

**A Foliated Seiberg-Witten Theory**

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by  
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The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

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presented by Andrew Renner, a candidate for the degree of Doctor of Philosophy of Mathematics, and hereby certify that in their opinion it is worthy of acceptance.

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# A Foliated Seiberg-Witten Theory

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## ABSTRACT

This thesis set out to investigate a generalization of Seiberg-Witten theory from four-dimensional manifolds to four-codimensional Riemannian foliations. The introduction of Sieberg-Witten theory has proved to have great utility for studying four-dimensional manifolds, a generalization to four-codimensional foliations could prove just as fruitful to the study of the transverse space, a (generally) non-Hausdorff topological space, associated to such a foliation. We begin by reviewing the fundamental theorems and definitions necessary for studying the transverse space of foliations, in particular Riemannian foliations. To then prepare for the Seiberg-Witten theory, the definitions and theory of Spin geometry and how they fit into the context of Riemannian foliations is discussed. And finally a transverse version of the Wittenboöck-Bochner-Lichnerowicz identities are established for  $\text{Spin}$  and  $\text{Spin}^c$  structures associated to a Riemannian foliation. Since subsequent analysis of the Sieberg-Witten equations require the use of Sobolev theory, we review a theory sufficient for a comparable transverse Sobolev theory for foliations. We then state the transverse Seiberg-Witten equations and determine the necessary bounds on the solutions to those equations. Subsequently, we investigate the analysis of the moduli space of solutions, in particular, the sequential compactness of the moduli space.

# Chapter 1

## Introduction

The study of foliations has its origins in the qualitative study of differential equations, often the study of the streamlines of a differential equation is enough to yield some insight into the global behavior of the system even without an explicit solution. These streamlines partition the space into one dimensional manifolds and it is this partition that is abstracted into a foliation, which is a partition of a manifold into equi-dimensional connected submanifolds called the leaves of the foliation. The study of foliations falls into largely two parts, one can study the leaf geometry or one can study the transverse geometry. The leaf geometry consists of studying the individual submanifolds and how they lie within the manifold whereas the transverse geometry is concerned with the quotient topology on the partition. To study the transverse geometry one studies a canonical vector bundle over the manifold called the transverse bundle that acts as proxy tangent space for the often topologically complicated quotient space. While studying the transverse bundle can be useful, often extra structure is required. Reinhart [Rei59] initiated the study of Riemannian foliations, which places a particular type of Riemannian metric on the transverse bundle which provides enough structure so that many of the theorems from Riemannian geometry have analogs for the Riemannian foliation. In particular, every Riemannian foliation of a

closed manifold (compact manifold without boundary) has a well-defined DeRham cohomology (called the basic cohomology) associated to the foliation, a well-defined transverse Laplacian, a corresponding Hodge-DeRham decomposition and a canonical Riemannian connection on the transverse bundle. Of course, classically any manifold has a well defined finite dimensional DeRham cohomology that is a topological invariant of the foliation, however this is not the case for an arbitrary foliation, the basic cohomology can fail on both counts. El Kacimi and Nicolau [EN93] were able to show that in the case of a Riemannian foliation of a closed manifold, the basic cohomology is a topological invariant (they also provide a non-Riemannian counterexample). Prior to that, Kamber and Tondeur, under the assumption that the mean curvature is basic, developed Hodge-DeRham theory for Riemannian foliations of compact oriented Riemannian manifolds [KT88] and El Kacimi and Hector also established the same decomposition [EH84; ESH85] but also established the finite dimensionality of the basic cohomology [EH86]. Park and Richardson [PR96] and separately, Alvarez [Alv92] showed that the assumption of basic mean curvature in the work of Kamber and Tondeur can be done using only the basic projection of the mean curvature instead. Finally, Domínguez [Dom95; Dom98] was able to show that without loss of generality, by modifying the metric in the leaf directions, one may assume that the mean curvature itself is basic. Furthermore, since the transverse bundle has a Riemannian structure, one can naturally investigate a corresponding Clifford bundle and in some cases a transverse spin geometry. Habib and Richardson [HR09] make a distinction between a transverse Dirac operator and a basic Dirac operator by twisting the non-self-adjoint transverse Dirac operator by the basic component of the mean

curvature to obtain the basic Dirac operator which is self-adjoint, they further prove the invariance of the basic Dirac operator for the Riemannian foliation. They further use this theory to study a twisted cohomological DeRham theory [HR13]. Prior to that Bruning, Kamber and Richardson [BKR11] developed an index theory for basic Dirac-type operator on Riemannian foliations.

In the early eighties, Donaldson established a new method for studying four-dimensional manifolds by using anti-self-dual connections to describe invariants, his work helped discover many homeomorphic yet not diffeomorphic manifolds. Later in the mid-nineties, Seiberg and Witten developed a theory that is substantially simpler and replicated many of Donaldson's results [SW94; Wit94]. In particular, this work helped established the existence of manifolds homeomorphic to  $\mathbb{R}^4$  but not diffeomorphic to the standard  $\mathbb{R}^4$ . Not only was the existence of these exotic  $\mathbb{R}^4$  established, but it was further shown that there is a continuum of such differentiable structures, which makes  $\mathbb{R}^4$  particularly exceptional since all other  $\mathbb{R}^n$  do have a unique differentiable structure. Arnold [Arn93] observed that for any exotic  $\mathbb{R}^4$ ,  $\mathbb{R}^4 \times \mathbb{R}$  must be both homeomorphic and diffeomorphic to  $\mathbb{R}^5$  since  $\mathbb{R}^5$  has a unique differentiable structure. Consequently, there is a 1-dimensional foliation on  $\mathbb{R}^5$  that has an exotic  $\mathbb{R}^4$  as it's transverse space. That is, every exotic  $\mathbb{R}^4$  can be realized as the transverse space of a foliation. Fan [Fan98] generalized this to consider a 5-dimensional manifold with a particular class of one-dimensional foliation and proceeds to develop a Seiberg-Witten theory for such spaces [Fan95]. He does not make the assumption that the foliation is Riemannian, rather he assumes what he calls self-duality of the foliation, which is weaker condition on the transverse metric. Since there is no trans-

verse Riemannian structure, his work relies on a particular representation of  $\text{Spin}(5)$  and is thus somewhat unnatural for the transverse space. Generalizing even further, it is natural to investigate Seiberg-Witten theory on higher dimensional manifolds foliated by a 4-codimensional Riemannian foliation. A foliated theory of this type is not a new idea, Wang [Wan15] has investigated a foliated Donaldson theory. In particular, there is a widely assumed theory for orbifolds (also called Satake manifolds or V-manifolds) which can often be realized as a particular class of foliations. Most recently, in the midst of work on this dissertation, Kordyukov, Lejmi and Weber [KLW16] published a general theory for Riemannian foliations of codimension four using Molino's structure theory for Riemannian foliations.

The main approach of this thesis is to define the basic Seiberg-Witten equations and, after assuming transverse theories for Sobolev embeddings, Rellich-Kondrakov compactness, and elliptic regularity, then to show that the moduli space is sequentially compact. Consequently, this generalizes the classical Seiberg-Witten theory to the more general case of 4-codimensional foliations, with the classical case being a 4-dimensional manifold with a zero dimensional foliation.

Chapter 2 recounts the main definitions of smooth foliations, foliated vector bundles, Riemannian foliations and their Hodge-DeRham theory and presents the theorems analogous to those for Riemannian manifolds, those analogous for vector bundles over Riemannian manifolds, the properties of the canonical connection and in particular, the mean curvature of a foliation of a Riemannian manifold.

In chapter 3 we discuss the definition of a foliated principal bundle and their relation to foliated vector bundles by way of the associated bundle construction. From

there we adapt many the definitions and theorems from spin geometry necessary for the notion of a transverse version for Riemannian foliations and thus the notion of transverse Clifford bundles, transverse  $\text{Spin}$  structures, transverse  $\text{Spin}^{\mathbb{C}}$  structures, transverse spinor bundles, transverse Dirac bundles, transverse Dirac operators are developed. The first results are the establishment of transverse Bochner, Lichnerowicz and Weitzenböck identities for a general foliated transverse Dirac bundle, in particular for the transverse bundle and transverse spinor bundles, giving us a transverse theory analogous to the classical theory.

Chapter 4 contains all the necessary analytic hypotheses necessary to prove sequential compactness of the moduli space of the Seiberg-Witten equations defined in the chapter 5. While the classical Sobolev theory was sufficient to establish DeRahm-Hodge theory in [KT88], the classic theory does not seem to be enough to prove sequential compactness since the proof uses Sobolev multiplication which is more sensitive to the grade of the Sobolev space, and will require a ‘basic Sobolev grade’ that uses the codimension of the foliation in the grade rather than the dimension of the ambient manifold as in the classical case. Similarly, we assume basic versions of Rellich-Kondrakov compactness and elliptic regularity for transversely elliptic operators.

In chapter 5, the basic Seiberg-Witten equations are defined, the necessary bounds on basic sections of the transverse spinor bundle and basic unitary connections are established which are independant of the assumptions made in the chapter 4. The Seiberg-Witten equations are reformulated as a functional, and with the analytic hypotheses from chapter 4 the gauge transformations are shown to act properly and

equivariantly on the domain and thus induces a function on the Hausdorff orbit space. Two things that differentiate Seiberg-Witten theory from Donaldson's theory is that the gauge transformations are abelian and the moduli space (solutions modulo the transformations) is always sequentially compact. For a foliated Seiberg-Witten theory the gauge transformations are still abelian, however to show sequential compactness, it is necessary to assume that a transverse version of Uhlenbeck's theorem regarding the bounds on the self-dual part of a  $U(1)$ -connection, which will establish, the following theorem 5.27.

**Theorem.** *Let  $M$  be a closed Riemannian manifold and  $\mathcal{F}$  be a 4-codimensional bundle-like oriented Riemannian foliation on  $M$  with mean curvature  $\tau$ . Assume that  $\mathcal{F}$  is a  $\text{Spin}^{\mathbb{C}}$  foliation. Then for any transverse  $\text{Spin}^{\mathbb{C}}$  structure with associated Spinor bundle  $S^{\mathbb{C}}$ , the space of solutions  $A \oplus \phi \in \mathcal{A}_B \oplus \Gamma_B(S^{\mathbb{C}}_+)$  (modulo gauge transformations) to*

$$F_+^A = \phi \otimes \phi^* - \frac{1}{2} |\phi|^2 \text{id}_{S^{\mathbb{C}}_+}$$

$$\nabla^A \phi = \frac{1}{2} \tau \cdot \phi$$

where  $\mathcal{A}_B$  are the basic connections on  $\Lambda^{\dim S^{\mathbb{C}}} (S^{\mathbb{C}})$  and  $\Gamma_B(S^{\mathbb{C}}_+)$  are the basic sections of  $\Gamma(S^{\mathbb{C}}_+)$ , is a sequentially compact space.

Lastly, we consider the elliptic complex given by the isotropy representation and the linearization of the Seiberg-Witten equations at a solution, and show that the index of such a complex can be calculated by using the theory developed in [BKR11]. Further developments of Seiberg-Witten theory, such as the moduli space being a generic finite dimensional manifold, the orientability of the moduli space and the definition of the Seiberg-Witten invariant are not pursued.

# Chapter 2

## Foliations

A foliation of a smooth manifold can be thought of as the same set with a weaker topology and, in general, some loss of dimension. For example, the plane  $\mathbb{R}^2$  is a two-dimensional manifold with one component, and this manifold can be partitioned into the disjoint union  $\mathcal{F}$  of vertical lines and this partition can be thought of as a one-dimensional manifold with a continuum of components. It is this disjoint union that is a foliation of  $M$ . Each vertical line can be thought of in two ways, from the perspective of  $\mathcal{F}$ , a vertical line is a connected component, and from the perspective of  $\mathbb{R}^2$  a vertical line is a submanifold.

**Definition 2.1** (Foliation). Let  $m = n+l$ . An atlas  $\mathcal{A}$  for an  $m$ -dimensional manifold  $M$  is an  $l$ -dimensional ( $n$ -codimensional) foliated atlas, if for any charts  $\chi', \chi \in \mathcal{A}$  the transition function has the form

$$\chi' \circ \chi^{-1}(x, y) = (u(x), v(x, y)) \tag{2.1}$$

for some continuous functions  $u : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $v : \mathbb{R}^m \rightarrow \mathbb{R}^l$ . A smooth foliation will require the foliated atlas to be a smooth atlas and thus the Jacobian of the transition functions must then satisfy

$$D(\chi' \circ \chi^{-1}) = \begin{bmatrix} \frac{\partial u}{\partial x} & 0 \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

at each point  $(x, y) \in \text{dom}(\chi' \circ \chi^{-1}) \subset \mathbb{R}^n \times \mathbb{R}^l$ . A foliation is then a maximal foliated atlas.

Locally, for each chart  $\chi$  of a smooth foliation we have the natural sequence of maps  $\text{dom}(\chi) \rightarrow \mathbb{R}^m \rightarrow \mathbb{R}^n$  and the preimage of each  $x \in \mathbb{R}^n$  partitions  $\text{dom} \chi$  into an  $l$ -dimensional submanifold of  $M$ , the connected components of these preimages are called the **plaques of the chart**  $\chi$ . The transition functions between any two charts preserve plaques and thus the plaques are independent of the chart, moreover the plaques for two overlapping charts will agree on the overlap but, in general, each plaque will extend outside of the overlapping domain. Consequently, the union of any two overlapping plaques is also a  $l$ -dimensional connected submanifold, each maximal such submanifold is a connected submanifold of  $M$  and is called a **leaf of the foliation**. Consequently, a foliated atlas determines a partition  $\mathcal{F}$  of the manifold  $M$  by connected  $l$ -dimensional submanifolds, this partition is also called a **foliation** of  $M$ . The foliation  $\mathcal{F}$  can be thought of as a topological space in two ways: with the quotient topology it is called the **transverse space** (or **leaf space**) and denoted  $M/\mathcal{F}$ , the other way is as the disjoint union of leaves which is called the **foliation manifold**  $\mathcal{F}$ . In the smooth case, each foliation chart generates an  $l$ -dimensional smooth distribution  $L$  of  $TM$  called the **foliation bundle**, such a distribution is involutive and thus determines an integrable subbundle; conversely by the Frobenius-Clebsch theorem an integral subbundle determines a foliation of  $M$ . Consequently, in the smooth case, there are three equivalent ways of viewing foliations: as determined

by a maximal foliated smooth atlas, as a partition of a manifold into connected smooth submanifolds, and as an integrable distribution. See [Mol88] for further details.

Let us now consider a few examples of  $l$ -dimensional (and  $n$ -codimensional) foliations for an  $m$ -dimensional manifold  $M$ . The **trivial foliation** (or **component foliation**) has  $m$ -dimensional leaves. By connectedness, the leaves of a trivial foliation are the connected components of  $M$  and consequently the foliation manifold  $\mathcal{F}$  is homeomorphic to the manifold  $M$  and the transverse space  $M/\mathcal{F}$  is homeomorphic to the discrete space that is in one-to-one correspondence with the connected components of  $M$ . At the other extreme is the **discrete foliation** (or **unfoliated manifold**) which has 0-dimensional leaves. By connectedness, the leaves of a discrete foliation are individual points of  $M$  and consequently the foliation manifold  $\mathcal{F}$  is homeomorphic to  $M$  with the discrete topology and the transverse space  $M/\mathcal{F}$  is homeomorphic to  $M$ . A slightly more interesting example is a **simple foliation** which is defined by a surjective submersion of an  $m$ -dimensional manifold onto an  $n$ -dimensional manifold,  $f : M \rightarrow N$ , with connected level sets. In this case, the leaves are the  $m - n$  dimensional level sets, and the transverse space  $M/\mathcal{F}$  is the same as  $N$ , in particular it is Hausdorff. More generally, a surjective submersion may not have connected level sets (in fact, by definition 2.1 this is the local picture of all foliations), and thus the transverse space may not be the same as  $N$ , in particular it may not be Hausdorff. Another classical example arises from an action of a Lie group on a manifold, the leaves are the connected components of the orbits, and again, the transverse space may not be Hausdorff. As a particular example of this, consider a non-vanishing smooth vector field on the torus, the leaves are the integral curves of

the flow; but if the flow is rational the orbits are periodic, consequently the leaves are closed and thus compact, however, if the flow is irrational then the orbits are dense in any open set.

Since we will be studying the transverse space of a smooth foliation, throughout we fix the notation  $M$  as an  $m = n + l$  dimensional smooth manifold foliated by an  $n$ -codimensional smooth foliation  $\mathcal{F}$  with an  $l$ -dimensional foliation bundle  $L$ . Moreover, in definition 2.1 we have adopted a ‘vertical leaves’ convention which is more amenable to studying the transverse space since the first  $n$  coordinates are the transverse coordinates, it is also common in the literature to see a ‘horizontal leaves’ convention.

## 2.1 Transverse Geometry

The continuous functions on the transverse space  $M/\mathcal{F}$  of a foliation are induced by the continuous functions on  $M$  which are constant on each leaf, such functions are called **basic functions**. In particular, the smooth basic functions,  $\Omega_B^0(M)$ , will serve as proxy for differentiable functions on the transverse space  $M/\mathcal{F}$ .

To determine the proxy for the tangent vector fields on the transverse space  $M/\mathcal{F}$ , we consider  $\Gamma(L)$  which is in involution, thus forms a Lie subalgebra of  $\Gamma(M)$ . We define  $\Gamma_{\mathcal{F}}(M)$  to be the Lie algebra idealizer of  $\Gamma(L)$  in  $\Gamma(M)$ , that is those  $V \in \Gamma(M)$  such that  $[V, X] \in \Gamma(L)$  for any  $X \in \Gamma(L)$ . In the literature one encounters a multitude of names for  $\Gamma_{\mathcal{F}}(M)$ : projectable [Ton97], foliate [Mol88], basic, base-like, foliated, etc, contrary to Molino we will reserve **foliate vector fields** for  $\Gamma(L)$  and following Tondeur we will use **projectable vector fields** for  $\Gamma_{\mathcal{F}}(M)$ . In particular, the projectable vector fields have flows that preserve leaves, moreover they locally

have the form  $V = \sum_{k=1}^n V^k(x) \frac{\partial}{\partial x^k} + \sum_{i=n+1}^m V^i(x, y) \frac{\partial}{\partial x^i}$ . Furthermore, there is the point-wise projection

$$\pi : TM \rightarrow TM/L \tag{2.2}$$

which defines a second bundle  $Q = TM/L$  called the **transverse bundle** which will serve as a proxy tangent bundle for the transverse space  $M/\mathcal{F}$ . The **transverse vector fields**  $\Gamma(Q)$  locally have the form  $V = \sum_{i=1}^n V^k(x, y) \frac{\partial}{\partial x^k}$  and thus have a general dependence on the plaque coordinate  $y$  and so will not be the proxy tangent vector fields for the transverse space. Instead, we define the **basic vector fields**  $\Gamma_B(Q)$  to be the subspace of  $\Gamma(Q)$  that is identified with  $\Gamma_{\mathcal{F}}(M)/\Gamma(L)$ . That identification makes  $\Gamma_B(Q)$  a Lie algebra and locally they have the form  $V = \sum_{i=1}^n V^k(x) \frac{\partial}{\partial x^k}$  which is independent of the plaque coordinate, thus these basic vector fields will be the proxy tangent vector fields for the transverse space. In fact,  $\Gamma_B(Q)$  is a module over the basic smooth functions  $\Omega_B^0(M)$  and naturally acts as derivations on  $\Omega_B^0(M)$ . The notion of a basic vector field can be reformulated by considering the transverse bundle  $Q$  restricted to each leaf.

**Theorem 2.2** (The Transverse Bott Connection). *Let  $Q$  be the transverse bundle,  $\mathcal{L}$  be the Lie derivative on  $M$ , and  $\pi : TM \rightarrow Q$ . Then the function  $\pi \circ \mathcal{L} : \Gamma(L) \times \Gamma(M) \rightarrow \Gamma(Q)$  descends to a function  $\overset{\circ}{\nabla} : \Gamma(L) \times \Gamma(Q) \rightarrow \Gamma(Q)$  which is smooth linear in  $\Gamma(L)$  and  $\Omega_B^0(M)$ -module derivation of  $\Gamma(Q)$ . That is,  $\overset{\circ}{\nabla}$  is a connection on  $Q$  restricted to each leaf, called the transverse Bott connection.*

*Proof.* Let  $X \in \Gamma(L)$  and  $V, V' \in \Gamma(M)$  such that  $\pi(V) = \pi(V')$ . Then  $V - V' \in \Gamma(L)$  and by involution, we have  $\mathcal{L}_X(V - V') \in \Gamma(L)$  and thus  $\pi \mathcal{L}_X V = \pi \mathcal{L}_X V'$ .

Consequently,  $\overset{\circ}{\nabla}$  is well defined. Furthermore, let  $f$  be a smooth function then  $\pi \mathcal{L}_{f \cdot X} V = -\pi \mathcal{L}_V(f \cdot X) = -\pi(\mathcal{L}_V f \cdot X - f \cdot \mathcal{L}_V X) = f\pi(\mathcal{L}_X V)$  because  $\pi(X) = 0$ . The remaining linearity and derivation properties of are inherited from the Lie derivative. ■

**Theorem 2.3.** *Let  $Q$  be the transverse bundle of a foliation on a manifold  $M$  and  $\overset{\circ}{\nabla}$  be the transverse Bott connection. Then the transverse Bott connection is flat.*

*Proof.* Let  $X, Y \in \Gamma(L)$  and  $V \in \Gamma(Q)$  then

$$\begin{aligned} & \overset{\circ}{\nabla}_X(\overset{\circ}{\nabla}_Y \pi V) - \overset{\circ}{\nabla}_Y(\overset{\circ}{\nabla}_X \pi V) - \overset{\circ}{\nabla}_{[X, Y]} \pi V \\ &= \overset{\circ}{\nabla}_X(\pi \mathcal{L}_Y V) - \overset{\circ}{\nabla}_Y(\pi \mathcal{L}_X V) - \pi \mathcal{L}_{[X, Y]} V \\ &= \pi \mathcal{L}_X \mathcal{L}_Y V - \pi \mathcal{L}_Y \mathcal{L}_X V - \pi \mathcal{L}_{[X, Y]} V \end{aligned}$$

which vanishes by the Jacobi Identity. Therefore,  $\overset{\circ}{\nabla}$  is flat. ■

Consequently, the basic vector fields are the transverse vector fields that are parallel with respect to this connection, it is this fact that is abstracted to determine basic sections of an arbitrary vector bundle over a foliated manifold. A **partial connection** on a vector bundle  $E$  over a foliated manifold is function  $\Gamma(L) \times \Gamma(E) \rightarrow \Gamma(E)$  that satisfies the usual properties for a connection. A flat partial connection will be called a **Bott connection** and because it is flat over each leaf, it defines an  $l$ -dimensional foliation of the vector bundle  $E$  called the **lifted foliation**. Consequently, if a Bott connection has been fixed for a vector bundle over a foliated manifold the bundle is said to be a **foliated vector bundle**.

Of course the standard pointwise constructions can be used to construct tensor product, direct sum, and Hom bundles. Furthermore, if each vector bundle has a Bott connection then the constructed vector bundle has an induced Bott connection, that is, those constructions with foliated vector bundles yields a foliated vector bundle. In particular, by the previous two theorems the transverse bundle  $Q$  is canonically a foliated vector bundle.

**Example 2.4** (Basic  $k$ -Forms). For any  $\omega \in \Omega^k(M)$  then  $\omega \in \Gamma(\Lambda^k Q^*)$  if and only if  $\iota_X \omega = 0$  for all  $X \in \Gamma(L)$ . Consequently, any such  $k$ -form will be called a **transverse  $k$ -form** and they locally have the form (using multi-index notation)  $\omega = \sum_{|\alpha|=k} \omega_\alpha(x, y) dx^\alpha$ , and thus have a general dependence on the plaque coordinate  $y$  and so will not act as proxy  $k$ -forms on the transverse space. The basic  $k$ -forms are then the transverse forms that vanish for the induced Bott connection, fortunately the induced Bott connection has a particularly simple form for transverse forms.

**Theorem 2.5.** *For any  $\omega \in \Omega^k(M)$  then  $\omega \in \Gamma_B(\Lambda^k Q^*)$  if and only if  $\iota_X \omega = 0$  and  $\mathcal{L}_X \omega = 0$  for all  $X \in \Gamma(L)$*

*Proof.* Let  $W \in \Gamma(\Lambda^k Q)$ ,  $\omega \in \Omega^k_B(M)$  and  $X \in \Gamma(L)$  then using the  $\overset{\circ}{\nabla}$  induced on  $\Lambda^k Q^*$  we have

$$\begin{aligned}
(\overset{\circ}{\nabla}_X \omega)W &= \mathcal{L}_X(\omega(W)) - \omega(\overset{\circ}{\nabla}_X W) \\
&= (\mathcal{L}_X \omega)(W) + \omega(\mathcal{L}_X W) - \omega(\overset{\circ}{\nabla}_X W) \\
&= (\mathcal{L}_X \omega)(W) + \omega(\mathcal{L}_X W - \overset{\circ}{\nabla}_X W) \\
&= (\mathcal{L}_X \omega)(W)
\end{aligned}$$

since  $\mathcal{L}_X W - \overset{\circ}{\nabla}_X W \in \Gamma(\Lambda^k L)$ . Consequently,  $\overset{\circ}{\nabla}_X \omega = \mathcal{L}_X \omega$  for  $X \in \Gamma(L)$ . ■

The basic  $k$ -forms are the proxy  $k$ -forms for the transverse space, and they locally have the form (using multi-index notation)  $\omega = \sum_{|\alpha|=k} \omega_\alpha(x) dx^\alpha$ .

**Theorem 2.6** (Basic DeRahm Complex). *Let  $\Omega_B^k(M)$  be the basic  $k$ -forms. Then  $\Omega_B^k(M)$  is a subcomplex of the DeRham complex  $\Omega^k(M)$ .*

*Proof.* Clearly for any basic  $k$ -form  $\omega$  we know  $d^2\omega = 0$ , thus it is sufficient to show that  $d\omega$  is basic. Cartan's formula  $\iota_X d\omega = \mathcal{L}_X \omega - d\iota_X \omega$  holds for any differential form  $\omega \in \Omega^k(M)$  and any vector field  $X \in \Gamma(M)$ . In particular, if  $X \in \Gamma(L)$ , applying Cartan's formula to  $d\omega$  yields  $0 = \mathcal{L}_X d\omega - d\iota_X d\omega$  thus, by example 2.4, it is sufficient to show  $\iota_X d\omega = 0$  when  $\omega$  is basic; which directly follows from Cartan's formula and example 2.4. ■

The cohomology of the above subcomplex is called the **basic DeRham cohomology** of the foliation. In general these cohomology groups may be infinite dimensional and may not be topological invariants of the foliation, however Riemannian foliations (to be discussed next) of a closed manifold always have finite dimensional basic cohomology [EH84; ESH85; EH86] and is a topological invariant [EN93].

Geometric structures defined by tensors on vector bundles have analogous definitions on foliated vector bundles, with the added requirement that the defining tensor be a basic tensor.

**Example 2.7** (Basic and Transverse Riemannian Structures). Recall that a real vector bundle  $E$  is said to be Riemannian if there exists a positive-definite form  $g^E \in \Gamma(E \odot E)$ , called a **Riemannian metric**. A foliated real vector bundle  $E$  is said to be **basically Riemannian** if there exists a *basic* Riemannian metric on  $E$ . In particular, if the transverse bundle  $Q$  is basically Riemannian, the foliation is said

**transversally Riemannian** or is called a **Riemannian foliation**. More generally, a complex vector bundle  $E$  is said to be Hermitian if there exists a positive-definite Hermitian form  $g^E$ , called a **Hermitian metric**. A foliated complex vector bundle  $E$  is said to be **basically Hermitian** if there exists a *basic* Hermitian metric on  $E$ .

**Example 2.8** (Transversely Oriented). Recall that a vector bundle  $E$  is said to be orientable if there exists a non-vanishing section of  $\Lambda^{\dim(E)}E$  and that such a choice is called an orientation of  $E$ . A foliated vector bundle  $E$  is said to be **basically orientable** if there exists a *basic* orientation on  $E$ . Recall, that if  $E$  is an orientable Riemannian vector bundle with Riemannian metric  $g^E$ , then the metric maps the orientation to a particular  $\mu^E \in \Gamma(\Lambda^{\dim(E)}E^*)$ , called the Riemannian volume form. In the case of an oriented Riemannian vector bundle, the Riemannian volume form will always be the assumed volume form. Furthermore, if  $E$  is both basically Riemannian and basically orientable then the Riemannian volume form is a basic  $n$ -form.

Let  $M$  be a Riemannian manifold with Riemannian metric  $g^M$ , if  $M$  is foliated then  $TM = L \oplus N$ , where the bundle  $N$  is orthonormal to  $L$  and the splitting induces metrics  $g^L$  and  $g^N$  on  $L$  and  $N$  respectively. The **characteristic form** of the foliation is defined as the determinant of  $g^L$ . Moreover, the transverse bundle  $Q$  can be identified with  $N$  and is thus a Riemannian vector bundle, but it may not be transversally Riemannian, that is,  $g^N$  may not be basic. Even if the foliation is Riemannian with basic metric  $g^Q$ , it is quite possible that  $g^Q \neq g^N$ . However, in the case that  $g^N = g^Q$  the foliation is said to be a **bundle-like Riemannian foliation**. To further ease terminology, if both the manifold and transverse bundle are orientable for a bundle-like Riemannian foliation then it will be called a **bundle-like oriented**

**Riemannian foliation.** In this case since  $TM = L \oplus Q$  and both  $TM$  and  $Q$  are oriented then  $L$  is oriented, and the characteristic form  $\chi$  is the Riemannian volume form for the leaves, consequently  $\mu^M = \mu^Q \wedge \chi$ . Notice that the transverse space of a Riemannian discrete foliation is the same as the manifold itself being Riemannian, consequently we will use this fact to compare a bundle-like Riemannian foliation to the underlying Riemannian manifold, by reserving, in all of our transverse definitions,  $M$  in place of  $Q$  for the discrete foliation.

**Definition 2.9** (Transverse Hodge Star). Let  $Q$  be the transverse bundle of a transversally oriented Riemannian foliation. Define the transverse Hodge star  $\star : \Lambda^k(Q^*) \rightarrow \Lambda^{n-k}(Q^*)$  to be the unique  $n - k$  form so that  $\eta \wedge \star \omega = g^Q(\eta, \omega) \mu^Q$  for all  $\eta \in \Lambda^k(Q^*)$ .

In the case of an bundle-like oriented Riemannian foliation we will use  $\star$  for the transverse Hodge star and  $*$  for the unfoliated Hodge star, furthermore we will use  $\bar{\star}$  is the inverse transverse Hodge star given by  $(-1)^{k(n-k)} \star$  and  $\bar{*}$  is the inverse Hodge star given by  $(-1)^{k(m-k)} *$ .

**Proposition 2.10** (Bundle-like Hodge Star). *Let  $Q$  be the transverse bundle of a bundle-like oriented Riemannian foliation. Then*

$$\star \alpha = (-1)^{l(n-k)} * (\alpha \wedge \chi) \qquad * \alpha = (\star \alpha) \wedge \chi$$

## 2.2 The Transverse Riemannian Connection

A foliated vector bundle has a Bott connection that determines which vector fields are basic fields, however to fully study the basic fields we require a (full) connection that is compatible with the Bott connection.

**Definition 2.11** (Adapted Connection). Let  $\overset{\circ}{\nabla}$  be the Bott connection for a foliated vector bundle  $E$ . Then a connection  $\nabla$  on  $E$  is **adapted** if  $\nabla_X = \overset{\circ}{\nabla}_X$  for any  $X \in \Gamma(L)$ .

Adapted connections will still annihilate basic sections in the leaf-wise directions, furthermore the existence of adapted connections is always assured [Mol88; Ton97]. Furthermore, the modeling of the local geometry of an unfoliated manifold by the tangent bundle is expressed through the torsion of a connection, we have the analogous tensor for the transverse bundle.

**Definition 2.12** (Transverse Torsion Tensor). Let  $\nabla$  be an adapted connection on the transverse bundle  $Q$ , the transverse torsion  $T^\nabla \in \Omega^2(\text{End } Q)$  is defined by

$$T^Q(U, V) = \nabla_U \pi V - \nabla_V \pi U - \pi[U, V]$$

for all  $U, V \in \Gamma(M)$ . The connection is said to be torsionless (or symmetric) if  $T^\nabla(U, V) = 0$  for all  $U, V \in \Gamma(M)$ .

As is the case for a Riemannian manifold, a Riemannian foliation has a canonical adapted connection.

**Proposition 2.13** (The Fundamental Theorem of Riemannian Foliations). *Let  $Q$  be the transverse bundle over a Riemannian foliation. Then there exists a unique torsionless metric adapted connection  $\nabla^Q$ . This connection is called the canonical transverse Riemannian connection or the transverse Levi-Civita connection.*

*Proof.* See [Ton97] Theorem 5.9 ■

We will only be interested in Riemannian foliations, consequently we will always assume that our transverse connection  $\nabla^Q$  is the canonical transverse Riemannian

connection, and it should be noted that the canonical transverse Riemannian connection for a discrete foliation is the the classical Levi-Civita connection for the underlying Riemannian foliation. The relation between transverse connectons for Riemannian foliation and Riemannian manifolds can be characterized by the following theorem.

**Proposition 2.14** (Riemannian Foliations of Riemannian Manifolds). *Let  $Q$  be the transverse bundle over a Riemannian foliation of a Riemannian manifold  $M$ . The induced transverse metric  $g^N$  from the Riemannian metric  $g^M$  is a basic metric if and only if the transverse connection defined by*

$$\nabla_X^Q V = \begin{cases} \overset{\circ}{\nabla} V & \text{for } X \in \Gamma(L) \\ \pi \nabla_X^M V & \text{for } X \in \Gamma(N) \end{cases} \quad (2.3)$$

*is a metric connection.*

*Proof.* See [Ton97] Theorem 5.8 ■

For an adapted connection to act as a proxy connection over the transverse space, the covariant derivative must map basic vector fields to basic vector fields when restricted to projectable vector fields.

**Definition 2.15** (Projectable and Basic Connections). Let  $\nabla$  be an adapted connection on a foliated vector bundle  $E$ . Then  $\nabla$  is said to be a **projectable connection** if the restriction  $\nabla : \Gamma_{\mathcal{F}}(M) \times \Gamma_B(E) \rightarrow \Gamma_B(E)$  exists. Furthermore,  $\nabla$  is said to be a **basic connection** if the curvature  $R^\nabla$  of  $\nabla$  is transverse, that is  $\iota_X R^\nabla = 0$ .

**Theorem 2.16.** *Let  $\nabla$  be an adapted connection on a foliated vector bundle  $E$  over a manifold  $M$  foliated by  $\mathcal{F}$ . Then if  $\nabla$  is basic it is projectable.*

*Proof.* Let  $X \in \Gamma(L)$ ,  $Y \in \Gamma_{\mathcal{F}}(M)$  and  $V \in \Gamma_B(E)$ . Let  $R^\nabla$  be the curvature of the basic connection  $\nabla$  then

$$0 = R^\nabla(X, Y)[V] = \nabla_X(\nabla_Y V) - \nabla_Y(\nabla_X V) - \nabla_{[X, Y]}V$$

and from the hypotheses  $\nabla_X V = 0$  and  $\nabla_{[X, Y]}V = 0$  thus  $\nabla_X(\nabla_Y V) = 0$ . Consequently,  $\nabla_Y V$  is a basic vector field, thus  $\nabla$  is projectable. ■

Although, the existence of adapted connections is assured, the existence of a basic connection has a topological obstruction [Mol88], however, in the case of a Riemannian foliation the transverse Riemannian connection is always basic.

**Proposition 2.17.** *The transverse Riemannian connection  $\nabla^Q$  is basic, thus projectable.*

*Proof.* See [Ton97] Corollary 5.12 ■

**Theorem 2.18** (Bundle-like Riemannian Curvature). *Let  $Q$  be the transverse bundle over a bundle-like Riemannian foliation. Then*

$$\pi R^M(\pi X, \pi Y)[Z] = R^Q(X, Y)[\pi Z]$$

for  $X, Y, Z \in \Gamma_{\mathcal{F}}(M)$ .

*Proof.* Let  $X, Y, Z \in \Gamma_{\mathcal{F}}(TM)$  then

$$R^Q(X, Y)[\pi Z] = \nabla_X^Q \nabla_Y^Q \pi Z - \nabla_Y^Q \nabla_X^Q \pi Z - \nabla_{[X, Y]}^Q \pi Z$$

and since  $\pi Z$  is basic, by proposition 2.17, we have

$$= \nabla_{\pi X}^Q \nabla_{\pi Y}^Q \pi Z - \nabla_{\pi Y}^Q \nabla_{\pi X}^Q \pi Z - \nabla_{\pi[X, Y]}^Q \pi Z$$

and since the difference  $\pi[X, Y] - [\pi X, \pi Y] \in \Gamma(L)$  we further have

$$= \nabla_{\pi X}^Q \nabla_{\pi Y}^Q \pi Z - \nabla_{\pi Y}^Q \nabla_{\pi X}^Q \pi Z - \nabla_{[\pi X, \pi Y]}^Q \pi Z$$

now by proposition 2.14 we finally have

$$\begin{aligned} &= \pi \nabla_{\pi X}^M \nabla_{\pi Y}^M Z - \pi \nabla_{\pi Y}^M \nabla_{\pi X}^M Z - \pi \nabla_{[\pi X, \pi Y]}^M Z \\ &= \pi R^M(\pi X, \pi Y)[Z] \end{aligned}$$

■

**Corollary 2.19** (Riemannian Curvature Symmetries). *Let  $Q$  be the transverse bundle over a bundle-like Riemannian foliation of  $M$ . Then for  $X, Y, Z, U, V \in \Gamma_B(Q)$ .*

$$R^Q(X, Y)[Z] + R^Q(Y, Z)[X] + R^Q(Z, X)[Y] = 0 \quad (2.4)$$

$$\langle R^Q(X, Y)[U], V \rangle = \langle R^Q(U, V)[X], Y \rangle \quad (2.5)$$

*Proof.* The theorems are classical for  $R^M$  and the basic version immediately follows from theorem 2.18. ■

As we have seen many of the ideas and definitions from the Riemannian geometry of manifolds can be adapted for the transverse bundle of a foliation. In particular, we can always use geodesic normal coordinates to provide a preferred frame for the transverse space as well and since the transverse Riemannian connection and the manifold's Riemannian connection agree in the transverse direction many of the local formulas involving the Riemannian manifold can be split into transverse and foliate directions.

Recall that  $\pi : TM \rightarrow Q$  is a linear bundle map and that on a bundle-like Riemannian foliation we have the canonical Riemannian connections on  $TM$  and on  $Q$  consequently there is an associated connection  $\nabla$  on  $\text{Hom}(TM, Q)$  which applied to  $\pi$  gives  $(\nabla_X \pi)Y = \nabla_X^Q \pi Y - \pi \nabla_X^M Y$  for all  $X, Y \in \Gamma(TM)$  this defines an associated field that makes it's presence felt in many global results.

**Definition 2.20** (Second Fundamental Form). Let  $Q$  be the transverse bundle over a bundle-like Riemannian foliation of  $M$ . The the second fundamental form

$$\text{II} : \Gamma(TM \otimes TM) \rightarrow \Gamma(Q) \tag{2.6}$$

is defined by  $\text{II}(X, Y) = -(\nabla_X \pi)(Y) = -\nabla_X^Q \pi Y + \pi \nabla_X^M Y$  for all  $X, Y \in \Gamma(TM)$ .

Moreover, the second fundamental form is symmetric and zero if either argument is transverse.

*Proof.* Choose any  $X, Y \in \Gamma(TM)$ , then

$$\text{II}(X, Y) - \text{II}(Y, X) = -\nabla_X^Q \pi Y + \pi \nabla_X^M Y + \nabla_Y^Q \pi X - \pi \nabla_Y^M X \tag{2.7}$$

since  $\nabla^Q$  is torsionless the first and third terms equal  $-\pi[X, Y]$  and similarly since  $\nabla^M$  is torsionless the second and fourth terms equal  $\pi[X, Y]$ , thus the second fundamental form is symmetric. Secondly, by directly appealing to the definition of  $\nabla^Q$  and of  $\text{II}$ , if either argument is a transverse vector field the value is zero. ■

**Definition 2.21** (Mean Curvature). The transverse vector field defined by

$$\tau = \text{trace}^M(\text{II}) = \sum_{j=1}^m \text{II}(e_j, e_j) = \sum_{k=n+1}^m \text{II}(e_k, e_k) = \text{trace}^L(\text{II})$$

is called the **mean curvature vector field** and the dual form  $\kappa = \iota_\tau g^Q$  is called the **mean curvature form**.

The mean curvature form and vector field will play a substantial role in the global analysis. This follows from the following useful formula.

**Proposition 2.22** (Rummler's Formula). *Let  $Q$  be the transverse bundle over a bundle-like oriented Riemannian foliation and  $V \in \Gamma(Q)$ . Then  $\mathcal{L}_V \chi + \kappa(V)\chi = \eta \in \Omega^l(M)$  where  $\iota_{X_1} \dots \iota_{X_l} \eta = 0$  for  $\{X_i\}_{i=1}^l \subset \Gamma(L)$ .*

*Proof.* See Corollary 4.20 and Equation 4.26 both from [Ton97] ■

**Definition 2.23** (Transversal Divergence). The transversal divergence  $\text{div}^Q : \Gamma_{\mathcal{F}}(M) \rightarrow \Omega^0(Q)$  is defined as the unique function that satisfies  $\mathcal{L}_V \mu^Q = (\text{div}^Q V)\mu^Q$ . Moreover,

**Theorem 2.24** (Bundle-like Divergence). *Let  $Q$  be the transverse bundle over a bundle-like oriented Riemannian foliation of  $M$ . Then  $\text{div}^M V = \text{div}^Q V - \kappa(V)$  for any transverse vector field  $V \in \Gamma(Q)$ .*

*Proof.*

$$(\text{div}^M V)\mu^M = \mathcal{L}_V \mu^M = \mathcal{L}_V \mu^Q \wedge \chi + \mu^Q \wedge \mathcal{L}_V \chi$$

using proposition 2.22 we have

$$\begin{aligned} &= (\text{div}^Q V)\mu^Q \wedge \chi + \mu^Q \wedge (-\kappa(V)\chi + \eta) \\ &= (\text{div}^Q V)\mu^Q \wedge \chi - \kappa(V)\mu^Q \wedge \chi \\ &= (\text{div}^Q V - \kappa(V))\mu^M \end{aligned}$$

The fact that  $\mu^Q \wedge \eta = 0$  follows from the fact that  $\mu^Q \wedge \eta$  must be proportional to  $\mu^M$  and the fact that since  $\iota_{X_1} \dots \iota_{X_l} (\mu^Q \wedge \eta) = \mu^Q \wedge \iota_{X_1} \dots \iota_{X_l} (\eta) = 0$  and  $\iota_{X_1} \dots \iota_{X_l} \mu^M \neq 0$  for any  $\{X_i\}_{i=1}^l \subset \Gamma(L)$  that the proportion must be zero. ■

Any bundle-like Riemannian foliation that has a basic mean curvature is said to be a **tense Riemannian foliation** and the bundle-like metric is a **tense metric**. Dominguez [Dom95; Dom98] showed that any Riemannian foliation of a closed manifold has a tense metric, and consequently we will work with these metrics. For such metrics proposition 2.22 has the following particularly nice consequence.

**Proposition 2.25** (Tense Foliations Have Closed Mean Curvature). *Let  $\kappa$  be the mean curvature of tense Riemannian foliation. Then  $d\kappa = 0$ .*

*Proof.* See [Ton97] Equation 7.5■

# Chapter 3

## Foliated Dirac Bundles

### 3.1 Foliated Principal Bundles

Let us recall the relationship between vector bundles and principal bundles, one may wish to consult [LM89]. Assume that  $\rho : G \rightarrow \text{Diff}(F)$  is a continuous homomorphism from the Lie group  $G$  to the group of diffeomorphisms of a manifold  $F$  (with the compact-open topology), and  $P$  is a smooth principal  $G$  bundle over  $M$ . Then we can construct a fiber bundle  $P \times_{\rho} F$ , called the **associated bundle** over  $M$  by identifying  $(p, f) \in P \times F$  with  $(pg^{-1}, \rho(g)f)$  for all  $g \in G$ , in particular we will only encounter vector spaces, algebras and modules for  $F$ .

**Example 3.1.** For our purposes, an important bundle associated to the transverse orthonormal frame bundle is the **transverse Clifford bundle**. We first summarize some general facts here concerning Clifford algebras and bundles (see [LM89] for the general details). An inner product space  $V$  over  $\mathbb{R}$  (or  $\mathbb{C}$ ) has a canonically associated algebra  $\text{Cl}(V)$ , called the Clifford algebra of  $V$ , constructed by creating an algebra of formal multiplication of vectors subject to the reduction rule

$$X \cdot Y + Y \cdot X = -2\langle X, Y \rangle. \tag{3.1}$$

Moreover, any orthonormal transformation  $T \in \text{O}(V)$  preserves the reduction

eq. (3.1) and so automatically extends to an automorphism of  $\text{Cl}(V)$  defined by  $E_1 \cdot E_2 \cdots E_k \mapsto T(E_1) \cdot T(E_2) \cdots T(E_k)$ . Consequently, there are two natural representations of  $\text{O}(V)$  the standard orthonormal transformations on  $V$  and the Clifford representation on  $\text{Cl}(V)$ . Thus any principal orthonormal bundle  $P_{\text{O}}$  determines two associated bundles: the vector bundle  $E$  associated to the standard orthonormal representation and the Clifford bundle  $\text{Cl}(E)$  associated to the Clifford representation. First, we notice that  $\text{Cl}(E) \approx \Lambda(E)$  as *vector bundles*, and the particular orthonormal transformation  $V \mapsto -V$  preserves eq. (3.1) and so it extends to an automorphism of  $\text{Cl}(E)$ , moreover it is an involution and consequently  $\text{Cl}(E)$  decomposes into the eigenbundles  $\text{Cl}_0(E)$  and  $\text{Cl}_1(E)$  that are isomorphic as vector spaces to  $\Lambda^{\text{even}}(E)$  and  $\Lambda^{\text{odd}}(E)$  respectively. Of course, the above also holds for a principal orthonormal bundle  $P_{\text{SO}}$ , in which case  $E$  is necessarily orientable. A choice of orientation  $\omega \in \Lambda^{\dim(E)} E$  can be thought of as an element of  $\Gamma(\text{Cl}(E))$ , if  $\dim E$  is even then  $\omega$  anticommutes with elements of  $E$  but if  $\dim E$  is odd then  $\omega$  commutes with elements of  $E$ , consequently  $\omega$  is in the center of  $\text{Cl}(E)$  for the latter case. Furthermore, as a Clifford element,  $\omega^2 = -1$  if  $n \equiv \{1, 2\} \pmod{4}$  and  $\omega^2 = 1$  if  $n \equiv \{3, 4\} \pmod{4}$ , consequently for the latter case we have another involution and  $\text{Cl}(Q)$  decomposes into eigenbundles  $\text{Cl}^+(E)$  and  $\text{Cl}^-(E)$  as *vector bundles*, if, in addition,  $\dim E$  is odd then the eigenbundles  $\text{Cl}^+(E)$  and  $\text{Cl}^-(E)$  are Clifford bundles.

In particular, there exists a **transverse Clifford bundle**  $\text{Cl}(Q)$  associated to the transverse orthonormal frame bundle  $F_{\text{SO}}(Q)$  of a transversally orientable Riemannian foliation. Consequently, we also have the bundles  $\text{Cl}_0(Q)$ ,  $\text{Cl}_1(Q)$  and  $\text{Cl}^+(Q)$ ,  $\text{Cl}^-(Q)$ .

For foliated manifolds the main interest lies in when an associated vector bundle

is foliated.

**Definition 3.2.** A principal  $G$ -bundle  $P$  over a manifold  $M$  foliated by  $\mathcal{F}$  is called a foliated principal bundle if there exists a foliation  $\mathcal{F}_P$  of  $P$ , such that the distribution associated to  $\mathcal{F}_P$ :

1. is invariant under right translations of  $G$
2. has zero intersection with the distribution associated to the fiber at each point
3. and projects onto the distribution associated to  $\mathcal{F}$

The foliation  $\mathcal{F}_P$  is called the **lifted foliation** of  $\mathcal{F}$  to  $P$  and the distribution associated to  $\mathcal{F}_P$  defines a flat principal connection when restricted to each leaf of  $\mathcal{F}$ , that is a flat partial principal connection (see [Mol88]). Consequently, a vector bundle is foliated if and only if it's frame bundle is a foliated principal bundle. In particular, since the frame bundle is foliated, a local section of the frame yields a basis consisting of basic vector fields.

**Example 3.3.** The principal bundle that will be at the heart of everything that follows is the **transverse orthonormal frame bundle**  $F_O(Q)$ . Such an orthonormal frame can always be established by performing Gram-Schmidt on any local basis. Furthermore, since the transverse orthonormal frame bundle is canonically a foliated principal bundle, any associated bundles are also canonically foliated.

**Example 3.4.** The transverse Clifford bundle itself is also a foliated algebra bundle.

## 3.2 Dirac Bundles

**Definition 3.5** (Transverse Dirac Bundle). Let  $Q$  be the transverse bundle over a Riemannian foliation. A transverse Dirac bundle is a foliated vector bundle  $E$  that has a smooth action of the transverse Clifford bundle  $\text{Cl}(Q)$  on  $E$ .

**Definition 3.6** (Self-Adjoint Transverse Dirac Bundle). Let  $g^E$  be the basic metric (Riemannian or Hermitian) on Dirac bundle  $E$  over a Riemannian foliation. Then  $E$  is called a self-adjoint transverse Dirac bundle if  $g^E(e \cdot \phi, \varphi) + g^E(\phi, e \cdot \varphi) = 0$  for any unit vector  $e \in Q$  and any  $\phi, \varphi \in E$ .

**Example 3.7** (Transverse Spinor Bundle). We summarize some facts here concerning spin groups, bundles, structures and spinor bundles (see [LM89] for the general details). The group  $\text{Spin}(n)$  is the simply connected double cover for  $\text{SO}(n)$  and is generated by the subgroup of units of  $\text{Cl}_0(n)$  that preserve  $\mathbb{R}^n \subset \text{Cl}(n)$ . For each  $n$  there is a canonical unitary representation of  $\text{Spin}(n)$  on  $\mathbb{C}^n$ . Consequently, for any principal  $\text{Spin}(n)$  bundle  $P_{\text{Spin}}$  there is the associated Hermitian vector bundle  $\mathbf{S}$  called the spinor bundle. A  $\text{Spin}$  structure is a map  $\sigma : P_{\text{Spin}} \rightarrow P_{\text{SO}}$  between a principal  $\text{Spin}$  bundle  $P_{\text{Spin}}$  and a principal orthonormal bundle  $P_{\text{SO}}$ , such that  $\sigma(\zeta p) = \sigma(p)\varpi(\zeta)$  for any  $p \in P_{\text{Spin}}$  and  $\zeta \in \text{Spin}$  where  $\varpi : \text{Spin} \rightarrow \text{SO}$  is the double cover. For any spin structure there are three associated vector bundles: the spinor bundle  $\mathbf{S}$  associated to  $P_{\text{Spin}}$ , the vector bundle  $E$  associated to  $P_{\text{SO}}$  and the Clifford bundle  $\text{Cl}(E)$  associated to the Clifford representation. Since  $E$  is necessarily orientable, let  $\omega \in \Lambda^{\dim(E)}(E)$  be an orientation, which can be thought of as an element of  $\Gamma(\text{Cl}(E))$  and consequently acts on  $\mathbf{S}$ . Furthermore,  $\omega^2 = -1$  if  $n \equiv \{1, 2\} \pmod{4}$  and  $\omega^2 = 1$  if  $n \equiv \{3, 4\} \pmod{4}$  consequently, for the latter case we have an involution

and thus the eigenbundles  $S_+$  and  $S_-$ .

In particular, for an orientable Riemannian foliation, if a **Spin** structure exists for the transverse orthonormal frame bundle  $P_{SO} = F_{SO}(Q)$ , the foliation is also said to be a **Spin foliation** ( or to have a **transverse Spin structure**) and any associated  $S$  bundle is called the **transverse spinor bundle** for that **Spin** structure. Furthermore, since the transverse orthonormal frame bundle is a foliated principal bundle, each  $P_{Spin}$  is canonically foliated for each **Spin** structure. Consequently, the transverse bundles  $S$ ,  $S_+$  and  $S_-$  are also canonically foliated basic Hermitian vector bundles.

**Example 3.8** (Tensor Bundles). Let  $S$  and  $E$  be a transverse Dirac bundle and vector bundle over the same Riemannian foliation. Then the foliated vector bundle  $S \otimes E$  is a transverse Dirac bundle with Clifford action locally defined by  $\psi(\phi \otimes V) = (\psi\phi) \otimes V$  moreover if  $S$  is self-adjoint and  $E$  is a Riemannian/Hermitian vector bundle over  $\mathcal{F}$  then  $S \otimes E$  is self-adjoint with respect to the the tensor metric.

**Example 3.9** (Transverse  $\text{Spin}^{\mathbb{C}}$  vector bundles). Of particular interest is the case of a  $\text{Spin}^{\mathbb{C}}$  foliation. The group  $\text{Spin}^{\mathbb{C}}(n) \approx \text{Spin}(n) \times_{\mathbb{Z}_2} \text{U}(1)$ . For each  $n$  there is a canonical unitary representation of  $\text{Spin}^{\mathbb{C}}(n)$  on  $\mathbb{C}^n$ . Consequently, for any principal  $\text{Spin}^{\mathbb{C}}(n)$  bundle  $P_{\text{Spin}^{\mathbb{C}}}$  there is an associated Hermitian vector bundle  $S^{\mathbb{C}}$  called the spinor bundle. A  $\text{Spin}^{\mathbb{C}}(n)$  structure is an equivariant map  $\sigma : P_{\text{Spin}^{\mathbb{C}}} \rightarrow P_{SO}(E) \times P_{\text{U}(1)}$  between a principal  $\text{Spin}^{\mathbb{C}}(n)$  bundle  $P_{\text{Spin}^{\mathbb{C}}}$ , a principal  $SO$  bundle and a principal  $\text{U}(1)$  bundle. For any  $\text{Spin}^{\mathbb{C}}$  structure there are three associated vector bundles: the spinor bundle  $S^{\mathbb{C}}$ , the vector bundle  $E$  associated to  $P_{SO}$  and the Clifford bundle  $\text{Cl}(E)$  associated to the Clifford representation. Locally,  $S^{\mathbb{C}} = S \otimes L$  where  $S$  is a local spinor bundle for a local spin structure and  $L$  is a local complex line bundle such that

$L \otimes L = \Lambda^{\dim(S^{\mathbb{C}})}(S^{\mathbb{C}})$  locally. In fact recall the converse: If  $K$  is a complex line bundle and there exists  $\text{Spin}^{\mathbb{C}}$  structures then there is a  $\text{Spin}^{\mathbb{C}}$  structure such that associated spinor bundle  $S^{\mathbb{C}}$ , has the associated bundle  $\Lambda^{\dim(S^{\mathbb{C}})}(S^{\mathbb{C}}) = K$ , locally the complex line bundles  $L$  equal  $\sqrt{K}$  (which may not exist globally). Now, if  $E$  is orientable, then locally an orientation  $\omega \in \Lambda^{\dim(E)}(E)$  has the same properties as in example 3.7, however, the extra latitude given by using complex line bundle  $L$  allows us to define  $\omega^{\mathbb{C}} = i^{[(n+1)/2]}\omega$  and so  $(\omega^{\mathbb{C}})^2 = 1$  is always the case. Consequently, we have an involution and thus the eigenbundles  $S_+^{\mathbb{C}}$  and  $S_-^{\mathbb{C}}$ .

In particular, for an orientable Riemannian foliation, if a  $\text{Spin}^{\mathbb{C}}$  structure exists for the transverse orthonormal frame bundle  $P_{\text{SO}} = F_{\text{SO}}(Q)$ , the foliation is also said to be a **transverse  $\text{Spin}^{\mathbb{C}}$  foliation** ( or to have a **transverse  $\text{Spin}^{\mathbb{C}}$  structure**) and any associated  $S^{\mathbb{C}}$  bundle is called the **transverse spinor bundle** for that  $\text{Spin}^{\mathbb{C}}$  structure. However, recall that the  $\text{Spin}^{\mathbb{C}}$  structures are determined by global complex line bundles  $K$  such that, locally, the associated spinor bundle has the form  $\text{Spin}^{\mathbb{C}} = S \otimes \sqrt{K}$ . Thus, since the transverse orthonormal frame bundle is a foliated principal bundle and each local  $S$  bundle is foliated, thus to foliated  $S^{\mathbb{C}}$  it is necessary to choose a local Bott connection for the local complex line bundles  $\sqrt{K}$ . Furthermore, the local Bott connections on  $\sqrt{K}$  are determined by a Bott connection on  $K$ . Consequently, the transverse spinor bundle  $S^{\mathbb{C}}$  is *not* canonically foliated, but are parameterized by the foliated complex line bundles, similarly for  $S_+^{\mathbb{C}}$  and  $S_-^{\mathbb{C}}$ .

### 3.3 Associated Connections

Recall that together a principal connection  $\omega$  and a representation  $\rho : G \rightarrow \text{Diff}(V)$  defines an associated connection on the associated bundle by  $\nabla_V \phi = \rho_*(\omega(V)) \curvearrowright \phi$

where  $\rho_*$  is the differential.

**Example 3.10** (Orthonormal Connection). Recall that  $\mathfrak{so}(n)$  is the space of skew-symmetric linear transformations. Consequently, for an orthonormal basis  $\{e_i\}_{i=1}^n$ ,  $\mathfrak{so}(n)$  has a basis  $\{e_i \wedge e_j\}_{i < j}$ . The orthogonal action of each base  $e_i \wedge e_j$  is given by

$$(e_i \wedge e_j) \curvearrowright V = \langle e_i, V \rangle e_j - \langle e_j, V \rangle e_i \quad (3.2)$$

Of course, from the Fundamental Theorem of Riemannian Foliations, proposition 2.13, we have an orthonormal connection on the the transverse bundle  $Q$ , we are therefore interested in how this connection induces other connections on bundles associated to the transverse orthonormal bundle.

**Example 3.11** (Transverse Clifford Connection). Recall that a Clifford bundle  $\text{Cl}(E)$  is a self-adjoint transverse Dirac bundle. Moreover, since any element of  $\text{SO}(n)$  naturally extends to an automorphism of  $\text{Cl}(n)$  any element of  $\mathfrak{so}(n)$  naturally extends to an endomorphism of  $\mathfrak{cl}(n)$  (the Lie algebra of the units of  $\text{Cl}(n)$ ) by

$$(e_i \wedge e_j) \curvearrowright \psi = \frac{1}{4}[[e_i, e_j], \psi] = \frac{1}{2}[e_i \cdot e_j, \psi] \quad (3.3)$$

for any  $\psi \in \mathfrak{cl}(n)$ . In particular, for a vector  $V \in E \subset \text{Cl}(E)$

$$\begin{aligned} (e_i \wedge e_j) \curvearrowright V &= \frac{1}{2}[e_i \cdot e_j, V] = \frac{1}{2}(e_i \cdot e_j \cdot V - V \cdot e_i \cdot e_j) \\ &= \frac{1}{2}(-e_i \cdot V \cdot e_j - 2\langle V, e_j \rangle e_i + e_i \cdot V \cdot e_j + 2\langle V, e_i \rangle e_j) \\ &= \langle V, e_i \rangle e_j - e_i \langle e_j, V \rangle \end{aligned}$$

which agrees with the orthonormal action. Consequently, the Clifford action and orthogonal action agree on  $E$  and no distinction will be made between the actions in this case.

In particular, there is a canonical connection on  $\text{Cl}(Q)$  induced by  $\nabla^Q$ . Locally, this means that the connection coefficients of  $\nabla^Q$  are  $\mathfrak{so}(n)$ -valued that are naturally extended by the above action to be  $\mathfrak{cl}(n)$ -valued. Because the action on  $Q$  is the same using either the orthogonal action or the Clifford covariant derivatives, we will abuse notation and use  $\nabla^Q$  for both the connection on  $\text{Cl}(Q)$  and on  $Q$ . Moreover, since the action is in  $\mathfrak{cl}(Q)$  the covariant derivative acts as a derivation on the Clifford bundle  $\text{Cl}(Q)$ , consequently, the covariant derivative restricts to the subbundles  $\text{Cl}^0(Q) \approx \Lambda^{\text{even}}(Q)$ ,  $\text{Cl}^1(Q) \approx \Lambda^{\text{odd}}(Q)$ . Furthermore, for a transversely orientable Riemannian foliation, the Riemannian volume form  $\mu^Q$  is globally parallel, and the connection restricts to  $\text{Cl}^\pm(Q)$ , moreover, the curvature of  $\nabla^Q$  also acts as a derivation on  $\text{Cl}(Q)$ . See [LM89] Section 4.8 for more details.

**Definition 3.12** (Dirac Connection). Let  $E$  be a self-adjoint transverse Dirac bundle. An adapted metric connection  $\nabla$  on  $E$  is called a Dirac connection if it acts as a Clifford module derivation, that is  $\nabla(\psi \cdot \phi) = (\nabla^Q \psi) \cdot \phi + \psi \cdot (\nabla \phi)$  for  $\psi \in \text{Cl}(Q)$  and  $\phi \in E$ .

**Definition 3.13** (Transverse Dirac Operator). Let  $E$  be a transverse Dirac bundle, and  $\nabla$  be a Dirac connection on  $E$ . Then the transverse Dirac operator  $\not{X} : \Gamma(E) \rightarrow \Gamma(E)$  is defined by

$$\not{X}\phi = \sum_i^n e_i \cdot \nabla_i \phi \tag{3.4}$$

for any transverse orthonormal frame  $\{e_i\}_{i=1}^n$  of  $Q$ .

**Example 3.14.** Let  $\text{Cl}(Q)$  be the transverse Clifford bundle of a Riemannian foliation. The transverse connection from example 3.11 is a Dirac connection, and

the corresponding transverse Dirac operator will be called the transverse DeRham operator.

**Example 3.15** (Spin connections). Recall that a  $\mathbf{Spin}(n)$  foliation has an associated spinor bundle  $\mathbf{S}$ . Since any element of  $\mathbf{SO}(n)$  is covered by two elements of  $\mathbf{Spin}(n)$ , both Lie groups have the same Lie algebra  $\mathfrak{so}(n) = \mathfrak{spin}(n)$ . Since  $\mathbf{Spin}(n)$  has a canonical unitary representation on  $\mathbb{C}^n$ , this determines the action of  $\mathfrak{spin}(n)$  by

$$(e_i \wedge e_j) \curvearrowright \phi = \frac{1}{4}[e_i, e_j] \curvearrowright \phi = \frac{1}{2}e_i \cdot e_j \curvearrowright \phi \quad (3.5)$$

where  $e_i \cdot e_j$  acts as a unitary matrix on the complex vector  $\phi$ . In particular, for  $n = 4$  the  $\mathbf{Spin}$  action to the matrices

$$\begin{aligned} e_1 \cdot e_2 &= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} & e_1 \cdot e_3 &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & e_1 \cdot e_4 &= \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \\ e_2 \cdot e_3 &= \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} & e_2 \cdot e_4 &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & e_3 \cdot e_4 &= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \end{aligned}$$

Furthermore, since  $\mu^Q$  is globally parallel the connection restricts to the subspaces  $\mathbf{Cl}^\pm(Q)$ , moreover the curvature also acts as a Clifford module derivation. See [LM89] Section 4.11 for more details.

Thus for a  $\mathbf{Spin}$  foliation there is a canonical connection on  $\mathbf{S}$  induced by  $\nabla^Q$  that will be called the transverse Atiyah-Singer connection and the corresponding Dirac operator is the transverse Atiyah-Singer operator. Locally this means that the connection coefficients of  $\nabla^Q$  are  $\mathfrak{spin}(n)$ -valued for the Atiyah-Singer connection.

**Example 3.16** (Generated Dirac Bundles). Recall from the last section that tensoring any self-adjoint Dirac bundle  $S$  that has Dirac connection  $\nabla^S$  with a Riemannian (Hermitian) vector bundle  $E$  that has a metric connection  $E$  results in a self-adjoint

Dirac bundle  $S \otimes E$ , and the induced connection  $\nabla^S \otimes \nabla^E$  is a Dirac connection for the tensored metrics.

**Example 3.17** ( $\text{Spin}^{\mathbb{C}}$  connections). Recall that a  $\text{Spin}^{\mathbb{C}}$  structure has an associated transverse spinor bundle  $S^{\mathbb{C}}$  that locally looks like  $S \otimes L$  for a local spinor bundle  $S$  and local complex line bundle  $L$ . Therefore, locally a connection on  $S^{\mathbb{C}}$  has the form  $\nabla \otimes A$  where locally,  $\nabla$  is the Atiyah-Singer connection and  $A$  is a connection on  $L$ , these local tensored connections patch together to give a global connection  $\nabla^A$  on  $S^{\mathbb{C}}$  that we will call a Seiberg-Witten connection and the corresponding transverse Dirac operator a Seiberg-Witten operator. Moreover, since the  $\text{Spin}^{\mathbb{C}}$  structures are parameterized by the complex line bundles, an adapted connection for a foliated complex line bundle  $K$  will induce a local connections on the local bundles  $L = \sqrt{K}$  and thus a global connection on  $S^{\mathbb{C}}$ . We will adapt the convention that any transverse definition where we have used the a superscript of  $M$  or  $Q$  then using any other symbol ( $A$  in particular) will correspond to the Seiberg-Witten connection that corresponds to a connection  $A$  on the corresponding complex line bundle.

### 3.4 Second Order Differential Operators

The a priori bounds in chapter 5 rely on a Bochner-Lichernowicz-Weitzenböck type identity that relates the square of the Dirac operator and the connection Laplacian defined in terms of a covariant second derivative. We develop the appropriate transverse notions for Riemannian foliations and show, in the case of a bundle-like Riemannian foliation, how it relates to the classical theory on the underlying Riemannian manifold.

**Definition 3.18** (Transverse Second Covariant Derivative). Let  $E$  be a vector bundle over a Riemannian foliation and  $\nabla$  be an adapted connection on  $E$ . The transverse covariant second derivative  $\nabla^{\text{II}} : \Gamma(E) \rightarrow \Gamma(T^*M \otimes T^*M \otimes E)$  is defined by

$$\nabla_{X,Y}^{\text{II}}\phi = \nabla_X \nabla_Y \phi - \nabla_{\nabla_X^{\mathcal{Q}} \pi Y} \phi$$

for all  $X, Y \in \Gamma(M)$ ,  $\phi \in \Gamma(E)$ . Moreover, if  $\nabla$  is projectable then  $\nabla_{X,Y}^{\text{II}}\phi \in \Gamma_B(E)$  for every  $X, Y \in \Gamma_{\mathcal{F}}(M)$  and  $\phi \in \Gamma_B(E)$ .

We will reserve the classical notation,  $\nabla^2$ , for the covariant second derivative of a vector bundle over Riemannian manifold and use  $\nabla^{\text{II}}$  for the transverse definition. This is reasonable notation, since, in the case of a bundle-like Riemannian foliation, they differ by the second fundamental form as the following theorem shows.

**Corollary 3.19** (Bundle-like Second Covariant Derivatives). *Let  $E$  be a vector bundle over a bundle-like Riemannian foliation and  $\nabla$  be an adapted connection on  $E$ . Then*

$$\nabla_{X,Y}^2\phi = \nabla_{X,Y}^{\text{II}}\phi - \nabla_{\text{II}(X,Y)}\phi$$

for every  $\phi \in \Gamma_B(E)$ .

*Proof.* Because  $\phi$  is basic,  $\nabla_{X,Y}^2\phi = \nabla_X \nabla_Y \phi - \nabla_{\pi \nabla_X^M Y} \phi = \nabla_X \nabla_Y \phi - \nabla_{\nabla_X^{\mathcal{Q}} \pi Y} \phi - \nabla_{\text{II}(X,Y)}\phi = \nabla_{X,Y}^{\text{II}}\phi - \nabla_{\text{II}(X,Y)}\phi$  from definition 2.20. ■

As in the classical case, the second covariant derivative, allows us to reformulate the curvature.

**Theorem 3.20.** *For any adapted connection  $\nabla$  on a foliated vector bundle  $E$  over a Riemannian foliation. Then*

$$R^{\nabla}(X, Y)[\phi] = \nabla_{X,Y}^{\text{II}}\phi - \nabla_{Y,X}^{\text{II}}\phi$$

*Proof.*

$$\begin{aligned}
R^\nabla(X, Y)[\phi] &= \nabla_X \nabla_Y \phi - \nabla_Y \nabla_X \phi - \nabla_{\pi[X, Y]} \phi \\
&= \nabla_X \nabla_Y \phi - \nabla_Y \nabla_X \phi - \nabla_{\nabla_X^Q \pi Y} \phi - \nabla_{\nabla_Y^Q \pi X} \phi \\
&= \nabla_{X, Y}^{\text{II}} \phi - \nabla_{Y, X}^{\text{II}} \phi
\end{aligned}$$

■

**Definition 3.21** (Transverse Connection Laplacian). Let  $E$  be a vector bundle over a Riemannian foliation and  $\nabla$  be an adapted connection on  $E$ . The transverse connection Laplacian  $\nabla^* \nabla : \Gamma(E) \rightarrow \Gamma(E)$  is defined by

$$\nabla^* \nabla(\phi) = -\text{trace}^Q(\nabla^{\text{II}} \phi)$$

for all  $\phi \in \Gamma(E)$  and the trace is with respect to the basic metric  $g^Q$ .

We will reserve the classical notation,  $\nabla^* \nabla$ , for the connection Laplacian of a vector bundle over Riemannian manifold and use  $\nabla^* \nabla$  for the transverse definition.

**Theorem 3.22** (Bundle-like Connection Laplacians). *Let  $E$  be a vector bundle over a bundle-like Riemannian foliation and  $\nabla$  be an adapted connection on  $E$ . Then  $\nabla^* \nabla \phi = \nabla^* \nabla \phi + \nabla_\tau \phi$  for  $\phi \in \Gamma_B(E)$ .*

*Proof.* Choose a foliated chart then  $M$  then

$$\begin{aligned}
\nabla^* \nabla \phi &= -\text{trace}^M(\nabla^* \nabla \phi) \\
&= -\sum_{j=1}^m \nabla_j \nabla_j \phi - \nabla_{\pi \nabla_j^M e_j} \phi \\
&= -\sum_{j=1}^m \nabla_j \nabla_j \phi - \nabla_{\nabla_j^Q \pi e_j} \phi - \nabla_{\text{II}(e_j, e_j)} \phi
\end{aligned}$$

$$\begin{aligned}
&= -\sum_{k=1}^n \nabla_k \nabla_k \phi - \nabla_{\nabla_k^Q e_k} \phi + \sum_{i=n+1}^m \nabla_{\Pi(e_i, e_i)} \phi \\
&= -\text{trace}^Q(\nabla^{\text{II}} \phi) + \text{trace}^L(\nabla_{\Pi}(e_i, e_i) \phi) \\
&= \nabla^* \nabla \phi + \nabla_{\tau} \phi
\end{aligned}$$

the last equality follows from definition 2.20. ■

### 3.5 Global Analysis

**Definition 3.23** ( $L^2$  inner product on  $\Gamma_B(E)$ ). Let  $E$  be a Riemannian (Hermitian) vector bundle over an bundle-like oriented Riemannian foliation of a compact manifold  $M$ . Then define the inner product  $\langle\langle \psi, \varphi \rangle\rangle_{L^2} = \int_M \langle \psi, \varphi \rangle \mu^M$  for any  $\psi, \varphi \in \Gamma_B(E)$ .

In particular, the transverse bundle  $Q^*$  of a bundle-like oriented Riemannian foliation of a closed manifold has an induced metric and so we can consider the associated inner product.

**Example 3.24.** Let  $Q$  be the transverse bundle of a bundle-like oriented Riemannian foliation of a closed manifold. Then

$$\langle\langle \alpha, \beta \rangle\rangle_{L^2} = \int_M g^Q(\alpha, \beta) \mu^Q \wedge \chi = \int_M \alpha \wedge (\star \beta) \wedge \chi = \int_M \alpha \wedge \star \beta = \int_M g^M(\alpha, \beta) \mu^M$$

for  $\alpha, \beta \in \Omega_B^k(Q^*)$  is an inner product.

In particular, Hodge theory has been developed for the basic DeRham cohomology of a tense bundle-like oriented Riemannian foliations of closed manifolds [KT88; KT83b; KT83a; KT84; EH84; ESH85; EH86]. Since the exterior derivative restricts to basic forms it is necessary to determine its formal adjoint the **basic codifferential**  $d^* : \Omega_B^k(M) \rightarrow \Omega_B^{k-1}(M)$  defined by  $(-1)^k(\bar{\star}d - \kappa \wedge \star) = (-1)^k(\bar{\star}d \star + \iota_{\tau})$ . The

transverse Laplacian is thus defined as  $\Delta_B = dd^* + d^*d$  and we have the orthogonal decomposition

$$\Omega_B^k(M) = \text{im } d \oplus \text{im } d^* \oplus \ker \Delta \quad (3.6)$$

with respect to the  $L^2$ -inner product.

Furthermore, letting  $\mathcal{H}^k = \ker \Delta$ , we have that  $\mathcal{H}^k \approx H_B^k(Q^*)$  and is finite dimensional for all  $k$ . The proof in [KT88] constructs an elliptic operator on all forms that restricts to the basic Laplacian  $\Delta_B$ , which is in general not the tangent Laplacian, nor is it self-adjoint, it is then shown that the Sobolev embedding and Rellich-Kodrakov compactness theorems for  $\Gamma(Q)$  restrict to basic sections. Furthermore, it ought to be noted that for the codifferential to map basic forms to basic forms, it is necessary that  $\kappa$  be basic.

Furthermore, there is another complex of basic forms that can be defined called the twisted complex. In this case we use  $d_k = d - \kappa$  and a corresponding codifferential  $d_k^*$ , cohomology  $H_\kappa^*$ , Laplacian  $\Delta_\kappa$  and Hodge theory. In particular there is a twisted Hodge duality between the basic DeRham cohomology and the twisted DeRham cohomology.

$$H_B^k(M) \otimes H_\kappa^{n-k}(M) \rightarrow \mathbb{R} \quad (3.7)$$

see [Ton97] Theorem 7.54, it particular it should be noted that in the case that  $\kappa$  is cohomologous to zero, in which case the foliation is said to be taut, then the twisted and basic cohomologies are isomorphic and the twisted duality is a true duality.

**Theorem 3.25** (Adjoint Connection Laplacians). *Let  $\nabla$  be an adapted connection*

on a vector bundle  $E$  over a bundle-like oriented Riemannian foliation. Then

$$\langle \langle \nabla^* \nabla \phi, \varphi \rangle \rangle = \langle \langle \nabla \phi, \nabla \varphi \rangle \rangle = \langle \langle \nabla^* \nabla \phi, \varphi \rangle \rangle + \langle \langle \nabla_\tau \phi, \varphi \rangle \rangle$$

for  $\phi \in \Gamma_B(E)$ .

*Proof.* Choose any local orthonormal frame of  $Q$ ,

$$\begin{aligned} \langle \nabla^* \nabla \phi, \varphi \rangle &= - \sum_{k=1}^n \langle \nabla_k \nabla_k \phi, \varphi \rangle - \langle \nabla_{\nabla_k^Q e_k} \phi, \varphi \rangle \\ &= - \sum_{k=1}^n \nabla_k \langle \nabla_k \phi, \varphi \rangle - \langle \nabla_k \phi, \nabla_k \varphi \rangle - \langle \nabla_{\nabla_k^Q e_k} \phi, \varphi \rangle \end{aligned}$$

The linear functional on  $\Gamma(Q)$  defined by  $X \mapsto \langle \nabla_X \phi, \varphi \rangle$  has a unique representative  $Y$ , that is  $\langle \nabla_X \phi, \varphi \rangle = \langle X, Y \rangle_Q$  for all  $X$ .

$$\begin{aligned} &= \langle \nabla \phi, \nabla \varphi \rangle - \sum_{k=1}^n \nabla_k \langle e_k, Y \rangle_Q - \langle \nabla_k^Q e_k, Y \rangle_Q \\ &= \langle \nabla \phi, \nabla \varphi \rangle - \sum_{k=1}^n \langle e_k, \nabla_k^Q Y \rangle_Q \\ &= \langle \nabla \phi, \nabla \varphi \rangle - \sum_{k=1}^n \langle e^k | \nabla_Y^Q e_k + \pi[e_k, Y] \rangle_Q \end{aligned}$$

consequently, since  $\mu^Q$  is parallel we have

$$\begin{aligned} &= \langle \nabla \phi, \nabla \varphi \rangle \mu^Q + \sum_{k=1}^n \langle \nabla_Y^Q e^k - \mathcal{L}_Y e^k, e^k \rangle \mu^Q \\ &= \langle \nabla \phi, \nabla \varphi \rangle \mu^Q + (\nabla_Y^Q - \mathcal{L}_Y) \mu^Q \\ &= \langle \nabla \phi, \nabla \varphi \rangle \mu^Q - \operatorname{div}^Q(Y) \mu^Q \\ &= \langle \nabla \phi, \nabla \varphi \rangle \mu^Q - \langle \nabla_\tau \phi, \varphi \rangle \mu^Q - \operatorname{div}^M(Y) \mu^Q \end{aligned}$$

Consequently, since  $M$  has no boundary the result follows from multiplying by  $\chi$ ,

integrating over the manifold and the divergence theorem. ■

**Theorem 3.26** (Adjoint of Transverse Dirac Operators). *Let  $E$  be a transverse self-adjoint Dirac bundle on a closed manifold  $M$  and  $\nabla$  be a Dirac connection on  $E$ . The formal adjoint of  $\not{A}$  with respect to  $L^2(E)$  is  $\not{A} - \tau$ .*

*Proof.* Choose any local orthonormal frame of  $Q$ , since  $E$  is self-adjoint

$$\begin{aligned} \langle \not{A}\phi, \varphi \rangle &= \sum_{k=1}^n \langle e_k \cdot \nabla_k \phi, \varphi \rangle = - \sum_{k=1}^n \langle \nabla_k \phi, e_k \cdot \varphi \rangle \\ &= - \sum_{k=1}^n \nabla_k \langle \phi, e_k \cdot \varphi \rangle - \langle \phi, \nabla_k^Q e_k \cdot \varphi \rangle - \langle \phi, e_k \cdot \nabla_k \varphi \rangle \end{aligned}$$

The linear functional on  $\Gamma(Q)$  defined by  $X \mapsto \langle \phi, X \cdot \varphi \rangle$  has a unique representative  $Y$ , that is  $\langle \phi, X \cdot \varphi \rangle = \langle X, Y \rangle_Q$  for all  $X$ , and since  $\nabla^Q$  is a transverse metric connection

$$\begin{aligned} &= \langle \phi, \not{A}\varphi \rangle - \sum_{k=1}^n \nabla_k \langle e_k, Y \rangle_Q - \langle \nabla_k^Q e_k, Y \rangle_Q \\ &= \langle \phi, \not{A}\varphi \rangle - \sum_{k=1}^n \langle e^k \mid \nabla_k^Q Y \rangle \\ &= \langle \phi, \not{A}\varphi \rangle - \sum_{k=1}^n \langle e^k \mid \nabla_Y^Q e_k + \pi[e_k, Y] \rangle \end{aligned}$$

consequently, since  $\mu^Q$  is parallel we have

$$\begin{aligned} &= \langle \phi, \not{A}\varphi \rangle \mu^Q + \sum_{k=1}^n \langle \nabla_Y^Q e^k - \mathcal{L}_Y e^k, e^k \rangle \mu^Q \\ &= \langle \phi, \not{A}\varphi \rangle \mu^Q + (\nabla_Y^Q - \mathcal{L}_Y) \mu^Q \\ &= \langle \phi, \not{A}\varphi \rangle \mu^Q - \operatorname{div}^Q(Y) \mu^Q \\ &= \langle \phi, \not{A}\varphi \rangle \mu^Q - \langle \phi, \tau \cdot \varphi \rangle \mu^Q - \operatorname{div}^M(Y) \mu^Q \end{aligned}$$

Consequently, since  $M$  has no boundary the result follows from multiplying by  $\chi$ , integrating over the manifold and the divergence theorem. ■

**Corollary 3.27** (Self-Adjoint Dirac Operator). *Let  $E$  be a transverse self-adjoint Dirac bundle on a closed manifold  $M$  and  $\nabla$  be a Dirac connection on  $E$ . Then the operator  $\not{A} - \frac{1}{2}\tau$  is self-adjoint on  $\Gamma(E)$ .*

*Proof.* Since the Dirac bundle is self-adjoint we have  $\langle \tau\phi, \varphi \rangle = \sum_{k=0}^n \tau_k \langle e_k \phi, \varphi \rangle = \sum_{k=0}^n -\tau_k \langle \phi, e_k \varphi \rangle = -\langle \phi, \tau\varphi \rangle$  since  $\tau$  is a real vector field. The result follows from theorem 3.26. ■

The self-adjoint Dirac operator was studied in [PR96; BKR11] under the name the basic Dirac operator.

**Corollary 3.28.** *If  $\tau \in \Gamma_B(M)$ , that is the Riemannian foliation is tense, then the operator  $\not{X} - \frac{1}{2}\tau$  is formally self-adjoint on  $\Gamma_B(E)$ .*

*Proof.* By corollary 3.27 it is sufficient to show that  $\tau$  preserves  $\Gamma_B(E)$  under Clifford multiplication when  $\tau$  is basic. Assume that  $X \in \Gamma(L)$  and  $\phi \in \Gamma_B(E)$  then since  $\nabla$  is a Dirac connection  $\nabla_X(\tau \cdot \phi) = \nabla_X \tau \cdot \phi + \tau \cdot \nabla_X \phi$  which vanishes since each term does. Thus  $\tau \cdot \phi$  is basic. ■

## 3.6 Weitzenböck Identities

The methods of Bochner, Weitzenböck and Lichnerowicz determines an identity between the connection Laplacian and the square of a Dirac operator, they in general differ by a lower order curvature terms. Such a method was used for **Spin** foliations by [Sle12] and [Jun07] to provide bounds for the eigenvalues of the the basic Dirac operator. In this section we will restrict the indices  $i, j, k, l$  in any summations to be  $1 \leq i, j, k, l \leq n$  (where  $n$  is still the codimension of the foliation) subject to any additional constraints for that particular summation.

**Definition 3.29** (Transverse Clifford Tensor). Let  $\text{Cl}(Q)$  be the transverse Clifford bundle for an  $n$ -codimensional Riemannian foliation. The Clifford tensor  $\mathfrak{R}^{\text{Cl}} \in$

$\Omega^2(\mathfrak{Cl}Q)$  is defined locally by

$$\mathfrak{R}^{\text{Cl}}(X, Y) = \frac{1}{4} \sum_{i,j} \langle R^Q(X, Y)[e_i], e_j \rangle e_i \cdot e_j$$

for  $X, Y \in \Gamma(M)$  and for any transverse orthonormal frame  $\{e_i\}_{i=1}^n$  of  $Q$ .

**Theorem 3.30** (Relation between  $R^\nabla$  and  $\mathfrak{R}^{\text{Cl}}$ ). *Let  $\text{Cl}(Q)$  be the transverse Clifford bundle for a Riemannian foliation and  $E$  be the transverse Clifford bundle or a transverse spinor bundle (if one exists) with their canonical Dirac connections. Then*

$$R^\nabla(X, Y)[\phi] = \mathfrak{R}^{\text{Cl}}(X, Y) \curvearrowright \phi$$

for any  $X, Y \in \text{T}(M)$  and  $\phi \in E$ .

**Definition 3.31** (The Weitzenböck Operator). Let  $E$  be a transverse Dirac bundle and let  $\nabla$  be a Dirac connection on  $E$ . The Weitzenböck operator  $\mathfrak{R}^W : \Gamma(E) \rightarrow \Omega^2(\text{End } E)$  is defined locally by

$$\mathfrak{R}^W[\phi] = \frac{1}{2} \sum_{i,j} e_i \cdot e_j \cdot R^\nabla(e_i, e_j)[\phi]$$

and for any transverse orthonormal frame  $\{e_i\}_{i=1}^n$  of  $Q$ .

**Definition 3.32** (Transverse Ricci Curvature). Let  $Q$  be the transverse bundle of an  $n$ -codimensional Riemannian foliation. The transverse Ricci curvature  $\text{ricc}^Q : Q \rightarrow Q$  is defined by

$$\text{ricc}^Q(V) = - \sum_i R^Q(V, e_i)[e_i]$$

and for any transverse orthonormal frame  $\{e_i\}_{i=1}^n$  of  $Q$ .

**Definition 3.33** (Transverse Scalar Curvature). Let  $Q$  be the transverse bundle of an  $n$ -codimensional Riemannian foliation. The transverse scalar curvature  $\text{scal}^Q : X \rightarrow \mathbb{R}$  is defined by

$$\text{scal}^Q(x) = - \sum_{i,j} \langle \mathbf{R}^Q(e_i, e_j)[e_i], e_j \rangle = - \sum_i \langle \text{ricc}(e_i), e_i \rangle$$

and for any transverse orthonormal frame  $\{e_i\}_{i=1}^n$  of  $Q$  at  $x$ .

**Theorem 3.34** (Transverse Weitzenböck Identity). *Let  $E$  be a transverse Dirac bundle and let  $\nabla$  be a Dirac connection on  $E$ . Then  $\not\partial \not\partial \phi = \nabla^* \nabla \phi + \mathfrak{R}^W[\phi]$ .*

*Proof.* Let us calculate  $\not\partial \not\partial \phi$  at a point  $x \in M$

$$\begin{aligned} \not\partial \not\partial \phi &= \sum_i e_i \cdot \nabla_i \sum_j e_j \cdot \nabla_j \phi \\ &= \sum_{i,j} e_i \cdot (\nabla_i e_j \cdot \nabla_j \phi + e_j \cdot \nabla_i \nabla_j \phi) \end{aligned}$$

without loss of generality we may assume normal coordinates at  $x$ , giving

$$\begin{aligned} &= \sum_{i=j} e_i \cdot e_j \cdot \nabla_i \nabla_j \phi + \sum_{i<j} e_i \cdot e_j \cdot \nabla_i \nabla_j \phi + \sum_{i>j} e_i \cdot e_j \cdot \nabla_i \nabla_j \phi \\ &= - \sum_k \nabla_k \nabla_k \phi + \sum_{i<j} e_i \cdot e_j \cdot (\nabla_i \nabla_j \phi - \nabla_j \nabla_i \phi) \end{aligned}$$

because we are in normal coordinates at  $x$  we have

$$\begin{aligned} &= - \sum_k \nabla_{k,k}^{\Pi} \phi + \sum_{i<j} \nabla_{i,j}^{\Pi} \phi - \nabla_{j,i}^{\Pi} \phi \\ &= \nabla^* \nabla \phi + \sum_{i<j} e_i \cdot e_j \cdot \mathbf{R}^{\nabla}(e_i, e_j)[\phi] \\ &= \nabla^* \nabla \phi + \mathfrak{R}^W[\phi] \end{aligned}$$

■

In particular, the Transverse Weitzenböck identity has particularly nice forms when applied to the transverse bundle and any transverse spinor bundles. The identities for transverse spinor bundles were established in [Sle12; Jun01; Jun07].

**Theorem 3.35** (Transverse Bochner Identity). *Let  $\text{Cl}(Q)$  be the transverse Clifford bundle for a transverse a Riemannian structure and let  $\nabla^Q$  be the canonical clifford connection on  $\text{Cl}(Q)$ . Then  $\not\partial\not\partial V = \nabla^* \nabla V + \text{ricc}^Q(V)$  for  $V \in \Gamma(Q) \subset \text{Cl}(Q)$ .*

*Proof.* Example 3.1 has shown that the transverse Clifford bundle is a Dirac bundle and  $\nabla^Q$  is a Dirac connection. Recall that  $\nabla^Q$  when restricted to  $\Gamma_B(Q)$  equals the transverse Levi-Civita connection on  $\Gamma_B(Q)$ , thus the second transverse covariant derivatives of vector fields are also equal. Consequently, it is sufficient to show that  $\mathfrak{R}^W[V] = \text{ricc}^Q(V)$  for  $V \in \Gamma(Q)$ . Let us first note that from Clifford action example 3.11

$$\begin{aligned} \mathfrak{R}^{\text{Cl}}(X, Y) \curvearrowright V &= \frac{1}{4} \sum_{i,j} \langle \text{R}^Q(X, Y)[e_i], e_j \rangle e_i \cdot e_j \curvearrowright V \\ &= \frac{1}{2} \sum_{i,j} \langle \text{R}^Q(X, Y)[e_i], e_j \rangle (\langle V, e_i \rangle e_j - \langle V, e_j \rangle e_i) \\ &= \frac{1}{2} \text{R}^Q(X, Y)[V] + \frac{1}{2} \text{R}^Q(X, Y)[V] \\ &= \text{R}^Q(X, Y)[V] \end{aligned}$$

Consequently, the Clifford curvature agrees with the Riemannian curvature when restricted to vector fields in  $\Gamma(\text{Cl}(Q))$ , and thus leaves  $\Gamma(\text{Cl}(Q))$  invariant. Since the Clifford transformation preserves  $\Gamma(Q)$ , we have the following

$$2\mathfrak{R}^W[V] = \sum_{i,j} e_i \cdot e_j \cdot \text{R}^\nabla(e_i, e_j)[V] = \sum_{i,j} \sum_k e_i \cdot e_j \cdot e_k \langle \text{R}^Q(e_i, e_j)[V], e_k \rangle \quad (3.8)$$

Furthermore, since if  $i = j$  the term is zero by the symmetry in the curvature, we can neglect them and rewrite

$$\begin{aligned}
2\mathfrak{A}^W[V] &= \sum_{i \neq j} e_i \cdot e_j \cdot e_i \langle \mathbf{R}^Q(e_i, e_j)[V], e_i \rangle + e_i \cdot e_j \cdot e_j \langle \mathbf{R}^Q(e_i, e_j)[V], e_j \rangle \\
&\quad + \sum_{i \neq j \neq k \neq i} e_i \cdot e_j \cdot e_k \langle \mathbf{R}^Q(e_i, e_j)[V], e_k \rangle
\end{aligned}$$

Let us rewrite the last sum over distinct indices as a sum over ordered indices over all permutations of those indices, i.e. using the action of  $S_3$  permute the indices. Let  $\sigma \in S_3$ , then by anti-commutivity of the Clifford multiplication for distinct orthonormal bases  $\sigma \curvearrowright e_i \cdot e_j \cdot e_k = |\sigma| e_i \cdot e_j \cdot e_k$ , thus we can write

$$\begin{aligned}
&\sum_{i \neq j, j \neq k, k \neq i} e_i \cdot e_j \cdot e_k \langle \mathbf{R}^Q(e_i, e_j)[V], e_k \rangle = \sum_{i < j < k} \sum_{\sigma \in S_3} |\sigma| e_i \cdot e_j \cdot e_k \langle \mathbf{R}^Q(e_{\sigma(i)}, e_{\sigma(j)})[V], e_{\sigma(k)} \rangle \\
&= \sum_{i < j < k} e_i \cdot e_j \cdot e_k \sum_{\sigma \in A_3} \langle \mathbf{R}^Q(e_{\sigma(i)}, e_{\sigma(j)})[V], e_{\sigma(k)} \rangle - \langle \mathbf{R}^Q(e_{\sigma(j)}, e_{\sigma(i)})[V], e_{\sigma(k)} \rangle \\
&= 2 \sum_{i < j < k} e_i \cdot e_j \cdot e_k \sum_{\sigma \in A_3} \langle \mathbf{R}^Q(e_{\sigma(i)}, e_{\sigma(j)})[V], e_{\sigma(k)} \rangle \\
&= -2 \sum_{i < j < k} e_i \cdot e_j \cdot e_k \langle \sum_{\sigma \in A_3} \mathbf{R}^Q(e_{\sigma(i)}, e_{\sigma(j)}) e_{\sigma(k)}, V \rangle \\
&= -2 \sum_{i < j < k} e_i \cdot e_j \cdot e_k \langle 0, V \rangle = 0
\end{aligned}$$

by the corollary 2.19. Continuing with the remaining remaining terms, by Clifford multiplication and corollary 2.19,

$$\begin{aligned}
&\sum_{i,j} e_j \langle \mathbf{R}^Q(V, e_i)[e_i], e_j \rangle + e_i \langle \mathbf{R}^Q(V, e_j)[e_j], e_i \rangle \\
&= \sum_{i,j} e_j \langle \mathbf{R}^Q(V, e_i)[e_i], e_j \rangle + \sum_{i,j} e_i \langle \mathbf{R}^Q(V, e_j)[e_j], e_i \rangle \\
&\quad = 2 \sum_{i,j} e_j \langle \mathbf{R}^Q(V, e_i)[e_i], e_j \rangle = -2 \sum_{i,j} \mathbf{R}^Q(e_i, V)[e_i] = 2\text{ricc}(V)
\end{aligned}$$

■

**Theorem 3.36** (Transverse Lichnerowicz Identity). *Let  $S$  be the transverse spinor bundle for a transverse  $\text{Spin}$  structure and let  $\nabla$  be the Atiyah-Singer connection on  $S$ . Then  $\not\partial \not\partial \phi = \nabla^* \nabla \phi + \frac{1}{4} \text{scal}^Q \cdot \phi$  for  $\phi \in \Gamma(S)$ .*

*Proof.* By construction the spinor bundle is a self-adjoint Dirac bundle and the spinor connection is a Dirac connection. The only thing that needs to be shown is that  $\mathfrak{R}^W = \frac{1}{4}\text{scal}$ . For any spinor  $\phi \in \Gamma(\mathbf{S})$  we have

$$\mathfrak{R}^W[\phi] = \frac{1}{2} \sum_{i,j} e_i \cdot e_j \cdot \mathbf{R}^\nabla(e_i, e_j)[\phi] = \frac{1}{2} \sum_{i,j} e_i \cdot e_j \cdot \mathfrak{R}^{\text{Cl}}(e_i, e_j) \curvearrowright \phi$$

Consequently, we will drop  $\phi$  and consider

$$\begin{aligned} 8\mathfrak{R}^W &= 4 \sum_{i,j} e_i \cdot e_j \cdot \mathfrak{R}^{\text{Cl}}(e_i, e_j) = \sum_{i,j} \sum_{k,l} e_i \cdot e_j \langle \mathbf{R}^Q(e_i, e_j)[e_k], e_l \rangle e_k \cdot e_l \\ &= - \sum_l \left( \sum_{i,j} \sum_k e_i \cdot e_j \cdot e_k \langle \mathbf{R}^Q(e_i, e_j)[e_l], e_k \rangle \right) \cdot e_l \\ &= -2 \sum_l \mathfrak{R}^W[e_l] \cdot e_l \\ &= -2 \sum_l \text{ricc}^Q(e_l) \cdot e_l \end{aligned}$$

by eq. (3.8) and theorem 3.35.

$$\begin{aligned} &= \sum_{l,k,j} e_k \cdot \langle \mathbf{R}^Q(e_l, e_j)[e_j], e_k \rangle \cdot e_l + \sum_{l,k,j} e_k \cdot \langle \mathbf{R}^Q(e_k, e_j)[e_j], e_l \rangle \cdot e_l \\ &= - \sum_l \text{ricc}^Q(e_l) \cdot e_l - \sum_k e_k \cdot \text{ricc}^Q(e_k) \\ &= -2 \sum_i \langle \text{ricc}^Q(e_i), e_i \rangle \\ &= 2\text{scal} \end{aligned}$$

■

**Corollary 3.37** (Transverse Lichernowicz Identity). *Let  $\mathbf{S}^{\mathbb{C}}$  be the transverse spinor bundle for a transverse  $\text{Spin}^{\mathbb{C}}$  structure and let  $\nabla^A$  be a Seiberg-Witten connection for  $\mathbf{S}^{\mathbb{C}}$ . Then  $\not\partial^A \not\partial^A \varphi = (\nabla^A)^* \nabla^A \varphi + \frac{1}{4} \text{scal}^Q \cdot \varphi + \frac{1}{2} \mathbf{F}^A \cdot \varphi$ .*

*Proof.* We do everything locally where  $\mathbf{S}^{\mathbb{C}} = \mathbf{S}^{\mathbb{C}} \otimes \mathbf{L}$  and we simply need to calculate  $\mathfrak{R}^W$  applied to a simple local section  $\phi \otimes \zeta$ . Then

$$\begin{aligned}
\mathfrak{R}^W[\phi \otimes \zeta] &= \frac{1}{2} \sum_{i,j} e_i \cdot e_j R^A(e_i, e_j)[\phi \otimes \zeta] \\
&= \frac{1}{2} \sum_{i,j} (e_i \cdot e_j \cdot R^\nabla(e_i, e_j)[\phi]) \otimes \zeta + \frac{1}{2} \sum_{i,j} e_i \cdot e_j \cdot \phi \otimes (F^A(e_i, e_j)[\zeta]) \\
&= \frac{1}{4} \text{scal}^Q \phi \otimes \zeta + \frac{1}{2} \left( \sum_{i,j} e_i \cdot e_j F^A(e_i, e_j) \right) \phi \otimes \zeta \\
&= \left( \frac{1}{4} \text{scal}^Q + \frac{1}{2} F^A \right) (\phi \otimes \zeta)
\end{aligned}$$

■

# Chapter 4

## Elliptic Theory

### 4.1 Sobolev Spaces

We collect some classical theorems of Sobolev theory that are useful in the analysis of the Seiberg-Witten equations in chapter 5, and investigate the criteria that is necessary for their adaptation to a transverse theory for foliations.

#### 4.1.1 Classical Theory

**Definition 4.1** ( $L^p$  norm on  $\Gamma(E)$ ). Let  $E$  be a Riemannian (Hermitian) vector bundle over a closed oriented Riemannian manifold  $M$  and  $p \geq 1$ . Then for any  $\phi \in \Gamma(E)$  the function  $\|\phi\|_{L^p} = \left(\int_M |\phi|^p \mu^M\right)^{1/p}$  is a norm on  $\Gamma(E)$ .

**Definition 4.2** (Sobolev Norms on  $\Gamma(E)$ ). Let  $E$  be Riemannian (Hermitian) vector bundle over a closed oriented Riemannian manifold  $M$ ,  $\nabla$  be a metric connection on  $E$  and  $p \geq 1$ . The  $k^{\text{th}}$ -order,  $p^{\text{th}}$  degree Sobolev norm of  $\phi \in \Gamma(E)$  is defined by  $\|\phi\|_{L_k^p} = \sum_{i=0}^k \|\nabla_i^k \phi\|_{L^p}$  and the  $k^{\text{th}}$ -order,  $p^{\text{th}}$  Sobolev sections  $L_k^p(\Gamma(E))$  is the completion of  $\Gamma(E)$  with respect to this norm.

While the definition of  $\|\cdot\|_{L_k^p}$  depends on a choice of metric and connection, it can be shown that the space  $L_k^p(E)$  does not depend on either [Aub98; Weh04]. The utility of such Sobolev spaces is that if  $D : \Gamma(E) \rightarrow \Gamma(F)$  is an  $m^{\text{th}}$ -order differential

operator then  $D$  extends, by density, to a continuous operator of  $L_k^q(E) \rightarrow L_{k-m}^q(F)$ . Furthermore, from the definition it is clear that if  $j \leq k$  then  $\|\phi\|_{L_j^q} \leq \|\phi\|_{L_k^q}$  and so the natural inclusion of  $L_k^q(E) \subset L_j^q(E)$  is continuous, and from the compactness of the manifold and Hölder's inequality, if  $1 \leq p \leq q$  then  $\|\phi\|_{L_k^p} \lesssim \|\phi\|_{L_k^q}$ , and so the natural inclusion  $L_k^q(E) \subset L_k^p(E)$  is continuous; the following theorems express a similar relationship when the degrees and the orders differ. For the proofs see Chapter 4 of [Wel08] for  $L_k^2$  sections of vector bundles over closed Riemannian manifolds, Chapter 2 of [Aub98] for  $L_k^p$  functions on Riemannian manifolds, and Chapter 5 of [Eva10] for  $L_k^p$  functions on specific domains of  $\mathbb{R}^n$  and for the Multiplication theorems see Appendix C of [Sal99] and Appendix B of [Weh04].

**Proposition 4.3** (Sobolev Embedding Theorem). *Let  $E$  be a Riemannian (Hermitian) vector bundle over a closed oriented Riemannian manifold  $M$ ,  $m = \dim M < \infty$ ,  $1 \leq j < k$  and  $1 \leq q, p < \infty$ . Then if  $k - m/q \geq j - m/p$  then  $L_k^q(\Gamma(E)) \hookrightarrow L_j^p(\Gamma(E))$  is continuous, that is  $\|\phi\|_{L_j^p} \lesssim \|\phi\|_{L_k^q}$*

**Proposition 4.4** (Rellich-Kondrakov Compactness Theorem). *Let  $E$  be a Riemannian (Hermitian) vector bundle over a closed oriented Riemannian manifold  $M$ . If  $1 \leq j < k$  and  $1 \leq q \leq p$  and*

1. *if  $k - m/q > j - m/p$  then  $L_k^q(\Gamma(E)) \hookrightarrow L_j^p(\Gamma(E))$  is a compact embedding.*

2. *if  $k - m/q > j$  then  $L_k^q(\Gamma(E)) \hookrightarrow C^j(M, E)$  is a compact embedding.*

It should be noted that there is a complementary norm on  $C^k(X, E)$  and a more refined version of embedding it into Hölder spaces, we will however, only be concerned that the functions in  $L_k^q$  are in fact differentiable, in particular this directly leads to

$\bigcap_{k=0}^{\infty} L_k^q(\Gamma(E)) = \Gamma(E)$ . Often it is convenient to refer to  $m/q - k$  as the grade of the Sobolev space  $L_k^q(E)$ , consequently proposition 4.3 says that  $L_k^q(E) \hookrightarrow L_j^p(E)$  is an embedding if the both the grade and the order of the former is larger than the grade and the order of the latter. To carry out the analysis of the classical Seiberg-Witten equations there are three main points, Rellich-Kondrakov proposition 4.4, Elliptic regularity and particular multiplicative properties of Sobolev spaces.

**Proposition 4.5** (Elliptic Regularity). *Let  $E$  be a Riemannian (Hermitian) vector bundle over an bundle-like oriented Riemannian foliation of a closed manifold  $M$ . Furthermore, let  $D$  be an elliptic operator on  $L_k^p(\Gamma(E)) \rightarrow L_{k-m}^p(\Gamma(E))$  then  $\|\phi\|_{L_{k+1}^p} \lesssim \|D\phi\|_{L_k^p} + \|\phi\|_{L_k^p}$  for any  $\phi \in L_k^p(\Gamma(E))$*

**Proposition 4.6** (Sobolev Multiplication). *Let  $\phi, \psi$  be complex valued functions on a closed oriented Riemannian manifold  $M$ ,  $m = \dim M < \infty$ ,  $1 \leq j \leq k$  and  $1 \leq q, p < \infty$  and  $j - m/p \leq k - m/q$ . Then*

1. if  $k - m/q > 0$  then  $\|\phi\psi\|_{L_j^p} \lesssim \|\phi\|_{L_k^q} \|\psi\|_{L_j^p}$
2. if  $k - m/q = 0$  and for fixed  $\epsilon > 0$  then  $\|\phi\psi\|_{L_j^p} \lesssim \|\phi\|_{L_k^q} \|\psi\|_{L_j^{p+\epsilon}}$ .
3. if  $j - m/p = k - m/q < 0$  then  $\|\phi\psi\|_{L_j^p} \lesssim \|\phi\|_{L_k^q} (\|\psi\|_{L_j^{m/j}} + \|\psi\|_{L^\infty})$

### 4.1.2 Basic Sobolev Theory

To consider a Seiberg-Witten theory for a codimension four foliation, we will postulate that a basic version of Sobolev embedding, multiplication, Rellich-Kondrakov compactness and elliptic regularity theorems hold. This necessity is due to the classical constructions of theorem 5.17 and theorem 5.27 which require delicate relationships

in the grade of the spaces involved, Classically, the grade depends on the dimension of the manifold, consequently, a ‘basic version’ of the Sobolev embedding, Sobolev multiplication and Rellich-Kondrakov compactness theorems ought to use a ‘transverse grade’ in it’s formulation, that is, a grade where the codimension of the foliation, rather than the dimension of the manifold, is used. As stated, the following hypotheses are known to hold only for the discrete foliation, simple foliations and compact Hausdorff foliations (foliations with a compact leaves and Hausdorff transverse space), which correspond, respectively, to the classical theory, Riemannian submersions (thus classical theory), and an orbifold theory (see [GN79; GN79]), we assume throughout the sequel the following ‘basic versions’ hold.

It is natural to consider under what circumstances the restriction of this classical case to the basic functions works. The main observation is that for an  $l$ -dimensional (codimension  $n$ ) foliation  $\mathcal{F}$ , the condition on the grades for Sobolev embedding is  $k - (n + l)/q \geq j - (n + l)/p$  leads us to consider  $l/p - l/q + k - n/q \geq j - n/p$  and consequently if we consider the ‘basic grades’  $k - n/q \geq j - n/p$  with the additional assumption of  $q \geq p$  the basic version implies the classical version.

**Hypothesis 4.7** (Basic Sobolev Theorem). *Let  $E$  be a basic Riemannian (Hermitian) vector bundle over an  $n$ -codimensional bundle-like oriented Riemannian foliation  $\mathcal{F}$  of a closed manifold  $M$ ,  $n = \text{codim } \mathcal{F}$ ,  $1 \leq j < k$  and  $1 \leq p \leq q < \infty$ . Then if  $k - n/q \geq j - n/p$  then  $L_k^q(\Gamma_B(E)) \hookrightarrow L_j^p(\Gamma_B(E))$  is continuous, that is  $\|\phi\|_{L_j^p} \lesssim \|\phi\|_{L_k^q}$*

**Hypothesis 4.8** (Basic Rellich-Kondrakov Compactness Theorem). *Let  $E$  be a basic Riemannian (Hermitian) vector bundle over an  $n$ -codimensional bundle-like oriented Riemannian foliation  $\mathcal{F}$  of a closed manifold  $M$ ,  $n = \text{codim } \mathcal{F}$ ,  $1 \leq j < k$  and*

$1 \leq p \leq q < \infty$ . Then

1. if  $k - n/q > j - n/p$  then  $L_k^q(\Gamma_B(E)) \hookrightarrow L_j^p(\Gamma_B(E))$  is a compact embedding.
2. if  $k - n/q > j$  then  $L_k^q(\Gamma_B(E)) \hookrightarrow C_B^j(M, E)$  is a compact embedding.

Where  $C_B^j(M, E)$  are the basic sections with  $E$  with  $j$  partial derivatives, moreover since the orders are integers we have the following observation: for a one-dimensional ( $l = 1$ ) foliation of an odd-dimensional manifold the inequality  $k - n/2 > j$  implies, since  $k, j$  are integers and  $n$  is even,  $k - m/2 > j$  and consequently the basic version implies the classical version.

Since a basic elliptic operator cannot be extended to an elliptic operator on the space of all sections, one wouldn't expect the following to hold except under the previous cases mentioned, or with some addition conditions satisfied by the foliation.

**Hypothesis 4.9** (Basic Elliptic Regularity). *Let  $E$  be a Riemannian (Hermitian) vector bundle over an bundle-like oriented Riemannian foliation of a compact manifold  $M$ . Furthermore, let  $D$  be an transversely elliptic operator on  $L_k^2(\Gamma_B(E))$  then*

$$\|\phi\|_{L_{k+1}^p} \lesssim \|D\phi\|_{L_k^p} + \|\phi\|_{L_k^p} \text{ for any } \phi \in L_k^p(\Gamma_B(E))$$

The multiplication theorems hold for basic sections of foliated vector bundles as modules over basic smooth functions, consequently we state the theorem for only basic functions which will extend to basic sections of other vector bundles.

**Hypothesis 4.10** (Basic Sobolev Multiplication). *Let  $\phi, \psi$  be complex valued basic functions on a bundle-like oriented Riemannian foliation  $\mathcal{F}$  of a compact manifold  $M$ ,  $n = \text{codim } \mathcal{F}$ ,  $1 \leq j \leq k$  and  $1 \leq q, p < \infty$  and  $j - n/p \leq k - n/q$  then*

1. if  $k - n/q > 0$  then  $\|\phi\psi\|_{L_j^p} \lesssim \|\phi\|_{L_k^q} \|\psi\|_{L_j^p}$

2. if  $k - n/q = 0$  and for fixed  $\epsilon > 0$  then  $\|\phi\psi\|_{L_j^p} \lesssim \|\phi\|_{L_k^q} \|\psi\|_{L_j^{p+\epsilon}}$ .

3. if  $j - n/p = k - n/q < 0$  then  $\|\phi\psi\|_{L_j^p} \lesssim \|\phi\|_{L_k^q} (\|\psi\|_{L_j^{n/j}} + \|\psi\|_{L^\infty})$

### 4.1.3 Examples of Basic Sobolev Theory

As has been mentioned, the hypotheses in the previous subsection do not hold for an arbitrary Riemannian foliation. Classically, the Seiberg-Witten equations are defined on a four-dimensional manifold, and consequently one does not necessarily need the full generalizations in the previous section, as we will see in [SW94] we will specifically need 4-codimensional versions of a few particular cases of the Sobolev multiplication, Sobolev embedding and of Rellich-Kondrakov compactness theorems. Any appropriate Seiberg-Witten theory that generalizes to 4-codimensional foliations ought to reduce to the classic theory for particularly nice foliations, in particular, these are cases where there is developed ‘basic theory’ for ellipticity and basic Sobolev theorems. In particular, recall that a simple foliation is one that can be defined by a surjective submersion with connected level sets. Of course, in the case of a simple foliation the transverse space is a manifold, since the range of the surjection equals the transverse space. Consequently, the transverse space is Hausdorff and each leaf has no leaf holonomy (a group associated to each leaf which describes how the leaves of the foliation ‘wind around’ another leaf see [Sha97; Ton97]). In this case the transverse space is a manifold and the classical theory applies. A particularly simple example of this is a zero-dimensional foliation in which case the transverse space *is the manifold*. Furthermore, there is a partial converse to this: if a foliation has a Hausdorff transverse space, each leaf has no holonomy, and the leaves are compact then the foliation is simple [Sha97]. Consequently, an interesting (non-classical) foliated Seiberg-Witten

theory ought to apply when one of these hypotheses is relaxed. When the condition of holonomy is relaxed, a foliation that has compact leaves and the transverse space is Hausdorff gives the transverse space a natural orbifold structure [GN80]. Orbifolds have partitions of unity [GN79] consequently the foliation has a Riemannian structure. Conversely, a Riemannian foliation with compact leaves has a Hausdorff transverse space and is consequently a Riemannian orbifold [Mol88]. Consequently, the basic elliptic, Sobolev and Rellich-Kondrakov theorems from the previous section all have a well developed theory from [GN79; GN80]. The approach by [KLW16] used Molino's structure theory [Mol88] theory for Riemannian foliations.

# Chapter 5

## Transverse Seiberg-Witten Theory

Suppose that  $\mathcal{F}$  is a 4-codimensional transversely oriented Riemannian foliation on a closed oriented Riemannian manifold. Without any loss of generality may assume that the foliation is bundle-like and tense, further suppose that the foliation is  $\text{Spin}^{\mathbb{C}}$ . For any  $\text{Spin}^{\mathbb{C}}$  structure we can consider the associated basic Seiberg-Witten equations defined by.

$$\begin{aligned} F_+^A &= \phi \otimes \phi^* - \frac{1}{2} |\phi|^2 \text{id}_{\mathbb{S}_+^{\mathbb{C}}} \\ \not{\nabla}^A \phi &= \frac{1}{2} \tau \cdot \phi \end{aligned} \tag{5.1}$$

where  $\phi \in \Gamma_B(\mathbb{S}^+)$  and  $A \in \mathcal{A}_B$  where  $\mathcal{A}_B$  are the basic connections on  $\det(\mathbb{S}^{\mathbb{C}})$  bundle. We first note that, since the dimension of  $Q$  is even,  $\mu^Q \cdot V = -V \cdot \mu^Q$  for any vector field  $V$  (see example 3.1) and consequently both  $\not{\nabla}^A \phi$  and  $\tau \cdot \phi$  are in  $\Gamma(\mathbb{S}_-^{\mathbb{C}})$ . The second equation is equivalent to  $(\not{\nabla}^A - \frac{1}{2} \tau) \phi = 0$  which is, since both  $\nabla^A$  and  $\tau$  are basic, self-adjoint according to corollary 3.27. It ought to be noted that the main difference between the classical equations and these basic equations is the appearance of the mean curvature and that they reduce to the classical equations in the case of discrete or harmonic ( $\tau = 0$ ) foliations, see [Wit94; SW94; Mor96; Sal99] for the classical equations and theory.

## 5.1 A Priori Estimates

Let  $\mathcal{F}$  be a bundle-like Riemannian 4-codimensional foliation of a closed manifold  $M$  with a transverse  $\text{Spin}^{\mathbb{C}}$  structure. Then from corollary 3.37 the Lichernowicz identity

$$\not\partial^A \not\partial^A \phi = (\nabla^A)^* \nabla^A \phi + \frac{1}{4} \text{scal}^Q \cdot \phi + \frac{1}{2} F^A \cdot \phi$$

must be satisfied for any  $A \oplus \phi \in \mathcal{A}_B \oplus \Gamma_B(\mathbb{S}_+^{\mathbb{C}})$ . Furthermore, if  $A \oplus \phi$  is a solution to eq. (5.1) then

$$\begin{aligned} \not\partial^A \not\partial^A \phi &= \frac{1}{2} \not\partial^A (\tau \cdot \phi) \\ &= \frac{1}{2} \sum_i e_i \cdot \nabla_i^A (\tau \cdot \phi) \\ &= \frac{1}{2} \sum_i e_i \cdot \nabla_i^Q \tau \cdot \phi + e_i \cdot \tau \cdot \nabla_i^A \phi \\ &= \frac{1}{2} \not\partial^Q \tau \cdot \phi - \frac{1}{2} \tau \cdot \not\partial^A \phi - \nabla_{\tau}^A \phi \\ &= \frac{1}{2} \not\partial^Q \tau \cdot \phi - \frac{1}{4} \tau^2 \cdot \phi - \nabla_{\tau}^A \phi \\ &= \frac{1}{2} \not\partial^Q \tau \cdot \phi + \frac{1}{4} |\tau|^2 \cdot \phi - \nabla_{\tau}^A \phi \end{aligned}$$

and since  $\phi \in \Gamma_B(\mathbb{S}_+^{\mathbb{C}})$

$$\begin{aligned} \frac{1}{2} F^A \curvearrowright \phi &= \frac{1}{2} F_+^A \curvearrowright \phi \\ &= \frac{1}{2} \left( \phi \otimes \phi^* - \frac{1}{2} |\phi|^2 \text{id}_{\mathbb{S}_+^{\mathbb{C}}} \right) \phi \\ &= \frac{1}{2} \phi |\phi|^2 - \frac{1}{4} |\phi|^2 \phi \\ &= \frac{1}{4} |\phi|^2 \phi \end{aligned}$$

which yields

$$\frac{1}{2} \not\partial^Q \tau \cdot \phi + \frac{1}{4} |\tau|^2 \cdot \phi - \nabla_{\tau}^A \phi = (\nabla^A)^* \nabla^A \phi + \frac{1}{4} \text{scal}^Q \cdot \phi + \frac{1}{4} |\phi|^2 \phi$$

theorem 3.22 then gives us

$$(\nabla^A)^* \nabla^A \phi = \frac{1}{4} (2\mathcal{X}^Q \tau + |\tau|^2 - \text{scal}^Q - |\phi|^2) \cdot \phi \quad (5.2)$$

It is from this equation, that all of the subsequent analysis in chapter 5 depends since they will provide bounds on the solutions, that will be developed shortly. Before we get to those bounds however, we make an observation that we will need for those bounds.

**Lemma 5.1.** *Let  $Q$  be the transverse bundle for a tense bundle-like Riemannian foliation and  $\tau$  be the mean curvature. Then  $\mathcal{X}^Q \tau$  is real-valued.*

*Proof.* Consider,

$$\begin{aligned} \mathcal{X}^Q \tau &= \sum_i^n e_i \cdot \nabla_i^Q \tau = \sum_{i,k} e_i \cdot e_k \langle e_k, \nabla_i^Q \tau \rangle \\ &= \sum_{i,k} e_i \cdot e_k \nabla_i^Q \langle e_k, \tau \rangle - \sum_{i,k} e_i \cdot e_k \langle \nabla_i^Q e_k, \tau \rangle \\ &= \sum_{i,k} e_i \cdot e_k \partial_i \kappa^k - \sum_{i,k} e_i \cdot e_k \langle \pi \nabla_i^M e_k, \tau \rangle \end{aligned}$$

without any loss of generality we may assume that we are in geodesic coordinates, so the second term vanishes and using the properties of Clifford multiplication

$$= \sum_{i < k \leq n} e_i \cdot e_k (\partial_i \kappa^k - \partial_k \kappa^i) - \sum_j^n \partial_j \kappa^j$$

and since the foliation is tense the first term vanishes by proposition 2.25. Consequently, at any point  $\mathcal{X}^Q \tau$  is real-valued. ■

**Theorem 5.2** (Pointwise Spinor Bound). *Let  $A \oplus \phi \in \mathcal{A}_B \oplus \Gamma_B(\mathbb{S}_+^{\mathbb{C}})$  be a solution to eq. (5.1). Then there exists a  $0 \leq K < \infty$  such that  $|\phi|^2 \leq K$ . Furthermore,  $\|\phi\|_{L^p} \leq \sqrt{K}(\text{vol}M)^{1/p}$  for all  $1 \leq p \leq \infty$ .*

*Proof.* First, since  $\phi$  is smooth on the compact manifold, let  $x_0 \in \operatorname{argmax}_{x \in M} |\phi(x)|^2$ .

The pointwise inner product of eq. (5.2) with  $\phi$  then yields

$$\langle (\nabla^A)^* \nabla^A \phi, \phi \rangle = \frac{1}{4} |\phi|^2 (2\mathcal{V}^Q \tau + |\tau|^2 - \operatorname{scal}^Q - |\phi|^2) \quad (5.3)$$

and the righthand side is real by lemma 5.1. Consequently, in normal coordinates at  $x_0$ ,

$$\begin{aligned} \Delta |\phi|^2 + |\nabla^A \phi|^2 &= \Re \langle (\nabla^A)^* \nabla^A \phi, \phi \rangle \\ &= \frac{1}{4} |\phi|^2 (2\mathcal{V}^Q \tau + |\tau|^2 - \operatorname{scal}^Q - |\phi|^2) \end{aligned}$$

and since  $0 \leq \Delta |\phi|^2$  at  $x_0$  we have

$$0 \leq \frac{1}{4} |\phi|^2 (2\mathcal{V}^Q \tau + |\tau|^2 - \operatorname{scal}^Q - |\phi|^2)$$

in which case there are two possibilities, if  $|\phi(x_0)|^2 = 0$  then we have that  $\phi = 0$ , thus we have the following pointwise bound.

$$|\phi(x)|^2 \leq (2\mathcal{V}^Q \tau + |\tau|^2 \cdot \phi - \operatorname{scal}^Q) \Big|_{x_0} \leq \max_{x \in M} \{0, 2\mathcal{V}^Q \tau + |\tau|^2 - \operatorname{scal}^Q\} = K \quad (5.4)$$

which establishes the first part. The second part follows from integrating an appropriate power over the compact  $M$ . ■

**Theorem 5.3** (Global Spinor Bound). *Let  $A \oplus \phi \in \mathcal{A}_B \oplus \Gamma_B(\mathbf{S}_+^{\mathbb{C}})$  be a solution to eq. (5.1). Then  $\|\nabla^A \phi\|_{L^2} \leq \frac{1}{2} K^2 \operatorname{vol} M$ .*

*Proof.* Integrate eq. (5.3) to get

$$\begin{aligned} \langle \nabla^A \phi, \nabla^A \phi \rangle &\leq \frac{1}{4} \max (|\phi|^2 |2\mathcal{V}^Q \tau + |\tau|^2 - \operatorname{scal}^Q| + |\phi|^4) \operatorname{vol} M \\ &\leq \frac{1}{2} K^2 \operatorname{vol} M \end{aligned}$$

■

**Theorem 5.4** (Pointwise Curvature Bounds). *Let  $A \oplus \phi \in \mathcal{A}_B \oplus \Gamma_B(\mathbf{S}_+^{\mathbb{C}})$  be a solution to eq. (5.1). Then  $|\mathbf{F}_+^A|^2 = 2|\phi|^4 \leq 2K^2$ . Furthermore,  $\|\mathbf{F}_+^A\|_{L^2} \leq 2K^2(\mathbf{vol}M)^{1/2}$*

*Proof.* For any self-dual two form  $\omega \in \Omega_{B^+}^2(M)$

$$\omega(e_1, e_2) = \omega(e_3, e_4), \quad \omega(e_1, e_3) = \omega(e_2, e_4), \quad \omega(e_1, e_4) = \omega(e_2, e_3)$$

and thus

$$|\omega|^2 = \sum_{i < j} \omega(e_i, e_j)^2 = 2(\omega(e_1, e_4)^2 + \omega(e_2, e_4)^2 + \omega(e_3, e_4)^2)$$

In particular, when  $\mathbf{F}_+^A = \phi \otimes \phi^* - \frac{1}{2}|\phi|^2$  is expanded with respect to the basis in example 3.9 we have

$$\mathbf{F}_+^A(e_1, e_4) = \phi_2 \bar{\phi}_1 + \phi_1 \bar{\phi}_2, \quad \mathbf{F}_+^A(e_2, e_4) = -\phi_2 \bar{\phi}_1 + \phi_1 \bar{\phi}_2,$$

$$\mathbf{F}_+^A(e_3, e_4) = -i|\phi_2|^2 - i|\phi_1|^2$$

and thus  $|\mathbf{F}_+^A|^2 = 2(|\phi_1|^2 + |\phi_2|^2)^2 = 2|\phi|^4 \leq 2K^2$ , establishes the first part. The second part, follows from integrating over the manifold. ■

**Theorem 5.5** (Global Curvature Bounds). *Let  $A \oplus \phi \in \mathcal{A}_B \times \Gamma_B(\mathbf{S}_+^{\mathbb{C}})$  be a solution to eq. (5.1). Then  $\|\mathbf{dF}_+^A\|_{L^2} \leq K^3(\mathbf{vol}M)^2 + \frac{1}{2}K^3\mathbf{vol}M$ .*

*Proof.* Let  $\mathbf{A}$  be skew-symmetrization. Since the connection is torsionless  $\mathbf{dF}_+^A = \mathbf{A}(\nabla^Q \mathbf{F}_+^A)$ . Consequently,  $|\mathbf{dF}_+^A|_{L^2} = |\mathbf{A}(\nabla^Q \mathbf{F}_+^A)|_{L^2} \leq |\nabla^Q \mathbf{F}_+^A|_{L^2}$  and since

$$\nabla^Q \mathbf{F}_+^A = \nabla^A \phi \otimes \phi^* + \phi \otimes \nabla^A \phi^* - \frac{1}{2} \nabla^Q |\phi|^2$$

Integrating, yields

$$\|\nabla^Q \mathbf{F}_+^A\|_{L^2} \leq \|\nabla^A \phi \otimes \phi^*\|_{L^2} + \|\phi \otimes \nabla^A \phi^*\|_{L^2} + \frac{1}{2} \|\nabla^A |\phi|^2\|_{L^2}$$

$$\begin{aligned}
&\leq 2\|\nabla^A\phi\|_{L^2}\|\phi\|_{L^2} + \|\nabla^A\phi\|\|\phi\|_{L^2} \\
&\leq 2K\|\nabla^A\phi\|_{L^2}\mathbf{vol}M + K\|\nabla^A\phi\|_{L^2} \\
&\leq K^3(\mathbf{vol}M)^2 + \frac{1}{2}K^3\mathbf{vol}M
\end{aligned}$$

■

We will use Sobolev spaces to perform the analysis in chapter 5 consequently we will use the previous bounds to obtain appropriate bounds that will be necessary in the sequel. Note that theorem 5.3 directly give bounds in terms of a Sobolev norm with respect to  $A$ , however in eq. (5.1),  $A$  is a variable and we certainly don't want our norm on  $\Gamma_B(\mathbf{S}_+^{\mathbb{C}})$  to vary. Recall that for any two unitary connections  $A, A_0$  on the determinant bundle then  $A - A_0 = \alpha \in \Omega_B^1(i\mathbb{R})$ , consequently we fix  $A_0$  and use that to express spinor solutions in terms of that Sobolev norm.

**Theorem 5.6** (Sobolev Spinor Bounds). *Let  $A \oplus \phi \in \mathcal{A}_B \oplus \Gamma_B(\mathbf{S}_+^{\mathbb{C}})$  be solutions to eq. (5.1) such that  $A - A_0 = \alpha$ . Then  $\|\phi\|_{L_1^2} \leq \frac{1}{2}K^2\mathbf{vol}M + (\|\alpha\|_{L^2} + 1)\sqrt{K}(\mathbf{vol}M)^{1/p}$ , where the  $L_1^2$  norm on  $\Gamma_B(\mathbf{S}_+^{\mathbb{C}})$  is defined using  $A_0$ .*

*Proof.* Recall that  $\nabla^{A_0} = \nabla^A - \alpha$  then repeatedly using the Cauchy-Schwarz inequality

$$\begin{aligned}
\|\nabla^{A_0}\phi\|_{L^2}^2 &= \langle \nabla^A\phi, \nabla^A\phi \rangle + 2\Re\langle \nabla^A\phi, \alpha\phi \rangle + \langle \alpha\phi, \alpha\phi \rangle \\
&\leq \|\nabla^A\phi\|_{L^2}^2 + 2\|\nabla^A\phi\|_{L^2}\|\alpha\phi\|_{L^2} + \|\alpha\phi\|_{L^2}^2 \\
&\leq \|\nabla^A\phi\|_{L^2}^2 + 2\|\nabla^A\phi\|_{L^2}\|\alpha\|_{L^2}\|\phi\|_{L^2} + \|\alpha\|_{L^2}^2\|\phi\|_{L^2}^2 \\
&\leq (\|\nabla^A\phi\|_{L^2} + \|\alpha\|_{L^2}\|\phi\|_{L^2})^2
\end{aligned}$$

Consequently,

$$\|\phi\|_{L_1^2} \leq \|\nabla^A\phi\|_{L^2} + (\|\alpha\|_{L^2} + 1)\|\phi\|_{L^2}$$

$$\leq \frac{1}{2}K^2 \mathbf{vol}M + (\|\alpha\|_{L^2} + 1)\sqrt{K}(\mathbf{vol}M)^{1/p}$$

■

## 5.2 Preliminaries

Let us summarize a number of tools necessary for the analysis of the Seiberg-Witten theory. Since there are a plethora of indices in the Sobolev multiplication theorems, we collect some general lemmas here for 4-codimensional foliations that will ease the discussion later. Throughout this section let  $M$  be an  $m$ -dimensional compact manifold foliated by a 4 codimensional bundle-like Riemannian foliation.

**Theorem 5.7** (Banach Modules).

1. if  $k \geq 3$  then the multiplication map  $L_k^2 \times L_j^2 \rightarrow L_j^2$  is continuous for all  $j \leq k$ .
2. if  $k = 2$  then the multiplication map  $(L_2^2 \cap L^\infty) \times L_2^2 \rightarrow L_2^2$  is continuous.
3. if  $k = 2$  then the multiplication map  $(L_2^2 \cap L^\infty) \times (L_2^2 \cap L^\infty) \rightarrow (L_2^2 \cap L^\infty)$  is continuous.
4. if  $k = 1$  then the multiplication map  $(L_1^2 \cap L^\infty) \times L_1^2 \rightarrow L_1^2$  is continuous.

*Proof.* If  $k \geq 3$ , then  $k - 4/2 > 0$  and the result follows directly from Sobolev multiplication item 1. For  $k = 2$ , assume that  $(\phi, \psi) \in (L_2^2 \cap L^\infty) \times L_2^2$  then item 3 yields  $\|\phi \cdot \psi\|_{L_2^2} \lesssim \|\phi\|_{L_2^3} \|\psi\|_{L_2^2} \lesssim \|\phi\|_{L^6} \|\psi\|_{L_2^2} \lesssim \|\phi\|_{L^\infty} \|\psi\|_{L_2^2}$  by Sobolev embedding hypothesis 4.7 and the compactness of  $M$  thus the multiplication is continuous. The fact that the multiplication  $(L_2^2 \cap L^\infty) \times (L_2^2 \cap L^\infty) \rightarrow (L_2^2 \cap L^\infty)$  follows from the previous and the fact that  $L^\infty \times L^\infty \rightarrow L^\infty$  holds on the compact manifold  $M$ .

For  $k = 1$ , assume that  $(\phi, \psi) \in (L_1^2 \cap L^\infty) \times L_1^2$  then item 3 yields  $\|\phi \cdot \psi\|_{L_2^2} \lesssim \|\psi\|_{L_1^2}(\|\phi\|_{L_1^2} + \|\phi\|_{L^\infty})$  which is finite by hypothesis. ■

A particular bootstrapping technique that is dependent on the Sobolev multiplication will be used regularly in the sequel. Since the proofs will often use the same chain of Sobolev multiplications we state that particular lemma here.

**Lemma 5.8** (Negative Grade Bootstrapping). *Let  $\{\phi_n\}_{n=1}^\infty$  and  $\{\psi_n\}_{n=1}^\infty$  be basic sequences such that*

$$\|\phi_n\|_{L_2^2} \lesssim \|\phi_n \cdot \psi_n\|_{L_1^2} + \|\phi_n\|_{L_1^2} \quad (5.5)$$

*If  $\|\psi_n\|_{L_1^2}$ ,  $\|\phi_n\|_{L_1^2}$  and  $\|\phi_n\|_{L^\infty}$  are all bounded, then  $\|\phi_n\|_{L_2^2}$  is bounded.*

*Proof.* Since the basic grade  $1 - 4/2 < 0$  we use Sobolev multiplication item 3

$$\begin{aligned} \|\phi_n\|_{L_2^2} &\lesssim \|\psi_n \cdot \phi_n\|_{L_1^2} + \|\phi_n\|_{L_1^2} \\ &\lesssim \|\psi_n\|_{L_1^2}(\|\phi_n\|_{L_1^2} + \|\phi_n\|_{L^\infty}) + \|\phi_n\|_{L_1^2} \end{aligned}$$

and the right hand side is bounded by hypothesis. ■

**Lemma 5.9** (Null Grade Bootstrapping). *Let  $\{\phi_n\}_{n=1}^\infty$  and  $\{\alpha_n\}_{n=1}^\infty$  be basic sequences such that*

$$\|\phi_n\|_{L_{k+1}^p} \lesssim \|\phi_n \cdot \alpha_n\|_{L_k^p} + \|\phi_n\|_{L_k^p} \quad (5.6)$$

*for all  $k \leq 2 \leq p$  such that  $p + k = 4$ . If  $\|\alpha_n\|_{L_2^2}$  and  $\|\phi_n\|_{L_2^2}$  are both bounded. Then  $\|\phi_n\|_{L_3^2}$  is bounded.*

*Proof.* Since the basic grade  $2 - 4/2 = 0$  we use Sobolev multiplication item 2

$$\|\phi_n\|_{L_3^2} \lesssim \|\phi_n \cdot \alpha_n\|_{L_2^2} + \|\phi_n\|_{L_2^2}$$

$$\lesssim \|\phi_n\|_{L_2^3} \|\alpha_n\|_{L_2^2} + \|\phi_n\|_{L_2^2}$$

thus it is sufficient to show  $\|\phi_n\|_{L_2^3}$  is bounded. Using Sobolev Multiplication item 2, consider

$$\begin{aligned} \|\phi_n\|_{L_2^3} &\lesssim \|\phi_n \cdot \alpha_n\|_{L_1^3} + \|\phi_n\|_{L_1^3} \\ &\lesssim \|\phi_n\|_{L_1^4} \|\alpha_n\|_{L_2^2} + \|\phi_n\|_{L_1^3} \end{aligned}$$

because the manifold is compact, Hölder's inequality yields  $\|\phi_n\|_{L_1^3} \lesssim \|\phi_n\|_{L_1^4}$ , thus it is sufficient to show  $\|\phi_n\|_{L_1^4}$  is bounded. This follows by Sobolev embedding since  $\|\phi_n\|_{L_1^4} \lesssim \|\phi_n\|_{L_2^2}$  which is bounded by hypothesis. ■

**Lemma 5.10** (Positive Grade Bootstrapping). *Let  $\{\phi_n\}_{n=1}^\infty$  and  $\{\alpha_n\}_{n=1}^\infty$  be basic sequences such that*

$$\|\phi_n\|_{L_{k+1}^2} \lesssim \|\phi_n \cdot \alpha_n\|_{L_k^2} + \|\phi_n\|_{L_k^2} \tag{5.7}$$

for all  $k \geq 3$ . If  $\|\alpha_n\|_{L_k^2}$  and  $\|\phi_n\|_{L_k^2}$  are bounded then  $\|\phi_n\|_{L_{k+1}^2}$  is bounded.

*Proof.* Since the basic grade  $k - 4/2 > 0$  we use Sobolev multiplication item 1

$$\begin{aligned} \|\phi_n\|_{L_{k+1}^2} &\lesssim \|\phi_n \cdot \alpha_n\|_{L_k^2} + \|\phi_n\|_{L_k^2} \\ &\lesssim \|\phi_n\|_{L_k^2} \|\alpha_n\|_{L_k^2} + \|\phi_n\|_{L_k^2} \end{aligned}$$

and the right hand side is bounded by hypothesis. ■

The purpose of these lemmas was to establish the following corollary, which will eliminate the burden of keeping track of indices in chapter 5.

**Corollary 5.11** (Nonnegative Grade Bootstrapping). *Let  $\{\phi_n\}_{n=1}^\infty$  and  $\{\alpha_n\}_{n=1}^\infty$  be basic sequences such that*

$$\|\phi_n\|_{L_{k+1}^p} \lesssim \|\phi_n \cdot \alpha_n\|_{L_k^p} + \|\phi_n\|_{L_k^p} \tag{5.8}$$

for all  $k \geq 1$  and  $p \geq 2$ . If  $\|\alpha_n\|_{L_k^2}$ ,  $\|\phi_n\|_{L_k^2}$  and  $\|\phi_n\|_{L^\infty}$  are all bounded then  $\|\phi_n\|_{L_{k+1}^2}$  is bounded.

*Proof.* The case  $k = 1$  is just lemma 5.8,  $k = 2$  is just lemma 5.9 and  $k \geq 3$  is lemma 5.10. ■

The key fact of the Seiberg-Witten equations is that they are a system of equations, there is a bootstrapping for this system as well.

**Lemma 5.12** (System Bootstrapping). *Let  $\{\phi_n\}_{n=1}^\infty$  and  $\{\alpha_n\}_{n=1}^\infty$  be basic sequences  $C_{k,p} \geq 0$  such that*

$$\begin{aligned}\|\phi_n\|_{L_{k+1}^p} &\lesssim \|\phi_n \cdot \alpha_n\|_{L_k^p} + \|\phi_n\|_{L_k^p} \\ \|\alpha_n\|_{L_{k+1}^p} &\lesssim \|\phi_n \cdot \phi_n\|_{L_k^p} + \|\alpha_n\|_{L_k^p} + C_{k,p}\end{aligned}$$

for all  $k \geq 1$  and  $p \geq 2$ . If  $\|\alpha_n\|_{L_k^2}$ ,  $\|\phi_n\|_{L_k^2}$  and  $\|\phi_n\|_{L^\infty}$  are all bounded then both  $\|\alpha_n\|_{L_{k+1}^2}$  and  $\|\phi_n\|_{L_{k+1}^2}$  are bounded.

*Proof.* Case  $k = 1$ : By lemma 5.8, the first inequality gives  $\|\phi_n\|_{L_2^2}$  is bounded, and then by Sobolev multiplication item 2, the second inequality yields

$$\begin{aligned}\|\alpha_n\|_{L_2^2} &\lesssim \|\phi_n \cdot \phi_n\|_{L_1^2} + \|\alpha_n\|_{L_1^2} \\ &\lesssim \|\phi_n\|_{L_2^3} \|\phi_n\|_{L_2^2} + \|\alpha_n\|_{L_1^2} + C_{1,2}\end{aligned}$$

$\|\phi_n\|_{L_2^3}$  is bounded by the same procedure in the proof of lemma 5.9.

Case  $k \geq 2$ : By lemma 5.9, the first inequality gives  $\|\phi_n\|_{L_{k+1}^2}$  is bounded, and then by Sobolev multiplication item 1, the second inequality yields

$$\|\alpha_n\|_{L_{k+1}^2} \lesssim \|\phi_n \cdot \phi_n\|_{L_k^2} + \|\alpha_n\|_{L_k^2}$$

$$\lesssim \|\phi_n\|_{L_k^2} \|\phi_n\|_{L_k^2} + \|\alpha_n\|_{L_k^2} + C_{k,2}$$

which is bounded by hypothesis. ■

### 5.3 The Configuration Space

To do the analysis of the basic Seiberg-Witten equations we know that the connections are parameterized by connections on the determinant bundle, and upon fixing a smooth basic Seiberg-Witten connection  $\nabla^{A_0}$  we have that  $\mathcal{A}_B \approx \Omega_B^1(i\mathbb{R})$ , the former being an affine space and the latter being a vector space, consequently we can expand our considerations to consider the connections that differ from our fixed connection by an element of  $L_k^2(\Omega_B^1(i\mathbb{R}))$ . Furthermore, since we are ultimately interested in smooth solutions, which by theorem 5.2 will always be in  $L^\infty(\Gamma_B(\mathbb{S}_+^{\mathbb{C}}))$  we will also consider the Sobolev space  $L_k^2(\Gamma_B(\mathbb{S}_+^{\mathbb{C}})) \cap L^\infty(\Gamma(\mathbb{S}_+^{\mathbb{C}}))$ . Together these will make the Hilbert space  $\mathcal{C}_k = L_k^2(\Omega_B^1(i\mathbb{R})) \oplus L_k^2(\Gamma_B(\mathbb{S}_+^{\mathbb{C}})) \cap L^\infty(\Gamma(\mathbb{S}_+^{\mathbb{C}}))$ , called the **configuration space** which will be the domain for our functionals. It will be necessary to do differential calculus on this infinite dimensional Hilbert space (see [Lan99; Lan95]).

**Proposition 5.13** (Sobolev Group). *Let  $k \geq 2$  then  $L_k^2(\Gamma_B(\mathbb{U}(1)))$  is an infinite dimensional Lie group, with Lie algebra  $L_k^2(\Omega_B^1(i\mathbb{R}))$ .*

*Proof.* When  $k \geq 3$  then  $L_k^2(\Gamma_B(\mathbb{U}(1))) \subset L_k^2(\Gamma_B(\mathbb{C}))$ , we first consider  $L_k^2(\Gamma_B(\mathbb{C}))$ . Now  $L_k^2(\Gamma_B(\mathbb{C}))$  is a Banach algebra, by theorem 5.7, and since the set of units of a Banach algebra are an open subset, the units are an infinite dimensional Lie group. The set  $L_k^2(\Gamma_B(\mathbb{U}(1))) = F^{-1}(1)$ , determined by the smooth functional  $F : L_k^2(\Gamma_B(\mathbb{C})) \rightarrow L_k^2(\Gamma_B(\mathbb{R}))$  defined by  $[F(\phi)](x) = |\phi(x)|$ , is a closed submanifold of the units by the Banach Inverse function theorem, and thus a smooth Lie subgroup.

For  $k = 1$ , a similar argument holds using  $L_k^2(\Gamma_B(\mathbb{C})) \cap L^\infty \Gamma_B(\mathbb{C})$ . ■

These Lie groups are continuous transformations of the bundle, the gauge action acts on the basic connections by pullback,  $g \curvearrowright \nabla^A = g^* \nabla^A = \nabla^{A+2g^{-1}dg}$ , and acts on the spinor bundle by multiplication,  $g \curvearrowright \phi = g^{-1}\phi$ . Since the gauge action on the connections uses one derivative of  $g$ , to make a smooth action, the configuration space must be of lower order by one. Consequently, we will be interested in the orbits by this gauge action. **From here on**  $k \geq 2$  will be a fixed number, indexing the order of the configuration space.

**Theorem 5.14** (Gauge Action). *Then the gauge action  $\mathcal{G}_{k+1} \times \mathcal{C}_k \rightarrow \mathcal{C}_k$  is smooth.*

*Proof.* Choose  $g \in \mathcal{G}_{k+1}$  and  $\alpha \oplus \phi \in \mathcal{C}_k$ . Since  $2g^{-1}dg \in L_k^2(i\mathbb{R})$  by theorem 5.7,  $g \curvearrowright \alpha = \alpha + 2g^{-1}dg$  is a continuous translation and is thus smooth. Since  $g^{-1}\phi \in L_k^2(\mathbf{S}_+^{\mathbb{C}})$  by Sobolev multiplication theorem 5.7, the action on  $g \curvearrowright \phi = g^{-1}\phi$  is continuous and linear in  $\phi$  and thus smooth. Lastly, let  $\gamma \in T_e \mathcal{G}_{k+1}$  be represented by the one parameter subgroup  $g(t) \in \mathcal{G}_{k+1}$  then the isotropy representation  $\text{iso}_{\alpha \oplus \phi} : T_e \mathcal{G}_{k+1} \rightarrow T_{\alpha \oplus \phi} \mathcal{C}_k$  is

$$\begin{aligned} \text{iso}_{\alpha \oplus \phi} \gamma &= \left. \frac{\partial}{\partial t} \right|_{t=0} (g \curvearrowright \alpha \oplus \phi) = \left. \frac{\partial}{\partial t} \right|_{t=0} (\alpha + 2g^{-1}dg \oplus g^{-1}\phi) \\ &= \left[ (-2g^{-2}\dot{g})dg + 2g^{-1}d\dot{g} \oplus -g^{-2}\dot{g}\phi \right]_{t=0} \\ &= 2d\gamma \oplus -\gamma\phi \end{aligned}$$

which has transverse symbol  $\sigma(\text{iso}_{\alpha \oplus \phi})$ . ■

We now want an infinite dimensional version of a the classically known Slice theorem for proper actions. Let us first recall that a continuous action of a topological

group on a topological space is proper if the map  $(g, x) \mapsto (gx, x)$  is a proper map, that is the preimage of compact sets is compact. In particular, for spaces with nicer structure we have other characterizations.

**Definition 5.15** (Proper Action). Let  $G \times X \rightarrow X$  be a continuous group action. If both  $G$  and  $M$  be Hausdorff and first-countable, such an action is said to be proper if for any sequence  $\{g_n\}_{n=1}^\infty \subset G$  and convergent sequence  $\{x_n\}_{n=1}^\infty \subset X$  such that  $\{g_n x_n\}_{n=1}^\infty \subset X$  converges, then there exists a convergent subsequence  $\{g_i\}_{i=1}^\infty \subset G$ .

**Proposition 5.16** (Slice Theorem). *Let  $G$  be a Lie group and  $G \times X \rightarrow X$  be a smooth proper action on the manifold  $X$ . Then for every point  $x \in X$  there exists a slice through  $x$ , moreover the orbit space  $X/G$  is Hausdorff.*

In particular, such a theorem holds when the Lie group and manifold are modeled on Hilbert spaces, that is, in the context of Hilbert manifolds. Such a theorem is developed Chapter 5 of [PT88]. Let us now show that our action is proper.

**Theorem 5.17** (Proper Action). *The gauge action  $\mathcal{G}_{k+1} \times \mathcal{C}_k \rightarrow \mathcal{C}_k$  is proper.*

*Proof.* It is sufficient to show that the action is proper on the first term of  $\mathcal{C}_k$ , that is on  $L_k^2(i\mathbb{R})$ . Assume that both sequences  $\{\alpha_n\}_{n=1}^\infty \subset L_k^2(i\mathbb{R})$  and  $\{g_n \curvearrowright \alpha_n\}_{n=1}^\infty \subset L_k^2(i\mathbb{R})$  converge. Consequently, both  $\|\alpha_n\|_{L_k^2}$  and  $\|g_n \curvearrowright \alpha_n\|_{L_k^2}$  are bounded, thus  $\|g_n \curvearrowright \alpha_n - \alpha_n\|_{L_k^2}$  is bounded. Now solving the action  $g_n \curvearrowright \alpha_n = \alpha_n + 2g_n^{-1}dg_n$  for  $dg_n$  yields

$$dg_n = \frac{1}{2}g_n(g_n \curvearrowright \alpha_n - \alpha_n).$$

Since  $g_n \in \mathcal{G}_k$ ,  $|g_n(x)| = 1$  for any  $x \in M$ ,  $\|g_n\|_{L^p} = \mu(M)^{1/p}$  for all  $1 \leq p \leq \infty$ . Thus

$\|g_n\|_{L^p_{k+1}} \lesssim \|g_n\|_{L^p} + \|dg_n\|_{L^p_k}$ . Consequently,

$$\|g_n\|_{L^p_{k+1}} \lesssim \|g_n(g_n \curvearrowright \alpha_n - \alpha_n)\|_{L^p_k} + \|g_n\|_{L^p_k} \quad (5.9)$$

by corollary 5.11 we know that  $\|g_n\|_{L^2_{k+1}}$  is bounded, thus by Rellich-Kondrakov compactness hypothesis 4.8 there exists a convergent subsequence  $\{g_{n_i}\}_{i=1}^\infty \subset \mathcal{G}_{k+1}$  which establishes that the action is proper. ■

**Corollary 5.18** (Orbit Space). *Each  $\alpha \oplus \phi \in \mathcal{C}_k$  has a local slice and  $\mathcal{C}_k/\mathcal{G}_{k+1}$  is a Hausdorff space,*

## 5.4 Seiberg-Witten Functional

Let us now define the Seiberg-Witten functional on our configuration space

**Proposition 5.19** (First Seiberg-Witten Functional). *The functional*

$$\text{sw}_k^1 : \mathcal{C}_k \rightarrow L^2_{k-1}(\text{End } \mathbb{S}_+^{\mathbb{C}})$$

*defined by*

$$\text{sw}_k^1(\alpha \oplus \phi) = F_+^{A_0} + d^+\alpha - \phi \otimes \phi^* + \frac{1}{2}|\phi|^2 \quad (5.10)$$

*is a smooth  $\mathcal{G}_{k+1}$ -invariant functional.*

*Proof.* Let  $g \in \mathcal{G}_{k+1}$ . Then

$$\begin{aligned} \text{sw}_k^1(g \curvearrowright \alpha \oplus g \curvearrowright \phi) &= F_+^{A_0} + d^+(\alpha + \frac{1}{2}g^{-1}dg) - (g^{-1}\phi) \otimes (g^{-1}\phi)^* + \frac{1}{2}|g^{-1}\phi|^2 \\ &= F_+^{A_0} + d^+\alpha - g^{-1}(g^{-1})^*\phi \otimes \phi^* + \frac{1}{2}|\phi|^2 \\ &= F_+^{A_0} + d^+\alpha - \phi \otimes \phi^* + \frac{1}{2}|\phi|^2 \\ &= \text{sw}_k^1(\alpha \oplus \phi) \end{aligned}$$

consequently,  $\text{sw}_k^1$  is  $\mathcal{G}_{k+1}$ -invariant. Lastly, let  $\beta \oplus \psi \in T_{\alpha \oplus \phi} \mathcal{C}_k$ . Then because the second term of  $\text{sw}_k^1$  is continuously linear in  $\alpha$ , it is smooth in  $\alpha$  with partial derivative

$$\left. \frac{\partial \text{sw}_k^1}{\partial \alpha} \right|_{\alpha \oplus \phi} \beta = d^+ \beta \quad (5.11)$$

and because the last two terms of  $\text{sw}_k^1$  is continuously quadratic in  $\phi$ , it is smooth in  $\phi$  with partial derivative

$$\left. \frac{\partial \text{sw}_k^1}{\partial \phi} \right|_{\alpha \oplus \phi} \psi = -\phi \otimes \psi^* - \psi \otimes \phi^* + \frac{\langle \psi, \phi \rangle + \langle \phi, \psi \rangle}{2} \quad (5.12)$$

which has transverse symbol  $\sigma(D_{\alpha \oplus \phi} \text{sw}_k^1)(\omega) = \alpha \wedge \omega$  ■

**Proposition 5.20** (Second Seiberg-Witten Functional). *The functional*

$$\text{sw}_k^2 : \mathcal{C}_k \rightarrow L_{k-1}^2(\mathbb{S}^{\mathbb{C}})$$

defined by

$$\text{sw}_k^2(\alpha \oplus \phi) = \not{\nabla}^{A_0} \phi + \frac{1}{2}(\alpha - \tau) \cdot \phi \quad (5.13)$$

is a smooth  $\mathcal{G}_{k+1}$ -equivariant functional.

*Proof.* Let  $g \in \mathcal{G}_{k+1}$ . Then

$$\begin{aligned} \text{sw}_k^2(g \curvearrowright \alpha \oplus g \curvearrowright \phi) &= (\not{\nabla}^{A_0} \phi + \frac{1}{2}(\alpha - \tau) + g^{-1} dg) g^{-1} \phi \\ &= \not{\nabla}^{A_0} (g^{-1} \phi) + \frac{1}{2}(\alpha - \tau) \cdot g^{-1} \phi + g^{-1} dg \cdot (g^{-1} \phi) \\ &= -g^{-2} dg \cdot \phi + \frac{1}{2}(\alpha - \tau) \cdot g^{-1} \phi + g^{-1} \not{\nabla}^{A_0} \phi + g^{-2} dg \cdot \phi \\ &= g^{-1} \text{sw}_k^2(\alpha \oplus \phi) \end{aligned}$$

thus  $\text{sw}_k^2$  is  $\mathcal{G}_{k+1}$ -equivariant. Lastly, let  $\beta \oplus \psi \in T_{\alpha \oplus \phi} \mathcal{C}_k$ . Then because the last term of  $\text{sw}_k^1$  is continuously linear in  $\alpha$ , it is smooth in  $\alpha$  with partial derivative

$$\left. \frac{\partial \text{sw}_k^2}{\partial \alpha} \right|_{\alpha \oplus \phi} \beta = \frac{1}{2} \beta \cdot \phi \quad (5.14)$$

and because  $\text{sw}_k^1$  is continuously linear in  $\phi$ , it is smooth in  $\phi$  with partial derivative

$$\left. \frac{\partial \text{sw}_k^2}{\partial \phi} \right|_{\alpha \oplus \phi} \psi = \not{\nabla}^{A_0} \psi + \frac{1}{2}(\alpha - \tau) \cdot \psi \quad (5.15)$$

which has transverse symbol  $\sigma(D_{\alpha \oplus \phi} \text{sw}_k^2)(\psi) = \phi \cdot \psi$ . ■

**Corollary 5.21** (Seiberg-Witten Functional). *The functional*

$$\text{sw}_k^1 \times \text{sw}_k^2 : \mathcal{C}_k \rightarrow L_{k-1}^2(\text{End } \mathbf{S}_+^{\mathbb{C}}) \times L_{k-1}^2(\mathbf{S}_-^{\mathbb{C}})$$

is a smooth  $\mathcal{G}_{k+1}$ -equivariant elliptic functional with differential

$$\begin{aligned} D_{\alpha \oplus \phi}(\text{sw}_k^1 \times \text{sw}_k^2)(\beta \oplus \psi) = \\ \left( d^+ \beta - \phi \otimes \psi^* - \psi \otimes \phi^* + \frac{\langle \psi, \phi \rangle + \langle \phi, \psi \rangle}{2}, \not{\nabla}^{A_0} \psi + \frac{1}{2}(\alpha - \tau) \cdot \psi + \frac{1}{2}\beta \cdot \phi \right) \end{aligned} \quad (5.16)$$

and  $\text{sw}_k^1 \times \text{sw}_k^2$  induces a continuous functional, called the Seiberg-Witten functional,

$$\text{sw}_k : \mathcal{C}_k / \mathcal{G}_{k+1} \rightarrow L_{k-1}^2(\text{End } \mathbf{S}_+^{\mathbb{C}}) \times L_{k-1}^2(\mathbf{S}_-^{\mathbb{C}})$$

on the orbit space  $\mathcal{C}_k / \mathcal{G}_{k+1}$ .

Furthermore, when working in  $\mathcal{C}_k / \mathcal{G}_{k+1}$  there is a preferred class of representatives: the Coulomb representatives.

**Theorem 5.22** (Coulomb Representatives). *Then for every  $[\alpha \oplus \phi] \in \mathcal{C}_k / \mathcal{G}_{k+1}$  there exists a representative such that  $d^* \alpha = 0$ , called a Coulomb representative. Moreover, any two Coulomb representatives differ by a basic harmonic gauge transformation.*

*Proof.* Let  $[\alpha \oplus \psi] \in \mathcal{C}_k / \mathcal{G}_{k+1}$  and define the gauge transformation  $g = \exp(-\frac{1}{2}\Delta^{-1}d^*\alpha)$  where  $\Delta^{-1}$  is a parametrix for the basic Laplacian  $\Delta_B$ . Then  $g \curvearrowright \alpha = \alpha + 2g^{-1}dg = \alpha - d\Delta^{-1}d^*\alpha$ . Consequently,  $d^*(g \curvearrowright \alpha) = d^*\alpha - d^*d\Delta^{-1}d^*\alpha = d^*\alpha - \Delta_B\Delta^{-1}d^*\alpha = 0$ .

A gauge transformation  $g$  is basic harmonic when the translation term  $2g^{-1}dg$  is harmonic in the basic DeRham cohomology. Clearly,  $d(g^{-1}dg) = 0$ , thus it suffices to show that  $d^*(g^{-1}dg) = 0$ . Let  $\alpha$  and  $\alpha'$  be any two Coulomb representatives, then there is a gauge transformation  $g$  between them and such that  $\alpha = \alpha' + g^{-1}dg$ . The transverse codifferential yields  $0 = d^*\alpha = d^*\alpha' + d^*(g^{-1}dg)$  which suffices for  $g$  to be basic harmonic. ■

The space that we wish to study consists of the solutions to the equation  $\text{sw}_k(\alpha \oplus \phi) = 0$ , the so-called moduli space  $\mathcal{M}_k$ . In particular, since smooth functions are dense in  $\mathcal{C}_k$  and there are slices through any point by corollary 5.18 without any loss of generality we may assume that a representative solution is on a slice through a smooth a solution. However, restricting the Seiberg-Witten equations to a slice through a smooth solution renders the Seiberg-Witten functional as elliptic with smooth coefficients, consequently every solution on this slice is also smooth. Furthermore, since  $\|g^{-1}\phi\|_{L^p} = \|\phi\|_{L^p}$  (including  $p = \infty$ ), that is, the norm is constant on orbits by  $\mathcal{G}_k$ . Consequently, the  $L^p$  norm of spinor solutions are uniformly bounded by the  $L^p$  norm of any smooth solution on that orbit.

**Corollary 5.23.** *For  $[\alpha \oplus \phi] \in \mathcal{M}_k$  and  $1 \leq p < \infty$  we have  $\|\phi\|_{L^p} < \sqrt{K}^p \text{vol}M$  and  $\|\phi\|_{L^\infty} < \sqrt{K}$*

*Proof.* It is sufficient to consider only the bounds of the smooth solutions which are uniformly bounded in  $L^p$  by theorem 5.6. ■

Moreover, the existence of smooth solutions is enough to show that every Coulomb solution is in fact smooth.

**Theorem 5.24** (Coulomb Solution Regularity). *Let  $\{\alpha_n \oplus \phi_n\}_{n=1}^\infty$  be Coulomb representatives of  $\{[\alpha_n \oplus \phi_n]\}_{n=1}^\infty \subset \mathcal{M}_k$ . Then if  $\|\alpha_n \oplus \phi_n\|_{L_k^2}$  is bounded then  $\|\alpha_n \oplus \phi_n\|_{L_k^2}$  is bounded for all  $k$ .*

*Proof.* Since  $\not\partial^{A_0}$  is elliptic,

$$\begin{aligned} \|\phi_n\|_{L_{k+1}^2} &\lesssim \|\not\partial^{A_0} \phi_n\|_{L_k^2} + \|\phi_n\|_{L_k^2} \\ &\lesssim \|(\alpha_n - \tau)\phi_n\|_{L_k^2} + \|\phi_n\|_{L_k^2} \end{aligned}$$

and since  $d^+ + d^*$  is elliptic,

$$\begin{aligned} \|\alpha_n\|_{L_{k+1}^2} &\lesssim \|(d^+ + d^*)\alpha_n\|_{L_k^2} + \|\alpha_n\|_{L_k^2} \\ &\lesssim \|\phi_n \otimes \phi_n - \frac{1}{2} |\phi_n|^2 + F_+^{A_0}\|_{L_k^2} + \|\alpha_n\|_{L_k^2} \\ &\lesssim \|\phi_n \otimes \phi_n\|_{L_k^2} + \|F_+^{A_0}\|_{L_k^2} + \|\alpha_n\|_{L_k^2} \end{aligned}$$

and, by corollary 5.23,  $\|\phi_n\|_{L^\infty}$  is bounded. Consequently, by system bootstrapping lemma 5.12  $\|\alpha_n \oplus \phi_n\|_{L_{k+1}^2}$  is bounded. ■

**Corollary 5.25** (Smooth Coulomb Solutions). *Let  $\alpha \oplus \phi \in \mathcal{C}_k$  be a Coulomb representative of  $[\alpha \oplus \phi] \in \mathcal{M}_k$ . Then  $\alpha \oplus \phi$  is a smooth representative.*

*Proof.* Consider  $\{\alpha_n \oplus \phi_n\}_{n=1}^\infty \subset \mathcal{C}_k$  defined by  $\alpha_n \oplus \phi_n = \alpha \oplus \phi$ . Then boundedness of  $\|\alpha_n \oplus \phi_n\|_{L_k^2}$  is equivalent to finiteness of  $\|\alpha \oplus \phi\|_{L_k^2}$ . Since,  $\|\alpha \oplus \phi\|_{L_1^2} \lesssim \|\alpha \oplus \phi\|_{L_k^2}$  is finite,  $\|\alpha\|_{L_1^2}$  is finite. Consequently,  $\|\alpha \oplus \phi\|_{L_k^2} \in \mathcal{C}_k$  for all  $k$ . That is  $\alpha \oplus \phi \in \bigcap_k^\infty \mathcal{C}_k = \Omega_B^1(i\mathbb{R}) \oplus \Gamma_B(\mathcal{S}_+^{\mathbb{C}})$ . ■

## 5.5 Sequential Compactness of the Moduli Space

**Hypothesis 5.26** (Uhlenbeck Representative). *There exists a finite  $C_k$ , such that for each element of the quotient space  $L_k^2(\Omega_B^1(i\mathbb{R}))/\mathcal{G}_{k+1}$  there is a Coulomb representative*

$\alpha$  such  $\|\alpha\|_{L^2_{k+1}} \lesssim \|F^A_+\|_{L^2_k} + C_k$ .

*Proof.* For the unfoliated case see [Mor96] or [Sal99]. ■

The main difficulty with establishing a basic version of such a theorem, is that the classic proof relies on the lattice  $H_1(M; \mathbb{Z})$  within  $H_1(M; \mathbb{R})$  and using Poincaré duality and Hodge duality to identify a lattice in the harmonic one-forms. The harmonic one-forms in that lattice can be used to provide a class of gauge transformations between Coulomb representatives. By this, we can choose a representative closest to the origin which is by necessity bounded. The problem for a basic theory is that such a gauge transformation may not actually be a basic gauge transformation, however when the foliation is taut, that is the mean curvature  $\kappa$  cohomologous to zero, then basic Hodge duality holds and the basic harmonic representatives are classically harmonic and such a theorem holds. In particular, for any sequence of solutions  $\alpha_n \oplus \phi_n \in \mathcal{C}_k$  we can, without loss of generality choose Uhlenbeck representatives such that  $L^2_1(\alpha_n)$  is a bounded sequence. This observation with theorem 5.6 and theorem 5.6 lead to the following.

**Theorem 5.27** (Sequentially Compact Moduli Space). *The moduli space  $\mathcal{M}_k$  is sequentially compact.*

*Proof.* Choose a sequence  $\{[\alpha_n \oplus \phi_n]\}_{n=1}^\infty \subset \mathcal{M}_k$ . Without any loss of generality, by hypothesis 5.26, we may assume that the elements of the sequence  $\{\alpha_n\}_{n=1}^\infty \subset \mathcal{C}_k$  are Uhlenbeck representatives, that is  $\|\alpha_n\|_{L^2_k}$  is bounded, and so  $\|\alpha_n\|_{L^2_1}$  is bounded. By theorem 5.24,  $\|\alpha_n \oplus \phi_n\|_{L^2_k}$  is bounded for all  $k \geq 2$ . In particular, for a fixed  $k$ ,  $\{\alpha_n \oplus \phi_n\}_{n=1}^\infty \subset \mathcal{C}_{k+1}$  is bounded, thus by Rellich-Kondrakov compactness hypothesis 4.8

there exists a convergent subsequence  $\{\alpha_{n_i} \oplus \phi_{n_i}\}_{i=1}^\infty \subset \mathcal{C}_k$ . Consequently,  $\{[\alpha_{n_i} \oplus \phi_{n_i}]\}_{i=1}^\infty \subset \mathcal{C}_k/\mathcal{G}_{k+1}$  is a convergent subsequence in the closed subset  $\mathcal{M}_k$ , and thus  $\mathcal{M}_k$  is sequentially compact. ■

## 5.6 The Associated Elliptic Complex

**Theorem 5.28.** *Let  $\text{iso}_{\alpha \oplus \phi}$  and  $D_{\alpha \oplus \phi}(\text{sw}_k^1 \times \text{sw}_k^2)$  respectively be the first and second map in the following sequence*

$$0 \rightarrow T_e(\mathcal{G}_k + 1) \longrightarrow T_{\alpha \oplus \phi}(\mathcal{C}_k) \longrightarrow T_{0 \oplus 0}(L_{k-1}^2(\text{End } \mathbf{S}_+^{\mathbb{C}}) \times L_{k-1}^2(\mathbf{S}_-^{\mathbb{C}})) \rightarrow 0$$

then the sequence is an elliptic complex if  $[\alpha \oplus \phi] \in \mathcal{M}_k$ .

*Proof.* Let  $\gamma \in T_e(\mathcal{G}_k)$

$$\begin{aligned} (D_{\alpha \oplus \phi} \text{sw}_k^1 \circ \text{iso}_{\alpha \oplus \phi})\gamma &= D_{\alpha \oplus \phi} \text{sw}_k^1(2d\gamma \oplus -\gamma\phi) \\ &= \frac{1}{2}(2d\gamma) \cdot \phi - \not{\nabla}^{A_0}(\gamma\phi) - \frac{1}{2}(\alpha - \tau) \cdot (\gamma\phi) \\ &= d\gamma \cdot \phi - d\gamma\phi - \gamma \cdot \not{\nabla}^{A_0}\phi - \gamma \frac{1}{2}(\alpha - \tau) \cdot \phi \\ &= -\gamma \cdot (\not{\nabla}^{A_0}\phi + \frac{1}{2}(\alpha - \tau)\phi) \end{aligned}$$

which vanishes when  $[\alpha \oplus \phi] \in \mathcal{M}_k$ . Considering the second factor, we have

$$\begin{aligned} (D_{\alpha \oplus \phi} \text{sw}_k^2 \circ \text{iso}_{\alpha \oplus \phi})\gamma &= D_{\alpha \oplus \phi} \text{sw}_k^2(2d\gamma \oplus -\gamma\phi) \\ &= d^+(2d\gamma) + \phi \otimes (\gamma\phi)^* + (\gamma\phi) \otimes \phi^* - \frac{\langle \gamma\phi, \phi \rangle + \langle \phi, \gamma\phi \rangle}{2} \\ &= \gamma^* \phi \otimes \phi^* + \gamma\phi \otimes \phi^* - \frac{\gamma |\phi|^2 + \gamma^* |\phi|^2}{2} \end{aligned}$$

since  $\gamma$  is purely imaginary, the conjugate  $\gamma^* = -\gamma$  thus the last line vanishes. ■

In particular, we have the following isomorphisms for the following spaces

$$T_e(\mathcal{G}_{k+1}) = L_{k+1}^2(\Omega_B^0(i\mathbb{R}))$$

$$T_{\alpha \oplus \phi}(\mathcal{C}_k) = L_k^2(\Omega_B^1(i\mathbb{R})) \oplus L_k^2(\Gamma_B(\mathbf{S}_+^{\mathbb{C}}))$$

$$T_{0 \oplus 0}(L_{k-1}^2(\text{End } \mathbf{S}_+^{\mathbb{C}}) \times L_{k-1}^2(\mathbf{S}_-^{\mathbb{C}})) = L_{k-1}^2(\Omega_{B+}^2(i\mathbb{R})) \oplus L_{k-1}^2(\Gamma_B(\mathbf{S}_-^{\mathbb{C}}))$$

and the sequence decomposes into the two sequences, and by homotoping lower order terms away the first sequence is

$$0 \longrightarrow L_{k+1}^2(\Omega_B(i\mathbb{R})) \longrightarrow L_k^2(\Omega_B^1(i\mathbb{R})) \longrightarrow L_{k-1}^2(\Omega_{B+}^2(i\mathbb{R})) \longrightarrow 0$$

where  $2d$  and  $d^+$  are the first and second map and  $\nabla^{A_0} + \frac{1}{2}(\alpha - \tau) = \nabla^A - \frac{1}{2}\tau$  is the second map for the second sequence

$$0 \longrightarrow 0 \longrightarrow L_k^2(\Gamma_B(\mathbf{S}_+^{\mathbb{C}})) \longrightarrow L_{k-1}^2(\Gamma_B(\mathbf{S}_-^{\mathbb{C}})) \longrightarrow 0$$

and clearly each is transversely elliptic. Consequently, their index can be calculated by [BKR11].

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## VITA

Andrew Renner was born and grew up in Grand Rapids, Minnesota. His first experience in secondary education was in 1996 at Itasca Community College in his hometown, which after a year he transferred to the University of Minnesota-Duluth where he earned a Bachelor of Science in Mathematics with a minor in Computer Science in 2000. In 2001, he enrolled at Utah State University and earned a Master of Science in Mathematics, under the direction of Dr. Mark Fels, in 2004. Subsequently, Andrew entered the workforce and, for two years, taught mathematics at Inver Hills Community College in Inver Grove Heights, Minnesota before working as an actuarial assistant at Allianz Life Insurance Company of North America in Golden Valley, Minnesota. In 2009, he enrolled at the University of Missouri and began studying differential topology under Dr. Shuguang Wang in 2011.