INERTIAL CHOW RINGS AND A NEW ASYMPTOTIC PRODUCT

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by

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INERTIAL CHOW RINGS AND A NEW ASYMPTOTIC PRODUCT

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and hereby certify that in their opinion it is worthy of acceptance.

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Professor Calin Chindris

Professor Zhenbo Qin

Professor Saku Aura

To my father, who introduced me to the

beauty and the depth of mathematics.

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ABSTRACT

For any toric Deligne-Mumford stack X and equivariant vector bundle V, we can define an two associative inertial products. We give a ring presentation for the inertial Chow ring of X under each of these products and compute these rings in the toric case. In particular, we make explicit distinctions between the contributions of the inertia of X and of the products themselves. We further show the existence of a new associative product on the inertia of X in which the rank of V asymptotically approaches infinity, and we compute its Chow ring.

Chapter 1

Introduction

1.1 Background and history

The orbifold cohomology ring was first introduced by W. Chen and Ruan in [6] and [7]. Fantechi and Göttsche showed in [16] how to calculate the orbifold cohomology when the orbifold $\mathscr{X}$ could be written as a global quotient by a finite group. Abramovich, Graber and Vistoli in [1] then produced an orbifold product on the Chow group $A^*(I\mathscr{X})$, where $I\mathscr{X}$ is the inertia of a smooth Deligne-Mumford stack $\mathscr{X}$ whose coarse moduli space is projective.

Earlier products involved taking the integral of an obstruction fundamental class on a moduli of stable maps to the orbifold. Jarvis, Kaufmann and Kimura gave a construction of the orbifold product in [22] which did not make reference to obstruction bundles, stable curves or moduli spaces. This construction was refined and generalized by Edidin, Jarvis and Kimura in [14] and [15]. Around the same time, Kaufmann showed in [25] and [26] that the Fantechi-Göttsche construction from [16] could be handled by an axiomatic approach.

Inertial products were defined by Edidin, Jarvis and Kimura in [15] as a generalization of the orbifold product on the inertia of a Deligne-Mumford stack. For every equivariant vector bundle V on $\mathcal{X}(\mathbf{\Sigma})$ they constructed two products on the inertia, called the $\star_{V^+}$ and $\star_{V^-}$ products.

Two special cases of inertial products are already well known. If V is the trivial bundle, the $\star_{V^+}$ product is the orbifold product of Chen and Ruan; if V is the tangent bundle, the $\star_{V^-}$ product is the virtual product of [21].

Cox developed a theory for global coordinates on a simplicial toric variety in [8]. In [3], Borisov, Chen and Smith generalized Cox's work to define toric stacks, their idea being to present them as quotient of an open set in a representation of a torus. They used this construction to give an explicit computation of the orbifold Chow ring of a toric Deligne-Mumford stacks. While restricting to the toric case limited the scope of the question, the relative simplicity of computations furthered the reach of the theory. Not long after, Jiang and Tseng in [23] adapted the results of [3] from rational coefficients to integer coefficients.

1.2 Main results

1.2.1 Inertial Chow rings

A toric stack is defined by the data of a fan Σ and a homomorphism $\beta : L \to N$, where L is a lattice and N is a finitely generated abelian group N. Together, this data $\mathbf{\Sigma} = (N, \Sigma, \beta)$ comprises what is known a *stacky fan*. Let $\mathcal{X}(\mathbf{\Sigma})$ be the toric stack associated to this stacky fan, as constructed in [3] or [19].

Let $A^*(\mathscr{X}, \star, R)$ denote the Chow ring of $\mathscr{X}$ under some inertial product $\star$ with coefficients in the ring R. Under this notation, the usual intersection ring of Fulton [17] on the inertia of $\mathscr{X}$ is written $A^*(\mathscr{X}, \cdot, \mathbb{Z})$, the orbifold Chow ring of [3] is notated $A^*(\mathscr{X}, \star_{orb}, \mathbb{Q})$ and the integral orbifold Chow ring of [23] is written $A^*(\mathscr{X}, \star_{orb}, \mathbb{Z})$.

The main result is a ring presentation for the inertial Chow ring of a toric stack associated to the stacky fan $\mathbf{\Sigma}$ in a way which explicitly describes and separates the contributions of $\mathbf{\Sigma}$ and $\star$.

Theorem 1.1. *If $\mathcal{X}(\mathbf{\Sigma})$ is a complete toric Deligne-Mumford stack and $\star$ is an inertial product on $\mathcal{X}(\mathbf{\Sigma})$, then there is an isomorphism of $\mathbb{Z}$-graded rings*

$$A^*(\mathcal{X}(\mathbf{\Sigma}), \star, \mathbb{Z}) \cong \frac{R_{\mathbf{\Sigma}}}{I(\mathbf{\Sigma}) + \mathrm{Cir}(\mathbf{\Sigma}) + \mathrm{CR}(\mathbf{\Sigma}) + \mathrm{BR}(\star, \mathbf{\Sigma})}.$$

In this presentation, $R_{\mathbf{\Sigma}}$ is a polynomial ring whose generators correspond to the rays of $\mathbf{\Sigma}$ and the twisted sectors of $\mathcal{X}(\mathbf{\Sigma})$. Three of the ideals which we quotient by can be written explicitly in terms of the combinatorics of the stacky fan, but the box relations ideal $\mathrm{BR}(\star, \mathbf{\Sigma})$ depends also on the choice of inertial product $\star$.

1.2.2 A new asymptotic product

In Section 6, we define a new product on the inertia of $\mathcal{X}(\mathbf{\Sigma})$.

Consider any G-equivariant bundle $V = \sum_{i=1}^n f_i(\mathcal{L}_i)$, where $f_i(\mathcal{L}_i)$ denotes a polynomial in $\mathcal{L}_i$ and $\mathcal{L}_i^{-1}$ with positive coefficients, and where $\mathcal{L}_i$ is a line bundle on $\mathscr{X}(\mathbf{\Sigma})$. We define the *asymptotic-minus product* of x and y to be

$$x \star_{V_\infty^-} y := \lim_{a\to\infty} \left(x \star_{(aV)^-} y \right),$$

and the *asymptotic-plus product* of x and y to be

$$x \star_{V^+_\infty} y := \lim_{a\to\infty} \left(x \star_{(aV)^+} y\right).$$

We show that both of these products are associative, and that in the Chow ring with rational coefficients the following relations hold:

(1a) $x \star_{V^+_\infty} y = 0$ if and only if $B^+_{\mathbf{\Sigma}}$ is nonempty, and

(1b) $x \star_{V^+_\infty} y = x \star_{V^+} y$ otherwise;

(2a) $x \star_{V^-_\infty} y = 0$ if and only if $B^-_{\mathbf{\Sigma}}$ is nonempty, and

(2b) $x \star_{V^-_\infty} y = x \star_{V^-} y$ otherwise;

where $B^+_{\mathbf{\Sigma}}$ and $B^-_{\mathbf{\Sigma}}$ are indexing sets which depend only on x and y, and not on V.

Chapter 2

Chow rings and equivariant Chow rings

2.1 Chow groups and Chow rings

A *k-cycle* on a variety X is a finite formal $\mathbb{Z}$-linear combination of k-dimensional irreducible varieties on X. These k-cycles on X generate an additive group which we denote Z_kX. Elements $\alpha \in Z_kX$ are of the form $\sum_i n_i[V_i]$, where the V_i are irreducible k-dimensional subvarieties of X, and $n_i \in \mathbb{Z}$.

Let W be a $(k+1)$-dimensional subvariety of X. Then the divisor of a rational function $f \in K(W)^*$ is denoted $[\mathrm{div}_W f]$, and is a k-cycle on W.

We define R_kX to be the subgroup of Z_kX generated by divisors of the form $[\mathrm{div}_W f]$, where W is a $(k+1)$-dimensional subvariety of X and f is a rational function on W. We say k-cycles α and β are *rationally equivalent* if $\alpha - \beta \in R_kX$.

We define A^iX to be the group of $(\dim(X) - i)$-cycles modulo the rational equivalence described above. When X is normal of dimension n, we have $A^1X = \mathrm{Cl}(X)$, the divisor class group of X.

Definition 2.1. The *Chow group* of X is the group

$$A^*X = \bigoplus_{i=0}^{\dim X} A^iX.$$

We only define the sum for $0 \leqslant i \leqslant \dim X$ because these are of course the only i for which there could possibly exist $(\dim X - i)$-dimensional subvarieties.

Of course, X is a subvariety of itself; we denote this by writing $[X] \in A^0X$. We call $[X]$ the *fundamental class* of X.

If $f : X \to Y$ is a proper morphism of varieties, we can define a homomorphism of cycles.

Definition 2.2 ([17])**.** Let V be a subvariety of a variety X, and let $f : X \to Y$ be a proper morphism. Then $W = f(V)$ is a closed subvariety of Y, and thus we have an induced imbedding $K(W)$ to $K(V)$ of function fields. We then define the *push-forward* of $[V]$ to be

$$f_*[V] = \begin{cases} [K(V) : K(W)][W] & \text{if } \dim V = \dim W \\ 0 & \text{otherwise} \end{cases}$$

The push-forward is a homomorphism, and is functorial in that $(gf)_* = g_*f_*$ for proper morphisms f and g.

We can also map cycles in the reverse direction of a morphism. For any flat morphism $f : X \to Y$ of flat dimension n with subvariety $V \subseteq Y$, we can define a *pull-back* of cycles $f^*: A^*Y \to A^*X$:

$$f^*[V] = [f^{-1}(V)].$$

The pullback is also defined when f is a regular embedding and, more generally, an l.c.i morphism.

The pull-back map, like the push-forward, is a homomorphism and functorial, with $(gf)^* = f^*g^*$ for any flat morphisms f and g.

For any open subset U of X, the sequence

$$A^*(X \setminus U) \to A^*X \to A^*U \to 0 \tag{2.1}$$

is exact.

If X is a smooth and separated, then the diagonal $\Delta : X \to X \times X$ is a regular embedding, enabling us to make the following definition.

Definition 2.3. If X is a smooth and separated variety, and if $[Z_1] \in A^{i_1}X$ and $[Z_2] \in A^{i_2}X$, we define the *intersection product* of $[Z_1]$ and $[Z_2]$ in $A^{i_1+i_2}X$ to be

$$[Z_1] \cdot [Z_2] = \Delta^*([Z_1 \times Z_2]).$$

Definition 2.4. The intersection product of Definition 2.3 makes A^*X into a graded ring which we call the *Chow ring* of X.

2.2 Chern classes

For every vector bundle V on a variety X, there are associated Chern classes $c_i(V)$ for $0 \leqslant i \leqslant \operatorname{rank} V$, as expounded in [17]. Chern classes are operations on Chow groups; the class $c_i(V)$ defines a homomorphism

$$A^k X \to A^{k+i} X$$

by the assignment $\alpha \mapsto c_i(V)\alpha$. By convention, we take c_0 to be the identity map.

Chern classes are compatible with pullbacks. That is, $c_i(f^*V)f^*\alpha = f^*(c_i(V)\alpha)$.

Given a vector bundle V on X, we can view its Chern class $c_i(V)$ as an element of the *operational Chow ring of* X, a ring graded by the codimension i which is isomorphic to the Chow ring of X, when X is smooth. As such, in the smooth case we will sometimes abuse terminology and say that $c_i(V)$ is an element of the Chow ring A^*X.

Proposition 2.5 ([17]). *If E is a vector bundle of rank r on a scheme X, with projection $\pi : E \to X$, then*

$$\pi^* : A^i X \to A^i E$$

is an isomorphism for any integer k.

2.3 Equivariant Chow groups

Equivariant Chow groups are defined in [12]. We will review the basics in this section.

Let X be an n-dimensional variety with the action of an algebraic group G. Then there exists an l-dimensional representation V of G which has an open set U on which

G acts freely, and furthermore the complement of V has codimension more than i. It is guaranteed in [12] that such a V exists.

The diagonal action on $X \times U$ is also free. As a result, we can define X_G to be the quotient

$$X \times U \to (X \times U)/G,$$

and we define the *i-th equivariant Chow group* to be $A^i_G(X) = A^i(X_G)$. Despite how it appears, $A^i_G(X)$ is not dependent on the choice of V or even upon the dimension of V, provided that $V \setminus U$ has codimension more than i (as we already required).

An equivariant Chow cohomology class $c \in A^i_G(X)$ is an assignment of an operation $c(t) : A^G_*(T) \to A^G_{*-i}(T)$ for every equivariant map $T \to X$ such that the $c(t)$ satisfy some reasonable compatibility requirements in equivariant intersection theory. (Here, as before, $A^G_i X \cong A^G_{\dim X - i} X$ when X is smooth.) We can then construct the graded ring

$$A^*_G(X) := \bigoplus_{i=0}^{\infty} A^i_G(X)$$

by composition. Note that if G acts freely, then $A^*_G(X) = A^*(X_G)$.

Equivariant Chow groups have the same functoriality as ordinary Chow groups for equivariant morphisms with any of the following properties: proper, flat, smooth, regular embedding or l.c.i. In particular, this means we may use push-forward and pull-back maps for equivariant Chow groups just as we do with ordinary Chow groups.

2.4 Chow rings of Deligne-Mumford stacks

A key property of equivariant Chow groups is that they are an invariant of a quotient stack. In the realm of Deligne-Mumford stacks, we have the following from [12].

Definition 2.6. If $\mathscr{X} = [X/G]$ is a Deligne-Mumford stack, then we define $A^*(\mathscr{X}) = A^*_G(X)$.

It is not the case that $A^i(\mathscr{X})$ is zero for arbitrarily large i, but we do have the following useful lemma from [11].

Lemma 2.7. *Let $\mathscr{X}$ be a Deligne-Mumford stack. Then $A^i(\mathscr{X}) \otimes \mathbb{Q} = 0$ for $i > \dim \mathscr{X}$.*

Chapter 3

Inertial products

In this chapter, we will introduce all the language we'll need to discuss inertia and inertial products. In this paper we are mainly concerned with Chow rings, but the general theory of inertial products is independent of Chow ring theory and we present it as such in this section.

In this section, $G \subseteq (\mathbb{C}^\times)^n$ will be a group acting on a variety X. Later, we'll narrow our focus to the case where X is a toric stack.

Definition 3.1. The *inertia variety* for X under the action of G is defined as

$$I_G X = \{(g, x) \mid gx = x\} \subset G \times X,$$

and the *l-th higher inertia variety* is the l-tuple fiber product over X

$$I_G^l X = \{(g_1, \dots, g_l, x) \mid g_i x = x \text{ for } i = 1, \dots, l\} \subset G^l \times X.$$

Definition 3.2. If $\mathscr{X}$ is defined to be the quotient stack $[X/G]$, then the *inertia* *stack* of $\mathscr{X}$ is

$$I\mathscr{X} = [I_G X/G],$$

and the *l-th higher inertia stack* is

$$\mathbb{I}^l\mathscr{X} = [\mathbb{I}^l_G X/G].$$

Definition 3.3. Let X be a variety with the action of a group G. For any l-tuple $\mathbf{g} = (g_1, \ldots, g_l) \in G^l$, define the twisted sector

$$X^{\mathbf{g}} = \{(g_1, \ldots, g_l, x) \in \mathbb{I}^l_G X\}.$$

Note that in the case $\mathbf{g} = \mathrm{id} \in G$, we do not consider the sector $X^{\mathbf{g}} = X$ to be "twisted".

Remark 3.4. For convenience purposes, we will occasionally identify $\mathbb{I}^l_G X$ with the open and closed subset

$$\{(g_1, \ldots, g_{l+1}, x) \,|\, g_1 g_2 \cdots g_{l+1} = 1\}$$

of $\mathbb{I}^{l+1}_G X$.

Definition 3.5. Take any $(g_1, g_2, g_3, x) \in \mathbb{I}^2_G X$ where, as in Remark 3.4, $g_1 g_2 g_3 = 1$. Then for $i = 1, 2, 3$, we define $e_i : \mathbb{I}^2_G X \to I_G X$ be the evaluation morphism $(g_1, g_2, g_3, x) \mapsto (g_i, x)$. Additionally, we define $\mu : \mathbb{I}^2_G X \to I_G X$ be the morphism $(g_1, g_2, g_3, x) \mapsto (g_1 g_2, x)$.

Let E be a rank-n vector bundle on X which is G-equivariant. Let g be an element of a group $G \subseteq (\mathbb{C}^\times)^n$ which acts trivially on X and whose action is an automorphism of the fibers of $E \to X$. Under these conditions, the eigenbundles for the action of g will be G-subbundles of E.

We'll denote the eigenvalues for the action of g on E by $\exp(2\pi\sqrt{-1}\lambda_k)$ for $1 \leqslant k \leqslant r$, and without loss of generality we can say $0 \leqslant \lambda_k < 1$ for each k. We will use E_k to denote the eigenbundle corresponding to λ_k.

Remark 3.6. It's important to note here that the group G acts on X^g for any $g \in G$. In general, the largest subgroup of G that can act on X^g is the centralizer of g. Of course, since $G \subseteq (\mathbb{C}^\times)^n$, G is its own centralizer. This simplifies Definitions 3.7 and 3.9 as compared to their more general counterparts in [15].

Definition 3.7. Let E be a rank-n vector bundle on X with the action of an algebraic group G, and let g be an element of G which acts trivially on X and whose action is an automorphism on the fibers of E. Then the *logarithmic trace* of E for the action of g is

$$L(g)(E) = \sum_{k=1}^{r} \lambda_k E_k \in K_G(X) \otimes \mathbb{R}$$

on each connected component of X.

Example 3.8. Let $Z = \mathbb{C}^4 - \{(0,0,0,0)\}$, and let $G = \mathbb{C}^\times$ act on Z, via

$$g \cdot (z_1, z_2, z_3, z_4) = \left(g^2 z_1, g^4 z_2, g^5 z_3, g^6 z_4\right).$$

Let V be a vector bundle on Z with subbundles L_i corresponding to the divisors where $z_i = 0$. That is, V is the tangent bundle $\mathbb{T} = L_1 + L_2 + L_3 + L_4$.

First, note that $I_G Z = \coprod Z^g$, where the union is over the twelve elements $g \in G = \mathbb{C}^\times$ which fix at least one point of Z. In order to fix a point of Z, g must be either a second, fourth, fifth or sixth root of unity.

Let's look at $g = e^{2\pi\sqrt{-1}/3} \in G$, and compute its logarithmic trace. Note that it only makes sense to consider the action of g on V if we restrict to $V|_{X^g}$.

Under the induced action of α on Z, we have

$$g \cdot (z_1, z_2, z_3, z_4) = \left(e^{4\pi\sqrt{-1}/3} z_1,\ e^{2\pi\sqrt{-1}/3} z_2,\ e^{4\pi\sqrt{-1}/3} z_3,\ z_4\right).$$

So g has three eigenvalues and eigenbundles for its action on $V|_{X^g}$:

$$\begin{aligned} \lambda_1 &= 1 && \text{for} \quad E_1 = L_4 \\ \lambda_2 &= e^{4\pi\sqrt{-1}/3} && \text{for} \quad E_2 = L_1 \text{ or } E_2 = L_3 \\ \lambda_3 &= e^{2\pi\sqrt{-1}/3} && \text{for} \quad E_3 = L_2 \end{aligned}$$

Thus, the logarithmic trace of g on $V|_{X^g}$ is

$$L(g)(V|_{X^g}) = \frac{2}{3}L_1 + \frac{1}{3}L_2 + \frac{2}{3}L_3.$$

Notice that the coefficient on L_4 in the logarithmic trace is 0; this is a direct consequence of the fact that $Z^g = \{(0,0,0,z_4) \mid z_4 \neq 0\}$.

Definition 3.9 ([14]). Let $\mathbf{g}$ be an l-tuple $(g_1, \dots g_l)$ such that there exists a finite subgroup of G containing every g_i for $1 \leqslant i \leqslant l$, and with the additional stipulation that $g_1 \cdots g_l = 1$. Then the *logarithmic restriction* of E is the class in $K_G(X^{\mathbf{g}})$ defined by the formula

$$E(\mathbf{g}) = \sum_{i=1}^{l} L(g_i)(E|_{X^{\mathbf{g}}}) + E^{\mathbf{g}} - E|_{X^{\mathbf{g}}}. \tag{3.1}$$

The assignment $LR : E \mapsto E(\mathbf{g})$ is called the *logarithmic restriction* map, and takes non-negative elements $E \in K_G(X)$ to non-negative elements $LR(E) \in K_G(X^{\mathbf{g}})$.

Example 3.10. We'll build on Example 3.8, with the same Z, G, α and V; additionally, let $\mathbf{g} = \left(e^{\pi\sqrt{-1}/3}, e^{2\pi\sqrt{-1}/3}, e^{\pi\sqrt{-1}}\right)$. Then we have $Z^{\mathbf{g}} = \{(0,0,0,z_4) \,|\, z_4 \neq 0\} \subseteq Z$, which in turn gives $V^{\mathbf{g}} = L_4$.

Then the logarithmic restriction of V at $\mathbf{g}$ is

$$\begin{aligned}
&V(\mathbf{g}) \\
&= L\left(e^{\pi\sqrt{-1}/3}\right)(V|_{Z^{\mathbf{g}}}) + L\left(e^{2\pi\sqrt{-1}/3}\right)(V|_{Z^{\mathbf{g}}}) + L\left(e^{\pi\sqrt{-1}}\right)(V|_{Z^{\mathbf{g}}}) + V^{\mathbf{g}} - V|_{Z^{\mathbf{g}}} \\
&= \left(\frac{1}{3}L_1 + \frac{2}{3}L_2 + \frac{5}{6}L_3\right) + \left(\frac{2}{3}L_1 + \frac{1}{3}L_2 + \frac{2}{3}L_3\right) + \frac{1}{2}L_3 \\
&+ L_4 - (L_1 + L_2 + L_3 + L_4) \\
&= L_3.
\end{aligned}$$

Proposition 3.11 ([15])**.** *Let E be a G-equivariant bundle on X, and let g_1, g_2 lie in a common compact subgroup of G. Then the virtual bundles*

$$E^+(g_1, g_2) = L(g_1)(E|_{X^{g_1,g_2}}) + L(g_2)(E|_{X^{g_1,g_2}}) - L(g_1g_2)(E|_{X^{g_1,g_2}}) \tag{3.2}$$

and

$$E^-(g_1, g_2) = L(g_1^{-1})(E|_{X^{g_1,g_2}}) + L(g_2^{-1})(E|_{X^{g_1,g_2}}) - L(g_1^{-1}g_2^{-1})(E|_{X^{g_1,g_2}}) \tag{3.3}$$

are represented by non-negative integral elements in $K_G(X^{g_1,g_2})$.

Definition 3.12. Let $\mathbf{g} = (g_1, g_2) \in G^2$. Then the classes R^+E and R^-E in $K_G(\mathbb{I}^2X)$ are defined by setting the component of R^+E (resp. R^-E) in $K_G(\mathbb{I}^2(X^{\mathbf{g}}))$ to be $E^+(X^{\mathbf{g}})$ (resp. $E^-(X^{\mathbf{g}})$).

Definition 3.13. Given a class $c \in A^*_G(\mathbb{I}^2 X)$, we define the *inertial product with respect to* c to be

$$x \star_c y := \mu_*(e_1^* x \cdot e_2^* y \cdot c),$$

where $x, y \in A^*_G(I_G X)$.

We say that an *associative vector bundle* on $\mathbb{I}^2 X$ is a vector bundle $\mathscr{R}$ such that the inertial product with respect to $c = \mathrm{eu}(\mathscr{R})$ is associate. We give two examples of associative vector bundles in the following theorem.

Theorem 3.14 ([15])**.** *Let $\mathscr{R}^+ E$ be the vector bundle $LR(\mathbb{T}) + R^+ E$ for any G-equivariant vector bundle E on X. Then the inertial product with respect to* $\mathrm{eu}(\mathscr{R}^+ E)$*, which we call $\star_{E^+}$, is associative. Similarly,* $\mathrm{eu}(\mathscr{R}^- E) = \mathrm{eu}(LR(\mathbb{T}) + R^- E)$*, called $\star_{E^-}$, also defines an associative inertial product.*

Remark 3.15. Edidin, Jarvis and Kimura show in [15] that the inertial products in Theorem 3.14 only depend on the K-theory class of the G-equivariant vector bundle E. Thus, it is useful to know the form of an arbitrary G-equivariant vector bundles on Z.

Let $\mathcal{L}_1, \ldots, \mathcal{L}_n$ be the line bundles associated to the divisors where $z_i = 0$. Then any G-equivariant vector bundle V has K-theory form

$$V = \sum_{i=1}^{n} f_i(\mathcal{L}_i), \tag{3.4}$$

where $f_i(\mathcal{L}_i)$ denotes a polynomial in $\mathcal{L}_i$ and $\mathcal{L}_i^{-1}$ with positive coefficients. For example, the rank 32 vector bundle $2{\mathcal{L}_1}^5 + 9\mathcal{L}_3 + 3{\mathcal{L}_2}^{-1} + 18{\mathcal{L}_1}^{-3}$ is G-equivariant.

Remark 3.16. The *orbifold product* $\star_{orb}$ of [3] can be alternatively defined by taking $E = 0$ in Definition 3.13 and Theorem 3.14 (with either $\mathscr{R}^+$ or $\mathscr{R}^-$). The *virtual* *product* $\star_{virt}$ of [21] can be alternatively defined by taking $E = \mathbb{T}$ in Definition 3.13 and Theorem 3.14 with $\mathscr{R}^{-}$.

It is our goal to examine what kind of ring structure an inertial product will give us in the Chow ring. In particular, we look to do so for an arbitrary equivariant vector bundle E. We'll show that some very nice presentations can be found in the toric case.

Chapter 4

Toric Deligne-Mumford stacks

We recap in this section the theory and construction of a toric stack from a stacky fan; in particular, we rely heavily on the results of [3], [9], [19] and [23], though we make a few notational adjustments.

4.1 Stacky fans and toric stacks

We'll set the stage for Chow ring calculations on toric stacks by introducing the full arsenal of terms and notations necessary to discuss stacky fans. For constructions involving cones and fans and general toric geometry, we use the language and notation of [3], with some influence from [19].

4.1.1 Definitions and setup

A stacky fan $\mathbf{\Sigma}$ is a triple (N, Σ, β), where N is a finitely generated abelian group of rank d, Σ is a simplicial fan[1] with n rays in the d-dimensional $\mathbb{Q}$-vector space

[1] Note the subtle notational distinction between Σ (fan) and $\mathbf{\Sigma}$ (stacky fan).

$N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$, and β is a homomorphism from $\mathbb{Z}^n$ to N which has finite cokernel.

We further require that there is a basis $\{e_1, \ldots, e_n\}$ of $\mathbb{Z}^n$, the image $b_i := \beta(e_i) \in N$, when tensored with $\mathbb{Q}$, is on the ray ρ_i for all $i = 1, \ldots, n$. We call $b_1, \ldots b_n$ the *distinguished points* of $\mathbf{\Sigma}$ in N.

Let X_Σ be the toric variety associated to the fan Σ. Let $T_N := \mathrm{Hom}(N^*, \mathbb{C}^\times)$ and $T_L := \mathrm{Hom}((\mathbb{Z}^n)^*, \mathbb{C}^\times) \cong (\mathbb{C}^\times)^n$. Then the natural map $\beta^* : N^* \to (\mathbb{Z}^n)^*$ induces a morphism of tori $T_\beta : T_L \to T_N$.

Let $N = \mathbb{Z}^d \oplus \mathbb{Z}/m_1 \oplus \cdots \oplus \mathbb{Z}/m_r$ be the invariant factor form of N. We define $\beta_{aug} : \mathbb{Z}^n \oplus \mathbb{Z}^r \to \mathbb{Z}^{d+r}$ to be the map represented by the matrix

$$\begin{bmatrix} b_{1,1} & & b_{n,1} & 0 & 0 & 0 \\ & \ddots & & 0 & \ddots & 0 \\ b_{1,d} & & b_{n,d} & 0 & 0 & 0 \\ b_{1,d+1} & & b_{n,d+1} & m_1 & 0 & 0 \\ & \ddots & & 0 & \ddots & 0 \\ b_{1,d+r} & & b_{n,d+r} & 0 & 0 & m_r \end{bmatrix}$$

where the i-th column of β_{aug} is a lift of b_i under the natural surjection $\mathbb{Z}^d \oplus \mathbb{Z}^r \to N$.

For the purpose of convenience, we will represent the integers $b_{i,d+j}$ (which are elements of $\mathbb{Z}/m_l$) using elements of the set $\{0, 1, \ldots, m_l - 1\}$ for $l = 1, \ldots, r$.

We define the map $E^\beta : (\mathbb{C}^\times)^{n+r} \to (\mathbb{C}^\times)^{d+r}$ by exponentiating β_{aug}. That is,

$$E^\beta(\gamma_1, \ldots, \gamma_n, s_1, \ldots, s_r) = \left(\prod_{i=1}^n {\gamma_i}^{b_{i,1}}, \ldots, \prod_{i=1}^n {\gamma_i}^{b_{i,d}},\ {s_1}^{m_1} \prod_{i=1}^n {\gamma_i}^{b_{i,d+1}}, \ldots, {s_r}^{m_r} \prod_{i=1}^n {\gamma_i}^{b_{i,d+r}} \right).$$

We define the group G to be the kernel of the map E^β, as a subgroup of $(\mathbb{C}^\times)^{n+r}$:

$$G = \left\{ (\gamma_1, \ldots, \gamma_n, s_1, \ldots, s_r) \,\middle|\, \begin{array}{c} \displaystyle\prod_{i=1}^n {\gamma_i}^{b_{i,j}} = 1 \text{ for } 1 \leqslant j \leqslant d, \text{ and} \\ \displaystyle {s_l}^{m_l} \cdot \prod_{i=1}^n {\gamma_i}^{b_{i,d+l}} = 1 \text{ for } 1 \leqslant l \leqslant r \end{array} \right\}. \tag{4.1}$$

Remark 4.1. If N is torsion-free, we can simplify this explicit construction. Since $r = 0$, we get

$$G = \left\{ (\gamma_1, \ldots, \gamma_n) \,\middle|\, \prod_{i=1}^n {\gamma_i}^{b_{i,j}} = 1 \text{ for } 1 \leqslant j \leqslant d \right\} \tag{4.2}$$

Let $\mathbb{C}[z_\rho \,|\, \rho \in \Sigma(1)] = \mathbb{C}[z_1, \ldots, z_n]$ be the total coordinate ring of X_Σ, and let Z be the quasi-affine subvariety of $\mathbb{A}^n$ cut out by the irrelevant ideal $J_\Sigma := \left\langle \prod_{\rho_i \not\subset \sigma} z_i \middle| \sigma \in \Sigma \right\rangle$.

We equip the open subset Z with the action of $G \subseteq (\mathbb{C}^\times)^{n+r}$, defined by

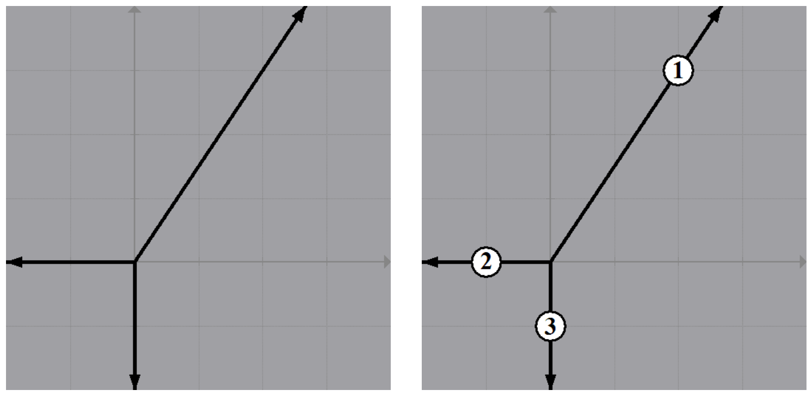
$$(\gamma_1, \ldots, \gamma_n, s_1, \ldots, s_n) \cdot (z_1, \ldots, z_n) = (\gamma_1 z_1, \ldots, \gamma_n z_n).$$

Note that since $V(J_\Sigma)$ is a union of coordinate subspaces, $Z = \mathbb{A}^n \setminus V(J_\Sigma)$ is G-invariant.

The *toric stack associated to* $\mathbf{\Sigma}$ is defined to be the stack quotient $\mathcal{X}(\mathbf{\Sigma}) := [X_\Sigma/G] \cong [Z/G]$ under this action. Further, under our setup $\mathcal{X}(\mathbf{\Sigma})$ will be a Deligne-Mumford stack.

We'll start with a more basic example for the reader who is unfamiliar with stacky fans.

Example 4.2. Let Σ be the fan pictured in Figure 4.1(a), with three rays and three two-dimensional cones (each generated by a pair of rays). Let $\mathbf{\Sigma}$ be the stacky fan



(a) The fan Σ (b) The stacky fan $\mathbf{\Sigma}$.

Figure 4.1: An illustration of the difference between the fan and the stacky fan in Example 4.2. The non-stacky fan does not have the distinguished points b_1, b_2 and b_3 (marked with circles on the stacky fan).

pictured in Figure 4.1(b), which can be defined by $\mathbf{\Sigma} = (N, \Sigma, \beta)$ with $N = \mathbb{Z}^2$ and

$$\beta = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 0 & -1 \end{bmatrix} : \mathbb{Z}^3 \to \mathbb{Z}^2.$$

Notice that the distinguished points, $b_1 = (2,3)$, $b_2 = (-1,0)$ and $b_3 = (0,-1)$, are the columns of β (see Remark 4.3) and are on the rays ρ_i.

The morphism of tori $T_\beta : (\mathbb{C}^\times)^3 \to (\mathbb{C}^\times)^2$ is given by

$$(\gamma_1, \gamma_2, \gamma_3) \mapsto ({\gamma_1}^2{\gamma_2}^{-1}, {\gamma_1}^3{\gamma_3}^{-1}).$$

Since N has no torsion, by (4.2) we have

$$G = \ker(E^\beta) = \{(\gamma_1, \gamma_2, \gamma_3) \in (\mathbb{C}^\times)^3 \mid {\gamma_1}^2{\gamma_2}^{-1} = 1, {\gamma_1}^3{\gamma_3}^{-1} = 1\}.$$

The action of G on X is induced by the diagonal action of $(\mathbb{C}^\times)^3$ on $\mathbb{C}^3$. However, because $G \cong \{(\lambda, \lambda^2, \lambda^3) \mid \lambda \in \mathbb{C}^\times\} \cong \mathbb{C}^\times$, we could equivalently describe this action of G on Z as the action of $\mathbb{C}^\times$ defined by

$$\lambda \cdot (z_1, z_2, z_3) = (\lambda z_1, \lambda^2 z_2, \lambda^3 z_3). \tag{4.3}$$

We also have $J_\Sigma = \langle z_1 z_2 z_3 \rangle \subset \mathbb{C}[z_1, z_2, z_3]$, so $Z = \mathbb{C}^3 - \{(0,0,0)\}$. So $\mathcal{X}(\mathbf{\Sigma}) = [Z/G]$, with the G action as described in (4.3). This stack $\mathcal{X}(\mathbf{\Sigma})$ is, of course, the weighted projective stack $\mathbb{P}(1,2,3)$.

Remark 4.3. In Example 4.2, instead of giving a matrix representation for β we could have just listed the distinguished points. This is more common in practice, and we'll use this approach in the next example.

Example 4.4. Let $N = \mathbb{Z}^3$ and Σ be the fan with rays generated over $N_\mathbb{Q} \cong \mathbb{Q}^3$ by the distinguished points $b_1 = (3, -2, 5)$, $b_2 = (0, 1, 0)$, $b_3 = (0, 0, 2)$ and $b_4 = (-1, 0, 0)$. Then the associated toric stack is $\mathcal{X}(\mathbf{\Sigma}) = \mathbb{P}(2, 4, 5, 6)$, which we saw in Examples 3.8 and 3.10.

Of course, not all toric stacks are weighted projective stacks, as we'll see in the next example.

Example 4.5. Let $N = \mathbb{Z}^2$ and Σ be the fan with rays generated in $N_\mathbb{Q} \cong \mathbb{Q}^2$ by the distinguished points $(1, 0)$ and $(1, 3)$, pictured in Figure 4.2. Since there is just one two-dimensional cone, we have $J_\Sigma = \emptyset$ and thus $Z = \mathbb{C}^2$. The map β is given by the matrix representation

$$\begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} : \mathbb{Z}^2 \to \mathbb{Z}^2,$$

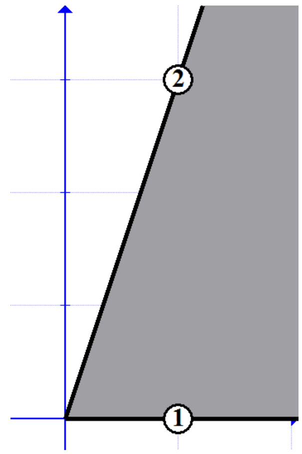


Figure 4.2: The stacky fan for $[\mathbb{C}^2/\mu_3]$, constructed in Example 4.5.

and $E^\beta : (\mathbb{C}^\times)^2 \to (\mathbb{C}^\times)^2$ is given by $(\gamma_1, \gamma_2) \mapsto (\gamma_1\gamma_2, {\gamma_2}^3)$. So by (4.2),

$$G = \{(\gamma_1, \gamma_2) \in (\mathbb{C}^\times)^2 \,|\, \gamma_1\gamma_2 = 1,\ \gamma_2^3 = 1\} \cong \{(\lambda^2, \lambda) \,|\, \lambda \in \mu_3\} \subseteq (\mathbb{C}^\times)^2,$$

where μ_p denotes the multiplicative group of p-th roots of unity. So we have $G \cong \mu_3$, and hence the toric stack $\mathcal{X}(\mathbf{\Sigma})$ is the quotient stack $[\mathbb{C}^2/\mu_3]$, with the action given by

$$\lambda \cdot (x_1, x_2) = (\lambda^2 x_1, \lambda x_2).$$

The reader who is familiar with toric varieties but not toric stacks might (correctly) point out that the "stacky" information of the distinguished points was not necessary to find the associated varieties of the last two examples. This is true, but only because the distinguished points happened to be minimal generators for the rays over N. While the distinguished points must be ray generators, they need not be minimal generators. Furthermore, the introduction of torsion to N renders the toric variety methods ineffective.

In the next example (seen in both [3] and [19]), we'll display the necessity of that extra data. We will use the example repeatedly in this paper as a concrete way to illustrate our abstract statements, as it is one of the simplest examples which illustrate "stackiness". It will also be our first example computing G using its full definition in (4.1).

Example 4.6. Let $N = \mathbb{Z} \oplus \mathbb{Z}/2$ and let Σ be the complete fan in $N_{\mathbb{Q}} \cong \mathbb{Q}$. Define β by the distinguished points $b_1 = (2,1)$ and $b_2 = (-3,0)$ in N. This defines a stacky fan $\mathbf{\Sigma} = (N, \Sigma, \beta)$. Notice that b_1 and b_2 are *not* minimal generators of ρ_1 and ρ_2 over $N_{\mathbb{Q}}$.

As N is not a lattice, we must compute G using (4.1). We have the homomorphism β_{aug}:

$$\begin{bmatrix} 2 & -3 & 0 \\ 1 & 0 & 2 \end{bmatrix} : \mathbb{Z}^3 \to \mathbb{Z}^2$$

Hence,

$$G = \ker(E^{\beta}) = \left\{ (\gamma_1, \gamma_2, s_1) \in (\mathbb{C}^{\times})^3 \,|\, {\gamma_1}^2 {\gamma_2}^{-3} = 1, \gamma_1 {s_1}^2 = 1 \right\}.$$

Computing J_{Σ} the usual way, we have $Z = \mathbb{C}^2 \setminus \{(0,0)\}$, so the G-action on Z is

$$(\gamma_1, \gamma_2, s_1) \cdot (z_1, z_2) = (\gamma_1 z_1, \gamma_2 z_2).$$

But since $G \cong \{ (\lambda^6, \lambda^4, \lambda^{-3}) \mid \lambda \in \mathbb{C}^{\times} \} \cong \mathbb{C}^{\times}$, the G-action is equivalent to the $\mathbb{C}^{\times}$-action

$$\lambda \cdot (z_1, z_2) = (\lambda^6 z_1, \lambda^4 z_2).$$

That is, $\mathcal{X}(\mathbf{\Sigma})$ is the weighted projective stack $\mathbb{P}(6,4)$, which of course is not a toric variety.

Remark 4.7. In Example 4.6, we made a specific choice of distinguished points $b_1 = (2,1)$ and $b_2 = (-3,0)$, which from might seem to contradict the fact that β_{aug} is independent of the choice of lift. If we had chosen any other equivalent lift under the surjection $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}/2$, we would have obtained the same stack. As an extreme example, we can pick $b_1 = (2,87)$ and $b_2 = (-3,54)$ which gives us

$$G \cong \left\{ (\lambda^6, \lambda^4, \lambda^{-369}) | \lambda \in \mathbb{C}^\times \right\},$$

which yields the same stack $\mathcal{X}(\mathbf{\Sigma}) = \mathbb{P}(6,4)$.

4.2 The box of a stacky fan

We must introduce $\mathrm{Box}(\mathbf{\Sigma})$, which plays a direct hand in our quest to find the simplest possible presentation for our Chow rings. In particular, the box elements will be the elements of minimal degree in the graded ring.

Define $\overline{N} := N/N_{tor}$. For any $v \in N$, we use $\overline{v}$ to denote the image of v in the natural projection $N \to \overline{N}$. In a slight abuse of notation, we will sometimes refer to $\overline{v}$ as being in Σ.

We use $\sigma(\overline{v})$ to denote the unique minimal cone $\sigma \in \Sigma$ which contains $\overline{v}$. More generally, $\sigma(\overline{v_1}, ..., \overline{v_k})$ is the minimal cone in Σ containing each of $\overline{v_1}, \dots, \overline{v_k}$.

Definition 4.8. For any cone $\sigma \in \Sigma$, we define $\mathrm{Box}(\boldsymbol{\sigma})$ to be the set of all elements $v \in N$ such that $\overline{v}$ is a linear combination of those $\overline{b_i}$ which are in the cone σ, using only rational coefficients in the range $[0,1)$. We define $\mathrm{Box}(\mathbf{\Sigma})$ to be the union of $\mathrm{Box}(\boldsymbol{\sigma})$ for all cones $\sigma \in \Sigma$.

Remark 4.9. We can make an equivalent formulation of Definition 4.8: For each

$v \in \mathrm{Box}(\boldsymbol{\sigma})$, there exist unique rational numbers $q_i \in [0,1)$ such that

$$\overline{v} = \sum q_i \overline{b_i}, \tag{4.4}$$

where the sum is over all i such that $\rho_i \in \sigma(\overline{v})$.

For any cone $\sigma \in \Sigma$, define

$$N_\sigma := \langle b_i \mid \rho_i \in \sigma \rangle \tag{4.5}$$

as a subgroup of N, and define $N(\sigma)$ be the finite quotient group N/N_σ. The set $\mathrm{Box}(\boldsymbol{\sigma})$ has a natural one-to-one correspondence with $N(\sigma)$, but the advantage to working with $\mathrm{Box}(\boldsymbol{\sigma})$ is that we are making a specific choice of coset representatives of the quotient group, choices which make the calculations (and eventually the Chow ring representations) simpler and more explicit.

Remark 4.10. When N has torsion, there can exist pairs $v_1, v_2 \in \mathrm{Box}(\boldsymbol{\sigma})$ for which $\overline{v_1} = \overline{v_2}$ but $v_1 \neq v_2$; this happens when v_1 and v_2 differ only by torsion. For a concrete example of this, see Example 4.12.

Example 4.11. Let $\mathcal{X}(\boldsymbol{\Sigma})$ be the toric stack $\mathbb{P}(6,4)$, constructed in Example 4.6. The squares in Figure 4.3 represent the eight elements of $\mathrm{Box}(\boldsymbol{\Sigma})$. For example, since $\overline{N} = \mathbb{Z}$, we have that $v = (1,1) \in N$ is in $\mathrm{Box}(\boldsymbol{\Sigma})$ because $\overline{v} = \frac{1}{2}\overline{b_1} \in \mathrm{Box}(\sigma_1)$, where σ_1 is the one-dimensional cone in the positive direction on the horizontal axis.

Notice also that $(5,1) \in N$, but $(5,1) \notin \mathrm{Box}(\boldsymbol{\Sigma})$. The box element which is equivalent to $(5,1)$ in $N(\sigma_1)$ is $(1,1)$, since $(5,1) = 2b_1 + (1,1) \in N$.

The set $\mathrm{Box}(\boldsymbol{\sigma})$ inherits the nice algebraic structure of $N(\sigma)$, but the minutia

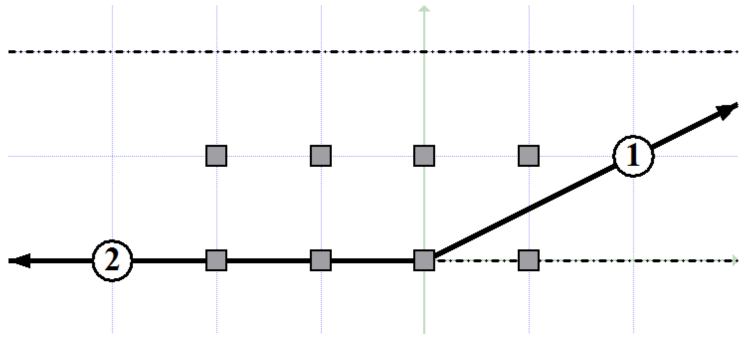


Figure 4.3: The stacky fan of Example 4.6, which yields the toric stack $\mathbb{P}(6,4)$. The vertical axis here is representing $\mathbb{Z}/2$; that is to say, the two dotted lines are identified. The squares are elements of $\mathrm{Box}(\mathbf{\Sigma})$.

surrounding addition are slightly different; one must be careful about remaining in the coset. We give an example to illustrate how simple this concept can be in practice.

Example 4.12. We'll revisit the stacky fan of Example 4.6 . We have $\mathrm{Box}(\sigma_1) = \{0, v_1, v_2, v_3\}$, where $0 = (0,0)$, $v_1 = (1,1)$, $v_2 = (1,0)$ and $v_3 = (0,1)$. Then we have the following addition table for $\mathrm{Box}(\boldsymbol{\sigma_1})$:

$$\begin{array}{c|cccc} + & 0 & v_1 & v_2 & v_3 \\ \hline 0 & 0 & v_1 & v_2 & v_3 \\ v_1 & v_1 & v_3 & 0 & v_2 \\ v_2 & v_2 & 0 & v_3 & v_1 \\ v_3 & v_3 & v_2 & v_1 & 0 \end{array} \tag{4.6}$$

The geometric intuition connecting Example 4.12 and Figure 4.3 is strong, and this is no coincidence. However, an arguably more important non-coincidence here is that the group of $\mathrm{Box}(\boldsymbol{\sigma_1})$ under addition (shown in (4.6)) is isomorphic to the group of the fourth roots of unity under usual multiplication.

The structure seen in this example is evidenced in the following proposition from

[3]; we provide an alternate proof in our notation.

Proposition 4.13. *If Σ is a complete fan, then the elements $v \in \mathrm{Box}(\boldsymbol{\Sigma})$ are in one-to-one correspondence with elements $g \in G$ which fix a point of Z.*

Proof. Let G_σ be the subgroup of G defined by the equations $\gamma_i = 1$ for all $\rho_i \notin \sigma$. That is,

$$G_\sigma = \left\{ (\gamma_1, \ldots, \gamma_n, s_1, \ldots, s_r) \;\middle|\; \begin{array}{c} \displaystyle\prod_{\rho_i \in \sigma} {\gamma_i}^{b_{i,j}} = 1 \text{ for } 1 \leqslant j \leqslant d, \text{ and} \\ \displaystyle {s_l}^{m_l} \cdot \prod_{\rho_i \in \sigma} {\gamma_i}^{b_{i,d+l}} = 1 \text{ for } 1 \leqslant l \leqslant r, \text{ and} \\ \gamma_i = 1 \text{ for all } \rho_i \notin \sigma \end{array} \right\}. \tag{4.7}$$

The set of elements of G which fix a point of Z will be $\bigcup_{\sigma \in \Sigma} G_\sigma$.

We'll prove the statement for $g \in G_\sigma$ and $v \in \mathrm{Box}(\boldsymbol{\sigma})$; taking unions will complete the proof.

For $1 \leqslant j \leqslant d$, recall that $b_{i,j}$ denotes the j-th entry of the distinguished point b_i corresponding to the ray ρ_i. We claim that for $g = (\gamma_1, \ldots, \gamma_n, s_1, \ldots, s_r) \in G_\sigma$, the correspondence is $g \leftrightarrow v$, where v is given by

$$\left(\sum_{i=1}^n b_{i,1} \frac{\mathrm{Log}(\gamma_i)}{2\pi\sqrt{-1}}, \ldots, \sum_{i=1}^n b_{i,d} \frac{\mathrm{Log}(\gamma_i)}{2\pi\sqrt{-1}}, \frac{m_1 \mathrm{Log}(s_1) - \mathrm{Log}({s_1}^{m_1})}{2\pi\sqrt{-1}}, \ldots, \frac{m_1 \mathrm{Log}(s_r) - \mathrm{Log}({s_r}^{m_r})}{2\pi\sqrt{-1}} \right), \tag{4.8}$$

with the convention that for any $\zeta \in \mathbb{C}^\times$ with $|\zeta| = 1$, we use the principal branch of the natural logarithm, $0 \leqslant \dfrac{\mathrm{Log}\,\zeta}{2\pi\sqrt{-1}} < 1$.

We first show that the way we defined v makes it an element of N. First, we have

$$\frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^n b_{i,j} \mathrm{Log}(\gamma_i) \equiv \frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^n \mathrm{Log}\left({\gamma_i}^{b_{i,j}}\right) \pmod 1 \equiv 0 \pmod 1. \tag{4.9}$$

We also have $(m_l \operatorname{Log}(s_l) - \operatorname{Log}({s_l}^{m_l}))/2\pi\sqrt{-1} \in \{0, \dots, m_l - 1\} = \mathbb{Z}/m_l$. So we have $v \in N$.

Next, we show that there exist rational numbers $q_i \in [0,1) \cap \mathbb{Q}$ such that $\overline{v} = \sum_{\rho_i \in \sigma} q_i \overline{b_i}$. Indeed,

$$\overline{v} = \left(\sum_{i=1}^{n} b_{i,1} \frac{\operatorname{Log}(\gamma_i)}{2\pi\sqrt{-1}}, \dots, \sum_{i=1}^{n} b_{i,d} \frac{\operatorname{Log}(\gamma_i)}{2\pi\sqrt{-1}} \right),$$

and if we set $q_i := \frac{\operatorname{Log}(\gamma_i)}{2\pi\sqrt{-1}}$, we have

$$\overline{v} = \left(\sum_{i=1}^{n} b_{i,1} q_i, \dots, \sum_{i=1}^{n} b_{i,d} q_i \right) = \sum_{i=1}^{n} q_i \overline{b_i} = \sum_{\rho_i \in \sigma} q_i \overline{b_i},$$

with the last equality due to the fact that $q_i = 0$ whenever $\rho_i \notin \sigma$. So $v \in \operatorname{Box}(\boldsymbol{\sigma})$.

It remains to show that $g \to v$ is onto. Take any $v \in \operatorname{Box}(\boldsymbol{\sigma})$, by (4.4) we know the first d components. So we have

$$v = \left(\sum_{\rho_i \in \sigma} q_i b_{i,1}, \dots, \sum_{\rho_i \in \sigma} q_i b_{i,d}, p_1, \dots, p_r \right),$$

for some $q_i \in [0,1) \cap \mathbb{Q}$, $1 \leqslant i \leqslant n$ and for some $p_l \in \{0, \dots, m_l - 1\} = \mathbb{Z}/m_l$, $1 \leqslant l \leqslant r$. We will show there is an element of G_σ which is sent to v under the correspondence in (4.8).

Set $\gamma_i := e^{2\pi\sqrt{-1}q_i}$ and $s_l := \left(\prod_{i=1}^{n} e^{-2\pi\sqrt{-1}q_i b_{i,d+l}}\right)^{1/m_l} e^{2\pi\sqrt{-1}p_l/m_l}$. We claim that $(\gamma_1, \dots, \gamma_n, s_1, \dots, s_r)$ is the element of G_σ such that $g \leftrightarrow v$.

Indeed, for $1 \leqslant j \leqslant d$ we have

$$\prod_{\rho_i \in \sigma} {\gamma_i}^{b_{i,j}} = \prod_{\rho_i \in \sigma} \left(e^{2\pi\sqrt{-1}q_i}\right)^{b_{i,j}} = \exp\left(2\pi\sqrt{-1} \sum_{\rho_i \in \sigma} q_i b_{i,j}\right) = 1;$$

for $1 \leqslant l \leqslant r$ we have

$$\begin{aligned}
{s_l}^{m_l} \prod_{\rho_i \in \sigma} {\gamma_i}^{b_{i,d+l}} &= \left(\left(\prod_{i=1}^{n} e^{-2\pi\sqrt{-1} q_i b_{i,d+l}} \right)^{1/m_l} e^{2\pi\sqrt{-1} p_l / m_l} \right)^{m_l} \prod_{\rho_i \in \sigma} {\gamma_i}^{b_{i,d+l}} \\
&= \left(\prod_{i=1}^{n} e^{-2\pi\sqrt{-1} q_i b_{i,d+l}} \right) e^{2\pi\sqrt{-1} p_l} \prod_{\rho_i \in \sigma} e^{2\pi\sqrt{-1} q_i b_{i,d+l}} \\
&= e^{2\pi\sqrt{-1} p_l} \\
&= 1;
\end{aligned}$$

and for $\rho_i \notin \sigma$ we clearly have $q_i = 0$ and thus $\gamma_i = 1$.

∎

Remark 4.14. Along with this statement, we have that every non-zero box element corresponds with a twisted sector in $I_G X$.

Example 4.15. Recall the diagram in Figure 4.3 which illustrates the stacky fan for the toric stack $\mathscr{X} = \mathbb{P}(6,4)$. By Proposition 4.13, we have the following correspondences under (4.8):

$$\begin{array}{lcl}
\underline{\mathrm{Box}(\boldsymbol{\Sigma})} & & \underline{G \subset (\mathbb{C}^\times)^3} \\
v_0 = (0,0) & \leftrightarrow & g_0 = (1,1,1) \\
v_1 = (1,0) & \leftrightarrow & g_1 = (-1,1,e^{\pi\sqrt{-1}/2}) \\
v_2 = (1,1) & \leftrightarrow & g_2 = (-1,1,e^{3\pi\sqrt{-1}/2}) \\
v_3 = (0,1) & \leftrightarrow & g_3 = (1,1,-1) \\
v_4 = (-1,0) & \leftrightarrow & g_4 = (1,e^{2\pi\sqrt{-1}/3},1) \\
v_5 = (-1,1) & \leftrightarrow & g_5 = (1,e^{2\pi\sqrt{-1}/3},-1) \\
v_6 = (-2,0) & \leftrightarrow & g_6 = (1,e^{4\pi\sqrt{-1}/3},1) \\
v_7 = (-2,1) & \leftrightarrow & g_7 = (1,e^{4\pi\sqrt{-1}/3},-1)
\end{array}$$

Note that $\boldsymbol{\Sigma}$ contains two one-dimensional cones and one zero-dimensional cone, and by definition

$$\begin{aligned}
\mathrm{Box}(\boldsymbol{\Sigma}) =& \mathrm{Box}(\boldsymbol{\sigma}_1) \bigcup \mathrm{Box}(\boldsymbol{\sigma}_2) \bigcup \mathrm{Box}(\mathbf{0}) \\
=& \{v_0, v_1, v_2, v_3\} \bigcup \{v_0, v_3, v_4, v_5, v_6, v_7\} \bigcup \{v_0, v_3\}.
\end{aligned}$$

4.3 Line bundles and Chern classes

Let $\mathcal{X}(\boldsymbol{\Sigma}) = [Z/G]$ be a the toric Deligne-Mumford stack associated to the stacky fan $\boldsymbol{\Sigma} = (\Sigma, \beta)$, and let L_i be the line bundle on the underlying toric variety $X(\Sigma)$ corresponding to the ray ρ_i.

A line bundle on $[Z/G]$ is a G-equivariant line bundle on Z. In the next lemma, accumulated from results in Section 3 of [23], we describe the nature of line bundles on $\mathcal{X}(\boldsymbol{\Sigma})$. In short, they are integral linear combinations of line bundles on $X(\Sigma)$.

Lemma 4.16. *We have*

$$\mathrm{Pic}(\mathcal{X}(\boldsymbol{\Sigma})) \cong G^{\vee} \cong A^1(\mathcal{X}(\boldsymbol{\Sigma}), \cdot, \mathbb{Z}).$$

In particular, the Picard group is generated by $L_1, \ldots, L_n$*, where each* L_i *corresponds to the ray* ρ_i*. Specifically, there exist integers* $f_{k,i}$ *for* $1 \leqslant i, k \leqslant n$ *such that*

$$\mathcal{L}_i = \bigotimes_k L_k{}^{\otimes f_{k,i}} \tag{4.10}$$

is the line bundle of $\mathcal{X}(\boldsymbol{\Sigma})$ *corresponding to the action of the* i*-th component of* G *on* Z*.*

Moreover, the class $\widetilde{y}^{b_i}$*, defined as*

$$\widetilde{y}^{b_i} = \sum_{k=1}^{n} f_{k,i} y^{b_k},$$

is the first Chern class of $\mathcal{L}_i$*, and if* $\{t_1, \ldots, t_{n-d}\}$ *are the generators of* $\overline{N}^{\vee}$*, we can write* $\widetilde{y}^{b_i}$ *as an integral linear combination of these generators:*

$$\widetilde{y}^{b_i} = \sum_{j=1}^{n-d} \alpha_{j,i} t_j, \tag{4.11}$$

where the integers $\alpha_{j,i}$ *for* $1 \leqslant i \leqslant n$ *and* $1 \leqslant j \leqslant n-r$ *are determined by the integers* $f_{k,i}$ *for* $1 \leqslant i, k \leqslant n$*.*

This relationship between $\tilde{y}^{b_1}, \ldots \tilde{y}^{b_n}$ and $y^{b_1}, \ldots, y^{b_n}$ and $t_1, \ldots, t_{n-d}$ is explicitly defined in Section 3 of [23], and is called the *associated formula of the stacky fan* $\boldsymbol{\Sigma}$. We illustrate it by revisiting the stack $\mathbb{P}(6,4)$ of Example 4.6.

Example 4.17. The stack $\mathcal{X}(\mathbf{\Sigma}) = \mathbb{P}(6,4)$ of Example 4.6 has distinguished points $b_1 = (2,1)$ and $b_2 = (-3,0)$ in $N = \mathbb{Z} \oplus \mathbb{Z}/2$. The underlying toric variety is $\mathbb{P}(3,2)$, with line bundles L_1 and L_2 having Chern classes $x_1 := c_1(L_1) = 3t$ and $x_2 := c_1(L_2) = 2t$. The associated formula of the stacky fan $\mathbf{\Sigma}$ is

$$\tilde{x}_1 = 2x_1 \qquad \tilde{x}_2 = 2x_2$$

where the coefficient 2 results from the torsion component of N. Thus, $c_1(\mathcal{L}_1) = 6t$ and $c_1(\mathcal{L}_2) = 4t$.

4.4 General Chow ring structure for toric Deligne-Mumford stacks

Here, we give some prior results in a notation which is more suitable to generalization. In particular, we give a nice ring representation for even the ordinary Chow ring. As in [23], we use the integer coefficients in place of the rational ones used in [3].

The construction begins with the polynomial ring R_β, which depends only on the distinguished points $\{b_1, \ldots b_n\} \subset N$ of $\mathbf{\Sigma}$:

$$R_\beta := \mathbb{Z}[y^{b_1}, ..., y^{b_n}] \tag{4.12}$$

In this context, y is a formal variable.

Later, we will be interested in the larger polynomial ring $R_{\mathbf{\Sigma}}$, which depends also on $\mathrm{Box}(\mathbf{\Sigma}) = \{v_1, \ldots, v_k\}$:

$$R_{\mathbf{\Sigma}} = \mathbb{Z}[y^{b_1}, \ldots, y^{b_n}, y^{v_1}, \ldots, y^{v_k}], \tag{4.13}$$

We introduce some important ideals in (both of) these rings. First, the Stanley-Reisner ideal $I(\mathbf{\Sigma})$:

$$I(\mathbf{\Sigma}) = \left\langle y^{b_{i_1}} \cdots y^{b_{i_l}} \mid \rho_{i_1} + \cdots + \rho_{i_l} \notin \Sigma \right\rangle \tag{4.14}$$

We also have the circuit ideal $\mathrm{Cir}(\mathbf{\Sigma})$:

$$\mathrm{Cir}(\mathbf{\Sigma}) = \left\langle \sum_{i=1}^{n} \theta\left(\overline{b_i}\right) \cdot y^{b_i} \,\middle|\, \theta \in \mathrm{Hom}(N, \mathbb{Z}) \right\rangle \tag{4.15}$$

If conditions are such that $\mathcal{X}(\mathbf{\Sigma})$ is a complete toric Deligne-Mumford stack, then due to [3] and [23], we have a nice ring presentation for the ordinary Chow ring with integer coefficients.

We will use the notation $A^*(X, \star, R)$ to denote the Chow ring of X under the inertial product $\star$ with coefficients in the ring R.

Proposition 4.18. *There is an isomorphism of $\mathbb{Z}$-graded rings*

$$A^*(\mathcal{X}(\mathbf{\Sigma}), \cdot, \mathbb{Z}) \cong \frac{R_\beta}{I_{\mathbf{\Sigma}} + \mathrm{Cir}_{\mathbf{\Sigma}}},$$

where $\cdot$ is the usual intersection product and the isomorphism is given by $c_1(L_i) \mapsto y^{b_i}$.

Explicitly, this says that y^{b_i} in our ring represents the first Chern class of the line bundle L_i.

We will have a nice result for the integral orbifold Chow ring in Section 5.2.

Chapter 5

The inertial Chow ring of a toric Deligne-Mumford stack

Borisov, L. Chen and Smith give a presentation for the orbifold Chow ring with rational coefficients (of a toric Deligne-Mumford stack) in [3] by giving generators and relations. In this section, we generalize the process to handle any of the inertial products in [15].

5.1 Toric computations

We lose generality by working with toric stacks (as opposed to stacks in general), but the benefit is the relative ease of computation. In this section, we'll go over the necessary combinatorial information.

Let $\mathbf{\Sigma} = (N, \Sigma, \beta)$ be a stacky fan with associated toric stack $\mathscr{X} = \mathcal{X}(\mathbf{\Sigma}) = [Z/G]$, as constructed in Section 4. Let $g = (\gamma_1, \ldots, \gamma_n, s_1, \ldots, s_r) \in G$ act on Z by the diagonal action of the first n components, as defined in Section 4.1. Recall that $\mathcal{L}_i$ is the line bundle on Z corresponding to the action of the i-th component of G.

Lemma 5.1. *Let $\mathscr{X}$ be the stack $\mathcal{X}(\mathbf{\Sigma}) = [Z/G]$ associated to the stacky fan $\mathbf{\Sigma} = (N, \Sigma, \beta)$. Let $I\mathscr{X} = [I_G X/G]$ be the inertia stack, and $\mathbb{T}_{I\mathscr{X}}$ be its tangent bundle. Then for any i, $1 \leqslant i \leqslant n$, and for any box element $v \in \mathrm{Box}(\mathbf{\Sigma})$ and its corresponding $g = (\gamma_1, \ldots, \gamma_n, s_1, \ldots, s_r) \in G$, the following statements are equivalent:*

1. $q_i \neq 0$
2. $\rho_i \subseteq \sigma(\overline{v})$
3. $\gamma_i \neq 1$
4. $z_i = 0$ *for all* $z = (z_1, ..., z_n) \in Z^g$
5. $\mathcal{L}_i$ *has coefficient 0 in* $(\mathbb{T}_{I\mathscr{X}})|_{Z^g}$

Proof. Lemma 4.6 of [3] shows that the first four statements are equivalent when Σ is a complete simplicial fan, so it will suffice for us to show that the fifth is equivalent to the fourth. The normal bundle to $I_G X$ is $\sum \mathcal{L}_i$, where the sum is over all i such that $z_i = 0$ for all $z \in Z^g$. Thus, the tangent bundle is $(\mathbb{T}_{I\mathscr{X}})|_{Z^g} = \sum \mathcal{L}_i$, where the sum is over all i such $z_i \neq 0$ for at least one $z \in Z^g$. That is, the coefficient on $\mathcal{L}_i$ is zero if and only if $z_i = 0$ for all $z \in Z^g$. ∎

Example 5.2. Consider the stack $\mathscr{X} = \mathbb{P}(6, 4)$ of Example 4.15. We have $\overline{v_5} = -1 = 0\overline{b_1} + \frac{1}{3}\overline{b_2}$, so $q_{5,1} = 0$ and $q_{5,2} \neq 0$ (where the notation $q_{j,i}$ describes the rational coefficient for $\overline{v_j}$ on b_i). That is to say, $\sigma(\overline{v_5})$ contains ρ_2 and not ρ_1. The corresponding group element $g_5 = (1, e^{2\pi\sqrt{-1}/3}, -1) \in G$ indeed has $\gamma_1 = 1$ and $\gamma_2 \neq 1$. Thus $Z^{g_5} = \{(z_1, 0) \,|\, z_1 \neq 0\}$. Finally, the tangent bundle here is just $\mathcal{L}_1$. In short, the five conditions of Lemma 5.1 are true for $i = 2$ and false for $i = 1$.

On the other hand, if we instead take the box element $v_2 = (1,1)$ corresponding to $g_2 = (-1, 1, e^{3\pi\sqrt{-1}/2})$, the five conditions of Lemma 5.1 are true for $i = 1$ and false for $i = 2$; accordingly, the tangent bundle is $\mathcal{L}_2$.

Lastly, if we take the box element $v_3 = (0,1)$ corresponding to $g_3 = (1,1,-1)$, we have all five conditions false for both $i = 1$ and $i = 2$, and the tangent bundle is $\mathcal{L}_1 + \mathcal{L}_2$. This last example illustrates an important type of box element, one in which $q_{j,i} = 0$ for all i. These correspond to elements of the stabilizer group of G.

Corollary 5.3. *If $e_1 : \mathbb{T}_{\mathbb{I}^2_G X} \to \mathbb{T}_{I_G X}$ and $e_2 : \mathbb{T}_{\mathbb{I}^2_G X} \to \mathbb{T}_{I_G X}$ are the evaluation maps and $(v_1, v_2) \in \mathrm{Box}(\mathbf{\Sigma}) \times \mathrm{Box}(\mathbf{\Sigma})$, then the coefficient of $\mathcal{L}_i$ in $(e_1^* \mathbb{T}_{I_G X})|_{X^{g_1,g_2}}$ (resp. $(e_2^* \mathbb{T}_{I_G X})|_{x^{g_1,g_2}}$) is 1 if and only if $q_{1,i} = 0$ (resp. $q_{2,i} = 0$).*

Corollary 5.4. *The coefficient of $\mathcal{L}_i$ in $\mathbb{T}_{I^2_G X}$ associated to $(v_1, v_2) \in \mathrm{Box}(\mathbf{\Sigma}) \times \mathrm{Box}(\mathbf{\Sigma})$ is 1 if and only if $q_{1,i} = q_{2,i} = 0$.*

Proof. Adapting Lemma 5.1, we have that a point $(z, g_1, g_2) \in \mathbb{I}^2_G X$ is in the fixed locus of both $g_1 = (\gamma_{1,1}, \dots, \gamma_{1,n}, s_{1,1}, \dots, s_{1,r})$ and $g_2 = (\gamma_{2,1}, \dots, \gamma_{2,n}, s_{2,1}, \dots, s_{2,r})$ if and only if

$$z \in \{(z_1, ..., z_n) \in Z \,|\, z_i = 0 \text{ if } \gamma_{1,i} \neq 1 \text{ or } \gamma_{2,i} \neq 1\}$$

$$= \{(z_1, ..., z_n) \in Z \,|\, z_i = 0 \text{ if } q_{1,i} \neq 0 \text{ or } q_{2,i} \neq 0\}$$

The first presentation of this set indicates that the tangent bundle to this set at (z, g_1, g_2) is

$$\mathbb{T}_{I^2_G X} = \sum_{\gamma_{1,i} = \gamma_{2,i} = 1} \mathcal{L}_i \ .$$

The second presentation indicates that the indexing set on this sum is equivalent to the set of i where $q_{1,i} = q_{2,i} = 0$. ∎

Example 5.5. Take our usual example, $X = \mathbb{P}(6,4)$, in the notation as in Example 4.15. Then for the pair $(v_4, v_3) \in \mathrm{Box}(\mathbf{\Sigma}) \times \mathrm{Box}(\mathbf{\Sigma})$, we have $e_1^*\mathbb{T}_{I_GX} = \mathcal{L}_2$ and $e_2^*\mathbb{T}_{I_GX} = \mathcal{L}_1 + \mathcal{L}_2$ by Corollary 5.3, and $\mathbb{T}_{I_G^2X} = \mathcal{L}_2$ by Corollary 5.4.

Lemma 5.6. *Let v be the box element corresponding to some $g \in G$, where $\overline{v} = \sum_{i=1}^n q_i\overline{b_i} \in \mathrm{Box}(\mathbf{\Sigma})$ for some $0 \leqslant q_i < 1$, $i = 1, 2, \ldots, n$. Then for all $i \in \{1, \ldots, n\}$,*

$$L(g)(\mathcal{L}_i) = q_i\mathcal{L}_i,$$

where $L(g)$ is the logarithmic trace of Definition 3.7.

Proof. For the usual action of $g = (\gamma_1, \ldots, \gamma_n, s_1, \ldots, s_r)$

$$g \cdot (z_1, \ldots, z_n) = (\gamma_1 z_1, \ldots, \gamma_n z_n),$$

the eigenvalue on the bundle $\mathcal{L}_i$ is γ_i. By Proposition 4.13, we have $\gamma_i = e^{2\pi\sqrt{-1}q_i}$. Thus $q_i\mathcal{L}_i$ is the logarithmic trace. ∎

Example 5.7. Once again recycling Example 4.15, we have $L(g_2)(\mathcal{L}_1) = \frac{1}{2}\mathcal{L}_1$ and $L(g_2)(\mathcal{L}_2) = 0$. Similarly, $L(g_7)(\mathcal{L}_1) = 0$ and $L(g_7)(\mathcal{L}_2) = \frac{2}{3}\mathcal{L}_2$. We also have $L(g_3)(\mathcal{L}_1) = L(g_3)(\mathcal{L}_2) = 0$, which nicely illustrates the fact that $L(g)$ depends on $\overline{v}$ but not on v. That is to say, logarithmic trace ignores torsion.

We now compute the classes $\mathcal{L}_i{}^+$ and $\mathcal{L}_i{}^-$ in $K_G(\mathbb{I}_G^2X)$ of Proposition 3.11.

Lemma 5.8. *Let $\mathcal{L}_i$ be the G-equivariant line bundle associated to the i-th component of the action of G on Z. For $j = 1, 2$, let $g_j = (\gamma_{j,1}, \ldots, \gamma_{j,n}, s_{j,1}, \ldots, s_{j,r})$ be an element of G, and let v_j be its corresponding element in $\mathrm{Box}(\mathbf{\Sigma})$, where $q_{j,i}$ is defined by $\overline{v_j} = \sum_{i=1}^n q_{j,i}\overline{b_{j,i}}$ (for $1 \leqslant i \leqslant n$ and $j = 1, 2$).*

Then for each i, $1 \leqslant i \leqslant n$, exactly one of the three cases holds:

1. (a) *At least one of $q_{1,i}, q_{2,i}$ is zero, and*

   (b) $\mathcal{L}_i{}^+(g_1, g_2) = 0$, *and*

   (c) $\mathcal{L}_i{}^-(g_1, g_2) = 0$.

2. (a) $q_{1,i} + q_{2,i} < 1$ *with $q_{1,i}, q_{2,i}$ both nonzero, and*

   (b) $\mathcal{L}_i{}^+(g_1, g_2) = 0$, *and*

   (c) $\mathcal{L}_i{}^-(g_1, g_2) = \mathcal{L}_i$.

3. (a) $q_{1,i} + q_{2,i} \geqslant 1$, *and*

   (b) $\mathcal{L}_i{}^+(g_1, g_2) = \mathcal{L}_i$, *and*

   (c) $\mathcal{L}_i{}^-(g_1, g_2) = 0$.

Proof. Since $0 \leqslant q_{i,j} < 1$ for any i and j, the cases 1(a), 2(a) and 3(a) are disjoint and encompass all possibilities. So we must only show that in each of the three cases, (a) implies both (b) and (c).

For case 1, without loss of generality we may assume $q_{1,i} = 0$. Then by Lemma 5.6, $\mathcal{L}_i(g_1) = \mathcal{L}_i(g_1^{-1}) = 0$, and thus we have

$$\mathcal{L}_i{}^+(g_1, g_2) = L(g_2)(\mathcal{L}_i|_{X^{g_1,g_2}}) - L(g_1 g_2)(\mathcal{L}_i|_{X^{g_1,g_2}})$$

and

$$\mathcal{L}_i{}^-(g_1, g_2) = L(g_2^{-1})(\mathcal{L}_i|_{X^{g_1,g_2}}) - L(g_2^{-1} g_1^{-1})(\mathcal{L}_i|_{X^{g_1,g_2}}).$$

Again by Lemma 5.6, $L(g_1 g_2)(\mathcal{L}_i) = q_{2,i}\mathcal{L}_i = L(g_2)(\mathcal{L}_i)$ and $L(g_2^{-1} g_1^{-1}) = (1-q_{2,i})\mathcal{L}_i = L(g_2^{-1})(\mathcal{L}_i)$, which completes the proof of case 1.

For cases 2 and 3, we have

$$L(g_1g_2)(\mathcal{L}_i) = \begin{cases} (q_{1,i}+q_{2,i})\mathcal{L}_i & \text{if} \quad 0 < q_{1,i}+q_{2,i} < 1 \\ (q_{1,i}+q_{2,i}-1)\mathcal{L}_i & \text{if} \quad q_{1,i}+q_{2,i} \geqslant 1 \end{cases}$$

and

$$L(g_2^{-1}g_1^{-1})(\mathcal{L}_i) = \begin{cases} (1-q_{1,i}-q_{2,i})\mathcal{L}_i & \text{if} \quad 0 < q_{1,i}+q_{2,i} \leqslant 1 \\ (2-q_{1,i}-q_{2,i})\mathcal{L}_i & \text{if} \quad q_{1,i}+q_{2,i} > 1 \end{cases}.$$

Thus,

$$\begin{aligned} {\mathcal{L}_i}^{+}(g_1,g_2) &= L(g_1)(\mathcal{L}_i|_{X^{g_1,g_2}}) + L(g_2)(\mathcal{L}_i|_{X^{g_1,g_2}}) - L(g_1g_2)(\mathcal{L}_i|_{X^{g_1,g_2}}) \\ &= \begin{cases} 0 & \text{if} \quad 0 < q_{1,i}+q_{2,i} < 1 \\ \mathcal{L}_i & \text{if} \quad q_{1,i}+q_{2,i} \geqslant 1 \end{cases} \end{aligned}$$

and

$$\begin{aligned} {\mathcal{L}_i}^{-}(g_1,g_2) &= L(g_1^{-1})(\mathcal{L}_i|_{X^{g_1,g_2}}) + L(g_2^{-1})(\mathcal{L}_i|_{X^{g_1,g_2}}) - L(g_2^{-1}g_1^{-1})(\mathcal{L}_i|_{X^{g_1,g_2}}) \\ &= \begin{cases} \mathcal{L}_i & \text{if} \quad 0 < q_{1,i}+q_{2,i} < 1 \\ 0 & \text{if} \quad q_{1,i}+q_{2,i} \geqslant 1 \end{cases}. \end{aligned}$$

∎

Definition 5.9. For any pair $v_1, v_2 \in \mathrm{Box}(\mathbf{\Sigma})$, define the indexing sets $B^{+}_{\mathbf{\Sigma}}(v_1,v_2)$ and $B^{-}_{\mathbf{\Sigma}}(v_1,v_2)$ to be the following subsets of $\{1,2,\ldots,n\}$:

$$B^{+}_{\mathbf{\Sigma}}(v_1,v_2) := \{i \,|\, q_{1,i}+q_{2,i} \geqslant 1\}$$

$$B^{-}_{\mathbf{\Sigma}}(v_1,v_2) := \{i \,|\, q_{1,i}+q_{2,i} < 1 \text{ and } q_{1,i}, q_{2,i} \neq 0\}.$$

Proposition 5.10. *Let V be the G-equivariant Z-bundle $V = \sum_{i=1}^{n} a_i\mathcal{L}_i$, where a_i is a non-negative integer for each i. Then for any pair $v_1, v_2 \in \mathrm{Box}(\mathbf{\Sigma})$,*

$$V^+(g_1, g_2) = \sum_{i=1}^{n} a_i {\mathcal{L}_i}^+(g_1, g_2) = \sum_{B_{\mathbf{\Sigma}}^+(v_1,v_2)} a_i\mathcal{L}_i$$

and

$$V^-(g_1, g_2) = \sum_{i=1}^{n} a_i {\mathcal{L}_i}^-(g_1, g_2) = \sum_{B_{\mathbf{\Sigma}}^-(v_1,v_2)} a_i\mathcal{L}_i.$$

Proof. Since the logarithmic trace of a vector bundle is additive, if $V = \sum_{i=1}^{n} a_i\mathcal{L}_i$ then for any $g \in G$,

$$L(g)(V) = L(g)\left(\sum_{i=1}^{n} a_i\mathcal{L}_i\right) = \sum_{i=1}^{n} L(g)(a_i\mathcal{L}_i) = \sum_{i=1}^{n} a_i L(g)(\mathcal{L}_i).$$

The statement then follows directly from Lemma 5.8. ∎

Example 5.11. The definitions of $B_{\mathbf{\Sigma}}^+$ and $B_{\mathbf{\Sigma}}^-$ may not seem particularly intuitive until we look at an example which includes a box diagram.

Consider the stacky fan $\mathbf{\Sigma}$ with $N = \mathbb{Z}^2$, with the rays of Σ generated by $b_1 = (2, 1)$, $b_2 = (0, 2)$ and $b_3 = (-3, -4)$. Figure 5.1 displays $\mathrm{Box}(\mathbf{\Sigma})$, in a way such that the elements in $\mathrm{Box}(\mathbf{\Sigma})$ (denoted by gray squares) are "boxed off" by the parallelograms formed by the b_i. Any element of $N = \mathbb{Z}^2$ outside the parallelograms is equivalent (modulo some $N(\boldsymbol{\sigma})$) to a box element lying inside one of the three parallelograms.

Consider the cone σ_1 formed by ρ_1 and ρ_2. We'll name the three non-zero box elements $v_1 = (0, 1)$, $v_2 = (1, 1)$ and $v_3 = (1, 2)$. Then $\overline{v_1} = \frac{1}{2}\overline{b_2}$, $\overline{v_2} = \frac{1}{2}\overline{b_1} + \frac{1}{4}\overline{b_2}$ and $\overline{v_3} = \frac{1}{2}\overline{b_1} + \frac{3}{4}\overline{b_2}$.

The intuitive way to think about $B_{\mathbf{\Sigma}}^+$ is that it lists the "directions" in which

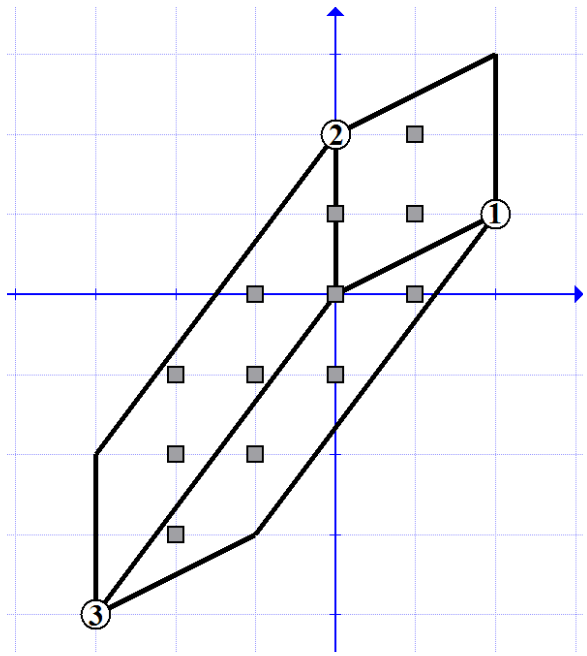


Figure 5.1: The box diagram for the stacky fan in Example 5.11 which produces $\mathbb{P}(6,5,4)$.

the sum of the box elements "leaves" these parallelograms. For instance, with the pair v_1 and v_2, $B^+_{\mathbf{\Sigma}}(v_1, v_2) = \emptyset$, because $(0,1) + (1,1) = (1,2)$ which is still inside the parallelogram. On the other hand, $B^+_{\mathbf{\Sigma}}(v_2, v_3) = \{1,2\}$, since $(1,1) + (1,2) = (2,3)$, which leaves the parallelogram in the direction of both b_1 and b_2. Similarly, $B^+_{\mathbf{\Sigma}}(v_2, v_2) = \{1\}$ because the sum $v_2 + v_2$ leaves the box only in the direction of b_1.

The other side of the coin is $B^-_{\mathbf{\Sigma}}$, which lists the ways in which the sum does *not* leave the parallelogram. For example, we have $B^-_{\mathbf{\Sigma}}(v_2, v_3) = \emptyset$ and $B^-_{\mathbf{\Sigma}}(v_2, v_2) = \{2\}$. An oddity occurs when one of the box elements happens to be on a ray, such as $B^-_{\mathbf{\Sigma}}(v_1, v_2) = \{2\}$; the index 1 is not included because v_1 is on the ray b_2 (and therefore $q_{1,1} = 0$).

Let $V = a_1\mathcal{L}_1 + a_2\mathcal{L}_2 + a_3\mathcal{L}_3$ for some non-negative integers a_1, a_2, a_3. Then we can compute the following, using Proposition 5.10 in conjunction with the above calculations:

$$\begin{aligned} V^+(v_1,v_2) &= 0 & V^-(v_1,v_2) &= a_2\mathcal{L}_2 \\ V^+(v_2,v_2) &= a_1\mathcal{L}_1 & V^-(v_2,v_2) &= a_2\mathcal{L}_2 \\ V^+(v_2,v_3) &= a_1\mathcal{L}_1 + a_2\mathcal{L}_2 & V^-(v_2,v_3) &= 0 \end{aligned}$$

5.2 The Chow ring for an inertial product

For any toric stack, finding a nice presentation of its orbifold Chow ring (with rational coefficients) is a task undertaken in [3]. In [23], the same problem is solved for integer coefficients. We will present the analogous statement for inertial Chow rings associated to vector bundles.

First, we recall some definitions from Sections 3 and 4.4.

Let $\mathcal{X}(\mathbf{\Sigma})$ be a toric stack with stacky fan $\mathbf{\Sigma} = (N, \Sigma, \beta)$, and let $b_1, \ldots, b_n$ be the distinguished points in N defined by β. Let $\mathrm{Box}(\mathbf{\Sigma}) = \{v_1, \ldots, v_k\}$. Then we have the ring $R_{\mathbf{\Sigma}}$ and its ideals $I(\mathbf{\Sigma})$, $\mathrm{Cir}(\mathbf{\Sigma})$ from Section 4.4:

$$R_{\mathbf{\Sigma}} = \mathbb{Z}[y^{b_1}, \ldots, y^{b_n}, y^{v_1}, \ldots, y^{v_k}],$$

$$I(\mathbf{\Sigma}) = \left\langle y^{b_{i_1}} \cdots y^{b_{i_l}} \,|\, \rho_{i_1} + \cdots + \rho_{i_l} \notin \Sigma \right\rangle,$$

$$\mathrm{Cir}(\mathbf{\Sigma}) = \left\langle \sum_{i=1}^{n} \theta\left(\overline{b_i}\right) \cdot y^{b_i} \,\middle|\, \theta \in \mathrm{Hom}(N, \mathbb{Z}) \right\rangle.$$

We also have one new ideal to define which is relevant for stacky fans with more than one top-dimensional cone.

Definition 5.12. We define the *cone relations ideal* in $R_{\mathbf{\Sigma}}$ to be

$$\mathrm{CR}(\mathbf{\Sigma}) := \langle y^{v_i} y^{v_j} \,|\, v_i, v_j \in \mathrm{Box}(\mathbf{\Sigma}) \text{ and no cone contains both } v_i \text{ and } v_j \rangle.$$

Recall from Section 4 that we can write $\mathcal{X}(\mathbf{\Sigma})$ as the quotient stack $[Z/G]$. Also recall from Theorem 3.14 that given a G-equivariant vector bundle V on $\mathcal{X}(\mathbf{\Sigma})$, the inertial product $\star_{V^+}$ is defined by

$$x \star_{V^+} y = \mu_*({e_1}^* x \cdot {e_2}^* y \cdot \mathrm{eu}(LR(\mathbb{T}) + R^+ V)),$$

where LR is the logarithmic restriction as in Definition 3.9 and R^+ is the class defined in Definition 3.12. Similarly, the inertial product $\star_{V^-}$ is defined by

$$x \star_{V^-} y = \mu_*({e_1}^* x \cdot {e_2}^* y \cdot \mathrm{eu}(LR(\mathbb{T}) + R^- V)).$$

With these in hand, we present the underlying module structure of the inertial Chow ring.

Proposition 5.13. *Let $\mathcal{X}(\mathbf{\Sigma})$ be a toric Deligne-Mumford stack associated to the stacky fan $\mathbf{\Sigma}$, and let $T = A^*(\mathcal{X}(\mathbf{\Sigma}), \cdot, \mathbb{Z})$, and let $\star$ be an inertial product. Then we have the following isomorphism of T-modules:*

$$A^*(\mathcal{X}(\mathbf{\Sigma}), \star, \mathbb{Z}) \cong \bigoplus_{v \in \mathrm{Box}(\mathbf{\Sigma})} T y^v$$

Proof. By Proposition 5.8 and Theorem 1.2 of [23], we have $T \cong \dfrac{\mathbb{Z}[y^{b_1}, \ldots, y^{b_n}]}{I(\mathbf{\Sigma}) + \mathrm{Cir}(\mathbf{\Sigma})}$, and that

$$A^*(\mathcal{X}(\mathbf{\Sigma}), \star_{orb}, \mathbb{Z}) \cong \frac{R_{\mathbf{\Sigma}}}{I(\mathbf{\Sigma}) + \mathrm{Cir}(\mathbf{\Sigma}) + \mathrm{CR}(\mathbf{\Sigma})} \cong \frac{T[\{y^v \mid v \in \mathrm{Box}(\mathbf{\Sigma})\}]}{\mathrm{CR}(\mathbf{\Sigma})},$$

where $\mathrm{Box}(\mathbf{\Sigma}) = \{v_1, \ldots, v_r\}$. Thus, as modules over T, we have

$$A^*(\mathcal{X}(\mathbf{\Sigma}), \star_{orb}, \mathbb{Z}) \cong \bigoplus_{v \in \mathrm{Box}(\mathbf{\Sigma})} T y^v$$

Finally, since it is only the multiplication operation on elements in $\{y^v | v \in \mathrm{Box}(\mathbf{\Sigma})\}$ which distinguishes inertial products from one another, and since this plays no role in T-module structure, we have that

$$A^*(\mathcal{X}(\mathbf{\Sigma}), \star, \mathbb{Z}) \cong A^*(\mathcal{X}(\mathbf{\Sigma}), \star_{orb}, \mathbb{Z}) \cong \bigoplus_{v \in \mathrm{Box}(\mathbf{\Sigma})} T y^v$$

as T-modules, for any inertial product $\star$. ∎

Of course, there are also multiplicative relations amongst the y^{v_i} in $\mathrm{CR}(\mathbf{\Sigma})$, but they do not affect the structure as a *module* over T.

As a consequence of this proposition, the calculations for the inertial products $\star_{V^+}$ and $\star_{V^-}$ can be made concrete. Indeed, for any $v_1, v_2 \in \mathrm{Box}(\mathbf{\Sigma})$, we have

$$y^{v_1} \star_{V^+} y^{v_2} = \mu_*({e_1}^* y^{v_1} \cdot {e_2}^* y^{v_2} \cdot \mathrm{eu}(LR(\mathbb{T}) + R^+V)),$$

where $LR(\mathbb{T}) + R^+V$ is calculated at (g_1, g_2), the pair of elements in G corresponding to v_1 and v_2 by Proposition 4.13. Despite the complexity of the formula, this is not difficult to compute on a case-by-case basis.

We can formalize the contribution of $\mathrm{eu}(LR(\mathbb{T}) + R^+V)$ to the inertial product. It depends only on the pair of box elements v_1, v_2, inspiring the following definition.

Definition 5.14. Let $\star$ be an inertial product on the toric stack $\mathscr{X} = \mathcal{X}(\mathbf{\Sigma})$ associated to the stacky fan $\mathbf{\Sigma} = (N, \Sigma, \beta)$. The *twisting function for the inertial product* $\star$

is a mapping

$$\mathrm{Tw}(\star) : \mathrm{Box}(\mathbf{\Sigma}) \times \mathrm{Box}(\mathbf{\Sigma}) \to A^*(\mathbb{I}^2_G \mathscr{X})$$

which sends a box pair (v_1, v_2) to the class completely determined by $\star$ in $A^*(\mathbb{I}^2_G \mathscr{X})$ at (g_1, g_2), where g_1 and g_2 are the elements of G corresponding to v_1 and v_2. Specifically, in writing out the $\star$ product for y^{v_1} and y^{v_2}, we define the twisting function for v_1 and v_2 as follows:

$$y^{v_1} \star y^{v_2} = y^{v_3} \cdot \mathrm{Tw}(\star)(v_1, v_2) \cdot \prod_{q_{1,i}+q_{2,i}=1} c_1(\mathcal{L}_i), \tag{5.1}$$

where v_3 is the unique box element such that $v_1 + v_2 = v_3$ under box addition. The additional factors of $c_1(\mathcal{L}_i)$ are the contribution from the normal bundle, by the projection formula of [17].

For any equivariant vector bundle E on X, if we let $c = \mathrm{eu}(LR(\mathbb{T}) + R^+E)$ we can explicitly calculate the twisting function on $\star_c$ from Definition 3.13. As a first example, we'll compute the twisting function for the orbifold product $\star_{orb}$ of Definition 3.16.

In this example and in the calculations which follow, keep in mind that the Euler class of a line bundle is just its first Chern class. We'll usually write $c_1(\mathcal{L}_i)$ instead of $\mathrm{eu}(\mathcal{L}_i)$, as a simplification. However, when the bundle has rank higher than one, we are forced to stick with the Euler class.

Example 5.15. By Definition 3.16, the orbifold product $\star_{orb}$ is given by $\star_c$ where $c = \mathrm{eu}(LR(\mathbb{T}) + R^+\mathbf{0}) = \mathrm{eu}(LR(\mathbb{T}))$. Then for any pair $(v_1, v_2) \in \mathrm{Box}(\mathbf{\Sigma}) \times \mathrm{Box}(\mathbf{\Sigma})$ with corresponding pair $g_1, g_2 \in G$, let $v_3 = v_1 + v_2$ under box addition. Then we have

$$y^{v_1} \star_{orb} y^{v_2} = y^{v_3} \cdot \mathrm{Tw}(\star_c)(v_1, v_2) \cdot \prod_{q_{1,i}+q_{2,i}=1} c_1(\mathcal{L}_i)$$

$$= y^{v_3} \cdot \mathrm{eu}(LR(\mathbb{T})(g_1, g_2, (g_1 g_2)^{-1})) \prod_{q_{1,i}+q_{2,i}=1} c_1(\mathcal{L}_i)$$

$$= y^{v_3} \cdot \prod_{q_{1,i}+q_{2,i}>1} c_1(\mathcal{L}_i) \prod_{q_{1,i}+q_{2,i}=1} c_1(\mathcal{L}_i)$$

$$= y^{v_3} \cdot \prod_{q_{1,i}+q_{2,i}\geqslant 1} c_1(\mathcal{L}_i)$$

Thus, by (5.1) the twisting function for the orbifold product is:

$$\mathrm{Tw}(\star_{orb})(v_1, v_2) = \prod_{q_{1,i}+q_{2,i}>1} c_1(\mathcal{L}_i) = \prod_{q_{1,i}+q_{2,i}>1} \widetilde{y}^{b_i}.$$

The purpose of the twisting function, illustrated in this example, is to give a succinct way of describing exactly how c affects the inertial product. We'll use this in the next definition as we continue our quest to provide a ring presentation for the inertial Chow ring.

Definition 5.16. Define $\mathrm{BR}(\star, \mathbf{\Sigma})$ to be the ideal

$$\left\langle y^{v_1} y^{v_2} - y^{v_3} \cdot \mathrm{Tw}(\star)(v_1, v_2) \cdot \prod_{q_{1,i}+q_{2,i}=1} \widetilde{y}^{b_i} \,\middle|\, v_1, v_2 \in \mathrm{Box}(\mathbf{\Sigma}) \right\rangle.$$

Of course, by definition of the $\star_{V^+}$ product, the listed generators of $\mathrm{BR}(\star_{V^+})$ are all 0 in the ring $A^*(\mathcal{X}(\mathbf{\Sigma}), \star_{V^+}, \mathbb{Z})$.

Theorem 5.17. *For any toric stack $\mathcal{X}(\mathbf{\Sigma})$ and any inertial product $\star$ of the form $\star_{V^+}$ or $\star_{V^-}$ for some G-equivariant vector bundle V, we have an isomorphism of $\mathbb{Z}$-graded rings*

$$A^*(\mathcal{X}(\mathbf{\Sigma}), \star, \mathbb{Z}) \cong \frac{R_{\mathbf{\Sigma}}}{I(\mathbf{\Sigma}) + \mathrm{Cir}(\mathbf{\Sigma}) + \mathrm{CR}(\mathbf{\Sigma}) + \mathrm{BR}(\star, \mathbf{\Sigma})},$$

where the isomorphism is given by $y^{b_i} \mapsto c_1(\mathcal{L}_i)$.

Proof. By Proposition 5.13, we know as an $A^*(\mathcal{X}(\mathbf{\Sigma}), \cdot, \mathbb{Z})$-module,

$$A^*(\mathcal{X}(\mathbf{\Sigma}), \star, \mathbb{Z}) \cong \frac{R_{\mathbf{\Sigma}}}{I(\mathbf{\Sigma}) + \mathrm{Cir}(\mathbf{\Sigma}) + \mathrm{CR}(\mathbf{\Sigma})}.$$

This completely describes the additive structure, but the multiplicative structure still remains to be described. We achieve this with the ideal $\mathrm{BR}(\star, \mathbf{\Sigma})$, since it describes completely all possible multiplication relations. Thus, taking the quotient by $\mathrm{BR}(\star, \mathbf{\Sigma})$ completes the proof. ∎

As the notation suggests, only the box relations ideal depends on the choice of product. We'll exemplify this as we compute the BR ideal for the inertial products defined in [15].

Proposition 5.18. *For any G-equivariant bundle $V = \sum f_i(\mathcal{L}_i)$ and stacky fan $\mathbf{\Sigma}$, the box relations ideal for the $\star_{V^+}$ product is*

$$\mathrm{BR}(\star_{V^-}, \mathbf{\Sigma}) = \left\langle y^{v_1} y^{v_2} - y^{v_3} \cdot \prod_{B^+_{\mathbf{\Sigma}}(v_1,v_2)} \mathrm{eu}(\mathcal{L}_i + f_i(\mathcal{L}_i)) \middle| v_1, v_2 \in \mathrm{Box}(\mathbf{\Sigma}) \right\rangle,$$

and the box relations ideal for the $\star_{V^-}$ product is

$$\mathrm{BR}(\star_{V^-}, \mathbf{\Sigma}) = \left\langle y^{v_1} y^{v_2} - y^{v_3} \cdot \prod_{B^-_{\mathbf{\Sigma}}(v_1,v_2)} \mathrm{eu}(f_i(\mathcal{L}_i)) \cdot \prod_{B^+_{\mathbf{\Sigma}}(v_1,v_2)} c_1(\mathcal{L}_i) \middle| v_1, v_2 \in \mathrm{Box}(\mathbf{\Sigma}) \right\rangle.$$

Proof. For any $v_1, v_2 \in \mathrm{Box}(\boldsymbol{\sigma})$ for some cone $\sigma \in \Sigma$, we have a corresponding $g_1, g_2 \in G$ and we can write $\overline{v_1} = \sum_{\rho_i \in \sigma} q_{1,i} \overline{b_i}$ and $\overline{v_2} = \sum_{\rho_i \in \sigma} q_{2,i} \overline{b_i}$. Let $v_3 = v_1 + v_2$ under addition in $\mathrm{Box}(\boldsymbol{\sigma})$.

We first compute the twisting function for the $\star_{V^+}$ product. To do so, we need the

Euler class of $\mathscr{R}^+V$, using Proposition 3.11 and Proposition 5.10. We have

$$\left(R^+V + LR(\mathbb{T})\right)(g_1, g_2) = V^+(g_1, g_2) + LR(\mathbb{T})(g_1, g_2) = \sum_{B^+_{\mathbf{\Sigma}}(v_1,v_2)} f_i(\mathcal{L}_i) + \sum_{q_{1,i}+q_{2,i}>1} \mathcal{L}_i,$$

and so

$$\begin{aligned}\mathrm{Tw}(\star_{V^+})(v_1, v_2) &= \mathrm{eu}\left(\sum_{B^+_{\mathbf{\Sigma}}(v_1,v_2)} f_i(\mathcal{L}_i) + \sum_{q_{1,i}+q_{2,i}>1} \mathcal{L}_i\right) \\ &= \prod_{B^+_{\mathbf{\Sigma}}(v_1,v_2)} \mathrm{eu}(f_i(\mathcal{L}_i)) \cdot \prod_{q_{1,i}+q_{2,i}>1} c_1(\mathcal{L}_i).\end{aligned}$$

The normal bundle for $\mathbf{g} = (g_1, g_2)$ is $\sum \mathcal{L}_i$, where the sum is over all i such that $\rho_i \in (\overline{v_1}, \overline{v_2})$ but $\rho_i \notin \sigma(\overline{v_3})$. Equivalently, we could say the sum is over all i such that $q_{1,i} + q_{2,i} = 1$.

To simplify notation a bit, let $w_j := y^{v_j}$ for $j = 1, 2, 3$. Then we have

$$\begin{aligned} w_1 \star_{V^+} w_2 =& \mu_*(e_1^* w_1 \cdot e_2^* w_2 \cdot \mathrm{eu}(\mathscr{R}^+V)) \\ =& w_3 \cdot \left(\prod_{B^+_{\mathbf{\Sigma}}(v_1,v_2)} \mathrm{eu}(f_i(\mathcal{L}_i)) \cdot \prod_{q_{1,i}+q_{2,i}>1} c_1(\mathcal{L}_i)\right) \cdot \prod_{q_{1,i}+q_{2,i}=1} c_1(\mathcal{L}_i) \\ =& w_3 \cdot \prod_{B^+_{\mathbf{\Sigma}}(v_1,v_2)} \mathrm{eu}(\mathcal{L}_i + f_i(\mathcal{L}_i)) \end{aligned}$$

Thus,

$$\mathrm{BR}(\star_{V^+}, \mathbf{\Sigma}) = \left\langle w_1 w_2 - w_3 \cdot \prod_{B^+_{\mathbf{\Sigma}}(v_1,v_2)} \mathrm{eu}(\mathcal{L}_i + f_i(\mathcal{L}_i)) \,\middle|\, v_1, v_2 \in \mathrm{Box}(\mathbf{\Sigma}) \right\rangle.$$

If we instead consider the $\star_{V^-}$ product, we instead have the twisting function

$$\mathrm{Tw}(\star_{V^-})(v_1,v_2)=\mathrm{eu}\left(\sum_{B^-_{\mathbf{\Sigma}}(v_1,v_2)} f_i(\mathcal{L}_i)+\sum_{q_{1,i}+q_{2,i}>1}\mathcal{L}_i\right)=\prod_{B^-_{\mathbf{\Sigma}}(v_1,v_2)}\mathrm{eu}(f_i(\mathcal{L}_i))\cdot\prod_{q_{1,i}+q_{2,i}>1}c_1(\mathcal{L}_i)$$

and so

$$\begin{aligned}
w_1\star_{V^-}w_2=&\mu_*(e_1^*w_1\cdot e_2^*w_2\cdot\mathrm{eu}(\mathscr{R}^-V))\\
=&w_3\cdot\left(\prod_{B^-_{\mathbf{\Sigma}}(v_1,v_2)}\mathrm{eu}(f_i(\mathcal{L}_i))\cdot\prod_{q_{1,i}+q_{2,i}>1}c_1(\mathcal{L}_i)\right)\cdot\prod_{q_{1,i}+q_{2,i}=1}c_1(\mathcal{L}_i)\\
=&w_3\cdot\prod_{B^-_{\mathbf{\Sigma}}(v_1,v_2)}\mathrm{eu}((\mathcal{L}_i))\cdot\prod_{q_{1,i}+q_{2,i}\geqslant 1}c_1(\mathcal{L}_i).
\end{aligned}$$

Therefore,

$$\mathrm{BR}(\star_{V^-},\mathbf{\Sigma})=\left\langle w_1w_2-w_3\cdot\prod_{B^-_{\mathbf{\Sigma}}(v_1,v_2)}\mathrm{eu}(f_i(\mathcal{L}_i))\cdot\prod_{B^+_{\mathbf{\Sigma}}(v_1,v_2)}c_1(\mathcal{L}_i)\,\middle|\,v_1,v_2\in\mathrm{Box}(\mathbf{\Sigma})\right\rangle.$$

∎

Corollary 5.19. *For the virtual product, we use* $V=\mathbb{T}=\sum_{i=1}^n\mathcal{L}_i$ *in the* $\star_{V^-}$*-product. Then*

$$\mathrm{Tw}(\star_{virt})(v_1,v_2)=\prod_{\rho_i\subseteq\sigma(\overline{v_1}),\sigma(\overline{v_2})}c_1(\mathcal{L}_i).$$

Example 5.20. Specifically, for the toric stack $\mathcal{X}(\mathbf{\Sigma})$ of Example 5.11, consider $v_1=(0,1)$ and $v_3=(1,2)$.

Then $q_{1,1}+q_{3,1}=0+\frac{1}{2}=\frac{1}{2}$ and $q_{1,2}+q_{3,2}=\frac{1}{2}+\frac{3}{4}=\frac{5}{4}$. So under the orbifold product, we have $\mathrm{Tw}(v_1,v_3)(\star_{orb})=c_1(\mathcal{L}_2)$.

On the other hand, we have $\rho_2\in\sigma(\overline{v_1})$ while $\rho_1,\rho_2\in\sigma(\overline{v_3})$. So under the virtual product, we have $\mathrm{Tw}(v_1, v_3)(\star_{virt}) = c_1(\mathcal{L}_2)$.

Now, we show the entire computation of a Chow ring for an inertial product.

Example 5.21. Let $\mathbf{\Sigma}$ be the stacky fan of Example 5.11, with $\mathcal{X}(\mathbf{\Sigma}) = \mathbb{P}(6, 5, 4)$. As before, let $\mathcal{L}_i$ be the G-equivariant line bundle associated to the i-th component of the action of G on $\mathcal{X}(\mathbf{\Sigma})$, and to the ray ρ_i, for $i = 1, 2, 3$. Let $V = a_1\mathcal{L}_1 + a_2\mathcal{L}_2 + a_3\mathcal{L}_3$, for non-negative integers a_1, a_2, a_3. We'll compute the Chow ring for the $\star_{V^+}$ product on $\mathcal{X}(\mathbf{\Sigma})$.

As stated previously, we can calculate the ring $R_{\mathbf{\Sigma}}$ and its ideals $I(\mathbf{\Sigma})$, $\mathrm{CR}(\mathbf{\Sigma})$ and $\mathrm{Cir}(\mathbf{\Sigma})$ without regard to the vector bundle V, so we'll do this first.

The distinguished points are $b_1 = (2, 1)$, $b_2 = (0, 2)$ and $b_3 = (-3, -4)$, so the box elements are

$$\begin{array}{llll} v_1 = (0,1) & v_4 = (-1,0) & v_7 = (-2,-2) & v_{10} = (0,-1) \\ v_2 = (1,1) & v_5 = (-1,-1) & v_8 = (-2,-3) & v_{11} = (1,0) \\ v_3 = (1,2) & v_6 = (-2,-1) & v_9 = (-1,-2) & \end{array}$$

with $q_{j,i}$ being defined by $\overline{v_j} = q_{j,1}\overline{b_1} + q_{j,2}\overline{b_2} + q_{j,3}\overline{b_3}$ as follows:

$$\begin{array}{llll} \overline{v_1} = \frac{1}{2}\overline{b_2} & \overline{v_4} = \frac{2}{3}b_2 + \frac{1}{3}\overline{b_3} & \overline{v_7} = \frac{1}{3}\overline{b_2} + \frac{2}{3}\overline{b_3} & \overline{v_{10}} = \frac{3}{5}\overline{b_1} + \frac{2}{5}\overline{b_3} \\ \overline{v_2} = \frac{1}{2}\overline{b_1} + \frac{1}{4}\overline{b_2} & \overline{v_5} = \frac{1}{6}\overline{b_2} + \frac{1}{3}\overline{b_3} & \overline{v_8} = \frac{1}{5}\overline{b_1} + \frac{4}{5}\overline{b_3} & \overline{v_{11}} = \frac{4}{5}\overline{b_1} + \frac{1}{5}\overline{b_3} \\ \overline{v_3} = \frac{1}{2}\overline{b_1} + \frac{3}{4}\overline{b_2} & \overline{v_6} = \frac{5}{6}\overline{b_2} + \frac{2}{3}\overline{b_3} & \overline{v_9} = \frac{2}{5}\overline{b_1} + \frac{3}{5}\overline{b_3} & \end{array}$$

along with $\mathbf{0} = (0, 0)$ (see Figure 5.1 for an intuition-aiding diagram). The fan $\mathbf{\Sigma}$ has three top-dimensional cones:

$$\mathrm{Box}(\boldsymbol{\sigma}_1) = \{\mathbf{0}, v_1, v_2, v_3\} \quad \mathrm{Box}(\boldsymbol{\sigma}_2) = \{\mathbf{0}, v_1, v_4, v_5, v_6, v_7\} \quad \mathrm{Box}(\boldsymbol{\sigma}_3) = \{\mathbf{0}, v_8, v_9, v_{10}, v_{11}\}$$

Let $w_j := y^{v_j}$ for $j = 1, \ldots, 11$, and let $x_i := y^{b_i}$ for $i = 1, 2, 3$. Then over coefficients in $\mathbb{Z}$, we have the generators of our ring:

$$R_{\mathbf{\Sigma}} = \mathbb{Z}[x_1, x_2, x_3, w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, w_9, w_{10}, w_{11}]$$

Based solely on the fan structure, we can compute the irrelevant ideal:

$$I(\mathbf{\Sigma}) = \langle x_1 x_2 x_3 \rangle$$

Using the distinguished points we compute the circuit ideal:

$$\mathrm{Cir}(\mathbf{\Sigma}) = \langle 2x_1 - 3x_3, x_1 + 2x_2 - 4x_3 \rangle$$

Finally, based on the boxes of the top-dimensional cones σ_1, σ_2 and σ_3, we have the cone relations ideal, writing $w_j := y^{v_j}$:

$$\mathrm{CR}(\mathbf{\Sigma}) = \left\langle \begin{array}{c} w_1w_8, w_1w_9, w_1w_{10}, w_1w_{11}, w_2w_4, w_2w_5, w_2w_6, w_2w_7, w_2w_8, w_2w_9, \\ w_2w_{10}, w_2w_{11}, w_3w_4, w_3w_5, w_3w_6, w_3w_7, w_3w_8, w_3w_9, w_3w_{10}, w_3w_{11}, \\ w_4w_8, w_4w_9, w_4w_{10}, w_4w_{11}, w_5w_8, w_5w_9, w_5w_{10}, w_5w_{11}, w_6w_8, w_6w_9, \\ w_6w_{10}, w_6w_{11}, w_7w_8, w_7w_9, w_7w_{10}, w_7w_{11} \end{array} \right\rangle$$

This leaves only box relations ideal, which depends on V. By Proposition 5.18, we have

$$\mathrm{BR}(\star_{V^+}, \mathbf{\Sigma}) = \left\langle w_{j_1} w_{j_2} - w_{j_3} \cdot \prod_{q_{1,i} + q_{2,i} \geqslant 1} c_1(\mathcal{L}_i)^{1+a_i} \,\middle|\, v_{j_1}, v_{j_2} \in \mathrm{Box}(\mathbf{\Sigma}) \right\rangle$$

For each of the three top-dimensional cones $\boldsymbol{\sigma}$, we must compute $w_{j_1}w_{j_2}$ for every pair $(v_{j_1}, v_{j_2}) \in \mathrm{Box}(\boldsymbol{\sigma})$. (Note that we may be lazy and not calculate for box elements which are not in the same top-dimensional cone, such as $y^{v_1}y^{v_8}$, since it's already an ideal generator in $\mathrm{CR}(\boldsymbol{\Sigma})$.)

So, the generators of $\mathrm{BR}(\star_{V^+}, \boldsymbol{\Sigma})$, or at least the ones that we need to pay attention to, are

$$
\begin{array}{ll}
w_1w_2 - w_3 & w_1w_3 - w_2 \cdot c_1(\mathcal{L}_2)^{a_2+1} \\
w_2w_3 - 1 \cdot c_1(\mathcal{L}_1)^{a_1+1} \cdot c_1(\mathcal{L}_2)^{a_2+1} & {w_1}^2 - 1 \cdot c_1(\mathcal{L}_2)^{a_2+1} \\
{w_2}^2 - w_1 \cdot c_1(\mathcal{L}_1)^{a_1+1} & {w_3}^2 - w_1 \cdot c_1(\mathcal{L}_1)^{a_1+1} \cdot c_1(\mathcal{L}_2)^{a_2+1} \\
w_1w_4 - w_5 \cdot c_1(\mathcal{L}_2)^{a_2+1} & w_1w_5 - w_4 \\
w_1w_6 - w_7 \cdot c_1(\mathcal{L}_2)^{a_2+1} & w_1w_7 - w_6 \\
w_4w_5 - w_6 & w_4w_6 - w_1 \cdot c_1(\mathcal{L}_2)^{a_2+1} \cdot c_1(\mathcal{L}_3)^{a_3+1} \\
w_4w_7 - 1 \cdot c_1(\mathcal{L}_2)^{a_2+1} \cdot c_1(\mathcal{L}_3)^{a_3+1} & w_5w_6 - 1 \cdot c_1(\mathcal{L}_2)^{a_2+1} \cdot c_1(\mathcal{L}_3)^{a_3+1} \\
w_5w_7 - w_1 \cdot c_1(\mathcal{L}_3)^{a_3+1} & w_6w_7 - w_5 \cdot c_1(\mathcal{L}_2)^{a_2+1} \cdot c_1(\mathcal{L}_3)^{a_3+1} \\
{w_4}^2 - w_7 \cdot c_1(\mathcal{L}_2)^{a_2+1} & {w_5}^2 - w_7 \\
{w_6}^2 - w_4 \cdot c_1(\mathcal{L}_2)^{a_2+1} \cdot c_1(\mathcal{L}_3)^{a_3+1} & {w_7}^2 - w_4 \cdot c_1(\mathcal{L}_3)^{a_3+1} \\
w_8w_9 - w_{10} \cdot c_1(\mathcal{L}_3)^{a_3+1} & w_8w_{10} - w_{11} \cdot c_1(\mathcal{L}_3)^{a_3+1} \\
w_8w_{11} - 1 \cdot c_1(\mathcal{L}_1)^{a_1+1} \cdot c_1(\mathcal{L}_3)^{a_3+1} & w_9w_{10} - 1 \cdot c_1(\mathcal{L}_1)^{a_1+1} \cdot c_1(\mathcal{L}_3)^{a_3+1} \\
w_9w_{11} - w_8 \cdot c_1(\mathcal{L}_1)^{a_1+1} & w_{10}w_{11} - w_9 \cdot c_1(\mathcal{L}_1)^{a_1+1} \\
{w_8}^2 - w_9 \cdot c_1(\mathcal{L}_3)^{a_3+1} & {w_9}^2 - w_{11} \cdot c_1(\mathcal{L}_3)^{a_3+1} \\
{w_{10}}^2 - w_8 \cdot c_1(\mathcal{L}_1)^{a_1+1} & {w_{11}}^2 - w_{10} \cdot c_1(\mathcal{L}_1)^{a_1+1}
\end{array}
$$

From these generator relations, we can eliminate w_3, w_4, w_6, w_7.

By the associated formula of $\mathbf{\Sigma}$ (see Lemma 4.16), we have the following:

$$c_1(\mathcal{L}_1) = x_1 = \widetilde{x}_1 = 6t$$

$$c_1(\mathcal{L}_2) = x_2 = \widetilde{x}_2 = 5t$$

$$c_1(\mathcal{L}_3) = x_3 = \widetilde{x}_3 = 4t$$

Finally, we have our ring presentation for $A^*(\mathcal{X}(\mathbf{\Sigma}), \star_{V^+}, \mathbb{Z})$:

$$\frac{\mathbb{Z}[t, w_1, w_2, w_5, w_8, w_9, w_{10}, w_{11}]}{\left\langle \begin{array}{c} 120t^3, w_1w_8, w_1w_9, w_1w_{10}, w_1w_{11}, w_2w_5, w_2w_8, w_2w_9, w_2w_{10}, w_2w_{11}, w_5w_8, w_5w_9, \\ w_5w_{10}, w_5w_{11}, {w_1}^2 - (5t)^{a_2+1}, {w_2}^2 - w_1(6t)^{a_1+1}, {w_5}^3 - w_1(4t)^{a_3+1}, \\ {w_8}^2 - w_9(4t)^{a_3+1}, w_8w_9 - w_{10}(4t)^{a_3+1}, w_8w_{10} - w_{11}(4t)^{a_3+1}, \\ w_8w_{11} - (6t)^{a_1+1}(4t)^{a_3+1}, {w_9}^2 - w_{11}(4t)^{a_3+1}, w_9w_{10} - (6t)^{a_1+1}(4t)^{a_3+1}, \\ {w_{10}}^2 - w_8(6t)^{a_1+1}, w_9w_{11} - w_8(6t)^{a_1+1}, {w_{11}}^2 - w_{10}(6t)^{a_1+1}, w_{10}w_{11} - w_9(6t)^{a_1+1} \end{array} \right\rangle}$$

Chapter 6

A new asymptotic product

In [15], a new class of inertial products are defined; every equivariant vector bundle gives rise to two such products. We introduce a new product which is not an inertial product, but is an asymptotic limit of inertial products.

Let $\mathcal{X}(\mathbf{\Sigma})$ be a toric stack with open set Z; we will need to consider arbitrary G-equivariant vector bundles on Z. Let $\mathcal{L}_1, \ldots, \mathcal{L}_n$ be the line bundles associated to the divisors where $z_i = 0$. Then any G-equivariant vector bundle V has K-theory form

$$V = \sum_{i=1}^{n} f_i(\mathcal{L}_i), \tag{6.1}$$

where $f_i(\mathcal{L}_i)$ denotes a polynomial in $\mathcal{L}_i$ and $\mathcal{L}_i^{-1}$ with positive coefficients. (For example, we could have $f_i(\mathcal{L}_i) = 2\mathcal{L}_i{}^5 + 9\mathcal{L}_i + 18\mathcal{L}_i{}^{-3}$.)

Definition 6.1. Consider a G-equivariant vector bundle V on Z of the form (3.4). Then for any positive number a, aV is also G-equivariant, and we define the *asymptotic-*

minus product of y_1 and y_2 to be

$$y_1 \star_{V_\infty^-} y_2 := \lim_{a\to\infty} \left(y_1 \star_{(aV)^-} y_2 \right),$$

and the *asymptotic-plus product* of y_1 and y_2 to be

$$y_1 \star_{V_\infty^+} y_2 := \lim_{a\to\infty} \left(y_1 \star_{(aV)^+} y_2 \right).$$

For reasons which will soon be made clear, we will work with Chow ring coefficients in $\mathbb{Q}$ and not $\mathbb{Z}$.

Lemma 6.2. *The asymptotic-plus and asymptotic-minus products are both associative for any G-equivariant bundle $V = \sum a\mathcal{L}_i$ on Z.*

Proof. As these products are limits of associative products, they are also associative. ∎

Theorem 6.3. *Let V be a G-equivariant vector bundle on Z for the toric stack $\mathcal{X}(\mathbf{\Sigma})$. Then for $y^{v_1}, y^{v_2} \in A_G^*(I_G X)$, we have the following identities for the asymptotic products of Definition 6.1 over rational coefficients:*

1. *$y^{v_1} \star_{V_\infty^+} y^{v_2} = 0$ if and only if $B_{\mathbf{\Sigma}}^+(v_1, v_2)$ is nonempty, and*

2. *$y^{v_1} \star_{V_\infty^+} y^{v_2} = y^{v_1} \star_{V^+} y^{v_2}$ otherwise.*

We also have the following identities for the $\star_{V_\infty^-}$ product:

1. *$y^{v_1} \star_{V_\infty^-} y^{v_2} = 0$ if and only if $B_{\mathbf{\Sigma}}^-(v_1, v_2)$ is nonempty, and*

2. *$y^{v_1} \star_{V_\infty^-} y^{v_2} = y^{v_1} \star_{V^-} y^{v_2}$ otherwise.*

Proof. By Lemma 2.7, $c_1(\mathcal{L}_i)$ is nilpotent for each i (after we tensor with $\mathbb{Q}$). Then for sufficiently large a, the high powers of the Chern and Euler classes vanish. As a result, box relations ideal of Proposition 5.18 for the V^+ product becomes

$$\begin{aligned}\mathrm{BR}(\star_{V_\infty^+}, \mathbf{\Sigma}) &= \left\langle w_1 w_2 - w_3 \cdot \prod_{q_{1,i}+q_{2,i}\geqslant 1} \mathrm{eu}(f_i(\mathcal{L}_i))^{1+a} \,\middle|\, w_1, w_2 \in \mathrm{Box}(\mathbf{\Sigma}) \right\rangle \\ &= \left\langle \{ w_1 w_2 |\, B^+_{\mathbf{\Sigma}}(v_1, v_2) \neq \emptyset \} \bigcup \{ w_1 w_2 - w_3 |\, B^+_{\mathbf{\Sigma}}(v_1, v_2) = \emptyset \} \right\rangle .\end{aligned}$$

The result for the V_∞^- product holds in a similar fashion. ∎

Example 6.4. Consider the stacky fan $\mathbf{\Sigma}$ of Example 5.21. We'll compute the Chow ring of $\mathcal{X}(\mathbf{\Sigma})$ under asymptotic-plus product associated to $V = a_1\mathcal{L}_1 + a_2\mathcal{L}_2 + a_3\mathcal{L}_3$. Note that the calculation for the inertial Chow ring $A^*(\mathcal{X}(\mathbf{\Sigma}), \star_{V_\infty^+}, \mathbb{Q})$ will be the same as with $A^*(\mathcal{X}(\mathbf{\Sigma}), \star_{V^+}, \mathbb{Q})$, except that all relations with a_i as exponents will vanish by Theorem 6.3. So, we have

$$A^*(\mathcal{X}(\mathbf{\Sigma}), \star_{V_\infty^+}, \mathbb{Q}) \cong \frac{\mathbb{Q}[t, w_1, w_2, w_5, w_8, w_9, w_{10}, w_{11}]}{\left\langle \begin{array}{c} t^3, w_1w_8, w_1w_9, w_1w_{10}, w_1w_{11}, w_2w_5, w_2w_8, w_2w_9, w_2w_{10}, w_2w_{11}, \\ w_5w_8, w_5w_9, w_5w_{10}, w_5w_{11}, {w_1}^2, {w_2}^2, {w_5}^3, {w_8}^2, w_8w_9, w_8w_{10}, \\ w_8w_{11}, {w_9}^2, w_9w_{10}, {w_{10}}^2, w_9w_{11}, {w_{11}}^2, w_{10}w_{11} \end{array} \right\rangle}$$

Remark 6.5. In Example 6.4, we saw that $I(\mathbf{\Sigma}) = \langle 120t^3 \rangle$ was reduced to $\langle t^3 \rangle$, a result available with $\mathbb{Q}$-coefficients but not $\mathbb{Z}$. This is a nice manifestation of the vital Lemma 2.7. With coefficients in $\mathbb{Z}$, powers such as $(5t)^a$ would never vanish under the ideal $\langle 120t^3 \rangle$, even as $a \to \infty$. With coefficients in $\mathbb{Q}$ we have nilpotency, as it vanishes for $a \geqslant 3$ (and thus vanishes in the asymptotic case).

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