beta := \Big(\sum_{j=1}^n h_j \partial_j u_\beta\Big)_\beta, \quad u \in \mathcal{E}, \tag{3.21}$$

with an analogous definition for $\nabla_h^{\mathcal{F}}$. In this context, $\nabla_h := \vec{h} \cdot \nabla$ is the usual direction derivative, in the direction of the vector h. It is useful to note that $\sigma(\nabla_h^{\mathcal{E}}; \xi) = i\langle \vec{h}, \xi \rangle I_{\mathcal{E}}$, where $I_{\mathcal{E}}$ stands for the identity operator on $\mathcal{E}$. Of course, a similar calculation applies to $\nabla_h^{\mathcal{F}}$.

The following Leibnitz formula is readily checked:

$$\nabla_h \langle u, w \rangle = \langle \nabla_h^{\mathcal{E}} u, w \rangle + \langle u, \nabla_h^{\mathcal{E}} w \rangle, \quad \forall\, u, w \in \mathcal{E}. \tag{3.22}$$

Of course, a similar Leibnitz formula holds for functions in $\mathcal{F}$.

If we now set $[D, \nabla_h] := D\nabla_h^{\mathcal{E}} - \nabla_h^{\mathcal{F}} D$, the symbol calculation

$$\sigma([D, \nabla_h]; \xi) = \sigma(D; \xi) i\langle \vec{h}, \xi \rangle I_{\mathcal{E}} - i\langle \vec{h}, \xi \rangle I_{\mathcal{F}}\, \sigma(D; \xi) = 0, \quad \forall\, \xi \in \mathbb{R}^n, \tag{3.23}$$

shows that $[D, \nabla_h]$ is a *first-order* differential operator. Integrating by parts then yields

$$\begin{aligned}
\int_{\partial\Omega} \langle \partial_\nu^A u, (\nabla_h^{\mathcal{E}} u)^c \rangle \, d\sigma &= \int_{\partial\Omega} \langle i\sigma(D^*; \nu) ADu, (\nabla_h^{\mathcal{E}} u)^c \rangle \, d\sigma \\
&= \int_\Omega \langle Lu, (\nabla_h^{\mathcal{E}} u)^c \rangle \, dx + \int_\Omega \langle ADu, (D\nabla_h^{\mathcal{E}} u)^c \rangle \, dx \\
&= \int_\Omega \langle Lu, (\nabla_h^{\mathcal{E}} u)^c \rangle \, dx + \int_\Omega \langle ADu, (\nabla_h^{\mathcal{F}} Du)^c \rangle \, dx \\
&\quad + \int_\Omega \langle ADu, ([D, \nabla_h] u)^c \rangle \, dx.
\end{aligned} \tag{3.24}$$

Next, observe that thanks to (3.22) and the fact that h has real-valued components, we have the sequence of identities

$$\begin{aligned}
\langle ADu, (\nabla_h^{\mathcal{F}} Du)^c \rangle &= \langle ADu, \nabla_h^{\mathcal{F}} (Du)^c \rangle \\
&= \nabla_h \langle ADu, (Du)^c \rangle - \langle \nabla_h^{\mathcal{F}} ADu, (Du)^c \rangle \\
&= \nabla_h \langle ADu, (Du)^c \rangle - \langle [\nabla_h^{\mathcal{F}}, A] Du, (Du)^c \rangle
\end{aligned}$$

$$-\langle A\nabla_h^{\mathcal{F}} Du, (Du)^c\rangle, \tag{3.25}$$

pointwise in Ω. In this connection, we note that

$$\sigma([\nabla_h^{\mathcal{F}}, A];\xi) = i\langle\vec{h},\xi\rangle I_{\mathcal{F}}A - A(i\langle\vec{h},\xi\rangle I_{\mathcal{F}}) = 0, \quad \forall\,\xi\in\mathbb{R}^n, \tag{3.26}$$

so we may conclude that $[\nabla_h^{\mathcal{F}}, A]$ is a *zero-order* operator. Moreover, (3.20) allows us to re-write the last term in (3.25) as $\langle\nabla_h^{\mathcal{F}} Du, (ADu)^c\rangle = (\langle ADu, (\nabla_h^{\mathcal{F}} Du)^c\rangle)^c$. Altogether, (3.25) becomes

$$2\,\mathrm{Re}\,\langle ADu, (\nabla_h^{\mathcal{F}} Du)^c\rangle = \nabla_h\langle ADu, (Du)^c\rangle + O(|Du|^2|[\nabla_h^{\mathcal{F}}, A]|). \tag{3.27}$$

Returning with this back in (3.24) then yields

$$\begin{aligned}\mathrm{Re}\int_{\partial\Omega}\langle\partial_\nu^A u, (\nabla_h^{\mathcal{F}} u)^c\rangle\,d\sigma &= \frac{1}{2}\int_\Omega \nabla_h\langle ADu, (Du)^c\rangle\,dx + \mathrm{Re}\int_\Omega\langle Lu, (\nabla_h^{\mathcal{F}} u)^c\rangle\,dx\\ &\quad + \int_\Omega O(|Du|^2|[\nabla_h^{\mathcal{F}}, A]|)\,dx\\ &\quad + \int_\Omega O(|A||Du||[D,\nabla_h]u|)\,dx.\end{aligned} \tag{3.28}$$

This completes the first round of integration by parts. Our approach involves a second round, based on the scalar Divergence Theorem, $\int_\Omega \nabla_h f\,dx = -\int_\Omega(\mathrm{div}\,h)f\,dx + \int_{\partial\Omega}\langle\vec{h},\nu\rangle f\,d\sigma$. Utilizing this in the context of (3.28), i.e. with $f := \langle ADu, (Du)^c\rangle$, gives a first version of a Rellich-type identity. To state this formally, we let $C_b^1(\bar{\Omega})$ denote the space of bounded, complex-valued functions of class C^1 in a neighborhood of $\bar{\Omega}$, with bounded first-order derivatives.

Theorem 3.1 *Assume that $\Omega\subset\mathbb{R}^n$ is a Lipschitz domain and let D be a first-order differential operator as in (3.19) with coefficients in $C_b^1(\bar{\Omega})$. Also, let the matrix-valued function A satisfy (3.20) and define L as in (3.10).*

Suppose next that $u\in C^2(\Omega)$ is a $\mathbb{R}^N$-valued function for which $M(\nabla u)\in L^2(\partial\Omega)$, the nontangential boundary trace $\nabla u\Big|_{\partial\Omega}$ exists pointwise almost everywhere, and ∇u and Lu are sufficiently well-behaved in Ω (e.g. being square integrable will do). Finally, fix an arbitrary vector field $\vec{h}\in C_b^1(\bar{\Omega})$ with real-valued components. Then there holds

$$
\begin{aligned}
2\,\mathrm{Re}\int_{\partial\Omega}&\langle\partial_\nu^A u,(\nabla_h^{\mathcal{F}}u)^c\rangle\,d\sigma\\
&=\int_{\partial\Omega}\langle\vec{h},\nu\rangle\langle ADu,(Du)^c\rangle\,d\sigma-\int_\Omega(\mathrm{div}\,h)\langle ADu,(Du)^c\rangle\,dx\\
&+2\,\mathrm{Re}\int_\Omega\langle Lu,(\nabla_h^{\mathcal{F}}u)^c\rangle\,dx+\int_\Omega O(|Du|^2|[\nabla_h^{\mathcal{F}},A]|)\,dx\\
&+\int_\Omega O(|A||Du||[D,\nabla_h]u|)\,dx.
\end{aligned}
\tag{3.29}
$$

In the second part of this section, we would like to further refine the above identity under the additional assumption that

$$
L \text{ is } \textit{strongly elliptic.} \tag{3.30}
$$

This entails that $\sigma(L;\xi)$ is an invertible matrix for any $\xi\neq 0$. Loosely speaking, this refinement is carried out by decomposing D into its tangential and normal component on $\partial\Omega$, analogously to the standard decomposition

$$
\nabla=\nabla_{tan}+\nu\,\partial_\nu \tag{3.31}
$$

of the full gradient operator in $\mathbb{R}^n$ into its tangential and normal components on $\partial\Omega$.

Let us describe a procedure which, given an arbitrary first-order differential operator P, allows one to decompose P as the sum of a tangential differential operator on $\partial\Omega$ and a suitable multiple of ∂_ν^A. The key observation is that the operator

$$
\tau:=P-i\sigma(P;\nu)\sigma(L;\nu)^{-1}\partial_\nu^A \tag{3.32}
$$

is tangential on $\partial\Omega$, in the sense that $\sigma(\tau;\nu)=0$, which follows readily from (3.13). In the case when this procedure is applied to D, the resulting tangential operator

$$
\tau_0:=D-i\sigma(D;\nu)\sigma(L;\nu)^{-1}\partial_\nu^A \tag{3.33}
$$

has the extra property that, on $\partial\Omega$,

$$
\sigma(D^*;\nu)A\tau_0=-i\partial_\nu^A-i\sigma(D^*;\nu)A\sigma(D;\nu)\sigma(L;\nu)^{-1}\partial_\nu^A=0. \tag{3.34}
$$

Now, writing $D = i\sigma(D;\nu)\sigma(L;\nu)^{-1}\partial_\nu^A + \tau_0$ and expanding $\langle ADu, (Du)^c\rangle$ yields

$$\begin{aligned}\langle ADu, (Du)^c\rangle &= \langle iA\sigma(D;\nu)\sigma(L;\nu)^{-1}\partial_\nu^A u, (Du)^c\rangle \\ &\quad +\langle A\tau_0 u, (i\sigma(D;\nu)\sigma(L;\nu)^{-1}\partial_\nu^A u)^c\rangle \\ &\quad +\langle A\tau_0 u, (\tau_0 u)^c\rangle \\ &=: I + II + III. \end{aligned} \tag{3.35}$$

Observe that

$$I = \langle \sigma(L;\nu)^{-1}\partial_\nu^A u, (-i\sigma(D^*;\nu)ADu)^c\rangle = \langle \sigma(L;\nu)^{-1}\partial_\nu^A u, (-\partial_\nu^A u)^c\rangle \tag{3.36}$$

and that, by (3.34),

$$II = \langle \sigma(D^*;\nu)A\tau_0 u, (i\sigma(L;\nu)^{-1}\partial_\nu^A u)^c\rangle = 0. \tag{3.37}$$

Thus, all in all,

$$\langle ADu, (Du)^c\rangle = \langle \sigma(-L;\nu)^{-1}\partial_\nu^A u, (\partial_\nu^A u)^c\rangle + \langle A\tau_0 u, (\tau_0 u)^c\rangle. \tag{3.38}$$

Similarly, we decompose

$$\nabla_h^{\mathcal{F}} = \langle \vec{h}, \nu\rangle\, \sigma(-L;\nu)^{-1}\partial_\nu^A + \tau_1, \tag{3.39}$$

where

$$\tau_1 := \nabla_h^{\mathcal{F}} - \langle \vec{h}, \nu\rangle \sigma(-L;\nu)^{-1}\partial_\nu^A \tag{3.40}$$

is tangential, by our previous discussion. Thus,

$$\begin{aligned}\operatorname{Re}\langle \partial_\nu^A u, (\nabla_h^{\mathcal{E}} u)^c\rangle &= \operatorname{Re}\langle \partial_\nu^A u, (\tau_1 u)^c\rangle + \langle \partial_\nu^A u, (\sigma(-L;\nu)^{-1}\partial_\nu^A u)^c\rangle\langle \vec{h}, \nu\rangle \\ &= \operatorname{Re}\langle \partial_\nu^A u, (\tau_1 u)^c\rangle + \langle \sigma(-L;\nu)^{-1}\partial_\nu^A u, (\partial_\nu^A u)^c\rangle\langle \vec{h}, \nu\rangle. \end{aligned} \tag{3.41}$$

Returning with (3.35)-(3.41) in (3.29) finally proves the following general Rellich-type identity.

Theorem 3.2 *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain, and let D be a first-order differential operator as in (3.19), with coefficients in $C^1_b(\bar{\Omega})$. Let the matrix-valued function A satisfy (3.20) and assume that the second-order operator L introduced in (3.10) is strongly elliptic. Next, assume that $u \in C^2(\Omega)$ is a $\mathbb{R}^N$-valued function such that $M(\nabla u) \in L^2(\partial\Omega)$, the nontangential boundary trace $\nabla u\Big|_{\partial\Omega}$ exists pointwise almost everywhere, and for which ∇u and Lu are sufficiently well-behaved in Ω (e.g. being square integrable will do). Finally, fix an arbitrary vector field $\vec{h} \in C^1_b(\bar{\Omega})$ with real-valued components. Then there holds*

$$\begin{aligned}
-\int_{\partial\Omega} &\langle \sigma(-L;\nu)^{-1}\partial^A_\nu u, (\partial^A_\nu u)^c\rangle \langle \vec{h}, \nu\rangle \, d\sigma \\
&= -\int_{\partial\Omega} \langle A\tau_0 u, (\tau_0 u)^c\rangle \langle \vec{h}, \nu\rangle \, d\sigma + 2\,\mathrm{Re}\int_{\partial\Omega} \langle \partial^A_\nu u, (\tau_1 u)^c\rangle \, d\sigma \\
&\quad -2\,\mathrm{Re}\int_{\Omega} \langle Lu, (\nabla^{\mathcal{F}}_h u)^c\rangle \, dx + \int_{\Omega} O(|Du|^2 |[\nabla^{\mathcal{F}}_h, A]|) \, dx \\
&\quad + \int_{\Omega} O(|A||Du||[D, \nabla_h]u|) \, dx,
\end{aligned} \tag{3.42}$$

where all O's involve only dimensional constants.

4 The Stokes system and hydrostatic potentials

4.1 Bilinear forms and conormal derivatives

For $\lambda \in \mathbb{R}$ fixed, let

$$a^{\alpha\beta}_{jk}(\lambda) := \delta_{jk}\delta_{\alpha\beta} + \lambda\,\delta_{j\beta}\delta_{k\alpha}, \qquad 1 \leq j, k, \alpha, \beta \leq n, \tag{4.1}$$

and, adopting the summation convention over repeated indices, consider the differential operator L_λ given by

$$(L_\lambda \vec{u})_\alpha := \partial_j(a^{\alpha\beta}_{jk}(\lambda)\partial_k u_\beta) = \Delta u_\alpha + \lambda\,\partial_\alpha(\mathrm{div}\,\vec{u}), \qquad 1 \leq \alpha \leq n. \tag{4.2}$$

The connection with the material in § 3.1 is as follows. Let $N := n$, $M := n^2$, and consider the first-order differential operator $Du := (\partial_k u_\beta)_{1\leq k,\beta\leq n}$ along with $Av := (a^{\alpha\beta}_{j,k}(\lambda)v_{k\beta})_{1\leq j,\alpha\leq n}$. Then $D^*v = -(\partial_k v_{k\beta})_{1\leq\beta\leq n}$ and, consequently,

$$L_\lambda u := -D^*ADu = \Big(\partial_j(a^{\alpha\beta}_{jk}(\lambda)\partial_k u_\beta)\Big)_{1\leq\alpha\leq n}. \tag{4.3}$$

Thus, all the results from § 3 apply to the operator (4.2). There is, however, one important nuance on which we would like to elaborate. Concretely, *as a whole*, the Stokes system does not fit into the general framework considered in § 3 because of the divergence-free condition imposed on $\vec{u}$ and because it involves a pressure function π which plays a different role than (the components of) $\vec{u}$. One of the aspects which is directly affected by this issue is the way we shall define the conormal derivative for the Stokes system. More specifically, various considerations dictate that the definition (3.11) should, in the case of the Stokes system, be altered to

$$\begin{aligned} \partial^\lambda_\nu(\vec{u},\pi) &:= \Big(\nu_j a^{\alpha\beta}_{jk}(\lambda)\partial_k u_\beta - \nu_\alpha\pi\Big)_{1\leq\alpha\leq n} \\ &= \Big[(\nabla\vec{u})^\top + \lambda(\nabla\vec{u})\Big]\nu - \pi\nu \quad \text{ on } \partial\Omega, \end{aligned} \tag{4.4}$$

where $\nabla\vec{u} = (\partial_k u_j)_{1\leq j,k\leq n}$ denotes the Jacobian matrix of the vector-valued function $\vec{u}$, and $\top$ stands for transposition of matrices.

To illustrate the fact that this definition is natural, consider the issue of Green's formulas, as discussed in § 3.1. Then, introducing the bilinear form

$$A_\lambda(\xi,\zeta) := a^{\alpha\beta}_{jk}(\lambda)\xi^\alpha_j\zeta^\beta_k, \qquad \forall\,\xi,\,\zeta\ n\times n \text{ matrices}, \tag{4.5}$$

we have the following useful integration by parts formulas:

$$\int_{\Omega_\pm}\langle L_\lambda\vec{u} - \nabla\pi, \vec{w}\rangle = \pm\int_{\partial\Omega}\langle\partial^\lambda_\nu(\vec{u},\pi),\vec{w}\rangle - \int_{\Omega_\pm} A_\lambda(\nabla\vec{u},\nabla\vec{w}) - \pi(\operatorname{div}\vec{w}), \tag{4.6}$$

and

$$\int_{\Omega_\pm} \langle L_\lambda \vec{u} - \nabla\pi, \vec{w}\rangle - \langle L_\lambda \vec{w} - \nabla\rho, \vec{u}\rangle = \pm \int_{\partial\Omega} \langle \partial^\lambda_\nu(\vec{u},\pi), \vec{w}\rangle - \langle \partial^\lambda_\nu(\vec{w},\rho), \vec{u}\rangle + \int_{\Omega_\pm} \pi(\operatorname{div}\vec{w}) - \rho(\operatorname{div}\vec{u}), \tag{4.7}$$

which should be compared with (3.15) and (3.18) respectively. Above, it is implicitly assumed that the functions involved are reasonably behaved near the boundary and at infinity (if the domain of integration is unbounded). Such considerations are going to be paid appropriate attention to in each specific application of these integration by parts formulas.

We next consider the issue of the (semi-) positiveness of the the bilinear form (4.5). As a preamble, we shall prove the following lemma.

Lemma 4.1 *For ξ an $n \times n$ matrix, $n \geq 2$, and $a, b, c \in \mathbb{R}$, let*

$$Q(\xi) = Q_{a,b,c}(\xi) := a\,|\xi|^2 + b\,|\tfrac{1}{2}(\xi + \xi^\top)|^2 + c\,|\mathrm{Tr}(\xi)|^2, \tag{4.8}$$

where Tr *stands for the usual matrix-trace operator,* $\top$ *denotes transposition, and* $|\xi| := [\mathrm{Tr}\,(\xi\,\xi^\top)]^{1/2}$. *Then*

$$(i) \qquad Q(\xi) \geq 0 \;\; \textit{for every } n \times n \textit{ matrix } \xi \qquad \Longleftrightarrow \quad \begin{cases} a \geq 0, \\ a + b \geq 0, \\ a + b + cn \geq 0, \end{cases}$$

$$(ii) \qquad \exists\, \kappa > 0 \;\; \textit{with} \;\; Q(\xi) \geq \kappa\, |\xi|^2 \qquad \forall\, \xi \qquad \Longleftrightarrow \quad \begin{cases} a > 0, \\ a + b > 0, \\ a + b + cn > 0, \end{cases}$$

$$(iii) \qquad \exists\, \kappa > 0 \textit{ with } Q(\xi) \geq \kappa\, |\tfrac{1}{2}(\xi + \xi^\top)|^2 \;\; \forall\, \xi \qquad \Longleftrightarrow \quad \begin{cases} a \geq 0, \\ a + b > 0, \\ a + b + cn > 0, \end{cases} \tag{4.9}$$

$$(iv) \qquad \exists\, \kappa > 0 \textit{ with } Q(\xi) \geq \kappa\, |\mathrm{Tr}\,(\xi)|^2 \qquad \forall\, \xi \qquad \Longleftrightarrow \quad \begin{cases} a \geq 0, \\ a + b \geq 0, \\ a + b + cn > 0, \end{cases}$$

$$(v) \qquad \exists\, \kappa > 0 \textit{ with } Q(\xi) \geq \kappa\, |\tfrac{1}{2}(\xi - \xi^\top)|^2 \;\; \forall\, \xi \qquad \Longleftrightarrow \quad \begin{cases} a > 0, \\ a + b \geq 0, \\ a + b + cn \geq 0. \end{cases}$$

Proof. Assume $Q(\xi) \geq 0$ for every $n \times n$ matrix ξ and define ξ^1, ξ^2, ξ^3 by

$$\xi^1_{jk} := \tfrac{1}{\sqrt{2}}\,(\delta_{j1}\delta_{k2} - \delta_{j2}\delta_{k1}), \qquad \xi^2_{jk} := \tfrac{1}{\sqrt{2}}\,(\delta_{j1}\delta_{k2} + \delta_{j2}\delta_{k1}), \qquad \text{and} \qquad \xi^3_{jk} := \tfrac{1}{\sqrt{n}}\,\delta_{jk}. \tag{4.10}$$

Then

$$Q(\xi^1) = a \geq 0, \qquad Q(\xi^2) = a + b \geq 0, \qquad \text{and} \qquad Q(\xi^3) = a + b + c\,n \geq 0. \tag{4.11}$$

Conversely, assume $a \geq 0$, $a + b \geq 0$ and $a + b + cn \geq 0$. Since

$$\tfrac{1}{n}|\mathrm{Tr}\,\xi|^2 \le |\tfrac{1}{2}(\xi+\xi^\top)|^2 \le |\xi|^2, \tag{4.12}$$

for every matrix ξ we may write

$$\begin{aligned}
\mathcal{Q}(\xi) &\ge a|\xi|^2 + b|\tfrac{1}{2}(\xi+\xi^\top)|^2 - (a+b)\tfrac{1}{n}|\mathrm{Tr}\,\xi|^2 \\
&= a\Big(|\xi|^2 - \tfrac{1}{n}|\mathrm{Tr}\,\xi|^2\Big) + b\Big(|\tfrac{1}{2}(\xi+\xi^\top)|^2 - \tfrac{1}{n}|\mathrm{Tr}\,\xi|^2\Big) \\
&\ge (a+b)\Big(|\tfrac{1}{2}(\xi+\xi^\top)|^2 - \tfrac{1}{n}|\mathrm{Tr}\,\xi|^2\Big) \\
&\ge 0.
\end{aligned} \tag{4.13}$$

Then (ii) follows from (i) once we notice that

$$Q_{a,b,c}(\xi) \ge \kappa\,|\xi|^2 \quad \forall\,\xi \iff Q_{a-\kappa,b,c}(\xi) \ge 0 \quad \forall\,\xi \iff \begin{cases} a \ge \kappa, \\ a+b \ge \kappa, \\ a+b+cn \ge \kappa. \end{cases} \tag{4.14}$$

Then (iii) and (iv) follow by similar arguments, and (v) also follows easily after noticing that

$$|\xi|^2 = |\tfrac{1}{2}(\xi+\xi^\top)|^2 + |\tfrac{1}{2}(\xi-\xi^\top)|^2. \tag{4.15}$$

This finishes the proof of the lemma. $\square$

Recall now the bilinear form (4.5).

Proposition 4.2 *For every $\lambda \in (-1,1]$ there exists $\kappa_\lambda > 0$ such that for every $n\times n$-matrix ξ*

$$A_\lambda(\xi,\xi) \ge \kappa_\lambda\,|\xi|^2 \;\; \textit{for} \;\; |\lambda| < 1 \qquad \textit{and} \qquad A_1(\xi,\xi) \ge \kappa_1\,|\xi+\xi^\top|^2. \tag{4.16}$$

Also, for $|\lambda| \le 1$, the Cauchy-Schwarz type inequality

$$A_\lambda(\xi,\zeta)^2 \le A_\lambda(\xi,\xi)A_\lambda(\zeta,\zeta) \tag{4.17}$$

holds for every $n\times n$-matrices ξ, ζ. Finally, for every $\lambda > -1$ there exists $\kappa_\lambda > 0$ such that

$$A_\lambda(\zeta,\zeta) \geq \kappa_\lambda |\zeta|^2 \quad \textit{for every matrix } \zeta \textit{ with entries of the form } \zeta_{jk} = \xi_j\eta_k. \tag{4.18}$$

Proof. Since $A_\lambda(\xi,\xi) = Q_{1-\lambda,2\lambda,0}(\xi)$, Lemma 4.1 readily gives (4.16). The same lemma also shows that, for $|\lambda| \leq 1$, the bilinear form (4.5) is nonnegative, hence the usual proof of the Cauchy-Schwarz inequality gives (4.17). As for (4.18), it suffices to notice that, if $\zeta = (\xi_j\eta_k)_{1\leq j,k\leq n}$, then $A_\lambda(\zeta,\zeta) = |\xi|^2|\eta|^2 + \lambda|\langle \xi,\eta\rangle|^2$. □

4.2 Hydrostatic layer potential operators

We continue to review background material by recalling the definitions and some basic properties of the layer potentials for the Stokes system in a Lipschitz domain $\Omega \subset \mathbb{R}^n, n \geq 2$. Let ω_{n-1} denote the surface measure of S^{n-1}, the unit sphere in $\mathbb{R}^n$, and let $E(x) = (E_{jk}(x))_{1\leq j,k\leq n}$ be the Kelvin matrix of fundamental solutions for the Stokes system, where

$$E_{jk}(x) := -\frac{1}{2\omega_{n-1}}\left(\frac{1}{n-2}\frac{\delta_{jk}}{|x|^{n-2}} + \frac{x_j x_k}{|x|^n}\right), \quad x \in \mathbb{R}^n \setminus \{0\}, \quad n \geq 3, \tag{4.19}$$

and corresponding to $n = 2$,

$$E_{jk}(x) := -\frac{1}{4\pi}\left(\delta_{jk}\log|x| + \frac{x_j x_k}{|x|^2}\right), \qquad x = (x_j)_j \in \mathbb{R}^2 \setminus \{0\}. \tag{4.20}$$

Let us also introduce a pressure vector $\vec{q}(x)$ given by

$$\vec{q}(x) = (q_j(x))_{1\leq j\leq n} := -\frac{1}{\omega_{n-1}}\frac{x}{|x|^n}, \quad x \in \mathbb{R}^n \setminus \{0\}. \tag{4.21}$$

Then we have

$$\partial_k E_{jk}(x) = 0 \ \text{ for } \ 1 \leq j \leq n \ \text{ and } \ \partial_j E_{jk}(x) = 0 \ \text{ for } \ 1 \leq k \leq n, \tag{4.22}$$

$$\Delta E_{jk}(x) = \Delta E_{kj}(x) = \partial_k q_j(x) = \partial_j q_k(x) \quad \text{for} \quad 1 \leq j,k \leq n. \tag{4.23}$$

Now, fix $-1 < \lambda \leq 1$, and define the single and double layer potential operators $\mathcal{S}$ and $\mathcal{D}_\lambda$ by

$$\mathcal{S}\vec{f}(x) := \int_{\partial\Omega} E(x-y)\,\vec{f}(y)\,d\sigma(y), \qquad x \notin \partial\Omega, \tag{4.24}$$

$$\mathcal{D}_\lambda\vec{f}(x) := \int_{\partial\Omega} [\partial^\lambda_{\nu(y)}\{E,\vec{q}\}(y-x)]^\top \vec{f}(y)\,d\sigma(y), \qquad x \notin \partial\Omega, \tag{4.25}$$

where $\partial^\lambda_{\nu(y)}\{E,\vec{q}\}$ is defined to be the matrix obtained by applying $\partial^\lambda_{\nu(y)}$ to each pair consisting of the j-th column in E and the j-th component of $\vec{q}$. More concretely,

$$(\partial^\lambda_{\nu(y)}\{E,\vec{q}\}(y-x))_{jk} := \nu_\alpha(y)\partial_\alpha E_{kj}(y-x) + \lambda\nu_\alpha(y)\partial_k E_{\alpha j}(y-x) - q_j(y-x)\nu_k(y). \tag{4.26}$$

Let us also define corresponding potentials for the pressure by

$$\mathcal{Q}\vec{f}(x) := \int_{\partial\Omega} \langle \vec{q}(x-y), \vec{f}(y)\rangle\,d\sigma(y) \qquad x \notin \partial\Omega, \tag{4.27}$$

$$\mathcal{P}_\lambda\vec{f}(x) := (1+\lambda)\int_{\partial\Omega} \nu_j(y)\langle(\partial_j\vec{q})(y-x), \vec{f}(y)\rangle\,d\sigma(y), \qquad x \notin \partial\Omega. \tag{4.28}$$

Then

$$\Delta\mathcal{S}\vec{f} - \nabla\mathcal{Q}\vec{f} = 0 \quad \text{and} \quad \operatorname{div}\mathcal{S}\vec{f} = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \partial\Omega, \tag{4.29}$$

and for each $\lambda \in \mathbb{R}$,

$$\Delta\mathcal{D}_\lambda\vec{f} - \nabla\mathcal{P}_\lambda\vec{f} = 0 \quad \text{and} \quad \operatorname{div}\mathcal{D}_\lambda\vec{f} = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \partial\Omega. \tag{4.30}$$

Let us also consider the fundamental solution for the Laplacian,

$$E_\Delta(x) := \begin{cases} -\frac{1}{(n-2)\omega_{n-1}|x|^{n-2}} \ \text{ if } \ n \geq 3, \\ \\ \frac{1}{2\pi}\log|x| \ \text{ if } \ n = 2, \end{cases} \tag{4.31}$$

and the corresponding single and double harmonic layer potentials

$$\mathcal{S}_\Delta f(x) := \int\limits_{\partial\Omega} E_\Delta(x-y) f(y)\, d\sigma(y), \qquad x \notin \partial\Omega, \tag{4.32}$$

$$\mathcal{D}_\Delta f(x) := \int\limits_{\partial\Omega} \partial_{\nu(y)} E_\Delta(x-y) f(y)\, d\sigma(y), \qquad x \notin \partial\Omega. \tag{4.33}$$

Then

$$\vec{q} = -\nabla E_\Delta \qquad \text{in } \ \mathbb{R}^n \setminus \{0\}, \tag{4.34}$$

and so

$$\mathcal{Q}\vec{f} = -\sum_{k=1}^{n} \partial_k(\mathcal{S}_\Delta f_k) = -\mathrm{div}\, \mathcal{S}_\Delta \vec{f}, \tag{4.35}$$

$$\mathcal{P}_\lambda \vec{f} = (1+\lambda)\mathrm{div}\, \mathcal{D}_\Delta \vec{f}. \tag{4.36}$$

Let us now record a basic result from the theory of singular integral operators of Calderón-Zygmund type on Lipschitz domains. To state it, recall that $\mathcal{F}$ denotes the Fourier transform in $\mathbb{R}^n$.

Proposition 4.3 *There exists a positive integer $N = N(n)$ with the following significance. Let Ω be as in (2.1), fix some function*

$$k \in C^N(\mathbb{R}^n \setminus \{0\}) \quad \textit{with} \quad k(-x) = -k(x) \quad \textit{and} \quad k(\lambda x) = \lambda^{-(n-1)} k(x) \ \ \forall\, \lambda > 0, \tag{4.37}$$

and define the singular integral operator

$$\mathcal{T} f(x) := \int_{\partial\Omega} k(x-y) f(y)\, d\sigma(y), \qquad x \in \mathbb{R}^n \setminus \partial\Omega. \tag{4.38}$$

Then for each $p \in (\frac{n-1}{n}, \infty)$ there exists a finite constant $C = C(p, n, \partial\Omega) > 0$ such that

$$\|M(\mathcal{T} f)\|_{L^p(\partial\Omega)} \leq C \|k|_{S^{n-1}}\|_{C^N} \|f\|_{H^p(\partial\Omega)}. \tag{4.39}$$

Furthermore, for each $p \in (1,\infty)$, $f \in L^p(\partial\Omega)$, the limit

$$Tf(x) := \text{p.v.} \int_{\partial\Omega} k(x-y)f(y)\,d\sigma(y) := \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} k(x-y)f(y)\,d\sigma(y) \tag{4.40}$$

exists for a.e. $x \in \partial\Omega$, and the jump-formula

$$\mathcal{T}f\Big|_{\partial\Omega}(x) := \lim_{\substack{z\to x \\ z\in\Gamma^{\pm}_{\kappa}(x)}} \mathcal{T}f(z) = \pm\tfrac{1}{2\sqrt{-1}}\mathcal{F}(k)(\nu(x))f(x) + Tf(x) \tag{4.41}$$

is valid at a.e. $x \in \partial\Omega$.

Let us now specialize (4.41) to the case of hydrostatic layer potentials.

Proposition 4.4 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a graph Lipschitz domain and assume that $1 < p < \infty$. Then for each $\lambda \in \mathbb{R}$, $\vec{f} \in L^p(\partial\Omega)$, and a.e. $x \in \partial\Omega$,*

$$\mathcal{Q}\vec{f}\Big|_{\partial\Omega_\pm}(x) = \pm\tfrac{1}{2}\langle\nu(x), \vec{f}(x)\rangle + \text{p.v.}\int_{\partial\Omega}\langle\vec{q}(x-y), \vec{f}(y)\rangle\,d\sigma(y), \tag{4.42}$$

$$\mathcal{D}_\lambda\vec{f}\Big|_{\partial\Omega_\pm}(x) = \Big(\pm\tfrac{1}{2}I + K_\lambda\Big)\vec{f}(x), \tag{4.43}$$

where I denotes the identity operator and

$$K_\lambda\vec{f}(x) := \text{p.v.}\int\limits_{\partial\Omega}[\partial^\lambda_{\nu(y)}\{E,\vec{q}\}(y-x)]^\top\vec{f}(y)\,d\sigma(y), \qquad x \in \partial\Omega. \tag{4.44}$$

Furthermore, if K^*_λ is the formal adjoint of K_λ, then

$$\partial^\lambda_\nu(\mathcal{S}\vec{f}, \mathcal{Q}\vec{f})\Big|_{\partial\Omega_\pm}(x) = \Big(\mp\tfrac{1}{2}I + K^*_\lambda\Big)\vec{f}(x). \tag{4.45}$$

Finally,

$$\nabla_{tan}\mathcal{S}\vec{f}\Big|_{\partial\Omega_+} = \nabla_{tan}\mathcal{S}\vec{f}\Big|_{\partial\Omega_-} \quad \textit{in} \quad L^p(\partial\Omega), \tag{4.46}$$

hence

$$S\vec{f} := \mathcal{S}\vec{f}\Big|_{\partial\Omega_+} = \mathcal{S}\vec{f}\Big|_{\partial\Omega_-} \quad in \quad \dot{L}^p_1(\partial\Omega). \tag{4.47}$$

In fact, analogous formulas hold in the case when $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain.

Proof. Recall that if m is an integer and P_j is a harmonic, homogeneous polynomial of degree $j \geq 0$ in $\mathbb{R}^n$ then, as is well-known (cf., e.g., p. 73 in [87]),

$$\mathcal{F}(Q_j)(x) = \frac{P_j(x)}{|x|^{j+n-m}} \tag{4.48}$$

where, with Γ denoting the standard Gamma function,

$$Q_j(x) := (-1)^j \gamma_{j,m} \frac{P_j(x)}{|x|^{j+m}} \quad \text{and} \quad \gamma_{j,m} := (-1)^{j/2} \pi^{\frac{n}{2}-m} \frac{\Gamma(\frac{j}{2}+\frac{m}{2})}{\Gamma(\frac{j}{2}+\frac{n}{2}-\frac{m}{2})}, \tag{4.49}$$

provided either $0 < m < n$, or $m \in \{0, n\}$ and $j \geq 1$. Based on this and (4.41), a straightforward calculation gives the following trace formulas

$$\partial_j \Big(\mathcal{S}_{\alpha\beta}\, g\Big)\Big|_{\partial\Omega_\pm}(x) = \mp\tfrac{1}{2}\nu_j(x)\Big(\delta_{\alpha\beta} - \nu_\alpha(x)\nu_\beta(x)\Big)g(x) + \partial_j S_{\alpha\beta}\, g(x) \tag{4.50}$$

valid at a.e. $x \in \partial\Omega$, for every $g \in L^p(\partial\Omega)$, $1 < p < \infty$, where for each $\alpha, \beta, j \in \{1, ..., n\}$, we have used the abbreviations

$$\mathcal{S}_{\alpha\beta}\, g(x) := \int_{\partial\Omega} E_{\alpha\beta}(x-y) g(y)\, d\sigma(y), \qquad x \in \mathbb{R}^n \setminus \partial\Omega, \tag{4.51}$$

$$\partial_j S_{\alpha\beta}\, g(x) := \text{p.v.} \int_{\partial\Omega} (\partial_j E_{\alpha\beta})(x-y) g(y)\, d\sigma(y), \qquad x \in \partial\Omega. \tag{4.52}$$

In particular, for $j \in \{1, ..., n\}$, we have

$$\partial_j \mathcal{S}\vec{f}\Big|_{\partial\Omega_\pm}(x) = \mp\tfrac{1}{2}\nu_j(x)\vec{f}_{tan}(x) + \text{p.v.}\int_{\partial\Omega} (\partial_j E)(x-y)\vec{f}(y)\, d\sigma(y), \tag{4.53}$$

at almost every $x \in \partial\Omega$, where $\vec{f}_{tan} := \vec{f} - \nu\langle\nu, \vec{f}\rangle$ is the tangential component of $\vec{f}$. In a similar fashion,

$$\partial_j \mathcal{S}_\Delta g\Big|_{\partial\Omega_\pm}(x) \;=\; \mp\tfrac{1}{2}\nu_j(x)\, g(x) + \text{p.v.}\int_{\partial\Omega} (\partial_j E_\Delta)(x-y) g(y)\, d\sigma(y), \tag{4.54}$$

for a.e. $x \in \partial\Omega$. Now, all the trace formulas in the statement of the proposition are direct corollaries of (4.53) and (4.54). $\square$

With the help of Proposition 4.3, we can now establish the following.

Proposition 4.5 *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a graph Lipschitz domain. Then for $\frac{n-1}{n} < p < \infty$, there exists $C = C(\partial\Omega, p)$ such that for any $\vec{f} = (f_1, ..., f_n)$ in $H^p(\partial\Omega)$,*

$$\|M(\nabla \mathcal{S}\vec{f})\|_{L^p(\partial\Omega)} + \|M(\mathcal{Q}\vec{f})\|_{L^p(\partial\Omega)} + \sum_{k=1}^{n} \|M(\nabla \mathcal{S}_\Delta\, f_k)\|_{L^p(\partial\Omega)} \leq C\|\vec{f}\|_{H^p(\partial\Omega)} \tag{4.55}$$

Moreover, for $\lambda \in \mathbb{R}$ and $1 < p < \infty$, there exists $C = C(\partial\Omega, p)$ such that for any $\vec{f} \in L^p(\partial\Omega)$,

$$\|M(\mathcal{D}_\lambda \vec{f})\|_{L^p(\partial\Omega)} \leq C\|\vec{f}\|_{L^p(\partial\Omega)}. \tag{4.56}$$

Similar results are also valid when $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, with $H^p(\partial\Omega)$ replaced by $h^p(\partial\Omega)$, its local version.

This result leads to the following corollary.

Corollary 4.6 *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain, and fix $\lambda \in \mathbb{R}$. Then the operators*

$$K_\lambda,\, K_\lambda^* : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega), \tag{4.57}$$

$$S : L^p(\partial\Omega) \longrightarrow L^p_1(\partial\Omega), \tag{4.58}$$

are well-defined, linear, and bounded whenever $1 < p < \infty$. A similar result holds when Ω is a graph Lipschitz domain, except in this case the Sobolev space $L^p_1(\partial\Omega)$ is replaced by its homogeneous version $\dot{L}^p_1(\partial\Omega)$.

We now turn to the action of layer potential operators on Sobolev spaces of negative smoothness. If $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, $p \in (1,\infty)$, and $\vec{f} = (f_1, ..., f_n)$ is a vector whose components are functionals in $L^p_{-1}(\partial\Omega) = \Big(L^{p'}_1(\partial\Omega)\Big)^*$, $1/p + 1/p' = 1$, we set

$$\mathcal{S}\vec{f}(x) := \Big(\sum_{k=1}^{n} \Big\langle E_{jk}(x-\cdot)\Big|_{\partial\Omega}, f_k\Big\rangle\Big)_{1\leq j\leq n}, \quad x \in \mathbb{R}^n \setminus \partial\Omega, \tag{4.59}$$

where in this context, $\langle\cdot,\cdot\rangle$ is the duality bracket between $L^p_{-1}(\partial\Omega)$ and $\Big(L^{p'}_1(\partial\Omega)\Big)^*$. It is then clear that this operator is compatible with (4.24), when the latter is considered acting on $L^p(\partial\Omega) \hookrightarrow L^p_{-1}(\partial\Omega)$. This justifies our retaining the same piece of notation for the single layer in (4.59). Similar considerations apply to the pressure potential

$$\mathcal{Q}\vec{f}(x) := \sum_{j=1}^{n} \Big\langle q_j(x-\cdot)\Big|_{\partial\Omega}, f_j\Big\rangle, \quad x \in \mathbb{R}^n \setminus \partial\Omega. \tag{4.60}$$

Proposition 4.7 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then the following hold for each $p \in (1,\infty)$:*

(i) For each $\vec{f} \in L^p_{-1}(\partial\Omega)$, the pair $(\mathcal{S}\vec{f}, \mathcal{Q}\vec{f})$ is a solution of the Stokes system in $\mathbb{R}^n \setminus \partial\Omega$ (i.e. the formulas in (4.29) continue to hold).

(ii) There exists $C = C(\Omega, p) > 0$ such that

$$\|M(\mathcal{S}\vec{f})\|_{L^p(\partial\Omega)} \leq C\|\vec{f}\|_{L^p_{-1}(\partial\Omega)}. \tag{4.61}$$

(iii) The boundary single layer operator

$$S\vec{f} := \mathcal{S}\vec{f}\Big|_{\partial\Omega_+} = \mathcal{S}\vec{f}\Big|_{\partial\Omega_-}, \tag{4.62}$$

is well-defined as a function in $L^p(\partial\Omega)$ for each $\vec{f} \in L^p_{-1}(\partial\Omega)$. Moreover,

$$S : L^p_{-1}(\partial\Omega) \longrightarrow L^p(\partial\Omega) \tag{4.63}$$

is a bounded operator, which is compatible with (4.58).

(iv) If $1/p+1/p'=1$, then the adjoint of (4.63) is

$$S : L^{p'}(\partial\Omega) \longrightarrow L_1^{p'}(\partial\Omega). \tag{4.64}$$

Proof. The claim in (i) is clear from (4.29) and (2.27). Next, if $\vec{f} \in L_{-1}^p(\partial\Omega)$, Corollary 2.2 gives that, for every $k = 1, 2, ..., n$, there exist functions g_k^0, g_k^{rs}, $1 \le r, s \le n$, such that

$$f_k = g_k^0 + \sum_{r,s=1}^n \partial_{\tau_{rs}} g_k^{rs}, \quad \|g_k^0\|_{L^p(\partial\Omega)} + \sum_{r,s=1}^n \|g_k^{rs}\|_{L^p(\partial\Omega)} \le 2\|f_k\|_{L_{-1}^p(\partial\Omega)}, \tag{4.65}$$

Based on this, the j-th component of $\mathcal{S}\vec{f}$ can be expressed as

$$\begin{aligned} \left(\mathcal{S}\vec{f}(x)\right)_j &= \sum_{k=1}^n \int_{\partial\Omega} E_{jk}(x-y) g_k^0(y)\, d\sigma(y) \\ &\quad - \sum_{k=1}^n \sum_{r,s=1}^n \int_{\partial\Omega} \partial_{\tau_{rs}}[E_{jk}(x-y)] g_k^{rs}(y)\, d\sigma(y), \end{aligned} \tag{4.66}$$

for each $x \in \mathbb{R}^n \setminus \partial\Omega$. This and Calderón-Zygmund theory then give

$$\|M(\mathcal{S}\vec{f})\|_{L^p(\partial\Omega)} \le C \sum_{k=1}^n \Big(\|g_k^0\|_{L^p(\partial\Omega)} + \sum_{r,s=1}^n \|g_k^{rs}\|_{L^p(\partial\Omega)} \Big) \le C\|\vec{f}\|_{L_{-1}^p(\partial\Omega)}, \tag{4.67}$$

justifying (4.61).

Formula (4.66) and Calderón-Zygmund theory also give that the pointwise nontangential traces in (4.62) exist. In fact, since

$$-\nu_r(x)(\widehat{\partial_s E_{jk}})(\nu(x)) + \nu_s(x)(\widehat{\partial_r E_{jk}})(\nu(x)) = 0, \tag{4.68}$$

it follows from (4.66) that there are no jump-terms when taking the boundary traces of $\mathcal{S}\vec{f}$ on $\partial\Omega_\pm$. In particular, $\mathcal{S}\vec{f}|_{\partial\Omega_+} = \mathcal{S}\vec{f}|_{\partial\Omega_-}$ and, in addition, the j-th component of $S\vec{f}$ is

$$\begin{aligned} (S\vec{f}(x))_j &= \sum_{k=1}^n \int_{\partial\Omega} E_{jk}(x-y) g_k^0(y)\, d\sigma(y) \\ &\quad - \sum_{k=1}^n \sum_{r,s=1}^n \text{p.v.} \int_{\partial\Omega} \partial_{\tau_{rs}}[E_{jk}(x-y)] g_k^{rs}(y)\, d\sigma(y), \end{aligned} \tag{4.69}$$

for a.e. $x \in \partial\Omega$. This also shows that the operator (4.63) is well-defined, bounded and compatible with (4.58). Finally, the claim in (iv) is easily justified based on the fact that S is self-adjoint as an operator on $L^2(\partial\Omega)$ plus a density argument. $\square$

In the study of the action of the hydrostatic layer potentials on Hardy-type spaces, the following standard result is going to be useful.

Lemma 4.8 *Let Ω be a graph Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, and consider a bounded, linear operator*

$$T : L^2(\partial\Omega) \longrightarrow L^2(\partial\Omega) \tag{4.70}$$

such that there exists a locally bounded function $k : \{(x, y) : x, y \in \partial\Omega, \ x \neq y\} \to \mathbb{R}$ with the following properties.

(i) For each $f \in L^2(\partial\Omega)$,

$$Tf(x) = \int_{\partial\Omega} k(x, y) f(y)\, d\sigma(y), \qquad x \in \partial\Omega \setminus \operatorname{supp} f. \tag{4.71}$$

(ii) There exist $C_0, C_1 > 0$ such that

$$|k(x, y)| \leq C_0 |x - y|^{-(n-1)} \quad \text{ if } x, y \in \partial\Omega, \ \ x \neq y, \tag{4.72}$$

$$|k(x, y) - k(x, y_0)| \leq C_0 \frac{|y - y_0|}{|x - y_0|^n}, \quad \text{ if } |y - y_0| < C_1 |x - y_0|. \tag{4.73}$$

Then there exists $\varepsilon > 0$ small and $\kappa > 0$ large such that if a is as in (2.30) then

$$m := Ta \Longrightarrow \kappa^{-1} m \text{ satisfies the last two conditions in (2.34).} \tag{4.74}$$

If, in addition to (i) and (ii) above, the operator T also satisfies $T^*(1) = 0$, in the sense that

$$f \in L^2(\partial\Omega) \text{ with compact support, } \int_{\partial\Omega} f\, d\sigma = 0 \Longrightarrow \int_{\partial\Omega} Tf\, d\sigma = 0, \tag{4.75}$$

then m is a fixed multiple of a (p,ε)-molecule Hence, in this latter case, T extends as a bounded operator

$$T : H^p_{at}(\partial\Omega) \longrightarrow H^p_{at}(\partial\Omega) \tag{4.76}$$

for every $\frac{n-1}{n} < p \leq 1$.

We can now establish the boundedness of the operator K^*_λ on atomic Hardy spaces.

Proposition 4.9 *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a graph Lipschitz domain and $\frac{n-1}{n} < p \leq 1$. Then*

$$K^*_\lambda : H^p_{at}(\partial\Omega) \longrightarrow H^p_{at}(\partial\Omega) \tag{4.77}$$

is a bounded operator for each $\lambda \in \mathbb{R}$. Moreover, a similar result holds when $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, provided $H^p_{at}(\partial\Omega)$ is replaced by its local version, $h^p_{at}(\partial\Omega)$.

Proof. This is a consequence of Lemma 4.8 once we check (4.75). To this end, assume that $\vec{f} \in L^2(\partial\Omega)$ has compact support and satisfies $\int_{\partial\Omega} \vec{f}\, d\sigma = 0$. Next, set $\vec{u} := \mathcal{S}\vec{f}$ and $\pi := \mathcal{Q}\vec{f}$ in Ω, so that from (4.45),

$$K^*_\lambda \vec{f} = \partial^\lambda_\nu(\mathcal{S}\vec{f}, \mathcal{Q}\vec{f})\Big|_{\partial\Omega} + \tfrac{1}{2}\vec{f}. \tag{4.78}$$

Thus, we need to establish that

$$\int_{\partial\Omega} \partial^\lambda_\nu(\vec{u}, \pi)\, d\sigma = 0. \tag{4.79}$$

Note that the vanishing moment condition for $\vec{f}$ ensures that the above integral is absolutely convergent and that

$$|\nabla\vec{u}(x)| + |\pi(x)| = O(|x|^{-n}) \qquad \text{at infinity.} \tag{4.80}$$

To prove (4.79), fix a function $\psi \in C^\infty_0(B(0,2))$ with $\psi \equiv 1$ on $B(0,1)$, and for each $R > 0$ set $\psi_R(x) := \psi(x/R)$. Then for each constant $\vec{c} \in \mathbb{R}^n$, using the integration by parts formula (4.6) with $\vec{w} := \psi_R\vec{c}$ gives

$$\begin{aligned}
\left|\Big\langle \int_{\partial\Omega} \partial_\nu^\lambda(\vec{u},\pi)\,d\sigma\,,\,\vec{c}\Big\rangle\right| &= \left|\lim_{R\to\infty}\int_{\partial\Omega}\langle\partial_\nu^\lambda(\vec{u},\pi),\psi_R\vec{c}\rangle\,d\sigma\right| \\
&= \left|\lim_{R\to\infty}\int_{\Omega}\Big\{A_\lambda(\nabla\vec{u},\nabla(\psi_R\vec{c}))-\pi\,\mathrm{div}\,(\psi_R\vec{c})\Big\}\,dx\right| \\
&= \overline{\lim_{R\to\infty}}\int_{x\in\Omega:\,R<|x|<2R}\Big(|\nabla\vec{u}(x)|+|\pi(x)|\Big)|\nabla\psi_R(x)|\,dx \\
&\leq C\lim_{R\to\infty}R^{-1}=0, \qquad (4.81)
\end{aligned}$$

by (4.80) and the fact that $|\nabla\psi_R(x)|\leq C/R$. Since $\vec{c}$ was arbitrary, this gives (4.79), thus finishing the proof of the proposition. □

Next, we wish to discuss the action of these various operators on Sobolev-Hardy spaces. To set the stage, we first note that, from (4.25)-(4.26), for each $\lambda\in\mathbb{R}$, $j\in\{1,...,n\}$, and $\vec{f}\in L^p(\partial\Omega)$, $1<p<\infty$,

$$\begin{aligned}
\Big(\mathcal{D}_\lambda\vec{f}\Big)_j(x) &= \int_{\partial\Omega}\Big(\nu_\alpha(y)(\partial_\alpha E_{jk})(y-x)+\lambda\,\nu_\alpha(y)(\partial_j E_{\alpha k})(y-x) \\
&\qquad -\nu_j(y)q_k(y-x)\Big)f_k(y)\,d\sigma(y), \qquad x\in\mathbb{R}^n\setminus\partial\Omega. \qquad (4.82)
\end{aligned}$$

Then for each $\vec{f}\in H_1^p(\partial\Omega)$, $\frac{n-1}{n}<p<\infty$, $r,j\in\{1,...,n\}$, and $x\in\mathbb{R}^n\setminus\partial\Omega$, we may write

$$\begin{aligned}
\partial_r(\mathcal{D}_\lambda\vec{f})_j(x) &= -\int_{\partial\Omega}\Big[\nu_\alpha(y)(\partial_r\partial_\alpha E_{jk})(y-x)+\lambda\nu_\alpha(y)(\partial_r\partial_j E_{\alpha k})(y-x) \\
&\qquad -\nu_j(y)(\partial_r q_k)(y-x)\Big]f_k(y)\,d\sigma(y) \qquad (4.83) \\
&= -\int_{\partial\Omega}\Big[\partial_{\tau_{\alpha r}(y)}(\partial_\alpha E_{jk})(y-x)+\lambda\partial_{\tau_{\alpha r}(y)}(\partial_j E_{\alpha k})(y-x)-\partial_{\tau_{jr}(y)}q_k(y-x)\Big]f_k(y)\,d\sigma(y) \\
&\quad -\int_{\partial\Omega}[\nu_r(y)\Delta E_{jk}(y-x)+\lambda\nu_r(y)(\partial_\alpha\partial_j E_{\alpha k})(y-x)-\nu_r(y)(\partial_j q_k)(y-x)]\,f_k(y)\,d\sigma(y).
\end{aligned}$$

From (4.22)-(4.23), it follows that the integrand in the last line of (4.83) vanishes. By further integrating by parts (cf. (2.9)) the tangential derivatives in (4.83) we arrive at the identity

$$\partial_r\left(\mathcal{D}_\lambda\vec{f}\right)_j(x) = \int\limits_{\partial\Omega}\Big[(\partial_\alpha E_{jk})(y-x)(\partial_{\tau_{\alpha r}} f_k)(y) + \lambda(\partial_j E_{\alpha k})(y-x)(\partial_{\tau_{\alpha r}} f_k)(y)$$
$$-q_k(y-x)(\partial_{\tau_{jr}} f_k)(y)\Big]\, d\sigma(y),$$

or equivalently,

$$\partial_r(\mathcal{D}_\lambda\vec{f})_j = -\partial_\alpha\mathcal{S}_{jk}(\partial_{\tau_{\alpha r}} f_k) - \lambda\partial_j\mathcal{S}_{\alpha k}(\partial_{\tau_{\alpha r}} f_k) - \partial_k\mathcal{S}_\Delta(\partial_{\tau_{jr}} f_k) \quad \text{in } \mathbb{R}^n\setminus\partial\Omega. \tag{4.84}$$

The same type of reasoning applies to (4.28). Specifically, we have for each $x\in\mathbb{R}^n\setminus\partial\Omega$,

$$\begin{aligned}
\mathcal{P}_\lambda\vec{f}(x) &= (1+\lambda)\int\limits_{\partial\Omega}\nu_r(y)(\partial_r q_k)(y-x)\, f_k(y)\, d\sigma(y)\\
&= -(1+\lambda)\int\limits_{\partial\Omega}\nu_r(y)(\partial_r\partial_k E_\Delta)(y-x) f_k(y)\, d\sigma(y)\\
&= -(1+\lambda)\int\limits_{\partial\Omega}(\partial_{\tau_{rk}}\partial_r E_\Delta)(y-x) f_k(y)\, d\sigma(y)\\
&= (1+\lambda)\int\limits_{\partial\Omega}(\partial_r E_\Delta)(y-x)(\partial_{\tau_{rk}} f_k)(y)\, d\sigma(y)\\
&= (1+\lambda)\partial_r\mathcal{S}_\Delta(\partial_{\tau_{rk}} f_k)(x),
\end{aligned} \tag{4.85}$$

whenever $\vec{f}\in H^p_1(\partial\Omega)$, $\frac{n-1}{n}<p<\infty$. With these identities in mind, we can prove the following results.

Proposition 4.10 *Fix $\lambda\in\mathbb{R}$. Then for each graph Lipschitz domain $\Omega\subseteq\mathbb{R}^n$, $n\geq 2$, and $\frac{n-1}{n}<p<\infty$, there exists a finite constant $C=C(\partial\Omega,p)>0$ such that*

$$\|M(\nabla\mathcal{D}_\lambda\vec{f})\|_{L^p(\partial\Omega)} + \|M(\mathcal{P}_\lambda\vec{f})\|_{L^p(\partial\Omega)} \leq C\|\vec{f}\|_{H^p_1(\partial\Omega)}, \quad \forall\vec{f}\in H^p_1(\partial\Omega). \tag{4.86}$$

Furthermore, an analogous estimate holds in the case when $\Omega\subset\mathbb{R}^n$ is a bounded Lipschitz domain, whenever $\vec{f}\in h^p_1(\partial\Omega)$.

Proof. This is a direct consequence of Proposition 4.3, (4.84), (4.85) and Lemma 2.3. $\square$

Proposition 4.11 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a graph Lipschitz domain. Then for every $\lambda \in \mathbb{R}$ and $\vec{f} \in \dot{L}_1^p(\partial\Omega)$, $1 < p < \infty$, there holds*

$$\partial_\nu^\lambda(\mathcal{D}_\lambda \vec{f}, \mathcal{P}_\lambda \vec{f})\Big|_{\partial\Omega_+} = \partial_\nu^\lambda(\mathcal{D}_\lambda \vec{f}, \mathcal{P}_\lambda \vec{f})\Big|_{\partial\Omega_-} \quad \text{in } L^p(\partial\Omega). \tag{4.87}$$

A similar identity is also valid when $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, whenever $\vec{f} \in L_1^p(\partial\Omega)$.

Proof. This follows from (4.84), (4.85), (4.50), and (4.54).

$\square$

Proposition 4.12 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a graph Lipschitz domain. Then for each $\lambda \in \mathbb{R}$,*

$$K_\lambda : H_1^p(\partial\Omega) \longrightarrow H_1^p(\partial\Omega) \tag{4.88}$$

is a well-defined, bounded operator for every $p \in (\frac{n-1}{n}, \infty)$. Moreover, a similar result holds in the case when $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, provided $H_1^p(\partial\Omega)$ is replaced by $h_1^p(\partial\Omega)$.

Proof. Assume first that $\frac{n-1}{n} < p \leq 1$. In this case, fix $p_o \in (1, \infty)$, $\varepsilon > 0$ sufficiently small, as well as $r, s \in \{1, ..., n\}$ arbitrary. Also, let $\vec{f}$ be a regular (p, p_o)-atom. By (2.47) and Lemma 4.8, it suffices to show that $\partial_{\tau_{rs}} K_\lambda \vec{f}$ is a (p, ε)-molecule. Since this issue is dilation invariant, there is no loss of generality in assuming that $0 \in \partial\Omega$,

$$\operatorname{supp} \vec{f} \subseteq S_1(0) \quad \text{and} \quad \|\nabla_{tan} \vec{f}\|_{L^{p_o}(\partial\Omega)} \leq 1. \tag{4.89}$$

Going further, we note that for each $j \in \{1, ..., n\}$,

$$\begin{aligned} \partial_{\tau_{rs}}(K_\lambda \vec{f})_j(x) &= \partial_{\tau_{rs}}(\tfrac{1}{2}\vec{f} + K_\lambda \vec{f})_j(x) - \tfrac{1}{2}\partial_{\tau_{rs}} f_j(x) \\ &= \nu_r(\partial_s \mathcal{D}_\lambda \vec{f})_j\Big|_{\partial\Omega}(x) - \nu_s(\partial_r \mathcal{D}_\lambda \vec{f})_j\Big|_{\partial\Omega}(x) - \tfrac{1}{2}\partial_{\tau_{rs}} f_j(x), \end{aligned} \tag{4.90}$$

at almost every $x \in \partial\Omega$. Now, if $\partial_j S_\Delta$ stands for the principal-value integral operator on $\partial\Omega$ with kernel $(\partial_j E_\Delta)(x-y)$, then at almost every point on $\partial\Omega$, we have from (4.84) and (4.50)

$$\begin{aligned}
\partial_s(\mathcal{D}_\lambda \vec{f})_j\Big|_{\partial\Omega} &= \tfrac{1}{2}\nu_\alpha(\delta_{jk}-\nu_j\nu_k)\partial_{\tau_{\alpha s}} f_k - \partial_\alpha S_{jk}(\partial_{\tau_{\alpha s}} f_k) \\
&\quad +\lambda\,\tfrac{1}{2}\nu_j(\delta_{\alpha k}-\nu_\alpha\nu_k)\partial_{\tau_{\alpha s}} f_k - \lambda\,\partial_j S_{\alpha k}(\partial_{\tau_{\alpha s}} f_k) \\
&\quad -\tfrac{1}{2}\nu_k\partial_{\tau_{sj}} f_k + \partial_k S_\Delta(\partial_{\tau_{sj}} f_k),
\end{aligned} \tag{4.91}$$

with a similar formula for $\partial_r(\mathcal{D}_\lambda \vec{f})_j\Big|_{\partial\Omega}$. Note that

$$\begin{aligned}
\nu_\alpha(\delta_{jk}-\nu_j\nu_k)\partial_{\tau_{\alpha s}} f_k &= \nu_\alpha(\delta_{jk}-\nu_j\nu_k)(\nu_\alpha(\nabla_{tan} f_k)_s - \nu_s(\nabla_{tan} f_k)_\alpha) \\
&= (\nabla_{tan} f_j)_s - \nu_j\nu_k(\nabla_{tan} f_k)_s,
\end{aligned} \tag{4.92}$$

and similarly,

$$\nu_j(\delta_{\alpha k}-\nu_\alpha\nu_k)\partial_{\tau_{\alpha s}} f_k = -\nu_j\nu_s(\nabla_{tan} f_k)_k, \tag{4.93}$$

$$\nu_k\partial_{\tau_{sj}} f_k = \nu_k\nu_s(\nabla_{tan} f_k)_j - \nu_k\nu_j(\nabla_{tan} f_k)_s. \tag{4.94}$$

Thus, the jump-terms in $\nu_r\partial_s(\mathcal{D}_\lambda \vec{f})_j\Big|_{\partial\Omega} - \nu_s\partial_r(\mathcal{D}_\lambda \vec{f})_j\Big|_{\partial\Omega}$ amount to $\frac{1}{2}J_1 + \frac{\lambda}{2}J_2 - \frac{1}{2}J_3$ where

$$\begin{aligned}
J_1 &:= \nu_r(\nabla_{tan} f_j)_s - \nu_s(\nabla_{tan} f_j)_r - \nu_r\nu_j\nu_k(\nabla_{tan} f_k)_s + \nu_s\nu_j\nu_k(\nabla_{tan} f_k)_r \\
&= \partial_{\tau_{rs}} f_j - \nu_j\nu_k\partial_{\tau_{rs}} f_k,
\end{aligned} \tag{4.95}$$

$$J_2 := -\nu_s\nu_j\nu_r(\nabla_{tan} f_k)_k + \nu_r\nu_j\nu_s(\nabla_{tan} f_k)_k = 0, \tag{4.96}$$

and

$$\begin{aligned}J_3 &:= \nu_r\nu_k\nu_s(\nabla_{tan} f_k)_j - \nu_r\nu_k\nu_j(\nabla_{tan} f_k)_s - \nu_s\nu_k\nu_r(\nabla_{tan} f_k)_j + \nu_s\nu_k\nu_j(\nabla_{tan} f_k)_r \\ &= -\nu_j\nu_k\partial_{\tau_{rs}} f_k. \end{aligned}\tag{4.97}$$

Thus, $\frac{1}{2}J_1 + \frac{\lambda}{2}J_2 - \frac{1}{2}J_3 = \frac{1}{2}\partial_{\tau_{rs}} f_j$, which cancels the last term in (4.90). In summary, all the jump-terms cancel out, and we arrive at the identity

$$\begin{aligned}\partial_{\tau_{rs}}(K_\lambda \vec{f})_j &= \nu_s\partial_\alpha S_{jk}(\partial_{\tau_{\alpha r}} f_k) + \lambda\,\nu_s\partial_j S_{\alpha k}(\partial_{\tau_{\alpha r}} f_k) - \nu_s\partial_k S_\Delta(\partial_{\tau_{rj}} f_k) \\ &\quad -\nu_r\partial_\alpha S_{jk}(\partial_{\tau_{\alpha s}} f_k) - \lambda\,\nu_r\partial_j S_{\alpha k}(\partial_{\tau_{\alpha s}} f_k) + \nu_r\partial_k S_\Delta(\partial_{\tau_{sj}} f_k),\end{aligned}\tag{4.98}$$

valid at almost every boundary point. Since $\partial_{\tau_{\alpha\beta}} f_k$ is a (p, p_o)-atom supported in $S_1(0)$, Lemma 4.8 ensures that, up to a fixed multiple, each term in the right hand-side of (4.98) satisfies the last two conditions in (2.34). There remains to show that $m := \partial_{\tau_{rs}} K_\lambda \vec{f}$ integrates to zero on $\partial\Omega$.

To justify this, fix a function $\psi \in C_0^\infty(B(0,2))$ such that $\psi \equiv 1$ on $B(0,1)$, and for each $k \in \mathbb{N}$ set $\psi_k(x) := \psi(2^{-k}x)$. Note that $\partial_{\tau_{sr}}\psi_k$ is supported in the annulus $\Delta_k := S_{2^{k+1}} \setminus S_{2^k}$ and satisfies $\|\partial_{\tau_{sr}}\psi_k\|_{L^\infty} \le C2^{-k}$. Also, $|K_\lambda \vec{f}(x)| \le C2^{-k(n-1)}$ for $x \in \Delta_k$. We can then use (2.16) in order to estimate

$$\Big|\int_{\partial\Omega} \psi_k(x)\partial_{\tau_{rs}} K_\lambda \vec{f}(x)\,d\sigma(x)\Big| = \Big|\int_{\partial\Omega} \partial_{\tau_{sr}}\psi_k(x) K_\lambda \vec{f}(x)\,d\sigma(x)\Big| \le C2^{-k}.\tag{4.99}$$

Thus,

$$\int_{\partial\Omega} \partial_{\tau_{rs}} K_\lambda \vec{f}(x)\,d\sigma(x) = \lim_{k\to\infty}\int_{\partial\Omega} \psi_k(x)\partial_{\tau_{rs}} K_\lambda \vec{f}(x)\,d\sigma(x) = 0,\tag{4.100}$$

as wanted. This finishes the proof of the proposition in the case when $\frac{n-1}{n} < p \le 1$. Finally, when $1 < p < \infty$, the desired conclusion follows from (4.90) and Proposition 4.10. $\square$

4.3 Traces of hydrostatic layer potentials in Hardy spaces

Consider the following general trace result.

Theorem 4.13 *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be the domain lying above the graph of a Lipschitz function and assume that $\frac{n-1}{n} < p < \infty$, $\lambda \in \mathbb{R}$. Then there exists a finite constant $C = C(\partial\Omega, p, \lambda) > 0$ with the following property. Whenever $\vec{u}$, π satisfy*

$$\begin{aligned} &\Delta\vec{u} = \nabla\pi, \quad \operatorname{div}\vec{u} = 0 \quad \text{in } \Omega, \\ &M(\nabla\vec{u}),\, M(\pi) \in L^p(\partial\Omega), \end{aligned} \tag{4.101}$$

then

$$\vec{u}\Big|_{\partial\Omega} \in H^p_1(\partial\Omega), \quad \partial^\lambda_\nu(\vec{u},\pi) \in H^p(\partial\Omega), \tag{4.102}$$

where the traces are taken in the sense described in § 11.6. Furthermore,

$$\|\vec{u}|_{\partial\Omega}\|_{H^p_1(\partial\Omega)} + \|\partial^\lambda_\nu(\vec{u},\pi)\|_{H^p(\partial\Omega)} \leq C\|M(\nabla\vec{u})\|_{L^p(\partial\Omega)} + C\|M(\pi)\|_{L^p(\partial\Omega)}. \tag{4.103}$$

Finally, similar results are valid in the case when Ω is a bounded Lipschitz domain in $\mathbb{R}^n$. In this case, (4.101) imply

$$\begin{aligned} &\vec{u}\Big|_{\partial\Omega} \in h^p_1(\partial\Omega), \quad \partial^\lambda_\nu(\vec{u},\pi) \in h^p(\partial\Omega), \quad \text{and} \\ &\|\vec{u}|_{\partial\Omega}\|_{h^p_1(\partial\Omega)} + \|\partial^\lambda_\nu(\vec{u},\pi)\|_{h^p(\partial\Omega)} \leq C\|M(\nabla\vec{u})\|_{L^p(\partial\Omega)} + C\|M(\pi)\|_{L^p(\partial\Omega)}. \end{aligned} \tag{4.104}$$

Proof. The well-posedness of the Dirichlet problem for the Stokes system in Lipschitz domains with data in $L^2(\partial\Omega)$, established in [34], and arguments which are well-understood by now (cf.the proof of Proposition 3.1 in [71] for details in similar circumstances) imply the following Fatou-type result:

$$(\vec{u},\pi) \text{ as in (4.101) and } M(\vec{u}) < \infty \text{ a.e. on } \partial\Omega \Longrightarrow \vec{u}\Big|_{\partial\Omega} \text{ exists a.e. on } \partial\Omega. \tag{4.105}$$

Moreover, since (4.101) imply that $\Delta\pi = \operatorname{div}\nabla\pi = \operatorname{div}\Delta\vec{u} = \Delta(\operatorname{div}\vec{u}) = 0$, we can utilize the following result established by B. Dahlberg in [18],

$$\Delta\pi = 0 \text{ in } \Omega \text{ and } M(\pi) < \infty \text{ a.e. on } \partial\Omega \Longrightarrow \pi\Big|_{\partial\Omega} \text{ exists a.e. on } \partial\Omega. \tag{4.106}$$

Then the theorem follows from (4.105) and (4.106) whenever $1 < p < \infty$. There remains to consider the case when $\frac{n-1}{n} < p \leq 1$. In this scenario, we introduce the vector fields

$$\vec{F}^r_{jk} := (\partial_k u_r)e_j - (\partial_j u_r)e_k \quad \text{in } \Omega, \qquad j,k,r \in \{1,...,n\}, \tag{4.107}$$

where $\{e_\ell\}_{1\leq\ell\leq n}$ is the standard orthonormal basis in $\mathbb{R}^n$. Note that, for each j,k,r,

$$\begin{aligned} & M(\vec{F}^r_{jk}) \in L^p(\partial\Omega), \ \vec{F}^r_{jk} \text{ has biharmonic components,} \\ & \operatorname{div} \vec{F}^r_{jk} = \partial_j\partial_k u_r - \partial_k\partial_j u_r = 0 \quad \text{in } \Omega, \\ & \langle \vec{F}^r_{jk}, \nu\rangle = \nu_j\partial_k u_r - \nu_k\partial_j u_r = \partial_{\tau_{jk}} u_r \quad \text{on } \partial\Omega. \end{aligned} \tag{4.108}$$

Then (2.43) and Corollary 11.14 give that

$$\|\vec{u}|_{\partial\Omega}\|_{H^p_1(\partial\Omega)} \approx \sum_{j,k=1}^n \|\partial_{\tau_{jk}}\vec{u}\|_{H^p(\partial\Omega)} \leq C\|M(\nabla\vec{u})\|_{L^p(\partial\Omega)}. \tag{4.109}$$

This proves the first membership in (4.102) and part of the estimate (4.103).

To bring in the conormal derivative, define

$$\vec{F}_j := \nabla u_j + \lambda\,\partial_j\vec{u} - \pi\, e_j, \qquad j \in \{1,...,n\}. \tag{4.110}$$

Then

$$\begin{aligned} & M(\vec{F}_j) \in L^p(\partial\Omega), \ \vec{F}_j \text{ has biharmonic components,} \\ & \operatorname{div} \vec{F}_j = (L_\lambda\vec{u})_j - \partial_j\pi = 0 \ \text{ in } \Omega, \\ & \langle \vec{F}_j, \nu\rangle = \big(\partial^\lambda_\nu(\vec{u},\pi)\big)_j \quad \text{on } \partial\Omega. \end{aligned} \tag{4.111}$$

Then Corollary 11.14 gives $\partial^\lambda_\nu(\vec{u},\pi) \in H^p(\partial\Omega)$ and

$$\|\partial^\lambda_\nu(\vec{u},\pi)\|_{H^p(\partial\Omega)} \leq C\|M(\nabla\vec{u})\|_{L^p(\partial\Omega)} + C\|M(\pi)\|_{L^p(\partial\Omega)}. \tag{4.112}$$

The argument for the case when Ω is a bounded Lipschitz domain is similar, and this finishes the proof of the theorem. $\square$

We can now state the following result regarding the traces of hydrostatic layer potentials.

Corollary 4.14 *Let Ω be a graph Lipschitz domain in $\mathbb{R}^n$, and assume that $\frac{n-1}{n} < p < \infty$, $\lambda \in \mathbb{R}$. Then*

$$\partial_\nu^\lambda(\mathcal{S}\vec{f}, \mathcal{Q}\vec{f})\Big|_{\partial\Omega_\pm} = \Big(\mp\tfrac{1}{2}I + K_\lambda^*\Big)\vec{f} \quad \text{in} \quad H^p(\partial\Omega), \quad \forall\, \vec{f} \in H^p(\partial\Omega), \tag{4.113}$$

$$\mathcal{D}_\lambda \vec{f}\Big|_{\partial\Omega_\pm} = \Big(\pm\tfrac{1}{2}I + K_\lambda\Big)\vec{f} \quad \text{in} \quad H_1^p(\partial\Omega), \quad \forall\, \vec{f} \in H_1^p(\partial\Omega), \tag{4.114}$$

$$\partial_{\tau_{jk}}\mathcal{S}\vec{f}\Big|_{\partial\Omega_+} = \partial_{\tau_{jk}}\mathcal{S}\vec{f}\Big|_{\partial\Omega_-} \quad \text{in} \quad H^p(\partial\Omega), \quad \forall\, \vec{f} \in H^p(\partial\Omega), \tag{4.115}$$

for every $j, k \in \{1, ..., n\}$. In particular,

$$\mathcal{S}\vec{f}\Big|_{\partial\Omega_+} = \mathcal{S}\vec{f}\Big|_{\partial\Omega_-} \quad \text{in} \quad H_1^p(\partial\Omega). \tag{4.116}$$

Moreover,

$$\partial_\nu^\lambda(\mathcal{D}_\lambda\vec{f}, \mathcal{P}_\lambda\vec{f})\Big|_{\partial\Omega_+} = \partial_\nu^\lambda(\mathcal{D}_\lambda\vec{f}, \mathcal{P}_\lambda\vec{f})\Big|_{\partial\Omega_-} \quad \text{in} \quad H^p(\partial\Omega), \quad \forall\, \vec{f} \in H_1^p(\partial\Omega). \tag{4.117}$$

Finally, analogous results hold in the case when $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, provided the Hardy spaces $H^p(\partial\Omega)$ and $H_1^p(\partial\Omega)$ are replaced by their local versions.

Proof. Consider formula (4.113). This is going to be a consequence of the fact that K_λ^* is bounded on $H^p(\partial\Omega)$ the observation that, by Theorem 4.13, the assignments

$$H^p(\partial\Omega) \ni \vec{f} \mapsto \partial_\nu^\lambda(\mathcal{S}\vec{f}, \mathcal{Q}\vec{f})\Big|_{\partial\Omega_\pm} \in H^p(\partial\Omega) \tag{4.118}$$

are bounded, plus the fact that (4.113) holds when $\vec{f}$ is an atom for $H^p(\partial\Omega)$, thanks to Proposition 4.4. All the other identities can be proved in a similar manner. $\square$

4.4 Integral representation formulas

We begin this section with the following useful representation formulas for solutions of the Stokes system.

Proposition 4.15 [`Green's Representation Formulas`]
Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be either a bounded Lipschitz domain, or a graph Lipschitz domain. For $1 \leq p < \infty$ fixed, assume that the functions $(\vec{u}, \pi)$ satisfy

$$\Delta\vec{u} - \nabla\pi = 0 \;\; in \;\; \Omega, \quad \operatorname{div}\vec{u} = 0 \;\; in \;\; \Omega, \quad and \quad M(\nabla\vec{u}), M(\pi) \in L^p(\partial\Omega). \tag{4.119}$$

Then $\vec{u}$ and π also satisfy the following integral representation formulas (modulo constants):

$$\vec{u}(x) = \mathcal{D}_\lambda\Big(\vec{u}\Big|_{\partial\Omega}\Big)(x) - \mathcal{S}\Big(\partial_\nu^\lambda(\vec{u}, \pi)\Big)(x), \qquad x \in \Omega, \tag{4.120}$$

$$\pi(x) = \mathcal{P}_\lambda\Big(\vec{u}\Big|_{\partial\Omega}\Big)(x) - \mathcal{Q}\Big(\partial_\nu^\lambda(\vec{u}, \pi)\Big)(x), \qquad x \in \Omega. \tag{4.121}$$

Proof. The identity (4.120) can be established, at least at the formal level, by specializing Green's formula (4.7) to the case when $\vec{w} := (E_{kj}(x - \cdot))_{1 \leq k \leq n}$, $\rho := q_j(x - \cdot)$, where $x \in \Omega$ is fixed and $j \in \{1, ..., n\}$ is arbitrary. If Ω is a bounded Lipschitz domain, (4.120) can be justified by writing (4.120) for a sequence of sub-domains Ω_j approximating the original Ω in the fashion described in Theorem 1.12 on p. 581 in [94], and then letting $j \to \infty$. Here, (4.105) and (4.106) are also used.

On the other hand, we also wish to establish (4.120) in the case when Ω is the upper-graph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$. In this case, we will show that

$$\partial_j\vec{u}(x) = \partial_j\mathcal{D}_\lambda\Big(\vec{u}\Big|_{\partial\Omega}\Big)(x) - \partial_j\mathcal{S}\Big(\partial_\nu^\lambda(\vec{u}, \pi)\Big)(x), \qquad x \in \Omega, \quad 1 \leq j \leq n, \tag{4.122}$$

which is enough to prove (4.120) modulo constants.

Fix $x \in \Omega$, $1 \leq j \leq n$, and for each $r, s > 0$, consider the bounded Lipschitz domain

$$D_{r,s} := \{y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \, |y'| < r, \; 0 < y_n - \varphi(y') < s\}. \tag{4.123}$$

Assume r and s are large enough so that $x \in D_{r,s}$ and $dist(x, \partial D_{r,s}) = dist(x, \partial\Omega)$. In particular, (4.122) holds for the domain $D_{r,s}$. Dividing the boundary of $D_{r,s}$ into its bottom, top, and vertical portions, we can write

$$\partial D_{r,s} = B_{r,s} \cup T_{r,s} \cup V_{r,s}, \tag{4.124}$$

where

$$\begin{aligned} B_{r,s} &:= \partial D_{r,s} \cap \partial\Omega, \\ T_{r,s} &:= \{y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \, |y'| \leq r, \; y_n = \varphi(x') + s\}, \\ V_{r,s} &:= \{y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \, |y'| = r, \; 0 < y_n - \varphi(y') < s\}. \end{aligned} \tag{4.125}$$

Consider the version of (4.122) written for the domain $D_{r,s}$, and let us break the right hand side into three separate terms corresponding to integrals over the bottom, top, and vertical portions of $\partial D_{r,s}$, In particular,

$$\partial_j \vec{u}(x) = I_{r,s} + II_{r,s} + III_{r,s}, \tag{4.126}$$

where the terms $I_{r,s}, II_{r,s}$, and $III_{r,s}$ correspond to integrals over $B_{r,s}, T_{r,s}$, and $V_{r,s}$ respectively. Next, we will monitor what happens to these terms as the parameters r, s approach infinity (in a suitable fashion).

We first claim that

$$\partial_j \, \mathcal{S}(\vec{f}\chi_{S_r(0)})(x) \longrightarrow \partial_j \, \mathcal{S}\vec{f}(x) \text{ as } r \to \infty \text{ for any } \vec{f} \in L^p(\partial\Omega), p \geq 1. \tag{4.127}$$

Since $x \in \Omega$ is fixed, for $y \in \partial\Omega$,

$$|\nabla E(x-y)| \leq \frac{C}{(1+|y|)^{n-1}} \in L^q(\partial\Omega) \quad \text{for every} \quad 1 < q \leq \infty, \tag{4.128}$$

and so (4.127) follows by the Lebesgue Dominated Convergence Theorem. Note that (4.127) also holds if we replace $\mathcal{S}$ with $\mathcal{S}_\Delta$. Now according to (4.84), we can rewrite derivatives on $\mathcal{D}_\lambda \vec{f}$ as a sum of derivatives on $\mathcal{S}$ and $\mathcal{S}_\Delta$ applied to tangential derivatives of $\vec{f}$. Then since $M(\nabla\vec{u}), M(\pi) \in L^p(\partial\Omega)$, it follows from (4.127) that the term $I_{r,s}$ converges to the right side of (4.122) as $r \to \infty$. By rewriting derivatives on the double layer as combinations of derivatives on single layers as before, we can also show that

$$\begin{aligned} |II_{r,s}| &\leq \int_{T_{r,s}} (|\nabla E(x-y)| + |\nabla E_\Delta(x-y)|)(|\nabla\vec{u}(y)| + |\pi(y)|)\, d\sigma_y, & (4.129)\\ |III_{r,s}| &\leq \int_{V_{r,s}} (|\nabla E(x-y)| + |\nabla E_\Delta(x-y)|)(|\nabla\vec{u}(y)| + |\pi(y)|)\, d\sigma_y. & (4.130) \end{aligned}$$

Estimating as in (4.128), for $q > 1$, we can write

$$\begin{aligned} \int_{T_{r,s}} |\nabla E(x-z)|^q \, d\sigma_z &\leq C \int_{T_{r,s}} \frac{1}{(1+|z|)^{(n-1)q}}\, d\sigma_z \leq C \int_{\partial D_{r,s}\cap\partial\Omega} \frac{1}{(1+|y+se_n|)^{(n-1)q}}\, d\sigma_y \\ &\leq C \int_{\mathbb{R}^{n-1}} \frac{1}{(s+|y'|)^{(n-1)q}}\, dy' \leq C s^{(n-1)(1-q)} \int_{\mathbb{R}^{n-1}} \frac{1}{(1+|w|)^{(n-1)q}}\, dw \\ &\leq C s^{(n-1)(1-q)}. \end{aligned} \tag{4.131}$$

In particular, repeating the same argument also for E_Δ,

$$\|\nabla E(x-\cdot) + \nabla E_\Delta(x-\cdot)\|_{L^q(T_{r,s})} \leq C s^{(n-1)(\frac{1}{q}-1)}, \quad \text{for any } 1 < q \leq \infty, \tag{4.132}$$

where the L^∞ estimate follows from (4.128). Then using (4.129), we can estimate $II_{r,s}$ by

$$|II_{r,s}| \leq C s^{-(n-1)\frac{1}{p}} \left(\|M(\nabla\vec{u})\|_{L^p(\partial\Omega)} + \|M(\pi)\|_{L^p(\partial\Omega)} \right) \longrightarrow 0 \quad \text{as} \quad s \to \infty. \tag{4.133}$$

Let us also note that if $z \in \partial\Omega$ is far away from $x \in \Omega$, then for any $w \in \Gamma(z)$, $|x-w| \sim |x-z|$, and so in fact

$$M(\nabla E(x-\cdot))(z) \leq \frac{C}{(1+|z|)^{n-1}}. \tag{4.134}$$

Then for r large,

$$\begin{aligned} \int\limits_{B_{2r,s}\setminus B_{r,s}} |M(\nabla E(x-\cdot))(z)|^q \, d\sigma_z &\leq C \int\limits_{B_{2r,s}\setminus B_{r,s}} \frac{1}{(1+|z|)^{(n-1)q}} \, d\sigma_z \\ &\leq Cr^{(n-1)(1-q)}, \end{aligned} \tag{4.135}$$

and so after repeating the argument for E_Δ, it follows that

$$\|M(\nabla E(x-\cdot)) + M(\nabla E_\Delta(x-\cdot))\|_{L^q(B_{2r,s}\setminus B_{r,s})} \leq Cr^{(n-1)(\frac{1}{q}-1)}, \quad \text{for any } 1 < q \leq \infty. \tag{4.136}$$

Then using (4.130), we can show that for R large,

$$\frac{1}{R}\int\limits_R^{2R} |III_{r,s}| \, dr \leq \frac{Cs}{R^{1+(n-1)\frac{1}{p}}} \left(\|M(\nabla \vec{u})\|_{L^p(\partial\Omega)} + \|M(\pi)\|_{L^p(\partial\Omega)} \right) \longrightarrow 0 \text{ as } R \to \infty. \tag{4.137}$$

Finally, (4.122) can be established by averaging (4.126) over $r \in [R, 2R]$ and then taking the limit as R and s approach infinity.

To establish (4.121), let $\{e_\ell\}_{1\leq \ell \leq n}$ be the standard orthonormal basis in $\mathbb{R}^n$ and for $x \in \Omega$, write

$$\begin{aligned} -\mathcal{Q}\Big(\partial_\nu^\lambda(\vec{u},\pi)\Big)(x) &= \int_{\partial\Omega} \Big\langle (\nabla E_\Delta)(x-y) \, , \, \partial_\nu^\lambda(\vec{u},\pi)(x) \Big\rangle \, d\sigma(y) \\ &= \partial_\ell \left[\int_{\partial\Omega} \Big\langle E_\Delta(x-y)e_\ell \, , \, \partial_\nu^\lambda(\vec{u},\pi)(x) \Big\rangle \, d\sigma(y) \right] \\ &= \partial_\ell \left[\int_\Omega A_\lambda\Big((\nabla\vec{u})(y), \nabla_y(E_\Delta(x-y)e_\ell) \Big) \, dy \right] - \partial_\ell \left[\int_\Omega \pi(y)(\partial_\ell E_\Delta)(x-y) \, dy \right] \\ &= -\partial_\ell \left[\int_\Omega \Big((\partial_j u_\ell)(y)(\partial_j E_\Delta)(x-y) + \lambda(\partial_\ell u_k)(y)(\partial_k E_\Delta)(x-y) \Big) \, dy \right] + \pi(x) \end{aligned}$$

$$= -\lim_{\varepsilon\to 0} \int\limits_{\substack{y\in\Omega\\|x-y|>\varepsilon}} \Big((\partial_j u_\ell)(y)(\partial_\ell\partial_j E_\Delta)(x-y) + \lambda(\partial_\ell u_k)(y)(\partial_\ell\partial_k E_\Delta)(x-y)\Big)\,dy$$

$$+\pi(x)$$

$$= -(1+\lambda)\lim_{\varepsilon\to 0} \int\limits_{\substack{y\in\Omega\\|x-y|>\varepsilon}} (\partial_j u_k)(y)(\partial_j\partial_k E_\Delta)(x-y)\,dy + \pi(x). \tag{4.138}$$

Above, (4.27) and (4.34) have been used in the first equality, (4.6) with $\vec{w} := E_\Delta(x-\cdot)e_\ell$ in the third, $\Delta E_\Delta = \delta$ and the identity

$$\begin{aligned} A_\lambda\Big(\nabla\vec{u}, \nabla_y(E_\Delta(x-\cdot)e_\ell\Big) &= \Big(\delta_{jk}\delta_{\alpha\beta} + \lambda\,\delta_{j\beta}\delta_{k\alpha}\Big)(\partial_j u_\alpha)(\partial_k E_\Delta)(x-\cdot)\delta_{\beta\ell} \\ &= -(\partial_j u_\ell)(\partial_j E_\Delta)(x-\cdot) - \lambda(\partial_\ell u_k)(\partial_k E_\Delta)(x-\cdot) \end{aligned} \tag{4.139}$$

in the fourth and, in the fifth, a well-know differentiation formula for singular integrals plus the fact that

$$\int_{S^{n-1}} (\partial_j\partial_k E_\Delta)(\omega)\,d\omega = 0, \qquad \forall\, j,k \in \{1,...,n\}. \tag{4.140}$$

On the other hand, since $\vec{u}$ is divergence-free, we have $\partial_{\tau_{jk}} u_k = -\nu_k(\partial_j u_k)|_{\partial\Omega}$, so (4.85) gives

$$\begin{aligned} \mathcal{P}_\lambda\Big(\vec{u}\Big|_{\partial\Omega}\Big)(x) &= (1+\lambda)\partial_j\mathcal{S}_\Delta(\partial_{\tau_{jk}}u_k)(x) = -(1+\lambda)\partial_j\mathcal{S}_\Delta\Big(\nu_k(\partial_j u_k)\Big|_{\partial\Omega}\Big)(x) \\ &= -(1+\lambda)\partial_j\left[\int_{\partial\Omega} E_\Delta(x-y)\nu_k(y)(\partial_j u_k)(y)\,d\sigma(y)\right] \\ &= (1+\lambda)\partial_j\left[\int_\Omega (\partial_k E_\Delta)(x-y)(\partial_j u_k)(y)\,dy\right] \\ &= (1+\lambda)\lim_{\varepsilon\to 0} \int\limits_{\substack{y\in\Omega\\|x-y|>\varepsilon}} (\partial_j\partial_k E_\Delta)(x-y)(\partial_j u_k)(y)\,dy, \end{aligned} \tag{4.141}$$

where we have integrated by parts and used div $\vec{u} = 0$ in the third equality and differentiated under the integral sign in the last step (here (4.140) was also used). Now, (4.121) follows from (4.138) and (4.141). Once this is established for nice domains, we can use the same approximation arguments from the proof of (4.120) to prove (4.121) for bounded Lipschitz domains and then also for graph Lipschitz domains. □

The previous representation formulas allow us to prove the following useful identities.

Proposition 4.16 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a either a bounded Lipschitz domain or the upper graph of a Lipschitz function. Then for any $\frac{n-1}{n} < p < \infty$,*

$$S(\partial_\nu^\lambda(\mathcal{D}_\lambda \vec{f}, \mathcal{P}_\lambda \vec{f})) = (\tfrac{1}{2}I + K_\lambda)(-\tfrac{1}{2}I + K_\lambda)\vec{f}, \qquad \forall \vec{f} \in h_1^p(\partial\Omega). \tag{4.142}$$

Proof. This follows by applying Green's formula (4.120) to the functions $\vec{u} = \mathcal{D}_\lambda \vec{f}$ and $\pi = \mathcal{P}_\lambda \vec{f}$ and then taking boundary traces. □

Proposition 4.17 *Let $\Omega_\pm \subseteq \mathbb{R}^n$, $n \geq 2$, be the domains lying above and below the graph of a Lipschitz function. Assume $(\vec{u}_\pm, \pi_\pm)$ solve the Stokes system in $\Omega_\pm$, and that $M(\nabla \vec{u}_\pm), M(\pi_\pm) \in L^p(\partial\Omega)$ for $p \in [1, \infty)$. Then the following boundary identities hold:*

$$\left(\mp\tfrac{1}{2}I + K_\lambda\right)(\vec{u}_\pm|_{\partial\Omega}) = S(\partial_\nu^\lambda(\vec{u}_\pm, \pi_\pm)) \quad \text{in } H_1^p(\partial\Omega), \tag{4.143}$$

and

$$\left(\pm\tfrac{1}{2}I + K_\lambda^*\right)(\partial_\nu^\lambda(\vec{u}_\pm, \pi_\pm)) = \partial_\nu^\lambda\left(\mathcal{D}(\vec{u}_\pm|_{\partial\Omega}), \mathcal{P}(\vec{u}_\pm|_{\partial\Omega})\right) \quad \text{in } H^p(\partial\Omega). \tag{4.144}$$

Proof. Since $\vec{\nu}_- = -\vec{\nu}$, applying (4.120) and (4.121) to $(\vec{u}_\pm, \pi_\pm)$ gives

$$\vec{u}_\pm(x) = \pm\mathcal{D}_\lambda\Big(\vec{u}_\pm\Big|_{\partial\Omega}\Big)(x) \mp \mathcal{S}\Big(\partial_\nu^\lambda(\vec{u}_\pm, \pi_\pm)\Big)(x), \qquad x \in \Omega_\pm; \tag{4.145}$$

$$\pi_\pm(x) = \pm\mathcal{P}_\lambda\Big(\vec{u}_\pm\Big|_{\partial\Omega}\Big)(x) \mp \mathcal{Q}\Big(\partial_\nu^\lambda(\vec{u}_\pm, \pi_\pm)\Big)(x), \qquad x \in \Omega_\pm. \tag{4.146}$$

Applying these identities in the definition of the conormal derivative, we can write

$$\partial_\nu^\lambda(\vec{u}_\pm,\pi_\pm) = \pm\partial_\nu^\lambda\Big(\mathcal{D}_\lambda(\vec{u}_\pm\,|_{\partial\Omega}),\mathcal{P}_\lambda(\vec{u}_\pm\,|_{\partial\Omega})\Big) \mp \partial_\nu^\lambda\Big(\mathcal{S}(\partial_\nu^\lambda(\vec{u}_\pm,\pi_\pm)),\mathcal{Q}(\partial_\nu^\lambda(\vec{u}_\pm,\pi_\pm)\Big) \tag{4.147}$$

The jump relation (4.45) then gives

$$\partial_\nu^\lambda(\vec{u}_\pm,\pi_\pm) = \pm\partial_\nu^\lambda\Big(\mathcal{D}_\lambda(\vec{u}_\pm\,|_{\partial\Omega}),\mathcal{P}_\lambda(\vec{u}_\pm\,|_{\partial\Omega})\Big) \mp (\mp\tfrac{1}{2}I+K_\lambda^*)\Big(\partial_\nu^\lambda(\vec{u}_\pm,\pi_\pm)\Big), \tag{4.148}$$

which is enough to establish (4.144). Similarly, taking boundary traces in (4.145) and using the jump relation (4.43) leads to

$$\vec{u}_\pm|_{\partial\Omega} = \pm(\pm\tfrac{1}{2}I+K_\lambda)(\vec{u}_\pm|_{\partial\Omega}) \mp S(\partial_\nu^\lambda(\vec{u}_\pm,\pi_\pm)), \tag{4.149}$$

from which (4.143) follows. □

4.5 Boundary integral operators and the transmission problem

In this section we assume that Ω is a graph Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. As usual, set $\Omega_+ := \Omega$, $\Omega_- := \mathbb{R}^n \setminus \bar{\Omega}$. We begin with the following uniqueness result.

Proposition 4.18 *Assume that $(\vec{u}_\pm,\pi_\pm)$ are solutions to the Stokes system*

$$\Delta\vec{u}_\pm = \nabla\pi_\pm, \quad \operatorname{div}\vec{u}_\pm = 0 \ \ \textit{in} \ \ \Omega_\pm, \qquad \textit{and} \qquad M(\nabla\vec{u}_\pm), M(\pi_\pm) \in L^p(\partial\Omega), \tag{4.150}$$

for some $\frac{n-1}{n} < p < \infty$, and that, in addition, they satisfy

$$\vec{u}_+|_{\partial\Omega} = \vec{u}_-|_{\partial\Omega} \qquad \textit{and} \qquad \partial_\nu^\lambda(\vec{u}_+,\pi_+) = \partial_\nu^\lambda(\vec{u}_-,\pi_-). \tag{4.151}$$

Then $\vec{u}_\pm$ and $\pi_\pm$ are constant.

Proof. Consider the functions

$$\vec{u} := \begin{cases} \vec{u}_+ \ \text{ in } \ \Omega_+, \\ \vec{u}_- \ \text{ in } \ \Omega_-, \end{cases} \qquad \text{and} \qquad \pi := \begin{cases} \pi_+ \ \text{ in } \ \Omega_+, \\ \pi_- \ \text{ in } \ \Omega_-. \end{cases} \tag{4.152}$$

Then $(\vec{u}, \pi)$ solves the Stokes system in $\mathbb{R}^n$. Let $M(\nabla\vec{u}) := \max\{M(\nabla\vec{u}_+), M(\nabla\vec{u}_-)\}$. Then for every fixed $x \in \mathbb{R}^n$ and R much larger than $\operatorname{dist}(x, \partial\Omega)$, interior estimates give

$$|\nabla\vec{u}(x)| \leq \left(\fint_{B_R(x)} |\nabla\vec{u}|^p\right)^{1/p} \leq CR^{-\frac{n-1}{p}} \|M(\nabla\vec{u})\|_{L^p(\partial\Omega)}. \tag{4.153}$$

After taking the limit as $R \to \infty$ in (4.153), it follows that $\nabla u \equiv 0$ in $\mathbb{R}^n$, and hence, $\vec{u}$ is a constant vector. Then since $\nabla\pi = \Delta\vec{u} \equiv 0$ in $\mathbb{R}^n$, we know that π must also be constant. $\square$

Suppose that

$$\vec{f} \in H^p(\partial\Omega), \qquad \vec{g} \in H_1^p(\partial\Omega), \tag{4.154}$$

are arbitrary, and for each $\mu \in [0, 1)$, consider the following transmission problems:

$$(T_\mu^+)^* \begin{cases} \vec{u}_\pm, \pi_\pm \ \text{ as in (4.150)}, \\ \vec{u}_+|_{\partial\Omega} - \vec{u}_-|_{\partial\Omega} = \vec{g}, \\ \partial_\nu^\lambda(\vec{u}_+, \pi_+) - \mu\partial_\nu^\lambda(\vec{u}_-, \pi_-) = \vec{f}, \end{cases} \qquad (T_\mu^-)^* \begin{cases} \vec{u}_\pm, \pi_\pm \ \text{ as in (4.150)}, \\ \vec{u}_+|_{\partial\Omega} - \vec{u}_-|_{\partial\Omega} = \vec{g}, \\ \mu\partial_\nu^\lambda(\vec{u}_+, \pi_+) - \partial_\nu^\lambda(\vec{u}_-, \pi_-) = \vec{f}, \end{cases} \tag{4.155}$$

$$(T_\mu^+) \begin{cases} \vec{u}_\pm, \pi_\pm \ \text{ as in (4.150)}, \\ \vec{u}_+|_{\partial\Omega} - \mu\vec{u}_-|_{\partial\Omega} = \vec{g}, \\ \partial_\nu^\lambda(\vec{u}_+, \pi_+) - \partial_\nu^\lambda(\vec{u}_-, \pi_-) = \vec{f}, \end{cases} \qquad (T_\mu^-) \begin{cases} \vec{u}_\pm, \pi_\pm \ \text{ as in (4.150)}, \\ \mu\vec{u}_+|_{\partial\Omega} - \vec{u}_-|_{\partial\Omega} = \vec{g}, \\ \partial_\nu^\lambda(\vec{u}_+, \pi_+) - \partial_\nu^\lambda(\vec{u}_-, \pi_-) = \vec{f}. \end{cases} \tag{4.156}$$

Let us remark that, given that Ω is a graph Lipschitz domain, a convenient interpretation of the boundary condition $\vec{u}_+|_{\partial\Omega} - \vec{u}_-|_{\partial\Omega} = \vec{g}$ in $(T_\mu^\pm)^*$ is $\partial_{\tau_{jk}}\vec{u}_+ - \partial_{\tau_{jk}}\vec{u}_- = \partial_{\tau_{jk}}\vec{g}$ on $\partial\Omega$, for every $j, k \in \{1, ..., n\}$. Similar considerations apply to $(T_\mu^\pm)$.

For any of the problems above and any $\frac{n-1}{n} < p < \infty$ fixed, we will say that problem is *well-posed* if for any data as in (4.154), there exists a solution $(\vec{u}_\pm, \pi_\pm)$ to the problem that must be unique (modulo constants) and which also satisfies the estimate

$$\|M(\nabla\vec{u}_\pm)\|_{L^p(\partial\Omega)} + \|M(\pi_\pm)\|_{L^p(\partial\Omega)} \leq C\left(\|\vec{f}\|_{H^p(\partial\Omega)} + \|\vec{g}\|_{H_1^p(\partial\Omega)}\right). \tag{4.157}$$

Notice that when $\mu = 1$, all of the above problems are identical and can be solved by the functions

$$\vec{u}_\pm := \mathcal{D}_\lambda \vec{g} - \mathcal{S}\vec{f} \text{ in } \Omega_\pm \quad \text{and} \quad \pi_\pm := \mathcal{P}_\lambda \vec{g} - \mathcal{Q}\vec{f} \text{ in } \Omega_\pm. \tag{4.158}$$

Furthermore, from Proposition 4.18, the solution is unique modulo constants. Now the following claims are obviously true:

$$(T_\mu^+)^* \text{ is well-posed} \Longleftrightarrow (T_\mu^-)^*, \text{ written with } \Omega_+ \text{ and } \Omega_- \text{ interchanged, is well-posed.} \tag{4.159}$$

$$(T_\mu^+) \text{ is well-posed} \Longleftrightarrow (T_\mu^-), \text{ written with } \Omega_+ \text{ and } \Omega_- \text{ interchanged, is well-posed.} \tag{4.160}$$

For $\mu > 0$ fixed, the following also hold:

$$\begin{aligned}
&(\vec{u}_+, \pi_+) \text{ and } (\vec{u}_-, \pi_-) \text{ solve } (T_\mu^+)^* \text{ for } (\vec{f}, \vec{g}) \\
&\qquad \Longleftrightarrow (\vec{u}_+, \pi_+) \text{ and } (\mu\vec{u}_-, \mu\pi_-) \text{ solve } (T_\mu^-) \text{ for } (\vec{f}, \mu\vec{g}),
\end{aligned} \tag{4.161}$$

$$\begin{aligned}
&(\vec{u}_+, \pi_+) \text{ and } (\vec{u}_-, \pi_-) \text{ solve } (T_\mu^-)^* \text{ for } (\vec{f}, \vec{g}) \\
&\qquad \Longleftrightarrow (\mu\vec{u}_+, \mu\pi_+) \text{ and } (\vec{u}_-, \pi_-) \text{ solve } (T_\mu^+) \text{ for } (\vec{f}, \mu\vec{g}),
\end{aligned} \tag{4.162}$$

$$\begin{aligned}
&(\vec{u}_+, \pi_+) \text{ and } (\vec{u}_-, \pi_-) \text{ solve } (T_\mu^+)^* \text{ for } (\vec{f}, \vec{g}) \\
&\qquad \Longleftrightarrow (\mu\vec{u}_+, \mu\pi_+) \text{ and } (\mu\vec{u}_-, \mu\pi_-) \text{ solve } (T_{1/\mu}^-)^* \text{ for } (\vec{f}, \mu\vec{g}),
\end{aligned} \tag{4.163}$$

$$\begin{aligned}
&(\vec{u}_+, \pi_+) \text{ and } (\vec{u}_-, \pi_-) \text{ solve } (T_\mu^+) \text{ for } (\vec{f}, \vec{g}) \\
&\qquad \Longleftrightarrow (\mu\vec{u}_+, \mu\pi_+) \text{ and } (\mu\vec{u}_-, \mu\pi_-) \text{ solve } (T_{1/\mu}^-) \text{ for } (\mu\vec{f}, \vec{g}).
\end{aligned} \tag{4.164}$$

From (4.163), we see that analyzing $(T_\mu^+)^*$ in the case $\mu > 1$ is equivalent to analyzing $(T_\mu^-)^*$ in the case when $\mu < 1$ and vice versa. Of course, from (4.164), there is also a similar connection between (T_μ^+) and (T_μ^-). With this in mind, in the sequel we will only deal with the case when $\mu < 1$. Further interconnections between the well-posedness of the four transmission boundary value problems in (4.155)-(4.156) are discussed below.

Proposition 4.19 *Assume that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a graph Lipschitz domain and that $\frac{n-1}{n} < p < \infty$, $-1 < \lambda \leq 1$. Then, for each (consistent) choice of the sign $\pm$ in the statements below, the following two claims are equivalent:*

(i) the transmission problem $(T^{\pm}_{\mu})^*$ is well-posed for every $\mu \in [0,1)$,

(ii) the operator

$$\pm\tfrac{1}{2}\tfrac{\mu+1}{\mu-1}I + K^*_\lambda : H^p(\partial\Omega) \longrightarrow H^p(\partial\Omega) \tag{4.165}$$

is an isomorphism for every $\mu \in [0,1)$.

Moreover, for each (consistent) choice of the sign $\pm$ in the statements below, the following two claims are also equivalent:

(iii) the transmission problem $(T^{\pm}_{\mu})$ is well-posed for every $\mu \in [0,1)$,

(iv) the operator

$$\pm\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K_\lambda : H^p_1(\partial\Omega) \longrightarrow H^p_1(\partial\Omega) \tag{4.166}$$

is an isomorphism for every $\mu \in [0,1)$.

Proof. By (4.159)-(4.160), it suffices to prove all the desired implications for just one fixed choice of the sign, since interchanging Ω_+ with Ω_- means that K_λ becomes $-K_\lambda$. In order to fix ideas, we shall carry out the proof for the choice 'plus' of the sign, with this convention being tacitly used throughout the proof.

As far as the implication $(ii) \Longrightarrow (i)$ is concerned, if the operator (4.165) is an isomorphism for every $\mu \in [0,1)$, set

$$\vec{f}_1 := \vec{f} - \partial^\lambda_\nu(\mathcal{D}^+_\lambda\vec{g}, \mathcal{P}^+_\lambda\vec{g}) + \mu\,\partial^\lambda_\nu(\mathcal{D}^-_\lambda\vec{g}, \mathcal{P}^-_\lambda\vec{g}) \in H^p(\partial\Omega), \tag{4.167}$$

$$\vec{f}_2 := \left(\tfrac{1}{2}\tfrac{\mu+1}{\mu-1}I + K^*_\lambda\right)^{-1}\vec{f}_1 \in H^p(\partial\Omega), \tag{4.168}$$

where the superscripts $\pm$ indicate that the layer potentials in questions are considered as mappings functions defined on $\partial\Omega$ into functions defined in $\Omega_\pm$. Then

$$\vec{u}_\pm := \tfrac{1}{1-\mu}\mathcal{S}^\pm \vec{f}_2 + \mathcal{D}^\pm_\lambda \vec{g}, \tag{4.169}$$

$$\pi_\pm := \tfrac{1}{1-\mu}\mathcal{Q}^\pm \vec{f}_2 + \mathcal{P}^\pm_\lambda \vec{g}, \tag{4.170}$$

solve $(T^+_\mu)^*$ and obey natural estimates, i.e.

$$\|M(\nabla\vec{u}_\pm)\|_{L^p(\partial\Omega)} + \|M(\pi_\pm)\|_{L^p(\partial\Omega)} \leq C\Big(\|\vec{f}\|_{L^p(\partial\Omega)} + \|\vec{g}\|_{H^p_1(\partial\Omega)}\Big). \tag{4.171}$$

Let us now consider the issue of uniqueness for $(T^+_\mu)^*$ under the assumption that (4.165) is an invertible operator. To this end, assume that $(\vec{u}_\pm, \pi_\pm)$ solve the homogeneous version of $(T^+_\mu)^*$. Subtracting the two versions of the identity (4.144) and keeping in mind that $\partial^\lambda_\nu(\vec{u}_+,\pi_+) = \mu\,\partial^\lambda_\nu(\vec{u}_-,\pi_-)$ and $\vec{u}_+|_{\partial\Omega} = \vec{u}_-|_{\partial\Omega}$ allows us to conclude that $(\frac{1}{2}\frac{\mu+1}{\mu-1}I + K^*_\lambda)(\partial^\lambda_\nu(\vec{u}_-,\pi_-)) = 0$. Thus, $\partial^\lambda_\nu(\vec{u}_-,\pi_-) = 0$ and, further, $\partial^\lambda_\nu(\vec{u}_+,\pi_+) = 0$. With this in hand, the desired conclusion follows from Proposition 4.18. This concludes the proof of $(ii) \Longrightarrow (i)$.

In the opposite direction, the *a priori* estimate associated with the version of $(T^+_\mu)^*$ when $\vec{g} = 0$ reads

$$\begin{aligned}\|\partial^\lambda_\nu(\vec{u}_+,\pi_+) - \mu\,\partial^\lambda_\nu(\vec{u}_-,\pi_-)\|_{H^p(\partial\Omega)} \;&\approx\; \|M(\nabla\vec{u}_+)\|_{L^p(\partial\Omega)} + \|M(\pi_+)\|_{L^p(\partial\Omega)} \\ &\quad + \|M(\nabla\vec{u}_-)\|_{L^p(\partial\Omega)} + \|M(\pi_-)\|_{L^p(\partial\Omega)},\end{aligned} \tag{4.172}$$

for any pair of functions $(\vec{u}_\pm, \pi_\pm)$ which solve the Stokes system in $\Omega_\pm$ and satisfy $\vec{u}_+|_{\partial\Omega} = \vec{u}_-|_{\partial\Omega}$, $M(\nabla\vec{u}_\pm)$, $M(\pi_\pm) \in L^p(\partial\Omega)$. Specializing this estimate to the case when $\vec{u}_\pm = \mathcal{S}\vec{h}$, $\pi_\pm := \mathcal{Q}\vec{h}$ in $\Omega_\pm$, with $\vec{h} \in H^p(\partial\Omega)$, then yields

$$\begin{aligned}\|\vec{h}\|_{H^p(\partial\Omega)} &= \|\partial^\lambda_\nu(\vec{u}_-,\pi_-) - \partial^\lambda_\nu(\vec{u}_+,\pi_+)\|_{L^p(\partial\Omega)} \\ &\leq C\left[\|M(\nabla\vec{u}_+)\|_{L^p(\partial\Omega)} + \|M(\pi_+)\|_{L^p(\partial\Omega)} + \|M(\nabla\vec{u}_-)\|_{L^p(\partial\Omega)} + \|M(\pi_-)\|_{L^p(\partial\Omega)}\right] \\ &\leq C\|\partial^\lambda_\nu(\vec{u}_+,\pi_+) - \mu\,\partial^\lambda_\nu(\vec{u}_-,\pi_-)\|_{L^p(\partial\Omega)} = C\|(\tfrac{1}{2}\tfrac{\mu+1}{\mu-1}I + K^*_\lambda)\vec{h}\|_{H^p(\partial\Omega)},\end{aligned} \tag{4.173}$$

where $C = C(\Omega, p, \mu) > 0$ is a finite constant. Thus, $\left\{\frac{1}{2}\frac{\mu+1}{\mu-1}I + K_\lambda^*\right\}_{0<\mu<1}$ is a continuously parametrized family of one-to-one operators with closed range (in particular, semi-Fredholm) on $H^p(\partial\Omega)$, which are invertible (via a Neumann series) when μ is sufficiently close to 1. The homotopic invariance of the index then gives that all the operators in question are invertible on $H^p(\partial\Omega)$.

Consider next the equivalence $(iii) \iff (iv)$. First, when the operator (4.166) is an isomorphism for each $\mu \in [0,1)$, a solution to (T_μ^+) which satisfies (4.171) is given by

$$\vec{u}_\pm := \mathcal{D}_\lambda^\pm\Big[\Big(\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K_\lambda\Big)^{-1}\Big(\tfrac{1}{1-\mu}\vec{g} + S\vec{f}\Big)\Big] - \mathcal{S}^\pm\vec{f} \text{ in } \Omega_\pm, \tag{4.174}$$

$$\pi_\pm := \mathcal{P}_\lambda^\pm\Big[\Big(\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K_\lambda\Big)^{-1}\Big(\tfrac{1}{1-\mu}\vec{g} + S\vec{f}\Big)\Big] - \mathcal{Q}^\pm\vec{f} \text{ in } \Omega_\pm. \tag{4.175}$$

Second, the *a priori* estimate associated with the problem (T_μ^+) implies that, for each $\mu \in [0,1)$,

$$\begin{aligned}\|\vec{u}_+|_{\partial\Omega} - \mu\,\vec{u}_-|_{\partial\Omega}\|_{H_1^p(\partial\Omega)} \;\approx\;& \|M(\nabla\vec{u}_+)\|_{L^p(\partial\Omega)} + \|M(\pi_+)\|_{L^p(\partial\Omega)} \\ &+\|M(\nabla\vec{u}_-)\|_{L^p(\partial\Omega)} + \|M(\pi_-)\|_{L^p(\partial\Omega)},\end{aligned} \tag{4.176}$$

for any pair of functions $(\vec{u}_\pm, \pi_\pm)$ which solve the Stokes system in $\Omega_\pm$ and satisfy $\partial_\nu^\lambda(\vec{u}_+, \pi_+) = \partial_\nu^\lambda(\vec{u}_-, \pi_-)$, as well as $M(\nabla\vec{u}_\pm), M(\pi_\pm) \in L^p(\partial\Omega)$. Specializing (4.176) to the case when $\vec{u}_\pm = \mathcal{D}_\lambda\vec{h}$, $\pi_\pm = \mathcal{P}_\lambda\vec{h}$ in $\Omega_\pm$, with $\vec{h} \in H_1^p(\partial\Omega)$, yields

$$\begin{aligned}\|\vec{h}\|_{H_1^p(\partial\Omega)} &= \|\vec{u}_+|_{\partial\Omega} - \vec{u}_-|_{\partial\Omega}\|_{H_1^p(\partial\Omega)} \\ &\leq \|M(\nabla\vec{u}_+)\|_{L^p(\partial\Omega)} + \|M(\nabla\vec{u}_-)\|_{L^p(\partial\Omega)} \\ &\leq C\|\vec{u}_+|_{\partial\Omega} - \mu\,\vec{u}_-|_{\partial\Omega}\|_{H_1^p(\partial\Omega)} = C\|(\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K_\lambda)\vec{h}\|_{H_1^p(\partial\Omega)},\end{aligned} \tag{4.177}$$

where $C = C(\Omega, p, \mu) > 0$ is a finite constant. With this in hand and arguing as before, we then conclude that the operator (4.166) is an isomorphism for every $\mu \in [0,1)$.

There remains the issue of proving uniqueness for (T_μ^+) when the operator (4.166) is an isomorphism for each $\mu \in [0,1)$. Once again, assume $(\vec{u}_\pm, \pi_\pm)$ is a solution of the homogeneous version of (T_μ^+). Then since $\vec{u}_+|_{\partial\Omega} = \mu\vec{u}_-|_{\partial\Omega}$ and $\partial_\nu^\lambda(\vec{u}_+, \pi_+) = \partial_\nu^\lambda(\vec{u}_-, \pi_-)$, subtracting the two versions of (4.143) yields after some simple algebra, $\left(\frac{1}{2}\frac{1+\mu}{1-\mu}I + K_\lambda\right)\left(\vec{u}_-\Big|_{\partial\Omega}\right) = 0$. Here, we have also made use of the fact that the single layer does not jump across $\partial\Omega$. Hence, $\vec{u}_-|_{\partial\Omega} = 0$, and so $\vec{u}_+|_{\partial\Omega} = 0$ as well. Then once again Proposition 4.18 may be invoked in order to conclude. □

An immediate corollary of the result above is the following.

Proposition 4.20 *Retain the same assumptions as in the statement of Proposition 4.19. Then, for each (consistent) choice of the sign, the operator*

$$\pm\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K_\lambda^* : H^p(\partial\Omega) \longrightarrow H^p(\partial\Omega) \tag{4.178}$$

is an isomorphism for each $\mu \in (0, 1)$ if and only if the operator

$$\pm\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K_\lambda : H_1^p(\partial\Omega) \longrightarrow H_1^p(\partial\Omega) \tag{4.179}$$

is an isomorphism for each $\mu \in (0, 1)$.

Proof. This is a consequence of the proof of Proposition 4.19 and (4.161)-(4.162). □

The above proposition does not cover the case when $\mu = 0$, which corresponds precisely to the operators which solve the Neumann problem (N) and the Regularity problem (R) in (1.3). This particular aspect is dealt with in in the next chapter, in Theorem 5.9. In order to better explain how the Neumann and Regularity problems are related to the transmission problems, we first need to introduce the following definition.

With $\frac{n-1}{n} < p < \infty$ fixed, we will say that (T_μ^+) is *semi-well-posed* if for any $\vec{f} \in H^p(\partial\Omega)$ and $\vec{g} \in H_1^p(\partial\Omega)$, there exists a solution $(\vec{u}_\pm, \pi_\pm)$ of (T_μ^+) such that the functions $\vec{u}_+$ and π_+ must be unique (modulo constants) and also satisfy the estimate

$$\|M(\nabla\vec{u}_+)\|_{L^p(\partial\Omega)} + \|M(\pi_+)\|_{L^p(\partial\Omega)} \leq C\left(\|\vec{f}\|_{H^p(\partial\Omega)} + \|\vec{g}\|_{H_1^p(\partial\Omega)}\right). \tag{4.180}$$

Similarly, we will say that (T_μ^-) is *semi-well-posed* if there exists a solution $(\vec{u}_\pm, \pi_\pm)$ such that $\vec{u}_-$ and π_- must be unique (modulo constants) and satisfy

$$\|M(\nabla\vec{u}_-)\|_{L^p(\partial\Omega)} + \|M(\pi_-)\|_{L^p(\partial\Omega)} \leq C\left(\|\vec{f}\|_{H^p(\partial\Omega)} + \|\vec{g}\|_{H_1^p(\partial\Omega)}\right). \tag{4.181}$$

With these definitions in mind, we can state and prove the following proposition that details the relationship between the transmission problems and the Neumann and Regularity problems.

Proposition 4.21 *Let $\Omega_\pm \subseteq \mathbb{R}^n$, $n \geq 2$, be a graph Lipschitz domains as before. Recall (1.3). For $\frac{n-1}{n} < p < \infty$ fixed, the following statements are equivalent:*

(1) (T_o^+) and $(T_o^-)^*$ are both semi-well-posed,

(2) (R) is well-posed in Ω_+ and (N) is well-posed in Ω_-,

(3) (T_o^+) and $(T_o^-)^*$ are both well-posed.

Moreover, a similar result holds in the case when the roles of $+$ and $-$ are reversed.

Proof. First, we will show (1) $\Longrightarrow$ (2). Assume (T_o^+) and $(T_o^-)^*$ are both semi-well-posed. For any $\vec{g} \in H_1^p(\partial\Omega)$, if $(\vec{u}_\pm, \pi_\pm)$ solves (T_o^+) with data $(0, \vec{g})$, then $(\vec{u}_+, \pi_+)$ will solve (R) in Ω_+ and also satisfy the appropriate estimate. For any $\vec{f} \in H^p(\partial\Omega)$, if $(\vec{u}_\pm, \pi_\pm)$ solves (T_o^-) with data $(\vec{f}, 0)$, then $(\vec{u}_-, \pi_-)$ will solve (N) in Ω_- and also satisfy the appropriate estimate.

To establish uniqueness for (R), assume $(\vec{u}_+, \pi_+)$ solves the homogeneous version of (R) in Ω_+. Let $(\vec{u}_-, \pi_-)$ be a solution to the Neumann problem (N) in Ω_- such that $\partial_\nu^\lambda(\vec{u}_-, \pi_-) = \partial_\nu^\lambda(\vec{u}_+, \pi_+)$. Then $(\vec{u}_\pm, \pi_\pm)$ will solve the homogeneous version of (T_o^+), which implies that $\vec{u}_+$ and π_+ must be constant. To establish uniqueness for (N), assume $(\vec{u}_-, \pi_-)$ solves the homogeneous version of (N) in Ω_-, and let $(\vec{u}_+, \pi_+)$ be a solution to the Regularity problem (R) in Ω_+ such that $\vec{u}_+|_{\partial\Omega} = \vec{u}_-|_{\partial\Omega}$. Then $(\vec{u}_\pm, \pi_\pm)$ will solve the homogeneous version of $(T_o^-)^*$, and so $\vec{u}_-$ and π_- must be constant.

Next, we will prove (2) $\Longrightarrow$ (3). Assume (R) is well-posed in Ω_+ and (N) is well-posed in Ω_-. For any $\vec{f} \in H^p(\partial\Omega)$ and $\vec{g} \in H_1^p(\partial\Omega)$, let $(\vec{u}_+, \pi_+)$ be the solution to (R) such that $\vec{u}_+|_{\partial\Omega} = \vec{g}$ and let $(\vec{u}_-, \pi_-)$ be the solution to (N) such that $\partial_\nu^\lambda(\vec{u}_-, \pi_-) = \partial_\nu^\lambda(\vec{u}_+, \pi_+) - \vec{f}$. Then $(\vec{u}_\pm, \pi_\pm)$ will solve (T_o^+) and satisfy the appropriate estimates. To establish uniqueness, assume $(\vec{u}_\pm, \pi_\pm)$ satisfies the homeogenous version of (T_o^+). Then from the uniqueness for (R), $\vec{u}_+$ and π_+ must be constant. In particular, since $M(\pi_+) \in L^p(\partial\Omega)$, it follows that $\pi_+ = 0$. Then $(\vec{u}_-, \pi_-)$ solves the homogeneous version of (N) in Ω_-, which means $\vec{u}_-$ and π_- must also be constant.

Similarly, if $(\vec{u}_-, \pi_-)$ is the solution to (N) such that $\partial_\nu^\lambda(\vec{u}_-, \pi_-) = \vec{f}$ and $(\vec{u}_+, \pi_+)$ is the solution to (R) that satisfies $\vec{u}_+|_{\partial\Omega} = \vec{u}_-|_{\partial\Omega} + \vec{g}$, then $(\vec{u}_\pm, \pi_\pm)$ will solve $(T_o^-)^*$ and also satisfy the appropriate estimates. To establish uniqueness, assume $(\vec{u}_\pm, \pi_\pm)$ satisfies the homeogenous version of $(T_o^-)^*$. Then $\vec{u}_-$ and π_- must be constant due to the uniqueness of solutions to (N). Then it follows that $\vec{u}_+|_{\partial\Omega} = 0$ in $H_1^p(\partial\Omega)$, and so from the uniqueness for (R), $\vec{u}_+$ and π_+ must also be constant. Since it is clear that $(3) \Longrightarrow (1)$, this finishes the proof of the equivalence of the statements $(1) - (3)$, and same result with the roles of $+$ and $-$ reversed follows similarly. $\square$

5 The L^p transmission problem with p near 2

5.1 Rellich identities and related estimates

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be either a graph Lipschitz domain or a bounded Lipschitz domain, and fix a vector field $\vec{h} \in C_b^1(\mathbb{R}^n)$ with real-valued components.

Proposition 5.1 *Assume that $\vec{u}_\pm = (u_k^\pm)_{1\leq k\leq n}$ are real-valued vector fields and $\pi_\pm$ are real-valued scalar functions such that*

$$L_\lambda \vec{u}_\pm = \nabla\pi_\pm, \quad \operatorname{div}\vec{u}_\pm = 0 \;\; in \;\; \Omega_\pm, \qquad M(\nabla\vec{u}_\pm),\, M(\pi_\pm) \in L^2(\partial\Omega). \tag{5.1}$$

Then for every $\lambda \in \mathbb{R}$,

$$\begin{aligned}\int_{\partial\Omega} A_\lambda(\nabla\vec{u}_\pm, \nabla\vec{u}_\pm)\langle\vec{h}, \nu\rangle d\sigma &= 2\int_{\partial\Omega} \langle\partial_\nu^\lambda(\vec{u}_\pm, \pi_\pm), \nabla_h\vec{u}_\pm\rangle\, d\sigma \pm \int_{\Omega_\pm} (\operatorname{div}\vec{h}) A_\lambda(\nabla\vec{u}_\pm, \nabla\vec{u}_\pm)\, dx \\ &\quad \pm 2\int_{\Omega_\pm} \Big[\pi_\pm(\partial_i u_k^\pm)(\partial_k h_i) - (\partial_i u_k^\pm)(\partial_j u_k^\pm + \lambda\partial_k u_j^\pm)(\partial_j h_i)\Big]\, dx\end{aligned}$$

$$= \quad 2\int_{\partial\Omega} \langle \partial_\nu^\lambda(\vec{u}_\pm, \pi_\pm), \nabla_h \vec{u}_\pm \rangle \, d\sigma + \int_{\Omega_\pm} \mathcal{O}_h^\pm \, dx, \tag{5.2}$$

and

$$\begin{aligned} \int_{\partial\Omega} (\pi_\pm)^2 \langle \vec{h}, \nu \rangle \, d\sigma \quad &= \quad -2\int_{\partial\Omega} \langle \partial_\nu^{-1}(\vec{u}_\pm, \pi_\pm), (\nabla \vec{u}_\pm)\vec{h} \rangle \, d\sigma \pm \int_{\Omega_\pm} (\operatorname{div} \vec{h})(\pi_\pm)^2 \, dx \\ &\quad \pm 2 \int_{\Omega_\pm} \Big[(\partial_k u_i^\pm)(\partial_j h_i)(\partial_j u_k^\pm - \partial_k u_j^\pm) - (\partial_j h_i)(\partial_j u_i^\pm)\pi_\pm \Big] \, dx \\ &= \quad -2\int_{\partial\Omega} \langle \partial_\nu^{-1}(\vec{u}_\pm, \pi_\pm), (\nabla \vec{u}_\pm)\vec{h} \rangle \, d\sigma + \int_{\Omega_\pm} \mathcal{O}_h^\pm \, dx, \end{aligned} \tag{5.3}$$

where $\mathcal{O}_h^\pm$ *denotes any function in* $\Omega_\pm$ *such that, for some finite, purely dimensional constant* $C > 0$*,*

$$\mathcal{O}_h^\pm \le C(|\nabla \vec{u}_\pm|^2 + |\pi_\pm|^2)|\nabla \vec{h}|. \tag{5.4}$$

Proof. As far as (5.2) is concerned, the idea is to start with (3.41) written for L_λ, $\Omega_\pm$ and $\vec{u}_\pm$ in place of L, Ω and u, respectively. Also, D and A are as discussed at the beginning of § 4.1.

Note that the second solid integral in the right hand-side of (3.42) contains Lu which, in our case, corresponds to $L_\lambda \vec{u}_\pm = \nabla \pi_\pm$. We now further integrate by parts this gradient operator and use the divergence-free condition on $\vec{u}_\pm$. The key aspect of this calculation is that resulting boundary term combines well with the first integral in (3.41), in the sense that it "completes" $\partial_\nu^A u$ to the correct conormal derivative $\partial_\nu^\lambda(\vec{u}_\pm, \pi_\pm)$ for the Stokes system.

This accounts for the form of the integrand in the first integral in the right hand-side of (5.2). The first integral on the second line in (5.2) is a byproduct of the integration by parts just described. Finally, all the other integrals in (5.2) can be easily traced back to (3.41), finishing the proof of (5.2).

The identity (5.3) is a rewriting of formula (1.5) on p. 775 of [34], in the terminology of conormal derivatives utilized in this work. This concludes the proof of the proposition. □

The Rellich identities (5.2) and (5.3) will play a vital role throughout. Our first application is the following estimate for the pressure term.

Proposition 5.2 *Assume that*

$$\Delta\vec{u}_\pm = \nabla\pi_\pm, \quad \operatorname{div}\vec{u}_\pm = 0 \ \ \textit{in} \ \ \Omega_\pm, \qquad M(\nabla\vec{u}_\pm),\, M(\pi_\pm) \in L^2(\partial\Omega). \tag{5.5}$$

Then there exists $C > 0$ such that for any $\varepsilon > 0$,

$$\begin{aligned}\int\limits_{\partial\Omega} |\pi_\pm|^2 \langle \vec{h}, \nu\rangle\, d\sigma \le C\varepsilon^{-1} \int\limits_{\partial\Omega} |\nabla\vec{u}_\pm^\top - \nabla\vec{u}_\pm|^2 |\vec{h}|\, d\sigma + \varepsilon \int\limits_{\partial\Omega} |\pi_\pm|^2 |\vec{h}|\, d\sigma \\ +C \int\limits_{\Omega_\pm} (|\nabla\vec{u}_\pm|^2 + |\pi_\pm|^2)|\nabla\vec{h}|\, dx.\end{aligned} \tag{5.6}$$

Proof. Combining (5.3) and (5.2) in the case $\lambda = -1$ gives

$$\begin{aligned}\int\limits_{\partial\Omega} |\pi_\pm|^2 \langle \vec{h}, \nu\rangle\, d\sigma &= -2 \int\limits_{\partial\Omega} \langle \partial_\nu^{-1}(\vec{u}_\pm, \pi_\pm), (\nabla\vec{u}_\pm)\vec{h}\rangle\, d\sigma + \int\limits_{\Omega_\pm} \mathcal{O}_h^\pm\, d\sigma \\ &= 2 \int\limits_{\partial\Omega} \langle \partial_\nu^{-1}(\vec{u}_\pm, \pi_\pm), (\nabla\vec{u}_\pm^\top - \nabla\vec{u}_\pm)\vec{h}\rangle\, d\sigma - \int\limits_{\partial\Omega} A_{-1}(\nabla\vec{u}_\pm, \nabla\vec{u}_\pm)\langle \vec{h}, \nu\rangle + \int\limits_{\Omega_\pm} \mathcal{O}_h^\pm\, d\sigma \\ &= 2 \int\limits_{\partial\Omega} \langle (\nabla\vec{u}_\pm^\top - \nabla\vec{u}_\pm)\nu - \pi_\pm\nu, (\nabla\vec{u}_\pm^\top - \nabla\vec{u}_\pm)\vec{h}\rangle\, d\sigma \\ &\qquad - \int\limits_{\partial\Omega} A_{-1}(\nabla\vec{u}_\pm, \nabla\vec{u}_\pm)\langle \vec{h}, \nu\rangle + \int\limits_{\Omega_\pm} \mathcal{O}_h^\pm\, d\sigma.\end{aligned} \tag{5.7}$$

Then since $A_{-1}(\nabla\vec{u}_\pm, \nabla\vec{u}_\pm) = \frac{1}{2}|\nabla\vec{u}_\pm^\top - \nabla\vec{u}_\pm|^2$, the result follows by using Cauchy's inequality with epsilon in (5.7). $\square$

Proposition 5.3 *For $\lambda \in [-1, 1]$, assume that*

$$L_\lambda\vec{u}_\pm = \nabla\pi_\pm, \quad \operatorname{div}\vec{u}_\pm = 0 \ \ \textit{in} \ \ \Omega_\pm, \qquad M(\nabla\vec{u}_\pm),\, M(\pi_\pm) \in L^2(\partial\Omega). \tag{5.8}$$

Then there exists $C > 0$ such that for any $\varepsilon > 0$ and any $\mu \in [0, 1)$,

$$\int_{\partial\Omega} \Big[A_\lambda(\nabla\vec{u}_+,\nabla\vec{u}_+) + \mu A_\lambda(\nabla\vec{u}_-,\nabla\vec{u}_-)\Big]\langle\vec{h},\nu\rangle\, d\sigma$$

$$\begin{aligned}
&\le \tfrac{C}{\varepsilon(1-\mu)^2}\int_{\partial\Omega} \Big[|\partial_\nu^\lambda(\vec{u}_+,\pi_+) - \mu\,\partial_\nu^\lambda(\vec{u}_-,\pi_-)|^2 + \mu|\nabla_{tan}\vec{u}_+ - \nabla_{tan}\vec{u}_-|^2\Big]\,|\vec{h}|\, d\sigma \\
&\quad +\varepsilon\int_{\partial\Omega} \big[|\nabla\vec{u}_+|^2 + |\pi_+|^2 + \mu|\nabla\vec{u}_-|^2 + \mu|\pi_-|^2\big]\,|\vec{h}|\, d\sigma \qquad (5.9)\\
&\quad +\tfrac{C}{1-\mu}\int_{\Omega_+}(|\nabla\vec{u}_+|^2 + |\pi_+|^2)|\nabla\vec{h}|\, dx + \tfrac{\mu C}{1-\mu}\int_{\Omega_-}(|\nabla\vec{u}_-|^2 + |\pi_-|^2)|\nabla\vec{h}|\, dx.
\end{aligned}$$

Proof. First, we point out that if $\operatorname{div}\vec{u}_\pm = 0$ in $\Omega_\pm$, then for every $j \in \{1, ..., n\}$,

$$\{(\nabla\vec{u}_\pm)\nu\}_j = \nu_k\partial_j u_k^\pm = \partial_{\tau_{kj}} u_k^\pm, \qquad (5.10)$$

and also

$$\langle\partial_\nu\vec{u}_\pm,\nu\rangle = \nu_k\,\nu_j\,\partial_j u_k^\pm = \nu_j\,\partial_{\tau_{kj}} u_k^\pm. \qquad (5.11)$$

Combining the Rellich identities in (5.2) for $\vec{u}_+$ and $\vec{u}_-$ gives

$$\int_{\partial\Omega} (A_\lambda(\nabla\vec{u}_+,\nabla\vec{u}_+) + \mu A_\lambda(\nabla\vec{u}_-,\nabla\vec{u}_-))\,\langle\vec{h},\nu\rangle\, d\sigma$$

$$\begin{aligned}
&= 2\int_{\partial\Omega}\Big(\langle\partial_\nu^\lambda(\vec{u}_+,\pi_+),\nabla_h\vec{u}_+\rangle + \mu\,\langle\partial_\nu^\lambda(\vec{u}_-,\pi_-),\nabla_h\vec{u}_-\rangle\Big)\, d\sigma + \int_{\Omega_+}\mathcal{O}_h^+\, dx + \mu\int_{\Omega_-}\mathcal{O}_h^-\, dx \\
&= -\tfrac{2\mu}{1-\mu}\int_{\partial\Omega}\Big\langle\partial_\nu^\lambda(\vec{u}_+,\pi_+) - \partial_\nu^\lambda(\vec{u}_-,\pi_-),\nabla_h\vec{u}_+ + \nabla_h\vec{u}_-\Big\rangle\, d\sigma \\
&\quad +\tfrac{2}{1-\mu}\int_{\partial\Omega}\Big\langle\partial_\nu^\lambda(\vec{u}_+,\pi_+) - \mu\,\partial_\nu^\lambda(\vec{u}_-,\pi_-),\nabla_h\vec{u}_+ + \mu\nabla_h\vec{u}_-\Big\rangle\, d\sigma \\
&\quad +\int_{\Omega_+}\mathcal{O}_h^+\, dx + \mu\int_{\Omega_-}\mathcal{O}_h^-\, dx. \qquad (5.12)
\end{aligned}$$

Using Cauchy's inequality with epsilon, the last two lines of (5.12) can be bounded by the right hand side of (5.9). From the definition of the conormal derivative, the third line in (5.12) can be written as

$$
\begin{aligned}
-\tfrac{2\mu}{1-\mu}\int_{\partial\Omega}&\langle \partial_\nu^\lambda(\vec{u}_+,\pi_+)-\partial_\nu^\lambda(\vec{u}_-,\pi_-),\nabla_h\vec{u}_++\nabla_h\vec{u}_-\rangle\,d\sigma\\
&=-\tfrac{2\mu}{1-\mu}\int_{\partial\Omega}\lambda\langle(\nabla\vec{u}_+)\nu-(\nabla\vec{u}_-)\nu,\nabla_h\vec{u}_++\nabla_h\vec{u}_-\rangle\,d\sigma\\
&\quad-\tfrac{4\mu}{1-\mu}\int_{\partial\Omega}(\langle\partial_\nu^0(\vec{u}_+,\pi_+),\nabla_h\vec{u}_+\rangle-\langle\partial_\nu^0(\vec{u}_-,\pi_-),\nabla_h\vec{u}_-\rangle)\,d\sigma\\
&\quad-\tfrac{2\mu}{1-\mu}\int_{\partial\Omega}\langle\partial_\nu^0(\vec{u}_-,\pi_-)+\partial_\nu^0(\vec{u}_+,\pi_+),\nabla_h\vec{u}_--\nabla_h\vec{u}_+\rangle\,d\sigma. \qquad (5.13)
\end{aligned}
$$

From (5.10), the second line of (5.13) can be bounded by the right side of (5.9). Applying the Rellich identity (5.2) in the case $\lambda=0$ to the third line of (5.13) gives

$$
\begin{aligned}
-\tfrac{4\mu}{1-\mu}\int_{\partial\Omega}&\Big(\langle\partial_\nu^0(\vec{u}_+,\pi_+),\nabla_h\vec{u}_+\rangle-\langle\partial_\nu^0(\vec{u}_-,\pi_-),\nabla_h\vec{u}_-\rangle\Big)\,d\sigma\\
&=-\tfrac{2\mu}{1-\mu}\int_{\partial\Omega}\Big(|\nabla\vec{u}_+|^2-|\nabla\vec{u}_-|^2\Big)\langle\vec{h},\nu\rangle\,d\sigma+\tfrac{\mu}{1-\mu}\int_{\Omega_+}\mathcal{O}_h^+\,dx+\tfrac{\mu}{1-\mu}\int_{\Omega_-}\mathcal{O}_h^-\,dx\\
&=-\tfrac{2\mu}{1-\mu}\int_{\partial\Omega}\Big(|\nabla_{tan}\vec{u}_+|^2-|\nabla_{tan}\vec{u}_-|^2\Big)\langle\vec{h},\nu\rangle\,d\sigma+\tfrac{\mu}{1-\mu}\int_{\Omega_+}\mathcal{O}_h^+\,dx+\tfrac{\mu}{1-\mu}\int_{\Omega_-}\mathcal{O}_h^-\,dx.\\
&\quad-\tfrac{2\mu}{1-\mu}\int_{\partial\Omega}\Big(|\partial_\nu\vec{u}_+|^2-|\partial_\nu\vec{u}_-|^2\Big)\langle\vec{h},\nu\rangle\,d\sigma. \qquad (5.14)
\end{aligned}
$$

Since $|\nabla_{tan}\vec{u}_+|^2-|\nabla_{tan}\vec{u}_-|^2=\langle\nabla_{tan}\vec{u}_+-\nabla_{tan}\vec{u}_-,\nabla_{tan}\vec{u}_++\nabla_{tan}\vec{u}_-\rangle$, the third line of (5.14) can also be bounded by the right side of (5.9). This leaves the last term of (5.14), which we will deal with in a moment. Splitting $\vec{h}$ into its normal and tangential components gives $\nabla_h=\nabla_{h_{tan}}+\langle\vec{h},\nu\rangle\,\partial_\nu$. Using this along with the definition of the conormal derivative in the last line of (5.13) gives

$$
-\tfrac{2\mu}{1-\mu}\int_{\partial\Omega}\Big\langle\partial_\nu^0(\vec{u}_+,\pi_+)+\partial_\nu^0(\vec{u}_-,\pi_-),\nabla_h\vec{u}_--\nabla_h\vec{u}_+\Big\rangle\,d\sigma
$$

$$= -\tfrac{2\mu}{1-\mu} \int\limits_{\partial\Omega} \Big\langle \partial_\nu^0(\vec{u}_+, \pi_+) + \partial_\nu^0(\vec{u}_-, \pi_-)\,,\, \nabla_{h_{tan}}\vec{u}_- - \nabla_{h_{tan}}\vec{u}_+ + (\partial_\nu \vec{u}_- - \partial_\nu \vec{u}_+)\langle \vec{h}, \nu\rangle \Big\rangle \, d\sigma$$

$$= -\tfrac{2\mu}{1-\mu} \int\limits_{\partial\Omega} \Big\langle \partial_\nu^0(\vec{u}_+, \pi_+) + \partial_\nu^0(\vec{u}_-, \pi_-), \nabla_{h_{tan}}\vec{u}_- - \nabla_{h_{tan}}\vec{u}_+ \Big\rangle \, d\sigma$$

$$-\tfrac{2\mu}{1-\mu} \int\limits_{\partial\Omega} (\pi_+ + \pi_-)\langle \nu, \partial_\nu \vec{u}_- - \partial_\nu \vec{u}_+\rangle \langle \vec{h}, \nu\rangle \, d\sigma$$

$$-\tfrac{2\mu}{1-\mu} \int\limits_{\partial\Omega} \Big(|\partial_\nu \vec{u}_-|^2 - |\partial_\nu \vec{u}_+|^2 \Big) \langle \vec{h}, \nu\rangle \, d\sigma. \tag{5.15}$$

Notice that the last term in (5.15) cancels the last term in (5.14). Using (5.11) and Cauchy's inequality with epsilon, it follows that the third and fourth lines of (5.15) can be bounded by the right side of (5.9). So combining (5.12), (5.13), (5.14), and (5.15) finishes the proof of Proposition 5.3. □

The previous estimate gives us a good upper bound for terms involving the quadratic form $A_\lambda(\nabla\vec{u}_\pm, \nabla\vec{u}_\pm)$. Our next result, which is specific to the case $\lambda = 1$, seeks to bound terms involving the full gradient, $\nabla\vec{u}_\pm$, by terms involving the symmetric part of the gradient, $\nabla\vec{u}_\pm^\top + \nabla\vec{u}_\pm$, plus other terms similar to those in the right hand side of (5.9).

Proposition 5.4 *Assume that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a Lipschitz domain and that*

$$\Delta\vec{u}_\pm = \nabla\pi_\pm, \quad \operatorname{div}\vec{u}_\pm = 0 \ \ in \ \ \Omega_\pm, \qquad M(\nabla\vec{u}_\pm), M(\pi_\pm) \in L^2(\partial\Omega). \tag{5.16}$$

Then there exists $C > 0$ such that for any $\varepsilon > 0$ and any $\mu \in [0, 1)$,

$$\int\limits_{\partial\Omega} \Big[|\nabla\vec{u}_+|^2 + \mu|\nabla\vec{u}_-|^2 + \tfrac{\mu(1+\mu)}{2(1-\mu)^2}|\pi_+ - \pi_-|^2 \Big] \langle \vec{h}, \nu\rangle \, d\sigma$$

$$\leq \tfrac{C}{\varepsilon(1-\mu)^2} \int\limits_{\partial\Omega} \Big[|\nabla\vec{u}_+^\top + \nabla\vec{u}_+|^2 + \mu|\nabla\vec{u}_-^\top + \nabla\vec{u}_-|^2 \Big] |\vec{h}| \, d\sigma$$

$$+\tfrac{C}{\varepsilon(1-\mu)^2} \int\limits_{\partial\Omega} \Big[|\partial_\nu^1(\vec{u}_+, \pi_+) - \mu\, \partial_\nu^1(\vec{u}_-, \pi_-)|^2 + \mu|\nabla_{tan}\vec{u}_+ - \nabla_{tan}\vec{u}_-|^2 \Big] |\vec{h}| \, d\sigma$$

$$+\varepsilon \int\limits_{\partial\Omega} \Big[|\nabla\vec{u}_+|^2 + |\pi_+|^2 + \mu|\nabla\vec{u}_-|^2 + \mu|\pi_-|^2 \Big] |\vec{h}| \, d\sigma$$

$$+\tfrac{C}{1-\mu}\int\limits_{\Omega_+}(|\nabla\vec{u}_+|^2+|\pi_+|^2)|\nabla\vec{h}|\,dx+\tfrac{\mu C}{1-\mu}\int\limits_{\Omega_-}(|\nabla\vec{u}_-|^2+|\pi_-|^2)|\nabla\vec{h}|\,dx. \tag{5.17}$$

Proof. Consider the following algebraic identity for $a, b \in \mathbb{R}$,

$$\tfrac{\mu}{1-\mu}(a-b)^2=\tfrac{1}{1-\mu}(a-\mu b)^2-a^2+\mu b^2. \tag{5.18}$$

Writing (5.18) with $a=\pi_+$ and $b=\pi_-$ and applying the Rellich identity (5.3) gives

$$\begin{aligned}
&\tfrac{\mu}{1-\mu}\int\limits_{\partial\Omega}|\pi_+-\pi_-|^2\langle\vec{h},\nu\rangle\,d\sigma\\
&\quad=\tfrac{1}{1-\mu}\int\limits_{\partial\Omega}(\pi_+-\mu\pi_-)^2\langle\vec{h},\nu\rangle\,d\sigma-\int\limits_{\partial\Omega}(\pi_+)^2\langle\vec{h},\nu\rangle\,d\sigma+\mu\int\limits_{\partial\Omega}(\pi_-)^2\langle\vec{h},\nu\rangle\,d\sigma\\
&\quad=\tfrac{1}{1-\mu}\int\limits_{\partial\Omega}(\pi_+-\mu\pi_-)^2\langle\vec{h},\nu\rangle\,d\sigma+2\int\limits_{\partial\Omega}\Big\langle\partial_\nu^{-1}(\vec{u}_+,\pi_+),(\nabla\vec{u}_+)\vec{h}\Big\rangle\,d\sigma\\
&\qquad-2\mu\int\limits_{\partial\Omega}\Big\langle\partial_\nu^{-1}(\vec{u}_-,\pi_-),(\nabla\vec{u}_-)\vec{h}\Big\rangle\,d\sigma+\int\limits_{\Omega_+}\mathcal{O}_h^+\,dx+\mu\int\limits_{\Omega_-}\mathcal{O}_h^-\,dx.\\
&\quad=\tfrac{1}{1-\mu}\int\limits_{\partial\Omega}(\pi_+-\mu\pi_-)^2\langle\vec{h},\nu\rangle\,d\sigma+2\int\limits_{\partial\Omega}\Big\langle\partial_\nu^{-1}(\vec{u}_+,\pi_+),(\nabla\vec{u}_++\nabla\vec{u}_+^\top)\vec{h}\Big\rangle\,d\sigma\\
&\qquad-2\mu\int\limits_{\partial\Omega}\Big\langle\partial_\nu^{-1}(\nabla\vec{u}_-,\pi_-),(\nabla\vec{u}_-+\nabla\vec{u}_-^\top)\vec{h}\Big\rangle\,d\sigma\\
&\qquad-2\int\limits_{\partial\Omega}\Big\langle\partial_\nu^{-1}(\vec{u}_+,\pi_+)-\mu\,\partial_\nu^{-1}(\nabla\vec{u}_-,\pi_-),\nabla_h\vec{u}_+\Big\rangle\,d\sigma\\
&\qquad+2\mu\int\limits_{\partial\Omega}\Big\langle\partial_\nu^{-1}(\vec{u}_-,\pi_-),\nabla_h\vec{u}_--\nabla_h\vec{u}_+\Big\rangle\,d\sigma+\int\limits_{\Omega_+}\mathcal{O}_h^+\,dx+\mu\int\limits_{\Omega_-}\mathcal{O}_h^-\,dx
\end{aligned} \tag{5.19}$$

Using the Rellich identity (5.2) in the case $\lambda=0$ along with the definition of the conormal derivative, we can write

$$\begin{aligned}
&\int\limits_{\partial\Omega}\left[|\nabla\vec{u}_+|^2+\mu|\nabla\vec{u}_-|^2\right]\langle\vec{h},\nu\rangle\,d\sigma\\
&\quad=\int\limits_{\partial\Omega}\left[2\langle\partial_\nu^0(\vec{u}_+,\pi_+),\nabla_h\vec{u}_+\rangle+2\mu\,\langle\partial_\nu^0(\vec{u}_-,\pi_-),\nabla_h\vec{u}_-\rangle\right]\,d\sigma+\int\limits_{\Omega_+}\mathcal{O}_h^+\,dx+\mu\int\limits_{\Omega_-}\mathcal{O}_h^-\,dx
\end{aligned}$$

$$= \int_{\partial\Omega} \left[2\langle \partial_\nu^1(\vec{u}_+, \pi_+), \nabla_h \vec{u}_+ \rangle + 2\mu \, \langle \partial_\nu^1(\vec{u}_-, \pi_-), \nabla_h \vec{u}_- \rangle \right] \, d\sigma + \int_{\Omega_+} \mathcal{O}_h^+ \, dx + \mu \int_{\Omega_-} \mathcal{O}_h^- \, dx$$

$$+ \tfrac{1+\mu}{1-\mu} \int_{\partial\Omega} \Big[\langle \partial_\nu^{-1}(\vec{u}_+, \pi_+) - \mu \, \partial_\nu^{-1}(\vec{u}_-, \pi_-), \nabla_h \vec{u}_+ \rangle - \langle \partial_\nu^1(\vec{u}_+, \pi_+) - \mu \, \partial_\nu^1(\vec{u}_-, \pi_-), \nabla_h \vec{u}_+ \rangle \Big] \, d\sigma$$

$$+ 2\mu \int_{\partial\Omega} \Big[\langle (\nabla \vec{u}_-)\nu, \nabla_h \vec{u}_+ - \nabla_h \vec{u}_- \rangle + \tfrac{2}{1-\mu} \langle (\nabla \vec{u}_+ - \nabla \vec{u}_-)\nu, \nabla_h \vec{u}_+ \rangle \Big] \, d\sigma. \tag{5.20}$$

If we multiply (5.19) by $\frac{1+\mu}{2(1-\mu)}$ and add it to (5.20) and also apply the Rellich identity (5.2) in the case $\lambda = 1$ to the first term in the third line of (5.20), we have

$$\int_{\partial\Omega} \left[|\nabla \vec{u}_+|^2 + \mu |\nabla \vec{u}_-|^2 + \tfrac{\mu(1+\mu)}{2(1-\mu)^2} \Big| \pi_+ - \pi_- \Big|^2 \right] \langle \vec{h}, \nu \rangle \, d\sigma$$

$$= \int_{\partial\Omega} \left[A_1(\nabla \vec{u}_+, \nabla \vec{u}_+) + \mu A_1(\nabla \vec{u}_-, \nabla \vec{u}_-) \right] \langle \vec{h}, \nu \rangle \, d\sigma + \tfrac{1}{1-\mu} \int_{\Omega_+} \mathcal{O}_h^+ \, dx + \tfrac{\mu}{1-\mu} \int_{\Omega_-} \mathcal{O}_h^- \, dx$$

$$- \tfrac{1+\mu}{1-\mu} \int_{\partial\Omega} \langle \partial_\nu^1(\vec{u}_+, \pi_+) - \mu \, \partial_\nu^1(\vec{u}_-, \pi_-), \nabla_h \vec{u}_+ \rangle \, d\sigma$$

$$+ 2\mu \int_{\partial\Omega} \Big[\langle (\nabla \vec{u}_-)\nu, \nabla_h \vec{u}_+ - \nabla_h \vec{u}_- \rangle + \tfrac{2}{1-\mu} \langle (\nabla \vec{u}_+ - \nabla \vec{u}_-)\nu, \nabla_h \vec{u}_+ \rangle \Big] \, d\sigma$$

$$+ \tfrac{1+\mu}{2(1-\mu)^2} \int_{\partial\Omega} (\pi_+ - \mu \pi_-)^2 \langle \vec{h}, \nu \rangle \, d\sigma + \tfrac{1+\mu}{1-\mu} \int_{\partial\Omega} \langle \partial_\nu^{-1}(\vec{u}_+, \pi_+), (\nabla \vec{u}_+ + \nabla \vec{u}_+^\top) \vec{h} \rangle \, d\sigma$$

$$- \tfrac{\mu(1+\mu)}{1-\mu} \int_{\partial\Omega} \langle \partial_\nu^{-1}(\vec{u}_-, \pi_-), (\nabla \vec{u}_- + \nabla \vec{u}_-^\top) \vec{h} \rangle \, d\sigma$$

$$+ \tfrac{\mu(1+\mu)}{1-\mu} \int_{\partial\Omega} \langle \partial_\nu^{-1}(\vec{u}_-, \pi_-), \nabla_h \vec{u}_- - \nabla_h \vec{u}_+ \rangle \, d\sigma. \tag{5.21}$$

Notice also that

$$\pi_+ - \mu \pi_- \Big|_{\partial\Omega} = (1-\mu) \Big\langle (\nabla \vec{u}_+^\top + \nabla \vec{u}_+)\nu, \nu \Big\rangle + \mu \Big\langle \partial_\nu \vec{u}_+ - \partial_\nu \vec{u}_-, \nu \Big\rangle$$

$$+ \mu \Big\langle (\nabla \vec{u}_+ - \nabla \vec{u}_-)\nu, \nu \rangle - \Big\langle \partial_\nu^1(\vec{u}_+, \pi_+) - \mu \, \partial_\nu^1(\vec{u}_-, \pi_-), \nu \Big\rangle. \tag{5.22}$$

Then using (5.10), (5.11), and (5.22), we can bound the first term of the fifth line of (5.21) as follows,

$$\tfrac{1+\mu}{2(1-\mu)^2}\int_{\partial\Omega}(\pi_+-\mu\pi_-)^2\langle\vec{h},\nu\rangle\,d\sigma \le C\int_{\partial\Omega}|\nabla\vec{u}_+^\top+\nabla\vec{u}_+|^2|\vec{h}|\,d\sigma$$

$$+\tfrac{C}{(1-\mu)^2}\int_{\partial\Omega}\Big[|\partial_\nu^1(\vec{u}_+,\pi_+)-\mu\,\partial_\nu^1(\vec{u}_-,\pi_-)|^2+\mu|\nabla_{tan}\vec{u}_+-\nabla_{tan}\vec{u}_-|^2\Big]|\vec{h}|\,d\sigma \tag{5.23}$$

The next step is to observe that

$$\nabla_h\vec{u}_\pm=\nabla_{h_{tan}}\vec{u}_\pm+(\partial_\nu\vec{u}_\pm)\langle\vec{h},\nu\rangle=\nabla_{h_{tan}}\vec{u}_\pm+\Big[(\nabla\vec{u}_\pm^\top+\nabla\vec{u}_\pm)\nu\Big]\langle\vec{h},\nu\rangle-\Big[(\nabla\vec{u}_\pm)\nu\Big]\langle\vec{h},\nu\rangle, \tag{5.24}$$

and therefore from (5.10),

$$\Big|\nabla_h\vec{u}_+-\nabla_h\vec{u}_-\Big|\le\Big|\nabla\vec{u}_+{}^\top+\nabla\vec{u}_+\Big||\vec{h}|+\Big|\nabla\vec{u}_-{}^\top+\nabla\vec{u}_-\Big||\vec{h}|+2\Big|\nabla_{tan}\vec{u}_+-\nabla_{tan}\vec{u}_-\Big||\vec{h}|. \tag{5.25}$$

Then the proposition follows by repeatedly applying Cauchy's inequality with epsilon in (5.21) while using (5.25) for the first term in the fourth line and the last term. Here, we also use the fact that $A_1(\nabla\vec{u}_\pm,\nabla\vec{u}_\pm)=\frac{1}{2}|\nabla\vec{u}_\pm{}^\top+\nabla\vec{u}_\pm|^2$. □

Using the previous two propositions, we can now prove our main estimates.

Corollary 5.5 *Let $\Omega\subset\mathbb{R}^n$, $n\ge 2$, be a Lipschitz domain. For $\lambda\in(-1,1]$, assume that*

$$L_\lambda\vec{u}_\pm=\nabla\pi_\pm,\quad \operatorname{div}\vec{u}_\pm=0\ \ in\ \ \Omega_\pm,\qquad M(\nabla\vec{u}_\pm),\,M(\pi_\pm)\in L^2(\partial\Omega). \tag{5.26}$$

Finally, let $\vec{h}\in C^\infty(\mathbb{R}^n)$ and $C_o>0$ be such that

$$1\le\langle\vec{h}(x),\vec{\nu}(x)\rangle\le C_o,\quad\forall\,x\in\partial\Omega. \tag{5.27}$$

Then there exists $C>0$ such that for $\mu\in[0,1)$,

$$\int_{\partial\Omega}\left[|\nabla\vec{u}_+|^2+\mu\,|\nabla\vec{u}_-|^2\right]\,d\sigma \tag{5.28}$$

$$\leq \tfrac{C}{(1-\mu)^6} \int\limits_{\partial\Omega} \left[|\partial_\nu^\lambda(\vec{u}_+, \pi_+) - \mu\, \partial_\nu^\lambda(\vec{u}_-, \pi_-)|^2 + \mu\, |\nabla_{tan}\vec{u}_+ - \nabla_{tan}\vec{u}_-|^2 \right] d\sigma$$

$$+\tfrac{C}{(1-\mu)^3} \int\limits_{\Omega_+} (|\nabla\vec{u}_+|^2 + |\pi_+|^2)|\nabla\vec{h}|\, dx + \tfrac{\mu C}{(1-\mu)^3} \int\limits_{\Omega_-} (|\nabla\vec{u}_-|^2 + |\pi_-|^2)|\nabla\vec{h}|\, dx \tag{5.29}$$

Proof. Choosing ε small enough in Proposition 5.2, we can show that

$$\int\limits_{\partial\Omega} |\pi_\pm|^2\, d\sigma \leq C \int\limits_{\partial\Omega} |\nabla\vec{u}_\pm|^2\, d\sigma + C \int\limits_{\Omega_\pm} (|\nabla\vec{u}_\pm|^2 + |\pi_\pm|^2)|\nabla\vec{h}|\, dx. \tag{5.30}$$

In the case $\lambda = 1$, since $A_1(\nabla\vec{u}_\pm, \nabla\vec{u}_\pm) = \frac{1}{2}|\nabla\vec{u}_\pm{}^\top + \nabla\vec{u}_\pm|^2$, combining Proposition 5.4, Proposition 5.3, and (5.30) gives

$$\begin{aligned}
\int\limits_{\partial\Omega} \left[|\nabla\vec{u}_+|^2 + \mu|\nabla\vec{u}_-|^2 \right] d\sigma &\leq \tfrac{C}{\varepsilon_1(1-\mu)^2} \int\limits_{\partial\Omega} \left[|\nabla\vec{u}_+{}^\top + \nabla\vec{u}_+|^2 + \mu|\nabla\vec{u}_-{}^\top + \nabla\vec{u}_-|^2 \right] d\sigma \\
&\quad + \tfrac{C}{\varepsilon_1(1-\mu)^2} \int\limits_{\partial\Omega} \left[|\partial_\nu^1(\vec{u}_+, \pi_+) - \mu\, \partial_\nu^1(\vec{u}_-, \pi_-)|^2 + \mu|\nabla_{tan}\vec{u}_+ - \nabla_{tan}\vec{u}_-|^2 \right] d\sigma \\
&\quad + \varepsilon_1 C \int\limits_{\partial\Omega} \left[|\nabla\vec{u}_+|^2 + \mu|\nabla\vec{u}_-|^2 \right] d\sigma, \\
&\quad + \tfrac{C}{1-\mu} \int\limits_{\Omega_+} (|\nabla\vec{u}_+|^2 + |\pi_+|^2)|\nabla\vec{h}|\, dx + \tfrac{\mu C}{1-\mu} \int\limits_{\Omega_-} (|\nabla\vec{u}_-|^2 + |\pi_-|^2)|\nabla\vec{h}|\, dx \\
&\leq \tfrac{C}{\varepsilon_2\varepsilon_1(1-\mu)^4} \int\limits_{\partial\Omega} \left[|\partial_\nu^1(\vec{u}_+, \pi_+) - \mu\, \partial_\nu^1(\vec{u}_-, \pi_-)|^2 + \mu|\nabla_{tan}\vec{u}_+ - \nabla_{tan}\vec{u}_-|^2 \right] d\sigma \\
&\quad + \left(\varepsilon_1 + \tfrac{\varepsilon_2}{\varepsilon_1(1-\mu)^2} \right) C \int\limits_{\partial\Omega} \left[|\nabla\vec{u}_+|^2 + \mu|\nabla\vec{u}_-|^2 \right] d\sigma \\
&\quad + \tfrac{C}{\varepsilon_1(1-\mu)^3} \int\limits_{\Omega_+} (|\nabla\vec{u}_+|^2 + |\pi_+|^2)|\nabla\vec{h}|\, dx \\
&\quad + \tfrac{\mu C}{\varepsilon_1(1-\mu)^3} \int\limits_{\Omega_-} (|\nabla\vec{u}_-|^2 + |\pi_-|^2)|\nabla\vec{h}|\, dx.
\end{aligned} \tag{5.31}$$

Then the corollary follows by letting $\varepsilon_2 = \varepsilon_1^2(1-\mu)^2$ and choosing ε_1 small enough. If $|\lambda| < 1$, there exists $C_\lambda > 0$ such that $|\nabla\vec{u}_\pm|^2 \leq C_\lambda A_\lambda(\nabla\vec{u}_\pm, \nabla\vec{u}_\pm)$, and so in this case, the corollary can be proved more directly using Proposition 5.3 and (5.30). $\square$

Corollary 5.6 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a Lipschitz domain and assume that, for some $\lambda \in (-1,1]$,*

$$L_\lambda \vec{u}_\pm = \nabla \pi_\pm, \quad \operatorname{div} \vec{u}_\pm = 0 \;\; in \;\; \Omega_\pm, \qquad M(\nabla \vec{u}_\pm),\, M(\pi_\pm) \in L^2(\partial\Omega). \tag{5.32}$$

Let $\vec{h} \in C^\infty(\mathbb{R}^n)$ and $C_o > 0$ be such that

$$1 \leq \langle \vec{h}(x), \vec{\nu}(x) \rangle \leq C_o, \quad \forall\, x \in \partial\Omega. \tag{5.33}$$

Then there exists $C > 0$ such that for $\mu \in [0,1)$,

$$\begin{aligned}
&\int_{\partial\Omega} \left[|\nabla \vec{u}_+|^2 + \mu\, |\nabla \vec{u}_-|^2 \right] d\sigma \\
&\qquad \leq \tfrac{C}{(1-\mu)^6} \int_{\partial\Omega} \Big[\mu |\partial^\lambda_\nu (\vec{u}_+, \pi_+) - \partial^\lambda_\nu (\vec{u}_-, \pi_-)|^2 + |\nabla_{tan} \vec{u}_+ - \mu \nabla_{tan} \vec{u}_-|^2 \Big]\, d\sigma \\
&\qquad\quad + \tfrac{C}{(1-\mu)^3} \int_{\Omega_+} (|\nabla \vec{u}_+|^2 + |\pi_+|^2) |\nabla \vec{h}|\, dx + \tfrac{\mu C}{(1-\mu)^3} \int_{\Omega_-} (|\nabla \vec{u}_-|^2 + |\pi_-|^2) |\nabla \vec{h}|\, dx
\end{aligned} \tag{5.34}$$

Proof. For $\mu \in (0,1)$, the corollary follows by applying Corollary 5.5 to the functions

$$\vec{v}_+ := \mu\, \vec{u}_-, \quad \vec{v}_- := \vec{u}_+, \quad \rho_+ := \mu\, \pi_-, \quad \rho_- := \pi_+, \tag{5.35}$$

and then dividing by μ. For $\mu = 0$, this follows by simply taking the limit as $\mu \to 0^+$. $\Box$

5.2 The case of a graph Lipschitz domain

In this section, we seek to establish the well-posedness of each of the various boundary value problems stated in § 1 in graph Lipschitz domains.

Lemma 5.7 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a graph Lipschitz domain as defined earlier. Then there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ such that whenever $2 - \varepsilon < p < 2 + \varepsilon$ and $\mu \in [0,1)$, the following hold:*

(i) The operators $\pm\frac{1}{2}\frac{1+\mu}{1-\mu}I + K_\lambda^*$ *are invertible on* $L^p(\partial\Omega)$*,*

(ii) The operators $\pm\frac{1}{2}\frac{1+\mu}{1-\mu}I + K_\lambda$ *are invertible on* $L^p(\partial\Omega)$*, on* $\dot{L}_1^p(\partial\Omega)$*, and on* $L_1^p(\partial\Omega)$*.*

Proof. It is enough to prove the lemma in the case $p = 2$, since the extension to $p \in (2-\varepsilon, 2+\varepsilon)$ is then a consequence of abstract stability results. For $\vec{f} \in L^2(\partial\Omega)$ fixed, let $\vec{u}_\pm := \mathcal{S}\vec{f}$ and $\pi_\pm := \mathcal{Q}\vec{f}$ in $\Omega_\pm$. Then $(\vec{u}_\pm, \pi_\pm)$ will satisfy

$$\begin{cases} \Delta\vec{u}_\pm = \nabla\pi_\pm, \ \ \operatorname{div}\vec{u}_\pm = 0 \ \text{ in } \ \Omega_\pm, \\ \vec{u}_+|_{\partial\Omega} = \vec{u}_-|_{\partial\Omega}, \\ \partial_\nu^\lambda(\vec{u}_+, \pi_+) - \mu\,\partial_\nu^\lambda(\vec{u}_-, \pi_-) = \left(-\frac{1}{2}(1+\mu)I + (1-\mu)K_\lambda^*\right)\vec{f} \ \text{ on } \ \partial\Omega, \\ M(\nabla\vec{u}_\pm), M(\pi_\pm) \in L^2(\partial\Omega). \end{cases} \tag{5.36}$$

Since $\Omega_\pm$ are graph Lipschitz domains, it is possible to select a *constant* vector field $\vec{h}$ that satisfies the hypothesis of Corollary 5.5. Applying Corollary 5.5 then gives

$$\int_{\partial\Omega} [|\nabla\vec{u}_+|^2 + \mu|\nabla\vec{u}_-|^2]\, d\sigma \le C \int_{\partial\Omega} |(-\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K_\lambda^*)\vec{f}|^2\, d\sigma. \tag{5.37}$$

Also, if we apply Corollary 5.6 in the case $\mu = 0$ with the roles of $\vec{u}_+$ and $\vec{u}_-$ reversed, we get

$$\int_{\partial\Omega} |\nabla\vec{u}_-|^2\, d\sigma \le C \int_{\partial\Omega} |\nabla_{tan}\vec{u}_-|^2\, d\sigma = C \int_{\partial\Omega} |\nabla_{tan}\vec{u}_+|^2\, d\sigma \le C \int_{\partial\Omega} |\nabla\vec{u}_+|^2\, d\sigma. \tag{5.38}$$

Then combining (5.37) and (5.38), and using (4.45) gives

$$\begin{aligned} \|\vec{f}\|_{L^2(\partial\Omega)} &= \|\partial_\nu^\lambda(\vec{u}_-, \pi_-) - \partial_\nu^\lambda(\vec{u}_+, \pi_+)\|_{L^2(\partial\Omega)} \\ &\le C\|\nabla\vec{u}_-\|_{L^2(\partial\Omega)} + C\|\nabla\vec{u}_+\|_{L^2(\partial\Omega)} \\ &\le C\|\nabla\vec{u}_+\|_{L^2(\partial\Omega)} \le C\|(-\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K_\lambda^*)\vec{f}\|_{L^2(\partial\Omega)}. \end{aligned} \tag{5.39}$$

From (5.39), it follows that $-\frac{1}{2}\frac{1+\mu}{1-\mu}I + K_\lambda^*$ is one-to-one and semi-Fredholm for every $\mu \in [0, 1)$. Also, if μ is sufficiently close to 1, we have that $-\frac{1}{2}\frac{1+\mu}{1-\mu}I + K_\lambda^*$ is invertible on

$L^2(\partial\Omega)$ via a Neumann series. It follows from the homotopic invariance of the index that $-\frac{1}{2}\frac{1+\mu}{1-\mu}I + K_\lambda^*$ is actually Fredholm with index zero for each $\mu \in [0,1)$, and therefore $-\frac{1}{2}\frac{1+\mu}{1-\mu}I + K_\lambda^*$ is invertible on $L^2(\partial\Omega)$. If we exchange the roles of $(\vec{u}_+, \pi_+)$ and $(\vec{u}_-, \pi_-)$ in the above argument, we can also show that $\frac{1}{2}\frac{1+\mu}{1-\mu}I + K_\lambda^*$ is invertible on $L^2(\partial\Omega)$. By duality, the operators $\pm\frac{1}{2}\frac{1+\mu}{1-\mu}I + K_\lambda$ must also be invertible on $L^2(\partial\Omega)$.

Now, for $\vec{g} \in \dot{L}_1^2(\partial\Omega)$, let $\vec{u}_\pm = \mathcal{D}_\lambda\vec{g}$ and $\pi_\pm = \mathcal{P}_\lambda\vec{g}$ in $\Omega_\pm$. Then $(\vec{u}_\pm, \pi_\pm)$ will satisfy

$$\begin{cases} \Delta\vec{u}_\pm = \nabla\pi_\pm, \ \operatorname{div}\vec{u}_\pm = 0 \ \text{ in } \ \Omega_\pm, \\ \vec{u}_+|_{\partial\Omega} - \mu\,\vec{u}_-|_{\partial\Omega} = \left(\frac{1}{2}(1+\mu)I + (1-\mu)K_\lambda\right)\vec{g} \ \text{ on } \partial\Omega, \\ \partial_\nu^\lambda(\vec{u}_+, \pi_+) = \partial_\nu^\lambda(\vec{u}_-, \pi_-), \\ M(\nabla\vec{u}_\pm), M(\pi_\pm) \in L^2(\partial\Omega). \end{cases} \tag{5.40}$$

Applying Corollary 5.6 gives

$$\int_{\partial\Omega} [|\nabla\vec{u}_+|^2 + \mu|\nabla\vec{u}_-|^2]\, d\sigma \le C \int_{\partial\Omega} |\nabla_{tan}[(\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K_\lambda)\vec{g}]|^2\, d\sigma. \tag{5.41}$$

Also, if we apply Corollary 5.5 in the case $\mu = 0$ with the roles of $\vec{u}_+$ and $\vec{u}_-$ reversed, we get

$$\int_{\partial\Omega} |\nabla\vec{u}_-|^2\, d\sigma \le C\int_{\partial\Omega} |\partial_\nu^\lambda(\vec{u}_-, \pi_-)|^2\, d\sigma = C\int_{\partial\Omega} |\partial_\nu^\lambda(\vec{u}_+, \pi_+)|^2\, d\sigma \le C\int_{\partial\Omega} |\nabla\vec{u}_+|^2\, d\sigma \tag{5.42}$$

Then combining (5.41) and (5.42), and using (4.43) gives

$$\begin{aligned} \|\vec{g}\|_{\dot{L}_1^2(\partial\Omega)} &= \|\vec{u}_+ - \vec{u}_-\|_{\dot{L}_1^2(\partial\Omega)} \\ &\le C\|\nabla\vec{u}_+\|_{L^2(\partial\Omega)} + C\|\nabla\vec{u}_-\|_{L^2(\partial\Omega)} \\ &\le C\|\nabla\vec{u}_+\|_{L^2(\partial\Omega)} \le C\|(\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K_\lambda)\vec{g}\|_{\dot{L}_1^2(\partial\Omega)}. \end{aligned} \tag{5.43}$$

From (5.43), it follows that $\frac{1}{2}\frac{1+\mu}{1-\mu}I+K_\lambda$ is one-to-one and semi-Fredholm for every $\mu \in [0,1)$, and repeating the same arguments as above leads to the conclusion that the operators $\pm\frac{1}{2}\frac{1+\mu}{1-\mu}I + K_\lambda$ are in fact invertible on $\dot{L}_1^2(\partial\Omega)$. Since these operators are invertible on $L^2(\partial\Omega)$ and $\dot{L}_1^2(\partial\Omega)$, we can establish

$$\|\vec{g}\|_{L^2_1(\partial\Omega)} \le C\|(\pm\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K_\lambda)\vec{g}\|_{L^2_1(\partial\Omega)}, \tag{5.44}$$

for any $\vec{g} \in L^2_1(\partial\Omega)$, which, after arguing as above, eventually allows us to conclude that the operators $\pm\frac{1}{2}\frac{1+\mu}{1-\mu}I + K_\lambda$ are also invertible on $L^2_1(\partial\Omega)$. □

The invertibility of these operators allows us to prove the well-posedness of the associated boundary value problems, as in the following theorem.

Theorem 5.8 *Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be a graph Lipschitz domain, and set $\Omega_+ := \Omega$, $\Omega_- := \mathbb{R}^n \setminus \bar{\Omega}$. Then there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ such that for any $p \in (2-\varepsilon, 2+\varepsilon)$, the transmission problems $(T^{\pm}_\mu)$ and $(T^{\pm}_\mu)^*$ (cf. (4.155)-(4.156)) are well-posed for any $\mu \in [0,1)$. Moreover, the Neumann problem (N) and the Regularity problem (R) (cf. (1.3)) are also well-posed in Ω_+ and Ω_- for any $p \in (2-\varepsilon, 2+\varepsilon)$.*

Proof. The well-posedness of $(T^{\pm}_\mu)$ and $(T^{\pm}_\mu)^*$ for any $\mu \in [0,1)$ follows directly from Lemma 5.7 and Proposition 4.19. Then Proposition 4.21 implies that (N) and (R) are also well-posed. □

With these results in mind, we can prove the following theorem.

Theorem 5.9 *Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be a graph Lipschitz domain and let $\frac{n-1}{n} < p_o < 2 < p_1 < \infty$. Then for $\lambda \in (-1,1]$, the following are equivalent:*

(1) the operators

$$\begin{gathered} \tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K^*_\lambda \text{ and } -\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K^*_\lambda \text{ are invertible on } H^p(\partial\Omega) \\ \text{for all } \mu \in [0,1) \text{ and for all } p \in (p_o, p_1), \end{gathered} \tag{5.45}$$

(2) the operators

$$\begin{gathered} \tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K_\lambda \text{ and } -\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K_\lambda \text{ are invertible on } H^p_1(\partial\Omega) \\ \text{for all } \mu \in [0,1) \text{ and for all } p \in (p_o, p_1). \end{gathered} \tag{5.46}$$

Proof. First, assume the operators $\pm\frac{1}{2}\frac{1+\mu}{1-\mu}I + K_\lambda^*$ are invertible on $H^p(\partial\Omega)$ for all $\mu \in [0,1)$ and for all $p \in (p_o, p_1)$. To prove the invertibility of $\frac{1}{2}\frac{1+\mu}{1-\mu}I + K_\lambda$ and $-\frac{1}{2}\frac{1+\mu}{1-\mu}I + K_\lambda$ on $H_1^p(\partial\Omega)$, from Proposition 4.19, it is enough to show that the transmission problems (T_μ^+) and (T_μ^-) are well-posed. In fact, given that (5.45) and (5.46) are invariant under changing the roles of Ω_+ and Ω_-, we may further conclude from (4.159)-(4.160) that it suffices to establish that just one of the problems (T_μ^+), (T_μ^-) is well-posed.

To prove the well-posedness of (T_μ^+), we can actually reduce matters to the case when $\vec{f} = 0$. To see this, let $(\vec{v}_\pm, \rho_\pm)$ solve the reduced transmission problem with datum $\vec{g} + (1-\mu)S\vec{f}$. Then $\vec{u}_\pm = \vec{v}_\pm - \mathcal{S}\vec{f}$, $\pi_\pm = \rho_\pm - \mathcal{Q}\vec{f}$ will solve (T_μ^+) and also satisfy the appropriate non-tangential maximal function estimates. For the rest of the proof, we will deal with the case when $\vec{f} = 0$.

Fix $p \in (p_o, p_1)$. First we claim that for $\vec{g} \in H_1^p(\partial\Omega)$,

$$\mathcal{S}\left[(\mp\tfrac{1}{2}I + K_\lambda^*)^{-1}\partial_\nu^\lambda(\mathcal{D}_\lambda\vec{g}, \mathcal{P}_\lambda\vec{g})\right] = \mathcal{D}_\lambda\vec{g} \quad \text{in } \Omega_\pm. \tag{5.47}$$

To prove this identity, it is enough to consider the case when $\vec{g}$ is in a dense subclass of $H_1^p(\partial\Omega)$. Assume $\vec{g} \in H_1^p(\partial\Omega) \cap \dot{L}_1^2(\partial\Omega)$. Using the jump formula (4.45), it can be shown that the left and right sides of (5.47) yield the same conormal derivative. Since the conormal derivatives of each side will be functions in $H^p(\partial\Omega) \cap L^2(\partial\Omega)$, it follows from the uniqueness for the L^2 Neumann problem that the left and right sides of (5.47) differ only by a constant. Finally, since each expression decays at infinity, the identity must hold. Moving to the boundary in (5.47) gives the useful identity

$$S\left[(\mp\tfrac{1}{2}I + K_\lambda^*)^{-1}\partial_\nu^\lambda(\mathcal{D}_\lambda\vec{g}, \mathcal{P}_\lambda\vec{g})\right] = (\pm\tfrac{1}{2}I + K_\lambda)\vec{g} \quad \text{on } \partial\Omega. \tag{5.48}$$

Next, we claim that the functions

$$\vec{u}_\pm := \tfrac{1}{1-\mu}\,\mathcal{S}\left[(\mp\tfrac{1}{2}I + K_\lambda^*)^{-1}(\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K_\lambda^*)^{-1}\partial_\nu^\lambda(\mathcal{D}_\lambda\vec{g}, \mathcal{P}_\lambda\vec{g})\right], \tag{5.49}$$

$$\pi_\pm := \tfrac{1}{1-\mu}\,\mathcal{Q}\left[(\mp\tfrac{1}{2}I + K_\lambda^*)^{-1}(\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K_\lambda^*)^{-1}\partial_\nu^\lambda(\mathcal{D}_\lambda\vec{g}, \mathcal{P}_\lambda\vec{g})\right], \tag{5.50}$$

satisfy the transmission problem (T^+_μ). The jump formula (4.45) gives

$$\begin{aligned}\partial^\lambda_\nu(\vec{u}_\pm,\pi_\pm) &= \tfrac{1}{1-\mu}(\mp\tfrac{1}{2}I+K^*_\lambda)(\mp\tfrac{1}{2}I+K^*_\lambda)^{-1}(\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I+K^*_\lambda)^{-1}\partial^\lambda_\nu(\mathcal{D}_\lambda\vec{g}\mathcal{P}_\lambda\vec{g}) \\ &= \tfrac{1}{1-\mu}(\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I+K^*_\lambda)^{-1}\partial^\lambda_\nu(\mathcal{D}_\lambda\vec{g},\mathcal{P}_\lambda\vec{g}), \qquad (5.51)\end{aligned}$$

and so $\partial^\lambda_\nu(\vec{u}_+,\pi_+) = \partial^\lambda_\nu(\vec{u}_-,\pi_-)$. For a bounded, linear operator T, assume $\eta I+T$ and $\gamma I+T$ are invertible operators for $\eta,\gamma\in\mathbb{R}$. The for $\mu\in\mathbb{R}$, the resolvent identity

$$(\eta I+T)^{-1}-\mu(\gamma I+T)^{-1} = (\eta I+T)^{-1}\Big((\gamma I+T)-\mu(\eta I+T)\Big)(\gamma I+T)^{-1} \qquad (5.52)$$

holds. By applying (5.52) twice and also using the boundary identity (5.48), we can write

$$\begin{aligned}&\vec{u}_+|_{\partial\Omega}-\mu\vec{u}_-|_{\partial\Omega} \\ &\quad= \tfrac{1}{1-\mu}\,S\left[\Big((-\tfrac{1}{2}I+K^*_\lambda)^{-1}-\mu(\tfrac{1}{2}I+K^*_\lambda)^{-1}\Big)(\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I+K^*_\lambda)^{-1}\partial^\lambda_\nu(\mathcal{D}_\lambda\vec{g},\mathcal{P}_\lambda\vec{g})\right] \\ &\quad= S\left[(-\tfrac{1}{2}I+K^*_\lambda)^{-1}(\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I+K^*_\lambda)(\tfrac{1}{2}I+K^*_\lambda)^{-1}(\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I+K^*_\lambda)^{-1}\partial^\lambda_\nu(\mathcal{D}_\lambda\vec{g},\mathcal{P}_\lambda\vec{g})\right] \\ &\quad= S\left[(-\tfrac{1}{2}I+K^*_\lambda)^{-1}(\tfrac{1}{2}I+K^*_\lambda)^{-1}\partial^\lambda_\nu(\mathcal{D}_\lambda\vec{g},\mathcal{P}_\lambda\vec{g})\right] \\ &\quad= S\left[\Big((-\tfrac{1}{2}I+K^*_\lambda)^{-1}-(\tfrac{1}{2}I+K^*_\lambda)^{-1}\Big)\partial^\lambda_\nu(\mathcal{D}_\lambda\vec{g},\mathcal{P}_\lambda\vec{g})\right] \\ &\quad= (\tfrac{1}{2}I+K_\lambda)\vec{g}-(-\tfrac{1}{2}I+K_\lambda)\vec{g}=\vec{g}. \qquad (5.53)\end{aligned}$$

To prove uniqueness for (T^+_μ), we will first prove uniqueness for the H^p Neumann problem (N).

Assume $(\vec{u}_+,\pi_+)$ satisfies the homogeneous version of the H^p Neumann problem in Ω_+. Define

$$\vec{u}_- := \mathcal{S}\left[\Big((-\tfrac{1}{2}I+K^*_\lambda)^{-1}-(\tfrac{1}{2}I+K^*_\lambda)^{-1}\Big)\partial^\lambda_\nu(\mathcal{D}_\lambda(\vec{u}_+|_{\partial\Omega}),\mathcal{P}_\lambda(\vec{u}_+|_{\partial\Omega}))\right] \quad \text{in } \Omega_-, \quad (5.54)$$

and

$$\pi_- := \mathcal{Q}\left[\Big((-\tfrac{1}{2}I + K_\lambda^*)^{-1} - (\tfrac{1}{2}I + K_\lambda^*)^{-1}\Big)\partial_\nu^\lambda(\mathcal{D}_\lambda(\vec{u}_+|_{\partial\Omega}), \mathcal{P}_\lambda(\vec{u}_+|_{\partial\Omega}))\right] \quad \text{in } \Omega_-. \quad (5.55)$$

Arguing as above using (5.48), it follows that $\vec{u}_-|_{\partial\Omega} = \vec{u}_+|_{\partial\Omega}$. Since $\vec{u}_-|_{\partial\Omega} = \vec{u}_+|_{\partial\Omega}$ and $\partial_\nu^\lambda(\vec{u}_+, \pi_+) = 0$, from (4.144) we have

$$(-\tfrac{1}{2}I + K_\lambda^*)\left(\partial_\nu^\lambda(\vec{u}_-, \pi_-)\right) = (\tfrac{1}{2}I + K_\lambda^*)\left(\partial_\nu^\lambda(\vec{u}_+, \pi_+)\right) = 0. \quad (5.56)$$

Since $-\frac{1}{2}I + K_\lambda^*$ is invertible on $H^p(\partial\Omega)$, it follows that

$$\partial_\nu^\lambda(\vec{u}_-, \pi_-) = 0 = \partial_\nu^\lambda(\vec{u}_+, \pi_+). \quad (5.57)$$

Then from Proposition 4.18, $\vec{u}_+$ and π_+ are constant. With a similar argument, we can also prove uniqueness for the H^p Neumann problem in Ω_-.

Let us return to the issue of uniqueness for the transmission problem (T_μ^-). Assume $(\vec{u}_\pm, \pi_\pm)$ solves the homogenous version of (T_μ^+). Multiplying the version of (4.144) corresponding to the sign minus by μ and subtracting it from the version of (4.144) corresponding to the sign plus and making use of the transmission conditions gives

$$(1-\mu)(\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K_\lambda^*)\left(\partial_\nu^\lambda(\vec{u}_+, \pi_+)\right) = 0. \quad (5.58)$$

Since the operator $\frac{1}{2}\frac{1+\mu}{1-\mu}I + K_\lambda^*$ is invertible, it follows that $\partial_\nu^\lambda(\vec{u}_+, \pi_+) = 0 = \partial_\nu^\lambda(\vec{u}_-, \pi_-)$. Now it follows from the uniqueness of the H^p Neumann problem that $\vec{u}_\pm$ and $\pi_\pm$ are constant. This finishes the proof of (1) $\Longrightarrow$ (2).

To prove (2) $\Longrightarrow$ (1), assume the operators $\pm\frac{1}{2}\frac{1+\mu}{1-\mu}I + K_\lambda$ are invertible on $H_1^p(\partial\Omega)$ for all $\mu \in [0,1)$ and for all $p \in (p_o, p_1)$. To prove the operators $\pm\frac{1}{2}\frac{1+\mu}{1-\mu}I + K_\lambda^*$ are invertible on $H^p(\partial\Omega)$ for all $\mu \in [0,1)$ and for all $p \in (p_o, p_1)$, it is enough to prove that $(T_\mu^\pm)^*$ are well-posed for all $\mu \in [0,1)$ and for all $p \in (p_o, p_1)$, and using a similar argument as before, this time we can reduce matters to the case when $\vec{g} = 0$. We will focus on $(T_\mu^+)^*$, as the result for $(T_\mu^-)^*$ follows similarly.

Fix $p \in (p_o, p_1)$. First, we claim that for $\vec{f} \in H^p(\partial\Omega)$,

$$\mathcal{D}_\lambda \left[(\pm\tfrac{1}{2}I + K_\lambda)^{-1} S\vec{f} \right] = \mathcal{S}\vec{f} \quad \text{in } \Omega_\pm. \tag{5.59}$$

To prove this identity, it is enough to consider the case when $\vec{f} \in H^p(\partial\Omega) \cap L^2(\partial\Omega)$. Using the jump formula (4.43), it can be shown that the left and right sides of (5.47) are equivalent on the boundary. Since the boundary version of each side is a function in $H_1^p(\partial\Omega) \cap \dot{L}_1^2(\partial\Omega)$, it follows from the uniqueness for the $\dot{L}_1^2$ Regularity problem that the left and right sides of (5.59) differ only by a constant. Then since each expression decays at infinity, the identity must hold. Computing the appropriate conormal derivative for each side in (5.47) gives the useful boundary identity

$$\partial_\nu^\lambda \Big(\mathcal{D}_\lambda((\pm\tfrac{1}{2}I + K_\lambda)^{-1} S\vec{f}), \mathcal{P}_\lambda((\pm\tfrac{1}{2}I + K_\lambda)^{-1} S\vec{f}) \Big) = (\mp\tfrac{1}{2}I + K_\lambda^*)\vec{f} \quad \text{on } \partial\Omega \tag{5.60}$$

Next, we claim that the functions

$$\vec{u}_\pm := \tfrac{1}{1-\mu}\, \mathcal{D}_\lambda \left[(\pm\tfrac{1}{2}I + K_\lambda)^{-1} (-\tfrac{1}{2}\tfrac{1+\mu}{1-\mu} I + K_\lambda)^{-1} S\vec{f} \right], \tag{5.61}$$

$$\pi_\pm := \tfrac{1}{1-\mu}\, \mathcal{P}_\lambda \left[(\pm\tfrac{1}{2}I + K_\lambda)^{-1} (-\tfrac{1}{2}\tfrac{1+\mu}{1-\mu} I + K_\lambda)^{-1} S\vec{f} \right], \tag{5.62}$$

will satisfy $(T_\mu^+)^*$ (with $\vec{g} = 0$, as agreed). On the boundary, we have

$$\vec{u}_\pm|_{\partial\Omega} = \tfrac{1}{1-\mu}(\pm\tfrac{1}{2}I + K_\lambda)(\pm\tfrac{1}{2}I + K_\lambda)^{-1} (-\tfrac{1}{2}\tfrac{1+\mu}{1-\mu} I + K_\lambda)^{-1} S\vec{f} = \tfrac{1}{1-\mu} (-\tfrac{1}{2}\tfrac{1+\mu}{1-\mu} I + K_\lambda)^{-1} S\vec{f},$$

and so $\vec{u}_+|_{\partial\Omega} = \vec{u}_-|_{\partial\Omega}$. Also, using (5.52) twice gives

$$\begin{aligned}
&\tfrac{1}{1-\mu} \Big((\tfrac{1}{2}I + K_\lambda)^{-1} - \mu(-\tfrac{1}{2}I + K_\lambda)^{-1} \Big) (-\tfrac{1}{2}\tfrac{1+\mu}{1-\mu} I + K_\lambda)^{-1} S\vec{f} \\
&\qquad = (\tfrac{1}{2}I + K_\lambda)^{-1} (-\tfrac{1}{2}\tfrac{1+\mu}{1-\mu} I + K_\lambda)(-\tfrac{1}{2}I + K_\lambda)^{-1} (-\tfrac{1}{2}\tfrac{1+\mu}{1-\mu} I + K_\lambda)^{-1} S\vec{f} \\
&\qquad = (\tfrac{1}{2}I + K_\lambda)^{-1} (-\tfrac{1}{2}I + K_\lambda)^{-1} S\vec{f} \\
&\qquad = \Big((-\tfrac{1}{2}I + K_\lambda)^{-1} - (\tfrac{1}{2}I + K_\lambda)^{-1} \Big) S\vec{f}.
\end{aligned} \tag{5.63}$$

Using (5.63) as well as the boundary identity (5.60), allows us to write

$$\begin{aligned}\partial_\nu^\lambda(\vec{u}_+,\pi_+)-\mu\partial_\nu^\lambda(\vec{u}_+,\pi_+) &= \partial_\nu^\lambda\Big(\mathcal{D}_\lambda\Big((-\tfrac{1}{2}I+K_\lambda)^{-1}S\vec{f}\Big),\mathcal{P}_\lambda\Big((-\tfrac{1}{2}I+K_\lambda)^{-1}S\vec{f}\Big)\Big)\\ &\quad -\partial_\nu^\lambda\Big(\mathcal{D}_\lambda\Big((\tfrac{1}{2}I+K_\lambda)^{-1}S\vec{f}\Big),\mathcal{P}_\lambda\Big((\tfrac{1}{2}I+K_\lambda)^{-1}S\vec{f}\Big)\Big)\\ &= (\tfrac{1}{2}I+K_\lambda^*)\vec{f}-(-\tfrac{1}{2}I+K_\lambda^*)\vec{f}=\vec{f}.\end{aligned} \tag{5.64}$$

This proves the existence of a solution to the transmission problem $(T_\mu^+)^*$. To prove uniqueness, we will first establish uniqueness for the H_1^p Regularity problem (R). Assume $(\vec{u}_+,\pi_+)$ solves the homogeneous version of the H_1^p Regularity problem and define

$$\vec{u}_- := \mathcal{D}_\lambda\Big[\Big((-\tfrac{1}{2}I+K_\lambda)^{-1}-(\tfrac{1}{2}I+K_\lambda)^{-1}\Big)S\Big(\partial_\nu^\lambda(\vec{u}_+,\pi_+)\Big)\Big] \quad \text{in } \Omega_-,$$

and

$$\pi_- := \mathcal{P}_\lambda\Big[\Big((-\tfrac{1}{2}I+K_\lambda)^{-1}-(\tfrac{1}{2}I+K_\lambda)^{-1}\Big)S\Big(\partial_\nu^\lambda(\vec{u}_+,\pi_+)\Big)\Big] \quad \text{in } \Omega_-.$$

Arguing as above using the boundary identity (5.60), it follows that $\partial_\nu^\lambda(\vec{u}_-,\pi_-)=\partial_\nu^\lambda(\vec{u}_+,\pi_+)$. Then since $\partial_\nu^\lambda(\vec{u}_-,\pi_-)=\partial_\nu^\lambda(\vec{u}_+,\pi_+)$ and $\vec{u}_+|_{\partial\Omega}=0$, using (4.143) gives

$$(\tfrac{1}{2}I+K_\lambda)(\vec{u}_-|_{\partial\Omega})=(-\tfrac{1}{2}I+K_\lambda)(\vec{u}_+|_{\partial\Omega})=0. \tag{5.65}$$

Since $\frac{1}{2}I+K_\lambda$ is invertible on $H_1^p(\partial\Omega)$, we have that $\vec{u}_-|_{\partial\Omega}=0=\vec{u}_+|_{\partial\Omega}$, and then it follows from Proposition 4.18 that $\vec{u}_+$ and π_+ must be constant.

Returning to the issue of uniqueness for $(T_\mu^+)^*$, assume $\vec{u}_\pm,\pi_\pm$ solve the homogeneous version of $(T_\mu^+)^*$. Multiplying the version of (4.143) corresponding to the sign minus by μ and subtracting it from the version corresponding to the sign plus, and also making use of the transmission conditions, gives

$$(1-\mu)(-\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I+K_\lambda)(\vec{u}_+|_{\partial\Omega})=0. \tag{5.66}$$

Since $-\frac{1}{2}\frac{1+\mu}{1-\mu}I+K_\lambda$ is invertible on $H_1^p(\partial\Omega)$, we have that $\vec{u}_+|_{\partial\Omega}=0=\vec{u}_-|_{\partial\Omega}$. Then from the uniqueness of the H_1^p Regularity problem, $\vec{u}_\pm$ and $\pi_\pm$ must be constant. This finishes the proof of the theorem. $\square$

We conclude this section with the following results.

Lemma 5.10 *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a graph Lipschitz domain. Then there exists $\varepsilon > 0$ such that for $p \in (2-\varepsilon, 2+\varepsilon)$, the operator*

$$S : L^p(\partial\Omega) \longrightarrow \dot{L}^p_1(\partial\Omega), \tag{5.67}$$

is an isomorphism.

Proof. For $\lambda \in (-1, 1]$ fixed, define the operator $S^{-1} : \dot{L}^2_1(\partial\Omega) \longrightarrow L^2(\partial\Omega)$ by

$$S^{-1}\vec{f} := (-\tfrac{1}{2}I + K^*_\lambda)^{-1}\left(\partial^\lambda_\nu(\mathcal{D}_\lambda[(\tfrac{1}{2}I + K_\lambda)^{-1}\vec{f}], \mathcal{P}_\lambda[(\tfrac{1}{2}I + K_\lambda)^{-1}\vec{f}])\right). \tag{5.68}$$

Using (5.48) and (5.60), it can be shown that (5.68) is in fact the inverse of (5.67). □

Lemma 5.11 *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a graph Lipschitz domain. If $\vec{u}$ and π satisfy*

$$\Delta\vec{u} = \nabla\pi, \quad \text{div}\,\vec{u} = 0 \text{ in } \Omega, \quad M(\nabla\vec{u}), M(\pi) \in L^2(\partial\Omega), \tag{5.69}$$

then there exists $\vec{f} \in L^2(\partial\Omega)$ and $\vec{c} \in \mathbb{R}^n$ such that $\vec{u} = \mathcal{S}\vec{f} + \vec{c}$ in Ω and $\pi = \mathcal{Q}\vec{f}$ in Ω.

Proof. This follows from Lemma 5.10 and the uniqueness (modulo constants) of the Regularity problem. In particular, $\vec{u} = \mathcal{S}(S^{-1}(\vec{u}|_{\partial\Omega})) + \vec{c}$ and $\pi = \mathcal{Q}(S^{-1}(\vec{u}|_{\partial\Omega}))$. □

5.3 Inverting the double layer on L^p for p near 2 on bounded Lipschitz domains

We debut with a few preliminaries. Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, for each $k \in \mathbb{N}$ we set

$$\mathbb{R}^k_{\partial\Omega} := \Big\{\sum_j c_j\chi_{\Sigma_j} : c_j \in \mathbb{R}^k \text{ and } \Sigma_j \text{ connected component of } \partial\Omega\Big\}, \tag{5.70}$$

$$\mathbb{R}^k_{\partial\Omega_\pm} := \Big\{\sum_j c_j\chi_{\partial\mathcal{O}_j} : c_j \in \mathbb{R}^k \text{ and } \mathcal{O}_j \text{ bounded connected component of } \Omega_\pm\Big\}, \tag{5.71}$$

$$\mathbb{R}^k_{\Omega_\pm} := \Big\{\sum_j c_j\chi_{\mathcal{O}_j} : c_j \in \mathbb{R}^k \text{ and } \mathcal{O}_j \text{ bounded connected component of } \Omega_\pm\Big\}, \tag{5.72}$$

with the convention that, when $k = 1$, we agree to drop it as a superscript. In particular, we have

$$\mathbb{R}^k_{\partial\Omega_\pm} = (\mathbb{R}^k_{\Omega_\pm})\Big|_{\partial\Omega} \tag{5.73}$$

and

$$\mathbb{R}^k_{\partial\Omega} = \mathbb{R}^k_{\partial\Omega_+} \oplus \mathbb{R}^k_{\partial\Omega_-}, \tag{5.74}$$

where the sum is direct but *not* orthogonal. For instance, we have

$$\Big[\mathbb{R}_{\partial\Omega_+}\Big]^\perp \cap \mathbb{R}_{\partial\Omega_-} = \{0\} \quad \text{and} \quad \Big[\mathbb{R}_{\partial\Omega_-}\Big]^\perp \cap \mathbb{R}_{\partial\Omega_+} = \{0\}, \tag{5.75}$$

where the orthogonal complements are taken in $L^2(\partial\Omega)$. Let us also point out here that

$$\begin{gathered} \dim \mathbb{R}^k_{\Omega_+} = \dim \mathbb{R}^k_{\partial\Omega_+} = k \cdot b_0, \qquad \dim \mathbb{R}^k_{\Omega_-} = \dim \mathbb{R}^k_{\partial\Omega_-} = k \cdot b_{n-1}, \\ \dim \mathbb{R}^k_{\partial\Omega} = k \cdot (b_0 + b_{n-1}), \end{gathered} \tag{5.76}$$

where the Betti numbers b_0, b_{n-1} represent the number of bounded connected components of Ω_+ and Ω_-, respectively. Therefore, the intuitive interpretation of b_{n-1} is the number of n-dimensional "holes" of Ω_+.

Lemma 5.12 *Let Ω be as above and fix $\lambda \in \mathbb{R}$. Then the following identities hold:*

$$\mathcal{S}(\nu\psi) = 0 \quad \text{in} \quad \Omega_\pm, \quad \forall\, \psi \in \mathbb{R}_{\partial\Omega}, \tag{5.77}$$

$$S(\nu\psi) = 0 \quad \text{on} \quad \partial\Omega, \qquad \forall \psi \in \mathbb{R}_{\partial\Omega}, \tag{5.78}$$

$$K^*_\lambda(\nu\psi) = \mp\tfrac{1}{2}\nu\psi \quad \text{on} \quad \partial\Omega, \qquad \forall\, \psi \in \mathbb{R}_{\partial\Omega_\pm}. \tag{5.79}$$

Proof. Let D be any bounded component of Ω_+ or Ω_-. For every $x \in \mathbb{R}^n \backslash \partial\Omega$ and $1 \leq j \leq n$, an integration by parts based on (4.29) gives

$$(\mathcal{S}(\nu\chi_{\partial D}))_j(x) = \int_{\partial D} E_{jk}(x-y)\nu_k(y)\,d\sigma(y) = -\int_D (\partial_k E_{jk})(x-y)\,dy = 0. \tag{5.80}$$

Thus, from (5.80) and (5.74),

$$\mathcal{S}(\nu\chi_{\partial D}) = 0 \quad \text{in} \quad \Omega_\pm, \tag{5.81}$$

which readily yields (5.77). This identity further yields (5.78) by taking boundary traces. Next, for any D, bounded, connected component of either Ω_+ or Ω_-,

$$\mathcal{Q}(\nu\chi_{\partial D})(x) = \int_{\partial D} (\partial_{\nu(y)} E_\Delta)(y-x)\,d\sigma(y) = \pm\chi_D(x), \quad \forall\, x \in \mathbb{R}^n \setminus \partial\Omega, \quad \text{if } D \subset \Omega_\pm. \tag{5.82}$$

In particular,

$$\begin{aligned} &\psi \in \mathbb{R}_{\partial\Omega_+} \Longrightarrow \mathcal{Q}(\nu\psi)\Big|_{\partial\Omega_+} = \psi \quad \text{and} \quad \mathcal{Q}(\nu\psi)\Big|_{\partial\Omega_-} = 0, \\ &\psi \in \mathbb{R}_{\partial\Omega_-} \Longrightarrow \mathcal{Q}(\nu\psi)\Big|_{\partial\Omega_+} = 0 \quad \text{and} \quad \mathcal{Q}(\nu\psi)\Big|_{\partial\Omega_-} = -\psi. \end{aligned} \tag{5.83}$$

Consequently,

$$(\mp\tfrac{1}{2}I + K^*_\lambda)(\nu\psi) = \partial^\lambda_\nu\Big(\mathcal{S}(\nu\psi)|_{\Omega_\pm}, \mathcal{Q}(\nu\psi)|_{\Omega_\pm}\Big) = \mp\nu\psi, \qquad \forall\, \psi \in \mathbb{R}_{\partial\Omega_\pm}, \tag{5.84}$$

which further entails (5.79). □

We continue to introduce notation which will be useful hereafter. Let Ψ be the $n(n+1)/2$-dimensional linear space of $\mathbb{R}^n$-valued functions $\psi = (\psi_j)_{1\leq j\leq n}$ defined in $\mathbb{R}^n$ and satisfying

$$\partial_j\psi_k + \partial_k\psi_j = 0, \qquad 1 \leq j, k \leq n, \tag{5.85}$$

and note that

$$\Psi = \Big\{\psi(x) = Ax + \vec{a} :\ A,\ n\times n \text{ antisymmetric matrix, and } \vec{a} \in \mathbb{R}^n\Big\}. \tag{5.86}$$

Now let

$$\Psi(\Omega_\pm) := \{\sum_j (\psi_j|_{\mathcal{O}_j})\chi_{\mathcal{O}_j} : \psi_j \in \Psi,\ \mathcal{O}_j \text{ bounded component of } \Omega_\pm\}. \tag{5.87}$$

Then for $\lambda \in (-1, 1]$, we can define

$$\Psi^\lambda(\Omega_\pm) := \begin{cases} \mathbb{R}^n_{\Omega_\pm}, & |\lambda| < 1, \\ \Psi(\Omega_\pm), & \lambda = 1, \end{cases} \tag{5.88}$$

and

$$\Psi^\lambda(\partial\Omega_\pm) := \Psi^\lambda(\Omega_\pm)|_{\partial\Omega_\pm}, \tag{5.89}$$

so that

$$\dim \Psi^\lambda(\partial\Omega_+) = \begin{cases} n \cdot b_0 \text{ if } |\lambda| < 1, \\ \frac{n(n+1)}{2} \cdot b_0 \text{ if } \lambda = 1, \end{cases} \qquad \dim \Psi^\lambda(\partial\Omega_-) = \begin{cases} n \cdot b_{n-1} \text{ if } |\lambda| < 1, \\ \frac{n(n+1)}{2} \cdot b_{n-1} \text{ if } \lambda = 1. \end{cases} \tag{5.90}$$

Finally, set

$$\begin{gathered} \Psi^1(\partial\Omega) := \{\sum_j (\psi_j|_{\Sigma_j})\chi_{\Sigma_j} : \psi_j \in \Psi,\ \Sigma_j \text{ component of } \partial\Omega\} \\ \text{and } \ \Psi^\lambda(\partial\Omega) := \mathbb{R}^n_{\partial\Omega} \text{ if } |\lambda| < 1, \end{gathered} \tag{5.91}$$

which implies

$$\dim \Psi^\lambda(\partial\Omega) = \begin{cases} n \cdot (b_0 + b_{n-1}) \ \text{ if } \ |\lambda| < 1, \\ \frac{n(n+1)}{2} \cdot (b_0 + b_{n-1}) \text{ if } \lambda = 1. \end{cases} \tag{5.92}$$

Lemma 5.13 *If Ω is as before, an alternate characterization of these spaces is*

$$\vec{u} \in \Psi^\lambda(\Omega_\pm) \iff \vec{u} \in C^2(\Omega_\pm) \text{ and } A_\lambda(\nabla\vec{u}, \nabla\vec{u}) = 0 \text{ in } \Omega_\pm. \tag{5.93}$$

Furthermore,

$$\vec{u}_{\pm} \in \Psi^{\lambda}(\Omega_{\pm}) \Longrightarrow \Delta\vec{u}_{\pm} = 0 \quad \text{and} \quad \operatorname{div}\vec{u}_{\pm} = 0 \quad \text{in } \Omega_{\pm}. \tag{5.94}$$

In particular, for every $\psi \in \Psi^{\lambda}(\Omega_{\pm})$,

$$(\psi, 0) \text{ solves the Stokes system in } \Omega_{\pm} \text{ and satisfies } \partial_{\nu}^{\lambda}(\psi, 0) = 0. \tag{5.95}$$

Conversely, if $\vec{u}_{\pm}$ *and* $\pi_{\pm}$ *satisfy the Stokes system in* $\Omega_{\pm}$ *and* $\vec{u}_{\pm} \in \Psi^{\lambda}(\Omega_{\pm})$, *then*

$$\pi_{\pm} \in \mathbb{R}_{\Omega_{\pm}} \qquad \text{and} \qquad \partial_{\nu}^{\lambda}(\vec{u}_{\pm}, \pi_{\pm}) \in \nu\mathbb{R}_{\partial\Omega_{\pm}}. \tag{5.96}$$

Finally,

$$\mathcal{D}_{\lambda}(\psi_{\pm}|_{\partial\Omega}) = \pm\psi_{\pm} \quad \text{in} \quad \Omega_{\pm}, \qquad \forall \psi_{\pm} \in \Psi^{\lambda}(\Omega_{\pm}), \tag{5.97}$$

and

$$(\mp\tfrac{1}{2}I + K_{\lambda})\psi_{\pm} = 0, \qquad \forall \psi_{\pm} \in \Psi^{\lambda}(\partial\Omega_{\pm}). \tag{5.98}$$

Proof. To see this, first assume $\psi_{\pm} \in \Psi^{\lambda}(\partial\Omega_{\pm})$. Then $(\tilde{\psi}_{\pm}, 0)$ satisfies the Stokes system in $\Omega_{\pm}$, where $\tilde{\psi}_{\pm}$ denotes the natural extension of $\psi_{\pm}$ to $\Omega_{\pm}$. Then (5.97) follows by invoking (4.120), (5.96) and (5.77). Finally, (5.98) is a direct consequence of (5.97) and the trace formula (4.43). $\square$

Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ and $p \in (\frac{n-1}{n}, \infty)$, set

$$h^p_{\Psi^{\lambda}_{\pm}}(\partial\Omega) := \Big\{\vec{f} \in h^p(\partial\Omega) : \langle \vec{f}, \psi \rangle = 0, \ \forall\, \psi \in \Psi^{\lambda}(\partial\Omega_{\pm})\Big\}, \tag{5.99}$$

$$h^p_{\Psi^{\lambda}}(\partial\Omega) := \Big\{\vec{f} \in h^p(\partial\Omega) : \langle \vec{f}, \psi \rangle = 0, \ \forall\, \psi \in \Psi^{\lambda}(\partial\Omega)\Big\}. \tag{5.100}$$

When $1 < p < \infty$, we shall write $L^p_{\Psi^{\lambda}_{\pm}}(\partial\Omega)$ and $L^p_{\Psi^{\lambda}}(\partial\Omega)$ in place of $h^p_{\Psi^{\lambda}_{\pm}}(\partial\Omega)$ and $h^p_{\Psi^{\lambda}}(\partial\Omega)$, respectively. For further use, we record here the following elementary lemma.

Lemma 5.14 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain. Then*

$$\Psi^\lambda(\partial\Omega) = \Psi^\lambda(\partial\Omega_+) \oplus \Psi^\lambda(\partial\Omega_-) \tag{5.101}$$

where the sum is direct. In addition,

$$\nu\,\mathbb{R}_{\partial\Omega} \hookrightarrow \Big[\Psi^\lambda(\partial\Omega)\Big]^\perp, \tag{5.102}$$

where the orthogonal complement is taken in $L^2(\partial\Omega)$. Also, for every $p \in (1,\infty)$,

$$L^p_{\Psi^\lambda_\pm}(\partial\Omega) \hookrightarrow L^p_0(\partial\Omega) := \Big\{\vec{f} \in L^p(\partial\Omega) : \int_{\partial\Omega} \vec{f}\, d\sigma = 0\Big\}, \tag{5.103}$$

$$\Big[\mathbb{R}^n_{\partial\Omega}\Big]^\perp \hookrightarrow \Big[\mathbb{R}^n_{\partial\Omega_\pm}\Big]^\perp \hookrightarrow L^p_0(\partial\Omega), \tag{5.104}$$

and

$$\nu\,\mathbb{R}_{\partial\Omega} \hookrightarrow L^p_0(\partial\Omega). \tag{5.105}$$

Proof. Consider the identity (5.101). In one direction, the right-to-left inclusion is a consequence of (5.74), (5.87), and (5.91). Since, by (5.90) and (5.92), the spaces whose equality we are trying to establish have the same (finite) dimension, there remains to show that the sum is direct. To this end, assume that $\psi \in \Psi^\lambda(\partial\Omega_+) \cap \Psi^\lambda(\partial\Omega_-)$ is arbitrary, and denote by $\psi_\pm \in \Psi^\lambda(\Omega_\pm)$ the natural extension of ψ in $\Omega_\pm$. Now, if we set $\tilde{\psi} := \psi_\pm$ in $\Omega_\pm$, the fact that $\psi_+|_{\partial\Omega} = \psi_-|_{\partial\Omega}$ ensures that (5.85) is satisfied by this function in $\mathbb{R}^n$, in the sense of distributions. Hence, $\tilde{\psi} \in \Psi$, and since it has compact support, ψ must vanish in $\mathbb{R}^n$. This forces $\psi = 0$ on $\partial\Omega$, finishing the proof of (5.101).

All the other formulas in the statement of the lemma follow more or less directly from definitions. The proof of the lemma is therefore complete. □

Moving on, for each $\vec{f} \in L^2(\partial\Omega)$, the functions

$$\vec{u}_\pm(x) := \mathcal{S}\vec{f}(x), \quad \pi_\pm(x) := \mathcal{Q}\vec{f}(x), \qquad x \in \Omega_\pm, \tag{5.106}$$

solve the Stokes system

$$\Delta\vec{u}_{\pm} - \nabla\pi_{\pm} = 0, \quad \operatorname{div}\vec{u}^{\pm} = 0 \quad \text{in } \Omega_{\pm}, \tag{5.107}$$

and satisfy

$$\|M(\nabla\vec{u}_{\pm})\|_{L^2(\partial\Omega)} + \|M(\pi_{\pm})\|_{L^p(\partial\Omega)} \leq C(\partial\Omega,p)\|\vec{f}\|_{L^2(\partial\Omega)}, \tag{5.108}$$

$$|\vec{u}_{-}(x)| + |x|\Big(|\nabla\vec{u}_{-}(x)| + |\pi_{-}(x)|\Big) = O(|x|^{2-n}) \ \text{ as } \ |x| \to \infty, \ \text{ if } n \geq 3. \tag{5.109}$$

Moreover, if $\int_{\partial\Omega}\vec{f}\,d\sigma = 0$, then for any $n \geq 2$ the decay condition (5.109) improves to

$$|\vec{u}_{-}(x)| + |x|\Big(|\nabla\vec{u}_{-}(x)| + |\pi_{-}(x)|\Big) = O(|x|^{1-n}) \ \text{ as } \ |x| \to \infty. \tag{5.110}$$

Consequently, Green's formula (4.6) gives

$$\int_{\Omega_+}\langle A_\lambda\nabla\vec{u}_+, \nabla\vec{u}_+\rangle\,dx = \int_{\partial\Omega}\Big\langle S\vec{f},\, \Big(-\tfrac{1}{2}I + K_\lambda^*\Big)\vec{f}\Big\rangle\,d\sigma, \tag{5.111}$$

and if either $n \geq 3$ or $\int_{\partial\Omega}\vec{f}\,d\sigma = 0$,

$$\int_{\Omega_-}\langle A_\lambda\nabla\vec{u}_-, \nabla\vec{u}_-\rangle\,dx = -\int_{\partial\Omega}\Big\langle S\vec{f},\, \Big(\tfrac{1}{2}I + K_\lambda^*\Big)\vec{f}\Big\rangle\,d\sigma. \tag{5.112}$$

For each $p \in (\frac{n-1}{n}, \infty)$, set

$$h^p_{1,\nu_\pm}(\partial\Omega) := \Big\{\vec{f} \in h^p_1(\partial\Omega) : \int_{\partial\Omega}\langle\psi, \vec{f}\rangle\,d\sigma = 0,\ \forall\,\psi \in \nu\,\mathbb{R}_{\partial\Omega_\pm}\Big\}, \tag{5.113}$$

and

$$h^p_{1,\nu}(\partial\Omega) := \Big\{\vec{f} \in h^p_1(\partial\Omega) : \int_{\partial\Omega}\langle\psi, \vec{f}\rangle\,d\sigma = 0,\ \forall\,\psi \in \nu\,\mathbb{R}_{\partial\Omega}\Big\}, \tag{5.114}$$

with the convention that, when $1 < p < \infty$, we shall write $L^p_{1,\nu_\pm}(\partial\Omega)$ in place of $h^p_{1,\nu_\pm}(\partial\Omega)$. For $1 < p < \infty$, let us also define

$$L^p_{\nu_\pm}(\partial\Omega) := \Big\{\vec{f} \in L^p(\partial\Omega) : \int_{\partial\Omega} \langle \psi, \vec{f} \rangle \, d\sigma = 0, \ \forall\, \psi \in \nu\, \mathbb{R}_{\partial\Omega_\pm}\Big\}, \tag{5.115}$$

$$L^p_{\nu}(\partial\Omega) := \Big\{\vec{f} \in L^p(\partial\Omega) : \int_{\partial\Omega} \langle \psi, \vec{f} \rangle \, d\sigma = 0, \ \forall\, \psi \in \nu\, \mathbb{R}_{\partial\Omega}\Big\}. \tag{5.116}$$

We can also prove the following.

Lemma 5.15 *For any $\lambda \in (-1, 1]$ and $p \in (1, \infty)$,*

$$\Psi^\lambda(\partial\Omega_+) \oplus \Psi^\lambda(\partial\Omega_-) = \Psi^\lambda(\partial\Omega) \hookrightarrow L^p_{1,\nu}(\partial\Omega) \hookrightarrow L^p_{1,\nu_\pm}(\partial\Omega). \tag{5.117}$$

Also, if $1 < p, p' < \infty$ satisfy $1/p + 1/p' = 1$, then

$$\Big(L^p_{\Psi^\lambda_\mp}(\partial\Omega)/\nu\mathbb{R}_{\partial\Omega_\pm}\Big)^* = L^{p'}_{\nu_\pm}(\partial\Omega)/\Psi^\lambda(\partial\Omega_\mp). \tag{5.118}$$

Proof. This can then be easily checked from definitions with the help of the general formula

$$\Big(\frac{Y_1}{Y_2}\Big)^* = \frac{Y_2^\perp}{Y_1^\perp}, \tag{5.119}$$

whenever X is a Banach space, $0 \hookrightarrow Y_2 \hookrightarrow Y_1 \hookrightarrow X$ are closed subspaces, and we have set $Y_j^\perp := \{\Lambda \in X^* : \Lambda(y) = 0, \ \forall\, y \in Y_j\}$, $j = 1, 2$. □

Finally, we are ready to state our next result. Before doing so, denote by $\mathrm{Ker}\,(T : A \to B)$ the null-space of a linear operator T from A into B.

Proposition 5.16 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. Then for each $\gamma \in \mathbb{R} \setminus [-\frac{1}{2}, \frac{1}{2}]$ and $\lambda \in (-1, 1]$, the operators*

$$\gamma I + K^*_\lambda : L^2(\partial\Omega) \longrightarrow L^2(\partial\Omega), \tag{5.120}$$

and

$$\gamma I + K_\lambda : L^2_1(\partial\Omega) \longrightarrow L^2_1(\partial\Omega), \tag{5.121}$$

are injective. Moreover, if $-1 < \lambda \leq 1$, the operators

$$\pm\tfrac{1}{2}I + K^*_\lambda : L^2_{\Psi^\lambda_\mp}(\partial\Omega)/\nu\mathbb{R}_{\partial\Omega_\pm} \longrightarrow L^2_{\Psi^\lambda_\mp}(\partial\Omega)/\nu\mathbb{R}_{\partial\Omega_\pm}, \tag{5.122}$$

as well as

$$\pm\tfrac{1}{2}I + K_\lambda : L^2_{\nu_\pm}(\partial\Omega)/\Psi^\lambda(\partial\Omega_\mp) \longrightarrow L^2_{\nu_\pm}(\partial\Omega)/\Psi^\lambda(\partial\Omega_\mp), \tag{5.123}$$

$$\pm\tfrac{1}{2}I + K_\lambda : L^2_{1,\nu_\pm}(\partial\Omega)/\Psi^\lambda(\partial\Omega_\mp) \longrightarrow L^2_{1,\nu_\pm}(\partial\Omega)/\Psi^\lambda(\partial\Omega_\mp), \tag{5.124}$$

are well-defined and injective. In addition,

$$\mathrm{Ker}\,(\pm\tfrac{1}{2}I + K_\lambda : L^2_{1,\nu_\pm}(\partial\Omega) \to L^2_{1,\nu_\pm}(\partial\Omega)) = \Psi^\lambda(\partial\Omega_\mp), \tag{5.125}$$

$$\mathrm{Ker}\,(\pm\tfrac{1}{2}I + K_\lambda : L^2_{\nu_\pm}(\partial\Omega) \to L^2_{\nu_\pm}(\partial\Omega)) = \Psi^\lambda(\partial\Omega_\mp), \tag{5.126}$$

$$\mathrm{Ker}\,(\pm\tfrac{1}{2}I + K^*_\lambda : L^2_{\Psi^\lambda_\mp}(\partial\Omega) \to L^2_{\Psi^\lambda_\mp}(\partial\Omega)) = \nu\mathbb{R}_{\partial\Omega_\pm}. \tag{5.127}$$

Finally,

$$\mathrm{Ker}\,(S : L^2(\partial\Omega) \to L^2_1(\partial\Omega)) = \begin{cases} \nu\mathbb{R}_{\partial\Omega} & \text{if } n \geq 3, \\ \nu\mathbb{R}_{\partial\Omega} \oplus \mathcal{W} & \text{if } n = 2, \end{cases} \tag{5.128}$$

where, for $n = 2$,

$$\mathcal{W} := \{\vec{f} \in L^2_{\nu_-}(\partial\Omega) : \, S\vec{f} = 0 \text{ on } \partial\Omega, \text{ and } \mathcal{Q}\vec{f} = 0 \text{ in } \Omega_+\} \tag{5.129}$$

also satisfies

$$\dim \mathcal{W} \leq 2. \tag{5.130}$$

Proof. Fix $\gamma \in \mathbb{R}$, $|\gamma| > \frac{1}{2}$, $-1 \leq \lambda \leq 1$, and assume that $\vec{f} \in L^2(\partial\Omega)$ is such that $(\gamma I + K^*_\lambda)\vec{f} = 0$. Also, let $(\vec{u}_\pm, \pi_\pm)$ be as in (5.106) and define $\vec{u}$, π in $\mathbb{R}^n$ as in (4.152). Since $\vec{u}_+|_{\partial\Omega} = \vec{u}_-|_{\partial\Omega}$, it follows that

$$u \in W^{1,2}_{loc}(\mathbb{R}^n). \tag{5.131}$$

Next, based on Green's formula (4.6), for each $\vec{c} \in \mathbb{R}^n$ we may write

$$\begin{aligned}
(\gamma + \tfrac{1}{2}) \Big\langle \int_{\partial\Omega} \vec{f} \, d\sigma \, , \, \vec{c} \Big\rangle &= (\gamma + \tfrac{1}{2}) \int_{\partial\Omega} \langle \vec{f}, \vec{c} \rangle \, d\sigma = - \int_{\partial\Omega} \Big\langle (-\tfrac{1}{2} I + K^*_\lambda) \vec{f} \, , \, \vec{c} \Big\rangle \, d\sigma \\
&= - \int_{\partial\Omega} \Big\langle \partial^\lambda_\nu (\mathcal{S}\vec{f}, \mathcal{Q}\vec{f}) \, , \, \vec{c} \Big\rangle \, d\sigma \\
&= - \int_\Omega \langle \Delta\vec{u} - \nabla\pi, \vec{c} \rangle - \int_\Omega A_\lambda(\nabla\vec{u}, \nabla\vec{c}) - \int_\Omega \pi \operatorname{div} \vec{c} \\
&= 0,
\end{aligned} \tag{5.132}$$

which shows that $\vec{f} \in L^2_0(\partial\Omega)$. In particular, the improved decay condition (5.110) holds which allow us to write

$$\begin{aligned}
0 &= \int_{\partial\Omega} \langle (\gamma I + K^*_\lambda) \vec{f}, S\vec{f} \rangle \, d\sigma \\
&= \int_{\partial\Omega} \Big\langle (-\gamma + \tfrac{1}{2})(-\tfrac{1}{2} I + K^*_\lambda) \vec{f} + (\gamma + \tfrac{1}{2})(\tfrac{1}{2} I + K^*_\lambda) \vec{f}, S\vec{f} \Big\rangle \, d\sigma \\
&= (-\gamma + \tfrac{1}{2}) \int_{\Omega_+} A_\lambda(\nabla\vec{u}, \nabla\vec{u}) \, dx + (-\gamma - \tfrac{1}{2}) \int_{\Omega_-} A_\lambda(\nabla\vec{u}, \nabla\vec{u}) \, dx.
\end{aligned} \tag{5.133}$$

Consequently,

$$\int_{\mathbb{R}^n} A_\lambda(\nabla\vec{u}, \nabla\vec{u}) \, dx = 0, \tag{5.134}$$

since $-\gamma - \frac{1}{2}$ and $-\gamma + \frac{1}{2}$ have the same sign and the integrands in the last line of (5.133) are nonnegative. Next, pick a function $\psi \in C^\infty_0(\mathbb{R}^n)$ which is identically one in a neighborhood of the origin and set $\psi_j(x) := \psi(x/j)$, $j \in \mathbb{N}$. We have

$$\begin{aligned}
\lim_{j\to\infty} \int_{\mathbb{R}^n} A_\lambda(\nabla(\psi_j \vec{u}), \nabla(\psi_j \vec{u})) \, dx &= \lim_{j\to\infty} \int_{\mathbb{R}^n} \psi_j^2 A_\lambda(\nabla\vec{u}, \nabla\vec{u}) \, dx \\
&\quad + \lim_{j\to\infty} \int_{\mathbb{R}^n} O\Big(|\psi_j| |\nabla\psi_j| |\vec{u}| |\nabla\vec{u}| + |\nabla\psi_j|^2 |\vec{u}|^2 \Big) \, dx \\
&= 0,
\end{aligned} \tag{5.135}$$

thanks to (5.134) and the improved decay of $\vec{u}$ at infinity. Since, by (5.131), $\psi_j u \in W^{1,2}(\mathbb{R}^n)$, Plancherel's formula (used twice) along with (4.18) then give

$$
\begin{aligned}
0 &= \lim_{j\to\infty} \int_{\mathbb{R}^n} A_\lambda(\nabla(\psi_j \vec{u}), \nabla(\psi_j \vec{u}))\, dx \geq \kappa \lim_{j\to\infty} \int_{\mathbb{R}^n} |\nabla(\psi_j \vec{u})|^2\, dx \\
&= \kappa \int_{\mathbb{R}^n} |\nabla \vec{u}|^2\, dx. \qquad (5.136)
\end{aligned}
$$

Thus, $\vec{u}$ is a constant in $\mathbb{R}^n$ and decays at infinity, hence ultimately $\vec{u} = 0$ in $\mathbb{R}^n$. In turn, this forces $\pi_\pm \in \mathbb{R}_{\Omega_\pm}$, prompting the conclusion that

$$
\vec{f} = \partial_\nu^\lambda(\vec{u}_-, \pi_-) - \partial_\nu^\lambda(\vec{u}_+, \pi_+) = \nu(\pi_+ - \pi_-) \in \nu\, \mathbb{R}_{\partial\Omega}. \qquad (5.137)
$$

Now, from (5.137), (5.79) and assumptions, we get

$$
0 = (\gamma I + K_\lambda^*)\vec{f} = (\gamma I + K_\lambda^*)(\nu\pi_+) - (\gamma I + K_\lambda^*)(\nu\pi_-) = (\gamma - \tfrac{1}{2})(\nu\pi_+) - (\gamma + \tfrac{1}{2})(\nu\pi_-) \qquad (5.138)
$$

Thus, π_+ is a multiple of π_-, and so (5.137) implies $\vec{f} \in \nu\, \mathbb{R}_{\partial\Omega_+} \cap \nu\, \mathbb{R}_{\partial\Omega_-}$. Then $\vec{f} = 0$, as wanted. This finishes the proof of the fact that the operator (5.120) is injective.

To see that the operator (5.121) is also injective, assume $\vec{f} \in L_1^2(\partial\Omega)$ is such that $(\gamma I + K_\lambda)\vec{f} = 0$. Let $\vec{u}_\pm = \mathcal{D}_\lambda \vec{f}$ in $\Omega_\pm$ and $\pi_\pm = \mathcal{P}_\lambda \vec{f}$ in $\Omega_\pm$. In particular,

$$
|\vec{u}_-(x)| = O(|x|^{1-n}) \text{ and } |\nabla \vec{u}_-(x)| + |\pi_-(x)| = O(|x|^{-n}), \text{ as } |x| \to \infty, \qquad (5.139)
$$

which ensures that the integration by parts formula (4.6) works in $\Omega_\pm$ to yield

$$
\begin{aligned}
0 &= \int_{\partial\Omega} \langle (\gamma I + K_\lambda)\vec{f}, \partial_\nu^\lambda(\mathcal{D}_\lambda \vec{f}, \mathcal{P}_\lambda \vec{f})\rangle\, d\sigma \\
&= \int_{\partial\Omega} \langle (\gamma + \tfrac{1}{2})(\tfrac{1}{2} I + K_\lambda)\vec{f} + (-\gamma + \tfrac{1}{2})(-\tfrac{1}{2} I + K_\lambda)\vec{f}, \partial_\nu^\lambda(\mathcal{D}_\lambda \vec{f}, \mathcal{P}_\lambda \vec{f})\rangle\, d\sigma \\
&= (\gamma + \tfrac{1}{2}) \int_{\Omega_+} A_\lambda(\nabla \vec{u}_+, \nabla \vec{u}_+)\, dx + (\gamma - \tfrac{1}{2}) \int_{\Omega_-} A_\lambda(\nabla \vec{u}_-, \nabla \vec{u}_-)\, dx. \qquad (5.140)
\end{aligned}
$$

Since $\gamma + \frac{1}{2}$ and $\gamma - \frac{1}{2}$ have the same sign, it follows from (5.93) that $\vec{u}_\pm \in \Psi^\lambda(\Omega_\pm)$ and therefore $\vec{u}_\pm|_{\partial\Omega} = \psi_\pm$ for some $\psi_\pm \in \Psi^\lambda(\partial\Omega_\pm)$. Then applying (5.98) gives

$$0 = (\gamma I + K_\lambda)\vec{f} = (\gamma I + K_\lambda)\psi_+ - (\gamma I + K_\lambda)\psi_- = (\gamma + \tfrac{1}{2})\psi_+ - (\gamma - \tfrac{1}{2})\psi_-. \quad (5.141)$$

This implies that ψ_+ is a multiple of ψ_-, and hence $\vec{f} \in \Psi^\lambda(\partial\Omega_+) \cap \Psi^\lambda(\partial\Omega_-) = \{0\}$.

Turning our attention to the operators in (5.122), we note that these are well-defined since

$$\Big(\pm\tfrac{1}{2}I + K_\lambda^*\Big)(\nu\varphi_\pm) = 0, \qquad \forall \varphi_\pm \in \mathbb{R}_{\partial\Omega_\pm}, \quad (5.142)$$

and, as a simple application of Green's formula (applied in the bounded components of $\Omega_\pm$) shows,

$$\Big(\pm\tfrac{1}{2}I + K_\lambda^*\Big)L^2(\partial\Omega) \subseteq L^2_{\Psi^\lambda_\mp}(\partial\Omega). \quad (5.143)$$

Consider next $\vec{f} \in L^2_{\Psi^\lambda_+}(\partial\Omega)$ such that $(-\frac{1}{2}I + K_\lambda^*)\vec{f} = \nu\varphi$, for some $\varphi \in \mathbb{R}_{\partial\Omega_-}$. Our goal is to show that $\vec{f} \in \nu\mathbb{R}_{\partial\Omega_-}$. To get started, we note that $\vec{f} \in L^2_0(\partial\Omega)$, thanks to (5.105). In turn, the fact that $\vec{f}$ has vanishing moment ensures that if $\vec{u}_\pm$, $\pi_\pm$ are as in (5.106) then (5.110) −and, hence, (5.112)− holds. Then

$$\int_{\Omega_+} A_\lambda(\nabla\vec{u}_+, \nabla\vec{u}_+)\,dx = \int_{\partial\Omega} \langle S\vec{f}, (-\tfrac{1}{2}I + K_\lambda^*)\vec{f}\rangle\,d\sigma = \int_{\partial\Omega} \langle S\vec{f}, \nu\varphi\rangle\,d\sigma = 0. \quad (5.144)$$

Thus from (5.93), $\vec{u}_+ \in \Psi^\lambda(\Omega_+)$. This implies $S\vec{f} = \vec{u}_+\Big|_{\partial\Omega} \in \Psi^\lambda(\partial\Omega_+)$ hence, from orthogonality considerations,

$$0 = \int_{\partial\Omega} \langle \vec{f}, S\vec{f}\rangle\,d\sigma \quad = \quad \int_{\partial\Omega} \langle (\tfrac{1}{2}I + K_\lambda^*)\vec{f}, S\vec{f}\rangle\,d\sigma = \int_{\Omega_-} A_\lambda(\nabla\vec{u}_-, \nabla\vec{u}_-)\,dx. \quad (5.145)$$

From (5.93), $\vec{u}_- \in \Psi^\lambda(\Omega_-)$, and in particular, this implies that $\vec{u}_-$ is harmonic in Ω_-. Thus π_- must be locally constant in Ω_- and vanish in the unbounded component of Ω_-. In other words, $\pi_- \in \mathbb{R}_{\Omega_-}$ and, as a result, we have

$$\vec{f} = (\tfrac{1}{2}I + K_\lambda^*)\vec{f} - (-\tfrac{1}{2}I + K_\lambda^*)\vec{f} = \partial_\nu^\lambda(\vec{u}_-, \pi_-) - \nu\varphi = -\nu[(\pi_-|_{\partial\Omega}) + \varphi] \in \nu\,\mathbb{R}_{\partial\Omega_-}. \quad (5.146)$$

We also need to show that if $\vec{f} \in L^2_{\Psi_-^\lambda}(\partial\Omega)$ is such that $(\frac{1}{2}I + K_\lambda^*)\vec{f} = \nu\varphi$ for some $\varphi \in \mathbb{R}_{\partial\Omega_+}$, then necessarily $\vec{f} \in \nu\mathbb{R}_{\partial\Omega_+}$. To this end, observe that $\vec{f} = \nu\varphi - (-\frac{1}{2}I + K_\lambda^*)\vec{f} \in L_0^2(\partial\Omega)$ by (5.143) and (5.105). With this in hand, the proof is carried out much as before.

Next, the operators in (5.124) are well-defined due to (5.98) and the fact that (as it can be checked using Green's formula in the bounded components of $\Omega_\pm$) ,

$$(\pm\tfrac{1}{2}I + K_\lambda)L_1^2(\partial\Omega) \subseteq L^2_{1,\nu_\pm}(\partial\Omega). \quad (5.147)$$

To see that these operators are injective, we will first show that

$$\vec{f} \in L^2_{1,\nu_-}(\partial\Omega) \text{ and } (-\tfrac{1}{2}I + K_\lambda)\vec{f} \in \Psi^\lambda(\partial\Omega_+) \Longrightarrow \vec{f} \in \Psi^\lambda(\partial\Omega_+). \quad (5.148)$$

To see this, let $\psi := (-\frac{1}{2}I + K_\lambda)\vec{f} \in \Psi^\lambda(\partial\Omega_+)$ and let $\vec{u}_\pm = \mathcal{D}_\lambda\vec{f}$ in $\Omega_\pm$ and $\pi_\pm = \mathcal{P}_\lambda\vec{f}$ in $\Omega_\pm$. Then (5.139) holds and (4.6) gives

$$\begin{aligned}\int_{\Omega_-} A_\lambda(\nabla\vec{u}_-, \nabla\vec{u}_-)\,dx &= -\int_{\partial\Omega} \langle\psi, \partial_\nu^\lambda(\vec{u}_-, \pi_-)\rangle\,d\sigma \\ &= -\int_{\partial\Omega} \langle\psi, \partial_\nu^\lambda(\vec{u}_+, \pi_+)\rangle\,d\sigma = -\int_{\partial\Omega} \langle u_+, \partial_\nu^\lambda(\tilde{\psi}, 0)\rangle\,d\sigma = 0 \quad (5.149)\end{aligned}$$

where $\tilde{\psi}$ denotes the extension of $\psi \in \Psi^\lambda(\partial\Omega_+)$ into Ω_+. It follows that $\vec{u}_- \in \Psi^\lambda(\Omega_-)$, and therefore, $\partial_\nu^\lambda(\vec{u}_+, \pi_+) = \partial_\nu^\lambda(\vec{u}_-, \pi_-) = -\nu\pi_- \in \nu\mathbb{R}_{\partial\Omega_-}$. Then

$$\int_{\Omega_+} A_\lambda(\nabla\vec{u}_+, \nabla\vec{u}_+)\,dx = -\int_{\partial\Omega} \langle u_+, \partial_\nu^\lambda(\vec{u}_+, \pi_+)\rangle\,d\sigma = \int_{\partial\Omega} \langle\psi + \vec{f}, \nu\pi_-\rangle\,d\sigma = 0, \quad (5.150)$$

since $\pi_- \in \mathbb{R}_{\partial\Omega_-}$ and ψ, $\vec{f} \in L^2_{1,\nu_-}(\partial\Omega)$. Thus $\vec{u}_+ \in \Psi^\lambda(\Omega_+)$, and so $\vec{f} = \vec{u}_+|_{\partial\Omega} - \psi \in \Psi^\lambda(\partial\Omega_+)$.

In a similar fashion, we can also show that

$$\vec{f} \in L^2_{1,\nu_+}(\partial\Omega) \text{ and } (\tfrac{1}{2}I + K_\lambda)\vec{f} \in \Psi^\lambda(\partial\Omega_-) \Longrightarrow \vec{f} \in \Psi^\lambda(\partial\Omega_-). \tag{5.151}$$

Here we only wish to remark that in place of (5.149) we write

$$\begin{aligned} \int_{\Omega_+} A_\lambda(\nabla\vec{u}_+, \nabla\vec{u}_+)\,dx &= \int_{\partial\Omega} \langle \psi, \partial^\lambda_\nu(\vec{u}_+, \pi_+)\rangle\,d\sigma \\ &= \int_{\partial\Omega} \langle \psi, \partial^\lambda_\nu(\vec{u}_-, \pi_-)\rangle\,d\sigma = \int_{\partial\Omega} \langle u_-, \partial^\lambda_\nu(\tilde{\psi}, 0)\rangle\,d\sigma = 0, \end{aligned} \tag{5.152}$$

where $\tilde{\psi} \in \Psi^\lambda(\Omega_-)$ is such that $\tilde{\psi}|_{\partial\Omega} = \psi := (\frac{1}{2}I + K_\lambda)\vec{f}$. The fact that there are no decay problems when using (4.7) in the next-to-last equality above is ensured by the fact that $\tilde{\psi}$ has, as any field in $\Psi^\lambda(\Omega_-)$, compact support. This finishes the proof of the claim made about the operators in (5.124).

Consider next (5.125). For this, the right-to-left inclusion has been already established in (5.98) (here (5.117) is also used), whereas the the opposite inclusion can be read off (5.148) and (5.151). Once (5.125) has been established, (5.126) follows from Lemma 11.41 in the Appendix, granted that

$$\pm\tfrac{1}{2}I + K_\lambda \text{ are Fredholm with index zero on } L^2(\partial\Omega) \text{ and } L^2_1(\partial\Omega). \tag{5.153}$$

However, this is proved in (5.166) and (5.168) below, independently of the current considerations. This finishes the proof of (5.126). As for (5.127), the right-to-left inclusion is a consequence of (5.79), while the left-to-right inclusion is implicit in the arguments just below (5.143) and (5.146).

Finally, to prove (5.128), consider first the case when $n \geq 3$. Then the right-to-left inclusion is contained in (5.78). To justify the remaining inclusion, assume that $\vec{f} \in L^2(\partial\Omega)$ is such that $S\vec{f} = 0$. Consider the functions $\vec{u}_\pm := \mathcal{S}\vec{f}$ in $\Omega_\pm$ and $\pi_\pm := \mathcal{Q}\vec{f}$ in $\Omega_\pm$. Then from (4.6),

$$\int_{\Omega_\pm} A_\lambda(\nabla\vec{u}_\pm, \nabla\vec{u}_\pm)\,dx = \pm\int_{\partial\Omega} \langle S\vec{f}, \partial^\lambda_\nu(\vec{u}_\pm, \pi_\pm)\rangle\,d\sigma = 0. \tag{5.154}$$

Then $\vec{u}_\pm \in \Psi^\lambda(\Omega_\pm)$, which implies that $\Delta\vec{u}_\pm = 0$ in $\Omega_\pm$, and so $\pi_\pm$ must be locally constant. Furthermore, we have

$$\vec{f} = \partial_\nu^\lambda(\vec{u}_-, \pi_-) - \partial_\nu^\lambda(\vec{u}_+, \pi_+) = \nu(\pi_+ - \pi_-) \in \nu\mathbb{R}_{\partial\Omega}, \tag{5.155}$$

which proves (5.128) when $n \geq 3$.

There remains to consider the case when $n = 2$, in which situation it may happen that there exist vector fields in $L^2(\partial\Omega)$ which do not belong to $\nu\mathbb{R}_{\partial\Omega}$, and yet are sent to zero by S. For example, if $\Omega = B(0, \sqrt{e})$ in $\mathbb{R}^2$, then $Se_j = 0$ for $j = 1, 2$; see, e.g. [62], p. 98. Nonetheless, any nonzero vector field $\vec{f} \in \mathcal{W}$ necessarily satisfies $\int_{\partial\Omega} \vec{f}\, d\sigma \neq 0$, otherwise the argument in the previous paragraph (in which we take into account that $\pi_+ = \mathcal{Q}\vec{f} = 0$ in Ω_+) places it in $\nu\mathbb{R}_{\partial\Omega_-}$, thus forcing $\vec{f} = 0$, from orthogonality considerations. This argument shows that the linear mapping $\mathcal{W} \ni \vec{\psi} \mapsto \int_{\partial\Omega} \vec{\psi}\, d\sigma \in \mathbb{R}^2$ is injective. Hence, $\dim \mathcal{W} \leq 2$, proving (5.130).

As for (5.128) when $n = 2$, the right-to-left inclusion is clear from (5.129) and (5.78). To prove the opposite inclusion, assume that $\vec{f} \in L^2(\partial\Omega)$ is such that $S\vec{f} = 0$ on $\partial\Omega$, and set $\vec{u} := \mathcal{S}\vec{f}$, $\pi := \mathcal{Q}\vec{f}$ in Ω_+. Then $\int_\Omega A_\lambda(\nabla\vec{u}, \nabla\vec{u})\, dx = \int_{\partial\Omega} \langle \partial_\nu^\lambda(\vec{u}, \pi), \vec{u}\rangle\, d\sigma = 0$, since $\vec{u}|_{\partial\Omega} = 0$. Consequently, $\vec{u} \in \Psi^\lambda(\Omega_+)$ hence, $\pi \in \mathbb{R}_{\Omega_+}$ by Lemma 5.13. This shows that for every connected component $\mathcal{O}_j$ of Ω_+, there exists a constant $c_j \in \mathbb{R}$ with the property that $\mathcal{Q}\vec{f}|_{\mathcal{O}_j} \equiv c_j$. If we now set

$$\vec{g} := \Big(\sum_{j=1}^{b_0} c_j \chi_{\partial\mathcal{O}_j}\Big)\nu \in \nu\mathbb{R}_{\partial\Omega_+} \hookrightarrow \mathrm{Ker}\,(S : L^2(\partial\Omega) \to L^2_1(\partial\Omega)), \tag{5.156}$$

then, by (5.82),

$$\mathcal{Q}\vec{g} = \sum_j c_j \chi_{\mathcal{O}_j} = \mathcal{Q}\vec{f} \;\text{ in }\; \Omega_+. \tag{5.157}$$

As a consequence, if $\vec{h} \in \nu\mathbb{R}_{\partial\Omega_-}$ denotes the projection of $\vec{f} - \vec{g}$ onto $\nu\mathbb{R}_{\partial\Omega_+}$, we may write $\vec{f} = (\vec{f} - \vec{g} - \vec{h}) + (\vec{g} + \vec{h})$, with $\vec{g} + \vec{h} \in \nu\mathbb{R}_{\partial\Omega_+} \oplus \nu\mathbb{R}_{\partial\Omega_-} = \mathbb{R}_{\partial\Omega}$ and $\vec{f} - \vec{g} - \vec{h} \in \mathcal{W}$, by (5.157), (5.82) and (5.78). We are therefore left with showing that $\mathcal{W} \cap \nu\mathbb{R}_{\partial\Omega} = 0$. Indeed, if $\varphi_\pm \in \mathbb{R}_{\partial\Omega_\pm}$ are such that $\mathcal{Q}(\nu\varphi_+ + \nu\varphi_-) = 0$ in Ω_+, then (5.82) shows that $\varphi_+ = 0$. Thus, if $\nu\varphi_+ + \nu\varphi_- \in \mathcal{W} \hookrightarrow L^2_{\nu_-}(\partial\Omega)$ to begin with, then necessarily $\varphi_- = 0$, and the desired conclusion follows. This last step finishes the proof of (5.128), and concludes the proof of the proposition. $\square$

We continue the discussion of the operators in question with the following results.

Theorem 5.17 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain. Then there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ with the property that for each $p \in (2-\varepsilon, 2+\varepsilon)$ the following statements are true. First, the operators*

$$\gamma I + K_\lambda,\ \gamma I + K^*_\lambda : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega), \tag{5.158}$$

$$\gamma I + K_\lambda : L^p_1(\partial\Omega) \longrightarrow L^p_1(\partial\Omega), \tag{5.159}$$

are invertible whenever $\lambda \in (-1, 1]$ and $\gamma \in \mathbb{R} \setminus [-\frac{1}{2}, \frac{1}{2}]$. Second, the operators

$$\pm\tfrac{1}{2} I + K^*_\lambda : L^p_{\Psi^\lambda_\mp}(\partial\Omega)/\nu\mathbb{R}_{\partial\Omega_\pm} \longrightarrow L^p_{\Psi^\lambda_\mp}(\partial\Omega)/\nu\mathbb{R}_{\partial\Omega_\pm}, \tag{5.160}$$

along with

$$\pm\tfrac{1}{2} I + K_\lambda : L^p_{1,\nu_\pm}(\partial\Omega)/\Psi^\lambda(\partial\Omega_\mp) \longrightarrow L^p_{1,\nu_\pm}(\partial\Omega)/\Psi^\lambda(\partial\Omega_\mp), \tag{5.161}$$

$$\pm\tfrac{1}{2} I + K_\lambda : L^p_{\nu_\pm}(\partial\Omega)/\Psi^\lambda(\partial\Omega_\mp) \longrightarrow L^p_{\nu_\pm}(\partial\Omega)/\Psi^\lambda(\partial\Omega_\mp) \tag{5.162}$$

are also invertible whenever $\lambda \in (-1, 1]$.

Proof. From known stability results, it suffices to deal with the case $p = 2$ only. In this scenario, pick a vector field $\vec{h} \in C^\infty_0(\mathbb{R}^n)$ with $\operatorname{supp} \vec{h} \subseteq D$ such that $\langle \vec{h}, \nu\rangle \geq \kappa$ a.e. on $\partial\Omega$, for some $\kappa = \kappa(\partial\Omega) > 0$. Fix $\vec{f} \in L^2(\partial\Omega)$ and consider $\vec{u}_\pm = \mathcal{S}\vec{f}, \pi_\pm = \mathcal{Q}\vec{f}$ in $\Omega_\pm$. Switching the roles of $\vec{u}_+$ and $\vec{u}_-$ in Corollary 5.6 and choosing $\mu = 0$ gives

$$\begin{aligned} \|\nabla \vec{u}_-\|_{L^2(\partial\Omega)} &\leq C\|\nabla_{tan}\vec{u}_-\|_{L^2(\partial\Omega)} + C\|\nabla\mathcal{S}\vec{f}\|_{L^2(\Omega_-\cap D)} + \|\mathcal{Q}\vec{f}\|_{L^2(\Omega_-\cap D)} \\ &= C\|\nabla_{tan}\vec{u}_+\|_{L^2(\partial\Omega)} + C\|\nabla\mathcal{S}\vec{f}\|_{L^2(\Omega_-\cap D)} + C\|\mathcal{Q}\vec{f}\|_{L^2(\Omega_-\cap D)} \end{aligned} \tag{5.163}$$

Combining (5.163) and Corollary 5.5 then gives

$$
\begin{aligned}
\|\vec{f}\|_{L^2(\partial\Omega)} &= \|\partial_\nu^\lambda(\vec{u}_-,\pi_-) - \partial_\nu^\lambda(\vec{u}_+,\pi_+)\|_{L^2(\partial\Omega)} \\
&\le C\|\nabla\vec{u}_+\|_{L^2(\partial\Omega)} + C\|\nabla\vec{u}_-\|_{L^2(\partial\Omega)} \\
&\le C\|\nabla\vec{u}_+\|_{L^2(\partial\Omega)} + C\|\nabla\mathcal{S}\vec{f}\|_{L^2(\Omega_-\cap D)} + C\|\mathcal{Q}\vec{f}\|_{L^2(\Omega_-\cap D)} \\
&\le C\|(-\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K_\lambda^*)\vec{f}\|_{L^2(\partial\Omega)} + C\|\nabla\mathcal{S}\vec{f}\|_{L^2(\Omega_+\cap D)} + C\|\mathcal{Q}\vec{f}\|_{L^2(\Omega_+\cap D)} \\
&\quad + C\|\nabla\mathcal{S}\vec{f}\|_{L^2(\Omega_-\cap D)} + C\|\mathcal{Q}\vec{f}\|_{L^2(\Omega_-\cap D)} \qquad (5.164)
\end{aligned}
$$

Since (5.164) holds for each $\mu \in [0,1)$ and the operators

$$\nabla\mathcal{S},\ \mathcal{Q} : L^2(\partial\Omega) \longrightarrow L^2(\Omega_\pm \cap D) \qquad (5.165)$$

are compact, the homotopic invariance of the index then proves

$$
\begin{aligned}
&\gamma I + K_\lambda^* : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \ \text{ is Fredholm with index zero} \\
&\text{whenever } \ 2-\varepsilon < p < 2+\varepsilon, \quad |\gamma| \ge \tfrac{1}{2}, \quad \text{and} \quad \lambda \in (-1,1],
\end{aligned} \qquad (5.166)
$$

first when $p = 2$ and then when $|p-2| < \varepsilon$ via perturbation results.

In a similar manner, if we consider $\vec{u}_\pm = \mathcal{D}_\lambda\vec{f}, \pi_\pm = \mathcal{P}_\lambda\vec{f}$ in $\Omega_\pm$ for $\vec{f} \in L_1^2(\partial\Omega)$, we can also show via Corollary 5.5 and Corollary 5.6 that given γ, λ as before, there exists $C = C(\partial\Omega,\gamma,\lambda) > 0$ such that

$$\|\vec{f}\|_{L_1^2(\partial\Omega)} \le C\|(\gamma I + K_\lambda)\vec{f}\|_{L_1^2(\partial\Omega)} + \text{residual terms}, \quad \forall\, \vec{f} \in L_1^2(\partial\Omega), \qquad (5.167)$$

where the residual terms yield compact operators from $L_1^2(\partial\Omega)$ into suitably chosen Banach spaces. Again using the homotopic invariance of the index and also perturbation results, it follows that

$$
\begin{aligned}
&\gamma I + K_\lambda : L_1^p(\partial\Omega) \longrightarrow L_1^p(\partial\Omega) \ \text{ is Fredholm with index zero} \\
&\text{whenever } \ 2-\varepsilon < p < 2+\varepsilon, \quad |\gamma| \ge \tfrac{1}{2}, \quad \text{and} \quad \lambda \in (-1,1].
\end{aligned} \qquad (5.168)
$$

Then the invertibility claims made in the statement of (5.158) and (5.159) follow from (5.166), (5.168), Proposition 5.16 and simple functional analysis. To also conclude that the operators in (5.160) and (5.161) are invertible, it is enough to establish that they are Fredholm operators of index zero.

First, let T_1 denote the operator $\frac{1}{2}I + K_\lambda^*$ acting from $L^p(\partial\Omega)$ to $L^p(\partial\Omega)$ and let T_2 denote the same operator acting instead from $L^p_{\Psi^\lambda_-}(\partial\Omega)/\nu\mathbb{R}_{\partial\Omega_+}$ to $L^p_{\Psi^\lambda_-}(\partial\Omega)/\nu\mathbb{R}_{\partial\Omega_+}$. Also, let

$$\iota : L^p_{\Psi^\lambda_-}(\partial\Omega) \longrightarrow L^p(\partial\Omega) \tag{5.169}$$

denote the natural inclusion operator, and let

$$\mathrm{pr} : L^p(\partial\Omega) \longrightarrow L^p_{\Psi^\lambda_-}(\partial\Omega) \tag{5.170}$$

be the projection operator given by

$$\mathrm{pr}\,\vec{f} := \vec{f} - \sum_i \Big(\int_{\partial\Omega} \langle \vec{f}, \psi_i \rangle \, d\sigma \Big) \psi_i \tag{5.171}$$

where the ψ_i's form an orthonormal basis of $\Psi^\lambda(\partial\Omega_-)$. Also, let

$$\widetilde{\mathrm{pr}} : L^p_{\Psi^\lambda_-}(\partial\Omega) \longrightarrow L^p_{\Psi^\lambda_-}(\partial\Omega)/\nu\mathbb{R}_{\partial\Omega_+} \tag{5.172}$$

denote the natural projection operator with regards to these spaces. Then using previous arguments, we can show that the following diagram commutes:

$$\begin{array}{ccccc} L^p_{\Psi^\lambda_-}(\partial\Omega) & \xrightarrow{\widetilde{\mathrm{pr}}} & L^p_{\Psi^\lambda_-}(\partial\Omega)/\nu\mathbb{R}_{\partial\Omega_+} & \xrightarrow{T_2} & L^p_{\Psi^\lambda_-}(\partial\Omega)/\nu\mathbb{R}_{\partial\Omega_+} \\ \big\downarrow \iota & & & & \big\uparrow \widetilde{\mathrm{pr}} \\ L^p(\partial\Omega) & \xrightarrow{T_1} & L^p(\partial\Omega) & \xrightarrow{\mathrm{pr}} & L^p_{\Psi^\lambda_-}(\partial\Omega) \end{array} \tag{5.173}$$

The estimate (5.164) shows that T_1 is a Fredholm operator of index zero. Since ι, pr, and $\widetilde{\mathrm{pr}}$ are also clearly Fredholm, it follows from (5.173) that T_2 must also be Fredholm.

Furthermore, since the Fredholm index of ι is the opposite of the Fredholm index of pr, it also follows from (5.173) that the index of T_2 must be zero. The rest of the cases in (5.160) and (5.161) follow similarly. Finally, that the operator in (5.162) is an isomorphism is a consequence of the corresponding statement for (5.160) and duality (cf. (5.118)). $\square$

5.4 Inverting the single layer on L^p for p near 2 on bounded Lipschitz domains

The goal of this first part of this section is to prove the following theorem.

Theorem 5.18 *For each bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ with $n \geq 3$ there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ with the property that*

$$S : L^p(\partial\Omega)\Big/\nu\,\mathbb{R}_{\partial\Omega} \longrightarrow L^p_{1,\nu}(\partial\Omega) \tag{5.174}$$

is an isomorphism for each $p \in (2-\varepsilon, 2+\varepsilon)$.

Proof. For starters, note that since $S(\nu\,\mathbb{R}_{\partial\Omega}) = 0$ and since for every bounded connected component D of $\Omega_\pm$,

$$\int_{\partial D} \langle S\vec{f}, \nu\rangle\, d\sigma = \int_D \operatorname{div} \mathcal{S}\vec{f}\, dx = 0, \qquad \forall\, \vec{f} \in L^p(\partial\Omega), \quad 1 < p < \infty, \tag{5.175}$$

the operator (5.174) is well-defined. Also, from known perturbation results, to prove the theorem, it suffices to consider the case when $p = 2$. To this end, recall the identity (4.142). From previous arguments, we know that $\pm\frac{1}{2}I + K_\lambda$ are Fredholm operators, and so from (4.142), the operator

$$S : L^2(\partial\Omega) \longrightarrow L^2_1(\partial\Omega) \tag{5.176}$$

must have a finite co-dimensional range, which further implies that its range is closed. Combining this with (5.128) confirms that the operator in (5.176) is Fredholm. To finish the proof, it is enough to establish that the Fredholm index of (5.176) is zero, since a similar argument as in the last paragraph of § 5.3 will then imply that (5.174) is also a Fredholm operator with index zero. Since, by (5.128), the operator (5.174) is injective, this would be enough to prove the theorem.

To show that (5.176) has index zero, consider the corresponding operator for the Lamé system

$$S_{\mu,\lambda} : L^2(\partial\Omega) \longrightarrow L^2_1(\partial\Omega), \tag{5.177}$$

defined in a similar manner as (5.176), except that the fundamental solution matrix $E = (E_{jk})_{j,k}$ is replaced by the fundamental solution $E^{\mu,\lambda} = (E^{\mu,\lambda}_{j,k})_{j,k}$ for the Lamé system of elastostatics, given by $L_{\mu,\lambda}\vec{u} = \mu\Delta\vec{u} + (\lambda+\mu)\nabla \mathrm{div}\,\vec{u}$, where

$$E^{\mu,\lambda}_{j,k}(x) := -\frac{1}{2\omega_{n-1}}\left(\frac{3\mu+\lambda}{\mu(2\mu+\lambda)}\frac{1}{n-2}\frac{\delta_{jk}}{|x|^{n-2}} + \frac{\mu+\lambda}{\mu(2\mu+\lambda)}\frac{x_j x_k}{|x|^n}\right), \quad x \in \mathbb{R}^n \setminus \{0\}. \tag{5.178}$$

Comparing (5.178) with (4.19) , it is clear that $E^{1,\lambda}_{j,k}(x) \to E_{j,k}(x)$ and $\nabla E^{1,\lambda}_{j,k}(x) \to \nabla E_{j,k}(x)$ as $\lambda \to \infty$, uniformly for x in compact sets, and so

$$\lim_{\lambda\to\infty} S_{1,\lambda} = S, \tag{5.179}$$

in the strong operator norm sense (as operators mapping $L^2(\partial\Omega)$ into $L^2_1(\partial\Omega)$). Since it is known that (5.177) is Fredholm with index zero when $\mu > 0$, $\lambda > -\frac{2\mu}{n}$ (cf., e.g., [33]), it follows from (5.179) that (5.176) has index zero as well. $\square$

Corollary 5.19 *For each bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^n$ with $n \geq 3$, there exists $\varepsilon > 0$ such that*

$$S : L^p_{-1}(\partial\Omega)/\nu\mathbb{R}_{\partial\Omega} \longrightarrow L^p_\nu(\partial\Omega) \tag{5.180}$$

is an isomorphism for each $p \in (2-\varepsilon, 2+\varepsilon)$.

Proof. Since (5.174) is a self-adjoint operator, Corollary 5.19 follows directly from Theorem 5.18 and duality. $\square$

In the second part of this section we treat the case $n = 2$. The main novelty is that, for two dimensional bounded Lipschitz domains, the structure of the null-space of the boundary single layer changes, compared to the higher dimensional case. Cf. (5.128)-(5.130).

Theorem 5.20 *Assume that $\Omega \subset \mathbb{R}^2$ is a bounded Lipschitz domain. Then there exists $\varepsilon > 0$ with the following properties. First, the space*

$$\{\vec{f} \in L^p_{\nu_-}(\partial\Omega) : S\vec{f} = 0 \text{ on } \partial\Omega, \text{ and } \mathcal{Q}\vec{f} = 0 \text{ in } \Omega_+\} \tag{5.181}$$

is independent of $p \in (2-\varepsilon, 2+\varepsilon)$. In particular, it agrees with the space defined in (5.129) and we shall keep denoting this by $\mathcal{W}$. Second, for any $p \in (2-\varepsilon, 2+\varepsilon)$, the operator

$$S : L^p(\partial\Omega)\Big/\nu\mathbb{R}_{\partial\Omega} \oplus \mathcal{W} \longrightarrow L^2_{1,\nu,\mathcal{W}}(\partial\Omega) := \Big\{\vec{f} \in L^2_{1,\nu}(\partial\Omega) : \textstyle\int_{\partial\Omega}\langle\vec{f}, \psi\rangle\, d\sigma = 0\ \forall\, \psi \in \mathcal{W}\Big\} \tag{5.182}$$

is an isomorphism.

Proof. Let $\varepsilon > 0$ be such that

$$S : L^p(\partial\Omega) \longrightarrow L^p_1(\partial\Omega) \tag{5.183}$$

is Fredholm with index zero whenever $p \in (2-\varepsilon, 2+\varepsilon)$. This can be arranged as before. Then, it follows from Lemma 11.41 that that the null-space of S in (5.183) is independent of $p \in (2-\varepsilon, 2+\varepsilon)$. As a consequence,

$$\mathrm{Ker}\,(S : L^p(\partial\Omega) \longrightarrow L^p_1(\partial\Omega)) = \nu\mathbb{R}_{\partial\Omega} \oplus \mathcal{W}, \qquad \forall\, p \in (2-\varepsilon, 2+\varepsilon), \tag{5.184}$$

where $\mathcal{W}$ is as in (5.129). Thus, if we temporarily denote the space (5.181) by $\mathcal{W}_p$, (5.184) implies $\mathcal{W}_p \subset \mathcal{W}_2$ for any $p \in (2-\varepsilon, 2+\varepsilon)$. On the other hand, the same type of argument which led to (5.128) gives the opposite inclusion so that, altogether, $\mathcal{W}_p = \mathcal{W}_2$ for each $p \in (2-\varepsilon, 2+\varepsilon)$. This proves the first claim in the statement of the theorem.

Going further, the fact that

$$\int_{\partial\Omega} \langle S\vec{f}, \psi \rangle \, d\sigma = \int_{\partial\Omega} \langle \vec{f}, S\psi \rangle \, d\sigma = 0, \qquad \forall\, \psi \in \mathcal{W}, \tag{5.185}$$

proves that the operator (5.182) is well-defined. Given that S in (5.183) is Fredholm with index zero if $p \in (2-\varepsilon, 2+\varepsilon)$ and that $\mathcal{W}$ is finite dimensional, it follows (similarly to what we have done in the proof of Theorem 5.18) that the operator (5.182) also has index zero. Since, as seen from (5.184), this is one-to-one, it ultimately follows that the operator in question is an isomorphism. □

We conclude this section with another important result involving the single layer in two dimensions.

Theorem 5.21 *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded Lipschitz domain, and define the operator*

$$\widetilde{S} : \Big(L^p(\partial\Omega)/\nu\mathbb{R}_{\partial\Omega} \Big) \oplus \mathbb{R}^2 \longrightarrow L^p_{1,\nu}(\partial\Omega) \oplus \mathbb{R}^2 \tag{5.186}$$

by setting

$$\widetilde{S}([\vec{g}], \vec{c}) := \left(S\vec{g} + \vec{c},\, ⨍_{\partial\Omega} \vec{g}\, d\sigma \right). \tag{5.187}$$

Then there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ such that $\widetilde{S}$ is an isomorphism for each $p \in (2-\varepsilon, 2+\varepsilon)$.

Proof. From stability results (cf. Theorem 11.44), it is enough to treat the case when $p = 2$. Consider the decomposition $\widetilde{S} = S_o + S_1$ where

$$S_o([\vec{g}], \vec{c}) := (S\vec{g}, 0) \qquad \text{and} \qquad S_1([\vec{g}], \vec{c}) := \left(\vec{c},\, ⨍_{\partial\Omega} \vec{g}\, d\sigma \right). \tag{5.188}$$

Note that S_1 is an operator of finite rank and is therefore compact. Then since $S_o \cong S$ is Fredholm with index zero when $p = 2$, it follows that $\widetilde{S} = S_o + S_1$ is also Fredholm with index zero when $p = 2$. Now to show that $\widetilde{S}$ is an isomorphism, it is enough to show that $\widetilde{S}$ is injective. Assume there exists $\vec{g} \in L^2(\partial\Omega)$ and $\vec{c} \in \mathbb{R}^2$ such that $\int_{\partial\Omega} \vec{g}\, d\sigma = 0$ and $S\vec{g} = -\vec{c}$. Set

$$\vec{u}_\pm = \mathcal{S}\vec{g} \text{ in } \Omega_\pm, \qquad \pi_\pm = \mathcal{Q}\vec{g} \text{ in } \Omega_\pm. \tag{5.189}$$

Using (5.111) and (5.112), for any $\lambda \in (-1, 1]$

$$\int_{\Omega_+} A_\lambda(\nabla\vec{u}_+, \nabla\vec{u}_+)\,dx + \int_{\Omega_-} A_\lambda(\nabla\vec{u}_-, \nabla\vec{u}_-)\,dx \tag{5.190}$$

$$= \int_{\partial\Omega} \Big\langle S\vec{g}, \left(-\tfrac{1}{2}I + K_\lambda^*\right)\vec{g} - \left(\tfrac{1}{2}I + K_\lambda^*\right)\vec{g}\Big\rangle\,d\sigma = -\int_{\partial\Omega} \langle S\vec{g}, \vec{g}\rangle\,d\sigma = \int_{\partial\Omega} \langle \vec{c}, \vec{g}\rangle\,d\sigma = 0.$$

Then from (5.93), we know that $\vec{u}_\pm \in \Psi^\lambda(\Omega_\pm)$ which further implies that $\pi_\pm \in \mathbb{R}_{\partial\Omega_\pm}$ and $\partial_\nu^\lambda(\vec{u}_\pm, \pi_\pm) \in \nu\mathbb{R}_{\partial\Omega_\pm}$. Then $\vec{g} = \partial_\nu^\lambda(\vec{u}_-, \pi_-) - \partial_\nu^\lambda(\vec{u}_+, \pi_+) \in \nu\mathbb{R}_{\partial\Omega}$ and so $\vec{c} = -S\vec{g} = 0$. This shows that $([\vec{g}], \vec{c}) = 0$ as desired, which establishes that $\widetilde{S}$ is an isomorphism when $p = 2$. $\square$

5.5 L^p-boundary value problems on bounded Lipschitz domains for p near 2

In this section we will focus on establishing well-posedness results for bounded Lipschitz domains. Our first result in this regard is the following.

Theorem 5.22 *Assume that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded Lipschitz domain and, as usual, set $\Omega_+ := \Omega$, $\Omega_- := \mathbb{R}^n \setminus \bar{\Omega}$. Also, fix $\mu \in (0, 1)$ and $\lambda \in (-1, 1]$. Then there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ such that for $p \in (2 - \varepsilon, 2 + \varepsilon)$, the transmission boundary value problem, concerned with finding two pairs of functions $(\vec{u}_\pm, \pi_\pm)$ in $\Omega_\pm$ satisfying*

$$\begin{cases} \Delta\vec{u}_\pm = \nabla\pi_\pm,\ \operatorname{div}\vec{u}_\pm = 0 \quad \text{in} \ \ \Omega_\pm, \\[2ex] M(\nabla\vec{u}_\pm),\ M(\pi_\pm) \in L^p(\partial\Omega), \\[2ex] \vec{u}_+\Big|_{\partial\Omega} - \vec{u}_-\Big|_{\partial\Omega} = \vec{g} \in L_1^p(\partial\Omega), \\[2ex] \partial_\nu^\lambda(\vec{u}_+, \pi_+) - \mu\,\partial_\nu^\lambda(\vec{u}_-, \pi_-) = \vec{f} \in L^p(\partial\Omega), \end{cases} \tag{5.191}$$

and the decay conditions

$$\vec{u}_-(x) = \begin{cases} O(|x|^{2-n}) \quad \textit{as} \;\; |x| \to \infty, \quad \textit{if} \;\; n \geq 3, \\ -\frac{1}{\mu} E(x)\Big(\int_{\partial\Omega} \vec{f}\, d\sigma\Big) + O(|x|^{-1}) \quad \textit{as} \;\; |x| \to \infty, \;\; \textit{if} \;\; n = 2, \end{cases} \tag{5.192}$$

$$\partial_j \vec{u}_-(x) = -\tfrac{1}{\mu}(\partial_j E)(x)\Big(\int_{\partial\Omega} \vec{f}\, d\sigma\Big) + O(|x|^{-n}) \;\; \textit{as} \;\; |x| \to \infty, \; 1 \leq j \leq n, \tag{5.193}$$

$$\pi_-(x) = \begin{cases} O(|x|^{1-n}) \quad \textit{as} \;\; |x| \to \infty, \quad \textit{if} \;\; n \geq 3, \\ \frac{1}{\mu}\Big\langle (\nabla E_\Delta)(x)\, , \, \int_{\partial\Omega} \vec{f}\, d\sigma \Big\rangle + O(|x|^{-2}) \quad \textit{as} \;\; |x| \to \infty, \;\; \textit{if} \;\; n = 2, \end{cases} \tag{5.194}$$

has a unique solution. In addition, there exists $C > 0$ such that

$$\|M(\nabla \vec{u}_\pm)\|_{L^p(\partial\Omega)} + \|M(\pi_\pm)\|_{L^p(\partial\Omega)} \leq C\|\vec{g}\|_{L^p_1(\partial\Omega)} + C\|\vec{f}\|_{L^p(\partial\Omega)}. \tag{5.195}$$

Furthermore, a similar result holds if (5.191) -(5.194) are replaced by

$$\begin{cases} \Delta \vec{u}_\pm = \nabla \pi_\pm, \;\; \operatorname{div} \vec{u}_\pm = 0 \quad \textit{in} \;\; \Omega_\pm, \\ M(\nabla \vec{u}_\pm), \, M(\pi_\pm) \in L^p(\partial\Omega), \\ \vec{u}_+\Big|_{\partial\Omega} - \mu \vec{u}_-\Big|_{\partial\Omega} = \vec{g} \in L^p_1(\partial\Omega), \\ \partial^\lambda_\nu(\vec{u}_+, \pi_+) - \, \partial^\lambda_\nu(\vec{u}_-, \pi_-) = \vec{f} \in L^p(\partial\Omega), \end{cases} \tag{5.196}$$

and the decay conditions

$$\vec{u}_-(x) = \begin{cases} O(|x|^{2-n}) \quad \textit{as} \;\; |x| \to \infty, \quad \textit{if} \;\; n \geq 3, \\ -E(x)\Big(\int_{\partial\Omega} \vec{f}\, d\sigma\Big) + O(|x|^{-1}) \quad \textit{as} \;\; |x| \to \infty, \;\; \textit{if} \;\; n = 2, \end{cases} \tag{5.197}$$

$$\partial_j \vec{u}_-(x) = -(\partial_j E)(x)\Big(\int_{\partial\Omega} \vec{f}\, d\sigma\Big) + O(|x|^{-n}) \;\; \textit{as} \;\; |x| \to \infty, \; 1 \leq j \leq n, \tag{5.198}$$

$$\pi_-(x) = \begin{cases} O(|x|^{1-n}) \quad \textit{as} \;\; |x| \to \infty, \quad \textit{if} \;\; n \geq 3, \\ \Big\langle (\nabla E_\Delta)(x)\, , \, \int_{\partial\Omega} \vec{f}\, d\sigma \Big\rangle + O(|x|^{-2}) \quad \textit{as} \;\; |x| \to \infty, \;\; \textit{if} \;\; n = 2. \end{cases} \tag{5.199}$$

Proof. Let $\varepsilon > 0$ be as in the statement of Theorem 5.17. Then for $p \in (2-\varepsilon, 2+\varepsilon)$, we know the operators

$$-\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K_\lambda^\star : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega), \quad \tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K_\lambda : L_1^p(\partial\Omega) \longrightarrow L_1^p(\partial\Omega) \tag{5.200}$$

are isomorphisms. Now, set

$$\vec{f}_1 := \vec{f} - \partial_\nu^\lambda(\mathcal{D}_\lambda^+\vec{g}, \mathcal{P}_\lambda^+\vec{g}) + \mu\, \partial_\nu^\lambda(\mathcal{D}_\lambda^-\vec{g}, \mathcal{P}_\lambda^-\vec{g}) \in L^p(\partial\Omega), \tag{5.201}$$

$$\vec{f}_2 := \Big(\tfrac{1}{2}\tfrac{\mu+1}{\mu-1}I + K_\lambda^*\Big)^{-1}\vec{f}_1 \in L^p(\partial\Omega), \tag{5.202}$$

where the superscripts $\pm$ indicate that the layer potentials in question are considered as mappings from functions defined on $\partial\Omega$ into functions defined in $\Omega_\pm$. Then

$$\vec{u}_\pm \;:=\; \tfrac{1}{1-\mu}\mathcal{S}^\pm\vec{f}_2 + \mathcal{D}_\lambda^\pm\vec{g}, \tag{5.203}$$

$$\pi_\pm \;:=\; \tfrac{1}{1-\mu}\mathcal{Q}^\pm\vec{f}_2 + \mathcal{P}_\lambda^\pm\vec{g}, \tag{5.204}$$

solve (9.31) and obey natural estimates, i.e.

$$\|M(\nabla\vec{u}_\pm)\|_{L^p(\partial\Omega)} + \|M(\pi_\pm)\|_{L^p(\partial\Omega)} \le C\Big(\|\vec{g}\|_{L_1^p(\partial\Omega)} + \|\vec{f}\|_{L^p(\partial\Omega)}\Big). \tag{5.205}$$

Let us now check the decay conditions (5.192)-(5.194). Clearly, (5.192) is a simple consequence of (5.203) if $n \ge 3$. Going further, we note that

$$\begin{aligned}\int_{\partial\Omega}\vec{f}_1\,d\sigma &= \int_{\partial\Omega}\vec{f}\,d\sigma - \int_{\partial\Omega}\partial_\nu^\lambda(\mathcal{D}_\lambda^+\vec{g}, \mathcal{P}_\lambda^+\vec{g})\,d\sigma + \mu\int_{\partial\Omega}\partial_\nu^\lambda(\mathcal{D}_\lambda^-\vec{g}, \mathcal{P}_\lambda^-\vec{g})\,d\sigma\\ &= \int_{\partial\Omega}\vec{f}\,d\sigma - (1-\mu)\int_{\partial\Omega}\partial_\nu^\lambda(\mathcal{D}_\lambda^+\vec{g}, \mathcal{P}_\lambda^+\vec{g})\,d\sigma\\ &= \int_{\partial\Omega}\vec{f}\,d\sigma,\end{aligned} \tag{5.206}$$

since

$$\partial_\nu^\lambda(\mathcal{D}_\lambda^+\vec{g}, \mathcal{P}_\lambda^+\vec{g}) = \partial_\nu^\lambda(\mathcal{D}_\lambda^-\vec{g}, \mathcal{P}_\lambda^-\vec{g}), \qquad \forall\,\vec{g} \in L_1^p(\partial\Omega). \tag{5.207}$$

On the other hand,

$$\begin{aligned}\int_{\partial\Omega} \vec{f}_1\,d\sigma &= \int_{\partial\Omega} \Big(\tfrac{1}{2}\tfrac{\mu+1}{\mu-1} I + K_\lambda^*\Big)\vec{f}_2\,d\sigma \\ &= \int_{\partial\Omega} \Big(-\tfrac{1}{2} I + K_\lambda^*\Big)\vec{f}_2\,d\sigma + \tfrac{\mu}{\mu-1}\int_{\partial\Omega} \vec{f}_2\,d\sigma \\ &= \tfrac{\mu}{\mu-1}\int_{\partial\Omega} \vec{f}_2\,d\sigma, \end{aligned} \tag{5.208}$$

so that

$$\int_{\partial\Omega} \vec{f}_2\,d\sigma = \tfrac{\mu-1}{\mu}\int_{\partial\Omega} \vec{f}_1\,d\sigma = \tfrac{\mu-1}{\mu}\int_{\partial\Omega} \vec{f}\,d\sigma. \tag{5.209}$$

Consequently, when $n = 2$,

$$\begin{aligned}\vec{u}_-(x) &= \tfrac{1}{1-\mu}\mathcal{S}^-\vec{f}_2(x) + \mathcal{D}_\lambda^-\vec{g}(x) \\ &= \tfrac{1}{1-\mu}\mathcal{S}^-\Big(\vec{f}_2 - {\textstyle\fint_{\partial\Omega}} \vec{f}_2\,d\sigma\Big)(x) + \tfrac{1}{1-\mu}E(x)\Big(\int_{\partial\Omega}\vec{f}_2\,d\sigma\Big) + O(|x|^{-1}) \\ &= -\tfrac{1}{\mu}E(x)\Big(\int_{\partial\Omega}\vec{f}\,d\sigma\Big) + O(|x|^{-1}) \quad \text{as } \ |x|\to\infty, \end{aligned} \tag{5.210}$$

in agreement with the case $n = 2$ of (5.192). Finally, that (5.203)-(5.204) satisfy the conditions (5.193)-(5.194) can be verified in a similar fashion.

Let us now consider the issue of uniqueness for (5.191)-(5.192). To this end, assume that $(\vec{u}_\pm, \pi_\pm)$ solves the homogeneous version of (5.191)-(5.194). The fact that $\vec{f} = 0$ implies that $\vec{u}_-$, π_- decay fast enough at infinity for the Green's formulas

$$\vec{u}_\pm = \pm\mathcal{D}_\lambda\Big(\vec{u}_\pm\Big|_{\partial\Omega}\Big) \mp \mathcal{S}(\partial_\nu^\lambda(\vec{u}_\pm, \pi_\pm)) \quad \text{in } \ \Omega_\pm, \tag{5.211}$$

to be valid. Based on (5.211), we may then write

$$\begin{aligned}\partial_\nu^\lambda(\vec{u}_\pm,\pi_\pm) + \Big(\pi_\pm\Big|_{\partial\Omega}\Big)\nu &= \pm\partial_\nu^\lambda\Big(\mathcal{D}_\lambda\Big(\vec{u}_\pm\Big|_{\partial\Omega}\Big), \mathcal{P}_\lambda\Big(\vec{u}_\pm\Big|_{\partial\Omega}\Big)\Big) \pm \Big(\mathcal{P}_\lambda\Big(\vec{u}_\pm\Big|_{\partial\Omega}\Big)\Big)\Big|_{\partial\Omega}\nu \\ &\quad \mp\partial_\nu^\lambda\Big(\mathcal{S}(\partial_\nu^\lambda(\vec{u}_\pm,\pi_\pm)), \mathcal{Q}(\partial_\nu^\lambda(\vec{u}_\pm,\pi_\pm))\Big) \\ &\quad \mp\Big(\mathcal{Q}(\partial_\nu^\lambda(\vec{u}_\pm,\pi_\pm))\Big)\Big|_{\partial\Omega}\nu, \end{aligned} \tag{5.212}$$

hence, invoking (4.121) and the jump-relations of hydrostatic layer potentials,

$$\begin{aligned}\partial_\nu^\lambda(\vec{u}_\pm,\pi_\pm) &= \pm\partial_\nu^\lambda\Big(\mathcal{D}_\lambda\Big(\vec{u}_\pm\Big|_{\partial\Omega}\Big),\mathcal{P}_\lambda\Big(\vec{u}_\pm\Big|_{\partial\Omega}\Big)\Big)\\ &\quad\mp\Big(\mp\tfrac{1}{2}I+K_\lambda^*\Big)\Big(\partial_\nu^\lambda(\vec{u}_\pm,\pi_\pm)\Big). \qquad (5.213)\end{aligned}$$

Adding the two versions of the identity (5.213) and keeping in mind that $\partial_\nu^\lambda(\vec{u}_+,\pi_+)=\mu\,\partial_\nu^\lambda(\vec{u}_-,\pi_-)$, $\vec{u}_+|_{\partial\Omega}=\vec{u}_-|_{\partial\Omega}$ and that (5.207) holds allows us to conclude that $(\frac{1}{2}\frac{\mu+1}{\mu-1}I+K_\lambda^*)(\partial_\nu^\lambda(\vec{u}_-,\pi_-))=0$. Since $\frac{\mu+1}{\mu-1}I+K_\lambda^*)$ is an invertible operator, $\partial_\nu^\lambda(\vec{u}_-,\pi_-)=0$, and further, $\partial_\nu^\lambda(\vec{u}_+,\pi_+)=0$. Moving to the boundary in each version of (5.211) then gives

$$(\tfrac{1}{2}I+K_\lambda)(\vec{u}_\pm|_{\partial\Omega})=\vec{u}_\pm|_{\partial\Omega}=-(-\tfrac{1}{2}I+K_\lambda)(\vec{u}_\pm|_{\partial\Omega}), \qquad (5.214)$$

from which it can be determined that $\vec{u}_\pm|_{\partial\Omega}=0$. Finally, it follows from returning to (5.211) again that $\vec{u}_\pm=0$ in $\Omega_\pm$. This forces $\pi_\pm$ to be locally constant, but since $\pi_+=\mu\,\pi_-$ on $\partial\Omega$ and π_- decays at infinity, we must have $\pi_\pm=0$ in $\Omega_\pm$ as well.

The result for (5.196)-(5.199) follows in a similar manner. More precisely, if

$$\begin{aligned}\vec{g}_1 &:= \vec{g}+(1-\mu)S\vec{f}\in L_1^p(\partial\Omega),\\ \vec{g}_2 &:= \Big(\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I+K_\lambda\Big)^{-1}\vec{g}_1\in L_1^p(\partial\Omega),\end{aligned} \qquad (5.215)$$

then

$$\vec{u}_\pm \;:=\; \tfrac{1}{1-\mu}\mathcal{D}_\lambda^\pm\vec{g}_2-\mathcal{S}^\pm\vec{f}\ \text{ in }\ \Omega_\pm, \qquad (5.216)$$

$$\pi_\pm \;:=\; \tfrac{1}{1-\mu}\mathcal{P}_\lambda^\pm\vec{g}_2-\mathcal{Q}^\pm\vec{f}\ \text{ in }\ \Omega_\pm, \qquad (5.217)$$

will satisfy (5.196)-(5.199) and also (5.195). As for uniqueness, it can be shown using (5.211) as above that solutions of the homogeneous version of (5.196)-(5.199) satisfy

$$\Big(-\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I+K_\lambda\Big)(\vec{u}_-|_{\partial\Omega})=0. \qquad (5.218)$$

It follows that $\vec{u}_-|_{\partial\Omega} = 0$ and therefore $\vec{u}_+|_{\partial\Omega} = 0$ as well. With this in mind, it can also be shown using (5.213) and the transmission conditions that $\partial_\nu^\lambda(\vec{u}_\pm, \pi_\pm) = 0$, and then uniqueness follows much as above. $\square$

Theorem 5.23 *Assume that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded Lipschitz domain. Then for $\lambda \in (-1, 1]$, there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ such that for $p \in (2 - \varepsilon, 2 + \varepsilon)$, the Neumann boundary value problem, concerned with finding functions $(\vec{u}, \pi)$ in Ω satisfying*

$$\begin{cases} \Delta\vec{u} = \nabla\pi, \ \operatorname{div}\vec{u} = 0 \quad \text{in} \ \ \Omega, \\ M(\nabla\vec{u}),\ M(\pi) \in L^p(\partial\Omega), \\ \partial_\nu^\lambda(\vec{u}, \pi) = \vec{f} \in L^p(\partial\Omega), \end{cases} \tag{5.219}$$

has a solution if and only if $\vec{f}$ satisfies $b_{n-1}(\Omega)$ linearly independent constraints. More specifically, (5.219) has a solution if and only if

$$\vec{f} \in \operatorname{Im}\Big(-\tfrac{1}{2}I + K_\lambda^* : L^p_{\Psi_+^\lambda}(\partial\Omega) \to L^p_{\Psi_+^\lambda}(\partial\Omega)\Big). \tag{5.220}$$

Whenever a solution of (5.219) exists, it is unique modulo adding to the velocity field functions from $\Psi^\lambda(\Omega)$. In addition, there exists $C > 0$ such that

$$\|M(\nabla\vec{u})\|_{L^p(\partial\Omega)} + \|M(\pi)\|_{L^p(\partial\Omega)} \leq C\|\vec{f}\|_{L^p(\partial\Omega)}, \tag{5.221}$$

for any solution $(\vec{u}, \pi)$ of (5.219).

Finally, a similar result holds for the exterior domain $\mathbb{R}^n \setminus \bar{\Omega}$ after including the decay conditions

$$\vec{u}(x) = \begin{cases} O(|x|^{2-n}) \quad \text{as} \ \ |x| \to \infty, \quad \text{if} \ \ n \geq 3, \\ E(x)\Big(\int_{\partial\Omega} \vec{f}\, d\sigma\Big) + O(|x|^{-1}) \quad \text{as} \ \ |x| \to \infty, \ \ \text{if} \ n = 2, \end{cases} \tag{5.222}$$

$$\partial_j\vec{u}(x) = (\partial_j E)(x)\Big(\int_{\partial\Omega} \vec{f}\, d\sigma\Big) + O(|x|^{-n}) \ \ \text{as} \ \ |x| \to \infty, \ 1 \leq j \leq n, \tag{5.223}$$

$$\pi(x) = \begin{cases} O(|x|^{1-n}) \quad \text{as} \ \ |x| \to \infty, \quad \text{if} \ \ n \geq 3, \\ \Big\langle(-\nabla E_\Delta)(x)\,,\, \int_{\partial\Omega} \vec{f}\, d\sigma\Big\rangle + O(|x|^{-2}) \quad \text{as} \ \ |x| \to \infty, \ \ \text{if} \ n = 2. \end{cases} \tag{5.224}$$

In particular, a solution to the exterior problem exists if and only if

$$\vec{f} \in \operatorname{Im}\Big(\tfrac{1}{2}I + K_\lambda^* : L^p_{\Psi_-^\lambda}(\partial\Omega) \to L^p_{\Psi_-^\lambda}(\partial\Omega)\Big), \tag{5.225}$$

and solutions are unique modulo adding to the velocity field functions from $\Psi^\lambda(\mathbb{R}^n \setminus \bar{\Omega})$.

Proof. Let $\varepsilon > 0$ be as in the statement of Theorem 5.17. Then for $p \in (2-\varepsilon, 2+\varepsilon)$, we know that the operator

$$-\tfrac{1}{2}I + K_\lambda^\star : L^p_{\Psi_+^\lambda}(\partial\Omega)/\nu\mathbb{R}_{\partial\Omega_-} \longrightarrow L^p_{\Psi_+^\lambda}(\partial\Omega)/\nu\mathbb{R}_{\partial\Omega_-} \tag{5.226}$$

is an isomorphism. Consider the claim that a solution for (5.219) exists if and only if (5.220) holds.

To justify the right-to-left implication, if (5.220) holds, say $\vec{f} = (-\frac{1}{2}I + K_\lambda^\star)\vec{g}$ for some $\vec{g} \in L^p_{\Psi_+^\lambda}(\partial\Omega)$, then

$$\vec{u} := \mathcal{S}\vec{g} \quad \text{and} \quad \pi := \mathcal{Q}\vec{g} \tag{5.227}$$

will satisfy (5.219) and (5.221).

In the opposite direction, assume that $\vec{f} \in L^p(\partial\Omega)$ is such that (5.219) has a solution $(\vec{u}, \pi)$. Then, if $\psi \in \Psi^\lambda(\partial\Omega_+)$, say $\psi = \tilde{\psi}|_{\partial\Omega}$ for some $\tilde{\psi} \in \Psi^\lambda(\Omega_+)$, we may write

$$\int_{\partial\Omega} \langle \psi, \vec{f}\rangle \, d\sigma = \int_{\partial\Omega} \langle \tilde{\psi}, \partial_\nu^\lambda(\vec{u}, \pi)\rangle \, d\sigma = \int_{\partial\Omega} \langle \partial_\nu^\lambda(\tilde{\psi}, 0), \vec{u}\rangle \, d\sigma = 0. \tag{5.228}$$

Hence, necessarily, $\vec{f} \in L^p_{\Psi_+^\lambda}(\partial\Omega)$.

Having established this, we now use the fact that (5.226) is an isomorphism in order to find $\vec{g} \in L^p_{\Psi_+^\lambda}(\partial\Omega)$ such that $(-\frac{1}{2}I + K_\lambda^\star)\vec{g} - \vec{f} = \nu\varphi$, for some $\varphi \in \mathbb{R}_{\partial\Omega_-}$. If we now set $w := \mathcal{S}\vec{g}$ and $\rho := \mathcal{Q}\vec{g}$ in Ω, then the pair $(w - u, \rho - \pi)$ solves the interior Neumann problem with datum $\nu\varphi$. We will now make a claim which implies that, necessarily, $\varphi = 0$. This, of course, entails $\vec{f} = (-\frac{1}{2}I + K_\lambda^\star)\vec{g}$, proving (5.220). The claim just alluded to above is the following:

$$\text{if } (\vec{u},\pi) \text{ solve (5.219) for } \vec{f} = \nu\varphi \text{ with } \varphi \in \mathbb{R}_{\partial\Omega_-}, \text{ then } \varphi = 0. \tag{5.229}$$

To justify this claim, write (4.120) and recall (5.78) to conclude that $\vec{u} = \mathcal{D}_\lambda(\vec{u}|_{\partial\Omega})$ in Ω. Going to the boundary then yields

$$\vec{u}\Big|_{\partial\Omega} \in \mathrm{Ker}\left(-\tfrac{1}{2}I + K_\lambda : L^p_{1,\nu_+}(\partial\Omega) \to L^p_{1,\nu_+}(\partial\Omega)\right) = \Psi^\lambda(\partial\Omega_+), \tag{5.230}$$

by (5.125). Utilizing this back into (4.120) and relying on (5.97) further gives $\vec{u} \in \Psi^\lambda(\Omega_+)$. Hence, $\nu\varphi = \partial^\lambda_\nu(\vec{u},\pi) \in \nu\mathbb{R}_{\partial\Omega_+}$ by (5.96) and, ultimately, $\varphi = 0$ given that the sum in (5.74) is direct. This concludes the proof of (5.229).

To establish uniqueness, if the functions $\vec{u}$ and π satisfy the homogeneous version of problem (5.219), then $\vec{u} = \mathcal{D}_\lambda(\vec{u}|_{\partial\Omega})$ in Ω, by (4.120). Going non-tangentially to the boundary then yields $(-\frac{1}{2}I + K_\lambda)(\vec{u}|_{\partial\Omega}) = 0$ on $\partial\Omega$ which shows that $\vec{u}|_{\partial\Omega} \in \mathrm{Ker}\,(-\frac{1}{2}I + K_\lambda : L^p_{1,\nu_+}(\partial\Omega) \to L^p_{1,\nu_+}(\partial\Omega)) = \Psi^\lambda(\partial\Omega_+)$, by (5.125), since $\vec{u}|_{\partial\Omega} \in L^p_{1,\nu_+}(\partial\Omega)$ to begin with. Hence, $\vec{u}|_{\partial\Omega} = \tilde{\psi}|_{\partial\Omega}$ for some function $\tilde{\psi} \in \Psi^\lambda(\Omega_+)$. It remains to invoke (4.120) once again in order to conclude that, by virtue of (5.97), $\vec{u} = \tilde{\psi}$ in Ω. This establishes the claim made about uniqueness for (5.219).

In the case of the exterior domain, a similar argument can be used to establish the existence of a solution. The key observation is that the decay conditions (5.222)-(5.224) are strong enough to guarantee that integral representation formulas analogous to (4.120)-(4.121) hold in $\mathbb{R}^n \setminus \bar{\Omega}$. More specifically, we have

$$\vec{u}(x) = -\mathcal{D}_\lambda\Big(\vec{u}\Big|_{\partial\Omega}\Big)(x) + \mathcal{S}\Big(\partial^\lambda_\nu(\vec{u},\pi)\Big)(x), \qquad x \in \mathbb{R}^n \setminus \bar{\Omega}, \tag{5.231}$$

$$\pi(x) = -\mathcal{P}_\lambda\Big(\vec{u}\Big|_{\partial\Omega}\Big)(x) + \mathcal{Q}\Big(\partial^\lambda_\nu(\vec{u},\pi)\Big)(x), \qquad x \in \mathbb{R}^n \setminus \bar{\Omega}. \tag{5.232}$$

These are proved starting with (4.120)-(4.121) written in $B_R \setminus \bar{\Omega}$, where B_R is a ball of radius R, large enough so that $\bar{\Omega} \subset B_R$, then passing to the limit as $R \to \infty$. The decay conditions (5.222)-(5.224) are then used to show that the contributions from ∂B_R tend to zero. With (5.231)-(5.232) in place, the proof of the uniqueness then proceeds as for the case of bounded domains. □

We can also state a similar result for the Regularity problem.

Theorem 5.24 *Assume that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded Lipschitz domain. Then there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ such that for $p \in (2-\varepsilon, 2+\varepsilon)$, the Regularity boundary value problem, concerned with finding functions $(\vec{u}, \pi)$ in Ω satisfying*

$$\begin{cases} \Delta\vec{u} = \nabla\pi, \ \ \operatorname{div}\vec{u} = 0 \quad \textit{in} \ \ \Omega, \\ M(\nabla\vec{u}), \, M(\pi) \in L^p(\partial\Omega), \\ \vec{u} = \vec{f} \in L^p_1(\partial\Omega), \end{cases} \tag{5.233}$$

has a solution if and only if

$$\vec{f} \in L^p_{1,\nu_+}(\partial\Omega). \tag{5.234}$$

In addition, the solution is unique modulo adding locally constant functions to the pressure, and there exists $C = C(\Omega, p) > 0$ such that

$$\|M(\nabla\vec{u})\|_{L^p(\partial\Omega)} + \|M(\pi)\|_{L^p(\partial\Omega)} \leq C\|\vec{f}\|_{L^p_1(\partial\Omega)}. \tag{5.235}$$

Furthermore, a similar result holds for the exterior domain $\mathbb{R}^n \setminus \bar{\Omega}$ after including the decay conditions

$$\vec{u}(x) = \begin{cases} O(|x|^{2-n}) \ \textit{as} \ |x| \to \infty, & \textit{if} \ \ n \geq 3, \\ E(x)\vec{A} + O(1) \ \textit{as} \ |x| \to \infty, & \textit{if} \ n = 2, \end{cases} \tag{5.236}$$

$$\partial_j\vec{u}(x) = \begin{cases} O(|x|^{1-n}) \ \textit{as} \ |x| \to \infty, & \textit{if} \ \ n \geq 3, \\ \partial_j E(x)\vec{A} + O(|x|^{-2}) \ \textit{as} \ |x| \to \infty, & \textit{if} \ n = 2, \end{cases} \tag{5.237}$$

$$\pi(x) = \begin{cases} O(|x|^{1-n}) \ \textit{as} \ |x| \to \infty, & \textit{if} \ \ n \geq 3, \\ \langle \nabla E_\Delta(x), \vec{A} \rangle + O(|x|^{-2}) \ \textit{as} \ |x| \to \infty, & \textit{if} \ n = 2, \end{cases} \tag{5.238}$$

where $\vec{A} \in \mathbb{R}^2$ is an arbitrary vector, specified a priori. In particular, a solution exists if and only if

$$\vec{f} \in L^p_{1,\nu_-}(\partial\Omega), \tag{5.239}$$

and solutions are unique modulo adding locally constant functions to the pressure.

Proof. Let $\varepsilon > 0$ be as in the statement of Theorem 5.17. Then for $p \in (2-\varepsilon, 2+\varepsilon)$, we know that for each $\lambda \in (-1, 1]$, the operator

$$\tfrac{1}{2}I + K_\lambda : L^p_{1,\nu_+}(\partial\Omega)/\Psi^\lambda_-(\partial\Omega) \longrightarrow L^p_{1,\nu_+}(\partial\Omega)/\Psi^\lambda_-(\partial\Omega) \quad \text{is an isomorphism.} \tag{5.240}$$

We now claim that, if $n \geq 3$,

$$\begin{aligned} &T : L^p_{1,\nu_+}(\partial\Omega) \oplus L^p(\partial\Omega) \longrightarrow L^p_{1,\nu_+}(\partial\Omega), \\ &T(\vec{g}_1, \vec{g}_2) := (\tfrac{1}{2}I + K_\lambda)\vec{g}_1 + S\vec{g}_2 \quad \text{ is onto.} \end{aligned} \tag{5.241}$$

To see that this is indeed the case, consider an arbitrary $\vec{f} \in L^p_{1,\nu_+}(\partial\Omega)$. It follows then from (5.240) that there exists $\vec{g}_1 \in L^p_{1,\nu_+}(\partial\Omega)$ with the property that $\vec{\psi} := \vec{f} - (\frac{1}{2}I + K_\lambda)\vec{g}_1 \in \Psi^\lambda_-(\partial\Omega)$. Using (5.117) and Theorem 5.18, we can then find $\vec{g}_2 \in L^p(\partial\Omega)$ with the property that $S\vec{g}_2 = \vec{\psi}$. Thus, $T(\vec{g}_1, \vec{g}_2) = \vec{f}$, proving the claim. In turn, (5.241) and (11.123) in the Appendix show that there exists $C = C(\Omega, p) > 0$ with the following property:

$$\begin{aligned} &\forall\, \vec{f} \in L^p_{1,\nu_+}(\partial\Omega)\ \ \exists\, (\vec{g}_1, \vec{g}_2) \in L^p_{1,\nu_+}(\partial\Omega) \oplus L^p(\partial\Omega) \quad \text{with} \\ &T(\vec{g}_1, \vec{g}_2) = \vec{f} \quad \text{and} \quad \|\vec{g}_1\|_{L^p_1(\partial\Omega)} + \|\vec{g}_2\|_{L^p(\partial\Omega)} \leq C\|\vec{f}\|_{L^p_1(\partial\Omega)}. \end{aligned} \tag{5.242}$$

Next, to show that (5.246) has a solution when $n \geq 3$ for every given $\vec{f} \in L^p_{1,\nu_+}(\partial\Omega)$, it suffices to observe that, if $(\vec{g}_1, \vec{g}_2) \in L^p_{1,\nu_+}(\partial\Omega) \oplus L^p(\partial\Omega)$ are as in the second line of (5.242), then

$$\vec{u} := \mathcal{D}_\lambda \vec{g}_1 + \mathcal{S}\vec{g}_2 \quad \text{and} \quad \pi := \mathcal{P}_\lambda \vec{g}_1 + \mathcal{Q}\vec{g}_2 \tag{5.243}$$

will satisfy (5.233) and (5.235). To establish uniqueness, again, when $n \geq 3$, assume that $\vec{u}$ and π satisfy the homogeneous version of (5.233). Then (4.120) implies $S(\partial^\lambda_\nu(\vec{u}, \pi)) = 0$ on

$\partial\Omega$. Hence, $\partial_\nu^\lambda(\vec{u},\pi) \in \nu\mathbb{R}_{\partial\Omega}$, by (5.128). Utilizing this back in (4.120) and invoking (5.77), we finally arrive at the conclusion that $\vec{u} = 0$ in Ω.

Turning our attention to the case when $n = 2$, consider in place of (5.241) the following claim:

$$\begin{gathered}\tilde{T} : L^p_{1,\nu_+}(\partial\Omega) \oplus L^p(\partial\Omega) \oplus \mathbb{R}^2 \longrightarrow L^p_{1,\nu_+}(\partial\Omega), \\ \tilde{T}(\vec{g}_1, \vec{g}_2, \vec{c}) := (\tfrac{1}{2}I + K_\lambda)\vec{g}_1 + S\vec{g}_2 + \vec{c} \quad \text{ is onto.}\end{gathered} \tag{5.244}$$

The first step in justifying this claim is as before. Namely, given $\vec{f} \in L^p_{1,\nu_+}(\partial\Omega)$, we can find some $\vec{g}_1 \in L^p_{1,\nu_+}(\partial\Omega)$ for which $\vec{\psi}_o := \vec{f} - (\frac{1}{2}I + K_\lambda)\vec{g}_1 \in \Psi^\lambda_-(\partial\Omega)$.

Since $\Psi^\lambda_-(\partial\Omega) \hookrightarrow L^p_{1,\nu}(\partial\Omega)$, it follows from Theorem 5.21 that there exists $\vec{g}_2 \in L^p(\partial\Omega)$ and $\vec{c} \in \mathbb{R}^2$ such that $S\vec{g}_2 + \vec{c} = \vec{\psi}_o$, and so the operator $\tilde{T}$ in (5.244) is onto, as claimed. With this in hand, the proof of the existence of a solution for (5.233), which satisfies natural estimates, proceeds as in the case $n \geq 3$, treated before.

To prove uniqueness for (5.233) when $n = 2$, we note that the same argument as in the case $n \geq 3$ shows that, if $\vec{u}$ and π satisfy the homogeneous version of (5.233), then

$$\partial_\nu^\lambda(\vec{u},\pi) = \nu\varphi + \psi, \quad \text{for some } \ \varphi \in \mathbb{R}_{\partial\Omega} \ \text{ and } \ \psi \in \mathcal{W}. \tag{5.245}$$

Plugging this back in (4.120) and keeping in mind (5.77) and (5.184), we may conclude that $\vec{u} = -\mathcal{S}\psi$ and $\pi = \mathcal{Q}(\nu\varphi)$ in Ω. In turn, this allows justifying the integration by parts formula $\int_\Omega A_\lambda(\nabla\vec{u}, \nabla\vec{u})\,dx = \int_{\partial\Omega}\langle\partial_\nu^\lambda(\vec{u},\pi), \vec{u}\rangle\,d\sigma$. Since $\vec{u}|_{\partial\Omega} = 0$, we finally conclude that $\vec{u} = 0$ in Ω, by invoking (5.93).

The exterior problem can be solved in much the same way. In this case, the decay conditions (5.236)-(5.238) with $\vec{A} = 0$ are crucial for justifying (5.231)-(5.232) for solutions of the homogeneous problem. Granted these identities, we once again arrive at (5.245), after which the solution proceeds much as before. $\square$

We conclude this section with a similar result for the Dirichlet problem.

Theorem 5.25 *Assume that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded Lipschitz domain. Then there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ such that for $p \in (2-\varepsilon, 2+\varepsilon)$, the Dirichlet boundary value problem, concerned with finding functions $(\vec{u}, \pi)$ in Ω satisfying*

$$
\begin{cases}
\Delta\vec{u} = \nabla\pi, \ \ \operatorname{div}\vec{u} = 0 \quad in \ \ \Omega, \\
M(\vec{u}) \in L^p(\partial\Omega), \\
\vec{u}\Big|_{\partial\Omega} = \vec{f} \in L^p_{\nu_+}(\partial\Omega),
\end{cases} \tag{5.246}
$$

has a solution which is unique modulo adding locally constant functions to the pressure. In addition, there exists $C > 0$ such that

$$
\|M(\vec{u})\|_{L^p(\partial\Omega)} \leq C\|\vec{f}\|_{L^p(\partial\Omega)}. \tag{5.247}
$$

Furthermore, a similar result holds for the exterior domain $\mathbb{R}^n \setminus \bar{\Omega}$ after including the decay conditions

$$
\vec{u}(x) = \begin{cases}
O(|x|^{2-n}) \ \ as \ |x| \to \infty, \quad \ if \ \ n \geq 3, \\
E(x)\vec{A} + O(1) \ \ as \ |x| \to \infty, \ \ if \ n = 2,
\end{cases} \tag{5.248}
$$

$$
\partial_j\vec{u}(x) = \begin{cases}
O(|x|^{1-n}) \ \ as \ |x| \to \infty, \quad \ if \ \ n \geq 3, \\
\partial_j E(x)\vec{A} + O(|x|^{-2}) \ \ as \ |x| \to \infty, \ \ if \ n = 2,
\end{cases} \tag{5.249}
$$

$$
\pi(x) = \begin{cases}
O(|x|^{1-n}) \ \ as \ |x| \to \infty, \quad \ if \ \ n \geq 3, \\
\langle \nabla E_\Delta(x), \vec{A}\rangle + O(|x|^{-2}) \ \ as \ |x| \to \infty, \ \ if \ n = 2,
\end{cases} \tag{5.250}
$$

where $\vec{A} \in \mathbb{R}^2$ is an arbitrary vector, specified a priori. In particular, a solution to the exterior problem exists if $\vec{f} \in L^p_{\nu_-}(\partial\Omega)$ and the solution is unique modulo adding locally constant functions to the pressure.

Proof. Let $\varepsilon > 0$ be as in the statement of Theorem 5.17, and fix $p \in (2-\varepsilon, 2+\varepsilon)$. Let us now assume that $n \geq 3$. Using (5.162) and (5.126), it can be checked (much as in the proof of Theorem 5.24), that

$$
\begin{aligned}
&T : L^p_{\nu_+}(\partial\Omega) \oplus L^p(\partial\Omega) \longrightarrow L^p_{\nu_+}(\partial\Omega), \\
&T(\vec{g}_1, \vec{g}_2) := (\tfrac{1}{2}I + K_\lambda)\vec{g}_1 + S\vec{g}_2 \quad \text{ is onto,}
\end{aligned} \tag{5.251}
$$

and

$$\begin{aligned}&\forall\, \vec{f} \in L^p_{\nu_+}(\partial\Omega)\ \exists\, (\vec{g}_1, \vec{g}_2) \in L^p_{\nu_+}(\partial\Omega) \oplus L^p(\partial\Omega) \quad \text{with} \\ &T(\vec{g}_1, \vec{g}_2) = \vec{f} \quad \text{and} \quad \|\vec{g}_1\|_{L^p(\partial\Omega)} + \|\vec{g}_2\|_{L^p(\partial\Omega)} \leq C\|\vec{f}\|_{L^p(\partial\Omega)}.\end{aligned} \tag{5.252}$$

Now, given an arbitrary $\vec{f} \in L^p_{\nu_+}(\partial\Omega)$, let $(\vec{g}_1, \vec{g}_2) \in L^p_{\nu_+}(\partial\Omega) \oplus L^p(\partial\Omega)$ be as in the second line of (5.252). Then

$$\vec{u} := \mathcal{D}_\lambda \vec{g}_1 + \mathcal{S}\vec{g}_2 \qquad \text{and } \pi := \mathcal{P}_\lambda \vec{g}_1 + \mathcal{Q}\vec{g}_2 \tag{5.253}$$

will satisfy (5.246) and (5.247).

To establish uniqueness, assume $\vec{u}$ and π satisfy the homogeneous version of (5.246). With $x_o \in \Omega$ fixed, let Ω_α be a sequence of sub-domains of Ω containing x_o that converge to Ω in the sense described in Lemma 11.54 in the Appendix. Define $\vec{E}_j(x) := \{E_{jk}(x)\}_k$ where E_{jk} is as in (4.20), and let q_j denote the jth component of $\vec{q}$ as defined in (4.21). Then for each $1 \leq j \leq n$ and each Ω_α, from Theorem 5.24, there exists $\vec{v}$ and q' such that

$$\begin{cases} \Delta\vec{v} = \nabla q', \ \ \operatorname{div}\vec{v} = 0 \quad \text{in} \quad \Omega, \\ M(\nabla\vec{v}), M(q') \in L^{p'}(\partial\Omega), \\ \vec{v} = \vec{E}_j(x_o - \cdot)|_{\partial\Omega_\alpha}. \end{cases} \tag{5.254}$$

Then for each $1 \leq j \leq n$ and each Ω_α, let

$$\vec{G}^\alpha_j := \vec{E}_j - \vec{v}, \qquad g^\alpha_j := q_j - q' \quad \text{in } \Omega_\alpha. \tag{5.255}$$

Then $\vec{G}^\alpha_j$ and g^α_j will satisfy

$$\operatorname{div}\vec{G}^\alpha_j = 0 \text{ in } \Omega_\alpha, \qquad \vec{G}^\alpha_j\Big|_{\partial\Omega_\alpha} = 0, \tag{5.256}$$

and

$$\int_{\Omega_\alpha} \langle \Delta \vec{G}_j^\alpha - \nabla g_j^\alpha, \vec{u} \rangle \, dx = u_j(x_o). \tag{5.257}$$

We now make the important claim that there exists a constant $C > 0$ independent of α such that

$$\|M(\nabla \vec{G}_j^\alpha)\|_{L^{p'}(\partial\Omega_\alpha)} + \|M(g_j^\alpha)\|_{L^{p'}(\partial\Omega_\alpha)} \leq C\|\vec{E}_j\|_{L_1^{p'}(\partial\Omega)}. \tag{5.258}$$

This is a consequence of the specific way in which the solution of the Regularity problem has been constructed in the proof of Theorem 5.24, Lemma 11.32 in the Appendix, in which we take T_α to be the operator (5.241) constructed for $\partial\Omega_\alpha$ in place of $\partial\Omega$, and the fact that the T_α's, after being appropriately identified with operators acting on functions defined on $\partial\Omega$, converge to T in the operator norm. See (11.207) and Lemma 11.54 in the Appendix for a proof of this latter claim.

Combining (5.257) with (4.7) and (5.256) then gives

$$u_j(x_o) = \int_{\partial\Omega_\alpha} \langle \partial_\nu^\lambda(\vec{G}_j^\alpha, g_j^\alpha), \vec{u} \rangle \, d\sigma. \tag{5.259}$$

Then since $M(\vec{u}) \in L^p(\partial\Omega)$ and $\vec{u}|_{\partial\Omega} = 0$, we can show via (5.259), (5.258), and the Lebesgue Dominated Convergence Theorem that $u_j(x_o) = 0$ (for this step, Lemma 11.54 is once again used to first replace the integral on $\partial\Omega_\alpha$ with one on $\partial\Omega$; cf. (11.192)-(11.194)). Since x_o was an arbitrary point in Ω, it follows that $\vec{u} = 0$ in Ω, as desired.

When $n = 2$, the same line of reasoning applies provided that, in place of (5.251), this time we use

$$\begin{aligned} &\tilde{T} : L^p_{\nu+}(\partial\Omega) \oplus L^p(\partial\Omega) \oplus \mathbb{R}^2 \longrightarrow L^p_{\nu+}(\partial\Omega), \\ &\tilde{T}(\vec{g}_1, \vec{g}_2, \vec{c}) := (\tfrac{1}{2}I + K_\lambda)\vec{g}_1 + S\vec{g}_2 + \vec{c}. \end{aligned} \tag{5.260}$$

The existence of a solution to the exterior Dirichlet problem can be established in much the same way. To prove uniqueness, assume $\vec{u}$ and π satisfy the homogeneous version of

(5.246) in the exterior domain $\mathbb{R}^n \setminus \bar{\Omega}$ and also satisfy (5.248)-(5.250). Fix $R > 0$ large enough that $\bar{\Omega} \subseteq B_R$, where $B_R := \{x \in \mathbb{R}^n : |x| < R\}$. Let D be the bounded Lipschitz domain given by $D := B_R \setminus \bar{\Omega}$. Since $\vec{u}$ and π satisfy the Stokes system in the exterior of Ω, it follows that $\vec{u}|_{\partial B_R} \in L_1^p(\partial B_R)$, and furthermore since $\vec{u}|_{\partial\Omega} = 0$, we can conclude that $\vec{u}|_{\partial D} \in L^p_{1,\nu_+}(\partial D)$. Theorem 5.24 applied for the domain D then guarantees that there exists a solution to (5.233) with data $\vec{f} = \vec{u}|_{\partial D}$. Due to the uniqueness portion of Theorem 5.25, the only possible solution is $\vec{u}$ and π, and therefore

$$M_D(\nabla\vec{u}),\, M_D(\pi) \in L^p(\partial D), \tag{5.261}$$

where M_D denotes the non-tangential maximal function associated with the domain D. This implies that

$$M(\nabla\vec{u}),\, M(\pi) \in L^p(\partial\Omega), \tag{5.262}$$

and then the uniqueness portion of Theorem 5.24 applied to the exterior domain forces $\vec{u} \equiv 0$, as desired. □

6 Local L^2 estimates

For the duration of this chapter we assume that Ω is a graph Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, and set $\Omega_+ := \Omega$, $\Omega_- := \mathbb{R}^n \setminus \bar{\Omega}$. Here, we will prove estimates of a local nature which will be useful throughout. For some fixed $x_o \in \partial\Omega$, let

$$S_R := S_R(x_o) = B_R(x_o) \cap \partial\Omega. \tag{6.1}$$

Also, define

$$D_R := D_R(x_o) = \{x + te_n : x \in S_R, |t| < \kappa R\}, \tag{6.2}$$

where $\kappa = \kappa(\partial\Omega) > 0$ is a fixed constant, and let

$$D_R^+ := D_R \cap \Omega_+ \quad \text{and} \quad D_R^- := D_R \cap \Omega_-. \tag{6.3}$$

If $S_R := S_R(x_o)$, for each $c > 0$ we also set $S_{cR} := S_{cR}(x_o)$, with a similar convention for D_{cR}.

6.1 Pressure, Caccioppoli, and local boundary estimates

For the duration of this section, assume $(\vec{u}_\pm, \pi_\pm)$ satisfy

$$\begin{cases} \Delta \vec{u}_\pm = \nabla \pi_\pm \quad \text{in} \;\; \Omega_\pm, \\ \operatorname{div} \vec{u}_\pm = 0 \quad \text{in} \;\; \Omega_\pm, \\ M(\nabla \vec{u}_\pm),\, M(\pi_\pm) \in L^2(\partial\Omega). \end{cases} \tag{6.4}$$

Our first local result is the following estimate for the pressure.

Lemma 6.1 *For any $q \geq 1$, there exists $C > 0$ such that*

$$\left(\fint_{D_R} |\pi_\pm|^2 \, dx \right)^{\frac{1}{2}} \leq C \left(\fint_{D_R} |\nabla \vec{u}_\pm|^2 \, dx \right)^{\frac{1}{2}} + \frac{C}{R} \left(\fint_{S_R} M(\vec{u}_\pm)^q \, d\sigma \right)^{\frac{1}{q}}. \tag{6.5}$$

Proof. Parametrize $D_R^\pm$ by $S_R \times (0, \kappa R) \ni (y,t) \mapsto y \pm t e_n \in D_R^\pm$ and fix two balls $B^\pm \subset D_R^\pm$ of radii comparable to R and such that $\operatorname{dist}(B^\pm, \partial D_R^\pm) \approx R$. For each $y \in S_R$ and $t \in (0, \kappa R)$ with $y \pm t e_n \in B^\pm$, using the fact that the pressure decays at infinity, the Fundamental Theorem of Calculus and interior estimates, we may write

$$\begin{aligned} |\pi_\pm(y \pm t e_n)| &\leq \int_t^\infty |(\nabla \pi_\pm)(y \pm s e_n)| \, ds \leq \int_{c_1 R}^\infty |(\Delta \vec{u}_\pm)(y \pm s e_n)| \, ds \\ &\leq \int_{c_1 R}^\infty \Big(\frac{C}{s^2} \fint_{B(y \pm s e_n, c_2 s)} |\vec{u}_\pm(z)| \, dz \Big) \, ds \\ &\leq C R^{-1} M(\vec{u}_\pm)(y). \end{aligned} \tag{6.6}$$

Hence,

$$\fint_{B^\pm} |\pi_\pm| \, dx \leq \frac{C}{R} \fint_{S_R} M(\nabla \vec{u}_\pm) \, d\sigma \leq \frac{C}{R} \left(\fint_{S_R} M(\vec{u}_\pm)^q \, d\sigma \right)^{\frac{1}{q}}. \tag{6.7}$$

According to the work of Bogovskiĭ [6], it is possible to construct two vector fields $\vec{w}_\pm$ in $D_R^\pm$ with the following properties:

$$\begin{aligned}
&(i) \quad \operatorname{div} \vec{w}_\pm = \pi_\pm - \tfrac{1}{|B^\pm|}\Big(\int_{D_R^\pm} \pi_\pm\Big)\chi_{B^\pm} \ \text{ in } \ D_R^\pm,\\
&(ii) \quad \vec{w}_\pm\Big|_{\partial D_R^\pm} \equiv 0, \\
&(iii) \quad \|\nabla \vec{w}_\pm\|_{L^2(D_R^\pm)} \le C\|\pi_\pm\|_{L^2(D_R^\pm)}.
\end{aligned} \tag{6.8}$$

Then integrating by parts, we have

$$\int\limits_{D_R^\pm} \pi_\pm(\operatorname{div} \vec{w}_\pm)\,dx = \int\limits_{D_R^\pm} A_\lambda(\nabla\vec{u}_\pm, \nabla\vec{w}_\pm)\,dx \mp \int\limits_{\partial D_R^\pm} \langle \partial_\nu^\lambda(\vec{u}_\pm, \pi_\pm), \vec{w}_\pm\rangle\,d\sigma, \tag{6.9}$$

and so using (6.8) and (6.7),

$$\begin{aligned}
\int\limits_{D_R^\pm} |\pi_\pm|^2\,dx \quad &= \quad \int\limits_{D_R^\pm} A_\lambda(\nabla\vec{u}_\pm, \nabla\vec{w}_\pm)\,dx + \left(\int\limits_{D_R^\pm} \pi_\pm\,dx\right)\left(\fint\limits_{B^\pm} \pi_\pm\,dx\right)\\
&\le \quad C\int\limits_{D_R^\pm} |\nabla\vec{u}_\pm||\nabla\vec{w}_\pm|\,dx + CR^{\frac{n}{2}}\left(\int\limits_{D_R^\pm}|\pi_\pm|^2\,dx\right)^{\frac12}\left(\fint\limits_{B^\pm}|\pi_\pm|\,dx\right)\\
&\le \quad C\left(\int\limits_{D_R^\pm}|\nabla\vec{u}_\pm|^2\,dx\right)^{\frac12}\left(\int\limits_{D_R^\pm}|\nabla\vec{w}_\pm|^2\,dx\right)^{\frac12}\\
&\qquad +CR^{\frac{n}{2}}\left(\int\limits_{D_R^\pm}|\pi_\pm|^2\,dx\right)^{\frac12} R^{-1}\left(\fint\limits_{S_R} M(\vec{u}_\pm)^q\,d\sigma\right)^{\frac1q}\\
&\le \quad C\left(\int\limits_{D_R^\pm}|\nabla\vec{u}_\pm|^2\,dx\right)^{\frac12}\left(\int\limits_{D_R^\pm}|\pi_\pm|^2\,dx\right)^{\frac12}\\
&\qquad +CR^{\frac{n}{2}-1}\left(\int\limits_{D_R^\pm}|\pi_\pm|^2\,dx\right)^{\frac12}\left(\fint\limits_{S_R} M(\vec{u}_\pm)^q\,d\sigma\right)^{\frac1q},
\end{aligned} \tag{6.10}$$

which is enough to prove the lemma. $\square$

Our next local result is the following Caccioppoli type estimate.

Lemma 6.2 *Let $\mu \in [0,1)$, $q \geq 1$, and $1 \leq s < t \leq 2$. Then there exists $C > 0$ such that*

$$\begin{aligned}
\int_{D^+_{sR}} |\nabla \vec{u}_+|^2 \, dx \quad &+ \quad \mu \int_{D^-_{sR}} |\nabla \vec{u}_-|^2 \, dx \\
&\leq \quad \frac{C}{R^2(t-s)} \left[\int_{D^+_{tR}} |\vec{u}_+|^2 \, dx + \mu \int_{D^-_{tR}} |\vec{u}_-|^2 \, dx \right] \\
&\quad + C R^{n-2} \left[\left(\fint_{S_{tR}} M(\vec{u}_+)^q \, d\sigma \right)^{\frac{2}{q}} + \mu \left(\fint_{S_{tR}} M(\vec{u}_-)^q \, d\sigma \right)^{\frac{2}{q}} \right] \\
&\quad + C \int_{S_{tR}} \left| \langle \partial^\lambda_\nu(\vec{u}_+, \pi_+), \vec{u}_+ \rangle - \mu \, \langle \partial^\lambda_\nu(\vec{u}_-, \pi_-), \vec{u}_- \rangle \right| d\sigma. \qquad (6.11)
\end{aligned}$$

Proof. Let $\eta \in C^\infty_0(\mathbb{R}^n)$ be such that $\eta \geq 0$ and $\operatorname{supp} \eta \subseteq D_{2R}$. Since $\Delta \vec{u}_\pm = \nabla \pi_\pm$ and $\operatorname{div} \vec{u}_\pm = 0$ in $\Omega_\pm$, using the integration by parts formula (4.6), we have that

$$\int_{D^\pm_{2R}} A_\lambda(\nabla \vec{u}_\pm, \nabla(\eta^2 \vec{u}_\pm)) \, dx = \pm \int_{S_{2R}} \langle \partial^\lambda_\nu(\vec{u}_\pm, \pi_\pm), \eta^2 \vec{u}_\pm \rangle \, d\sigma + \int_{D^\pm_{2R}} \pi_\pm \operatorname{div}(\eta^2 \vec{u}_\pm) \, dx. \qquad (6.12)$$

Multiplying the minus version of (6.12) by μ and adding it the plus version gives

$$\begin{aligned}
\int_{D^+_{2R}} A_\lambda(\nabla \vec{u}_+, \nabla(\eta^2 \vec{u}_+)) \, dx \quad &+ \quad \mu \int_{D^-_{2R}} A_\lambda(\nabla \vec{u}_-, \nabla(\eta^2 \vec{u}_-)) \, dx \\
&= \int_{D^+_{2R}} \pi_+ \operatorname{div}(\eta^2 \vec{u}_+) \, dx + \mu \int_{D^-_{2R}} \pi_- \operatorname{div}(\eta^2 \vec{u}_-) \, dx \qquad (6.13) \\
&\quad + \int_{S_{2R}} \eta^2 \left(\langle \partial^\lambda_\nu(\vec{u}_+, \pi_+), \vec{u}_+ \rangle - \mu \, \langle \partial^\lambda_\nu(\vec{u}_-, \pi_-), \vec{u}_- \rangle \right) d\sigma.
\end{aligned}$$

Expanding the terms $\nabla(\eta^2 \vec{u}_\pm)$ and $\operatorname{div}(\eta^2 \vec{u}_\pm)$ in (6.13) and using Cauchy's inequality with epsilon leads to the following estimate,

$$\int_{D_{2R}^+} \eta^2 A_\lambda(\nabla \vec{u}_+, \nabla \vec{u}_+)\, dx + \mu \int_{D_{2R}^-} \eta^2 A_\lambda(\nabla \vec{u}_-, \nabla \vec{u}_-)\, dx$$

$$\leq C_\varepsilon \left[\int_{D_{2R}^+} |\nabla \eta|^2 |\vec{u}_+|^2\, dx + \mu \int_{D_{2R}^-} |\nabla \eta|^2 |\vec{u}_-|^2\, dx \right]$$

$$+\varepsilon \left[\int_{D_{2R}^+} \eta^2 \left(|\nabla \vec{u}_+|^2 + |\pi_+|^2 \right) dx + \mu \int_{D_{2R}^-} \eta^2 \left(|\nabla \vec{u}_-|^2 + |\pi_-|^2 \right) dx \right]$$

$$+ \int_{S_{2R}} \eta^2 \left| \langle \partial_\nu^\lambda (\vec{u}_+, \pi_+), \vec{u}_+ \rangle - \mu \langle \partial_\nu^\lambda (\vec{u}_-, \pi_-), \vec{u}_- \rangle \right| d\sigma. \tag{6.14}$$

Now for any $1 \leq s < t \leq 2$, let η have the following properties

$$\begin{cases} \eta \equiv 1 \text{ on } D_{sR} \\ \text{supp } \eta \subseteq D_{tR} \\ 0 \leq \eta \leq 1 \\ \|\nabla \eta\|_{L^\infty} \leq \frac{C}{R(t-s)}. \end{cases} \tag{6.15}$$

Using (6.15) and Lemma 6.1 in (6.14) then gives

$$\int_{D_{sR}^+} A_\lambda(\nabla \vec{u}_+, \nabla \vec{u}_+)\, dx + \mu \int_{D_{sR}^-} A_\lambda(\nabla \vec{u}_-, \nabla \vec{u}_-)\, dx$$

$$\leq \frac{C_\varepsilon}{R^2(t-s)^2} \left[\int_{D_{tR}^+} |\vec{u}_+|^2\, dx + \mu \int_{D_{tR}^-} |\vec{u}_-|^2\, dx \right]$$

$$+\varepsilon\, C \left[\int_{D_{tR}^+} |\nabla \vec{u}_+|^2\, dx + \mu \int_{D_{tR}^-} |\nabla \vec{u}_-|^2\, dx \right]$$

$$+\varepsilon\, C R^{n-2} \left[\left(\fint_{S_{tR}} M(\vec{u}_+)^q\, d\sigma \right)^{\frac{2}{q}} + \mu \left(\fint_{S_{tR}} M(\vec{u}_-)^q\, d\sigma \right)^{\frac{2}{q}} \right]$$

$$+ \int_{S_{tR}} \left| \langle \partial_\nu^\lambda (\vec{u}_+, \pi_+), \vec{u}_+ \rangle - \mu \langle \partial_\nu^\lambda (\vec{u}_-, \pi_-), \vec{u}_- \rangle \right| d\sigma. \tag{6.16}$$

Next, we claim that (6.16) can be improved to

$$
\begin{aligned}
&\int_{D^+_{sR}} |\nabla \vec{u}_+|^2\,dx + \mu \int_{D^-_{sR}} |\nabla \vec{u}_-|^2\,dx \\
&\qquad \le \frac{C_\varepsilon}{R^2(t-s)^2}\left[\int_{D^+_{tR}} |\vec{u}_+|^2\,dx + \mu \int_{D^-_{tR}} |\vec{u}_-|^2\,dx\right] + \varepsilon\, C\left[\int_{D^+_{tR}} |\nabla \vec{u}_+|^2\,dx + \mu \int_{D^-_{tR}} |\nabla \vec{u}_-|^2\,dx\right] \\
&\qquad +\varepsilon\, C R^{n-2}\left[\left(\fint_{S_{tR}} M(\vec{u}_+)^q\,d\sigma\right)^{\frac{2}{q}} + \mu\left(\fint_{S_{tR}} M(\vec{u}_-)^q\,d\sigma\right)^{\frac{2}{q}}\right] \\
&\qquad + \int_{S_{tR}} \Big|\langle \partial^\lambda_\nu(\vec{u}_+,\pi_+),\vec{u}_+\rangle - \mu\,\langle \partial^\lambda_\nu(\vec{u}_-,\pi_-),\vec{u}_-\rangle\Big|\,d\sigma. \qquad (6.17)
\end{aligned}
$$

For $|\lambda| < 1$, this follows by (4.16). For $\lambda = 1$, (6.17) can be justified using the following version of Korn's inequality which we will prove in § 11.4.

Lemma 6.3 [`Korn's inequality`]

Let $D \subset \mathbb{R}^n$, $n \ge 2$, be a bounded Lipschitz domain of diameter R and assume that $1 < p < \infty$. Then there exists a finite, positive constant C which depends on p and the Lipschitz character of D but not on R, such that

$$\|\nabla \vec{u}\|_{L^p(D)} \le C\Big\{\|\nabla \vec{u} + \nabla \vec{u}^\top\|_{L^p(D)} + R^{-1}\|\vec{u}\|_{L^p(D)}\Big\}, \qquad (6.18)$$

uniformly for $\vec{u} \in L^p_1(D)$.

Next, we state another useful result.

Lemma 6.4 [`Hole Filling Lemma`]

For any $0 < \theta < 1$, $\alpha > 0$, and any non-decreasing functions A and B, if f is locally bounded and

$$f(s) \le (t-s)^{-\alpha}A(t) + B(t) + \theta f(t) \quad \text{whenever} \;\; \tau_0 \le s < t \le \tau_1, \qquad (6.19)$$

then

$$f(s) \leq C\Big[(t-s)^{-\alpha} A(t) + B(t)\Big] \quad \textit{whenever } \tau_0 \leq s < t \leq \tau_1. \tag{6.20}$$

For a proof of the Hole Filling Lemma, see the Appendix. Now Lemma 6.2 follows by choosing ε small enough in (6.17) and applying the Hole Filling Lemma. □

Our next result is a local estimate for $\nabla \vec{u}_\pm$ on the boundary.

Lemma 6.5 *Let $\mu \in [0,1)$. Then there exists $C > 0$ such that*

$$\begin{aligned}
&\int_{S_R} \left(|\nabla \vec{u}_+|^2 + \mu |\nabla \vec{u}_-|^2\right) d\sigma \\
&\qquad \leq \tfrac{C}{(1-\mu)^6} \int_{S_{2R}} \Big(\mu |\nabla_{tan} \vec{u}_+ - \nabla_{tan} \vec{u}_-|^2 + |\partial_\nu^\lambda(\vec{u}_+, \pi_+) - \mu\, \partial_\nu^\lambda(\vec{u}_-, \pi_-)|^2\Big)\, d\sigma \\
&\qquad + \tfrac{C}{R(1-\mu)^3}\Big[\int_{D_{2R}^+} (|\nabla \vec{u}_+|^2 + |\pi_+|^2)\, dx + \mu \int_{D_{2R}^-} (|\nabla \vec{u}_-|^2 + |\pi_-|^2)\, dx\Big].
\end{aligned} \tag{6.21}$$

Proof. For any $1 \leq s < t \leq 2$, there exists a smooth vector field $\vec{h}_s^t$ such that

$$\langle \vec{h}_s^t, \nu \rangle \geq 1 \text{ on } S_{sR}, \quad |\vec{h}_s^t| \leq C(\partial\Omega), \quad \operatorname{supp} \vec{h}_s^t \subseteq D_{tR}, \quad |\nabla \vec{h}_s^t| \leq \frac{C}{R(t-s)}. \tag{6.22}$$

Then by applying Proposition 5.2 with $\vec{h} = \vec{h}_s^t$ and ε chosen small enough, we can show that

$$\int_{S_{sR}} |\pi_\pm|^2\, d\sigma \leq \frac{C}{R(t-s)} \int_{D_{tR}^\pm} \left[|\nabla \vec{u}_\pm|^2 + |\pi_\pm|^2\right] dx + C \int_{S_{tR}} |\nabla \vec{u}_\pm|^2\, d\sigma + \tfrac{1}{2} \int_{S_{tR}} |\pi_\pm|^2\, d\sigma. \tag{6.23}$$

Then from the Hole Filling Lemma, it follows that for any $1 \leq s < t \leq 2$,

$$\int_{S_{sR}} |\pi_\pm|^2\, d\sigma \leq \frac{C}{R(t-s)} \int_{D_{tR}^\pm} \left[|\nabla \vec{u}_\pm|^2 + |\pi_\pm|^2\right] dx + C \int_{S_{tR}} |\nabla \vec{u}_\pm|^2\, d\sigma. \tag{6.24}$$

Applying Proposition 5.3 with $\vec{h} = \vec{h}_s^t$ also gives

$$\int_{S_{sR}} \left[A_\lambda(\nabla\vec{u}_+,\nabla\vec{u}_+) + \mu A_\lambda(\nabla\vec{u}_-,\nabla\vec{u}_-)\right]\, d\sigma$$

$$\leq \tfrac{C}{\varepsilon(1-\mu)^2} \int_{S_{tR}} \Big[|\partial^\lambda_\nu(\vec{u}_+,\pi_+) - \mu\,\partial^\lambda_\nu(\vec{u}_-,\pi_-)|^2 + \mu|\nabla_{tan}\vec{u}_+ - \nabla_{tan}\vec{u}_-|^2\Big]\, d\sigma$$

$$+\,\varepsilon \int_{S_{tR}} \Big[|\nabla\vec{u}_+|^2 + |\pi_+|^2 + \mu|\nabla\vec{u}_-|^2 + \mu|\pi_-|^2\Big]\, d\sigma$$

$$+\tfrac{C}{(1-\mu)}\tfrac{1}{R(t-s)}\Big[\int_{D^+_{tR}} \left(|\nabla\vec{u}_+|^2 + |\pi_+|^2\right)\, dx + \mu \int_{D^-_{tR}} \left(|\nabla\vec{u}_-|^2 + |\pi_-|^2\right)\, dx\Big], \tag{6.25}$$

which holds for any $1 \leq s < t \leq 2$. Consider the case $\lambda = 1$. Now, fix $1 \leq s < t \leq 2$, and let $t' := \frac{1}{2}(s+t)$ and $s' = \frac{1}{2}(s+t')$. Then $1 \leq s < s' < t' < t \leq 2$, and also $s' - s \sim t' - s' \sim t - t' \sim t - s$. Then since $A_1(\nabla\vec{u}_\pm,\nabla\vec{u}_\pm) = \frac{1}{2}|\nabla\vec{u}^\top_\pm + \nabla\vec{u}_\pm|^2$, applying Proposition 5.4 with $\vec{h} = \vec{h}^{s'}_s$ gives

$$\int_{S_{sR}} \left[|\nabla\vec{u}_+|^2 + \mu|\nabla\vec{u}_-|^2\right]\, d\sigma \leq \tfrac{C}{\varepsilon(1-\mu)^2} \int_{S_{s'R}} \left[A_1(\nabla\vec{u}_+,\nabla\vec{u}_+) + \mu A_1(\nabla\vec{u}_-,\nabla\vec{u}_-)^2\right]\, d\sigma$$

$$+\tfrac{C}{\varepsilon(1-\mu)^2} \int_{S_{s'R}} \Big[|\partial^1_\nu(\vec{u}_+,\pi_+) - \mu\,\partial^1_\nu(\vec{u}_-,\pi_-)|^2 + \mu|\nabla_{tan}\vec{u}_+ - \nabla_{tan}\vec{u}_-|^2\Big]\, d\sigma$$

$$+\,\varepsilon \int_{S_{s'R}} \Big[|\nabla\vec{u}_+|^2 + |\pi_+|^2 + \mu|\nabla\vec{u}_-|^2 + \mu|\pi_-|^2\Big]\, d\sigma$$

$$+\tfrac{C}{(1-\mu)}\tfrac{1}{R(s'-s)}\Big[\int_{D^+_{s'R}} \left(|\nabla\vec{u}_+|^2 + |\pi_+|^2\right)\, dx + \mu \int_{D^-_{s'R}} \left(|\nabla\vec{u}_-|^2 + |\pi_-|^2\right)\, dx\Big]. \tag{6.26}$$

Combining (6.26) with (6.25) where s and t are replaced by s' and t' and ε is replaced by $\varepsilon^2(1-\mu)^2$ and also invoking (6.24) with s replaced by t' gives

$$\int_{S_{sR}} \left[|\nabla\vec{u}_+|^2 + \mu|\nabla\vec{u}_-|^2\right]\, d\sigma \leq \tfrac{C}{\varepsilon^3(1-\mu)^6} \int_{S_{t'R}} \Big[|\partial^1_\nu(\vec{u}_+,\pi_+) - \mu\,\partial^1_\nu(\vec{u}_-,\pi_-)|^2 + \mu|\nabla_{tan}\vec{u}_+ - \nabla_{tan}\vec{u}_-|^2\Big]\, d\sigma$$

$$+\,\varepsilon C \int_{S_{t'R}} \Big[|\nabla\vec{u}_+|^2 + |\pi_+|^2 + \mu|\nabla\vec{u}_-|^2 + \mu|\pi_-|^2\Big]\, d\sigma$$

$$+\tfrac{C}{\varepsilon(1-\mu)^3}\tfrac{1}{R(t'-s)}\Big[\int\limits_{D^+_{t'R}}\left(|\nabla\vec{u}_+|^2+|\pi_+|^2\right)\,dx+\mu\int\limits_{D^-_{t'R}}\left(|\nabla\vec{u}_-|^2+|\pi_-|^2\right)\,dx\Big]$$

$$\leq \tfrac{C}{\varepsilon^3(1-\mu)^6}\int\limits_{S_{tR}}\Big[|\partial^1_\nu(\vec{u}_+,\pi_+)-\mu\,\partial^1_\nu(\vec{u}_-,\pi_-)|^2+\mu|\nabla_{tan}\vec{u}_+-\nabla_{tan}\vec{u}_-|^2\Big]\,d\sigma$$

$$+\varepsilon C\int\limits_{S_{tR}}\Big[|\nabla\vec{u}_+|^2+\mu|\nabla\vec{u}_-|^2\Big]\,d\sigma$$

$$+\tfrac{C}{\varepsilon(1-\mu)^3}\tfrac{1}{R(t-s)}\Big[\int\limits_{D^+_{tR}}\left(|\nabla\vec{u}_+|^2+|\pi_+|^2\right)\,dx+\mu\int\limits_{D^-_{tR}}\left(|\nabla\vec{u}_-|^2+|\pi_-|^2\right)\,dx\Big]. \quad (6.27)$$

Since (6.27) holds for every $1\leq s<t\leq 2$, after choosing ε small enough, applying the Hole Filling Lemma gives

$$\int\limits_{S_{sR}}\left[|\nabla\vec{u}_+|^2+\mu|\nabla\vec{u}_-|^2\right]$$

$$\leq \tfrac{C}{(1-\mu)^6}\int\limits_{S_{tR}}\Big[|\partial^1_\nu(\vec{u}_+,\pi_+)-\mu\,\partial^1_\nu(\vec{u}_-,\pi_-)|^2+\mu|\nabla_{tan}\vec{u}_+-\nabla_{tan}\vec{u}_-|^2\Big]\,d\sigma$$

$$+\tfrac{C}{(1-\mu)^3R(t-s)}\Big[\int\limits_{D^+_{tR}}\left(|\nabla\vec{u}_+|^2+|\pi_+|^2\right)\,dx+\mu\int\limits_{D^-_{tR}}\left(|\nabla\vec{u}_-|^2+|\pi_-|^2\right)\,dx\Big] \quad (6.28)$$

which holds for any $1\leq s<t\leq 2$. This is enough to prove the lemma in the case $\lambda=1$. For $|\lambda|<1$, there exists $C_\lambda>0$ such that $|\nabla\vec{u}_\pm|^2\leq C_\lambda A_\lambda(\nabla\vec{u}_\pm,\nabla\vec{u}_\pm)$. In this case, (6.28) is not needed, and the lemma follows more directly by combining (6.25) and (6.24) and using the Hole Filling Lemma as above. □

The previous lemma also implies the following.

Lemma 6.6 *Let $\mu\in[0,1)$. Then there exists $C>0$ such that*

$$\int\limits_{S_R}\left(|\nabla\vec{u}_+|^2+\mu|\nabla\vec{u}_-|^2\right)\,d\sigma$$

$$\leq \tfrac{C}{(1-\mu)^6}\int\limits_{S_{2R}}\Big(|\nabla_{tan}\vec{u}_+-\mu\nabla_{tan}\vec{u}_-|^2+\mu|\partial^\lambda_\nu(\vec{u}_+,\pi_+)-\,\partial^\lambda_\nu(\vec{u}_-,\pi_-)|^2\Big)\,d\sigma$$

$$+\frac{C}{R(1-\mu)^3}\left[\int_{D_{2R}^+}(|\nabla\vec{u}_+|^2+|\pi_+|^2)\,dx+\mu\int_{D_{2R}^-}(|\nabla\vec{u}_-|^2+|\pi_-|^2)\,dx\right]. \tag{6.29}$$

Proof. For $\mu \in (0,1)$, this lemma follows by reversing the roles of Ω_+ and Ω_-, applying Lemma 6.5 to the functions

$$\vec{v}_+ = \mu\vec{u}_-, \quad \rho_+ = \mu\pi_-, \quad \vec{v}_- = \vec{u}_+, \quad \rho_- = \pi_+, \tag{6.30}$$

and then dividing by μ. For $\mu = 0$, the lemma follows by simply taking the limit as $\mu \to 0^+$.

6.2 Reverse Hölder estimates

This section will be devoted to proving the following result.

Lemma 6.7 [`Reverse Hölder Inequality`]

Let $a \in (1,2]$ and let $D_s \subseteq \mathbb{R}^n$, $n \geq 2$, be a family of Lipschitz domains such that

$$diam(D_s) \sim s \sim |D_s|^{\frac{1}{n}} \qquad and \qquad D_s \subseteq D_t \;\; for \;\; s < t. \tag{6.31}$$

If $u \in C^1(\mathbb{R}^n)$ satisfies

$$\int_{D_s} |\nabla u|^2\,dx \leq \frac{C}{(t-s)^2}\int_{D_t}|u|^2\,dx \;\; for\ every \;\; \tau \leq s < t \leq a\tau, \tag{6.32}$$

then for any $p > 0$ and there exists $C = C(p,a) > 0$ such that

$$\Big(\fint_{D_\tau}|u|^2\,dx\Big)^{\frac{1}{2}} \leq C\Big(\fint_{D_{a\tau}}|u|^p\,dx\Big)^{\frac{1}{p}}. \tag{6.33}$$

Proof. For $p \geq 2$, the lemma follows from Hölder's inequality. Assume $0 < p < 2$. By dilation, it is enough to consider the case when

$$\fint_{D_1}|u|^p\,dx = 1, \tag{6.34}$$

and to show that there exists a constant $C > 0$ such that

$$\fint_{D_{1/a}} |u|^2\,dx \le C. \tag{6.35}$$

Assume

$$\int_{D_{1/a}} |u|^2\,dx \ge 1. \tag{6.36}$$

Fix $\frac{2n}{n+2} < q < 2$. By the Gagliardo-Nirenberg-Sobolev inequality, there exists a finite, positive constant $C = C(n,q)$ such that

$$\Big(\fint_{D_s} |u|^{\frac{nq}{n-q}}\,dx\Big)^{\frac{n-q}{nq}} \le C\Big[s\Big(\fint_{D_s} |\nabla u|^q\,dx\Big)^{\frac{1}{q}} + \Big(\fint_{D_s} |u|^q\,dx\Big)^{\frac{1}{q}}\Big]. \tag{6.37}$$

After dilation, we are in the case when $\tau = \frac{1}{a}$, and so after applying Hölder's inequality and (6.32) in (6.37), we have for $\frac{1}{a} < s < t < 1$,

$$\begin{aligned}\Big(\fint_{D_s} |u|^{\frac{nq}{n-q}}\,dx\Big)^{\frac{n-q}{nq}} &\le C\Big[s\Big(\fint_{D_s} |\nabla u|^2\,dx\Big)^{\frac{1}{2}} + \Big(\fint_{D_s} |u|^2\,dx\Big)^{\frac{1}{2}}\Big] \\ &\le C\Big[s^2\frac{1}{s^n}\frac{1}{(t-s)^2}\int_{D_t} |u|^2\,dx + \frac{1}{s^n}\int_{D_t} |u|^2\,dx\Big]^{\frac{1}{2}}. \end{aligned} \tag{6.38}$$

Using the fact that $\frac{1}{a} < s < 1$ in (6.38) then gives

$$\begin{aligned}\Big(\int_{D_s} |u|^{\frac{nq}{n-q}}\,dx\Big)^{\frac{n-q}{nq}} &\le Ca^{\frac{n-2}{2}}\Big(\frac{1}{(t-s)^2} + \frac{1}{s^2}\Big)^{\frac{1}{2}}\Big(\int_{D_t} |u|^2\,dx\Big)^{\frac{1}{2}} \\ &\le Ca^{\frac{n-2}{2}}\Big(\frac{1}{(t-s)^2}\Big(\frac{t}{s}\Big)^2\Big)^{\frac{1}{2}}\Big(\int_{D_t} |u|^2\,dx\Big)^{\frac{1}{2}} \\ &\le \frac{Ca^{\frac{n}{2}}}{(t-s)}\Big(\int_{D_t} |u|^2\,dx\Big)^{\frac{1}{2}}. \end{aligned} \tag{6.39}$$

Define $I(s) := \Big(\int_{D_s} |u|^2\,dx\Big)^{\frac{1}{2}}$, and choose $\alpha \in (0, \frac{2(n-q)}{nq})$ such that $\frac{nq}{n-q}\,\alpha + p\,(1-\alpha) = 2$. Then by Hölder's inequality,

$$\begin{aligned} I(s)^2 &= \int_{D_s} |u|^2\,dx = \int_{D_s} |u|^{\frac{nq}{n-q}\alpha}|u|^{p(1-\alpha)}\,dx \\ &\le \Big(\int_{D_s} |u|^{\frac{nq}{n-q}}\,dx\Big)^{\alpha}\Big(\int_{D_s} |u|^p\,dx\Big)^{1-\alpha} \le C\Big(\int_{D_s} |u|^{\frac{nq}{n-q}}\,dx\Big)^{\alpha}, \end{aligned} \tag{6.40}$$

and so by (6.39),

$$I(s)^{\frac{2(n-q)}{nq\alpha}} \leq C\Big(\int_{D_s} |u|^{\frac{nq}{n-q}}\,dx\Big)^{\frac{n-q}{nq}} \leq \frac{C}{t-s}\Big(\int_{D_t} |u|^2\,dx\Big)^{\frac{1}{2}} = \frac{C}{t-s}\,I(t). \tag{6.41}$$

From (6.41), it follows that

$$\ln I(s) \leq C\theta + \theta\,\ln I(t) - \theta\,\ln(t-s). \tag{6.42}$$

where $\theta := \frac{nq\alpha}{2(n-q)} \in (0,1)$. In particular, if we let $t = s^\gamma$ for some $\theta < \gamma < 1$, then

$$\ln I(s) \leq C\theta + \theta\,\ln I(s^\gamma) - \theta\,\ln(s^\gamma - s). \tag{6.43}$$

Integrating (6.43) over $s \in [\frac{1}{a}, 1]$ against $\frac{ds}{s}$ gives

$$\int_{1/a}^1 \ln I(s)\frac{ds}{s} \leq C\theta + \theta\int_{1/a}^1 \ln I(s^\gamma)\frac{ds}{s} - \theta\int_{1/a}^1 \ln(s^\gamma - s)\,\frac{ds}{s}. \tag{6.44}$$

By a change of variables, we can write

$$\theta\int_{1/a}^1 \ln I(s^\gamma)\frac{ds}{s} = \gamma^{-1}\theta\int_{(1/a)^\gamma}^1 \ln I(s)\frac{ds}{s} \leq \gamma^{-1}\theta\int_{1/a}^1 \ln I(s)\,\frac{ds}{s}, \tag{6.45}$$

after which (6.44) becomes

$$(1-\gamma^{-1}\theta)\int_{1/a}^1 \ln I(s)\,\frac{ds}{s} \leq C(\theta,\gamma). \tag{6.46}$$

Since $I(s)$ is non-decreasing,

$$(1-\gamma^{-1}\theta)(1-\tfrac{1}{a})\ln I(\tfrac{1}{a}) \leq (1-\gamma^{-1}\theta)\int_{1/a}^1 \ln I(s)\,\frac{ds}{s} \leq C(\theta,\gamma), \tag{6.47}$$

which implies that

$$I(\tfrac{1}{a}) \leq e^{\frac{C(\theta,\gamma,a)}{(1-\gamma^{-1}\theta)}} = C(\partial\Omega,p,a). \tag{6.48}$$

Thus, the lemma holds. $\square$

7 The transmission problem in two and three dimensions

The goal of this chapter is to establish the atomic theory for the transmission problems (4.155), (4.156) in the case when Ω is a graph Lipschitz domain in $\mathbb{R}^2$ or $\mathbb{R}^3$. In practice, proving that (T_μ^+) is well-posed for arbitrary graph Lipschitz domains automatically implies that (T_μ^-) is well-posed for arbitrary graph Lipschitz domains because of the symmetry of the geometry. With this in mind, in subsequent work we will often drop the sign and just refer to the transmission problems as $(T_\mu) := (T_\mu^+)$ and $(T_\mu)^* := (T_\mu^+)^*$.

Assume $\Omega \subset \mathbb{R}^n$ is a graph Lipschitz domain, and set $\Omega_+ := \Omega$, $\Omega_- := \mathbb{R}^n \setminus \bar{\Omega}$. We will prove that there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ such that (T_μ) and $(T_\mu)^*$ are well-posed for every $\mu \in [0, 1)$ and for

$$\tfrac{2}{3} - \varepsilon < p < 2 + \varepsilon, \quad n = 2, \tag{7.1}$$

$$1 - \varepsilon < p < 2 + \varepsilon, \quad n = 3. \tag{7.2}$$

With the case when p is near 2 well understood, we will first establish well-posedness for $p \leq 1$, and then use interpolation to handle the case $1 < p < 2$.

7.1 Uniqueness

Recall (4.155), (4.156). In this section, we will prove a few uniqueness results.

Theorem 7.1 *Let Ω be as above, $n \geq 3$, $\mu \in [0, 1)$, and fix $\frac{n-1}{n} < p < n - 1$. Assume that there exists $1 < q < n - 1$ with the following properties:*

$$(i) \qquad \frac{n}{n-1} < \frac{1}{p} + \frac{1}{q} \leq \frac{n+1}{n-1}; \tag{7.3}$$

$$(ii) \qquad \text{for any } \vec{f} \in L^q(\partial\Omega), \vec{g} \in \dot{L}_1^q(\partial\Omega), \text{ a solution of } (T_\mu)^* \text{ with data } (\vec{f}, \vec{g}) \text{ exists.} \tag{7.4}$$

Then if $(\vec{u}_\pm, \pi_\pm)$ solves the homogeneous version of $(T_\mu)^*$, the functions $\vec{u}_+$, π_+, $\mu\vec{u}_-$, and $\mu\pi_-$ must be constant. Moreover, the same result holds if we replace $(T_\mu)^*$ with (T_μ).

First, we record an auxiliary result, whose proof is given in the appendix.

Lemma 7.2 [Hardy's estimate]

Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be the domain lying above the graph of a Lipschitz function φ. Assume w is biharmonic in Ω and $M(\nabla w) \in L^p(\partial\Omega)$ for some $p < n-1$. Then there exist constants $c = c(w) \in \mathbb{R}$ and $C = C(\partial\Omega) > 0$ such that

$$\|M(w-c)\|_{L^{p^*}(\partial\Omega)} \leq C\|M(\nabla w)\|_{L^p(\partial\Omega)} \quad \textit{where} \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n-1}. \tag{7.5}$$

Proof of Theorem 7.1. Assume $(\vec{u}_\pm, \pi_\pm)$ satisfy

$$\Delta\vec{u}_\pm = \nabla\pi_\pm, \quad \operatorname{div}\vec{u}_\pm = 0 \text{ in } \Omega_\pm, \qquad M(\nabla\vec{u}_\pm), M(\pi_\pm) \in L^p(\partial\Omega), \tag{7.6}$$

along with

$$\vec{u}_+\Big|_{\partial\Omega} = \vec{u}_-\Big|_{\partial\Omega}, \qquad \partial_\nu^\lambda(\vec{u}_+, \pi_+) = \mu\,\partial_\nu^\lambda(\vec{u}_-, \pi_-) \quad \text{on } \partial\Omega. \tag{7.7}$$

Applying Lemma 7.2 to $\vec{u}_\pm$, there exists $\vec{c}_\pm \in \mathbb{R}^n$ such that $M(\vec{u}_\pm - \vec{c}_\pm) \in L^{p^*}(\partial\Omega)$ where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n-1}$. Using the first transmission boundary condition in (7.7),

$$\vec{c}_- - \vec{c}_+ = (\vec{u}_+ - \vec{c}_+)\Big|_{\partial\Omega} - (\vec{u}_- - \vec{c}_-)\Big|_{\partial\Omega} \in L^{p^*}(\partial\Omega), \tag{7.8}$$

and so $\vec{c}_+ = \vec{c}_- =: \vec{c}$. Let us re-denote $\vec{u}_\pm - \vec{c}$ by $\vec{u}_\pm$ and then we will show that $\vec{u}_+ \equiv 0$ and $\mu\vec{u}_- \equiv 0$. Fix $x_o \in \mathbb{R}^n \setminus \partial\Omega$ and $\vec{b} \in \mathbb{R}^n$. Also, let

$$\vec{v} := E(\,\cdot - x_o)\vec{b} \quad \text{and} \quad \widetilde{q} := \vec{q}(\,\cdot - x_o) \cdot \vec{b}, \tag{7.9}$$

where E and $\vec{q}$ are as before. Then $(\vec{v}, \widetilde{q})$ satisfies

$$\begin{cases} \Delta\vec{v} - \nabla\widetilde{q} = 0 \text{ in } \mathbb{R}^n \setminus \{x_o\}, \\ \\ \operatorname{div}\vec{v} = 0 \text{ in } \mathbb{R}^n. \end{cases} \tag{7.10}$$

We also have that $\partial_\nu^\lambda(\vec{v}, \widetilde{q}) \in \bigcap\limits_{r>1} L^r(\partial\Omega)$ and so by (7.4), we can find $(\vec{w}_\pm, \rho_\pm)$ that solves

$$
\begin{cases}
\Delta \vec{w}_\pm = \nabla \rho_\pm \quad \text{in} \quad \Omega_\pm, \\
\operatorname{div} \vec{w}_\pm = 0 \quad \text{in} \quad \Omega_\pm, \\
\vec{w}_+\Big|_{\partial\Omega} = \vec{w}_-\Big|_{\partial\Omega}, \\
\partial_\nu^\lambda(\vec{w}_+, \rho_+) - \mu\, \partial_\nu^\lambda(\vec{w}_-, \rho_-) = (1-\mu)\, \partial_\nu^\lambda(\vec{v}, \widetilde{q}) \in L^q(\partial\Omega), \\
M(\nabla \vec{w}_\pm),\ M(\rho_\pm) \in L^q(\partial\Omega).
\end{cases}
\tag{7.11}
$$

Notice also that, by subtracting an appropriate constant as before, we can even choose $\vec{w}_\pm$ so that $M(\vec{w}_\pm) \in L^{q^*}(\partial\Omega)$. Then the functions

$$
\vec{G}_\pm := \vec{w}_\pm - \vec{v} \quad \text{and} \quad g_\pm := \rho_\pm - \widetilde{q},
\tag{7.12}
$$

must satisfy

$$
\begin{cases}
\Delta \vec{G}_\pm = \nabla g_\pm \quad \text{in} \quad \Omega_\pm \setminus \{x_o\}, \\
\operatorname{div} \vec{G}_\pm = 0 \quad \text{in} \quad \Omega_\pm \\
\vec{G}_+\Big|_{\partial\Omega} = \vec{G}_-\Big|_{\partial\Omega}, \\
\partial_\nu^\lambda(\vec{G}_+, g_+) = \mu\, \partial_\nu^\lambda(\vec{G}_-, g_-), \\
M(\nabla \vec{G}_\pm),\ M(g_\pm) \in L^q(\partial\Omega).
\end{cases}
\tag{7.13}
$$

Fix $R > 0$, and let $\psi \in C^\infty$ be such that

$$
\begin{cases}
\operatorname{supp} \psi \subseteq B_{2R}(x_o), \\
\psi \equiv 1 \quad \text{on} \quad B_R(x_o), \\
\|\nabla \psi\|_{L^\infty} \leq \frac{C}{R}, \\
\|\nabla^2 \psi\|_{L^\infty} \leq \frac{C}{R^2}.
\end{cases}
\tag{7.14}
$$

Applying the integration by parts formula (4.7) to $(\vec{G}_\pm, g_\pm)$ and $(\psi\vec{u}_\pm, \psi\pi_\pm)$ gives

$$
\int_{\Omega_\pm} \langle L_\lambda \vec{G}_\pm - \nabla g_\pm \,,\, \psi \vec{u}_\pm \rangle \, dx
$$

$$= \pm \int_{\partial\Omega} \Big[\langle \partial_\nu^\lambda(\vec{G}_\pm, g_\pm), \psi\vec{u}_\pm\rangle - \langle \partial_\nu^\lambda(\psi\vec{u}_\pm, \psi\pi_\pm), \vec{G}_\pm\rangle \Big] \, d\sigma + \int_{\Omega_\pm} \langle L_\lambda(\psi\vec{u}_\pm) - \nabla(\psi\pi_\pm), \vec{G}_\pm\rangle \, dx$$

$$+ \int_{\Omega_\pm} \Big[g_\pm \operatorname{div}(\psi\vec{u}_\pm) - \pi\psi(\operatorname{div}\vec{G}) \Big] \, dx$$

$$= \pm \int_{\partial\Omega} \Big[\langle \partial_\nu^\lambda(\vec{G}_\pm, g_\pm), \psi\vec{u}_\pm\rangle - \langle (\partial_\nu\psi)\vec{u}_\pm + \psi\partial_\nu^\lambda(\vec{u}_\pm, \pi_\pm), \vec{G}_\pm\rangle + \int_{\Omega_\pm} \langle L_\lambda\vec{u} - \nabla\pi_\pm, \psi\vec{G}_\pm\rangle \Big] \, d\sigma$$

$$+ \int_{\Omega_\pm} \Big\langle 2(\nabla\vec{u}_\pm)^\top \nabla\psi + (\Delta\psi)\vec{u}_\pm + \lambda\Big[(\operatorname{div}\vec{u}_\pm)\nabla\psi + \nabla\vec{u}_\pm\nabla\psi + (\nabla^2\psi)\vec{u}_\pm \Big], \vec{G}_\pm \Big\rangle \, dx$$

$$+ \int_{\Omega_\pm} \Big\{ -\langle \pi_\pm\nabla\psi, \vec{G}_\pm\rangle + g_\pm\Big[\psi(\operatorname{div}\vec{u}_\pm) + \langle \vec{u}_\pm, \nabla\psi\rangle \Big] - \pi_\pm\psi(\operatorname{div}\vec{G}_\pm) \Big\} \, dx. \tag{7.15}$$

Let us set $\vec{u} := \vec{u}_\pm$ in $\Omega_\pm$, $\pi := \pi_\pm$ in $\Omega_\pm$, with similar conventions for $\vec{G}$, g and $\vec{w}$, ρ. If we now multiply the minus version of (7.15) by μ and add it to the plus version, and then use (7.7) and (7.13), we obtain

$$\Big| \int_{\Omega_+} \langle L_\lambda\vec{G}_+ - \nabla g_+ \,,\, \psi\vec{u}_+\rangle \, dx + \mu \int_{\Omega_-} \langle L_\lambda\vec{G}_- - \nabla g_- \,,\, \psi\vec{u}_-\rangle \, dx \Big|$$

$$\leq \int_{\partial\Omega} |\langle (\partial_\nu\psi)\vec{u}, \vec{G}\rangle| \, d\sigma + \int_{\mathbb{R}^n\setminus\partial\Omega} \Big| \Big\langle 2(\nabla\vec{u})^\top\nabla\psi + \vec{u}\Delta\psi + \lambda\Big[(\nabla\vec{u})\nabla\psi + (\nabla^2\psi)\vec{u} \Big], \vec{G} \Big\rangle \Big| \, dx$$

$$+ \int_{\mathbb{R}^n\setminus\partial\Omega} \Big[|\langle \pi\nabla\psi, \vec{G}\rangle| + |g||\langle \vec{u}, \nabla\psi\rangle| \Big] \, dx. \tag{7.16}$$

Define $A_R := B_{2R}(x_o) \setminus B_R(x_o)$ and $S_R := A_R \cap \partial\Omega$. Then using (7.14),

$$\Big| \int_{\Omega_+} \langle L_\lambda\vec{G}_+ - \nabla g_+ \,,\, \psi\vec{u}_+\rangle \, dx + \mu \int_{\Omega_-} \langle L_\lambda\vec{G}_- - \nabla g_- \,,\, \psi\vec{u}_-\rangle \, dx \Big|$$

$$\leq \frac{C}{R}\int_{S_R} |\vec{u}||\vec{G}| + \frac{C}{R}\int_{A_R} |\nabla\vec{u}||\vec{G}| + \frac{C}{R}\int_{A_R} |\pi||\vec{G}| + \frac{C}{R}\int_{A_R} |\vec{u}||g| + \frac{C}{R^2}\int_{A_R} |\vec{u}||\vec{G}|$$

$$=: I + II + III + IV + V. \tag{7.17}$$

It also follows by direct calculation that

$$|\vec{v}| \leq \frac{C}{R^{n-2}} \quad \text{and} \quad |\widetilde{q}| \leq \frac{C}{R^{n-1}} \quad \text{on} \;\; A_R. \tag{7.18}$$

We will also need the following lemma which is proved in [31].

Lemma 7.3 *For every Lipschitz domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$ (assumed to be either bounded or of graph type) and any number $p > 0$, there exists a finite constant $C = C(\Omega, p) > 0$ such that the estimate*

$$\|u\|_{L^{p\,n/(n-1)}(\Omega)} \leq C\|M(u)\|_{L^p(\partial\Omega)}, \tag{7.19}$$

holds for every continuous function u in Ω.

Applying Lemma 7.3 to the functions $\vec{u}, \nabla\vec{u}, \pi, \vec{w}, \nabla\vec{w}$, and ρ allows us to conclude that

$$\nabla\vec{u}, \pi \in L^{\frac{p\,n}{n-1}}(\Omega_\pm),\;\; \nabla\vec{w}, \rho \in L^{\frac{qn}{n-1}}(\Omega_\pm),\;\; \vec{u} \in L^{\frac{p^*n}{n-1}}(\Omega_\pm), \;\; \text{and } \vec{w} \in L^{\frac{q^*n}{n-1}}(\Omega_\pm). \tag{7.20}$$

Combining (7.18) and (7.20), we see that there exists $C > 0$ independent of R such that for $R > 1$, the following estimates hold:

$$\begin{aligned}
\|\vec{G}\|_{L^{\frac{q^*n}{n-1}}(A_R)} &\leq \|\vec{w}\|_{L^{\frac{q^*n}{n-1}}(A_R)} + \|\vec{v}\|_{L^{\frac{q^*n}{n-1}}(A_R)} \\
&\leq C + \frac{C}{R^{n-2}}(R^n)^{\frac{n-1}{q^*n}} \leq C(1 + R^{(n-1)(\frac{1}{q}-1)}) \leq C, && (7.21) \\
\|\vec{G}\|_{L^{q^*}(S_R)} &\leq \|M(\vec{w})\|_{L^{q^*}(S_R)} + \|\vec{v}\|_{L^{q^*}(S_R)} \\
&\leq C + \frac{C}{R^{n-2}}(R^{n-1})^{\frac{1}{q^*}} \leq C(1 + R^{(n-1)(\frac{1}{q}-1)}) \leq C, && (7.22) \\
\|g\|_{L^{\frac{qn}{n-1}}(A_R)} &\leq \|\rho\|_{L^{\frac{qn}{n-1}}(A_R)} + \|\widetilde{q}\|_{L^{\frac{qn}{n-1}}(A_R)} \\
&\leq C + \frac{C}{R^{n-1}}(R^n)^{\frac{n-1}{qn}} \leq C(1 + R^{(n-1)(\frac{1}{q}-1)}) \leq C. && (7.23)
\end{aligned}$$

It follows from (7.3) that

$$\frac{1}{p^*} + \frac{1}{q} = \frac{1}{p} + \frac{1}{q^*} \; \leq \; \frac{n}{n-1} \;\; \text{and} \;\; \frac{1}{p^*} + \frac{1}{q^*} \; \leq \; 1, \tag{7.24}$$

and so we can define $\beta > 0$ by

$$\beta := \frac{1}{p} + \frac{1}{q} - \frac{n}{n-1} = \frac{1}{p^*} + \frac{1}{q} - 1 = \frac{1}{p} + \frac{1}{q^*} - 1 = \frac{1}{p^*} + \frac{1}{q^*} - \frac{n-2}{n-1}. \tag{7.25}$$

Returning to (7.17), by (7.20)-(7.23) and Hölder's inequality, we have that as $R \to \infty$,

$$\begin{aligned}
I &\leq \frac{C}{R}\left(\int_{S_R} |M(\vec{u})|^{p^*}\right)^{\frac{1}{p^*}} \left(\int_{S_R} |\vec{G}|^{q^*}\right)^{\frac{1}{q^*}} (R^{n-1})^{1-\frac{1}{p^*}-\frac{1}{q^*}} \leq CR^{-\beta(n-1)} \to 0,\\
II &\leq \frac{C}{R}\left(\int_{A_R} |\nabla\vec{u}|^{\frac{pn}{n-1}}\right)^{\frac{n-1}{pn}} \left(\int_{A_R} |\vec{G}|^{\frac{q^*n}{n-1}}\right)^{\frac{n-1}{q^*n}} (R^{n})^{1-\frac{n-1}{pn}-\frac{n-1}{q^*n}} \leq CR^{-\beta(n-1)} \to 0,\\
III &\leq \frac{C}{R}\left(\int_{A_R} |\pi|^{\frac{pn}{n-1}}\right)^{\frac{n-1}{pn}} \left(\int_{A_R} |\vec{G}|^{\frac{q^*n}{n-1}}\right)^{\frac{n-1}{q^*n}} (R^{n})^{1-\frac{n-1}{pn}-\frac{n-1}{q^*n}} \leq CR^{-\beta(n-1)} \to 0,\\
IV &\leq \frac{C}{R}\left(\int_{A_R} |\vec{u}|^{\frac{p^*n}{n-1}}\right)^{\frac{n-1}{p^*n}} \left(\int_{A_R} |g|^{\frac{qn}{n-1}}\right)^{\frac{n-1}{qn}} (R^{n})^{1-\frac{n-1}{p^*n}-\frac{n-1}{qn}} \leq CR^{-\beta(n-1)} \to 0, \quad \text{and}\\
V &\leq \frac{C}{R^2}\left(\int_{A_R} |\vec{u}|^{\frac{p^*n}{n-1}}\right)^{\frac{n-1}{p^*n}} \left(\int_{A_R} |\vec{G}|^{\frac{q^*n}{n-1}}\right)^{\frac{n-1}{q^*n}} (R^{n})^{1-\frac{n-1}{p^*n}-\frac{n-1}{q^*n}} \leq CR^{-\beta(n-1)} \to 0.
\end{aligned} \tag{7.26}$$

Hence, from (7.16),

$$\int_{\Omega_+} \langle L_\lambda \vec{G}_+ - \nabla g_+ \,,\, \psi\vec{u}_+\rangle\, dx + \mu \int_{\Omega_-} \langle L_\lambda \vec{G}_- - \nabla g_- \,,\, \psi\vec{u}_-\rangle\, dx = 0. \tag{7.27}$$

As a direct consequence of the particular construction of the functions $(\vec{G}, g)$ as a fundamental solution for the Stokes system, it follows that

$$\int_{\mathbb{R}^n \setminus \partial\Omega} \langle L_\lambda \vec{G} - \nabla g \,,\, \vec{u}\psi\rangle\, dx = \langle \vec{u}(x_o), \vec{b}\rangle. \tag{7.28}$$

If $x_o \in \Omega_+$, then $L_\lambda \vec{G}_- - \nabla g_- = 0$ in Ω_- and so from (7.27) and (7.28), $\langle \vec{u}_+(x_0), \vec{b}\rangle = 0$. Then since this holds for every $x_o \in \Omega_+$ and $\vec{b} \in \mathbb{R}^n$, we must have $\vec{u}_+ \equiv 0$. Similarly, if we instead consider the case when $x_o \in \Omega_-$, it follows that $\mu\vec{u}_- \equiv 0$.

If we instead assume that $(\vec{u}_\pm, \pi_\pm)$ solves the homogeneous version of (T_μ), then (7.7) will be replaced by

$$\vec{u}_+\Big|_{\partial\Omega} = \mu\,\vec{u}_-\Big|_{\partial\Omega}, \qquad \partial_\nu^\lambda(\vec{u}_+,\pi_+) = \partial_\nu^\lambda(\vec{u}_-,\pi_-) \quad \text{on } \partial\Omega. \tag{7.29}$$

Proceeding in a similar fashion as before, this time we can use the hypothesis to construct functions $(\vec{G}_\pm, g_\pm)$ that satisfy

$$\begin{cases} \Delta\vec{G}_\pm = \nabla g_\pm \quad \text{in} \quad \Omega_\pm \setminus \{x_o\}, \\ \operatorname{div}\vec{G}_\pm = 0 \quad \text{in} \quad \Omega_\pm \\ \vec{G}_+\Big|_{\partial\Omega} = \mu\,\vec{G}_-\Big|_{\partial\Omega}, \\ \partial_\nu^\lambda(\vec{G}_+, g_+) = \partial_\nu^\lambda(\vec{G}_-, g_-), \\ M(\nabla\vec{G}_\pm),\ M(g_\pm) \in L^q(\partial\Omega), \end{cases} \tag{7.30}$$

along with (7.28). The rest of the proof follows similarly to the previous argument, except this time we use (7.29) and (7.30) in place of (7.7) and (7.13). This concludes the proof. □

Although the previous theorem is stated for $n \geq 3$, it will be most useful when $n = 3$, since in this case, if $\frac{2}{3} < p \leq 1$, we can always find q close enough to 2 that satisfies (7.3)-(7.4). Since we are also concerned with the two dimensional case, we will need the following result (the reader is advised to revisit the conventions made at the beginning of § 7):

Lemma 7.4 *Let $\Omega \subset \mathbb{R}^2$ be a graph Lipschitz domain and set $\Omega_+ := \Omega$, $\Omega_- := \mathbb{R}^n \setminus \bar{\Omega}$. For $\mu \in [0,1)$ and $\frac{1}{2} < p < 1$ fixed, assume that $(\vec{u}_\pm, \pi_\pm)$ solve the homogeneous version of either (T_μ) or $(T_\mu)^*$. Then the functions $\vec{u}_+$, π_+, $\mu\vec{u}_-$, and $\mu\pi_-$ are constant.*

Proof. Since $M(\nabla\vec{u}_\pm) \in L^p(\partial\Omega)$, after subtracting a suitable constant from $\vec{u}_\pm$, we can conclude from Lemma 7.2 that $M(\vec{u}_\pm) \in L^{p^*}(\partial\Omega)$ where $\frac{1}{p^*} = \frac{1}{p} - 1$. Then by Lemma 7.3, the locally integrable function $\vec{u} := \vec{u}_\pm$ in $\Omega_\pm$ satisfies $\vec{u} \in L^q(\mathbb{R}^2)$, where $1/q = 1/(2p) - 1/2$. Note that $\frac{1}{2} < p < 1$ forces $q \in (2,\infty)$. In the same context as that of (6.6), we now have

$$\begin{aligned} |\pi_\pm(y \pm te_n)| &\leq \int_{cR}^\infty \frac{C}{s^2}\Big(⨍_{B(y\pm se_n, c_2 s)} |\vec{u}_\pm(z)|^q\,dz\Big)^{1/q} ds \\ &\leq C\|\vec{u}\|_{L^q(\mathbb{R}^2)} \int_{cR}^\infty \frac{ds}{s^{2+2/q}} = CR^{-1-2/q}, \end{aligned} \tag{7.31}$$

(where C depends on $\vec{u}$), leading to

$$\fint_{B^{\pm}} |\pi_{\pm}|\, dx \leq CR^{-1-2/q}, \tag{7.32}$$

in place of (6.7) and, further, to

$$\left(\fint_{B(0,R)\cap\Omega_{\pm}} |\pi_{\pm}|^2\, dx \right)^{\frac{1}{2}} \leq C \left(\fint_{B(0,R)} |\nabla \vec{u}|^2\, dx \right)^{\frac{1}{2}} + CR^{-1-2/q}, \tag{7.33}$$

in place of (6.5). With this in hand and by proceeding as in the proof of Lemma 6.2 we obtain that, whenever $\mu \in [0,1)$,

$$\int_{B(0,R)\cap\Omega_{+}} |\nabla \vec{u}_{+}|^2\, dx + \mu \int_{B(0,R)\cap\Omega_{-}} |\nabla \vec{u}_{-}|^2\, dx \leq \frac{C}{R^2} \int_{B(0,2R)} |\vec{u}|^2\, dx + CR^{-4/q}, \tag{7.34}$$

which should be compared to (6.11). Using the fact that $\vec{u} \in L^q(\mathbb{R}^2)$ for some $q > 2$, allows us to estimate

$$\frac{C}{R^2} \int_{B(0,2R)} |\vec{u}|^2\, dx \leq CR^{-4/q}, \tag{7.35}$$

hence altogether

$$\int_{B(0,R)\cap\Omega_{+}} |\nabla \vec{u}_{+}|^2\, dx + \mu \int_{B(0,R)\cap\Omega_{-}} |\nabla \vec{u}_{-}|^2\, dx \leq CR^{-4/q}, \tag{7.36}$$

by (7.34)-(7.35), where C is independent of R. Letting $R \to \infty$ then proves that $\vec{u}_{+}$ is a constant in Ω_{+} and that $\mu\vec{u}_{-}$ is a constant in Ω_{-}. $\square$

7.2 Atomic estimates

This section will be devoted to proving the following two results. Recall the conventions made at the beginning of § 7.

Proposition 7.5 *Assume $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a graph Lipschitz domain and fix $\lambda \in (-1,1]$ and $\mu \in [0,1)$. As usual, set $\Omega_{+} := \Omega$, $\Omega_{-} := \mathbb{R}^n \setminus \bar{\Omega}$. Assume there exists $1 < q < \frac{n-1}{n-2}$ such* *that the operators* $\pm\frac{1}{2}\frac{1+\mu}{1-\mu}I + K_\lambda$ *are invertible on* $L^q(\partial\Omega)$ *and the* L^q *Dirichlet problem is well-posed. Then for* $\frac{(n-1)q}{n-1+q} < p \leq 1$, *there exists* $C > 0$ *such that for any* $\vec{f} \in H^p_{at}(\partial\Omega)$ *and* $\vec{g} \in H^{1,p}_{at}(\partial\Omega)$, *there exist functions* $(\vec{u}_\pm, \pi_\pm)$ *that solve* $(T_\mu)^*$ *(cf. (4.155) and the discussion in the beginning of § 7) and satisfy*

$$\begin{aligned} &\|M(\nabla\vec{u}_+)\|_{L^p(\partial\Omega)} + \|M(\pi_+)\|_{L^p(\partial\Omega)} \\ &\qquad +\mu\|M(\nabla\vec{u}_-)\|_{L^p(\partial\Omega)} + \mu\|M(\pi_-)\|_{L^p(\partial\Omega)} \leq C\left(\|\vec{g}\|_{H^{1,p}_{at}(\partial\Omega)} + \|\vec{f}\|_{H^p_{at}(\partial\Omega)}\right) \end{aligned} \tag{7.37}$$

Proposition 7.6 *Let* $\Omega \subset \mathbb{R}^n$, $n \geq 2$, *be a graph Lipschitz domain in* $\mathbb{R}^n$, $n \geq 2$, *and set* $\Omega_+ := \Omega$, $\Omega_- := \mathbb{R}^n \setminus \bar{\Omega}$. *Also, fix* $\lambda \in (-1, 1]$ *and* $\mu \in [0, 1)$. *Assume there exists* $1 < q < \frac{n-1}{n-2}$ *such that the operators* $\pm\frac{1}{2}\frac{1+\mu}{1-\mu}I + K_\lambda$ *are invertible on* $L^q_1(\partial\Omega)$. *Then for* $\frac{(n-1)q}{n-1+q} < p \leq 1$, *there exists* $C > 0$ *such that for any* $\vec{f} \in H^p_{at}(\partial\Omega)$ *and* $\vec{g} \in H^{1,p}_{at}(\partial\Omega)$, *there exist functions* $(\vec{u}_\pm, \pi_\pm)$ *that solve* (T_μ) *and satisfy*

$$\begin{aligned} &\|M(\nabla\vec{u}_+)\|_{L^p(\partial\Omega)} + \|M(\pi_+)\|_{L^p(\partial\Omega)} \\ &\qquad +\mu\|M(\nabla\vec{u}_-)\|_{L^p(\partial\Omega)} + \mu\|M(\pi_-)\|_{L^p(\partial\Omega)} \leq C\left(\|\vec{g}\|_{H^{1,p}_{at}(\partial\Omega)} + \|\vec{f}\|_{H^p_{at}(\partial\Omega)}\right) \end{aligned} \tag{7.38}$$

Arguing as in the proof of Theorem 5.9, to prove Proposition 7.5, we can reduce matters to considering the case when $\vec{g} = 0$. We will first consider the case when $\vec{f}$ is a (p, ∞)-atom as defined in (2.30). Fix p such that $\frac{(n-1)q}{n-1+q} < p \leq 1$, and let $\vec{a}$ be a (p, ∞)-atom. Since $\vec{a} \in L^2(\partial\Omega)$, from Lemma 5.7, we can define

$$\begin{aligned} \vec{u}_\pm &:= \tfrac{1}{1-\mu}\,\mathcal{S}\Big((-\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K^*_\lambda)^{-1}\vec{a}\Big) \quad \text{in } \ \Omega_\pm, \\ \pi_\pm &:= \tfrac{1}{1-\mu}\,\mathcal{Q}\Big((-\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K^*_\lambda)^{-1}\vec{a}\Big) \quad \text{in } \ \Omega_\pm. \end{aligned} \tag{7.39}$$

By Proposition 4.5, (4.29), (4.47), and (4.45), the functions $(\vec{u}_\pm, \pi_\pm)$ will satisfy

$$\begin{cases} \Delta \vec{u}_\pm = \nabla \pi_\pm \quad \text{in} \quad \Omega_\pm, \\ \operatorname{div} \vec{u}_\pm = 0 \quad \text{in} \quad \Omega_\pm, \\ \vec{u}_+ \Big|_{\partial\Omega} = \vec{u}_- \Big|_{\partial\Omega}, \\ \partial_\nu^\lambda(\vec{u}_+, \pi_+) - \mu\, \partial_\nu^\lambda(\vec{u}_-, \pi_-) = \vec{a} \quad \text{on} \quad \partial\Omega, \\ \|M(\nabla \vec{u}_\pm)\|_{L^2(\partial\Omega)} + \|M(\pi_\pm)\|_{L^2(\partial\Omega)} \le C\|\vec{a}\|_{L^2(\partial\Omega)}. \end{cases} \tag{7.40}$$

Our goal is to show there exists $C = C(\partial\Omega) > 0$ such that

$$\|M(\nabla \vec{u}_+)\|_{L^p(\partial\Omega)} + \|M(\pi_+)\|_{L^p(\partial\Omega)} + \mu\|M(\nabla \vec{u}_-)\|_{L^p(\partial\Omega)} + \mu\|M(\pi_-)\|_{L^p(\partial\Omega)} \le C. \tag{7.41}$$

By dilation, it is enough to consider the case when $\vec{a}$ satisfies

$$\operatorname{supp} \vec{a} \subseteq S_1(0), \qquad \|\vec{a}\|_{L^\infty(\partial\Omega)} \le 1, \qquad \text{and} \qquad \int_{\partial\Omega} \vec{a}\, d\sigma = 0. \tag{7.42}$$

To begin, we will need the following auxiliary result.

Lemma 7.7 *Assume Ω is a graph Lipschitz domain in $\mathbb{R}^n$, $n \ge 2$, and let $\vec{a}$ be as in (7.42). Then for $1 < p < \infty$, there exists $C = C(\partial\Omega, p)$ such that*

$$\|M(\mathcal{S}\vec{a})\|_{L^p(\partial\Omega)} \le C. \tag{7.43}$$

Proof. First, notice that there exists $C_0(\partial\Omega, \kappa) > 0$ such that

$$|x - y| \le C_0|z - y|, \quad \forall x, y \in \partial\Omega, \quad z \in \Gamma(x). \tag{7.44}$$

Fix $x = (x', x_n) \in \partial\Omega$ and $z \in \Gamma(x)$. Then from (7.42), we can write

$$\mathcal{S}\vec{a}(z) = \int_{\partial\Omega} E(z - y)\vec{a}(y)\, d\sigma(y) = \int_{S_1(0)} (E(z - y) - E(z))\vec{a}(y)\, d\sigma(y). \tag{7.45}$$

Then

$$|E(z-y)-E(z)| \le C|y||(\nabla E)(z-\theta y)| \le C\frac{|y|}{|z-\theta y|^{n-1}}, \tag{7.46}$$

for some $0<\theta<1$. In particular, if $y \in S_1(0)$ and $x \in \partial\Omega \setminus S_{2C_0}(0)$, then

$$|z-\theta y| \ge |z| - \theta|y| \ge \frac{1}{C_0}|x| - \theta|y| > \frac{1}{2C_0}|x'|, \tag{7.47}$$

and so from (7.45) and (7.46),

$$|\mathcal{S}\vec{a}(z)| \le \frac{C}{|x'|^{n-1}}, \qquad \forall x \in \partial\Omega \setminus S_{2C_0}(0). \tag{7.48}$$

Thus

$$\int_{\partial\Omega\setminus S_{2C_0}(0)} |M(\mathcal{S}\vec{a})|^p \, d\sigma \le C \int_{\mathbb{R}^{n-1}\setminus B_1(0)} \frac{C_0}{|x'|^{(n-1)p}} \, dx' \le C. \tag{7.49}$$

Also if $n \ge 3$, from (7.44),

$$|M(\mathcal{S}\vec{a})(x)| \le C \int_{S_1(0)} \frac{C_0}{|x-y|^{n-2}} \, |\vec{a}(y)| \, d\sigma. \tag{7.50}$$

A similar estimate holds in the case $n=2$ when the term $|x-y|^{-(n-2)}$ is replaced by $1+|\log|x-y||$. In either case, it follows by Schur's Lemma that

$$\int_{S_{2C_0}(0)} |M(\mathcal{S}\vec{a})|^p \, d\sigma \le C \int_{S_{2C_0}(0)} |\vec{a}|^p \, d\sigma \le C, \tag{7.51}$$

which, combined with (7.49), finishes the proof. □

The previous lemma allows us to prove the following useful estimate.

Lemma 7.8 *Retain the same setting as in Proposition 7.5. Let the function $\vec{a}$ be as in (7.42) and $(\vec{u}_\pm, \pi_\pm)$ be as in (7.39). Assume that there exists some $q>1$ with the property that the operator $-\frac{1}{2}\frac{1+\mu}{1-\mu}I + K_\lambda$ is invertible on $L^q(\partial\Omega)$ and the L^q Dirichlet problem is well-posed. Then there exists $C = C(q, \partial\Omega) > 0$ such that*

$$\|M(\vec{u}_\pm)\|_{L^q(\partial\Omega)} \le C. \tag{7.52}$$

Proof. First, since $|S\vec{a}(x)| \le M(\mathcal{S}\vec{a})(x)$ for every $x \in \partial\Omega$, using the previous lemma we have

$$\|S\vec{a}\|_{L^p(\partial\Omega)} \le C(\partial\Omega, p) \quad \text{for} \quad 1 < p < \infty. \tag{7.53}$$

Since $\vec{u}_+|_{\partial\Omega} = \vec{u}_-|_{\partial\Omega}$, multiplying the minus version of (4.143) by μ and adding it to the plus version gives

$$(1-\mu)\left(-\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K_\lambda\right)(\vec{u}_\pm|_{\partial\Omega}) = S\left(\partial_\nu^\lambda(\vec{u}_+, \pi_+) - \mu\,\partial_\nu^\lambda(\vec{u}_-, \pi_-)\right) = S\vec{a}. \tag{7.54}$$

Since $-\frac{1}{2}\frac{1+\mu}{1-\mu}I + K_\lambda$ is an invertible operator on $L^q(\partial\Omega)$, from (7.54) we have

$$\vec{u}_\pm\Big|_{\partial\Omega} = \tfrac{1}{1-\mu}\left(-\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K_\lambda\right)^{-1}(S\vec{a}). \tag{7.55}$$

Then from the well-posedness of the Dirichlet problem, we have

$$\begin{aligned} \|M(\vec{u}_\pm)\|_{L^q(\partial\Omega)} &\le C\|\vec{u}_\pm|_{\partial\Omega}\|_{L^q(\partial\Omega)} \\ &\le C\|(-\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K_\lambda)^{-1}\|_{\mathcal{L}(L^q(\partial\Omega))} \cdot \|S\vec{a}\|_{L^q(\partial\Omega)} \le C, \end{aligned} \tag{7.56}$$

where, for a linear, bounded operator T mapping a quasi-Banach space X into itself, $\|T\|_{\mathcal{L}(X)}$ denotes the operator norm. This finishes the proof of the lemma. $\square$

Next, define the boundary annulus

$$\Lambda_R := \{(x', \varphi(x')) : x' \in \mathbb{R}^{n-1},\, R \le |x'| \le 2R\} \subseteq \partial\Omega. \tag{7.57}$$

For u defined in $\Omega_\pm$, let

$$
\begin{aligned}
M_R^0(u)(x) &:= \sup\{|u(y)| : y \in \Gamma^{\pm}(x),\, |x-y| < R\}, \qquad x \in \partial\Omega,\\
M_R^\infty(u)(x) &:= \sup\{|u(y)| : y \in \Gamma^{\pm}(x),\, |x-y| \geq R\}, \qquad x \in \partial\Omega.
\end{aligned} \tag{7.58}
$$

For any real homogenous constant coefficient elliptic operator L and a function u satisfying $Lu = 0$ in a domain $\mathcal{D} \subset \mathbb{R}^n$, we have the well-known interior estimate

$$
|D^\alpha u(x)| \leq C\delta^{-|\alpha|}(x) \max_{|z-x|<\frac{\delta(x)}{2}} |u(z)|, \tag{7.59}
$$

where $\delta(x) =$ dist$(x, \partial\mathcal{D})$ and α is any multi-index (cf. [73]). Now there exists constants $\eta > 0$ and $\kappa^* > 0$, depending on $\partial\Omega$ and κ such that for any $x \in \partial\Omega$ and $y \in \Gamma^{\pm}(x) \setminus B_R(x)$, it holds that $B_{\eta R}(y) \subset \Gamma^{\pm}_{\kappa^*}(x) \subset \Omega_{\pm}$. Fix $x \in \partial\Omega$ and let $y \in \Gamma^{\pm}(x) \setminus B_R(x)$. Specializing (7.59) to the case when the domain $\mathcal{D} = B_{\eta R}(y)$ gives

$$
|\nabla \vec{u}_{\pm}(y)| \leq \frac{C}{\eta R} \max_{|z-y|<\frac{\eta R}{2}} |\vec{u}_{\pm}(z)|, \tag{7.60}
$$

and then since $B_{\eta R}(y) \subset \Gamma^{\pm}_{\kappa^*}(x)$, it follows that

$$
|\nabla \vec{u}_{\pm}(y)| \leq \frac{C}{\eta R} M^*(\vec{u}_{\pm})(x), \tag{7.61}
$$

where M^* is the non-tangential maximal function associated with the cones $\Gamma^{\pm}_{\kappa^*}(x)$. Taking the supremum over both sides for $y \in \Gamma(x) \setminus B_R(x)$, we see that for any $x \in \partial\Omega$,

$$
M_R^\infty(\nabla \vec{u}_{\pm})(x) \leq \frac{C}{\eta R} M^*(\vec{u}_{\pm})(x). \tag{7.62}
$$

Next, we need a similar estimate for the function π. Fix an $x \in \partial\Omega$ and $y \in \Gamma^{\pm}(x) \setminus B_R(x)$. Let $\omega = \frac{y-x}{|y-x|}$, and then for any t, $|y + t\omega - x| = |y - x| + t$. Since we know the pressure decays at infinity, the Fundamental Theorem of Calculus gives us that

$$
|\pi_{\pm}(y)| \leq \int_0^\infty |(\nabla \pi_{\pm})(y + t\omega)|\, dt = \int_0^\infty |(\Delta \vec{u}_{\pm})(y + t\omega)|\, dt. \tag{7.63}
$$

Now since $y + t\omega \in \Gamma^{\pm}(x) \setminus B_{t+R}(x)$, for the same η and κ^* as before, we have $B_{\eta(t+R)}(y + t\omega) \subset \Gamma^{\pm}_{\kappa^*}(x)$ and using a similar estimate as before gives

$$|(\Delta\vec{u}_\pm)(y+t\omega)| \le \frac{C}{(\eta(t+R))^2} M^*(\vec{u}_\pm)(x). \tag{7.64}$$

Then for any $y \in \Gamma^\pm(x) \setminus B_R(x)$,

$$|\pi_\pm(y)| \le \frac{C}{\eta^2} M^*(\vec{u}_\pm)(x) \int_0^\infty \frac{1}{(t+R)^2}\, dt \le \frac{C}{R} M^*(\vec{u}_\pm)(x). \tag{7.65}$$

Taking the supremum of both sides then gives

$$M_R^\infty(\pi_\pm)(x) \le \frac{C}{R} M^*(\vec{u}_\pm)(x). \tag{7.66}$$

Since $\frac{(n-1)q}{n-1+q} < p \le 1$, we have that $q < \frac{(n-1)p}{n-1-p} \le \frac{n-1}{n-2}$. Define

$$\gamma := \frac{(n-1)p}{q} - (n-1-p) > 0. \tag{7.67}$$

Then using (7.62), (7.66), Hölder's inequality, and Lemma 7.8, we can conclude that

$$\int_{\Lambda_R} M_R^\infty(\nabla\vec{u}_\pm)^p + M_R^\infty(\pi_\pm)^p \le \frac{C}{R^p}\left[\left(\int_{\partial\Omega} M^*(\vec{u}_\pm)^q\, d\sigma\right)^{\frac{1}{q}} \cdot (R^{n-1})^{\frac{1}{p}-\frac{1}{q}}\right]^p \le CR^{-\gamma}. \tag{7.68}$$

We need to prove a similar estimate for $M_R^0(\nabla\vec{u}_\pm)$ and $M_R^0(\pi_\pm)$. The first step will be to establish the following estimate.

Lemma 7.9 *Let $\vec{a}$ be as in (7.42) and $(\vec{u}_\pm, \pi_\pm)$ be as in (7.39). If $S_{2R} \cap S_1(0) = \emptyset$, then*

$$\int_{D_R^+} \left[|\nabla\vec{u}_+|^2 + |\pi_+|^2\right] dx + \mu \int_{D_R^-} \left[|\nabla\vec{u}_-|^2 + |\pi_-|^2\right] dx \le CR^{n-2-\frac{2}{q}(n-1)}. \tag{7.69}$$

Proof. Combining Lemma 6.1 and Lemma 7.8 gives

$$\int_{D_R^\pm} |\pi_\pm|^2\, dx \le C \int_{D_R^\pm} |\nabla\vec{u}_\pm|^2\, dx + CR^{n-2-\frac{2}{q}(n-1)}, \tag{7.70}$$

and so to prove the lemma, it is enough to show that

$$\int_{D_R^+} |\nabla \vec{u}_+|^2\,dx + \mu \int_{D_R^-} |\nabla \vec{u}_-|^2\,dx \le CR^{n-2-\frac{2}{q}(n-1)}. \tag{7.71}$$

From (7.40), it is clear that

$$\langle \partial_\nu^\lambda(\vec{u}_+, \pi_+), \vec{u}_+\rangle - \mu\,\langle \partial_\nu^\lambda(\vec{u}_-, \pi_-), \vec{u}_-\rangle = \langle \vec{a}, \vec{u}_+\rangle = 0 \quad \text{on} \quad S_{2R}, \tag{7.72}$$

and so combining Lemma 6.2 and Lemma 7.8 leads to the estimate

$$\int_{D_{sR}^+} |\nabla \vec{u}_+|^2 + \mu \int_{D_{sR}^-} |\nabla \vec{u}_-|^2 \le \frac{C}{R^2(t-s)^2}\Big[\int_{D_{tR}^+} |\vec{u}_+|^2\,dx + \mu \int_{D_{tR}^-} |\vec{u}_-|^2\,dx\Big] + CR^{n-2-\frac{2}{q}(n-1)} \tag{7.73}$$

for every $1 \le s < t \le 2$. Note that we can assume that

$$R^{n-2-\frac{2}{q}(n-1)} \le \frac{1}{R^2(t-s)^2}\Big[\int_{D_{tR}^+} |\vec{u}_+|^2\,dx + \mu \int_{D_{tR}^-} |\vec{u}_-|^2\,dx\Big] \quad \text{whenever} \;\; 1 \le s < t \le 2, \tag{7.74}$$

otherwise we can prove (7.71) directly by using (7.73). Now, using (7.74) along with Lemma 7.73, we have

$$\int_{D_{sR}^+} |\nabla \vec{u}_+|^2\,dx + \mu \int_{D_{sR}^-} |\nabla \vec{u}_-|^2\,dx \le \frac{2C}{R^2(t-s)^2}\Big[\int_{D_{tR}^+} |\vec{u}_+|^2\,dx + \mu \int_{D_{tR}^-} |\vec{u}_-|^2\,dx\Big]. \tag{7.75}$$

Define

$$\vec{u} := \begin{cases} \vec{u}_+ \text{ in } \Omega_+, \\ \vec{u}_- \text{ in } \Omega_-. \end{cases} \tag{7.76}$$

Then if $\mu \in (0,1)$, we can rewrite (7.75) as

$$\int_{D_{sR}} |\nabla \vec{u}|^2\,dx \le \frac{2C}{\mu R^2(t-s)^2} \int_{D_{tR}} |\vec{u}|^2\,dx, \tag{7.77}$$

and so applying Lemma 6.7 and using Lemma 7.8, we can conclude that

$$\Big(\fint_{D_R} |\vec{u}|^2 \, dx\Big)^{\frac{1}{2}} \le C\Big(\fint_{D_{aR}} |\vec{u}|^q \, dx\Big)^{\frac{1}{q}} \le C\Big(\fint_{S_{aR}} M(\vec{u})^q \, dx\Big)^{\frac{1}{q}} \le CR^{-\frac{1}{q}(n-1)}. \tag{7.78}$$

Combining (7.77) and (7.78) finally gives

$$\int_{D_R^{\pm}} |\nabla \vec{u}_{\pm}|^2 \, dx \le \frac{C}{R^2} \int_{D_{\frac{3}{2}R}} |\vec{u}|^2 \, dx \le CR^{n-2-\frac{2}{q}(n-1)}, \tag{7.79}$$

as desired. The analogous result follows similarly when $\mu = 0$, although in this case, we can apply Lemma 6.7 more directly using (7.75). This finishes the proof of the lemma. □

Now assume $S_{6R} \cap S_1(0) = \emptyset$. Then $\partial_\nu^\lambda(\vec{u}_+, \pi_+) - \mu\, \partial_\nu^\lambda(\vec{u}_-, \pi_-) = 0$ on S_{6R}. Using the well-posedness of the L^2 Regularity problem, we have for each $s \in [1, \frac{3}{2}]$,

$$\begin{aligned}\int_{S_R} \Big[M_R^0(\nabla \vec{u}_+)^2 + \mu\, M_R^0(\nabla \vec{u}_-)^2\Big] \, d\sigma &\le C\Big[\int_{\partial D_{sR}^+} M_{D_{sR}^+}(\nabla \vec{u}_+)^2 \, d\sigma + \mu \int_{\partial D_{sR}^-} M_{D_{sR}^-}(\nabla \vec{u}_-)^2 \, d\sigma\Big] \\ &\le C\Big[\int_{\partial D_{sR}^+} |\nabla_{tan} \vec{u}_+|^2 \, d\sigma + \mu \int_{\partial D_{sR}^-} |\nabla_{tan} \vec{u}_-|^2 \, d\sigma\Big] \\ &\le C\int_{S_{sR}} \Big[|\nabla \vec{u}_+|^2 + \mu |\nabla \vec{u}_-|^2\Big] \, d\sigma \\ &\quad + C\Big[\int_{\partial D_{sR}^+ \setminus \partial\Omega} |\nabla \vec{u}_+|^2 \, d\sigma + \mu \int_{\partial D_{sR}^- \setminus \partial\Omega} |\nabla \vec{u}_-|^2 d\sigma\Big] \end{aligned} \tag{7.80}$$

Integrating (7.80) over $s \in [1, \frac{3}{2}]$ and applying Lemma 6.5 and Lemma 7.9 then gives

$$\begin{aligned}&\int_{S_R} \Big[M_R^0(\nabla \vec{u}_+)^2 + \mu M_R^0(\nabla \vec{u}_-)^2\Big] \, d\sigma \\ &\quad \le \frac{C}{R}\Big[\int_{D_{3R}^+} (|\nabla \vec{u}_+|^2 + |\pi_+|^2) \, d\sigma + \mu \int_{D_{3R}^-} (|\nabla \vec{u}_-|^2 + |\pi_-|^2) \, dx\Big] \\ &\quad \le CR^{(n-3)-\frac{2}{q}(n-1)}. \end{aligned} \tag{7.81}$$

After covering Λ_R with a finite number of appropriate surface balls, we can then conclude that

$$\int\limits_{\Lambda_R} [M_R^0(\nabla\vec{u}_+)^p + \mu M_R^0(\nabla\vec{u}_-)^p]\, d\sigma$$

$$\leq CR^{(n-1)(1-\frac{p}{2})}\Big[\Big(\int\limits_{\Lambda_R} M_R^0(\nabla\vec{u}_+)^2\, d\sigma\Big)^{\frac{p}{2}} + \mu\Big(\int\limits_{\Lambda_R} M_R^0(\nabla\vec{u}_-)^2\, d\sigma\Big)^{\frac{p}{2}}\Big]$$

$$\leq CR^{(n-1-p)-\frac{p}{q}(n-1)} = CR^{-\gamma}. \tag{7.82}$$

Analogous estimates for $M_R^0(\pi_\pm)$ follow via a similar argument. These estimates along with (7.68) then guarantee that

$$\int\limits_{\Lambda_R} \Big[M(\nabla\vec{u}_+)^p + M(\pi_+)^p + \mu M(\nabla\vec{u}_-)^p + \mu M(\pi_-)^p\Big]\, d\sigma \leq CR^{-\gamma}. \tag{7.83}$$

Finally, using (7.83) along with the L^2 theory leads to the estimate,

$$\int\limits_{\partial\Omega} \Big[M(\nabla\vec{u}_+)^p + M(\pi_+)^p + \mu M(\nabla\vec{u}_-)^p + \mu M(\pi_-)^p\Big]\, d\sigma$$

$$\leq \int\limits_{S_8(0)} \Big[M(\nabla\vec{u}_+)^p + M(\pi_+)^p + \mu M(\nabla\vec{u}_-)^p + \mu M(\pi_-)^p\Big]\, d\sigma$$

$$+\sum_{j=3}^{\infty} \int\limits_{\Lambda_{2^j}} \Big[M(\nabla\vec{u}_+)^p + M(\pi_+)^p + \mu M(\nabla\vec{u}_-)^p + \mu M(\pi_-)^p\Big]\, d\sigma$$

$$\leq C\Big[\Big(\int\limits_{\partial\Omega} M(\nabla\vec{u}_+)^2\, d\sigma\Big)^{\frac{p}{2}} + \Big(\int\limits_{\partial\Omega} M(\pi_+)^2\, d\sigma\Big)^{\frac{p}{2}}\Big]$$

$$+\mu C\Big[\Big(\int\limits_{\partial\Omega} M(\nabla\vec{u}_-)^2\, d\sigma\Big)^{\frac{p}{2}} + \Big(\int\limits_{\partial\Omega} M(\pi_-)^2\, d\sigma\Big)^{\frac{p}{2}}\Big] + C\sum_{j=3}^{\infty}(2^j)^{-\gamma}$$

$$\leq C\Big(\int\limits_{\partial\Omega} |\vec{a}|^2\, d\sigma\Big)^{\frac{p}{2}} + C\sum_{j=3}^{\infty} 2^{-j\gamma} \leq C, \tag{7.84}$$

which proves (7.41). With this in mind, we can finish the

Proof of Proposition 7.5. For any $\vec{f} \in H^p_{at}(\partial\Omega)$, we can write $\vec{f} = \sum_{j=1}^{\infty} \lambda_j \, \vec{a}_j$ such that each $\vec{a}_j$ is a p-atom and $\left(\sum_{j=1}^{\infty} |\lambda_j|^p\right)^{\frac{1}{p}} \leq 2\|\vec{f}\|_{H^p_{at}(\partial\Omega)}$. For each $\vec{a}_j$ we can find $\vec{u}^j_\pm$ and $\pi^j_\pm$ that solve (7.40) with datum $\vec{a}_j$ and also satisfy (7.41). Then the functions $\vec{u}_\pm := \sum_{j=1}^{\infty} \lambda_j \, \vec{u}^j_\pm$ and $\pi_\pm := \sum_{j=1}^{\infty} \lambda_j \pi^j_\pm$ will satisfy

$$\begin{cases} \Delta \vec{u}_\pm = \nabla \pi_\pm \ \text{ in } \ \Omega_\pm, \\ \operatorname{div} \vec{u}_\pm = 0 \ \text{ in } \ \Omega_\pm, \\ \vec{u}_+ \Big|_{\partial\Omega} = \vec{u}_- \Big|_{\partial\Omega}, \\ \partial^\lambda_\nu(\vec{u}_+, \pi_+) - \mu \, \partial^\lambda_\nu(\vec{u}_-, \pi_-) = \vec{f} \ \text{ on } \ \partial\Omega, \\ \|M(\nabla \vec{u}_+)\|_{L^p(\partial\Omega)} + \|M(\pi_+)\|_{L^p(\partial\Omega)} \\ \quad + \mu\|M(\nabla \vec{u}_-)\|_{L^p(\partial\Omega)} + \mu\|M(\pi_-)\|_{L^p(\partial\Omega)} \leq C\|f\|_{H^p_{at}(\partial\Omega)}. \end{cases} \tag{7.85}$$

Since we have reduced matters to the case when $\vec{g} = 0$, Proposition 7.5 follows. ☐

Next, Proposition 7.6 can be established in a similar fashion. Here, we can reduce matters to considering the case when $\vec{f} = 0$ and $\vec{g} = \vec{a}$ where $\vec{a}$ is a *regular* (p, ∞)-atom satisfying

$$\operatorname{supp} a \subseteq S_1(0), \qquad \vec{a}(0) = 0, \qquad \|\nabla_{tan} a\|_{L^\infty(\partial\Omega)} \leq 1. \tag{7.86}$$

We need to prove that there exists a solution that satisfies (7.41). Now since $\vec{a} \in \dot{L}^2_1(\partial\Omega)$, we can define

$$\begin{aligned} \vec{u}_\pm &:= \tfrac{1}{1-\mu} \, \mathcal{D}_\lambda\Big((-\tfrac{1}{2}\tfrac{1+\mu}{1-\mu} I + K_\lambda)^{-1} \vec{a}\Big) \quad \text{in } \ \Omega_\pm, \\ \pi_\pm &:= \tfrac{1}{1-\mu} \, \mathcal{P}_\lambda\Big((-\tfrac{1}{2}\tfrac{1+\mu}{1-\mu} I + K_\lambda)^{-1} \vec{a}\Big) \quad \text{in } \ \Omega_\pm. \end{aligned} \tag{7.87}$$

By Proposition 4.5, (4.29), (4.47) and (4.45), the functions $\vec{u}_\pm$, $\pi_\pm$ will satisfy

$$\begin{cases} \Delta \vec{u}_{\pm} = \nabla \pi_{\pm} \quad \text{in} \quad \Omega_{\pm}, \\ \operatorname{div} \vec{u}_{\pm} = 0 \quad \text{in} \quad \Omega_{\pm}, \\ \vec{u}_{+}\Big|_{\partial\Omega} - \mu\, \vec{u}_{-}\Big|_{\partial\Omega} = \vec{a}, \\ \partial_{\nu}^{\lambda}(\vec{u}_{+}, \pi_{+}) = \partial_{\nu}^{\lambda}(\vec{u}_{-}, \pi_{-}) \quad \text{on} \quad \partial\Omega, \\ \|M(\nabla \vec{u}_{\pm})\|_{L^2(\partial\Omega)} + \|M(\pi_{\pm})\|_{L^2(\partial\Omega)} \leq C\|\vec{a}\|_{\dot{L}^2_1(\partial\Omega)}. \end{cases} \tag{7.88}$$

Since we also have $\vec{a} \in L^q(\partial\Omega)$, it follows from Proposition 4.5 that

$$\|M(\vec{u})\|_{L^q(\partial\Omega)} \leq \|\vec{a}\|_{L^q(\partial\Omega)} \leq C, \tag{7.89}$$

which we will use in place of Lemma 7.8. We can also replace (7.72) with

$$\langle \partial_{\nu}^{\lambda}(\vec{u}_{+}, \pi_{+}), \vec{u}_{+}\rangle - \mu\, \langle \partial_{\nu}^{\lambda}(\vec{u}_{-}, \pi_{-}), \vec{u}_{-}\rangle = \langle \partial_{\nu}^{\lambda}(\vec{u}_{+}, \pi_{+}), \vec{a}\rangle = 0 \quad \text{on} \quad S_{2R}. \tag{7.90}$$

The rest of the proof of (7.41) follows as before except this time, we use Lemma 6.6 in place of Lemma 6.5 to establish (7.81) from (7.80). This is enough to establish Proposition 7.6. We can now prove the following result regarding $p < 1$. Before stating it, recall (1.3), (4.155), (4.156) and the conventions made at the beginning of § 7.

Lemma 7.10 *Let $n = 2$ or 3, and let $\Omega \subset \mathbb{R}^n$ be a graph Lipschitz domain. Also, set $\Omega_+ := \Omega$, $\Omega_- := \mathbb{R}^n \setminus \bar{\Omega}$ and fix $\lambda \in (-1, 1]$ along with $\mu \in [0, 1)$. Then there exists $\varepsilon > 0$ such that the boundary value problems (T_μ), $(T_\mu)^*$, (N), and (R) are well-posed for every $\frac{2(n-1)}{n+1} - \varepsilon < p < 1$.*

Proof. For $\mu \in (0, 1)$, the well-posedness of (T_μ) and $(T_\mu)^*$ follows by choosing q sufficiently close to 2 and applying either Proposition 7.5 or Proposition 7.6 followed by either Theorem 7.1 or Lemma 7.4. In the case $\mu = 0$, the same argument proves that (T_o) and $(T_o)^*$ are semi-well-posed, and since this will also hold when the roles of Ω_+ and Ω_- are reversed, we can conclude from Proposition 4.21 that (T_o), $(T_o)^*$, (N), and (R) are also well-posed. $\square$

7.3 Interpolation arguments

Throughout this section, assume that $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, is a graph Lipschitz domain, and set $\Omega_+ := \Omega$, $\Omega_- := \mathbb{R}^n \setminus \bar{\Omega}$. Recall from Lemma 5.7 that the operators

$$\Big(\pm\tfrac{1}{2}\tfrac{\mu+1}{\mu-1}I + K_\lambda^*\Big)^{-1} : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \tag{7.91}$$

are well-defined, linear, and bounded for each $\mu \in [0, 1)$, whenever $2-\varepsilon < p < 2+\varepsilon$. Let us denote by $T_\pm$ the version of (7.91) corresponding to $p = 2$. We aim to show that whenever $\frac{2(n-1)}{n+1} - \varepsilon < p < 1$, there exists $C = C(\Omega, \mu, p) > 0$ such that

$$\|T_\pm \vec{a}\|_{H^p_{at}(\partial\Omega)} \leq C, \qquad \forall\, \vec{a}\ H^p_{at}(\partial\Omega) - \text{atom}. \tag{7.92}$$

Consider the case of T_+ (the claim about T_- is handled similarly) and fix an $H^p_{at}(\partial\Omega)$-atom $\vec{a}$. From the arguments in § 7.2, we know the functions

$$\vec{u}_\pm := \tfrac{1}{1-\mu}\mathcal{S}(T_+\vec{a}) \text{ in } \Omega_\pm \quad \text{ and } \quad \pi_\pm := \tfrac{1}{1-\mu}\mathcal{Q}(T_+\vec{a}) \text{ in } \Omega_\pm \tag{7.93}$$

solve $(T_\mu)^*$ with data $(0, \vec{a})$ and satisfy the estimate

$$\|M(\nabla\vec{u}_+)\|_{L^p(\partial\Omega)} + \|M(\nabla\pi_+)\|_{L^p(\partial\Omega)} + \mu\|M(\nabla\vec{u}_-)\|_{L^p(\partial\Omega)} + \mu\|M(\nabla\pi_-)\|_{L^p(\partial\Omega)} \leq C \tag{7.94}$$

where C is independent of $\vec{a}$. From the well-posedness of the Regularity problem, we also have

$$\begin{aligned}&\|M(\nabla\vec{u}_-)\|_{L^p(\partial\Omega)} + \|M(\nabla\pi_-)\|_{L^p(\partial\Omega)}\\ &\qquad \leq C\|\vec{u}_-\|_{H^{1,p}_{at}(\partial\Omega)} = C\|\vec{u}_+\|_{H^{1,p}_{at}(\partial\Omega)} \leq C\|M(\nabla\vec{u}_+)\|_{L^p(\partial\Omega)},\end{aligned} \tag{7.95}$$

and so (7.94) can be improved to

$$\|M(\nabla\vec{u}_+)\|_{L^p(\partial\Omega)} + \|M(\nabla\pi_+)\|_{L^p(\partial\Omega)} + \|M(\nabla\vec{u}_-)\|_{L^p(\partial\Omega)} + \|M(\nabla\pi_-)\|_{L^p(\partial\Omega)} \leq C \tag{7.96}$$

Thus,

$$\begin{aligned}
\|T_+\vec{a}\|_{H^p_{at}(\partial\Omega)} &= \|\partial^\lambda_\nu(\vec{u}_+,\pi_+) - \partial^\lambda_\nu(\vec{u}_-,\pi_-)\|_{H^p_{at}(\partial\Omega)} \\
&\leq \|M(\nabla\vec{u}_+)\|_{L^p(\partial\Omega)} + \|M(\nabla\pi_+)\|_{L^p(\partial\Omega)} \\
&\quad + \|M(\nabla\vec{u}_-)\|_{L^p(\partial\Omega)} + \|M(\nabla\pi_-)\|_{L^p(\partial\Omega)} \\
&\leq C, \qquad (7.97)
\end{aligned}$$

by jump-relations, Theorem 4.13, and (7.96).

Our next claim is that if $\vec{f} \in H^p_{at}(\partial\Omega) \cap L^2(\partial\Omega)$ then $T_\pm\vec{f} \in L^2(\partial\Omega)$ satisfies

$$\|T_\pm\vec{f}\|_{H^p_{at}(\partial\Omega)} \leq C\|\vec{f}\|_{H^p_{at}(\partial\Omega)}, \qquad (7.98)$$

where $C > 0$ is independent of $\vec{f}$. To see this, we shall invoke an observation made in (6.5) on p. 948 of [74], which we state here in a slightly more general form than we need in the current context. Specifically, if $\frac{n-1}{n} < p \leq 1$ and $\vec{f} \in H^p_{at}(\partial\Omega) \cap L^2(\partial\Omega)$, there exist a sequence of coefficients $(\lambda_j)_j \in \ell^1$ and a sequence of $H^p_{at}(\partial\Omega)$-atoms $\vec{a}_j$, such that

$$\begin{gathered}
\vec{f} = \textstyle\sum_{j=1}^\infty \lambda_j\,\vec{a}_j \text{ in } H^p_{at}(\partial\Omega), \quad \sum_{j=1}^\infty |\lambda_j| \leq C\|\vec{f}\|_{H^p_{at}(\partial\Omega)}, \text{ and} \\
\vec{f}_N := \textstyle\sum_{j=1}^N \lambda_j\,\vec{a}_j \text{ converges to } \vec{f} \text{ in } L^2(\partial\Omega) \text{ as } N \to \infty.
\end{gathered} \qquad (7.99)$$

Now if we consider such a decomposition of $\vec{f}$, on the one hand, $T_\pm\vec{f}_N$ is Cauchy in $H^p_{at}(\partial\Omega)$, hence convergent in $H^p_{at}(\partial\Omega)$ to some $\vec{g}_\pm$ for which $\|\vec{g}_\pm\|_{H^p_{at}(\partial\Omega)} \leq C\|\vec{f}\|_{H^p_{at}(\partial\Omega)}$, thanks to (7.92). On the other hand, $T_\pm\vec{f}_N \to T_\pm\vec{f}$ in $L^2(\partial\Omega)$. Consequently, for any vector-valued function $\vec{\psi} \in \mathrm{Lip}\,(\partial\Omega)$ with compact support,

$$\int_{\partial\Omega} \vec{\psi}\cdot T_\pm\vec{f}\,d\sigma = \lim_{N\to\infty}\int_{\partial\Omega} \vec{\psi}\cdot T_\pm\vec{f}_N\,d\sigma = \langle\vec{\psi},\vec{g}_\pm\rangle, \qquad (7.100)$$

where $\langle\cdot,\cdot\rangle$ stands for the distributional paring on $\partial\Omega$ (i.e., the pairing between $\mathrm{Lip}_{comp}(\partial\Omega)$ and its topological dual). This proves that $T_\pm\vec{f} = \vec{g}_\pm$, from which the estimate (7.98) follows. This establishes that

$$\left(\pm\tfrac{1}{2}\tfrac{\mu+1}{\mu-1}I + K_\lambda^*\right)^{-1} : H^p_{at}(\partial\Omega) \longrightarrow H^p_{at}(\partial\Omega) \tag{7.101}$$

are well-defined, linear, and bounded whenever $\frac{2(n-1)}{n+1} - \varepsilon < p < 1$, and further, by interpolating (7.101) with (7.91), that

$$\left(\pm\tfrac{1}{2}\tfrac{\mu+1}{\mu-1}I + K_\lambda^*\right)^{-1} : H^p(\partial\Omega) \longrightarrow H^p(\partial\Omega) \tag{7.102}$$

are well-defined, linear, and bounded whenever $\frac{2(n-1)}{n+1} - \varepsilon < p < 2+\varepsilon$.

In summary, the above reasoning shows that for $\mu \in [0,1)$,

$$\pm\tfrac{1}{2}\tfrac{\mu+1}{\mu-1}I + K_\lambda^* : H^p(\partial\Omega) \longrightarrow H^p(\partial\Omega) \text{ isomorphically, for } \tfrac{2(n-1)}{n+1} - \varepsilon < p < 2+\varepsilon. \tag{7.103}$$

With (7.103) in hand, we can prove the following theorem.

Theorem 7.11 *Let $n = 2$ or 3 and $\Omega \subset \mathbb{R}^n$ be a graph Lipschitz domain. As usual, set $\Omega_+ := \Omega$, $\Omega_- := \mathbb{R}^n \setminus \bar{\Omega}$. Then there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ such that for $\lambda \in (-1,1]$, $\mu \in [0,1)$, and $\frac{2(n-1)}{n+1} - \varepsilon < p < 2+\varepsilon$, the boundary value problems (T_μ), $(T_\mu)^*$ in (4.155)-(4.156) as well as (N) and (R) in (1.3) are well-posed.*

Proof. The well-posedness of (T_μ) and $(T_\mu)^*$ follows from (7.103), Theorem 5.9, and Theorem 4.19. Since this result will also hold if the roles of Ω_+ and Ω_- are reversed, the well-posedness of (N) and (R) follow from Proposition 4.21. $\square$

8 Higher dimensions

In this chapter, we adapt the arguments of Z. Shen from [83] and [84] in order to extend our results to the case when $n \geq 4$. Specifically, our goal is to prove the following theorem.

Theorem 8.1 *Assume that $\Omega \subseteq \mathbb{R}^n$, $n \geq 4$, is a graph Lipschitz domain and set $\Omega_+ := \Omega$, $\Omega_- := \mathbb{R}^n \setminus \bar{\Omega}$. Then there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ such that the transmission problems $(T_\mu^\pm)$ and $(T_\mu^\pm)^*$ from (4.155)-(4.156) are well-posed for any $\mu \in [0,1)$ and any $\frac{2(n-1)}{n+1} - \varepsilon < p < 2+\varepsilon$. Moreover, the Neumann problem (N) and the Regularity problem (R) in (1.3) are well-posed for $\frac{2(n-1)}{n+1} - \varepsilon < p < 2+\varepsilon$.*

To accomplish this, we will consider the following auxiliary problem,

$$
(T^*)\quad \begin{cases} \Delta \vec{u}_\pm = \nabla \pi_\pm \quad \text{in} \quad \Omega_\pm, \\ \operatorname{div} \vec{u}_\pm = 0 \quad \text{in} \quad \Omega_\pm, \\ \vec{u}_+\Big|_{\partial\Omega} - \mu\, \vec{u}_-\Big|_{\partial\Omega} = \vec{g} \in L^p(\partial\Omega), \\ \partial_\nu^\lambda(\vec{u}_+, \pi_+) = \partial_\nu^\lambda(\vec{u}_-, \pi_-), \\ M(\vec{u}_\pm) \in L^p(\partial\Omega). \end{cases} \tag{8.1}
$$

Above, the equality $\partial_\nu^\lambda(\vec{u}_+, \pi_+) = \partial_\nu^\lambda(\vec{u}_-, \pi_-)$ has to be (suitably) understood in $L^p_{-1}(\partial\Omega)$, when p is near 2. Since the operator $\frac{1}{2}\frac{1+\mu}{1-\mu}I + K_\lambda$ is invertible on $L^p(\partial\Omega)$ for p near 2, we can show that the functions

$$
\vec{u}_\pm := \mathcal{D}_\lambda\Big((\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K_\lambda)^{-1}\vec{g}\Big) \quad \text{and} \quad \pi_\pm := \mathcal{P}_\lambda\Big((\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K_\lambda)^{-1}\vec{g}\Big) \quad \text{in} \ \ \Omega_\pm \tag{8.2}
$$

solve (8.1) and also satisfy the estimate

$$
\|M(\vec{u}_\pm)\|_{L^p(\partial\Omega)} \leq C\|\vec{g}\|_{L^p(\partial\Omega)}, \tag{8.3}
$$

as long as p is near 2. In this chapter, we will extend this result to include $2 - \varepsilon < p < \frac{2(n-1)}{n-3} + \varepsilon$. A key step is to prove the following Reverse Hölder estimate for the non-tangential maximal operator.

Lemma 8.2 [`Reverse Hölder estimates`]
Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 4$, be a graph Lipschitz domain. As usual, set $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^n \setminus \bar{\Omega}$. Assume $\Delta\vec{u}_\pm = \nabla\pi_\pm$, $\operatorname{div}\vec{u}_\pm = 0$ in $\Omega_\pm$, and define $M(\vec{u}) := \max\{M(\vec{u}_+), M(\vec{u}_-)\}$ and $p_n := \frac{2(n-1)}{n-3}$. If $M(\nabla\vec{u}_\pm), M(\pi_\pm) \in L^2(\partial\Omega)$ and $\vec{u}_+ - \mu\,\vec{u}_- = 0$ on S_{128R} for $\mu \in [0, 1)$, then

$$
\begin{aligned}
\left(\fint_{S_R} M(\vec{u})^{p_n}\, d\sigma\right)^{\frac{1}{p_n}} \quad &\leq \quad C\left(\fint_{S_{256R}} M(\vec{u})^2\, d\sigma\right)^{\frac{1}{2}} \\
&\quad + CR\left(\fint_{S_{256R}} |\partial_\nu^\lambda(\vec{u}_+, \pi_+) - \partial_\nu^\lambda(\vec{u}_-, \pi_-)|^2\, d\sigma\right)^{\frac{1}{2}}.
\end{aligned} \tag{8.4}
$$

The Proof of Lemma 8.2 is going to be presented in the next section.

8.1 Preliminary estimates

Recall the definitions of S_R and $D_R^\pm$ from (6.1)-(6.3). We will start with the following result.

Lemma 8.3 *If $\Delta\vec{u}_\pm = \nabla\pi_\pm$, $\operatorname{div}\vec{u}_\pm = 0$ in $\Omega_\pm$ and $M(\nabla\vec{u}_\pm)$, $M(\pi_\pm) \in L^2(\partial\Omega)$, then*

$$\int_{D_R^+} |\nabla\vec{u}_+|^2\,dx + \mu\int_{D_R^-} |\nabla\vec{u}_-|^2\,dx$$
$$\leq \frac{C}{R}\int_{S_{2R}} \left[M(\vec{u}_+)^2 + \mu M(\vec{u}_-)^2\right]\,d\sigma + CR\int_{S_{2R}} \mu|\partial_\nu^\lambda(\vec{u}_+,\pi_+) - \partial_\nu^\lambda(\vec{u}_-,\pi_-)|^2\,d\sigma$$
$$+C\int_{S_{2R}} |\partial_\nu^\lambda(\vec{u}_+,\pi_+)||\vec{u}_+ - \mu\vec{u}_-|\,d\sigma. \tag{8.5}$$

Proof. From Cauchy's inequality, we have that

$$\int_{S_{2R}} \left|\langle\partial_\nu^\lambda(\vec{u}_+,\pi_+),\vec{u}_+\rangle - \mu\,\langle\partial_\nu^\lambda(\vec{u}_-,\pi_-),\vec{u}_-\rangle\right|\,d\sigma$$
$$= \int_{S_{2R}} \left|\langle\partial_\nu^\lambda(\vec{u}_+,\pi_+),\vec{u}_+ - \mu\vec{u}_-\rangle + \mu\,\langle\partial_\nu^\lambda(\vec{u}_+,\pi_+) - \partial_\nu^\lambda(\vec{u}_-,\pi_-),\vec{u}_-\rangle\right|\,d\sigma \tag{8.6}$$
$$\leq \int_{S_{2R}} \Big(|\partial_\nu^\lambda(\vec{u}_+,\pi_+)||\vec{u}_+ - \mu\vec{u}_-| + \mu\,R\,|\partial_\nu^\lambda(\vec{u}_+,\pi_+) - \partial_\nu^\lambda(\vec{u}_-,\pi_-)|^2 + \frac{\mu}{4R}M(\vec{u}_-)^2\Big)\,d\sigma.$$

Utilizing (8.6) in Lemma 6.2 along with the estimate

$$\int_{D_R^\pm} |\vec{u}_\pm|^2\,dx \leq CR\int_{S_R} M(\vec{u}_\pm)^2\,d\sigma \tag{8.7}$$

is enough to verify (8.5). □

Let $M_{D_R^\pm}$ denote the non-tangential maximal functions associated with the bounded domains $D_R^\pm$. Consider the following lemma.

Lemma 8.4 *Assume $\Delta\vec{u}_\pm = \nabla\pi_\pm$, $\operatorname{div}\vec{u}_\pm = 0$ in $\Omega_\pm$. If $M(\vec{u}_\pm)$, $M(\nabla\vec{u}_\pm)$ belong to $L^2(\partial\Omega)$ and $\vec{u}_+ - \mu\vec{u}_- = 0$ on S_{8R}, then*

$$\int_{S_R} (M_{D_R^+}(\nabla \vec{u}_+)^2 + M_{D_R^+}(\pi_+)^2)\, d\sigma + \mu \int_{S_R} (M_{D_R^-}(\nabla \vec{u}_-)^2 + M_{D_R^-}(\pi_-)^2)\, d\sigma \tag{8.8}$$
$$\leq C \int_{S_{8R}} \mu |\partial_\nu^\lambda(\vec{u}_+, \pi_+) - \partial_\nu^\lambda(\vec{u}_-, \pi_-)|^2\, d\sigma + \frac{C}{R^2} \int_{S_{8R}} \left(M(\vec{u}_+)^2 + \mu M(\vec{u}_-)^2\right)\, d\sigma.$$

Proof. Using the well-posedness of the L^2 Regularity problem on bounded domains, it follows that for $s \geq 1$,

$$\int_{S_R} (M_{D_R^\pm}(\nabla \vec{u}_\pm)^2 + M_{D_R^\pm}(\pi_\pm)^2)\, d\sigma \leq C \int_{S_{sR}} |\nabla_{tan} \vec{u}_\pm|^2\, d\sigma + C \int_{\partial D_{sR}^\pm \cap \Omega_\pm} |\nabla_{tan} \vec{u}_\pm|^2\, d\sigma. \tag{8.9}$$

Integrating (8.9) over $s \in [1, 2]$ gives

$$\int_{S_R} (M_{D_R^\pm}(\nabla \vec{u}_\pm)^2 + M_{D_R^\pm}(\pi_\pm)^2)\, d\sigma \leq C \int_{S_{2R}} |\nabla \vec{u}_\pm|^2\, d\sigma + \frac{C}{R} \int_{D_{2R}^\pm} |\nabla \vec{u}_\pm|^2\, d\sigma. \tag{8.10}$$

Applying Lemma 6.6 and Lemma 6.1 and using the assumption that $\vec{u}_+ - \mu \vec{u}_- = 0$ on S_{8R} leads to the estimate

$$\int_{S_R} (M_{D_R^+}(\nabla \vec{u}_+)^2 + M_{D_R^+}(\pi_+)^2)\, d\sigma + \mu \int_{S_R} (M_{D_R^-}(\nabla \vec{u}_-)^2 + M_{D_R^-}(\pi_-)^2)\, d\sigma$$
$$\leq \frac{C}{R} \left[\int_{D_{4R}^+} |\nabla \vec{u}_+|^2\, d\sigma + \mu \int_{D_{4R}^-} |\nabla \vec{u}_-|^2\, d\sigma \right] + C\mu \int_{S_{4R}} |\partial_\nu^\lambda(\vec{u}_+, \pi_+) - \partial_\nu^\lambda(\vec{u}_-, \pi_-)|^2\, d\sigma$$
$$+ \frac{C}{R^2} \int_{S_{4R}} \left(M(\vec{u}_+)^2 + \mu M(\vec{u}_-)^2\right)\, d\sigma. \tag{8.11}$$

Then applying Lemma 8.3 and using the fact that $\vec{u}_+ - \mu \vec{u}_- = 0$ on S_{8R} gives

$$\int_{S_R} (M_{D_R^+}(\nabla \vec{u}_+)^2 + M_{D_R^+}(\pi_+)^2)\, d\sigma + \mu \int_{S_R} (M_{D_R^-}(\nabla \vec{u}_-)^2 + M_{D_R^-}(\pi_-)^2)\, d\sigma \tag{8.12}$$
$$\leq C\mu \int_{S_{8R}} |\partial_\nu^\lambda(\vec{u}_+, \pi_+) - \partial_\nu^\lambda(\vec{u}_-, \pi_-)|^2\, d\sigma + \frac{C}{R^2} \int_{S_{8R}} \left(M(\vec{u}_+)^2 + \mu M(\vec{u}_-)^2\right)\, d\sigma,$$

which finishes the proof. $\square$

At this point, we can proceed with the

Proof of Lemma 8.2. Let $x \in S_R$ and $y \in \Gamma_{\pm}(x)$ be such that $|y-x| > cR$. Then interior estimates yield

$$|\vec{u}_{\pm}(y)| \leq C \fint_{B_{cR}(y)} |\vec{u}_{\pm}|\, dz \leq C \fint_{S_{2R}} M(\vec{u}_{\pm})\, d\sigma. \tag{8.13}$$

From (8.13), it follows that for any $p > 0$,

$$\begin{aligned} \left(\fint_{S_R} M_R^{\infty}(\vec{u}_{\pm})^p \, d\sigma \right)^{\frac{1}{p}} &\leq \sup_{x \in S_R} M_R^{\infty}(\vec{u}_{\pm})(x) \\ &\leq C \fint_{S_{2R}} M(\vec{u}_{\pm})\, d\sigma \leq C \left(\fint_{S_{2R}} M(\vec{u}_{\pm})^2 \, d\sigma \right)^{\frac{1}{2}}. \end{aligned} \tag{8.14}$$

Then to prove the lemma, it is enough to show that

$$\begin{aligned} \left(\fint_{S_R} M_R^{0}(\vec{u}_{\pm})^{p_n} \, d\sigma \right)^{\frac{1}{p_n}} \leq\ & CR \left(\fint_{S_{128R}} |\partial_\nu^\lambda(\vec{u}_+, \pi_+) - \partial_\nu^\lambda(\vec{u}_-, \pi_-)|^2 \, d\sigma \right)^{\frac{1}{2}} \\ &+ C \left(\fint_{S_{128R}} M(\vec{u})^2 \, d\sigma \right)^{\frac{1}{2}}. \end{aligned} \tag{8.15}$$

Next, we claim that for $x \in S_R$,

$$M_R^0(\vec{u}_{\pm})(x) \leq C \int_{S_{2R}} \frac{M_{D_{2R}^{\pm}}(\nabla \vec{u}_{\pm})(z)}{|x-z|^{n-2}} \, d\sigma(z) + C \fint_{S_{2R}} M(\vec{u}_{\pm})\, d\sigma. \tag{8.16}$$

Let $y \in \Gamma_+(x)$ such that $|y-x| < cR$. Let $\omega := \frac{y-x}{|y-x|}$, and $y' = y + cR\omega$. Then $y' \in \Gamma_+(x)$ and $cR < |y'-x| < 2cR$, and

$$|\vec{u}_+(y') - \vec{u}_+(y)| = \left| \int_0^{cR} \frac{d}{dt} [\vec{u}_+(y+t\omega)]\, dt \right| \leq \int_0^{cR} |\nabla \vec{u}_+(y+t\omega)|\, dt. \tag{8.17}$$

From interior estimates, for $0 < t < cR$,

$$|\nabla \vec{u}_+(y+t\omega)| \quad \leq \quad C \fint_{B_{ct}(y+t\omega)} |\nabla \vec{u}_+(z)|\, dz \quad \leq \quad C \fint_{S_{ct}(x)} M_{D_{2R}^+}(\nabla \vec{u}_+)(z)\, d\sigma(z). \tag{8.18}$$

Then combining (8.17) and (8.18), and using Fubini's theorem yields

$$
\begin{aligned}
|\vec{u}_+(y') - \vec{u}_+(y)| &\leq C \int_0^{cR} \int\limits_{S_{ct}(x)} t^{-(n-1)} M_{D^+_{2R}}(\nabla \vec{u}_+)(z)\, d\sigma(z)\, dt \\
&\leq C \int\limits_{S_{2R}(x)} \int_{c|x-z|}^{\infty} t^{-(n-1)} M_{D^+_{2R}}(\nabla \vec{u}_+)(z)\, dt\, d\sigma(z) \\
&\leq C \int_{S_{2R}(x)} \frac{M_{D^+_{2R}}(\nabla \vec{u}_+)(z)}{|x-z|^{n-2}}\, d\sigma(z). \qquad (8.19)
\end{aligned}
$$

Then using (8.19) and (8.13) for y' gives

$$
\begin{aligned}
|\vec{u}_+(y)| &\leq |\vec{u}_+(y') - \vec{u}_+(y)| + |\vec{u}_+(y')| \\
&\leq C \int\limits_{S_{2R}} \frac{M_{D^+_{2R}}(\nabla \vec{u}_+)(z)}{|x-z|^{n-2}}\, d\sigma(z) + C ⨍_{S_{2R}} M(\vec{u}_+)\, d\sigma. \qquad (8.20)
\end{aligned}
$$

Taking the supremum over y proves the plus version of (8.16). The minus version follows similarly. Multiplying the minus version of (8.16) by $\mu^{1/2}$ and adding it to the plus version gives

$$
\begin{aligned}
M^0_R(\vec{u}_+)(x) + \mu^{1/2} M^0_R(\vec{u}_-)(x) &\leq C \int\limits_{S_{2R}} \frac{M_{D^+_{2R}}(\nabla \vec{u}_+)(z) + \mu^{1/2} M_{D^-_{2R}}(\nabla \vec{u}_-)(z)}{|x-z|^{n-2}}\, d\sigma(z) \\
&\quad + C ⨍\limits_{S_{2R}} \Big(M(\vec{u}_+) + \mu^{1/2} M(\vec{u}_-) \Big)\, d\sigma. \qquad (8.21)
\end{aligned}
$$

Then by Fractional Integration Theorem, it follows that

$$
\begin{aligned}
&\left(⨍_{S_R} \Big(M^0_R(\vec{u}_+) + \mu^{1/2} M^0_R(\vec{u}_-) \Big)^{p_n}\, d\sigma \right)^{\frac{1}{p_n}} \\
&\leq CR \left(⨍_{S_{2R}} \Big(M_{D^+_{2R}}(\nabla \vec{u}_+) + \mu^{1/2} M_{D^-_{2R}}(\nabla \vec{u}_-) \Big)^2\, d\sigma \right)^{\frac{1}{2}} \\
&+ C ⨍_{S_{2R}} \Big(M(\vec{u}_+) + \mu^{1/2} M(\vec{u}_-) \Big)\, d\sigma
\end{aligned}
$$

$$\leq CR\left(\fint_{S_{2R}} \left(M_{D^+_{2R}}(\nabla \vec{u}_+)^2 + \mu M_{D^-_{2R}}(\nabla \vec{u}_-)^2\right)\, d\sigma\right)^{\frac{1}{2}}$$

$$+C\left(\fint_{S_{2R}} \left(M(\vec{u}_+)^2 + \mu\, M(\vec{u}_-)^2\right)\, d\sigma\right)^{\frac{1}{2}}. \tag{8.22}$$

Applying Lemma 8.4 gives

$$\left(\fint_{S_R} \left(M^0_R(\vec{u}_+) + \mu^{1/2} M^0_R(\vec{u}_-)\right)^{p_n}\, d\sigma\right)^{\frac{1}{p_n}}$$

$$\leq C\left(\fint_{S_{16R}} \left(M(\vec{u}_+)^2 + \mu\, M(\vec{u}_-)^2\right)\, d\sigma\right)^{\frac{1}{2}}$$

$$+CR\left(\fint_{S_{16R}} \mu\, |\partial^\lambda_\nu(\vec{u}_+,\pi_+) - \partial^\lambda_\nu(\vec{u}_-,\pi_-)|^2\, d\sigma\right)^{\frac{1}{2}}. \tag{8.23}$$

For $\mu \in (0,1)$, this is enough to establish (8.15) and prove the lemma. In the case $\mu = 0$, the estimate (8.23) gives that

$$\left(\fint_{S_R} M^0_R(\vec{u}_+)^{p_n}\, d\sigma\right)^{\frac{1}{p_n}} \leq C\left(\fint_{S_{16R}} M(\vec{u}_+)^2\, d\sigma\right)^{\frac{1}{2}}. \tag{8.24}$$

Therefore to finish the proof, we still need to show that if $\vec{u}_+ = 0$ on S_{128R}, then

$$\left(\fint_{S_R} M^0_R(\vec{u}_-)^{p_n}\, d\sigma\right)^{\frac{1}{p_n}} \leq CR\left(\fint_{S_{128R}} |\partial^\lambda_\nu(\vec{u}_+,\pi_+) - \partial^\lambda_\nu(\vec{u}_-,\pi_-)|^2\, d\sigma\right)^{\frac{1}{2}}$$

$$+C\left(\fint_{S_{128R}} M(\vec{u})^2\, d\sigma\right)^{\frac{1}{2}}. \tag{8.25}$$

Since $\vec{u}_+ = 0$ on S_{128R}, we can apply Lemma 8.4 with $\mu = 0$ and get

$$\int_{S_{16R}} |\partial^\lambda_\nu(\vec{u}_+,\pi_+)|^2\, d\sigma \leq C \int_{S_{16R}} (M_{D^+_{16R}}(\nabla\vec{u}_+)^2 + M_{D^+_{16R}}(\pi_+)^2)\, d\sigma \leq \frac{C}{R^2}\int_{S_{128R}} M(\vec{u}_+)^2\, d\sigma. \tag{8.26}$$

Arguing as before using fractional integration estimates, we have

$$\left(\fint_{S_R} M^0_R(\vec{u}_-)^{p_n}\, d\sigma\right)^{\frac{1}{p_n}} \leq CR\left(\fint_{S_{2R}} M_{D^-_{2R}}(\nabla\vec{u}_-)^2\, d\sigma\right)^{\frac{1}{2}} + C\left(\fint_{S_{2R}} M(\vec{u}_-)^2\, d\sigma\right)^{\frac{1}{2}}. \tag{8.27}$$

Now, applying Lemma 6.5 with $\mu = 0$ and $\vec{u}_+$ exchanged with $\vec{u}_-$ leads to the estimate

$$\int\limits_{S_R} |\nabla \vec{u}_-|^2 \, d\sigma \leq C \int\limits_{S_{2R}} |\partial_\nu^\lambda(\vec{u}_-, \pi_-)|^2 \, d\sigma + \frac{C}{R} \int\limits_{D_{2R}^-} (|\nabla \vec{u}_-|^2 + |\pi_-|^2) \, dx. \tag{8.28}$$

Similarly, using Lemma 6.1 and apply Lemma 8.3 with $\mu = 0$ and $\vec{u}_+$ exchanged with $\vec{u}_-$ gives

$$\begin{aligned}\int\limits_{D_R^-} (|\nabla \vec{u}_-|^2 + |\pi_-|^2) \, dx &\leq \frac{C}{R} \int\limits_{S_{2R}} M(\vec{u}_-)^2 \, d\sigma + C \int\limits_{S_{2R}} |\partial_\nu^\lambda(\vec{u}_-, \pi_-)||\vec{u}_-| \, d\sigma \\ &\leq \frac{C}{R} \int\limits_{S_{2R}} M(\vec{u}_-)^2 \, d\sigma + CR \int\limits_{S_{2R}} |\partial_\nu^\lambda(\vec{u}_-, \pi_-)|^2 \, d\sigma. \end{aligned} \tag{8.29}$$

Combining (8.10) with (8.28) and then using (8.29) yields

$$\int\limits_{S_{2R}} M_{D_{2R}^-}(\nabla \vec{u}_-)^2 \, d\sigma \leq \frac{C}{R^2} \int\limits_{S_{16R}} M(\vec{u}_-)^2 \, d\sigma + C \int\limits_{S_{16R}} |\partial_\nu^\lambda(\vec{u}_-, \pi_-)|^2 \, d\sigma. \tag{8.30}$$

Then using (8.30) in (8.27) gives

$$\begin{aligned}\left(\fint_{S_R} M_R^0(\vec{u}_-)^{p_n} \, d\sigma \right)^{\frac{1}{p_n}} &\leq CR \left(\fint_{S_{16R}} |\partial_\nu^\lambda(\vec{u}_-, \pi_-)|^2 \, d\sigma \right)^{\frac{1}{2}} + C \left(\fint_{S_{16R}} M(\vec{u}_-)^2 \, d\sigma \right)^{\frac{1}{2}} \\ &\leq CR \left(\fint_{S_{16R}} |\partial_\nu^\lambda(\vec{u}_+, \pi_+) - \partial_\nu^\lambda(\vec{u}_-, \pi_-)|^2 \, d\sigma \right)^{\frac{1}{2}} \\ &\quad + C \left(\fint_{S_{16R}} M(\vec{u}_-)^2 \, d\sigma \right)^{\frac{1}{2}} + CR \left(\fint_{S_{16R}} |\partial_\nu^\lambda(\vec{u}_+, \pi_+)|^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{8.31}$$

Combining (8.26) with (8.31) is enough to establish (8.25) and finish the proof. □

We will also need the following technical lemma which is proved by Z. Shen in [84].

Lemma 8.5 *Assume $0 < \beta < 1 < \alpha$ and $1 < q < p$. Also, let Q_0 be a cube in $\mathbb{R}^n$ and $F \in L^1(2Q_0)$, $f \in L^q(2Q_0)$. Suppose that there exist $C_1, C_2 > 0$ with the property that for each dyadic sub-cube Q of Q_0 with $|Q| \leq \beta|Q_0|$, there exist two integrable functions F_Q and R_Q on $2Q$ such that $|F| \leq |F_Q| + |R_Q|$ on $2Q$, and*

$$\left(\fint_{2Q} |R_Q|^p \, dx \right)^{\frac{1}{p}} \leq C_1 \left[\fint_{\alpha Q} |F| \, dx + \fint_{Q} |f| \, dx \right], \tag{8.32}$$

$$\fint_{2Q} |F_Q| \, dx \leq C_2 \fint_{Q} |f| \, dx. \tag{8.33}$$

Then

$$\left(\fint_{Q_0} |F|^q \, dx \right)^{\frac{1}{q}} \leq C \fint_{2Q_0} |F| \, dx + C \left(\fint_{2Q_0} |f|^q \, dx \right)^{\frac{1}{q}}, \tag{8.34}$$

where $C = C(p, q, C_1, C_2, \alpha, \beta, n) > 0$.

The following version of Gehring's Lemma is also necessary.

Lemma 8.6 [`Gehring's Lemma`]

Fix $p > 1$, *and let* $1 \leq q < p$. *Assume there exists functions* $g, h \in L^p(\partial\Omega)$ *and* $K > 0$ *such that for any surface ball* S_R,

$$\left(\fint_{S_R} |g|^p \, dx \right)^{\frac{1}{p}} \leq K \left(\fint_{S_{2R}} |g|^q \, d\sigma \right)^{\frac{1}{q}} + \left(\fint_{S_{2R}} |h|^p \, d\sigma \right)^{\frac{1}{p}}. \tag{8.35}$$

Then there exist $\varepsilon_o > 0$ *and* $C > 0$, *depending only on* K, p *and* q, *such that if* $0 \leq \varepsilon < \varepsilon_o$, *then*

$$\int\limits_{\partial\Omega} |g|^{p+\varepsilon} \, d\sigma \leq C \int\limits_{\partial\Omega} |h|^{p+\varepsilon} \, d\sigma. \tag{8.36}$$

For a proof of this lemma, see the Appendix. Our next lemma will show that that the estimate (8.3) for solutions of (8.1) continues to hold for larger values of p.

Lemma 8.7 *Let* $\Omega \subset \mathbb{R}^n$, $n \geq 4$, *be a Lipschitz domain, and set* $p_n := \frac{2(n-1)}{n-3}$. *Then there exists* $\varepsilon = \varepsilon(\Omega) > 0$ *such that for any* $\vec{g} \in L^2_1(\partial\Omega) \hookrightarrow L^{p_n}(\partial\Omega)$ *then the* L^2*-solution* $(\vec{u}_\pm, \pi_\pm)$ *of (8.1) satisfies the estimate*

$$\int\limits_{\partial\Omega} M(\vec{u})^p \, d\sigma \leq C(\Omega, p) \int\limits_{\partial\Omega} |\vec{g}|^p \, d\sigma \quad \textit{for every } p \in (2, p_n + \varepsilon), \tag{8.37}$$

where, as before, $M(\vec{u}) := \max\{M(\vec{u}_+), M(\vec{u}_-)\}$.

Proof. First, let $(\vec{u}_\pm, \pi_\pm)$ be as in (8.2). Since $\vec{g} \in L^2_1(\partial\Omega)$, we have $M(\vec{u}_\pm)$, $M(\nabla \vec{u}_\pm)$, $M(\pi_\pm) \in L^2(\partial\Omega)$. Applying Lemma 7.2 then gives that $M(\vec{u}_\pm) \in L^{p_n}(\partial\Omega)$. We need to show that $\vec{u}_\pm$ satisfies (8.37). Fix $S_R \subset \partial\Omega$. Choose $\phi \in C^\infty_o(\mathbb{R}^n)$ such that $\phi \equiv 1$ on S_{128R}, $\phi \equiv 0$ on $\partial\Omega \setminus S_{256R}$, $|\phi| \leq 1$ and $|\nabla\phi| \leq \frac{C}{R}$. Define $\vec{v}^\pm := \mathcal{D}_\lambda \left((\frac{1}{2}\frac{1+\mu}{1-\mu} I + K_\lambda)^{-1}(\phi\vec{g}) \right)$ in $\Omega_\pm$ and set $\eta^\pm := \mathcal{P}_\lambda \left((\frac{1}{2}\frac{1+\mu}{1-\mu} I + K_\lambda)^{-1}(\phi\vec{g}) \right)$ in $\Omega_\pm$. Set $M(\vec{v}) := \max\{M(\vec{v}_+), M(\vec{v}_-)\}$. Using the L^2 well-posedness estimate for $\vec{v}_\pm$, we have

$$\int_{\partial\Omega} M(\vec{v})^2 \, d\sigma \leq C \int_{S_{256R}} |\vec{g}|^2 \, d\sigma. \tag{8.38}$$

Let $\vec{w}_\pm := \vec{u}_\pm - \vec{v}_\pm$ and $\rho_\pm := \pi_\pm - \eta_\pm$. Then we have $\vec{w}_+ - \mu\,\vec{w}_- = g - \phi g = 0$ on S_{128R} and $\partial^\lambda_\nu(\vec{w}_+, \rho_+) = \partial^\lambda_\nu(\vec{w}_-, \rho_-)$ on $\partial\Omega$. Set $M(\vec{w}) := \max\{M(\vec{w}_+), M(\vec{w}_-)\}$. Applying Lemma 8.2 we then obtain

$$\left(\fint_{S_R} M(\vec{w})^{p_n} \, d\sigma \right)^{\frac{1}{p_n}} \leq C \left(\fint_{S_{128R}} M(\vec{w})^2 \, d\sigma \right)^{\frac{1}{2}}. \tag{8.39}$$

Combining (8.39) and (8.38) then gives

$$\begin{aligned} \left(\fint_{S_R} M(\vec{w})^{p_n} \, d\sigma \right)^{\frac{1}{p_n}} &\leq C \left(\fint_{S_{128R}} \left(M(\vec{u})^2 + M(\vec{v})^2 \right) d\sigma \right)^{\frac{1}{2}} \\ &\leq C \left(\fint_{S_{256R}} \left(M(\vec{u})^2 + |\vec{g}|^2 \right) d\sigma \right)^{\frac{1}{2}}. \end{aligned} \tag{8.40}$$

Then applying Lemma 8.5 with

$$F := M(\vec{u})^2, \quad F_{S_R} := M(\vec{v})^2, \quad R_{S_R} := M(\vec{w})^2, \quad f := |\vec{g}|^2, \quad \text{and} \quad q \in (1, p_n/2), \tag{8.41}$$

we obtain, with $p := 2q \in (2, p_n)$,

$$\left(\fint_{S_R} M(\vec{u})^p \, d\sigma \right)^{\frac{1}{p}} \leq C \left(\fint_{S_{2R}} M(\vec{u})^2 \, d\sigma \right)^{\frac{1}{2}} + C \left(\fint_{S_{2R}} |\vec{g}|^p \, d\sigma \right)^{\frac{1}{p}}. \tag{8.42}$$

Since this holds for every $2 < p < p_n$ and $M(\vec{u})$, $\vec{g} \in L^q(\partial\Omega)$ for every $2 \leq q \leq p_n$, it follows from Lemma 8.6 that there exists $\varepsilon > 0$ such that

$$\int_{\partial\Omega} M(\vec{u})^p \, d\sigma \leq C_p \int_{\partial\Omega} |\vec{g}|^p \, d\sigma \quad \text{whenever} \quad 2 < p < p_n + \varepsilon. \tag{8.43}$$

This finishes the proof. □

The previous estimate allows us to establish the invertibility of the boundary integral operators in the following theorem.

Theorem 8.8 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 4$, be a graph Lipschitz domain and fix $\mu \in [0, 1)$. There exists $\varepsilon > 0$ such that for $2 - \varepsilon < p < \frac{2(n-1)}{n-3} + \varepsilon$, the operators $\pm\frac{1}{2}\frac{1+\mu}{1-\mu}I + K_\lambda$ are invertible on $L^p(\partial\Omega)$.*

Proof. This has already been established in the case when p is near 2. Let $\varepsilon > 0$ be as in Lemma 8.7 and fix $2 < p < \frac{2(n-1)}{n-3} + \varepsilon$. Let $\vec{g} \in L^p(\partial\Omega)$. Then there exists $\vec{g}_j \in L^p(\partial\Omega) \cap L^2_1(\partial\Omega)$ ($j \in \mathbb{N}$) such that $\vec{g}_j$ converges to $\vec{g}$ in $L^p(\partial\Omega)$, as $j \to \infty$. Since $\frac{1}{2}\frac{1+\mu}{1-\mu}I + K_\lambda$ is an invertible operator on $L^2_1(\partial\Omega)$, for each $j \in \mathbb{N}$, there exists $\vec{f}_j \in L^2_1(\partial\Omega)$ such that

$$(\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K_\lambda)\vec{f}_j = \vec{g}_j. \tag{8.44}$$

For j fixed, let $\vec{u}_\pm = \mathcal{D}_\lambda \vec{f}_j$ in $\Omega_\pm$ and $\pi_\pm = \mathcal{P}_\lambda \vec{f}_j$ in $\Omega_\pm$. Then $(\vec{u}_\pm, \pi_\pm)$ solves (8.1) with datum $\vec{g}_j$. Then by Lemma 8.7,

$$\int_{\partial\Omega} |\vec{f}_j|^p \, d\sigma = \int_{\partial\Omega} |\vec{u}_+ - \vec{u}_-|^p \, d\sigma \leq 2^p \int_{\partial\Omega} M(\vec{u})^p \, d\sigma \leq C \int_{\partial\Omega} |\vec{g}_j|^p \, d\sigma, \tag{8.45}$$

which proves that $\vec{f}_j \in L^p(\partial\Omega)$. Repeating the above argument with the functions $\vec{f}_j - \vec{f}_k$ and $\vec{g}_j - \vec{g}_k$, $j, k \in \mathbb{N}$, we can conclude that

$$\|\vec{f}_j - \vec{f}_k\|_{L^p(\partial\Omega)} \leq C\|\vec{g}_j - \vec{g}_k\|_{L^p(\partial\Omega)} \quad \forall\, j, k \in \mathbb{N}. \tag{8.46}$$

Since $\{\vec{g}_j\}_j$ is a Cauchy sequence in $L^p(\partial\Omega)$, it follows that $\{\vec{f}_j\}_j$ is a Cauchy sequence in $L^p(\partial\Omega)$, and so there exists $\vec{f} \in L^p(\partial\Omega)$ such that $\vec{f}_j$ converges to $\vec{f}$ in $L^p(\partial\Omega)$. Then, for every $j \in \mathbb{N}$, formula (8.44) gives

$$\|(\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K_\lambda)\vec{f} - \vec{g}\|_{L^p(\partial\Omega)} \leq \|(\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K_\lambda)(\vec{f} - \vec{f}_j)\|_{L^p(\partial\Omega)} + \|\vec{g}_j - \vec{g}\|_{L^p(\partial\Omega)}, \tag{8.47}$$

so letting $j \to \infty$ yields that $(\frac{1}{2}\frac{1+\mu}{1-\mu}I + K_\lambda)\vec{f} = \vec{g}$. Thus, the operator $\frac{1}{2}\frac{1+\mu}{1-\mu}I + K_\lambda$ maps onto $L^p(\partial\Omega)$, and is therefore semi-Fredholm on $L^p(\partial\Omega)$ for every $\mu \in [0, 1)$. For μ close enough to 1, the operator $\frac{1}{2}\frac{1+\mu}{1-\mu}I + K_\lambda$ is invertible on $L^p(\partial\Omega)$ via a Neumann series, so it has index zero. Then $\frac{1}{2}\frac{1+\mu}{1-\mu}I + K_\lambda$ has index zero on $L^p(\partial\Omega)$ for all $\mu \in [0, 1)$, so it is, in fact, invertible on $L^p(\partial\Omega)$ for all $\mu \in [0, 1)$. If we reverse the roles of $\vec{u}_+$ and $\vec{u}_-$ and repeat the argument, we can show that the operator $-\frac{1}{2}\frac{1+\mu}{1-\mu}I + K_\lambda$ is also invertible on $L^p(\partial\Omega)$. This completes the proof. □

We conclude this section with

Proof of Theorem 8.1. Since the operators $\pm\frac{1}{2}\frac{1+\mu}{1-\mu}I + K_\lambda$ are invertible on $L^p(\partial\Omega)$ for $\mu \in [0, 1)$ and $2 - \varepsilon < p < \frac{2(n-1)}{n-3} + \varepsilon$, by duality, the operators $\pm\frac{1}{2}\frac{1+\mu}{1-\mu}I + K^*_\lambda$ are invertible on $L^p(\partial\Omega)$ for $\mu \in [0, 1)$ and $\frac{2(n-1)}{n+1} - \varepsilon < p < 2 + \varepsilon$. Then the theorem follows from Proposition 4.19 and Theorem 5.9. □

8.2 The Dirichlet problem

This section will be devoted to proving the following result.

Theorem 8.9 *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a graph Lipschitz domain. Then there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ such that for each*

$$\begin{aligned} &2 - \varepsilon < p < \infty, && \text{if} \quad n = 2, 3, \\ &2 - \varepsilon < p < \tfrac{2(n-1)}{n-3} + \varepsilon, && \text{if} \quad n \geq 4, \end{aligned} \tag{8.48}$$

the Dirichlet problem

$$\begin{cases} \Delta\vec{u} = \nabla\pi, \ \ \operatorname{div}\vec{u} = 0 \quad \text{in} \ \ \Omega, \\ M(\vec{u}) \in L^p(\partial\Omega), \\ \vec{u}\Big|_{\partial\Omega} = \vec{f} \in L^p(\partial\Omega), \end{cases} \tag{8.49}$$

has a solution, which is unique modulo adding functions which are constant in Ω to the pressure term. In addition, there exists a finite constant $C > 0$ such that

$$\|M(\vec{u})\|_{L^p(\partial\Omega)} \leq C\|\vec{f}\|_{L^p(\partial\Omega)}. \tag{8.50}$$

Proof. Let $\lambda \in (-1,1]$. From Theorem 8.8, (7.103), and duality it follows that the operator

$$\tfrac{1}{2}I + K_\lambda : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \tag{8.51}$$

is an isomorphism for each p as in (8.48). Then the functions

$$\vec{u} = \mathcal{D}_\lambda((\tfrac{1}{2}I + K_\lambda)^{-1}\vec{f}) \qquad \text{and} \qquad \pi = \mathcal{P}_\lambda((\tfrac{1}{2}I + K_\lambda)^{-1}\vec{f}) \tag{8.52}$$

will solve (8.49) and satisfy (8.50).

Turning our attention to the issue of uniqueness, let $(\vec{u}, \pi)$ solve the homogeneous version of (8.50) for some $p \in (2-\varepsilon, \frac{2(n-1)}{n-3}+\varepsilon)$. To fix ideas, assume that Ω is the upper-graph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfying $\varphi(0) = 0$, and for each $R > 0$, consider the bounded Lipschitz domain

$$D_R := \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \; |x'| < 2R, \; 0 < x_n - \varphi(x') < 2R\}. \tag{8.53}$$

As it will be shown in § 9.2, via arguments which are independent of the present considerations, there exists some finite constant $C > 0$ which depends only on p and the Lipschitz character of Ω, such that

$$\int_{\partial D_R} M_{D_R}(\vec{u})^p \, d\sigma \leq C \int_{\partial D_R} |\vec{u}|^p \, d\sigma, \tag{8.54}$$

where M_{D_R} stands for the nontangential maximal operator associated with the domain D_R. In order to continue, set $S_R := B(0,R) \cap \partial\Omega$ and denote by $V_R := \partial D_R \setminus \Big(S_R \cup (S_R + Re_n)\Big)$ the lateral side of the boundary of the domain D_R. Then, with M_R^0 as in (7.58), we may write

$$
\begin{aligned}
\int_{S_R} M_R^0(\vec{u})^p\, d\sigma &\leq \int_{\partial D_R} M_{D_R}(\vec{u})^p\, d\sigma \leq C\int_{\partial D_R} |\vec{u}|^p\, d\sigma \\
&= C\int_{V_R} |\vec{u}|^p\, d\sigma + C\int_{S_R} |\vec{u}(\cdot + Re_n)|^p\, d\sigma + C\int_{S_R} |\vec{u}|^p\, d\sigma \\
&\leq C\int_{V_R} |\vec{u}|^p\, d\sigma + C\int_{\partial\Omega} |\vec{u}(\cdot + Re_n)|^p\, d\sigma \\
&=: I_R + II_R, \qquad (8.55)
\end{aligned}
$$

since $\vec{u}$ vanishes on $\partial\Omega$. Next, observe that if $\eta > 0$ is a sufficiently small constant depending only on $\partial\Omega$, then for each $x \in \partial\Omega$, interior estimates and Lemma 7.3 give

$$
\begin{aligned}
|\vec{u}(x + Re_n)| &\leq C\Big(⨍_{B(x+Re_n,\eta R)} |\vec{u}|^{\frac{pn}{n-1}}\Big)^{\frac{n-1}{pn}} \\
&\leq CR^{-\frac{n-1}{p}} \|\vec{u}\|_{L^{pn/(n-1)}(\Omega)} \leq CR^{-\frac{n-1}{p}} \|M(\vec{u})\|_{L^p(\partial\Omega)}. \qquad (8.56)
\end{aligned}
$$

In particular,

$$
\begin{aligned}
&\lim_{R\to\infty} |\vec{u}(x + Re_n)| = 0 \text{ for each } x \in \partial\Omega, \\
&\text{and } |\vec{u}(\cdot + Re_n)| \leq M(\vec{u}) \text{ for each } R > 0,
\end{aligned}
\qquad (8.57)
$$

so that,

$$
\lim_{R\to\infty} II_R = 0, \qquad (8.58)
$$

by Lebesgue's Dominated Convergence Theorem. Let us now replace R by τR in (8.55) and then integrate the resulting inequality for $\tau \in [1, 3/2]$. If we consider the pipe-like region

$$
P_R := \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} :\ R/2 < |x'| < 4R,\ 0 < x_n - \varphi(x') < 4R\}, \qquad (8.59)
$$

then, on account of (8.58), we obtain

$$\begin{aligned}\int_{S_R} M_R^0(\vec{u})^p\, d\sigma &\leq C\int_1^{3/2} I_{\tau R}\, d\tau + C\int_1^{3/2} II_{\tau R}\, d\tau \\ &\leq CR^{-1}\int_{P_R} |\vec{u}|^p\, dx + o(1) \leq C\int_{S_{4R}\setminus S_{R/2}} M(\vec{u})^p\, d\sigma + o(1) \quad (8.60)\end{aligned}$$

as $R \to \infty$. However, since $M(\vec{u}) \in L^p(\partial\Omega)$, we also have $\int_{S_{4R}\setminus S_{R/2}} M(\vec{u})^p\, d\sigma = o(1)$ as $R \to \infty$. Hence, by Lebesgue's Monotone Convergence Theorem,

$$\int_{\partial\Omega} M(\vec{u})^p\, d\sigma = \lim_{R\to\infty} \int_{S_R} M_R^0(\vec{u})^p\, d\sigma = 0. \tag{8.61}$$

From this we may, of course, conclude that $\vec{u}$ vanishes in Ω. □

9 Boundary value problems in bounded Lipschitz domains

9.1 Localization arguments

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and consider an open, finite cover of $\partial\Omega$ with coordinate charts (Z_i, φ_i), $i = 1, ..., m$. Also, for each i, denote by Σ_i the graph of φ_i in the system of coordinates induced by Z_i.

For fixed $\mu \in [0, 1)$, $-1 < \lambda \leq 1$, denote by T the operator $\pm\frac{1}{2}\frac{1+\mu}{1-\mu}I + K_\lambda$ on $\partial\Omega$, where K_λ is as in (4.44), and let T_i stands for $\pm\frac{1}{2}\frac{1+\mu}{1-\mu}I + K^i_\lambda$ on Σ_i, where K^i_λ is as in (4.44) but with $\partial\Omega$ replaced by Σ_i. In particular, for each $p \in (\frac{n-1}{n}, 1]$ (which we shall henceforth assume) there exists $C = C(\lambda, \mu, p) > 0$ such that

$$\|f\|_{\widetilde{H}^{1,p}_{at}(\Sigma_i)} \leq C\|T_i f\|_{\widetilde{H}^{1,p}_{at}(\Sigma_i)}, \qquad \forall\, f \in \widetilde{H}^{1,p}_{at}(\Sigma_i), \quad 1 \leq i \leq m. \tag{9.1}$$

Next, let $\{\xi_i\}_{1\leq i\leq m}$ be a family of smooth functions with compact support in Z_i which form a partition of unity in a neighborhood of $\partial\Omega$. Also, for each i, let $\zeta_i \in C^\infty_0(Z_i)$ be such that $\zeta_i \equiv 1$ in a neighborhood of $\operatorname{supp}\xi_i$. Then, with λ and p as above, for any $f \in h^{1,p}_{at}(\partial\Omega)$, we may write

$$\|f\|_{h^{1,p}_{at}(\partial\Omega)} \leq C\sum_{i=1}^m \|\xi_i f\|_{h^{1,p}_{at}(\partial\Omega)} \leq C\sum_{i=1}^m \|\widetilde{\xi_i f}\|_{\widetilde{H}^{1,p}_{at}(\Sigma_i)} \leq C\sum_{i=1}^m \|T_i(\xi_i f)\|_{\widetilde{H}^{1,p}_{at}(\Sigma_i)}$$

$$
\begin{aligned}
&\leq \quad C\sum_{i=1}^{m}\|\zeta_i T_i(\xi_i f)\|_{\widetilde{H}^{1,p}_{at}(\Sigma_i)} + C\sum_{i=1}^{m}\|(1-\zeta_i)T_i(\xi_i f)\|_{\widetilde{H}^{1,p}_{at}(\Sigma_i)} \qquad (9.2)\\
&\leq \quad C\sum_{i=1}^{m}\sum_{j=1}^{n-1}\|\partial_{\tau^i_{jn}}[\zeta_i T_i(\xi_i f)]\|_{H^p_{at}(\Sigma_i)} + C\sum_{i=1}^{m}\sum_{j=1}^{n-1}\|\partial_{\tau^i_{jn}}[(1-\zeta_i)T_i(\xi_i f)]\|_{H^p_{at}(\Sigma_i)}.
\end{aligned}
$$

Above, the first inequality uses the fact that $f = \sum_{i=1}^{m} \xi_i f$ on $\partial\Omega$, the second one follows from Lemma 2.10 (here, tilde denotes the extension by zero outside the support to a function defined on Σ_i), the third is based on (9.1), while the fourth one is implied by Lemma 2.7. Finally, the fifth inequality is a consequence of (2.61) (here, the tangential derivative operator $\partial_{\tau^i_{jn}}$ is defined as before, but relative to the system of coordinates induced by Z_i in $\mathbb{R}^n$).

We adopt the following terminology. Call an expression of the form $\|\mathcal{R}f\|_{\mathcal{X}}$ *residual* if $\mathcal{R}$ maps $h^{1,p}_{at}(\partial\Omega)$ compactly into the quasi-Banach space $\mathcal{X}$. Recall the index p^* from (2.48) and observe that for each $q \in (1, p^*)$, the operator of multiplication by ξ_i is compact from $h^{1,p}_{at}(\partial\Omega)$ into $L^q(\Sigma_i)$. This and Lemma 2.9 show that the terms in the last double sum in (9.2) are residual. In order to continue, note that there exists a family of 'nice' singular integral operators $\{R_k\}_{1\leq k\leq n}$ on $\partial\Omega$, such that

$$
\partial_{\tau_{jn}} T = \pm\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}\,\partial_{\tau_{jn}} + \sum_{k=1}^{n} R_k \partial_{\tau_{jk}}. \tag{9.3}
$$

In fact, from the identity (4.98), the R_k's can be taken to be principal-value singular integral operators on $\partial\Omega$ whose kernels are of the form $\partial_k E(x-y)$ or $\partial_k E_\Delta(x-y)$, $1 \leq k \leq n$. Furthermore, we also have

$$
\partial_{\tau^i_{jn}} T_i = \pm\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}\,\partial_{\tau^i_{jn}} + \sum_{k=1}^{n} R^i_k \partial_{\tau^i_{jk}}, \qquad 1 \leq i \leq m, \tag{9.4}
$$

where R^i_k is the version of R_k written for Σ_i in place of $\partial\Omega$. Consider now a typical term in the next-to-the-last double sum in (9.2), and for a fixed $q \in (1, p^*)$, note that

$$
\begin{aligned}
\|\partial_{\tau^i_{jn}}[\zeta_i T_i(\xi_i f)]\|_{H^p_{at}(\Sigma_i)} \quad &\leq \quad C\|\zeta_i T_i(\xi_i f)\|_{h^{1,p}_{at}(\Sigma_i)} \leq C\|\zeta_i T_i(\xi_i f)\|_{h^{1,p}_{at}(\partial\Omega)}\\
&\approx \quad \|\zeta_i T_i(\xi_i f)\|_{L^q(\partial\Omega)} + \|\partial_{\tau^i_{jn}}[\zeta_i T_i(\xi_i f)]\|_{h^p_{at}(\partial\Omega)}
\end{aligned}
$$

$$= \quad \|\zeta_i T(\xi_i f)\|_{L^q(\partial\Omega)} + \|\partial_{\tau^i_{jn}}[\zeta_i T(\xi_i f)]\|_{h^p_{at}(\partial\Omega)}, \tag{9.5}$$

thanks to (2.83), (2.91), (2.93), and the fact that the integral operators T_i and T have the same kernel. Since

$$q \in (1, p^*) \implies h^{1,p}_{at}(\partial\Omega) \hookrightarrow L^q(\partial\Omega) \text{ compactly}, \tag{9.6}$$

and since $\zeta_i T \xi_i$ maps $L^q(\partial\Omega)$ boundedly into itself, we may conclude that the first term in the bottom line of (9.5) is residual. Regarding the second term, using (9.4) we may write

$$\begin{aligned} \partial_{\tau^i_{jn}}[\zeta_i T(\xi_i f)] &= (\partial_{\tau^i_{jn}}\zeta_i) T(\xi_i f) + \sum_{k=1}^{n} \zeta_i R_k((\partial_{\tau^i_{jk}}\xi_i) f) \pm \tfrac{1}{2}\tfrac{1+\mu}{1-\mu}(\partial_{\tau^i_{jn}}\xi_i) f \\ &\quad \pm \tfrac{1}{2}\tfrac{1+\mu}{1-\mu}\xi_i \partial_{\tau^i_{jn}} f + \sum_{k=1}^{n} \zeta_i R_k(\xi_i(\partial_{\tau^i_{jk}} f)). \end{aligned} \tag{9.7}$$

Again, granted (9.6) and the fact that the operators $(\partial_{\tau^i_{jn}}\zeta_i)T\xi_i$, $\zeta_i R_k(\partial_{\tau^i_{jk}}\xi_i)$ map $L^q(\partial\Omega)$ boundedly into itself, we may further deduce that the first three terms in the right hand-side of (9.7) give rise to residual expressions. There remains to consider the terms in the last line in (9.7) which, with the help of (9.3), we further transform as

$$\begin{aligned} \pm\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}\xi_i \partial_{\tau^i_{jn}} f + \sum_{k=1}^{n} \zeta_i R_k(\xi_i(\partial_{\tau^i_{jk}} f)) &= \sum_{k=1}^{n} \zeta_i [R_k, \xi_i](\partial_{\tau^i_{jk}} f) \pm \tfrac{1}{2}\tfrac{1+\mu}{1-\mu}\xi_i \partial_{\tau^i_{jn}} f + \sum_{k=1}^{n} \xi_i R_k(\partial_{\tau^i_{jk}} f) \\ &= \sum_{k=1}^{n} \zeta_i [R_k, \xi_i](\partial_{\tau^i_{jk}} f) + \xi_i \partial_{\tau^i_{jn}}(Tf). \end{aligned} \tag{9.8}$$

Since for every $p \in (\frac{n-1}{n}, 1]$ there exist $q > 1$ and $s \in (0,1)$ such that $h^p_{at}(\partial\Omega) \hookrightarrow B^{q,q}_{-s}(\partial\Omega)$ compactly and since $L^q(\partial\Omega) \hookrightarrow h^p(\partial\Omega)$, Lemma 2.23 shows that each $[R_k, \xi_i]\partial_{\tau^i_{jk}}$ gives rise to a residual expression. If we also note that

$$\|\xi_i \partial_{\tau^i_{jn}}(Tf)\|_{h^p_{at}(\partial\Omega)} \leq C \|\partial_{\tau^i_{jn}}(Tf)\|_{h^p_{at}(\partial\Omega)} \leq C \|Tf\|_{h^{1,p}_{at}(\partial\Omega)}, \tag{9.9}$$

then the above reasoning proves that, whenever $\frac{n-1}{n} < p \leq 1$, $\mu \in [0,1)$ and $-1 < \lambda \leq 1$, there exists a constant $C > 0$ such that

$$\|f\|_{h^{1,p}_{at}(\partial\Omega)} \leq C\|(\pm\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K_\lambda)f\|_{h^{1,p}_{at}(\partial\Omega)} + \text{residual expressions}, \tag{9.10}$$

for every $f \in h^{1,p}_{at}(\partial\Omega)$.

The estimate (9.10) leads to the following results.

Proposition 9.1 *For $n = 2, 3$, let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and assume that $\mu \in [0, 1)$ and $-1 < \lambda \leq 1$. Then there exists $\varepsilon > 0$ such that*

$$\pm\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K_\lambda : h^{1,p}_{at}(\partial\Omega) \longrightarrow h^{1,p}_{at}(\partial\Omega) \tag{9.11}$$

are Fredholm operators of index zero for each $\frac{2(n-1)}{n+1} - \varepsilon < p \leq 1$.

Proof. The estimate (9.10) shows that the operators $\pm\frac{1}{2}\frac{1+\mu}{1-\mu}I + K_\lambda$ are bounded from below modulo compact operators on $h^{1,p}_{at}(\partial\Omega)$ for each $\mu \in [0, 1)$. In particular, (9.11) are semi-Fredholm operators. Since they are invertible when μ is sufficiently close to 1, the homotopic invariance of the index may be invoked in order to conclude that this one-parameter family of operators (indexed by μ) consists of Fredholm operators with index zero. □

Corollary 9.2 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain and assume that $\mu \in [0, 1)$ and $-1 < \lambda \leq 1$. Then there exists $\varepsilon > 0$ such that for $p \in (\frac{2(n-1)}{n+1} - \varepsilon, 2 + \varepsilon)$,*

$$\pm\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K_\lambda : h^{p}_{1}(\partial\Omega) \longrightarrow h^{p}_{1}(\partial\Omega) \tag{9.12}$$

are Fredholm operators of index zero.

Proof. The case $p \leq 1$ is covered by the previous proposition. When $p > 1$, we can derive an estimate corresponding to (9.10) in a similar fashion as before, although in this case, since we are dealing with classic Sobolev spaces $L^p_1(\partial\Omega)$, the argument is a little more straightforward. Again, this type of estimate is enough to prove that the operators in question are Fredholm with index zero. □

As a result of the previous theorem when $\mu = 0$, it can also be shown that the operators

$$\pm\tfrac{1}{2}I + K_\lambda : h^p_{1,\nu_\pm}(\partial\Omega) \to h^p_{1,\nu_\pm}(\partial\Omega) \tag{9.13}$$

are Fredholm with index zero. In particular, using Lemma 11.41 and (5.125) then gives

$$\mathrm{Ker}\Big(\pm\tfrac{1}{2}I + K_\lambda : h^p_{1,\nu_\pm}(\partial\Omega) \to h^p_{1,\nu_\pm}(\partial\Omega)\Big) = \Psi^\lambda(\partial\Omega_\mp), \tag{9.14}$$

for each $p \in \Big(\frac{2(n-1)}{n+1} - \varepsilon, 2+\varepsilon\Big)$. We can now prove the following theorem.

Theorem 9.3 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain and assume $-1 < \lambda \leq 1$. Then there exists $\varepsilon > 0$ such that for $p \in (\frac{2(n-1)}{n+1} - \varepsilon, 2+\varepsilon)$, the operators*

$$\pm\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K_\lambda : h^p_1(\partial\Omega) \longrightarrow h^p_1(\partial\Omega) \tag{9.15}$$

are isomorphisms for all $\mu \in (0,1)$. Moreover, corresponding to the case $\mu = 0$, the operators

$$\pm\tfrac{1}{2}I + K_\lambda : h^p_{1,\nu_\pm}(\partial\Omega)/\Psi^\lambda(\partial\Omega_\mp) \longrightarrow h^p_{1,\nu_\pm}(\partial\Omega)/\Psi^\lambda(\partial\Omega_\mp) \tag{9.16}$$

are also isomorphisms.

Proof. From Theorem 5.17, we know the above operators are isomorphisms when p is near 2. Then since $L^2_1(\partial\Omega)$ is dense in $h^p_1(\partial\Omega)$, the operators in (9.15) must have dense range. From Corollary 9.2, the range is also closed, and so the operators are surjective. Since they are also Fredholm with index zero, this implies that the operators in (9.15) are in fact isomorphisms.

Arguing as in the last paragraph of § 5.3, it follows from Corollary 9.2 that the operators in (9.16) are Fredholm with index zero. Since we know that (9.16) are isomorphisms when p is near 2 and $L^2_{1,\nu_\pm}(\partial\Omega)$ is dense in $h^p_{1,\nu_\pm}(\partial\Omega)$, these operators must have dense range for each p in the desired range. Since the range is also closed, the operators in (9.16) must be onto, and therefor they are in fact isomorphisms. □

At this point, we are ready to prove the following result with regards to the invertibility of the single layer.

Theorem 9.4 *For each bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ with $n \geq 3$, there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ with the property that*

$$S : h^p(\partial\Omega)\Big/\nu\,\mathbb{R}_{\partial\Omega} \longrightarrow h^p_{1,\nu}(\partial\Omega) \tag{9.17}$$

is an isomorphism for each $p \in (\frac{2(n-1)}{n+1} - \varepsilon, 2+\varepsilon)$.

Proof. First, note that the operator (9.17) is well-defined due to (5.78) and (5.175). We will show that

$$\mathrm{Ker}\Big(S : h^p(\partial\Omega) \to h^p_1(\partial\Omega)\Big) = \nu\,\mathbb{R}_{\partial\Omega}. \tag{9.18}$$

Assume $\vec{f} \in h^p(\partial\Omega)$ is such that $S\vec{f} = 0$. Then $\vec{u}_\pm := \mathcal{S}\vec{f}$ in $\Omega_\pm$ and $\pi_\pm := \mathcal{Q}\vec{f}$ in $\Omega_\pm$ satisfy

$$\begin{cases} \Delta\vec{u}_\pm = \nabla\pi_\pm \text{ in } \Omega_\pm, \\ \mathrm{div}\,\vec{u}_\pm = 0 \text{ in } \Omega_\pm, \\ \vec{u}_\pm|_{\partial\Omega} = 0, \\ M(\nabla\vec{u}_\pm), M(\pi_\pm) \in L^p(\partial\Omega). \end{cases} \tag{9.19}$$

Since $M(\nabla\vec{u}_\pm) \in L^p(\partial\Omega)$, by Lemma 11.9, it follows that $M(\vec{u}) \in L^{p^*}(\partial\Omega)$ where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n-1}$. Then since $p^* > 2 - \varepsilon$, uniqueness for the L^2 Dirichlet problem guarantees that $\vec{u}_\pm$ are locally constant. Then $\pi_\pm$ are also locally constant, and so it follows that

$$\vec{f} = \partial^\lambda_\nu(\vec{u}_-, \pi_-) - \partial^\lambda_\nu(\vec{u}_+, \pi_+) = \nu(\pi_+ - \pi_-) \in \nu\mathbb{R}_{\partial\Omega}, \tag{9.20}$$

which proves (9.18). From (4.142), we know that

$$S \circ (\partial^\lambda_\nu(\mathcal{D}_\lambda(\cdot), \mathcal{P}_\lambda(\cdot))) = (\tfrac{1}{2}I + K_\lambda) \circ (-\tfrac{1}{2}I + K_\lambda), \tag{9.21}$$

as operators on $h^p_1(\partial\Omega)$. Although the identity (4.142) was originally proven for $p \geq 1$, by a density argument, it must also hold for $\frac{n-1}{n} < p < 1$. Now from Corollary 9.2, we know that the operators $\pm\frac{1}{2}I + K_\lambda$ are Fredholm for $p \in (\frac{2(n-1)}{n+1} - \varepsilon, 2+\varepsilon)$, and hence from (4.142), we can conclude that the operator

$$S : h^p(\partial\Omega) \to h^p_1(\partial\Omega) \tag{9.22}$$

has a finite codimensional range, which in turn implies that its range is closed. Now since the operator in (9.22) has closed range and (9.18) holds for all $\frac{2(n-1)}{n+1} - \varepsilon < p < 2+\varepsilon$, it follows that (9.17) is injective and has closed range for all values of p in this range. Furthermore, from Theorem 5.18, the operator in (9.17) is an isomorphism when p is near 2, and so applying Theorem 11.47 from the Appendix, it must be an isomorphism for all $\frac{2(n-1)}{n+1} - \varepsilon < p < 2+\varepsilon$. □

Since (9.17) is a self-adjoint operator, the following corollary follows immediately by duality.

Corollary 9.5 *For each bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ with $n \geq 3$ there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ with the property that for each*

$$2 - \varepsilon < p < \infty \quad \textit{if} \;\; n = 3, \tag{9.23}$$

$$2 - \varepsilon < p < \tfrac{2(n-1)}{n-3} + \varepsilon \quad \textit{if} \;\; n \geq 4, \tag{9.24}$$

the operator

$$S : L^p_{-1}(\partial\Omega)\Big/\nu\,\mathbb{R}_{\partial\Omega} \longrightarrow L^p_\nu(\partial\Omega) \tag{9.25}$$

is an isomorphism.

We also have the following results for $n = 2$.

Theorem 9.6 *For each bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ with the property that the operator*

$$\widetilde{S} : \Big(h^p(\partial\Omega)/\nu\mathbb{R}_{\partial\Omega}\Big) \oplus \mathbb{R}^2 \longrightarrow h^p_{1,\nu}(\partial\Omega) \oplus \mathbb{R}^2, \tag{9.26}$$

given by

$$\widetilde{S}([\vec{g}],\vec{c}) := \left(S\vec{g}+\vec{c},\ \fint_{\partial\Omega}\vec{g}\,d\sigma\right), \tag{9.27}$$

is an isomorphism for each $p \in (\frac{2}{3}-\varepsilon, 2+\varepsilon)$.

Proof. Arguing as in the proof of Theorem 9.4, we can establish that (9.22) is a Fredholm operator for each $p \in (\frac{2}{3}-\varepsilon, 2+\varepsilon)$. Recall the decomposition $\widetilde{S} = S_o + S_1$ as defined in (5.188). Since we know $S_o \cong S$ is Fredholm, and S_1 is compact (being an operator of finite rank), it follows that $\widetilde{S}$ is also Fredholm, and therefore has closed range for all $p \in (\frac{2}{3}-\varepsilon, 2+\varepsilon)$. Since $\widetilde{S}$ is an isomorphism for $p = (2-\varepsilon, 2+\varepsilon)$ according to Theorem 5.21, it has dense range for all $p \in (\frac{2}{3}-\varepsilon, 2+\varepsilon)$, and therefore it is onto for all p in this range. Applying Theorem 11.47 from the Appendix, we can conclude that $\widetilde{S}$ is an isomorphism for each p in the desired range. □

It can also be shown that (9.26) is a self-adjoint operator, and so the following corollary follows immediately by duality.

Corollary 9.7 *For each bounded Lipschitz domain* $\Omega \subset \mathbb{R}^2$ *there exists* $\varepsilon = \varepsilon(\partial\Omega) > 0$ *with the property that the operator*

$$\widetilde{S} : \left(L^p_{-1}(\partial\Omega)/\nu\mathbb{R}_{\partial\Omega}\right) \oplus \mathbb{R}^2 \longrightarrow L^p_\nu(\partial\Omega) \oplus \mathbb{R}^2 \tag{9.28}$$

as in (9.27) is an isomorphism for each $2-\varepsilon < p < \infty$.

Next, we state another result involving the single layer in two dimensions.

Theorem 9.8 *For each bounded Lipschitz domain* $\Omega \subset \mathbb{R}^2$ *there exists* $\varepsilon = \varepsilon(\partial\Omega) > 0$ *with the property that*

$$S : h^p(\partial\Omega)\Big/\nu\,\mathbb{R}_{\partial\Omega} \oplus \mathcal{W} \longrightarrow h^p_{1,\nu,\mathcal{W}}(\partial\Omega) := \left\{\vec{f} \in h^p_{1,\nu} : \int_{\partial\Omega}\langle\vec{f},\varphi\rangle\,d\sigma = 0\ \forall\varphi\in\mathcal{W}\right\} \tag{9.29}$$

is an isomorphism for each $p \in (\frac{2}{3}-\varepsilon, 2+\varepsilon)$, *where* $\mathcal{W}$ *is as in (5.129).*

Proof. From Theorem 9.6, we know $\widetilde{S}$ is an isomorphism for each $p \in (\frac{2}{3} - \varepsilon, 2 + \varepsilon)$. In particular, $\widetilde{S}$ has index zero, and so since $S \cong \widetilde{S} - S_1$ where S_1 as in (5.188) is compact, it follows that S must have index zero for each $p \in (\frac{2}{3} - \varepsilon, 2 + \varepsilon)$. Using (5.184) and applying Theorem 11.41 then gives

$$\operatorname{Ker}\left(S : h^p(\partial\Omega) \longrightarrow h_1^p(\partial\Omega)\right) = \nu\mathbb{R}_{\partial\Omega} \oplus \mathcal{W}, \qquad \forall\, p \in (\tfrac{2}{3} - \varepsilon, 2 + \varepsilon), \tag{9.30}$$

and therefore (9.29) is indeed an isomorphism for each p in the desired range. $\square$

Consider now the following transmission boundary value problem for the Stokes system:

$$\begin{cases} \Delta\vec{u}_\pm - \nabla\pi_\pm = 0 \quad \text{in} \quad \Omega_\pm, \\ M(\nabla\vec{u}_\pm), M(\pi_\pm) \in L^p(\partial\Omega), \\ \vec{u}_+\Big|_{\partial\Omega} - \vec{u}_-\Big|_{\partial\Omega} = \vec{f} \in h_1^p(\partial\Omega), \\ \partial_\nu^\lambda(\vec{u}_+, \pi_+) - \mu\, \partial_\nu^\lambda(\vec{u}_-, \pi_-) = \vec{g} \in h^p(\partial\Omega), \end{cases} \tag{9.31}$$

along with the decay conditions

$$\vec{u}_-(x) = \begin{cases} O(|x|^{2-n}) \quad \text{as} \quad |x| \to \infty, \quad \text{if} \quad n \geq 3, \\ -\frac{1}{\mu}E(x)\Big(⨍_{\partial\Omega} \vec{g}\, d\sigma\Big) + O(|x|^{-1}) \quad \text{as} \quad |x| \to \infty, \quad \text{if} \quad n = 2, \end{cases} \tag{9.32}$$

$$\partial_j\vec{u}_-(x) = -\tfrac{1}{\mu}(\partial_j E)(x)\Big(⨍_{\partial\Omega} \vec{g}\, d\sigma\Big) + O(|x|^{-n}) \quad \text{as} \quad |x| \to \infty, \ 1 \leq j \leq n, \tag{9.33}$$

$$\pi_-(x) = \begin{cases} O(|x|^{1-n}) \quad \text{as} \quad |x| \to \infty, \quad \text{if} \quad n \geq 3, \\ \frac{1}{\mu}\Big\langle \nabla E_\Delta)(x)\,,\, ⨍_{\partial\Omega} \vec{g}\, d\sigma \Big\rangle + O(|x|^{-2}) \quad \text{as} \quad |x| \to \infty, \quad \text{if} \quad n = 2. \end{cases} \tag{9.34}$$

Above, $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, $\mu \in (0, 1)$ is the transmission parameter and we have set $\Omega_+ := \Omega$, $\Omega_- := \mathbb{R}^n \setminus \bar{\Omega}$. Also, when $\frac{n-1}{n} < p < 1$, the integral $\int_{\partial\Omega} \vec{g}\, d\sigma$ should be interpreted as $\Big(\langle g_\ell, \chi_{\partial\Omega}\rangle\, e_\ell\Big)_{1 \leq \ell \leq n}$, with $\langle\cdot,\cdot\rangle$ denoting the duality pairing between $h^p(\partial\Omega)$ and $C^{(n-1)(1/p-1)}(\partial\Omega)$.

We can now prove the following result.

Theorem 9.9 *Assume that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded Lipschitz domain and that $\frac{n-1}{n} < p < \infty$, $-1 < \lambda \leq 1$. Then the following claims are equivalent:*

(i) the problem (9.31)-(9.34) is well-posed for every $\mu \in (0,1)$;

(ii) the operator

$$\tfrac{1}{2}\tfrac{\mu+1}{\mu-1}I + K_\lambda^* : h^p(\partial\Omega) \longrightarrow h^p(\partial\Omega) \tag{9.35}$$

is an isomorphism for every $\mu \in (0,1)$;

(iii) the operator

$$\tfrac{1}{2}\tfrac{\mu+1}{\mu-1}I + K_\lambda : h_1^p(\partial\Omega) \longrightarrow h_1^p(\partial\Omega) \tag{9.36}$$

is an isomorphism for every $\mu \in (0,1)$.

Proof. The proof of the implication $(ii) \Longrightarrow (i)$ follows exactly as in the proof of the first part of Theorem 5.22. In the opposite direction, the *a priori* estimate associated with the version of (9.31) when $\vec{f} = 0$ reads

$$\begin{aligned}\|\partial_\nu^\lambda(\vec{u}_+, \pi_+) - \mu\, \partial_\nu^\lambda(\vec{u}_-, \pi_-)\|_{h^p(\partial\Omega)} \quad &\approx \quad \|M(\nabla\vec{u}_+)\|_{L^p(\partial\Omega)} + \|M(\pi_+)\|_{L^p(\partial\Omega)} \\ &\quad + \|M(\nabla\vec{u}_-)\|_{L^p(\partial\Omega)} + \|M(\pi_-)\|_{L^p(\partial\Omega)} \end{aligned} \tag{9.37}$$

for any pair of functions $(\vec{u}_\pm, \pi_\pm)$ which solve the Stokes system in $\Omega_\pm$ and satisfy $\vec{u}_+|_{\partial\Omega} = \vec{u}_-|_{\partial\Omega}$, $M(\nabla\vec{u}_\pm)$, $M(\pi_\pm) \in L^p(\partial\Omega)$. Specializing this estimate to the case when $\vec{u}_\pm = \mathcal{S}\vec{h}$, $\pi_\pm := \mathcal{Q}\vec{h}$ in $\Omega_\pm$, with $\vec{h} \in h^p(\partial\Omega)$, and arguing as in (4.173) then yields

$$\|\vec{h}\|_{h^p(\partial\Omega)} \leq C\|(\tfrac{1}{2}\tfrac{\mu+1}{\mu-1}I + K_\lambda^*)\vec{h}\|_{h^p(\partial\Omega)}, \tag{9.38}$$

where $C = C(\Omega, p, \mu) > 0$ is a finite constant. Thus, $\left\{\frac{1}{2}\frac{\mu+1}{\mu-1}I + K_\lambda^*\right\}_{0<\mu<1}$ is a continuously parametrized family of one-to-one operators with closed range (in particular, semi-Fredholm) on $h^p(\partial\Omega)$, which are invertible (via a Neumann series) when μ is sufficiently close to 1. The homotopic invariance of the index then gives that all the operators in question are invertible on $h^p(\partial\Omega)$.

Consider next the equivalence $(i) \iff (iii)$. First, when the operator (9.36) is an isomorphism for each $\mu \in (0,1)$, a solution to (9.31)-(9.34) which satisfies (5.205) is given by

$$\begin{aligned}
\vec{u}_+ &:= -\mathcal{S}\vec{g} + \mathcal{D}_\lambda\Big[\Big(\tfrac{1}{2}\tfrac{\mu+1}{\mu-1}I + K_\lambda\Big)^{-1}\Big(S\vec{g} + \tfrac{\mu}{\mu-1}\vec{f}\Big)\Big] \ \text{ in } \ \Omega_+, && (9.39)\\
\pi_+ &:= -\mathcal{Q}\vec{g} + \mathcal{P}_\lambda\Big[\Big(\tfrac{1}{2}\tfrac{\mu+1}{\mu-1}I + K_\lambda\Big)^{-1}\Big(S\vec{g} + \tfrac{\mu}{\mu-1}\vec{f}\Big)\Big] \ \text{ in } \ \Omega_+, && (9.40)\\
\vec{u}_- &:= -\frac{1}{\mu}\mathcal{S}\vec{g} + \frac{1}{\mu}\mathcal{D}_\lambda\Big[\Big(\tfrac{1}{2}\tfrac{\mu+1}{\mu-1}I + K_\lambda\Big)^{-1}\Big(S\vec{g} + \tfrac{\mu}{\mu-1}\vec{f}\Big)\Big] \ \text{ in } \ \Omega_-, && (9.41)\\
\pi_- &:= -\frac{1}{\mu}\mathcal{Q}\vec{g} + \frac{1}{\mu}\mathcal{P}_\lambda\Big[\Big(\tfrac{1}{2}\tfrac{\mu+1}{\mu-1}I + K_\lambda\Big)^{-1}\Big(S\vec{g} + \tfrac{\mu}{\mu-1}\vec{f}\Big)\Big] \ \text{ in } \ \Omega_-. && (9.42)
\end{aligned}$$

Second, if the problem (9.31) is well-posed for each $\mu \in (0,1)$, then

$$\begin{aligned}
\|\mu\,\vec{u}_+|_{\partial\Omega} - \vec{u}_-|_{\partial\Omega}\|_{h^p_1(\partial\Omega)} &\approx \|M(\nabla\vec{u}_+)\|_{L^p(\partial\Omega)} + \|M(\pi_+)\|_{L^p(\partial\Omega)}\\
&\quad + \|M(\nabla\vec{u}_-)\|_{L^p(\partial\Omega)} + \|M(\pi_-)\|_{L^p(\partial\Omega)}, \qquad (9.43)
\end{aligned}$$

for any pair of functions $(\vec{u}_\pm, \pi_\pm)$ which solve the Stokes system in $\Omega_\pm$ and satisfy $\partial^\lambda_\nu(\vec{u}_+, \pi_+) = \partial^\lambda_\nu(\vec{u}_-, \pi_-)$, as well as $M(\nabla\vec{u}_\pm), M(\pi_\pm) \in L^p(\partial\Omega)$. Indeed, this is the *apriori* estimate associated with the version of (9.31) in which we multiply by μ the first boundary condition, re-denote $\mu\vec{u}_-$ by $\vec{u}_-$, and take $\vec{g} = 0$. Now, specializing (9.43) to the case when $\vec{u}_\pm = \mathcal{D}_\lambda\vec{h}$, $\pi_\pm = \mathcal{P}_\lambda\vec{h}$ in $\Omega_\pm$, with $\vec{h} \in h^p_1(\partial\Omega)$, yields

$$\begin{aligned}
\|\vec{h}\|_{h^p_1(\partial\Omega)} &= \|\vec{u}_+|_{\partial\Omega} - \vec{u}_-|_{\partial\Omega}\|_{h^p_1(\partial\Omega)}\\
&\leq \|M(\nabla\vec{u}_+)\|_{L^p(\partial\Omega)} + \|M(\pi_+)\|_{L^p(\partial\Omega)} + \|M(\nabla\vec{u}_-)\|_{L^p(\partial\Omega)} + \|M(\pi_-)\|_{L^p(\partial\Omega)}\\
&\leq C\|\mu\,\vec{u}_+|_{\partial\Omega} - \vec{u}_-|_{\partial\Omega}\|_{h^p_1(\partial\Omega)} = C\|(\tfrac{1}{2}\tfrac{\mu+1}{\mu-1}I + K^*_\lambda)\vec{h}\|_{h^p_1(\partial\Omega)}, \qquad (9.44)
\end{aligned}$$

where $C = C(\Omega, p, \mu) > 0$ is a finite constant. With this in hand and arguing as before, we then conclude that the operator (9.36) is an isomorphism for every $\mu \in (0,1)$.

There remains the issue of proving uniqueness for (9.31) when, say, the operator (9.36) is an isomorphism for each $\mu \in (0,1)$. Once again, if $(\vec{u}_\pm, \pi_\pm)$ is a solution of the homogeneous version of (9.31)-(9.34), Green's formulas (5.211) hold. Multiplying the version of (5.211) corresponding to the sign minus by ν, then adding it to the the version of (5.211) corresponding to the sign plus yields, after taking boundary traces

$$\vec{u}_+\Big|_{\partial\Omega} + \mu\,\vec{u}_-\Big|_{\partial\Omega} = \Big(\tfrac{1}{2}I + K_\lambda\Big)\Big(\vec{u}_+\Big|_{\partial\Omega}\Big) - \mu\Big(-\tfrac{1}{2}I + K_\lambda\Big)\Big(\vec{u}_-\Big|_{\partial\Omega}\Big), \tag{9.45}$$

since the single layer does not jump across $\partial\Omega$ and $\partial_\nu^\lambda(\vec{u}_+, \pi_+) = \mu\,\partial_\nu^\lambda(\vec{u}_-, \pi_-)$. Thus, keeping in mind that $\vec{u}_+|_{\partial\Omega} = \vec{u}_-|_{\partial\Omega}$ yields, after some algebra, $\Big(\frac{1}{2}\frac{\mu+1}{\mu-1}I + K_\lambda\Big)\Big(\vec{u}_+\Big|_{\partial\Omega}\Big) = 0$. Hence, $\vec{u}_+|_{\partial\Omega} = 0$, and so $\vec{u}_-|_{\partial\Omega} = 0$ as well. If in place of (4.152), we now set

$$\vec{u} := \begin{cases} \vec{u}_+ \ \text{ in } \ \Omega_+, \\ \mu\,\vec{u}_- \ \text{ in } \ \Omega_-, \end{cases} \qquad \text{and} \qquad \pi := \begin{cases} \pi_+ \ \text{ in } \ \Omega_+, \\ \mu\,\pi_- \ \text{ in } \ \Omega_-, \end{cases} \tag{9.46}$$

then the pair $(\vec{u}, \pi)$ solves the Stokes system in $\mathbb{R}^n$ and decay at infinity. Interior estimates then force that $\vec{u} = 0$ from which the desired conclusion follows. $\square$

Running the same type of argument as above, but for the transmission problem

$$\begin{cases} \Delta\vec{u}_\pm - \nabla\pi_\pm = 0 \quad \text{in} \quad \Omega_\pm, \\ M(\nabla\vec{u}_\pm), M(\pi_\pm) \in L^p(\partial\Omega), \\ \vec{u}_+\Big|_{\partial\Omega} - \mu\vec{u}_-\Big|_{\partial\Omega} = \vec{g} \in h_1^p(\partial\Omega), \\ \partial_\nu^\lambda(\vec{u}_+, \pi_+) - \partial_\nu^\lambda(\vec{u}_-, \pi_-) = \vec{f} \in h^p(\partial\Omega), \end{cases} \tag{9.47}$$

with decay conditions

$$\vec{u}_-(x) = \begin{cases} O(|x|^{2-n}) \quad \text{as} \quad |x| \to \infty, \quad \text{if} \quad n \geq 3, \\ E(x)\Big(\int_{\partial\Omega} \vec{f}\,d\sigma\Big) + O(|x|^{-1}) \quad \text{as} \quad |x| \to \infty, \quad \text{if} \quad n = 2, \end{cases} \tag{9.48}$$

$$\partial_j\vec{u}_-(x) = (\partial_j E)(x)\Big(\int_{\partial\Omega} \vec{f}\,d\sigma\Big) + O(|x|^{-n}) \quad \text{as} \quad |x| \to \infty, \ 1 \leq j \leq n, \tag{9.49}$$

$$\pi_-(x) = \begin{cases} O(|x|^{1-n}) \quad \text{as} \quad |x| \to \infty, \quad \text{if} \quad n \geq 3, \\ \Big\langle \nabla E_\Delta)(x)\,,\, \int_{\partial\Omega} \vec{f}\, d\sigma \Big\rangle + O(|x|^{-2}) \quad \text{as} \quad |x| \to \infty, \quad \text{if} \;\; n = 2. \end{cases} \tag{9.50}$$

in place of (9.31)-(9.34), yields the following result.

Theorem 9.10 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain and assume that $\frac{n-1}{n} < p < \infty$, $-1 < \lambda \leq 1$. Then the fact that the transmission problem (9.47)-(9.50) is well-posed for each $\mu \in (0,1)$ is equivalent with each of the following two conditions:*

$$-\tfrac{1}{2}\tfrac{\mu+1}{\mu-1} I + K_\lambda^* : h^p(\partial\Omega) \longrightarrow h^p(\partial\Omega) \text{ isomorphically, } \forall\, \mu \in (0,1), \tag{9.51}$$

$$-\tfrac{1}{2}\tfrac{\mu+1}{\mu-1} I + K_\lambda : h_1^p(\partial\Omega) \longrightarrow h_1^p(\partial\Omega) \text{ isomorphically, } \forall\, \mu \in (0,1). \tag{9.52}$$

We can now also prove the following theorem.

Theorem 9.11 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain and assume that $-1 < \lambda \leq 1$. Then there exists $\varepsilon > 0$ such that for $p \in (\frac{2(n-1)}{n+1} - \varepsilon, 2 + \varepsilon)$,*

$$\pm\tfrac{1}{2}\tfrac{1+\mu}{1-\mu} I + K_\lambda^* : h^p(\partial\Omega) \longrightarrow h^p(\partial\Omega) \tag{9.53}$$

are isomorphisms for all $\mu \in (0,1)$. Furthermore, corresponding to the case $\mu = 0$, the operators

$$\pm\tfrac{1}{2} I + K_\lambda^* : h^p_{\Psi^\lambda_\mp}(\partial\Omega)/\nu\mathbb{R}_{\partial\Omega_\pm} \longrightarrow h^p_{\Psi^\lambda_\mp}(\partial\Omega)/\nu\mathbb{R}_{\partial\Omega_\pm} \tag{9.54}$$

are also isomorphisms.

Proof. Let $p \in (\frac{2(n-1)}{n+1} - \varepsilon, 2 + \varepsilon)$. If $\mu \in (0,1)$, it follows from Theorem 9.3, Theorem 9.9, and Theorem 9.10 that the operators in (9.53) are isomorphisms. If we can show that the operators in (9.54) are Fredholm with index zero, then we can finish the proof by arguing as in the proof of Theorem 9.3.

From Theorem 9.4, we know that (9.22) is a Fredholm operator of index zero. Now, returning to the identity (9.21) and using Corollary 9.2, we can conclude that

$$\partial_\nu^\lambda(\mathcal{D}_\lambda(\cdot),\mathcal{P}_\lambda(\cdot)) : h_1^p(\partial\Omega) \longrightarrow h^p(\partial\Omega) \tag{9.55}$$

is also a Fredholm operator of index zero.

For $\vec{f} \in h^p(\partial\Omega)$, let $\vec{u}_\pm := \mathcal{S}\vec{f}$ in $\Omega_\pm$ and $\pi_\pm := \mathcal{Q}\vec{f}$ in $\Omega_\pm$. Applying (4.144) to these functions leads to the identity

$$\partial_\nu^\lambda(\mathcal{D}_\lambda(S\vec{f}),\mathcal{P}_\lambda(S\vec{f})) = (\tfrac{1}{2}I + K_\lambda^*)(-\tfrac{1}{2}I + K_\lambda^*)\vec{f}, \qquad \forall\, \vec{f} \in h^p(\partial\Omega). \tag{9.56}$$

Although (4.144) only holds as stated for $p \geq 1$, the identity (9.56) still holds for $\frac{n-1}{n} < p < 1$ by virtue of a density argument. Now, since the operators (9.55) and (9.22) in the left hand side of (9.56) are Fredholm and the operators in the right side commute, it follows that the operators

$$\pm\tfrac{1}{2}I + K_\lambda^* : h^p(\partial\Omega) \longrightarrow h^p(\partial\Omega) \tag{9.57}$$

both have a closed, finite co-dimensional range as well as a finite dimensional kernel. Hence, they are both Fredholm. Now that we know the operators

$$\pm\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K_\lambda^* : h^p(\partial\Omega) \longrightarrow h^p(\partial\Omega) \tag{9.58}$$

are Fredholm for all $\mu \in [0,1)$, it follows that the Fredholm index must be constant for all μ in this range. Thus the operators in (9.57), which correspond to the case $\mu = 0$, are Fredholm with index zero. Finally, arguing in a similar fashion as in the last paragraph of § 5.3, we can show that the operators in (9.54) are also Fredholm with index zero, as desired. □

We conclude this section with two corollaries.

Corollary 9.12 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain and assume that $-1 < \lambda \leq 1$. Then there exists $\varepsilon > 0$ such that for each*

$$\begin{aligned} &2-\varepsilon < p < \infty \qquad \text{if } n = 2,3, \\ &2-\varepsilon < p < \tfrac{2(n-1)}{n-3} + \varepsilon \qquad \text{if } n \geq 4, \end{aligned} \tag{9.59}$$

and each $\mu \in (0,1)$, *the operators*

$$\pm\tfrac{1}{2}\tfrac{1+\mu}{1-\mu} I + K_\lambda : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \tag{9.60}$$

are isomorphisms for all $\mu \in (0,1)$*. Moreover, corresponding to the case* $\mu = 0$*, the operators*

$$\pm\tfrac{1}{2} I + K_\lambda : L^p_{\nu_\pm}(\partial\Omega)/\Psi^\lambda(\partial\Omega_\mp) \longrightarrow L^p_{\nu_\pm}(\partial\Omega)/\Psi^\lambda(\partial\Omega_\mp) \tag{9.61}$$

are also isomorphisms.

Proof. This follows from Theorem 9.11 and duality. □

To state our second corollary, we need some preparations. Recall the duality result from (2.68). The dual of $h^1_{at}(\partial\Omega)$ involves the local BMO space. which we briefly review. For some fixed $0 < r_o < \operatorname{diam}(\partial\Omega)$, the space $\mathrm{bmo}\,(\partial\Omega)$ is then introduced as

$$f \in \mathrm{bmo}\,(\partial\Omega) \overset{def}{\Longleftrightarrow} f \in L^2(\partial\Omega) \quad \text{and} \quad \sup_{\substack{\Delta_r \text{ surface ball} \\ \text{with } r \leq r_o}} ⨍_{\Delta_r} |f - f_{\Delta_r}|\, d\sigma < \infty \tag{9.62}$$

(with $f_{\Delta_r} := ⨍_{\Delta_r} f\, d\sigma$, where the barred integral indicates averaging), and is equipped with the natural norm. Then (cf. [17])

$$\Big(h^1_{at}(\partial\Omega)\Big)^* = \mathrm{bmo}\,(\partial\Omega) \quad \text{and} \quad h^1_{at}(\partial\Omega) = \Big(\mathrm{vmo}\,(\partial\Omega)\Big)^*, \tag{9.63}$$

where

$$f \in \mathrm{vmo}\,(\partial\Omega) \overset{def}{\Longleftrightarrow} f \in \mathrm{bmo}\,(\partial\Omega) \;\; \text{and} \;\; \lim_{R\to 0} \left(\sup_{\substack{\Delta_r \text{ surface ball} \\ \text{with } r \leq R}} ⨍_{\Delta_r} |f - f_{\Delta_r}|\, d\sigma \right) = 0 \tag{9.64}$$

is Sarason's space of functions of vanishing mean oscillation. Define the spaces $\mathrm{bmo}_{\nu_\pm}(\partial\Omega)$, $\mathrm{vmo}_{\nu_\pm}(\partial\Omega)$ and $C^\alpha_{\nu_\pm}(\partial\Omega)$ in an analogous fashion to (5.115).

Corollary 9.13 *Suppose that $\Omega \subset \mathbb{R}^n$, $n \in \{2,3\}$, is a bounded Lipschitz domain and assume that $-1 < \lambda \leq 1$. Then, for each $\mu \in (0,1)$, the operators*

$$\pm\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K_\lambda : \mathrm{bmo}\,(\partial\Omega) \longrightarrow \mathrm{bmo}\,(\partial\Omega), \tag{9.65}$$

$$\pm\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K_\lambda : \mathrm{vmo}\,(\partial\Omega) \longrightarrow \mathrm{vmo}\,(\partial\Omega), \tag{9.66}$$

are isomorphisms. In addition, corresponding to the case $\mu = 0$, the operators

$$\pm\tfrac{1}{2}I + K_\lambda : \mathrm{bmo}_{\nu_\pm}(\partial\Omega)/\Psi^\lambda(\partial\Omega_\mp) \longrightarrow \mathrm{bmo}_{\nu_\pm}(\partial\Omega)/\Psi^\lambda(\partial\Omega_\mp), \tag{9.67}$$

$$\pm\tfrac{1}{2}I + K_\lambda : \mathrm{vmo}_{\nu_\pm}(\partial\Omega)/\Psi^\lambda(\partial\Omega_\mp) \longrightarrow \mathrm{vmo}_{\nu_\pm}(\partial\Omega)/\Psi^\lambda(\partial\Omega_\mp), \tag{9.68}$$

are isomorphisms. Finally, there exists $\varepsilon > 0$ such that

$$\begin{aligned} &0 < \alpha < \tfrac{1}{2} + \varepsilon \quad \text{if } n = 2, \\ &0 < \alpha < \varepsilon \quad \text{if } n = 3, \end{aligned} \tag{9.69}$$

the operators

$$\pm\tfrac{1}{2}\tfrac{1+\mu}{1-\mu}I + K_\lambda : C^\alpha(\partial\Omega) \longrightarrow C^\alpha(\partial\Omega), \qquad \mu \in (0,1), \tag{9.70}$$

$$\pm\tfrac{1}{2}I + K_\lambda : C^\alpha_{\nu_\pm}(\partial\Omega)/\Psi^\lambda(\partial\Omega_\mp) \longrightarrow C^\alpha_{\nu_\pm}(\partial\Omega)/\Psi^\lambda(\partial\Omega_\mp) \tag{9.71}$$

are also isomorphisms.

Proof. This follows from Theorem 9.11, the above discussion and duality. □

9.2 Main well-posedness results with nontangential maximal function estimates

We can now state some of our main results. The first involves the transmission problem.

Theorem 9.14 *Assume that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded Lipschitz domain and set $\Omega_+ := \Omega$, $\Omega_- := \mathbb{R}^n \backslash \bar{\Omega}$. Also, fix $\mu \in (0,1)$ and $\lambda \in (-1,1]$. Then there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ such that for each*

$$\tfrac{2(n-1)}{n+1} - \varepsilon < p < 2 + \varepsilon \tag{9.72}$$

the transmission boundary value problem, concerned with finding two pairs of functions $(\vec{u}_\pm, \pi_\pm)$ *in* $\Omega_\pm$ *satisfying*

$$\begin{cases} \Delta \vec{u}_\pm = \nabla \pi_\pm, \ \operatorname{div} \vec{u}_\pm = 0 \quad in \ \ \Omega_\pm, \\ M(\nabla \vec{u}_\pm), \, M(\pi_\pm) \in L^p(\partial\Omega), \\ \vec{u}_+\Big|_{\partial\Omega} - \vec{u}_-\Big|_{\partial\Omega} = \vec{g} \in h_1^p(\partial\Omega), \\ \partial_\nu^\lambda(\vec{u}_+, \pi_+) - \mu\, \partial_\nu(\vec{u}_-, \pi_-) = \vec{f} \in h^p(\partial\Omega), \end{cases} \tag{9.73}$$

and the decay conditions

$$\vec{u}_-(x) = \begin{cases} O(|x|^{2-n}) \quad as \ \ |x| \to \infty, \quad if \ \ n \geq 3, \\ -\tfrac{1}{\mu} E(x) \Big(\int_{\partial\Omega} \vec{f}\, d\sigma \Big) + O(|x|^{-1}) \quad as \ \ |x| \to \infty, \quad if \ n = 2, \end{cases} \tag{9.74}$$

$$\partial_j \vec{u}_-(x) = -\tfrac{1}{\mu} (\partial_j E)(x) \Big(\int_{\partial\Omega} \vec{f}\, d\sigma \Big) + O(|x|^{-n}) \ \ as \ \ |x| \to \infty, \ 1 \leq j \leq n, \tag{9.75}$$

$$\pi_-(x) = \begin{cases} O(|x|^{1-n}) \quad as \ \ |x| \to \infty, \quad if \ \ n \geq 3, \\ \tfrac{1}{\mu} (\nabla E_\Delta)(x) \cdot \Big(\int_{\partial\Omega} \vec{f}\, d\sigma \Big) + O(|x|^{-2}) \quad as \ \ |x| \to \infty, \quad if \ n = 2, \end{cases} \tag{9.76}$$

has a unique solution. In addition, there exists $C > 0$ *such that*

$$\|M(\nabla \vec{u}_\pm)\|_{L^p(\partial\Omega)} + \|M(\pi_\pm)\|_{L^p(\partial\Omega)} \leq C \|\vec{g}\|_{h_1^p(\partial\Omega)} + C \|\vec{f}\|_{h^p(\partial\Omega)}. \tag{9.77}$$

Proof. This follows directly from Theorem 9.3 and Theorem 9.9.

□

This leads us to our next result for the Dirichlet problem.

Theorem 9.15 *Assume that* $\Omega \subset \mathbb{R}^n$, $n \geq 2$, *is a bounded Lipschitz domain. Then there exists* $\varepsilon = \varepsilon(\partial\Omega) > 0$ *such that for each*

$$2 - \varepsilon < p < \infty \quad \textit{if} \;\; n = 2, 3, \tag{9.78}$$

$$2 - \varepsilon < p < \tfrac{2(n-1)}{n-3} + \varepsilon \quad \textit{if} \;\; n \geq 4, \tag{9.79}$$

the interior Dirichlet boundary value problem

$$\begin{cases} \Delta\vec{u} = \nabla\pi, \;\; \operatorname{div}\vec{u} = 0 \quad \textit{in} \;\; \Omega, \\ \\ M(\vec{u}) \in L^p(\partial\Omega), \\ \\ \vec{u}\Big|_{\partial\Omega} = \vec{f} \in L^p_{\nu_+}(\partial\Omega), \end{cases} \tag{9.80}$$

has a solution, which is unique modulo adding functions which are locally constant in Ω to the pressure term. In addition, there exists a finite constant $C > 0$ such that

$$\|M(\vec{u})\|_{L^p(\partial\Omega)} \leq C\|\vec{f}\|_{L^p(\partial\Omega)}. \tag{9.81}$$

Similar results are valid for the exterior Dirichlet problem, formulated much as (9.80) with the additional decay conditions

$$\vec{u}(x) = \begin{cases} O(|x|^{2-n}) \;\; \textit{as} \; |x| \to \infty, \quad \textit{if} \;\; n \geq 3, \\ \\ E(x)\vec{A} + O(1) \;\; \textit{as} \; |x| \to \infty, \;\; \textit{if} \; n = 2, \end{cases} \tag{9.82}$$

$$\partial_j\vec{u}(x) = \begin{cases} O(|x|^{1-n}) \;\; \textit{as} \; |x| \to \infty, \quad \textit{if} \;\; n \geq 3, \\ \\ \partial_j E(x)\vec{A} + O(|x|^{-2}) \;\; \textit{as} \; |x| \to \infty, \;\; \textit{if} \; n = 2, \end{cases} \tag{9.83}$$

$$\pi(x) = \begin{cases} O(|x|^{1-n}) \;\; \textit{as} \; |x| \to \infty, \quad \textit{if} \;\; n \geq 3, \\ \\ \langle \nabla E_\Delta(x), \vec{A} \rangle + O(|x|^{-2}) \;\; \textit{as} \; |x| \to \infty, \;\; \textit{if} \; n = 2, \end{cases} \tag{9.84}$$

for some a priori given constant $\vec{A} \in \mathbb{R}^2$. Also, the standard nontangential maximal operator in (9.81) should be replaced by its truncated version.

Proof. Fix $\lambda \in (-1, 1]$. From Corollary 9.12, for any $\vec{f} \in L^p_{\nu_+}(\partial\Omega)$, there exists $\vec{g}_1 \in L^p_{\nu_+}(\partial\Omega)$ and $\vec{\psi}_o \in \Psi^\lambda(\partial\Omega_-)$ such that $(\frac{1}{2}I + K_\lambda)\vec{g}_1 + \vec{\psi}_o = \vec{f}$. Since $\vec{\psi}_o \in L^p_\nu(\partial\Omega)$, according to Corollary 9.5, when $n \geq 3$ there exists $\vec{g}_2 \in L^p_{-1}(\partial\Omega)$ such that $S\vec{g}_2 = \vec{\psi}_o$. Then

$$\vec{u} := \mathcal{D}_\lambda \vec{g}_1 + \mathcal{S}\vec{g}_2 \qquad \text{and} \qquad \pi := \mathcal{P}_\lambda \vec{g}_1 + \mathcal{Q}\vec{g}_2 \tag{9.85}$$

will satisfy (9.80) and (9.81). The case $n = 2$ can be treated in a similar manner. In this case, using Corollary 9.7, we can instead find $\vec{g}_2 \in L^p_{-1}(\partial\Omega)$ and $\vec{c} \in \mathbb{R}^2$ such that $S\vec{g}_2 + \vec{c} = \vec{\psi}_o$. Then

$$\vec{u} := \mathcal{D}_\lambda \vec{g}_1 + \mathcal{S}\vec{g}_2 + \vec{c} \qquad \text{and} \qquad \pi := \mathcal{P}_\lambda \vec{g}_1 + \mathcal{Q}\vec{g}_2 \tag{9.86}$$

will satisfy (9.80) and (9.81). Existence of solutions for the exterior Dirichlet problem can be established in a similar fashion. This time, when $n = 2$, we can invoke Theorem 9.6 in order to be able to choose $\vec{g}_2$ such that

$$\int_{\partial\Omega} \vec{g}_2 \, d\sigma = \vec{A}, \tag{9.87}$$

which, in turn, will guarantee that the solution just constructed has the appropriate decay, as prescribed in (9.82)-(9.84). Finally, uniqueness in the case $p > 2$ follows from uniqueness for the case when p is near 2, which is guaranteed by Theorem 5.25. □

Theorem 9.16 *Assume that $\Omega \subset \mathbb{R}^n$, $n \in \{2, 3\}$, is a bounded Lipschitz domain. Then there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ such that if (9.69) holds then the interior Dirichlet boundary value problem*

$$\begin{cases} \Delta \vec{u} = \nabla \pi, \ \ \operatorname{div} \vec{u} = 0 \quad \text{in} \ \ \Omega, \\ \vec{u} \in C^\alpha(\bar{\Omega}), \\ \vec{u}\Big|_{\partial\Omega} = \vec{f} \in C^\alpha_{\nu_+}(\partial\Omega), \end{cases} \tag{9.88}$$

has a solution, which is unique modulo adding functions which are locally constant in Ω to the pressure term. In addition, there exists a finite constant $C > 0$ such that

$$\|\vec{u}\|_{C^\alpha(\bar{\Omega})} + \sup_{x \in \Omega} \Big[\operatorname{dist}(x, \partial\Omega)^{1-\alpha} |\nabla \vec{u}(x)| \Big] \le C \|\vec{f}\|_{C^\alpha(\partial\Omega)}. \tag{9.89}$$

Similar results are valid for the exterior Dirichlet problem with the additional decay conditions (9.82) imposed.

Proof. This is proved much as Theorem 9.15, with the help of Corollary 9.13. $\square$

We next discuss the case of the Dirichlet problem with data from BMO and VMO spaces. A few preliminaries are necessary. Given a Lipschitz domain $\Omega \subset \mathbb{R}^n$, define the set of *Carleson measures*, $Car\,(\Omega)$, as the subclass of Borelian measures μ on Ω satisfying

$$\|\mu\|_{Car\,(\Omega)} := \sup\left\{\frac{\mu(B(x,r)\cap\Omega)}{r^{n-1}} : x \in \partial\Omega,\ 0 < r < \operatorname{diam}(\partial\Omega)\right\} < \infty. \tag{9.90}$$

We shall also make use of a distinguished subclass, $Car_*(\Omega)$, of the space of Carleson measures in Ω, defined by

$$\mu \in Car_*(\Omega) \overset{def}{\Longleftrightarrow} \mu \in Car\,(\Omega) \quad \text{and} \quad \lim_{\delta\to 0}\left(\sup_{\substack{x\in\partial\Omega \\ 0<r<\delta}} \frac{\mu(B(x,r)\cap\Omega)}{r^{n-1}}\right) = 0. \tag{9.91}$$

Theorem 9.17 *Assume that $\Omega \subset \mathbb{R}^n$, $n \in \{2,3\}$, is a bounded Lipschitz domain. Then the interior Dirichlet boundary value problem*

$$\begin{cases} \Delta\vec{u} = \nabla\pi, \ \ \operatorname{div}\vec{u} = 0 \quad in \ \ \Omega, \\ |\nabla\vec{u}|^2 \operatorname{dist}(\cdot,\partial\Omega)\,dx \in Car\,(\Omega), \\ \vec{u}\Big|_{\partial\Omega} = \vec{f} \in \operatorname{bmo}_{\nu_+}(\partial\Omega), \end{cases} \tag{9.92}$$

has a solution, which is unique modulo adding functions which are locally constant in Ω to the pressure term. In addition, there exists a finite constant $C > 0$ such that

$$\|\,|\nabla\vec{u}|^2 \operatorname{dist}(\cdot,\partial\Omega)\,dx\|_{Car\,(\Omega)} \le C\|\vec{f}\|_{\operatorname{bmo}(\partial\Omega)}. \tag{9.93}$$

and

$$|\nabla\vec{u}|^2 \operatorname{dist}(\cdot,\partial\Omega)\,dx \in Car_*(\Omega) \Longleftrightarrow \vec{f} \in \operatorname{vmo}(\partial\Omega). \tag{9.94}$$

Similar results are valid for the exterior Dirichlet problem with the additional decay conditions (9.82) imposed.

Proof. The invertibility of the relevant boundary integral operators has been established in Corollary 9.13. With this in hand, the we proceed largely as in the proof of Theorem 9.15. The only novel aspect is that, in the current context, we need to know that the double layer operator $\mathcal{D}_\lambda$ maps functions from BMO on the boundary into densities of Carleson measures. This, however, is covered by the following general result. Let $k \in C^\infty(\mathbb{R}^n \setminus \{0\})$ be an odd function which is homogeneous of degree $-(n-1)$. Also, fix some $b \in L^\infty(\partial\Omega)$ and assume that the operator

$$\mathcal{T} f(x) := \int_{\partial\Omega} k(x-y)b(y)f(y)\, d\sigma(x), \quad x \in \Omega, \tag{9.95}$$

satisfies

$$\mathcal{T} 1 \equiv const \quad \text{ in } \ \Omega. \tag{9.96}$$

Then

$$\|\, (\mathcal{T} f)|_{\partial\Omega}\|_{\mathrm{bmo}\,(\partial\Omega)} + \|\, |\nabla \mathcal{T} f|^2 \mathrm{dist}\,(\cdot, \partial\Omega)\, dx\|_{Car\,(\Omega)} \leq C\|f\|_{\mathrm{bmo}\,(\partial\Omega)}. \tag{9.97}$$

See [69] for a proof of this claim. The proof of the theorem is therefore finished. $\square$

We now turn to the following result for the Regularity problem.

Theorem 9.18 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain. Then there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ such that for each p as in (9.72), the interior Regularity boundary value problem*

$$\begin{cases} \Delta \vec{u} = \nabla \pi, \ \ \mathrm{div}\, \vec{u} = 0 \quad in \ \ \Omega, \\ M(\nabla \vec{u}),\, M(\pi) \in L^p(\partial\Omega), \\ \vec{u}\Big|_{\partial\Omega} = \vec{f} \in h^p_{1,\nu_+}(\partial\Omega), \end{cases} \tag{9.98}$$

has a solution, which is unique modulo adding functions which are locally constant in Ω to the pressure.

In addition, there exists a finite constant $C > 0$ such that

$$\|M(\nabla \vec{u})\|_{L^p(\partial\Omega)} + \|M(\pi)\|_{L^p(\partial\Omega)} \leq C\|\vec{f}\|_{h^p_{1,\nu}(\partial\Omega)}. \tag{9.99}$$

Similar results are valid for the exterior Regularity problem, formulated much as (9.98) with the additional decay conditions (9.82)-(9.84).

Proof. Since the operator

$$\tfrac{1}{2}I + K_\lambda : h^p_{1,\nu_+}/\Psi^\lambda(\partial\Omega_-) \longrightarrow h^p_{1,\nu_+}/\Psi^\lambda(\partial\Omega_-) \tag{9.100}$$

is an isomorphism for each p as in (9.72), we can find $\vec{g}_1 \in h^p_{1,\nu_+}(\partial\Omega)$ and $\vec{\psi}_o \in \Psi^\lambda(\partial\Omega_-)$ such that $(\frac{1}{2}I + K_\lambda)\vec{g}_1 + \vec{\psi}_o = \vec{f}$. Since $\psi_o \in h^p_{1,\nu}$, if $n \geq 3$, it follows from Theorem 9.4 that there exists $\vec{g}_2 \in h^p(\partial\Omega)$ such that $S\vec{g}_2 = \psi_o$. Then

$$\vec{u} := \mathcal{D}_\lambda \vec{g}_1 + \mathcal{S}\vec{g}_2 \qquad \text{and} \qquad \pi := \mathcal{P}_\lambda \vec{g}_1 + \mathcal{Q}\vec{g}_2 \tag{9.101}$$

will satisfy (9.98) and (9.99). When $n = 2$, it follows from Theorem 9.6 that there exists $\vec{g}_2 \in h^p(\partial\Omega)$ and $\vec{c} \in \mathbb{R}^2$ such that $S\vec{g}_2 + \vec{c} = \psi_o$. In this case,

$$\vec{u} := \mathcal{D}_\lambda \vec{g}_1 + \mathcal{S}\vec{g}_2 + \vec{c} \qquad \text{and} \qquad \pi := \mathcal{P}_\lambda \vec{g}_1 + \mathcal{Q}\vec{g}_2 \tag{9.102}$$

will satisfy (9.98) and (9.99). Existence of solutions for the exterior regularity problem can be established in a similar fashion. Much as in the case of the Dirichlet problem, when $n = 2$, it is possible to choose $\vec{g}_2$ such that (9.87) holds. This guarantees that our solution has the appropriate decay, as prescribed in (9.82)-(9.84). As for uniqueness, an inspection of the corresponding argument in the proof of Theorem 5.24 shows that the same technique can be used in the current context as well. □

We finish this section with the following result for the Neumann problem.

Theorem 9.19 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain and fix $\lambda \in (-1,1]$. Then there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ such that for each p as in (9.72), the interior Neumann boundary value problem*

$$\begin{cases} \Delta \vec{u} = \nabla \pi, \;\; \operatorname{div} \vec{u} = 0 \quad in \;\; \Omega, \\ M(\nabla \vec{u}),\, M(\pi) \in L^p(\partial\Omega), \\ \partial_\nu^\lambda(\vec{u},\pi) = \vec{f} \in h^p(\partial\Omega), \end{cases} \tag{9.103}$$

has a solution if and only if

$$\vec{f} \in \operatorname{Im}\Big(-\tfrac{1}{2}I + K_\lambda^* : h^p_{\Psi_+^\lambda}(\partial\Omega) \to h^p_{\Psi_+^\lambda}(\partial\Omega)\Big). \tag{9.104}$$

Moreover, this solution is unique modulo adding to the velocity field functions from $\Psi^\lambda(\Omega)$. In addition, there exists a finite constant $C > 0$ such that

$$\|M(\nabla \vec{u})\|_{L^p(\partial\Omega)} + \|M(\pi)\|_{L^p(\partial\Omega)} \leq C\|\vec{f}\|_{h^p(\partial\Omega)}. \tag{9.105}$$

Finally, a similar result holds for the exterior domain $\mathbb{R}^n \setminus \bar{\Omega}$ if we include the decay conditions

$$\vec{u}(x) = \begin{cases} O(|x|^{2-n}) \quad as \;\; |x| \to \infty, \quad if \;\; n \geq 3, \\ E(x)\Big(\int_{\partial\Omega} \vec{f}\, d\sigma\Big) + O(|x|^{-1}) \quad as \;\; |x| \to \infty, \;\; if \;\; n = 2, \end{cases} \tag{9.106}$$

$$\partial_j \vec{u}(x) = (\partial_j E)(x)\Big(\int_{\partial\Omega} \vec{f}\, d\sigma\Big) + O(|x|^{-n}) \;\; as \;\; |x| \to \infty, \; 1 \leq j \leq n, \tag{9.107}$$

$$\pi(x) = \begin{cases} O(|x|^{1-n}) \quad as \;\; |x| \to \infty, \quad if \;\; n \geq 3, \\ \Big\langle (-\nabla E_\Delta)(x)\,,\, \int_{\partial\Omega} \vec{f}\, d\sigma \Big\rangle + O(|x|^{-2}) \quad as \;\; |x| \to \infty, \;\; if \;\; n = 2. \end{cases} \tag{9.108}$$

More precisely, a solution to the exterior problem satisfying the above decay conditions exists if and only if

$$\vec{f} \in \operatorname{Im}\Big(\tfrac{1}{2}I + K_\lambda^* : L^p_{\Psi_-^\lambda}(\partial\Omega) \to L^p_{\Psi_-^\lambda}(\partial\Omega)\Big), \tag{9.109}$$

and solutions are unique modulo adding to the velocity field functions from $\Psi^\lambda(\mathbb{R}^n \setminus \bar{\Omega})$.

Proof. Since we have established in Theorem 9.11 that the operators (9.54) are isomorphisms and also that (9.14) holds for each p in the desired range, the proof that a solution exists if and only if $\vec{f}$ is as in (9.104) follows exactly as in the proof of Theorem 5.23. The claim for the exterior Neumann problem, along with the corresponding uniqueness statement, follows similarly. $\square$

10 The Poisson problem for the Stokes system

10.1 Stokes-Besov and Stokes-Triebel-Lizorkin spaces

Here we shall adapt the standard Triebel-Lizorkin and Besov scales to the Stokes system. Concretely, for a bounded Lipschitz domain Ω in $\mathbb{R}^n$, $n \geq 2$, and $0 < p, q \leq \infty$, $\alpha \in \mathbb{R}$, we set

$$SB^{p,q}_\alpha(\Omega) := \Big\{(\vec{u}, \pi) \in B^{p,q}_\alpha(\Omega) \oplus B^{p,q}_{\alpha-1}(\Omega) : \Delta\vec{u} - \nabla\pi = 0, \text{ div}\,\vec{u} = 0 \text{ in } \Omega\Big\}, \quad (10.1)$$

$$SF^{p,q}_\alpha(\Omega) := \Big\{(\vec{u}, \pi) \in F^{p,q}_\alpha(\Omega) \oplus F^{p,q}_{\alpha-1}(\Omega) : \Delta\vec{u} - \nabla\pi = 0, \text{ div}\,\vec{u} = 0 \text{ in } \Omega\Big\}, \quad (10.2)$$

(with the convention that $p < \infty$ in the latter case) equipped with the norms $\|\cdot\|_{SF^{p,q}_\alpha(\Omega)}$, $\|\cdot\|_{SB^{p,q}_\alpha(\Omega)}$, naturally induced by $B^{p,q}_\alpha(\Omega) \oplus B^{p,q}_{\alpha-1}(\Omega)$ and $F^{p,q}_\alpha(\Omega) \oplus F^{p,q}_{\alpha-1}(\Omega)$, respectively. In particular,

$$SF^{p,p}_\alpha(\Omega) = SB^{p,p}_\alpha(\Omega) \text{ for every } \alpha \in \mathbb{R}, \;\; 0 < p < \infty. \quad (10.3)$$

Our next few results focus on some of the properties of these spaces.

Theorem 10.1 *Let* $\Omega \subset \mathbb{R}^n$, $n \geq 2$, *be a bounded Lipschitz domain. Then for every* $\alpha \in \mathbb{R}$, $0 < p < \infty$,

$$SF^{p,q}_\alpha(\Omega) \textit{ is independent of } q \in (0, \infty). \quad (10.4)$$

Furthermore, for any $p \in (\frac{n-1}{n}, 2]$, $q \in (0, \infty)$ *there exists* $C = C(\Omega, p, q) > 0$ *such that*

$$\|M(\nabla\vec{u})\|_{L^p(\partial\Omega)} + \|M(\pi)\|_{L^p(\partial\Omega)} \leq C\|(\vec{u}, \pi)\|_{SF^{p,q}_{1+1/p}(\Omega)}. \tag{10.5}$$

Proof. If $q_1, q_2 \in (0, \infty)$, we have

$$\begin{aligned} SF^{p,q_1}_{\alpha}(\Omega) &\subset \Big[F^{p,q_1}_{\alpha}(\Omega) \cap \operatorname{Ker}\Delta^2\Big] \oplus \Big[F^{p,q_1}_{\alpha-1}(\Omega) \cap \operatorname{Ker}\Delta\Big] \\ &= \Big[F^{p,q_2}_{\alpha}(\Omega) \cap \operatorname{Ker}\Delta^2\Big] \oplus \Big[F^{p,q_2}_{\alpha-1}(\Omega) \cap \operatorname{Ker}\Delta\Big], \end{aligned} \tag{10.6}$$

by Theorem 11.15. Thus, $SF^{p,q_1}_{\alpha}(\Omega) \subset F^{p,q_2}_{\alpha}(\Omega) \oplus F^{p,q_2}_{\alpha-1}(\Omega)$ and, hence, $SF^{p,q_1}_{\alpha}(\Omega) \subseteq SF^{p,q_2}_{\alpha}(\Omega)$. Similarly, $SF^{p,q_2}_{\alpha}(\Omega) \subseteq SF^{p,q_1}_{\alpha}(\Omega)$, so ultimately, $SF^{p,q_1}_{\alpha}(\Omega) = SF^{p,q_2}_{\alpha}(\Omega)$, proving (10.4). Finally, (10.5) is a consequence of (10.6) and Theorem 11.16. □

Corollary 10.2 *Let* $\Omega \subset \mathbb{R}^n$, $n \geq 2$, *be a bounded Lipschitz domain. Then for each* $\lambda \in \mathbb{R}$, *the conormal derivative assignment* $(\vec{u}, \pi) \mapsto \partial^{\lambda}_{\nu}(\vec{u}, \pi)$ *induces a bounded operator*

$$\partial^{\lambda}_{\nu} : SF^{p,q}_{1+1/p}(\Omega) \longrightarrow h^p(\partial\Omega) \tag{10.7}$$

whenever $\frac{n-1}{n} < p \leq 2$ *and* $0 < q < \infty$.

Proof. This follows directly from (10.5) and Theorem 4.13. □

Recall that $(\cdot, \cdot)_{\theta,p}$ and $[\cdot, \cdot]_{\theta}$ stand, respectively, for the real and the complex method of interpolation.

Theorem 10.3 *Let* $\Omega \subset \mathbb{R}^n$, $n \geq 2$, *be a bounded Lipschitz domain and assume that* $0 < q_0, q_1, q \leq \infty$, $\alpha_0, \alpha_1 \in \mathbb{R}$, $\alpha_0 \neq \alpha_1$, $0 < \theta < 1$. *Also, set* $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$. *Then, if* $0 < p < \infty$,

$$\Big(SF^{p,q_0}_{\alpha_0}(\Omega)\,,\, SF^{p,q_1}_{\alpha_1}(\Omega)\Big)_{\theta,q} = SB^{p,q}_{\alpha}(\Omega), \tag{10.8}$$

and if $0 < p \leq \infty$,

$$\left(SB^{p,q_0}_{\alpha_0}(\Omega)\,,\,SB^{p,q_1}_{\alpha_1}(\Omega)\right)_{\theta,q} = SB^{p,q}_{\alpha}(\Omega). \tag{10.9}$$

Let $0 < p_0, p_1 < \infty$, $0 < q_0, q_1 \leq \infty$ *with* $\min\{q_0, q_1\} < \infty$, $\alpha_0, \alpha_1 \in \mathbb{R}$, $0 < \theta < 1$ *and set* $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, *and* $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. *Then*

$$\left[SF^{p_0,q_0}_{\alpha_0}(\Omega)\,,\,SF^{p_1,q_1}_{\alpha_1}(\Omega)\right]_{\theta} = SF^{p,q}_{\alpha}(\Omega). \tag{10.10}$$

Finally, if $\alpha_0, \alpha_1 \in \mathbb{R}$, $0 < p_0, p_1, q_0, q_1 \leq \infty$ *with* $\min\{q_0, q_1\} < \infty$, *then*

$$\left[SB^{p_0,q_0}_{\alpha_0}(\Omega)\,,\,SB^{p_1,q_1}_{\alpha_1}(\Omega)\right]_{\theta} = SB^{p,q}_{\alpha}(\Omega), \tag{10.11}$$

where $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, *and* $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

Proof. Fix an open cube $Q \subset \mathbb{R}^n$ containing $\overline{\Omega}$, and for and $i = 0, 1$, set

$$\begin{aligned} X_i &:= F^{p_i,q_i}_{\alpha_i}(\Omega) \oplus F^{p_i,q_i}_{\alpha_i - 1}(\Omega), \quad Z_i := F^{p_i,q_i}_{\alpha_i - 2,0}(Q) \oplus F^{p_i,q_i}_{\alpha_i - 1,0}(Q), \\ Y_i &:= F^{p_i,q_i}_{\alpha_i - 2,0}(Q \setminus \overline{\Omega}) \oplus F^{p_i,q_i}_{\alpha_i - 1,0}(Q \setminus \overline{\Omega}) \hookrightarrow Z_i. \end{aligned} \tag{10.12}$$

As discussed in [50], the spaces $X_0 + X_1$ and $Y_0 + Y_1$ are analytically convex (cf. the discussion preceding (11.144) for a definition). Let E^Q_Ω denote Rychkov's extension operator truncated near $\overline{\Omega}$ so that it maps the distributions from the Triebel-Lizorkin scale in Ω to distributions supported in the cube Q, with preservation of smoothness. Also, set $L(\vec{u}, \pi) := (\Delta\vec{u} - \nabla\pi, \operatorname{div}\vec{u})$ and

$$\Pi\vec{u}(x) := \int_{\mathbb{R}^n} E(x-y)\vec{u}(y)\,dy, \qquad x \in \mathbb{R}^n, \tag{10.13}$$

$$\Theta\vec{u}(x) := \int_{\mathbb{R}^n} \langle q(x-y), \vec{u}(y)\rangle\,dy, \qquad x \in \mathbb{R}^n, \tag{10.14}$$

$$\Pi_\Delta f(x) := \int_{\mathbb{R}^n} E_\Delta(x-y) f(y)\,dy, \qquad x \in \mathbb{R}^n. \tag{10.15}$$

In particular,

$$\Delta\Pi - \nabla\Theta = I, \qquad \operatorname{div}\Pi = 0, \qquad \Delta\Pi_\Delta = I, \tag{10.16}$$

where I stands for the identity operator. The intention is to use Lemma 11.43 with $D := L\circ E^Q_\Omega$ and

$$G(\vec{w},f) := \Big(\mathcal{R}_\Omega\Big(\Pi\vec{w}+\nabla\Pi_\Delta f\Big)\,,\,\mathcal{R}_\Omega\Big(\Theta\vec{w}+f\Big)\Big), \tag{10.17}$$

where $\mathcal{R}_\Omega$ is the operator of restriction to Ω. Note that, in the notation of Lemma 11.43, $X_i(D) = SF^{p_i,q_i}_{\alpha_i}(\Omega)$ for $i=0,1$. There remains to check that $K := D\circ G - I$, as a bounded linear operator from Z_i into itself, actually maps Z_i into Y_i, $i=0,1$. To this end, for every pair of test functions $(\vec{\phi},\psi)\in C^\infty_c(\Omega)\oplus C^\infty_c(\Omega)$, and every $(\vec{w},f)\in Z_i$, we compute

$$\begin{aligned}
&\langle (D\circ G - I)(\vec{w},f),(\vec{\phi},\psi)\rangle\\
&\quad = \Big\langle \Big(\Delta\Big[\Pi\vec{w}+\nabla\Pi_\Delta f\Big]\Big|_\Omega - \nabla\Big[\Theta\vec{w}+f\Big]\Big|_\Omega, \operatorname{div}\Big[\Pi\vec{w}+\nabla\Pi_\Delta f\Big]\Big|_\Omega\Big)\,,\,(\vec{\phi},\psi)\Big\rangle\\
&\qquad - \Big\langle (\vec{w},f),(\vec{\phi},\psi)\Big\rangle = 0.
\end{aligned}\tag{10.18}$$

Hence, $K(\vec{w},f)=0$ in Ω which proves that K maps Z_i into Y_i. Then (10.8) and (10.10) follow from Lemma 11.43. A similar argument works for the Besov scale and this finishes the proof of the theorem. $\square$

10.2 Conormal derivatives on Stokes-Besov and Stokes-Triebel-Lizorkin scales

Let X be a Banach space with dual X^*. For every $n\times n$ matrix $F = (F^\alpha_j)_{\alpha,j}$ with entries from X, and every $n\times n$ matrix $G=(G^\beta_k)_{\beta,k}$ with entries from X^*, and each $\lambda\in\mathbb{R}$, we set

$$\mathbb{A}_\lambda(F,G) := a^{\alpha\beta}_{jk}(\lambda)\langle F^\alpha_j, G^\beta_k\rangle, \tag{10.19}$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing between X and X^*, and $a^{\alpha\beta}_{jk}(\lambda)$ are as in (4.1). While our notation does not emphasize the dependence of $\langle\cdot,\cdot\rangle$ and $\mathbb{A}_\lambda$ on X, the particular nature of X should be clear from the context in each case.

The main results of this section are as follows.

Proposition 10.4 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain and assume that $0 < s < 1$, $1 < p, q < \infty$, $\lambda \in \mathbb{R}$. Then*

$$\partial_\nu^\lambda : SB^{p,q}_{s+1/p}(\Omega) \longrightarrow B^{p,q}_{s-1}(\partial\Omega) \tag{10.20}$$

given by

$$\Big\langle \partial_\nu^\lambda(\vec{u}, \pi)\,,\,\vec{\psi}\Big\rangle := \mathbb{A}_\lambda\Big(\nabla\vec{u}, \nabla\mathrm{Ex}(\vec{\psi})\Big) - \Big\langle \pi, \mathrm{div}\,\mathrm{Ex}(\vec{\psi})\Big\rangle, \qquad \forall\,\vec{\psi} \in B^{p',q'}_{1-s}(\partial\Omega), \tag{10.21}$$

is a well-defined, bounded operator, where Ex *is the extension operator introduced in Theorem 2.18 and $1/p + 1/p' = 1$, $1/q + 1/q' = 1$.*

Furthermore, for every $(\vec{u}, \pi) \in SB^{p,q}_{s+1/p}(\Omega)$ and $\vec{w} \in B^{p',q'}_{1-s+1/p'}(\Omega)$, the following integration by parts formula holds:

$$\mathbb{A}_\lambda\Big(\nabla\vec{u}, \nabla\vec{w}\Big) = \Big\langle \pi, \mathrm{div}\,\vec{w}\Big\rangle + \Big\langle \partial_\nu^\lambda(\vec{u}, \pi)\,,\,\mathrm{Tr}\,\vec{w}\Big\rangle. \tag{10.22}$$

Proof. Assume that $(\vec{u}, \pi) \in SB^{p,q}_{s+1/p}(\Omega)$. Then $\vec{u} \in B^{p,q}_{s+\frac{1}{p}}(\Omega)$, $\pi \in B^{p,q}_{s+\frac{1}{p}-1}(\Omega)$ and we have $\Delta\vec{u} - \nabla\pi = 0$, $\mathrm{div}\,\vec{u} = 0$ in Ω. Also, $\vec{\psi} \in B^{p',q'}_{1-s}(\partial\Omega)$ forces $\mathrm{Ex}(\vec{\psi}) \in B^{p',q'}_{1-s+1/p'}(\Omega)$. Consequently, thanks to Proposition 2.15, the matrix $\nabla\mathrm{Ex}(\vec{\psi}) \in B^{p',q'}_{1-s-1/p}(\Omega) = \Big(B^{p,q}_{s+1/p-1}(\Omega)\Big)^*$ pairs well with $\nabla\vec{u} \in B^{p,q}_{s+\frac{1}{p}-1}(\Omega)$. In a similar fashion, $\mathrm{div}\,\mathrm{Ex}(\vec{\psi}) \in \Big(B^{p,q}_{s+1/p-1}(\Omega)\Big)^*$ pairs well with $\pi \in B^{p,q}_{s+\frac{1}{p}-1}(\Omega)$. This shows that $\partial_\nu^\lambda(\vec{u}, \pi) \in \Big(B^{p',q'}_{1-s}(\partial\Omega)\Big)^* = B^{p,q}_{s-1}(\partial\Omega)$ and

$$\|\partial_\nu^\lambda(\vec{u}, \pi)\|_{B^{p,q}_{s-1}(\partial\Omega)} \leq C\|\vec{u}\|_{B^{p,q}_{s+\frac{1}{p}}(\Omega)} + C\|\pi\|_{B^{p,q}_{s+\frac{1}{p}-1}(\Omega)}. \tag{10.23}$$

This finishes the proof of the well-posedness and boundedness of the operator (10.20)-(10.21).

Going further, what we have proved up to this point yields

$$\Big\langle \partial_\nu^\lambda(\vec{u}, \pi)\,,\,\mathrm{Tr}\,\vec{w}\Big\rangle = \mathbb{A}_\lambda\Big(\nabla\vec{u}, \nabla\mathrm{Ex}(\mathrm{Tr}\,\vec{w})\Big) - \Big\langle \pi, \mathrm{div}\,\mathrm{Ex}(\mathrm{Tr})\,\vec{w}\Big\rangle \tag{10.24}$$

so (10.22) follows as soon as we establish that

$$\mathbb{A}_\lambda\Big(\nabla\vec{u},\nabla\vec{w}\Big)-\Big\langle\pi,\operatorname{div}\vec{w}\Big\rangle=0,\qquad\forall\,\vec{w}\in B^{p'q'}_{1-s+1/p'}(\Omega)\text{ with }\operatorname{Tr}\vec{w}=0.\tag{10.25}$$

Since, by Theorem 2.19, $C^\infty_c(\Omega)$ is dense in $\{\vec{w}\in B^{p'q'}_{1-s+1/p'}(\Omega):\operatorname{Tr}\vec{w}=0\}$, it suffices to prove (10.25) when $\vec{w}\in C^\infty_c(\Omega)$. However, in this scenario, the identity in (10.25) follows from the fact that $\Delta\vec{u}-\nabla\pi=0$ in the sense of distributions in Ω. □

Proposition 10.5 *Assume that $\Omega\subset\mathbb{R}^n$, $n\geq 2$, is a bounded Lipschitz domain and that $0<s<1$, $1<p,q<\infty$, $\lambda\in\mathbb{R}$. Then*

$$\partial^\lambda_\nu:SF^{p,q}_{s+1/p}(\Omega)\longrightarrow B^{p,p}_{s-1}(\partial\Omega)\tag{10.26}$$

given by

$$\Big\langle\partial^\lambda_\nu(\vec{u},\pi)\,,\,\vec{\psi}\Big\rangle:=\mathbb{A}_\lambda\Big(\nabla\vec{u},\nabla\mathrm{Ex}(\vec{\psi})\Big)-\Big\langle\pi,\operatorname{div}\mathrm{Ex}(\vec{\psi})\Big\rangle,\qquad\forall\,\vec{\psi}\in B^{p',p'}_{1-s}(\partial\Omega),\tag{10.27}$$

is a well-defined, bounded operator, where Ex *is the extension operator introduced in Theorem 2.18 and $1/p+1/p'=1$, $1/q+1/q'=1$.*

In addition, the following identity holds for any $(\vec{u},\pi)\in SF^{p,q}_{s+1/p}(\Omega)$, $\vec{w}\in F^{p',q'}_{1-s+1/p'}(\Omega)$:

$$\mathbb{A}_\lambda\Big(\nabla\vec{u},\nabla\vec{w}\Big)=\Big\langle\pi,\operatorname{div}\vec{w}\Big\rangle+\Big\langle\partial^\lambda_\nu(\vec{u},\pi)\,,\,\operatorname{Tr}\vec{w}\Big\rangle.\tag{10.28}$$

Proof. This closely parallels that of Proposition 10.4. □

Note that the definitions (10.21)-(10.27) correspond to a formal application of Green's formula (4.6). The applicability of this point of view is limited to the range $1<p,q<\infty$, as $B^{p,q}_{s-1}(\partial\Omega)$ fails to be a dual space if $\min\{p,q\}\leq 1$. We nonetheless have:

Theorem 10.6 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n\geq 2$. Also, assume that $\lambda\in\mathbb{R}$. Then the conormal operator from Proposition 10.4 extends to a bounded mapping*

$$\begin{gathered}\partial^\lambda_\nu:SB^{p,q}_{s+1/p}(\Omega)\longrightarrow B^{p,q}_{s-1}(\partial\Omega),\quad\textit{whenever}\\ \tfrac{n-1}{n}<p<\infty,\;0<q\leq\infty,\;(n-1)\Big(\tfrac{1}{p}-1\Big)_+<s<1.\end{gathered}\tag{10.29}$$

Analogously, the conormal operator from Proposition 10.5 extends to a bounded mapping

$$\begin{aligned} &\partial_\nu^\lambda : SF^{p,q}_{s+1/p}(\Omega) \longrightarrow B^{p,p}_{s-1}(\partial\Omega), \quad \textit{whenever} \\ &\tfrac{n-1}{n} < p < \infty,\ 0 < q < \infty,\ (n-1)\Big(\tfrac{1}{p} - 1\Big)_+ < s < 1. \end{aligned} \tag{10.30}$$

Proof. Call a point in $\mathbb{R}^3$ with coordinates $(s, 1/p, 1/q)$ "good" if

$$\partial_\nu^\lambda : SF^{p,q}_{s+1/p}(\Omega) \longrightarrow F^{p,q}_{s-1}(\partial\Omega) \quad \text{is well-defined and bounded.} \tag{10.31}$$

Furthermore, call a region $E \subset \mathbb{R}^3$ "good" if all points in E are good. Then by Propositions 10.4-10.5 and Corollary 10.2, the following set is good:

$$\Big\{\Big(s, \tfrac{1}{p}, \tfrac{1}{p}\Big) : 1 < p < \infty,\ 0 < s < 1\Big\} \quad \text{and} \quad \Big\{\Big(1, \tfrac{1}{p}, \tfrac{1}{2}\Big) : \tfrac{n-1}{n} < p \leq 2\Big\}. \tag{10.32}$$

Also, by Theorem 10.3 and Proposition 2.24,

$$E \text{ good} \implies \text{the convex hull of } E \text{ is good.} \tag{10.33}$$

Finally, if for any $E \subset \mathbb{R}^3$ we denote by $\mathrm{Pr}_{xy}E$ the projection of E onto the (horizontal) xy-plane, we note that

$$\begin{aligned} E \text{ good open set in } \mathbb{R}^3 \implies \partial_\nu^\lambda : SF^{p,q}_{s+1/p}(\Omega) \longrightarrow B^{p,p}_{s-1}(\partial\Omega) \text{ is bounded} \\ \text{whenever } (s, 1/p) \in \mathrm{Pr}_{xy}E \text{ and } 0 < q < \infty. \end{aligned} \tag{10.34}$$

Indeed, this is a consequence of (10.8) and (2.162) (with $p = q$), plus (10.4) and the fact that diagonal Besov and Triebel-Lizorkin spaces coincide.

With this information available, the end-game in the proof of the theorem is as follows. First, by (10.32)-(10.33), the interior of the parallelogram with vertices at

$$O(0,0,0), \quad A(1,0,0), \quad B(1,1,1), \quad C(0,1,1) \tag{10.35}$$

is a good set, and so is the segment with end-points

$$P(1, \tfrac{1}{2}, \tfrac{1}{2}), \qquad Q(1, \tfrac{n}{n-1}, \tfrac{1}{2}). \tag{10.36}$$

See picture below:

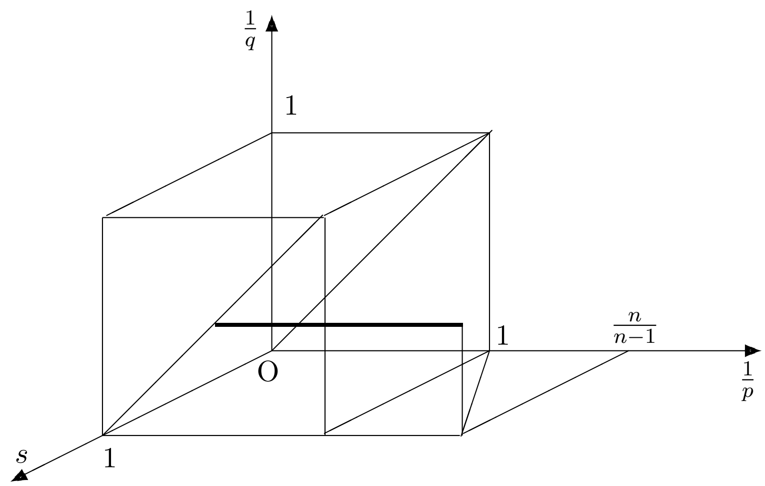

Figure 5

By (10.33), it follows that the pyramid with vertex at Q (given in (10.36)) and whose base is the parallelogram with vertices as in (10.35) is good. Since the projection of this pyramid on the $(s, 1/p)$-plane is the region described by

$$\Big\{\Big(s, \tfrac{1}{p}\Big) : 0 < p < \infty,\ (n-1)\Big(\tfrac{1}{p} - 1\Big)_+ < s < 1\Big\}, \tag{10.37}$$

it follows that the conormal derivative operator is bounded under the conditions specified in (10.30).

Finally, the corresponding claim about (10.29) is a consequence of what we have just proved, (10.8) and (2.162). This finishes the proof of the theorem. $\square$

10.3 The conormal derivative of the Stokes-Newtonian potentials

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain and assume that $\frac{n-1}{n} < p \leq 1$, $(n-1)(\frac{1}{p} - 1) < s < 1$. Call $m_S \in L^\infty(\partial\Omega)$ a $B^{p,p}_{s-1}(\partial\Omega)$ molecule if there exist $M > \frac{n-1}{p}$ and a surface ball S centered at $x_S \in \partial\Omega$ and having radius $r \in (0, \operatorname{diam}\Omega)$ such that

$$(1)\ |m_S(x)| \leq r^{s-1-\frac{n-1}{p}}(1+r^{-1}|x-x_S|)^{-M+s-1} \quad \text{for } x \in \partial\Omega, \tag{10.38}$$

$$(2)\ \int_{\partial\Omega} m_S(x)\, d\sigma_x = 0 \ \text{ if } \ r < \eta. \tag{10.39}$$

The molecular theory developed by M. Frazier and B. Jawerth in the Euclidean setting can be adapted to the case of Lipschitz surfaces. In particular, we have (see [64] for a proof):

Proposition 10.7 *Let* $(n-1)/n < p \leq 1$ *and* $(n-1)(\frac{1}{p}-1) < s < 1$*. Then, given an arbitrary bounded Lipschitz domain* $\Omega \subset \mathbb{R}^n$*,* $n \geq 2$*, there exists* $\eta = \eta(\partial\Omega) > 0$ *such that*

$$\begin{aligned} \|f\|_{B^{p,p}_{s-1}(\partial\Omega)} \ \approx \ \inf\Big\{ \Big(\sum_S |\lambda_S|^p\Big)^{1/p} : \\ f = \sum_S \lambda_S m_S,\ m_S\text{'s are } B^{p,p}_{s-1}(\partial\Omega) \text{ molecules},\ \{\lambda_S\}_S \in \ell^p \Big\} \end{aligned} \tag{10.40}$$

uniformly for $f \in B^{p,p}_{s-1}(\partial\Omega)$.

Conversely, there exists $C = C(\partial\Omega, s, p, M, n) > 0$ *such that for any countable family* $\{m_S\}_S$ *of* $B^{p,p}_{s-1}(\partial\Omega)$ *molecules and any numerical sequence* $\{\lambda_S\}_S \in \ell^p$*,*

$$\Big\|\sum_S \lambda_S m_S\Big\|_{B^{p,p}_{s-1}(\partial\Omega)} \leq C\|\{\lambda_S\}_S\|_{\ell^p}. \tag{10.41}$$

Assume that $s \in \mathbb{R}$, $0 < p \leq 1$, $p \leq q \leq \infty$, and $p < p_1 < +\infty$, define $J := \frac{n}{p}$, and fix an integer $L \geq \max\{[J-n-s], -1\}$. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, $\beta \in \mathbb{N}_o$ and $\rho > 1$ are constants depending on Ω. Under these circumstances, call a function A_Q a *rough atom for* $F^{p,q}_{s,0}(\Omega)$ if

$$(1)\ \exists\, Q \in \mathbb{R}^n \text{ such that } \operatorname{supp} A \subseteq Q \subset \Omega \text{ and } \rho\, Q \subset \Omega, \tag{10.42}$$

$$(2)\ \|A\|_{F^{p_1,q}_s(\mathbb{R}^n)} \leq |Q|^{1/p_1 - 1/p}, \tag{10.43}$$

$$(3)\ \int_{\mathbb{R}^n} x^\gamma A(x)\, dx = 0 \ \text{ if } \ |\gamma| \leq L \ \text{ and } \ l(Q) < 2^{-\beta}. \tag{10.44}$$

The following result has been proved in [64].

Theorem 10.8 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, and assume that p, q, s, p_1, J, L are as above. Then there exist $\beta \in \mathbb{N}_o$ and $\rho > 1$ such that any $f \in F^{p,q}_{s,0}(\Omega)$ can be expanded in a series*

$$f = \sum_{k \in \mathbb{Z}} \lambda_k A_k \quad \textit{with convergence in} \quad S'(\mathbb{R}^n), \tag{10.45}$$

where the atoms A_k satisfy (10.42)-(10.44) and $\{\lambda_k\}_{k\in\mathbb{Z}} \in \ell^p$. Furthermore,

$$\|f\|_{F^{p,q}_{s,0}(\Omega)} \approx \inf\Big\{ \|\{\lambda_k\}_k\|_{\ell^p};\, f = \sum \lambda_k A_k \Big\}, \tag{10.46}$$

where the infimum is taken over all possible representations of f in a series of atoms satisfying (10.42)-(10.44).

We are now in a position to discuss the main result of this section.

Theorem 10.9 *Consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and supposed p, q, s are fixed such that $\frac{n-1}{n} < p \leq \infty$, $(n-1)(1/p-1)_+ < s < 1$ and $0 < q \leq \infty$. Then, for each $\lambda \in \mathbb{R}$,*

$$\partial^\lambda_\nu(\Pi, \Theta) : B^{p,q}_{s+1/p-2,0}(\Omega) \longrightarrow B^{p,q}_{s-1}(\partial\Omega), \tag{10.47}$$

$$\partial^\lambda_\nu(\Pi, \Theta) : F^{p,q}_{s+1/p-2,0}(\Omega) \longrightarrow B^{p,p}_{s-1}(\partial\Omega), \quad \textit{if } p \neq \infty, \tag{10.48}$$

are well-defined, linear, and bounded operators.

Proof. We start with implication (10.48) for $\frac{n-1}{n} < p \leq 1$, $(n-1)(1/p-1) < s < 1$ and $p \leq q \leq \infty$. By Proposition 10.7 and Theorem 10.8, it is enough to show that $\partial^\lambda_\nu(\Pi, \Theta)$ maps rough $F^{p,q}_{s+1/p-2,0}(\Omega)$-atoms to $B^{p,p}_{s-1}(\partial\Omega)$-molecules.

Note that current restrictions on indices imply that rough $F^{p,q}_{s+1/p-2,0}(\Omega)$-atoms satisfy (10.42)–(10.44) with $L \geq 0$. Consider first such a rough atom A supported in a Whitney cube $Q \subset \Omega$, with center $x_Q \in Q$ and pick $x_S \in \partial\Omega$ such that $|x_Q - x_S| = \operatorname{dist}(x_Q, \partial\Omega)$. Then set $m := \partial^\lambda_\nu(\Pi(A), \Theta(A))$ on $\partial\Omega$ which, so we claim, is a molecule for $B^{p,p}_{s-1}(\partial\Omega)$ concentrated about the surface ball $S := B(x_S, l(Q)) \cap \partial\Omega$.

The claim will be justified by checking (10.38)-(10.39). Take the vanishing moment condition, required when $l(Q)$ is small. Assuming that this is the case, A has one vanishing moment and, for every $\vec{c}\in\mathbb{R}^n$,

$$
\begin{aligned}
\Big\langle \int_{\partial\Omega} m\,d\sigma\,,\,\vec{c}\Big\rangle &= \int_{\partial\Omega}\langle m,\vec{c}\rangle\,d\sigma = \int_{\partial\Omega}\langle\partial^{\lambda}_{\nu}(\Pi A,\Theta A),\vec{c}\rangle\,d\sigma \\
&= \int_{\Omega}\langle\Delta\Pi A-\nabla\Theta A,\vec{c}\rangle\,dx = \int_{\Omega}\langle A,\vec{c}\rangle\,dx \\
&= \Big\langle\int_{\mathbb{R}^n}A\,dx\,,\,\vec{c}\Big\rangle = 0, \qquad (10.49)
\end{aligned}
$$

by Green's formula (4.6), written with $\vec{u}=\Pi A$, $\pi=\Theta A$, $\vec{w}=\vec{c}$, $\rho=0$, the first identity in (10.16) and the support condition on A. Thus, $\int_{\partial\Omega}m\,d\sigma=0$, as desired.

Turning to size estimates, we observe that m can be expressed in the form (recall that x_Q is the center of Q),

$$
m(x)=\int_Q\Big(\partial^{\lambda}_{\nu(x)}\{E,\vec{q}\}(y-x)-\partial^{\lambda}_{\nu(x)}\{E,\vec{q}\}(x_Q-x)\Big)\xi(y)A(y)\,dy, \qquad (10.50)
$$

for some $\xi\in C^{\infty}_c(\Omega)$ such that $\xi\equiv 1$ on Q, ξ vanishes outside some small neighborhood cQ, $c=c(\Omega)>1$, $0\le\xi\le 1$, and $|\nabla\xi|\le Cl(Q)^{-1}$.

For the range of indices we are currently working with,

$$
F^{p_1,q}_{s+1/p-2}(\mathbb{R}^n)\hookrightarrow L^{p_2}_{-1}(\mathbb{R}^n), \qquad \text{if} \quad s+\tfrac{1}{p}-2-\tfrac{n}{p_1}=-1-\tfrac{n}{p_2}, \qquad (10.51)
$$

where $p_1>1$ is the index appearing in (10.43), chosen sufficiently close to 1, and $p_2>p_1$. Also, $(L^{p_2}_{-1}(\mathbb{R}^n))^*=L^{p_2'}_{1}(\mathbb{R}^n)$, so that (10.50) together with (10.51) and (10.43) imply

$$
|m(x)|\le C\|F_x\|_{L^{p_2'}_1(\mathbb{R}^n)}\|A\|_{L^{p_2}_{-1}(\mathbb{R}^n)}\le C|Q|^{\frac{1}{p_1}-\frac{1}{p}}\|F_x\|_{L^{p_2'}_1(\mathbb{R}^n)}, \qquad (10.52)
$$

where

$$
F_x(y):=\Big(\partial^{\lambda}_{\nu(x)}\{E,\vec{q}\}(y-x)-\partial^{\lambda}_{\nu(x)}\{E,\vec{q}\}(x_Q-x)\Big)\xi(y), \quad y\in\mathbb{R}^n. \qquad (10.53)
$$

We can see that

$$
\begin{aligned}
|\nabla F_x(y)| &\leq C\,\frac{|\xi(y)|}{|x-y|^n} \\
&\quad + C\Big|\Big(\partial^{\lambda}_{\nu(x)}\{E,\vec{q}\}(y-x) - \partial^{\lambda}_{\nu(x)}\{E,\vec{q}\}(x_Q - x)\Big)\Big|\,|\nabla\xi(y)| \\
&=: I + II. && (10.54)
\end{aligned}
$$

By the Mean Value Theorem,

$$
\begin{aligned}
II &\leq C|y-x_Q| \sup_{z\in[y,x_Q]} \Big|\nabla_z[\partial^{\lambda}_{\nu(x)}\{E,\vec{q}\}(z-x)]^{\top}\Big|\;|\nabla\xi(y)| \\
&\leq Cl(Q) \sup_{z\in[y,x_Q]} \frac{1}{|x-z|^n}\,|\nabla\xi(y)|, && (10.55)
\end{aligned}
$$

so that

$$
II \leq C \sup_{z\in[y,x_Q]} \frac{1}{|x-z|^n}, \tag{10.56}
$$

since $|\nabla\xi| \leq \frac{C}{l(Q)}$. Using the property that Q is a Whitney cube for Ω and keeping in mind that $y\in cQ$, $x\in\partial\Omega$, $z\in[y,x_Q]$, some elementary geometry leads to the conclusion that $|x-x_Q| \leq C|x-z|$. Consequently,

$$
II \leq Cl(Q)^{-n}\left(1+\frac{|x-x_Q|}{l(Q)}\right)^{-n}. \tag{10.57}
$$

The same reasoning shows that a similar estimate holds for I, so that altogether,

$$
\|\nabla F_x\|_{L^{p_2'}(\mathbb{R}^n)} \leq Cl(Q)^{-\frac{n}{p_2}}\left(1+\frac{|x-x_Q|}{l(Q)}\right)^{-n}. \tag{10.58}
$$

Similarly,

$$
\|F_x\|_{L^{p_2'}(\mathbb{R}^n)} \leq Cl(Q)^{1-\frac{n}{p_2}}\left(1+\frac{|x-x_Q|}{l(Q)}\right)^{-n} \leq Cl(Q)^{-\frac{n}{p_2}}\left(1+\frac{|x-x_Q|}{l(Q)}\right)^{-n}, \tag{10.59}
$$

where the last inequality rests on the observation that $l(Q)$ is bounded by the diameter of the domain Ω. Then by (10.52), (10.58), and (10.59),

$$|m(x)| \leq Cl(Q)^{s-1-\frac{n-1}{p}} \left(1 + \frac{|x - x_Q|}{l(Q)}\right)^{-n}. \tag{10.60}$$

Now, by definition, $|x_Q - x_S| = \operatorname{dist}(x_Q, \partial\Omega)$, so that $|x - x_S| \leq |x - x_Q| + |x_Q - x_S| \leq 2\,|x - x_Q|$ for every $x \in \partial\Omega$. If we now set $r := l(Q)$, then

$$1 + \frac{|x - x_Q|}{r} \geq 1 + \frac{1}{2}\,\frac{|x - x_S|}{r} \geq \frac{1}{2}\left(1 + \frac{|x - x_S|}{r}\right), \tag{10.61}$$

which entails

$$|m(x)| \leq C\, r^{s-1-\frac{n-1}{p}} \left(1 + \frac{|x - x_S|}{r}\right)^{-n}. \tag{10.62}$$

This proves (10.38) with $M := n+s-1 > \frac{n-1}{p}$ and justifies the claim that m is a molecule for $B^{p,p}_{s-1}(\partial\Omega)$ concentrated about the surface ball $S = S_r(x_S)$. At this stage, Proposition 10.7 applies and yields that, for $\frac{n-1}{n} < p \leq 1$ and $(n-1)(1/p - 1) < s < 1$, the operator (10.48) is well-defined and bounded, first for $p \leq q \leq \infty$, and then for the complementary range, $0 < q \leq p$, by embeddings.

To further expand this range, we shall rely on the observation that

$$\int_{\partial\Omega} \langle \partial^{\lambda}_{\nu}(\Pi\vec{u}, \Theta\vec{u})\,,\, \vec{f}\rangle\, d\sigma = \int_{\Omega} \langle \vec{u}, \mathcal{D}_{\lambda}\vec{f}\rangle\, dx, \tag{10.63}$$

i.e., the conormal derivative of Newtonian potential can be viewed as the adjoint of the double layer. Then, Proposition 10.11, the duality results in (2.118)-(2.119) and interpolation with what we have just proved allows us to cover the range of indices described in the statement of the theorem.

Finally, the claim made about the operator (10.47) is a consequence of the boundedness of (10.48), the duality reasoning described in the paragraph above (in particular, contributing to the case $p = \infty$) and interpolation. □

10.4 The conormal on Besov and Triebel-Lizorkin spaces: the general case

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain and assume that $1 < p, q < \infty$, $0 < s < 1$. If $\vec{u} \in B^{p,q}_{s+\frac{1}{p}}(\Omega)$, $\pi \in B^{p,q}_{s+\frac{1}{p}-1}(\Omega)$ and $\vec{f} \in B^{p,q}_{s+\frac{1}{p}-2,0}(\Omega)$ are such that $\Delta\vec{u} - \nabla\pi = \vec{f}|_{\Omega}$ in

Ω, then as suggested by (4.7), it is natural to define $\partial_\nu^\lambda(\vec{u},\pi)_{\vec{f}} \in B^{p,q}_{s-1}(\partial\Omega) = \Big(B^{p',q'}_{1-s}(\partial\Omega)\Big)^*$, $1/p + 1/p' = 1$, $1/q + 1/q' = 1$, $\lambda \in \mathbb{R}$, by setting

$$\Big\langle \partial_\nu^\lambda(\vec{u},\pi)_{\vec{f}}, \vec{\psi} \Big\rangle := \Big\langle \vec{f}, \mathrm{Ex}(\vec{\psi}) \Big\rangle + \mathbb{A}_\lambda\Big(\nabla\vec{u}, \nabla\mathrm{Ex}(\vec{\psi})\Big) - \Big\langle \pi, \operatorname{div}\mathrm{Ex}(\vec{\psi}) \Big\rangle, \quad \forall\, \vec{\psi} \in B^{p',q'}_{1-s}(\partial\Omega), \tag{10.64}$$

where Ex is the extension operator introduced in Theorem 2.18. The conditions on the indices p, q, s ensure that all duality pairings in the right-hand side of (10.64) are well-defined. Similar considerations apply to the case of Triebel-Lizorkin spaces. As before, this duality-based approach is restricted to the case when $1 < p, q < \infty$, as $B^{p,q}_{s-1}(\partial\Omega)$ fails to be a dual space if $\min\{p,q\} \le 1$. We nonetheless have:

Theorem 10.10 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \ge 2$, and assume that $\frac{n-1}{n} < p \le \infty$ and $(n-1)(1/p-1)_+ < s < 1$, $0 < q \le \infty$. Also, assume that $\lambda \in \mathbb{R}$. Then one can define a concept of conormal derivative, i.e. a bounded, linear application*

$$\begin{aligned} &(\vec{u},\pi,\vec{f}) \mapsto \partial_\nu^\lambda(\vec{u},\pi)_{\vec{f}} \textit{ mapping } \mathcal{B}^{p,q}_s(\Omega) \textit{ onto } B^{p,q}_{s-1}(\partial\Omega), \textit{ where} \\ &\mathcal{B}^{p,q}_s(\Omega) := \Big\{(\vec{u},\pi,\vec{f}) \in B^{p,q}_{s+\frac{1}{p}}(\Omega) \oplus B^{p,q}_{s+\frac{1}{p}-1}(\Omega) \oplus B^{p,q}_{s+\frac{1}{p}-2,0}(\Omega) : \\ &\qquad\qquad \Delta\vec{u} - \nabla\pi = \vec{f}|_\Omega \textit{ and } \operatorname{div}\vec{u} = 0 \textit{ in } \Omega\Big\}, \end{aligned} \tag{10.65}$$

which is compatible with (10.64) when $1 < p, q < \infty$. Furthermore, there exists a linear, bounded, right-inverse of (10.65).

Similar conclusions are valid in the context of Triebel-Lizorkin spaces, i.e. for the application

$$\begin{aligned} &(\vec{u},\pi,\vec{f}) \mapsto \partial_\nu^\lambda(\vec{u},\pi)_{\vec{f}} \textit{ mapping } \mathcal{F}^{p,q}_s(\Omega) \textit{ onto } B^{p,p}_{s-1}(\partial\Omega), \textit{ where} \\ &\mathcal{F}^{p,q}_s(\Omega) := \Big\{(\vec{u},\pi,\vec{f}) \in F^{p,q}_{s+\frac{1}{p}}(\Omega) \oplus F^{p,q}_{s+\frac{1}{p}-1}(\Omega) \oplus F^{p,q}_{s+\frac{1}{p}-2,0}(\Omega) : \\ &\qquad\qquad \Delta\vec{u} - \nabla\pi = \vec{f}|_\Omega \textit{ and } \operatorname{div}\vec{u} = 0 \textit{ in } \Omega\Big\}, \end{aligned} \tag{10.66}$$

assuming that $p \ne \infty$.

Proof. Set

$$\partial_\nu^\lambda(\vec{u},\pi)_{\vec{f}} := \partial_\nu^\lambda\Big(\vec{u} - \big[\Pi\vec{f}\big]\Big|_\Omega, \pi - \big[\Theta\vec{f}\big]\Big|_\Omega\Big) + \partial_\nu^\lambda\Big(\Pi\vec{f}, \Theta\vec{f}\Big), \tag{10.67}$$

where, in the right-hand side of the above equality, the first conormal derivative is taken in the sense of (10.29) in Theorem 10.6, while the second one is taken in the sense of (10.47) in Theorem 10.9. The properties of this conormal derivative claimed in the statement of the theorem then follows from this. □

Remark. In what follows, we agree to simplify the notation by writing $\partial_\nu^\lambda(\vec{u},\pi)$ in place of $\partial_\nu^\lambda(\vec{u},\pi)_{\vec{0}}$, whenever $\Delta\vec{u} - \nabla\pi = 0$ in Ω.

10.5 Layer potentials on Besov and Triebel-Lizorkin spaces

In this section we establish mapping properties for the hydrostatic layer potentials on Besov and Triebel-Lizorkin spaces in Lipschitz domains.

Proposition 10.11 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, and assume that $\lambda \in \mathbb{R}$, $\frac{n-1}{n} < p \leq \infty$, $(n-1)(\frac{1}{p}-1)_+ < s < 1$, and $0 < q \leq \infty$. Then*

$$\mathcal{D}_\lambda : B_s^{p,q}(\partial\Omega) \longrightarrow B_{s+\frac{1}{p}}^{p,q}(\Omega), \tag{10.68}$$

$$\mathcal{S} : B_{s-1}^{p,q}(\partial\Omega) \longrightarrow B_{s+\frac{1}{p}}^{p,q}(\Omega), \tag{10.69}$$

$$\mathcal{P}_\lambda : B_s^{p,q}(\partial\Omega) \longrightarrow B_{s+\frac{1}{p}-1}^{p,q}(\Omega), \tag{10.70}$$

$$\mathcal{Q} : B_{s-1}^{p,q}(\partial\Omega) \longrightarrow B_{s+\frac{1}{p}-1}^{p,q}(\Omega), \tag{10.71}$$

are well-defined, bounded operators. Furthermore,

$$\mathcal{D}_\lambda : B_s^{p,p}(\partial\Omega) \longrightarrow F_{s+\frac{1}{p}}^{p,q}(\Omega), \tag{10.72}$$

$$\mathcal{S} : B_{s-1}^{p,p}(\partial\Omega) \longrightarrow F_{s+\frac{1}{p}}^{p,q}(\Omega), \tag{10.73}$$

$$\mathcal{P}_\lambda : B_s^{p,p}(\partial\Omega) \longrightarrow F_{s+\frac{1}{p}-1}^{p,q}(\Omega), \tag{10.74}$$

$$\mathcal{Q} : B_{s-1}^{p,p}(\partial\Omega) \longrightarrow F_{s+\frac{1}{p}}^{p,q}(\Omega), \tag{10.75}$$

are also well-defined and bounded provided s, p, q are as before and $p \neq \infty$.

Proof. From Theorem 11.18 and Theorem 11.15 it follows that

$$\mathcal{D}_\lambda : B_s^{p,p}(\partial\Omega) \longrightarrow \mathbb{H}^p_{s+\frac{1}{p}}(\Omega;\Delta^2) = F^{p,q}_{s+\frac{1}{p}}(\Omega) \cap \operatorname{Ker}\Delta^2 \tag{10.76}$$

is well-defined and bounded whenever $0 < p, q \leq \infty$, $(n-1)(\frac{1}{p}-1)_+ < s < 1$, provided $q = \infty$ if $p = \infty$. This and real interpolation (cf. Proposition 2.20 and Theorem 2.13) then justify (10.68) and (10.72) (in the latter case, we also use monotonicity of the Triebel-Lizorkin scale to cover the case $q = \infty$). That the operators in (10.70)-(10.71) and (10.74)-(10.75) are also well-defined and bounded is a consequence of (4.35)-(4.36) and the mapping properties of the harmonic layer potentials on the Besov-Triebel-Lizorkin scale proved in [64].

As regards $\mathcal{S}$, Theorem 11.19 and Theorem 11.15 give that

$$\mathcal{S} : B_{s-1}^{p,p}(\partial\Omega) \longrightarrow \mathbb{H}^p_{s+\frac{1}{p}-1}(\Omega;\Delta^2) = F^{p,q}_{s+\frac{1}{p}-1}(\Omega) \cap \operatorname{Ker}\Delta^2 \tag{10.77}$$

is well-defined and bounded for $0 < p, q \leq \infty$, $(n-1)(\frac{1}{p}-1)_+ < s < 1$, granted that $q = \infty$ if $p = \infty$. Then, much as before, the operators (10.69), (10.73) are seen to be well-defined and bounded. $\square$

Recall next the boundary layer potential operators K_λ defined in (4.44), its formal adjoint K_λ^*, and S introduced in (4.47).

Proposition 10.12 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. If $(n-1)/n < p \leq \infty$ and $(n-1)(\frac{1}{p}-1)_+ < s < 1$, $0 < q \leq \infty$, $\lambda \in \mathbb{R}$, then the operators*

$$K_\lambda : B_s^{p,q}(\partial\Omega) \longrightarrow B_s^{p,q}(\partial\Omega), \tag{10.78}$$

$$K_\lambda^* : B_{s-1}^{p,q}(\partial\Omega) \longrightarrow B_{s-1}^{p,q}(\partial\Omega), \tag{10.79}$$

$$S : B_{s-1}^{p,q}(\partial\Omega) \longrightarrow B_s^{p,q}(\partial\Omega), \tag{10.80}$$

are well-defined, linear, and bounded.

Proof. Since

$$\mathrm{Tr}\circ\mathcal{D}_\lambda=\tfrac{1}{2}I+K_\lambda,\qquad \mathrm{Tr}\circ\mathcal{S}=S, \tag{10.81}$$

the claims about (10.78) and (10.80) are consequences of Proposition 10.11 and Theorem 2.18. Finally, using the fact that

$$\partial_\nu^\lambda\circ(\mathcal{S},\mathcal{Q})=-\tfrac{1}{2}I+K_\lambda^*, \tag{10.82}$$

together with Theorem 10.6 and Proposition 10.11, the claim about the operator (10.79) follows as well. □

For a given bounded Lipschitz domain Ω in $\mathbb{R}^n$, $n\geq 2$, the range of indices for which the boundary layer potentials for the Stokes system are invertible on the Besov scale considered on $\partial\Omega$ depends on the dimension n of the ambient space and the Lipschitz character of Ω. The latter is manifested by a parameter $\varepsilon\in(0,1]$ which can be thought of as the measuring the degree of roughness of Ω (thus, the larger ε the milder the Lipschitz nature of Ω, and the smaller ε, the more acute Lipschitz nature of Ω). To best describe these regions, for each $n\geq 2$ and $\varepsilon>0$ we let $\mathcal{R}_{n,\varepsilon}$ denote the following sets. For $n=2$, $\mathcal{R}_{2,\varepsilon}$ is the collection of all pairs of numbers s,p with the property that either one of the following two conditions below is satisfied:

$$\begin{aligned}(I_2):&\quad 0\leq\tfrac{1}{p}<s+\tfrac{1+\varepsilon}{2}\quad\text{and}\quad 0<s\leq\tfrac{1+\varepsilon}{2},\\ (II_2):&\quad -\tfrac{1+\varepsilon}{2}<\tfrac{1}{p}-s<\tfrac{1+\varepsilon}{2}\quad\text{and}\quad \tfrac{1+\varepsilon}{2}<s<1.\end{aligned} \tag{10.83}$$

Corresponding to $n=3$, $\mathcal{R}_{3,\varepsilon}$ is the collection of all pairs s,p with the property that either of the following two conditions holds:

$$\begin{aligned}(I_3):&\quad 0\leq\tfrac{1}{p}<\tfrac{s}{2}+\tfrac{1+\varepsilon}{2}\quad\text{and}\quad 0<s<\varepsilon,\\ (II_3):&\quad -\tfrac{\varepsilon}{2}<\tfrac{1}{p}-\tfrac{s}{2}<\tfrac{1+\varepsilon}{2}\quad\text{and}\quad \varepsilon\leq s<1.\end{aligned} \tag{10.84}$$

Finally, corresponding to $n\geq 4$, we let $\mathcal{R}_{n,\varepsilon}$ denote the collection of all pairs s,p with the property that

$$(I_n): \quad \tfrac{n-3}{2(n-1)} - \varepsilon < \tfrac{1}{p} - \tfrac{s}{n-1} < \tfrac{1}{2} + \varepsilon \text{ and } 0 < s < 1,\ 1 < p < \infty. \tag{10.85}$$

To proceed, we shall now introduce some versions of the boundary Besov spaces which are well-suited for the formulation and treatment of boundary value problems for the Stokes system in Lipschitz domains. Concretely, if Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, and $(n-1)/n < p \leq \infty$, $(n-1)(\frac{1}{p}-1)_+ < s < 1$, $0 < q \leq \infty$, we set:

$$B^{p,q}_{s,\nu_\pm}(\partial\Omega) := \Big\{\vec{f} \in B^{p,q}_s(\partial\Omega) : \int_{\partial\Omega} \langle\psi, \vec{f}\rangle\, d\sigma = 0,\ \forall\, \psi \in \nu\, \mathbb{R}_{\partial\Omega_\pm}\Big\}, \tag{10.86}$$

$$B^{p,q}_{s,\nu}(\partial\Omega) := \Big\{\vec{f} \in B^{p,q}_s(\partial\Omega) : \int_{\partial\Omega} \langle\psi, \vec{f}\rangle\, d\sigma = 0,\ \forall\, \psi \in \nu\, \mathbb{R}_{\partial\Omega}\Big\}, \tag{10.87}$$

$$B^{p,q}_{s-1,\Psi^\lambda_\mp}(\partial\Omega) := \Big\{\vec{f} \in B^{p,q}_{s-1}(\partial\Omega) : \int_{\partial\Omega} \langle\psi, \vec{f}\rangle\, d\sigma = 0,\ \forall\, \psi \in \Psi^\lambda(\partial\Omega_\pm)\Big\}, \tag{10.88}$$

$$B^{p,q}_{s,\nu,\mathcal{W}}(\partial\Omega) := \Big\{\vec{f} \in B^{p,q}_{s,\nu}(\partial\Omega) : \int_{\partial\Omega} \langle\psi, \vec{f}\rangle\, d\sigma = 0,\ \forall\, \psi \in \mathcal{W}\Big\} \text{ if } n = 2. \tag{10.89}$$

On these spaces, below we show that the boundary hydrostatic layer potentials are invertible for suitable indices p, q, s. We have:

Theorem 10.13 *Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. Then there exists $\varepsilon = \varepsilon(\Omega) \in (0,1]$ with the following property. If $(n-1)/n < p \leq \infty$, $(n-1)(\frac{1}{p}-1)_+ < s < 1$, $0 < q \leq \infty$, and $\lambda \in (-1,1]$, then the operators*

$$\pm\tfrac{1}{2}I + K_\lambda : B^{p,q}_{s,\nu_\pm}(\partial\Omega)/\Psi^\lambda(\partial\Omega_\pm) \longrightarrow B^{p,q}_{s,\nu_\pm}(\partial\Omega)/\Psi^\lambda(\partial\Omega_\pm), \tag{10.90}$$

$$\pm\tfrac{1}{2}I + K^*_\lambda : B^{p,q}_{s-1,\Psi^\lambda_\mp}(\partial\Omega)/\nu\mathbb{R}_{\partial\Omega_\pm} \longrightarrow B^{p,q}_{s-1,\Psi^\lambda_\mp}(\partial\Omega)/\nu\mathbb{R}_{\partial\Omega_\pm}, \tag{10.91}$$

$$S : B^{p,q}_{s-1}(\partial\Omega)/\nu\mathbb{R}_{\partial\Omega} \longrightarrow B^{p,q}_{s,\nu}(\partial\Omega) \quad \text{if } n \geq 3, \tag{10.92}$$

$$S : B^{p,q}_{s-1}(\partial\Omega)/\nu\mathbb{R}_{\partial\Omega} \oplus \mathcal{W} \longrightarrow B^{p,q}_{s,\nu,\mathcal{W}}(\partial\Omega) \quad \text{if } n = 2, \tag{10.93}$$

$$\widetilde{S} : \Big(B^{p,q}_{s-1}(\partial\Omega)/\nu\mathbb{R}_{\partial\Omega}\Big) \oplus \mathbb{R}^2 \longrightarrow B^{p,q}_{s,\nu}(\partial\Omega) \oplus \mathbb{R}^2 \quad \text{if } n = 2, \tag{10.94}$$

are invertible whenever the pair (s,p) belongs to the region $\mathcal{R}_{n,\varepsilon}$, described in (10.83)-(10.85).

Proof. This follows from the invertibility results on Hardy spaces from § 9.1 and repeated applications of the complex and real method of interpolation. $\square$

10.6 The Poisson problem with Dirichlet and Neumann boundary conditions

Here our goal is to describe the ranges of indices for which the Poisson problem for the Stokes system equipped with Dirichlet or Neumann boundary conditions is well-posed for data in Besov and Triebel-Lizorkin spaces in bounded Lipschitz domains. As a preamble, we record some useful integral representation formulas.

Proposition 10.14 *Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, $\frac{n-1}{n} < p \leq \infty$, $(n-1)(1/p-1)_+ < s < 1$, and $0 < q \leq \infty$. Then for every number $\lambda \in \mathbb{R}$ and every pair $(\vec{u}, \pi) \in SB^{p,q}_{s+\frac{1}{p}}(\Omega)$ there holds*

$$\begin{aligned} \vec{u} &= \mathcal{D}_\lambda(\operatorname{Tr}\vec{u}) - \mathcal{S}(\partial^\lambda_\nu(\vec{u},\pi)) \quad \text{in} \quad \Omega, \\ \pi &= \mathcal{P}_\lambda(\operatorname{Tr}\vec{u}) - \mathcal{Q}(\partial^\lambda_\nu(\vec{u},\pi)) \quad \text{in} \quad \Omega. \end{aligned} \tag{10.95}$$

Similar integral representation formulas are valid in the context of Triebel-Lizorkin spaces, i.e. when $(\vec{u}, \pi) \in SF^{p,q}_{s+\frac{1}{p}}(\Omega)$, granted that $p \neq \infty$.

Proof. These formulas follow from (4.120)-(4.121), a density argument, and the mapping properties of the operators involved (established earlier). $\square$

We are now ready to state and prove the first main result of this section, dealing with the inhomogeneous problem for the Stokes system with Dirichlet boundary condition.

Theorem 10.15 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, and for $\frac{n-1}{n} < p \leq \infty$, $0 < q \leq \infty$, $(n-1)(\frac{1}{p}-1)_+ < s < 1$, consider the following boundary value problem,*

$$\begin{aligned} &\Delta\vec{u} - \nabla\pi = \vec{f} \in B^{p,q}_{s+\frac{1}{p}-2}(\Omega), \quad \operatorname{div}\vec{u} = g \in B^{p,q}_{s+\frac{1}{p}-1}(\Omega), \\ &\vec{u} \in B^{p,q}_{s+\frac{1}{p}}(\Omega), \quad \pi \in B^{p,q}_{s+\frac{1}{p}-1}(\Omega), \quad \operatorname{Tr}\vec{u} = \vec{h} \in B^{p,q}_{s}(\partial\Omega), \end{aligned} \tag{10.96}$$

subject to the (necessary) compatibility condition

$$\int_{\partial\mathcal{O}} \langle \nu, \vec{h}\rangle \, d\sigma = \int_{\mathcal{O}} g(x)\, dx, \quad \textit{for every component } \mathcal{O} \textit{ of } \Omega. \tag{10.97}$$

Then there exists $\varepsilon = \varepsilon(\Omega) \in (0,1]$ such that (10.96) is well-posed (with uniqueness modulo locally constant functions in Ω for the pressure), if the pair (s,p) belongs to the region $\mathcal{R}_{n,\varepsilon}$, described in (10.83)-(10.85).

Furthermore, the solution has an integral representation formula in terms of hydrostatic layer potential operators and satisfies natural estimates. Concretely, there exists a finite, positive constant $C = C(\Omega, p, q, s, n)$ such that

$$\|\vec{u}\|_{B^{p,q}_{s+\frac{1}{p}}(\Omega)} + \|\pi\|_{B^{p,q}_{s+\frac{1}{p}-1}(\Omega)/\mathbb{R}_{\Omega_+}} \leq C\|\vec{f}\|_{B^{p,q}_{s+\frac{1}{p}-2}(\Omega)} + C\|g\|_{B^{p,q}_{s+\frac{1}{p}-1}(\Omega)} + C\|\vec{h}\|_{B^{p,q}_{s}(\partial\Omega)}. \tag{10.98}$$

Moreover, analogous well-posedness results hold on the Triebel-Lizorkin scale, i.e. for the problem

$$\begin{aligned} &\Delta\vec{u} - \nabla\pi = \vec{f} \in F^{p,q}_{s+\frac{1}{p}-2}(\Omega), \quad \operatorname{div}\vec{u} = g \in F^{p,q}_{s+\frac{1}{p}-1}(\Omega), \\ &\vec{u} \in F^{p,q}_{s+\frac{1}{p}}(\Omega), \quad \pi \in F^{p,q}_{s+\frac{1}{p}-1}(\Omega), \quad \operatorname{Tr}\vec{u} = \vec{g} \in B^{p,p}_{s}(\partial\Omega), \end{aligned} \tag{10.99}$$

where the data is, once again, made subject to (10.97). This time, in addition to the previous conditions imposed on the indices p, q, it is also assumed that $p, q < \infty$.

Proof. Let $\vec{v}$ be such that

$$\vec{v} \in B^{p,q}_{s+\frac{1}{p}-1}(\Omega), \qquad \operatorname{div}\vec{v} = g \text{ in } \Omega. \tag{10.100}$$

For example, we may take

$$\vec{v} := \nabla\Pi_{\Delta} g \tag{10.101}$$

where $\Pi_{\Delta} : B^{p,q}_{s+\frac{1}{p}-1}(\Omega) \to B^{p,q}_{s+\frac{1}{p}+1}$ is the harmonic Newtonian potential in Ω (i.e., the operator of convolution with E_{Δ} from (4.31)). Next, consider $\vec{w}$, ρ for which

$$(\vec{w},\rho)\in B^{p,q}_{s+\frac{1}{p}}(\Omega)\oplus B^{p,q}_{s+\frac{1}{p}-1}(\Omega),\quad \Delta\vec{w}-\nabla\rho=\vec{f}-\Delta\vec{v}\ \text{ and }\ \mathrm{div}\,\vec{w}=0\ \text{ in }\Omega. \tag{10.102}$$

For this, we may take $\vec{w}:=\Pi(\vec{f}-\Delta\vec{v})$ and $\rho:=\Theta(\vec{f}-\Delta\vec{v})$, where Π, Θ are as in (10.13)-(10.14). We now claim that

$$\mathrm{Tr}\,\vec{v}+\mathrm{Tr}\,\vec{w}-\vec{h}\in B^{p,q}_{s,\nu_+}(\partial\Omega). \tag{10.103}$$

To see this, we first observe that $\mathrm{Tr}\,\vec{v}+\mathrm{Tr}\,\vec{w}-\vec{h}\in B^{p,q}_{s}(\partial\Omega)$. To check the orthogonality condition on $\nu\mathbb{R}_{\partial\Omega_+}$, by virtue of (5.73) it suffices to note that for every $\psi\in\mathbb{R}_{\Omega_+}$ we have

$$\begin{aligned}\int_{\partial\Omega}\langle(\mathrm{Tr}\,\vec{v}+\mathrm{Tr}\,\vec{w})\,,\,\nu\rangle\,\psi\,d\sigma &= \int_{\Omega}\psi\,\mathrm{div}\,(\vec{v}+\vec{w})\,dx\\ &= \int_{\Omega}g\,\psi\,dx=\int_{\partial\Omega}\langle\nu,\vec{h}\rangle\psi\,d\sigma,\end{aligned} \tag{10.104}$$

by (10.97). This proves the claim made in (10.103).

Next, we make the claim that if $n\geq 3$, then

$$\begin{aligned}&T:B^{p,q}_{s,\nu_+}(\partial\Omega)\oplus B^{p,q}_{s-1}(\partial\Omega)\longrightarrow B^{p,q}_{s,\nu_+}(\partial\Omega),\\ &T(\vec{g}_1,\vec{g}_2):=(\tfrac{1}{2}I+K_\lambda)\vec{g}_1+S\vec{g}_2\qquad\text{is onto.}\end{aligned} \tag{10.105}$$

To justify this claim, consider an arbitrary $\vec{f}\in B^{p,q}_{s,\nu_+}(\partial\Omega)$. Then (10.90) gives that there exists $\vec{g}_1\in B^{p,q}_{s,\nu_+}(\partial\Omega)$ such that $\vec{\psi}:=\vec{f}-(\frac{1}{2}I+K_\lambda)\vec{g}_1\in\Psi^\lambda_-(\partial\Omega)$. This, (5.117), and (10.92) then guarantee the existence of some $\vec{g}_2\in B^{p,q}_{s-1}(\partial\Omega)$ with the property that $S\vec{g}_2=\vec{\psi}$. Consequently, $T(\vec{g}_1,\vec{g}_2)=\vec{f}$, proving the claim.

Having established (10.103) and (10.105), we can now produce a solution for (10.96) in the form

$$\vec{u}:=\vec{v}+\vec{w}+\mathcal{D}_\lambda\vec{g}_1+\mathcal{S}\vec{g}_2,\qquad \pi:=\rho+\mathcal{P}_\lambda\vec{g}_1+\mathcal{Q}\vec{g}_2, \tag{10.106}$$

where

$$(\vec{g}_1, \vec{g}_2) \in B^{p,q}_{s,\nu_+}(\partial\Omega) \oplus B^{p,q}_{s-1}(\partial\Omega) \text{ are such that } T(\vec{g}_1, \vec{g}_2) = \vec{h} - \operatorname{Tr}\vec{v} - \operatorname{Tr}\vec{w}. \quad (10.107)$$

Furthermore, it is implicit in the above construction that (10.98) holds. The case $n = 2$ is handled analogously, so we omit the details.

To prove uniqueness, assume that $\vec{u}, \pi$ solve the homogeneous version of (10.96). We may then conclude that $(\vec{u}, \pi) \in SB^{p,q}_{s+\frac{1}{p}}(\Omega)$ and Proposition 10.14 gives

$$\vec{u} = -\mathcal{S}(\partial^\lambda_\nu(\vec{u}, \pi)) \text{ in } \Omega. \quad (10.108)$$

Taking boundary traces of both sides then yields

$$S(\partial^\lambda_\nu(\vec{u}, \pi)) = 0 \text{ in } B^{p,q}_{s-1}(\partial\Omega), \quad (10.109)$$

so that $\partial^\lambda_\nu(\vec{u}, \pi) \in \nu\mathbb{R}_{\partial\Omega}$. Returning with this in (10.108) and invoking (5.77), (5.83), then gives $\vec{u} = 0$ in Ω and $\pi \in \mathbb{R}_{\Omega_+}$, as desired.

For the Triebel-Lizorkin scale a very similar approach works as well. Thus, the proof of the theorem is complete at this point. $\square$

Our second main result in this section pertains to the Poisson problem for the Stokes system with Neumann boundary conditions.

Theorem 10.16 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, and for $\frac{n-1}{n} < p \leq \infty$, $0 < q \leq \infty$, and $(n-1)(\frac{1}{p} - 1)_+ < s < 1$, consider the following boundary value problem:*

$$\begin{aligned} &\Delta\vec{u} - \nabla\pi = \vec{f}\Big|_\Omega, \quad \vec{f} \in B^{p,q}_{s+\frac{1}{p}-2,0}(\Omega), \quad \operatorname{div}\vec{u} = 0 \text{ in } \Omega, \\ &\vec{u} \in B^{p,q}_{s+\frac{1}{p}}(\Omega), \quad \pi \in B^{p,q}_{s+\frac{1}{p}-1}(\Omega), \quad \partial^\lambda_\nu(\vec{u}, \pi)_{\vec{f}} = \vec{h} \in B^{p,q}_{s-1}(\partial\Omega), \end{aligned} \quad (10.110)$$

where the data are assumed to satisfy the necessary compatibility condition

$$\partial^\lambda_\nu\Big(\Pi\vec{f}, \Theta\vec{f}\Big) - \vec{h} \in \operatorname{Im}\Big(-\tfrac{1}{2}I + K^*_\lambda : B^{p,q}_{s-1,\Psi^\lambda_+}(\partial\Omega) \to B^{p,q}_{s-1,\Psi^\lambda_+}(\partial\Omega)\Big). \quad (10.111)$$

Then there exists $\varepsilon = \varepsilon(\Omega) \in (0, 1]$ such that (10.110) has a unique solution (modulo adding to the velocity functions from $\Psi^\lambda(\Omega)$) if the pair s, p belongs to the region $\mathcal{R}_{n,\varepsilon}$ described *in (10.83)-(10.85). In addition, the solution (normalized so that* $\int_\Omega \langle \vec{u}(x), \psi(x)\rangle\, dx = 0$ *for every* $\psi \in \Psi^\lambda(\Omega)$*) satisfies the estimate*

$$\|\vec{u}\|_{B^{p,q}_{s+\frac{1}{p}}(\Omega)} + \|\pi\|_{B^{p,q}_{s+\frac{1}{p}-1}(\Omega)} \leq C\|\vec{f}\|_{B^{p,q}_{s+\frac{1}{p}-2,0}(\Omega)} + C\|\vec{h}\|_{B^{p,q}_{s-1}(\partial\Omega)}. \tag{10.112}$$

An analogous well-posedness result holds for the problem

$$\begin{aligned} &\Delta\vec{u} - \nabla\pi = \vec{f}\Big|_\Omega, \quad \vec{f} \in F^{p,q}_{s+\frac{1}{p}-2,0}(\Omega), \quad \operatorname{div}\vec{u} = 0 \text{ in } \Omega, \\ &\vec{u} \in F^{p,q}_{s+\frac{1}{p}}(\Omega), \quad \pi \in F^{p,q}_{s+\frac{1}{p}-1}(\Omega), \quad \partial^\lambda_\nu(\vec{u},\pi)_{\vec{f}} = \vec{h} \in B^{p,p}_{s-1}(\partial\Omega), \end{aligned} \tag{10.113}$$

assuming that $p, q < \infty$*, and*

$$\partial^\lambda_\nu\Big(\Pi\vec{f}, \Theta\vec{f}\Big) - \vec{h} \in \operatorname{Im}\Big(-\tfrac{1}{2}I + K^*_\lambda : B^{p,p}_{s-1,\Psi^\lambda_+}(\partial\Omega) \to B^{p,p}_{s-1,\Psi^\lambda_+}(\partial\Omega)\Big). \tag{10.114}$$

Proof. The fact that (10.111) is a necessary condition for the solvability of (10.110) can be proved following the same set of ideas as in the case of (9.104), after observing that

$$\vec{w} := \vec{u} - \Pi\vec{f}, \qquad \rho := \pi - \Theta\vec{f} \tag{10.115}$$

solve

$$\begin{aligned} &\Delta\vec{w} - \nabla\rho = 0 \text{ in } \Omega, \quad \operatorname{div}\vec{w} = 0 \text{ in } \Omega, \\ &\vec{w} \in B^{p,q}_{s+\frac{1}{p}}(\Omega), \quad \rho \in B^{p,q}_{s+\frac{1}{p}-1}(\Omega), \\ &\partial^\lambda_\nu(\vec{w},\rho) = \vec{h} - \partial^\lambda_\nu(\Pi\vec{f}, \Theta\vec{f}) \in B^{p,q}_{s-1}(\partial\Omega). \end{aligned} \tag{10.116}$$

In turn, granted (10.111), existence is seen by taking

$$\vec{u} := \Pi\vec{f} - \mathcal{S}(-\tfrac{1}{2}I + K^*_\lambda)^{-1}(\partial^\lambda_\nu(\Pi\vec{f}, \Theta\vec{f}) - \vec{h}), \tag{10.117}$$

$$\pi := \Theta\vec{f} - \mathcal{Q}(-\tfrac{1}{2}I + K^*_\lambda)^{-1}(\partial^\lambda_\nu(\Pi\vec{f}, \Theta\vec{f}) - \vec{h}). \tag{10.118}$$

Given our earlier results on the mapping properties of the hydrostatic layer potentials plus the current assumptions on the indices s, p, q, this is easily seen to solve (10.110).

To establish uniqueness, if the functions $\vec{u}$ and π satisfy the homogeneous version of problem (10.110), then $\vec{u} = \mathcal{D}_\lambda(\operatorname{Tr}\vec{u})$ in Ω, by (10.95). Taking boundary traces (in the sense of Besov spaces) then yields $(-\frac{1}{2}I + K_\lambda)(\operatorname{Tr}\vec{u}) = 0$ on $\partial\Omega$. This shows that $\operatorname{Tr}\vec{u} \in \Psi^\lambda(\partial\Omega_+)$, by a variant of (5.125). Hence, $\operatorname{Tr}\vec{u} = \psi|_{\partial\Omega}$ for some function $\psi \in \Psi^\lambda(\Omega_+)$. It remains to invoke (10.95) once again in order to conclude that, by virtue of (5.97), $\vec{u} = \psi$ in Ω. This establishes the claim made about uniqueness for (10.110).

The treatment of (10.113) is analogous, and this finishes the proof of the theorem. $\square$

A less precise formulation of Theorem 10.16 is that *problems (10.110), (10.113) have solutions for data* $(\vec{f}, \vec{h})$ *belonging to a finite co-dimensional subspace of* $B^{p,q}_{s+1/p-2,0}(\Omega) \oplus B^{p,q}_{s-1}(\partial\Omega)$ *and* $F^{p,q}_{s+1/p-2,0}(\Omega) \oplus B^{p,p}_{s-1}(\partial\Omega)$, *respectively, and uniqueness holds up to a finite dimensional space.*

To see this, let us rephrase condition (10.111) as

$$(\vec{f}, \vec{h}) \in \Phi^{-1}\mathrm{Im}\left(-\tfrac{1}{2}I + K^*_\lambda : B^{p,q}_{s-1,\Psi^\lambda_+}(\partial\Omega) \to B^{p,q}_{s-1,\Psi^\lambda_+}(\partial\Omega)\right), \tag{10.119}$$

where Φ is the bounded, linear application given by

$$\Phi : B^{p,q}_{s+1/p-2,0}(\Omega) \oplus B^{p,q}_{s-1}(\partial\Omega) \ni (\vec{f}, \vec{h}) \mapsto \partial^\lambda_\nu\left(\Pi\vec{f}, \Theta\vec{f}\right) - \vec{g} \in B^{p,q}_{s-1}(\partial\Omega). \tag{10.120}$$

Since $\operatorname{Ker}(-\frac{1}{2}I + K^*_\lambda : B^{p,q}_{s-1,\Psi^\lambda_+}(\partial\Omega) \to B^{p,q}_{s-1,\Psi^\lambda_+}(\partial\Omega))$ is, thanks to (10.91), a space of finite codimension in $B^{p,q}_{s-1}(\partial\Omega)$, the desired conclusion now follows from Lemma 11.42 in the Appendix.

In the case when $\mathbb{R}^n \setminus \bar{\Omega}$ is connected, we can further rephrase Theorem 10.16 in the following fashion.

Theorem 10.17 *Assume that* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^n$, $n \geq 2$, *with connected complement and that* $\frac{n-1}{n} < p \leq \infty$, $0 < q \leq \infty$, *and* $(n-1)(\frac{1}{p} - 1)_+ < s < 1$. *Then there exists* $\varepsilon = \varepsilon(\Omega) \in (0, 1]$ *such that the Poisson problem for the Stokes system with Neumann boundary condition*

$$\begin{aligned} &\Delta\vec{u} - \nabla\pi = \vec{f}\Big|_{\Omega}, \quad \vec{f} \in B^{p,q}_{s+\frac{1}{p}-2,0}(\Omega), \quad \operatorname{div}\vec{u} = 0 \ \textit{in}\ \Omega, \\ &\vec{u} \in B^{p,q}_{s+\frac{1}{p}}(\Omega), \quad \pi \in B^{p,q}_{s+\frac{1}{p}-1}(\Omega), \quad \partial^{\lambda}_{\nu}(\vec{u},\pi)_{\vec{f}} = \vec{h} \in B^{p,q}_{s-1}(\partial\Omega), \end{aligned} \tag{10.121}$$

has a unique solution (modulo adding to the velocity functions from $\Psi^{\lambda}(\Omega)$) if the pair s,p belongs to the region $\mathcal{R}_{n,\varepsilon}$ described in (10.83)-(10.85) and the data $(\vec{f},\vec{h})$ satisfy the necessary compatibility condition

$$\int_{\Omega} \langle \vec{f}, \psi\rangle\, dx = \int_{\partial\Omega} \langle \vec{h}, \psi\rangle\, d\sigma, \qquad \forall\, \psi \in \Psi^{\lambda}(\Omega). \tag{10.122}$$

In addition, the solution (normalized so that $\int_{\Omega} \vec{u}\cdot\psi = 0$ for every $\psi \in \Psi^{\lambda}(\Omega)$) satisfies the estimate

$$\|\vec{u}\|_{B^{p,q}_{s+\frac{1}{p}}(\Omega)} + \|\pi\|_{B^{p,q}_{s+\frac{1}{p}-1}(\Omega)} \leq C\|\vec{f}\|_{B^{p,q}_{s+\frac{1}{p}-2,0}(\Omega)} + C\|\vec{h}\|_{B^{p,q}_{s-1}(\partial\Omega)}. \tag{10.123}$$

Moreover, an analogous well-posedness result holds for the problem

$$\begin{aligned} &\Delta\vec{u} - \nabla\pi = \vec{f}\Big|_{\Omega}, \quad \vec{f} \in F^{p,q}_{s+\frac{1}{p}-2,0}(\Omega), \quad \operatorname{div}\vec{u} = 0 \ \textit{in}\ \Omega, \\ &\vec{u} \in F^{p,q}_{s+\frac{1}{p}}(\Omega), \quad \pi \in F^{p,q}_{s+\frac{1}{p}-1}(\Omega), \quad \partial^{\lambda}_{\nu}(\vec{u},\pi)_{\vec{f}} = \vec{h} \in B^{p,p}_{s-1}(\partial\Omega), \end{aligned} \tag{10.124}$$

assuming that $p,q < \infty$.

Proof. Given that we are assuming that Ω_- is connected, it follows that $\mathbb{R}_{\partial\Omega_-} = 0$. Thus, in the current context, (10.91) becomes

$$-\tfrac{1}{2}I + K^*_{\lambda} : B^{p,q}_{s-1,\Psi^{\lambda}_+}(\partial\Omega) \longrightarrow B^{p,q}_{s-1,\Psi^{\lambda}_+}(\partial\Omega) \ \text{ isomorphically,} \tag{10.125}$$

if s,p,q are as in the statement of Theorem 10.13. As a consequence, the image of the operator $-\frac{1}{2}I + K^*_{\lambda}$ acting on $B^{p,q}_{s-1,\Psi^{\lambda}_+}(\partial\Omega)$ is the entire space $B^{p,q}_{s-1,\Psi^{\lambda}_+}(\partial\Omega)$. In turn, this implies that the compatibility condition (10.111) takes the form

$$\partial^{\lambda}_{\nu}\Big(\Pi\vec{f}, \Theta\vec{f}\Big) - \vec{h} \in B^{p,q}_{s-1,\Psi^{\lambda}_+}(\partial\Omega). \tag{10.126}$$

In other words,

$$\int_{\partial\Omega} \Big\langle \partial_\nu^\lambda \Big(\Pi\vec{f}, \Theta\vec{f}\Big)\,,\,\psi\Big\rangle\,d\sigma = \int_{\partial\Omega} \langle \vec{h}, \psi\rangle\,d\sigma, \qquad \forall\,\psi \in \Psi_+^\lambda(\partial\Omega). \tag{10.127}$$

At this point, there remains to observe that

$$\int_{\partial\Omega} \Big\langle \partial_\nu^\lambda \Big(\Pi\vec{f}, \Theta\vec{f}\Big)\,,\,\psi\Big\rangle\,d\sigma = \int_{\Omega} \langle \vec{f}(x), \psi(x)\rangle\,dx, \qquad \forall\,\psi \in \Psi^\lambda(\Omega), \tag{10.128}$$

as is clear from (4.7) and (5.95). This proves that, in the current context, (10.111) reduces precisely to (10.122), finishing the proof of the theorem. □

11 Appendix

11.1 Smoothness spaces in the Euclidean setting

Here we briefly review Besov and Triebel-Lizorkin scales in $\mathbb{R}^n$. One convenient point of view is offered by the classical Littlewood-Paley theory (cf., e.g., [79], [90]). More specifically, let Ξ be the collection of all systems $\{\zeta_j\}_{j=0}^\infty$ of Schwartz functions with the following properties:

(i) there exist positive constants A, B, C such that

$$\begin{cases} \operatorname{supp}(\zeta_0) \subset \{x:\,|x| \le A\}; \\ \operatorname{supp}(\zeta_j) \subset \{x:\,B2^{j-1} \le |x| \le C2^{j+1}\} \quad \text{if } j \in \mathbb{N}; \end{cases} \tag{11.1}$$

(ii) for every multi-index α there exists a positive, finite constant C_α such that

$$\sup_{x\in\mathbb{R}^n}\sup_{j\in\mathbb{N}} 2^{j|\alpha|}|\partial^\alpha\zeta_j(x)| \le C_\alpha; \tag{11.2}$$

(iii)

$$\sum_{j=0}^\infty \zeta_j(x) = 1 \text{ for every } x \in \mathbb{R}^n. \tag{11.3}$$

Let $s \in \mathbb{R}$ and $0 < q \le \infty$ and fix some family $\{\zeta_j\}_{j=0}^\infty \in \Xi$. Also, let $\mathcal{F}$ and $S'(\mathbb{R}^n)$ denote, respectively, the Fourier transform and the class of tempered distributions in $\mathbb{R}^n$. Then Triebel-Lizorkin space $F_s^{p,q}(\mathbb{R}^n)$ is defined for each $0 < p < \infty$ as

$$F_s^{p,q}(\mathbb{R}^n) := \Big\{ f \in S'(\mathbb{R}^n) : \quad \|f\|_{F_s^{p,q}(\mathbb{R}^n)} := \Big\| \Big(\sum_{j=0}^{\infty} |2^{sj}\mathcal{F}^{-1}(\zeta_j \mathcal{F} f)|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} < \infty \Big\}. \tag{11.4}$$

If $0 < p \leq \infty$ then the Besov space $B_s^{p,q}(\mathbb{R}^n)$ can be defined as

$$B_s^{p,q}(\mathbb{R}^n) := \Big\{ f \in S'(\mathbb{R}^n) : \quad \|f\|_{B_s^{p,q}(\mathbb{R}^n)} := \Big(\sum_{j=0}^{\infty} \|2^{sj}\mathcal{F}^{-1}(\zeta_j \mathcal{F} f)\|^q_{L^p(\mathbb{R}^n)} \Big)^{1/q} < \infty \Big\}. \tag{11.5}$$

A different choice of the system $\{\zeta_j\}_{j=0}^{\infty} \in \Xi$ yields the same spaces (11.4)-(11.5), albeit equipped with equivalent norms. Furthermore, the class of Schwartz functions in $\mathbb{R}^n$ is dense in both $B_s^{p,q}(\mathbb{R}^n)$ and $F_s^{p,q}(\mathbb{R}^n)$ provided $s \in \mathbb{R}$ and $0 < p, q < \infty$.

As far as the real method of interpolation is concerned, we note the following classical result.

Theorem 11.1 *(cf.* **[90]***) Let* $\alpha_0, \alpha_1 \in \mathbb{R}$*,* $\alpha_0 \neq \alpha_1$*,* $0 < q_0, q_1, q \leq \infty$*,* $0 < \theta < 1$*,* $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$*. Then*

$$(F_{\alpha_0}^{p,q_0}(\mathbb{R}^n), F_{\alpha_1}^{p,q_1}(\mathbb{R}^n))_{\theta,q} = B_{\alpha}^{p,q}(\mathbb{R}^n), \quad 0 < p < \infty, \tag{11.6}$$

$$(B_{\alpha_0}^{p,q_0}(\mathbb{R}^n), B_{\alpha_1}^{p,q_1}(\mathbb{R}^n))_{\theta,q} = B_{\alpha}^{p,q}(\mathbb{R}^n), \quad 0 < p \leq \infty. \tag{11.7}$$

Turning to the complex method of interpolation, we have:

Theorem 11.2 *Let* $\alpha_0, \alpha_1 \in \mathbb{R}$*,* $0 < p_0, p_1 \leq \infty$*, and* $0 < q_0, q_1 \leq \infty$ *with the property that either* $\max\{p_0, q_0\} < \infty$*, or* $\max\{p_1, q_1\} < \infty$*. Then*

$$[F_{\alpha_0}^{p_0,q_0}(\mathbb{R}^n), F_{\alpha_1}^{p_1,q_1}(\mathbb{R}^n)]_\theta = F_{\alpha}^{p,q}(\mathbb{R}^n), \tag{11.8}$$

where $0 < \theta < 1$*,* $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$*,* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$*, and* $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$*.*

Furthermore, if $\alpha_0, \alpha_1 \in \mathbb{R}$*,* $0 < p_0, p_1, q_0, q_1 \leq \infty$ *and* $\min\{q_0, q_1\} < \infty$*, then also*

$$[B_{\alpha_0}^{p_0,q_0}(\mathbb{R}^n), B_{\alpha_1}^{p_1,q_1}(\mathbb{R}^n)]_\theta = B_{\alpha}^{p,q}(\mathbb{R}^n), \tag{11.9}$$

where $0 < \theta < 1$*,* $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$*,* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$*, and* $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$*.*

When $p, q \geq 1$, this is well-known; cf. [35], [89]. For the *entire* scale $p, q > 0$, the result has been established in [66], [50].

11.2 Gehring's lemma

Let us first recall the definition of a space of homogeneous type, as introduced by R. Coifman and G. Weiss in [17]. Assume that Σ is a set equipped with a quasi-distance, i.e. a function $d : \Sigma \times \Sigma \to [0, \infty)$ satisfying $d(x, y) = 0 \Leftrightarrow x = y$, $d(x, y) = d(y, x)$ and such that there exists $\kappa \geq 1$, called concavity constant, for which

$$d(x, y) \leq \kappa \, (d(x, z) + d(z, y)), \qquad \forall \, x, y, z \in \Sigma. \tag{11.10}$$

In turn, a choice of a quasi-distance naturally induces a topology on Σ for which the balls $B(x, r) := \{y \in \Sigma : \, d(x, y) < r\}$ (which, unlike the case of a metric space, are not necessarily open when $\kappa > 1$) form a base. A well-known theorem of Macías and Segovia ([61]) asserts that the original quasi-distance function on Σ can be replaced by an equivalent one which has the additional property that the associated balls are open. It is also well-known that Σ is compact if and only if $\mu(\Sigma) < +\infty$.

A space of homogeneous type is a structure (Σ, d, μ), where d is a quasi-distance on the set Σ and μ is a measure defined on the minimal sigma-algebra containing all Borel sets and all balls, and which is doubling, i.e., there exists a $A > 1$, called the doubling constant, such that

$$0 < \mu(B(x, 2r)) \leq A \, \mu(B(x, r)) < \infty, \qquad \forall \, x \in \Sigma, \;\; \forall \, r > 0. \tag{11.11}$$

In the sequel, if $\lambda > 0$ and $B = B(x, r)$, we shall use the notation $\lambda B := B(x, \lambda r)$. Also, the symbol $⨍$ indicates integral average, and $L^p(\Sigma, d\mu)$ stands for the Lebesgue space of μ-measurable, p-th power integrable functions on Σ. The following Calderón-Zygmund decomposition result and Vitali covering lemma are well known. See, e.g., [2], [17].

Lemma 11.3 *Given a space of homogeneous type (Σ, d, μ), there exists $c > 1$ depending only the concavity constant κ such that the following holds. If $\mathcal{B} = \{B_\alpha\}_{\alpha \in \mathcal{A}}$ is a family* *of balls and* $E := \bigcup_\alpha B_\alpha$ *is* μ*-measurable and* $\mu(E) < \infty$*, then there exists a sequence of mutually disjoint balls* $\{B_j\}_{j\in\mathbb{N}} \subset \mathcal{B}$ *such that any* $B \in \mathcal{B}$ *is contained in some* cB_j*. In particular,* $E \subset \bigcup_j cB_j$*.*

Lemma 11.4 *For every space of homogeneous type* (Σ, d, μ) *with the property that the balls are open sets there exists a finite constant* $c > 1$*, depending only on the concavity constant* κ *(in fact, the same constant as in Lemma 11.3) with the following significance. Assume that* $f \in L^1(\Sigma, d\mu)$ *is a nonnegative function and that* $\lambda > ⨍_\Sigma f\,d\mu$*. Then there exists a sequence of mutually disjoint balls* $B_j = B(x_j, r_j)$*,* $j \in \mathbb{N}$*, such that*

$$⨍_{cB_j} f\,d\mu \le \lambda < ⨍_{B_j} f\,d\mu \qquad \forall\, j \in \mathbb{N}, \tag{11.12}$$

$$f \le \lambda \ \ \textit{pointwise } \mu\textit{-a.e. on } \ \Sigma \setminus \bigcup_{j\in\mathbb{N}} cB_j. \tag{11.13}$$

We are now ready to state the main result in this section which is a version of the celebrated Gehring's lemma [38], proved here via an approach more akin to the work in [43].

Proposition 11.5 *Assume that* (Σ, d, μ) *is a non-compact space of homogeneous type and that* $1 \le q < p$*. Also, suppose* g, h *are two non-negative functions,* $g \in L^p(\Sigma, d\mu)$*, and there exist* $K \ge 0$ *and* $\eta > 1$ *such that*

$$\left(⨍_B g^p\,d\mu\right)^{\frac{1}{p}} \le K\left(⨍_{\eta B} g^q\,d\mu\right)^{\frac{1}{q}} + \left(⨍_{\eta B} h^p\,d\mu\right)^{\frac{1}{p}} \qquad \textit{for every ball } B \subset \Sigma. \tag{11.14}$$

Then there exists $\varepsilon_o > 0$*, depending only on* p, q, K, η *and* κ*,* A *(the concavity and doubling constants for* (Σ, d, μ)*, respectively), such that whenever* $0 \le \varepsilon < \varepsilon_o$*,*

$$\int_\Sigma g^{p+\varepsilon}\,d\mu \le C \int_\Sigma h^{p+\varepsilon}\,d\mu, \tag{11.15}$$

where $C > 0$ *depends only on* p, q, K, η*,* κ*,* A *and* ε*.*

Proof. From an earlier discussion, by eventually replacing the original quasi-distance on Σ, there is no loss of generality in assuming that the balls in Σ are open sets. Assume that this is the case, and for each $r > 0$, set

$$G_r := \{x \in \Sigma : g(x) > r\} \quad \text{and} \quad H_r := \{x \in \Sigma : h(x) > r\}. \tag{11.16}$$

For each fixed $t > 0$ we now perform a Calderón-Zygmund decomposition for the function g^p at level $(\lambda t)^p$, with $\lambda > 1$ to be specified later. This gives a sequence of mutually disjoint balls $\{B_j\}_{j \in \mathbb{N}}$ and a constant $c > 1$ such that

$$⨍_{cB_j} g^p \leq (\lambda t)^p < ⨍_{B_j} g^p \quad \text{and} \quad g^p \leq (\lambda t)^p \quad \mu\text{-a.e. on } \Sigma \setminus \bigcup_{j \in \mathbb{N}} cB_j. \tag{11.17}$$

Cf. Lemma 11.4 above. In particular, $G_{\lambda t} \subseteq \bigcup_j cB_j$ so by (11.17) we have

$$\int_{G_{\lambda t}} g^p \, d\mu \leq \sum_j \int_{cB_j} g^p \, d\mu \leq (\lambda t)^p \sum_j \mu(cB_j). \tag{11.18}$$

Next, $\frac{1}{\mu(\eta B_j)} \int_{\eta B_j \setminus G_t} g^q \, d\mu \leq t^q$, so we may write

$$\begin{aligned} \left(⨍_{\eta B_j} g^q \, d\mu \right)^{\frac{1}{q}} &= \left(\frac{1}{\mu(\eta B_j)} \int_{\eta B_j \cap G_t} g^q \, d\mu + \frac{1}{\mu(\eta B_j)} \int_{\eta B_j \setminus G_t} g^q \, d\mu \right)^{\frac{1}{q}} \\ &\leq \left(\frac{1}{\mu(\eta B_j)} \int_{\eta B_j \cap G_t} g^q \, d\mu \right)^{\frac{1}{q}} + t \\ &\leq 2t + \frac{1}{t^{q-1}} \cdot \frac{1}{\mu(\eta B_j)} \int_{\eta B_j \cap G_t} g^q \, d\mu, \end{aligned} \tag{11.19}$$

where, in the second and third inequalities, use has been made of the elementary estimates $(a+b)^{\frac{1}{q}} \leq a^{\frac{1}{q}} + b^{\frac{1}{q}}$ valid for any $a, b \geq 0$ and $M^{\frac{1}{q}} \leq t + \frac{M}{t^{q-1}}$ valid for any $M \geq 0$, $t > 0$ (here $q \geq 1$ is used). Going further, a similar argument gives

$$\left(⨍_{\eta B} h^p \, d\mu \right)^{\frac{1}{p}} \leq \left(\int_{\eta B_j \cap H_t} h^p \, d\mu \right)^{\frac{1}{p}} + t \leq 2t + \frac{1}{t^{p-1}} \cdot \frac{1}{\mu(\eta B_j)} \int_{\eta B_j \cap H_t} h^p \, d\mu. \tag{11.20}$$

A combination of (11.14), (11.17), (11.19) and (11.20), now gives

$$\begin{aligned} \lambda t \leq \left(⨍_{B_j} g^p \, d\mu \right)^{\frac{1}{p}} &\leq (2K+2)t + \left(\frac{K}{t^{q-1}} \cdot \frac{1}{\mu(\eta B_j)} \int_{\eta B_j \cap G_t} g^q \, d\mu \right) \\ &\quad + \left(\frac{1}{t^{p-1}} \cdot \frac{1}{\mu(\eta B_j)} \int_{\eta B_j \cap H_t} h^p \, d\mu \right). \end{aligned} \tag{11.21}$$

Hence,

$$(\lambda - 2K - 2)\mu(\eta B_j) \leq \frac{K}{t^q}\int_{\eta B_j \cap G_t} g^q \, d\mu + \frac{1}{t^p}\int_{\eta B_j \cap H_t} h^p \, d\mu. \tag{11.22}$$

At this stage, we fix $\lambda > 2K + 2$ (so that $\lambda > 1$) for the remainder of the proof.

Next, Lemma 11.3 and the doubling property (11.11) ensure that there exists a set $\mathbb{N}' \subseteq \mathbb{N}$ such that

$$\begin{aligned} &\text{the balls } \{\eta B_{j'}\}_{j' \in \mathbb{N}'} \text{ are mutually disjoint,} \\ &\text{and} \quad \mu\Big(\bigcup_{j \in \mathbb{N}} \eta B_j\Big) \leq C' \sum_{j' \in \mathbb{N}'} \mu(\eta B_{j'}), \end{aligned} \tag{11.23}$$

where C' depends only on A and κ. In concert with (11.11), (11.22) and the fact that the balls in the family $\{B_j\}_{j \in \mathbb{N}}$ are mutually disjoint, this estimate allows us to write, for some C'' depending only on A and κ,

$$\begin{aligned} \sum_{j \in \mathbb{N}} \mu(cB_j) &\leq C'' \sum_{j \in \mathbb{N}} \mu(B_j) = C'' \mu\Big(\bigcup_{j \in \mathbb{N}} B_j\Big) \leq C'' \mu\Big(\bigcup_{j \in \mathbb{N}} \eta B_j\Big) \leq C'C'' \sum_{j' \in \mathbb{N}'} \mu(\eta B_{j'}) \\ &\leq \frac{C}{\lambda - 2K - 2} \sum_{j' \in \mathbb{N}'} \left(\frac{K}{t^q} \int_{\eta B_{j'} \cap G_t} g^q \, d\mu + \frac{1}{t^p} \int_{\eta B_{j'} \cap H_t} h^p \, d\mu \right) \\ &\leq \frac{C}{\lambda - 2K - 2} \left[\frac{K}{t^q} \int_{G_t} g^q \, d\mu + \frac{1}{t^p} \int_{H_t} h^p \, d\mu \right], \end{aligned} \tag{11.24}$$

where $C := C'C''$ depends only on A and κ. Note that the last step above uses the first condition in (11.23). From this and (11.18) we then obtain

$$\int_{G_{\lambda t}} g^p \, d\mu \leq \frac{C\lambda^p}{\lambda - 2K - 2} \left[\frac{K}{t^{q-p}} \int_{G_t} g^q \, d\mu + \int_{H_t} h^p \, d\mu \right]. \tag{11.25}$$

Recall that $\lambda > 1$ and $p - q \geq 0$, so that $G_{\lambda t} \subset G_t$, and further,

$$\int_{G_t \setminus G_{\lambda t}} g^p \, d\mu = \int_{G_t \setminus G_{\lambda t}} g^q g^{p-q} \, d\mu \leq \lambda^{p-q} t^{p-q} \int_{G_t} g^q \, d\mu. \tag{11.26}$$

By adding (11.25) and (11.26) we arrive at

$$\int_{G_t} g^p\, d\mu \le \left(\frac{CK\lambda^p}{\lambda - 2K - 2} + \lambda^{p-q}\right) t^{p-q} \int_{G_t} g^q\, d\mu + \left(\frac{C\lambda^p}{\lambda - 2K - 2}\right) \int_{H_t} h^p\, d\mu. \tag{11.27}$$

Multiplying both sides of this last inequality by t^α, for some $\alpha \in \mathbb{R}$ to be chosen momentarily, and then integrating with respect to t in the interval $(0, T)$, with $T > 0$ an arbitrary, fixed number, yields an estimate of the form

$$\begin{aligned} \int_0^T \left(\int_{G_t} t^\alpha g^p\, d\mu\right) dt \;&\le\; C_0 \int_0^T \left(\int_{G_t} t^{p-q+\alpha} g^q\, d\mu\right) dt \\ &\quad + C_1 \int_0^\infty \left(\int_{H_t} t^\alpha h^p\, d\mu\right) dt, \end{aligned} \tag{11.28}$$

where

$$C_0 := \lambda^{p-q} + \frac{CK\lambda^p}{\lambda - 2K - 2}, \qquad C_1 := \frac{C\lambda^p}{\lambda - 2K - 2}. \tag{11.29}$$

Let us now fix $\alpha > -1$ and use Fubini's theorem to compute

$$\begin{aligned} \int_0^T \left(\int_{G_t} t^\alpha g^p\, d\mu\right) dt &= \int_\Sigma \left(\int_0^T t^\alpha \chi_{G_t}\, dt\right) g^p\, d\mu \\ &= \int_\Sigma \left(\int_0^{\min\{g(x),T\}} t^\alpha\, dt\right) g(x)^p\, d\mu(x) = \frac{1}{\alpha+1} \int_\Sigma g^p \Big[\min\{g, T\}\Big]^{1+\alpha} d\mu, \end{aligned} \tag{11.30}$$

since $\chi_{G_t}(x) = 1$ if and only if $g(x) > t$. Similarly,

$$\int_0^\infty \left(\int_{H_t} t^\alpha h^p\, d\mu\right) dt = \frac{1}{\alpha+1} \int_\Sigma h^{p+\alpha+1}\, d\mu. \tag{11.31}$$

Finally, $\alpha > -1$ and $p \ge q$ force $p - q + \alpha > -1$ and the same type of argument as before gives

$$\begin{aligned} \int_0^T \left(\int_{G_t} t^{p-q+\alpha} g^q\, d\mu\right) dt \;&=\; \frac{1}{p-q+\alpha+1} \int_\Sigma g^q \Big[\min\{g, T\}\Big]^{p-q+\alpha+1} d\mu \\ &\le\; \frac{1}{p-q+\alpha+1} \int_\Sigma g^p \Big[\min\{g, T\}\Big]^{\alpha+1} d\mu. \end{aligned} \tag{11.32}$$

Altogether, for each $T > 0$ we obtain

$$\begin{aligned}\int_\Sigma g^p \Big[\min\{g,T\}\Big]^{\alpha+1} d\mu &\leq \frac{C_0(\alpha+1)}{p-q+\alpha+1}\int_\Sigma g^p \Big[\min\{g,T\}\Big]^{\alpha+1} d\mu \\ &\quad + C_1 \int_\Sigma h^{p+\alpha+1}\, d\mu, \end{aligned} \tag{11.33}$$

with C_0, C_1 as in (11.29). Note that the integral in the left-hand side matches the first integral in the right-hand side and is finite for each $T > 0$ since

$$\int_\Sigma g^p \Big[\min\{g,T\}\Big]^{\alpha+1} d\mu \leq T^{\alpha+1}\int_\Sigma g^p\, d\mu < +\infty, \tag{11.34}$$

given that the function g belongs to $L^p(\Sigma, d\mu)$. Consequently, in order to absorb the first term from the right-hand side into the left-hand side we need to choose $\alpha > -1$ such that $p - q + \alpha + 1 > (\alpha+1)C_0$. If $C_0 > 1$, this requirement becomes $0 < \alpha + 1 < \frac{p-q}{C_0-1}$. However, if $\lambda > \max\{2K+2, 1\}$ then $C_0 > 1$, as is visible from (11.29). We obtain

$$\int_\Sigma g^p \Big[\min\{g,T\}\Big]^{\alpha+1} d\mu \leq C_2 \int_\Sigma h^{p+\alpha+1}\, d\mu, \tag{11.35}$$

where C_2 is independent of T. By letting $T \to \infty$ and invoking Lebesgue's Monotone Convergence Theorem, we may now conclude that (11.15) holds whenever $0 < \varepsilon < \varepsilon_o := \frac{p-q}{C_0-1}$.

Finally, the case $\varepsilon = 0$ follows directly from (11.14) by writing

$$\left(\int_{B_R} g^p\, d\mu\right)^{\frac{1}{p}} \leq K \frac{\mu(B_R)^{1/p}}{\mu(\eta B_R)^{1/q}} \left(\int_{\eta B_R} g^q\, d\mu\right)^{\frac{1}{q}} + \left(\frac{\mu(B_R)}{\mu(\eta B_R)}\right)^{1/p} \left(\int_{\eta B_R} h^p\, d\mu\right)^{\frac{1}{p}} \tag{11.36}$$

where $R > 0$ is arbitrary and $B_R := B(x_o, R)$ for some fixed point $x_o \in \Sigma$, and then letting R approach infinity. Since $q < p$, the coefficient of the first integral in the right-hand side goes to zero, whereas the the coefficient of the second one stays bounded. This finishes the proof of the proposition. $\square$

11.3 Hole-filling lemma

Lemma 11.6 *Let f be an arbitrary locally bounded function on $\mathbb{R}$ with the property that there exist real numbers θ_0, θ_1, nondecreasing functions A and B, $\alpha > 0$, and $\theta \in (0,1)$ such that*

$$f(s) \leq [A(t)(t-s)^{-\alpha} + B(t)] + \theta f(t) \quad \text{for all } \theta_0 \leq s < t \leq \theta_1. \tag{11.37}$$

Then there exists $C > 0$ such that

$$f(r) \leq C[A(R)(R-r)^{-\alpha} + B(R)] \quad \text{for all } \theta_0 \leq r < R \leq \theta_1. \tag{11.38}$$

Proof. Fix $\sigma \in (0,1)$ arbitrary and let $t_0 = r$, $t_{i+1} = t_i + (1-\sigma)(R-r)\sigma^i$, for each $i \geq 0$. Then $t_\infty = R$, and

$$t_n - r = t_n - t_0 = \sum_{i=0}^{n-1}(t_{i+1} - t_i) = (1-\sigma)(R-r)\sum_{i=0}^{n-1}\sigma^i = (R-r)(1-\sigma^n). \tag{11.39}$$

Thus, for each i,

$$\begin{aligned} f(t_i) &\leq [A(t_{i+1})(1-\sigma)^{-\alpha}(R-r)^{-\alpha}\sigma^{-i\alpha} + B(t_{i+1})] + \theta f(t_{i+1}) \\ &\leq [A(R)(1-\sigma)^{-\alpha}(R-r)^{-\alpha}\sigma^{-i\alpha} + B(R)] + \theta f(t_{i+1}). \end{aligned} \tag{11.40}$$

Multiplying (11.40) by θ^i we obtain that

$$\theta^i f(t_i) \leq I(\theta\sigma^{-\alpha})^i + \theta^i B(R) + \theta^{i+1} f(t_{i+1}), \tag{11.41}$$

where $I := A(R)(1-\sigma)^{-\alpha}(R-r)^{-\alpha}$. Summing up (11.41) over i, we obtain

$$\sum_{i=0}^{n}\theta^i f(t_i) \leq I\sum_{i=0}^{n}(\theta\sigma^{-\alpha})^i + B(R)\sum_{i=0}^{n}\theta^i + \sum_{i=1}^{n+1}\theta^i f(t_i). \tag{11.42}$$

Hence, after subtracting $\sum\limits_{i=1}^{n}\theta^i f(t_i)$ from (11.42), we see that

$$f(r) \leq I\sum_{i=0}^{n}(\theta\sigma^{-\alpha})^i + B(R)\sum_{i=0}^{n}\theta^i + \theta^{n+1} f(t_{n+1}). \tag{11.43}$$

Now we select $\sigma \in (0,1)$ so that $\theta\sigma^{-\alpha} < 1$. Then, after letting $n \to \infty$ in (11.43), since $f(t_{n+1})$ stays bounded, we get that

$$f(r) \leq I \cdot \frac{1}{1-\theta\sigma^{-\alpha}} + \frac{1}{1-\theta} B(R). \tag{11.44}$$

If now $C := \max\{\frac{1}{1-\theta\sigma^{-\alpha}}, \frac{1}{1-\theta}\}$, we have that

$$\begin{aligned} f(r) &\leq C(I + B(R)) = C[A(R)(1-\sigma)^{-\alpha}(R-r)^{-\alpha} + B(R)] \\ &\leq C[A(R)(R-r)^{-\alpha} + B(R)]. \end{aligned}$$

11.4 Korn's inequality

The goal of this section is to prove Lemma 6.3. For a Lipschitz domain D in $\mathbb{R}^n$ and $1 < p < \infty$, we set $L^p_1(D)$ to be the L^p-based Sobolev space of order one in D, let $L^p_{1,0}(D)$ denote the closure of $C^\infty_o(D)$ in $L^p_1(D)$, and let $L^p_{-1}(D)$ be the dual of $L^{p'}_{1,0}(D)$, where $1/p + 1/p' = 1$.

We start with a result of independent interest.

Lemma 11.7 *Let $D \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain and suppose that $1 < p < \infty$. Then there exists a finite constant $C > 0$ depending only on n, p, the diameter of D, and the Lipschitz character of D such that every distribution $u \in L^p_{-1}(D)$ with $\nabla u \in L^p_{-1}(D)$ has the property that $u \in L^p(D)$ and*

$$\|u\|_{L^p(D)} \leq C\|\nabla u\|_{L^p_{-1}(D)} + C\|u\|_{L^p_{-1}(D)} \tag{11.45}$$

holds.

Proof. The problem is local in character, and hence, there is no loss of generality assuming that $D \subset B(0,1)$ is a Lipschitz domain which is starlike with respect to some ball $B \subset D$, of radius comparable to the diameter of D via constants which, in turn, depend only on the diameter and the Lipschitz character of D. Assuming that this is the case, fix a function $\theta \in C^\infty_o(B)$ with $\int \theta = 1$. In this context, Bogovskiı has constructed a linear operator $\mathcal{J}$ with the following properties. First, for each $1 < q < \infty$,

$$\mathcal{J} : L^q(D) \to L^q_{1,0}(D) \tag{11.46}$$

is bounded, and if $R := \operatorname{diam}(D)$, then

$$\text{the operator norm of } \mathcal{J} \text{ in (11.46) is } \leq C(\partial D, q, R). \tag{11.47}$$

Second,

$$\mathcal{J}\varphi \in C^\infty_o(D) \quad \text{whenever} \quad \varphi \in C^\infty_o(D), \tag{11.48}$$

and third,

$$\operatorname{div} \mathcal{J}\varphi = \varphi - \theta \Big(\int \varphi(x)\, dx \Big) \qquad \text{for any} \quad \varphi \in C^\infty_o(D). \tag{11.49}$$

Then, for any $\varphi \in C^\infty_o(D)$, we may write

$$\begin{aligned}
|\langle u, \varphi\rangle| &\leq |\langle u, \operatorname{div} \mathcal{J}\varphi\rangle| + |\langle u, \theta\rangle|\, |\langle \varphi, 1\rangle| \\
&\leq |\langle \nabla u, \mathcal{J}\varphi\rangle| + |\langle u, \theta\rangle| \|\varphi\|_{L^{p'}(D)} \\
&\leq \|\nabla u\|_{L^p_{-1}(D)} \|\mathcal{J}\varphi\|_{L^{p'}_{1,0}(D)} + |\langle u, \theta\rangle| \|\varphi\|_{L^{p'}(D)} \\
&\leq C(\|\nabla u\|_{L^p_{-1}(D)} + |\langle u, \theta\rangle|) \|\varphi\|_{L^{p'}(D)}.
\end{aligned} \tag{11.50}$$

Since $C^\infty_c(D)$ is dense in $L^{p'}(D)$, we see that $u \in \Big(L^{p'}(D)\Big)^* = L^p(D)$. Finally, since $|\langle u, \theta\rangle| \leq \|u\|_{L^p_{-1}(D)} \|\theta\|_{L^{p'}_{1,0}(D)} \leq C(\theta) \|u\|_{L^p_{-1}(D)}$, we also see that (11.45) holds. □

Next, the goal is to prove the following Korn type estimate.

Proposition 11.8 *Let D be a Lipschitz domain of diameter R and assume that $1 < p < \infty$. Then there exists a finite constant $C > 0$ which depends only on p and the Lipschitz character of D such that*

$$\|\vec{u}\|_{L^p_1(D)} \leq C \Big\{ \|\nabla \vec{u} + \nabla \vec{u}^\top\|_{L^p(D)} + C R^{-1} \|\vec{u}\|_{L^p(D)} \Big\}, \tag{11.51}$$

uniformly for $\vec{u} \in L^p_1(D)$.

Proof. Given how the estimate (11.45) dilates with respect to R, matters can be readily reduced to the case when $R = 1$. Next, for each $j, k \in \{1, ..., n\}$, we set

$$\varepsilon_{jk}(\vec{u}) := \tfrac{1}{2}(\partial_j u_k + \partial_k u_j). \tag{11.52}$$

so that $(\nabla\vec{u} + \nabla\vec{u}^{\top})_{jk} = 2\varepsilon_{jk}(\vec{u})$. A direct calculation then shows that

$$\partial_i\partial_j u_k = \partial_i\varepsilon_{jk}(\vec{u}) + \partial_j\varepsilon_{ik}(\vec{u}) - \partial_k\varepsilon_{ij}(\vec{u}), \qquad \forall\, i, j, k. \tag{11.53}$$

In particular, by Lemma 11.7 and the fact that $\nabla : L^p(D) \to L^p_{-1}(D)$ is bounded,

$$\begin{aligned}
\sum_{j,k} \|\partial_j u_k\|_{L^p(D)} &\leq C \sum_{j,k}\sum_i \|\partial_i\partial_j u_k\|_{L^p_{-1}(D)} + C\sum_{j,k} \|\partial_j u_k\|_{L^p_{-1}(D)} \\
&\leq C \sum_{i,j,k} \|\partial_i\varepsilon_{jk}(\vec{u})\|_{L^p_{-1}(D)} + C\sum_k \|u_k\|_{L^p(D)} \\
&\leq C \sum_{j,k} \|\varepsilon_{jk}(\vec{u})\|_{L^p(D)} + C\|\vec{u}\|_{L^p(D)} \\
&\leq C\|\nabla\vec{u} + \nabla\vec{u}^{\top}\|_{L^p(D)} + C\|\vec{u}\|_{L^p(D)}.
\end{aligned} \tag{11.54}$$

Now (11.51) readily follows from this. □

11.5 Hardy's estimate

Let L be a homogeneous, constant coefficient, elliptic operator. The aim of this section is to present a result which can, in essence, be attributed to Hardy.

Lemma 11.9 [`Hardy's estimate`]
Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be the domain lying above the graph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$. Assume w is a null-solution of L in Ω and that $M(\nabla w) \in L^p(\partial\Omega)$ for some $p < n-1$. Then there exist constants $c = c(w) \in \mathbb{R}$ and $C = C(\partial\Omega) > 0$ such that

$$\|M(w-c)\|_{L^{p^*}(\partial\Omega)} \leq C\|M(\nabla w)\|_{L^p(\partial\Omega)} \;\; \textit{where} \;\; \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n-1}. \tag{11.55}$$

Prior to presenting the proof of this proposition we isolate one technical aspect.

Lemma 11.10 *Assume that Ω is a graph Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, and that $u \in C^1(\Omega)$, $C > 0$ and $\alpha > 1$ are such that*

$$|\nabla u(x)| \leq C \operatorname{dist}(x, \partial\Omega)^{-\alpha}, \qquad \forall\, x \in \Omega. \tag{11.56}$$

Then for each $x \in \Omega$, the limit

$$c := \lim_{t \to \infty} u(x + te_n) \tag{11.57}$$

exists, is independent of x, and, moreover

$$|u(x) - c| \leq C \operatorname{dist}(x, \partial\Omega)^{1-\alpha}, \qquad \forall\, x \in \Omega. \tag{11.58}$$

Proof. For every $x \in \Omega$ and $t \geq 0$ set

$$c(x,t) := u(x + te_n) + \int_t^\infty (\partial_n u)(x + se_n)\, ds. \tag{11.59}$$

By (11.56), the integral in (11.59) is absolutely convergent, and, obviously, the expression in the right hand-side is independent of $t \geq 0$. We may thus abbreviate $c(x) := c(x,t)$. Hence, the limit

$$\lim_{t \to \infty} u(x + te_n) = \lim_{t \to \infty} c(x) = c(x) \quad \text{ exists for every } x \in \Omega. \tag{11.60}$$

To prove that this limit is actually independent of x, observe that if $x, y \in \Omega$ are arbitrary, fixed, and $t \geq 0$ is sufficiently large, then every $z \in [x + te_n, y + te_n]$ belongs to Ω and $\operatorname{dist}(z, \partial\Omega) \geq Ct$. Therefore, by (11.56) and the Mean Value Theorem,

$$|u(x + te_n) - u(y + te_n)| \leq C(\partial\Omega, x, y, u)\, t^{-\alpha} \to 0 \ \text{ as } \ t \to \infty, \tag{11.61}$$

which shows that $c(x) = c(y)$, for every $x, y \in \Omega$. If we now let $c \in \mathbb{R}$ be $c(x)$, $x \in \Omega$, then

$$
\begin{aligned}
|u(x) - c| &\leq \int_0^\infty |(\partial_n u)(x + se_n)|\, ds \\
&\leq C \int_0^\infty \Big[\mathrm{dist}\,(x, \partial\Omega) + s\Big]^{-\alpha} ds = C\, \mathrm{dist}\,(x, \partial\Omega)^{1-\alpha},
\end{aligned}
\tag{11.62}
$$

proving (11.58). □

In applications, we typically start with a null-solution u of an elliptic operator in Ω which satisfies $M(\nabla u) \in L^p(\partial\Omega)$ for some $0 < p < n-1$. Fix $x \in \Omega$ and set $R := \mathrm{dist}\,(x, \partial\Omega)$. Then by interior estimates and (11.64) below,

$$
|\nabla u(x)| \leq C \Big(⨍_{B(x,R/2)} |\nabla u|^p \Big)^{1/p} \leq C R^{-\frac{n-1}{p}} \|M(\nabla u)\|_{L^p(\partial\Omega)}. \tag{11.63}
$$

Note that $0 < p < n-1$ implies $\alpha := (n-1)/p > 1$, so the previous discussion about the decay of u applies.

Lemma 11.11 *For every Lipschitz domain Ω (bounded, or of graph type) in $\mathbb{R}^n$, $n \geq 2$, there exists a finite constant $C = C(\Omega) > 0$ with the following property. For every measurable set $E \subset \Omega$ and every measurable function $u : \Omega \to \mathbb{R}$, one has*

$$
\int_E |u(x)|\, dx \leq C[\mathrm{dist}\,(E, \partial\Omega) + \mathrm{diam}\,(E)] \int_{\mathcal{U}(E)} M(u)\, d\sigma, \tag{11.64}
$$

where

$$
\mathcal{U}(E) := \{x \in \partial\Omega :\, \Gamma_\kappa^+(x) \cap E \neq \emptyset\}. \tag{11.65}
$$

Proof. For every $\delta > 0$, set $\mathcal{O}_\delta := \{x \in \Omega :\, \mathrm{dist}\,(x, \partial\Omega) < \delta\}$. As shown in [41], for a class of domains containing those which are Lipschitz, there exists $C = C(\Omega) > 0$ such that, for every measurable function $v : \Omega \to \mathbb{R}$,

$$
\int_{\mathcal{O}_\delta} |v(x)|\, dx \leq C\delta \int_{\partial\Omega} M(v)\, d\sigma, \tag{11.66}
$$

uniformly in $\delta > 0$. Let us specialize this to the case when $\delta := \operatorname{dist}(E, \partial\Omega) + \operatorname{diam}(E)$ and $v := u\chi_E$. Since, in this scenario, $E \subset \mathcal{O}_\delta$, we may write

$$\int_E |u(x)|\,dx = \int_{\mathcal{O}_\delta} |(u\chi_E)(x)|\,dx \leq C\delta \int_{\partial\Omega} M(u\chi_E)\,d\sigma \leq C\delta \int_{\mathcal{U}(E)} M(u)\,d\sigma, \tag{11.67}$$

as desired. $\square$

We are now ready to discuss the

Proof of Lemma 11.9. The argument below is due to Russell Brown [9] and we are most grateful to him for allowing us to include it here. According to [27], for any $\alpha > 0$, we have interior estimates of the form

$$|w(x)|^\alpha \leq C \fint_{B(x,\delta(x)/2)} |w|^\alpha, \tag{11.68}$$

where $\delta(x) := \operatorname{dist}(x, \partial\Omega)$. Let $x = (x', x_n)$ and $\bar{x} = (x', \varphi(x'))$. Then since by Lemma 11.11

$$\fint_{B(x,R)} |w|^\alpha\,dx \leq C \fint_{S_{cR}(\bar{x})} |M(w)|^\alpha\,d\sigma, \qquad \text{if} \quad R \approx \delta(x), \tag{11.69}$$

we have that

$$|w(x)| \leq C \left(\fint_{S_{c\delta(x)}(\bar{x})} |M(w)|^\alpha\,d\sigma \right)^{\frac{1}{\alpha}}, \tag{11.70}$$

hence, further,

$$|w(x)| \leq C\delta(x)^{-\frac{n-1}{\alpha}} \|M(w)\|_{L^\alpha(\partial\Omega)}. \tag{11.71}$$

Now since the components of ∇w are also null-solutions of L in Ω, we can conclude that

$$|\nabla w(x)| \leq C\delta(x)^{-\frac{n-1}{p}} \|M(\nabla w)\|_{L^p(\partial\Omega)}. \tag{11.72}$$

In particular, by Lemma 11.10, we can choose $c \in \mathbb{R}$ such that $u := w - c$ vanishes at infinity (in the quantitative sense described there). Fix $x \in \partial\Omega$ and let $y = (y', y_n) \in \Gamma(x)$. Then

$$|u(y)| = \left|\int_{y_n}^{\infty} \partial_n u(y',t)\,dt\right| \leq \int_{y_n}^{\infty} |\nabla u(y',t)|dt = \int_{y_n}^{\infty} |\nabla w(y',t)|dt. \tag{11.73}$$

Choose α so that $\frac{p}{n-1} < \alpha <$min$\{1,p\}$. Now applying (11.70) with ∇w in place of w gives

$$|u(y)| \leq C\int_{y_n}^{\infty} \left(\fint_{S_{ct}(x)} |M(\nabla w)|^{\alpha}\,d\sigma\right)^{\frac{1}{\alpha}} dt. \tag{11.74}$$

Let $\mathcal{M}$ denote the Hardy-Littlewood maximal function on $\partial\Omega$. Then by definition,

$$\fint_{S_{Ct}(x)} |M(\nabla w)|^{\alpha}\,d\sigma \leq \mathcal{M}(M(\nabla w)^{\alpha})(x), \tag{11.75}$$

and so from (11.74),

$$|u(y)| \leq C\mathcal{M}(M(\nabla w)^{\alpha})^{\frac{1}{\alpha}-1}(x)\int_{y_n}^{\infty}\int\limits_{S_{ct}(x)} \frac{1}{t^{n-1}}M(\nabla w)^{\alpha}(z)\,d\sigma(z)\,dt. \tag{11.76}$$

Notice that if $z \in S_{ct}(x)$, then $|x-z| < ct$. So by switching the order of integration, we get

$$\begin{aligned}
|u(y)| &\leq C\mathcal{M}(M(\nabla w)^{\alpha})^{\frac{1}{\alpha}-1}(x)\int\limits_{\partial\Omega} M(\nabla w)^{\alpha}(z)\left(\int_{\frac{1}{c|x-z|}}^{\infty} \frac{1}{t^{n-1}}\,dt\right)d\sigma(z)\\
&\leq C\mathcal{M}(M(\nabla w)^{\alpha})^{\frac{1}{\alpha}-1}(x)\int\limits_{\partial\Omega} \frac{M(\nabla w)^{\alpha}(z)}{|x-z|^{n-2}}\,d\sigma(z)\\
&\leq C\mathcal{M}(M(\nabla w)^{\alpha})^{\frac{1}{\alpha}-1}(x)\,I_1(M(\nabla w)^{\alpha})(x),
\end{aligned} \tag{11.77}$$

where, for $0 < \theta < n-1$, I_θ denotes the fractional integration operator given by

$$I_\theta h(x) := \int\limits_{\partial\Omega} \frac{h(z)}{|x-z|^{n-1-\theta}}\,d\sigma(z), \qquad x \in \partial\Omega. \tag{11.78}$$

Taking the supremum over all $y \in \Gamma(x)$ in (11.77), we have

$$M(u)(x) \leq C\mathcal{M}(M(\nabla w)^{\alpha})^{\frac{1}{\alpha-1}}(x)\,I_1(M(\nabla w)^{\alpha})(x), \tag{11.79}$$

and so

$$\int_{\partial\Omega} M(u)^{p^*}\, d\sigma \le C \int_{\partial\Omega} \left(\mathcal{M}(M(\nabla w)^\alpha)\right)^{p^*(\frac{1}{\alpha}-1)} \left(I_1(M(\nabla w)^\alpha)\right)^{p^*}\, d\sigma. \tag{11.80}$$

Choose $r > 1$ so that

$$(1-\alpha)r = 1 - \frac{p}{n-1} = \frac{p}{p^*}. \tag{11.81}$$

Then by Hölder's inequality,

$$\int_{\partial\Omega} M(u)^{p^*}\, d\sigma \le C \left(\int_{\partial\Omega} \left(\mathcal{M}(M(\nabla w)^\alpha)\right)^{p^*(\frac{1-\alpha}{\alpha})r}\, d\sigma \right)^{\frac{1}{r}} \left(\int_{\partial\Omega} \left(I_1(M(\nabla w)^\alpha)\right)^{p^* r'}\, d\sigma \right)^{\frac{1}{r'}}. \tag{11.82}$$

Let $q := \frac{p}{\alpha}$, so that $1 < q < n-1$, and pick q^* such that $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{n-1}$. Then from our choice of r in (11.81), we have the following:

$$\begin{aligned}
&(a) \quad p^*\Big(\frac{1-\alpha}{\alpha}\Big)r = \frac{p^*}{\alpha}\cdot\frac{p}{p^*} = \frac{p}{\alpha} = q,\\
&(b) \quad \frac{1}{p^* r'} = \frac{1}{p^*}\Big(1-\frac{1}{r}\Big) = \frac{1}{p^*} - \frac{1}{p^* r} = \frac{1}{p} - \frac{1}{n-1} - \frac{1-\alpha}{p} = \frac{\alpha}{p} - \frac{1}{n-1} = \frac{1}{q^*},\\
&(c) \quad \frac{1}{r} + \frac{q^*}{qr'} = \frac{p^*(1-\alpha)}{p} + \frac{p^* r'}{qr'} = \frac{p^*(1-\alpha)}{p} + \frac{p^*}{q} = \frac{p^*(1-\alpha)}{p} + \frac{\alpha p^*}{p} = \frac{p^*}{p}.
\end{aligned} \tag{11.83}$$

Applying the identities to (11.80) gives

$$\int_{\partial\Omega} M(u)^{p^*}\, d\sigma \le C \left(\int_{\partial\Omega} \left(\mathcal{M}(M(\nabla w)^\alpha)\right)^{q}\, d\sigma \right)^{\frac{1}{r}} \left(\int_{\partial\Omega} \left(I_1(M(\nabla w)^\alpha)\right)^{q^*}\, d\sigma \right)^{\frac{1}{r'}}. \tag{11.84}$$

It is well known that for $1 < q < n-1$, $\mathcal{M}$ is a bounded operator from $L^q(\partial\Omega)$ to $L^q(\partial\Omega)$, and I_1 is bounded from $L^q(\partial\Omega)$ to $L^{q^*}(\partial\Omega)$. Then since $M(\nabla w)^\alpha \in L^q(\partial\Omega)$, it follows that

$$\begin{aligned}
\int_{\partial\Omega} M(u)^{p^*}\, d\sigma &\le C \left(\int_{\partial\Omega} (M(\nabla w)^\alpha)^q\, d\sigma \right)^{\frac{1}{r}} \left(\left(\int_{\partial\Omega} (M(\nabla w)^\alpha)^q\, d\sigma \right)^{\frac{q^*}{q}} \right)^{\frac{1}{r'}}\\
&= C \left(\int_{\partial\Omega} M(\nabla w)^p\, d\sigma \right)^{\frac{1}{r}+\frac{q^*}{qr'}} = C \left(\int_{\partial\Omega} M(\nabla w)^p\, d\sigma \right)^{\frac{p^*}{p}},
\end{aligned} \tag{11.85}$$

and so finally we can conclude

$$\|M(u)\|_{L^{p^*}(\partial\Omega)} \leq C\|M(\nabla w)\|_{L^p(\partial\Omega)}, \tag{11.86}$$

finishing the proof of the lemma. □

11.6 Traces in Hardy spaces

Here we record some useful trace theorems in Hardy spaces for functions in Lipschitz domains, which have been recently proved in [44]. The first such result reads as follows.

Theorem 11.12 *Let Ω be a graph Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, with outward unit normal ν, and fix*

$$0 < p, q < \infty, \quad \frac{n-1}{n} < r \leq 1 \quad \textit{such that} \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r}. \tag{11.87}$$

Consider also $D : C^1(\Omega, \mathbb{C}^N) \to C^0(\Omega, \mathbb{C}^M)$ a homogeneous, first-order differential operator with constant, complex coefficients (i.e., as in (3.1) for $m = 1$), and denote by D^* its (formal) adjoint and by $\sigma(D;\xi) \in \mathbb{C}^{M\times N}$, $\xi \in \mathbb{R}^n$, its symbol (cf. (3.5)).

Assume that $F \in C^1(\Omega, \mathbb{C}^N)$ and $G \in C^1(\Omega, \mathbb{C}^M)$ are two functions which satisfy

$$DF = 0 \quad \textit{and} \quad D^*G = 0 \quad \textit{in } \Omega, \tag{11.88}$$

$$M(F) \in L^p(\partial\Omega), \qquad M(G) \in L^q(\partial\Omega), \tag{11.89}$$

and which are null-solutions of certain strongly elliptic, self-adjoint, second-order, homogeneous, (real) constant coefficient, differential operators. Let $\langle\cdot,\cdot\rangle$ denote the canonical inner product in $\mathbb{C}^M$, and for every $\varepsilon > 0$, define

$$F_\varepsilon(x) := F(x + \varepsilon e_n), \quad G_\varepsilon(x) := G(x + \varepsilon e_n), \quad x \in \overline{\Omega}, \tag{11.90}$$

where $e_n = (0, ..., 0, 1) \in \mathbb{R}^n$.

Then $\langle\sigma(D;\nu)F_\varepsilon, G_\varepsilon\rangle \in H^r_{at}(\partial\Omega)$ for each $\varepsilon > 0$, the limit

$$\langle\sigma(D;\nu)F, G\rangle := \lim_{\varepsilon\to 0^+} \langle\sigma(D;\nu)F_\varepsilon, G_\varepsilon\rangle \tag{11.91}$$

exists in $H^r_{at}(\partial\Omega)$, and there exists a finite constant $C = C(\partial\Omega, n, p, q) > 0$ such that

$$\|\langle\sigma(D;\nu)F, G\rangle\|_{H^r_{at}(\partial\Omega)} \le C\,\|M(F)\|_{L^p(\partial\Omega)}\|M(G)\|_{L^q(\partial\Omega)}. \tag{11.92}$$

Furthermore, when $r = 1$, one can define the trace $\langle\sigma(D;\nu)F, G\rangle \in H^1_{at}(\partial\Omega) \subset L^1(\partial\Omega)$ in a non-tangential pointwise sense, as

$$\langle\sigma(D;\nu(x))F(x), G(x)\rangle = \lim_{\substack{y\to x\\ y\in\Gamma(x)}} \langle\sigma(D;\nu(x))F(y), G(y)\rangle, \qquad \text{at a.e. } x \in \partial\Omega. \tag{11.93}$$

Finally, in the case when G (F, respectively) is a constant function, one can allow the index q (p, respectively) in (11.87) to take the value ∞ as well.

A suitable version of the above theorem holds for bounded Lipschitz domains, in which scenario it is natural to employ the local Hardy spaces $h^r_{at}(\partial\Omega)$, introduced in § 2.3. Concretely, we have the following.

Theorem 11.13 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \ge 2$, and fix $0 < p, q < \infty$ and $\frac{n-1}{n} < r \le 1$ such that $1/p + 1/q = 1/r$. Consider also a homogeneous, first-order differential operator D with constant coefficients and two functions*

$$F \in C^1(\Omega, \mathbb{C}^N), \quad G \in C^1(\Omega, \mathbb{C}^M), \tag{11.94}$$

which are null-solutions of certain strongly elliptic, self-adjoint, second-order, homogeneous, (real) constant coefficient, differential operators in Ω, and such that

$$DF = 0 \quad \text{and} \quad D^*G = 0 \quad \text{in } \Omega, \tag{11.95}$$

$$M(F) \in L^p(\partial\Omega), \qquad M(G) \in L^q(\partial\Omega). \tag{11.96}$$

Then there exists a finite constant $C = C(\partial\Omega, n, p, q) > 0$ and a function in $h^r_{at}(\partial\Omega)$, denoted by $\langle\sigma(D;\nu)F, G\rangle$, for which

$$\|\langle\sigma(D;\nu)F, G\rangle\|_{h^r_{at}(\partial\Omega)} \le C\,\|M(F)\|_{L^p(\partial\Omega)}\|M(G)\|_{L^q(\partial\Omega)} \tag{11.97}$$

and such that the following holds. Let Z be a coordinate cylinder for $\partial\Omega$, with axis in the direction of a unit vector (pointing into Ω) denoted by e_n, and pick a function $\zeta \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\zeta \subset Z$*. Then*

$$\lim_{\varepsilon\to 0^+}\int_{Z\cap\partial\Omega}\langle\sigma(D;\nu(x))F(x+\varepsilon e_n), G(x+\varepsilon e_n)\rangle\,\zeta(x)\,d\sigma(x)$$
$$= \int_{\partial\Omega}\langle\sigma(D;\nu)F, G\rangle\,\zeta\,d\sigma, \tag{11.98}$$

where the last integral above stands for the paring between $h^r_{at}(\partial\Omega)$ and $\mathrm{Lip}\,(\partial\Omega)$*.*

Finally, in the case when G (F, respectively) is a constant function, one can allow the index q (p, respectively) to take the value ∞.

The case when F is the gradient of a harmonic function u with $M(\nabla u) \in L^p(\partial\Omega)$, $G \equiv 1$, and $D = \mathrm{div}$ has been proved by B. Dahlberg and C. Kenig in [20], based on duality and a refinement of an extension theorem due to N. Varopoulos [93]. The approach in [44] is more akin to the work of M. Wilson [96]. In applications to the Stokes system in Lipschitz domains, the following particular case of Theorem 11.12, Theorem 11.13 is going to be of particular importance.

Corollary 11.14 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a graph Lipschitz domain, with outward unit normal ν, and assume that $\frac{n-1}{n} < r \leq 1$. Then there exists a finite constant $C = C(\partial\Omega, r) > 0$ such that for any divergence-free vector field $\vec{F} : \Omega \to \mathbb{R}^n$ with biharmonic components for which $M(\vec{F}) \in L^p(\partial\Omega)$ there holds*

$$\langle\nu, \vec{F}\rangle \in H^r_{at}(\partial\Omega) \quad \textit{and} \quad \|\langle\nu, \vec{F}\rangle\|_{H^r_{at}(\partial\Omega)} \leq C\|M(\vec{F})\|_{L^p(\partial\Omega)}. \tag{11.99}$$

Above, $\langle\nu, \vec{F}\rangle$ on $\partial\Omega$ is considered in the sense of Theorem 11.12. Furthermore, a similar result is valid in the case of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, in which case (11.99) reads

$$\langle\nu, \vec{F}\rangle \in h^r_{at}(\partial\Omega) \quad \textit{and} \quad \|\langle\nu, \vec{F}\rangle\|_{h^r_{at}(\partial\Omega)} \leq C\|M(\vec{F})\|_{L^p(\partial\Omega)}, \tag{11.100}$$

with $\langle\nu, \vec{F}\rangle$ on $\partial\Omega$ defined in the sense of Theorem 11.13.

Proof. Consider $\vec{F}$ as above, $G \equiv 1$, $q = \infty$, $p = r$ and $D := \mathrm{div}$ (so that $D^* = -\nabla$). In particular, $D\vec{F} = 0$, $D^*G = 0$, $M(\vec{F}) \in L^p(\partial\Omega)$, $M(G) \in L^\infty(\partial\Omega)$ and $\langle \sigma(D;\nu)\vec{F}, G\rangle = i\,\langle \nu, \vec{F}\rangle$. Then (11.99), (11.100) follow directly from Theorem 11.12 and Theorem 11.13, respectively. $\square$

11.7 Spaces of null-solutions of elliptic operators

Let $L = \sum_{|\gamma|=m} a_\gamma \partial^\gamma$ be a constant coefficient, elliptic differential operator of order $m \in 2\,\mathbb{N}$ in $\mathbb{R}^n$. For a fixed, bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, denote by $\mathrm{Ker}\, L$ the space of functions satisfying $Lu = 0$ in Ω. Then, for $0 < p \leq \infty$ and $\alpha \in \mathbb{R}$, introduce $\mathbb{H}^p_\alpha(\Omega; L)$ the space of functions $u \in \mathrm{Ker}\, L$ subject to the condition

$$\|u\|_{\mathbb{H}^p_\alpha(\Omega;L)} := \|\delta^{\langle\alpha\rangle-\alpha}|\nabla^{\langle\alpha\rangle}u|\|_{L^p(\Omega)} + \sum_{j=0}^{\langle\alpha\rangle-1} \|\nabla^j u\|_{L^p(\Omega)} < +\infty. \tag{11.101}$$

Above, ∇^j stands for vector of all mixed-order partial derivatives of order j and $\langle\alpha\rangle$ is the smallest nonnegative integer greater than or equal to α, i.e.,

$$\langle\alpha\rangle := \begin{cases} \alpha, \ \text{if } \alpha \text{ is a nonnegative integer}, \\ [\alpha] + 1, \ \text{if } \alpha > 0, \ \alpha \notin \mathbb{N}, \\ 0, \ \text{if } \alpha < 0, \end{cases} \tag{11.102}$$

where $[\cdot]$ is the integer-part function. Parenthetically, let us point out that an equivalent quasi-norm on $\mathbb{H}^p_\alpha(\Omega; L)$ is given by

$$\|\delta^{\langle\alpha\rangle-\alpha}|\nabla^{\langle\alpha\rangle}u|\|_{L^p(\Omega)} + \sup_{x\in\mathcal{O}} |u(x)|, \tag{11.103}$$

where $\mathcal{O}$ denotes some fixed compact subset of Ω. The following result has essentially been established in [64]; see also [50], [70]. It extends results from [46], where the authors have dealt with the case $1 < p, q < \infty$, $s > 0$, $L = \Delta$, and [1] where the case $1 < p, q < \infty$, $s > 0$, $L = \Delta^2$ is treated.

Theorem 11.15 *Assume that L is a homogeneous, constant coefficient, elliptic differential operator and that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded Lipschitz domain. Then*

$$\mathbb{H}^p_\alpha(\Omega; L) = F^{p,q}_\alpha(\Omega) \cap \mathrm{Ker}\, L \tag{11.104}$$

for every $\alpha \in \mathbb{R}$, $0 < p < \infty$, and $0 < q < \infty$. In particular, for each fixed $\alpha \in \mathbb{R}$ and $0 < p < \infty$, the space $F_\alpha^{p,q}(\Omega) \cap \operatorname{Ker} L$ is independent of $q \in (0, \infty)$.

Furthermore, corresponding to $p = \infty$, there holds

$$\mathbb{H}^\infty_{k+s}(\Omega; L) = B^{\infty,\infty}_{k+s}(\Omega) \cap \operatorname{Ker} L \tag{11.105}$$

for each $k \in \mathbb{N}_0$ and $s \in (0, 1)$.

Our next result is as follows.

Theorem 11.16 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, and assume that L is a homogeneous, constant (real) coefficient, symmetric, strongly elliptic differential operator of order $2m$, $m \in \mathbb{N}$. Then if $u \in F^{p,q}_{m-1+1/p}(\Omega)$ for some $\frac{n-1}{n} < p \leq 2$, $0 < q < \infty$, and $Lu = 0$ in Ω, it follows that $M(\nabla^{m-1} u) \in L^p(\partial\Omega)$ and a natural estimate holds.*

In the proof of this theorem, the following result from [64] is going to be useful.

Lemma 11.17 *Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, and that L is as above. Also, fix $k \in \mathbb{N}_0$, $0 < p < \infty$, and $s \in \mathbb{R}$ with $sp > -1$. Then there exists a relatively compact subset $\mathcal{O}$ of Ω and $C > 0$ such that*

$$\Big(\int_\Omega (\delta(x)^s |u(x)|)^p \, dx\Big)^{1/p} \leq C \left[\Big(\int_\Omega (\delta(x)^{s+k} |\nabla^k u(x)|)^p \, dx\Big)^{1/p} + \sup_{x \in \mathcal{O}} |u(x)|\right], \tag{11.106}$$

uniformly for $u \in \operatorname{Ker} L$.

We now present the

Proof of Theorem 11.16. Recall the area function

$$\mathcal{A}(u)(x) := \Big(\int_{\Gamma(x)} \delta(y)^{2-n} |\nabla u(y)|^2 \, dy\Big)^{\frac{1}{2}}, \quad x \in \partial\Omega. \tag{11.107}$$

As proved by Dahlberg-Kenig-Pipher-Verchota in [22], for every $0 < p < \infty$, there exists $C > 0$ such that

$$\|M(\nabla^{m-1}u)\|_{L^p(\partial\Omega)} \leq C\|\mathcal{A}(\nabla^{m-1}u)\|_{L^p(\partial\Omega)} + C\sum_{j=0}^{m-1}\|\nabla^j u\|_{L^1(\Omega)}. \tag{11.108}$$

If $\{Q_j\}_j$ is a Whitney decomposition of Ω into Euclidean cubes Q_j of side-length $l(Q_j)$, we may then estimate

$$\begin{aligned}
\int_{\partial\Omega}\Big(\mathcal{A}(\nabla^{m-1}u)(x)\Big)^p d\sigma_x &= \int_{\partial\Omega}\Big(\int_\Omega \delta(y)^{2-n}|\nabla^m u(y)|^2\chi_{\{y\in\Gamma(x)\}}dy\Big)^{\frac{p}{2}}d\sigma_x \qquad (11.109)\\
&= \int_{\partial\Omega}\Big(\sum_j\int_{Q_j}\delta(y)^{2-n}|\nabla^m u(y)|^2\chi_{\{y\in\Gamma(x)\}}dy\Big)^{\frac{p}{2}}d\sigma_x =: I.
\end{aligned}$$

If $y\in Q_j$ and $x\in\partial\Omega$ such that $y\in\Gamma(x)$, then $x\in\Delta_j$, where Δ_j is the "cone shadow" of Q_j on $\partial\Omega$, i.e., $\Delta_j := \{x\in\partial\Omega:\ \Gamma(x)\cap Q_j\neq\emptyset\}$. In particular, $\sigma(\Delta_j)\approx l(Q_j)^{n-1}$, uniformly in j.

Assume that $0<p\leq 2$. Then

$$\begin{aligned}
I &\leq \int_{\partial\Omega}\sum_j\Big(\int_{Q_j}\delta(y)^{2-n}|\nabla^m u(y)|^2\chi_{\{y\in\Gamma(x)\}}dy\Big)^{\frac{p}{2}}d\sigma_x\\
&\leq \sum_j\int_{\Delta_j}\left[l(Q_j)\Big(⨍_{Q_j}|\nabla^m u|^2\Big)^{\frac{1}{2}}\right]^p d\sigma\\
&\leq C\sum_j l(Q_j)^{n-1+p}⨍_{Q_j^*}|\nabla^m u|^p \leq C\int_\Omega[\delta^{1-\frac{1}{p}}|\nabla^m u|]^p\\
&\leq C\|u\|^p_{\mathbb{H}^p_{m-1+1/p}(\Omega;L)} \leq C\|u\|_{F^{p,q}_{m-1+1/p}(\Omega)}, \qquad (11.110)
\end{aligned}$$

provided $\frac{1}{p}>n(\frac{1}{p}-1)$ (or, equivalently, $p>\frac{n-1}{n}$). For the second inequality in (11.110), we have used the fact that the function $\nabla^m u\in\operatorname{Ker}L$ satisfies the reverse Hölder inequality

$$\Big(⨍_{Q_j}|\nabla^m u|^2\Big)^{\frac{1}{2}} \leq C\Big(⨍_{Q_j^*}|\nabla^m u|^p\Big)^{\frac{1}{p}}, \tag{11.111}$$

where Q_j^* is concentric double of Q_j. Let us also point out that the next-to-last estimate in (11.110) follows straight from definitions when $1\leq p\leq 2$ and is a consequence of Lemma 11.17 when $\frac{n-1}{n}<p<1$. Finally, the last estimate in (11.110) is implied by Theorem 11.15.

The above argument shows that $\|\mathcal{A}(\nabla^{m-1}u)\|_{L^p(\partial\Omega)} \leq C\|u\|_{F^{p,q}_{m-1+1/p}(\Omega)}$. Since we also have $F^{p,q}_{1/p}(\Omega) \hookrightarrow L^{np/(n-1)}(\Omega)$, the desired conclusion now follows from (11.108). $\square$

11.8 Singular integral operators on Sobolev-Besov spaces

We start with a result describing mapping properties on Besov spaces of integral operator modeled upon the hydrostatic double layer.

Theorem 11.18 *Let Ω be a (bounded or graph) Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. Consider the integral operator*

$$Tf(x) = \int_{\partial\Omega} k(x,y)f(y)d\sigma_y, \qquad x \in \Omega, \tag{11.112}$$

satisfying the following conditions:

$$(1) \quad T1 = const, \tag{11.113}$$

$$(2) \quad |\nabla_x^k k(x,y)| \leq C|x-y|^{-(n+k-1)}, \quad k = 1, ..., N, \tag{11.114}$$

for some positive integer N. Then, with $\delta := \mathrm{dist}\,(\cdot, \partial\Omega)$,

$$\|\delta^{k-\frac{1}{p}-s}|\nabla^k Tf|\|_{L^p(\Omega)} + \sum_{j=0}^{k-1} \|\nabla^j Tf\|_{L^p(\Omega)} \leq C\|f\|_{B^{p,p}_s(\partial\Omega)}, \tag{11.115}$$

granted that $k \in \{1, ..., N\}$, $\frac{n-1}{n} < p \leq \infty$, and $(n-1)(\frac{1}{p}-1)_+ < s < 1$.

For a proof of Theorem 11.18 see [64]. The next result gives an analogue of Theorem 11.18 for single layer-like integral operators.

Theorem 11.19 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, and consider the integral operator*

$$Rf(x) = \int_{\partial\Omega} k(x,y)f(y)d\sigma_y, \qquad x \in \Omega, \tag{11.116}$$

whose kernel satisfies the conditions

$$|\nabla_x^k \nabla_y^j k(x,y)| \leq C|x-y|^{-(n-2+k+j)}, \quad j = 0, 1, \tag{11.117}$$

for $k = 1, 2, ..., N$, where N is some positive integer. Then

$$\|\delta^{k-\frac{1}{p}-s}|\nabla^k Rf|\|_{L^p(\Omega)} + \sum_{j=0}^{k-1} \|\nabla^j Rf\|_{L^p(\Omega)} \leq C\|f\|_{B^{p,p}_{s-1}(\partial\Omega)}, \quad k = 1, 2, ..., N, \tag{11.118}$$

granted that $\frac{n-1}{n} < p \leq \infty$ and $(n-1)(\frac{1}{p} - 1)_+ < s < 1$.

Once again, see [64] for a proof.

11.9 Functional analysis on quasi-Banach spaces

In the first part of this section we discuss a number of results related to Fredholm theory on quasi-Banach spaces. Since such a topic has intrinsic interest, we adopt a slightly more general point of view and record a body of results which is richer than the one strictly required by the applications to the kind of partial differential equations pursued in this work.

The following useful results appear in [81].

Theorem 11.20 (Finite Dimensional Extension Theorem) *Assume that Y is a closed subspace of a Hausdorff linear topological space X, and that M is a finite dimensional subspace of X. Then $Y + M$ is closed in X.*

Theorem 11.21 (Finite Codimension Theorem) *If Y is a closed subspace, of finite codimension in a Hausdorff linear topological space X, and M is any algebraic complement of Y, then $X = Y \oplus M$.*

Proposition 11.22 *Assume that X is a closed subspace of a Hausdorff linear topological space. If Y and Z are two linear subspaces of X which complement each other (i.e., $Y \oplus Z = X$) then Y and Z are closed in X.*

Theorem 11.23 *Assume that X is a closed subspace of a Hausdorff linear topological space. Then X is finite dimensional if and only if X is locally compact.*

Proposition 11.24 *If $S : Y \to Z$ and $T : X \to Y$ are linear transformations acting on vector spaces, both of which have finite dimensional kernels, then the composition $ST : X \to Z$ also has finite dimensional kernel and, moreover,*

$$\begin{aligned}\dim \operatorname{Ker}(ST : X \to Z) &= \dim \operatorname{Ker}(T : X \to Y) \\ &\quad + \dim \Big[\operatorname{Ker}(S : Y \to Z) \cap \operatorname{Im}(T : X \to Y)\Big]. \quad (11.119)\end{aligned}$$

To be precise, this is stated and proved in § 8 of [81] in the case when $X = Y = Z$, but the same elementary reasoning applies in the slightly more generality above.

Definition 11.25 *Let X be a vector space. A quasi-norm is a nonnegative real-valued function $\|\cdot\|$ on X such that*

$$\|x\| = 0 \iff x = 0, \qquad \|\alpha x\| = |\alpha|\|x\|, \qquad \|x + y\| \leq \kappa(\|x\| + \|y\|), \qquad (11.120)$$

where $x, y \in X$, α is any scalar, and $\kappa \geq 1$ is independent of x and y.

Call X a quasi-Banach space if there exists a quasi-norm for which this X complete.

Theorem 11.26 (Aoki-Rolewicz Theorem) *Let X be a quasi-Banach space. Then there exists $0 < p \leq 1$ and an equivalent quasi-norm $\|\cdot\|$ on X such that*

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p, \qquad \forall x, y \in X. \qquad (11.121)$$

Definition 11.27 *If X and Y are quasi-Banach spaces, denote by $\mathcal{L}(X, Y)$ the space of linear, continuous operators from X to Y. An operator $T \in \mathcal{L}(X, Y)$ is said to be compact if the image under T of any bounded subset of X is a relatively compact subset of Y. Finally, denote by $\mathcal{K}(X, Y)$ the space of compact operators from X into Y.*

We equip $\mathcal{L}(X, Y)$ with the natural quasi-norm $\|T\|_{\mathcal{L}(X,Y)} := \sup\{\|Tx\|_Y : x \in X,\ \|x\|_X \leq 1\}$.

Suppose that X is a quasi-Banach space and $T \in \mathcal{L}(X, X)$. We claim that the operator $\lambda I + T$ is invertible (with I denoting the identity) on X for any $\lambda \in \mathbb{R}$ with $|\lambda|$ large enough. Indeed, the inverse can be given in the form of a Neumann series

$$(\lambda I + T)^{-1} = \sum_{j=0}^{\infty} (-1)^j \lambda^{-j-1} T^j, \qquad (11.122)$$

which converges in the operator norm if $|\lambda|$ is large enough. To see this, by the Aoki-Rolewicz Theorem, there is no loss of generality in assuming that X is a p-Banach space, for some $p \in (0,1]$. Then $\|\sum_{j=M}^{N}(-1)^j\lambda^{-j-1}T^j\|_{\mathcal{L}(X,X)}^p \leq \sum_{j=M}^{N}|\lambda|^{-j-1}\|T^j\|_{\mathcal{L}(X,X)}^p \leq |\lambda|^{-1}\sum_{j=M}^{N}(|\lambda|^{-1/p}\|T\|_{\mathcal{L}(X,X)})^{jp}$ which is a piece of a convergent geometric series if $\|T\|_{\mathcal{L}(X,X)} < |\lambda|$.

Theorem 11.28 *Let X and Y be quasi-Banach spaces. Then $\mathcal{L}(X,Y)$ is a quasi-Banach space and $\mathcal{K}(X,Y)$ is a closed, two-sided ideal in $\mathcal{L}(X,Y)$.*

When $X = Y$, this follows from the discussion in § 3 (p. 3.1) in [81]; see also Proposition 9.5 on p. 9.3 in [81]. Once again, having $X = Y$ is inessential for the current purposes.

Next, we record a result proved in [51]; cf. Proposition 7.8 on p. 132, and Proposition 7.9 on p. 134. To state it, given two quasi-Banach spaces, we let $G_1(X,Y)$ denote the set of isomorphic embeddings of X into Y, and $G_2(X,Y)$ the set of open mappings of X into Y.

Proposition 11.29 *For any two quasi-Banach spaces X and Y, the set $G_j(X,Y)$ is open in $\mathcal{L}(X,Y)$, $j = 1,2$, and $G_1(X,Y) \cap G_2(X,Y)$ is both closed and open in either of $G_1(X,Y)$, $G_2(X,Y)$.*

The result below is contained in Lemma 4.11 on p. 74 of [51].

Proposition 11.30 *Suppose that X, Y are two quasi-Banach spaces. Then $A + K$ has closed range for any $A \in G_1(X,Y)$ and $K \in \mathcal{K}(X,Y)$.*

Consider next two quasi-Banach spaces $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and let $T : X \to Y$ be a linear, bounded operator. Define $\kappa(T;\, X,Y)$ to be the smallest constant so that if $y \in Y$ then there exists $x \in X$ so that $Tx = y$ and $\|x\|_X \leq \kappa(T;\, X,Y)\|y\|_Y$. Note that, by the Open Mapping Theorem (which remains valid in the context of quasi-Banach spaces; cf. Theorem 1.4 in [51]),

$$\kappa(T;\, X,Y) \text{ is finite if and only if } T \text{ maps } X \text{ onto } Y. \tag{11.123}$$

We also let $\eta(T;\, X,Y)$ be the largest constant so that $\eta(T;\, X,Y)\|x\|_X \leq \|Tx\|_Y$ for each $x \in X$. Once again by virtue of the Open mapping Theorem,

$$\eta(T;\,X,Y)>0 \text{ if and only if } T \text{ is injective with closed range.} \tag{11.124}$$

The result below has been proved in [48].

Lemma 11.31 *Suppose that $(X,\|\cdot\|_X)$, $(Y,\|\cdot\|_Y)$ are two quasi-normed spaces such that X is complete. Also, suppose that $T:X\to Y$ is a linear, bounded operator for which the following property is true: there exist $0<C_0<+\infty$ and $0<a<1$ such that for each y in the unit sphere of Y one can find $x\in X$ with $\|x\|_X\leq C_0$ and $\|y-Tx\|_Y\leq a$.*

Then T is onto and $\kappa(T;\,X,Y)\leq C_1$ for some C_1 depending exclusively on C_0, the quasi-norm constant of X and a.

We shall also need a variant of Lemma 11.31 for sequences of operators.

Lemma 11.32 *Assume that X, Y are Banach spaces and that $(T_\alpha)_{\alpha\in\mathbb{N}}$ is a sequence of bounded, linear operators, mapping X into Y, converging to some $T:X\to Y$ in the operator norm. If T is onto, then there exists $C>0$ and α_0 such that*

$$\forall\,\alpha\geq\alpha_0,\ \forall y\in Y\Longrightarrow\exists\,x\in X\ \text{ so that } T_\alpha x=y,\ \|x\|_X\leq C\|y\|_Y. \tag{11.125}$$

Proof. This is a consequence of Lemma 11.31. Specifically, there exists C_0 such that if $y\in Y$ has $\|y\|_Y=1$ then there exists $x\in X$ with $\|x\|_X\leq C_0$ and $Tx=y$. Then we may write $\|T_\alpha x-y\|_Y=\|T_\alpha x-Tx\|_Y\leq\|x\|_X\|T_\alpha-T\|_{\mathcal{L}(X,Y)}\leq C_0\|T_\alpha-T\|_{\mathcal{L}(X,Y)}$ which shows that, for sufficiently large α, we always have "good" approximate solutions to $T_\alpha x=y$ and this, by Lemma 11.31, gives an actual solution with the desired control of the quasi-norm. $\square$

Definition 11.33 *Let X and Y be quasi-Banach spaces. Call $T\in\mathcal{L}(X,Y)$ Fredholm if:*

(1) T has a closed range,

(2) T has finite codimensional range,

(3) $\operatorname{Ker}T$ is finite dimensional and topologically complemented in X.

Set $\Phi(X,Y) := \{T \in \mathcal{L}(X,Y) : T$ *Fredholm*$\}$ *and define the index function*

$$\mathrm{ind} : \Phi(X,Y) \longrightarrow \mathbb{Z}, \qquad \mathrm{ind}\, T := \dim(\mathrm{Ker}\, T) - \mathrm{codim}\,(\mathrm{Im}\, T). \tag{11.126}$$

.

Occasionally, if we wish to stress the spaces on which the operator T is considered, we may write index $(T : X \to Y)$, Ker $(T : X \to Y)$, etc. When $X = Y$, the above definition becomes a particular case of that in § 6 in [81]. Again, $X = Y$ has been assumed there merely for convenience, and that removing this assumption does not affect the subsequent analysis.

As pointed out in § 6 of [81], it is not always the case that a finite dimensional subspace E of a Hausdorff, linear topological space X is necessarily topologically complemented. However, this does happen whenever X^* separates X.

Definition 11.34 *If* X *and* Y *are two quasi-Banach spaces, set*

$$\begin{aligned} \Phi_+(X,Y) \;\; := \;\; & \{T \in \mathcal{L}(X,Y) : T \textit{ has closed range and a finite dimensional} \\ & \textit{kernel, which is topologically complemented in } X\}, \end{aligned} \tag{11.127}$$

and

$$\Phi_-(X,Y) := \{T \in \mathcal{L}(X,Y) : T \textit{ has closed range and finite dimensional cokernel}\}. \tag{11.128}$$

The set of semi-Fredholm operators is then defined as $\Phi_-(X,Y) \cup \Phi_+(X,Y)$. *The index function (11.126) can then be extended to the set of all semi-Fredholm operators by setting*

$$\begin{aligned} & \mathrm{index} : \Phi_-(X,Y) \cup \Phi_+(X,Y) \longrightarrow \mathbb{Z} \cup \{\pm\infty\}, \\ & \mathrm{index}\, T := \dim(\mathrm{Ker}\, T) - \dim(\mathrm{coker}\, T) \end{aligned} \tag{11.129}$$

Clearly,

$$\Phi(X,Y) = \Phi_-(X,Y) \cap \Phi_+(X,Y). \tag{11.130}$$

As shown below, the demand of "having closed range" is superfluous (and, hence, it may be omitted) in the above definitions of semi-Fredholmness and Fredholmness.

Lemma 11.35 *Let X, Y be two quasi-Banach spaces and assume that $T \in \mathcal{L}(X,Y)$ is such that TX has finite codimension in Y (i.e. there exists M, finite dimensional subspace of Y such that $M + TX = Y$). Then TX is closed in Y.*

Before presenting the proof, let us note that if X, Y are quasi-Banach then for any $T \in \mathcal{L}(X,Y)$,

$$TX \text{ has finite codimension in } Y \Longleftrightarrow \dim\left(\frac{Y}{TX}\right) < +\infty. \tag{11.131}$$

Furthermore, the codimension of TX in Y is equal to the dimension of the space Y/TX.

Proof of Lemma 11.35. Let M be a finite dimensional subspace of Y such that $M+TX = Y$. By further refining it (e.g., replacing it by a complement of $M \cap TX$ in M), it can be also assumed that $M \cap TX = \{0\}$. Being finite dimensional, M is closed. Consider then $T_1 : X \times M \to Y$, defined by $T_1(x,y) := Tx + y$, which is linear, continuous, and onto. Since $\operatorname{Ker} T_1 = \operatorname{Ker} T \times \{0\} \hookrightarrow X \times \{0\}$, it follows that $TX = T_1(X \times \{0\})$ is closed in Y, by invoking the next lemma. $\square$

Here is the result alluded to above:

Lemma 11.36 *Let X, Y be two quasi-Banach spaces and assume that $T \in \mathcal{L}(X,Y)$ is such that TX is closed. If X_o is a closed subspace of X with the property that $\operatorname{Ker} T \subset X_o$, then TX_o is closed in Y.*

Proof. Since X_o is closed in X, then $X_o/\operatorname{Ker} T$ is closed in $X/\operatorname{Ker} T$. However, $T : X/\operatorname{Ker} T \to TX$ is an algebraical and topological isomorphism, and TX_o can be identified with the image of this latter operator of the closed subspace $X_o/\operatorname{Ker} T$. Thus, TX_o is closed in TX and, further, in Y. $\square$

The following lemmas further summarize various properties of Fredholm and semi-Fredholm operators which we will find useful later on.

Theorem 11.37 *Let X and Y be Banach spaces and let $T \in \mathcal{L}(X,Y)$. Then the following assertions hold.*

(1) If $T \in \Phi_{\pm}(X,Y)$ and $S \in \Phi_{\pm}(Y,Z)$ then $ST \in \Phi_{\pm}(X,Z)$ and

$$\mathrm{index}\,(ST) = \mathrm{index}\,(S) + \mathrm{index}\,(T). \tag{11.132}$$

(2) If X and Y have reasonable dual spaces, then $T \in \Phi_{\pm}(X,Y)$ if and only if $T^* \in \Phi_{\mp}(Y^*,X^*)$. Moreover, $\mathrm{index}\,(T) = -\mathrm{index}\,(T^*)$.

(3) $T \in \Phi_{+}(X,Y)$ if and only if T is bounded from below modulo compact operators. That is, there exist a quasi-Banach space Z, a compact operator $K : X \to Z$, and a positive constant C such that

$$\|x\|_X \leq C\|Tx\|_Y + \|Kx\|_Z \quad \textit{for any } x \in X. \tag{11.133}$$

In particular, $\Phi_{+}(X,Y)$ is open in $\mathcal{L}(X,Y)$ and $\Phi_{+}(X,Y)$ is stable under addition of compact operators.

(4) The set $\Phi_{-}(X,Y)$ is open in $\mathcal{L}(X,Y)$ and $\Phi_{-}(X,Y)$ is stable under addition of compact operators.

(5) If X_0 is a closed subspace of X and $T \in \Phi_{+}(X,X)$ with $TX_0 \subseteq X_0$, then $T|_{X_0} \in \Phi_{+}(X_0,X_0)$.

(6) $T \in \Phi(X,Y)$ if and only if there exist $S_1, S_2 \in \mathcal{L}(Y,X)$ and $K_1 \in \mathcal{K}(Y,Y)$ and $K_2 \in \mathcal{K}(X,X)$, such that

$$TS_1 = I_Y + K_1, \quad S_2T = I_X + K_2. \tag{11.134}$$

In fact, we may take $S_1 = S_2 \in \Phi(X,Y)$ (i.e., T is Fredholm if and only if it is invertible modulo compact operators).

(7) *The index function (11.129) is continuous.*

Proof. The claims in (1) and (6) appear in § 6 and § 8 of [81], at least when $X = Y$, and an inspection of the proof shows that this restriction can be easily removed.

Let us consider (3). In one direction, if T is bounded from below, modulo compact operators, introduce $A = (T, K) : X \to Y \oplus Z$ (with the latter space equipped with the natural quasi-norm $\|(y,z)\|_{Y\oplus Z} := \|y\|_Y + \|z\|_Z$). Then (11.133) amounts to $\eta(A; X, Y \oplus Z) > 0$, i.e. $A \in G_1(X, Y \oplus Z)$ (in the terminology of Proposition 11.29) . Since $(0, -K) \in \mathcal{K}(X, Y \oplus Z)$, Proposition 11.30 then gives that $(T, 0) = A + (0, -K)$ has closed range. Thus, T has closed range, as desired. To show that $N := \operatorname{Ker} T$, which is a closed subspace of X, is finite dimensional, it suffices to check that its unit ball is sequentially relatively compact (here, Theorem 11.23 is used). To this end, fix an arbitrary sequence $\{x_j\}_j$ of vectors in X with $\|x_j\|_X \leq 1$ and $Tx_j = 0$. Without loss of generality, it can be assumed that $\{Kx_j\}_j$ converges in Z. Writing (11.133) for $x = x_j - x_k$, then proves that $\{x_j\}_j$ is Cauchy, hence, convergent in X. This concludes the proof of the fact that, for an operator in $\mathcal{L}(X,Y)$, being bounded from below modulo compact operators entails membership to $\Phi_+(X,Y)$.

Conversely, if $T \in \Phi_+(X,Y)$ and Z is a topological complement of $\operatorname{Ker} T$ (which, by Proposition 11.22, means that Z is closed in X), define $K : X = \operatorname{Ker} T \oplus Z \to Z$ by $K(x,y) := x$. Since K has finite rank, $K \in \mathcal{K}(X,Z)$. Then, since $T : Z \to \operatorname{Im} T$ is an isomorphism, for each $x \in X$ with $x = x_o + y$, $x_o \in Z$, $y \in \operatorname{Ker} T$, we may write $\|x\|_X \leq \kappa(\|x_o\|_X + \|y\|_X) \leq \kappa(\|Tx_o\|_Y + \|Kz\|_Z) = \kappa(\|Tx\|_Y + \|Kz\|_Z)$. Thus, (11.133) follows.

Next we consider (4). Let $T \in \Phi_-(X,Y)$. Then there exists $M \subseteq Y$ such that $Y = TX \oplus M$ and $\dim M < +\infty$. Define $\tilde{T} : X \oplus M \to Y$ by $\tilde{T}(x,m) := Tx + m$. Then $\tilde{T}$ is onto, and hence from (11.123), $C_o := \kappa(\tilde{T}; X \oplus M, Y) < +\infty$. Let $R \in \mathcal{L}(X,Y)$ be such that $\|R\|_{\mathcal{L}(X,Y)} < \frac{1}{2C_o}$. Define $\tilde{R} : X \oplus M \to Y$ by $\tilde{R}(x,y) = Rx$, and so $\|\tilde{R}\|_{\mathcal{L}(X\oplus M,Y)} < \frac{1}{2C_o}$. Then from the definition of $\kappa(\tilde{T}; X \oplus M, Y)$, for any $y \in Y, \|y\|_Y \leq 1$, there exists $(x,m) \in X \oplus M$ such that $\tilde{T}(x,m) = y$ and $\|(x,m)\|_{X\oplus M} \leq C_o$. Then

$$\|y - (\tilde{T} + \tilde{R})(x, m)\|_Y \leq \|\tilde{R}\|_{\mathcal{L}(X \oplus M, Y)} \|(x, m)\|_{X \oplus M} \leq \tfrac{1}{2}, \tag{11.135}$$

and so it follows from Lemma 11.31 that $\tilde{T} + \tilde{R}$ is onto. Then

$$Y = \mathrm{Im}\,(\tilde{T} + \tilde{R}) = \{Tx + m + Rx :\, x \in X,\, m \in M\} = (T + R)X + M, \tag{11.136}$$

and so the range of $T + R$ has finite codimension in Y. From Lemma 11.35, $T + R$ has closed range, and so $T + R \in \Phi_-(X, Y)$. Therefore $\Phi_-(X, Y)$ is open in $\mathcal{L}(X, Y)$.

To see that $\Phi_-(X, Y)$ is stable under addition of compact operators, let $T \in \Phi_-(X, Y)$ and $K \in \mathcal{K}(X, Y)$, and we will show that $T + K \in \Phi_-(X, Y)$. First we will treat the case when T is onto. Using (11.123), define $C_1 := \kappa(T; X, Y)$. Since $K \in \mathcal{K}(X, Y)$, there exists an operator $K_1 \in \mathcal{K}(X, Y)$ of finite rank such that $\|K - K_1\|_{\mathcal{L}(X,Y)} \leq \frac{1}{2C_1}$. Define $T_1 := T + (K - K_1)$, and let $y \in Y, \|y\|_Y \leq 1$. From the definition of $\kappa(T; X, Y)$, there exists $x \in X$ such that $y = Tx$ and $\|x\|_X \leq C_1$. Then

$$\|y - T_1 x\|_Y = \|y - Tx - (K - K_1)x\| \leq \|K - K_1\|_{\mathcal{L}(X,Y)} \|x\|_X \leq \tfrac{1}{2}, \tag{11.137}$$

and so Lemma 11.31 implies that T_1 is onto. Then since $T + K = T_1 + K_1$ and K_1 has finite rank, it follows that $T + K$ has finite codimensional range, and then Lemma 11.35 implies that the range of $T + K$ is closed. This establishes that $T + K \in \Phi_-(X, Y)$ under the assumption that T is onto.

Next, we consider the general case. Let $M \subseteq Y$ be such that $Y = TX \oplus M$ and $\dim M < +\infty$. Define $\tilde{T}, \tilde{K} : X \oplus M \longrightarrow Y$ by

$$\tilde{T}(x, y) := Tx + y \qquad \text{and} \qquad \tilde{K}(x, y) := Kx. \tag{11.138}$$

Since $\tilde{T}$ is onto and $\tilde{K}$ is compact, using the previous case, we know that $\tilde{T} + \tilde{K}$ has closed range of finite codimension in Y. Then since

$$\mathrm{Im}\,(\tilde{T} + \tilde{K}) = \{Tx + y + Kx :\, x \in X,\, y \in M\} = \mathrm{Im}\,(T + K) + M, \tag{11.139}$$

it follows that the range of $T+K$ has finite codimension in $\mathrm{Im}\,(\tilde{T}+\tilde{K})$. Then the range of $T+K$ also has finite codimension in Y. Lemma 11.35 then implies that the range of $T+K$ is also closed, and hence $T+K \in \Phi_-(X,Y)$. This finishes the proof of (4). For the remaining items, the interested reader is referred to [49]. $\square$

As a consequence of (6) above, we have the following. Consider $\mathcal{U}$ a topological space and let $\mathcal{U} \ni \lambda \mapsto T_\lambda \in \Phi_+(X,Y) \cup \Phi_-(X,Y)$ be a continuous mapping. Then the function $\mathcal{U} \ni \lambda \mapsto \dim(\mathrm{Ker}\, T_\lambda) - \dim(\mathrm{coker}\, T_\lambda) \in \mathbb{Z} \cup \{\pm\infty\}$ is locally constant. In particular, $\lambda \mapsto \mathrm{index}\,(T_\lambda)$ is constant on each connected component of $\mathcal{U}$.

In the next corollary we single out a consequence of the last point in the above lemma which is particularly relevant for us in applications.

Corollary 11.38 *If $T \in \mathcal{L}(X,X)$ is such that $\lambda I + T$ is a semi-Fredholm operator for any $\lambda \in \mathbb{R}$, $|\lambda| \geq \frac{1}{2}$, then $\lambda I + T$ is actually a Fredholm operator with index zero for any λ in the indicated range.*

Proof. Recalling that for $|\lambda|$ large enough the operator $\lambda I + T$ is invertible (see the discussion preceding Theorem 11.28), the point (6) in Lemma 11.37 gives that $\mathrm{index}\,(\lambda I + T) = 0$ for any $\lambda \in \mathbb{R}$ with $|\lambda| \geq \frac{1}{2}$. Hence, the conclusion follows. $\square$

Lemma 11.39 *Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ be quasi-Banach spaces and consider the commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{X} & \longrightarrow & \mathcal{Y} & \longrightarrow & \mathcal{Z} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{X} & \longrightarrow & \mathcal{Y} & \longrightarrow & \mathcal{Z} & \longrightarrow & 0 \end{array} \tag{11.140}$$

where all arrows are linear and bounded and the horizontal sequences are exact. Then the following hold:

(a) If two vertical arrows are isomorphisms then so is the third one.

(b) If two vertical arrows are Fredholm operators then so is the third one. Moreover, the index of the middle vertical arrow is the sum of the indexes of the other two vertical arrows.

Lemma 11.40 *Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, $\mathcal{W}$ be quasi-Banach spaces and consider the commutative diagram*

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathcal{Z} & \longrightarrow & \mathcal{W} \end{array} \tag{11.141}$$

where all arrows are linear and bounded. If three of the four arrows are Fredholm operators then so is the fourth one.

The following result is going to be of importance for us.

Lemma 11.41 *Let X_j, Y_j, $j = 1, 2$, be two quasi-Banach spaces such that the inclusions $X_1 \hookrightarrow X_2$, $Y_1 \hookrightarrow Y_2$ are continuous, and the second one has dense range. If $T \in \Phi(X_1, Y_1) \cap \Phi(X_2, Y_2)$ is such that* $\text{index}\,(T : X_1 \to Y_1) = \text{index}\,(T : X_2 \to Y_2)$ *then* $\text{Ker}\,(T : X_1 \to Y_1) = \text{Ker}\,(T : X_2 \to Y_2)$.

Proof. Since TX_1 has finite codimension in Y_1, there exists a finite dimensional subspace M of Y_1 such that $TX_1 \oplus M = Y_1$ (direct, non-orthogonal sum). We claim that $TX_2 + M = Y_2$. To prove the claim, observe that $Y_1 = TX_1 + M \subseteq TX_2 + M$. Hence, since Y_1 is densely embedded into Y_2, so is $TX_2 + M$. Moreover, because TX_2 is closed and M is finite dimensional, Theorem 11.20 implies that $TX_2 + M$ is closed in Y_2. Combining these results, the claim follows. Going further, by using the claim we obtain that $\dim\left(\frac{Y_1}{TX_1}\right) = \dim M \geq \dim\left(\frac{Y_2}{TX_2}\right)$ which, in turn, implies that $\dim \text{coker}\,(T : X_1 \to Y_1) \geq \dim \text{coker}\,(T : X_2 \to Y_2)$. The latter inequality together with the fact that the index of T is the same when acting from X_j onto Y_j for $j = 1$ and $j = 2$ give that $\dim \text{Ker}\,(T : X_1 \to Y_1) \geq \dim \text{Ker}\,(T : X_2 \to Y_2)$. The reversed inequality is obvious, thus the conclusion follows. □

Lemma 11.42 *Let X, Y be quasi-Banach spaces and assume that $T \in \mathcal{L}(X, Y)$. If $Z \hookrightarrow Y$ is a closed subspace of finite codimension, then $T^{-1}Z$ is a closed subspace of finite codimension in X.*

Proof. Since T is continuous and Z is closed, it follows that $T^{-1}Z$ is closed as well. Next, consider the linear operator

$$\hat{T} : X\Big/T^{-1}Z \longrightarrow Y\Big/Z, \qquad \hat{T}[x] := [Tx], \tag{11.142}$$

where for each $x \in X$, $[x]$ stands for the class of x in $X/T^{-1}Z$, and $[Tx]$ stands for the class of Tx in Y/Z. Clearly, $\hat{T}$ is one-to-one which then entails

$$\dim\left(X\Big/T^{-1}Z\right) \leq \dim\left(Y\Big/Z\right) < +\infty. \tag{11.143}$$

Thus, $T^{-1}Z$ is a space of finite codimension in X. $\square$

We conclude this section with several stability results proved in [48], [50]. First, we need to recall some definitions. A quasi-Banach space X is called *analytically convex* if there is a constant C such that for every polynomial $P : \mathbb{C} \to X$ we have $\|P(0)\|_X \leq C \max_{|z|=1} \|P(z)\|_X$. It is shown in [47] that if X is analytically convex it has an equivalent quasi-norm which is plurisubharmonic (i.e. we can insist that the constant C above can be taken to be 1). Let us also point out that being analytically convex is equivalent to the condition that

$$\max_{0<\Re e\, z<1} \|f(z)\|_X \leq C \max_{\Re e\, z=0,1} \|f(z)\|_X, \tag{11.144}$$

for any analytic function $f : \{z \in \mathbb{C} : 0 < \Re e\, z < 1\} \to X$ which is continuous on the closed strip $\{z \in \mathbb{C} : 0 \leq \Re e\, z \leq 1\}$.

Clearly, any Banach space is analytically convex. Other useful criteria for analytic convexity can be found in [47], [26], [50]. The relevance of this concept stems from the fact that Calderón's complex method of interpolation, originally devised for Banach spaces, can be most naturally adapted to analytically convex quasi-Banach spaces. A more thorough discussion in this regard can be found in [50]. Here, we only wish to quote a result which has been proved in [50].

Lemma 11.43 *Let X_i, Y_i, Z_i, $i = 0, 1$, be quasi-Banach spaces such that $X_0 \cap X_1$ is dense in both X_0 and X_1, and similarly for Z_0, Z_1. Suppose that $Y_i \hookrightarrow Z_i$, $i = 0, 1$ and there exists a linear operator D such that $D : X_i \to Z_i$ boundedly for $i = 0, 1$. Define the spaces*

$$X_i(D) := \{u \in X_i : Du \in Y_i\}, \quad i = 0, 1, \tag{11.145}$$

equipped with the graph norm, i.e. $\|u\|_{X_i(D)} := \|u\|_{X_i} + \|Du\|_{Y_i}$, $i = 0, 1$. *Finally, suppose that there exist continuous linear mappings* $G : Z_i \to X_i$ *and* $K : Z_i \to Y_i$ *with the property* $D \circ G = I + K$ *on the spaces* Z_i *for* $i = 0, 1$. *Then, for each* $0 < \theta < 1$ *and* $0 < q \leq \infty$,

$$(X_0(D), X_1(D))_{\theta,q} = \{u \in (X_0, X_1)_{\theta,q} : Du \in (Y_0, Y_1)_{\theta,q}\}. \tag{11.146}$$

Furthermore, if the spaces $X_0 + X_1$ *and* $Y_0 + Y_1$ *are analytically convex, then*

$$[X_0(D), X_1(D)]_\theta = \{u \in [X_0, X_1]_\theta : Du \in [Y_0, Y_1]_\theta\}, \qquad \theta \in (0, 1). \tag{11.147}$$

We continue with a very useful result which essentially asserts that, on a complex interpolation scales of quasi-Banach spaces, the property of being invertible is stable and the inverses are compatible. The Banach space version can be found in [13], [86], [3], [85], [95]. The theorem below was proved in [50], following earlier work in [48].

Theorem 11.44 *Let* X_0, X_1 *and* Y_0, Y_1 *be two compatible couples of quasi-Banach spaces and assume that* $X_0 + X_1$ *and* $Y_0 + Y_1$ *are analytically convex. Also, consider a bounded, linear operator* $T : X_j \to Y_j$, $j = 0, 1$. *If* $X_\theta := [X_0, X_1]_\theta$ *and* $Y_\theta := [Y_0, Y_1]_\theta$, *then for each* $\theta \in (0, 1)$, *then* T *induces a bounded linear operator*

$$T_\theta : X_\theta \longrightarrow Y_\theta, \qquad \theta \in (0, 1), \tag{11.148}$$

in a natural fashion. Moreover,

$$\|T_\theta\|_{X_\theta \to Y_\theta} \leq \|T\|^{1-\theta}_{X_0 \to X_0} \|T\|^{\theta}_{X_1 \to X_1}, \qquad \theta \in (0, 1). \tag{11.149}$$

Assume next that there exists $\theta_o \in (0, 1)$ *such that* T_{θ_o} *is an isomorphism. Then there exists* $\varepsilon > 0$ *such that* T_θ *continues to be isomorphism whenever* $|\theta - \theta_o| < \varepsilon$.

Furthermore, if I *is any open subinterval of* $(0, 1)$ *with the property that* T_θ^{-1} *exists for every* $\theta \in I$, *then* T_θ^{-1} *agrees with* $T_{\theta'}^{-1}$ *on* $Y_\theta \cap Y_{\theta'}$ *for any* $\theta, \theta' \in I$.

Theorem 11.45 *Under the hypotheses of Theorem 11.44, if T_{θ_o} is surjective and has finite-dimensional kernel then there exists $\varepsilon > 0$ so that* $\dim \ker T_\theta$ *is constant for $|\theta - \theta_o| < \varepsilon$.*

Theorem 11.46 *Retain the same hypotheses as in Theorem 11.44 and assume that $Y_0 \cap Y_1$ is dense in each Y_θ for $0 < \theta < 1$ (which is automatic for the case of inner complex interpolation). Then if T_{θ_o} is Fredholm, there exists $\varepsilon > 0$ so that T_θ is Fredholm for $|\theta - \theta_0| < \varepsilon$ and the index is constant.*

Our last result in this section is a global stability theorem from [48].

Theorem 11.47 *Retain the same hypotheses as in Theorem 11.44 and, in addition, assume that there exists $\theta_o \in I$ such that $T_{\theta_o} : X_{\theta_o} \to Y_{\theta_o}$ is an isomorphism. Then, if $\eta(T_\theta) > 0$ for all $\theta \in I$ or if $\kappa(T_\theta) < \infty$ for all $\theta \in I$, it follows that $T_\theta : X_\theta \to Y_\theta$ is an isomorphism for all $\theta \in I$.*

11.10 Surface to surface change of variables

The following result, of general nature, from [42] is going to be useful for us.

Proposition 11.48 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $\mathcal{O}$ an open neighborhood of $\overline{\Omega}$, and let $F : \mathcal{O} \to \mathbb{R}^n$ be an orientation preserving C^∞-diffeomorphism. Then $\widetilde{\Omega} := F(\Omega)$ is a Lipschitz domain and if $\nu, \widetilde{\nu}$ and $\sigma, \widetilde{\sigma}$ are, respectively, the outward unit normals and surface measures on $\partial\Omega$ and $\partial\widetilde{\Omega}$, then*

$$\widetilde{\nu} = \frac{(DF^{-1})^\top (\nu \circ F^{-1})}{|(DF^{-1})^\top (\nu \circ F^{-1})|}, \tag{11.150}$$

$$\widetilde{\sigma} = |(DF^{-1})^\top (\nu \circ F^{-1})| \, (|\det DF| \circ F^{-1}) \, F_* \sigma, \tag{11.151}$$

where $(DF^{-1})^\top$ denotes the transposed of the Jacobian matrix of F^{-1}, and $F_*\sigma$ is the push-forward of the measure σ.

Below, we study how tangential derivatives transform under changing variables in the ambient Euclidean space.

Proposition 11.49 *In the context of Proposition 11.48, and assuming* $1 < p < \infty$*, one has*

$$\|f\|_{L^p(\partial\Omega)} \approx \|f \circ F^{-1}\|_{L^p(\partial\widetilde{\Omega})}, \qquad \|f\|_{L^p_1(\partial\Omega)} \approx \|f \circ F^{-1}\|_{L^p_1(\partial\widetilde{\Omega})}. \tag{11.152}$$

Furthermore, for every $j, k \in \{1, ..., n\}$*,*

$$\partial_{\widetilde{\tau}_{jk}}(f \circ F^{-1}) = \frac{\Big[(DF^{-1})^\top[(\nabla_{tan} f \otimes \nu - \nu \otimes \nabla_{tan} f) \circ F^{-1}](DF^{-1})\Big]_{kj}}{|(DF^{-1})^\top(\nu \circ F^{-1})|}. \tag{11.153}$$

Proof. The first equivalence in (11.152) is a direct consequence of Proposition 11.48, whereas the second follows from (11.153) and Proposition 11.48.

Consider now the identity (11.153). For each $j, k \in \{1, \dots, n\}$, denote by $\partial_{\widetilde{\tau}_{jk}}$ the tangential derivative on $\partial\widetilde{\Omega}$ given by $\widetilde{\nu}_j \partial_k - \widetilde{\nu}_k \partial_j$. We then have

$$\begin{aligned}
\partial_{\widetilde{\tau}_{jk}}(f \circ F^{-1}) &= \widetilde{\nu}_j \partial_k (f \circ F^{-1}) - \widetilde{\nu}_k \partial_j (f \circ F^{-1}) \\
&= \widetilde{\nu}_j ((\partial_\ell f) \circ F^{-1}) \partial_k F^{-1}_\ell - \widetilde{\nu}_k ((\partial_r f) \circ F^{-1}) \partial_j F^{-1}_r.
\end{aligned} \tag{11.154}$$

Employing Proposition 11.48 we further write

$$\begin{aligned}
\widetilde{\nu}_j ((\partial_\ell f) \circ F^{-1}) \partial_k F^{-1}_\ell &= \frac{((DF^{-1})^\top(\nu \circ F^{-1}))_j (\nabla f \circ F^{-1})_\ell (DF^{-1})_{\ell k}}{|(DF^{-1})^\top(\nu \circ F^{-1})|} \\
&= \frac{[(DF^{-1})^\top((\nabla f \circ F^{-1}) \otimes (\nu \circ F^{-1}))(DF^{-1})]_{kj}}{|(DF^{-1})^\top(\nu \circ F^{-1})|},
\end{aligned} \tag{11.155}$$

where for two vectors $a, b \in \mathbb{R}^n$ with $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$, we have set $a \otimes b$ to stand for the $n \times n$ matrix whose ij entry is given by

$$(a \otimes b)_{ij} := a_i b_j, \qquad i, j \in \{1, \dots, n\}. \tag{11.156}$$

Thus, based on (11.154) and (11.155),

$$\partial_{\widetilde{\tau}_{jk}}(f\circ F^{-1}) = \frac{[(DF^{-1})^{\top}((\nabla f\circ F^{-1})\otimes(\nu\circ F^{-1}))(DF^{-1})]_{kj}}{|(DF^{-1})^{\top}(\nu\circ F^{-1})|} - \frac{[(DF^{-1})^{\top}((\nabla f\circ F^{-1})\otimes(\nu\circ F^{-1}))(DF^{-1})]_{jk}}{|(DF^{-1})^{\top}(\nu\circ F^{-1})|}. \tag{11.157}$$

This further gives,

$$\partial_{\widetilde{\tau}_{jk}}(f\circ F^{-1}) = \frac{[(DF^{-1})^{\top}(a\otimes b - b\otimes a)(DF^{-1})]_{kj}}{|(DF^{-1})^{\top}(\nu\circ F^{-1})|}, \tag{11.158}$$

where

$$a := \nabla f\circ F^{-1} \qquad \text{and} \qquad b := \nu\circ F^{-1}. \tag{11.159}$$

Since, generally speaking, $a\otimes b - b\otimes a = a_b\otimes b - b\otimes a_b$ where $a_b := a-(a\cdot b)b$, we may finally conclude that, for every j,k, (11.153) holds. $\square$

11.11 Truncating singular integrals

Recall that a function $\varphi : U\to\mathbb{R}$, U open subset of $\mathbb{R}^n$ is called *Lipschitz* provided that there exists $M>0$ such that $|\varphi(x)-\varphi(y)|\leq M|x-y|$ for all $x,y\in U$. The best constant in the above inequality is called *the Lipschitz constant of* φ.

The following is an old result of Rademacher (cf.[77]).

Lemma 11.50 *Let φ be a real-valued, Lipschitz function defined in an open set U of $\mathbb{R}^n$. Then for each $1\leq j\leq n$, $\frac{\partial\varphi}{\partial x_j}$ exists at almost every point x in U and $\frac{\partial\varphi}{\partial x_j}\in L^\infty(U,\mathbb{R})$. In fact, $\|\nabla\varphi\|_{L^\infty}$ is the Lipschitz constant of φ and for almost every $x\in\mathbb{R}^n$ there exists a vector $\nabla\varphi(x)$ such that*

$$\lim_{|y|\downarrow 0}\frac{|\varphi(x+y)-\varphi(x)-\langle\nabla\varphi(x),y\rangle\,|}{|y|}=0. \tag{11.160}$$

If $U\subseteq\mathbb{R}^n$, call $\Phi : U\to\mathbb{R}^m$ bi-Lipschitz if there exist $0<M_1\leq M_2<\infty$ such that

$$M_1|x-y|\leq|\Phi(x)-\Phi(y)|\leq M_2|x-y|, \qquad \forall\, x,y\in U. \tag{11.161}$$

When U is an open set, it is known (cf. [77]) that necessarily $m \geq n$, Φ is an open mapping, the Jacobian matrix $D\Phi = (\partial_j \Phi_k)_{1\leq j\leq n,\, 1\leq k\leq m}$ exists a.e. in U and

$$\operatorname{rank} D\Phi(x) = n \text{ for a.e. } x \in U. \tag{11.162}$$

Our goal here is to establish the following.

Proposition 11.51 *Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz function with Lipschitz constant M, and assume that $F : \mathbb{R}^m \to \mathbb{R}$, $F \in C^N(\mathbb{R}^m)$, for some sufficiently large $N \in \mathbb{N}$, F is odd function. For $x, y \in \mathbb{R}^n$ with $x \neq y$ we set $K(x,y) := \frac{1}{|x-y|^n} F\left(\frac{A(x)-A(y)}{|x-y|}\right)$, and for $\varepsilon > 0$, define the truncated operator*

$$T_\varepsilon f(x) := \int_{|x-y|>\varepsilon} K(x,y) f(y)\, dy, \qquad x \in \mathbb{R}^n. \tag{11.163}$$

As is well-known (cf., e.g., [67]), if $1 < p < \infty$ and $f \in L^p(\mathbb{R}^n)$ then the limit $\lim_{\varepsilon\to 0} T_\varepsilon f(x)$ exists for almost every $x \in \mathbb{R}^n$ and the operator

$$Tf(x) := \lim_{\varepsilon\to 0} T_\varepsilon f(x), \qquad x \in \mathbb{R}^n, \tag{11.164}$$

is bounded on $L^p(\mathbb{R}^n)$.

Assume that $B : \mathbb{R}^n \to \mathbb{R}^{m'}$, $m' \geq n$, is a functions satisfying

$$M^{-1}|x-y| \leq |B(x) - B(y)| \leq M|x-y|, \quad \forall\, x, y \in \mathbb{R}^n, \tag{11.165}$$

for some $M > 1$. Then if $1 < p < \infty$ and $f \in L^p(\mathbb{R}^n)$, the limit

$$\lim_{\varepsilon\to 0} \int_{|B(x)-B(y)|>\varepsilon} K(x,y) f(y)\, dy, \tag{11.166}$$

exists and is equal to $Tf(x)$ (as defined in (11.164)) for almost every $x \in \mathbb{R}^n$. In other words, for any function B as in (11.165), one has the representation

$$Tf(x) = \lim_{\varepsilon\to 0} \int_{|B(x)-B(y)|>\varepsilon} K(x,y) f(y)\, dy, \tag{11.167}$$

for almost every $x \in \mathbb{R}^n$.

To prove it, we isolate the key technical step in the form of a lemma, stated below.

Lemma 11.52 *Let $A : \mathbb{R}^n \to \mathbb{R}^m$ and $B : \mathbb{R}^n \to \mathbb{R}^{m'}$, $m' \geq n$, be functions satisfying*

$$|A(x) - A(y)| \leq M|x - y|, \qquad \textit{and} \tag{11.168}$$

$$M^{-1}|x - y| \leq |B(x) - B(y)| \leq M|x - y|, \quad \forall\, x, y \in \mathbb{R}^n, \tag{11.169}$$

for some constant $M > 1$. Also let $F : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ be a C^1, odd function. Fix $x \in \mathbb{R}^n$ and for each $\varepsilon > 0$ consider

$$U(\varepsilon) := \{y \in \mathbb{R}^n : 1 > |x - y| > \varepsilon\}, \tag{11.170}$$

$$V(\varepsilon) := \{y \in \mathbb{R}^n : |(DB)(x)(x - y)| > \varepsilon,\ |x - y| < 1\}, \tag{11.171}$$

$$W(\varepsilon) := \{y \in \mathbb{R}^n : |B(x) - B(y)| > \varepsilon,\ |x - y| < 1\}. \tag{11.172}$$

Then

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \int_{U(\varepsilon)} \frac{1}{|x - y|^n} F\Big(\frac{A(x) - A(y)}{|x - y|}\Big)\, dy &= \lim_{\varepsilon \downarrow 0} \int_{V(\varepsilon)} \frac{1}{|x - y|^n} F\Big(\frac{A(x) - A(y)}{|x - y|}\Big)\, dy \\ &= \lim_{\varepsilon \downarrow 0} \int_{W(\varepsilon)} \frac{1}{|x - y|^n} F\Big(\frac{A(x) - A(y)}{|x - y|}\Big)\, dy \end{aligned} \tag{11.173}$$

provided the Jacobian matrices $(DA)(x)$ and $(DB)(x)$ exist, $\operatorname{rank}(DB)(x) = n$, *and one of the above three limits exists and is finite.*

Proof. Without loss of generality we can take $x = 0$, $A(0) = 0$, $B(0) = 0$. By Lemma 11.50 there exist nonnegative functions $\eta_A(t)$ and $\eta_B(t)$ defined for $t > 0$, so that $\eta_A(t) \downarrow 0$, $\eta_B(t) \downarrow 0$ as $t \downarrow 0$ and

$$|A(y) - (DA)(0)y| \leq |y|\, \eta_A(|y|), \tag{11.174}$$

$$|B(y) - (DB)(0)y| \leq |y|\, \eta_B(|y|), \tag{11.175}$$

for $y \in \mathbb{R}^n$. If, for each $\varepsilon > 0$, we now introduce $\Delta(\varepsilon) := \{y \in \mathbb{R}^n : \varepsilon > |y| > \varepsilon \|(DB)(0)\|^{-1}\}$ then $V(\varepsilon) \setminus U(\varepsilon) \subseteq \Delta(\varepsilon)$. Employing the properties of F, the fact that $V(\varepsilon) \setminus U(\varepsilon)$ is symmetric with respect to the origin and the estimate (11.174), the absolute value of the difference of the first two limits in (11.173) is estimated by

$$
\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\left|\int_{V(\varepsilon)\setminus U(\varepsilon)}\frac{1}{|y|^n}F\Big(\frac{A(y)}{|y|}\Big)\,dy\right| \\
&\quad = \lim_{\varepsilon\downarrow 0}\frac{1}{2}\left|\int_{V(\varepsilon)\setminus U(\varepsilon)}\frac{1}{|y|^n}\Big[F\Big(\frac{A(y)}{|y|}\Big)+F\Big(\frac{A(-y)}{|y|}\Big)\Big]\,dy\right| \\
&\quad = \lim_{\varepsilon\downarrow 0}\frac{1}{2}\left|\int_{V(\varepsilon)\setminus U(\varepsilon)}\frac{1}{|y|^n}\Big[F\Big(\frac{A(y)}{|y|}\Big)-F\Big(-\frac{A(-y)}{|y|}\Big)\Big]\,dy\right| \\
&\quad \leq \Big[\sup_{|\xi|\leq M}|(DF)(\xi)|\Big]\lim_{\varepsilon\downarrow 0}\int_{\Delta(\varepsilon)}\eta_A(|y|)|y|^{-n}\,dy \\
&\quad \leq C\lim_{\varepsilon\downarrow 0}\eta_A(\varepsilon)=0, \qquad (11.176)
\end{aligned}
$$

which proves the first equality in (11.173).

In order to prove the second equality in (11.173), observe that for each point $y\in V(\varepsilon)\setminus W(\varepsilon)$ we have $M^{-1}|y|\leq|B(y)|<\varepsilon$, so that $|y|<\varepsilon M$. That is,

$$
y\in V(\varepsilon)\setminus W(\varepsilon)\Longrightarrow |y|<\varepsilon M \quad\text{and}\quad |B(y)|<\varepsilon. \qquad (11.177)
$$

Based on this, we may conclude that

$$
y\in V(\varepsilon)\setminus W(\varepsilon)\Longrightarrow |(DB)(0)y|\leq|(DB)(0)y-B(y)|+|B(y)|\leq\varepsilon M\,\eta_B(\varepsilon M)+\varepsilon \qquad (11.178)
$$

and, further,

$$
y\in V(\varepsilon)\setminus W(\varepsilon)\Longrightarrow \varepsilon<|(DB)(0)y|\leq\varepsilon M\,\eta_B(\varepsilon M)+\varepsilon. \qquad (11.179)
$$

From (11.177) and (11.179) we may therefore conclude that

$$
V(\varepsilon)\setminus W(\varepsilon)\subseteq Z(\varepsilon;M\,\eta_B(\varepsilon M);(DB)(0)) \qquad (11.180)
$$

where we have set

$$Z(\varepsilon;a;R) := \{y \in \mathbb{R}^n : \varepsilon < |Ry| \leq \varepsilon a + \varepsilon\},$$
$$\text{if } \varepsilon > 0,\ a > 0, \text{ and } R \text{ is a } m' \times n \text{ matrix of rank } n. \tag{11.181}$$

Let $\mathcal{H}^k_N$ be the k-dimensional Hausdorff measure in $\mathbb{R}^N$. To estimate the size of $Z(\varepsilon;a;R)$, we first note that

$$Z(\varepsilon;a;R) = \varepsilon\, Z(1;a;R), \qquad \forall\, \varepsilon > 0. \tag{11.182}$$

On the other hand, if we set $H_n := \{Ry : y \in \mathbb{R}^n\}$ then, since R is a rank n matrix, H_n is an n-dimensional plane in $\mathbb{R}^{m'}$ and $R : \mathbb{R}^n \to H_n$ is a linear isomorphism. Hence,

$$\begin{aligned} \mathcal{H}^n_n\Big(Z(1;a;R)\Big) &= \mathcal{H}^n_n\Big(\{y \in \mathbb{R}^n : 1 < |Ry| \leq a+1\}\Big) \\ &\leq C\,\mathcal{H}^n_{m'}\Big(\{Y \in H_n : 1 < |Y| \leq a+1\}\Big). \end{aligned} \tag{11.183}$$

Simple geometric considerations show that the

$$\lim_{a\to 0} \mathcal{H}^n_{m'}\Big(\{Y \in H_n : 1 < |Y| \leq a+1\}\Big) = 0. \tag{11.184}$$

From this, (11.182), (11.180) and the fact that $\eta_B(\varepsilon\, M) \to 0$ as $\varepsilon \to 0$, we finally deduce that

$$\lim_{\varepsilon\to 0} \frac{\mathcal{H}^n_n\Big(V(\varepsilon)\backslash W(\varepsilon)\Big)}{\varepsilon^n} = 0. \tag{11.185}$$

Since the expression $\frac{1}{|x-y|^n} F\Big(\frac{A(x)-A(y)}{|x-y|}\Big)$ restricted to $V(\varepsilon)\backslash W(\varepsilon)$ (itself, a subset of $\{y \in \mathbb{R}^n : \varepsilon M > |y| > \varepsilon\|(DB)(0)\|^{-1}\}$) is pointwise of the order ε^{-n}, we conclude that the integral of this function over the set $V(\varepsilon)\backslash W(\varepsilon)$ converges to zero as $\varepsilon \to 0$.

Moving on, an argument analogous to (11.179) gives that

$$\varepsilon - \varepsilon M\, \eta_B(\varepsilon M) < |(DB)(0)y| \leq \varepsilon, \tag{11.186}$$

uniformly for $y \in W(\varepsilon)\backslash V(\varepsilon)$. Thus, for reasons similar to those discussed above, the integral of $\frac{1}{|x-y|^n} F\left(\frac{A(x)-A(y)}{|x-y|}\right)$ over $W(\varepsilon)\backslash V(\varepsilon)$ also vanishes as $\varepsilon \downarrow 0$, which completes the proof of the second equality in the conclusion of the lemma. □

After this preamble, it is straightforward to carry out the

Proof of Proposition 11.51. The claim in (11.167) is an immediate corollary of Lemma 11.52 and (11.162). □

11.12 Approximating Lipschitz domains

For various purposes, it convenient to approximate, in a suitable sense, a given Lipschitz domain with a sequence of sub-domains. Several variants can be found in the literature. See, for example, [72] and [94] for such approximating schemes involving C^∞-smooth sub-domains. For us here, however, the following approximation result, proved by A.P. Calderón in [11], is particularly useful.

Lemma 11.53 *Consider a bounded Lipschitz domain Ω in $\mathbb{R}^n$, $n \geq 2$, with surface measure σ and outward unit normal ν, along with a Lipschitz vector field $\vec{h}$ on $\partial\Omega$, satisfying*

$$|\vec{h}(x)| = 1 \quad \textit{and} \quad \langle \vec{h}(x), \nu(x)\rangle \geq \kappa \quad \textit{for a.e. } x \in \partial\Omega, \tag{11.187}$$

where $\kappa \in (0,1)$ is a fixed constant. Let Ω_t be the subset of Ω defined by

$$\Omega_t := \Omega \setminus \{x - s\vec{h}(x) : x \in \partial\Omega,\ 0 < s < t\}. \tag{11.188}$$

Then there exists a small positive number t_o, depending only on the Lipschitz character of Ω, the Lipschitz constant of $\vec{h}$, n, and κ, such that the following hold.

(i) Whenever $0 < t < t_o$, Ω_t is a Lipschitz domain and

$$\partial\Omega_t = \{x - t\vec{h}(x) : x \in \partial\Omega\}. \tag{11.189}$$

(ii) *There exists a covering of $\partial\Omega$ with finitely many coordinate cylinders which also form a family of coordinate cylinders for $\partial\Omega_t$, for each $t \in (0, t_o)$. Moreover, for each such cylinder $C(r, h)$, if φ and φ_t are the corresponding Lipschitz functions whose graphs describe the boundaries of Ω and Ω_t respectively in $C(r, h)$, then $\|\nabla\varphi_t\|_{L^\infty} \leq \|\nabla\varphi\|_{L^\infty}$ and $\nabla\varphi_t \to \nabla\varphi$ pointwise a.e. as $t \to 0^+$.*

(iii) *Consider the mapping $F_t : \mathbb{R}^n \to \mathbb{R}^n$ defined by $F_t(x) := x - t\vec{h}(x)$. Then F_t is bi-Lipschitz, uniformly in $t \in (0, t_o)$. As a consequence,*

$$\Lambda_t : \partial\Omega \longrightarrow \partial\Omega_t, \qquad \Lambda_t(x) := x - t\vec{h}(x), \quad x \in \partial\Omega, \tag{11.190}$$

is a bi-Lipschitz function for each $t \in (0, t_o)$ and the Lipschitz constants of Λ_t and ${\Lambda_t}^{-1}$ are uniformly bounded in t.

(iv) *For every $t \in (0, t_o)$ and every $x \in \partial\Omega$, there holds $\Lambda_t(x) \in \Gamma(x)$ and*

$$\sup_{x\in\partial\Omega} |x - \Lambda_t(x)| \leq Ct, \tag{11.191}$$

for some finite, positive constant $C = C(\Omega, \vec{h})$.

(v) *For each $t \in (0, t_o)$, there exist positive functions $\omega_t : \partial\Omega \to \mathbb{R}_+$, bounded away from zero and infinity uniformly in t, such that, for any measurable set $F \subset \partial\Omega$,*

$$\int_F \omega_t \, d\sigma = \int_{\Lambda_t(F)} d\sigma_t, \tag{11.192}$$

where $d\sigma_t$ denotes the surface measure on $\partial\Omega_t$. In addition,

$$\sup_{x\in\partial\Omega} |1 - \omega_t(x)| \leq Ct, \qquad \forall\, t \in (0, t_o), \tag{11.193}$$

where C is as before.

(vi) If ν_t is the outward unit normal vector to $\partial\Omega_t$, then, with C as above,

$$\sup_{x\in\partial\Omega} |\nu(x) - \nu_t(\Lambda_t(x))| \leq Ct, \qquad \forall\, t \in (0, t_o). \tag{11.194}$$

We wish to complement this lemma with several related results (working in the same context as above). First, consider a function

$$k \in C^N(\mathbb{R}^n \setminus \{0\}), \quad k(-x) = -k(x), \quad k(\lambda x) = \lambda^{1-n} k(x) \text{ if } \lambda > 0, \tag{11.195}$$

where $N = N(n)$ is a sufficiently large integer. To this, we associate the singular integral operator

$$Tf(x) := \lim_{\varepsilon\to 0^+} \int\limits_{\substack{|x-y|>\varepsilon \\ y\in\partial\Omega}} k(x-y) f(y)\, d\sigma(y), \qquad x \in \partial\Omega. \tag{11.196}$$

Furthermore, let T_t, $t \in (0, t_o)$, denote the version of the integral operator (11.196) written for $\partial\Omega_t$ in place of $\partial\Omega$.

We claim that for each $p \in (1, \infty)$, there exists $C(\Omega, \vec{h}, k, p) > 0$ with the property that

$$\|[T_t(f \circ \Lambda_t^{-1})] \circ \Lambda_t - Tf\|_{L^p(\partial\Omega)} \leq C\, t \|f\|_{L^p(\partial\Omega)}, \qquad \forall\, t \in (0, t_o). \tag{11.197}$$

To prove this claim, for $x \in \partial\Omega$ and $t \in (0, t_o)$ we write

$$\begin{aligned}
T_t(f \circ \Lambda_t^{-1})(\Lambda_t(x)) &= \lim_{\varepsilon\to 0^+} \int\limits_{\substack{|\Lambda_t(x)-y'|>\varepsilon \\ y'\in\partial\Omega_t}} k(\Lambda_t(x) - y') f(\Lambda_t^{-1}(y'))\, d\sigma_t(y') \\
&= \lim_{\varepsilon\to 0^+} \int\limits_{\substack{|\Lambda_t(x)-\Lambda_t(y)|>\varepsilon \\ y\in\partial\Omega}} k(\Lambda_t(x) - \Lambda_t(y)) f(y) \omega_t(y)\, d\sigma(y) \\
&= \lim_{\varepsilon\to 0^+} \int\limits_{\substack{|F_t(x)-F_t(y)|>\varepsilon \\ y\in\partial\Omega}} k(F_t(x) - F_t(y)) f(y) \omega_t(y)\, d\sigma(y) \\
&= \lim_{\varepsilon\to 0^+} \int\limits_{\substack{|x-y|>\varepsilon \\ y\in\partial\Omega}} k(\Lambda_t(x) - \Lambda_t(y)) f(y) \omega_t(y)\, d\sigma(y).
\end{aligned} \tag{11.198}$$

Above, the first equality follows from (11.196), the second from (11.192), the third uses the definition of F_t introduced in (iii) in Lemma 11.53, and the fourth is a consequence of results in § 11.11. Consequently,

$$T_t(f \circ \Lambda_t^{-1})(\Lambda_t(x)) - Tf(x) = R_t^1 f(x) + R_t^2 f(x), \tag{11.199}$$

where, for $x \in \partial\Omega$ and $t \in (0, t_o)$, we have set

$$\begin{aligned} R_t^1 f(x) &:= \lim_{\varepsilon \to 0^+} \int\limits_{\substack{|x-y|>\varepsilon \\ y \in \partial\Omega}} k(\Lambda_t(x) - \Lambda_t(y)) f(y) [\omega_t(y) - 1] \, d\sigma(y), && (11.200) \\ R_t^2 f(x) &:= \lim_{\varepsilon \to 0^+} \int\limits_{\substack{|x-y|>\varepsilon \\ y \in \partial\Omega}} [k(\Lambda_t(x) - \Lambda_t(y)) - k(x-y)] f(y) \, d\sigma(y). && (11.201) \end{aligned}$$

The operator R_t^1 is amenable to Calderón-Zygmund theory (either directly, or after changing variables back to $\partial\Omega_t$) and, by (11.193), we thus obtain

$$\|R_t^1 f\|_{L^p(\partial\Omega)} \leq C \|[\omega_t - 1] f\|_{L^p(\partial\Omega)} \leq C\, t \|f\|_{L^p(\partial\Omega)}, \tag{11.202}$$

uniformly for $t \in (0, t_o)$. As for the contribution from $R_t^2 f$, first note that, by the Mean Value Theorem,

$$\begin{aligned} R_t^2 f(x) &= \lim_{\varepsilon \to 0^+} \int\limits_{\substack{|x-y|>\varepsilon \\ y \in \partial\Omega}} [k(\Lambda_t(x) - \Lambda_t(y)) - k(x-y)] f(y) \, d\sigma(y) \\ &= t \int_0^1 R_{t,\theta}^2 f(x) \, d\theta, && (11.203) \end{aligned}$$

where, for $x \in \partial\Omega$, $t \in (0, t_0)$ and $\theta \in [0,1]$, we have set

$$R_{t,\theta}^2 f(x) := \lim_{\varepsilon \to 0^+} \int\limits_{\substack{|x-y|>\varepsilon \\ y \in \partial\Omega}} (\nabla k)(x - y - \theta t(h(x) - h(y)))(h(x) - h(y)) f(y) \, d\sigma(y). \tag{11.204}$$

By Calderón-Zygmund theory, we have

$$\|R^2_{t,\theta} f\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)}, \tag{11.205}$$

uniformly for $t \in (0, t_o)$ and $\theta \in [0, 1]$. From this and (11.203), we then obtain

$$\|R^2_t f\|_{L^p(\partial\Omega)} \leq C\, t\|f\|_{L^p(\partial\Omega)}, \tag{11.206}$$

uniformly for $t \in (0, t_o)$. In concert, (11.202), (11.206) and (11.199) prove (11.197).

Next, we claim that if $1 \leq j, k \leq n$ and $1 < p < \infty$, then there exists $C > 0$ such that

$$\|\partial_{\tau_{jk}} f - [\partial_{\tau^t_{jk}}(f \circ \Lambda_t^{-1})] \circ \Lambda_t\|_{L^p(\partial\Omega)} \leq C\, t\|\nabla_{tan} f\|_{L^p(\partial\Omega)}, \qquad \forall\, t \in (0, t_o), \tag{11.207}$$

where $\partial_{\tau_{jk}}$ is the tangential derivative operator on $\partial\Omega$ introduced in (2.14), and $\partial_{\tau^t_{jk}}$ is its version relative to $\partial\Omega_t$. Of course, it suffices to prove the pointwise inequality

$$|\partial_{\tau_{jk}} f - [\partial_{\tau^t_{jk}}(f \circ \Lambda_t^{-1})] \circ \Lambda_t| \leq C\, t|\nabla_{tan} f| \text{ on } \partial\Omega, \quad \forall\, t \in (0, t_o), \tag{11.208}$$

where ∇_{tan} is the tangential gradient on $\partial\Omega$. To see this, bring in (11.153) written for the change of variable mapping $F_t(x) = x - t\,\vec{h}(x)$. Using the fact that

$$DF_t = I + O(t), \quad DF_t^{-1} = I + O(t), \quad (DF_t^{-1})^\top = I + O(t), \qquad t \in (0, t_o), \tag{11.209}$$

and recalling (11.194), we obtain from (11.153) and (11.150) that

$$\begin{aligned}
\partial_{\tau^t_{jk}}(f \circ \Lambda_t^{-1}) \quad &= \quad \frac{\Big[(\nabla_{tan} f) \circ F_t^{-1} \otimes (DF_t^{-1})^\top (\nu \circ F_t^{-1})\Big]_{kj}}{|(DF_t^{-1})^\top(\nu \circ F_t^{-1})|} \\
&\quad - \frac{\Big[(DF^{-1})(\nu \circ F_t^{-1}) \otimes (\nabla_{tan} f) \circ F_t^{-1}]\Big]_{kj}}{|(DF_t^{-1})^\top(\nu \circ F_t^{-1})|} + O(t|(\nabla_{tan} f) \circ F_t^{-1}|) \\
&= \quad \Big[(\nabla_{tan} f) \circ \Lambda_t^{-1} \otimes \nu_t\Big]_{kj} - \Big[\nu_t \otimes (\nabla_{tan} f) \circ \Lambda_t^{-1}\Big]_{kj} + O(t|(\nabla_{tan} f) \circ \Lambda_t^{-1}|)
\end{aligned}$$

$$
\begin{aligned}
&= (\nu_t)_j(\nabla_{tan}f)_k \circ \Lambda_t^{-1} - (\nu_t)_k(\nabla_{tan}f)_j \circ \Lambda_t^{-1} + O(t|(\nabla_{tan}f)\circ\Lambda_t^{-1}|) \\
&= (\nu\circ\Lambda_t^{-1})_j(\nabla_{tan}f)_k \circ \Lambda_t^{-1} - (\nu\circ\Lambda_t^{-1})_k(\nabla_{tan}f)_j \circ \Lambda_t^{-1} + O(t|(\nabla_{tan}f)\circ\Lambda_t^{-1}|) \\
&= (\partial_{\tau_{jk}}f)\circ\Lambda_t^{-1} + O(t|(\nabla_{tan}f)\circ\Lambda_t^{-1}|). \qquad (11.210)
\end{aligned}
$$

This clearly implies (11.208).

Lemma 11.54 *In the context of Lemma 11.53, let K_λ be the double layer potential operator for the Stokes system on $\partial\Omega$, and denote by K_λ^t the corresponding operator considered on $\partial\Omega_t$. Then for each $p\in(1,\infty)$,*

$$
\|K_\lambda f - [K_\lambda^t(f\circ\Lambda_t^{-1})]\circ\Lambda_t\|_{L_1^p(\partial\Omega)} \le C\,t\|f\|_{L_1^p(\partial\Omega)}, \qquad \forall\, t\in(0,t_o), \qquad (11.211)
$$

where $C>0$ depends only on Ω and p.

Proof. Fix $f\in L_1^p(\partial\Omega)$ with $\|f\|_{L_1^p(\partial\Omega)}=1$. Also, recall from (4.98) that there exist Calderón-Zygmund type operators T_{jkrs} on $\partial\Omega$, along with their counterparts T_{jkrs}^t on $\partial\Omega_t$, for which the following commutation identities hold:

$$
\partial_{\tau_{jk}}K_\lambda = T_{jkrs}\partial_{\tau_{rs}}, \qquad \partial_{\tau_{jk}}K_\lambda^t = T_{jkrs}^t\partial_{\tau_{rs}^t}, \qquad \forall\, j,k\in\{1,\ldots,n\}. \qquad (11.212)
$$

Turning to (11.211) in the earnest, we first note that

$$
\|K_\lambda f - [K_\lambda^t(f\circ\Lambda_t^{-1})]\circ\Lambda_t\|_{L^p(\partial\Omega)} \le C\,t, \qquad \forall\, t\in(0,t_o), \qquad (11.213)
$$

by (11.197) (and (11.194)). Fix now $j,k\in\{1,\ldots,n\}$ and consider

$$
\|\partial_{\tau_{jk}}(Kf) - \partial_{\tau_{jk}}([K^t(f\circ\Lambda_t^{-1})]\circ\Lambda_t)\|_{L^p(\partial\Omega)}. \qquad (11.214)
$$

Given the goal we have in mind, it is permissible to replace terms in (11.214) with other expressions that differ from these by residues whose L^p norm on $\partial\Omega$ is $O(t)$. With this convention in mind, $\partial_{\tau_{jk}}([K^t(f\circ\Lambda_t^{-1})]\circ\Lambda_t)$ can then be replaced, thanks to (11.207) and (11.212), by

$$[\partial_{\tau^t_{jk}} K^t(f \circ \Lambda_t^{-1})] \circ \Lambda_t = [T^t_{jkrs}(\partial_{\tau^t_{rs}}(f \circ \Lambda_t^{-1}))] \circ \Lambda_t. \tag{11.215}$$

Going further, recall that $\partial_{\tau_{jk}} Kf = T_{jkrs}(\partial_{\tau_{rs}} f)$ and note that this last term can be replaced by $[T^t_{jkrs}((\partial_{\tau_{rs}} f) \circ \Lambda_t^{-1})] \circ \Lambda_t$, by (11.197). This matches the last expression in (11.215), up to an error that can be estimated as follows:

$$\begin{aligned} \|(\partial_{\tau_{jk}} f) \circ \Lambda_t^{-1} - \partial_{\tau^t_{jk}}(f \circ \Lambda_t^{-1})\|_{L^p(\partial\Omega_t)} &\approx \|\partial_{\tau_{jk}} f - (\partial_{\tau^t_{jk}}(f \circ \Lambda_t^{-1})) \circ \Lambda_t\|_{L^p(\partial\Omega)} \\ &= O(t), \end{aligned} \tag{11.216}$$

by (11.192) and (11.207). Thus, all errors have been shown to have proper control, and the estimate (11.211) is proved. □

References

[1] V. Adolfsson and J. Pipher, *The inhomogeneous Dirichlet problem for Δ^2 in Lipschitz domains*, J. Funct. Anal., 159 (1998), no. 1, 137–190.

[2] H. Aimar, *Singular integrals and approximate identities on spaces of homogeneous type*, Trans. Amer. Math. Soc., 292 (1985), no. 1, 135–153.

[3] E. Albrecht, V. Müller, *Spectrum of interpolated operators*, Proc. Amer. Math. Soc., 129 (2001), no. 3, 807–814.

[4] C. Amrouche and V. Girault, *On the existence and regularity of the solution of Stokes problem in arbitrary dimension*, Proc. Japan Acad. Ser. A Math. Sci., 67 (1991), no. 5, 171–175.

[5] C. Amrouche and V. Girault, *Decomposition of vector spaces and application to the Stokes problem in arbitrary dimension*, Czechoslovak Math. J., 44(119) (1994), no. 1, 109–140.

[6] M.E. Bogovskiĭ, *Solutions of some problems of vector analysis, associated with the operators* div *and* grad, Trudy Sem. S.L. Soboleva, No. 1, 1980, Akad. Nauk SSSR Sibirsk. Otdel., Inst. Mat., Novosibirsk, 1980, 5–40.

[7] M. Bownik, *Boundedness of operators on Hardy spaces via atomic decompositions*, Proc. Amer. Math. Soc., 133 (2005), no. 12, 3535–3542.

[8] R.M. Brown, *The Neumann problem on Lipschitz domains in Hardy spaces of order less than one*, Pacific Journal of Math., 171 (1995), no. 2, 389–406.

[9] R.M. Brown, personal communication.

[10] R.M. Brown and Z. Shen, *Estimates for the Stokes operator in Lipschitz domains*, Indiana Univ. Math. J., 44 (1995), no. 4, 1183–1206.

[11] A.P. Calderón, *Boundary value problems for the Laplace equation in Lipschitzian domains*, "Recent Progress in Fourier Analysis," I. Peral and J. Rubio de Francia, eds., Elsevier/North-Holland, Amsterdam, 1985.

[12] A.P. Calderón and A. Zygmund, *On a problem of Mihlin*, Trans. Amer. Math. Soc., 78 (1955), 209–224.

[13] W. Cao and Y. Sagher, *Stability of Fredholm properties on interpolation scales*, Ark. för Math., 28 (1990), 249–258.

[14] L. Cattabriga, *Su un problema al contorno relativo al sistema di equazioni di Stokes*, Rend. Sem. Mat. Univ. Padova, 31 (1961) 308–340.

[15] D.-C. Chang, S.G. Krantz and E.M. Stein, *H^p theory on a smooth domain in R^N and elliptic boundary value problems*, J. Funct. Anal., 114 (1993), no. 2, 286–347.

[16] D.-C. Chang, S.G. Krantz and E.M. Stein, *Hardy spaces and elliptic boundary value problems*, The Madison Symposium on Complex Analysis (Madison, WI, 1991), 119–131, Contemp. Math., 137, Amer. Math. Soc., Providence, RI, 1992.

[17] R.R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc., 83 (1977), no. 4, 569–645.

[18] B.E.J. Dahlberg, *Estimates of harmonic measure*, Arch. Rational Mech. Anal., 65 (1977), no. 3, 275–288.

[19] B. Dahlberg, *L^q-estimates for Green potentials in Lipschitz domains*, Math. Scand., 44 (1979), no. 1, 149–170.

[20] B. Dahlberg and C. Kenig, *Hardy spaces and the Neumann problem in L^p for Laplace's equation in Lipschitz domains*, Annals of Math., 125 (1987), 437–465.

[21] B.E.J. Dahlberg and C.E. Kenig, *L^p estimates for the three-dimensional systems of elastostatics on Lipschitz domains*, Analysis and Partial Differential Equations, pp. 621–634, Lecture Notes in Pure and Appl. Math., Vol. 122, Dekker, New York, 1990.

[22] B.E.J. Dahlberg, C.E. Kenig, J. Pipher and G.C. Verchota, *Area integral estimates for higher order elliptic equations and systems*, Ann. Inst. Fourier (Grenoble), 47 (1997), no. 5, 1425–1461.

[23] B.E.J. Dahlberg, C.E. Kenig and G.C. Verchota, *Boundary value problems for the systems of elastostatics in Lipschitz domains*, Duke Math. J., 57 (1988), no. 3, 795–818.

[24] M. Dauge, *Stationary Stokes and Navier-Stokes systems on two- or three-dimensional domains with corners. I. Linearized equations*, SIAM J. Math. Anal., 20 (1989), no. 1, 74–97.

[25] R. Dautray and J.-L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, Vol. 6, Springer-Verlag, Berlin, 1993.

[26] W. J. Davis, D. J. H. Garling and N. Tomczak-Jaegermann, *The complex convexity of quasi-normed linear spaces*, J. Funct. Anal., 55 (1984), 110–150.

[27] J. Détraz *Proprietes de moyenne pour les solutions de systemes elliptiques*, Publicacions Matematiques, 37 (1993), 83–89.

[28] P. Deuring, *The Stokes system in 3D-Lipschitz domains: a survey of recent results*, Progress in Partial Differential Equations: the Metz Surveys, 4, pp. 187–204, Pitman Res. Notes Math. Ser., Vol. 345, Longman, Harlow, 1996.

[29] P. Deuring, *L^p-theory for the Stokes system in 3D domains with conical boundary points*, Indiana Univ. Math. J., 47 (1998), no. 1, 11–47.

[30] P. Deuring, *The 3D Stokes systems in domains with conical boundary points*, Theory of the Navier-Stokes Equations, pp. 1–10, Ser. Adv. Math. Appl. Sci., Vol. 47, World Sci. Publ., River Edge, NJ, 1998.

[31] M. Dindoš and M. Mitrea, *Semilinear Poissson problems in Sobolev-Besov spaces on Lipschitz domains*, Publ. Mat., 46 (2002), no. 2, 353–403.

[32] M. Dindoš and M. Mitrea, *The stationary Navier-Stokes system in nonsmooth manifolds: the Poisson problem in Lipschitz and C^1 domains*, Arch. Ration. Mech. Anal., 174 (2004), no. 1, 1–47.

[33] L. Escauriaza, M. Mitrea, J. Pipher and M. Wright, *The transmission problem for the Lamé system in Lipschitz domains*, preprint (2008).

[34] E.B. Fabes, C.E. Kenig and G.C. Verchota *The Dirichlet problem for the Stokes system on Lipschitz domains*, Duke Math. J. 57 (1988), no. 3, 769–793.

[35] M. Frazier and B. Jawerth, *A discrete transform and decompositions of distribution spaces*, J. Funct. Anal., 93 (1990) no. 1, 34–170.

[36] M. Frazier, B. Jawerth and G. Weiss, *Littlewood-Paley Theory and the Study of Function Spaces*, CBMS Regional Conference Series in Mathematics, Vol. 79, AMS, Providence, RI, 1991.

[37] G.P. Galdi, C.G. Simader and H. Sohr, *On the Stokes problem in Lipschitz domains*, Ann. Mat. Pura Appl., (4) 167 (1994), 147–163.

[38] F.W. Gehring, *The L^p-integrability of the partial derivatives of a quasiconformal mapping*, Acta Math., 130 (1973), 265–277.

[39] Y. Giga, *Analyticity of the semigroup generated by the Stokes operator in L^r spaces*, Math. Z., 178 (1981), no. 3, 297–329.

[40] V. Girault and P.-A. Raviart, *Finite Element Approximation of the Navier-Stokes Equations*, Lecture Notes in Mathematics, No. 749, Springer-Verlag, Berlin-New York, 1979.

[41] S. Hofmann, M. Mitrea and M. Taylor, *Singular integrals and elliptic boundary problems on regular Semmes-Kenig-Toro domains*, preprint, (2008).

[42] S. Hofmann, M. Mitrea and M. Taylor, *Geometric and transformational properties of Lipschitz domains, Semmes-Kenig-Toro domains, and other classes of finite perimeter domains*, Journal of Geometric Analysis, 17 (2007), 593–647.

[43] T. Iwaniec, *The Gehring lemma*, Quasiconformal Mappings and Analysis, pp. 181–204, Springer, New York, 1998.

[44] T. Jakab, I. Mitrea and M. Mitrea, *Traces of functions in Hardy and Besov spaces on Lipschitz domains with applications to compensated compactness and the theory of Hardy and Bergman type spaces*, preprint, (2005).

[45] S. Janson, *On the interpolation of sublinear operators*. Studia Math., 75 (1982), no. 1, 51–53.

[46] D. Jerison and C. Kenig, *The inhomogeneous problem in Lipschitz domains*, J. Funct. Anal., 130 (1995), 161–219.

[47] N.J. Kalton, *Plurisubharmonic functions on quasi-Banach spaces*, Studia Math., 84 (1986), 297–324.

[48] N. Kalton and M. Mitrea *Stability results on interpolation scales of quasi-Banach spaces and applications*, Trans. Amer. Math. Soc., 350 (1998), no. 10, 3903–3922.

[49] N. Kalton and M. Mitrea, in preparation.

[50] N. Kalton, S. Mayboroda and M. Mitrea *Interpolation of Hardy-Sobolev-Besov-Triebel-Lizorkin spaces and applications to problems in partial differential equations*, to appear, (2007).

[51] N.J. Kalton, N.T. Peck and J.W. Roberts, *An F-Space Sampler*, London Math. Soc., Lecture Notes Series, No. 89, 1984.

[52] R.B. Kellogg and J.E. Osborn, *A regularity result for the Stokes problem in a convex polygon*, J. Functional Analysis, 21 (1976), no. 4, 397–431.

[53] C. E. Kenig, *Harmonic analysis techniques for second order elliptic boundary value problems*, CBMS Regional Conference Series in Mathematics, No. 83, AMS, Providence, RI, 1994.

[54] J. Kilty, *The L^p Dirichlet problem for the Stokes system in Lipschitz domains*, preprint, 2008.

[55] M. Kohr and I. Pop, *Viscous Incompressible Flow for Low Reynolds Numbers*, Advances in Boundary Elements, Vol. 16, WIT Press, Southampton, 2004.

[56] V. Kozlov and V. Maz'ya, *Boundary behavior of solutions to linear and nonlinear elliptic equations in plane convex domains*, Math. Res. Lett., 8 (2001), no. 1-2, 189–193.

[57] V.A. Kozlov and V.G. Maz'ya, *Spectral properties of the operator bundles generated by elliptic boundary-value problems in a cone*, Funktsional. Anal. i Prilozhen., 22 (1988), no. 2, 38–46; translation in Functional Anal. Appl., 22 (1988), 114–121.

[58] V.A. Kozlov and V.G. Maz'ya, *On the spectrum of the operator pencil generated by the Dirichlet problem in a cone*, Mat. Sb., 182 (1991), no. 5, 638–660; translation in Math. USSR-Sb., 73 (1992), no. 1, 27–48.

[59] V.A. Kozlov, V.G. Maz'ya and J. Rossmann, *Spectral Problems Associated with Corner Singularities of Solutions to Elliptic Equations*, Amer. Math. Soc., Providence, RI, 2001.

[60] V.A. Kozlov, V. Maz'ya and C. Schwab, *On singularities of solutions to the Dirichlet problem of hydrodynamics near the vertex of a cone*, J. Reine Angew. Math., 456 (1994), 65–97.

[61] R.A. Macías and C. Segovia, *Lipschitz functions on spaces of homogeneous type*, Adv. in Math., 33 (1979), no. 3, 257–270.

[62] P. Maremonti, R. Russo and G. Starita, *On the Stokes equations: the boundary value problem*, Advances in fluid dynamics, pp. 69–140, Quad. Mat., Vol. 4, Dept. Math., Seconda Univ. Napoli, Caserta, 1999.

[63] S. Mayboroda and M. Mitrea, *Sharp estimates for Green potentials on non-smooth domains*, Mathematical Research Letters, 11 (2004), 481–492.

[64] S. Mayboroda and M. Mitrea, *The solution of the Chang-Krein-Stein conjecture*, pp.,61-154 in the Proceedings of the Conference in Harmonic Analysis and its Applications, Tokyo Woman's Cristian University, March 24-26, 2007, Tokyo, Japan, Akihiko Miyachi editor.

[65] V. Maz'ya and J. Rossmann, *Mixed boundary value problems for the Navier-Stokes system in polyhedral domains*, preprint (2008).

[66] O. Mendez and M. Mitrea, *The Banach envelopes of Besov and Triebel-Lizorkin spaces and applications to partial differential equations*, J. Fourier Anal. Appl., 6 (2000), 503–531.

[67] Y. Meyer, *Ondelettes et opérateurs. II. Opérateurs de Calderón-Zygmund*, Actualités Mathématiques, Hermann, Paris, 1990.

[68] D. Mitrea, M. Mitrea and M. Taylor, *Layer Potentials, the Hodge Laplacian and Global Boundary Problems in Non-Smooth Riemannian Manifolds*, Memoirs of the American Mathematical Society, Vol. 150, No. 713, 2001.

[69] I. Mitrea, M. Mitrea, J. Pipher and M. Wright, *Optimal estimates for the inhomogeneous problem for the bi-Laplacian in three-dimensional Lipschitz domains*, preprint (2008).

[70] I. Mitrea and M. Mitrea, *Multiple layer potentials for higher order elliptic boundary value problems*, preprint (2008).

[71] M. Mitrea and M. Taylor, *Potential theory on Lipschitz domains in Riemannian manifolds: L^p, Hardy and Holder type results*, Communications in Analysis and Geometry, 9 (2001), 369–421.

[72] J. Nečas, *Les méthodes directes en théorie des équations élliptique*, Academia, Prague, 1967.

[73] J. Pipher and G.C. Verchota *Dilation invariant estimates and the boundary Grding inequality for higher order elliptic operators*, Ann. of Math., 142 (1995), no. 1, 1–38.

[74] J. Pipher and G. Verchota, *The Dirichlet problem in L_p for the biharmonic equation on Lipschitz domains*, Amer. J. Math., 114 (1992), 923–972.

[75] J. Pipher and G. Verchota, *A maximum principle for biharmonic functions in Lipschitz and C^1 domains*, Comment. Math. Helv., 68 (1993), no. 3, 385–414.

[76] H. Power and L.C. Wrobel, *Boundary Integral Methods in Fluid Mechanics*, Computational Mechanics Publications, Southampton, 1995.

[77] H. Rademacher, *Über partielle und totale Differenzierbarkeit von Funktionen mehrerer Variablen und über die Transformation der Doppelintegrale*, Math. Ann., 79 (1918), 340–359.

[78] J.M. Rallison, *The deformation of small viscous drops and bubbles in shear flows*, Ann. Rev. Fluid Mech., 16 (1984), 45–66.

[79] T. Runst and W. Sickel, *Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Operators*, de Gruyter, Berlin, New York, 1996.

[80] V. Rychkov, *On restrictions and extensions of the Besov and Triebel-Lizorkin spaces with respect to Lipschitz domains*, J. London Math. Soc., (2) 60 (1999), no. 1, 237–257.

[81] J.H. Shapiro, *The Riesz and Fredholm Theories in Linear Topological Spaces*, preprint, 1995.

[82] Z. Shen, *A note on the Dirichlet problem for the Stokes system in Lipschitz domains*, Proc. Amer. Math. Soc., 123 (1995), no. 3, 801–811.

[83] Z. Shen, *The L^p Dirichlet problem for elliptic systems on Lipschitz domains*, Math. Research Letters, 13 (2006), 143–159.

[84] Z. Shen, *The L^p Boundary Value Problems on Lipschitz domains*, Adv. Math., 216 (2007), 212–254.

[85] Z. Slodkowski, *A generalization of Vesentini and Wermer's theorems*, Rend. Sem. Mat. Univ. Padova, 75 (1986), 157–171.

[86] I. Ya.Šneiberg, *Spectral properties of linear operators in interpolation families of Banach spaces*, Mat. Issled., 9 (1974), 214–229.

[87] E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J. 1970.

[88] R. Temam, *Navier-Stokes Equations. Theory and Numerical Analysis*, Studies in Mathematics and its Applications, Vol. 2, North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977.

[89] H. Triebel, *Interpolation theory, function spaces, differential operators*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.

[90] H. Triebel, *Theory of Function Spaces*, Birkhäuser, Berlin, 1983.

[91] H. Triebel, *Function spaces on Lipschitz domains and on Lipschitz manifolds. Characteristic functions as pointwise multipliers*, Rev. Mat. Complut., 15 (2002), 475–524.

[92] W. Varnhorn, *The Stokes Equations*, Mathematical Research, Vol. 76, Akademie-Verlag, Berlin, 1994.

[93] N. Th. Varopoulos, *Zeros of H^p functions in several complex variables*, Pacific J. Math., 88 (1980), no. 1, 189–246.

[94] G. Verchota, *Layer potentials and boundary value problems for Laplace's equation in Lipschitz domains*, J. Funct. Anal., 59 (1984), 572–611.

[95] A. T. Vignati and M. Vignati, *Spectral theory and complex interpolation*, J. Funct. Anal., 80 (1988), 387–397.

[96] J.M. Wilson *On the Atomic Decomposition for Hardy Spaces*, Pacific Journal of Math., 116 (1985), no. 1, 201–207.

VITA

Matthew Wright was born June 6, 1980 in Kansas City, Missouri. After graduating from Raytown High School in 1998, Mattthew attended Truman State University in Kirksville, Missouri, where he received a Bachelors degree in mathematics and physics. He entered the graduate program in mathematics at the University of Missouri in 2002 and expects to receive his Doctor of Philosophy degree in May of 2008.