SOME RESULTS IN CONVEX GEOMETRY

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ABSTRACT

This thesis is divided into four parts. The first part is about proving that the unit ball of the Lorentz space $\ell_{w,q}^n$ is not an intersection body for $n \geq 5$ and $q > 2$. We go on to explain the connection with this result to the Busemann-Petty problem in convex geometry.

The second section proves separation for an inequality by V. Yaskin and M. Yaskina for polar centroid bodies. We prove separation for $p \geq 0$ and go on to find the corresponding "hyperplane" inequalities which resemble the inequalities connected with the hyperplane conjecture for convex bodies.

The third section is about a hyperplane-type inequality involving arbitrary measures and subspaces of unconditional spaces. This is an extension of A. Koldobsky's inequality from 2013 for unconditional bodies.

In the fourth section we find rough upper and lower bounds of volumes of central cross-sections of rectangular boxes in $\mathbb{R}^n$.

Chapter 1

Introduction

1.1 Preliminaries

The purpose of this thesis is to explain some new results on the study of convex bodies. We start off with standard definitions used throughout the paper. If $x, y \in \mathbb{R}^n$ are $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ respectively then $x \cdot y$ denotes the scalar product on $\mathbb{R}^n$ defined by $x \cdot y := \sum_{i=1}^{n} x_i y_i$ and $|\cdot|_2$ is the standard Euclidean norm on $\mathbb{R}^n$ defined by $|x|_2 := (x \cdot x)^{1/2}$. We let $S^{n-1} := \{x \in \mathbb{R}^n : |x|_2 = 1\}$.

For any compact set $E \subseteq \mathbb{R}^n$ we define the *Minkowski functional* of E on $\mathbb{R}^n$, denoted $\|\cdot\|_E$, as

$$\|x\|_E := \min\{a \geq 0 : x \in aE\} \quad \text{for all} \quad x \in \mathbb{R}^n, \tag{1.1.1}$$

and the radial function $\rho_E(x)$ of E as

$$\rho_E(x) := \max\{\lambda \geq 0 : \lambda x \in K\} = \frac{1}{\|x\|_E} \quad \text{for all} \quad x \in \mathbb{R}^n \setminus 0. \tag{1.1.2}$$

We say a compact set $E \subseteq \mathbb{R}^n$ has continuous boundary if $\|\cdot\|_E$ is continuous on $\mathbb{S}^{n-1}$.

We define an n-dimensional *body* K to be any compact subset of $\mathbb{R}^n$ with non-empty interior and whose boundary is continuous.

An n-dimensional body K is called a *star body* if the origin is an interior point of K and if for every $x \in K$ each point on the line $[0, x)$ is an interior point of K.

We can define vector space operations on subsets of $\mathbb{R}^n$ in an intuitive sence. If $A, B \subseteq \mathbb{R}^n$, $\lambda \in \mathbb{R}$ we define $A+B := \{a+b : a \in A, b \in B\}$ and $\lambda A := \{\lambda a : a \in A\}$. We can use this to define a body $K \subseteq \mathbb{R}^n$ as *origin-symmetric* if and only if $K = -K$.

The main object of study in this thesis is the notion of symmetric, convex bodies. A body K is a origin-symmetric, convex body in $\mathbb{R}^n$ if and only if its Minkowski functional is a norm. Likewise, for every norm $\|\cdot\|$ on $\mathbb{R}^n$ there exists an origin-symmetric convex body K such that $\|\cdot\| = \|\cdot\|_K$ on $\mathbb{R}^n$. The body K is the unit ball of the n-dimensional Banach space $(\mathbb{R}^n, \|\cdot\|)$. Thus an n-dimensional, origin-symmetric, convex body K is in one to one correspondence with an n-dimensional Banach space. Moreover, an $(n-1)$-dimensional subspace of a n-dimensional Banach space X corresponds to an $(n-1)$-dimensional central cross section of the body K and an $(n-1)$-dimensional coset of a space X corresponds to a projection of K onto an $(n-1)$-dimensional subspace of X.

1.2 Volume Comparison Problems

The motivation for much of this thesis lies in volume comparision problems. We start off with a well-known geometric question called the *Busemann-Petty problem* posed in 1956 [8]:

Question 1.2.1. *Suppose K and L are two origin-symmetric convex bodies in $\mathbb{R}^n$.*

If $vol_{n-1}(K \cap \xi^\perp) \leq vol_{n-1}(L \cap \xi^\perp)$, for all $\xi \in S^{n-1}$, does it follow that $vol_n(K) \leq vol_n(L)$?

The answer is yes if $n \leq 4$ and no if $n \geq 5$. The first counter-example was found by Larman and Rogers [35] who used probabilistic techniques to construct a counter example in dimension 12. The full solution was completed in the late 1990's as a result of a series of papers [35], [2], [13], [5], [37], [42], [9], [10], [53], [51], [52], and [12]; see [11, pg. 343] or [25, pg. 3] for a history of the problem and its solution. The crucial step was an observation of E. Lutwak about an important family of bodies called intersection bodies. Following Lutwak [37] we say an origin-symmetric star body K is called an *intersection body of a star body* L if the radius of K in every direction $\xi \in S^{n-1}$ is equal to the $(n-1)$-dimensional volume of the intersection of L and the $(n-1)$-dimensional central hyperplane $\xi^\perp$ orthogonal to ξ. In other words,

$$\rho_K(\xi) = \mathrm{vol}_{n-1}(L \cap \xi^\perp),$$

for all $\xi \in S^{n-1}$, where $\rho_K(\xi) := \|\xi\|_K^{-1}$ is the radial function of K. Since we can express $\mathrm{vol}_{n-1}(L \cap \xi^\perp)$ as

$$\frac{1}{n-1}\int_{S^{n-1}} \|x\|_L^{-n+1} dx = R\left((n-1)^{-1}\|x\|_L^{-n+1}\right)$$

where

$$\xi \mapsto Rf(\xi) := \int_{S^{n-1}\cap\xi^\perp} f(x)dx, \qquad \xi \in S^{n-1}$$

denotes the spherical Radon transform of the function f, then we can generalize our definition of an intersection body, as in [14], as follows: An origin-symmetric star body K is an *intersection body* if there exists a finite Borel measure μ on S^{n-1} such that $\|\cdot\|_K^{-1} = R\mu$ in the sense of linear functionals on the space $C(S^{n-1})$ of continuous function on the sphere, i.e.

$$\int_{S^{n-1}} \|x\|_K^{-1} f(x)dx = \int_{S^{n-1}} R(f)(x)d\mu$$

for all $f \in C(S^{n-1})$.

The concept of an intersection body is important in the dual Brunn-Minkowski theory, introduced by Lutwak [37].

By a result of Lutwak [37], if K is an intersection body, the answer to the question of the Busemann-Petty problem is affirmative for any origin-symmetric star body L. On the other hand, if L is a non-intersection body it can be perturbed to construct a body K giving, together with L, a counterexample to the Busemann-Petty problem. As shown in [10], [52] and [12], every origin-symmetric, convex body in dimension $n \leq 4$ is an intersection body. In dimensions 5 and higher there exist origin-symmetric, convex, non-intersection bodies, each of which provides a counterexample to the Busemann-Petty problem in these dimensions. The first example of a non-intersection body was given by Gardner [9] as a certain rounded cylinder in $\mathbb{R}^5$. Other examples of non-intersection bodies in dimensions $n \geq 5$ were given in [51], [12], [21], [24], and [50]. In particular, the unit balls of the spaces ℓ_q^n where $n \geq 5,\ q > 2$ are not intersection bodies, as it was proved in [21], [24] (or [25, sec. 4.4]) provides a characterization of intersection bodies in terms of the second derivative of the norm. For details and more results on intersection bodies see either the book [11] or the book [25, ch. 4].

Chapter 2

Lorentz Balls as Non-intersection Bodies

2.1 Intro

The contents of this chapter are from the published paper [48]. As mentioned in section 2 of chapter 1, the solution to the Busemann-Petty problem shows that for every non-intersection body K one can construct another body L such that K and L provide a counterexample to the Busemann-Petty problem. Also in this section it was mentioned that A. Koldobsky showed in [21] that the unit balls of the spaces ℓ_q^n where $n \geq 5,\ q > 2$ are non-intersection bodies. In this section we consider a generalization of this observation by showing that the unit ball of the n-dimensional Lorentz space $\ell_{w,q}^n$ is not an intersection body for $q > 2$ and $n \geq 5$. Our proof is based on a modification of the second derivative test for intersection bodies found in [24] or [25, pg. 83]. This second derivative test is applicable only when $q > 2$. This is because the proof relies on the fact that the Lorentz norm has continuous first and second derivatives for these values of q. This can be seen by direct computation of the derivatives, as we have done in (2.3.2) and (2.3.3). It should be noted that, as of the writing of this paper, it is unknown to the author whether the unit ball of $\ell^n_{w,q}$ is an intersection body for $0 < q \leq 2$ and $n \geq 5$.

2.2 Main Theorem

We now go on to defining the main subject of this chapter which is the idea of a Lorentz ball. Let $w = (a_1, \ldots, a_n) \in \mathbb{R}^n$ be such that $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$ and define a norm $\|\cdot\|_{w,q}$ on $\mathbb{R}^n$ as

$$\|x\|_{w,q} = \Big(a_1(x_1^*)^q + \ldots + a_n(x_n^*)^q\Big)^{1/q}, \tag{2.2.1}$$

where $x_1^*, \ldots, x_n^*$ is the non-increasing permutation of $|x_1|, \ldots, |x_n|$. We call $\ell^n_{w,q} := (\mathbb{R}^n, \|.\|_{w,q})$ the n-dimensional Lorentz space. We call w the weight corresponding to the space. Notice if this weight w is equal to $(1, 0, \ldots, 0)$ then, in the definition of the Lorentz norm, all components $a_i(x_i^*)^p$, except for the $i = 1$ case, cancel and we are left with

$$\|x\|_{w,q} = (a_1 \max_{1 \leq i \leq n} |x_i|^p)^{1/p}. \tag{2.2.2}$$

On the other hand, if w has equal components, say $w = (c, c, \ldots, c)$ where c is some constant, then

$$\|x\|_{w,q} = c((x_1^*)^q + \ldots + (x_n^*)^q)^{1/q} = c\|x\|_q \tag{2.2.3}$$

So $\ell^n_{w,q}$ can be interpreted as an intermediate space between ℓ^n_∞ and ℓ^n_q which motivates the study of this subject.

Now we appeal to the theory of distributions, which one can read more about in the book [25, sec. 2.5]. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$, let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and

$$\mathbb{N}_0^m = \{(a_1, \ldots, a_m) : a_i \in \mathbb{N}_0, 1 \leq i \leq m\}.$$

For $\alpha \in \mathbb{N}_0^m$, let $|\alpha| = \sum_{i=1}^n \alpha_k$ and D^α be a differential operator defined for every $f \in C^{|\alpha|}(\mathbb{R}^n)$ by

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha} \ldots \partial x_n^{\alpha_n}}. \tag{2.2.4}$$

Let $\mathscr{S}^n$ denote the set of all complex valued functions $\phi \in C^\infty(\mathbb{R}^n)$ converging to zero at infinity together with all their derivatives faster than any negative power of $|\cdot|_2$, i.e.

$$p_k(\phi) := \sup_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^n} (1 + |x|_2)^k |D^\alpha \phi(x)| < \infty, \quad \text{for all } k \in \mathbb{N}. \tag{2.2.5}$$

A function $\phi \in \mathscr{S}^n$ is called a *test function*. $\mathscr{S}^n$ is a vector space which has a locally convex basis generated by the seminorms (2.2.5) (for more discussion on this see [44]). The dual of $\mathscr{S}^n$, denoted $(\mathscr{S}^n)'$, is the collection of linear, continuous functions on $\mathscr{S}^n$. A function $\Lambda \in (\mathscr{S}^n)'$ is called a *distribution* on $\mathscr{S}^n$. If f is a locally integrable, complex-valued function on $\mathbb{R}^n$ and $\phi \in \mathscr{S}^n$ we define

$$\langle f, \phi \rangle := \int_{\mathbb{R}^n} f(x)\phi(x)dx.$$

This definition makes the function $\langle f, \cdot \rangle : \mathscr{S}^n \to \mathbb{R}$, defined by $\phi \mapsto \langle f, \phi \rangle$, a distribution on $\mathscr{S}^n$ (for a proof see section 6.11 of [44]). For $\phi \in \mathscr{S}^n$ we define the Fourier transform of ϕ at ξ as

$$\widehat{\phi}(\xi) := \int_{\mathbb{R}^n} \phi(x) e^{-ix\cdot\xi} dx. \tag{2.2.6}$$

If $\phi \in \mathscr{S}^n$ then $\widehat{\phi} \in \mathscr{S}^n$ (see [44] theorem 7.4). Equation (2.2.6) is usually defined first for functions f in $L_1(\mathbb{R}^n)$ and then extended to $L_2(\mathbb{R}^n)$ functions using a limiting process. The integral might not exist if f is not $L_1(\mathbb{R}^n)$ but rather only locally integrable i.e. that the integral of f is finite on compact sets. The Fourier transform does exists for $\phi \in \mathscr{S}^n$ because of the decay property (2.2.5) of such functions so this suggests that we can weaken the definition of the Fourier transform by defining it in terms of distributions. If f is a locally integrable function on $\mathbb{R}^n$ then we define the Fourier transform $\widehat{f}$, in the sense of distributions, as the action of integrating f against the Fourier transform of a test function $\phi \in \mathscr{S}^n$, i.e. as the linear transformation $\phi \to \int_{\mathbb{R}^n} f(x)\widehat{\phi}(x)dx$. Another way to write this is $\left\langle \widehat{f}, \phi \right\rangle := \left\langle f, \widehat{\phi} \right\rangle$. So $\widehat{f}$ is not seen as a function from $\mathbb{R}^n$ to $\mathbb{R}$ but from $\mathscr{S}^n$ to $\mathbb{R}$. This definition of the Fourier transform allows us to take the Fourier transform of functions which are not necessarily integrable. An example of such functions is the family defined as $\|\cdot\|_K^{-n+p}$ where K is an origin-symmetric, convex body and $p \in (0, n-1)$. This family of functions is studied more in A. Koldobsky's book [25].

In [22, thm. 1] (also see [25, thm. 4.1]) A. Koldobsky showed that an origin-symmetric star body K is an intersection body if and only if $\|\cdot\|_K^{-1}$ is a positive definite distribution, i.e. the Fourier transform of $\|\cdot\|_K^{-1}$ is a positive function in the sense of distributions. This means that $\left\langle (\|\cdot\|_K^{-1})^\wedge, \phi \right\rangle = \left\langle \|\cdot\|_K^{-1}, \widehat{\phi} \right\rangle \geq 0$ for every non-negative $\phi \in \mathscr{S}$. So in order to show that the unit balls of the spaces $\ell_{w,q}^n$ are not intersection bodies it is enough to show the following theorem which, as a corollary, includes the value $p = n-1$:

Theorem 2.2.1. *Let $n \geq 5$ and $q > 2$. The function $\|\cdot\|_{w,q}^{-p}$ represents a positive* *definite distribution if and only if $p \in [n-3, n)$.*

Let $B^n_{w,q}$ denote the unit ball of $\ell^n_{w,q}$. We can use theorem (2.2.1) along with the fact that $\|x\|_{w,q} = \|x\|_{B^n_{w,q}}$ to get the following:

Corollary 2.2.2. *The unit ball of $\ell^n_{w,q}$ is not an intersection body for $q > 2$ and $n \geq 5$.*

As originally shown by G. Zhang in [52] (or see corollary 4.2 in [25] for a proof) every origin symmetric convex body in $\mathbb{R}^n$ for $n \leq 4$ is not an intersection body so we do not worry about these values of n in the above theorem.

As for the other values of q it is still unknown. In order to prove theorem (2.2.1) we investigate the second derivative of the Lorentz norm with respect to the first coordinate. We can write formulas for the first and second derivatives of the Lorentz norm when $q > 2$ but when $0 < q \leq 2$ the norm $\|\cdot\|_{w,q}$ fails to be twice differentiable. This is why we cannot use the same argument for the other values of q. It is not known by the author whether $\|\cdot\|_{w,q}^{-p}$ is positive definite for $0 < q \leq 2$.

Also, when we say that $q > 2$ this does not include the case $q = \infty$. A. Koldobsky showed in [21] that the function $\|\cdot\|_\infty^{-p}$ is a positive definite distribution if $p \in [n-3, n)$ and and is not positive definite if $p \in (0, n-3)$. So, for the case when $n \geq 5$, and $p = -1$, the unit cube of $\mathbb{R}^n$ is not an intersection body. Since $\|\cdot\|_{w,\infty}$ is equivalent to $\|\cdot\|_\infty$.

The proof of this theorem follows the second derivative test found in [24] or [25, thm. 4.19], and as such requires Lemma 2.4.1 below, which is itself a modification of Lemma 1 of [24] (or [25, lemma 4.20]). The original proof of this lemma relies on the fact that the function $x_1 \to \|x\|_{w,q}$ has continuous first and second derivative on $\mathbb{R}$

for a fixed $(x_2, \ldots, x_n) \in \mathbb{R}^{n-1} \setminus \{0\}$. The Lorentz norm doesn't have this property on all of $\mathbb{R}^n$ but fortunately the set on which this fails has measure zero.

2.3 Properties of the Lorentz Norm

Fix $q > 2$ and w as above. If $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ then y will denote $(x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$. Let $e_1, e_2, \ldots, e_n$ be an orthonormal basis of $\mathbb{R}^n$. Throughout the paper $\|x\|_{w,q}$ denotes $\|x_1 e_1 + \ldots + x_n e_n\|_{w,q}$ and $\|y\|_{w,q}$ denotes $\|x_2 e_2 + \ldots + x_n e_n\|_{w,q}$. We let $|\cdot|_2$ and B_2^n denote the norm and unit ball of ℓ_2^n respectively.

Let S_n be the group of permutations on n symbols, let

$$A_\sigma = \{x \in \mathbb{R}^n : |x_{\sigma(1)}| > |x_{\sigma(2)}| > \ldots > |x_{\sigma(n)}|\}$$

for all $\sigma \in S_n$. For each such σ, we note that A_σ is open and that $\mathbb{R}^n \setminus (\bigcup_\sigma A_\sigma)$ is of Lebesgue measure zero.

For a given $\sigma \in S_n$ and $y = (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$, let $\|x\|'_{x_1}(\cdot, y)$ and $\|x\|''_{x_1^2}(\cdot, y)$ denote the first and second derivative of the function

$$x_1 \to \|x_1 e_1 + x_2 e_2 + \ldots + x_n e_n\|_{w,q}$$

respectively and let $A_\sigma^1(y) = \{\xi \in \mathbb{R} : \xi \cdot y \in A_\sigma\}$. $A_\sigma^1(y)$ is the 1-dimensional preimage of the function $\xi \to \xi \cdot y$ into A_σ.

Proposition 2.3.1. *For $q > 2$, $\sigma \in S_n$ and a fixed $y = (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$ with $y \neq 0$, the function $x_1 \to \|x_1 e_1 + x_2 e_2 + \ldots + x_n e_n\|$ has a continuous first and second derivative on the set $A_\sigma^1(y)$.*

Moreover, for every $(x_1, y) \in A_\sigma$ with $\|y\| = 1$ we have

$$\|x\|'_{x_1}(0, y) = \|x\|''_{x_1^2}(0, y) = 0, \tag{2.3.1}$$

and there exists a $K = K(q, \sigma) > 0$ such that $\|x\|''_{x_1^2}(x_1, y) < K$ for every $y \in \mathbb{R}^{n-1}$ with $\|y\| = 1$, and every $x_1 \in A^1_\sigma(y)$.

Proof. Fix $\sigma \in S_n$ and $y \in \mathbb{R}^{n-1} \setminus \{0\}$. We will now consider the function $x_1 \to \|x_1e_1 + x_2e_2 + \ldots + x_ne_n\|$ defined on the set $A^1_\sigma(y)$. This function is even so it suffices to assume $x_1 \geq 0$, which implies $(|x_1|^q)' = qx_1^{q-1}$. Let $i \in \{1, \ldots, n\}$ be such that $x_i^* = |x_1|$. Since we are only considering $(x_1, y) \in A_\sigma$, the index i is constant. Let $C_{\sigma,y} = \sum_{j \neq i} a_j(x_j^*)^q$, which is constant for a fixed σ and y. A direct calculation shows

$$\|x\|'_{x_1}(x_1, y) = \frac{a_ix_1^{q-1}}{\left(a_ix_1^q + C_{\sigma,y}\right)^{1-\frac{1}{q}}} \tag{2.3.2}$$

and

$$\|x\|''_{x_1^2}(x_1, y) = \frac{(q-1)a_ix_1^{q-2}C_{\sigma,y}}{\left(a_ix_1^q + C_{\sigma,y}\right)^{2-\frac{1}{q}}}. \tag{2.3.3}$$

Since $y \neq 0$ then $\|x\|'_{x_1}$ and $\|x\|''_{x_1^2}$ are both well defined, continuous functions on $A^1_\sigma(y)$. Notice also that

$$\|x\|'_{x_1}(0, y) = \|x\|''_{x_1^2}(0, y) = 0.$$

Now suppose $\|y\| = 1$. Notice that since by our definition of $\|y\| = \|x_2e_2 + \ldots + x_ne_n\| = (a_2(x_2^*)^q + \ldots + a_n(x_n^*)^q)^{1/q}$ we have $C_{\sigma,y} = \|y\|^q$. Since we are supposing

$\|y\| = 1$ then $C_{\sigma,y} = 1$. Then (2.3.3) becomes

$$\|x\|''_{x_1^2}(x_1, y) = \frac{(q-1)a_i x_1^{q-2}}{\left(a_i x_1^q + 1\right)^{2-\frac{1}{q}}}. \tag{2.3.4}$$

If $x_1 \geq 1$ then, remembering that $x_1 \geq 0$,

$$\begin{aligned}\|x\|''_{x_1^2}(x_1, y) &= \frac{(q-1)a_i x_1^{q-2}}{\left(a_i x_1^q + 1\right)^{2-\frac{1}{q}}} \leq \frac{(q-1)a_i x_1^{q-2}}{\left(a_i x_1^q\right)^{2-\frac{1}{q}}} \\ &= \frac{(q-1)}{a_i^{q-1} x_1^{q-1}} \leq \frac{(q-1)}{a_i^{q-1}} \end{aligned} \tag{2.3.5}$$

If $0 \leq x_1 < 1$ then

$$\|x\|''_{x_1^2}(x_1, y) \frac{(q-1)a_i x_1^{q-2}}{\left(a_i x_1^q + 1\right)^{2-\frac{1}{q}}} \leq (q-1)a_i x_1^{q-2} \leq (q-1)a_i$$

Now remember that since $x \in A_\sigma$ and σ is fixed then the weight a_i, next to x_1, is fixed. In either case we have bounded $\|x\|''_{x_1^2}(x_1, y)$ above by a constant dependent on q and σ only. So there exists a constant $K = K(q, \sigma)$ as Proposition 2.3.1 claims exists. □

Proposition 2.3.1 shows that the functions $\|x\|'_{x_1}(\cdot, y)$ and $\|x\|''_{x_1^2}(\cdot, y)$ exist on $A^1_\sigma(y)$ for every fixed $\sigma \in S_n$ and every $y \in \mathbb{R}^{n-1} \setminus \{0\}$, which is why we defined the sets A_σ as we did.

We now state the following remarks that will be important in our proof.

Remark 2.3.2. *Let $0 < p < n-3$ and fix $\sigma \in S_n$.*

(i) The function $y \mapsto \|y\|^{-p-2}$ is locally integrable on $\mathbb{R}^{n-1}$.

(ii) *For every fixed* $y \in \mathbb{R}^{n-1}$*, the function* $x_1 \mapsto \|x_1e_1 + \ldots + x_ne_n\|$ *is a convex differentiable function whose derivative at* $x_1 = 0$ *is zero and therefore* $\|x\| \geq \|y\|$ *for every* $x = (x_1, y) \in A_\sigma$.

(iii) *The function* $\|x\|''_{x_1^2}$ *is nonnegative, homogeneous of degree* -1*. If* K *is the upperbound found in Proposition 2.3.1 then for every* $(x_1, y) \in A_\sigma$ *with* $y \neq 0$ *we have*

$$\|x\|''_{x_1^2}(x) = \|x\|''_{x_1^2}\left(\|y\|\frac{x}{\|y\|}\right) = \frac{1}{\|y\|}\|x\|''_{x_1^2}\left(\frac{x}{\|y\|}\right) \leq \frac{K}{\|y\|}. \tag{2.3.6}$$

(iv) *Convergence in the limit* $\lim_{x_1 \to 0}\|x\|''_{x_1^2}(x_1, y) = 0$ *is uniform with respect to* $y \in \mathbb{R}^{n-1}$ *when* $\|y\| = 1$.

Proof.

(i) Let $B^n_{w,q}$, k_1, and k_2 be as in the proof of Proposition 2.3.1 so that $k_1|y|_2 \leq \|y\|_{w,q} \leq k_2|y|_2$ for all $y \in \mathbb{R}^{n-1}$. When we switch to polar coordinates we get

$$\int_{\lambda B_2^n} \|y\|^{-p-2}\, dy \leq 2\pi k_1^{-p-2}\frac{\lambda^{n-3-p}}{n-3-p}. \tag{2.3.7}$$

This shows (i) holds because the right hand side of (2.3.7) is finite for a given value of λ.

(ii) A direct computation using the definition of convexity shows the function $x_1 \mapsto \|x_1e_1 + \ldots + x_ne_n\|$ is convex on the set $A^1_\sigma(y)$. By Proposition 2.3.1 we know that $\|x\|'_{x_1}(0, y) = 0$ so, by convexity, $\|x\| \geq \|y\|$.

(iii) The homogeneity of $\|x\|''_{x_1^2}$ follows from the fact that $\|x\|'_{x_1}$ is homogeneous of degree 0.

(iv) Notice that the expression for $\|x\|''_{x_1^2}$ in (2.3.4) does not depend on y when $\|y\| = 1$ so claim (iv) holds.

$\square$

2.4 Lemma

Define a family of even Schwartz functions $(\phi_m)_{m\in\mathbb{N}}$, as

$$\phi_m(x) = h_m(x_1)u(x_2, \ldots, x_n), \quad x \in \mathbb{R}^n, \tag{2.4.1}$$

where for each $m \in \mathbb{N}$, $h_m : \mathbb{R} \to \mathbb{R}$ is defined as

$$h_m(x_1) = \frac{m}{\sqrt{2\pi}} \exp\left(-\frac{x_1^2 m^2}{2}\right),$$

and $u : \mathbb{R}^{n-2} \to \mathbb{R}$ is defined as

$$u(x_2, \ldots, x_n) = \frac{1}{(2\pi)^{(n-1)/2}} \exp\left(-\frac{x_2^2 + \ldots + x_n^2}{2}\right).$$

Lemma 2.4.1. *Let $p \in (0, n-3)$. For every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $-\epsilon \leq \left\langle (\|\cdot\|^{-p})''_{x_1}, \phi_m \right\rangle$ whenever $m \geq N$.*

Proof. We will first show this theorem holds when $\left\langle (\|\cdot\|^{-p})''_{x_1}, \phi_m \right\rangle$ is replaced with

$$\int_{A_\sigma} \|x\|^{-p} \frac{d^2}{dx_1^2} \phi_m(x)\, dx$$

where $\sigma \in S_n$ is arbitrary.

Fix $\sigma \in S_n$. By Fubini's theorem we have

$$\int_{A_\sigma} \|x\|^{-p} \frac{d^2}{dx_1^2}\phi_m(x)\, dx = \int_{\mathbb{R}^{n-1}} u(y) \left(\int_{A^1_\sigma(y)} \|x\|^{-p} \frac{d^2}{dx_1^2} h_m(x_1)\, dx_1 \right) dy. \qquad (2.4.2)$$

Since $(\|x\|^{-p})''_{x_1^2}(\cdot, y)$ exists on each $A^1_\sigma(y)$ and since $h_m(x_1) \to 0$ as $x_1 \to \pm\infty$ for all $m \in \mathbb{N}$ we can integrate by parts twice to see that (2.4.2) is equal to

$$\int_{\mathbb{R}^{n-1}} u(y) \left(\int_{A^1_\sigma(y)} (\|x\|^{-p})''_{x_1^2}(x_1, y)\, h_m(x_1)\, dx_1 \right) dy. \qquad (2.4.3)$$

Now notice

$$(\|x\|^{-p})''_{x_1^2}(x) = p(p+1)\|x\|^{-p-2}(\|x\|'_{x_1}(x))^2 - p\|x\|^{-p-1}\|x\|''_{x_1^2}(x),$$

so that (2.4.3) breaks into the sum of two integrals, the first of which is postive and the second is negative. Therefore, in order to prove the lemma, it suffices to show that the integral of the second term

$$\int_{\mathbb{R}^{n-1}} u(y) \left(\int_{A^1_\sigma(y)} (\|x\|^{-p-1}\|x\|''_{x_1^2}(x_1, y)\, h_m(x_1)\, dx_1 \right) dy \qquad (2.4.4)$$

converges to 0 as m grows large.

Let $\epsilon > 0$. By Remark 2.3.2 (i), $\|y\|^{-p-2}$ is locally integrable on $\mathbb{R}^{n-1}$ hence

$$L = \int_{\mathbb{R}^{n-1}} \|y\|^{-p-2} u(y) dy < \infty.$$

By (2.3.7) we can choose $d > 0$ small enough so that

$$\int_{\|y\|<d} \|y\|^{-p-2} u(y) dy < \frac{\epsilon}{3K}, \tag{2.4.5}$$

where K is the constant from Propostion 2.3.1. By Remark (ii) and (iii) we have

$$\begin{aligned}
&\int_{\|y\|<d} u(y) \left(\int_{A^1_\sigma(y)} (\|x\|^{-p-1} \|x\|''_{x_1^2}(x_1, y)\, h_m(x_1)\, dx_1 \right) dy \\
&\leq \int_{\|y\|<d} u(y) \left(\int_{A^1_\sigma(y)} (\|y\|^{-p-2}\, h_m(x_1)\, dx_1 \right) dy \\
&\leq K \int_{\mathbb{R}} h_m(x_1) dx_1 \int_{\mathbb{R}^{n-1}} \|y\|^{-p-2} u(y) dy \leq \frac{\epsilon}{3}.
\end{aligned} \tag{2.4.6}$$

By Remark (iv) there exists a $\delta > 0$ such that

$$\|x\|''_{x_1^2}(x_1, y) < \frac{\epsilon}{3L} \tag{2.4.7}$$

whenever $|x_1| < \delta$ and $\|y\| = 1$. By Remark (iii) the second derivative of $\|x\|''_{x_1^2}$ is homogeneous of degree -1 so this, along with uniform convergence, implies that

$$\|x\|''_{x_1^2}(x_1, y) = \frac{1}{\|y\|} \|x\|''_{x_1^2} \left(\frac{x_1}{\|y\|}, \frac{y}{\|y\|} \right) \leq \frac{\epsilon}{3L\|y\|} \tag{2.4.8}$$

whenever $|x_1| / \|y\| < \delta$.

Now consider the sets

$$E_1(y) := \{x_1 \in A^1_\sigma(y) : |x_1| < \delta d\}$$

$$E_2(y) := \{x_1 \in A^1_\sigma(y) : |x_1| > \delta d\}.$$

If $\|y\| \geq d$ and $x_1 \in E_2(y)$ then $|x_1| / \|y\| < \delta$ so we can use (2.4.8) to get

$$\int_{\|y\|>d} u(y) \left(\int_{E_1(y)} \|x\|^{-p-1} \|x\|''_{x_1^2}(x_1, y)\, h_m(x_1)\; dx \right) dy$$
$$\leq \frac{\epsilon}{3L} \int_{\mathbb{R}} h_m(x_1) dx_1 \int_{\mathbb{R}^{n-1}} \|y\|^{-p-2} u(y) dy \leq \frac{\epsilon}{3}. \tag{2.4.9}$$

Finally, notice for a given $\eta > 0$

$$\lim_{m\to\infty} \int_{\{x_1 : |x_1| > \eta\}} h_m(x_1) dx_1 = 0,$$

hence we can choose $N \in \mathbb{N}$ so that for all $m > N$, the following holds:

$$\int\limits_{\{x_1\,:\,|x_1| > \delta d\}} h_m(x_1)\; dx_1 < \frac{\epsilon}{3KL}. \tag{2.4.10}$$

This along with Remark (ii) and (iii) imply for all $m > N$

$$\int_{\|y\|\geq d} u(y) \left(\int_{E_2(y)} \|x\|^{-p-1} \|x\|''_{x_1^2}(x_1, y)\, h_m(x_1)\; dx \right) dy$$
$$\leq K \frac{\epsilon}{3L} \int_{\mathbb{R}} h_m(x_1) dx_1 \int_{\mathbb{R}^{n-1}} \|y\|^{-p-2} u(y) dy \leq \frac{\epsilon}{3}. \tag{2.4.11}$$

Combining (2.4.6), (2.4.9), and (2.4.11) gives us

$$\lim_{m\to\infty} \int_{\mathbb{R}^{n-1}} u(y) \left(\int_{A^1_\sigma(y)} (\|x\|^{-p-1} \|x\|''_{x_1^2}(x_1, y)\, h_m(x_1)\, dx_1 \right) dy = 0. \tag{2.4.12}$$

Since ϵ and σ were arbitrary we must have that this theorem holds when $\left\langle (\|\cdot\|^{-p})''_{x_1}, \phi_m \right\rangle$ is replaced with $\int_{A_\sigma} \|x\|^{-p} \frac{d^2}{dx_1^2} \phi_m(x)\, dx$ where $\sigma \in S_n$. Notice

$$\left\langle (\|x\|^{-p})''_{x_1^2}, \phi_m \right\rangle = \int_{\mathbb{R}^n} \|x\|^{-p} \frac{d^2}{dx_1^2} \phi_m(x)\, dx$$

$$= \sum_{\sigma \in S_n} \int_{A_\sigma} \|x\|^{-p} \frac{d^2}{dx_1^2} \phi_m(x)\, dx. \tag{2.4.13}$$

S_n is finite so (2.4.13) along with our previous work completes the lemma. $\square$

2.5 Proof of the Main Theorem

Proof. Let $n \geq 5$. With $B^n_{w,q}$ still denoting the unit ball of $\ell^n_{w,q}$ we have that $B^n_{w,q}$ is an origin-symmetric convex body in $\mathbb{R}^n$, so Corollary 4.9 of [25] shows that if $p \in [n-3, n)$ then $\|x\|^{-p}_{B^n_{w,q}} = \|\cdot\|^{-p}_{w,q}$ represents a positive definite distribution.

Now suppose $p \in (0, n-3)$ and that, by way of contradiction, $\|\cdot\|^{-p}_{w,q}$ is a positive definite distribution. By Corollary 2.26 of [25] there exists a finite Borel measure μ on S^{n-1} such that for every even test function ϕ,

$$\langle \|\cdot\|^{-p}, \phi \rangle = \int_{\mathbb{R}^n} \|x\|^{-p} \phi(x)\, dx = \int_{S^{n-1}} \left(\int_0^\infty t^{p-1} \widehat{\phi}(t\xi)\, dt \right) d\mu(\xi) \tag{2.5.1}$$

Since $\left(\frac{d^2}{dx_1^2}\phi\right)^\wedge(\xi) = -\xi_1^2 \widehat{\phi}(\xi)$ we have

$$\left\langle \left(\|\cdot\|^{-p}\right)''_{x_1^2}, \phi \right\rangle = \left\langle \|\cdot\|^{-p}, \frac{d^2}{dx_1^2}\phi \right\rangle = - \int_{S^{n-1}} \left(\int_{\mathbb{R}} |t|^{p+1}\, \widehat{\phi}(t\xi)\, dt \right) \xi_1^2 \, d\mu(\xi). \tag{2.5.2}$$

This holds for all even test function ϕ so let us consider the functions ϕ_m as defined in (2.4.1).

We now recall a few properties of the Fourier transform. Remember we defined this as the following: If $f \in \mathscr{S}(\mathbb{R}^n)$ the Fourier transform of f, denoted $\widehat{f}$, is

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(\xi) e^{-ix\cdot\xi} dx. \tag{2.5.3}$$

Let δ_λ be the multiplication function defined by $\delta_\lambda(f)(x) = f(\lambda x)$ where $\lambda \in \mathbb{R}$. The properties of the Fourier transform we will use are the following:

1. $(\delta_\lambda(f))^\wedge = \frac{1}{\lambda^n}\widehat{f}\left(\frac{1}{\lambda}x\right)$

2. If $f_1 : \mathbb{R}^m \to \mathbb{R}$ and $f_2 : \mathbb{R}^{n-m} \to \mathbb{R}$ then

$$\Big(f_1(x_1, \ldots, x_m) f_2(x_{m+1}, \ldots, x_n)\Big)^\wedge = (f_1(x_1, \ldots, x_m)^\wedge)(f_2(x_{m+1}, \ldots, x_n)^\wedge)$$

3. $\left(\exp\left(-\frac{|x|_2^2}{2}\right)\right)^\wedge(\xi) = (2\pi)^{\frac{n}{2}} \exp\left(-\frac{|\xi|_2^2}{2}\right)$

Combining these properties, we can now calculate the Fourier transform of the family of functions ϕ_m in (2.4.1) to get

$$\widehat{\phi}_m(\xi) = \exp\left(-\frac{\xi_1^2}{2m^2}\right)\exp\left(-\frac{\xi_2^2 + \ldots + \xi_n^2}{2}\right). \tag{2.5.4}$$

Using (2.5.1), (2.5.2), and (2.5.4) along with the change of variable

$$x = \frac{t^2}{2}\exp\left(\frac{\xi_1^2}{m^2} + \xi_2^2 + \ldots + \xi_n^2\right)$$

gives us

$$\begin{aligned}
\Big\langle (\|\cdot\|^{-p})''_{x_1}, \phi_m \Big\rangle &= -2\int_{S^{n-1}} \left(\int_0^\infty t^{p+1}\widehat{\phi_m}(t\xi)\, dt\right) \xi_1^2 \, d\mu(\xi) \\
&= -2^{(p+1)/2}\Gamma\left(\frac{p+2}{2}\right)\int_{S^{n-1}} \xi_1^2\left(\frac{\xi_1^2}{m^2} + \xi_2^2 + \ldots + \xi_n^2\right) d\mu(\xi) \\
&\leq -2^{(p+1)/2}\Gamma\left(\frac{p+2}{2}\right)\int_{S^{n-1}} \xi_1^2 \, d\mu(\xi), \qquad (2.5.5)
\end{aligned}$$

where the last inequality follows because $\xi \in S^{n-1}$. By Lemma 2.4.1, we can make the left side of (2.5.5) as close to zero as we like by taking m large enough. The

Gamma function is positive on $(0, \infty)$ so the integral on the right side of (2.5.5) is non-positive hence

$$\int_{S^{n-1}} \xi_1^2 \, d\mu(\xi) = 0. \tag{2.5.6}$$

This implies μ is supported on the intersection of the hyperplane $\xi_1 = 0$ and the sphere S^{n-1}. By (2.5.2) we have $(\|x\|^{-p})''_{x_1}(x) = 0$ on each A_σ which implies for each $\sigma \in S_n$ there exist two functions, each depending on our permutation σ, $g_{1,\sigma}(x_2, \ldots, x_n)$ and $g_{1,\sigma}(x_2, \ldots, x_n)$ such that $\|x\|^{-p} = g_{1,\sigma}(x_2, \ldots, x_n) + x_1 g_{2,\sigma}(x_2, \ldots, x_n)$ on A_σ. It is important to note that $g_{2,\sigma}$ cannot be identically zero everywhere or else $\|(x_1, x_2, \ldots, x_n)\|_{w,q}$ would not depend on x_1 which cannot be since $\|\cdot\|_{w,q}$ is a norm and must depend on all its coordinates. Thus $\|\cdot\|_{w,q}$ has a representation as a linear funtion of x_1 on each A_σ and it is this representation that leads to a contradiction because we know $\|\cdot\|_{w,q}$ is a norm so $\|(x_1, x_2, \ldots, x_n)\|_{w,q} \to \infty$ as $x_1 \to \infty$ for all fixed $(x_2, \ldots, x_n) \in \mathbb{R}^n \setminus \{0\}$ hence $\|x\|^{-p} \to 0$ as $x_1 \to \infty$. But, by the linear representation of $\|x\|^{-p}$ we know that $\|x\|^{-p} \to \infty$ on each A_σ as $x_1 \to \infty$. Thus our original assumption that $\|\cdot\|_{w,q}^{-p}$ is a positive definite function was false. In particular we see that the unit ball of the Lorentz norm is not an intersection body for $q > 2$ and $n \geq 5$. This completes the proof of the main theorem. $\square$

Chapter 3

Separation of Centroid Bodies

3.1 Introduction

The main results of this section are from the paper [47] which is currently in review. We denote by $\mathscr{K}_{os}^n$ the class of origin-symmetric convex bodies K in $\mathbb{R}^n$, i.e. K is a compact set with non-empty interior and such that $K = -K$. For every $K \in \mathscr{K}_{os}^n$ we let $|K|$ denote the volume of K which is the n-dimensional Lebesgue measure of K. We let L_p denote $L_p([0,1], dx)$ the space of p-summable functions on $[0,1]$ with the one dimensional Lebesgue measure. If c is a constant, we use $c = c(n,p)$ to denote that c depends on n and p. We use the notation $X \hookrightarrow Y$ to denote the space X embeds isometrically in Y, i.e. there exists a one to one linear map $T : X \to Y$ such that $\|Tx\|_Y = \|x\|_X$ for all $x \in X$. If K is a convex body then its support function is defined as $h(K,\xi) = \max\{\xi \cdot x : x \in K\}$ where $x \cdot y$ is the scalar product of x and y. If K is an origin-symmetric star body then its *polar body* K^* is the body defined by $K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K\}$. If $p \geq 1$ then the L_p-centroid body of K, denoted $\Gamma_p(K)$, is the origin symmetric star body defined by

$$h\Big(\Gamma_p(K), \xi\Big) = \left(\frac{1}{|K|}\int_K |x\cdot\xi|^p\, dx\right)^{\frac{1}{p}}, \quad \text{for all} \quad \xi \in S^{n-1}. \tag{3.1.1}$$

If $p < 1$ the right hand side of (3.1.1) is not necessarily convex and thus cannot be used as a support function for a body. Following V. Yaskin and M. Yaskina [49], for $0 < p < 1$ we still use the term polar L_p-centroid body for the star body $\Gamma_p^*(K)$ defined by

$$\|\xi\|_{\Gamma_p^*(K)} = \left(\frac{1}{|K|}\int_K |x\cdot\xi|^p\, dx\right)^{\frac{1}{p}}, \quad \text{for all} \quad \xi \in S^{n-1}. \tag{3.1.2}$$

If we send p to zero then the definition of a polar L_0-centroid body looks as follows:

$$\|\xi\|_{\Gamma_0^*(K)} = \exp\left(\frac{1}{|K|}\int_K \ln|x\cdot\xi|\, dx\right), \quad \text{for all} \quad \xi \in S^{n-1}. \tag{3.1.3}$$

A volume comparison problem is usually in the following form: If $K, D \in \mathscr{K}_{os}^n$, is it true that the inequality $f_K(\xi) \le f_D(\xi)$, $\forall \xi \in S^{n-1}$ implies $|K| \le |D|$? Here f_K is some function on S^{n-1} which describes a certain geometric characteristic of K.

If the volume comparison problem is affirmative then we can ask if a corresponding separation inequality holds: if $\varepsilon > 0$ and

$$f_K(\xi) \le f_D(\xi) - \varepsilon, \quad \text{for all} \quad \xi \in S^{n-1} \tag{3.1.4}$$

is it true there exists a constant c, not depending on the dimension or ε, so that

$$|K|^q \le |D|^q - c\varepsilon? \tag{3.1.5}$$

Here q is a power which could change depending on f. In our paper it turns out to be p/n for a certain range of p. The corresponding stability inequality asks whether

$$f_K(\xi) \leq f_D(\xi) + \varepsilon, \quad \text{for all} \quad \xi \in S^{n-1} \tag{3.1.6}$$

implies

$$|K|^q \leq |D|^q + c\varepsilon. \tag{3.1.7}$$

For example the Busemann-Petty problem asks the following: Let $\xi^{\perp} = \{x \in \mathbb{R}^n : x \cdot \xi = 0\}$ be the $(n-1)$-dimensional hyperplane orthogonal to $\xi \in S^{n-1}$. If $\left|K \cap \xi^{\perp}\right| \leq \left|D \cap \xi^{\perp}\right|$ for every $\xi \in S^{n-1}$ is it true that $|K| \leq |D|$? The solution was the result of a long series of papers. A unified, Fourier analytic solution was given in [12]. A complete history can be found in [25, pg 3] or [11, pg 343] and the list of references therein. The projection counterpart of the Busemann-Petty problem is called Shephard's problem and it reads as follows: Let K and D be origin-symmetric, convex bodies in $\mathbb{R}^n$ and suppose, for every $\xi \in S^{n-1}$,

$$\left|K|\xi^{\perp}\right| \leq \left|D|\xi^{\perp}\right|. \tag{3.1.8}$$

Does it follow that $|K| \leq |D|$? The solution was found independently by C.M. Petty [43] and R. Schneider [45]. Later, a Fourier analytic proof was given in [33] by A. Koldobksy, D. Ryabogin, and A. Zvavitch.

In [26], [29], and [32] A. Koldobsky proved the corresponding stability and separation results for the Busemann-Petty problem as well as Shephard's problem and, from these results, deduced new volume comparison inequalities using differentiation techniques.

Recently, J. Hu [16] showed stability and separation results for a modified L_p-Shephard problem. In what follows we will need the following notation: $\widetilde{\Gamma}_p(K)$ will be the body whose support function is

$$h\left(\widetilde{\Gamma}_p(K), \xi\right) = \left(\frac{1}{|K|}\int_{S^{n-1}} |x\cdot\xi|^p\, S_p(K,x)\right)^{1/p}.$$

Here $S_p(K,\cdot)$ is the p-th surface area measure of K defined by the following: Define the *p-th Firey-summation* $K +_p L$ as $h(\alpha K +_p \beta L, \cdot)^p = \alpha h(K,\cdot)^p + \beta h(L,\cdot)^p$. In the papers [40] and [39] E. Lutwak defined the p-mixed volume $V_p(K,L)$, $p \geq 1$ as

$$\frac{n}{p} V_p(K,L) = \lim_{\varepsilon\to 0} \frac{V(K +_p \varepsilon L) - V(K)}{\varepsilon}.$$

Lutwak proved in these papers that there exists a measure $S_p(K,\cdot)$ on S^{n-1} such that

$$V_p(K,L) = \frac{1}{n}\int_{S^{n-1}} h(L,\xi)^p dS_p(K,\xi).$$

Notice this formula is an L_p extension of the well known ([46]) formula for the mixed volume K and L:

$$V(K,L) = \frac{1}{n}\int_{S^{n-1}} h(L,\xi) dS(K,\xi),$$

Here $S(K,\cdot)$ is the surface area measure of K.

Now we get to J. Hu's stability and separation result for the modified L_p Shephard problem:

Proposition 3.1.1. *(J. Hu [16]) Let $p \geq 1$, where p is not an even integer. Suppose that $\varepsilon > 0$, let $K, L \in \mathscr{K}^n_{os}$, and suppose $(\mathbb{R}^n, \|\cdot\|_{L^*})$ embeds isometrically into L_p.*

There exists constants $C_1(p,n)$ and $C_2(p,n)$ that satisfy the following: If for every $\xi \in S^{n-1}$,

$$h(\widetilde{\Gamma}_p(K),\xi)^p \leq h(\widetilde{\Gamma}_p(L),\xi)^p + \varepsilon$$

then

$$|K|^{-\frac{p}{n}} \leq |L|^{-\frac{p}{n}} + \varepsilon C_1(p,n).$$

If for every $\xi \in S^{n-1}$,

$$h(\widetilde{\Gamma}_p(K),\xi)^p \leq h(\widetilde{\Gamma}_p(L),\xi)^p - \varepsilon$$

then

$$|K|^{-\frac{p}{n}} \leq |L|^{-\frac{p}{n}} - \varepsilon C_2(p,n).$$

Here L^* is the volume normalized polar projection body of L.

These results by J. Hu are similar to ours but the exponents are negative and the proof uses Fourier analytic techniques whereas the results presented here deal with positive exponents and use classical inequalities related to convex bodies.

The purpose of this current chapter is to show separation, when $p \geq 0$, for the following L_p-centroid volume comparison result proved by V. Yaskin and M. Yaskina:

Proposition 3.1.2 ([49])**.** *Let $p \in (-1,0) \cup (0,1)$ and $K, D \in \mathscr{K}^n_{os}$ be such that $(\mathbb{R}^n, \|\cdot\|_D)$ embeds in L_p and $\Gamma^*_p D \subseteq \Gamma^*_p K$. Then $|K| \leq |D|$.*

Here $\Gamma^*_p(K)$ is the polar centroid body of K which we described above in section 3.1 of chapter 3.

From our separation results we obtain new results using the techniques described by A. Koldobsky in [29].

We can now talk briefly about the techniques developed by A. Koldobsky in [29] that we intend to use. If separation holds for a volume comparision problem, i.e. if $f_K(\xi) \leq f_D(\xi) - \varepsilon$ for all $\xi \in S^{n-1}$ implies $|K|^q \leq |D|^q - c\varepsilon$ for some constant c, then one can choose

$$\varepsilon := \min_{\xi \in S^{n-1}} |f_K(\xi) - f_D(\xi)|,$$

so that both inequality (3.1.4) and $f_D(\xi) \leq f_K(\xi) - \varepsilon \quad \text{for all} \quad \xi \in S^{n-1}$ hold. If this were true we would have both $|K|^q \leq |D|^q - c\varepsilon$ and $|D|^q \leq |K|^q - c\varepsilon$ hence

$$c \min_{\xi \in S^{n-1}} |f_K(\xi) - f_D(\xi)| \leq ||K|^q - |D|^q|. \tag{3.1.9}$$

If we can choose a sequence of $(D_m)_{m=1}^{\infty} \subseteq \mathscr{K}_{os}^n$ so $(\mathbb{R}^n, \|\cdot\|_{D_m})$ embeds in L_p for each m and both $|D_m|$ and f_{D_m} go to zero as $m \to \infty$ then (3.1.9) turns into

$$c \min_{\xi \in S^{n-1}} |f_K(\xi)| \leq |K|^q. \tag{3.1.10}$$

If stability holds then one can choose

$$\varepsilon := \max_{\xi \in S^{n-1}} |f_K(\xi) - f_D(\xi)| \tag{3.1.11}$$

and arrive at

$$|K|^q \leq c \max_{\xi \in S^{n-1}} \left| K \cap \xi^{\perp} \right|. \tag{3.1.12}$$

Inequalities (3.1.10) and (3.1.12) are called slicing inequalities. The name alludes to their similarity to the well-known slicing problem initially asked by J. Bourgain in [4] and has been the subject of much research since then. An equivalent version of the slicing problem is the following: Is there a universal constant c so that

$$|K|^{\frac{n-1}{n}} \leq c \max_{\xi \in S^{n-1}} \left| K \cap \xi^{\perp} \right| \tag{3.1.13}$$

whenever K is an origin-symmetric convex body in $\mathbb{R}^n$? Taking $|K| = 1$ gives us the following question: Is there a universal constant C so that

$$C \leq \max_{\xi \in S^{n-1}} \left| K \cap \xi^{\perp} \right| ? \tag{3.1.14}$$

whenever $K \in \mathscr{K}_{os}^n$. The constant C does not depend on the dimension of the ambient space nor the body K. The best known constant to date is $c = O(n^{1/4})$ due to B. Klartag [20], who improved the previous estimate of $O(n^{1/4}) \log(n)$ by Bourgain [6]. For more partial results see [30], [3], [6], [17], [20], [34], and [41]. The book [7] contains a thorough exposition of the problem.

The outline of this paper is as follows: In section 2 we prove separation inequalities for Proposition 3.1.2 for $p > 1$. In section 3 we prove separation inequalities for $p = 0$. It is worth noting that the author was unable to prove the corresponding stability result.

3.2 Separation Inequalities

In this section we consider the separation inequality corresponding to Proposition 3.1.2. We will need the following well-known proposition which goes back to P. Lévy (see Lemma 6.4 of [25] for a proof):

Proposition 3.2.1. *A normed space* $(\mathbb{R}^n, \|\cdot\|)$ *embeds in* L_p*,* $p \in (0, \infty)$ *if and only* *if there exits a finite, Borel measure μ on S^{n-1} so that*

$$\|\xi\|^p = \int_{S^{n-1}} |(x,\xi)|^p \, d\mu(x), \quad \textit{for all} \quad \xi \in S^{n-1}. \tag{3.2.1}$$

In V. Yaskin and M. Yaskina's original result, stated in Proposition 3.1.2, they used the notation $\Gamma_p^* D \subseteq \Gamma_p^* K$. We use the equivalent formulation $\|\xi\|_{\Gamma_p^* K} \leq \|\xi\|_{\Gamma_p^* D}$ for all $\xi \in S^{n-1}$ to formulate our main separation result:

Theorem 3.2.2. *Let $p > 0$ and suppose $K, D \in \mathcal{K}^n$ and $(\mathbb{R}^n, \|\cdot\|_D) \hookrightarrow L_p$. If $\varepsilon > 0$ is small enough so that*

$$\|\xi\|^p_{\Gamma_p^*(K)} \leq \|\xi\|^p_{\Gamma_p^*(D)} - \varepsilon, \quad \textit{for all} \quad \xi \in S^{n-1} \tag{3.2.2}$$

then

$$|K|^{\frac{p}{n}} \leq |D|^{\frac{p}{n}} - c\varepsilon \tag{3.2.3}$$

where

$$c = |B_2^n|^{\frac{n+p}{n}} (n+p) \frac{\Gamma\left(\frac{n+p}{2}\right)}{2\pi^{\frac{n-1}{2}} \Gamma\left(\frac{p+1}{2}\right)}. \tag{3.2.4}$$

The following result is crucial for our proof. It comes from section 2.2 of [41].

Proposition 3.2.3. *If $K, D \in \mathscr{K}^n_{os}$ and $p > 0$ then*

$$\left(\frac{|K|}{|D|}\right)^{\frac{1}{n}} \left(\frac{n}{n+p}\right)^{\frac{1}{p}} \leq \left(\frac{1}{|K|} \int_K \|x\|_D^p \, dx\right)^{\frac{1}{p}}.$$

Sending $p \to 0$ gives us

$$\frac{1}{n} \ln\left(\frac{|K|}{|D|}\right) - \frac{1}{n} \leq \frac{1}{|K|} \int_K \ln\|x\|_D^p \, dx. \tag{3.2.5}$$

Proof of theorem 3.2.2. Equation (3.2.2) can be rewritten as

$$\frac{1}{|K|}\int_K |(x,\xi)|^p \, dx \le \frac{1}{|D|}\int_D |(x,\xi)|^p \, dx - \varepsilon. \tag{3.2.6}$$

Let μ_D be the measure arising from our assumptions that $(\mathbb{R}^n, \|\cdot\|_D) \hookrightarrow L_p$, as in Proposition 3.2.1. Integrating both sides of (3.2.6) with respect to this new measure and then using Fubini's theorem gives us

$$\begin{aligned}&\frac{1}{|K|}\int_K \int_{S^{n-1}} |(x,\xi)|^p \, d\mu_D(\xi)\, dx \le \\ &\frac{1}{|D|}\int_D \int_{S^{n-1}} |(x,\xi)|^p \, d\mu_D(\xi)\, dx - \varepsilon \int_{S^{n-1}} d\mu_D(\xi).\end{aligned} \tag{3.2.7}$$

We can use Proposition 3.2.1 to rewrite (3.2.7) as

$$\frac{1}{|K|}\int_K \|x\|_D^p \, dx \le \frac{1}{|D|}\int_D \|x\|_D^p \, dx - \varepsilon \int_{S^{n-1}} d\mu_D(\xi). \tag{3.2.8}$$

Using polar coordinates we have

$$\begin{aligned}\int_D \|x\|_D^p \, dx &= \int_{S^{n-1}} \int_0^{\|\theta\|_K^{-1}} \|r\theta\|_K^p r^{n-1}\, d\theta \\ &= \frac{1}{n+p}\int_{S^{n-1}} \|\theta\|_K^{-n}\, d\theta = \frac{n}{n+p}\, |D|\, .\end{aligned} \tag{3.2.9}$$

Here we used the fact that

$$|D| = \frac{1}{n}\int_{S^{n-1}} \|x\|_D^{-n} dx \tag{3.2.10}$$

which can be seen by switching to polar coordinates in the integral representation of the n-dimensional volume of D. By Proposition 3.2.3 we have

$$\frac{n}{n+p}\left(\frac{|K|}{|D|}\right)^{\frac{p}{n}} \leq \frac{1}{|K|}\int_K \|x\|_D^p dx. \tag{3.2.11}$$

Combining (3.2.8), (3.2.9), and (3.2.11) we get

$$\frac{n}{n+p}\left(\frac{|K|}{|D|}\right)^{\frac{p}{n}} \leq \frac{n}{n+p} - \varepsilon\int_{S^{n-1}} d\mu_D(\xi). \tag{3.2.12}$$

Now we wish to bound the integral in (3.2.12) from below. Lemma 3.12 of [25] says that if $p > -1$ then

$$c_0\int_{S^{n-1}} |(x,\xi)|^p\, dx = 1, \quad \text{for all} \quad \xi \in S^{n-1},$$

where

$$c_0 = \frac{\Gamma\left(\frac{n+p}{2}\right)}{2\pi^{\frac{n-1}{2}}\Gamma\left(\frac{p+1}{2}\right)}. \tag{3.2.13}$$

This, along with using polar coordinates, the definition of μ_D, and Fubini's theorem, gives us

$$\begin{aligned}\int_{S^{n-1}} d\mu_D(\xi) &= c_0\int_{S^{n-1}}\int_{S^{n-1}} |(x,\xi)|^p\, dx d\mu_D(\xi).\\ &= c_0\int_{S^{n-1}} \|x\|_D^p dx\\ &= (n+p)c_0\int_{B_2^n} \|x\|_D^p dx.\end{aligned} \tag{3.2.14}$$

Using (3.2.14) and Proposition 3.2.3 we have

$$c_0\,|B_2^n|\,n\left(\frac{|B_2^n|}{|D|}\right)^{\frac{p}{n}} \leq \int_{S^{n-1}} d\mu_D(\xi). \tag{3.2.15}$$

Combining (3.2.12) and (3.2.15) we get

$$\frac{n}{n+p}\left(\frac{|K|}{|D|}\right)^{\frac{p}{n}} \leq \frac{n}{n+p} - \varepsilon c_0 \left|B_2^n\right| n \left(\frac{|B_2^n|}{|D|}\right)^{\frac{p}{n}}$$

which simplifies to

$$|K|^{\frac{p}{n}} \leq |D|^{\frac{p}{n}} - c\varepsilon.$$

where $c = c(p,n)$ is the constant from (3.2.4). ☐

Corollary 3.2.4. *Let $p > 0$. Suppose $K, D \in \mathscr{K}_{os}^n$ are such that $(\mathbb{R}^n, \|\cdot\|_K) \hookrightarrow L_p$ and $(\mathbb{R}^n, \|\cdot\|_D) \hookrightarrow L_p$. Then*

$$c \min_{\xi\in S^{n-1}} \left| \|\xi\|^p_{\Gamma_p^*(K)} - \|\xi\|^p_{\Gamma_p^*(D)} \right| \leq \left| \, |K|^{\frac{p}{n}} - |D|^{\frac{p}{n}} \, \right|, \tag{3.2.16}$$

where the constant c is as in (3.2.4).

Proof. Let

$$\varepsilon := \min_{\xi\in S^{n-1}} \left| \, \|\xi\|^p_{\Gamma_p^*(K)} - \|\xi\|^p_{\Gamma_p^*(D)} \, \right|.$$

This implies

$$\|\xi\|^p_{\Gamma_p^*(K)} \leq \|\xi\|^p_{\Gamma_p^*(D)} - \varepsilon, \quad \text{for all} \quad \xi \in S^{n-1}. \tag{3.2.17}$$

By Theorem 3.2.2 we have

$$|D|^{\frac{p}{n}} \leq |K|^{\frac{p}{n}} - c\varepsilon, \tag{3.2.18}$$

where c is as in (3.2.4).

Our choice of ε also gives us the same inequality as (3.2.17) but with K and D

interchanged, so, by Theorem 3.2.2, we have

$$|K|^{\frac{p}{n}} \le |D|^{\frac{p}{n}} - c\varepsilon. \tag{3.2.19}$$

Notice the constant c in both (3.2.19) and (3.2.18) are the same because each constant come from Theorem 3.2.2 and the constant there does not depend on the bodies K and D. Solving for ε in both (3.2.18) and (3.2.19) gives us (3.2.16). $\square$

Corollary 3.2.5. *If $p > 0$ and $K \in \mathscr{K}_{os}^n$ is such that $(\mathbb{R}^n, \|\cdot\|_K) \hookrightarrow L_p$ then*

$$\min_{\xi \in S^{n-1}} \|\xi\|^p_{\Gamma_p^*(K)} \le c\,|K|^{\frac{p}{n}} \tag{3.2.20}$$

where the constant c is as in (3.2.4).

Proof. Take $D = \delta B$ where B is the unit ball of any n-dimensional subspace of L_p so that $(\mathbb{R}^n, \|\cdot\|_D) \hookrightarrow L_p$. Thus K and D are as in Corollary 3.2.4 and we can let $\delta \to 0$ to get (3.2.20). $\square$

3.3 Embeddings in L_0

In the past sections we formulated a separation inequality for p-centroid bodies when $p > 0$. In this section we formulate the corresponding separation inequality for 0-centroid bodies. We then go on to form a new slicing-type inequality. Specifically we will prove the following theorem and its corollary:

Theorem 3.3.1. *Suppose $K, D \in \mathscr{K}_{os}^n$ and $(\mathbb{R}^n, \|\cdot\|_D) \hookrightarrow L_0$. If*

$$\|\xi\|_{\Gamma_0^*(K)} \le \|\xi\|_{\Gamma_0^*(D)} - \varepsilon, \qquad \textit{for all} \quad \xi \in S^{n-1} \tag{3.3.1}$$

then there exists a constant $c = c(n, p, D)$ *so that*

$$\ln |K| \leq \ln |D| - c\varepsilon \tag{3.3.2}$$

where

$$c = n\pi^{(1-n)/2} \left[\frac{\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} - \sqrt{\pi} \frac{\Gamma'\left(\frac{n}{2}\right)}{\Gamma^2\left(\frac{n}{2}\right)} \right]^{-1} \left[\ln\left(\frac{|B_2^n|}{e\,|D|} \right) + \frac{|S^{n-1}|}{n} - \lambda_0 \left| S^{n-1} \right| \right] \tag{3.3.3}$$

and

$$\lambda_0 = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} \ln\|\theta\|_D d\theta - \frac{1}{2\sqrt{\pi}} \Gamma'\left(\frac{1}{2}\right) + \frac{1}{2} \frac{\Gamma'\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}. \tag{3.3.4}$$

Remark 3.3.2. *The constant* λ_0 *was calculated in section 2 [19]. Unlike the* $p \neq 0$ *case, the constant* c *in* (3.3.2) *depends on the body* D*. This cannot be avoided with our current technique because the definition of the space* $(\mathbb{R}^n, \|\cdot\|_D)$ *embedding into* L_0 *(Definition 3.3.4) produces a constant which depends on* D*.*

Corollary 3.3.3. *Suppose* $K, D \in \mathscr{K}_{os}^n$ *are such that* $(\mathbb{R}^n, \|\cdot\|_K) \hookrightarrow L_0$ *and* $(\mathbb{R}^n, \|\cdot\|_D) \hookrightarrow L_0$*. Then there exists a constant* $\lambda = \lambda(n, K, D)$ *so that*

$$\min_{\xi \in S^{n-1}} \left| \|\xi\|_{\Gamma_0^*(K)} - \|\xi\|_{\Gamma_0^*(D)} \right| \leq \lambda \left| \ln |K| - \ln |D| \right|. \tag{3.3.5}$$

Before we go on we must define what it means for a body to embed in L_0. The following comes from [19]. It is an extension of the definition of a body embedding in L_p when p is sent to 0.

Definition 3.3.4. *Let* $D \in \mathscr{K}_{os}^n$*. We say the space* $(\mathbb{R}^n, \|\cdot\|_D)$ *embeds into* L_0 *if there* *exists a finite Borel measure μ on S^{n-1} and a constant λ_0 so that*

$$\ln\|x\|_D = \int_{S^{n-1}} \ln|(\theta, x)|\, d\mu(\theta) + \lambda_0, \quad \textit{for all} \quad x \in \mathbb{R}^n \backslash \{0\}. \tag{3.3.6}$$

The measure μ is necessarily a probability measure and the constant λ_0 depends on the body D and can actually be calculated (see sec. 2 of [19]). It is exactly the constant λ_0 from (3.3.4).

In what follows, we will need the next proposition.

Proposition 3.3.5 (sec. 2 of [19])**.** *There exists a constant $c_1 = c_1(n) \neq 0$ so that*

$$c_1 \int_{S^{n-1}} \ln|(\theta, \xi)|\, d\theta = 1, \quad \textit{for all} \quad \xi \in S^{n-1}. \tag{3.3.7}$$

In fact

$$c_1 = \pi^{(n-1)/2} \left(\frac{\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} - \sqrt{\pi} \frac{\Gamma'\left(\frac{n}{2}\right)}{\Gamma^2\left(\frac{n}{2}\right)} \right). \tag{3.3.8}$$

We define the 0-centroid body $\Gamma_0 K$ of K as the body with the support function

$$\|\xi\|_{\Gamma_0^*(K)} = \frac{1}{|K|} \int_K \ln|(x, \xi)|\, dx \quad \text{for all} \quad \xi \in S^{n-1}. \tag{3.3.9}$$

Proof of Theorem 3.3.1. We can rewrite (3.3.1) as

$$\frac{1}{|K|} \int_K \ln|(x, \xi)|\, dx \leq \frac{1}{|D|} \int_D \ln|(x, \xi)|\, dx - \varepsilon, \quad \text{for all} \quad \xi \in S^{n-1}. \tag{3.3.10}$$

Since $(\mathbb{R}^n, \|\cdot\|_D) \hookrightarrow L_0$ there exists a finite Borel measure μ on S^{n-1} as in (3.3.6). Integrating both sides of (3.3.10) with this measure and using Fubini's theorem, we get

$$\begin{aligned}&\frac{1}{|K|}\int_K\int_{S^{n-1}}\ln|(x,\xi)|\,d\mu(\xi)dx\\&\leq\frac{1}{|D|}\int_D\int_{S^{n-1}}\ln|(x,\xi)|\,d\mu(\xi)dx-\varepsilon\int_{S^{n-1}}d\mu.\end{aligned}\tag{3.3.11}$$

Using the definition of μ we get that (3.3.11) is actually

$$\frac{1}{|K|}\int_K\Big(\ln\|x\|_D-\lambda_0\Big)dx\leq\frac{1}{|D|}\int_D\Big(\ln\|x\|_D-\lambda_0\Big)dx-\varepsilon\int_{S^{n-1}}d\mu,\tag{3.3.12}$$

where λ_0 is the constant in (3.3.6). Using Fubini's theorem along with Proposition 3.3.5 and Proposition (3.3.8) we have

$$\begin{aligned}\int_{S^{n-1}}d\mu&=\int_{S^{n-1}}|\xi|_2\,d\mu(\xi)\\&=c_1\int_{S^{n-1}}\int_{S^{n-1}}\ln|(\theta,\xi)|\,d\mu(\xi)\,d\theta\\&=c_1\left(\int_{S^{n-1}}\ln\|\theta\|_Dd\theta-\lambda_0\left|S^{n-1}\right|\right).\end{aligned}\tag{3.3.13}$$

Where c_1 is as in (3.3.8). By the second part of Proposition 3.2.3 we have

$$\frac{|B_2^n|}{n}\Big(\ln|B_2^n|-\ln|D|\Big)-\frac{|B_2^n|}{n}\leq\int_{B_2^n}\ln\|x\|_Ddx.\tag{3.3.14}$$

Expressing the right hand side in polar coordinates and integrating by parts gives us

$$\int_{B_2^n}\ln\|x\|_Ddx=-\frac{|S^{n-1}|}{n^2}+\frac{1}{n}\int_{S^{n-1}}\ln\|\theta\|_Dd\theta.\tag{3.3.15}$$

Combining (3.3.14) and (3.3.15) we get

$$|B_2^n|\left(\ln|B_2^n|-\ln|D|\right)-|B_2^n|+\frac{|S^{n-1}|}{n}\leq\int_{S^{n-1}}\ln\|\theta\|_D d\theta \tag{3.3.16}$$

Combining (3.3.13) and (3.3.16) we get

$$c_1\left(|B_2^n|\Big(\ln|B_2^n|-\ln|D|\Big)-|B_2^n|+\frac{|S^{n-1}|}{n}-\lambda_0\left|S^{n-1}\right|\right)\leq\int_{S^{n-1}}d\mu \tag{3.3.17}$$

Let

$$c_2:=c_1\left(|B_2^n|\Big(\ln|B_2^n|-\ln|D|\Big)-|B_2^n|+\frac{|S^{n-1}|}{n}-\lambda_0\left|S^{n-1}\right|\right) \tag{3.3.18}$$

where, as a reminder, c_1 is as in (3.3.8). Combining (3.3.12) and (3.3.17)

$$\frac{1}{|K|}\int_K\ln\|x\|_D dx\leq\frac{1}{|D|}\int_D\ln\|x\|_D dx-\varepsilon c_2 \tag{3.3.19}$$

By Proposition 3.2.3

$$\frac{1}{n}\Big(\ln|K|-\ln|D|\Big)-\frac{1}{n}\leq\frac{1}{|K|}\int_K\ln\|x\|_D dx. \tag{3.3.20}$$

One can also show

$$\frac{1}{|D|}\int_D\ln\|x\|_D dx=-\frac{1}{n}. \tag{3.3.21}$$

Combining this with (3.3.19) and (3.3.20) we get

$$\ln|K|\leq\ln|D|-\varepsilon(nc_2), \tag{3.3.22}$$

which is what we wanted to prove. □

Proof of Corollary 3.3.3. Let

$$\varepsilon := \min_{\xi \in S^{n-1}} \Big| \|\xi\|_{\Gamma_0^*(K)} - \|\xi\|_{\Gamma_0^*(D)} \Big|. \tag{3.3.23}$$

By Theorem 3.3.1 this implies

$$\|\xi\|_{\Gamma_0^*(K)} \leq \|\xi\|_{\Gamma_0^*(D)} - \varepsilon \quad \text{for all} \quad \xi \in S^{n-1}, \tag{3.3.24}$$

as well as the same inequality but with D and K switched. Since $(\mathbb{R}^n, \|\cdot\|_D) \hookrightarrow L_0$ we can use Theorem 3.3.1 to get that there exists $\lambda_D = \lambda(n, D)$ so that

$$\ln|K| \leq \ln|D| - \lambda_D \epsilon. \tag{3.3.25}$$

By our choice of ϵ we also have

$$\|\xi\|_{\Gamma_0^*(D)} \leq \|\xi\|_{\Gamma_0^*(K)} - \varepsilon \quad \text{for all} \quad \xi \in S^{n-1}. \tag{3.3.26}$$

Since $(\mathbb{R}^n, \|\cdot\|_K) \hookrightarrow L_0$ we can use Theorem 3.3.1 to get that there exists $\lambda_K = \lambda_K(n, K)$

$$|K| \leq |D| - \lambda_K \epsilon. \tag{3.3.27}$$

Solving for ϵ gives us that there exists a constant $\lambda = \min\{\lambda_K, \lambda_D\}$ so that

$$\min_{\xi \in S^{n-1}} \Big| \|\xi\|_{\Gamma_0^*(K)} - \|\xi\|_{\Gamma_0^*(D)} \Big| \leq \lambda \Big| \ln|K| - \ln|D| \Big|. \tag{3.3.28}$$

□

Corollary 3.3.6. *Suppose $K \in \mathscr{K}_{os}^n$ is such that $(\mathbb{R}^n, \|\cdot\|_D) \hookrightarrow L_0$. Then there exists* *a constant* $\lambda = \lambda(n, K, D)$ *so that*

$$\min_{\xi \in S^{n-1}} \Big| \|\xi\|_{\Gamma_0^*(K)} - 1 \Big| \leq \lambda \Big| \ln |K| \Big|. \tag{3.3.29}$$

Proof. Take $K, D \in \mathscr{K}_{os}^n$ be such that $(\mathbb{R}^n, \|\cdot\|_K) \hookrightarrow L_0$ and $D = \delta B$ where B is the unit ball of any n-dimensional subspace of L_0 so that $(\mathbb{R}^n, \|\cdot\|_B)$ embeds in L_0. Using Corollary 3.3.3 with this choice of K and D we can let $\delta \to 1$ to get (3.3.29). $\square$

As an example of such bodies K, mentioned in the statement of Corollary 3.3.6, and B used in the proof, we can use the following facts:

Proposition 3.3.7 (Thm 6.17 of [25])**.** *Every* n*-dimensional subspace of* L_q, $0 < q \leq 2$*, embeds in* L_{-p} *for every* $0 < p < n$*.*

Proposition 3.3.8 (Thm 6.4 of [19])**.** *Let* $K \in \mathscr{K}_{os}^n$ *and let* $\varepsilon > 0$ *be given. If* $(\mathbb{R}^n, \|\cdot\|_K) \hookrightarrow L_{-p}$ *for every* $p \in (0, \epsilon)$ *then* $(\mathbb{R}^n, \|\cdot\|_K) \hookrightarrow L_0$*.*

So if $K \in \mathscr{K}_{os}^n$ and $X = (\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_q for $0 < q \leq 2$ then X also embeds in L_0. This completes the L_0 embedding section.

Chapter 4

A Hyperplane Inequality for Subspaces of Unconditional Spaces

4.1 Intro

In this third chapter we investigate the properties of subspaces of unconditional spaces by proving the following inequality which we classify as a "hyperplane inequality" for reasons which we describe shortly.

Theorem 4.1.1. *Let X be an n-dimensional unconditional space, let E be a k-dimensional subspace of X with unit ball B_E. There exists a universal constant C so for every measure μ with even, positive, continuous density on B_E*

$$\mu(B_E) \leq C \left(\frac{n}{k}\right)^2 \max_{\xi \in S^n} \mu(B_E \cap \xi^\perp) \left|B_E\right|^{1/k}. \tag{4.1.1}$$

To make sense of this, we recall a few definitions.

Definition 4.1.2. *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a function. A measure μ on $\mathbb{R}^n$ is said to* *have density f if*

$$\mu(A) = \int_A f(x)dx.$$

for every measurable set A.

Definition 4.1.3. *Let K be an origin symmetric convex body in $\mathbb{R}^n$ and let $u_1, \ldots, u_n$ be the standard basis of $\mathbb{R}^n$. An origin symmetric convex body K is said to be an unconditional body if*

$$\prod_{i=1}^{n} [-|a_i|, |a_i|] \subset K$$

for every $a = (a_1, \ldots, a_n) \in K$.

A finite-dimensional normed space $X = (\mathbb{R}^n, \|\cdot\|)$ is called an unconditional space *if its unit ball $\{x \in \mathbb{R}^n : \|x\| \leq 1\}$ is an unconditional body.*

Definition 4.1.3 says that unconditional bodies are those which are symmetric about every coordinate planes which explains why these bodies might have predictable properties.

Theorem 4.1.1 is actually a generalization of a theorem by A. Koldobsky in [27] where he proved the case when $k = n$ i.e. inequality (4.1.1) holds when E is an unconditional space and not just a subspace of unconditional space. A. Koldobsky's theorem is itself a variant of a result by J. Bourgain which shows the theorem when $k = n$, μ is the Lebesgue measure, and K is an unconditional body. This result by Bourgain was the beginning of the now well-known Hyperplane Conjecture which asks if there is a universal constant C such that

$$\mathrm{vol}_n(K) \leq C \max_{\xi \in S^{n-1}} \Big(\mathrm{vol}_{n-1}(K \cap \xi^{\perp}) \Big) \mathrm{vol}_n(K)^{1/n}. \tag{4.1.2}$$

for all origin-symmetric convex bodies K in $\mathbb{R}^n$ (see [41] or the book [7] for more details about the Hyperplane Inequality). Comparing equations (4.1.1) and (4.1.2) we see that our original inequality (4.1.1) is what might be called a hyperplane inequality for arbitrary measures.

From what the author can tell, the first examination of these hyperplane inequalities for arbitrary measures was by A. Koldobsky in [26] where he showed the existence of a constant C where (4.1.1) holds when K is an intersection body. Later, in [27], he showed the same thing but when K is an unconditional body.

4.2 Proof of the main theorem

The proof of the main theorem relies on the following generalization by M. Junge [18] of Lozanovskii's classical result about unconditional spaces.

Proposition 4.2.1 (M. Junge)**.** *Let X be an n-dimensional Banach space with unconditional basis and let E be a k-dimensional subspace of X. There exists a diagonal, linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ so that*

$$B_E \subseteq T(B_1^k), \quad \left(\frac{|T(B_1^n)|}{|B_E|}\right)^{1/k} \leq \left(e\frac{n}{k}\right)^2. \tag{4.2.1}$$

Junge's result corresponds to Lozanovskii's theorem when $k = n$.

We will also need the following lemma:

Proposition 4.2.2 (A. Koldobsky [28])**.** *Suppose that K is an intersection body in $\mathbb{R}^n$, f is an even, continuous function on K, $f \geq 1$ everywhere on K, and $\varepsilon > 0$. If*

$$\int_{K\cap\xi^\perp} f(x)dx \leq \left|K \cap \xi^\perp\right| + \varepsilon, \quad \forall\xi \in S^{n-1},$$

then

$$\int_K f(x)dx \le |K| + \frac{n}{n-1} c_n |K|^{1/n} \varepsilon$$

where

$$c_n = \frac{|B_2^n|^{\frac{n-1}{n}}}{\left|B_2^{n-1}\right|}. \tag{4.2.2}$$

We will use this proposition when f is an indicator function of symmetric, convex bodies which is allowable since we can approximate such function with even, continuous functions.

For completeness, we recall the definition of an intersection body.

Definition 4.2.3. *An origin-symmetric star body K is called an* intersection body *if there exists a finite Borel measure μ on S^{n-1} such that*

$$\int_{S^{n-1}} (R\phi)(\xi) d\mu(\xi) = \int_{S_{n-1}} \|\xi\|_K \phi(\xi) d\xi$$

for every Schwartz function $\phi : \mathbb{R}^n \to \mathbb{R}$. Here R denotes the spherical Radon transform.

This definition says that a body K is an intersection body if its norm is the spherical Radon transform of a measure on S^{n-1} in the sense of distributions on Schwartz functions. Intersection bodies provide a unified, Fourier analytic solution to the Busemann-Petty Problem. You can read more about them in the book [25]. For now we only need the following propositions.

Proposition 4.2.4. *(A. Koldobsky [23]) The unit ball of any n dimensional subspace of $L_q([0,1])$ for $0 < q \le 2$ is an intersection body.*

Proposition 4.2.5. *(E. Lutwak [38]) Any linear transformation of an intersection body is again an intersection body.*

Combining these two we see that any linear transformation of the cross-polytope B_1^n is again an intersection body. We now come to the proof of the main theorem.

Proof of Theorem 4.1.1. Let T be the operator from Proposition 4.2.1 and let $K :=T(B_1^k)$. By our observation after Proposition 4.2.5, K is an intersection body. Let g be the density of the given measure μ and let $f := \chi_K + g\chi_L$ where χ_K and χ_L are the indicator functions of K and L. Choose

$$\varepsilon := \max_{\xi \in S^{n-1}} \left(\int_{K \cap \xi^\perp} f(x)dx - \left| K \cap \xi^\perp \right| \right) = \max_{\xi \in S^{n-1}} \int_{L \cap \xi^\perp} g(x)dx \tag{4.2.3}$$

so that ε, f, and μ satisfy the requirements of Proposition 4.2.2 and therefore

$$\begin{aligned} \mu(B_E) &= \int_{B_E} g(x)dx = \int_K f(x)dx - |K| \\ &\leq \frac{n}{n-1} c_n \, |K|^{1/k} \max_{\xi \in S^{n-1}} \int_{B_E \cap \xi^\perp} g(x)dx \\ &= \frac{n}{n-1} c_n \, |K|^{1/k} \max_{\xi \in S^{n-1}} \mu(B_E \cap \xi^\perp) \\ &\leq \frac{n}{n-1} c_n \, |B_E|^{1/k} \left(e\frac{n}{k} \right)^2 \max_{\xi \in S^{n-1}} \mu(B_E \cap \xi^\perp). \end{aligned} \tag{4.2.4}$$

The second inequality follows from Proposition 4.2.2 and the last inequality follows from Proposition 4.2.1. It is known $c_n \in \left(\frac{1}{\sqrt{e}}, 1 \right)$ (see, for example, [31]) so we can take $C = e^2$. This concludes the proof. $\square$

Chapter 5

Extremal Sections of Boxes

5.1 Intro

This chapter is about upper and lower bounds for the cross-sectional volume of a box in $\mathbb{R}^n$. By a box in $\mathbb{R}^n$ we mean an origin-symmetric convex body defined as a product of the form $\prod_{j=1}^{n}[-a_j, a_j]$ whose volume is 1 and by cross sectional volume we mean the $(n-1)$-dimensional Lebesgue measure of the union of K with the hyperplane $\xi^{\perp}$ where $\xi \in S^{n-1}$. Here $\xi^{\perp} = \{x \in \mathbb{R} : x \cdot \xi = 0\}$ i.e. $\xi^{\perp}$ is the $(n-1)$-dimensional hyperplane orthogonal to the unit vector $\xi \in S^{n-1}$.

We start with a brief history of these types of problems. The investigation of these problems started with the cube in $\mathbb{R}^n$ of volume 1, i.e. the body defined by $\prod_{j=1}^{n}\left[-\frac{1}{2}, \frac{1}{2}\right]$. H. Hadwiger showed the following:

Proposition 5.1.1 (H. Hadwiger)**.** *Let* $Q_n = \prod_{j=1}^{n}\left[-\frac{1}{2}, \frac{1}{2}\right]$*. For every* $\xi \in S^{n-1}, 1 \leq \left|Q_n \cap \xi^{\perp}\right|$*. Moreover the exact minimum value is* 1.

A simple proof of the inequality can be found in A. Koldobsky's book [25] as Lemma 7.5 and Theorem 7.6. The exact equality can actually be explained with the following argument: consider the hyperplane of the form $(x_k = 0) = \{(x_i) \in \mathbb{R}^n : x_k = 0\}$. This hyperplane can be represented as $\xi^\perp$, $\xi = (0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 is in the k-th component. Since $K \cap (x_k = 0)$ is just $\prod_{j=1}^n \left[-\frac{1}{2}, \frac{1}{2}\right]$ then $\left|K \cap \xi^\perp\right| = |K \cap (x_i = 0)| = \mathrm{vol}_n \left(\prod_{j=1}^n \left[-\frac{1}{2}, \frac{1}{2}\right]\right) = 1$.

Years after H. Hadwiger found the minimal cross-section of the unit cube, Keith Ball found the least upper bound:

Proposition 5.1.2 (K. Ball [1])**.** *For every $\xi \in S^{n-1}$, $\left|Q \cap \xi^\perp\right| \leq \sqrt{2}$.*

The exact maximum is achieved at $\sqrt{2} = \left|K \cap \xi^\perp\right|$ where $\xi = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \ldots, 0\right)$. This result is remarkable because the constant does not depend on the dimension n. We would expect an upper bound to depend on the dimension because the diagonal of the cube in $\mathbb{R}^n$ grows like $\sqrt{n}$ as $n \to \infty$ yet this shows the central cross-sectional volumes are concentrated away from the corners of the cube and more towards the center as the dimension grows larger.

We now get to the main part of this chapter. The original goal was to explore the upper and lower bounds for the box of volume 1 in $\mathbb{R}^n$ i.e. the convex set of the form $K = \prod_{j=1}^n [-a_j, a_j]$ where $\mathrm{vol}_n(K) = 1$. Then we wanted to find the exact upper and lower bounds of the box. We found upper and lower bounds for the central cross-sectional volumes but half way through the investigation the author received a private communitation from Dr. Galyna Livshyts with a proof of the exact values. In fact, the largest central cross-section of a unit rectangle is of the form $\xi_{j,k}^\perp$ where $\xi_{j,k}$ is the unit vector in $\mathrm{span}\{e_j, e_k\}$ which is perpendicular to $a_j' e_j + a_k' e_k$, where $a_j' := \frac{a_j}{\sqrt{a_k^2 + a_j^2}}$ and $a_k' := \frac{a_k}{\sqrt{a_k^2 + a_j^2}}$. $\xi_{j,k}^\perp \cap \mathrm{span}\{e_j, e_k\}$ is just the diagonal of the two dimensional box $K \cap \mathrm{span}\{e_j, e_k\}$. The maximum volume achieved by such a hyperplane is the maximum of the quantity

$$\sqrt{\frac{1}{a_j^2}+\frac{1}{a_k^2}} \tag{5.1.1}$$

as a_j and a_k vary over all such lengths which make up the rectangle $K=\prod_{j=1}^{n}[-a_i,a_i]$. More recently, G. Livshyts, G. Pauoris, and P. Pivovarov extended this result to get sharp bounds for marginal densities of product measures in [36].

The sections that follow do not explain the calculations for the exact upper and lower bounds obtained by G. Livshyts or from [36] but rather they explain how the current author got the non-sharp upper and lower bounds using techniques from A. Koldobsky's book [25].

5.2 Bounds for the hyperplane volumes of the box

In this section we will find upper and lower bounds for the the central cross-sectional volume of the unit box $\left|K\cap\xi^{\perp}\right|$. In particular, we will prove the following:

Theorem 5.2.1. *Let K be a box in $\mathbb{R}^n$. Then*

$$\frac{1}{2\max(a_j)} \leq \left|K\cap\xi^{\perp}\right| \leq \frac{1}{\sqrt{2}\min(a_j)}. \tag{5.2.1}$$

This is an extension of the next two results. Q_n refers to the cube in $\mathbb{R}^n$ with volume 1. We will also need the following proposition:

Proposition 5.2.2. *Let $e_1,\dots,e_n$ be the standard coordinate vectors in $\mathbb{R}^n$, let K be a box in $\mathbb{R}^n$, let P_i be the orthogonal projection onto the hyperplane $x_i=0$ and let θ_i be the angle between ξ_i and e_i. Then $vol_{n-1}(P_i(K\cap\xi^{\perp}))=\cos(\theta_i)vol_{n-1}(K\cap\xi)$.*

proof of theorm 5.2.1. We start off with proving the upper bound. Fix $\xi\in S^{n-1}$.

First suppose there exists ξ_i where $1 \leq \xi_i\sqrt{2}$. The set $K \cap (x_i = 0)$ has volume $\prod_{j\neq i} 2a_j$ so the projection $P_i(K\cap\xi^\perp)$ has volume at most $\prod_{j\neq i} 2a_j$. The cosine of the angle between $P_i(K\cap\xi_i)$ and $K\cap\xi_i^\perp$ is ξ_i, hence using Proposition 5.2.2 along with our assumption on the upper bound of $1/\xi_i$ gives us

$$\begin{aligned}\mathrm{vol}_{n-1}(P_i(K\cap\xi^\perp)) = \xi_i \mathrm{vol}_{n-1}(K\cap\xi^\perp) \leq \prod_{j\neq i} 2a_j = \frac{1}{2a_i}\\ \Leftrightarrow \mathrm{vol}_{n-1}(K\cap\xi^\perp) \leq \frac{\sqrt{2}}{2a_i} \leq \frac{1}{\sqrt{2}\min(a_j)}.\end{aligned} \tag{5.2.2}$$

Now suppose $\xi_j\sqrt{2} \leq 1$ for all $j = 1, \ldots, n$. Define $A_{k,\xi}(t) = \mathrm{vol}_{n-1}(K\cap(\xi^\perp + t\xi))$. We follow the argument stated in [1] along with a suitable change of variables to get

$$\widehat{A_{K,\xi}}(r) = \prod_{j=1}^n \frac{\sin(a_j r\xi_j)}{a_j r\xi_j}$$

and hence, taking the inverse fourier transform, we get

$$\mathrm{vol}_{n-1}(K\cap\xi^\perp) = A_{K,\xi}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\prod_{j=1}^n \frac{\sin(a_j t\xi_j)}{a_j t\xi_j}dt. \tag{5.2.3}$$

Hölder's inequality gives us

$$\mathrm{vol}_{n-1}(K\cap\xi^\perp) \leq \prod_{j=1}^n \left(\frac{1}{2a_j\xi_j}\right)^{\xi_j^2}\left(\int_{\mathbb{R}^n}\left|\frac{\sin(\pi t)}{\pi t}\right|^{\frac{1}{\xi_j^2}} dt\right)^{\xi_j^2}. \tag{5.2.4}$$

We will need the following:

Proposition 5.2.3 (K. Ball [1])**.**

$$\int_{-\infty}^{\infty} \left| \frac{\sin(\pi t)}{\pi t} \right|^p dt \leq \sqrt{\frac{2}{p}} \qquad \textit{if} \quad p \geq 2, \tag{5.2.5}$$

and there is equality if and only if $p = 2$*.*

Using this last proposition in (5.2.4) gives

$$A_{k,\xi}(0) \leq \prod_{j=1}^{n} \left(\frac{1}{2a_j \xi_j} \right)^{\xi_j^2} \left(\sqrt{2\xi_j^2} \right)^{\xi_j^2} = \frac{1}{\sqrt{2} a_1^{\xi_j^2} \dots a_n^{\xi_n^2}}.$$

Now we must find the minimum of the function $f(\xi_1, \dots, \xi_n) = a_1^{\xi_1^2} \dots a_n^{\xi_n^2}$ on the unit sphere. S^{n-1} is compact so the minimum exists at some point which we will call $\eta \in S^{n-1}$. We can use Lagrange multipliers to say the derivative of f at η is a constant multiple of the derivative of $x \mapsto x_1^2 + \ldots + x_n^2$. This means there exists a $\lambda \in \mathbb{R}$ so that

$$\left. \frac{d}{dx_i} f \right|_{\eta} = \lambda \left. \frac{d}{dx_i} g \right|_{\eta} \Leftrightarrow 2\eta_i f(\eta) \log(a_i) = \lambda 2\eta_i \tag{5.2.6}$$

for every $i = 1, \dots, n$. Solving for a_i in the above equation implies that, at the minimum η, the bases a_i in the expression $a_1^{\eta_1^2} \dots a_n^{\xi_n^2}$ are equal whenever the corresponding η_i are non-zero. Thus the minumum occurs at a point η such that the bases of the non-zero η_i are all equal, say to some a_0, i.e. if $\eta_{i(1)}, \dots, \eta_{i(m)}$ are all the non-zero η_i then

$$f(\eta) = a_0^{\eta_{i(1)}^2} \cdots a_0^{\eta_{i(m)}^2} = a_0^{\eta_{i(1)}^2 + \ldots + \eta_{i(m)}^2} = a_0^{\eta_1^2 + \ldots + \eta_n^2} = a_0.$$

So the minimum occurs at one of the a_i. All of these values are obtainable on the set S^{n-1} just by plugging in the standard orthonormal basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$ into f so the minimum of f is the minimum of all a_i, $i \leq i \leq n$. Hence we have found the upperbound $\frac{1}{\sqrt{2}\min(a_j)}$.

We now consider the lower bound in theorem 5.2.1. We follow a modified argument given in [25] as Lemma 7.5 and theorem 7.6. The following lemma is due to D. Hensley [15].

Lemma 5.2.4. *let f be a non-negative non-increasing integrable continuous function on $[0, \infty)$ such that $\int_0^\infty t^2 f(t)dt < \infty$. then*

$$3(f(0))^2 \int_0^\infty t^2 f(t)dt \geq \left(\int_0^\infty f(t)dt\right)^3. \tag{5.2.7}$$

Now fix $\xi \in S^{n-1}$ and let $f(t) = A_{K,\xi}(t)$. By Brunn's theorem in convex geometry and the definition of K we have that this choice of f satisfies the requirements in Lemma 5.2.4. Therefore

$$\begin{aligned}\int_0^\infty t^2 f(t)dt &= \frac{1}{2}\int_K (x,\xi)^2 dx = \frac{1}{2}\sum_{j=1}^n \int_K x_i^2 \xi_i^2 dx_i \\ &= \frac{1}{2}\sum_{i=1}\left(\prod_{j\neq i} 2a_j\right) 2\frac{(a_i)^3}{3} \leq \frac{\left(\max(a_j)\right)^2}{6}.\end{aligned} \tag{5.2.8}$$

Here we used the fact that $1 = \mathrm{vol}_n(K) = \prod_{j=1}^n 2a_j$ and that $\int_K x_i x_j dx = 0$ for all $i \neq j$. This integral equality is just the observation that $x_i x_j$ is an odd function and that K is symmetric with respect to every coordinate plane, i.e. K is an unconditional body. Now using lemma 5.2.4 and the fact that $\int_0^\infty f(t)dt = \frac{1}{2}$ we get

$$(\mathrm{vol}_{n-1}(K \cap \xi^\perp))^2 \geq \frac{1}{3} \frac{6}{\big(\max(a_j)\big)^2} \left(\frac{1}{8}\right) = \left(\frac{1}{2\max(a_j)}\right)^2 \tag{5.2.9}$$

which shows the lower bound and completes the proof. $\square$

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