ALGEBRAIC RESOLUTION OF FORMAL IDEALS ALONG A VALUATION

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and hereby certify that in their opinion it is worthy of acceptance.

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RESOLUTION OF FORMAL IDEALS
ALONG A VALUATION

ABSTRACT

Let X be a possibly singular complete algebraic variety, defined over a field k of characteristic zero. X is nonsingular at $p \in X$ if $O_{X,p}$ is a regular local ring. The problem of resolution of singularities is to show that there exists a nonsingular complete variety X, which birationally dominates X. Resolution of singularities (in characteristic zero) was proved by Hironaka in 1964. The valuation theoretic analogue to resolution of singularities is local uniformization.

Let ν be a valuation of the function field of X, ν dominates a unique point p, on any complete variety Y, which birationally dominates X. The problem of local uniformization is to show that, given a valuation ν of the function field of X, there exists a complete variety Y, which birationally dominates X such that the center of ν on Y, is a regular local ring. Zariski proved local uniformization (in characteristic zero) in 1944. His proof gives a very detailed analysis of rank 1 valuations, and produces a resolution which reflects invariants of the valuation.

We extend Zariski's methods to higher rank to give a proof of local uniformization which reflects important properties of the valuation. We simultaneously resolve the centers of all the composite valuations, and resolve certain formal ideals associated to the valuation.

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Chapter 1

Introduction

1.1 History of the problem

One of the fundamental problems in Algebraic Geometry is the problem of Resolution of Singularities. If Y is a singular algebraic variety, a resolution of singularities of Y is a proper mapping $\phi : X \to Y$ which is an isomorphism on a dense open subset of Y, such that X is nonsingular. Hironaka [29] proved in 1964 that there exists a resolution of Y if Y is defined over a field of characteristic zero. The proof uses the existence of a hypersurface of maximal contact to reduce to an induction on the dimension of Y. There have been significant simplifications of this theorem in recent years, including Abramovich and de Jong [5], Bierstone and Milman [7], Bogomolov and Pantev [8], Bravo, Encinas and Villamayor [9], Encinas and Hauser [21], Encinas and Villamayor [22], Hauser [26], Kollár [35], Villamayor [44], and Wlodarczyk [45].

Since Zariski introduced general valuation theory into algebraic geometry, valuations have been important in addressing resolution problems.

Suppose that K is an algebraic function field over a base field k, and V is a valuation ring of K. V determines a unique center on a proper variety X whose function field is K. The valuation gives a way of reducing a global problem on X, such as resolution, to a local problem, studying the local rings of centers of V on different varieties X whose function field is K.

The valuation theoretic analogue of resolution of singularities is local uniformization. A variety X is nonsingular at a point p if and only if the local ring $O_{X,p}$ is a regular local ring.

The problem of local uniformization is to find a regular local ring R essentially of finite type over k with quotient field K such that the valuation ring V dominates R. That is, $R \subset V$ and $m_V \cap R = m_R$. In 1944, Zariski [48] proved local uniformization over fields of characteristic zero. To be precise, he proved:

Theorem 1.1.1. *(Zariski) Suppose that $f \in R$. Then there exists a birational extension of regular local rings $R \to R_1$ such that R_1 is dominated by ν, and $ord_{R_1}\overline{f} \leq 1$ where $\overline{f}$ is the strict transform of f in R_1. If ν has rank 1, then there exists R_1 such that $\overline{f}$ is a unit in R_1.*

Zariski first proved local uniformization for two-dimensional function fields over an algebraically closed field of characteristic zero in [47]. He later proved local uniformization for algebraic function fields of characteristic 0 in [48].

Abhyankar has proven local uniformization in positive characteristic for two dimensional function fields, surfaces and three dimensional varieties [1], [2].

One of the most important techniques in studying resolution problems is to pass to the completion of a local ring (the germ of a singularity). This allows us to reduce local questions on singularities to problems on power series.

The first question which arises on completion is if the following generalization of local uniformization is true:

Question 1.1.2. *Given $f \in \hat{R}$, does there exist a birational extension $R \to R_1$ where R_1 is a regular local ring dominated by V such that $ord_{\hat{R}_1}\overline{f} \leq 1$, where $\overline{f}$ is the strict transform of f in $\hat{R}_1$?*

The answer is surprisingly NO! We give a simple counter example to our question in Chapter 3, which comes from a discrete valuation.

The example can be understood in terms of an extension of our valuation ring V to a valuation ring dominating $\hat{R}$. It is a fact that the rank (page 7) of a valuation V dominating R often increases when extending the valuation to a valuation ring $\hat{V}$ dominating $\hat{R}$. Some papers where this is studied are Spivakovsky [41], Heinzer and Sally [28], and Cutkosky and Ghezzi [19].

The first measure of this increase of rank is the prime ideal

$$Q_{\hat{R}} = \{f \in \hat{R} \mid \nu(f) \geq n \text{ for all } n \in \mathbb{N}\}.$$

This prime has been previously defined and studied by Teissier [43], Cutkosky [16] and Spivakovsky.

If the rank $V = 1$, then there is a unique extension of V to the quotient field of $\hat{R}/Q_{\hat{R}}$ dominating $\hat{R}/Q_{\hat{R}}$. Thus the rank of the extension does not increase, and the prime ideal $Q_{\hat{R}}$ led to this obstruction.

In spite of the fact that we cannot resolve the singularity of $f = 0$ by a birational extension of R, we can resolve the formal singularity, whose local ring is $\hat{R}/Q_{\hat{R}}$, by a birational extension of R. This is proven for valuations of rank 1 by Cutkosky and Ghezzi [19].

Theorem 1.1.3. *(Cutkosky, Ghezzi) Suppose that rank $V = 1$. Then there exists a birational extension $R \to R_1$ where R_1 is a regular local ring dominated by ν such that $Q_{\hat{R}_1}$ is a regular prime*

Zariski proves local uniformization by constructing special birational extensions $R \to R_1$ dominated by rank 1 valuation rings which he calls Perron transforms. Cutkosky and Ghezzi make essential use of Perron transforms and Zariski's resolution algorithm in their proof of Theorem 1.1.3.

We extend Perron transforms to arbitrary rank in Chapter 5, and prove a strong form of local uniformization, which generalizes both Theorem 1.1.1 and Theorem 1.1.3.

1.2 Statement of the main result

First we introduce some notation which is necessary for the statement of out theorem. Let

$$(0) = P_V^t \subset \cdots \subset P_V^1 \subset P_V^0$$

be the chain of prime ideals in V. Let

$$(0) = P_R^t \subset \cdots \subset P_R^1 \subset P_R^0$$

be the induced chain of prime ideals in R, where $P_R^i = P_V^i \cap R$. Let ν_i be a valuation whose valuation ring is $V_i = V_{P_V^i}$.

Consider the following condition (1.1) on a Cauchy sequence $f = \{f_n\}$ in $\widehat{R_{P_R^i}}$.

$$\text{For all } l \in \mathbb{N}, \text{ there exists } n_l \in \mathbb{N} \text{ such that } \nu_i(f_n) \geq l\nu(m_{\widehat{R_{P_R^i}}}) \text{ if } n \geq n_l. \quad (1.1)$$

Define the prime ideal $Q_{\widehat{R_{P_R^i}}} \subset \widehat{R_{P_R^i}}$ for the valuation ring $V_{P_V^i}$ by

$$Q_{\widehat{R_{P_R^i}}} = \left\{ \; f \in \widehat{R_{P_R^i}} \mid \; f \text{ satisfies (1.1).} \; \right\}$$

Theorem 1.2.1. *Suppose that R is a local domain which is essentially of finite type over a field k of characteristic zero, and V is a valuation ring of the quotient field of R which dominates R. Let*

$$(0) = P_V^t \subset \cdots \subset P_V^0$$

be the chain of prime ideals of V*. Then there exists a birational extension* $R \to R_1$ *such that* R_1 *is a regular local ring and* V *dominates* R_1*. Further,* $P_{R_1}^i = P_V^i \cap R_1$ *are regular primes for all* i*, and*

$$Q_{\widehat{{R_1}_{P_{R_1}^i}}} \subset \widehat{{R_1}_{P_{R_1}^i}}$$

are regular primes for all i*.*

Chapter 2

Preliminaries

In this section, we introduce some notations and assumptions that will hold in chapters 4 and 5.

Suppose that T is a regular local ring of dimension q, which is essentially of finite type over a field k of characteristic zero, with maximal ideal m_T, such that T/m_T is an algebraic extension of k. Suppose that ν is a valuation of the quotient field of T which dominates T. Let $QF(T)$ be the quotient field of T.

By definition ν is a homomorphism $\nu : QF(T)^* \rightarrow \Gamma$ from the multiplicative group of $QF(T)$ onto an ordered abelian group Γ such that:

1. $\nu(ab) = \nu(a) + \nu(b)$ for $a, b \in QF(T)^*$,
2. $\nu(a+b) \geq \min \{\nu(a), \nu(b)\}$ for $a, b \in QF(T)^*$,
3. $\nu(c) = 0$ for $0 \neq c \in k$.

We extend ν to $QF(T)$ by setting $\nu(0) = \infty$.

Let $V = \{a \in QF(T) | \nu(a) \geq 0\}$. Then V is a ring and is called the valuation ring of ν.

Let $m_V = \{a \in QF(T) | \nu(a) > 0\}$. Then m_V is the unique maximal ideal of V.

The rank of a valuation ring V is the number of proper prime ideals in V, which are necessarily ordered by inclusion.

The rational rank of ν is the maximal number of rationally independent elements in Γ, which is bounded above by the dimension of T.

Let

$$(0) \subset \cdots \subset P_V \subset m_V$$

be the chain of prime ideals in V.

Let $\Gamma_1 \subset \Gamma_V$ be the rank 1 isolated subgroup of the valuation ring V/P_V. We have an embedding $\Gamma_1 \subset \mathbb{R}$ of ordered groups, such that $\mathbb{Z} \subset \Gamma_1$. In this way we identify the integers with a subgroup of Γ_1. We will sometimes say that an element $\gamma \in \Gamma_V$ is ∞ if $\gamma \notin \Gamma_1$, and $\gamma < \infty$ if $\gamma \in \Gamma_1$. If $\gamma \in \Gamma_1$, then there exists $n \in \mathbb{N}$ such that $\gamma \leq n$.

The maximal ideal of T is $m_T = m_V \cap T$.

In chapters 4 and 5, we assume that

$$\text{trdeg}_{T/m_T} V/m_V = 0. \tag{2.1}$$

Let $P_T = P_V \cap T$.

Suppose that V/P_V has rational rank s.

In chapters 4 and 5, we assume that

$$\text{trdeg}_{T_{P_T}/P_T T_{P_T}} V_{P_V}/P_V V_{P_V} = \text{trdeg}_{QF(T/P_T)} QF(V/P_V) = 0. \tag{2.2}$$

Chapter 3

Example

Example 3.0.2. *There exists a discrete valuation ring V dominating a regular local ring R of dimension 3 such that for all $r \geq 2$, there exists $f \in \hat{R}$ such that for all birational extensions $R \to R_1$ of regular local rings dominated by V, the strict transform of f has order $\geq r$ in $\hat{R}_1$.*

Proof. Let $t, \phi(t), \psi(t) \in k[[t]]$ be algebraically independent elements of positive order.

We have the inclusion $k(x, y, z) \hookrightarrow k((t))$ defined by $x = t, y = \phi(t)$ and $z = \psi(t)$. The order valuation on $k((t))$ (with valuation ring $k[[t]]$) restricts to a discrete rank 1 valuation on $k(x, y, z)$, dominating $R = k[x, y, z]_{(x,y,z)}$.

$Q_{\hat{R}} = (y - \phi(x), z - \psi(x)) \subset \hat{R}$ is a regular prime of height 2, and it defines a curve $\gamma \subset \text{Spec}\ (\hat{R})$.

$Q_{\hat{R}} \cap R = (0)$.

Suppose that $r \in \mathbb{N}(r \geq 2)$.

Let $f = (y - \phi(x))^r + (z - \psi(x))^{r+1} \in \hat{R}$.

$ord_{\hat{R}} f = r$ and $ord_\gamma f = ord_{(y-\phi(x), z-\psi(x))} f = r$.

Suppose that $R \to R_1$ is a birational extension where R_1 is a regular local ring dominated by ν.

In $\hat{R}_1$, write $y - \phi(x) = h_1 g_1$ and $z - \psi(x) = h_2 g_2$ where g_1 is the strict transform of $y - \phi(x)$ in $\hat{R}_1$, $h_1 \in R_1$ and g_2 is the strict transform of $z - \psi(x)$ in $\hat{R}_1$, $h_2 \in R_1$.

$\infty = \nu(y - \phi(x)) = \nu(h_1) + \nu(g_1)$ and $\nu(h_1) < \infty$ thus $\nu(g_1) = \infty$.

Similarly, $\nu(g_2) = \infty$.

Let $\overline{\gamma}$ be the strict transform of γ in $Spec(\hat{R}_1)$, $I_{\overline{\gamma}} = (g_1, g_2) = Q_{\hat{R}_1} \subset m_{\hat{R}_1}$

$$ord_{\hat{R}_1} \overline{f} \geq ord_{\overline{\gamma}} \overline{f} = ord_\gamma f = r$$

□

In the previous example, we have $f \in Q_{\hat{R}}$, and has value larger than any element in the value group.

This means that the rank of the valuation ring V must increase when passing to the completion $\hat{R}$.

Chapter 4

Resolution in highest height

4.1 Perron Transforms

Throughout this chapter we assume that the assumptions of chapter 2 hold. We will define two types of Peron Transforms.

Perron Transforms of type $(1, 0)$**:**

Suppose that

$$x_1, \ldots, x_s, \ldots, x_p, \ldots, x_q$$

is a regular system of parameters in T, such that $s \leq p$, $x_1, \ldots, x_p \notin P_T$, $x_{p+1}, \ldots, x_q \in P_T$, and $\nu(x_1), \ldots, \nu(x_s)$ are rationally independent.

Let $\tau_i = \nu(x_i)$ for $1 \leq i \leq s$.

We define two types of transforms of type $(1, 0)$:

- Transforms of type I.

  Set $\tau_i(0) = \tau_i$ for $1 \leq i \leq s$. For each positive integer h define s positive, rationally independent real numbers $\tau_1(h), \ldots, \tau_s(h)$ by the "Algorithm of

Perron" [48].

$$\begin{array}{rcl} \tau_1(h-1) & = & \tau_s(h) \\ \tau_2(h-1) & = & \tau_1(h) + a_2(h-1)\tau_s(h) \\ \vdots & & \\ \tau_s(h-1) & = & \tau_{s-1}(h) + a_s(h-1)\tau_s(h) \end{array}$$

Where

$$a_j(h-1) = \left[\frac{\tau_j(h-1)}{\tau_1(h-1)}\right], 2 \leq j \leq s$$

the greatest integer in $\frac{\tau_j(h)}{\tau_1(h)}$. There are nonnegative integers $A_i(h)$ such that

$$\tau_i = A_i(h)\tau_1(h) + A_i(h+1)\tau_2(h) + \cdots + A_i(h+s-1)\tau_s(h)$$

for $1 \leq i \leq s$.

Then $Det(A_i(h+j)) = (-1)^{h(s-1)}$ (See [48] page 385).

These numbers have the important property that

$$lim_{h\to\infty}\frac{A_i(h)}{A_1(h)} = \frac{\nu(x_i)}{\nu(x_1)} \tag{4.1}$$

we refer to [48] page 385.

Let $A_i(h+j) = a_{ij+1}$, and define:

$$\begin{array}{rcl} x_1 & = & x_1(1)^{a_{11}} \ldots x_s(1)^{a_{1s}} \\ \vdots & & \\ x_s & = & x_1(1)^{a_{s1}} \ldots x_s(1)^{a_{ss}} \\ x_{s+1} & = & x_{s+1}(1) \\ \vdots & & \\ x_q & = & x_q(1). \end{array} \tag{4.2}$$

Then $Det(a_{ij}) = \pm 1$ and $\nu(x_1(1)) = \tau_1(h), \ldots, \nu(x_s(1)) = \tau_s(h)$ are rationally independent. We necessarily have that $x_1(1), \ldots, x_s(1) \notin P_{T_1}$.

Define a transformation $T \to T_1$ of type I along ν by

$$T_1 = T[x_1(1), \ldots, x_s(1)]_{T[x_1(1),\ldots,x_s(1)]\cap m_V}.$$

T_1 is a regular local ring, $QF(T) = QF(T_1)$ and ν dominates T_1.

- Transforms of type II_r.

  Now we define $T \to T_1$ of type II_r along ν, we refer to [48] (with the restriction that $s+1 \le r \le p$), as follows :

Set $\nu(x_r) = \tau_r$. τ_r is rationally dependent on $\tau_1, \dots, \tau_s$ since the rational rank of ν is s. There are thus integers $\lambda, \lambda_1, \dots, \lambda_s$ such that $\lambda > 0, (\lambda, \lambda_1, \dots, \lambda_s) = 1$ and

$$\lambda\tau_r = \lambda_1\tau_1 + \cdots + \lambda_s\tau_s. \tag{4.3}$$

We first perform a transform $T \to \tilde{T}(1)$ of type I along ν where $\tilde{T}(1)$ has regular parameters $\tilde{x}_1(1), \dots, \tilde{x}_s(1)$ defined as in (4.2). Then $\nu(\tilde{x}_i(1)) = \tau_i(h)$ for $1 \le i \le s$, $v(\tilde{x}_r(1)) = \tau_r$. Set

$$\lambda_i(h) = \lambda_1 A_1(h+i-1) + \lambda_2 A_2(h+i-1) + \cdots + \lambda_s A_s(h+i-1)$$

for $1 \le i \le s$. Then

$$\lambda\tau_r = \lambda_1(h)\tau_1(h) + \cdots + \lambda_s(h)\tau_s(h).$$

Take h sufficiently large that all $\lambda_i(h) > 0$. This is possible by (4.1), since $\lambda_1\tau_1 + \cdots + \lambda_s\tau_s > 0$. We still have $(\lambda, \lambda_1(h), \dots, \lambda_s(h)) = 1$ since $\mathrm{Det}(A_i(h+j-1) = \pm 1$. After re-indexing the $\tilde{x}_i(1)$, we may suppose that $\lambda_1(h)$ is not divisible by λ. Let $\lambda_1(h) = \lambda\mu + \lambda'$, with $0 < \lambda' < \lambda$. Now perform the following transform $\tilde{T}(1) \to \tilde{T}(2)$ along ν defined by $\tilde{x}_1(1) = \tilde{x}_r(2)$,

$\tilde{x}_r(1) = \tilde{x}_1(2)\tilde{x}_r(2)^{\mu}$, and $\tilde{x}_i(1) = \tilde{x}_i(2)$ otherwise. Set $\tau_i' = \nu(\tilde{x}_i(2))$ for all i. $\tau_1', \ldots, \tau_s', \tau_r'$ are positive and

$$\lambda'\tau_r' = \lambda_1'\tau_1' + \cdots + \lambda_s'\tau_s'$$

where $\lambda_1' = \lambda, \lambda_i' = -\lambda_i(h)$ for $2 \leq i \leq s$. Thus we have achieved a reduction on λ, and by repeating the above procedure, we reduce to the case

$$\tau_r = \lambda_1\tau_1 + \cdots + \lambda_s\tau_s$$

In this case we define

$$\begin{array}{lcl} x_1 & = & N_1 \\ \vdots & & \\ x_s & = & N_s \\ x_r & = & N_1^{\lambda_1} \ldots N_s^{\lambda_s} N_r. \end{array}$$

Thus in the general case, there exists $a_{ij} \in \mathbb{N}$, $1 \leq i, j \leq s+1$ such that

$$\begin{array}{lcl} x_1 & = & N_1^{a_{11}} \ldots N_s^{a_{1s}} N_r^{a_{1s+1}} \\ \vdots & & \\ x_s & = & N_1^{a_{s1}} \ldots N_s^{a_{ss}} N_r^{a_{ss+1}} \\ x_r & = & N_1^{a_{s+11}} \ldots N_s^{a_{s+1s}} N_r^{a_{s+1s+1}}. \end{array}$$

where $Det(a_{ij}) = \pm 1$ and $\nu(N_1), \ldots, \nu(N_s)$ are positive and rationally independent, and $\nu(N_r) = 0$.

Define a transformation $T \to T_1$ of type II_r along ν by

$$T_1 = T[N_1, \ldots, N_s, N_r]_{T[N_1,\ldots,N_s,N_r]\cap m_V}.$$

We necessarily have that $N_1, \ldots, N_s, N_r \notin P_{T_1}$, $QF(T) = QF(T_1)$ and ν dominates T_1.

Perron Transforms of type $(2,0)$.

Suppose that

$$x_1, \ldots, x_s, \ldots, x_p, \ldots, x_q$$

is a regular system of parameters in T such that $\nu(x_1), \ldots, \nu(x_s)$ are rationally independent.

Suppose that $d_1, \ldots, d_s \in \mathbb{N}$ and that $x_j \in P_T$

Define a transformation $T \rightarrow T_1$ of type $(2,0)$ along ν by

$$T_1 = T[\frac{x_j}{x_1^{d_1} \ldots x_s^{d_s}}]_{T[\frac{x_j}{x_1^{d_1} \ldots x_s^{d_s}}] \cap m_V}.$$

In all cases, by (2.1), we have that $dim(T_1) = dim(T)$ and T_1/m_{T_1} is a finite extension of T/m_T.

For an ideal $I \subset T$, let

$$\nu(I) = \min\{\nu(f) \mid f \in I\}.$$

Consider the following condition (4.4) on a Cauchy sequence $\{f_n\}$ in T.

$$\text{For all } l \in \mathbb{N}, \text{ there exists } n_l \in \mathbb{N} \text{ such that } \nu(f_n) \geq l\nu(m) \text{ if } n \geq n_l. \qquad (4.4)$$

Lemma 4.1.1. *1. Suppose that $\{f_n\}$ and $\{g_n\}$ are two Cauchy sequences in T converging to $f \in \hat{T}$ and $\{f_n\}$ satisfies (4.4). Then $\{g_n\}$ satisfies (4.4).*

2. Let

$$Q_{\hat{T}} = \left\{ \begin{array}{ll} f \in \hat{T} \mid & \text{A Cauchy sequence } \{f_n\} \text{ in } T \text{ which} \\ & \text{converges to } f \text{ satisfies (4.4)} \end{array} \right\}.$$

Then $Q_{\hat{T}}$ is a prime ideal in $\hat{T}$.

3. $Q_{\hat{T}} \cap T = P_T$.

Suppose that $f \in \hat{T}$ is not in $Q_{\hat{T}}$. Let $\{f_n\}$ be a Cauchy sequence in T which converges to f. There exists an $l \in \mathbb{N}$ such that there are arbitrarily large n with $\nu(f_n) < l\nu(m)$. We can thus choose n_0 such that $f_n - f_{n_0} \in m^l$ if $n \geq n_0$ and $\nu(f_{n_0}) < l\nu(m)$. For $n \geq n_0$, we have $f_n = f_{n_0} + h$ with $h \in m^l$. $\nu(h) \geq l\nu(m) > \nu(f_{n_0})$ implies $\nu(f_n) = \nu(f_{n_0})$ for $n \geq n_0$.

A similar calculation shows that the above value $\nu(f_{n_0})$ is independent of choice of n_0 satisfying the above conditions, and is independent of choice of Cauchy sequence converging to f.

We may thus define $\nu(f) = \nu(f_n)$ for sufficiently large n, if $f \in \hat{T} - Q_{\hat{T}}$. We will sometimes write $\nu(f) = \infty$ if $f \in Q_{\hat{T}}$. Observe that an extension of ν to a valuation of $QF(\hat{T})$ which dominates $\hat{T}$ is uniquely determined on elements of $\hat{T} - Q_{\hat{T}}$. We now identify ν with an extension of ν to QF($\hat{T}$) which dominates $\hat{T}$.

Let $\sigma(T) = dim(\hat{T}/Q_{\hat{T}})$ and $\tau(T) = dim(T/P_T)$.

We have $\sigma(T) \leq \tau(T)$ with equality if $Q_{\hat{T}} = P_T\hat{T}$.

Let

$$\omega(T) = dim(T) - dim_{T/m_T}(P_T/m_T^2 \cap P_T).$$

We have

$$dim_{T/m_T}(P_T/m_T^2 \cap P_T) \leq \text{height } P_T$$

with equality if and only if T/P_T is a regular local ring.

Thus $w(T) \geq dim(T) - \text{height } P_T = dim(T/P_T)$ and $w(T) = dim(T/P_T)$ if and only if T/P_T is a regular local ring.

4.2 Etale Perron Transforms

Suppose that $x_1, \ldots, x_s, \ldots, x_p, \ldots, x_q$ is a system of regular parameters in T, such that $x_1, \ldots, x_s, \ldots, x_p \notin P_T$, $x_{p+1}, \ldots, x_q \in P_T$ and $\nu(x_1), \ldots, \nu(x_s)$ are rationally independent.

Let $\beta_0 \in \hat{T}$ be a primitive element of T/m_T over k. Let

$$U = T[\beta_0]_{m_{\hat{T}} \cap T_{[\beta_0]}} \subset \hat{T}.$$

We thus have that the subfield $k[\beta_0] \subset U$ is a coefficient field of U. We will identify $k[\beta_0]$ with U/m_U.

ν extends to a valuation of $\mathrm{QF}(U)$ which dominates U by restricting our extension of ν to $\mathrm{QF}(\hat{T})$ to $\mathrm{QF}(U)$. ν is uniquely determined on elements of U which are not in $Q_{\hat{U}} = Q_{\hat{T}}$.

For the moment, let us call our extension of ν to $QF(\hat{T})$ $\hat{\nu}$ and let $\hat{V}$ be the valuation ring of $\hat{\nu}$. Let ν' be the restriction of $\hat{\nu}$ to $QF(U)$ with valuation ring $V' = \hat{V} \cap QF(U)$.

Since $QF(U)$ is finite over $QF(T)$, ν and ν' both have the same rank and rational rank.

Thus the chain of prime ideals in V' is

$$(0) \subset \cdots \subset p_{V'} \subset m_{V'}$$

where $p_{V'} \cap QF(T) = p_V$, and the rank 1 valuation rings $V'/p_{V'}$ and V/p_V have the same rational rank s.

Let $p_U = p_{V'} \cap U$.

We have $\tau(U) = \tau(T)$ by (2.2) and $w(U) \leq w(T)$ with equality if $P_U = P_T U$.

Let $p = \omega(T)$ and $p_0 = \omega(U)$, we have $p_0 \leq p$.

We also define two types of etale Perron transforms:

- Etale Perron transform of type I:

  We have that $x_1, \ldots, x_s \notin P_U$.

  Let

$$U \to U_1 = U[x_1(1), \ldots, x_s(1)]_{m_V \cap U[x_1(1), \ldots, x_s(1)]}$$

  be a transformation of type I along ν. Define $x_1(1), \ldots, x_s(1)$ by:

$$\begin{array}{lcl} x_1 & = & x_1(1)^{a_{11}} \ldots x_s(1)^{a_{1s}} \\ \vdots & & \\ x_s & = & x_1(1)^{a_{s1}} \ldots x_s(1)^{a_{ss}} \\ x_{s+1} & = & x_{s+1}(1) \\ \vdots & & \\ x_q & = & x_q(1) \end{array}$$

  Where $Det(a_{ij}) = \pm 1$.

  We have that the field $k[\beta_0] \subset U_1$ has the property that $k[\beta_0] = U/m_U = U_1/m_{U_1}$. We further have that $U \to U_1$ is a birational extension.

  We identify ν with an extension of ν to the Quotient field of $\hat{U}_1$ which dominates $\hat{U}_1$. Let $p_1 = \omega(U_1)$. We have that

$$s \leq p_1 = \omega(U_1) \leq \omega(U) = p_0 \leq w(T) = p.$$

  We have that

$$x_1(1), \ldots, x_s(1), x_{s+1}, \ldots, x_q$$

a regular system of parameters in U_1, such that $\nu(x_1(1)),\ldots,\nu(x_s(1))$ are rationally independent.

$$T \to U \to U_1$$

(with the regular parameters $x_1,\ldots,x_s,\ldots,x_q$ and $x_1(1),\ldots,x_s(1),\ x_{s+1},\ldots,x_q$) is called an etale Perron transform of type I along ν.

- Etale Perron transform of type $II_r : (s+1 \leq r \leq p)$

Let $\lambda(t_1,\ldots,t_{r-1})$ be a polynomial in the polynomial ring $U/m_U[t_1,\ldots,t_{r-1}]$, in the variables $t_1,\ldots,t_{r-1}$.

We have $x_1,\ldots,x_s \notin P_U$ and we assume that

$$x'_r = x_r - \lambda(x_1,\ldots,x_{r-1}) \notin Q_{\hat{U}}.$$

Let

$$U \to U' = U[N_1,\ldots,N_s,N_r]_{m_V \cap U[N_1,\ldots,N_s,N_r]}$$

be a transformation of type II_r along ν. Define $N_1,\ldots,N_s,N_r$ by

$$\begin{aligned}
x_1 &= N_1^{a_{11}}\ldots N_s^{a_{1s}} N_r^{a_{1s+1}}\\
&\vdots\\
x_s &= N_1^{a_{s1}}\ldots N_s^{a_{ss}} N_r^{a_{ss+1}}\\
x'_r &= N_1^{a_{s+11}}\ldots N_s^{a_{s+1s}} N_r^{a_{s+1s+1}}
\end{aligned}$$

where $Det(a_{ij}) = \pm 1$, $\nu(N_1), \ldots, \nu(N_s)$ are rationally independent and $\nu(N_r) = 0$. Define $x_1(1) = N_1, \ldots, x_s(1) = N_s, \overline{x}_r(1) = N_r$.

Let $\beta_1 \in \hat{U}'$ be a primitive element of $U'/m_{U'}$ over k. Let

$$U_1 = U'[\beta_1]_{m_{\hat{U}'} \cap U'[\beta_1]}.$$

We identify ν with an extension of ν to the Quotient field of $\hat{U}_1$ which dominates $\hat{U}_1$.

Let $\overline{\alpha}$ be the residue of $\overline{x}_r(1)$ in U_1/m_{U_1}. Let $\tilde{x}_i(1) = x_i$ if $s < i \le p$, $i \ne r$ and $\tilde{x}_r(1) = \overline{x}_r(1) - \overline{\alpha}$.

$x_1(1), \ldots, x_s(1), \tilde{x}_{s+1}(1), \ldots, \tilde{x}_r(1), \ldots, \tilde{x}_p(1)$, $x_{p+1}, \ldots, x_q$ are regular parameters in U_1. We necessarily have that $\nu(x_1(1)), \ldots, \nu(x_s(1))$ are rationally independent and $x_1(1), \ldots, x_s(1) \notin P_{U_1}$.

Let $p_1 = \omega(U_1)$. We have that

$$s \le p_1 = \omega(U_1) \le \omega(U) = p_0 \le \omega(T) = p.$$

We choose a regular system of parameters

$$x_1(1), \ldots, x_{p_1}(1), \ldots, x_p(1), x_{p+1}, \ldots, x_q \tag{4.5}$$

in U_1 such that

$x_1(1), \ldots, x_{p_1}(1) \notin P_{U_1}$, $x_{p_1+1}(1), \ldots, x_p(1), x_{p+1}, \ldots, x_q \in P_{U_1}$,

and there exists a 1-1 map $\sigma : \{s+1, \ldots, p_1\} \to \{s+1, \ldots, p\}$ such that $x_i(1) = \tilde{x}_{\sigma(i)}(1)$ for $s+1 \le i \le p_1$.

$$T \to U \to U_1$$

(with the regular parameters $x_1, \ldots, x_s, \ldots, x_p, \ldots, x_q$ and $x_1(1), \ldots,\ x_s(1), \ldots, x_p(1), x_{p+1}, \ldots, x_q$) is called an etale Perron transform of type II_r along ν.

We say that $T \to U \to U_1$ is an etale Perron transform along ν, if it is an etale Perron transform of type I or II_r.

If $T \to U \to U_1$ is an etale Perron transform along ν, we have $dim(T) = dim(U) = dim(U_1)$ and U_1/m_{U_1} is finite over T/m_T by (2.1), $\sigma(T) = \sigma(U) \geq \sigma(U_1)$, $\tau(T) = \tau(U) = \tau(U_1)$, by (2.2) and $\omega(T) \geq \omega(U) \geq \omega(U_1)$.

Suppose that

$$T \to U \to U_1 \to U_2 \to \cdots \to U_n$$

is a sequence of etale Perron transforms along ν. Further suppose that $\sigma(U_n) = \sigma(T)$ and $\tau(U_n) = \tau(T)$. Let $g = 0$ be a local equation of the exceptional locus of $\text{spec}(U_n) \to \text{spec}(T)$. Suppose that $J \subset T$ is an ideal. We define the strict transform $\overline{J}$ of J in U_n by

$$\overline{J} = \cup_{j=1}^{\infty}(JU_n : g^j U_n).$$

For an ideal $I \subset \hat{T}$, we define the strict transform $\overline{I}$ of I in $\hat{U}_n$ by

$$\overline{I} = \cup_{j=1}^{\infty}(I\hat{U}_n : g^j \hat{U}_n).$$

Definition 4.2.1. We will say that T has property (A) if whenever T_0 is an algebraic regular local ring of K, such that V dominates T, T dominates T_0, and

$$T_0 \to U \to U_1 \to \cdots \to U_n \tag{4.6}$$

is a sequence of etale Perron transforms along ν, then

$$\sigma(U_n) = \sigma(T).$$

Given a sequence (4.6), we have that $\tau(U_n) = \tau(T)$ by (2.2).

From now on in this section, we assume that T satisfies property (A). Then for a sequence (4.6), we have that P_{U_n} is an irreducible component of the strict transform of P_T in U_n and $Q_{\hat{U}_n}$ is an irreducible component of the strict transform of $Q_{\hat{T}}$ in $\hat{U}_n$.

4.3 Structure theorems

Theorem 4.3.1. *Let $p = \omega(T)$. Suppose that $x_1, \ldots, x_p, x_{p+1}, \ldots, x_q$ are regular parameters in T such that $x_1, \ldots, x_p \notin P_T$, $x_{p+1}, \ldots, x_q \in P_T$ and $\nu(x_1), \ldots, \nu(x_s)$ are rationally independent. Suppose that*

$$T \to U \to U_1 \to \cdots \to U_n$$

is a sequence of etale Perron transforms along ν (where we have identified ν with an extension of ν to $QF(U_n)$ which dominates U_n). Let $k_n \cong U_n/m_{U_n}$ be an coefficient field of U_n. Suppose that the prescribed regular parameters of U_n are $x_1(n), \ldots, x_s(n), \ldots, x_{p_n}(n), \ldots, x_p(n), x_{p+1}, \ldots, x_q$, where $p_n = \omega(U_n)$, and $\nu(x_1(n)), \ldots, \nu(x_s(n))$ are rationally independent which satisfy

$$x_i = \phi_i^n(x_1(n), \ldots, x_p(n), x_{p+1}, \ldots, x_q)$$

for $1 \leq i \leq p$*, where* $\phi_i^n \in k_n[[x_1(n), \ldots, x_p(n), x_{p+1}, \ldots, x_q]]$*.*

Then there exists $l_0 \in \mathbb{N}$ *such that for all* $l \geq l_0$*, there exists a sequence of transforms* $T \to T_{n,l}$ *along* ν *of type* $(1,0)$ *with the following properties:*

1. $T_{n,l}/m_{T_{n,l}} \cong k_n$*.*

2. $T_{n,l}$ *has regular parameters* $x_{1,l}(n), \ldots, x_{p_n,l}(n), \ldots, x_{p,l}(n), x_{p+1}, \ldots, x_q$ *such that there exists* $h_i \in m^l_{\hat{T}_{n,l}}$ *satisfying*

$$x_i = \phi_i^n(x_{1,l}(n), \ldots, x_{p,l}(n), x_{p+1}, \ldots, x_q) + h_i$$

   for $1 \leq i \leq p$*.*

3. *For* $1 \leq i \leq p_n$*,* $\nu(x_{i,l}(n)) = \nu(x_i(n)) < l_0$

4. $\nu(x_{i,l}(n)) > l$ *if* $p_n < i \leq p$*.*

5. $\nu(x_i(n) - x_{i,l}(n)) > l$ *for* $1 \leq i \leq p$*.*

Proof. We prove the theorem by induction on the length of a factorization of $U \to U_n$ into a sequence of etale Perron transforms along ν.

We thus assume that the conclusions of the theorem are true for a sequence of etale Perron transforms $U \to U_1$ along ν (which could be more than one), and $U_1 \to U_2$ is a single etale Perron transform along ν. Let $k_1 = U_1/m_{U_1}$ and $k_2 = U_2/m_{U_2}$.

We have prescribed regular parameters $x_1, \ldots, x_p, \ldots, x_q$ in U. Let $p_1 = \omega(U_1)$. Assume that $x_1(1), \ldots, x_{p_1}(1), \ldots, x_p(1), x_{p+1}, \ldots, x_q$ are the prescribed regular parameters in U_1. There exist

$$\phi_i^1 \in U_1/m_1[[x_1(1), \ldots, x_p(1), x_{p+1}, \ldots, x_q]]$$

such that

$$x_i = \phi_i^1(x_1(1), \ldots, x_p(1), x_{p+1}, \ldots, x_q)$$

for $1 \le i \le p$.

By induction, there exists a positive integer $l_0(1)$ such that for all $l(1) \ge l_0(1)$, there exist $T_{1,l(1)}$ such that (1) - (5) of the conclusions of the theorem hold for $U \to U_1$ and $T \to T_{1,l(1)}$.

(3) implies that $l_0(1) > \max\{\nu(x_1(1)), \ldots, \nu(x_{p_1}(1))\}$.

It also follows that $\nu(x_{i,l(1)}(1)) = \nu(x_i(1))$ for $1 \le i \le p_1$ and $\nu(m_{U_1}) = \nu(m_{T_{1,l(1)}})$.

We assume that $U_1 \to U_2$ is of type II_r. The case when $U_1 \to U_2$ is of type I is similar. $U_1 \to U_2$ is then defined as follows. There exists j with $s+1 \le j \le p_1$ and

$$\begin{aligned}\lambda(x_1(1), \ldots, &x_{j-1}(1), x_{j+1}(1), \ldots, x_p(1)) \\ &\in U_1/m_{U_1}[x_1(1), \ldots, x_{j-1}(1), x_{j+1}(1), \ldots, x_p(1)]\end{aligned}$$

such that

$$x_j(1)' = x_j(1) - \lambda(x_1(1), \ldots, x_{j-1}(1), x_{j+1}(1), \ldots, x_p(1))$$

and we have regular parameters

$x_1(2), \ldots, x_s(2), x_{s+1}(1), \ldots, x_{j-1}(1), \tilde{x}_j(2), x_{j+1}(1), \ldots, x_{p_1}(1), \ldots, x_p(1),$

$x_{p+1}, \ldots, x_q$ in U_2 where $x_1(2), \ldots, x_s(2)$ and $\tilde{x}_j(2)$ are defined by

$$\begin{array}{lll} x_1(1) & = & x_1(2)^{a_{11}} \ldots x_s(2)^{a_{1s}}(\tilde{x}_j(2)+\overline{\alpha}_2)^{a_{1s+1}} \\ \vdots & & \\ x_s(1) & = & x_1(2)^{a_{s1}} \ldots x_s(2)^{a_{ss}}(\tilde{x}_j(2)+\overline{\alpha}_2)^{a_{ss+1}} \\ x_j(1)' & = & x_1(2)^{a_{s+11}} \ldots x_s(2)^{a_{s+1s}}(\tilde{x}_j(2)+\overline{\alpha}_2)^{a_{s+1s+1}} \end{array} \tag{4.7}$$

for appropriate $0 \neq \overline{\alpha}_2 \in U_2/m_2$ where $Det(a_{ij}) = \pm 1$.

Let $(b_{ij}) = (a_{ij})^{-1}$.

Let $d = \nu(x'_j(1)) < \infty$. There exists n such that $n\nu(m_{\hat{U}_1}) > d$.

Let $l_0(2)$ be chosen so that

$$l_0(2) > max\ \{\nu(x_{p_1+1}(1)), \ldots, \nu(x_s(1)), \nu(\tilde{x}_j(2)), l_0(1), l_0(1)\sum_{k=1}^{s} b_{ik}; i = 1, \ldots, s\}.$$

Suppose $l \geq l_0(2)$.

Let $\xi_1 = max\{l - [\sum_{i=1, i\neq k}^{s} b_{s+1i}\nu(x_i(1)) + (b_{s+1k} - 1)\nu(x_k(1))], k = 1, \ldots, s\}$.

Let $\xi_2 = max\{l - [b_{ts+1}\nu(x_j(1)') + \sum_{i=1, i\neq k}^{s} b_{ti}x_i(1) + (b_{tk} - 1)\nu(x_k(1))]\}$

where the maximum is over $k = 1, \ldots, s$ and $t = 1, \ldots, s$.

Let $\xi_3 = max\{l - [(b_{ks+1} - 1)\nu(x_j(1)') + \sum_{i=1}^{s}(b_{ki})\nu(x_i(1))], k = 1, \ldots, s\}$.

Choose $l(1)$ such that $l(1) \geq max\{l_0(2), d, l_0(1), \xi_1, \xi_2, \xi_3\}$.

Choose $n_0 \in \mathbb{N}$ such that $n_0\nu(m_{U_1}) > l(1)$, and $n_0 > l$.

Choose $n_1 \in \mathbb{N}$ such that $n_1\nu(m_{U_2}) > l$.

There exists $\lambda_l \in T_{1,l(1)}$ such that

$$\lambda_l = \lambda(x_{1,l(1)}(1), \ldots, x_{j-1,l(1)}(1), x_{j+1,l(1)}(1), \ldots, x_{p_1,l(1)}(1), \ldots\ldots, x_{p,l(1)}(1)) + h'$$

with $h' \in m^{n_0}_{\hat{T}_{1,l(1)}}$. Let $x_{j,l(1)}(1)' = x_{j,l(1)}(1) - \lambda_l$. There exist $g_{i,l(1)} \in QF(U_1)$ such that $x_{i,l(1)}(1) = x_i(1) + g_{i,l(1)}$ for $1 \leq i \leq p$, and $\nu(g_{i,l(1)}) > l(1)$.

$$\begin{aligned}
x_{j,l(1)}(1)' &= x_{j,l(1)}(1) - \lambda_l \\
&= x_{j,l(1)}(1) - \lambda(x_{1,l(1)}(1), \ldots, x_{j-1,l(1)}(1), x_{j+1,l(1)}(1), \ldots, x_{p,l(1)}(1)) - h' \\
&= x_j(1) + g_{j,l(1)} - \lambda(x_1(1) + g_{1,l(1)}, \ldots, x_p(1) + g_{p,l(1)}) - h' \\
&= x_j(1) - \lambda(x_1(1), \ldots, x_p(1)) + \textstyle\sum_{i=1}^{p} g_{i,l(1)}\Lambda_i - h' \\
&= x_j(1)' + \textstyle\sum_{i=1}^{p} g_{i,l(1)}\Lambda_i - h',
\end{aligned}$$

where $\Lambda_i \in k_1[g_{1,l(1)}, \ldots, g_{p,l(1)}, x_1(1), \ldots, x_p(1)]$. Thus $\nu(\Lambda_i) \geq 0$ for $1 \leq i \leq p$.

We have

$$\nu(\sum_{i=1}^{p} g_{i,l(1)}\Lambda_i) > min\{g_{i,l(1)}\} > l(1) > d,$$

and

$$\nu(h') > n_0\nu(m_{T_{1,l(1)}}) = n_0\nu(m_{U_1}) > l(1) > d.$$

Thus $\nu(x_{j,l(1)}(1)') = d = \nu(x_j(1)')$.

Let $g'_{j,l(1)} = x_{j,l(1)}(1)' - x_j(1)'$.

$$x_{j,l(1)}(1)' - x_j(1)' = \sum_{i=1}^{p} g_{i,l(1)}\Lambda_i - h'.$$

Thus $\nu(g'_{j,l(1)}) > l(1)$.

With a_{ij} as defined in (4.7), set

$$\begin{array}{rcl} x_{1,l(1)}(1) & = & x_{1,l}(2)^{a_{11}} \ldots x_{s,l}(2)^{a_{1s}} \overline{x}_{j,l}(2)^{a_{1s+1}} \\ \vdots & & \\ x_{s,l(1)}(1) & = & x_{1,l}(2)^{a_{s1}} \ldots x_{s,l}(2)^{a_{ss}} \overline{x}_{j,l}(2)^{a_{ss+1}} \\ x_{j,l(1)}(1)' & = & x_{1,l}(2)^{a_{s+11}} \ldots x_{s,l}(2)^{a_{s+1s}} \overline{x}_{j,l}(2)^{a_{s+1s+1}}. \end{array}$$

Let $(b_{ij}) = (a_{ij})^{-1}$.

We have

$$\begin{array}{rcl} x_{1,l}(2) & = & x_{1,l(1)}(1)^{b_{11}} \ldots x_{s,l(1)}(1)^{b_{1s}} [x_{j,l(1)}(1)']^{b_{1s+1}} \\ \vdots & & \\ x_{s,l}(2) & = & x_{1,l(1)}(1)^{b_{s1}} \ldots x_{s,l(1)}(1)^{b_{ss}} [x_{j,l(1)}(1)']^{b_{ss+1}} \\ \overline{x}_{j,l}(2) & = & x_{1,l(1)}(1)^{b_{s+11}} \ldots x_{s,l(1)}(1)^{b_{s+1s}} [x_{j,l(1)}(1)']^{b_{s+1s+1}}. \end{array}$$

$\nu(x_{j,l(1)}(1)') = \nu(x_j(1)')$ and $\nu(x_{i,l(1)}(1)) = \nu(x_i(1))$ for $i = 1, \ldots, s$.

Thus $\nu(x_{i,l}(2)) = \nu(x_i(2)) > 0$ for $i = 1, \ldots, s$ and $\nu(\overline{x}_{j,l}(2)) = 0$.

Thus $T_{1,l(1)}[x_{1,l}(2), \ldots, x_{s,l}(2), \overline{x}_{j,l}(2)] \subset V$.

Let

$$T_{2,l} = T_{1,l(1)}[x_{1,l}(2), \ldots, x_{s,l}(2), \overline{x}_{j,l}(2)]_{T_{1,l(1)}[x_{1,l}(2), \ldots, x_{s,l}(2), \overline{x}_{j,l}(2)] \cap m_V}.$$

Let $\overline{V}$ be the valuation ring of an extension of ν to $\mathrm{QF}(U_2)$ which dominates U_2. Let $L = T_{2,l}/m_{T_{2,l}} = k_1[\beta] \subset \overline{V}/m_{\overline{V}}$, where β is the residue of $\overline{x}_{j,l}(2)$ in L.

$$\begin{aligned}\beta &= \left[\frac{[x_{j,l(1)}(1)']^{b_{s+1s+1}}}{x_{1,l(1)}(1)^{b_{s+11}}\dots x_{s,l(1)}(1)^{b_{s+1s}}}\right]\\ &= \left[\frac{(x_j(1)'+g'_{j,l(1)})^{b_{s+1s+1}}}{(x_1(1)+g_{1,l(1)})^{b_{s+11}}\dots(x_s(1)+g_{s,l(1)})^{b_{s+1s}}}\right]\\ &= \left[\frac{(x_j(1)')^{a_1}}{x_1(1)^{b_{s+11}}\dots x_s(1)^{b_{s+1s}}}\right] = \overline{\alpha}_2,\end{aligned}$$

since

$$\nu(g_{i,l(1)}) > l(1) > l_0(2) > \nu(x_i(1)),$$

for $i = 1\dots, s$.

and

$$\nu(g'_{j,l(1)}) > l(1) > d = \nu(x_j(1)').$$

Thus

$$T_{2,l}/m_{T_{2,l}} = k_1[\overline{\alpha}_2] = U_2/m_{U_2} = k_2 \subset \overline{V}/m_{\overline{V}}.$$

And $x_{1,l}(2), \dots, x_{s,l}(2), x_{s+1,l(1)}(1), \dots, x_{j-1,l(1)}(1), \tilde{x}_{j,l}(2), x_{j+1,l(1)}(1), \dots, x_{p_1,l(1)}(1), \dots, x_{p,l(1)}(1), x_{p+1}, \dots, x_q$ are regular parameters in $\hat{T}_{2,l}$, where $\tilde{x}_{j,l}(2) = \overline{x}_{j,l}(2) - \overline{\alpha}$.

Let $p_2 = \omega(U_2)$. We have a prescribed regular system of parameters

$$x_1(2), \dots, x_{p_2}(2), x_{p_2+1}(2), \dots, x_{p_1}(2), x_{p_1+1}(1), \dots, x_p(1), x_{p+1}, \dots, x_q$$

of U_2, where

$$x_{p_2+1}(2), \dots, x_{p_1}(2), x_{p_1+1}(1), \dots, x_p(1), x_{p+1}, \dots, x_q$$

are a basis of $P_{U_2}/P_{U_2} \cap m^2_{U_2}$. There exists a 1-1 map

$$\sigma : \{s+1, \dots, p_2\} \to \{s+1, \dots, p_1\}$$

such that $x_i(2) = x_{\sigma(i)}(1)$ if $s+1 \leq i \leq p_2$ and $\sigma(i) \neq j$, and $x_i(2) = \tilde{x}_j(2)$ if $\sigma(i) = j$.

Let

$$\psi_i(x_1(2), \ldots, x_s(2), x_{s+1}(1), \ldots, x_{j-1}(1), \tilde{x}_j(2), \ldots, x_p(1))$$
$$\in U_2/m_2[[x_1(2), \ldots, x_s(2), x_{s+1}(1), \ldots, \tilde{x}_j(2), \ldots, x_p(1), x_{p+1}, \ldots, x_q]]$$

be defined by

$$\psi_i = \phi_i^1(x_1(2)^{a_{11}} \ldots x_s(2)^{a_{1s}}(\tilde{x}_j(2)+\overline{\alpha}_2)^{a_{1s+1}}, \ldots, x_1(2)^{a_{s1}} \ldots x_s(2)^{a_{ss}}$$

$$(\tilde{x}_j(2)+\overline{\alpha}_2)^{a_{ss+1}}, x_{s+1}(1), \ldots, x_{j-1}(1), x_1(2)^{a_{s+11}} \ldots x_s(2)^{a_{s+1s}}(\tilde{x}_j(2)+$$

$$\overline{\alpha}_2)^{a_{s+1s+1}} + \lambda(x_1(2)^{a_{11}} \ldots x_s(2)^{a_{1s}}(\tilde{x}_j(2)+\overline{\alpha}_2)^{a_{1s+1}}, \ldots, x_1(2)^{a_{s1}} \ldots x_s(2)^{a_{ss}}$$

$$(\tilde{x}_j(2)+\overline{\alpha}_2)^{a_{ss+1}}, x_{s+1}(1), \ldots, x_{j-1}(1), x_{j+1}(1), \ldots, x_p(1)), x_{j+1}(1), \ldots, x_p(1),$$

$$x_{p+1}, \ldots, x_q).$$

There exist series

$$\Delta_i(x_1(2), \ldots, x_{p_2}(2), x_{p_2+1}(2), \ldots, x_q) \in U_2/m_{U_2}[[x_1(2), \ldots, x_p(2), x_{p+1}, \ldots, x_q]]$$

for $1 \leq i \leq p$, such that

$$x_1(2) = \Delta_1, \ldots, x_s(2) = \Delta_s,$$

$$x_{s+1}(1) = \Delta_{s+1}, \ldots, x_{j-1}(1) = \Delta_{j-1},$$

$$\tilde{x}_j(2) = \Delta_j,$$

$$x_{p+1}(1) = \Delta_{p+1}, \ldots, x_p(1) = \Delta_p.$$

We have

$$x_i = \phi_i^2(x_1(2), x_2(2), \ldots, x_p(2), x_{p+1}, \ldots, x_q)$$

for $1 \le i \le p$ are defined by

$$\phi_i^2(x_1(2), x_2(2), \ldots, x_p(2), x_{p+1}, \ldots, x_q) = \psi_i(\Delta_1, \Delta_2, \ldots, \Delta_p, x_{p+1}, \ldots, x_q)$$
$$\in U_2/m_2[[x_1(2), \ldots, x_{p_2}(2), \ldots, x_p(2), x_{p+1}, \ldots, x_q]].$$

Invert the series Δ_i to get series

$$E_i(x_1(2), \ldots, x_{s+1}(2), x_{s+1}(1), \ldots, x_{j-1}(1), \tilde{x}_j(2), x_{j+1}(1), \ldots, x_p(1), x_{p+1}, \ldots, x_q)$$

such that $x_i(2) = E_i$ for $1 \le i \le p$.

There exist $h_i \in m_{\hat{T}_{2,l}}^{n_1}$ such that

$$E_i(x_{1,l}(2), \ldots, x_{s,l}(2), x_{s+1,l(1)}(1), \ldots, x_{j-1,l(1)}(1), \tilde{x}_{j,l}(2),$$
$$x_{j+1,l(1)}(1), \ldots, x_{p,l(1)}(1), x_{p+1}, \ldots, x_q) - h_i \in T_{2,l}.$$

Define regular parameters $x_{1,l}(2), \ldots, x_{p,l}(2), x_{p+1}, \ldots, x_q$ in $T_{2,l}$ by

$$x_{i,l}(2) = E_i(x_{1,l}(2), \ldots, x_{s,l}(2), x_{s+1,l(1)}(1), \ldots, x_{j-1,l(1)}(1),$$
$$\tilde{x}_{j,l}(2), x_{j+1,l(1)}(1), \ldots, x_{p,l(1)}(1), x_{p+1}, \ldots, x_q) - h_i \text{ for } 1 \le i \le p.$$

Now we show that for $1 \le i \le p$, $x_i = \phi_i^2(x_{1,l}(2), \ldots, x_{p,l}(2), x_{p+1}, \ldots, x_q) + g_i$, with $g_i \in m_{\hat{T}_{2,l}}^{l}$, and thus proving (2) of the conclusions of the Theorem.

By induction, there exists $H_i \in m_{\hat{T}_{1,l(1)}}^{l(1)}$ such that

$$x_i = \phi_i^1(x_{1,l(1)}(1), x_{2,l(1)}(1), \ldots, x_{j,l(1)}(1), \ldots, x_{p,l(1)}(1), x_{p+1}, \ldots, x_q) + H_i$$

for $1 \le i \le p$.

Thus

$$\begin{aligned}
x_i = {} & \phi_i^1(x_{1,l}(2)^{a_{11}} \dots x_{s,l}(2)^{a_{1s}}(\tilde{x}_{j,l}(2)+\overline{\alpha}_2)^{a_{1s+1}}, \dots, x_{1,l}(2)^{a_{s1}} \dots x_{s,l}(2)^{a_{ss}} \\
& (\tilde{x}_{j,l}(2)+\overline{\alpha}_2)^{a_{ss+1}}, x_{s+1,l(1)}(1), \dots, x_{j-1,l(1)}(1), x_{1,l}(2)^{a_{s+11}} \dots x_{s,l}(2)^{a_{s+1s}} \\
& (\tilde{x}_{j,l}(2)+\overline{\alpha}_2)^{a_{s+1s+1}} + \lambda_l, x_{j+1,l(1)}(1), \dots, x_{p,l(1)}(1), x_{p+1}, \dots, x_q) + H_i \\
= {} & \phi_i^1(x_{1,l}(2)^{a_{11}} \dots x_{s,l}(2)^{a_{1s}}(\tilde{x}_{j,l}(2)+\overline{\alpha}_2)^{a_{1s+1}}, \dots, x_{1,l}(2)^{a_{s1}} \dots x_{s,l}(2)^{a_{ss}} \\
& (\tilde{x}_{j,l}(2)+\overline{\alpha}_2)^{a_{ss+1}}, x_{s+1,l(1)}(1), \dots, x_{j-1,l(1)}(1), x_{1,l}(2)^{a_{s+11}} \dots x_{s,l}(2)^{a_{s+1s}} \\
& (\tilde{x}_{j,l}(2)+\overline{\alpha}_2)^{a_{s+1s+1}} + \lambda(x_{1,l}(2)^{a_{11}} \dots x_{s,l}(2)^{a_{1s}}(\tilde{x}_{j,l}(2)+\overline{\alpha}_2)^{a_{1s+1}}, \\
& \dots, x_{1,l}(2)^{a_{s1}} \dots x_{s,l}(2)^{a_{ss}}(\tilde{x}_{j,l}(2)+\overline{\alpha}_2)^{a_{ss+1}}, x_{s+1,l(1)}(1), \dots, x_{j-1,l(1)}(1), \\
& x_{j+1,l(1)}(1), \dots, x_{p,l(1)}(1)) + h', x_{j+1,l(1)}(1), \dots, x_{p,l(1)}(1), x_{p+1}, \dots, x_q) + H_i.
\end{aligned}$$

Where $h' \in m^{n_0}_{\hat{T}_{1,l(1)}}$ thus $h' \in m^{n_0}_{\hat{T}_{2,l}}$ and thus $h' \in m^{l}_{\hat{T}_{2,l}}$ since $n_0 > l$,

and similarly $H_i \in m^{l}_{\hat{T}_{2,l}}$ since $l(1) > l$.

Thus the expression of x_i becomes:

$$\begin{aligned}
x_i = {} & \phi_i^1(x_{1,l}(2)^{a_{11}} \dots x_{s,l}(2)^{a_{1s}}(\tilde{x}_{j,l}(2)+\overline{\alpha}_2)^{a_{1s+1}}, \dots, x_{1,l}(2)^{a_{s1}} \dots x_{s,l}(2)^{a_{ss}} \\
& (\tilde{x}_{j,l}(2)+\overline{\alpha}_2)^{a_{ss+1}}, x_{s+1,l(1)}(1), \dots, x_{j-1,l(1)}(1), x_{1,l}(2)^{a_{s+11}} \dots x_{s,l}(2)^{a_{s+1s}} \\
& (\tilde{x}_{j,l}(2)+\overline{\alpha}_2)^{a_{s+1s+1}} + \lambda(x_{1,l}(2)^{a_{11}} \dots x_{s,l}(2)^{a_{1s}}(\tilde{x}_{j,l}(2)+\overline{\alpha}_2)^{a_{1s+1}}, \dots, \\
& x_{1,l}(2)^{a_{s1}} \dots x_{s,l}(2)^{a_{ss}}(\tilde{x}_{j,l}(2)+\overline{\alpha}_2)^{a_{ss+1}}, x_{s+1,l(1)}(1), \dots, x_{j-1,l(1)}(1), \\
& x_{j+1,l(1)}(1), \dots, x_{p,l(1)}(1)), x_{j+1,l(1)}(1), \dots, x_{p,l(1)}(1), x_{p+1}, \dots, x_q) + H_i \\
& + h'\overline{\Delta}_1.
\end{aligned}$$

For some $\overline{\Delta}_1 \in \hat{T}_{2,l}$.

Let $\hat{H}_i = H_i + h'\overline{\Delta}_1$.

Then $\hat{H}_i \in m^{l}_{\hat{T}_{2,l}}$.

On the other hand:

$\psi_i(x_{1,l}(2), \dots, x_{s,l}(2), x_{s+1,l(1)}(1), \dots, x_{j-1,l(1)}(1), \tilde{x}_{j,l(1)}(2), \dots, x_{p,l(1)}(1), x_{p+1}, \dots,$

$x_q) = \phi_i^1(x_{1,l}(2)^{a_{11}} \dots x_{s,l}(2)^{a_{1s}}(\tilde{x}_{j,l}(2)+\overline{\alpha}_2)^{a_{1s+1}}, \dots, x_{1,l}(2)^{a_{s1}} \dots x_{s,l}(2)^{a_{ss}}(\tilde{x}_{j,l}(2)+$

$\overline{\alpha}_2)^{a_{ss+1}}, x_{s+1,l(1)}(1), \dots, x_{j-1,l(1)}(1), x_{1,l}(2)^{a_{s+11}} \dots x_{s,l}(2)^{a_{s+1s}}(\tilde{x}_{j,l}(2)+\overline{\alpha}_2)^{a_{s+1s+1}}$

$+ \lambda(x_{1,l}(2)^{a_{11}} \dots x_{s,l}(2)^{a_{1s}}(\tilde{x}_{j,l}(2)+\overline{\alpha}_2)^{a_{1s+1}}, \dots, x_{1,l}(2)^{a_{s1}} \dots x_{s,l}(2)^{a_{ss}}$

$(\tilde{x}_{j,l}(2)+\overline{\alpha}_2)^{a_{ss+1}}, x_{s+1,l(1)}(1), \dots, x_{j-1,l(1)}(1), x_{j+1,l(1)}(1), \dots, x_{p,l(1)}(1)) + h',$

$x_{j+1,l(1)}(1), \dots, x_{p,l(1)}(1), x_{p+1}, \dots, x_q)$

$= \phi_i^1(x_{1,l}(2)^{a_{11}} \dots x_{s,l}(2)^{a_{1s}}(\tilde{x}_{j,l}(2)+\overline{\alpha}_2)^{a_{1s+1}}, \dots, x_{1,l}(2)^{a_{s1}} \dots x_{s,l}(2)^{a_{ss}}$

$(\tilde{x}_{j,l}(2)+\overline{\alpha}_2)^{a_{ss+1}}, x_{s+1,l(1)}(1), \dots, x_{j-1,l(1)}(1), x_{1,l}(2)^{a_{s+11}} \dots x_{s,l}(2)^{a_{s+1s}}$

$(\tilde{x}_{j,l}(2)+\overline{\alpha}_2)^{a_{s+1s+1}}+\lambda(x_{1,l}(2)^{a_{11}}\ldots x_{s,l}(2)^{a_{1s}}(\tilde{x}_{j,l}(2)+\overline{\alpha}_2)^{a_{1s+1}},\ldots,x_{1,l}(2)^{a_{s1}}\ldots$

$x_{s,l}(2)^{a_{ss}}(\tilde{x}_{j,l}(2)+\overline{\alpha}_2)^{a_{ss+1}},x_{s+1,l(1)}(1),\ldots,x_{j-1,l(1)}(1),x_{j+1,l(1)}(1),\ldots,x_{p,l(1)}(1)),$

$x_{j+1,l(1)}(1),\ldots,x_{p,l(1)}(1),x_{p+1},\ldots,x_q)+h'\overline{\phi}_i$

Where $\overline{\phi}_i\in\hat{T}_{2,l}$.

This implies

$$\psi_i(x_{1,l}(2),\ldots,x_{s,l}(2),x_{s+1,l(1)}(1),\ldots,x_{j-1,l(1)}(1),\tilde{x}_{j,l}(2),x_{j,l(1)}(1),\ldots,x_{p,l(1)}(1),$$
$$x_{p+1},\ldots,x_q)=x_i-\hat{H}_i+h'\overline{\phi}_i.$$

Let $g_i=-\hat{H}_i+h'\overline{\phi}_i$ thus $g_i\in m^l_{\hat{T}_{2,l}}$ and $\psi_i(x_{1,l}(2),\ldots,x_{s,l}(2),x_{s+1,l(1)}(1),\ldots,$ $x_{j-1,l(1)}(1),\tilde{x}_{j,l}(2),x_{j,l(1)}(1)\ldots,x_{p,l(1)}(1),x_{p+1},\ldots,x_q)=x_i+g_i$. Now, we have that $x_{i,l}(2)=E_i(x_{1,l}(2),\ldots,x_{s,l}(2),x_{s+1,l(1)}(1),\ldots,x_{j-1,l(1)}(1),\tilde{x}_{j,l}(2),$ $x_{j,l(1)}(1)\ldots,x_{p,l(1)}(1),x_{p+1},\ldots,x_q)-h_i$ for $1\le i\le p$.

Thus $x_{i,l}(2)+h_i=E_i(x_{1,l}(2),x_{2,l(1)}(1),\ldots,\tilde{x}_{j,l}(2),\ldots,x_{p,l(1)}(1),x_{p+1},\ldots,x_q)$ for $1\le i\le p$.

This implies that

$$x_{1,l}(2)=\Delta_1(x_{1,l}(2)+h_1,x_{2,l}(2)+h_2,\ldots,x_{p,l}(2)+h_p,x_{p+1},\ldots,x_q)$$
$$\vdots$$
$$x_{s,l}(2)=\Delta_s(x_{1,l}(2)+h_1,x_{2,l}(2)+h_2,\ldots,x_{p,l}(2)+h_p,x_{p+1},\ldots,x_q)$$
$$x_{s+1,l(1)}(1)=\Delta_{s+1}(x_{1,l}(2)+h_1,x_{2,l}(2)+h_2,\ldots,x_{p,l}(2)+h_p,x_{p+1},\ldots,x_q)$$
$$\vdots$$
$$x_{j-l,l(1)}(1)=\Delta_{j-1}(x_{1,l}(2)+h_1,x_{2,l}(2)+h_2,\ldots,x_{p,l}(2)+h_p,x_{p+1},\ldots,x_q)$$
$$\tilde{x}_{j,l}(2)=\Delta_j(x_{1,l}(2)+h_1,x_{2,l}(2)+h_2,\ldots,x_{p,l}(2)+h_p,x_{p+1},\ldots,x_q)$$
$$x_{j+1,l(1)}(1)=\Delta_{j+1}(x_{1,l}(2)+h_1,x_{2,l}(2)+h_2,\ldots,x_{p,l}(2)+h_p,x_{p+1},\ldots,x_q)$$

$$\vdots$$

$$x_{p,l(1)}(1) = \Delta_p(x_{1,l}(2) + h_1, x_{2,l}(2) + h_2, \ldots, x_{p,l}(2) + h_p, x_{p+1}, \ldots, x_q)$$

Thus

$$\begin{aligned}
x_{1,l}(2) &= \Delta_1(x_{1,l}(2), x_{2,l}(2), \ldots, x_{p,l}(2), x_{p+1}, \ldots, x_q) + \overline{H}_1 \\
&\vdots \\
x_{s,l}(2) &= \Delta_s(x_{1,l}(2), x_{2,l}(2), \ldots, x_{p,l}(2), x_{p+1}, \ldots, x_q) + \overline{H}_s \\
x_{s+1,l(1)}(1) &= \Delta_{s+1}(x_{1,l}(2), x_{2,l}(2), \ldots, x_{p,l}(2), x_{p+1}, \ldots, x_q) + \overline{H}_{s+1} \\
&\vdots \\
x_{j-l,l(1)}(1) &= \Delta_{j-1}(x_{1,l}(2), x_{2,l}(2), \ldots, x_{p,l}(2), x_{p+1}, \ldots, x_q) + \overline{H}_{j-1} \\
\tilde{x}_{j,l}(2) &= \Delta_j(x_{1,l}(2), x_{2,l}(2), \ldots, x_{p,l}(2), x_{p+1}, \ldots, x_q) + \overline{H}_j \\
x_{j+1,l(1)}(1) &= \Delta_{j+1}(x_{1,l}(2), x_{2,l}(2), \ldots, x_{p,l}(2), x_{p+1}, \ldots, x_q) + \overline{H}_{j+1} \\
&\vdots \\
x_{p,l(1)}(1) &= \Delta_p(x_{1,l}(2), x_{2,l}(2), \ldots, x_{p,l}(2), x_{p+1}, \ldots, x_q) + \overline{H}_p.
\end{aligned}$$

Where $\overline{H}_i \in m^l_{\hat{T}_{2,l}}$ for $1 \leq i \leq p$.

Thus $\psi_i(x_{1,l}(2), \ldots, x_{s,l}(2), x_{s+1,l(1)}(1), \ldots, \tilde{x}_{j,l}(2), \ldots, x_{p,l(1)}(1), x_{p+1}, \ldots, x_q)$
$= x_i + g_i,$

becomes $\psi_i(\Delta_1 + \overline{H}_1, \ldots, \Delta_p + \overline{H}_p, x_{p+1}, \ldots, x_q)$

$$= x_i + g_i(\Delta_1 + \overline{H}_1, \ldots, \Delta_p + \overline{H}_p, x_{p+1}, \ldots, x_q).$$

This implies

$$\begin{aligned}
x_i &= \psi_i(\Delta_1, \ldots, \Delta_p, x_{p+1}, \ldots, x_q) + G_i \text{ where } G_i \in m^l_{\hat{T}_{2,l}} \\
&= \phi_i^2(x_{1,l}(2), \ldots, x_{s,l}(2), \ldots, x_{p,l}(2), x_{p+1}, \ldots, x_q) + G_i.
\end{aligned}$$

Thus this proves (2) of the conclusions of the theorem.

Now we proceed to prove conclusion (3) of the theorem:

We have $\hat{S}_1 : x_{1,l}(2), \ldots, x_{s,l}(2) x_{s+1,l(1)}(1), \ldots, x_{j-1,l(1)}(1), \tilde{x}_{j,l}(2),$

$x_{j+1,l(1)}(1), \ldots, x_{p,l(1)}(1), x_{p+1}, \ldots, x_q$ a system of regular parameters of $\hat{T}_{2,l}$ where

$$\begin{array}{rcl} x_{1,l}(2) & = & x_{1,l(1)}(1)^{b_{11}} \ldots x_{s,l(1)}(1)^{b_{1s}}[x_{j,l(1)}(1)']^{b_{1s+1}} \\ \vdots & & \\ x_{s,l}(2) & = & x_{1,l(1)}(1)^{b_{s1}} \ldots x_{s,l(1)}(1)^{b_{ss}}[x_{j,l(1)}(1)']^{b_{ss+1}} \\ \overline{x}_{j,l}(2) & = & x_{1,l(1)}(1)^{b_{s+11}} \ldots x_{s,l(1)}(1)^{b_{s+1s}}[x_{j,l(1)}(1)']^{b_{s+1s+1}} \end{array}$$

and $S_1 : x_1(2), \ldots, x_s(2), x_{s+1}(1), \ldots, x_{j-1}(1), \tilde{x}_j(2), x_{j+1}(1), \ldots, x_p(1), x_{p+1}, \ldots, x_q$ a system of regular parameters of U_2 where

$$\begin{array}{rcl} x_1(2) & = & x_1(1)^{b_{11}} \ldots x_s(1)^{b_{1s}}[x_j(1)']^{b_{1s+1}} \\ \vdots & & \\ x_s(2) & = & x_1(1)^{b_{s1}} \ldots x_s(1)^{b_{ss}}[x_j(1)']^{b_{ss+1}} \\ \overline{x}_j(2) & = & x_1(1)^{b_{s+11}} \ldots x_s(1)^{b_{s+1s}}[x_j(1)']^{b_{s+1s+1}} \end{array}$$

where $\tilde{x}_j(2) = \overline{x}_j(2) + \overline{\alpha}_2$.

By induction we have that $\nu(x_{i,l(1)}(1)) = \nu(x_i(1)) < l_0(1)$.

Notice that $\nu(x_i(2)) = \sum_{k=1}^{s} b_{ki}\nu(x_i(1)) < \sum_{k=1}^{s} b_{ki} l_0(1) < l_0(2)$ for $i = 1, \ldots, s$. Thus $\nu(x_i(2)) = \nu(x_{i,l}(2)) < l_0(2)$ for $i = 1, \ldots, s$.

$$\begin{aligned} \tilde{x}_{j,l}(2) &= \overline{x}_{j,l}(2) - \overline{\alpha}_2 \\ &= x_{1,l(1)}(1)^{b_{s+11}} \ldots x_{s,l(1)}(1)^{b_{s+1s}}[\overline{x}_{j,l(1)}(1)']^{b_{s+1s+1}} - \overline{\alpha}_2 \\ &= (x_1(1) + g_{1,l(1)})^{b_{s+11}} \ldots (x_s(1) + g_{s,l(1)})^{b_{s+1s}}(x_j(1)' + g'_{j,l(1)})^{b_{s+1s+1}} - \overline{\alpha}_2 \\ &= x_1(1)^{b_{s+11}}[1 + \tfrac{g_{1,l(1)}}{x_1(1)}]^{b_{s+11}} \ldots x_s(1)^{b_{s+1s}}[1 + \tfrac{g_{s,l(1)}}{x_s(1)}]^{b_{s+1s}}(x_j(1)' + g'_{j,l(1)})^{b_{s+1s+1}} \\ &\quad - \overline{\alpha}_2 \\ &= x_1(1)^{b_{s+11}} \ldots x_s(1)^{b_{s+1s}}[1 + \tfrac{g_{1,l(1)}}{x_1(1)}]^{b_{s+11}} \ldots [1 + \tfrac{g_{s,l(1)}}{x_s(1)}]^{b_{s+1s}}(x_j(1)' + \\ &\quad g'_{j,l(1)})^{b_{s+1s+1}} - \overline{\alpha}_2 \\ &= x_1(1)^{b_{s+11}} \ldots x_s(1)^{b_{s+1s}}[1 + \tfrac{g_{1,l(1)}}{x_1(1)}\Omega_1] \ldots [1 + \tfrac{g_{s,l(1)}}{x_s(1)}\Omega_s](x_j(1)' + g'_{j,l(1)})^{b_{s+1s+1}} \\ &\quad - \overline{\alpha}_2 \\ &= x_1(1)^{b_{s+11}} \ldots x_s(1)^{b_{s+1s}}[1 + \tfrac{g_{1,l(1)}}{x_1(1)}\Omega'_1 + \cdots + \tfrac{g_{s,l(1)}}{x_s(1)}\Omega'_s](x_j(1)' + g'_{j,l(1)})^{b_{s+1s+1}} \\ &\quad - \overline{\alpha}_2 \\ &= x_1(1)^{b_{s+11}} \ldots x_s(1)^{b_{s+1s}}(x_j(1)' + g'_{j,l(1)})^{b_{s+1s+1}} - \overline{\alpha}_2 + \\ &\quad x_1(1)^{b_{s+11}} \ldots x_s(1)^{b_{s+1s}}[\tfrac{g_{1,l(1)}}{x_1(1)}\Omega'_1 + \cdots + \tfrac{g_{s,l(1)}}{x_s(1)}\Omega'_s] \\ &= \tilde{x}_j(2) + x_1(1)^{b_{s+11}} \ldots x_s(1)^{b_{s+1s}}[\tfrac{g_{1,l(1)}}{x_1(1)}\Omega'_1 + \cdots + \tfrac{g_{s,l(1)}}{x_s(1)}\Omega'_s] \end{aligned}$$

where $\Omega_i, \Omega'_i \in \mathbb{Z}[\frac{g_{1,l(1)}}{x_1(1)}, \dots, \frac{g_{s,l(1)}}{x_s(1)}]$, for $i = 1, \dots, s$, are of value greater or equal to zero, by choice of $l(1)(\geq \xi_1)$. Now $\nu(x_1(1)^{b_{s+11}} \dots x_i(1)^{b_{s+1i}} \dots x_s(1)^{b_{s+1s}} \frac{g_{i,l(1)}}{x_i(1)} \Omega'_1)$ $= \nu(x_1(1)^{b_{s+11}} \dots x_i(1)^{b_{s+1i}-1} \dots x_s(1)^{b_{s+1s}} g_{i,l(1)} \Omega'_1) > l > \nu(\tilde{x}_j(2))$ by choice of $l(1)$, for $i = 1, \dots, s$.

Thus

$$\nu(\tilde{x}_{j,l}(2)) \;=\; \nu(\tilde{x}_j(2)) < l_0(2).$$

[Moreover, we also conclude that $\nu(\tilde{x}_{j,l}(2) - \tilde{x}_j(2)) > l$.]

And by induction we have:

- $\nu(x_i(1)) = \nu(x_{i,l(1)}(1)) < l_0(1) < \infty$ for $s+1 \leq i \leq p_1, \sigma(i) \neq j$.
- $\nu(x_{i,l}(1)) > l$ and $\nu(x_i(2)) = \infty$ for $p_2 < i \leq p$.
- $\nu(x_i) = \infty$ for $p < i \leq q$.

This implies that $\nu(m_{\hat{T}_{2,l}}) = \nu(\hat{m}_{U_2})$.

This also implies that $\nu(m_{T_{2,l}}) = \nu(m_{U_2})$.

Now for $s+1 \leq i \leq p_2$ and $\sigma(i) \neq j$, we have $x_i(2) = E_i(x_1(2), \dots,$ $x_s(2), x_{s+1}(1), \dots, \tilde{x}_j(2), \dots, x_p(1), x_{p+1}, \dots, x_q) = x_{\sigma(i)}(1)$ and we have defined the regular parameters $x_{1,l}(2), \dots, x_{p,l}(2), x_{p+1}, \dots, x_q$ in $T_{2,l}$ by

$$x_{i,l}(2) = E_i(x_{1,l}(2), \dots, x_{s,l}(2), x_{s+1,l(1)}(1), \dots, \overline{x}_{j,l}(2) - \overline{\alpha}_2, \dots, x_{p,l(1)}(1), x_{p+1}, \dots, x_q) - h_i.$$

Thus $x_{i,l}(2) = x_{\sigma(i),l(1)}(1) - h_i$ where $h_i \in m_{\hat{T}_{2,l}}^{n_1}$ and $n_1 v(m_{\hat{T}_{2,l}}) = n_1 v(m_{U_2}) >$ $l_0(2) > l_0(1) > \nu(x_{\sigma(i),l(1)}(1)) = \nu(x_{\sigma(i)}(1))$ since $\sigma(i) \in \{s+1, \dots, p_1\}$ for

$i \in \{s+1, \ldots, p_2\}$.

This implies $\nu(h_i) \geq n_1 v(m_{\hat{T}_{2,l}}) > \nu(x_{\sigma(i),l(1)}(1))$,

thus $\nu(x_{i,l}(2)) = \nu(x_{\sigma(i),l(1)}(1) - h_i) = \nu(x_{\sigma(i),l(1)}(1)) = \nu(x_\sigma(i)(1)) = \nu(x_i(2)) < l_0(2)$.

Now if $\sigma(i) = j$ then

$$\begin{aligned} x_i(2) &= \tilde{x}_j(2) \\ &= \overline{x_j(2)} - \bar{\alpha}_2 \\ &= x_1(1)^{b_{s+11}} \ldots x_s(1)^{b_{s+1s}} [\overline{x_j(1)'}]^{b_{s+1s+1}} - \bar{\alpha}_2 \end{aligned}$$

thus

$$\begin{aligned} x_{i,l}(2) &= x_{1,l(1)}(1)^{b_{s+11}} \ldots x_{s,l(1)}(1)^{b_{s+1s}} [\overline{x}_{j,l(1)}(1)']^{b_{s+1s+1}} - \overline{\alpha}_2 - h_i \\ &= \overline{x}_{j,l}(2) - \overline{\alpha}_2 - h_i \end{aligned}$$

where $\nu(h_i) > \nu(\tilde{x}_j(2))$ by the choice of n_1, thus $\nu(h_i) > \nu(\tilde{x}_j(2)) = \nu(\overline{x}_{j,l}(2) - \overline{\alpha}_2)$.

Thus, $\nu(x_{i,l}(2)) = \nu(\overline{x}_{j,l}(2) - \overline{\alpha}_2 - h_i) = \nu(\overline{x}_{j,l}(2) - \overline{\alpha}_2) = \nu(\tilde{x}_j(2))$

$= \nu(x_i(2)) < l_0(2)$ and thus (3) of the conclusions of the theorem is proved.

To prove (4) and (5) we first compare the values of the regular parameters of $\hat{S}_1$ and S_1.

We already showed that $\nu(\tilde{x}_{j,l}(2) - \tilde{x}_j(2)) > l$ and we get by induction that $\nu(x_{i,l(1)}(1) - x_i(1)) > l(1) > l$ for $1 \leq i \leq p$ and $i \neq j$.

Now, for $1 \leq i \leq s$

$$\begin{aligned}
x_{i,l}(2) &= x_{1,l(1)}(1)^{b_{i1}} \dots x_{s,l(1)}(1)^{b_{is}} x'_{j,l(1)}(1)^{b_{is+1}} \\
&= (x_1(1) + g_{1,l(1)})^{b_{i1}} \dots (x_s(1) + g_{s,l(1)})^{b_{is}} (x'_j(1) + g'_{j,l(1)})^{b_{is+1}} \\
&= x_1(1)^{b_{i1}} (1 + \tfrac{g_{1,l(1)}}{x_1(1)})^{b_{i1}} \dots x_s(1)^{b_{is}} (1 + \tfrac{g_{s,l(1)}}{x_s(1)})^{b_{is}} x'_j(1)^{b_{is+1}} (1 + \tfrac{g'_{j,l(1)}}{x'_j(1)})^{b_{is+1}} \\
&= x_1(1)^{b_{i1}} \dots x_s(1)^{b_{is}} x'_j(1)^{b_{is+1}} (1 + \tfrac{g_{1,l(1)}}{x_1(1)} \Omega_{1i}) \dots (1 + \tfrac{g_{s,l(1)}}{x_s(1)} \Omega_{si})(1 + \tfrac{g'_{j,l(1)}}{x'_j(1)} \\
&\quad \Omega_{s+1i}) \\
&= x_1(1)^{b_{i1}} \dots x_s(1)^{b_{is}} x'_j(1)^{b_{is+1}} (1 + \tfrac{g_{1,l(1)}}{x_1(1)} \Omega'_{1i} + \dots + \tfrac{g_{s,l(1)}}{x_s(1)} \Omega'_{si} + \tfrac{g'_{j,l(1)}}{x'_j(1)} \Omega'_{s+1i}) \\
&= x_1(1)^{b_{i1}} \dots x_s(1)^{b_{is}} x'_j(1)^{b_{is+1}} + x_1(1)^{b_{i1}-1} \dots x_s(1)^{b_{is}} x'_j(1)^{b_{is+1}} g_{1,l(1)} \Omega'_{1i} \\
&\quad + \dots + x_1(1)^{b_{i1}} \dots x_s(1)^{b_{is}} x'_j(1)^{b_{is+1}-1} g'_{j,l(1)} \Omega'_{s+1i} \\
&= x_i(2) + x_1(1)^{b_{i1}-1} \dots x_s(1)^{b_{is}} x'_j(1)^{b_{is+1}} g_{1,l(1)} \Omega'_{1i} \\
&\quad + \dots + x_1(1)^{b_{i1}} \dots x_s(1)^{b_{is}} x'_j(1)^{b_{is+1}-1} g'_{j,l(1)} \Omega'_{s+1i}
\end{aligned}$$

where $\Omega_{ki}, \Omega'_{ki} \in \mathbb{Z}\ [\frac{g_{1,l(1)}}{x_1(1)}, \dots, \frac{g_{s,l(1)}}{x_s(1)}, \frac{g'_{j,l(1)}}{x'_j(1)}]$ of value greater or equal to zero, by choice of $l(1)$, for $1 \leq k \leq s+1$ and thus $\nu(x_{i,l}(2) - x_i(2)) > l$, by choice of $l(1)(\geq \xi_2$ and $\xi_3)$.

Let $\hat{s}_i$ and s_i be the i-th regular parameter in $\hat{S}_1$ and S_1 respectively, in the given order, for $1 \leq i \leq p$ and let $d_i = s_i - \hat{s}_i$ for $1 < i \leq p$. Then $s_i = d_i + \hat{s}_i$ and $\nu(d_i) > l$.

For all $1 \leq i \leq p$ we have

$$\begin{aligned}
x_i(2) - x_{i,l}(2) \quad = \quad & E_i(s_1, s_2, \dots, s_j, \dots, s_p, x_{p+1}, \dots, x_q) \\
& - E_i(\hat{s}_1, \hat{s}_2, \dots, \hat{s}_j, \dots, \hat{s}_p, x_{p+1}, \dots, x_q) + h_i.
\end{aligned}$$

We write $E_i(t_1, \dots, t_q) \in k_2[[t_1, \dots, t_q]]$ as a sum of it's monomials:

$$E_i = \sum_n M_n$$

where $M_n = a_n t_1^{n_1} \dots t_q^{n_q}$ with $a_n \in k_2$ and $n_1, \dots, n_q$ are non-negative integers. Let $M_n(2) = M_n(s_1, s_2, \dots, s_j, \dots, s_p, x_{p+1}, \dots, x_q)$ then

$$x_i(2) = \sum_n M_n(2)$$

Let $M_{n,l}(2) = M_n(\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_j, \ldots, \hat{s}_p, x_{p+1}, \ldots, x_q)$ then

$$x_{i,l}(2) = \sum_n M_{n,l}(2) - h_i.$$

This implies $x_i(2) - x_{i,l}(2) = \sum_n [M_n(2) - M_{n,l}(2)] + h_i$, where by the choice of $n_1 \nu(h_i) > l$.

Now,

$$\begin{aligned} M_n(2) - M_{n,l}(2) &= a_n[s_1^{n_1} s_2^{n_2} \ldots s_j^{n_j} \ldots s_p^{n_p} x_{p+1}^{s_{p+1}} \ldots x_q^{s_q} - \hat{s}_1^{n_1} \hat{s}_2^{n_2} \ldots \hat{s}_j^{n_j} \\ &\quad \ldots \hat{s}_p^{n_p} x_{p+1}^{s_{p+1}} \ldots x_q^{s_q}] \\ &= a_n[(\hat{s}_1 + d_1)^{n_1} (\hat{s}_2 + d_2)^{n_2} \ldots (\hat{s}_j + d_j)^{n_j} \ldots (\hat{s}_p + d_p)^{n_p} \\ &\quad x_{p+1}^{s_{p+1}} \ldots x_q^{s_q} - \hat{s}_1^{n_1} \hat{s}_2^{n_2} \ldots \hat{s}_j^{n_j} \ldots \hat{s}_p^{n_p} x_{p+1}^{s_{p+1}} \ldots x_q^{s_q}] \\ &= a_n[\hat{s}_1^{n_1} \hat{s}_2^{n_2} \ldots \hat{s}_j^{n_j} \ldots \hat{s}_p^{n_p} x_{p+1}^{s_{p+1}} \ldots x_q^{s_q} + d_1 \hat{\xi}_1 + \cdots + d_p \hat{\xi}_p \\ &\quad - \hat{s}_1^{n_1} \hat{s}_2^{n_2} \ldots \hat{s}_j^{n_j} \ldots \hat{s}_p^{n_p} x_{p+1}^{s_{p+1}} \ldots x_q^{s_q}], \end{aligned}$$

where $\hat{\xi}_i \in \mathbb{Z}\ [\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_j, \ldots, \hat{s}_p, x_{p+1}, \ldots, x_q, d_1, \ldots, d_p]$, thus $\nu(\hat{\xi}_i) \geq 0$, and thus $M_n(2) - M_{n,l}(2) = a_n[d_1 \hat{\xi}_1 + \cdots + d_p \hat{\xi}_p]$ so $\nu(M_n(2) - M_{n,l}(2)) > l$ and thus $\nu(x_{i,l}(2) - x_i(2)) > l$, for $1 \leq i \leq p$ and this proves (5) of the conclusions of the theorem.

Moreover, for $p_2 < i \leq p, \nu(x_i(2)) = \infty$ this implies that $\nu(x_{i,l}(2)) > l$ and thus proving (4).

□

Theorem 4.3.2. *Suppose that* $x_1, \ldots, x_s, \ldots, x_t, x_{t+1}, \ldots, x_q$ *is a regular system of parameters in* T *such that* $\nu(x_1), \ldots, \nu(x_s)$ *are rationally independent of finite value,* $t \geq s$ *and* $Q_{\hat{T}} \cap T/m_T[[x_1, \ldots, x_t]] = (0)$*. Suppose that* $0 \neq f \in T/m_T[[x_1, \ldots, x_t]]$*. Then there exists a sequence of etale Perron transforms*

$$T \to U \to U_1 \to \cdots \to U_n$$

along ν such that U_n has a system of regular parameters $x_1(n) \ldots, x_t(n), x_{t+1}, \ldots, x_q$ such that $\nu(x_1(n)), \ldots, \nu(x_s(n))$ are rationally independent and one of the following two cases holds:

1. $f = x_1(n)^{a_1} \ldots x_s(n)^{a_s} \gamma(x_1(n), \ldots, x_s(n), x_t(n))$,

   where $\gamma \in U_n/m_{U_n}[[x_1(n), \ldots, x_s(n), x_t(n)]]$ is a unit series.

   or

2. $Q_{\hat{U}_n} \cap U_n/m_{U_n}[[x_1(n), \ldots, x_t(n)]] \neq (0)$.

Proof. We first consider the case where $t = s$.

$f \in T/m_T[[x_1, \ldots, x_s]]$ and we write $f = \sum_{i \geq 1} a_{i_1 \ldots i_s} x_1^{i_1} \ldots x_s^{i_s}$ where $a_{i_1 \ldots i_s} \in T/m_T$.

Notice that if $a_{i_1 \ldots i_s}$ and $a_{j_1 \ldots j_s} \neq 0$, and $\nu(a_{i_1 \ldots i_s} x_1^{b_1^i} \ldots x_s^{b_s^i}) = \nu(a_{j_1 \ldots j_s} x_1^{b_1^j} \ldots x_s^{b_s^j})$ then $b_1^i = b_1^j, \ldots, b_s^i = b_s^j$ since $\nu(x_1(1)), \ldots, \nu(x_s(1))$ are rationally independent. Thus we may assume that $a_{\bar{i}_1 \ldots \bar{i}_s} x_1^{\bar{i}_1} \ldots x_s^{\bar{i}_s}$ is of minimum value ($a_{\bar{i}_1 \ldots \bar{i}_s} \neq 0$), and that $\nu(a_{i_1 \ldots i_s} x_1^{i_1} \ldots x_s^{i_s}) > \nu(a_{\bar{i}_1 \ldots \bar{i}_s} x_1^{\bar{i}_1} \ldots x_s^{\bar{i}_s})$ for $i \neq \bar{i}$.

Let $I = \left\langle a_{i_1 \ldots i_s} x_1^{i_1} \ldots x_s^{i_s}, i \geq 1 \right\rangle \subseteq T/m_T[[x_1, \ldots, x_s]]$.
Then there exists a finite set $S \subset \mathbb{N}$ such that $I = \left\langle a_{i_1 \ldots i_s} x_1^{i_1} \ldots x_s^{i_s}, i \in S \right\rangle$ and we necessarily have that $\bar{i} \in S$.

Thus $f = x_1^{\bar{i}_1} \ldots x_s^{\bar{i}_s} \overline{\gamma} + \sum_{i \in S - \{\bar{i}\}} x_1^{i_1} \ldots x_s^{i_s} f'$.

Where $f' \in T/m_T[[x_1, \ldots, x_s]]$ and necessarily $\overline{\gamma} \in T/m_T$ is a unit.

moreover we still have that $\nu(x_1^{\bar{i}_1}\dots x_s^{\bar{i}_s}) < \nu(x_1^{i_1}\dots x_s^{i_s})$ for $i \in S-\{\bar{i}\}$.

Let $T \to U \to U_1$ be a sequence of etale Perron transforms of type I along ν such that U_1 has regular parameters $x_1(1),\dots,x_s(1),x_{s+1},\dots,x_q$ where

$$\begin{array}{rcl} x_1 & = & x_1(1)^{a_{11}}\dots x_s(1)^{a_{1s}} \\ \vdots & & \\ x_s & = & x_1(1)^{a_{s1}}\dots x_s(1)^{a_{ss}} \end{array}$$

and $\nu(x_1(1)),\dots,\nu(x_s(1))$ are rationally independent.

Thus we have $f = x_1(1)^{b_1^{\bar{i}}}\dots x_s(1)^{b_s^{\bar{i}}}\overline{\gamma}+\sum_{i\in S-\{\bar{i}\}} x_1(1)^{b_1^i}\dots x_s(1)^{b_s^i} f'(x_1(1)\dots x_s(1))$ and $\nu(x_1(1)^{b_1^{\bar{i}}}\dots x_s(1)^{b_s^{\bar{i}}}) < \nu(x_1(1)^{b_1^i}\dots x_s(1)^{b_s^i})$.

Then by Lemma 4.2 [16] there exists a finite sequence of etale Perron transforms of type I along ν $U_1 \to U_2$ such that U_2 has regular parameters $x_1(2),\dots,x_s(2),x_{s+1},\dots,x_q$ such that

$$f = x_1(2)^{a_1}\dots x_s(2)^{a_s}\gamma$$

where $\gamma = \overline{\gamma}+\sum_{i\in S-\{\bar{i}\}} x_1(2)^{d_1^i}\dots x_s(2)^{d_s^i} f'(x_1(2),\dots,x_s(2))$ is a unit in $U_2/m_{U_2}[[x_1(2),\dots,x_s(2)]]$ since $\overline{\gamma}\neq 0$ and $d_l^i \neq 0$ for some $1\le l\le s$.

Now consider the case $t > s$.

Assume by induction that the theorem is true for $t-1$. We will prove it for t.

Let $r = \text{order}(f(0,\dots,0,x_t))$. We have $0\le r\le\infty$. If $r=0$, then 1 of the conclusions of the theorem hold in $T/m_T[[x_1,\dots,x_t]]$. Assume that $r\ge 1$. We have an expression

$$f = \sum_{i=1}^{m} a_i(x_1, \ldots, x_{t-1})x_t^{d_i} + \sum_{m<i\le n} a_i(x_1, \ldots, x_{t-1})x_t^{d_i} + \sum_{n<i} a_i(x_1, \ldots, x_{t-1})x_t^{d_i}$$

where $a_i \in T/m_T[[x_1, \ldots, x_{t-1}]]$ for all i, and the the first sum is over the terms $a_i x_t^{d_i}$ of minimal value ρ,

$$d_1 < d_2 < \cdots < d_m,$$

and we have that $d_i\nu(x_t) > \rho$ if $i > n$, and $d_{n+1} < d_i$ if $i > n+1$. We necessarily have that $\nu(x_t^r) \ge \rho$.

Thus

$$d_i \le r \text{ for } 1 \le i \le m. \tag{4.8}$$

By induction, applied to $a_i(x_1, \ldots, x_{t-1})$, for $1 \le i \le n$, we either construct a sequence of etale Perron transforms along ν satisfying 2 of the conclusions of the theorem, or we have a sequence of etale Perron transforms $T \to U \to U_1$ along ν such that U_1 has regular parameters $x_1(1), \ldots, x_{t-1}(1), x_t, \ldots, x_q$ such that there exist unit series

$$\overline{a}_i(x_1(1), \ldots, x_{t-1}(1)) \in U_1/m_{U_1}[[x_1(1), \ldots, x_{t-1}(1)]],$$

and $c_i^j \in \mathbb{N}$ satisfying

$$a_i(x_1, \ldots, x_{t-1}) = x_1(1)^{c_i^1} \ldots x_s(1)^{c_i^s}\overline{a}_i(x_1(1), \ldots, x_{t-1}(1))$$

for $1 \le i \le n$, and there exist series

$$b_i(x_1(1), \ldots, x_{t-1}(1)) \in U_1/m_{U_1}[[x_1(1), \ldots, x_{t-1}(1)]],$$

satisfying

$$a_i(x_1, \ldots, x_{t-1}) = b_i(x_1(1), \ldots, x_{t-1}(1))$$

for $n < i$. Thus we have

$$f = \sum_{i=1}^{n} \overline{a}_i x_1(1)^{c_i^1} \ldots x_s(1)^{c_i^s} x_t^{d_i} + \sum_{n<i} b_i x_t^{d_i}. \tag{4.9}$$

We further may assume that

$$Q_{\hat{U}_1} \cap U_1/m_{U_1}[[x_1(1), \ldots, x_{t-1}(1), x_t]] = (0).$$

Now we define an etale Perron transform of type II_t $U_1 \to U_2$ by

$$\begin{array}{lcl} x_1(1) & = & x_1(2)^{a_{11}} \ldots x_s(2)^{a_{1s}} \overline{x}_t(2)^{a_{1,s+1}} \\ \vdots & & \\ x_s(1) & = & x_1(2)^{a_{s1}} \ldots x_s(2)^{a_{ss}} \overline{x}_t(2)^{a_{s,s+1}} \\ x_t & = & x_1(2)^{a_{s+1,1}} \ldots x_s(2)^{a_{s+1,s}} \overline{x}_t(2)^{a_{s+1,s+1}} \end{array} \tag{4.10}$$

We have that $\nu(x_1(2)), \ldots, \nu(x_s(2))$ are rationally independent and $\nu(\overline{x}_t(2)) = 0$. Let $\overline{\alpha}$ be the residue of $\overline{x}_t(2)$ in U_2/m_{U_2}. Let $\tilde{x}_t(2) = \overline{x}_t(2) - \overline{\alpha}$. Then

$$x_1(2), \ldots, x_s(2), x_{s+1}(1), \ldots, x_{t-1}(1), \tilde{x}_t(2), x_{t+1}, \ldots, x_q$$

are regular parameters in U_2. We may assume that

$$Q_{\hat{U}_2} \cap U_2/m_{U_2}[[x_1(2), \ldots, x_s(2), x_{s+1}(1), \ldots, x_{t-1}(1), \tilde{x}_t(2)]] = (0),$$

since otherwise we have achieved 2. of the conclusions of the theorem.

We will now show that the following formula holds

$$f = \sum_{i=1}^{m} \overline{a}_i x_1(2)^{b_1^i} \ldots x_s(2)^{b_s^i} (\tilde{x}_t(2) + \overline{\alpha})^{b_{s+1}^i} + \tag{4.11}$$

$$\sum_{i=m+1}^{n} \overline{a}_i x_1(2)^{b_1^i} \dots x_s(2)^{b_s^i} (\tilde{x}_t(2) + \overline{\alpha})^{b_{s+1}^i} + (x_1(2)^{a_{s+1,1}} \dots x_s(2)^{a_{s+1,s}})^{d_{n+1}} \Lambda$$

with $\Lambda \in U_2/m_{U_2}[[x_1(2), \dots, x_s(2), \tilde{x}_t(2)]]$,

$b_j^i = c_i^1 a_{1j} + \dots + c_i^s a_{sj} + d_i a_{s+1,j}$ for $1 \le i \le m$ and $1 \le j \le s+1$,

$b_1^i = b_1^1, \dots, b_s^i = b_s^1$, for $1 \le i \le m$,

$\nu(x_1(2)^{b_1^i} \dots x_s(2)^{b_s^i}) > \rho$ for $m < i \le n$ and $\nu((x_1(2)^{a_{s+1,1}} \dots x_s(2)^{a_{s+1,s}})^{d_{n+1}}) > \rho$.

To establish (4.11), we first recall that we have

$$f = \sum_{i=1}^{n} \overline{a}_i x_1(1)^{c_i^1} \dots x_s(1)^{c_i^s} x_t^{d_i} + \sum_{n<i} b_i x_t^{d_i}.$$

and since $d_i > d_{n+1}$ for $i > n+1$ then:

$$f = \sum_{i=1}^{n} \overline{a}_i x_1(1)^{c_i^1} \dots x_s(1)^{c_i^s} x_t^{d_i} + x_t^{d_{n+1}} \sum_{n<i} b_i x_t^{d_i - d_{n+1}}.$$

Substituting (4.10) into f, we have

$$f = \sum_{i=1}^{m} \overline{a}_i x_1(2)^{b_1^i} \dots x_s(2)^{b_s^i} (\tilde{x}_t(2) + \overline{\alpha})^{b_{s+1}^i} + \sum_{i=m+1}^{n} \overline{a}_i x_1(2)^{b_1^i} \dots x_s(2)^{b_s^i}$$
$$(\tilde{x}_t(2) + \overline{\alpha})^{b_{s+1}^i} + (x_1(2)^{a_{s+1,1}} \dots x_s(2)^{a_{s+1,s}})^{d_{n+1}} \sum_{n<i} b_i x_1(2)^{a_{s+1,1}(d_i - d_{n+1})} \dots$$
$$x_s(2)^{a_{s+1,s}(d_i - d_{n+1})} (\tilde{x}_t(2) + \overline{\alpha})^{a_{s+1,s+1 d_i}},$$

where $b_j^i = c_i^1 a_{1j} + \dots + c_i^s a_{sj} + d_i a_{s+1,j}$ for $1 \le i \le m$ and $1 \le j \le s+1$.

We have that:

$\rho = \nu(\overline{a}_1 x_1(2)^{b_1^1} \dots x_s(2)^{b_s^1} (\tilde{x}_t(2) + \overline{\alpha})^{b_{s+1}^1}) = \nu(\overline{a}_i x_1(2)^{b_1^i} \dots x_s(2)^{b_s^i} (\tilde{x}_t(2) + \overline{\alpha})^{b_{s+1}^i})$

for $1 \le i \le m$.

Thus $\nu(x_1(2)) b_1^1 + \dots + \nu(x_s(2)) b_s^1 = \nu(x_1(2)) b_1^i + \dots + \nu(x_s(2)) b_s^i$.

Hence $\nu(x_1(2))(b_1^1 - b_1^i) + \dots + \nu(x_s(2))(b_s^1 - b_s^i) = 0$, and thus $b_1^1 = b_1^i, \ldots, b_s^1 = b_s^i$ since $\nu(x_1(2)), \ldots, \nu(x_s(2))$ are rationally independent.

Moreover $\nu(\overline{a}_i x_1(2)^{b_1^i} \ldots x_s(2)^{b_s^i}(\tilde{x}_t(2)+\overline{\alpha})^{b_{s+1}^i}) > \rho$ for $m < i \leq n$,

and since $\nu(x_t^{d_{n+1}}) > \rho$ then $\nu((x_1(2)^{a_{s+1,1}} \ldots x_s(2)^{a_{s+1,s}})^{d_{n+1}}) > \rho$.

Let $\Lambda = \sum_{n<i} b_i x_1(2)^{a_{s+1,1}(d_i-d_{n+1})} \ldots x_s(2)^{a_{s+1,s}(d_i-d_{n+1})}(\tilde{x}_t(2)+\overline{\alpha})^{a_{s+1,s+1}d_i}$, then f has the desired form (4.11).

Now we will show that there exists an etale Perron transform of type I:

$$\begin{array}{rcl} x_1(2) & = & x_1(3)^{a'_{11}} \ldots x_s(3)^{a'_{1s}} \\ \vdots & & \\ x_s(2) & = & x_1(3)^{a'_{s1}} \ldots x_s(3)^{a'_{ss}} \end{array} \tag{4.12}$$

such that $(x_1(2)^{b_1^1} \ldots x_s(2)^{b_s^1})x_1(3) \ldots x_s(3)$ divides $x_1(2)^{b_1^i} \ldots x_s(2)^{b_s^i}$ for $m < i \leq n$ and $(x_1(2)^{b_1^1} \ldots x_s(2)^{b_s^1})x_1(3) \ldots x_s(3)$ divides $(x_1(2)^{a_{s+1,1}} \ldots x_s(2)^{a_{s+1,s}})^{d_{n+1}}$ in $U_2/m_{U_2}[[x_1(3), \ldots, x_s(3), \tilde{x}_t(2)]]$.

To establish (4.12), we first observe that $\nu(x_1(2)^{b_1^i} \ldots x_s(2)^{b_s^i}) =$

$\nu(\overline{a}_i x_1(2)^{b_1^i} \ldots x_s(2)^{b_s^i}(\tilde{x}_t(2)+\overline{\alpha})^{b_{s+1}^i}) > \rho = \nu(x_1(2)^{b_1^1} \ldots x_s(2)^{b_s^1})$

for $m < i \leq n$, and $\nu((x_1(2)^{a_{s+1,1}} \ldots x_s(2)^{a_{s+1,s}})^{d_{n+1}}) > \rho = \nu(x_1(2)^{b_1^1} \ldots x_s(2)^{b_s^1})$.

Then by Lemma 4.2 of [16] there exists a finite sequence of etale Perron transform along ν of type $I, U_2 \to U_3$ such that U_3 has regular parameters $x_1(3), \ldots, x_s(3)$, $x_{s+1}(2), \ldots, \tilde{x}_t(2), x_{t+1}, \ldots, x_q$ defined by

$$\begin{array}{rcl} x_1(2) & = & x_1(3)^{a'_{11}} \ldots x_s(3)^{a'_{1s}} \\ \vdots & & \\ x_s(2) & = & x_1(3)^{a'_{s1}} \ldots x_s(3)^{a'_{ss}} \end{array} \tag{4.13}$$

such that $x_1(2)^{b_1^1} \ldots x_s(2)^{b_s^1}$ divides $x_1(2)^{b_1^i} \ldots x_s(2)^{b_s^i}$ for $m < i \leq n$, and $x_1(2)^{b_1^1} \ldots x_s(2)^{b_s^1}$ divides $(x_1(2)^{a_{s+1,1}} \ldots x_s(2)^{a_{s+1,s}})^{d_{n+1}}$ in

$U_2/m_{U_2}[[x_1(3),\ldots,x_s(3),\tilde{x}_t(2)]]$.

We have $x_1(2)^{b_1^1}\ldots x_s(2)^{b_s^1}=x_1(3)^{b'^1_1}\ldots x_s(3)^{b'^1_s}$ and $x_1(2)^{b_1^i}\ldots x_s(2)^{b_s^i}=x_1(3)^{b'^i_1}\ldots x_s(3)^{b'^i_s}$ where b'^j_i are natural numbers, and since ρ is the minimal value, then for every $i>m$, there exists j, with $1\le j\le s$ such that $b'^i_j-b'^1_j>0$.

Since the exponents a'_{ij} in an etale Perron transform of type I (4.12) are all positive integers, after possibly performing a finite etale Perron transform of type I (4.12), we obtain that $b'^i_j-b'^1_j>0$ for all j with $1\le j\le s$ and $i>m$.

Define $A_{ij}\in\mathbb{N}$ by

$$\begin{pmatrix} a_{11} & \ldots & a_{1,s+1}\\ \ldots & \ldots & \ldots\\ a_{s+1,1} & \ldots & a_{s+1,s+1}\end{pmatrix}\begin{pmatrix} a'_{11} & \ldots & a'_{1s} & 0\\ \ldots & \ldots & \ldots & \\ a'_{s1} & \ldots & a'_{ss} & 0\\ 0 & \ldots & 0 & 1\end{pmatrix}=\begin{pmatrix} A_{11} & \ldots & A_{1,s+1}\\ \ldots & \ldots & \ldots\\ A_{s+1,1} & \ldots & A_{s+1,s+1}\end{pmatrix}.$$

After making the substitution (4.10) in (4.12) we get

$$\begin{array}{lcl} x_1(1) & = & x_1(3)^{a'_{11}a_{11}+\cdots+a'_{s1}a_{1s}}\ldots x_s(3)^{a'_{1s}a_{11}+\cdots+a'_{ss}a_{1s}}\overline{x}_t(2)^{a_{1,s+1}}\\ \vdots & & \\ x_s(1) & = & x_1(3)^{a'_{11}a_{s1}+\cdots+a'_{s1}a_{ss}}\ldots x_s(3)^{a'_{1s}a_{s1}+\cdots+a'_{ss}a_{ss}}\overline{x}_t(2)^{a_{s,s+1}}\\ x_t & = & x_1(3)^{a'_{11}a_{s+1,1}+\cdots+a'_{s1}a_{s+1,s}}\ldots x_s(3)^{a'_{1s}a_{s+1,1}+\cdots+a'_{ss}a_{s+1,s}}\overline{x}_t(2)^{a_{s+1,s+1}}.\end{array}$$

Thus by the definition of A_{ij}:

$$\begin{array}{lcl} x_1(1) & = & x_1(3)^{A_{11}}\ldots x_s(3)^{A_{1s}}\overline{x}_t(2)^{A_{1,s+1}}\\ \vdots & & \\ x_s(1) & = & x_1(3)^{A_{s1}}\ldots x_s(3)^{A_{ss}}\overline{x}_t(2)^{A_{s,s+1}}\\ x_t & = & x_1(3)^{A_{s+1,1}}\ldots x_s(3)^{A_{s+1,s}}\overline{x}_t(2)^{A_{s+1,s+1}}.\end{array}\tag{4.14}$$

Now we show that we have an expression

$$f=\sum_{i=1}^{m}\overline{a}_i x_1(3)^{B_1^i}\ldots x_s(3)^{B_s^i}(\tilde{x}_t(2)+\overline{\alpha})^{B_{s+1}^i}+x_1(3)^{B_1^1+1}\ldots x_s(3)^{B_s^1+1}\Omega \tag{4.15}$$

with $\Omega \in U_2/m_{U_2}[[x_1(3), \ldots, x_s(3), \tilde{x}_t(2)]]$,

$B^i_j = c^1_i A_{1j} + \cdots + c^s_i A_{sj} + d_i A_{s+1,j}$ for $1 \le i \le m$ and $1 \le j \le s+1$,

$B^i_1 = B^1_1, \ldots, B^i_s = B^1_s$, for $1 \le i \le m$.

To establish (4.15) let $B^i_j = c^1_i A_{1j} + \cdots + c^s_i A_{sj} + d_i A_{s+1,j}$ for $1 \le i \le m$ and $1 \le j \le s+1$,

then

$$(B^i_1 \ldots B^i_s) \quad = \quad (b^i_1 \ldots b^i_s) \quad \begin{pmatrix} a'_{11} & ... & a'_{1s} \\ ... & ... & ... \\ a'_{s1} & ... & a'_{ss} \end{pmatrix}$$

and making the substitution (4.14) into (4.11), we have:

$$f = \sum_{i=1}^{m} \overline{a}_i x_1(3)^{B^i_1} \ldots x_s(3)^{B^i_s} (\tilde{x}_t(2) + \overline{\alpha})^{B^i_{s+1}} + \sum_{i=m+1}^{n} \overline{a}_i x_1(3)^{B^i_1} \ldots x_s(3)^{B^i_s}$$
$$(\tilde{x}_t(2) + \overline{\alpha})^{B^i_{s+1}} + (x_1(3)^{A_{s+1,1}} \ldots x_s(3)^{A_{s+1,s+1}})^{d_{n+1}} \Omega'.$$

Where $\Omega' = \Lambda(x_1(3) \ldots x_s(3)) \in U_2/m_{U_2}[[x_1(3), \ldots, x_s(3), \tilde{x}_t(2)]]$,

and

$$f = \sum_{i=1}^{m} \overline{a}_i x_1(3)^{B^i_1} \ldots x_s(3)^{B^i_s} (\tilde{x}_t(2) + \overline{\alpha})^{B^i_{s+1}} + x_1(3)^{B^1_1+1} \ldots x_s(3)^{B^1_s+1} \Omega$$

.

Moreover, since $\nu(x_1(3)), \ldots, \nu(x_s(3))$ are rationally independent, then $B^i_j = B^1_j$, for $1 \le i \le m$, and $1 \le j \le s$.

By the above analysis, we see that we may thus replace if necessary (a_{ij}) with (A_{ij}), and we have

$$f = \sum_{i=1}^{m} \overline{a}_i x_1(2)^{b^i_1} \ldots x_s(2)^{b^i_s} (\tilde{x}_t(2) + \overline{\alpha})^{b^i_{s+1}} + x_1(2)^{b^1_1+1} \ldots x_s(2)^{b^1_s+1} \Omega$$

, where $b_j^i = c_i^1 a_{1j} + \cdots + c_i^s a_{sj} + d_i a_{s+1j}$ for $1 \le i \le m$ and $1 \le j \le s+1$,

and $b_1^1 = b_1^i, \ldots, b_s^1 = b_s^i$ for $1 \le i \le m$.

Define

$$f = x_1(2)^{b_1^1} \ldots x_s(2)^{b_s^1} (\tilde{x}_t(2) + \overline{\alpha})^{b_{s+1}^1} f_1,$$

where

$$f_1 = \sum_{i=1}^{m} \overline{a}_i (\tilde{x}_t(2) + \overline{\alpha})^{b_{s+1}^i - b_{s+1}^1} + x_1(2) \ldots x_s(2) \Omega$$

. Now $b_1^1 = b_1^i, \ldots, b_s^1 = b_s^i$ for $1 \le i \le m$ which gives the following

$$\begin{pmatrix} a_{11} & \ldots & a_{s+1,1} \\ \ldots & \ldots & \ldots \\ a_{1s} & \ldots & a_{s+1,s} \\ a_{1,s+1} & \ldots & a_{s+1,s+1} \end{pmatrix} \begin{pmatrix} c_i^1 - c_1^1 \\ \ldots \\ c_i^s - c_1^s \\ d_i - d_1 \end{pmatrix} = \begin{pmatrix} b_1^i - b_1^1 \\ \ldots \\ \ldots \\ b_{s+1}^i - b_{s+1}^1 \end{pmatrix} = \begin{pmatrix} 0 \\ \ldots \\ 0 \\ b_{s+1}^i - b_{s+1}^1 \end{pmatrix}.$$

By Cramer's rule

$$\begin{aligned} d_i - d_1 &= \frac{Det \begin{pmatrix} a_{11} & \ldots & a_{s1} & 0 \\ \ldots & \ldots & \ldots & 0 \\ a_{1,s+1} & \ldots & a_{s,s+1} & b_{s+1}^i - b_{s+1}^1 \end{pmatrix}}{Det(a_{ij})} \\ &= \frac{(b_{s+1}^i - b_{s+1}^1) Det \begin{pmatrix} a_{11} & \ldots & a_{s1} \\ \ldots & \ldots & \ldots \\ a_{1s} & \ldots & a_{ss} \end{pmatrix}}{Det(a_{ij})}. \end{aligned}$$

This implies

$$b_{s+1}^i - b_{s+1}^1 = \frac{Det(a_{ij})(d_i - d_1)}{Det \begin{pmatrix} a_{11} & \ldots & a_{s1} \\ \ldots & \ldots & \ldots \\ a_{1s} & \ldots & a_{ss} \end{pmatrix}} = \frac{d_i - d_1}{a} \text{ for } 1 \le i \le m,$$

$$\text{where } a = \pm Det \begin{pmatrix} a_{11} & \ldots & a_{s1} \\ \ldots & \ldots & \ldots \\ a_{1s} & \ldots & a_{ss} \end{pmatrix} \in \mathbb{Z}.$$

Moreover $a | d_i - d_1$ for $1 \le i \le m$.

From now on, we assume $a > 0$. The case $a < 0$ is similar.

Let $e_i = \frac{d_i - d_1}{a}$.

Then

$$f_1 = \sum_{i=1}^{m} \overline{a}_i(\tilde{x}_t(2) + \overline{\alpha})^{\frac{d_i - d_1}{a}} + x_1(2) \dots x_s(2)\Omega.$$

Let $r_1 = \text{ord} f_1(0, \dots, 0, \tilde{x}_t(2)) = \text{ord} \sum_{i=1}^{m} \tilde{a}_i(\tilde{x}_t(2) + \overline{\alpha})^{e_i}$ where $\tilde{a}_i = \overline{a}_i(0, \dots, 0)$ $\in U_2/m_{U_2}$.

Then $r_1 \le \frac{d_m - d_1}{a} \le r$ since $d_m \le r$ by (4.8).

Assume that $r_1 = r$.

Then $d_1 = 0, d_m = r$ and $a = 1$, so that $d_i = e_i$ for $1 \le i \le m$.

ord $f(0, \dots, 0, x_t) = r$ and $d_m = r$ implies a_m is a unit in $T/m_T[[x_1, \dots, x_{t-1}]]$ and thus $a_m = \overline{a}_m$.

So $c_m^1 = \dots = c_m^s = 0$.

Define

$$\xi(t) = f_1(0, \dots, 0, t - \overline{\alpha}) = \sum_{i=1}^{m} \tilde{a}_i t^{e_i},$$

$\text{ord}\xi(t + \overline{\alpha}) = \text{ord} f_1(0, \dots, 0, t) = r = \deg\xi(t + \overline{\alpha})$ since $e_m = r$.

Thus $\xi(t + \overline{\alpha}) = \tilde{a}_m t^r$.

Now

$$\begin{aligned}
\tilde{a}_m t^r =& \xi(t + \overline{\alpha}) = \sum_{i=1}^{m} \tilde{a}_i (t + \overline{\alpha})^{e_i} \\
=& \tilde{a}_m (t + \overline{\alpha})^r + \sum_{i=1}^{m-1} \tilde{a}_i (t + \overline{\alpha})^{e_i} \\
=& \tilde{a}_m t^r + r\overline{\alpha}\tilde{a}_m t^{r-1} + \tilde{a}_{m-1} t^{e_{m-1}} + \text{terms of order} < r - 1.
\end{aligned}$$

Thus $e_{m-1} = r - 1$.

And moreover

$$r\overline{\alpha}\tilde{a}_m + \tilde{a}_{m-1} = 0 \in U_2/m_{U_2}. \tag{4.16}$$

Thus

$$e_m = d_m = r \text{ and } e_{m-1} = d_{m-1} = r - 1, \tag{4.17}$$

since $d_1 = 0$ and $a = 1$.

Now, we compute using (4.16) and (4.17)

$$\begin{aligned} \tfrac{a_m x_t}{a_{m-1}} &= \tfrac{a_m x_t^r}{a_{m-1} x_t^{r-1}} \\ &= \tfrac{a_m x_t^{d_m}}{a_{m-1} x_t^{d_{m-1}}} \\ &= \tfrac{\overline{a}_m x_1(2)^{b_1^m} \ldots x_s(2)^{b_s^m} (\tilde{x}_t(2)+\overline{\alpha})^{b_{s+1}^m}}{\overline{a}_{m-1} x_1(2)^{b_1^{m-1}} \ldots x_s(2)^{b_s^{m-1}} (\tilde{x}_t(2)+\overline{\alpha})^{b_{s+1}^{m-1}}} \\ &= \tfrac{\overline{a}_m (\tilde{x}_t(2)+\overline{\alpha})^{b_{s+1}^m - b_{s+1}^{m-1}}}{\overline{a}_{m-1}}. \end{aligned}$$

We have

$$\begin{aligned} b_{s+1}^m - b_{s+1}^{m-1} &= (b_{s+1}^m - b_{s+1}^1) - (b_{s+1}^{m-1} - b_{s+1}^1) \\ &= e_m - e_{m-1} = r - (r-1) = 1. \end{aligned}$$

Then $\frac{x_t}{a_{m-1}} = \frac{\tilde{x}_t(2)+\overline{\alpha}}{\overline{a}_{m-1}}$.

Taking the residue $\overline{\lambda}'$ of $\frac{x_t}{a_{m-1}}$ in U_2/m_{U_2}, using (4.16) we have that

$$\overline{\lambda}' = \left[\frac{x_t}{a_{m-1}}\right] = \frac{\overline{\alpha}}{\tilde{a}_{m-1}} = -\frac{1}{r\tilde{a}_m} \in T/m_T.$$

Thus $\nu(x_t - \overline{\lambda}' a_{m-1}) > \nu(x_t)$.

$x_t - \overline{\lambda}' a_{m-1} \notin Q_{\hat{T}}$ since $x_t - \overline{\lambda}' a_{m-1} \in T/m_T[[x_1, \ldots, x_t]]$.

Let $\beta = \nu(x_t - \overline{\lambda}_1' a_{m-1})$. There exists $\gamma \in \mathbb{N}$ such that $\nu(m_T^\gamma) > \beta$ and there

exists $\varphi_1 \in T/m_T[x_1, \ldots, x_{t-1}] \subset U$ such that

$\overline{\lambda}' a_{m-1} - \varphi_1 \in (x_1, \ldots, x_{t-1})^\gamma T/m_T[[x_1, \ldots, x_{t-1}]]$.

Then $\beta = \nu(x_t - \varphi_1)$.

Let $x_t' = x_t - \varphi_1$.

Let

$$f'(x_1, \ldots, x_{t-1}, x_t') = f(x_1, \ldots, x_{t-1}, x_t' + \varphi_1) = f(x_1, \ldots, x_t).$$

$\mathrm{ord}(f'(0, \ldots, 0, x_t')) = r$. We repeat the above construction to either achieve a reduction $r_1 < r$, or we obtain a new change of variables in U with $\nu(x_t'') > \nu(x_t')$ and $\nu(x_t) < \nu(x_t') < \nu(x_t'') \leq \nu(f)$.

Since we cannot have an infinite sequence of this kind, we eventually either achieve 2 of the conclusions of the theorem, or find a change of variables in x_t in U, which leads to a reduction $r_1 < r$.

Suppose that $r_1 < r$.

$$x_1(2), \ldots, x_s(2), x_{s+1}, \ldots, x_{t-1}, \tilde{x}_t(2), x_{t+1}, \ldots, x_q$$

are then a regular system of parameters in U_2, and we have an expression

$$f = x_1(2)^{b_1^1} \ldots x_s(2)^{b_s^1} f_1(x_1(2), \ldots, x_s(2), x_{s+1}, \ldots, x_{t-1}, \tilde{x}_t(2))$$

in $\hat{U}_2$, where $r_1 = \mathrm{ord}(f_1(0, \ldots, 0, \tilde{x}_t(2))) < r$.

We iterate the above construction to achieve either 2. of the conclusions of the theorem, or a reduction to $r = 0$, so that the conclusions of the theorem hold for t, and by the induction, the conclusion of the theorem hold. □

Theorem 4.3.3. *Suppose that $x_1, \ldots, x_s, \ldots, x_t, x_{t+1}, \ldots, x_q$ is a regular sequence in T, such that $\nu(x_1), \ldots, \nu(x_s)$ are rationally independent and*

$Q_{\hat{T}} \cap T/m_T[[x_1, \ldots, x_{t+1}]] \neq (0)$ and $Q_{\hat{T}} \cap T/m_T[[x_1, \ldots, x_t]] = (0)$. Then there exists a sequence of etale Perron transforms

$$T \to U \to U_1 \to \cdots \to U_n$$

along ν such that U_n has regular parameters $x_1(n), \ldots, x_{t+1}(n), x_{t+2}, \ldots, x_q$ such that $\nu(x_1(n)), \ldots, \nu(x_s(n))$ are rationally independent and and one of the following holds:

1. *There exists a series $\phi(x_1(n), \ldots, x_t(n)) \in U_n/m_{U_n}[[x_1(n), \ldots, x_t(n)]]$ such that $x_{t+1}(n) - \phi \in Q_{\hat{U}_n}$, and $Q_{\hat{U}_n} \cap U_n/m_{U_n}[[x_1(n), \ldots, x_t(n)]] = (0)$.*

2. *$Q_{\hat{U}_n} \cap U_n/m_{U_n}[[x_1(n), \ldots, x_t(n)]] \neq (0)$.*

Proof. With our assumptions, there exists $0 \neq f \in Q_{\hat{T}} \cap T/m_T[[x_1, \ldots, x_{t+1}]]$. We have that $\mathrm{ord}(f(x_1, \ldots, x_{t+1})) \geq 1$.

Let $r = \mathrm{order}(f(0, \ldots, 0, x_{t+1}))$. If $r = 1$, then 1. of the conclusions of the theorem holds in $\hat{T}$, by the Weierstrass preparation theorem. We may thus assume that $2 \leq r \leq \infty$. Since $\nu(x_1), \ldots, \nu(x_s)$ are rationally independent, $Q_{\hat{T}} \cap T/m_T[[x_1, \ldots, x_s]] = (0)$ so $t \geq s$.

We have an expression

$$f = \sum_{i=1}^{m} a_i(x_1, \ldots, x_t)x_{t+1}^{d_i} + \sum_{m<i\leq n} a_i(x_1, \ldots, x_t)x_{t+1}^{d_i} + \sum_{n<i} a_i(x_1, \ldots, x_t)x_{t+1}^{d_i}$$

where $a_i \in T/m_T[[x_1, \ldots, x_t]]$ and the first sum is over the terms $a_i x_{t+1}^{d_i}$ of minimal value ρ,

$$d_1 < d_2 < \cdots < d_m$$

and we have that $d_i\nu(x_{t+1}) > \rho$ if $i > n$ and $d_{n+1} > d_i$ if $i > n$. By our assumption, $\nu(a_i) < \infty$ for all i. We have that $\nu(x_{t+1}^r) \geq \rho$. Thus

$$d_i \leq r \text{ for } 1 \leq i \leq m. \tag{4.18}$$

Then by Theorem 4.3.2, we have a sequence of etale Perron transforms $T \to U \to U_1$ along ν such that U_1 has regular parameters $x_1(1), \ldots, x_t(1), x_{t+1}, \ldots, x_q$ and such that we have either achieved 2. of the conclusions of the theorem, or there exist unit series

$$\overline{a}_i(x_1(1), \ldots, x_t(1)) \in U_1/m_{U_1}[[x_1(1), \ldots, x_t(1)]],$$

and $c_i \in \mathbb{N}$ satisfying

$$a_i(x_1, \ldots, x_t) = x_1(1)^{c_i^1} \ldots x_s(1)^{c_i^s}\overline{a}_i(x_1(1), \ldots, x_t(1))$$

for $1 \leq i \leq n$, and there exist series

$$b_i(x_1(1), \ldots, x_t(1)) \in U_1/m_{U_1}[[x_1(1), \ldots, x_t(1)]],$$

satisfying

$$a_i(x_1, \ldots, x_t) = b_i(x_1(1), \ldots, x_t(1))$$

for $n < i$. Thus we have

$$f = \sum_{i=1}^{n} \overline{a}_i x_1(1)^{c_i^1} \ldots x_s(1)^{c_i^s} x_{t+1}^{d_i} + \sum_{n<i} b_i x_{t+1}^{d_i} \tag{4.19}$$

We further assume that

$$Q_{\hat{U}_1} \cap U_1/m_{U_1}[[x_1(1), \ldots, x_t(1)]] = (0).$$

Now we define an etale Perron transform of type II_{t+1} $U_1 \rightarrow U_2$ by

$$
\begin{array}{lll}
x_1(1) & = & x_1(2)^{a_{11}} \dots x_s(2)^{a_{1s}} \overline{x}_{t+1}(2)^{a_{1,s+1}} \\
\vdots & & \\
x_s(1) & = & x_1(2)^{a_{s1}} \dots x_s(2)^{a_{ss}} \overline{x}_{t+1}(2)^{a_{s,s+1}} \\
x_{t+1} & = & x_1(2)^{a_{s+1,1}} \dots x_s(2)^{a_{s+1,s}} \overline{x}_{t+1}(2)^{a_{s+1,s+1}}.
\end{array}
\tag{4.20}
$$

We have that $\nu(x_1(2)), \dots, \nu(x_s(2))$ are rationally independent and

$\nu(\overline{x}_{t+1}(2)) = 0$.

Let $\overline{\alpha}$ be the residue of $\overline{x}_{t+1}(2)$ in U_2/m_{U_2}. Let $\tilde{x}_{t+1}(2) = \overline{x}_{t+1}(2) - \overline{\alpha}$. Then

$$x_1(2), \dots, x_s(2), x_{s+1}(1), \dots, x_t(1), \tilde{x}_{t+1}(2), x_{t+2}, \dots, x_q$$

are regular parameters in U_2, and

$$\hat{U}_2 = U_2/m_{U_2}[[x_1(2), \dots, x_s(2), x_{s+1}(1), \dots, x_t(1), \tilde{x}_{t+1}(2), x_{t+2}, \dots, x_q]].$$

We may assume that

$$Q_{\hat{U}_2} \cap U_2/m_{U_2}[[x_1(2), \dots, x_s(2), x_{s+1}(1), \dots, x_t(1)]] = (0)$$

since otherwise we have achieved 2. of the conclusions of the theorem.

We will now show that the following formula holds

$$
\begin{aligned}
f = \sum_{i=1}^{m} \overline{a}_i x_1(2)^{b_1^i} \dots x_s(2)^{b_s^i} (\tilde{x}_{t+1}(2) + \overline{\alpha})^{b_{s+1}^i} + \\
\sum_{i=m+1}^{n} \overline{a}_i x_1(2)^{b_1^i} \dots x_s(2)^{b_s^i} (\tilde{x}_{t+1}(2) + \overline{\alpha})^{b_{s+1}^i} + (x_1(2)^{a_{s+1,1}} \dots x_s(2)^{a_{s+1,s}})^{d_{n+1}} \Lambda
\end{aligned}
\tag{4.21}
$$

with $\Lambda \in U_2/m_{U_2}[[x_1(2), \dots, x_s(2), \tilde{x}_{t+1}(2)]]$,

$b_j^i = c_i^1 a_{1j} + \dots + c_i^s a_{sj} + d_i a_{s+1,j}$ for $1 \leq i \leq m$ and $1 \leq j \leq s+1$,

$b_1^i = b_1^1, \dots, b_s^i = b_s^1$, for $1 \leq i \leq m$,

$\nu(x_1(2)^{b_1^i} \ldots x_s(2)^{b_s^i}) > \rho$ for $m < i \leq n$ and $\nu((x_1(2)^{a_{s+1,1}} \ldots x_s(2)^{a_{s+1,s}})^{d_{n+1}}) > \rho$.

To establish (4.21), we first recall that we have

$$f = \sum_{i=1}^{n} \overline{a}_i x_1(1)^{c_i^1} \ldots x_s(1)^{c_i^s} x_{t+1}^{d_i} + \sum_{n<i} b_i x_{t+1}^{d_i}.$$

and since $d_i > d_{n+1}$ for $i > n+1$ then:

$$f = \sum_{i=1}^{n} \overline{a}_i x_1(1)^{c_i^1} \ldots x_s(1)^{c_i^s} x_{t+1}^{d_i} + x_{t+1}^{d_{n+1}} \sum_{n<i} b_i x_{t+1}^{d_i - d_{n+1}}.$$

Substituting (4.19) into f, we have

$$f = \sum_{i=1}^{m} \overline{a}_i x_1(2)^{b_1^i} \ldots x_s(2)^{b_s^i} (\tilde{x}_{t+1}(2) + \overline{\alpha})^{b_{s+1}^i} + \sum_{i=m+1}^{n} \overline{a}_i x_1(2)^{b_1^i} \ldots x_s(2)^{b_s^i}$$
$$(\tilde{x}_{t+1}(2) + \overline{\alpha})^{b_{s+1}^i} + (x_1(2)^{a_{s+1,1}} \ldots x_s(2)^{a_{s+1,s}})^{d_{n+1}} \sum_{n<i} b_i x_1(2)^{a_{s+1,1}(d_i - d_{n+1})}$$
$$\ldots x_s(2)^{a_{s+1,s}(d_i - d_{n+1})} (\tilde{x}_{t+1}(2) + \overline{\alpha})^{a_{s+1,s+1} d_i},$$

where $b_j^i = c_i^1 a_{1j} + \cdots + c_i^s a_{sj} + d_i a_{s+1,j}$ for $1 \leq i \leq m$ and $1 \leq j \leq s+1$.

We have that

$$\rho = \nu(\overline{a}_1 x_1(2)^{b_1^1} \ldots x_s(2)^{b_s^1} (\tilde{x}_{t+1}(2) + \overline{\alpha})^{b_{s+1}^1})$$
$$= \nu(\overline{a}_i x_1(2)^{b_1^i} \ldots x_s(2)^{b_s^i} (\tilde{x}_{t+1}(2) + \overline{\alpha})^{b_{s+1}^i}) \text{ for } 1 \leq i \leq m,$$

thus $\nu(x_1(2))b_1^1 + \cdots + \nu(x_s(2))b_s^1 = \nu(x_1(2))b_1^i + \cdots + \nu(x_s(2))b_s^i$,

thus $\nu(x_1(2))(b_1^1 - b_1^i) + \cdots + \nu(x_s(2))(b_s^1 - b_s^i) = 0$,

and hence $b_1^1 = b_1^i, \ldots, b_s^1 = b_s^i$ since $\nu(x_1(2)), \ldots, \nu(x_s(2))$ are rationally independent.

Moreover $\nu(\overline{a}_i x_1(2)^{b_1^i} \ldots x_s(2)^{b_s^i} (\tilde{x}_{t+1}(2) + \overline{\alpha})^{b_{s+1}^i}) > \rho$ for $m < i \leq n$, and since $\nu(x_{t+1}^{d_{n+1}}) > \rho$ then $\nu((x_1(2)^{a_{s+1,1}} \ldots x_s(2)^{a_{s+1,s}})^{d_{n+1}}) > \rho$.

Let

$$\Lambda = \sum_{n<i} b_i x_1(2)^{a_{s+1,1}(d_i-d_{n+1})} \dots x_s(2)^{a_{s+1,s}(d_i-d_{n+1})} (\tilde{x}_{t+1}(2)+\overline{\alpha})^{a_{s+1,s+1}d_i},$$

then f has the desired form (4.21).

Now we will show that there exists an etale Perron transform of type I:

$$\begin{array}{lcl} x_1(2) & = & x_1(3)^{a'_{11}} \dots x_s(3)^{a'_{1s}} \\ \vdots & & \\ x_s(2) & = & x_1(3)^{a'_{s1}} \dots x_s(3)^{a'_{ss}} \end{array} \tag{4.22}$$

such that $(x_1(2)^{b_1^1} \dots x_s(2)^{b_s^1}) x_1(3) \dots x_s(3)$ divides $x_1(2)^{b_1^i} \dots x_s(2)^{b_s^i}$ for $m < i \le n$

and $(x_1(2)^{b_1^1} \dots x_s(2)^{b_s^1}) x_1(3) \dots x_s(3)$ divides $(x_1(2)^{a_{s+1,1}} \dots x_s(2)^{a_{s+1,s}})^{d_{n+1}}$

in $U_2/m_{U_2}[[x_1(3), \dots, x_s(3), \tilde{x}_{t+1}(2)]]$.

To establish (4.22) we first observe that

$$\nu(x_1(2)^{b_1^i} \dots x_s(2)^{b_s^i}) = \nu(\overline{a}_i x_1(2)^{b_1^i} \dots x_s(2)^{b_s^i} (\tilde{x}_{t+1}(2)+\overline{\alpha})^{b_{s+1}^i}) > \rho$$

for $m < i \le n$,

and $\rho = \nu(x_1(2)^{b_1^1} \dots x_s(2)^{b_s^1})$, thus $\nu(x_1(2)^{b_1^i} \dots x_s(2)^{b_s^i}) > \nu(x_1(2)^{b_1^i} \dots x_s(2)^{b_s^i})$ for

$m < i \le n$,

and $\nu((x_1(2)^{a_{s+1,1}} \dots x_s(2)^{a_{s+1,s}})^{d_{n+1}}) > \rho = \nu(x_1(2)^{b_1^1} \dots x_s(2)^{b_s^1})$.

Then by Lemma 4.2 [16] there exists a finite sequence of etale Perron transform along ν of type $I, U_2 \to U_3$ such that U_3 has regular parameters

$x_1(3), \dots, x_s(3), x_{s+1}(2), \dots, \tilde{x}_{t+1}(2), x_{t+2} \dots, x_q$ defined by

$$\begin{array}{lcl} x_1(2) & = & x_1(3)^{a'_{11}} \dots x_s(3)^{a'_{1s}} \\ \vdots & & \\ x_s(2) & = & x_1(3)^{a'_{s1}} \dots x_s(3)^{a'_{ss}} \end{array} \tag{4.23}$$

such that $x_1(2)^{b_1^1} \dots x_s(2)^{b_s^1}$ divides $x_1(2)^{b_1^i} \dots x_s(2)^{b_s^i}$ for $m < i \le n$ and

$x_1(2)^{b_1^1} \dots x_s(2)^{b_s^1}$ divides $(x_1(2)^{a_{s+1,1}} \dots x_s(2)^{a_{s+1,s}})^{d_{n+1}}$ in

$U_2/m_{U_2}[[x_1(3),\dots,x_s(3),\tilde{x}_{t+1}(2)]]$.

We have $x_1(2)^{b_1^1}\dots x_s(2)^{b_s^1} = x_1(3)^{b'^1_1}\dots x_s(3)^{b'^1_s}$ and $x_1(2)^{b_1^i}\dots x_s(2)^{b_s^i} = x_1(3)^{b'^i_1}\dots x_s(3)^{b'^i_s}$, where b'^j_i are natural numbers, and since ρ is the minimal value, then for all $i > m$ there exists j; $1 \le j \le s$ such that $b'^i_j - b'^1_j > 0$.

Since the exponents a'_{ij} in an etale Perron transform of type I (4.22) are all positive integers, after possibly performing a finite etale Perron transform of type I (4.22), we obtain that $b'^i_j - b'^1_j > 0$ for all j with $1 \le j \le s$ and $i > m$.

Define $A_{ij} \in \mathbb{N}$ by

$$\begin{pmatrix} a_{11} & \dots & a_{1,s+1} \\ \dots & \dots & \dots \\ a_{s+1,1} & \dots & a_{s+1,s+1} \end{pmatrix} \begin{pmatrix} a'_{11} & \dots & a'_{1s} & 0 \\ \dots & \dots & \dots & \\ a'_{s1} & \dots & a'_{ss} & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix} = \begin{pmatrix} A_{11} & \dots & A_{1,s+1} \\ \dots & \dots & \dots \\ A_{s+1,1} & \dots & A_{s+1,s+1} \end{pmatrix}.$$

After making the substitution (4.19) in (4.22) we get

$$\begin{array}{lll} x_1(1) & = & x_1(3)^{a'_{11}a_{11}+\dots+a'_{s1}a_{1s}}\dots x_s(3)^{a'_{1s}a_{11}+\dots+a'_{ss}a_{1s}}\overline{x}_{t+1}(2)^{a_{1,s+1}} \\ \vdots & & \\ x_s(1) & = & x_1(3)^{a'_{11}a_{s1}+\dots+a'_{s1}a_{ss}}\dots x_s(3)^{a'_{1s}a_{s1}+\dots+a'_{ss}a_{ss}}\overline{x}_{t+1}(2)^{a_{s,s+1}} \\ x_t & = & x_1(3)^{a'_{11}a_{s+1,1}+\dots+a'_{s1}a_{s+1,s}}\dots x_s(3)^{a'_{1s}a_{s+1,1}+\dots+a'_{ss}a_{s+1,s}}\overline{x}_{t+1}(2)^{a_{s+1,s+1}}. \end{array}$$

Thus by the definition of A_{ij}:

$$\begin{array}{lll} x_1(1) & = & x_1(3)^{A_{11}}\dots x_s(3)^{A_{1s}}\overline{x}_{t+1}(2)^{A_{1,s+1}} \\ \vdots & & \\ x_s(1) & = & x_1(3)^{A_{s1}}\dots x_s(3)^{A_{ss}}\overline{x}_{t+1}(2)^{A_{s,s+1}} \\ x_t & = & x_1(3)^{A_{s+1,1}}\dots x_s(3)^{A_{s+1,s}}\overline{x}_{t+1}(2)^{A_{s+1,s+1}}. \end{array} \tag{4.24}$$

Now we show that we have an expression

$$f = \sum_{i=1}^{m} \overline{a}_i x_1(3)^{B_1^i}\dots x_s(3)^{B_s^i}(\tilde{x}_{t+1}(2)+\overline{\alpha})^{B_{s+1}^i} + x_1(3)^{B_1^1+1}\dots x_s(3)^{B_s^1+1}\Omega \tag{4.25}$$

with $\Omega \in U_2/m_{U_2}[[x_1(3),\dots,x_s(3),\tilde{x}_{t+2}(2)]]$, $B_j^i = c_i^1A_{1j}+\dots+c_i^sA_{sj}+d_iA_{s+1,j}$ for $1 \le i \le m$ and $1 \le j \le s+1$, and $B_1^i = B_1^1,\dots,B_s^i = B_s^1$, for $1 \le i \le m$.

To establish (4.25) let $B_j^i = c_i^1 A_{1j} + \cdots + c_i^s A_{sj} + d_i A_{s+1,j}$ for $1 \leq i \leq m$ and $1 \leq j \leq s+1$.

Then

$$(B_1^i \dots B_s^i) \;=\; (b_1^i \dots b_s^i) \begin{pmatrix} a'_{11} & ... & a'_{1s} \\ ... & ... & ... \\ a'_{s1} & ... & a'_{ss} \end{pmatrix}$$

Making the substitution (4.24) into (4.21), we have:

$$f = \sum_{i=1}^{m} \overline{a}_i x_1(3)^{B_1^i} \dots x_s(3)^{B_s^i} (\tilde{x}_{t+1}(2) + \overline{\alpha})^{B_{s+1}^i} + \sum_{i=m+1}^{n} \overline{a}_i x_1(3)^{B_1^i} \dots x_s(3)^{B_s^i} (\tilde{x}_{t+1}(2) + \overline{\alpha})^{B_{s+1}^i} + (x_1(3)^{A_{s+1,1}} \dots x_s(3)^{A_{s+1,s+1}})^{d_{n+1}} \Omega'$$

where $\Omega' = \Lambda(x_1(3) \dots x_s(3)) \in U_2/m_{U_2}[[x_1(3), \dots, x_s(3), \tilde{x}_{t+1}(2)]]$,

and

$$f = \sum_{i=1}^{m} \overline{a}_i x_1(3)^{B_1^i} \dots x_s(3)^{B_s^i} (\tilde{x}_{t+1}(2) + \overline{\alpha})^{B_{s+1}^i} + x_1(3)^{B_1^1+1} \dots x_s(3)^{B_s^1+1} \Omega.$$

Moreover, since $\nu(x_1(3)), \dots, \nu(x_s(3))$ are rationally independent, then $B_j^i = B_j^1$ for $1 \leq i \leq m$, and $1 \leq j \leq s$.

By the above analysis, we see that we may thus replace if necessary (a_{ij}) with (A_{ij}), and we have

$$f = \sum_{i=1}^{m} \overline{a}_i x_1(2)^{b_1^i} \dots x_s(2)^{b_s^i} (\tilde{x}_{t+1}(2) + \overline{\alpha})^{b_{s+1}^i} + x_1(2)^{b_1^1+1} \dots x_s(2)^{b_s^1+1} \Omega,$$

where $b_j^i = c_i^1 a_{1j} + \cdots + c_i^s a_{sj} + d_i a_{s+1j}$ for $1 \leq i \leq m$ and $1 \leq j \leq s+1$,

and $b_1^1 = b_1^i, \dots, b_s^1 = b_s^i$ for $1 \leq i \leq m$.

Define

$$f = x_1(2)^{b_1^1} \dots x_s(2)^{b_s^1} (\tilde{x}_{t+1}(2) + \overline{\alpha})^{b_{s+1}^1} f_1$$

where

$$f_1 = \sum_{i=1}^{m} \overline{a}_i(\tilde{x}_{t+1}(2) + \overline{\alpha})^{b^i_{s+1} - b^1_{s+1}} + x_1(2) \ldots x_s(2)\Omega.$$

Now $b^1_1 = b^i_1, \ldots, b^1_s = b^i_s$ for $1 \leq i \leq m$ which gives the following

$$\begin{pmatrix} a_{11} & \ldots & a_{s+1,1} \\ \ldots & \ldots & \ldots \\ a_{1s} & \ldots & a_{s+1,s} \\ a_{1,s+1} & \ldots & a_{s+1,s+1} \end{pmatrix} \begin{pmatrix} c^1_i - c^1_1 \\ \ldots \\ c^s_i - c^s_1 \\ d_i - d_1 \end{pmatrix} = \begin{pmatrix} b^i_1 - b^1_1 \\ \ldots \\ \ldots \\ b^i_{s+1} - b^1_{s+1} \end{pmatrix} = \begin{pmatrix} 0 \\ \ldots \\ 0 \\ b^i_{s+1} - b^1_{s+1} \end{pmatrix}.$$

By Cramer's rule

$$\begin{aligned} d_i - d_1 &= \frac{Det\begin{pmatrix} a_{11} & \ldots & a_{s1} & 0 \\ \ldots & \ldots & \ldots & 0 \\ a_{1,s+1} & \ldots & a_{s,s+1} & b^i_{s+1} - b^1_{s+1} \end{pmatrix}}{Det(a_{ij})} \\ &= \frac{(b^i_{s+1} - b^1_{s+1})Det\begin{pmatrix} a_{11} & \ldots & a_{s1} \\ \ldots & \ldots & \ldots \\ a_{1s} & \ldots & a_{ss} \end{pmatrix}}{Det(a_{ij})}. \end{aligned}$$

Thus $b^i_{s+1} - b^1_{s+1} = \frac{Det(a_{ij})(d_i - d_1)}{Det\begin{pmatrix} a_{11} & \ldots & a_{s1} \\ \ldots & \ldots & \ldots \\ a_{1s} & \ldots & a_{ss} \end{pmatrix}} = \frac{d_i - d_1}{a}$ for $1 \leq i \leq m$ where

$$a = \pm Det\begin{pmatrix} a_{11} & \ldots & a_{s1} \\ \ldots & \ldots & \ldots \\ a_{1s} & \ldots & a_{ss} \end{pmatrix} \in \mathbb{Z}.$$

Moreover $a | d_i - d_1$ for $1 \leq i \leq m$.

From now on, we assume $a > 0$. The case $a < 0$ is similar.

Let $e_i = \frac{d_i - d_1}{a}$.

This implies

$$f_1 = \sum_{i=1}^{m} \overline{a}_i(\tilde{x}_{t+1}(2) + \overline{\alpha})^{\frac{d_i - d_1}{a}} + x_1(2) \ldots x_s(2)\Omega.$$

Let $r_1 = \text{ord} f_1(0,\dots,0,\tilde{x}_{t+1}(2)) = \text{ord} \sum_{i=1}^{m} \tilde{a}_i(\tilde{x}_{t+1}(2)+\overline{\alpha})^{e_i}$ where

$\tilde{a}_i = \overline{a}_i(0,\dots,0) \in U_2/m_{U_2}$.

Then $r_1 \leq \frac{d_m - d_1}{a} \leq r$ since $d_m \leq r$ by (4.18).

Assume that $r_1 = r$**.**

Then $d_1 = 0, d_m = r$ and $a = 1$, so that $d_i = e_i$ for $1 \leq i \leq m$.

ord $f(0,\dots,0,x_{t+1}) = r$ and $d_m = r$ imply that a_m is a unit in $T/m_T[[x_1,\dots,x_t]]$ thus $a_m = \overline{a}_m$ and $c_m^1 = \cdots = c_m^s = 0$.

Define $\xi(t) = f_1(0,\dots,0,t-\overline{\alpha}) = \sum_{i=1}^{m} \tilde{a}_i t^{e_i}$.

$\text{ord}\xi(t+\overline{\alpha}) = \text{ord} f_1(0,\dots,0,t) = r = \deg\xi(t+\overline{\alpha})$, since $e_m = r$.

Thus $\xi(t+\overline{\alpha}) = \tilde{a}_m t^r$.

Now

$$\begin{aligned}\tilde{a}_m t^r =& \xi(t+\overline{\alpha}) = \sum_{i=1}^{m} \tilde{a}_i (t+\overline{\alpha})^{e_i} \\ =& \tilde{a}_m (t+\overline{\alpha})^r + \sum_{i=1}^{m-1} \tilde{a}_i (t+\overline{\alpha})^{e_i} \\ =& \tilde{a}_m t^r + r\overline{\alpha}\tilde{a}_m t^{r-1} + \tilde{a}_{m-1} t^{e_{m-1}} + \text{terms of order} < r-1.\end{aligned}$$

This implies that $e_{m-1} = r-1$

and moreover

$$r\overline{\alpha}\tilde{a}_m + \tilde{a}_{m-1} = 0 \in U_2/m_{U_2}. \tag{4.26}$$

Thus

$$e_m = d_m = r \text{ and } e_{m-1} = d_{m-1} = r-1, \tag{4.27}$$

since $d_1 = 0$ and $a = 1$.

Now, we compute using (4.26) and (4.27)

$$\begin{array}{rcl} \frac{a_m x_{t+1}}{a_{m-1}} & = & \frac{a_m x_{t+1}^r}{a_{m-1} x_{t+1}^{r-1}} \\ & = & \frac{a_m x_{t+1}^{d_m}}{a_{m-1} x_{t+1}^{d_{m-1}}} \\ & = & \frac{\overline{a}_m x_1(2)^{b_1^m} \ldots x_s(2)^{b_s^m} (\tilde{x}_{t+1}(2)+\overline{\alpha})^{b_{s+1}^m}}{\overline{a}_{m-1} x_1(2)^{b_1^{m-1}} \ldots x_s(2)^{b_s^{m-1}} (\tilde{x}_{t+1}(2)+\overline{\alpha})^{b_{s+1}^{m-1}}} \\ & = & \frac{\overline{a}_m (\tilde{x}_{t+1}(2)+\overline{\alpha})^{b_{s+1}^m - b_{s+1}^{m-1}}}{\overline{a}_{m-1}}. \end{array}$$

We have

$$\begin{array}{rcl} b_{s+1}^m - b_{s+1}^{m-1} & = & (b_{s+1}^m - b_{s+1}^1) - (b_{s+1}^{m-1} - b_{s+1}^1) \\ & = & e_m - e_{m-1} = r - (r-1) = 1. \end{array}$$

Then $\frac{x_{t+1}}{a_{m-1}} = \frac{\tilde{x}_{t+1}(2)+\overline{\alpha}}{\overline{a}_{m-1}}$.

Taking the residue $\overline{\lambda}'$ of $\frac{x_{t+1}}{a_{m-1}}$ in U_2/m_{U_2}, we have that

$$\overline{\lambda}' = \left[\frac{x_{t+1}}{a_{m-1}}\right] = \frac{\overline{\alpha}}{\tilde{a}_{m-1}} = -\frac{1}{r\tilde{a}_m} \in T/m_T.$$

Thus $\nu(x_{t+1} - \overline{\lambda}' a_{m-1}) > \nu(x_{t+1})$. If $x_{t+1} - \overline{\lambda}' a_{m-1} \in Q_{\hat{T}}$, then we have reached the conclusions of 1. of the theorem. Assume that $x_{t+1} - \overline{\lambda}' a_{m-1} \notin Q_{\hat{T}}$. Let $\beta = \nu(x_{t+1} - \overline{\lambda}' a_{m-1})$.

There exists $\gamma \in \mathbb{N}$ such that $\nu(m_T^\gamma) > \beta$ and there exists $\varphi_1 \in T/m_t[x_1, \ldots, x_t] \subset U$ such that $\overline{\lambda}' a_{m-1} - \varphi_1 \in (x_1, \ldots, x_t)^\gamma T/m_T[[x_1, \ldots, x_t]]$.

Then $\beta = \nu(x_{t+1} - \varphi_1)$.

Let $x'_{t+1} = x_{t+1} - \varphi_1$.

$$f'(x_1, \ldots, x_t, x'_{t+1}) = f(x_1, \ldots, x_t, x'_{t+1} + \varphi_1) = f(x_1, \ldots, x_{t+1}).$$

$\mathrm{ord}(f'(0, \ldots, 0, x'_{t+1})) = r$. We repeat the above construction to either achieve a reduction $r_1 < r$, or we obtain a new change of variables with

$$\nu(x''_{t+1}) > \nu(x'_{t+1}) > \nu(x_{t+1}).$$

Iterating this step, we either construct an infinite sequence

$$0 < \nu(x_{t+1}) < \nu(x'_{t+1}) < \cdots \nu(x^{(n)}_{t+1}) < \cdots \tag{4.28}$$

or we find a change of variables in x_{t+1} which leads to a reduction $r_1 < r$. Suppose that (4.28) holds, then by construction, there exist polynomials $a^{(n)} \in T/m_T[x_1, \ldots, x_t]$, such that $x^{(n+1)}_{t+1} = x^{(n)}_{t+1} - a^{(n)}, \nu(a^{(n)}) = \nu(x^{(n)}_{t+1})$ for $n \geq 0$ where $x^{(1)}_{t+1} = x'_{t+1}$, $x^{(0)}_{t+1} = x_{t+1}$.

Let $\tau_n = \nu(x^{(n)}_{t+1})$ for $n \geq 0$ and $\tau_0 = \nu(x_{t+1})$.

Then we have an infinite sequence $\tau_0 < \tau_1 < \cdots < \tau_n < \ldots$

and thus

$$\lim_{n\to\infty} \tau_n = \infty \text{ by lemma 2.3 of [16].}$$

Let $A = T/m_T[[x_1, \ldots, x_t]]$.

Let

$$\alpha^{(n)} = \sum_{j=0}^{n-1} a^{(j)} \in A,$$

then $x^{(n)}_{t+1} = x_{t+1} - \alpha^{(n)}$ and $\nu(\alpha^{(n)}) = \nu(x^{(n)}_{t+1}) = \tau_n$.

Let $\delta_n = \{f \in A/\nu(f) \geq \tau_n\} \subseteq A$.

This implies that

$$\alpha^{(n)} - \alpha^{(n+i)} = x^{(n)}_{t+1} - x^{(n+i)}_{t+1} = \sum_{j=n}^{n+i-1} a^{(j)} \in \delta_n.$$

If $f \in \cap_{n=0}^{\infty}\delta_n$ then $\nu(f) = \infty$ and thus $f \in Q_{\hat{T}} \cap A = (0)$ by assumption, thus proving that $\cap_{n=0}^{\infty}\delta_n = (0)$. By Theorem (13) p.270 of [46] there exists an integral valued function $S(n)$ such that $S(n) \to \infty$ as $n \to \infty$ and such that $\delta_n \subseteq m_A^{S(n)}$. This implies that $\alpha^{(n)} - \alpha^{(n+i)} \in m_A^{S(n)}$ for all $i \geq 0$ and thus $\{\alpha^{(n)}\}$ is a Cauchy Sequence in A.

So $\{x_{t+1}^{(n)}\}$ is a Cauchy sequence in $\hat{T}$.

Then

$$\lim_{n\to\infty} x_{t+1}^{(n)} = x_{t+1} - \lim_{n\to\infty} \alpha^{(n)}$$

where

$$\nu(\lim_{n\to\infty} x_{t+1}^{(n)}) = \infty.$$

Let

$$\phi_{t+1}(x_1, \ldots, x_t) = \lim_{n\to\infty} \alpha^{(n)}$$

then $x_{t+1} - \phi_{t+1} \in Q_{\hat{T}}$.

Thus 1. of the conclusions of the theorem holds.

Suppose that $r_1 < r$.

$$x_1(3), \ldots, x_s(3), x_{s+1}(1), \ldots, x_t(1), \tilde{x}_{t+1}(2), x_{t+2}, \ldots, x_q$$

are then a regular system of parameters in U_3, and we have an expression

$$f = x_1(3)^{B_1^1} \ldots x_s(3)^{B_s^1} f_1$$

in $\hat{U}_3$, where $r_1 = \text{ord}(f_1(0, \ldots, 0, \tilde{x}_{t+1}(2))) < r$.

Iteration We iterate the above construction, applied to $f_1 \in \hat{U}_3$.

We must eventually reach a local ring U_n where either 2. of the conclusions of the theorem hold, or $f = x_1(n)^{a_1} \ldots x_s(n)^{a_s} f_n$ where $r_n =$ $\text{ord}(f_n(x_1(n), \ldots, x_{t+1}(n))) \leq 1$. $f \in Q_{\hat{T}}$ implies that $r_n = 1$. By the Weierstrass preparation theorem, there exists a series $\phi_{t+1}(x_1(n), \ldots, x_t(n))$ such that

$x_{t+1} - \phi_{t+1}(x_1(n), \ldots, x_t(n)) \in Q_{\hat{U}_n}$, so that 1. of the conclusions of the theorem holds.

$\square$

Theorem 4.3.4. *Suppose that $x_1, \ldots, x_s, \ldots, x_a, \ldots x_q$ is a regular system of parameters in T such that $\nu(x_1), \ldots, \nu(x_s)$ are rationally independent, and there exist $\phi_i \in T/m_T[[x_1, \ldots, x_{i-1}]]$ such that $x_i - \phi_i \in Q_{\hat{T}}$ for $a+1 \leq i \leq q$. Further suppose that $Q_{\hat{T}} \cap T/m_T[[x_1, \ldots, x_a]] \neq (0)$. Then there exists a sequence of etale Perron transforms*

$$T \to U \to U_1 \to \cdots \to U_n$$

along ν such that U_n has a regular system of parameters $x_1(n), \ldots, x_a(n), x_{a+1}, \ldots,$ x_q, and $\phi_a \in U_n/m_{U_n}[[x_1(n), \ldots, x_{a-1}(n)]]$ such that $\nu(x_1(n)), \ldots, \nu(x_s(n))$ are rationally independent and such that $x_a(n) - \phi_a \in Q_{\hat{U}_n}$. We further have

$$\phi_i \in U_n/m_{U_n}[[x_1(n), \ldots, x_a(n), x_{a+1}, \ldots, x_{i-1}]]$$

for $a + 1 \leq i \leq q$.

Proof. Set $a(T) = a$. Let $t = t(T)$ be defined from the (ordered) regular sequence $x_1, \ldots, x_a, \ldots, x_q$ by the condition that $Q_{\hat{T}} \cap T/m_T[[x_1, \ldots, x_t]] = (0)$ and $Q_{\hat{T}} \cap T/m_T[[x_1, \ldots, x_{t+1}]] \neq (0)$. We have $1 \leq t(T) \leq a - 1$.

By Theorem 4.3.3, we can construct a sequence of etale Perron transforms $T \to U \to U_1$ along ν such that U_1 has regular parameters

$$x_1(1), \ldots, x_t(1), x_{t+1}(1), \ldots, x_{a+1}, \ldots, x_q$$

such that either 1. or 2. of the conclusions of Theorem 4.3.3 hold.

Suppose that 1. of the conclusions of Theorem 4.3.3 holds. Then after interchanging $x_{t+1}(1)$ and x_a (if $t+1 \neq a$) we have attained the conclusions of the theorem.

Suppose that 2 of the conclusions of Theorem 4.3.3 holds. Then the ordered regular sequence $x_1(1), \ldots, x_t(1), x_{t+1}(1), \ldots, x_{a+1}, \ldots, x_q$ in U_1 satisfies the assumptions of Theorem 4.3.4, with $a(U_1) < a(T)$.

We now apply Theorem 4.3.3 to this regular sequence in U_1, with $t(U_1) \leq a(U_1) - 1$. After finitely many iterations of Theorem 4.3.3 we must achieve the conclusions of Theorem 4.3.4. $\square$

Lemma 4.3.5. *Suppose that $x_1, \ldots, x_t, \ldots x_q$ is a regular system of parameters in T, and there exist $\phi_i \in T/m_T[[x_1, \ldots, x_{i-1}]]$ such that $x_i - \phi_i \in Q_{\hat{T}}$ for $t+1 \leq i \leq q$. Then there exists $\overline{\phi}_i \in T/m_T[[x_1, \ldots, x_t]]$ such that $x_i - \overline{\phi}_i \in Q_{\hat{T}}$ for $t+1 \leq i \leq q$.*

Proof. We prove this by induction:

Set $\overline{\phi}_{t+1} = \phi_{t+1} \in T/m_T[[x_1, \ldots, x_t]]$.

Then

$$\begin{aligned} x_{t+2} - \phi_{t+2}(x_1, \ldots, x_{t+1}) &= x_{t+2} - \phi_{t+2}(x_1, \ldots, x_t, (x_{t+1} - \overline{\phi}_{t+1}) + \overline{\phi}_{t+1}) \\ &= x_{t+2} - \phi_{t+2}(x_1, \ldots, x_t, \overline{\phi}_{t+1}) + \Omega^{(1)}_{t+2}[x_{t+1} - \overline{\phi}_{t+1}]. \end{aligned}$$

Set $\overline{\phi}_{t+2} = \phi_{t+2}(x_1, \ldots, x_t, \overline{\phi}_{t+1}) \in U_1/m_{U_1}[[x_1, \ldots, x_t]]$ since $\overline{\phi}_{t+1} \in T/m_T[[x_1, \ldots, x_t]]$. Then $x_{t+2} - \overline{\phi}_{t+2} \in Q_{\hat{T}}$ since $x_{t+2} - \phi_{t+2}$ and $x_{t+1} - \overline{\phi}_{t+1}$ are in $Q_{\hat{T}}$.

Assume that it is true up to $j-1$.

Then

$$\begin{aligned}x_{t+j} - \phi_{t+j}(x_1, \ldots, x_{t+j-1}) &= x_{t+j} - \phi_{t+j}(x_1, \ldots, x_t, (x_{t+1} - \overline{\phi}_{t+1}) + \\ &\quad \overline{\phi}_{t+1}, \ldots, (x_{t+j-1} - \overline{\phi}_{t+j-1}) + \overline{\phi}_{t+j-1}) \\ &= x_{t+j} - \phi_{t+j}(x_1, \ldots, x_t, \overline{\phi}_{t+1}, \ldots, \overline{\phi}_{t+j-1}) + \\ &\quad \Omega^{(1)}_{t+j}[x_{t+1} - \overline{\phi}_{t+1}] + \cdots + \Omega^{(j-1)}_{t+j}[x_{t+j-1} - \overline{\phi}_{t+j-1}].\end{aligned}$$

Set $\overline{\phi}_{t+j} = \phi_{t+j}(x_1, \ldots, x_t, \overline{\phi}_{t+1}, \ldots, \overline{\phi}_{t+j-1}) \in T/m_T[[x_1, \ldots, x_t]]$, since $\overline{\phi}_{t+1}, \ldots, \overline{\phi}_{t+j-1} \in T/m_T[[x_1, \ldots, x_t]]$.

Then $x_{t+j} - \overline{\phi}_{t+j} \in Q_{\hat{T}}$, since $x_{t+j} - \phi_{t+j}, x_{t+1} - \overline{\phi}_{t+1}, \ldots, x_{t+j-1} - \overline{\phi}_{t+j-1} \in Q_{\hat{T}}$. □

Lemma 4.3.6. *Suppose that $x_1, \ldots, x_a, \ldots x_q$ is a regular system of parameters in T, and there exist $\phi_i \in T/m_T[[x_1, \ldots, x_a]]$ such that $x_i - \phi_i \in Q_{\hat{T}}$ for $a+1 \leq i \leq q$. Suppose also that $Q_{\hat{T}} \cap T/m_T[[x_1, \ldots, x_a]] = (0)$, then $Q_{\hat{T}} = (x_{a+1} - \phi_{a+1}, \ldots, x_q - \phi_q)$.*

Proof. Let $f \in Q_{\hat{T}} \subseteq \hat{T}$ and write $f = \sum_I a_I x_1^{i_1} \ldots x_a^{i_a} x_{a+1}^{i_{a+1}} \cdots x_q^{i_q}$ where $a_I \in \hat{T}/m_{\hat{T}}$.

Expanding

$$f = \sum_I a_I x_1^{i_1} \ldots x_a^{i_a} ([x_{a+1} - \phi_{a+1}] + \phi_{a+1})^{i_{a+1}} \ldots ([x_q - \phi_q] + \phi_q)^{i_q}$$

we obtain

$$f = (x_{a+1} - \phi_{a+1})\Omega_{a+1} + \cdots + (x_q - \phi_q)\Omega_q + \Omega$$

where $\Omega_i \in T/m_T[[x_1, \ldots, x_q]]$ and $\Omega \in T/m_T[[x_1, \ldots, x_a]]$ since $\phi_i \in T/m_T[[x_1, \ldots, x_a]]$.

Since $f, x_i - \phi_i \in Q_{\hat{T}}$ for $a + 1 \leq i \leq q$ and $\Omega \in T/m_T[[x_1, \ldots, x_a]]$ where $Q_{\hat{T}} \cap T/m_T[[x_1, \ldots, x_a]] = (0)$ this implies that $\Omega = 0$. Thus $Q_{\hat{T}} = (x_{a+1} - \phi_{a+1}, \ldots, x_q - \phi_q)$ □

Corollary 4.3.7. *Suppose that $x_1, \ldots, x_s, \ldots, x_a, \ldots x_q$ is a regular system of parameters in T, such that $\nu(x_1), \ldots, \nu(x_s)$ are rationally independent, and there exist $\phi_i \in T/m_T[[x_1, \ldots, x_{i-1}]]$ such that $x_i - \phi_i \in Q_{\hat{T}}$ for $a+1 \leq i \leq q$. Then there exists a sequence of etale Perron transforms $T \to U \to U_1$ along ν such that U_1 has a regular system of parameters $x_1(1), \ldots, x_t(1), \ldots, x_a(1), x_{a+1}, \ldots, x_q$, and $\overline{\phi}_i \in U_1/m_{U_1}[[x_1(1), \ldots, x_a(1)]]$ for $t+1 \leq i \leq q$ such that*

$$Q_{\hat{U}_1} = (x_{t+1}(1) - \overline{\phi}_{t+1}, \ldots, x_a(1) - \overline{\phi}_a, x_{a+1} - \overline{\phi}_{a+1}, \ldots, x_q - \overline{\phi}_q).$$

Proof. Suppose that $Q_{\hat{T}} \cap T/m_T[[x_1, \ldots, x_a]] = (0)$. Then there exists $\overline{\phi}_i \in T/m_T[[x_1, \ldots, x_a]]$ for $a+1 \leq i \leq q$ such that $Q_{\hat{T}} = (x_{a+1} - \overline{\phi}_{a+1}, \ldots, x_q - \overline{\phi}_q)$ by Lemma 4.3.5 and Lemma 4.3.6.

Suppose that $Q_{\hat{T}} \cap T/m_T[[x_1, \ldots, x_a]] \neq (0)$, then by Theorem 4.3.4, there exists a sequence of etale Perron transforms $T \to U \to U_1$, along ν such that U_1 has regular parameters $x_1(1), \ldots, x_a(1), x_{a+1}, \ldots, x_q$ and there exist series $\phi_i^1 \in U_1/m_{U_1}[[x_1(1), \ldots, x_{i-1}(1)]]$ for $a \leq i \leq q$ such that $x_a(1) - \phi_a^1 \in Q_{\hat{U}_1}$ and $x_i - \phi_i^1 \in Q_{\hat{U}_1}$ for $a < i \leq q$.

If $Q_{\hat{U}_1} \cap U_1/m_{U_1}[[x_1(1), \ldots, x_a(1)]] \neq (0)$, then apply Theorem 4.3.4 again.

In this way we construct a sequence of etale Perron transforms along ν

$$T \to U \to U_1 \to \ldots$$

The sequence must terminate in U_r with

$x_{a-r+1}(r) - \phi_{a-r+1}^r, \ldots, x_a(r) - \phi_a^r, x_{a+1} - \phi_{a+1}^r, \ldots, x_q - \phi_q^r \in Q_{\hat{U}_r}$,

$\phi_i^r \in U_r/m_{U_r}[[x_1(r), \ldots, x_{i-1}(r)]]$ for $a - r + 1 \leq i \leq q$, and

$Q_{\hat{U}_r} \cap U_r/m_{U_r}[[x_1(r), \ldots, x_a(r)]] = (0).$

The conclusions of the corollary follow from the first part of the proof.

□

Theorem 4.3.8. *Suppose that $x_1, \ldots, x_s, \ldots, x_p, \ldots x_q$ is a regular system of parameters in T, such that $\nu(x_1), \ldots, \nu(x_s)$ are rationally independent and $x_i \in P_T$ for $p+1 \leq i \leq q$, and $l > 0$. Then there exists a sequence of transforms of type $(1,0)$ and $(2,0)$ $T \to T_1$ along ν such that T_1 has a regular system of parameters*

$$y_1, \ldots, y_s, \ldots, y_t, \ldots, y_p, \ldots, y_q,$$

and $\phi_i \in T_1/m_{T_1}[[y_1, \ldots, y_{i-1}]]$, $\tau_i \in m^l_{\hat{T}_{1,l}}$ for $t+1 \leq i \leq p$ such that $\nu(y_1), \ldots, \nu(y_s)$ are rationally independent, and

$$Q_{\hat{T}_1} = (y_{t+1} - \phi_{t+1} - \tau_{t+1}, \ldots, y_p - \phi_p - \tau_p, y_{p+1}, \ldots, y_q).$$

Further, there exist $d_1, \ldots, d_s \in \mathbb{N}$ such that $y_1^{d_1} \ldots y_s^{d_s} y_i = x_i$ for $p+1 \leq i \leq q$ and $T_{P_T} = (T_1)_{P_{T_1}}$.

Proof. By Corollary 4.3.7 (with $\phi_i = 0$ for $p+1 \leq i \leq q$) and Theorem 4.3.2, there exists a sequence of etale Perron transforms $T \to U \to U_1$ such that

$$Q_{\hat{U}_1} = (x_{t+1}(1) - \overline{\phi}_{t+1}, \ldots, x_p(1) - \overline{\phi}_p, x_{p+1}, \ldots, x_q)$$

has the form of the conclusions of Corollary 4.3.7.

In particular

$$\overline{\phi}_i \in U_1/m_{U_1}[[x_1(1), \ldots, x_t(1)]], \text{ for every } i. \tag{4.29}$$

Let $\alpha = \max\{\nu(x_1(1)^{c_{i1}} \ldots x_s(1)^{c_{is}})\}$ for $t+1 \leq i \leq p$. Where the c_{ij} are defined on page 67. There exists $n \in \mathbb{N}$ such that

$$\alpha < nv(x_1(1)). \tag{4.30}$$

Choose $m_0 \in \mathbb{N}$ so that $m_0 > nv(x_2(1))$, and $m \in \mathbb{N}$ such that $m > \max\{\frac{\alpha}{\nu(m_{T_1})}, m_0, l_0\}$. Where l_0 is the constant of the conclusion of Theorem 4.3.1.

Let $T \to T_{1,m}$ be the corresponding sequence of transforms of type $(1,0)$ of the conclusions of Theorem 4.3.1.

By property (A), $Q_{\hat{U}_1}$ is a component of the strict transform of $Q_{\hat{T}}$ in $\hat{U}_1$. Since $x_1(1)\ldots x_s(1) = 0$ is a local equation of the exceptional locus of $\mathrm{spec}(U_1) \to \mathrm{spec}(T)$, we see that there exist $f_{t+1}, \ldots, f_p \in Q_{\hat{T}}$ and $c_{i1}, \ldots, c_{is} \in \mathbb{N}$ such that

$$\frac{f_{t+1}}{x_1(1)^{c_{t+1,1}} \ldots x_s(1)^{c_{t+1,s}}}, \ldots, \frac{f_p}{x_1(1)^{c_{p1}} \ldots x_s(1)^{c_{ps}}}, x_{p+1}, \ldots, x_q$$

is a basis of $Q_{\hat{U}_1}$.

Since $\nu(x_i(1) - \overline{\phi}_i) = \infty$ for $t+1 \leq i \leq p$, we may replace $x_i(1)$ with the subtraction of the first part of the series $\overline{\phi}_i$ from $x_i(1)$ to obtain

$$\nu(x_i(1)) = \nu(\overline{\phi}_i) > m_0 \tag{4.31}$$

for $t+1 \leq i \leq p$, and

$$\overline{\phi}_i \in m_{T_{1,m}}^m \quad \text{for every } i. \tag{4.32}$$

Let

$$\sigma_i = \begin{cases} \frac{f_{t+i}}{x_1(1)^{c_{t+i,1}} \ldots x_s(1)^{c_{t+i,s}}} & \text{for } 1 \leq i \leq p-t \\ x_{i+t} & \text{for } p-t+1 \leq i \leq q-t. \end{cases}$$

$$\psi_i = \begin{cases} x_{t+i}(1) - \overline{\phi}_{t+i} & \text{for } 1 \leq i \leq p-t \\ x_{i+t} & \text{for } p-t+1 \leq i \leq q-t. \end{cases}$$

There exist $b_{ij} \in \hat{U}_1$ such that

$$\sigma_i = \sum_{j=1}^{q-t} b_{ij}\psi_j$$

for $1 \le i \le q-t$, with

$$b_{ij} = \delta_{ij}$$

for $p-t+1 \le i$, such that $\mathrm{Det}(b_{ij})$ is a unit in $\hat{U}_1$. There exist $h_i \in m^m_{\hat{T}_{1,m}}$ such that

$$\begin{aligned} f_{t+i} = x_{1,m}(1)^{c_{t+i,1}} \dots x_{1,m}(s)^{c_{t+i,s}}[\sum_{j=1}^{p-t} b_{ij}(x_{1,m}(1), \dots, x_{1,m}(s), \dots, x_{p,m}(1), \\ x_{p+1}, \dots, x_q)\psi_j(x_{1,m}(1), \dots, x_{1,m}(s), \dots, x_{p,m}(1), x_{p+1}, \dots, x_q) + \\ \sum_{j=p-t+1}^{q} b_{ij}(x_{1,m}(1), \dots, x_{1,m}(s), \dots, x_{p,m}(1), x_{p+1}, \dots, x_q)x_j] + h_i \end{aligned}$$

for $1 \le i \le p-t$.

Perform successive sequences of transforms of type II_r $T_{i-1,m} \to T_{i,m}$, defined for $2 \le i \le t-s+1$, by

$$\begin{array}{lcl} x_{1,m}(i-1) & = & x_{1,m}(i)^{a_{11}^{(i)}} \dots x_{s,m}(i)^{a_{1s}^{(i)}} N_{s+i-1,m}(i) \\ \vdots & & \vdots \\ x_{s,m}(i-1) & = & x_{1,m}(i)^{a_{s1}^{(i)}} \dots x_{s,m}(i)^{a_{ss}^{(i)}} N_{s+i-1,m}(i) \\ x_{s+i-1,m}(i-1) & = & x_{1,m}(i)^{a_{s+1,1}^{(i)}} \dots x_{s,m}(i)^{a_{s+1,s}^{(i)}} N_{s+i-1,m}(i) \end{array}$$

and $x_{k,m}(i) = x_{k,m}(i-1)$ if $k \notin \{1, \dots, s, s+i-1\}$.

Let $x_{s+i-1,m}(i) \in T_{i+1,m}$ be such that

$$x_{1,m}(i), \dots, x_{s+i-1,m}(i), x_{s+i,m}(1), x_{p,m}(1), \dots, x_{p+1}, \dots, x_q$$

are regular parameters in $T_{i+1,m}$. In $T_{t-s+1,m}$,

$$x_{1,m}(t-s+1), \dots, x_{s,m}(t-s+1), \dots, x_{t,m}(t-s+1), x_{t+1,m}(1), \dots, x_{p,m}(1), \\ x_{p+1}, \dots, x_q$$

are then regular parameters. For all i, we have finite sums

$$h_i = \sum_j x_{1,m}(t-s+1)^{d^i_{j1}} \ldots x_{s,m}(t-s+1)^{d^i_{js}}\Lambda_j + \sum_{t+1\le k\le p} \Omega_k x_{k,m}(1)^2 + \sum_{p+1\le k\le q} \Psi_k x_k \tag{4.33}$$

where $\nu(x_{1,m}(t-s+1)^{d^i_{j1}} \ldots x_{s,m}(t-s+1)^{d^i_{js}}) > \alpha$ for all j, and $\Lambda_i, \Omega_k, \Psi_k \in T_{t-s+1,m}$, for all i, k.

Let $\phi_i = \overline{\phi}_i(x_{1,m}(1), \ldots, x_{t,m}(1))$.

We further have by (4.29) and (4.32) that there are finite sums $\phi_i = \sum_j e_{ij} x_{1,m}(t-s+1)^{f^i_{j1}} \ldots x_{s,m}(t-s+1)^{f^i_{js}}$ for $1 \le i \le p-1$.

with $\nu(x_{1,m}(t-s+1)^{f^i_{j1}} \ldots x_{s,m}(t-s+1)^{f^i_{js}}) > \alpha$, for all i and j.

We have an expression

$$x_{1,m}(1) = x_{1,m}(t-s+1)^{g_1} \ldots x_{s,m}(t-s+1)^{g_s}\Omega$$

where Ω is a unit in $T_{t-s+1,m}$.

For $t+1 \le k \le p$, by (4.31) and Theorem 4.3.1 we have that $\nu(x_{k,m}(1)) > m_0$, so by (4.30) $\nu(x_{k,m}(1)) > \nu((x_{1,m}(t-s+1)^{g_1} \ldots x_{s,m}(t-s+1)^{g_s})^n) > \alpha$. Recall that $\nu(x_i) = \infty$ for $p+1 \le i \le q$. We may then define a transformation

$T_{t-s+1,m} \rightarrow T_{t-s+2,m}$ by

$x_{k,m}(1) = (x_{1,m}(t-s+1)^{g_1} \ldots x_{s,m}(t-s+1)^{g_s})^n - x_{k,m}(t-s+1)$ for $t+1 \le k \le q$.

Now substitute into (4.33), and perform a final transform $T_{t-s+2,m} \rightarrow T_{t-s+3,m}$ of type I, to obtain by Lemma 4.2 [16] regular parameters $x_{1,m}(t-s+3), \ldots, x_{t+1,m}(t-s+3) = x_{t+1,m}(t-s+1), \ldots, x_{p,m}(t-s+3) = x_{p,m}(t-s+1), x_{p+1}(t-s+3), \ldots, x_q(t-s+3)$ in $T_{t-s+3,m}$.

Let $G_{t+1},\ldots,G_p$ be the strict transforms of $f_{t+1},\ldots,f_p$ in $\hat{T}_{t-s+3,m}$.

We have that $G_{t+i}-\sum_{j=1}^{p-t} b_{ij}(0,\ldots,0)x_{t+j,m}(t-s+3)\in m^2_{\hat{T}_{t-s+3,m}}$ for $1\le i\le p-t$ and

$$\text{Det}\begin{pmatrix} b_{11}(0,\ldots,0) & ... & b_{1,p-t}(0,\ldots,0)\\ ... & ... & ...\\ b_{p-t,1}(0,\ldots,0) & ... & b_{p-t,p-t}(0,\ldots,0)\end{pmatrix}\neq 0.$$

Thus by property (A), Lemma (4.3.6) and the Weierstrass preparation Theorem, there exists a regular system of parameters in $T_{t-s+3,m}$ of the form of the conclusion of Theorem 4.3.8.

□

Theorem 4.3.9. *Suppose that ν has rank ≥ 2, and $x_1,\ldots,x_s,\ldots,x_p,x_{p+1},\ldots,x_q$ is a regular sequence in T such that $\nu(x_1),\ldots,\nu(x_s)$ are rationally independent, and*

$$(x_{p+1},\ldots,x_q)\subset_{\neq} P_T.$$

Then there exists a sequence of etale Perron transforms along ν

$$T\to U\to U_1\to\cdots\to U_e$$

such that U_e has regular parameters $x_1(e),\ldots,x_s(e),\ldots,x_p(e),x_{p+1},\ldots,x_q$ such that $\nu(x_1(e)),\ldots,\nu(x_s(e))$ are rationally independent, and such that

$$(x_p(e),x_{p+1},\ldots,x_q)\subset P_{U_e}.$$

Proof. If we achieve

$$\omega(U_i)=dim(U_i)-dim_{U_i/m_{U_i}}(P_{U_i}/m^2_{U_i}\cap P_{U_i})<p$$

in some U_i, then we terminate the algorithm, as we can make a change of variables in U_i so that the conclusions of the theorem hold in U_i. We may thus assume that

$$\omega(U_i) = p \tag{4.34}$$

throughout the proof.

By Corollary 4.3.7, there exists a sequence of etale Perron transforms $U \to U_1$ along ν such that there exists $m \geq s$ such that there exist regular parameters

$$x_1(1), \ldots, x_s(1), \ldots, x_m(1), \ldots, x_p(1), x_{p+1}, \ldots, x_q$$

in U_1, series $\phi_i \in (U_1/m_{U_1})[[x_1(1), \ldots, x_m(1)]]$ for $m+1 \leq i \leq p$ such that

$$Q_{\hat{U}_1} = (x_{m+1}(1) - \phi_{m+1}, \ldots, x_p(1) - \phi_p, x_{p+1}, \ldots, x_q).$$

Let $\hat{x}_i = x_i(1) - \phi_i$ for $m+1 \leq i \leq p$.

By our assumptions, there exists $g \in P_{U_1} - (x_{p+1}, \ldots, x_q)U_1$. In $\hat{U}_1$ we have an expansion:

$$g = \sum_{i_{m+1}+\cdots i_p > 0} a_{i_{m+1},\ldots,i_p}(x_1(1), \ldots, x_m(1))\hat{x}_{m+1}^{i_{m+1}} \cdots \hat{x}_p^{i_p} + \sum_{i=p+1}^{q} x_i\Omega_i. \tag{4.35}$$

Since some $a_{i_{m+1},\ldots,i_p}(x_1(1), \ldots, x_m(1)) \neq 0$, differentiating with respect to the partials computed from the regular system of parameters

$$x_1(1), \ldots, x_p(1), x_{p+1}, \ldots, x_q,$$

we see that for some i with $m+1 \leq i \leq p$,

$$\frac{\partial g}{\partial x_i(1)}(x_1(1), \ldots, x_p(1), 0 \ldots, 0)$$

has smaller order with respect to the ideal

$$(\hat{x}_{m+1},\ldots,\hat{x}_p)\subset T/m[[x_1(1),\ldots,\hat{x}_p]]\subset T/m[[x_1(1),\ldots,x_p(1)]]$$

than $g(x_1(1),\ldots,x_p(1),0,\ldots,0)$ does.

Suppose that $\frac{\partial g}{\partial x_i(1)}\in P_{U_1}=Q_{\hat{U}_1}\cap U_1$. Then we repeat the above argument with g replaced with $\frac{\partial g}{\partial x_i(1)}$. After a finite number of iterations we find $g\in P_{U_1}$ such that $\frac{\partial g}{\partial x_i(1)}\not\in Q_{\hat{U}_1}$. Thus $\frac{\partial g}{\partial x_i(1)}\not\in P_{U_1}$. After possibly interchanging $x_i(1)$ and $x_p(1)$, we may assume that $i=p$. Thus $\frac{\partial g}{\partial x_p(1)}\not\in P_{U_1}$. In particular, $a_{0\cdots 01}\neq 0$.

We have $Q_{\hat{U}_1}\cap U_1/m_{U_1}[[x_1(1),\ldots,x_m(1)]]=(0)$.

We have $\nu(x_p(1))=\nu(\phi_p)$ since $\hat{x}_p\in Q_{\hat{U}_1}$ ($x_p(1)\not\in Q_{\hat{U}_1}$ by (4.34)). There exists a polynomial

$$h_p(x_1(1),\ldots,x_m(1))\in U_1/m_{U_1}[x_1(1),\ldots,x_m(1)]$$

and a series $\Psi_p\in U_1/m_{U_1}[[x_1(1),\ldots,x_m(1)]]$ such that $\phi_p=h_p+\Psi_p$ with $\nu(\Psi_p)>\nu(a_{0\cdots 01})$. We make a change of variables in U_1, replacing $x_p(1)$ with $x_p(1)-h_p$. We now have $\hat{x}_p=x_p(1)-\Psi_p$, so we have $\nu(x_p(1))=\nu(\Psi_p)>\nu(a_{0,\ldots,0,1})$.

In the same way, we make changes of variables in $x_{m+1}(1),\ldots,x_{p-1}(1)$, so we may assume that

$$\begin{array}{l}\nu(x_i(1))=\nu(\Psi_i)>\nu(a_{0,\ldots,0,1}x_p(1))\text{ for }m+1\le i\le p-1,\text{ and}\\ \nu(x_p(1))>\nu(a_{0,\ldots,0,1}).\end{array}\tag{4.36}$$

By Theorem 4.3.2, and since we are assuming that T satisfies property (A) of Definition (4.2.1), there exists a sequence of etale Perron transforms $U_1\to U_2$ along ν such that U_2 has regular parameters

$$x_1(2),\ldots,x_s(2),\ldots,x_m(2),x_{m+1}(1),\ldots,x_p(1),x_{p+1},\ldots,x_q$$

with

$$a_{0,\dots,0,1} = x_1(2)^{c_1} \dots x_s(2)^{c_s}\gamma(x_1(2),\dots,x_m(2))$$

where $\gamma \in \hat{U}_2$ is a unit series, and

$$\Psi_i = x_1(2)^{d_{i1}} \dots x_s(2)^{d_{is}}\delta_i(x_1(2),\dots,x_m(2))$$

for $m+1 \le i \le p$, where γ and $\delta_i \in \hat{U}_2$ are unit series.

For $2 \le i \le p-m+1$, now perform etale Perron transforms along ν, $U_i \to U_{i+1}$, defined by

$$\begin{array}{lll} x_1(i) & = & x_1(i+1)^{a_{11}^{(i+1)}} \dots x_s(i+1)^{a_{1s}^{(i+1)}} (x_{m+i-1}(i+1)+\alpha_{i+1})^{a_{1,s+1}^{(i+1)}} \\ \vdots & & \vdots \\ x_s(i) & = & x_1(i+1)^{a_{s1}^{(i+1)}} \dots x_s(i+1)^{a_{ss}^{(i+1)}} (x_{m+i-1}(i+1)+\alpha_{i+1})^{a_{s,s+1}^{(i+1)}} \\ x_{m+i-1}(1) & = & x_1(i+1)^{a_{s+1,1}^{(i+1)}} \dots x_s(i+1)^{a_{s+1,s}^{(i+1)}} (x_{m+i-1}(i+1)+\alpha_{i+1})^{a_{s+1,s+1}^{(i+1)}} \end{array}$$

and $x_k(i+1) = x_k(i)$ if $k \notin \{1,\dots,s,m+i-1\}$.

In U_{j-m+1},

$$x_1(j-m+1),\dots,x_s(j-m+1),\dots,x_{j-1}(j-m+1),x_j(1),\dots,x_p(1),x_{p+1},\dots,x_q$$

are regular parameters, and there exists a unit series

$$\epsilon_j(x_1(j-m+1),\dots,x_{j-1}(j-m+1))$$
$$\in U_{j-m+1}/m_{U_{j-m+1}}[[x_1(j-m+1),\dots,x_{j-1}(j-m+1)]]$$

and $\overline{d}_{1j},\dots,\overline{d}_{sj} \in \mathbb{N}$ such that

$$\hat{x}_j = x_j(1) - x_1(j-m+1)^{\overline{d}_{1j}} \dots x_s(j-m+1)^{\overline{d}_{sj}}\epsilon_j.$$

In $\hat{U}_{j-m+2}$,

$$\hat{x}_j = x_1(j-m+2)^{a_{s+1,1}^{(j-m+2)}} \dots x_s(j-m+2)^{a_{s+1,s}^{(j-m+2)}} (x_j(j-m+2)+ \alpha_{j-m+2})^{a_{s+1,s+1}^{(j-m+2)}} - [x_1(j-m+2)^{a_{11}^{(j-m+2)}} \dots x_s(j-m+2)^{a_{1s}^{(j-m+2)}} (x_j(j-m+2)+\alpha_{j-m+2})^{a_{1,s+1}^{(j-m+2)}}]^{\overline{d}_{1j}} \dots x_1(j-m+2)^{a_{s1}^{(j-m+2)}} \dots [x_s(j-m+2)^{a_{ss}^{(j-m+2)}} (x_j(j-m+2)+\alpha_{j-m+2})^{a_{s,s+1}^{(j-m+2)}}]^{\overline{d}_{sj}} \epsilon_j.$$

Set $\overline{x}_i = x_i(j-m+2)$ for $1 \leq i$, $\alpha = \alpha_{j-m+2}$, $a_{ij} = a_{ij}(j-m+2)$. There exists $0 \neq \beta \in U_{j-m+2}/m_{U_{j-m+2}}$ and $\Lambda_i \in U_{j-m+2}/m_{U_{j-m+2}}[[\overline{x}_1, \dots, \overline{x}_j]]$ such that

$$\epsilon_j = \beta + \Lambda_1 \overline{x}_1 + \dots + \Lambda_{j-1} \overline{x}_{j-1}.$$

Since $\nu(\overline{x}_1), \dots, \nu(\overline{x}_s)$ are rationally independent we have :

$$\begin{aligned} a_{s+1,1} &= a_{11}\overline{d}_{1j} + \dots + a_{s1}\overline{d}_{sj} \\ &\vdots \\ a_{s+1,s} &= a_{1s}\overline{d}_{1j} + \dots + a_{ss}\overline{d}_{sj} \end{aligned} \tag{4.37}$$

and thus

$$\hat{x}_j = \overline{x}_1^{a_{s+1,1}} \dots \overline{x}_s^{a_{s+1,s}} [(\overline{x}_j + \alpha)^{a_{s+1,s+1}} - \beta(\overline{x}_j+\alpha)^{a_{1,s+1}\overline{d}_{1j} + \dots + a_{s,s+1}\overline{d}_{sj}} + \overline{\Lambda}_1 \overline{x}_1 + \dots + \overline{\Lambda}_{j-1}\overline{x}_{j-1}]$$

for some

$$\overline{\Lambda}_1, \dots, \overline{\Lambda}_{j-1} \in U_{j-m+2}/m_{U_{j-m+2}}[[\overline{x}_1, \dots, \overline{x}_j]].$$

Let $\overline{d}_j = a_{1,s+1}\overline{d}_{1j} + \dots + a_{s,s+1}\overline{d}_{sj}$.

We expand

$$(\overline{x}_j+\alpha)^{a_{s+1,s+1}} - \beta(\overline{x}_j+\alpha)^{\overline{d}_j} = (\alpha^{a_{s+1,s+1}} - \beta\alpha^{\overline{d}_j}) + (a_{s+1,s+1}\alpha^{a_{s+1,s+1}-1} - \beta\overline{d}_j\alpha^{\overline{d}_j-1})\overline{x}_j + \overline{x}_j^2\Sigma.$$

$\alpha^{a_{s+1,s+1}} - \beta\alpha^{\overline{d}_j} = 0$ since $\hat{x}_j \in Q_{\hat{U}_{j-m+2}}$.

Thus $\beta = \alpha^{a_{s+1,s+1} - \overline{d}_j}$

$$a_{s+1,s+1}\alpha^{a_{s+1,s+1}-1} - \beta\overline{d}_j\alpha^{\overline{d}_j-1} = \beta\alpha^{\overline{d}_j-1}[a_{s+1,s+1} - \overline{d}_j].$$

By (4.37), we have :

$$\begin{pmatrix} a_{11} & ... & a_{s+1,1} \\ ... & ... & ... \\ a_{1s} & ... & a_{s+1,s} \\ a_{1,s+1} & ... & a_{s+1,s+1} \end{pmatrix} \begin{pmatrix} \overline{d}_{1j} \\ ... \\ \overline{d}_{sj} \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ ... \\ 0 \\ \overline{d}_j - a_{s+1,s+1} \end{pmatrix}.$$

By Cramer's rule

$$-1 = \frac{Det\begin{pmatrix} a_{11} & ... & a_{s1} & 0 \\ ... & ... & ... & 0 \\ a_{1,s+1} & ... & a_{s,s+1} & \overline{d}_j - a_{s+1,s+1} \end{pmatrix}}{Det(a_{ij})}.$$

Since $Det(a_{ij}) = \pm 1$, we have
$\pm 1 = Det\begin{pmatrix} a_{11} & ... & a_{s1} \\ ... & ... & \\ a_{1s} & & a_{ss} \end{pmatrix}(\overline{d}_j - a_{s+1,s+1})$, and thus $\overline{d}_j - a_{s+1,s+1} \neq 0$.

By the Weierstrass preparation theorem, there exists series Ω_j and ψ_j such that Ω_j is a unit series in $U_{j-m+2}/m_{U_{j-m+2}}[[\overline{x}_1, \ldots, \overline{x}_j]]$ and ψ_j is a series in $U_{j-m+2}/m_{U_{j-m+2}}[[\overline{x}_1, \ldots, \overline{x}_{j-1}]]$, such that

$$\hat{x}_j = \overline{x}_1^{a_{s+1,1}} \ldots \overline{x}_s^{a_{s+1,s}} \Omega_j(\overline{x}_j - \psi_j(\overline{x}_1, \ldots, \overline{x}_{j-1})).$$

Thus in $\hat{U}_{p-m+2}$, we have an expression

$$\hat{x}_j = x_1(p-m+2)^{c_{1j}} \ldots x_s(p-m+2)^{c_{sj}} \overline{\Omega}_j(x_j(p-m+2) -$$

$$\sigma_j(x_1(p-m+2), \ldots, x_{j-1}(p-m+2), x_{j+1}(p-m+2), \ldots, x_p(p-m+2)))$$

where $\overline{\Omega}_j \in U_{j-m+2}/m_{U_{j-m+2}}[[x_1(p-m+2), \ldots, x_p(p-m+2)]]$ is a unit series and

$$\nu(x_1(p-m+2)^{c_{1j}} \ldots x_s(p-m+2)^{c_{sj}}) = \nu(x_j(1)). \tag{4.38}$$

We have

$$a_{0,\ldots,0,1} = x_1(2)^{c_1} \ldots x_s(2)^{c_s} \gamma(x_1(2), \ldots, x_m(2)) = x_1(p-m+2)^{\overline{c}_1} \ldots x_s(p-m+2)^{\overline{c}_s} \overline{\gamma}$$

where $\overline{\gamma} \in U_{j-m+2}/m_{U_{j-m+2}}[[x_1(p-m+2), \ldots, x_p(p-m+2)]]$ is a unit series.

There exists $F, G_i \in U_1/m_{U_1}[[x_1(1), \ldots, x_p(1)]]$ such that

$$g = a_{0,\ldots,0,1}\hat{x}_p + F\hat{x}_p^2 + \sum_{i=m+1}^{p-1} G_i\hat{x}_i + \sum_{j=p+1}^{q} x_j\Omega_j.$$

In $\hat{U}_{p-m+2}$, we have $\overline{F}, \overline{G}_j, \overline{\Sigma} \in U_{j-m+2}/m_{U_{j-m+2}}[[x_1(p-m+2), \ldots, x_p(p-m+2)]]$ such that $\overline{\Sigma}$ is a unit and

$$g = x_1(p-m+2)^{\overline{c}_1+c_{1p}} \ldots x_s(p-m+2)^{\overline{c}_s+c_{sp}}(x_p(p-m+2)-\sigma_p)\overline{\Sigma}+$$
$$x_1(p-m+2)^{2c_{1p}} \ldots x_s(p-m+2)^{2c_{sp}}\overline{F} + \sum_{j=m+1}^{p-1} x_1(p-m+2)^{c_{1j}} \ldots$$
$$x_s(p-m+2)^{c_{sj}}\overline{G}_j + \sum_{j=p+1}^{q} x_i\Omega_i.$$

By (4.38) and (4.36), and Lemma 4.2 [16], after possibly performing an etale Perron transform of type I along ν, we have that

$x_1(p-m+2)^{\overline{c}_1+c_{1p}} \ldots x_s(p-m+2)^{\overline{c}_s+c_{sp}}$ properly divides $x_1(p-m+2)^{2c_{1p}} \ldots x_s(p-m+2)^{2c_{sp}}$ and $x_1(p-m+2)^{c_{1j}} \ldots x_s(p-m+2)^{c_{sj}}$ $m+1 \leq j \leq p-1$.

Let $a = p-m+2$, $c_i = \overline{c}_i + c_{ip}$ for $1 \leq i \leq s$. By the Weierstrass Preparation Theorem, we have an expression

$$g = x_1(a)^{c_1} \ldots x_s(a)^{c_s}\gamma(x_p(a) - \Sigma(x_1(a), \ldots, x_{p-1}(a))) + \sum_{j=p+1}^{q} x_j\Omega_j,$$

where $\gamma \in U_a/m_{U_a}[[x_1(a), \ldots, x_p(a)]]$ is a unit series. Extend $x_{p+1}, \ldots, x_q$ to a minimal system of generators $x_{p+1}, \ldots, x_q, f_1, \ldots, f_t$ of P_{U_a}. We have

$$x_1(a)^{c_1} \ldots x_s(a)^{c_s}(x_p(a) - \Sigma) \in P_{U_a}\hat{U}_a.$$

Since we are assuming that T satisfies property (A) of Definition (4.2.1), P_{U_a} is a component of the strict transform of P_U in U_a.

Since $x_1(a)\dots x_s(a)=0$ is a local equation of the exceptional locus of $\mathrm{spec}(U_a)\to\mathrm{spec}(U)$, we have that $x_1(a)\dots x_s(a)\notin P_{U_a}$.

Thus $x_p(a)-\Sigma\in P_{U_a}\hat{U}_a$ We have an expression

$$x_p(a)-\Sigma=\sum_{i=p+1}^{q}b_ix_i+\sum_{j=1}^{t}d_if_i$$

with $b_i,d_i\in\hat{U}_a$. Thus $\frac{\partial f_i}{\partial x_p(a)}(0,\dots,0)\neq 0$ for some $1\leq i\leq t$, which implies that

$$\dim_{U_a/m_{U_a}}(P_{U_a}/m_{U_a}^2\cap P_{U_a})>q-p$$

and thus

$$\omega(U_a)=dim(U_a)-\dim_{U_a/m_{U_a}}(P_{U_a}/m_{U_a}^2\cap P_{U_a})<p.$$

$\square$

Corollary 4.3.10. *Suppose that ν has rank ≥ 2, and $x_1,\dots,x_s,\dots,x_p,x_{p+1},\dots,x_q$ is a regular sequence in T such that $\nu(x_1),\dots,\nu(x_s)$ are rationally independent, and such that*

$$(x_{p+1},\dots,x_q)\subset_{\neq}P_T.$$

Then there exists a sequence of etale Perron transforms along ν

$$T\to U\to U_1\to\cdots\to U_e$$

such that U_e has regular parameters

$$x_1(e),\dots,x_s(e),\dots,x_{\sigma(1)}(e),x_{\sigma(1)+1}(e),\dots,x_p(e),x_{p+1},\dots x_q$$

such that $\nu(x_1(e)), \ldots, \nu(x_s(e))$ *are rationally independent and such that*

$$(x_{\sigma(1)+1}(e), \ldots, x_p(e), x_{p+1}, \ldots, x_q) = P_{U_e}.$$

Proof. By Theorem 4.3.9, there exists a sequence of etale Perron transforms along ν.

$$T \to U \to U_1$$

such that U_1 has regular parameters

$$x_1(1), \ldots, x_s(1), \ldots, x_p(1), x_{p+1}, \ldots x_q$$

such that

$$(x_p(1), x_{p+1}, \ldots, x_q) \subset P_{U_1}.$$

and we have a regular sequence of length $q - p + 1$ in P_{U_1}.

If $P_{U_1} = (x_p(1), x_{p+1}, \ldots, x_q)$, then we are done; otherwise, we repeat the algorithm of Theorem 4.3.9, each time getting a longer regular sequence in P_{U_i}. Since the length of a regular sequence is $\leq q$, this process must terminate after finitely many steps, and thus achieving the conclusion of the Corollary.

□

Theorem 4.3.11. *Suppose that* ν *has rank* ≥ 2, *and* $x_1, \ldots, x_s, \ldots, x_p, x_{p+1}, \ldots, x_q$ *is a regular sequence in* T *such that* $\nu(x_1), \ldots, \nu(x_s)$ *are rationally independent and such that*

$$(x_{p+1}, \ldots, x_q) \subset_{\neq} P_T.$$

Then there exists a sequence of transforms of types $(1,0)$ *and* $(2,0)$ *along* ν

$$T \to T_1 \to \cdots \to T_e$$

such that T_e *has regular parameters*

$$x_1(e), \ldots, x_s(e), \ldots, x_{\sigma(1)}(e), x_{\sigma(1)+1}(e), \ldots, x_{p+1}, \ldots, x_q(e)$$

such that $\nu(x_1(e)), \ldots, \nu(x_s(e))$ *are rationally independent and*

$$(x_{\sigma(1)+1}(e), \ldots x_{p+1}(e), \ldots, x_q(e)) = P_{T_e}.$$

Further, there exist $n_1, \ldots, n_s \in \mathbb{N}$ *such that* $x_1(e)^{n_1} \ldots x_s(e)^{n_s} x_i(e) = x_i$ *for* $p+1 \leq i \leq q$ *and* $T_{P_T} = (T_e)_{P_{T_e}}$.

Proof. By Corollary 4.3.10 there exists a sequence of etale Perron transforms $T \to U \to U_1$ such that U_1 had regular parameters

$$x_1(1), \ldots, x_t(1), \ldots, x_p(1), x_{p+1}, \ldots, x_q,$$

such that:

$$P_{U_1} = (x_{t+1}(1), \ldots, x_p(1), x_{p+1}, \ldots, x_q).$$

Let $T \to T_{1,m}$ be the corresponding sequence of transforms of type $(1,0)$ of the conclusions of Theorem 4.3.1.

By property (A), P_{U_1} is a component of the strict transform of P_T in U_1 (and P_{T_1} is a component of the strict transform of P_T in T_1). Since $x_1(1) \ldots x_s(1) = 0$ is a local equation of the exceptional locus of $\mathrm{spec}(U_1) \to \mathrm{spec}(T)$, we see that there exist $f_{t+1}, \ldots, f_p \in P_T$ and $c_{i1}, \ldots, c_{is} \in \mathbb{N}$ such that

$$\frac{f_{t+1}}{x_1(1)^{c_{t+1,1}} \ldots x_s(1)^{c_{t+1,s}}}, \ldots, \frac{f_p}{x_1(1)^{c_{p1}} \ldots x_s(1)^{c_{ps}}}, x_{p+1}, \ldots, x_q$$

is a basis of P_{U_1}.

Let

$$\sigma_i = \begin{cases} \frac{f_{t+i}}{x_1(1)^{c_{t+i,1}} \ldots x_s(1)^{c_{t+i,s}}} & \text{for } 1 \le i \le p-t \\ x_{i+t} & \text{for } p-t+1 \le i \le q-t. \end{cases}$$

$$\psi_i = \begin{cases} x_{t+i}(1) - \overline{\phi}_{t+i} & \text{for } 1 \le i \le p-t \\ x_{i+t} & \text{for } p-t+1 \le i \le q-t. \end{cases}$$

There exist $b_{ij} \in \hat{U}_1$ such that

$$\sigma_i = \sum_{j=1}^{q-t} b_{ij}\psi_j$$

for $1 \le i \le p-t$, with

$$b_{ij} = \delta_{ij}$$

for $p-t+1 \le i$, such that $\mathrm{Det}(b_{ij})$ is a unit in $\hat{U}_1$. There exist $h_i \in m^m_{\hat{T}_{1,m}}$ such that

$$f_{t+i} = x_{1,m}(1)^{c_{t+i,1}} \ldots x_{1,m}(s)^{c_{t+i,s}}[\sum_{j=1}^{p-t} b_{ij}(x_{1,m}(1), \ldots, x_{1,m}(s), \ldots, x_{p,m}(1),$$
$$x_{p+1}, \ldots, x_q)\psi_j(x_{1,m}(1), \ldots, x_{1,m}(s), \ldots, x_{p,m}(1), x_{p+1}, \ldots, x_q) +$$
$$\sum_{j=p-t+1}^{q} b_{ij}x_j] + h_i$$

for $1 \le i \le p-t$.

As in the proof of Theorem 4.3.8, there exists a sequence of transforms $T \to T_{1,m} \to T_2$ along ν satisfying the conclusion of Theorem 4.3.11.

□

Theorem 4.3.12. *Suppose that $x_1, \ldots, x_s, \ldots, x_q$ are regular parameters in T such that $\nu(x_1), \ldots, \nu(x_s)$ are rationally independent. Suppose that $f \in \hat{T}$.*

1. *If* $f \notin Q_{\hat{T}}$*, then there exists a sequence of transforms of type* $(1,0)$ *and* $(2,0)$*,* $T \to T_1$ *such that*

$$f = x_1(1)^{c_1} \dots x_s(1)^{c_s}\gamma$$

where $\gamma \in \hat{T}_1$ *is a unit.*

2. *If* $f \in Q_{\hat{T}}$ *and* $d > 0$*, then there exists a sequence of transforms of type* $(1,0)$ *and* $(2,0)$*,* $T \to T_1$ *such that*

$$f = x_1(1)^{c_1} \dots x_s(1)^{c_s}\gamma$$

where $\gamma \in \hat{T}_1$*, and* $\nu(x_1(1)^{c_1} \dots x_s(1)^{c_s}) > d$*.*

Proof. If $f \in \hat{T} - Q_{\hat{T}}$ then there exists $l \in \mathbb{N}$ such that $\nu(f) < l v(m)$, thus $f = g + h$ where $h \in m^l$ and $\nu(f) = \nu(g) < l v(m)$. By the methods presented in this section, there exists a sequence of transforms of type $(1,0)$ followed by a sequence of transforms of type $(2,0)$, $T \to T_1$ such that

$$f = x_1(1)^{c_1} \dots x_s(1)^{c_s}\gamma$$

where $\gamma \in \hat{T}_1$ is a unit.

If $f \in Q_{\hat{T}}$ then, $\nu(f) > l v(m)$ for all $l \in \mathbb{N}$, thus given $d > 0$, there exists a sequence of transforms of type $(1,0)$ followed by a sequence of transforms of type $(2,0)$, $T \to T_1$ such that

$$f = x_1(1)^{c_1} \dots x_s(1)^{c_s}\gamma$$

where $\gamma \in \hat{T}_1$ and $\nu(x_1(1)^{c_1} \dots x_s(1)^{c_s}) > d$. □

Chapter 5

Resolution in all height

Let

$$(0) = P_V^t \subset \cdots \subset P_V^1 \subset P_V^0 = m_V$$

be the chain of prime ideals in V. Let

$$(0) = P_T^t \subset \cdots \subset P_T^1 \subset P_T^0 = m_T$$

be the chain of prime ideals in T, where $P_T^i = P_V^i \cap T$.

Let ν_i be a valuation whose valuation ring is $V_i = V_{P_V^i}$.

Consider the following condition (5.1) on a Cauchy sequence $\{f_n\}$ in $T_{P_T^i}$.

$$\text{For all } l \in \mathbb{N}, \text{ there exists } n_l \in \mathbb{N} \text{ such that } \nu_i(f_n) \geq l\nu(m_{T_{P_T^i}}) \text{ if } n \geq n_l. \quad (5.1)$$

Define $Q_{\widehat{T_{P_T^i}}} \subset \widehat{T_{P_T^i}}$ for the valuation ring $V_{P_V^i}$ by the following:

$Q_{\widehat{T_{P_T^i}}} = \{f \in \widehat{T_{P_T^i}} |$ A Cauchy sequence$\{f_n\}$ in $T_{P_T^i}$ which converges to f satisfies (5.1)$\}$.

We assume throughout this section that

$$\text{trdeg}_{T_{P_T^i}/P_T^i T_{P_T^i}}(V_{P_V^i}/P_V^i V_{P_V^i}) = \text{trdeg}_{QF(T/P_T^i)} QF(V/P_V^i) = 0$$

for all i, and $T_{P_T^i}$ satisfies property (A) for $V_{P_V^i}$ for all i.

Let

$$\tau(i) = dim(T/P_T^i),\ \sigma(i) = dim(\widehat{T_{P_T^i}}/Q_{\widehat{T_{P_T^i}}})$$

for $0 \leq i \leq t$.

Let s_i be the rational rank of $(V/P_V^{i+1})_{P_V^i}$ for $0 \leq i \leq t$. We will write $s = s_0$ and $\nu = \nu_0$.

Lemma 5.0.13. *Suppose there exists a regular system of parameters $z_1, \ldots, z_q$ in T such that $\nu(z_i) < \infty$ and $\nu(z_1), \ldots, \nu(z_s)$ are rationally independent. Let $R = T_{P_T^1}$. Suppose that P_T^1 is a regular prime in T and*

$$y_{\tau(1)+1}, \ldots, y_q$$

are regular parameters in R. Then there exists a sequence of Perron transforms of types $(1,0)$ and $(2,0)$ $T \rightarrow T_1$ along ν such that

$$(T_1)_{P_{T_1}^1} = T_{P_T^1},$$

and T_1 has a regular system of parameters

$$x_1(1), \ldots, x_s(1), \ldots, x_{\tau(1)+1}(1), \ldots, x_q(1)$$

such that $\nu(x_1(1)), \ldots, \nu(x_s(1))$ are rationally independent and

$$P_{T_1}^1 = (x_{\tau(1)+1}(1), \ldots, x_q(1))$$

and

$$y_i = x_1(1)^{d_i^1} \ldots x_s(1)^{d_i^s} x_i(1)$$

for some $d_i^j \in \mathbb{N}$ *and* $\tau(1)+1 \leq i \leq q$.

Proof. There are regular parameters

$$x_1, \ldots, x_{\tau(1)+1}, \ldots, x_q$$

in T such that

$$P_T^1 = (x_{\tau(1)+1}, \ldots, x_q).$$

There exist $\lambda_{ij} \in R$ such that

$$y_{\tau(1)+i} = \sum_{j=1}^{q-\tau(1)} \lambda_{ij} x_{\tau(1)+j}$$

for $1 \leq i \leq q - \tau(1)$.

For $1 \leq i \leq q - \tau(1)$, there exist $f_{ij}, g_{ij} \in T$ with $g_{ij} \notin P_T^1$ such that

$$\lambda_{ij} = \frac{f_{ij}}{g_{ij}}.$$

By Theorem 4.3.12, there exists a sequence of transforms of types $(1,0)$ and $(2,0)$ $T \to T_1$ such that T_1 has regular parameters

$$x_1(1), \ldots, x_{\sigma(1)}(1), x_{\sigma(1)+1}, \ldots, x_q,$$

such that

$$g_{ij} = x_1(1)^{d_{ij}^1} \ldots x_s(1)^{d_{ij}^s} \gamma_{ij}$$

where $\gamma_{ij} \in T_1$ are units for all i, j. Now we perform a transform of type $(2,0)$ $T_1 \to T_1'$, defined by

$$x_i = x_1(1)^{d_i^1} \ldots x_s(1)^{d_i^s} x_i(1)$$

for $\tau(1)+1 \leq i \leq q$, and appropriately large d_i^j, followed by a transform of type I, so that (by Lemma 4.2 [16]) T_1' has regular parameters

$$x_1(1), \ldots, x_q(1)$$

such that there are expansions

$$y_{\tau(1)+i} = \sum_{j=1}^{q-\tau(1)} \lambda_{ij}' x_{\tau(1)+j}(1) \tag{5.2}$$

for $1 \leq i \leq q-\tau(1)$, with $\lambda_{ij}' \in T_1'$.

We will now show that we can make a further sequence of transforms $T_1' \to T_2$ of types $(1,0)$ and $(2,0)$ such that T_2 has regular parameters

$$x_1(2), \ldots, x_q(2)$$

with $P_{T_2}^1 = (x_{\tau(1)+1}(2), \ldots, x_q(2))$, and there are expressions

$$y_{\tau(1)+i} = x_1(2)^{d_i^1} \ldots x_s(2)^{d_i^s} x_{\tau(1)+i}(2) \tag{5.3}$$

for $1 \leq i \leq q-\tau(1)$.

We prove this as follows. Since $\mathrm{Det}(\lambda_{ij}')$ is a unit in R, there exists a λ_{ik}' such that $\lambda_{ik}' \notin P_{T_1'}^1$. Without loss of generality, $i=1$. By Theorem 4.3.12, there exists a sequence of transforms of types $(1,0)$ and $(2,0)$ $T_1' \to T_1''$ such that T_1'' has a regular system of parameters

$$x_1(1)'', \ldots, x_{\tau(1)}(1)'', x_{\tau(1)+1}(1), \ldots, x_q(1)$$

such that for $1 \leq j \leq q-\tau(1)$ $\lambda_{1j}' = [x_1(1)'']^{c_j^1} \ldots [x_s(1)'']^{c_j^s}\gamma_j$, where $\gamma_j \in T_1''$ is a unit if $\lambda_{1j}' \notin P_{T_1'}^1$, and $\gamma_j \in T_1''$ and $\nu([x_1(1)'']^{c_j^1} \ldots [x_s(1)'']^{c_j^s})$ is arbitrarily large if

$\lambda'_{1j} \in P^1_{T'_1}$. After permuting $x_{\tau(1)+1}, \ldots, x_q$, we may assume that

$$\nu([x_1(1)'']^{c^1_j} \ldots [x_s(1)'']^{c^s_j}) \text{ has minimal value for } 1 \leq j \leq q - \tau(1).$$

After performing a transform of type I, we then have an expression

$$\begin{aligned} y_{\tau(1)+1} = & [x_1(1)'']^{c^1_1} \ldots [x_s(1)'']^{c^1_s}[(\gamma_1 x_{\tau(1)+1}(1)) + \sum_{j=2}^{q-\tau(1)} [x_1(1)'']^{c^1_j - c^1_1} \ldots \\ & [x_s(1)'']^{c^s_j - c^s_1} \gamma_j x_{\tau(1)+j}(1)]. \end{aligned}$$

We may make a change of variables in T''_1, replacing $x_{\tau(1)+1}$ with $x_{\tau(1)}(1)'' = \frac{y_{\tau(1)+1}}{[x_1(1)'']^{c^1_1} \ldots [x_s(1)'']^{c^1_s}}$. We now have an expression similar to (5.2)

$$y_{\tau(1)+i} = \sum_{j=1}^{q-\tau(1)} \lambda''_{ij} x_{\tau(1)+j}(1) \tag{5.4}$$

for $1 \leq i \leq q - \tau(1)$ in T''_1, with $y_{\tau(1)+1} = [x_1(1)'']^{c^1_1} \ldots [x_s(1)'']^{c^1_s} x_{\tau(1)+1}(1)''$. Since $\text{Det}(\lambda''_{ij})$ is a unit in $P^1_{T''_1}$, we have $\lambda''_{ij} \notin P^1_{T''_1}$ for some $2 \leq i$.

We iterate our construction of $T'_1 \to T''_1$ to construct $T''_1 \to T_2$ such that (5.3) holds. □

5.1 Perron Transforms

Perron Transforms of type $(1, m)$

Suppose that $1 \leq m \leq n$ (with $n \leq t$) and there exists a regular system of parameters

$$x_1, \ldots, x_q$$

in T such that

$$P^i_T = (x_{\tau(i)+1}, \ldots, x_q)$$

for $0 \leq i \leq n$, and $v_i(x_{\tau(i)+1}), \ldots, v_i(x_{\tau(i)+s_i})$ are rationally independent for $0 \leq i \leq n$.

Analogous to a transform of type $(1,0)$ in Chapter 4, we define two types of such transforms along ν $T \rightarrow T_1$, transforms of type I and type II_r

Type I is defined by a transformation

$$\begin{array}{lcl} x_{\tau(m)+1} & = & x_{\tau(m)+1}(1)^{a_{11}} \ldots x_{\tau(m)+s_m}(1)^{a_{1s_m}} \\ \vdots & & \\ x_{\tau(m)+s_m} & = & x_{\tau(m)+1}(1)^{a_{s_m1}} \ldots x_{\tau(m)+s_m}(1)^{a_{s_ms_m}} \end{array} \tag{5.5}$$

where Det $(a_{ij}) = \pm 1$ and $v_m(x_{\tau(m)+1}(1)), \ldots, v_m(x_{\tau(m)+s_m}(1))$ are rationally independent.

We define $T_1 = T[x_{\tau(m)+1}(1), \ldots, x_{\tau(m)+s_m}(1)]_{T[x_{\tau(m)+1}(1),\ldots,x_{\tau(m)+s_m}(1)]\cap m_V}$.

Suppose that $\tau(m) + s_m < r$ and $v_m(x_r) < \infty$. We will say that $x_{\tau(m)+1}, \ldots, x_{\tau(m)+s_m}, x_r$ are permissible of type m if there exists a transformation

$$\begin{array}{lcl} x_{\tau(m)+1} & = & N_{\tau(m)+1}^{a_{11}} \ldots N_{\tau(m)+s_m}^{a_{1s_m}} N_r^{a_{1,s_m+1}} \\ \vdots & & \\ x_{\tau(m)+s_m} & = & N_{\tau(m)+1}^{a_{s_m1}} \ldots N_{\tau(m)+s_m}^{a_{s_m,s_m}} N_r^{a_{s_m,s_m+1}} \\ x_r & = & N_{\tau(m)+1}^{a_{s_m+1,1}} \ldots N_{\tau(m)+s_m}^{a_{s_m+1,s_m}} N_r^{a_{s_m+1,s_m+1}} \end{array} \tag{5.6}$$

where Det $(a_{ij}) = \pm 1$, $0 < v_m(N_i) < \infty$ for $\tau(m) + 1 \leq i \leq \tau(m) + s_m$, $v_m(N_{\tau(m)+1}), \ldots, v_m(N_{\tau(m)+s_m})$ are rationally independent, $v_m(N_r) = 0$, and $\nu(N_r) \geq 0$.

If such a transformation exists, we define

$T_1 = T[N_{\tau(m)+1}, \ldots, N_{\tau(m)+s_m}, N_r]_{T[N_{\tau(m)+1},\ldots,N_{\tau(m)+s_m},N_r]\cap m_V}$ and call $T \rightarrow T_1$ a transformation of type II_r.

Let $(b_{ij}) = (a_{ij})^{-1}$. We have

$$N_i = x_{\tau(m)+1}^{b_{i1}} \dots x_{\tau(m)+s_m}^{b_{i,s_m}} x_r^{b_{i,s_m+1}} \tag{5.7}$$

for $\tau(m)+1 \leq i \leq \tau(m)+s_m$ and $i = r$.

Remark 5.1.1. *1. We could have that $P_{T_1}^j$ are not regular primes if $j \leq m$.*

2. We have $P_{T_1}^i = (x_{\tau(i)+1}, \dots, x_q)$ are regular primes for $m < i \leq n$.

3. Type $(1, m)$ is a generalization to higher rank of the transform of type $(1, 0)$ of Chapter4.

Transforms of type $(2, m)$

Suppose that $x_j \in P_T^{m+1}$ and $d_1, \dots, d_{s_m} \in \mathbb{N}$.

Define a transformation $T \to T_1$ of type $(2, m)$ by

$$T_1 = T[\frac{x_j}{x_{\tau(m)+1}^{d_1} \dots x_{\tau(m)+s_m}^{d_{s_m}}}]_{T[\frac{x_j}{x_{\tau(m)+1}^{d_1} \dots x_{\tau(m)+s_m}^{d_{s_m}}}] \cap m_V}.$$

Let $x_j(1) = \frac{x_j}{x_{\tau(m)+1}^{d_1} \dots x_{\tau(m)+s_m}^{d_{s_m}}}$, and $x_i(1) = x_i$ if $i \neq j$.

Remark 5.1.2. *1. We have $P_{T_1}^i = (x_{\tau(i)+1}(1), \dots, x_q(1))$ for $1 \leq i \leq n$.*

2. Type $(2, m)$ is a generalization to higher rank of the transform of type $(2, 0)$ of Chapter 4.

5.2 Extension of results to higher rank

Lemma 5.2.1. *Suppose there exists a regular system of parameters $z_1, \ldots, z_q$ in T such that $\nu(z_i) < \infty$ and $\nu(z_1), \ldots, \nu(z_s)$ are rationally independent. Suppose that $R = T_{P_T^1}$ has regular parameters*

$$y_{\tau(1)+1}, \ldots, y_{\tau(2)+1}, \ldots, y_{\tau(m)+1}, \ldots, y_q$$

such that $v_m(y_{\tau(m)+1}), \ldots, v_m(y_{\tau(m)+s_m})$ are rationally independent, and $\tau(m) + s_m < r$ is such that $v_m(y_r) < \infty$, and $y_{\tau(m)+1}, \ldots, y_{\tau(m)+s_m}, y_r$ are permissible of type m. Let $R \to R_1$ be the transformation of type $(1, m-1)$ and type II_r defined by (5.6) so that

$$R_1 = R[N_{\tau(m)+1}, \ldots, N_r]_B$$

Where $B = R[N_{\tau(m)+1}, \ldots, N_r] \cap m_{V_{P_V^1}}$.

Then there exists a sequence of transformations of type $(1,0)$ and $(2,0)$ $T \to T_1$ such that $(T_1)_{P_{T_1}^1} = R_1$ and T_1 has regular parameters

$$x_1(1), \ldots, x_{\tau(1)}(1), x_{\tau(1)+1}(1), \ldots, x_{\tau(2)+1}(1), \ldots, x_{\tau(m)+1}(1), \ldots, x_q(1)$$

such that $\nu(x_1(1)), \ldots, \nu(x_s(1))$ are rationally independent, $P_{T_1}^1 = (x_{\tau(1)+1}(1), \ldots, x_q(1))$, there exist $d_i^j \in \mathbb{N}$ such that $y_i = x_1(1)^{d_i^1} \ldots x_s(1)^{d_i^s} x_i(1)$ for $\tau(1) + 1 \le i$, and $x_{\tau(m)+1}(1), \ldots, x_{\tau(m)+s_m}(1), x_r(1)$ are permissible of type m.

Let $b_{ij} = (a_{ij}^{-1})$ as in (5.6), and let

$$\overline{N}_{\tau(m)+i} = x_{\tau(m)+1}(1)^{bi1} \ldots x_{\tau(m)+s_m}(1)^{b_{i,s_m}} x_r(1)^{b_{i,s_m+1}}$$

for $1 \le i \le s_m$ and $\overline{N}_r = x_{\tau(m)+1}(1)^{b_{s_m+1,1}} \ldots x_r(1)^{b_{s_m+1,s_m+1}}$

Let $T_1 \rightarrow T_2$ be the resulting transform of type $(1, m)$ where

$$T_2 = T_1[\overline{N}_{\tau(m)+1}, \ldots, \overline{N}_{\tau(m)+s_m}, \overline{N}_r]_{B_1}$$

where $B_1 = T_1[\overline{N}_{\tau(m)+1}, \ldots, \overline{N}_{\tau(m)+s_m}, \overline{N}_r] \cap m_V$.

Then $(T_2)_{P^1_{T_2}} = R_1$.

Proof. We first apply Theorem 4.3.11 to construct $T \rightarrow T_1$ such that $P^1_{T_1}$ is a regular prime and $(T_1)_{P^1_{T_1}} = R$. Then we apply Lemma 5.0.13 to construct $T_1 \rightarrow T_2$ such that $(T_2)_{P^1_{T_2}} = (T_1)_{P^1_{T_1}}$ and T_2 has a regular system of parameters

$$x_1(2), \ldots, x_{\tau(1)+1}(2), \ldots, x_{\tau(m)+1}(2), \ldots, x_q(2)$$

satisfying $y_i = x_1(2)^{d_i^1} \ldots x_s(2)^{d_i^s} x_i(2)$ for $\tau(1) + 1 \leq i \leq q$ and some $d_i^j \in \mathbb{N}$. We have that $b_{s_m+1,i} < 0$ for some i (as in (5.7)). After possibly permuting the $x_j(1)$, we may assume that $i \neq 1$.

Now make the transformation of type $(2, m)$ $T_2 \rightarrow T_3$ defined by

$$x_{\tau(m)+i}(2) = x_1(3)^{c_1^i} \ldots x_s(1)^{c_s^i} x_{\tau(m)+i}(3)$$

for $1 \leq i \leq s_m$, where c_i^j is chosen so that

$$\nu(x_{\tau(m)+1}(3)^{b_{s_m+1,1}} \ldots x_r(3)^{b_{s_m+1,s_m+1}}) > 0.$$

We have that $(T_3)_{P^1_{T_3}} = (T_2)_{P^1_{T_2}}$.

We now have that $x_{\tau(m)+1}(3), \ldots, x_{\tau(m)+s_m}(3), x_r(3)$ are permissible of type m. Define $M_{\tau(m)+i} = x_{\tau(m)+1}(3)^{b_{i1}} \ldots x_{\tau(m)+s_m}(3)^{b_{i,s_m}} x_r(3)^{b_{i,s_m+1}}$ for $1 \leq i \leq s_m$ and $M_r = x_{\tau(m)+1}(3)^{b_{s_m+1,1}} \ldots x_{\tau(m)+s_m}(3)^{b_{s_m+1,s_m}} x_r(3)^{b_{s_m+1,s_m+1}}$.

We may define the transformation of type $(1,m)$ $T_3 \to T_4$, where

$$T_4 = T_3[M_{\tau(m)+1}, \ldots, M_{\tau(m)+s_m}, M_r]_{B_3}$$

with $B_3 = T_3[M_{\tau(m)+1}, \ldots, M_{\tau(m)+s_m}, M_r] \cap m_V$.

There exists $n_i^j \in \mathbb{Z}$ such that $M_i = N_i x_1(3)^{n_1^i} \ldots x_s(3)^{n_s^i}$ for all i.

Since $P^1_{T_4} \cap T_3 = P^1_{T_3}$, $R = (T_3)_{P^1_{T_3}}$ and $x_1(3), \ldots, x_s(3) \notin P^1_{T_3}$, we have that

$$(T_4)_{P^1_{T_4}} = (T_3)_{P^1_{T_3}}[M_{\tau(m)+1}, \ldots, M_{\tau(m)+s_m}, M_r]_{P^1_{T_4}}$$

$$= R[N_{\tau(m)+1}, \ldots, N_{\tau(m)+s_m}, N_r]_{R[N_{\tau(m)+1}, \ldots, N_{\tau(m)+s_m}, N_r] \cap m_{V_{P^1_V}}} = R_1.$$

□

Remark 5.2.2. *A similar but simpler statement holds for transformations $R \to R_1$ of type $(1, m-1)$ and type I.*

Lemma 5.2.3. *Suppose there exists a regular system of parameters $z_1, \ldots, z_q$ in T such that $\nu(z_i) < \infty$ and $\nu(z_1), \ldots, \nu(z_s)$ are rationally independent. Suppose that $R = T_{P^1_T}$ has regular parameters*

$$y_{\tau(1)+1}, \ldots, y_{\tau(m)+1}, \ldots, y_{\sigma(m)+s_m}, \ldots, y_j, \ldots, y_q$$

such that $\nu(y_{\tau(m)+1}), \ldots, \nu(y_{\tau(m)+s_m})$ are rationally independent, and $y_j \in P_R^{m+1}$.

Suppose that $R \to R_1$ is the transformation of type $(2, m-1)$ defined by

$$R_1 = R\left[\frac{y_j}{y^{d_1}_{\tau(m)+1} \cdots y^{d_{s_m}}_{\tau(m)+s_m}}\right]_{R\left[\frac{y_j}{y^{d_1}_{\tau(m)+1} \cdots y^{d_{s_m}}_{\tau(m)+s_m}}\right] \cap m_{V_{P^1_V}}}$$

for some $d_1, \ldots, d_{s_m} \in \mathbb{N}$. Then there exists a sequence of transformations of types $(1,0)$ and $(2,0)$, $T \to T_1$ such that $(T_1)_{P^1_{T_1}} = R$ and T_1 has regular parameters

$$x_1(1), \ldots, x_{\tau(1)+1}(1), \ldots, x_{\tau(m)+1}(1), \ldots x_q$$

such that $\nu(x_1(1)),\ldots,\nu(x_s(1))$ *are rationally independent and there exists* $d_i^j\in\mathbb{N}$ *with* $y_i=x_1(1)^{d_i^1}\ldots x_s(1)^{d_i^s}x_i(1)$ *for* $\tau(1)+1\le i$*. Let* $T_1\to T_2$ *where*

$$T_2=T_1[\frac{x_j(1)}{x_{\tau(m)+1}(1)^{d_1}\ldots x_{\tau(m)+s_m}(1)^{d_{s_m}}}]_{T_1[\frac{x_j(1)}{x_{\tau(m)+1}(1)^{d_1}\ldots x_{\tau(m)+s_m}(1)^{d_{s_m}}}]\cap m_V}$$

be the resulting transformation of type $(2,m)$*. Then* $(T_2)_{P^1_{T_2}}=R_1$*.*

Proof. The proof is similar to the proof of Lemma 5.2.1, but is a little simpler. □

Theorem 5.2.4. *Suppose that* T *satisfies the assumptions of this section, and that* T *has a regular system of parameters* $x_1,\ldots,x_q$ *such that* $x_{\tau(i)+1},\ldots,x_{\tau(i)+s_i}\in P^i_T-P^{i+1}_T$*, for every* i *and* $v_i(x_{\tau(i)+1}),\ldots,v_i(x_{\tau(i)+s_i})$ *are rationally independent for all* i*. Then there exists a sequence of transformations* $T\to T_1$ *(of types* $(1,a)$ *and* $(2,b)$ *for appropriate* $a,b\in\mathbb{N}$*) such that*

$$(0)=P^t_{T_1}\subset\cdots\subset P^0_{T_1}=m_{T_1}$$

are regular primes.

Proof. The proof is by induction on t. The case $t=0$ is trivial. Assume that the theorem is true for $t-1$. Let $R=T_{P^1_T}$. By induction, there exists a sequence of transformations $R\to R_1$ such that the conclusions of the theorem hold for ν_1 on R_1, so that R_1 has a system of regular parameters

$$y_{\tau(1)+1},\ldots,y_{\tau(2)+1},\ldots,y_{\tau(t-1)+1},\ldots,y_q$$

satisfying the conclusions of the theorem.

By Lemma 5.2.1 , Remark 5.2.2 and Lemma 5.2.3, there exists a sequence of transformations $T\to T_1$ such that $(T_1)_{P^1_{T_1}}=R_1$.

By Theorem 4.3.11, there exists a sequence of transformations of type $(1,0)$ and $(2,0)$ $T_1 \to T_2$ such that $P^1_{T_2}$ is a regular prime and $(T_2)_{P^1_{T_2}} = R_1$.

By Lemma 5.0.13, there exists a sequence of transformations of types $(1,0)$ and $(2,0)$ $T_2 \to T_3$ such that $(T_3)_{P^1_{T_3}} = R_1$ and T_3 has a system of regular parameters

$$x_1(3), \ldots, x_{\tau(1)+1}(3), \ldots, x_q(3)$$

such that $y_i = x_1(3)^{d_i^1} \ldots x_s(3)^{d_i^s} x_i(3)$ for $\tau(1)+1 \leq i$, and some $d_i \in \mathbb{N}$. Thus the conclusions of the theorem hold for T_3. $\square$

Chapter 6

Local Uniformization

Normal uniformization transformation sequences (NUTS) are defined in Definition 6.1 of [19]. Etale Perron transforms, and any birational extensions of normal local rings dominated by a valuation are examples of NUTS.

Lemma 6.0.5. *Suppose that K is an algebraic function field over a field k of characteristic zero, V is a valuation ring of K, and R is an algebraic local ring of K such that V dominates R. Let*

$$(0) = P_V^t \subset \cdots \subset P_V^1 \subset P_V^0 = m_V$$

be the chain of prime ideals in V. Let $V_i = V_{P_V^i}$ for $0 \leq i$, ν_i be a valuation whose valuation ring is V_i. Then there exists a normal algebraic local ring S of K, such that V dominates S and S dominates R, with the property that for all i, if $S_{P_V^i \cap S} \to S_1$ is a NUTS along ν_i, then $\sigma(S_{P_V^i \cap S}) = \sigma(S_1)$ and $\tau(S_{P_V^i \cap S}) = \tau(S_1)$. Thus $S_{P_V^i \cap S}$ satisfies property (A) for all i. Further,

$$trdeg_{QF(S_1/m_{V_i} \cap S_1)} QF(V/P_V^i) = 0$$

for all i.

Proof. We choose $\lambda_1, \ldots, \lambda_n \in V$ such that

$$\mathrm{trdeg}_{QF(R[\lambda_1,\ldots,\lambda_n]/P_V^i \cap R[\lambda_1,\ldots,\lambda_n])} QF(V/P_V^i) = 0$$

for all i. Let

$$\tilde{S} = R[\lambda_1, \ldots, \lambda_n]_{R[\lambda_1,\ldots,\lambda_n] \cap m_V}.$$

By Theorem 6.3 [19], there exists a normal algebraic local ring S of K, with quotient field K such that V dominates S and S dominates $\tilde{S}$, with the property that for all i, if $S_{P_V^i \cap S} \to S_1$ is a NUTS along ν_i then $\sigma(S_{P_V^i \cap S}) = \sigma(S_1)$. Since $S_{P_V^i \cap S}$ dominates $\tilde{S}_{P_V^i \cap \tilde{S}}$, we have that

$$\mathrm{trdeg}_{\mathrm{QF}(S_1/m_{V_i} \cap S_1)} \mathrm{QF}(V/P_V^i) = \mathrm{trdeg}_{QF(\tilde{S}/P_V^i \cap \tilde{S})} QF(V/P_V^i) = 0,$$

and $\tau(S_1) = \tau(S_{P_V^i \cap S})$ for all i. □

The following theorem is stronger than Zariski's local uniformization theorem.

Theorem 6.0.6. *Suppose that R is a local domain which is essentially of finite type over a field k of characteristic zero, and ω is a valuation of the quotient field of R which dominates R. Let W be the valuation ring of ω, and let*

$$(0) = P_W^t \subset \cdots \subset P_W^0 = m_W$$

be the chain of prime ideals of W. Then there exists a birational extension $R \to R_1$ such that R_1 is a regular local ring and W dominates R_1. Further, $P_{R_1}^i = P_W^i \cap R_1$ are regular primes for all i, and $Q_{\widehat{R_1}_{P_{R_1}^i}} \subset \widehat{R_1}_{P_{R_1}^i}$ are regular primes for all i.

Proof. Let S be the extension of R satisfying the conclusions of Lemma 6.0.5. Since S is essentially of finite type over k, there exists a regular local ring T, which is essentially of finite type over k, and a prime ideal P in T such that $T/P = S$. Let ν_t be the PT_P adic valuation of the regular local ring T_P, and let ν be the composite of ω and ν_t. Let V be the valuation ring of ν.

$$\text{trdeg}_{QF(T/P_T^i)} QF(V/P_V^i) = 0$$

for all i and $T_{P_T^i}$ satisfies property (A) for all i.

Let

$$(0) \subset P_V^t \subset \cdots \subset P_V^0 = m_V$$

be the chain of prime ideals in V. We have $V/P_V^i \cong W/P_W^i$ for $0 \leq i \leq t$. Let s_i be the rational rank of $(V/P_V^{i+1})_{P_V^i}$ for all i. Let $\tau(i) = dim(T/P_V^i \cap T)$. By adjoining appropriate elements of k to S, we may further assume that T is a localization of a polynomial ring, $T = k[x_1, \ldots, x_q]_{(x_1,\ldots,x_q)}$ where $x_{\tau(i)+1}, \ldots, x_{\tau(i)+s_i} \in P_T^i - P_T^{i+1}$, for every i, and $v_i(x_{\tau(i)+1}), \ldots, v_i(x_{\tau(i)+s_i})$ are rationally independent for all i.

By Theorem 5.2.4, There exists a sequence of transforms $T \to T_1$ such that T_1 is a regular local ring, $P_{T_1}^i$ are regular primes for all i, and $T_P \cong T_{P_T^t} \cong (T_1)_{P_{T_1}^t}$. Now by applying the methods of Chapter 4 and by Theorem 4.3.8, we achieve $Q_{\widehat{(T_1)_{P_{T_1}^i}}} \subset \widehat{(T_1)_{P_{T_1}^i}}$ are regular primes for all i. Thus $R \to R_1 = T_1/P_{T_1}^t$ is a birational extension of R with the desired properties. $\square$

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