DIFFERENCES IN CONCEPTIONS OF VARIABLES AMONG STUDENTS WITH TYPICAL ACHIEVEMENT, LOW ACHIEVEMENT AND MATHEMATICS LEARNING DISABILITIES

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ABSTRACT

This study investigated the differences in conceptions of variable among the groups of students identified as having a mathematics learning disability (MLD), other low mathematics achievement (LMA) students and typical mathematics achievement (TMA) students. This was done by analyzing the responses of these students on items involving the comparison of expressions involving generalized quantities. The theoretical framework for this study was based upon the learning trajectory (LT) of the levels of sophistication of students' conceptions of variable. Data analysis included descriptive and inferential statistics.

Results from the data analysis revealed little evidence of differences between MLD and LMA students. Differences between all students with mathematics difficulties (MD), including MLD and LMA students, and their TMA peers were primarily limited to items with higher procedural or operational complexity. The lack of significant differences between the MD and TMA students on other items suggests that most students' conceptions of variables are at a low level of sophistication. Furthermore, that students characterized by low-achievement and typical-achievement labels can have similarly low-levels of sophistication of conceptions of variables suggests that not only do the current measures of achievement provide an incomplete picture of students' understandings of algebra, but they also disproportionately disadvantage those labeled as low-achieving.

CHAPTER 1: INTRODUCTION

Background

In recent major reports on the teaching and learning of mathematics (e.g., National Mathematics Advisory Panel, 2008), proficiency in mathematics has been identified as requisite for the success of individuals. Success in mathematics opens doors for individuals by providing them with more college and career options and thus a higher potential salary (U.S. Department of Education, 1997). Conversely, those who are not successful in mathematics are substantially limited by the inability to proficiently use basic mathematics skills and concepts such as arithmetic, measurement, and simple algebraic skills. The Every Child a Chance Trust (as cited in Geary, Hoard, Nugent, & Bailey, 2012, p. 206) reported that large-scale studies of representative samples of adults show as many as 25% of adults lack these mathematical competencies, "making them functionally innumerate".

Algebra is often identified as an exemplar of these basic mathematical abilities. For example, Moses and Cobb (2001) described algebra as necessary for all students in order to understand and succeed in a world with ever-increasing technological integration by characterizing algebra as "the gatekeeper for citizenship; and people who don't have it are like the people who couldn't read and write in the industrial age" (p.14). Algebra may also be chosen as an exemplar because of its role as both a prerequisite for post-secondary education and a common post-secondary topic of study. These requirements are also reflected in secondary curricula where algebra comprises a substantial portion of middle and high school students' mathematics requirements. Furthermore, many state legislatures have taken the position that students must complete at least one, and often two, courses in algebra to earn a high school diploma. Because of the long-lasting effects of students' experience with mathematics in K–12 schools (National Mathematics Advisory Panel, 2008; U.S. Department of Education, 1997), and algebra's substantial role in school mathematics, algebra is referred to as a gateway and a barrier to higher mathematics and post-education endeavors (Dede, 2004; National Mathematics Advisory Panel, 2008; Subramaniam & Banerjee, 2011). Finally, opportunities to learn and be successful in algebra have been called equity and civil rights issues because of the limitations that low algebra achievement places upon individuals in their career opportunities (Kaput, 1998; Moses, 1993; Moses & Cobb, 2001; Stein, Kaufman, Sherman, & Hillen, 2011).

Because of the widespread difficulty that students have with algebra, some have called for the removal of algebra as a universal requirement for all students. Hacker (2012) suggests that alternatives to traditional algebra courses be offer that focus on the applications of arithmetic, numeracy and quantitative literacy would be more beneficial for students not ending in a STEM or other field that does require algebra and more advanced mathematics. A student taking such an applied mathematics course instead of algebra in grades eight, nine, or ten effectively reduces the opportunity for that student to choose a mathematics required path later in their education. Hacker also reports that there is no research support for the claim that proficiency in mathematics improves critical thinking or analysis of current situations. This may have more to do with the ways in which algebra is currently taught than the potential it has to impact students' abilities that transfer beyond the mathematics classroom.

Kaput and Blanton (2000) referred to “America's Algebra Problem,” which they described as the “intolerably poor outcomes of our highly dysfunctional, late, abrupt, isolated, and superficial approach to school algebra, and our collective inability to move beyond it” (p. 2). They described secondary students’ disappointing algebra understandings, public dislike of mathematics, and the inequity of academic tracking as aspects of this problem. These aspects can be linked to U.S. teaching practices and curricula that are focused on procedural skills instead of conceptual understanding (Kaput & Blanton, 2000). More recently, Schneider, Rittle-Johnson, and Star (2011) reported that the relationship between conceptual and procedural knowledge is still unclear. Kieran (2013) described the separation of procedural and conceptual understanding as a “false dichotomy” that has especially impacted the field of school algebra. She states that despite traditional instructional practices, even the symbolic aspects of algebra, like the conceptions students have about variables, have conceptual foundations that students need to develop.

The lack of agreement of an appropriate balance between conceptual understanding and procedural skills may stem in part from differences in the perceived purpose and scope of algebra. For example, those who see algebra primarily as a tool for solving for unknown values in equations would focus much more on the procedural aspects of algebra than someone who sees algebra as the study of relationships between quantities. Usiskin (1988) asserted that these perceived differences in the purpose and scope of algebra are tied to different conceptions of algebra. Usiskin extended these conceptions to include the use of variables stating that, “purposes for algebra are determined by, or are related to, different conceptions of algebra, which correlate with the different relative importance given to various uses of variables" (p. 9). Since variable use is a major component of algebra and students' understanding of algebra has been identified as problematic, it is reasonable to hypothesize that students' understanding and use of variable may also be problematic.

Early research found that most students had a limited view of variables. In a seminal study, Küchemann (1978) identified three constricted uses and perceptions of variables as including assigning it an arbitrary value, ignoring the variable, or using the variable as the name of an object. For example, in the following item from the Concept of Variable (CoV) measure created by the *Algebra Screening and Progress Monitoring* (ASPM) project (Foegen & Dougherty, 2010), a student who is treating a variable as the name of an object might state that the expression $4c + 3d$ represents four cupcakes plus three doughnuts.

CoV Item:

Cupcakes cost c cents each and doughnuts cost d cents each. I buy 4 cupcakes and 3 doughnuts. What does $4c + 3d$ stand for or represent?

Küchemann also identified that a small minority of students had a more sophisticated view of variable and could think of and use variables as a specific unknown number or as a generalized number. More recent studies have identified a substantial portion of students see variables as representing a single unknown value (Booth, 1984; Malisani & Spagnolo, 2009; Stacey & MacGregor, 1999). Additionally, students who see a variable as representing a single unknown value had more limited algebraic thinking, problem solving, equation solving, and making generalizations than those with a more sophisticated conception of variable. (Cai, Moyer, Wang, & Nie, 2011; Knuth, Alibali, McNeil, Weinberg, & Stephens, 2011).

While the work done in this area is substantial, there is a gap in the research around examining specifically how the uses and conceptions of variables differ between successful algebra students and those who have difficulties in algebra. Most of the research on students' understanding of variable has focused on identifying and exploring students' uses and conceptions of variables, leaving a gap around the characteristics of the disaggregated groups of students with mathematics difficulties in algebra. A substantial amount of work has been done around the characteristics of students who have difficulties in mathematics. Much of this work typically focuses on elementary grade mathematics. Specifically, this focus has been primarily on arithmetic skills and fact recall (L. S. Fuchs, Fuchs, & Hollenbeck, 2007; Murphy, Mazzocco, Hanich, & Early, 2007) or on the persistence or progression of the performance of those skills from kindergarten or first grade to one to three years later (Geary et al., 2012).

Research has suggested that students who have difficulties in mathematics differ from their typically-achieving peers in both mathematics-specific and more general characteristics. Geary et al. (2010) reported that many students with mathematics difficulties often have low expectations for success and inadequate instruction and support (Jones, Wilson, & Bhojwani, 1997; Misquitta, 2011). In addition to these environmental factors, Hecht and Vagi (2010) described that when compared to their typically-achieving peers, students with mathematics difficulties have a more limited working memory and attention span.

Mathematics-specific characteristics of students who are low achieving in mathematics include difficultly both storing and recalling basic arithmetic facts, which impact students' ability to achieve fluency in computation (Butler, Beckingham, & Lauscher, 2005; Geary et al., 2012; Vanbinst, Ghesquière, & De Smedt, 2014). These children also experience delays in achieving accuracy and fluency in arithmetic operations and procedures and typically have a limited conceptual understanding of how and why the procedures work (Geary, 2004; Hecht & Vagi, 2010) which often results in their reliance upon less advanced solution strategies than their typically-achieving peers (Butler et al., 2005; Geary, 2004). Murphy et al. (2007) reported that students were less successful in identifying arithmetic errors when presented with incorrect work than their typically-achieving peers.

In the special education literature, researchers often differentiate between two groups that have mathematics difficulties. Geary et al. (2012) described one group of students with mathematics as having a mathematics learning disability (MLD) and claimed that they comprise the lowest 7–10% of all students. MLDs are defined as mathematics learning difficulties having a cognitive origin (Lewis, 2014; Mazzocco, 2007). This cognitive origin is considered the feature that distinguishes MLD students from other low-achieving students whose difficulties stem from other factors (Lewis, 2014; Wong, Ho, & Tang, 2014). Students who have persistent learning difficulties in mathematics, although not as severe as those with MLDs make up the second group these students are often referred to as students with low mathematical achievement (LMA). Geary et al. stated that an additional 10% to 15% of all students fall into this category. Students who are not in the MLD and LMA groups are considered to be typical mathematics achievement (TMA) students. Due to differences in classification procedures across studies, these percentages can vary substantially. Often, all students with mathematics difficulties are grouped together to comprise the lowest 25% of students on a measure of mathematics achievement. For the purpose of this study, when referring to the group of students that includes both the MLD and LMA students, I will describe them as students with *mathematics difficulties* (MD). This method of creating one low-achieving and one typically achieving group of students is referred to as the single cut-off method.

Despite a general agreement about the origins, definitions of and differences between students with MLDs and LMA, there is not consensus about how to identify these students (Mazzocco, 2007). Murphy et al. (2007) analyzed journal articles published in English from 1985 to 2006 to gauge the range and typical cut-off score used by researchers as well as to compile a list of common characteristics of these students with MD. They found that typical cut-off scores for these groups of students clustered around the 10^{th} and 25^{th} percentile for students with MLDs and LMA, respectively. When multiple cut-off scores are used to create three or more achievement groups of students, a multiple cut-off method is being used.

Wong et al. (2014) expressed concern that because these cut-off scores are arbitrary, there is still danger of not accurately grouping students with common characteristics. For example, students who score slightly better than the cut-off score could have more in common with students whose scores are below the cut-off than those who are well above the cut-off score, but still in the same category. Wong et al. also warned that this cut-off method only considers the achievement of students at a particular instance in time and neglects the persistence of the difficulty. For example, Murphy et al.

(2007) identified a case in which students started out with lower initial mathematical knowledge of a topic, but then caught up with their TMA peers over time. These students would have been classified as a LMA student at the first assessment, but because the difficulty was not persistent as at a later assessment, the student achieved at a similar level with her TMA peers.

Despite the limitations of the multiple cut-off method of grouping students by achievement, it is still one of the most commonly used approaches to identify students who experience mathematical difficulties (Geary et al., 2012). Due to restriction of resources (e.g., time, funding, access to students) and the importance of identifying these students, the multiple cut-off method is considered sufficient by many researchers (Geary et al., 2012; Hecht & Vagi, 2010; Murphy et al., 2007).

Researchers have called for more studies that investigate characteristics of MLD and LMA students in more complex mathematical domains and with older children (L. S. Fuchs et al., 2007; Geary et al., 2012). Lewis (2014) warned the limited focus of research on these students of elementary-aged students' speed and accuracy on basic arithmetic has resulted in the use of limited automaticity of arithmetic facts as a de facto defining characteristic of students with MLDs. This neglects the more complex and conceptually-based aspects of mathematics. Specifically, Hecht and Vagi (2010) hypothesized that based on the difficulties that MLD and LMA students have with conceptual aspects of arithmetic, it is reasonable to predict that these students would also have more difficulty with conceptual understandings of other mathematical domains than their TMA peers. Furthermore, Lewis (2014) and Mazzocco and Devlin (2008) suggested that differences between students with MLDs, LMA, and TMA could be found in comparing conceptual and procedural understanding for some mathematical domains.

Statement of the Problem

Geary et al. (2012) confirmed that students who have low mathematics achievement in school are at a high risk of innumeracy in adulthood. Because of the prominence of algebra within school mathematics and the central role that variables play in algebra, understanding the characteristics of how students with mathematics difficulty use and conceive of variable is foundational to success in algebra. Students' concepts of variable and their misconceptions around expressions containing variables have been studied for students in general and characteristics of students with MD have been studied for elementary level mathematics. This substantial work leaves a significant gap in our understanding of specific differences between conceptions and use of variables amongst students with MD and TMA students. The purpose of this study is to investigate the differences between the responses of low- and typically-achieving students in an Algebra 1 course on items designed to assess students' ability to compare expressions involving variables.

Research Questions

The general question under investigation is *How do students' responses to items designed to assess their ability to compare expressions involving generalized quantities at the beginning and end of an Algebra 1 course vary in relation to their performance on a standardized algebra assessment?* I will consider this variation with respect to both students' responses on individual items and student growth across administrations. Specifically:

1. Do the responses of students who are at or below the 10^{th} percentile on the Iowa

End of Course assessment (IEOC), and those of students who are above the 10th percentile and at or below the 25th percentile differ on eight items designed to assess students' ability to compare expressions involving generalized quantities on a conceptual progress-monitoring tool at the beginning of an algebra course?

2. How do responses on these items differ among the MLD, LMA and TMA groups of students at the beginning of an algebra course?
3. How do responses on these items differ among these groups of students at the end of an algebra course?
4. How does student growth differ for these groups of students from the beginning of an algebra course to the end of an algebra course?

Theoretical Framework

The theoretical framework guiding this study builds on the notion that different uses of variables suggest corresponding levels of sophistication of conceptions of variable. A student's conception of variable is the student's idea of what a variable is, how it acts, and what it can represent. A student's use of variables refers to the actions that the student does in connection with this conception of variable. While we cannot know exactly what a student's conception of variable is, we can make inferences about that conception based on how the student uses variables. Blanton, Brizuela, Gardiner, Sawrey, and Newman-Owens (2015) described a progression of these conceptions and uses of variable in their learning trajectory (LT) that characterized increasingly sophisticated levels in students' thinking about variable and variable notation as summarized in Table 1.

Table 1

Progression in Students' Thinking About Variable and Variable Notation (Blanton et al., 2015)

Trajectory Levels	Characteristics
1. Pre-Variable/Pre-Symbolic	Does not recognize a variable quantity (unknown) in a mathematical situation; Does not use or accept the use of any symbolic notation (literal or non-literal) to represent a quantity.
2. Pre-Variable/Letter as Label or as Representing Object	Tacitly accepts the use of a literal symbol to represent something not known, but not an unknown mathematical quantity (hence, an indication that child does not yet recognize a variable quantity).
3. Letter as Representing Variable with Fixed, Deterministic Value	Recognizes a variable quantity and sees the literal symbol as representing the quantity; Value of the variable quantity is deterministic, often linked to the letter used to symbolize the quantity and its ordinal position in the alphabet.
4. Letter as Representing Variable with Fixed but Arbitrarily Chosen Value	Recognizes a variable quantity as an unknown with a single, fixed value *that can be randomly chosen*; Sees literal symbol as representing "any number," but conceptualizes "any number" as *any fixed number* that is arbitrarily chosen.
5. Letter as Representing Variable as Varying Unknown	Recognizes variable quantity as varying unknown and literal symbol as representing a varying, unknown quantity.
6. Letter as Representing Variable as Mathematical Object	Able to construct functional relationships that appropriately represent variables and constants; Able to act on variables, represented with literal symbols, as tools for representing generalized functional relationships.

In addition to different conceptions and uses of variable, students can also have misconceptions about variables and their use in expressions. Misconceptions differ from the alternate conceptions described above by being inconsistent with the conceptions of variable and algebra that are being employed. For example, thinking of a variable as a single unknown value is not a misconception, but a less sophisticated conception than considering multiple possible values for that variable. Alternatively, a student who states that you cannot compare $t + 3$ and $5 + t$ because t could be a different number in $t + 3$ than in $5 + t$ is exhibiting a misconception. Russell, O'Dwyer, and Miranda (2009) differentiated between misconceptions and errors by describing errors as inconsistent mistakes, whereas misconceptions are when students consistently make the same mistakes. This consistency suggests that these mistakes occur when the student incorrectly applies previously learned strategies or concepts.

In light of variables' central role in algebra, misconceptions and limited conceptions about variables impact students' ability to reason and operate algebraically in addition to the specific errors they perpetuate (Clement, 1982; Schwartzman, 1996; Stacey & MacGregor, 1997). MacGregor and Stacey (1997) described common misconceptions that students have when working with expressions that include variables. These misconceptions are summarized in Table 2. Christou, Vosniadou, and Vamvakoussi (2007) described an additional misconception that students have about variables. They found that students often believe that the value that a variable can take on is limited to the natural numbers. This misconception could be generalized to describe any limitiation that students percieve about the sets of numbers from which a variable's value can come.

By combining the work of Blanton et al. (2015), MacGregor and Stacey (1997), and Christou et al. (2007) together, I created a map of the conceptions of variable in Table 3 that are suggested in these misconceptions of variables. It is not surprising that as the conception of variable as a constant or fixed unknown has more identified variable uses and misconceptions than other conceptions of variables due to its more common occurrence in studied populations.

Table 2

Misconceptions About Expressions and Variables from MacGregor and Stacey (1997)

Misconception	Example Problem	Student Reasoning
Different variables must be different values	Can $a + b$ be equal to $a + c$?	These expressions cannot be equal because b and c are different
The same variable may not be the same value	Which is larger: $3 + x$ or $x + 5$?	Since x could be 4 in $3 + x$ but 1 in $x + 5$, we cannot say which expression is larger
Arithmetic conventions overgeneralized	Simplify $2(h + 10)$	Since $h + 10$ is equivalent to $h10$, this expression could simplify to $h20$
Unknown value prevents comparison	*Is* a or b is larger in the equation $a = 28 + b$?	Since the a and b are unknown, it is impossible to tell which is greater

It is important to note that since these misconceptions have not been empirically linked to these levels of the LT, the strongest connections that can be made between the misconceptions and the LT levels is that of consistency. I propose that the connections in Table 3 present the LT levels that the misconceptions seem to be consistent with.

While the connections between some of these conceptions, uses, and misconceptions are fairly straightforward, others are more complicated. For example, if a student states that two expressions cannot be compared because the value of the variable is unknown, it follows that this student is thinking of variable as a fixed unknown similar to the first two misconceptions in Table 3. However, in the case of students overgeneralizing arithmetic operations, it is possible that they could be thinking of variable in any of the levels 3-6, but because of the focus on the operation, too little evidence is given of the conception of variable being drawn upon. Thus it is not clear which LT level this misconception is consistent with.

Table 3

Map of Uses and Misconceptions of Variable to Conceptions of Variable

Misconceptions	Trajectory Level	Description
Different variables must be different values	3 or 4	Students are viewing variables as having a fixed value, although without further information it would be difficult to determine if that fixed value is arbitrary or deterministic
The same variable may not be the same value	3 or 4	
Unknown value prevents comparison	4	Students are viewing the variable as having a specific value that they do not know, so they cannot calculate the value of the expression
Variable may only represent natural numbers or integers	5	Students are considering that the variable could be a limited range of numbers.
Arithmetic conventions overgeneralized	None	Students are focused on the operation rather than the variable

In addition to this perspective about the connections between students' uses, misconceptions and conceptions of variables, the theoretical perspective taken with respect to students with MD is influential in the structure of my study. In this study, I view all learners as part of a normal distribution of mathematics achievement and growth as described by Geary et al. (2012). Although this means that groups of learners are not independent of each other, it does not preclude different, but overlapping, sets of characteristics. Furthermore, the perspective that all students are part of this distribution does not require that all students have the same factors in their mathematical achievement and growth. Specifically, I take the perspective that the mathematics difficulties that students with MLDs experience are the result of cognitive differences in comparison with their TMA peers (Lewis, 2014; Mazzocco, 2007). These differences are not deficits in the sense that students with MLDs are deficient in the ability, or are unable, to learn the content with which they are experiencing difficulty. Rather, students with MLDs have different connections than their TMA peers within the foundational aspects of the content which inhibit the typical connections from these foundational topics to next level of content. Thus these few who exhibit abnormal connections can be thought of as the left tail of the normal distribution described by Gearyet al. (2012). Other students who experience low mathematics achievement also may have persistent learning difficulties in mathematics, although not as severe as those with MLDs. LMA students' mathematical difficulties stem from other, including environmental, factors (Lewis, 2014; Wong et al., 2014).

Methodology

Using data collected as part of the *Algebra Screening and Progress Monitoring* (ASPM) project (Foegen & Dougherty, 2010), I analyzed students' responses to a subset of items on a conceptual algebra progress monitoring measure to identify and describe differences among the MLD, LMA, and TMA groups of students. The ASPM project (funded by the U.S. Department of Education Institute of Education Sciences, award number: R324A100087) provided the context for my study. ASPM created, refined, and gauged the technical adequacy of three conceptual and three procedural algebra measures for screening and progress monitoring over the course of four years. The focus of the project was to "investigate the validity and reliability of six types of algebra measures for screening and progress monitoring students with, or at risk for, disabilities" (Foegen & Dougherty, 2010, p. 2). Data used for the current study were from the third year of the project (2013–2014).

From the literature on students with MD (e.g., Geary et al., 2012), students were separated into three groups: TMA, LMA and MLD. Students were assigned to one of these groups based on the percentile rank of their Iowa End of Course (IEOC) assessment score. Students with scores at or below the 10th percentile on this assessment were considered MLD. Those with scores at or below the 25th percentile but above the 10th percentile were considered LMA. Students with scores higher than the 25th percentile were considered TMA. The 10th and 25th percentiles were chosen to allow for comparison with the mathematics learning difficulties research, which most commonly used these cut-off scores (Murphy et al. (2007).

Using the levels of sophistication of conceptions of variable from Blanton et al.'s (2015) LT summarized in Table 1 and the misconceptions about variables and expressions with variables identified by MacGregor and Stacey (1997) and Christou et al. (2007) as summarized in Table 3, I coded student responses and multiple-choice options as consistent with of a specific TL level or misconception. In my study, these different descriptions of how students think of and use variables were used to classify students' responses on items designed to assess students' ability to compare expressions involving generalized quantities. These classifications were then compared across the MLD, LMA and TMA groups using descriptive and inferential statistics and coded for the concepts of variable and common misconceptions identified in Table 3.

Significance of the Study

This study is an effort to bridge the gap between the work in the field of mathematics education and that of special education. A majority of the work that has been done in the field of special education around students with low mathematical achievement has focused on elementary grades. The ideas of algebra and variable have been extensively addressed by the field of mathematics education, but researchers have not specifically examined the differences between MLD, LMA, and TMA students. This study draws from both bodies of research to provide an understanding of how students with MD in algebra think about and use variables. It also provides opportunities for further collaborations across these two rich fields.

Limitations of the Study

Because this study represents a secondary data analysis, the types of data available were limited. Specifically, as the data that were available did not include interviews with students about how they used or thought of variables, only inferences can be made about how they thought about variables. Furthermore, as this study looked at one component of students' algebra understanding, no claims can be made regarding the validity of these results for other aspects of algebra or other mathematical domains. Finally, due to the small number of items analyzed, this study cannot speak to the variability of students' conception of variable and misconceptions around variables across different contexts or problem types.

CHAPTER 2: REVIEW OF LITERATURE

The purpose of this study is to investigate the differences between the responses of low- and typically-achieving students in an Algebra 1 course on items designed to assess students' ability to compare expressions involving variables. Students' conceptions, use, and misconceptions of expressions containing variables have been studied relative to students in general. Characteristics of students with MD have been studied at elementary levels of mathematics. These substantial bodies of work leave a significant gap in our understanding of specific differences between students with MD and TMA students' conceptions and use of variables.

This review of literature frames this gap in our understanding by situating it as the intersection of the research that has been done in mathematics education around students' uses, conceptions, and misconceptions of variables, and the special education research regarding the identification and characteristics of students with MD. In particular, in this review I first will consider the connection of students' conceptions of variable with aspects of algebra, then report results around students' uses, conceptions, and misconceptions of variables and finally discuss the gaps in this body of literature as they relate to my study. I will then describe research that has been done regarding the identification of students with MD, their characteristics, and gaps in that body of work. The literature reviewed from mathematics education and special education will frame the gap between them, which is the focus of this study.

Conception of Variable as an Important Aspect of Algebra

Conceptions of variables, including the roles they play and the meaning each role entails, are intrinsically tied to perspectives of algebra. Algebra is often described as having a variety of perspectives. Kaput (1995) identified five common aspects of algebra as summarized in Table 4.

Table 4

Kaput's (1995) Five Aspects of Algebra (p.6–9)

Aspect of Algebra	Description	Example
Generalized Arithmetic	Represent general number relationships and properties and use symbols in arithmetic operations	$a + b = b + a$
Generalized Quantitative Reasoning	Represent some attribute of a situation that is regarded as measurable (or countable)	How many and what size of groups (units) are needed to add to 16? (e.g. 2 groups of 8 or 4 groups of 4)
Syntactically-Guided Manipulation of Formalisms or Symbols	Manipulate symbols based on syntactic rules	Solve $10 = 2x + 1$ by applying the rule to subtract 1 then divide by 2
Study of Structures Abstracted from Computations and Relations	Extend algebraic thinking beyond numbers to other sets of objects	Operations on matrices: What is the sum of matrices A and B?
Study of Functions, Relations and Joint Variation	Generalize relationships between jointly varying quantities	$f(x) = 3x + 1$

Kaput (1995) commented that while generalized arithmetic reasoning is an important aspect of algebra, it constitutes a foundation for more advanced views of algebra rather than a complete description of algebra in general. He asserted that generalized quantitative reasoning is also central and foundational to all other aspects of algebra, but differs from generalized arithmetic reasoning in that a numerical value need not be assigned or it may consist of the reasoning with and generalization of groups of numbers. Kaput (1995) described this aspect of algebra as "encompassing" (p. 7) generalized arithmetic reasoning. He also suggested that generalized quantitative reasoning provides a better foundation for further algebraic reasoning than arithmetic reasoning alone because of the larger variety of situations from which it can draw (e.g., comparing lengths).

Kaput (1995) clarified that syntactically-guided manipulation can be a powerful aspect of algebra as a tool when connected to the corresponding mathematical concepts so that it can be done flexibly, or it can constitute little more than surface level engagement when the manipulation is performed as memorized steps in a structured algorithm. For example, when asked to solve the equation $x + 2 = 5$, the student may have simply memorized the procedure to subtract 2 from each side of the equation. A different student may have an understanding that this equation represents a relationship of equivalency between the quantity 5 and $x + 2$ and from this understanding be able to recognize that the relationship $5 – 2$ is also equivalent to $x + 2 – 2$ and therefore x is equivalent to 3, even if this recognition is not employed for each manipulation. Kaput identified algebra as the study of structures and systems abstracted from computations and relations (1) to enrich understanding of the systems that they are abstracted from, (2) to provide intrinsically useful structures for computations freed of the particulars that they once were tied to, and (3) to provide the base for yet higher levels of abstraction and formalization. Algebra as the study of functions, relations and joint variation grows out of, and intertwines with, generalized quantitative reasoning with roots in causality and joint variation. Kaput (1995) and other researchers (e.g. Sfard & Linchevski, 1994) have suggested that this aspect of algebra as the study of functions could be the central topic around which all of algebra can be organized. Kaput and Blanton (2000) suggested that these multiple aspects of algebra should be viewed as components that together make up algebra rather than separate purposes. They advocated that this wider view of algebra benefits students across K–12 and beyond.

Central to each of these identified aspects of algebra is some level of generalization or abstraction. Kaput, Blanton, and Moreno (2008) described generalization as the compressing of multiple instances into a single statement. This statement is then a representation, or a symbol, for the commonalities among those compressed instances. While this symbol can take many forms, including a statement in one's natural language (Radford, 2011), this review of literature and study will focus on the more conventional symbolic notation of these generalized quantities, often referred to as a variable and represented with a letter (Kaput, 2008; Kline, 1972).

Usiskin (1988) described how different uses and conceptions of variables are connected to the different conceptions of algebra, as summarized in Table 5.

Table 5

Summary of Connections Between Conceptions of Algebra and Conceptions of Variable Described by Usiskin (1988)

Conception of Algebra	Conception of Variable	Variable Usage	Example of Usage
Generalized arithmetic	Pattern generalizers	Generalize patterns	$a + b = b + a$
A means to solve certain problems	Constant or unknown values	Solve for an unknown and simplify problems	Solve: $4x + 2 = 10$
The study of relationships	Arguments and parameters	Relate quantities and graph relationships	$f(x) = 3x + 2$
The study of structure	Arbitrary symbols or objects	Manipulate and justify	Factor: $15x^3 + 10x^2y - 5x^2y^2$

Usiskin included what Kaput (1995) identified as the syntactically-guided manipulation of formalisms or symbols in his description of algebra as a means to solve certain problems. Both algebra in terms of functions and Kaput's algebra as a generalized quantitative reasoning were included in the study of relationships by Usiskin.

Despite the integral connection of variables and algebra, Radford (2006) asserted that while variables are an important aspect of algebra, they do not define what it is. Many students use variables in 'unalgebraic' ways even though other mathematics students have reasoned algebraically without the use of variables. For example, students who use a guess-and-check method to write a formula for a pattern such as $n \times 2 + 3$ may not have considered any generalized aspects of the pattern, but merely happened to write an expression with a variable that is correct. On the other hand, students using algebra tiles to represent the relationship between quantities may not use a written symbol to represent the quantity they are working with (Radford, 2006). While not all students who use algebra tiles are connecting that physical representation to a relationship between quantities, students could provide evidence of this underling conceptual grounding with a description of the relationship in natural language. Thus how students think about variables is important to be studied as a component of algebra.

Students' conception of variable is inexorably connected with algebra and mathematics in general from elementary school through college (Hirsch & Lappan, 1989; Philipp, 1992; Rivera & Becker, 2011). As Usiskin (1988) identified, the ways in which variables are conceived and used are connected to different conceptions of what algebra is. Thus if a student's conception of variable is limited, it stands to reason that the student's conception of algebra will also be somewhat limited. For example, a student who conceives of a variable as representing a fixed unknown value may not be able to make sense of the instruction to determine if the quantity $n + 3$ or $5 + n$ is larger without replacing n with a specific value. This student may be able to solve equations with a single variable, but he is thinking of algebra not as a tool to investigate relationships between quantities, but as a set of steps to solve problems. Since algebra has been identified as a gatekeeper to higher mathematics and greater opportunities in general, any limitations on a student's ability to use algebra in meaningful and substantive ways have long-lasting effects. In this way, students' conception of variable is an important aspect of algebra.

Conceptions, Uses, and Misconceptions of Variable

In an effort to understand how students conceive of and use variables, Küchemann (1978) characterized students' variable use by describing how letters were used as variables in specific problems, as illustrated in Table 6 below. He found that many high school age students engaging with contextualized tasks struggled to produce an appropriate algebraic cost equation for two differently priced colors of pencil because they viewed the variables as labels for the sets of pencils (i.e., "*r*" for red and "*b*" for blue) rather than as variable quantities.

In addition to his findings that most 13- to 15-year-old students considered variables to be labels or objects, Küchemann (1978) also found that only a few students conceived of variables as specific unknowns, and fewer still thought of them as generalized numbers or as representing any or all members of a set. Furthermore, some students believed that the variable not only represents a "concrete entity", but also simultaneously represents multiple attributes of that concrete entity (e.g., the variable b stands for the books, the price of the books and the quantity of books).

Table 6

Uses of Variable from Küchemann (1978)

Use of Variable	Example	Description of Variable Use
Letter evaluated	$a + 5 = 8$	The value of a can be immediately identified
Letter ignored	$a + b = 34$ $a + b + 2 = ?$	The value of $a + b$ can be ignored since it is in both statements
Letter as an object or label	$3a$ represents the number of dollars you paid for some apples. What does a represent?	a stands for apples
Letter as a specific unknown	A figure has n sides each with length 2. What is the perimeter of the figure?	Since n is fixed unknown number, the perimeter cannot be evaluated beyond the expression $2n$
Letter as a generalized number	$c + d = 10$ $c < d$ What could c be?	The letter c represents any one of a set of numbers, rather than a single unknown value
Letter as a variable	Which is larger, $2n$ or $n + 2$?	The quantity $2n$ must be compared with $n + 2$ as n varies

In describing students' use of variable, I am referring to the ways in which students treat, or employ, the variable. For example, in the description provided by Küchemann (1978) of "letter ignored" the student's use of the variable could be to ignore it and not use it at all. In this case the use is actually the lack of use.

Rosnick (1980) described similar categories of conception and use of variable and included variable as a constant value (e.g., π). His findings support those of Küchemann, suggesting that many students do not perceive variables in algebraic expressions as standing for a number, rather they see them as a label that stands for a "concrete entity rather than a more abstract 'number' of things" (p. 2). In subsequent research studying student misunderstandings around the simplification of algebraic expressions, Booth (1984) found that many students view variables as unknown specific numbers, rather than quantities that can vary or as generalized numbers. With the identification of these common conceptions and uses of variables by Küchemann, Rosnick and Booth as a foundation, more recent research worked to deepen researchers' understandings of students' conceptions and uses of variable. It is upon this foundation that the theoretical framework for the current study is built.

Conceptions and Uses of Variable

Based upon the work of Küchemann, Rosnick, and Booth, researchers have identified advantages of conceptions of variables that allow for multiple values when compared with conceptions of values that only allow for variables representing a single value. These advantages include stronger algebraic reasoning as well as a stronger ability to generalize and represent situations mathematically.

Knuth et al. (2011) demonstrated that middle grades students' conception of variable is connected to students' abilities to reason algebraically. The researchers found that just under half of sixth grade students (increasing to over 75% of eighth grade students) conceived of variable as having more than one possible value. Students with this conception of variable were significantly more likely to correctly compare and justify the comparison of two algebraic expressions. Knuth et al. (2011) claimed that "the importance of fostering a multiple-values interpretation of [variable] and suggested that efforts to foster such an interpretation will likely contribute to students' preparation for algebra and algebraic ways of thinking" (p. 275).

Cai et al. (2011) reported results from the LieCal Project, which provides an example of research that supports the idea that students with a conception of variable that allows for multiple values are more likely to be successful in algebra. This project was a longitudinal project that investigated the effects of the Connected Mathematics Project (CMP) curriculum (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1997, 2009) on middle school students' algebraic development and compared them to the effects of other middle school mathematics curricula (non-CMP). The authors characterized CMP as having a functional approach to algebra that emphasizes relationships among quantities. They characterize the non-CMP curriculum as having a procedural approach that focused on variable as an object or placeholder. The authors compared students' algebraic thinking related to representing situations, equation solving, and making generalizations. They found that while both CMP and non-CMP students showed similar improvement in their ability to solve equations, CMP students had a significantly higher increase in their ability to generalize and represent situations mathematically. As the focus of the CMP curriculum is on functions, students who experienced it would be more likely to develop a conception of variable that allowed for multiple values according to the connections identified by Usiskin (1988).

In addition to the research which suggested that students with a conception of variable that accommodates multiple values are more likely to have stronger algebraic reasoning skills as well as a stronger ability to generalize and represent situations mathematically, research also suggested that a conception of variable that is limited to a single fixed value tends to be connected to arithmetic reasoning and not algebraic reasoning (Küchemann, 1978; Stacey & MacGregor, 1999). Stacey and MacGregor (1999) focused on the difference between how variables are used when problems are solved using arithmetic and algebraic approaches in interviews of students aged 13 to 16. In arithmetic, problems are solved by calculating with known numbers and working towards the answer. The basic logic of the algebraic method is different: Relationships are described using both knowns and unknowns in a similar way and then equivalent relationships are derived from them until an answer to the problem is obtained. For example, consider a student asked to find the length of each side of a triangle if the perimeter is 44 cm and the side lengths are labeled as x cm, $2x$ cm and 14 cm. Stacey and MacGregor (1999) described a student's solution using a primarily arithmetic approach would consist of a string of operations whereas an algebraic approach could involve the manipulation of an equation. The authors concluded from their results that the compulsion to calculate prevents students looking for, selecting, and naming the appropriate unknown or unknowns. This arithmetic disposition prevents some students from attempting an algebraic approach, and deflects others away from an algebraic method that they have started. Stacey and MacGregor (1999) described students who either never write an equation (even when prompted to) and instead solve a problem as a sequence of operations or that write an equation, but still solve it using the same arithmetic logic or even a guess-and-check strategy. These results supported the idea that conceptions of variables primarily as single unknown values tend to be connected with an arithmetic problem-solving approach.

Stacey and MacGregor (1999) explained that arithmetic reasoning tends to lead students to think of the variables as "unknown values." This conception of variable inhibits students as they consider the relationship between two variables, such as $4 + x = y - 2$, and other more sophisticated concepts of variable (Smith, 2011; Stacey & MacGregor, 1999). If a variable is seen as an "unknown value" in this arithmetic reasoning there is nothing to do, which leads students to the conclusion that we cannot consider the relationship unless we know what the values of x and y are. Thus, students who conceive of variables as being limited to a single unknown value will be less likely to reason algebraically in situations which variables can have multiple values.

While these conceptions of variables have been linked to specific tendencies, some research suggested that the conception of variable used may depend on the context and representations used in the problem. Malisani and Spagnolo (2009) described that when students with a less sophisticated conception of variable were given a problem with visual aids (e.g., graphs, tables) they were more likely to consider that a variable could have multiple values. The authors administered a problem in which students can set up an equation in two variables to describe the relationship between the amount of money two individuals bet. When presented in this context, students tended to consider only one possible solution. But when a similar mathematical situation requires students to graph the line before answering the question, students were more likely to consider multiple values for the variables. This implies that the conception of variable used may depend to some degree upon the context of the problem or the representations used in the problem.

Development of Conceptions of Variable

Research which relied upon the work of Küchemann, Rosnick, and Booth as foundational has not only described advantages and disadvantages of conceptions of variables that allow for multiple versus single values respectively, but also has worked to describe when the development of these different conceptions of variables occur (Britt & Irwin, 2011; Malisani & Spagnolo, 2009; Sfard & Linchevski, 1994). Malisani and Spagnolo (2009) created a hierarchy of difficulty for students' concepts of variable from the earlier work. They identified the "letter as an unknown" conception is relatively simple and elementary students can attain a certain understanding of the algebraic unknown (Carpenter & Levi, 2000; Kaput & Blanton, 2000). The conceptions of the generalized number and the variable in functional relations seem to be more difficult for students. Relying heavily on the work of Küchemann (1981), they showed that most students between 13 and 15 years of age treated the letters in expressions or in equations as specific unknowns before they treat them as generalized numbers or variables in functional relations.

Malisani and Spagnolo (2009) also observed that not only do students primarily use the conception of the variable as an unknown, those who use this conception do not use the variable in functional relations. They suggested that this limited conception of variable precedes the more useful multiple-value conception of variable. Sfard and Linchevski (1994) and Britt and Irwin (2011) provided support to the idea that there is a progression of concepts of variable, but suggested that earlier exposure to all concepts of variable is helpful. Specifically, the students should have opportunities to work with several concepts of variable, and at various levels of generality, in all areas of their mathematics curriculum.

Blanton et al. (2015) provided further evidence of this progression by building a learning trajectory (LT) using classroom teaching experiments (CTEs) with second grade students. The lessons used in the CTEs were based on tasks that focused on children's functional thinking. The LT was specifically designed to characterize increasingly sophisticated levels in students' thinking about variable and variable notation in functional relationships as they progress through the prepared instructional sequence. The authors provided Table 1 in Chapter 1 to summarize the levels of sophistication that their LT describes. Blanton et al. (2015) noted their LT is specifically linked to variable quantities associated with functional relationships and connected to the instructional sequence from which it was developed. Because of these connections, this LT should be seen as a possible progression of students' conceptions of variable and variable notation and may not apply in other algebraic contexts.

Blanton et al. (2015) observed that in levels 1 and 2 students were still thinking in terms of counting and measuring to find unknown values indicating that they did not think about the variable quantity as an indeterminate quantity, but rather treated variable as having known values. In levels 3–5, students develop the idea of a variable as a unknown quantity. This progression begins as the conception of the quantity as a fixed value then as a single value from a range of values and finally to a varying unknown representing multiple values simultaneously. Level 6 is the culmination of the LT in which students "could mathematize unknown quantities and act on these quantities as objects in and of themselves, or even combine them with other symbols (operations, numerals) to represent functional relationships" (p. 35).

Despite the work suggesting that students' conception of variable progresses from single-value unknown to a function or multiple-valued conception, Dede (2004) suggested that this perceived progression could be because of the way variables are initially introduced to students as an unknown value in equations with subsequent steps taken to ascertain its value. Thus many students begin with this conception of what a variable is and struggle to develop other, more sophisticated, concepts of variable. He suggested that initially teaching a functional conception of variable instead of building upon variables as unknowns in equations may facilitate students' understanding of the concept of function, which is one of the primary concepts in mathematics. Although introducing students to variables from a functional or multiple-valued conception does not preclude the possibility that students would still develop a single unknown value conception of variable before multiple-valued conception, it does provide students with exposure to these ideas at an earlier age.

Blanton and Kaput (2011) also advocated for exposing young children to a variety of conceptions of variable and that children can develop a sense of variable as they have opportunities to use symbolic notation in meaningful ways. Blanton and Kaput found that, when curriculum and instruction provide opportunity for thinking about functional relationships, students can transition from talking about generalizations with natural language at grades PreK–1 to symbolic notational systems by grade 3 (Blanton & Kaput, 2004). The authors noted that while students' understanding of variable at these young ages are certainly in the early stages (for example, care must be taken to ensure that the student does not confuse the variable as representing the object and not the quantity), the purpose of opportunities to interact with variables is to begin using symbolic representations. As students have exposure to these more basic ideas in the early grades, they will have greater cognitive ability to investigate more complex ideas in later elementary grades and beyond (Blanton & Kaput, 2011).

Other studies have also supported the idea that elementary-aged children can begin to reason with variables by using multiple values. Working with 1st and 2nd grade students, Carpenter and Levi (2000) showed that even young students could begin to reason with classes of solutions rather than only specific solutions. In their study, Carpenter and Levi looked at how some students reason about whether $78 - 49 + 49 = 78$ is true; some students were able to rationalize that subtracting 49 and then adding 49 creates a 0 leading to the generalization that $78 - x + x$ would equal 78, even if x varied, showing an early understanding of variable.

Misconceptions About Variables

In addition to the extensive work that has been done around conceptions of variable and their development, explorations into students' misconceptions around variables have also been conducted based on Küchemann, Rosnick, and Booth's research. Russell et al. (2009) differentiated between misconceptions and errors by describing errors as inconsistent mistakes, whereas misconceptions are when students consistently make the same mistakes. This consistency suggests that these mistakes occur when the student incorrectly applies previously learned strategies or concepts. In light of variables' central role in algebra, misconceptions and limited conceptions about variables affect students' ability to reason and operate algebraically in addition to the specific errors they perpetuate (Clement, 1982; Schwartzman, 1996; Stacey & MacGregor, 1997).

One example of a misconception about variables was described by McNeil et al. (2010) when they examined the relationship of conceptions and uses of variables with middle school students' understanding of algebraic expressions. They asked students to interpret an expression (e.g., 4*c,* 3*b*) in a story problem in which each variable represented the price of an item. The authors found that when the variable was the first letter of an object referenced in the problem, students were more likely to misinterpret the variables as labels for objects (e.g., *c* stands for *cake*) than when the variables were not letters from the names of the objects used in the problem.

Another example of such a misconception is provided by Fujii (2003) concerning conditions under which the same variable represents the same or different values. Often in mathematics classrooms, the variable *x* is used repeatedly in one after another unconnected problem. Fujii (2003) documented misconceptions of interpreting variable, namely that the same variable in the same problem may not stand for the same value, and that different variables must represent different values. He commented that in light of the ways in which variables are commonly used in classrooms, these misconceptions are not surprising.

MacGregor and Stacey (1997) identified another misconception associated with variables in expressions. Students tend to simplify expressions such as $5 + n$ to a single entity such as $5n$. They interpret operational signs to mean that something must be done, a computation must be made to lead to an 'answer.' The authors suggested that this misconception stems from the inability of students to see a variable expression as quantity, but rather they are seen as a process to be completed (Hewitt, 2012; Sfard, 1991).

Misconceptions about variables can also come from overgeneralizing previously learned concepts and conventions. Students have learned that when writing mixed numbers, the whole number and the fraction are being added together. This can lead students to assume that the expression $h + 10$ is equivalent to $h10$. Furthermore, since students have already constructed a meaning for $h10$, $4 \times h$ is sometimes written by students as h^4 (MacGregor & Stacey, 1997). MacGregor and Stacey suggested that students may "remember certain surface features and spatial displays without the associated meaning and context. Consequently, when new concepts and notation are introduced, students are unable to link these" (1997, p. 17)

Other misconceptions about algebraic expressions that MacGregor and Stacey (1997) identified included assigning variables an arbitrary value or a reasonable value. As mentioned previously, students also experience confusion when using variables that are the same letters used to label the units of measure (e.g., g represents grams) leading to an inconsistent interpretation of g as grams and number of grams (MacGregor & Stacey, 1997; Rosnick, 1980; Stacey & MacGregor, 1997). Blanton et al. (2015) provided an alternative perspective to students' use of arbitrary values to variables. They viewed this as an "emerging conception that reflects their attempts to make sense of a new notational system whose symbols have another inherent and non-mathematical meaning" (p. 21–22) in young children.

MacGregor and Stacey (1997) also identified a direct link between thinking of variable as a single unknown value when they asked students if a or b is larger in the equation $a = 28 + b$. Many students responded that because the a and b are unknown, it is impossible to tell which is greater. Other students interpreted this equation as a process to be carried out, and took the equation to mean "*a* equals 28 and then add *b*".

Christou and colleagues (Christou & Vosniadou, 2005; Christou et al., 2007) investigated eighth- and ninth-grade students' understanding of variable and discovered another misconception. When asked to list numbers that could be assigned to a variable in given expressions, very few students responded with any numbers other than natural numbers. Students were then asked to list numbers that could not be assigned to a variable found in the same given expressions. Most students only listed negative integers. The researchers reported that students also tend to interpret variables as not able to represent fractional quantities. This restriction on the values that a variable can represent restricts students from making generalizations about variable relationships and inhibits their understanding and learning of algebra.

Summary

Based on the foundational work of Küchemann (1978), Rosnick (1980), and Booth (1984), more recent research suggested that students who think of variables representing multiple values have stronger algebraic reasoning as well as a stronger ability to generalize and represent situation mathematically. Elementary students can develop a multiple-value conception of variable (Blanton et al., 2015). Researchers suggested that by introducing students to more sophisticated concepts of variable at a young age will allow them to be more successful in algebra in their future. By contrast, students limiting variables to a single value tend to use arithmetic reasoning over algebraic reasoning. Some research suggested that students' conception of variable follows a progression from a single unknown value to a functional conception, while other researchers have suggested that this is a consequence of the instructional approach experienced by the student. Furthermore, common misconceptions about variables and their use have been documented by MacGregor and Stacey (1997) and Christou et al. (2007). This work on the levels of sophistication of students' conceptions of variables and their misconceptions about variables for the basis for the frameworks for the current study.

Gaps in the Literature About Concepts of Variable

The research that has been done around students' conceptions, uses, and misconceptions of variables provides a solid base for understanding the types of concepts about variable that students may have. Despite this strong base of research, there are still some unanswered questions. The literature on students' conceptions of variable is silent on the similarities and differences that LMA students and students with MLDs exhibit when working with variables. In particular, this study will compare the similarities and differences between MLD, LMA, and TMA algebra students' concepts of variable and those of the general population of students. As described in the above literature, conceptions of variables that allow for multiple values have been tied to increased algebraic reasoning and stronger abilities to generalize and represent problems mathematically. However, these discussions have fallen short of connecting these advantages to students' success in algebra classes as typically measured by assessments that do not gauge these attributes, but focus almost exclusively on procedural knowledge of algebraic manipulations.

Despite the gap in research on MLD and LMA students' concepts of variable, a substantial amount of work has been done with MLD and LMA students' numeric and arithmetic concepts and skills. In the following sections I first discuss who these students are and how they are identified. Next, I discuss what we know about these students with regard to their mathematical understandings. Finally, I address gaps and limitations of this body of literature.

Students with Difficulties in Mathematics

Geary et al. (2012) described students with low achievement in mathematics as two groups of students. The first group, which the authors estimated comprises 7–10% of all students, including those students with a mathematics learning disability (MLD). The second group included those students who do not have a MLD, but still had persistent learning difficulties in mathematics. As these difficulties were less severe than students with MLD, these students are often referred to as students with low mathematical achievement (LMA). Geary et al. estimated that approximately an additional 10% of all students fall into this category Students who are not in the MLD and LMA groups are considered typically-achieving (TMA) students. Due to differences in classification procedures, these percentages can vary substantially. Often, students with mathematics difficulties are grouped together to comprise the lowest 25% of students. The percentile cut-off scores typically come from student scores on a measure of mathematics achievement.

MLDs are defined as having mathematics learning difficulties with a cognitive origin (Lewis, 2014; Mazzocco, 2007). The cognitive origin is considered the distinguishing feature of students with MLDs from other students with LMA whose difficulties stem from other, including environmental, factors (Lewis, 2014; Wong et al., 2014). Geary (2004) noted, “In theory, a learning disability can result from deficits in the ability to represent or process information in one or all of the many mathematical domains (e.g., geometry) or in one or a set of individual competencies within each domain" (p. 4). MLDs may only impact a subset of mathematical skills, concepts or domains for any given child. Geary (2004) reported "children with MLD often have severe deficits in some of these areas and average or better competencies in others." (p. 5). In addition to the variability in these students' mathematical difficulties, many children with MLDs may also show signs of other disorders, including reading disabilities and attention-deficit/ hyperactivity disorder (ADHD).

Identifying Students with Mathematics Difficulties

Despite a general agreement about the origins, definitions of, and differences between students with MLDs and LMA, there is not consensus about how to identify these students (Mazzocco, 2007). Additionally, different studies used the terms 'mathematics learning difficulty,' 'mathematics learning disability,' 'low achievement,' and 'struggling learners' to refer to students with low mathematics achievement, a subset of this group, or to a group that includes students with both MLDs and LMA. This complicates the identification of common characteristics of students with low achievement in mathematics because the characteristics identified in any one study may not be generalizable to another group of students with low mathematical achievement as it is dependent on the criteria used to group the students.

Historically, a common strategy used to identify students with MLDs was to identify a discrepancy between a student's IQ and mathematical performance level. Recently, substantial issues have been raised concerning the reliability, predictive power, stability, and the ability to reliably differentiate between children with MLD and their LMA peers (e.g. Francis et al., 2005; Jordan, Hanich, & Kaplan, 2003b; Vaughn &

Fuchs, 2003). An example of one of these issues relates to the fact that these tests are only a snapshot of a student's mathematical performance and may not be an accurate depiction of overall abilities, thus compromising the stability of this method of identification.

Currently, one of the most common strategies used across many studies focusing on students with MLDs is to group all LMA students using one performance cut-off criteria below which all students are considered MLD (or as having mathematics learning difficulties, to be struggling mathematically, etc.) (Wong et al., 2014). Murphy et al. (2007) analyzed journal articles from 1985 to 2006 to gauge the range and typical cut-off score used by researchers as well as to compile a list of common characteristics of these LMA students. They found that while cut-off scores used in the analyzed studies typically clustered around the 10th and 25th percentile for students with MLDs and LMA, respectively, there was a large range of cut-off scores used, from the 5th to the 46th percentile. Researchers justified this wide range of cut-off scores by citing differences in tests, as being necessary to obtain a sufficient sample size or to reflect the samples' proportion of other risk factors (e.g., eligibility for free or reduced lunch). Murphy et al. (2007) observed that while these differences do raise valid concerns, the wide range of cut-off scores introduces the possibility of identifying students who have different characteristics than the claimed focus of the study of students with MLDs.

Landerl, Bevan, and Butterworth (2004) observed that studies that used higher cut-off scores may provide more insight into the characteristics of LMA students than of those who have an MLD. Using the high end of Geary et al.'s (2012) estimate that 10% of all students have a MLD, then a group of students who were at or below the 25th

percentile could contain more students without MLDs (about 15% of the students included) than those with MLDs. This possible majority of non-MLD students could skew the sample to have characteristics more consistent of students with LMA without MLDs than like students with MLDs (Thouless, 2014).

Wong et al. (2014) reported that many recent studies have attempted to reduce this effect by grouping students into three categories, rather than two, based on their mathematical achievement as illustrated in Table 7.

Table 7

Percentile Rank and Standard Deviation Cut-Off Scores

Group	Percentile Range	Range of Standard Deviations Below the Mean
Students with a Mathematics Learning Disability (MLD)	Below the10th	1.5 or greater
Other Low-Achieving Students (LMA)	At or above the 10th and below the 25th	Between 1.5 and 0.67
Typically Achieving Students (TMA)	At or above the 25th	Above 0.67

While using multiple cut off scores is an improvement upon the one cut-off score approach by differentiating between students with MLDs and LMA, Wong et al. (2014) expressed concern that because these cut-off scores are arbitrary, there is still danger of not accurately grouping students with common characteristics. For example, students who score slightly better than the cut-off score could have more in common with students whose scores are below the cut-off than those who are well above the cut-off score, but still in the same category. Applying this reasoning to other numbers of groups and despite the number of groups introduced, any use of arbitrary cut-off values introduces the possibility of the assigned groups to contain students who have substantially different mathematical characteristics.

Wong et al. (2014) also commented that any version of the cut-off approach that only considers the achievement of students at a particular instant in time neglects the component of mathematical growth that occurs over time. The persistence over time of mathematics difficulties is considered by some to be a requisite of MLD in addition to the cognitive origin requirement (Vanbinst et al., 2014). A cut-off score at only one point in time may identify some students as having a MLD when their difficulties are not persistent and thus they should be classified as students with LMA instead of with MLD. For example, Murphy et al. (2007) identified a case in which students started out with lower initial mathematical knowledge of a topic, but then caught up with their TMA peers over time. Thus, any method relying upon cut-off scores will not perfectly identify students with MLDs.

Despite the limitations of using multiple cut-off scores, it is still one of the most commonly used approaches to identify students who experience mathematical difficulties (Geary et al., 2012). Due to restriction of resources (e.g., time, funding, access to students) and the importance of identifying these students, the multiple cut-off method is considered sufficient by many researchers (Geary et al., 2012; Hecht & Vagi, 2010; Murphy et al., 2007).

Characteristics of Students with Mathematics Difficulties

Because of the many different possible sources of difficulties, identifying common characteristics of these students is challenging. The common use of a cut-off score at the 25th percentile (Murphy et al., 2007) may in effect combine the MLD and LMA groups of students (Landerl et al., 2004). This creates inconsistencies in differentiating between the groups of MLD and LMA students when interpreting results.

This situation can lead to complications as these groups of students could have different sources of difficulty.

Studies have shown that there are differences between the sets of characteristics of LMA and MLD students. Lewis (2014) explained that the difficulties of students with MLDs "stem from a cognitive origin, leading to qualitatively different error patterns than those experienced by their low-achieving peers" (p.351). In their longitudinal study of students in grades K–3, Murphy et al. (2007) provided evidence that students who are grouped in the MLD classification do have different characteristics than those grouped into the LMA classification. While both the LMA and MLD groups of students started at a lower level of proficiency than the TMA students group, the growth of the students with LMA paralleled the TMA student, while the students with MLDs had a slower, and decreasing, rate of growth.

Other researchers also identified a difference between students with LMA and MLD (Misquitta, 2011). Most studies that differentiated between students with LMA and MLD identified different but overlapping sets of characteristics for both of these groups. A longitudinal study of rational number sense by Mazzocco and Devlin (2008) of middle school students also showed this pattern of having different, but overlapping sets of characteristics. The researchers found that while MLD and LMA students all scored lower than their TMA peers, students with MLD struggled for different reasons than LMA students. For example, MLD students failed to correctly read decimals while LMA students lacked conceptual understanding of rational numbers despite being able to accurately read decimals. These results support the hypothesis that there are two distinct groups of children who struggle with low mathematics achievement.

Although research suggests that there are differences between students with MLDs and LMA students, there is also evidence that these groups of students share attributes. In a longitudinal study of elementary school children, Geary et al. (2012) grouped students into MLD, LMA, and TMA classifications and compared these groups' IQ, working memory, processing speed, in-class attention, and numeracy skills. Although these groups did show general differences, Geary et al. (2012) did not find clear-cut distinctions between the groups of students with MLD and LMA. The authors suggested that this finding implies that children in these groups are part of a normal distribution of mathematics achievement and growth. Geary et al. (2012) summarized that their results suggest that the low start point and slow growth of MLD and LMA students do not result from the same factors, but these factors do overlap to some extent and suggest that the sets of the characteristics of these groups of students are different but may not be mutually exclusive.

The remainder of this section reports the findings on characteristics of students with low achievement in mathematics. When discussing studies that differentiated between LMA and MLD students, these differences are noted. Because of the similarities of the groups of students with MLD and LMA, results from either group or the combined group can inform our understanding of students who struggle as they are all part of the same continuum. Hecht and Vagi (2010) reported that some attributes of students with low mathematics achievement may be specific to a mathematical concept, skill, or domain while other attributes are not specific to mathematics. As such this section is organized by first discussing the characteristics of students' mathematics difficulties that are not specific to mathematics followed by those characteristics of these students that are specific to mathematics.

Characteristics not specific to mathematics. Geary et al. (2010) reported that many students with low achievement in mathematics share similar domain independent characteristics, impacting their mathematics achievement. Despite the definition of MLDs having a cognitive origin, environmental characteristics can affect students with and without MLDs. Similarly, the mathematics achievement of students without MLDs can also be impacted by cognitive characteristics, which are related to, if not as severe as, those of students with MLDs. These findings provided further evidence to support the theory of students with MD as part of the normal curve of students' achievement in mathematics (Geary et al., 2010).

Environmental characteristics. Despite defining MLDs as having MD stemming from cognitive factors, researchers reported that students with MD are limited by factors such as prior low achievement, which leads to deficits in prerequisite knowledge for subsequent topics, low expectations for success, and inadequate instruction and support (Jones et al., 1997; Misquitta, 2011). Other researchers (Geary et al., 2012; Wong et al., 2014) observed that the students within the LMA group had the lowest socioeconomic status (SES) and that it was significantly and substantially lower than the students with MLD group. They suggested that a lack of parental involvement in learning among the low SES families could be one of the reasons for the slow development of this group.

Another environmental factor that can affect the growth of students with learning difficulties is poor or insufficient classroom instruction (Butler et al., 2005; Carnine, 1997; Stein et al., 2011). Butler et al. (2005) reported that due to the procedurally-based, memorization-heavy instructional strategies traditionally used in special education classrooms, students in these classrooms may have constructed less useful conceptions about what mathematics is and how to learn it. This is not to suggest that all students identified as having a MLD necessarily have been identified as needing an individualized education plan (IEP) and receiving service in a special education classroom, although many of these students are served in such a setting because of their IEP status.

Cognitive characteristics. In addition to environmental factors affecting students with low mathematical achievement, cognitive characteristics also influence these students' mathematical growth. Research has described that when compared to their TMA peers, students with MD have a more limited working memory (Andersson, 2010; Geary et al., 2012; Misquitta, 2011). Hecht and Vagi (2010) defined working memory as the "concurrent storage and manipulation of the information necessary to perform a mental task" (p. 845). For a student with poor working memory, information represented in working memory is more likely to decay than for a student with typical working memory. For example, "two minus one-third equals" might decay into "$2\frac{1}{3} =$" while the student is converting the 2 into six-thirds before taking away one-third.

In addition to working memory, low attention is another cognitive characteristic that has been associated with students who have MD. Students with LMA and MLDs tend to be less attentive during instruction than their TMA peers. Further, children with MLDs and LMA require a longer time to reengage their attention after being distracted (Geary et al., 2012; Hecht & Vagi, 2010; Misquitta, 2011). This longer time not attending to the contents of the working memory increases the likelihood of decay of the information held there (Hecht & Vagi, 2010; Misquitta, 2011). The related cognitive characteristics of poor working memory and low attention in students with LMA and

MLDs negatively impact mathematical achievement (Geary et al., 2012; Hecht & Vagi, 2010).

Mathematics-specific characteristics. In addition to the work that has investigated the characteristics of students with MLDs and LMA that are independent of mathematics, studies have also considered how these students learn specific mathematics skills and concepts. Due to the heavy focus of these studies on the elementary grades, especially the early grades, there are many results about numeracy skills, factual recall, and computation but, few about other mathematical domains.

Numeracy skills, factual recall and arithmetic. Students with low mathematical achievement, with and without MLDs, tend to have difficultly both storing and recalling basic arithmetic facts; these difficulties can persist over time despite instructional interventions. These difficulties then affect students' ability to achieve computational fluency (Butler et al., 2005; Geary et al., 2012; Vanbinst et al., 2014). Geary et al. (2012) clarified that these students can recall some facts, but they remember significantly and substantially fewer facts than their TMA peers and commit more errors when they do recall these facts. These deficits tend to diminish over years, but few of the students with MD catch up to their TMA peers. These children also experience similar delays in achieving accuracy and fluency in arithmetic operations and procedures. In both factual recall and development of procedural fluency, LMA children fall between children with MLD and TMA children on initial assessments. Despite this lower starting point, the students with LMA catch up to TMA students over the course of time while the MLD students do not (Geary et al., 2012).

Not only do these difficulties exist when considering the group of students with

MD as a whole and specifically for students with MLDs, but Jordan, Hanich, and Kaplan (2003a) found that some LMA students also exhibited these factual retrieval deficits that persisted over time. Thus while results from Geary et al. suggest that LMA students have a developmental delay, Jordan et al.'s work showed that some LMA students may have retrieval deficits as persistent as those of children with MLD (Jordan et al., 2003a).

In addition to identifying these deficits of factual storage and retrieval and fluency in executing arithmetic procedures, students with mathematical difficulties also exhibit differences in their rate of growth in these areas when compared to their TMA peers (Geary et al., 2012; Murphy et al., 2007). Geary et al. (2012) reported that retrieval of mathematics facts by students in the MLD group was characterized by slow growth, in relation both to their TMA and LMA peers. Due to this slow growth rate, the MLD students did not catch up to their TMA or LMA peers over the course of the longitudinal study and their trajectory suggested that they would not do so. Students in the LMA group showed a similarly low beginning point as the students with MLD group, but experienced a growth trajectory similar to that of TMA children. Because of this parallel growth, students in the LMA group performed in fifth grade similar to the TMA group in second grade. Murphy et al. (2007) added that not only did the students with MLDs have a slower growth rate than all of their peers, but they also appeared to have a decreasing growth rate, suggesting that they may plateau in their arithmetic growth at some point in the future.

Murphy et al. (2007) reported an exception to the trends noted above on visual-spatial skills. LMA and MLD students all started at the same level, which was less than TMA students, but the students with LMA caught up to the TMA students while the students with MLDs showed substantially slower growth than the LMA or TMA students. This deviation from the typically reported comparison patterns between students with MLDs, LMA and TMA further suggests that students' mathematical learning difficulties may be dependent upon the specific mathematical domain or skill set.

Conceptual understandings. In addition to making frequent errors when preforming procedures, students with mathematical difficulties also typically have a limited conceptual understanding of how and why the procedures work (Butler et al., 2005; Geary, 2004; Hecht & Vagi, 2010; Jones et al., 1997). Jones et al. (1997) reported that MLD students tend to have higher achievement on the application of procedures of arithmetic skills than the application of their associated concepts. Students with MLDs and LMA also perform lower on other more advanced concepts. Hecht and Vagi (2010) reported that students with mathematical difficulties (both MLD and LMA) are less likely to make sense of operations with fractions than their TMA peers. Students with mathematical difficulties are also less likely to provide a valid explanation of why procedures with fractions work and how to use them to solve problems involving fractions.

In another example of the conceptual struggles of students with MD, a longitudinal study of rational number sense by Mazzocco and Devlin (2008) found that MLD students typically failed to correctly read and identify decimals, make basic comparisons with rational numbers, and explain the concepts of rational numbers and their basic operations. In comparison, LMA students could accurately read decimals and had some limited proficiency performing operations with rational numbers, but they lacked conceptual understanding of rational numbers. These results suggested that one difference between students with MLDs and LMA could be found in comparing conceptual and procedural understanding for some mathematical domains (Lewis, 2014).

Solution strategies. Both the mathematic specific and non-specific characteristics of students with MD influence the strategies they use when solving both computational and conceptual problems. Students with mathematical difficulties typically have a limited conceptual understanding of how and why the procedures work. This limited conceptual understanding of the procedures may contribute to a delay in building or adopting more complex procedures, which can increase the difference in achievement and strategy use between them and their TMA peers (Butler et al., 2005; Geary, 2004).

In general, students with MLDs and LMA tend to rely on less advanced strategies and are typically delayed in their development of more advanced strategies when compared with their peers. These students who have difficulties in mathematics often experience a developmental delay in obtaining proficiency with the same procedures that their TMA peers use (Geary, 2004). For example, Hecht and Vagi (2010) reported that a common characteristic of students with MD is the tendency to use more error-prone and slower counting-based strategies to find the answers to simple arithmetic problems rather than retrieving from long-term memory the relevant factual knowledge. In this way the cognitive characteristics of these students influence their solution strategies.

Gaps in the Literature About Concepts of Variable

While the work that has been done to understand the characteristics of students with low achievement in mathematics is invaluable, there is still more work to be done. Most of the research that has been done at the elementary level focused on arithmetic skills and fact recall (L. S. Fuchs et al., 2007; Murphy et al., 2007) or on the persistence or progression of the performance of those skills from kindergarten or first grade to 1 to 3 years later (Geary et al., 2012). Studies have also considered the effects of working memory and other cognitive processes on students' mathematical achievement, and the studies have also typically focused on elementary students and been short term in duration.

In light of these limitations, researchers have called for more studies that investigate these characteristics in more complex mathematical domains and with older children (L. S. Fuchs et al., 2007; Geary et al., 2012). Lewis (2014) warned that the limited focus of students with MLDs and LMA on elementary-aged students' speed and accuracy on basic arithmetic has resulted in limited automaticity of arithmetic facts being used as a de facto defining characteristic of students with MLD. This neglects the more complex and conceptually-based aspects of mathematics and the characteristics of students with MLDs and LMA when interacting with these aspects.

Specifically, Hecht and Vagi (2010) hypothesized that based on the difficulties that MLDs and LMA students have with conceptual aspects of mathematics, it is reasonable to predict that these students would also have more difficulty with conceptual understandings of other mathematical domains. Thus future work should focus on middle or high school students' conceptual understanding of mathematics. The current study will work to add to this literature base by describing the differences in MD and TMA students' conceptions of variables.

Summary of the Literature About Students with Mathematics Difficulties

Low-achieving students are typically classified into two groups: students with MLDs and students with LMA based on cut-off sores. While these groups share many characteristics, they also differ in other ways. These groups can be thought of as representing portions of the normal curve of student learning of mathematics rather than distinct groups. These cut-off scores can be supplemented with longitudinal studies and examination of students' progress as a response to planned and structured interventions. Characteristics of these students are complicated to interpret because of the difficulty in identifying these groups of students and the similarity of their characteristics.

In general, students with MD who score at or below the 25^{th} percentile in mathematics achievement exhibit characteristics associated with environmental aspects including low expectations for success and inadequate instruction and support. They also exhibit cognitive characteristics of poor working memory and low attention. In addition to these general characteristics, the students with MD typically are challenged with storing and retrieving mathematical facts and performing basic arithmetic. They also are more limited in their conceptual understandings when compared to their TMA peers. These difficulties often result in delays or deficits in students' development of more sophisticated solution strategies.

Conclusion

This review of literature of research on student conceptions of variables, including situating conceptions of variables within the larger context of algebra, and identifying different conceptions of variable that students have. Many students conceive of variables as representing a single unknown value (Malisani & Spagnolo, 2009). Students with a conception of variable that does not allow for multiple values for the variable are limited in their algebraic thinking, problem solving and generalizations. (Cai, et al., 2011; Knuth, et al., 2011).

Additionally, research regarding the identification and characteristics of students with MD were summarized. As extensively discussed above, at the intersection of the research on students' conception of variable and the exploration of the characteristics of students' with MD, there is a significant gap. The description of these students conceive of and apply knowledge of variables is important in both the study of students with MD and understanding students' conceptions of variable. As Hecht and Vagi (2010) suggested, it is reasonable to suppose students with MD will be more limited in their conceptual understandings of more complex mathematical domains. Specifically, these students' conceptions of variables may be substantially more limited than their TMA peers. In an effort to address this hypothesis, the current study examined students' responses on items designed to assess students' ability to compare expressions involving generalized quantities and focused on differences in these responses between TMA, LMA, and MLD students.

CHAPTER 3: METHODOLOGY

The goal of this study is to characterize the differences in students' uses, conceptions, and misconceptions of variable among students with a mathematics learning disability (MLD), non-MLD students with low mathematical achievement (LMA), and typically-achieving students (TMA). The study lies at the intersection of the research from the field of mathematics education on students' uses, conceptions, and misconceptions of variable and research from the field of special education on students with MLDs and other LMA students. In particular, I examined the differences in students' responses to items designed to assess their ability to compare expressions involving generalized quantities among MLD, LMA, and TMA students both at the beginning and end of an Algebra 1 course as well as the students' growth across the Algebra 1 course. These responses come from data collected by the *Algebra Screening and Progress Monitoring* (Foegen & Dougherty, 2010) project (funded by the U.S. Department of Education Institute of Education Sciences, award number: R324A100087), and were analyzed by coding student responses for uses, conceptions, and misconceptions of variables then using descriptive and inferential statistics to detect differences in students' responses between MLD, LMA, and TMA students.

This chapter provides a detailed description of the data I analyzed and methods that I employed to execute the analysis. It begins with an overview of *Algebra Screening and Progress Monitoring* (ASPM) and a detailed discussion of the ASPM project's procedures and activities during the third project year, from which the data for this analysis were obtained. A discussion of the theoretical framework, which provided a structure for the analysis, is then followed by a description of the data and analysis of student assessment responses.

Context of the Study

The ASPM project (funded by the U.S. Department of Education Institute of Education Sciences, award number: R324A100087) provided the context for the current study. The overall goals of the four-year ASPM project were to create, refine, and gauge the technical adequacy of three conceptual and three procedural algebra measures for screening and progress monitoring. The focus of the project was to "investigate the validity and reliability of six types of algebra measures for screening and progress monitoring students with, or at risk for, disabilities" (Foegen & Dougherty, 2010, p. 2). Data for the current study were drawn from the third year of the project (2013–2014). The following descriptions of the ASPM project include the procedures and activities of the project up to and including the third project year.

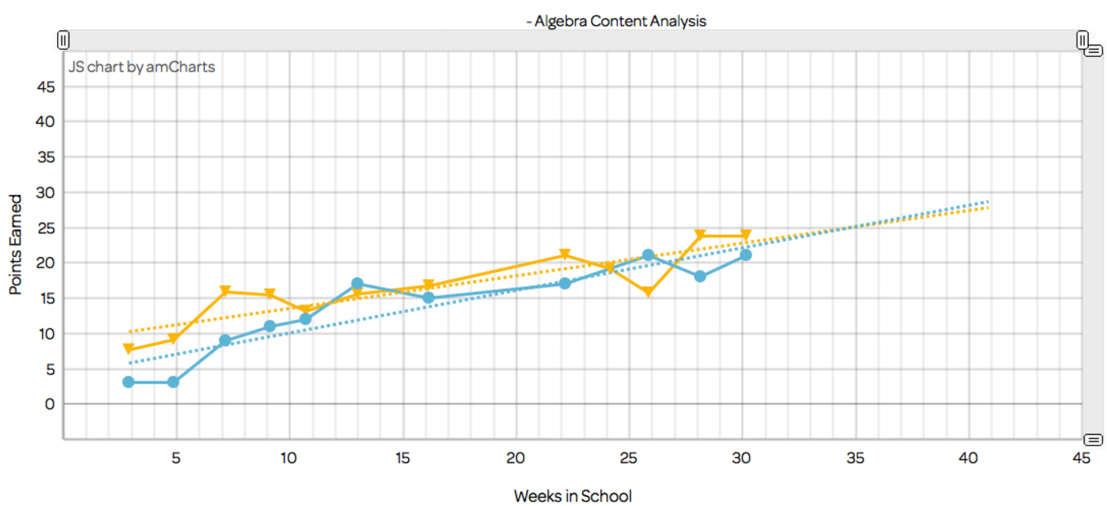
Project Description

In an effort to remedy a lack of assessment tools to screen students at risk of low achievement in algebra, including those with mathematics learning disabilities, and to monitor their progress, Foegen and Dougherty (2010) created the ASPM project. Their work began with three algebra assessments focused on procedural proficiencies that had been previously developed by an earlier funded project (Project AAIMS, Foegen, 2003). These three assessments were revised under the ASPM project. Additionally, the project developed three new conceptually-focused algebra screening and progress monitoring assessments.

Progress monitoring assessments are a systematic type of timed formative assessment that are short in duration (typically less than 10 minutes), administered frequently (typically weekly or bi-weekly) using standardized administration protocols and scored with a scoring guide. The data obtained from these assessments are often displayed as a graph of a student's growth over time. These graphs can provide evidence of the effectiveness of student interventions by showing a change in the slope of the line connecting the student's assessment scores. Figure 1 provides an example of a hypothetical student's progress monitoring scores compared to the average class scores.

Figure 1

Hypothetical Student's Progress Monitoring Scores and Average Class Scores



On this graph, the student's actual scores on a progress monitoring assessment administered bi-weekly are indicated with the yellow triangles and then the line of best fit is calculated and displayed as the yellow dotted line. The data points indicated by the blue circles represent the average class scores with a blue dotted line indicating the line of best fit. Thus a student's growth (slope of the trend line) can be compared to the growth of the class. Additionally, a vertical line can be inserted to indicate the introduction of a specific intervention and the data that come after this line can be used to find a new line of best fit and used as a measure of the effectiveness of the intervention.

Representing a type of assessment known as Curriculum-Based Measurement (Deno, 1985), these assessments generally include items that are not restricted to a unit or specific topic, but sample a wide range of a curriculum or end-of-year expectations in order to provide a picture of a student's growth across time. Multiple equivalent forms of the assessment are used to avoid students memorizing a sequence of responses. An additional characteristic of progress monitoring measures is that they are tested for technical adequacy (i.e., reliability and validity). While the initial development of progress monitoring measures focused on measuring basic reading skills for students with disabilities, their scope has been expanded to include class-wide monitoring of the effectiveness of any interventions or other instruction in special education and general education mathematics classrooms (L. Fuchs, Fuchs, Phillips, Hamlett, & Karns, 1995) as well as a screening tool for early identification of students in need of academic support (Good & Kaminski, 1996).

There were six progress-monitoring measures used by the ASPM project: (a) Algebra Basic Skills; (b) Algebra Foundations; (c) Algebra Content Analysis; (d) Concept of Variable; (e) Proportional Reasoning; (f) Translations, Functions, and Graphs. One form of each of the six measures as they were used in project year 3, along with the corresponding scoring guide and standardized administration protocol are included in the Appendices (Appendices A-F). The three existing progress-monitoring measures were revised based on recommendations from an expert review panel. These revisions included wording changes to be more mathematically appropriate or accurate, answer placement and formatting of graphs (Dougherty, DeLeeuw, & Foegen, 2014). After these initial revisions to the existing progress monitoring measures and the creation of the new measures, The ASPM project administered the measures first as screening tools and then later used as progress monitoring assessments in subsequent years of the project. Based on the student response patterns and feedback from expert review panels, teachers, and members of the ASPM team, all six measures were revised between each project year, which corresponded to consecutive school years. The measures were initially implemented as screening tools in order to refine them before the more extensive progress monitoring administrations. As screening tools, the measures were administered at the beginning and end of the school year using two equivalent forms of the measures. When employed as progress monitoring assessments, they were administered bi-weekly, rotating through 12 equivalent forms of the measures.

The previously developed progress monitoring assessments focused on procedural proficiencies (Project AAIMS, Foegen, 2003). One of these measures, Algebra Basic Skills (ABS), consisted of items that asked students to solve simple equations, apply the distributive property, evaluate integer expressions, combine like terms, and calculate missing terms in proportions. Another of these measures, Algebra Foundations (AF), contained items that asked students to write and evaluate expressions, solve simple equations, graph inequalities on a number line, identify the slope and y-intercept of a line, and find missing values in tables. The final procedural measure, Algebra Content Analysis (ACA) included items in which students were asked to solve systems of linear equations, evaluate and simplify expressions, and write linear equations.

In addition to the work with the three existing procedural assessments, ASPM also created three new algebra progress-monitoring assessments that focused on conceptual understandings. One of these measures, Concept of Variable (CoV), consisted of items that focused on the behavior of variables as generalized quantities in expressions and equations, relationships represented by equations and expressions, and the effect of varying the value of the variable. Another of these measures, Proportional Reasoning (PR), included items that elicited students' reasoning about relationships between and among equations, expressions, and quantities and to apply multiplicative reasoning. The final conceptual measure, Translations, Functions, and Graphs (TFG), contained items that asked students to represent relationships in multiple ways by linking tabular, symbolic, and graphical representations and transforming functions or relationships based on given criteria. Table 8 provides a summary of the six progress monitoring measures used in the ASPM project. Each of the six measures focused on a different aspect or component of algebra, each of which is described in detail in the following section.

In order to provide a more complete description of the development and revision of these measures, this process will be described using the measure entitled "Concept of Variable" (CoV) as an exemplar. The development and revisions of the other conceptual measures mirrored the development of the CoV measure. The topics around which items were built came from the Major Topics in School Algebra identified by the National Mathematics Advisory Panel (2008) and the Common Core State Standards in Mathematics (CCSSM, National Governors Association Center for Best Practices and Council of Chief State School Officers (NGA Center and CCSSO), 2010). For the CoV measure, these topics included students' understanding of relationships within and among expressions, equations, inequalities, functions, and polynomials.

Table 8

Attributes of ASPM Measures

Measure	Number of Items	Duration of Assessment	Assessment Focus
ABS	52	5 Minutes	• Solve simple equations • Apply the distributive property • Evaluate integer expressions • Combine like terms • Calculate proportions
AF	42	5 Minutes	• Write and evaluate expressions • Solve simple equations • Graph inequalities on a number line • Identify the slope and *y*-intercept of a line • Identify missing values in tables
ACA	16	7 Minutes	• Solve systems of linear equations • Evaluate and simplify expressions • Write linear equations
CoV	19	7 Minutes	• The behavior of variables as generalized quantities in expressions and equations • Relationships between and among equations and expressions • The effect of varying the value of a variable
PR	18	7 Minutes	• The relationships between and among equations, expressions, and quantities • Apply multiplicative reasoning
TFG	34	7 Minutes	• Represent relationships in multiple ways • Move among these representations • Transform functions into different representations

After these topics were identified, items were created by the project team members that drew upon Krütetskii's (1976) framework of three types of questions that require thinking beyond computational aspects. These three types of questions are reversibility, flexibility, and generalization questions. Using this framework, questions were created in each of these categories that require students to draw upon their conceptual understandings (Dougherty, Bryant, Bryant, Darrough, & Pfannenstiel, 2014; Rachlin, 1995). A brief discussion of each of these question types follows.

Krütetskii (1976) described reversibility as the ability to reverse the typical direction of a mental process. A common form that these problems take is to provide students the answer to a problem and ask them to create a problem that fits that answer (Dougherty, Bryant, et al., 2014). For example, a reversibility question might ask students to write an equation that has a solution of 3. These types of problems help draw upon students' conceptual understanding by asking students to do something for which they do not have a prescribed procedure and for which there are multiple correct solutions.

Flexibility was identified by Krütetskii (1976) as the ability to switch between multiple strategies of employing relationships between and among problems, solutions, and solution strategies. Asking students to be flexible in using different solution strategies for a given problem and across problems helps them to develop connections across and among solution strategies and problems. A common form that these types of problems take is to ask students solve a problem in multiple ways (Dougherty, Bryant, et al., 2014). In the ASPM project assessments, items were created that provided opportunities for students to solve in multiple ways. For example, a flexibility item on the CoV measure was "If $d - 243 = 542$, what does $d - 245$ equal?". This represents a flexibility item because students can solve it in multiple ways, including using the relationship of the subtrahend to the difference or solving for d in the first equation.

Questions that focus on students' ability to generalize typically ask students to make conjectures about observed patterns and relationships (Dougherty, Bryant, et al., 2014). For example, a generalization item on the CoV measure was "Jon said, '$m - 1$ is always greater than $1 - m$.' Do you agree with Jon?" Students are asked to interpret the relationship between the two expressions by considering the variable as a generalized quantity, rather than a specific value.

Using these types of questions as a framework, the initial design of the CoV measure included, almost exclusively, items that were extended or open response. Low student response rates and low reliability in scoring in the first two years of the project prompted revisions of these items. By year 3, these items had become, almost exclusively, multiple-choice items. In transforming items from extended or open response to multiple-choice items, the team made only minor revisions to item stems and added four options for students to choose. These response choices were constructed from common student misconceptions and influenced by the most common student responses in project years one and two.

In addition to administering these progress-monitoring measures to students, the ASPM project also administered two criterion-referenced algebra assessments and collected teacher ratings of students' algebra proficiency. The first criterion-referenced assessment was The Iowa End of Course (IEOC) Assessment – Algebra I (IEOC; Iowa Testing Programs, 2012) which was intended to assess the student knowledge and skills typically emphasized in beginning formal algebra classes. The IEOC is a 30-item multiple-choice instrument that is aligned with the Iowa Core Curriculum and contains primarily procedurally-focused items. Using its standardized administration procedure, students had 40 minutes to take the IEOC using a test booklet and Scantron. In 2008 and 2009, the IEOC was administered to nearly 9,000 students; the internal consistency coefficient was .80, with a standard error of measurement of 2.40 (D. Henkhaus, Iowa

Testing Programs, personal communication, September 10, 2010). . Mean scores on the four strands of the IEOC form were: Basic Algebra (11 items, *m* = 6.77, *SD* = 2.10); Linear Equations and Inequalities (4 items, *m* = 1.74, *SD* = 1.01); Systems of Linear Equations and Inequalities (5 items, *m* = 2.37, *SD* = 1.35), and Polynomials (10 items, *m* = 4.8, *SD* = 2.26).

The second criterion-referenced assessment was the Formative Assessment in a Networked Classroom (FANC) with 29 items. This measure was developed by a team of researchers the University of Hawai'i as part of a National Science Foundation funded project. It was designed to be a more conceptually-focused assessment as it incorporated adapted items from National Assessment of Educational Progress (NAEP). The final version of the assessment included 29 items, and the mean point-biserial correlation coefficient, as a measure of items' correlation with an overall assessment results, was .37 (*SD* = .08). The development team administered a total of 1,838 assessments at the beginning and end of the school year. Of the 1,838 students, 1,696 completed both the pretest and posttest. The posttest internal consistency reliability was .90 (P. Brandon, personal communication, December 14, 2011).

Technical Adequacy Data for the Progress-Monitoring Assessments

Evidence of technical adequacy, including alternate form reliability, test-retest reliability, concurrent validity, and predictive validity, was available from the ASPM project year 2 data. These data do not include any TFG results at it was created between years 2 and 3 of the project. Also, data on CoV and PR are more limited than for AF, ACA, and ABS as CoV and PR were in earlier stages and only had two forms during year 2.

Alternate form reliability. Table 9 below provides alternate form reliability correlation coefficients for ABS, AF, and ACA. Alternate form reliabilities were calculated by correlating Form 2 with Form 3 within a single round for each procedural measure. Based on the conventional standard of .80, the single forms of the ABS and AF measures demonstrated adequate alternate form reliability, while ACA did not. Round 2 reliability coefficients for each of the measures were stronger than the coefficients obtained for Rounds 1 or 3. Round 2 being conducted within a week of Round 1 could explain this, and may suggest that performance is somewhat dependent upon students' familiarity with the measures (Foegen & Dougherty, 2016).

Table 9

Alternate Form Reliabilities for ABS, AF, and ACA (Foegen & Dougherty, 2016)

Measure	Round 1	Round 2	Round 3
Algebra Basic Skills	.84	.88	.86
Algebra Foundations	.82	.86	.82
Algebra Content Analysis	.72	.81	.77

The alternate form reliability for CoV and for PR was not evaluated for these measures because of the structure of the data collection in this project year 2. Students took Form A of each CoV and PR at the beginning of the course and Form B of each of the measures at the end of the course. Thus there would be an expected difference in student scores across a course (Foegen & Dougherty, 2016).

Test-retest reliability. Test-retest reliabilities (Table 10) were calculated by correlating students' scores on the same form across Rounds 1 and 2. Strong test-retest reliability values imply that students obtain consistent scores on the same form of a measure across a brief time period. For all measures, the correlation based on the mean of the two forms from Round 1 to Round 2 resulted reliability coefficients that exceeded .80. This result suggests that the mean of two forms of a measure will produce the most reliable estimate of student performance (Foegen & Dougherty, 2016). As CoV and PR were only administered at the beginning and end of the course in year 2, there is no test-retest data available for these measures.

Table 10

Test/Retest Reliabilities for ABS, AF, and ACA Full Sample (Foegen & Dougherty, 2016)

Measures	Mean	Form 2	Form 3
Algebra Basic Skills	.91	.88	.86
Algebra Foundations	.90	.84	.86
Algebra Content Analysis	.84	.77	.78

Concurrent validity. Round 3 (end of the course) progress monitoring measures were correlated with the end of the year criterion measures (IEOC and FANC). Table 11 below contains the concurrent validity results. In addition to comparing the correlation between Forms 2 and 3 of Round 3 to the criterion measures, correlations between the mean of students' scores on Forms 2 and 3 (labeled as ABS, AF, and ACA in Table 11) were correlated with the criterion measures. Overall, correlations were weak for the conceptual measure and moderate for the procedural measures suggesting a closer relationship between the procedural and the criterion measures than with the conceptual and criterion measures (Foegen & Dougherty, 2016).

Predictive validity results. Table 11 below also contains the correlations coefficients between the measures taken at the beginning of the course and criterion measures taken at the end of the course. The correlation involving the average of two procedural forms with the criterion measures was also considered. Correlating the mean of two forms with the end criterion measures generally resulted in higher predictive validities than correlating an individual form with criterion measures. Similar to the concurrent validities, the predictive validities were also weak to moderate suggesting that the conceptual measures have less power to predict performance on the criterion measures than the procedural measures (Foegen & Dougherty, 2016).

Table 11

Predictive and Concurrent Validity (Foegen & Dougherty, 2016)

Measure 1	Measure 2	Predictive Validity	Concurrent Validity
CoV_A	IEOC	.31	.42
PR_A	IEOC	.18	.13
CoV_A	FANC	.33	.44
PR_A	FANC	.24	.22
ABS 2	IEOC	.43	.48
ABS 3	IEOC	.42	.46
ABS	IEOC	.45	.48
AF 2	IEOC	.50	.54
AF 3	IEOC	.52	.61
AF	IEOC	.53	.60
ACA 2	IEOC	.52	.59
ACA 3	IEOC	.52	.61
ACA	IEOC	.56	.63
ABS 2	FANC	.46	.54
ABS 3	FANC	.49	.49
ABS	FANC	.50	.54
AF 2	FANC	.49	.55
AF 3	FANC	.54	.60
AF	FANC	.53	.60
ACA 2	FANC	.44	.56
ACA 3	FANC	.42	.57
ACA	FANC	.47	.59

Setting and Participants ASPM Project Year 3 (2013–2014)

Students in grades 9 through 12 who were enrolled in an Algebra 1 course or a course with similar curriculum comprised the target population for the ASPM project. This includes students enrolled in a variety of courses including Algebra 1 taught as a year-long course, Algebra 1 taught as a semester course, Algebra 1A, and Algebra 1B. In the Algebra 1A and 1B courses, the Algebra 1 curriculum is taught over the course of two years. The Algebra 1 taught as a semester course occurred in a semester block format with 400 minutes of instruction weekly, resulting in approximately the same amount of instruction time as the year-long Algebra 1 course. Research sites for the project in the third year included schools in Mississippi, Iowa and Missouri. Table 12 provides demographics for the schools that served as research sites for ASPM in the 2013–2014 school year. Approximate school enrollments are provided to protect the anonymity of the schools participating in the study.

Table 12

School Level Data

State	School	Number of Teachers involved with Project	Approximate School Enrollment	Percent of Students on Free/Reduced Lunch
Iowa	A	2	1400	50
	B	3	1400	66
	C	2	1200	71
Missouri	D	1	<100	61
	E	2	1600	42
	F	4	900	77
	G	1	1500	61
Mississippi	H	4	1600	52*
	J	3	900	52*

* District Free/Reduced Lunch rate, as school level data were not available

After IRB processes were followed and approval attained, consent was requested of students and parents to collect demographic and other school record data. Out of 1109 students, consent was obtained for 52% of the students in the participating classes. Demographic data for those students for whom consent was obtained included student age, IEP status, free/reduced lunch status, gender, ethnicity, course grade, and state test score for the end-of-course assessment as required by the corresponding state. Of the students for whom consent was received, 48% were male and 52% were female. Table 13 provides ethnicity, grade level, and IEP and English as a Second Language (ESL) status demographics of the students for whom consent was obtained.

Table 13

Demographic Data

	Grade					
Ethnicity	9	10	11	12	Total	%
Asian	4	2	2	0	8	2
Black	64	14	0	0	78	15
Hispanic	53	22	4	1	80	15
Indian	0	0	3	0	3	<1
Multiple Races	11	3	0	0	14	3
Pacific Islander	1	0	0	0	1	<1
White	296	45	5	0	346	65
IEP Status						
Yes	37	14	2	0	53	14
No	272	55	4	0	331	86
ESL Status						
Yes	89	18	3	0	110	24
No	297	54	2	0	353	76

Procedures ASPM Project Year 3 (2013-2014)

Because analysis focused on students' responses from the 2013–2014 school year (project year 3) on two forms of the ASPM CoV measure, a more detailed description of the activities and procedures of the ASPM project associated with the CoV measure is provided. Data collected from all students in participating teachers' classes included scores on the six ASPM measures and student scores on IEOC and FANC. In the 2013–2014 school year, CoV was used as a screening measure and two forms of the CoV measure were each administered three times to all students involved in the project. The first administration served as a baseline for students' performance and occurred at the beginning of the year (or the semester for semester-long courses). The second administration followed within two weeks of the first administration and served to gauge test-retest and alternate form reliability of students' scores on the measure. The final administration occurred at the end of the course. At each administration, all students took both CoV Form A and CoV Form B. In addition to these administrations of CoV, all students also took the IEOC and FANC at the end of the course.

Classroom teachers administered all assessments using standardized administration protocols. Teachers had previously received training on the administration of the measures using the protocols. Students' responses to IEOC and FANC items were recorded on scantron sheets while their responses to the items on the project-created assessments were recorded on the students' copy of the assessment. Students' CoV measures were scored by members of the ASPM project team in order to develop and refine the scoring guides. Reliability in scoring was obtained by scoring each class set of student measures by at least two project team members until class sets of measures were scored with more than 95% agreement for multiple class sets. Once reliability at this level had been obtained, project team members scored the remainder of the student probes independently and reliability was checked for 20% of the class sets.

The Current Study

In an effort to understand the differences among MLD, LMA, and TMA students' conceptions, uses, and misconceptions of variables, I analyzed students' responses to items on the CoV Form A and Form B measures collected by the ASPM project in the third year of that project. This section describes the data that were selected for analysis in this study and the methods of analysis of the data.

Data Selection

Eight items from the CoV measures (four items from Form A and the corresponding four items from Form B) provide insight into students' comparisons of expressions with generalized quantities. As cited in the literature review, students' comparison of expressions involving variables can provide valuable insight into the possible concepts variable that students hold (MacGregor & Stacey, 1997). These items are reported in Table 14 below. Each item is given a label that describes the form of the CoV measure it came from as well as its position on the form for ease of reference. For example, item CoV_B19 is the 19th item on the Form B of the CoV measure. Items CoV_A09, CoV_B19, CoV_A15, and CoV_B07 each have a multiple-choice component and an open-ended component. When these components are referred to separately from each other, each component will be identified as item CoV_A09.1 and CoV_A09.2 respectively.

These items were selected by first identifying all items on the CoV measure in which students compared expressions that involved variables. Of the 16 items identified (eight items on Form A and the corresponding eight items on Form B), six were eliminated because students' explanations almost exclusively focused on operations rather than the comparison of the two expressions with variables. For example, one of these items asked students to determine if $3x - 6y = -3(2y - x)$ was *Always, Sometimes,* or *Never true* and to explain their answer. Most students who responded to this item focused on the multiplication of -3 and $2y$ and -3 and $-x$ without mentioning anything about comparing the expressions or values of the variables. While this could be interpreted as students using variables as arbitrary symbols or objects (Usiskin, 1988), I felt that there was not enough evidence to justify any inference about students' conceptions of the variables.

Table 14

CoV Items Analyzed

Label	Item
CoV_A01	Jon said, "m – 1 is always greater than 1 – m." Do you agree with Jon? A. Yes, Jon is correct because m is a positive number. B. Yes, Jon is correct because you cannot substitute a negative number for m. C. No, Jon is not correct because 1 – m is greater than m – 1 when m is negative integer. D. No, Jon is not correct because these expressions are equivalent.
CoV_B05	Mari said, "2t is always greater than t + 2." Do you agree with Mari? A. Yes, because multiplication always gives you a larger answer than addition. B. Yes, because t is a positive number. C. No, because multiplication is not the inverse of addition. D. No, because it is possible that 2t can be equal to or less than t + 2.
CoV_A09	Bart said, "t + 3 is less than 5 + t." Circle one: Always true, Sometimes true, Never true. Explain your answer
CoV_B19	Anne said, "m + (–7) is less than –5 + m." Circle one: Always true, Sometimes true, Never true. Explain your answer
CoV_A15	$B + E + M = B + R + S$ Circle one: Always true, Sometimes true, Never true. Explain your answer.
CoV_B07	$B + E + M = M + B + E$ Circle one: Always true, Sometimes true, Never true. Explain your answer.
CoV_A17	Sam said, "2n + 1 represents a greater quantity than 2(n + 1)." Do you agree with Sam? A. Disagree, the two expressions are equivalent because they have the same numbers and variable. B. Disagree, the distributive property produces a constant of 2 in the second expression as opposed to 1 in the first expression, and the variable has a coefficient of 2 in both expressions. C. Agree, 2n + 1 is greater than 2(n + 1) because the 2 is multiplying the n in the first expression, not the 1. The value of n could be greater than 1. D. You cannot tell which is larger because it depends on the value of n.
CoV_B02	Char said, "3n – 1 represents a greater quantity than 3(n – 1)." Do you agree with Char? A. Disagree, the two expressions are equivalent because they have the same numbers and variable. B. Disagree, 3n – 1 is less than 3(n – 1) because the 3 is multiplying the n in the first expression, not the 1. The value of n could be less than 1. C. Agree, the distributive property produces a constant of –3 in the second expression as opposed to –1 in the first expression, and the variable has a coefficient of 3 in both expressions. D. You cannot tell which is larger because it depends on the value of n.

As students were primarily explaining how to us the distributive property to multiply and not referencing the variables as quantities, there was inconclusive evidence of students' ideas about variables.

Two other items were also eliminated due to students' focus on the operations of the items. One of these items asked students to compare expressions with exponents and the other asked students to compare rational expressions. In both cases, student responses provided more evidence of misconceptions about the notation of the expressions than of uses of variables in expressions. For example, when asked to compare g^3 and $3g$, most students' responses focused on the exponential notation meaning to use g as a factor three times. Similar to the other eliminated items, I felt that these responses did not provide sufficient evidence of a particular use of variables in comparing these expressions.

In addition to the items that were chosen from the CoV measure, students' IEOC score data were used to create the MLD, LMA, and TMA groups. Students were assigned to one of these groups based on the national percentile rank that corresponded with their IEOC score. The group of students with scores that corresponded with or below the 10th percentile was labeled as the MLD group. Students' IEP status was not taken into consideration in creating these groups because of the low number of students for whom data was available on their IEP status (IEP status was known for about 35% of the students). Those with scores that corresponded with or below the 25th percentile but above the 10th percentile were labeled as the LMA group. Students with scores that corresponded to percentiles higher than the 25th percentile were labeled as the TMA group. Due to limitations of the IEOC data available, it was not possible to consider other cut-off scores to define these groups. The IEOC provided only the conversions between raw scores and percentiles for the 10th, 25th, 50th, 75th, and 90th percentiles. If other raw scores to percentile conversions were available, a buffer zone between these groups or different cut off values would have been investigated to better fit the characteristics of students.

The IEOC was chosen as the measure to group students into the MLD, LMA, and TMA groups for three reasons. First, the IEOC's items were created and revised using a quality-control processes including a multistage, multipurpose review for content and fairness; field testing with approximately 250 students; and use of item statistics and technical specifications in the item selection process. Second, the assessment was administered to over 3000 students in 2010 and scores were normed to provide percentile ranks. Raw scores on the IEOC of 9, 12, 15, 19 and 23 corresponded to percentile ranks of 10th, 25th, 50th, 75th and 90th respectively (Iowa Testing Programs, 2010). Third, the content of the IEOC is focused on the procedures and skills of algebra, which provided a contrast to the content of the CoV measure and is more aligned with the content of the assessments used in the MLD and LMA literature that are typically skill- and procedurally-focused (L. S. Fuchs et al., 2007; Murphy et al., 2007). Figure 2 provides

Figure 2

Items Similar to Items on the IEOC

Following the order of operations, what is the first step in evaluating the following expression? $10(3 - 6) \div 2 + 1$	Simplify the following expression completely. $(2xy)^3(x^0y^2)$
What is the value of $3x^2 + 10x$ when $x = 5$?	Factor the following expression completely. $x^4 + 7x^3 - 8x^2$

the stem of four multiple-choice items that are similar in format, but with different numbers than items on the IEOC as examples of the nature of these items.

It is important to establish the alignment with the assessments used in previous studies because by using a similar measure of student achievement, the validity of those findings increase for the groups of students that I have identified. Table 15 provides further details about the composition of the IEOC (Iowa Testing Programs, 2010). It is also important to note the differences between the IEOC, a procedural- or skill-focused measure and the CoV measure with a conceptual focus.

Table 15

Number and Description of IEOC Item Types (Iowa Testing Programs, 2010)

Domain	Number of Items	Descriptions of Items
Basic Algebra	11	Topics may include number systems and properties, variable expressions, equations, inequalities, and functions
Linear Equations and Inequalities	4	Topics may include equation of a line, parallel and perpendicular slopes, and graphs of lines and inequalities on the coordinate plane
Systems of Linear Equations and Inequalities	5	Topics may include systems of equations that can be solved by substitution, elimination, or graphing; and graphs of systems of equations and inequalities on the coordinate plane
Polynomials	10	Topics may include arithmetic with polynomials, simplification of polynomials, solving polynomials by factoring or the Quadratic Formula, and graphs of polynomials

Data Coding

The coding of the data was based on a framework that drew from multiple sources, as described in Chapter 1. By connecting the work of Blanton et al. (2015), MacGregor and Stacey (1997), and Vosniadou and Vamvakoussi (2007), I created a map of the conceptions of variable in Table 3 in Chapter 1 that are evident in misconceptions of variables. Student responses were coded as consistent with a specific conception or misconception of variable. As discussed previously, without empirical evidence, it cannot be known that these misconceptions are indicative of a specific level of the LT, but we can propose that they appear to be consistent with a LT level.

As described in the descriptions of the first two misconceptions in Table 3, differentiating between levels 3 and 4 of Blanton et al.'s learning trajectory (LT) is difficult in the context of this study. To remedy this difficulty, for the purposes of coding student responses reliably, I collapsed levels 3 and 4 into one code which I called level 3/4. This modification is reflected in Table 16.

I coded student responses to the four multiple-choice items (CoV_A01, CoV_A17, CoV_B05, and CoV_B02) by connecting each multiple-choice option to the conceptions and misconceptions of variable from Tables 3 and 16. Table 17 provides the codes from Tables 3 and 16 that were assigned to each of the multiple-choice options for these items. A code of LT X indicates that it has been identified as being consistent with level X of my modified version of Blanton et al.'s learning trajectory. After the LT code, a description of misconceptions is also provided.

Table 16

Modified (Blanton et al., 2015) Variable LT

Trajectory Levels	Characteristics
1. Pre-Variable/Pre-Symbolic	Does not recognize a variable quantity (unknown) in a mathematical situation; Does not use or accept the use of any symbolic notation (literal or non-literal) to represent a quantity.
2. Pre-Variable/Letter as Label or as Representing Object	Tacitly accepts the use of a literal symbol to represent something not known, but not an unknown mathematical quantity (hence, an indication that child does not yet recognize a variable quantity).
3/4. Letter as Representing Variable with Fixed Value	Recognizes a variable quantity and sees the variable as representing as a single fixed value; Value of the variable quantity can be deterministic or can be chosen randomly.
5. Letter as Representing Variable as Varying Unknown	Recognizes variable quantity as varying unknown and literal symbol as representing a varying, unknown quantity.
6. Letter as Representing Variable as Mathematical Object	Able to construct functional relationships that appropriately represent variables and constants; Able to act on variables, represented with literal symbols, as tools for representing generalized functional relationships.

Table 17

Multiple-Choice Items, Labels and Codes

Item	Codes
CoV_A01 Jon said, "m – 1 is always greater than 1 – m." Do you agree with Jon?	
A. Yes, Jon is correct because m is a positive number.	LT 3/4, Variable may only represent natural numbers or integers
B. Yes, Jon is correct because you cannot substitute a negative number for m.	LT 3/4, Variable may only represent natural numbers or integers
C. No, Jon is not correct because 1 – m is greater than m – 1 when m is a negative integer.	LT 5
D. No, Jon is not correct because these expressions are equivalent.	Arithmetic conventions overgeneralized
CoV_B05 Mari said, "2t is always greater than t + 2." Do you agree with Mari?	
A. Yes, because multiplication always gives you a larger answer than addition.	Arithmetic conventions overgeneralized
B. Yes, because t is a positive number.	LT 3/4, Variable may only represent natural numbers or integers
C. No, because multiplication is not the inverse of addition.	Misunderstanding of the relationship between addition and multiplication
D. No, because it is possible that 2t can be equal to or less than t + 2.	LT 5

Multiple-Choice Items, Labels and Codes Continued

Item	Codes
CoV_A17 Sam said, "$2n + 1$ represents a greater quantity than $2(n + 1)$." Do you agree with Sam?	
A. Disagree, the two expressions are equivalent because they have the same numbers and variable.	Disregards the relationship between quantities and focuses only on the structure of the problem
B. Disagree, the distributive property produces a constant of 2 in the second expression as opposed to 1 in the first expression, and the variable has a coefficient of 2 in both expressions.	LT 5
C. Agree, $2n + 1$ is greater than $2(n + 1)$ because the 2 is multiplying the n in the first expression, not the 1. The value of n could be greater than 1.	Misunderstanding of the distributive property
D. You cannot tell which is larger because it depends on the value of n.	LT 3/4, Unknown value prevents comparison
CoV_B02 Char said, "$3n – 1$ represents a greater quantity than $3(n – 1)$." Do you agree with Char?	
A. Disagree, the two expressions are equivalent because they have the same numbers and variable.	Disregards the relationship between quantities and focuses only on the structure of the problem
B. Disagree, $3n – 1$ is less than $3(n – 1)$ because the 3 is multiplying the n in the first expression, not the 1. The value of n could be less than 1.	Misunderstanding of the distributive property
C. Agree, the distributive property produces a constant of –3 in the second expression as opposed to –1 in the first expression, and the variable has a coefficient of 3 in both expressions.	LT 5
D. You cannot tell which is larger because it depends on the value of n.	LT 3/4, Unknown value prevents comparison

Items not assigned a LT code suggest misconceptions similar to the overgeneralization of arithmetic operations (as in Table 3), in which evidence to support the identification of a level of sophistication of a student's conception of variable is not provided because of the student's focus on the operation. For example, a response of C on item CoV_B05 provides no information as to how the student is thinking about variables because that response is focused entirely on a rule that a student has remembered about multiplication and addition. Because of this lack of attention to the variable, it does not make sense to talk about this response as providing information about the student's ideas about variables.

In order to substantiate the use of the LT in these and subsequent codes, the lead author of the LT was contacted and these connections were discussed and refined. The lead author agreed in general with the consistency of the student responses with the identified LT levels, but felt that the connections made were too strong. Based on this discussion and her recommendation, the connections that were made between item responses and LT levels were changed. Initially these responses were coded as suggesting that the students who gave those responses were at the indicated LT level. These connections were changed so that the responses were consistent with student thinking at the indicated learning trajectory level. This change was important because LTs must be empirically verified and since these responses were not empirically validated at these levels, we can only say that the responses are consistent with the LT levels.

I also coded student responses on items CoV_A09, CoV_A15, CoV_B07, and CoV_B19 through a multi-step process. I first randomly selected 58 of the explanations to develop my coding scheme. I wrote a description of the ways in with students used variables in these explanations for each student. Then I looked across these descriptions and I identified themes that were common in multiple students' responses. I connected these themes to the conceptions and misconceptions of variables identified in Tables 3 and 16. From connections, seven codes emerged for each open-ended item. Tables 18–20 provide the codes used for each of the open-ended portion of each of these four items as well as samples of students' responses that will be coded in each category. For each student response a code is connected to a level of the LT described in Table 16.

Table 18

Codes and Examples of Student Responses for Items CoV_A09 and CoV_B19

CoV_A09		CoV_B19	
Bart said, "*t* + 3 is less than 5 + *t*." Circle one: Always true, Sometimes true, Never true. Explain your answer		Anne said, "*m* + (–7) is less than –5 + *m*." Circle one: Always true, Sometimes true, Never true. Explain your answer	
Code	Example	Code	Example
1: Explicit variable as multiple values simultaneously, LT 5	"Because 3 is less than 5 no matter what '*t*' is"	1: Explicit variable as multiple values simultaneously, LT 5	"No matter what number, the *m* + (–7) will always be less"
2: Implied variable as multiple values simultaneously, LT 5	"Because 3 is a smaller number compared with 5"	2: Implied variable as multiple values simultaneously, LT 5	"B/c –7 is less than –5"
3: Substitution suggesting variable as a specific unknown value, LT 3/4	"*t* = 10 10 + 3 = 13"	3: Substitution suggesting variable as a specific unknown value, LT 3/4	"10 + (–7) < –5 + 10 because 3 is not less than 5"
4: Unknown value prevents comparison, LT 3/4	"Depends on what the variables stand for"	4: Unknown value prevents comparison, LT 3/4	"Depend on if there is a positive/negative integer"
5: The same variable may not be the same value, LT 3/4	"In t + 3 the *t* could be 10 and be greater than 5 + *t* if the *t* was 2"	5: The same variable may not be the same value, LT 3/4	"Because *t* could be different in *t* + 3 than in 5 + *t*"
6: Inconclusive, incorrect or does not address the question	"Depends on how you work the problem"	6: Inconclusive, incorrect or does not address the question	"–7 is different than –5"
7: No Explanation (did select always, sometimes or never)		7: No Explanation (did select always, sometimes or never)	

Table 19

Codes and Examples of Student Responses for Item CoV_A15

CoV_A15	
$B + E + M = B + R + S$ Circle one: Always true, Sometimes true, Never true. Explain your answer.	
Code	Example
1: Implies that each variable must have the same value in each instance and recognizes that different pairs of numbers can have the same sum or that multiple variables could have the same value, LT 5	“This is only true if $E + M$ has the same value as $R + S$.”
2: Implies that each variable has the same value in each instance and does not consider that different pairs of numbers can have the same sum or that different variables could have the same value, LT 5	“Never, because the variable are not the same”
3: Substitution suggesting variable as a specific unknown value, LT 3/4	“Because each letter means 1 so if they got 3 on each side then they are equal”
4: Unknown value prevents comparison, LT 3/4	“*R* is unknown”
5: The same variable may not be the same value, LT 3/4	“3 + 5 + 4 = 1 + 2 + 9”
6: Inconclusive, incorrect or does not address the question	“Because it doesn't matter what variable you use it is still a letter.”
7: No Explanation (did select always, sometimes or never)	

Table 20

Codes and Examples of Student Responses for Item CoV_B07

CoV_B07	
$B + E + M = M + B + E$ Circle one: Always true, Sometimes true, Never true. Explain your answer.	
Code	Example
1: Explicit variable as multiple values simultaneously by stating that for any value of the variable it is true, LT 5	"No matter what order the numbers are in, they'll always equal the same thing"
2a: Implied variable as multiple values simultaneously by stating that 1) each variable has the same value, 2) in addition the 3) order doesn't matter (needs all 3 of these requirements) LT 5	"No matter what order the numbers are when adding they'll always equal the same thing"
2b: Implied variable as multiple values simultaneously by stating that 1) each variable has the same value, 2) in addition the 3) order doesn't matter (answer contains 2 of these 3 requirements), LT 5	"Variables are all the same just in different order"
2c: Implied variable as multiple values simultaneously by stating that 1) each variable has the same value, 2) in addition the 3) order doesn't matter (answer contains 1 of these 3 requirements), LT 5	"Because the same letter on each side has to be same number"
3: Substitution suggesting variable as a specific unknown value, LT 3/4	"2 + 1 + 3 = 3 + 2 + 1"
4: Unknown value prevents comparison, LT 3/4	"Depends on the number"
5: The same variable may not be the same value, LT 3/4	"2 + 1 + 3 = 4 + 2 + 1"
6: Inconclusive, incorrect or does not address the question	"It is always the same because all numbers have to be equivalent to the variable"
7: No Explanation (did select always, sometimes or never)	

Similar to the multiple-choice items, because many different types of student responses could be consistent with the thinking of a student who is at a particular LT level, multiple codes were created that were connected to some of the LT levels to provide additional insight into the differences in student responses. For example, responses coded as 1 and

2a are both identified as being consistent with conceptions of variable at the LT level 5. But a response that explicitly states that this is true for all values of the variable provide stronger evidence of that LT level than a response that does not do this explicitly and thus is coded as 2a.

Before I began to code, I asked a fellow PhD student to code 20% with me to verify the reliability of my coding. This PhD student was well qualified to verify the reliability of the coding with a master's degree in mathematics and 2 years of experience working on the ASPM project. For each open-ended item in each round, at least 20% of the responses were randomly identified. The specific numbers of items coded are reported in Table 21.

Table 21

Number of Responses Coded

Item	Round	Total Responses	Responses Coded	Percent Agreement
CoV_A09	1	272	71	89
CoV_A09	3	246	65	85
CoV_B19	1	72	30	87
CoV_B19	3	78	30	90
CoV_A15	1	131	35	86
CoV_A15	3	152	40	83
CoV_B07	1	385	100	83
CoV_B07	3	356	99	81

After providing the PhD student with the code descriptions and examples of coded student work in Tables 17–20, we both coded this 20% sample of items independently. The a priori rate of agreement of our codes were set at 80%. After comparing our codes, I found that we had a range of 81% to 90% agreement on each of the 8 sets of student responses as shown in Table 21. After the reliability of the codes was established, I coded all of the explanations by reading each of the student responses and then assigning a code to it based on the code descriptions and examples of coded student work in Tables 17–20.

Data Analysis

I used a combination of descriptive and inferential statistics to compare the responses from each group of MLD, LMA, and TMA students with each other group to identify differences between groups of students for each item at each time point. I first determined if there was a difference in response between students in the MLD and LMA group on each item at each time point. I did this using a two-proportion *z*-test to determine if the proportion of students who chose each multiple-choice option (or who provided an explanation that was assigned a code) was different for students in the MLD and LMA groups. These analyses were used to answer research question 1.

For each multiple-choice item at each time point, I first tabulated how many students in each group selected each option. I then performed a chi-squared test for independence to determine if the multiple-choice option that a student selects was independent of the group that she is in for each item at each time point. For the chi-squared tests, only student responses were considered and the students who skipped the item were not. This was done in order to be able to look for differences across items. Because the items were placed in different sequences on each form and because it was a timed assessment, an item at the beginning of a measure would be expected to have a higher response rate than an item at the end of the measure. The chi-squared test assumes that the expected value for each cell is greater than 5 for at least 20% of the cells. Due to low response rates on two items, this threshold was not met. Therefore, two items, CoV_B19.2_R1 and CoV_A15.1, were omitted from the chi-squared analysis. In order to meet this threshold for a few other items, some response categories were pooled. These are noted in the discussion of the chi-squared results in Chapter 4.

Similarly, for each open-ended item, I first tabulated how many students in each group provided an explanation that was coded with each particular code. I then performed a chi-squared test for independence to determine if codes that have been given to students' explanations are independent of the group that the student was in for each item at each time point. As there are eight items and two time points each (Round 1 and Round 3), I have 16 comparisons of students' responses. These analyses using the chi-squared test for independence were used to answer research questions 2 and 3.

In addition to comparing differences in student responses among these groups of students, I also investigated the change in responses across time points for each group on each item. For each item, a two-proportions z-test was also conducted to compare the proportion of the student group who selected each response option or whose response was assigned each code in the first administration of the items to that same proportion in the final administration of the items. *Z*-tests were performed for the changes in the number of students who did not respond to each item. While this was not the case with the chi-squared tests, because the *z*-tests looked at the difference between the responses controlled for the varied item location on the measure. A 2 proportions *z*-test was chosen as the statistical test to compare the proportion of student responses across time provided because it was the best fit of the options considered, but was still not ideal. Because the data was a pre- and post-test format, with the same students taking each Round, a *t*-test was not appropriate because it assumes independence of the samples. A common fix for this is to perform a paired-samples *t*-test, but this also was not suitable for this data because of the high number of students who only took the Round 1 or Round 3 measures, but not both. The *z*-test assumes some knowledge of the population, which was not available. This problem was resolved by treating the whole sample as the population. This limits the generalizability of the results to a larger population, but it allows for some inferences to be made about this group of students. These analyses were used to answer research question 4.

Summary

Analysis of student responses to progress monitoring items designed to assess students' ability to compare expressions involving generalized quantities provides the basis for understanding the differences in students' response between MLD, LMA, and TMA groups of students. Data were drawn from the ASPM project year 3. These data furnished a picture of how students with MLD and LMA differ from their TMA peers. I identified relevant items from the ASPM project measures and coded student responses for levels of sophistication of conceptions and misconceptions about variable. Descriptive and inferential statistics were used to describe differences between MLD, LMA, and TMA students' responses on these items.

CHAPTER 4: FINDINGS

This chapter describes results obtained from implementation of the research design and analysis procedures described in the previous chapter. The goal of this research is to describe how students' responses to items designed to assess their ability to compare expressions involving generalized quantities at the beginning and end of an Algebra 1 course vary in relation to their performance on a standardized algebra assessment.

Using data collected as part of the *Algebra Screening and Progress Monitoring* (ASPM) project (Foegen & Dougherty, 2010), I analyzed students' responses to the items listed in Table 14 in Chapter 3 on a conceptual algebra progress monitoring measure to identify and describe differences among the mathematics learning disability (MLD), other non-MLD low mathematics achievement (LMA), and typical mathematics achievement (TMA) students. Student responses and multiple-choice options were coded based on the levels of sophistication of conceptions of variable from Blanton et al.'s (2015) learning trajectory (LT) and the misconceptions about variables and expressions with variables identified by MacGregor and Stacey (1997) and Christou et al. (2007). The codes I used are listed in Tables 17–20 in Chapter 3. These coded responses were then compared across the MLD, LMA and TMA groups using descriptive and inferential statistics.

The findings are organized by specific research question with a report of the descriptive and inferential statistics and with conclusions based on both sets statistics for each question. For each research question, the descriptive statistics will report the number of students being considered and the numbers of students for each achievement category.

The inferential statistics, either the chi-squared statistic or z-score, will be reported with their associated p values. Finally, conclusions that can be made based on the statistics provided will be presented.

Due to the large number of data tables that will be included in this chapter, a few general comments about these tables will be included here to avoid redundancy in the following sections. In many of the following tables, the column headings of CoV_A_R1 indicate the form and Round for each entry. Other sub-headings of MLD, LMA, MD, and TMA indicate the group of students. The row label of the symbol "#" refers to the total number of students who fall into each category. Significant results are noted. Cells with a value of 0 indicate that no students provided the indicated response or that there was no change, while the entry of 0.0 indicates that the value of the cell is less than 0.05 and was thus rounded down to 0.0. An entry of "-" in a cell indicates that the indicated code was not used for the indicated item or that the indicated test was not performed for the indicated item.

Research Question 1

Do responses differ between students' that are at or below the 10th percentile on the Iowa End of Course assessment (IEOC), and those of students who are between the 10th percentile and the 25th percentile on eight items designed to assess students' ability to compare expressions involving generalized quantities on a conceptual progress-monitoring tool at the beginning of an algebra course?

For this question, only data from students who were grouped into the MLD and LMA groups were considered. These student responses to the items and the results of a chi-squared test for independence were used to determine if the students' responses were independent of the group that she is in for each item at each time point. This required 22 chi-squared tests for independence as reported in Table 22 below, which provides the number of students in each sample, the chi-squared, and the degrees of freedom value for each test. In the following table and the other similar tables in this chapter, the row labels of A01, B05 are the question from the specified form. The questions that are grouped together for each row are done so because those items were designed to be comparable across the two forms of the measure.

Table 22

Results of Chi-Squared Tests for Independence of MLD and LMA groups

Item		CoV_A_R1	CoV_A_R3	CoV_B_R1	CoV_B_R3
A01, B05	χ^2	5.7693	3.2980	0.3988	6.0876
	df	4	4	3	3
	#	215	255	268	277
A09.1, B19.1	χ^2	1.2705	1.3317	0.7200	1.9288
	df	2	2	2	3
	#	150	166	72	94
A09.2, B19.2	χ^2	5.3119	7.4701	-	11.0996**
	df	5	4	-	4
	#	150	170	73	97
A15.1, B07.1	χ^2	-	1.4208	6.6892**	3.5113
	df	-	2	2	4
	#	78	101	203	198
A15.2, B07.2	χ^2	7.8443*	4.1679	6.6779	6.1279
	df	4	4	4	5
	#	79	100	206	199
A17, B02	χ^2	1.2364	5.1912	1.1202	2.8620
	df	3	3	3	3
	#	79	127	253	265

* $p < 0.1$, ** $p < 0.05$

Based on these results, there is very little support for the hypothesis that students' responses to these items are dependent on their membership in the MLD or LMA group. As noted in Table 22, students' responses were found to be independent of their group on all but 3 of 24 administrations of the items. Items CoV_B19.2_R3, CoV_B07.1_R1, and

CoV_A15.2 provide some evidence that there is a significant difference ($0.0255 < p < 0.0975$) between responses of students in the MLD and LMA groups on these three items only, but on the majority of the items, and therefore in general, there was not a significant difference. It is interesting to note that all of the items on which there was a significant difference are part of an open-ended item. CoV_A15.2 is significant in Round 1 with a *p*-value of 0.0975, while the other two significant items have a stronger significance (p = 0.0255 and $p = 0.0353$). Other than the open-ended format of the item, it is difficult to find similarities across these three items. The two items that were not included in the chi-squared analysis due to low numbers of responses could have provided some additional insight into commonalities as to the specific ways in which MLD and LMA students respond to these specific items, but without more significant data, it is difficult to say what these different areas could be. From a holistic perspective, looking across all of the items, the lack of significance on over 85% of the items provides very little evidence of differences between the responses of the MLD and LMA groups.

Changes from Round 1 to Round 3 administrations of each item were also compared to determine if there were significant changes in how students from each group responded to each item. These differences were identified by considering the significant changes in proportion of each group of students as described by a two proportions *z*-test. This *z*-test was performed for the proportion of the group (LMA, TMA, or MD) that selected a particular response on the first administration and the proportion of the same group who chose the same response on the final administration. For example, the percent of LMA students who responded to item CoV_A01 by selecting option A in Round 1 was compared with the percent of students in the LMA group who chose option A on

CoV_A01 in Round 3. Following this procedure for each option on each item and each group, I conducted 210 two-proportion *z*-tests. The results of these tests are summarized in Table 23.

The number in each cell of Table 23, and similar tables throughout this chapter, represents the difference between the percent of students in the indicated group who selected the indicated response in Round 3 and Round 1. A negative difference indicates that the percent of students in the indicated group who selected the given option in Round 3 was smaller than the percent of students in that same group who selected that option in Round 1. A row label of "Bl" or "Blank" refers to the students for whom the answer was left blank, or in other words, students who did not attempt the item. The row labels of A, B, C, and D refer to the corresponding multiple-choice option on the multiple-choice items. The row labels of A, S, and N refer to student responses of Always, Sometimes, and Never on the first part of the open response items. On the open-ended items, row labels of 1, 2, 2a, …, 6, and 7 refer to the code assigned from Tables 18–20 in Chapter 3. Bolded entries are the most correct responses of the responses that have been identified as being consistent with a conception of variable of a student and the level 5 of the LT. Response options or codes that are not bolded indicate some limitation in how students are thinking about the variable.

These results also provide little evidence that there are differences between the MLD and LMA groups. Looking across all of the items, responses of students in the MLD and/or students in the LMA group significantly changed on eight items.

Table 23

Change in Percent of Group Responses for MLD, LMA, and MD

Item		CoV_A			CoV_B		
		MLD	LMA	MD	MLD	LMA	MD
A01, B05	A	1.2	-8.5*	-3.7	-13.7**	-4.7	-9.1**
	B	6.7	3.4	5.0	4.0	1.9	2.9
	C	**-11.3***	**10.3***	**-0.3**	3.6	-5.2	-0.9
	D	0.9	1.3	1.1	**9.2***	**15.8*****	**12.5*****
	Bl	2.6	-6.5	-2.0	-3.0	-7.8**	-5.4**
A09.1, B19.1	A	**-3.2**	**1.1**	**-1.0**	**1.1**	**0.6**	**0.8**
	S	-4.6	-1.6	-3.1	3.8	2.5	3.1
	N	4.3	0.1	2.2	5.0	1.9	3.4
	Bl	3.5	0.4	1.9	-9.9**	-4.9	-7.4**
A09.2, B19.2	1	**2.6**	**-2.8**	**-0.2**	**0.0**	**0.6**	**0.3**
	2	**-1.1**	**1.3**	**0.2**	**-0.6**	**1.3**	**0.4**
	3	0.6	-1.1	-0.3	0.0	0.6	0.3
	4	-6.5	4.0	-1.1	1.4	-1.9	-0.2
	5	-0.1	2.0	0.9	1.3	0.6	1.0*
	6	-1.1	2.0	0.5	-1.6	-3.2	-2.4
	7	3.4	-5.4	-1.1	9.8**	8.2**	8.9***
	Bl	2.2	0.1	1.1	-10.4**	-6.2	-8.2**
A15.1, B07.1	A	4.0	1.6	2.8	**5.6**	**-3.8**	**1.0**
	S	**-0.6**	**1.6**	**0.5**	-7.8**	-1.9	-4.9
	N	1.0	0.3	0.6	5.4*	1.3	3.3*
	Bl	-4.4	-3.5	-4.0	-3.2	4.5	0.6
A15.2, B07.2	1	**-1.9**	**2.4**	**0.3**	**0**	**0**	**0**
	2	**3.7**	**-0.7**	**1.4**	-	-	-
	2a	-	-	-	**0.7**	**1.3**	**1.0**
	2b	-	-	-	**1.1**	**-4.5**	**-1.7**
	2c	-	-	-	**-3.9**	**-3.9**	**-3.9**
	3	-0.8	0.0	-0.4	-0.6	3.2**	1.3*
	4	1.4	0.0	0.7	-0.6	2.5	1.0
	5	0	0	0.3	0.1	0.0	0.0
	6	-3.1	3.7	0.7	-0.7	-7.1*	-3.9
	7	4.2	-2.7	1.4	7.1	3.7	5.3
	Bl	-3.5	-2.7	-3.1	-3.2	4.9	0.8
A_17, B02	A	2.2	8.8**	5.6**	5.0	-3.3	0.8
	B	**6.7**	**4.7**	**5.7***	6.4	3.0	4.6
	C	5.9	-3.4	1.2	**-0.8**	**1.7**	**0.5**
	D	-1.4	1.6	0.2	-3.1	3.8	0.3
	Bl	-13.5**	-11.7*	-12.6***	-7.4*	-5.2	-6.2**

* $p < 0.1$, ** $p < 0.05$, ***$p < 0.01$

Six of these eight items (CoV_B05, CoV_B19.1, CoV_B19.2, CoV_B7.1, CoV_B7.2, CoV_A17, and CoV_B02) did not provide evidence of differences between these groups because responses to items that changed significantly either both increased or both decreased for each group. For example, on item CoV_B_05, responses of choice A and no response from students in both the MLD and LMA groups decreased from Round 1 to Round 3 while the choice of option D significantly increased for both groups. Because the percent of each group selecting each item with significant change increased or decreased at the same time on these six items, these items provide no evidence of difference between the MLD and LMA groups. In other words, in general, the same significant changes occurred for both groups on these six items. Also, the groups were the same in that there were not significant changes on most of the rest of the items.

There were two items that did suggest some difference in the changes in student responses between the MLD and LMA groups. On item CoV_B07.2, the percent of LMA students who responses were coded as a 3 and 6 (see Table 20) changed significantly from Round 1 to Round 3 but these significant changes were not mirrored in the MLD group. From Table 20, code 3 on CoV_B07.2 indicates that the student's justification was based on the substitution of a single value for the variable, which is consistent with a LT level 3/4. Also a code of 6 indicates that the student's explanation was inconclusive, incorrect or does not address the question. The percent of LMA students who responded with code 6 decreased and with code 3 increased. This suggests that the change in the responses of LMA was progress, from an incomplete or invalid response to one that is consistent with a LT level 3/4 conception of variable. While an increase of responses consistent with a LT level 5 conception of variable would have shown greater growth in the level of sophistication of students' conception of variable, this change is still movement in the positive direction. On the other hand, the MLD students on this administration of this item did not demonstrate any significant evidence of growth in the sophistication of their conceptions of variable on this item, suggesting the possibility of a lower growth rate for the MLD than the LMA group, although this will have to be tested.

The other item that provides some evidence of a difference between the responses of students in the MLD and LMA was CoV_A01. While these differences were significant ($0.0693 < p < 0.0560$), the percentage of MLD students that chose C decreased significantly while the percentage of LMA who chose C increased significantly from Round 1 to Round 3.

Overall, there is little evidence that there are differences in the responses of students in the MLD and LMA groups. Students' responses were not different enough to claim that the MLD and LMA groups were not independent of each other on three of the 24 items administered. Also, the change in percent of each group' responses only suggested a difference on one of the items. Because of this lack of evidenced of a difference between these groups, students in the MLD and MLA groups will be combined into one MD group. This may result in the loss of some information, as in CoV_A01 in Table 23 were both the MLD and MLA groups had a significant change in percent of student responses but not significant change is indicated in the MD group because these differences offset each other. Despite this possible loss of information, there is not enough evidence to suggest that there is a difference in level of sophistication of students' conceptions of variables between these two groups.

Research Question 2

How do responses on these items differ among the MLD, LMA and TMA groups of students at the beginning of an algebra course?

Student responses to the 12 items differed on some items and had similarities on other items at the beginning of their algebra course. Table 24 provides the results of 10 chi-squared tests for independence results for the MD and TMA groups for responses to the first administration (Round 1).

Table 24

Results of Chi-Squared Tests for Independence of MD and TMA groups in Round 1

Item		CoV_A_R1	CoV_B_R1
A01, B05	χ^2	5.1104	30.0060****
	df	4	3
	#	519	612
A09.1, B19.1	χ^2	3.7374	0.3136
	df	2	2
	#	395	134
A09.2, B19.2	χ^2	8.5662	-
	df	5	-
	#	397	137
A15.1, B07.1	χ^2	-	32.3348***
	df	-	3
	#	203	520
A15.2, B07.2	χ^2	16.7485***	13.0642**
	df	4	4
	#	205	523
A17, B02	χ^2	6.6108*	4.1681
	df	3	3
	#	170	550

* $p < 0.1$, ** $p < 0.05$, ***$p < 0.01$, ****$p<0.001$

Two items, CoV_A15.1 and CoV_B19.2 were omitted from the chi-squared test due to the lower response rates and low frequencies of a few of the codes for that item leading to expected frequencies that were too low for the test to produce valid results (more than 20% of the cells had expected values of less than 5).

Of the 10 items analyzed with the chi-squared test, only five items (CoV_B05, CoV_B07.1, CoV_A15.2, CoV_B07.2, and CoV_A17) showed significant differences ($p < 0.0854$) between the responses of MD and TMA students. These five items will be discussed individually to illuminate the differences between the groups of students. The remaining items will then be discussed to identify similarities among the groups of MD and TMA students. Table 25 presents the percent of each group that gave each response to each item. These data are used in the subsequent discussions. When discussing a particular question, I zoom in on Table 25 by breaking out a particular item's results. For example, the portion of Table 25 that contains the data for item CoV_A01 would be Table 25.1, CoV_B05 would be Table 25.2, CoV_A09.1 would be Table 25.3, etc.

Table 25

Percent of Group Who Responded (Resp.) and Percent of Total Group in Round 1

		CoV_A				CoV_B			
		MD		TMA		MD		TMA	
Item		Resp.	Total	Resp.	Total	Resp.	Total	Resp.	Total
A01, B05	A	22.3	19.6	20.7	16.3	50.4	42.6	52.9	45.4
	B	16.7	14.7	14.1	11.1	10.8	9.1	3.5	3.0
	C	**37.7**	**33.1**	**34.2**	**26.9**	17.2	14.5	8.7	7.5
	D	23.3	20.4	30.9	24.3	**21.6**	**18.3**	**34.9**	**29.9**
	Bl	-	12.2	-	21.4	-	15.5	-	14.2
	#	215	245	304	387	268	317	344	401
A09.1, B19.1	A	**36.7**	**22.4**	**46.5**	**29.4**	**34.7**	**7.9**	**38.7**	**5.9**
	S	50.0	30.5	41.6	26.3	34.7	7.9	30.6	4.7
	N	13.3	8.1	11.8	7.5	30.6	6.9	30.6	4.7
	Bl	-	39.0	-	36.9	-	77.4	-	84.7
	#	150	246	245	388	72	318	62	405

Percent of Group Who Responded (Resp.) and Percent of Total Group in Round 1 Continued

		CoV_A				CoV_B			
		MD		TMA		MD		TMA	
Item		Resp.	Total	Resp.	Total	Resp.	Total	Resp.	Total
A09.2, B19.2	1	**6.7**	**4.1**	**10.5**	**6.7**	**0.0**	**0.0**	**0.0**	**0.5**
	2	**15.3**	**9.3**	**13.4**	**8.5**	**8.2**	**1.9**	**8.2**	**1.5**
	3	3.3	2.0	4.9	3.1	1.4	0.3	1.4	0.0
	4	22.7	13.8	27.9	17.8	15.1	3.5	15.1	1.2
	5	1.3	0.8	3.2	2.1	0.0	0.0	0.0	0.2
	6	11.3	6.9	13.0	8.2	28.8	6.6	28.8	4.4
	7	39.3	24.0	27.1	17.3	46.6	10.7	46.6	7.9
	Bl	-	39.0	-	36.3	-	77.0	-	84.2
	#	150	246	247	388	73	318	64	405
A15.1, B07.1	A	9.0	2.8	0.8	0.3	**59.8**	**38.4**	**82.3**	**64.4**
	S	**38.5**	**12.2**	**36.8**	**11.9**	33.3	21.4	13.9	10.9
	N	52.6	16.7	62.4	20.1	6.9	4.4	3.8	3.0
	Bl	-	68.3	-	67.8	-	35.8	-	21.8
	#	78	246	125	388	204	318	316	404
A15.2, B07.2	1	**8.9**	**2.8**	**18.3**	**5.9**	-	-	-	-
	2	**24.1**	**7.7**	**43.7**	**14.2**	-	-	-	-
	2a	-	-	-	-	**1.5**	**0.9**	**1.9**	**1.5**
	2b	-	-	-	-	**18.0**	**11.6**	**22.7**	**17.8**
	2c	-	-	-	-	**22.3**	**14.5**	**32.8**	**25.7**
	3	1.3	0.4	1.6	0.5	0.5	0.3	1.3	1.0
	4	0.0	0.0	0.8	0.3	1.9	1.3	0.6	0.5
	5	-	-	-		1.5	0.9	0.9	0.7
	6	16.5	5.3	6.3	2.1	20.9	13.5	17.0	13.3
	7	49.4	15.9	29.4	9.5	33.5	21.7	22.7	17.8
	Bl	-	67.9	-	67.5	-	35.2	-	21.7
	#	79	246	126	388	206	318	317	405
A17, B02	A	17.7	5.7	22.0	5.2	20.2	16.1	18.5	13.6
	B	**31.6**	**10.2**	**46.2**	**10.8**	31.2	24.9	24.9	18.4
	C	38.0	12.2	22.0	5.2	**34.4**	**27.4**	**38.0**	**28.0**
	D	12.7	4.1	9.9	2.3	14.2	11.4	18.5	13.6
	Bl	-	67.9	-	76.5	-	20.2	-	26.3
	#	79	246	91	338	253	317	297	403

In Table 25 and the other tables that are similar to it in this chapter, the column headings of Resp. and Total refer to the percent of students from the indicated group who provided a response to each item and the percent of the total number of students completed any item on the CoV measure. For example, while there were 215 MD students who provided a response to CoV_A01, there were a total of 245 MD students who responded to at least one item on the CoV_A measure.

Differences

In general, the differences observed on each item that resulted in significant differences between the response patterns of MD and TMA students were not surprising. Students in the TMA group more often provided responses consistent with LT level 5 conceptions of variable, while MD students more frequently provided responses that corresponded to lower LT levels or skipped the item. Specific discussion for each item that provided significant chi-squared results are now presented.

CoV_B05. Recall from Table 14 that item CoV_B05 is:

Mari said, "2t is always greater than $t + 2$." Do you agree with Mari?
A. Yes, because multiplication always gives you a larger answer than addition.
B. Yes, because t is a positive number.
C. No, because multiplication is not the inverse of addition.
D. No, because it is possible that 2t can be equal to or less than $t + 2$.

A chi-squared test showed that the groups MD and TMA are not independent of the response given at $p < 0.001$ on the item CoV_B05 in Round 1. Table 25.2 provides insight into the differences among the MD and TMA group responses.

While the chi-squared test rejected the hypothesis that the responses of these two groups were independent of their group membership, it is interesting to note that for both groups the most common response was choice A, which was an overgeneralization of the arithmetic conventions. Also, both groups had a similar proportion that chose not to respond to this item. This leaves the difference between the MD and MLD groups to be among the distribution of the responses of B, C, and D. A larger proportion of MD students choose B or C than students from the TMA group. Choice B suggests that students have a limited view of what values the variable can take on, specifically that the variable can only represent positive numbers. Choice C suggests that students may not have a solid understanding of the relationship between addition and multiplication.

Table 25.2

Percent of Group who Responded and Percent of Total Group in Round 1 on Item CoV_B05

	CoV_B05			
	MD		TMA	
Item Option	Responded	Total	Responded	Total
A	50.4	42.6	52.9	45.4
B	10.8	9.1	3.5	3.0
C	17.2	14.5	8.7	7.5
D	**21.6**	**18.3**	**34.9**	**29.9**
Blank	-	15.5	-	14.2
#	268	317	344	401

Choice D was selected by a higher proportion of students in the TMA group than in the MD group. This suggests that more of the TMA students are thinking about variables in a way that is consistent with a LT level 5 conception of variable. These students are not distracted by overgeneralizations of the operations, but instead are able to focus on the variable as a quantity that represents multiple values.

CoV_B07.1 and CoV_B7.2. Recall from Table 14 that item CoV_B07 is:

$$B + E + M = M + B + E$$

Circle one: Always true Sometimes true Never true.
Explain your answer.

A chi-squared test showed that the groups MD and TMA are also not independent of the response given at $p < 0.001$ on the item CoV_B07.1 in Round 1. Table 25.8 illustrates the differences among the MD and TMA group responses.

While the proportion of both the MD and TMA students who chose Always True was higher than any other response, there are still substantial differences in these groups' response patterns. A much higher proportion of MD students chose to skip this item or chose the response of Sometimes True than the proportion of TMA students who made the same selection. It is interesting to note that although the highest proportion of both the MD and TMA students responded with the most correct answer the difference on the proportions of each group substantially contributed to the statistically significant difference in how the two groups responded on this item. These differences are also seen in how the groups of students responded to the explanation portion of the item, CoV_B07.2.

Table 25.8

Percent of Group Who Responded and Percent of Total Group in Round 1 on Item CoV_B07.1

	CoV_B07.1			
	MD		TMA	
Item Option	Responded	Total	Responded	Total
A	**59.8**	**38.4**	**82.3**	**64.4**
S	33.3	21.4	13.9	10.9
N	6.9	4.4	3.8	3.0
Blank	-	35.8	-	21.8
#	204	318	316	404

A chi-squared test showed that the groups MD and TMA are not independent of the response given at $p = 0.0110$ on the item CoV_B07.2 in Round 1 provides the evidence that students from the MD and TMA groups responded differently on this item.

In order to have less then 20% of the cells with an expected value of less than 5, some codes were pooled to form larger groups. Specifically, responses that were coded as 2a and 2b were combined into one column that represented codes in which students' responses implied that variables could have multiple values simultaneously by stating at least 2 of the following statements: each variable has the same value, it was addition, and the order does not matter. Also codes 3, 4, and 5 were combined into one column to represent student responses that suggest some form of a limited conception of variable and/or a misconception about variables. Table 25.10 provides the proportions of the MD and TMA group for each individual code.

Table 25.10

Percent of Group Who Responded and Percent of Total Group in Round 1 on Item CoV_B07.2

Response Code	CoV_B07.2			
	MD		TMA	
	Responded	Total	Responded	Total
2a	**1.5**	**0.9**	**1.9**	**1.5**
2b	**18.0**	**11.6**	**22.7**	**17.8**
2c	**22.3**	**14.5**	**32.8**	**25.7**
3	0.5	0.3	1.3	1.0
4	1.9	1.3	0.6	0.5
5	1.5	0.9	0.9	0.7
6	20.9	13.5	17.0	13.3
7	33.5	21.7	22.7	17.8
Blank	-	35.2	-	21.7
#	206	318	317	405

From the chi-squared tests, we can identify the cells that contributed the most to the significant result in favor of the hypothesis that the MD and MLD groups' responses were statistically different from each other. This is done by identifying the largest chi-squared standardized residual values, which represent the largest differences from the values expected under the null hypothesis that the responses of students in the MD and MLD groups are independent of their group membership. For this item, the largest standardized residuals are found for the cells of Table 26.10 representing both MD and TMA students whose responses were coded as 2c and 7. These standardized residuals, along with the percentages reported in Table 25.10, suggest that the more frequent coding of TMA students' responses as 2c than MD students' responses as 2c and the more frequent skipping of the explanation after choosing *Always, Sometimes,* or *Never* of the MD students than the TMA students were major contributors to the statistical difference between these two groups.

Table 26.10

Chi-Squared Standardized Residual Values for Item CoV_B07.2 Round 1

Group	2a, 2b	2c	3, 4, 5	6	7
MD	-1.3870	-2.5886	0.6581	1.1037	2.7151
TMA	1.3870	2.5886	-0.6581	-1.1037	-2.7151

Despite these differences, there are some similarities that are worth noting. First, there were no student responses that were coded as a 1, indicating that no students explicitly stated this was true for all values of the variables. While this type of response would have provided the strongest evidence of a conception of variable that is consistent with the thinking of a student at the level 5 of the LT, it may not be surprising that students did not explicitly state this because of the lack of attention to conceptions of variable as multiple values these classrooms, as observed by ASPM project personnel during classroom observations (B. Dougherty, personal communication, January 27, 2015). Second, very few of the students' responses in either group were coded as a 3, 4, or 5 for this item (2.5% and 2.2% of the total number of MD and TMA respectively), which suggests some limitation in how students are thinking about the variable. Finally, about half (56.9% of MD and 49.5% of TMA) of the students in both the TMA and MD groups did not attempt an explanation (by either skipping the item altogether or skipping the explanation part of the item).

CoV_A15.2. Recall from Table 14 that item CoV_A15 is:

$$B + E + M = B + R + S$$

Circle one: Always true Sometimes true Never true.
Explain your answer.

A chi-squared test showed that the groups MD and TMA are not independent of the response given at $p = 0.022$ on the item CoV_A15.2 in Round 1. In order to have less than 20% of the cells with an expected value of less than 5, some codes were pooled to form larger groups. Specifically, codes as 3, 4, and 6 were pooled. This column represented student responses that suggest some form of a limited conception of variable, a misconception about variables or other error in their response. Similarly to Table 25.10, Table 25.9 provides the proportions of the MD and TMA group for each individual code.

Table 25.9

Percent of Group Who Responded and Percent of Total Group in Round 1 on Item CoV_A15.2

Response Code	CoV_A_15.2			
	MD		TMA	
	Responded	Total	Responded	Total
1	**8.9**	**2.8**	**18.3**	**5.9**
2	**24.1**	**7.7**	**43.7**	**14.2**
3	1.3	0.4	1.6	0.5
4	0.0	0.0	0.8	0.3
5	0	0	0	0
6	16.5	5.3	6.3	2.1
7	49.4	15.9	29.4	9.5
Blank	-	67.9	-	67.5
#	79	246	126	388

The largest chi-squared standardized residual values for this item are found for the cells in Table 26.9 representing both MD and TMA students whose responses were coded as 2 and 7.

Table 26.9

Chi-Squared Standardized Residual Values for Item CoV_B15.2.2 Round 1

Group	1	2	3, 4, 6	6	7
MD	-1.5987	-2.9927	1.8492	0.4757	2.8857
TMA	1.5987	2.9927	-1.8492	-0.4757	-2.8857

These standardized residuals, along with the percentages reported in Table 25.9, suggest that the more frequent coding of TMA students' responses as 2 than MD students' responses as 2 and the more frequent skipping of the explanation after choosing *Always, Sometimes,* or *Never* of the MD students than the TMA students were major contributors to the statistical difference between these two groups.

Interestingly, there were no student responses that were coded as a 5, indicating that no students provided evidence of the misconceptions that two different instances of the same variable could be defined as two different values simultaneously. Also, over three-fourths (83.8% of MD and 77% of TMA) of the students in both the TMA and MD groups did not attempt an explanation (by either skipping the item altogether or skipping the explanation part of the item). While this number is higher than for CoV_B07.2, which could be explained by this item coming later on this timed measure (15 out of 19 versus 7 out of 19 on CoV_B), this represents an alarming number of students who did not even attempt an explanation.

CoV_A17. Recall from Table 14 that item CoV_A17 is:

Sam said, "$2n + 1$ represents a greater quantity than $2(n + 1)$." Do you agree with Sam?

A. Disagree, the two expressions are equivalent because they have the same numbers and variable.
B. Disagree, the distributive property produces a constant of 2 in the second expression as opposed to 1 in the first expression, and the variable has a coefficient of 2 in both expressions.
C. Agree, $2n + 1$ is greater than $2(n + 1)$ because the 2 is multiplying the n in the first expression, not the 1. The value of n could be greater than 1.
D. You cannot tell which is larger because it depends on the value of n.

A chi-squared test showed that the groups MD and TMA are not independent of the response given at $p = 0.0854$ on the item CoV_A17 in Round 1. While this is a weaker statement than the other significant chi-squared results, it is worth discussing because it follows a similar pattern (see Table 25.11) as the other significant Round 1 items.

Table 25.11

Percent of Group Who Responded and Percent of Total Group in Round 1 on Item CoV_A17

	CoV_A17			
	MD		TMA	
Item Option	Responded	Total	Responded	Total
A	17.7	5.7	22.0	5.2
B	**31.6**	**10.2**	**46.2**	**10.8**
C	38.0	12.2	22.0	5.2
D	12.7	4.1	9.9	2.3
Blank	-	67.9	-	76.5
#	79	246	91	338

While the chi-squared test results support the hypothesis that there is a difference between the responses of the MD and TMA students, there are some similarities that are worth noting. Specifically, the proportion of MD students who responded A and C was similar to the proportion of TMA students who made the same choice. Also, as with the previous two items, there are a high number of students from both groups who did not attempt the item. Despite this similarity, the fact that a larger proportion of TMA students did not respond than of the MD students does break from the patterns seen of the previous items.

The largest chi-squared standardized residual values for this item are found for the cells in Table 26.11 representing both MD and TMA students' responses of B and C.

Table 26.11

Chi-Squared Standardized Residual Values for Item CoV_A17 Round 1

Group	A	B	C	D
MD	-0.6920	-1.9307	2.2830	0.5713
TMA	0.6920	1.9307	-2.2830	-0.5713

These standardized residuals, along with the percentages reported in Table 25.11, suggest that the more frequent TMA students' responses of B than MD students' responses and the more frequent response of C by the MD students than the TMA students were major contributors to the statistical difference between these two groups.

Summary of differences. From the differences between the MD and MLD groups of students discussed in the previous sections, several common differences between these groups are evident. First, on all of the multiple-choice items (CoV_B05, CoV_B07.1, and CoV_A17) that showed a significant chi-squared result, a higher proportion of the students from the TMA group chose the most correct response than the MD students. This difference was also identified in each case as being among the largest contributors to the significant chi-squared result, suggesting that TMA students are more likely to choose the most correct answer than their MD counterparts on these items.

Interestingly, this pattern was slightly different in the explanation portion of the open-ended items (CoV_A15.2 and CoV_B07.2). For both of these items the differences in the second-to-highest category of responses (2 and the combination of 2b and 2c respectively) were important contributors to the significance of the chi-squared test. This varies from the multiple-choice responses in that the multiple-choice item responses saw the corresponding difference in the highest choice. Additionally, in the open-ended items, similar magnitude (but opposite in direction) differences were seen in the lower-end responses, specifically, the TMA students' lower response of codes 7 and 6 in items CoV_A15.2 and CoV_B07.2, respectively. This suggests that MD students are more likely to provide an explanation that is inconclusive, incorrect or does not address the question or to skip the item than their TMA counterparts across both types of items (open-ended and multiple-choice). For example, TMA students would more likely provide a response of 'Always True because in addition order doesn't matter' on item CoV_B07, while a MD student would be more likely to only circle Always True and not attempt an explanation. Implications of these results will be discussed in Chapter 5. These results of the MD students being more likely to respond with lower-level answer and TMA students being more likely to respond with higher-level response are not surprising.

Similarities

While the differences identified in the student responses to the items provided evidence that there is a significant difference between how TMA and MD students responded to these items were not surprising, it was surprising to see the lack of difference between the responses of these groups of students on the other items. On five of the ten items in Round 1 on which a chi-square test was preformed, there was not a significant result (CoV_A01, CoV_A09.1, CoV_A09.2, CoV_B19.1, and CoV_B02). These results fail to reject the hypothesis that there is not a difference in the responses of the students in the MD and MLD groups on these items.

In all but one of the items that did not show significance (CoV_A01, CoV_A09.1, CoV_A09.2, and CoV_B19.1), the item asked the students to compare two expressions of the form $x \pm a$ and $b \pm x$, where a and b are some integer. The other insignificant item had a similar format, comparing two expressions of $bx \pm a$ and $b(x \pm a)$, where a is some integer and b is a natural number. The lack of significance in the difference in the responses between these groups of students suggests that MD and TMA students may not respond differently on these types of items.

Summary of Research Question 2

In attempting to describe the differences between responses on these items differ among the MD and TMA groups of students at the beginning of an algebra course, identifying differences across all of the items is difficult. Half of the items analyzed with a chi-squared test suggest that there is a difference between the responses form the students in these groups, while the other half suggest no difference. Of the two items that were not analyzed with a chi-squared test, the descriptive statistics reported in Table 25 of CoV_A15.1 seem to support the hypothesis that there are differences between the responses of the MD and TMA groups in the differences in percentage of group selecting the choices of Always and Never. The corresponding statistics for CoV_B19.2 show very little difference between the percentage of each groups' responses that were assigned a particular code. Thus by the numbers of significant results, it cannot be said that there are differences in the responses of students from the MD and TMA groups that stretch across all of these items.

It is interesting to note that all of the parts of the open-ended items that were designed to be equivalent in content and difficulty that were tested were either all significant or all not significant. Items Cov_A09 and CoV_B19 were designed to be corresponding items on the A and B forms of the CoV measure and all parts of these two items that were tested showed a lack of significance. On the other hand, all of the parts of items CoV_A15 and CoV_B07 that were analyzed showed significant differences between the MD and TMA groups of student responses. This suggests that differences in student responses are not necessarily because of the type of question. In other words this data does not support the claim that the difference between MD and TMA student responses are because of the open-ended nature of these items.

Similarly, these data do not support the claim that the differences between the responses of the MD and TMA students are due to the multiple-choice format of these items. Significant differences were seen in some multiple-choice items (e.g. CoV_A01) while not in others (e.g. CoV_B05). Furthermore, because corresponding items on the alternate forms of the CoV measure were not consistently significant or not significant (e.g. CoV_A01 and CoV_B05), these data do not support that the difference in responses between MD and MLD students cannot be described in terms of the common content represented by these items that were designed to be equivalent in content and difficulty.

Despite these difficulties in specifically identifying differences between the responses of students in the MD and MLD groups, one of the similarities across these groups can also provide some insight into a possible difference between these groups. As described above, most of the items that did not provide significant results on the chi-squared test had a common structure of the expressions to be compared. Therefore, most of the items with expressions not of the structure $x \pm a$ and $b \pm x$, did show significant results. The only items that do not necessarily follow this pattern are CoV_A17 and CoV_B02. These ask students to compare expressions of the form $bx \pm a$ and $b(x \pm a)$, which is a little more complicated than those that did not show significant results, but not as reliant upon operations, algorithms, or properties as the other items (e.g., CoV_A15, CoV_B05). Of these two items, CoV_A17 produced significant results ($p = 0.0854$) and CoV_B02 did not ($p = 0.2439$). These items may show a transition from the less to more complicated items in which there is little difference between students' responses among groups for the less complex items but the differences increase as the complexity of the structure of the expression of the item increases.

Research Question 3

How do responses on these items differ among these groups of students at the end of an algebra course?

As Research Question 3 is very similar to Research Question 2, the only difference being the measures being administered at the end of the algebra course, the structure of this section will parallel that of the previous. Student responses to the 12 items differed on some items and had similarities on other items at the end of their algebra course. Table 27 provides 12 chi-squared tests for independence results for the MD and TMA groups for responses to the final administration (Round 3).

Of the 12 items analyzed with the chi-squared test, only four items (CoV_B05, CoV_B07.1, CoV_B07.2, and CoV_A17) showed significant differences ($p < 0.0053$) between the responses of MD and TMA students. One additional item, CoV_A15.2, showed results ($p = 0.1109$) that were worth discussing.

Table 27

Results of Chi-Squared Tests for Independence of MD and TMA groups in Round 3

Item		CoV_A_R3	CoV_B_R3
A01, B05	χ^2	6.1596	19.7676****
	df	4	3
	#	519	615
A09.1, B19.1	χ^2	2.1412	4.1633
	df	3	3
	#	376	184
A09.2, B19.2	χ^2	4.1953	2.4735
	df	4	4
	#	381	189
A15.1, B07.1	χ^2	2.0492	24.4449****
	df	2	4
	#	243	497
A15.2, B07.2	χ^2	7.5164*	33.6036****
	df	4	5
	#	245	501
A17, B02	χ^2	12.7132***	2.4013
	df	3	3
	#	269	571

* $p = 0.1109$, ** $p < 0.05$, ***$p < 0.01$, ****$p <0.001$

These five items are discussed individually to illuminate the differences between the groups of students. The remaining items are then discussed to identify similarities among the groups of MD and TMA students. Table 28 presents the percent of each group that gave each response to each item. These data are used in the subsequent discussions.

Table 28

Percent of Group Who Responded (percent of total group) Round 3

		CoV_A				CoV_B			
		MD		TMA		MD		TMA	
Item		Resp.	Total	Resp.	Total	Resp.	Total	Resp.	Total
A01,	A	17.6	15.8	20.1	17.8	37.2	33.4	48.2	43.9
B05	B	22.0	19.7	15.2	13.5	13.4	12.0	5.0	4.6
	C	**36.5**	**32.7**	**42.4**	**37.7**	15.2	13.6	10.1	9.2
	D	23.9	21.5	22.3	19.9	**34.3**	**30.8**	**36.7**	**33.4**
	Bl	-	10.2	-	11.1	-	10.1	-	8.9
	#	255	284	264	297	277	308	338	371
A09.1,	A	**36.1**	**21.4**	**41.4**	**29.4**	**29.0**	**8.7**	**27.5**	**6.7**
B19.1	S	46.4	27.4	44.3	31.4	36.6	11.0	48.4	11.8
	N	17.5	10.3	14.3	10.1	34.4	10.3	24.2	5.9
	Bl	-	40.9	-	29.1	-	70.0	-	75.7
	#	166	281	210	296	93	310	91	374
A09.2,	**1**	**6.5**	**3.9**	**12.3**	**8.8**	**1.0**	**0.3**	**1.1**	**0.3**
B19.2	**2**	**15.9**	**9.5**	**14.2**	**10.1**	**7.2**	**2.3**	**8.7**	**2.1**
	3	2.9	1.8	3.3	2.4	2.1	0.6	2.2	0.5
	4	21.2	12.7	22.3	15.8	10.3	3.2	18.5	4.5
	5	2.9	1.8	1.9	1.3	3.1	1.0	0.0	0.0
	6	12.4	7.4	12.8	9.1	13.4	4.2	14.1	3.5
	7	38.2	22.9	33.2	23.6	62.9	19.6	55.4	13.6
	Bl	-	40.1	-	29.0	-	68.8	-	75.4
	#	170	284	211	297	97	311	92	374
A15.1,	A	15.8	5.7	10.5	5.1	**61.9**	**39.4**	**79.9**	**63.9**
B07.1	S	**35.6**	**12.7**	**39.9**	**19.3**	25.9	16.5	16.1	12.8
	N	48.5	17.3	49.7	24.0	12.2	7.7	4.0	3.2
	Bl	-	64.3	-	51.7	-	36.5	-	20.1
	#	101	283	143	296	197	310	299	374

Percent of Group Who Responded (percent of total group) Round 3 Continued

Item		CoV_A				CoV_B			
		MD		TMA		MD		TMA	
		Resp.	Total	Resp.	Total	Resp.	Total	Resp.	Total
A15.2, B07.2	1	**9.0**	**3.2**	**19.3**	**9.4**	0	0	0	0
	2	**26.0**	**9.2**	**30.3**	**14.8**	-	-	-	-
	2a	-	-	-	-	**3.0**	**1.9**	**3.6**	**2.9**
	2b	-	-	-	-	**15.6**	**10.0**	**20.9**	**16.8**
	2c	-	-	-	-	**16.6**	**10.6**	**32.8**	**26.5**
	3	0.0	0.0	2.1	1.0	2.5	1.6	3.0	2.4
	4	2.0	0.7	0.7	0.3	3.5	2.3	2.0	1.6
	5	0	0	0	0	1.5	1.0	1.7	1.3
	6	16.0	5.6	14.5	7.1	15.1	9.6	15.2	12.3
	7	47.0	16.5	33.1	16.2	42.2	27.0	20.9	16.8
	Bl	-	64.8	-	51.2	-	36.0	-	19.3
	#	100	284	145	297	199	311	302	374
A_17, B02	A	25.2	11.3	21.1	10.1	19.6	16.9	17.0	14.0
	B	**35.4**	**15.8**	**54.2**	**26.0**	34.3	29.5	32.0	26.3
	C	29.9	13.4	14.8	7.1	**32.5**	**27.9**	**38.6**	**31.7**
	D	9.4	4.2	9.9	4.7	13.6	11.7	12.4	10.2
	Bl	-	55.3	-	52.0	-	14.0	-	17.7
	#	127	284	142	296	265	308	306	372

Differences

Similar to the differences identified in Round 1, the differences observed on each item that resulted in significant differences between the response patterns of MD and TMA students were expected. Students in the TMA group more often provided responses consistent with LT level 5 conceptions of variable, while MD students more frequently provided responses that corresponded to lower LT levels or skipped the item. While there were five items that were identified as providing evidence of differences between the MD and TMA groups of students in both Rounds 1 and 3, in Round 1 it was five out of 10 items analyzed but it was five out of 12 in Round 3. In both rounds, the same five items were identified as providing evidence of differences between the two groups. Specific discussion for each item that provided significant chi-squared results are now presented.

CoV_B05. Recall from Table 14 that item CoV_B05 is:

> Mari said, "2*t* is always greater than *t* + 2." Do you agree with Mari?
> A. Yes, because multiplication always gives you a larger answer than addition.
> B. Yes, because *t* is a positive number.
> C. No, because multiplication is not the inverse of addition.
> D. No, because it is possible that 2*t* can be equal to or less than *t* + 2.

A chi-squared test showed that the groups MD and TMA are not independent of the response given at $p < 0.001$ on the item CoV_B05 in Round 3. Table 28.2 provides the proportions of each group for each response option.

Table 28.2

Percent of Group Who Responded and Percent of Total Group in Round 3 on Item CoV_B05

	CoV_B05			
	MD		TMA	
Item Option	Responded	Total	Responded	Total
A	37.2	33.4	48.2	43.9
B	13.4	12.0	5.0	4.6
C	15.2	13.6	10.1	9.2
D	**34.3**	**30.8**	**36.7**	**33.4**
Blank	-	10.1	-	8.9
#	277	308	338	371

While the chi-squared test supported the hypothesis that the responses of these two groups are dependent on their group membership, it is interesting to note that the TMA group had a higher response rate of A, an overgeneralization of an arithmetic rule, than the MD group. Also the MD group had a higher proportion of students that chose B and C than the TMA group. These differences were identified as the primary contributors to the significant chi-squared result in Table 29.2. Both groups had a similar proportion that chose the most correct answer of D and chose not to respond to this item.

Table 29.2

Chi-Squared Standardized Residual Values for Item CoV_B005 Round 3

Group	A	B	C	D
MD	-2.7497	3.6307	1.9133	-0.6159
TMA	2.7497	-3.6307	-1.9133	0.6159

These interesting patterns may be explained by the overreliance of TMA students on rules and procedures while MD students have other influences in their selection of a response. For both groups, only about a third of students chose the most correct answer, which suggests that most of the students (both MD and TMA) selected a multiple-choice option that was consistent with a conception of variables at the LT level 3/4 (Table 17 in Chapter 3 provides the codes for CoV_B05). These response patterns were similar to the response patters on CoV_B05 in Round 1.

CoV_B07.1 and CoV_B07.2. Recall from Table 14 that item CoV_B07 is:

$$B + E + M = M + B + E$$

Circle one: Always true Sometimes true Never true.
Explain your answer.

A chi-squared test showed that the groups MD and TMA are also not independent of the response given at $p < 0.001$ on the item CoV_B07.1 in Round 3. Table 28.8 illustrates the differences among the MD and TMA group responses.

While the proportion of MD and TMA students who chose Always True was higher than any other response, there are still substantial differences in these groups' response patterns. The standardized residuals for this chi-squared test showed that the differences between the proportions of students in each group that chose each response substantially contributed to this highly significant result. Similarly to the Round 1 administration of this item, a higher proportion of MD students did not respond to this item than TMA students.

Table 28.8

Percent of Group Who Responded and Percent of Total Group in Round 3 on Item CoV_B07.1

Item Option	CoV_B07.1			
	MD		TMA	
	Responded	Total	Responded	Total
A	**61.9**	**39.4**	**79.9**	**63.9**
S	25.9	16.5	16.1	12.8
N	12.2	7.7	4.0	3.2
Blank	-	36.5	-	20.1
#	197	310	299	374

A chi-squared test showed that the groups MD and TMA are not independent of the response given at $p < 0.001$ on the item CoV_B07.2 in Round 3. In order to have less than 20% of the cells with an expected value of less than 5, codes were pooled to form larger groups just as in CoV_07.2 for Round 1. Specifically, responses that were coded as 2a and 2b were pooled and codes 3, 4, and 5 were also pooled. Table 28.10 give the individual proportions of the MD and TMA responses for each code.

From the chi-squared standardized residual values for the cells in Table 29.10 representing both MD and TMA students whose responses were coded as 2c and 7 contributed most to the significant chi-squared result.

These standardized residuals, along with the percentages reported in Table 28.10, suggest that the more frequent coding of TMA students' responses as 2c than MD students' responses as 2c and the more frequent skipping of the explanation after choosing *Always, Sometimes,* or *Never* of the MD students than the TMA students were major contributors to the statistical difference between these two groups.

Table 28.10

Percent of Group Who Responded and Percent of Total Group in Round 3 on Item CoV_B07.2

Response Code	CoV_B07.2			
	MD		TMA	
	Responded	Total	Responded	Total
2a	**3.0**	**1.9**	**3.6**	**2.9**
2b	**15.6**	**10.0**	**20.9**	**16.8**
2c	**16.6**	**10.6**	**32.8**	**26.5**
3	2.5	1.6	3.0	2.4
4	3.5	2.3	2.0	1.6
5	1.5	1.0	1.7	1.3
6	15.1	9.6	15.2	12.3
7	42.2	27.0	20.9	16.8
Blank	-	36.0	-	19.3
#	199	311	302	374

Similarly to the responses received for this item in Round 1, there were no student responses that were coded as a 1, indicating that they explicitly stated that this was true for all values of the variables. Very few of the students' responses (about 5% of the total number of students) in either group were coded as a 3, 4, or 5 for this item, which codes suggest some limitation in how students are thinking about the variable.

Table 29.10

Chi-Squared Standardized Residual Values for Item CoV_B07.2 Round 3

Group	2a, 2b	2c	3, 4, 5	6	7
MD	-1.5587	-4.1463	0.3932	-0.0478	5.1356
TMA	1.5587	4.1463	-0.3932	0.0478	-5.1356

Upon closer inspection, more than 40% of students selected an answer that could be consistent with a LT level 5 conception of variable, the coding of this item is very generous as shown in Table 20 in Chapter 3. Under this coding scheme, responses that provided an explanation coded as a 1, 2a, 2b, or 2c were considered to be consistent with a LT level 5 conception of variable. This was giving the benefit of the doubt to these students. A response of “because of the commutative property”, which was taken to be equivalent to stating that the order doesn’t matter, was coded as 2c and thus connected with a LT level 5 conception of variable. As shown in Table 28.10, while 46.2% of the TMA student responses (in Round 3) were coded as consistent with LT level 5, the majority of these responses fell into the 2c code category.

It is not unreasonable to consider that some students who provided a response that was coded as 2c were not thinking about and using variables in a way that was consistent with a LT level 5 conception of variable and may have been simply ignoring the variables and matching the pattern with the pattern that they had seen associated with the commutative property. Using this approach, the name of the property is used as a label of a pattern and not as a mathematical idea. Because this study could not differentiate between how students were thinking about the commutative property when citing it as an explanation and when they were thinking about the variables at a LT level 5, all such explanations were connected with a LT level 5. Other codes across all items similarly gave the benefit of the doubt to student responses, resulting in a higher probability that students with a lower level of sophistication of conception of variable were labeled with a higher level of sophistication. Implications of this are discussed in Chapter 5.

CoV_A17. Recall from Table 14 that item CoV_A17 is:

Sam said, “$2n + 1$ represents a greater quantity than $2(n + 1)$.” Do you agree with Sam?

A. Disagree, the two expressions are equivalent because they have the same numbers and variable.

B. Disagree, the distributive property produces a constant of 2 in the second expression as opposed to 1 in the first expression, and the variable has a coefficient of 2 in both expressions.

C. Agree, $2n + 1$ is greater than $2(n + 1)$ because the 2 is multiplying the n in the first expression, not the 1. The value of n could be greater than 1.
D. You cannot tell which is larger because it depends on the value of n.

A chi-squared test showed that the groups MD and TMA are not independent of the response given at $p = 0.0053$ on the item CoV_A17 in Round 3. Table 28.11 provides the proportions of each groups' responses.

Table 28.11

Percent of Group Who Responded and Percent of Total Group in Round 3 on Item CoV_A17

	CoV_A17			
	MD		TMA	
Item Option	Responded	Total	Responded	Total
A	25.2	11.3	21.1	10.1
B	**35.4**	**15.8**	**54.2**	**26.0**
C	29.9	13.4	14.8	7.1
D	9.4	4.2	9.9	4.7
Blank	-	55.3	-	52.0
#	127	284	142	296

The largest chi-squared standardized residual values for this item are found for the cells in Table 29.11 representing both MD and TMA students' responses of B and C.

Table 29.11

Chi-Squared Standardized Residual Values for Item CoV_A17 Round 3

Group	A	B	C	D
MD	0.7913	-3.0907	2.9943	-0.1137
TMA	-0.7913	3.0907	-2.9943	0.1137

These standardized residuals, along with the percentages reported in Table 28.11, suggest that the more frequent TMA students' responses of B than MD students' responses and the more frequent response of C by the MD students than the TMA students were major contributors to the statistical difference between these two groups.

A similar proportion of students chose the choices of A and D from both groups.

In addition to the similar percentages, the low chi-squared standardized residuals suggest that these small differences contribute little to the significance of the chi-squared result. As listed in Table 18, neither choice A or D refers to any procedure, algorithm, or property. Choice A focuses on the variable and the numbers as independent objects without attending to the operations that connect them or the quantities they represent. Choice D focuses on the variable as a specific unknown value. On the other hand, both choice B and C explicitly reference operations.

CoV_A15.2. Recall from Table 14 that item CoV_A15 is:

$$B + E + M = B + R + S$$

Circle one: Always true Sometimes true Never true.
Explain your answer.

A chi-squared test did not show that the groups MD and TMA are not independent of the response given ($p = 0.1110$) on the item CoV_A15.2 in Round 3. In order to have less than 20% of the cells with an expected value of less than 5, some codes were pooled to form larger groups. Specifically, responses that were coded as 3 and 4 were combined into one column. This column represented student responses that suggest some form of a limited conception of variable or a misconception about variables.

Table 28.9 provides insight into the differences among the MD and TMA group responses. While this chi-squared result was higher than the typical level for significance, it is worth discussing because it is the only other item that was close to significant at this level and it follows a similar pattern as the other significant Round 3 items.

Table 28.9

Percent of Group Who Responded and Percent of Total Group in Round 3 on Item CoV_A15.2

Response Code	CoV_A15.2			
	MD		TMA	
	Responded	Total	Responded	Total
1	**9.0**	**3.2**	**19.3**	**9.4**
2	**26.0**	**9.2**	**30.3**	**14.8**
3	0.0	0.0	2.1	1.0
4	2.0	0.7	0.7	0.3
5	0	0	0	0
6	16.0	5.6	14.5	7.1
7	47.0	16.5	33.1	16.2
Blank	-	64.8	-	51.2
#	100	284	145	297

The largest chi-squared standardized residual values for this item are found for the cells in Table 29.9 representing both MD and TMA students whose responses were coded as 1 and 7.

Table 29.9

Chi-Squared Standardized Residual Values for Item CoV_A15.2 Round 3

Group	1	2	3, 4	6	7
MD	-2.2748	-0.5117	-0.3776	0.3260	2.1942
TMA	2.2748	0.5117	0.3776	-0.3260	-2.1942

These standardized residuals, along with the percentages reported in Table 28.9, suggest that the more frequent coding of TMA students' responses as 1 than MD students' responses as 1 and the more frequent skipping of the explanation after choosing *Always, Sometimes,* or *Never* of the MD students than the TMA students were major contributors to the statistical difference between these two groups.

Again, there were no student responses that were coded as a 5, indicating that students had the misconceptions that two different instances of the same variable could be defined as two different values simultaneously. Also, only a few of the students' responses in either group were coded as a 3 or 4 for this item, which is similar to CoV_B07.2. Additionally, over two-thirds (81.3% of MD and 67.4% of TMA) of the students in both the TMA and MD groups did not attempt an explanation (by either skipping the item altogether or skipping the explanation part of the item). This represents an alarming number of students who did not even attempt an explanation.

Summary of differences. From the differences between the MD and MLD groups of students discussed in the previous sections, several common differences between these groups are evident, most of which mirror the results found for Round 1. First, on all of the multiple-choice items (CoV_B05, CoV_B07.1, and CoV_A17) TMA students are more likely to choose the most correct answer than their MD counterparts on these items. Also, MD students were more likely to skip these items.

A similar pattern was observed on the open-ended items (CoV_A15.2 and CoV_B07.2). For both of these items the differences in the second to highest category of responses (2 and the combination of 2b and 2c respectively) were important contributors to significance of the chi-squared test. As in for Round 1, these differences were expected. Two variations in the response patterns between the Round 1 and Round 3 included the lack of significance ($p = 0.1110$) of item CoV_B15.2 and the lower proportion of items showing differences between the MD and TMA groups in Round 3 than in Round 1. These similarities and differences in the response patterns in Rounds 1 and 3 will be discussed further in the section addressing Research Question 4.

Similarities

While the differences identified in the student responses to the items provided evidence that there is a significant difference between how TMA and MD students responded to these items were not surprising, the lack of difference between the responses of these groups of students on the other items is. On the seven (of 12) items in Round 3 that were not discussed in the differences section (CoV_A01, CoV_A09.1, CoV_A09.2, CoV_B19.1, CoV_B19.2, CoV_A15.1 and CoV_B02) on the other items was surprising. These results fail to reject the hypothesis that there is not a difference in the responses of the students in the MD and MLD groups on these items.

Similar to the results from Round 1, in all but two of the items that were not used as evidence of a difference (CoV_A01, CoV_A09.1, CoV_A09.2, CoV_B19.1, and CoV_B19.2) each item asked the students to compare two expressions of the form $x \pm$ a and b $\pm x$, where a and b are some integer. CoV_A17 had a similar format of comparing two expressions of $bx \pm$ a and $b(x \pm$ a), where a is some integer and b is a natural number. Also, different from Round 1, item CoV_B15 did not show significant differences between the MD and MLD groups. The lack of significance in the difference in the responses between these groups of students suggests that MD and TMA students may not respond differently on these types of items.

Summary of Research Question 3

In attempting to describe the differences between responses on these items among the MD and TMA groups of students at the end of an algebra course, identifying differences across all of the items is difficult. Only one-third of the items analyzed with a chi-squared test suggest that there is a difference between the responses form the students in these groups, while the other half suggest no difference. Thus by the numbers of significant results alone, it cannot be said that there are differences in the responses of students from the MD and TMA groups that stretch across all of these items. As with Research Question 2, this data does not support differences between the MD and TMA groups of students based on question format or content strand as there were questions of each type that showed significant differences and questions of each type that did not show significant differences.

Despite the lack of evidence to claim that TMA and MD student responses differ on all of these items in general, on multiple-choice or open-ended items or on the content strands identified in the creation of the items, one similarity stands out. Again, in a similar fashion to the data from Round one, there is some evidence that the format of the expressions being compared may provide some differentiation between the responses of MD and MLD students. As described above, most of the items that did not provide significant results on the chi-squared test had a common format of the expressions to be compared. Therefore, most of the items with expressions not of the format $x \pm a$ and $b \pm x$, did show significant results.

Again, similarly to Round 1, items that did not necessarily follow this pattern are CoV_A17 and CoV_B02. These ask students to compare expressions of the form $bx \pm a$ and $b(x \pm a)$, which is a little more complicated than those that did not show significant results, but not as reliant upon operations, algorithms, or properties as the other items. Of these two items one produced significant results and the other did not. Again, these items may show a transition from the less to more complicated items in which there is little difference between students' responses among groups for the less complex items but the differences increase as the complexity of the item increases.

Research Question 4

How does student growth differ for these groups of students from the beginning of an algebra course to the end of an algebra course?

In addition to describing how MD and TMA students differ at the beginning and end of an algebra course, an important description of how these groups of students' responses differ on items designed to assess students' ability to compare expressions involving generalized quantities is how their responses change over the course of an Algebra 1 course. I discuss these differences by first comparing the similarities and differences identified in the previous two sections of the students' responses at the beginning and end of their Algebra 1 course. In addition to this discussion of differences, I also investigate differences in how students responded to these items by considering the significant changes in proportion of each group of students as described by a two proportions *z*-test.

Changes Identified from the Descriptions of Rounds 1 and 3

While there were many similarities and differences across the descriptions in the Round 1 and Round 3 descriptions, one that stands out is the apparent decrease in the difference between the MD and TMA groups as evidenced by the results of the chi-squared tests. In Round 1, five out of ten items showed some level of significant difference in the response patterns of students between the MD and TMA groups. In Round 3, only four of 12 items showed this same significance. These data seem to suggest that the difference between the responses of these groups decrease over the course of the Algebra 1 course. The cause of this decrease in difference is investigated in the next section, but this seems to suggest that the rate of growth of the MD group was actually larger than that of the TMA group. This is surprising, as many studies have documented TMA students as having a larger rate of growth than their MD peers.

Besides this surprising general picture, the individual item and smaller-scope responses appear to be consistent across the two administrations. On almost all items, a larger proportion of TMA students selected answers connected to the higher levels of the LT (see Tables 25 and 28) than the MD students. There were seven items that did not follow this general trend, but none of them produced a significant chi-squared result, suggesting that there was not enough evidence to claim that there was a significant difference in how the MD and TMA students responded to the item in general.

Across both administrations, the lack of evidence supporting differences stemming from the item format (i.e., multiple-choice and open-ended items) in the response patterns of students in the MD and TMA groups. This was evidenced by occurrence of both significant and insignificant chi-squared results on multiple-choice items and both significant and insignificant chi-squared results on open-ended items as reported in Tables 24 and 27.

Finally, the significance of the differences in student response patterns between the MD and TMA groups seem to be consistently related to the complexity of the structure of the expression being compared in the item across both Round 1 and Round 3. Items with a simpler structure of $x \pm a$ and $b \pm x$, did not show significant results, while the items that involved formats with more procedural, algorithmic or properties-based aspect did show significant differences in both rounds. The lack of growth of the MD students on these types of items in relation to their TMA peers could be explained by MD students not receiving needed instruction or help to develop a better understanding of these types of formats. This implication will be discussed further in Chapter 5.

Changes Identified from *z*-Test Results

Despite the consistency of the general observations made from Round 1 to Round 3, there were some significant differences between the change in responses of students in the MD and TMA groups. These differences were identified by considering the significant changes in proportion of each group of students as described by a two proportions *z*-test. This *z*-test was performed for the proportion of the group (MD or TMA) that selected a particular response on the first administration and the proportion of the same group who chose the same response on the final administration. For example, the percent of MD students who responded to item CoV_A01 by selecting option A in Round 1 was compared with the percent of students in the MD group who chose option A on CoV_A01 in Round 3. Following this procedure for each option on each item and each group, I conducted 138 two-proportion *z*-tests. The results of these tests are summarized in Table 30. The number in each cell of Table 30 represents the difference between the percent of students in the indicated group who selected the indicated response in Round 3 and Round 1.

All of the items showed significant change in proportion of at least one of the groups for at least one of the response options or codes ($p < 0.1$). In seven items, change in proportion of TMA students was significant for at least one response choice. In five items change in proportion of MD students was significant for at least one response choice. In four items did both the proportion of MD students and the proportion of TMA students change.

Table 30

Change in Percent of Group Responses for MD and TMA

Item		CoV_A		CoV_B	
		MD	TMA	MD	TMA
A01, B05	A	-3.7	1.6	-9.1**	-1.5
	B	5.0	2.4	2.9	1.6
	C	**-0.3**	**10.8******	-0.9	1.7
	D	1.1	-4.4	**12.5******	**3.5**
	Bl	-2.0	-10.3***	-5.4**	-5.3**
A09.1, B19.1	A	**-1.0**	**0.0**	**0.8**	**0.8**
	S	-3.1	2.7	3.1	1.2
	N	2.2	5.1	3.4	7.1***
	Bl	1.9	-7.8**	-7.4**	-9.0***
A09.2, B19.2	1	**-0.2**	**2.1**	**0.3**	**-0.2**
	2	**0.2**	**1.6**	**0.4**	**0.7**
	3	-0.3	-0.7	0.3	0.5
	4	-1.1	-2.0	-0.2	3.3***
	5	0.9	-0.7	1.0*	-0.2
	6	0.5	0.8	-2.4	-1.0
	7	-1.1	6.3**	8.9***	5.7***
	Bl	1.1	-7.4**	-8.2**	-8.8***
A15.1, B07.1	A	2.8	4.8****	**1.0**	**-0.5**
	S	**0.5**	**3.9**	-4.9	0.2
	N	0.6	7.4***	3.3*	1.9
	Bl	-4.0	-16.1****	0.6	-1.7
A15.2, B07.2	1	**0.3**	**3.5***	**0**	**0**
	2	**1.4**	**0.6**	-	-
	2a	-	-	**1.0**	**1.5**
	2b	-	-	**-1.7**	**-0.9**
	2c	-	-	**-3.9**	**0.8**
	3	-0.4	0.5	1.3*	1.4
	4	0.7	0.1	1.0	1.1
	5	0.3	0	0.0	0.6
	6	0.7	5.0***	-3.9	-1.0
	7	1.4	6.6***	5.3	-0.9
	Bl	-3.1	-16.3****	0.8	-2.5
A_17, B02	A	5.6**	5.0**	0.8	0.3
	B	**5.7***	**15.2******	4.6	8.0**
	C	1.2	1.9	**0.5**	**3.7**
	D	0.2	2.4*	0.3	-3.4
	Bl	-12.6***	-24.5****	-6.2**	-8.6**

* $p < 0.1$, ** $p < 0.05$, ***$p < 0.01$, ****$p < 0.001$

In order to understand the differences between the changes of the response of students in the MD and TMA groups, a description of each group is provided followed by a comparison of the changes of the two groups. In general, a significant positive change in the proportions of students with a response that is identified as being consistent with a conception of variable of LT level 5 is considered growth. A significant negative change in the proportion of students with a response that is not identified as being consistent with a conception of variable of LT level 5, cannot be considered growth unless it is accompanied by a significant positive change in the proportions of students with a response that is identified as being consistent with a conception of variable of LT level 5. This is true because it could be that these students changed their response from a response that suggests one misconception to a different misconception. In other words, it is not enough to decrease responses that are consistent with lower-level conceptions of variable. Growth requires students increasing responses that are consistent with more sophisticated conceptions of variable. A significant decrease in the proportion of students who left an item blank is also considered growth, although it cannot be considered as strong of a statement as an increase in the TL level 5 responses. While it is a positive outcome anytime a student who did not attempt an item subsequently makes an attempt, because the measures were timed, the fact that a student does not attempt an item may have more to do with the placement of the item on the measure and the student's understanding of the previous items than of the item that is actually skipped.

MD Students. Of these eight items, MD student responses showed a significant change from Round 1 to Round 3 on at least one choice or code in the proportion of students who selected that response on five items. Based on the codes assigned and the

LT levels identified in Tables 18–20 each item with at least one response option with a significant change will be discussed individually.

CoV_B05. Recall from Table 14 that item CoV_B05 is:

Mari said, "2*t* is always greater than *t* + 2." Do you agree with Mari?
A. Yes, because multiplication always gives you a larger answer than addition.
B. Yes, because *t* is a positive number.
C. No, because multiplication is not the inverse of addition.
D. No, because it is possible that 2*t* can be equal to or less than *t* + 2.

The proportion of MD students who selected A or left this item blank significantly decreased from Round 1 to Round 3 and those who selected the most correct answer of D significantly increased. The change in the responses of the MD students on this item shows growth because of the increase of students who selected D (LT 5) and the decrease of students who selected A (overgeneralized arithmetic conventions) or did not attempt the item. While there was a slight increase in students selecting B, which suggests a misconception about the values that the variable can be, this change was not found to be significant ($p = 0.2440$).

CoV_B19.1 and CoV_B19.2. Recall from Table 14 that item CoV_B19 is:

Anne said, "*m* + (–7) is less than –5 + *m*."
Circle one: Always true Sometimes true Never true.
Explain your answer.

Due to the significant change identified for this item of a decrease in MD students who skipped the item, we can cautiously say that this item provides some evidence that the MD students showed growth. Because this item was the last item on the Form B of the Concept of Variable measure, the likelihood of some students attempting it in Round 3 but not in Round 1 because of its placement is high. Despite the decrease in MD students who skipped this item, there was also a significant increase in students who only circled

Always, Sometimes or Never but did not provide an explanation. Any engagement with an item is considered a positive change, but that engagement was likely limited as evidenced by the increase in the proportion of students who did not attempt an explanation. There was also an increase in the proportion of students who provided an explanation coded as a 5 that was significant (p = 0.0792). It is not possible to determine if this increase was represented growth.

CoV_B7.1 and CoV_B7.2. Recall from Table 14 that item CoV_B07 is:

$$B + E + M = M + B + E$$

Circle one: Always true Sometimes true Never true.
Explain your answer.

Because of the small percent change on this item, it is not possible to determine if the change on this item represented growth for the MD students. While a significant change (p = 0.0793) was identified in the number of MD students who selected Never, there was also a decrease in students who selected the most correct response of Sometimes True that was not significant (p = 0.1147). Similarly, the small change in percent and the relatively high p values of the codes for the student explanations for this item limit the conclusions that can be drawn from the changes in responses on this item.

CoV_A17. Recall from Table 14 that item CoV_A17 is:

Sam said, "$2n + 1$ represents a greater quantity than $2(n + 1)$." Do you agree with Sam?

A. Disagree, the two expressions are equivalent because they have the same numbers and variable.
B. Disagree, the distributive property produces a constant of 2 in the second expression as opposed to 1 in the first expression, and the variable has a coefficient of 2 in both expressions.
C. Agree, $2n + 1$ is greater than $2(n + 1)$ because the 2 is multiplying the n in the first expression, not the 1. The value of n could be greater than 1.
D. You cannot tell which is larger because it depends on the value of n.

With a significant decrease ($p = 0.0030$) in the proportion of students that did not attempt this item and the significant increase ($p = 0.0540$) of students who selected the response of B, which has been connected with the LT level 5, there is evidence to support that MD student growth between rounds 1 and 3. Regardless of the apparent change of some of the students who skipped the item in Round 1 to option A, which represents a disregard of the quantities in the item, or B both types of change represent growth.

CoV_B02. Recall from Table 14 that item CoV_B02 is:

Char said, "$3n - 1$ represents a greater quantity than $3(n - 1)$." Do you agree with Char?

A. Disagree, the two expressions are equivalent because they have the same numbers and variable.
B. Disagree, $3n - 1$ is less than $3(n - 1)$ because the 3 is multiplying the n in the first expression, not the 1. The value of n could be less than 1.
C. Agree, the distributive property produces a constant of -3 in the second expression as opposed to -1 in the first expression, and the variable has a coefficient of 3 in both expressions.
D. You cannot tell which is larger because it depends on the value of n.

Due to the significant change ($p = 0.0388$) identified for this item of a decrease in MD students who skipped the item, we can cautiously say that this item provides some evidence that the MD students showed growth. The lack of significant increases ($0.1940 < p < 0.8966$) in selection of another option, there are limited in describing this change in more detail.

TMA Students. Of these eight items, TMA student responses showed a significant change from Round 1 to Round 3 on at least one choice or code in the proportion of students who selected that response on seven items. In keeping with the structure of the previous section, each item with at least one response option with a significant change is discussed individually with respect to the codes assigned and the LT levels identified in Tables 18–20.

CoV_A01. Recall from Table 14 that item CoV_A01 is:

Jon said, "$m - 1$ is always greater than $1 - m$." Do you agree with Jon?

A. Yes, Jon is correct because *m* is a positive number.
B. Yes, Jon is correct because you cannot substitute a negative number for *m*.
C. No, Jon is not correct because $1 - m$ is greater than $m - 1$ when m is negative integer.
D. No, Jon is not correct because these expressions are equivalent.

The significant increase ($p = 0.0025$) in the proportion of TMA students who selected the most correct option and the significant decrease ($p = 0.0004$) of these students who left this item blank suggests growth for the TMA students on this item.

CoV_B05. Recall from Table 14 that item CoV_B05 is:

Mari said, "$2t$ is always greater than $t + 2$." Do you agree with Mari?

A. Yes, because multiplication always gives you a larger answer than addition.
B. Yes, because *t* is a positive number.
C. No, because multiplication is not the inverse of addition.
D. No, because it is possible that $2t$ can be equal to or less than $t + 2$.

Due to the significant change ($p < 0.05$) identified for this item of a decrease in TMA students who skipped the item, we can cautiously say that this item provides some evidence that the TMA students showed growth. The lack of significant increases ($p > 0.1$) in selection of another option, there are limited in describing this change in more detail.

CoV_A9.1 and CoV_A9.2. Recall from Table 14 that item CoV_A09 is:

Bart said, "$t + 3$ is less than $5 + t$."
Circle one: Always true Sometimes true Never true.
Explain your answer.

Due to the significant change identified for this item of a decrease in TMA students who skipped the item ($p < 0.05$), we can cautiously say that this item provides some evidence that the TMA students showed growth. Despite the decrease in TMA students who skipped this item, the was also a significant increase in students who only circled Always, Sometimes or Never but did not provide an explanation ($p < 0.05$). Any engagement is considered a positive change, but that engagement was likely limited as evidenced by the increase in the proportion of students who did not attempt an explanation.

CoV_B19.1 and CoV_B19. 2. Recall from Table 14 that item CoV_B19 is:

Anne said, "$m + (-7)$ is less than $-5 + m$."
Circle one: Always true Sometimes true Never true.
Explain your answer.

The significant change ($p = 0.0022$) identified for this item of a decrease in TMA students who skipped the item, and the significant increase ($p = 0.0053$) of students who selected a response that was not connected with a LT level of 5 suggest some growth. While a significant increase in the proportion of students providing a response that was connected with a LT level 5 could have suggested more growth, any engagement with an item is considered a positive change.

CoV_A15.1 and CoV_A15.2. Recall from Table 14 that item CoV_A15 is:

$$B + E + M = B + R + S$$

Circle one: Always true Sometimes true Never true.
Explain your answer.

As evidenced by the significant change ($p < 0.001$) identified for this item of a decrease in TMA students who skipped the item, we can cautiously say that this item provides some evidence that the TMA students showed growth. Despite the decrease in TMA students who skipped this item, the was also a significant increase ($p = 0.0013$) in students who only circled Always, Sometimes or Never but did not provide an explanation or provided an incorrect or incomplete answer. While there was an increase in the proportion of students who provided an explanation that was connected with the

LT level 5, the small percentage and the minimal significance ($p = 0.0838$) limit the conclusions that can be drawn from this result.

CoV_A17. Recall from Table 14 that item CoV_A17 is:

Sam said, “$2n + 1$ represents a greater quantity than $2(n + 1)$.” Do you agree with Sam?

A. Disagree, the two expressions are equivalent because they have the same numbers and variable.
B. Disagree, the distributive property produces a constant of 2 in the second expression as opposed to 1 in the first expression, and the variable has a coefficient of 2 in both expressions.
C. Agree, $2n + 1$ is greater than $2(n + 1)$ because the 2 is multiplying the n in the first expression, not the 1. The value of n could be greater than 1.
D. You cannot tell which is larger because it depends on the value of n.

The significant increase ($p < 0.001$) in the proportion of TMA students who selected the most correct option and the significant decrease ($p < 0.001$) of these students who left this item blank suggests growth for the TMA students on this item.

CoV_B02. Recall from Table 14 that item CoV_B02 is:

Char said, “$3n – 1$ represents a greater quantity than $3(n – 1)$.” Do you agree with Char?

A. Disagree, the two expressions are equivalent because they have the same numbers and variable.
B. Disagree, $3n – 1$ is less than $3(n – 1)$ because the 3 is multiplying the n in the first expression, not the 1. The value of n could be less than 1.
C. Agree, the distributive property produces a constant of –3 in the second expression as opposed to –1 in the first expression, and the variable has a coefficient of 3 in both expressions.
D. You cannot tell which is larger because it depends on the value of n.

Due to the significant change ($p = 0.0042$) identified for this item of a decrease in TMA students who skipped the item, we can cautiously say that this item provides some evidence that the TMA students showed growth. Despite the decrease in TMA students who skipped this item, there was also a significant increase ($p = 0.0076$) in students who responded with option B, which suggests a misunderstanding of the distributive property.

Any engagement with an item is considered a positive change, but the growth on this item was less than if there had been an increase of students selecting the most correct response of option C.

Comparison of MD and TMA. From this discussion of the changes in the proportion of MD and TMA students' responses, I now compare the growth of the MD and MLD groups. Table 31 summarizes the discussion of the previous two sections by labeling each item as providing evidence of growth, providing limited evidence of growth, providing evidence of change but not growth, and no significant change.

Table 31

Summary of Evidence of Growth by Item

Item	MD	TMA
CoV_A01	No Significant Change	Growth
CoV_B05	Growth	Limited Growth
CoV_A9.1 and CoV_A9.2	No Significant Change	Limited Growth
CoV_B19.1 and CoV_B19.2	Limited Growth	No Significant Change
CoV_A15.1 and CoV_A15.2	No Significant Change	Limited Growth
CoV_B7.1 and CoV_B7.2	Change but Not Growth	No Significant Change
CoV_A17	Growth	Growth
CoV_B02	Limited Growth	Limited Growth

It is important to recognize that the category of 'No significant change' here indicates that there were not significant changes in the responses from Round 1 to 3. This implies that there was not substantial evidence of growth for these students at the end of an algebra 1 course compared to the beginning of that course as measured by these items. Furthermore, almost all items in the category of "Change but Not Growth" provided evidence of more attempts of the item, but provided no evidence that the more students' responses were connected to a LT level 5 conception of variable. The one exception to this was CoV_A15.2 with a significant increase of code 1 (p = 0.0838). Because of this low *p*-value and the small size of the increase the amount of evidence it provided was questioned.

Based on these discussions of evidence of growth, these data only suggest that the MD and TMA groups both experienced limited growth. The MD students only grew significantly on items CoV_B05 and CoV_A17 and the TMA students only grew significantly on items CoV_A01 and CoV_A17. Based on two items for which growth is evidenced, it is difficult to make many claims about the differences or similarities of the change in student responses.

It is interesting to note that open-ended items provided at best limited evidence of growth, and the multiple-choice items are the only format on which evidence of growth of conceptions of variable were obtained. Contrary to the evidence provided in Research Question 2 and Research Question 3, the format of the item may be a factor in the student responses. It is also interesting that the distinction that was found to be useful in Research Question 2 and Research Question 3 between the format of the expressions to be compared was not useful in describing differences between the changes in the responses of the MD and TMA groups.

Summary

Research question 1: Differences between MLD and LMA students' responses. Overall, the data considered in this study contained insufficient evidence to claim that there are differences in the responses of students in the MLD and LMA groups. Based on both a chi-squared analysis of student response patterns in the MLD and LMA groups, only three of 24 items provided statistically significant evidence that the response patterns of these two groups were different. Furthermore, all but one of 12 items failed to show significant differences in the change of the proportions of student responses that suggested a difference in the change in student response patterns from Round 1 to Round 3. Because of this lack of evidenced of a difference between these groups, students in the MLD and MLA groups were combined into one mathematics difficulties (MD) group.

Research questions 2 and 3: Differences between MD and TMA students' responses. In attempting to describe the differences between how responses on these items differ among the MD and TMA groups of students at the beginning and also at the end of an algebra course, identifying differences across all of the items was difficult. There were not found to be differences in the ways that students from the MD and TMA groups responded to open-ended or multiple-choice items.

Despite these difficulties in specifically identifying differences between the responses of students in the MD and MLD groups, a possible difference between these groups was identified. Items with expressions not of the format $x \pm a$ and $b \pm x$, did show significant results, while those of this format did not show significant results. These items may show a transition from the less to more complicated items in which there is little difference between students' responses among groups for the less complex items but the differences increase as the complexity of the item increases by relying more upon procedures, algorithms and properties. Thus, while the differences are limited, the format of the expression may have more to do with the difference than the conceptual understanding of comparing two expressions containing variables.

Research question 4: Differences between MD and TMA students' growth. Similarly, any differences between the growth of MD and TMA students over the course of the algebra 1 class were difficult to identify. The evidence presented seemed to suggest that the growth of both groups of students was similar across Round 1 and Round 3 based on the results of the *z*-tests. A comparison of the results of the chi-squared tests in Round 1 and Round 3 presents an apparent decrease in the difference between the MD and TMA groups as evidenced by the results of the chi-squared tests. These data seem to suggest that the difference between the responses of these groups decrease over the course of the Algebra 1 course, which in turn suggest that the MD group has a higher growth rate than the TMA group, while the *z*-test data provides little evidence of a difference in growth rate. Because of these mixed results, further research needs to be done to investigate this question.

Conclusion

Findings from this investigation are based on examination of students' responses to items designed to assess their ability to compare expressions involving generalized quantities at the beginning and end of an Algebra 1 course. The data indicated that difference between MLD and MLA students' responses were extremely limited. When comparing the differences between MD and TMA students' responses, more similarities than differences were found, although there did seem to be some difference between MD and TMA students' responses on items with expressions with a more complex format that relied upon procedural, algorithmic and properties. Finally, the growth experienced by both the MD and TMA groups of students seemed to be very similar.

In Chapter 5, the data presented and summarized in this chapter related to differences between MD and TMA students' responses to these items is utilized to respond to the research questions that formed the basis for this study. This is done to address the gap in our understanding of specific differences between low- (both MLD and LMA) and TMA students' conceptions and use of variables. Because of the central role of variable in algebra and the featured role of algebra in school mathematics, it is vital to understand how students, especially MD students, think about and use variables.

CHAPTER 5: DISCUSSION AND RECOMMENDATIONS

Chapter 5 first presents an overview of the study and findings to set the context for discussion of limitations of the study, findings and implications, and future research.

Overview of Study

Individuals who are not successful in algebra are limited in their post-secondary and career options. Algebra is described as “the gatekeeper for citizenship; and people who don’t have it are like the people who couldn’t read and write in the industrial age” (Moses & Cobb, 2001, p. 14). Students’ conceptions of variables are foundational to conceptual understandings in algebra (Usiskin, 1988). There is currently is a gap in our understanding of specific differences between students with mathematics difficulties (MD) and typically mathematics achieving (TMA) students’ conceptions and use of variables in algebra. The purpose of this study is to investigate the differences between the responses of mathematics learning disability (MLD), other low mathematics achieving (LMA) and TMA students in an Algebra 1 course on items designed to assess students’ ability to compare expressions involving variables.

Research Questions

The general question under investigation is *How do students’ responses to items designed to assess their ability to compare expressions involving generalized quantities at the beginning and end of an Algebra 1 course vary in relation to their performance on a standardized algebra assessment?* I will consider this variation with respect to both students’ responses on individual items and student growth across administrations. Specifically:

1. Do the responses of students who are at or below the 10th percentile on the Iowa

End of Course assessment (IEOC), and those of students who are above the 10^{th} percentile and at or below the 25^{th} percentile differ on eight items designed to assess students' ability to compare expressions involving generalized quantities on a conceptual progress-monitoring tool at the beginning of an algebra course?

2. How do responses on these items differ among the MLD, LMA and TMA groups of students at the beginning of an algebra course?
3. How do responses on these items differ among these groups of students at the end of an algebra course?
4. How does student growth differ for these groups of students from the beginning of an algebra course to the end of an algebra course?

Method

This investigation is based upon work done by Blanton et al. (2015) in which they described a progression of conceptions and uses of variable as a learning trajectory (LT) that characterized the levels of sophistication in students' thinking about variable. Based on this framework, I analyzed students' responses to identify and describe differences among the MLD, LMA, and TMA groups of students by coding for the concepts of variable and common misconceptions and using descriptive and inferential statistics and.

In order to answer each research question, data were gathered for each question from the following source(s):

- Research Question 1: Chi-squared tests of the independence of the MLD and MLA groups of student responses for each item in each round and two-proportions *z*-test to determine if the proportion of students in each group in

Round 1 is different than in Round 3 for each response option or code for each item.

- Research Question 2: Chi-squared tests of the independence of the MD and TLA groups of student responses for each item in Round 1
- Research Question 3: Chi-squared tests of the independence of the MD and TLA groups of student responses for each item in Round 3
- Research Question 4: Two-proportions *z*-test to determine if the proportion of students in each group in Round 1 is different than in Round 3 for each response option or code for each item.

Summary of Findings

A summary of the findings detailed in Chapter 4 is provided here to provide context for the discussions of the limitations and implications that follow. This section is organized by research question.

Research Question 1

Do responses differ between students' that are at or below the 10th percentile on the Iowa End of Course assessment (IEOC), and those of students who are between the 10th percentile and the 25th percentile on eight items designed to assess students' ability to compare expressions involving generalized quantities on a conceptual progress-monitoring tool at the beginning of an algebra course?

Overall, the data considered in this study contained insufficient evidence to claim that there are differences in the responses of students in the MLD and LMA groups. Based on chi-squared tests of independence of student response patterns in the MLD and LMA groups, only three of 24 administrations of the items provided statistically significant evidence that the response patterns of these two groups were different. Furthermore, all but one of 12 items failed to show significant differences in the change of the proportions of student responses from Round 1 to Round 3. Because of this lack of evidence of differences between these groups, students in the MLD and MLA groups were combined into one MD group for the remaining research questions.

Research Question 2 and 3

How do responses on these items differ among the MLD, LMA and TMA groups of students at the beginning of an algebra course? At the end of an algebra course?

In attempting to describe the differences between responses on these items among the MD and TMA groups of students at the beginning and also at the end of an algebra course, identifying such differences across all of the items was difficult. There were not found to be differences in the ways that students from the MD and TMA groups responded to open-ended items, multiple-choice items, or on items designed to assess the same content. Despite these difficulties, a possible difference between these groups was identified. Items with expressions of the structure $x \pm a$ and $b \pm x$ (for a and b integers and x is a variable), did not show significant differences, while those not of this structure did show significant differences.

Research Question 4

How does student growth differ for these groups of students from the beginning of an algebra course to the end of an algebra course?

Similarly, any differences between the growth of MD and TMA students over the course of the algebra 1 class were difficult to identify. The evidence presented seemed to suggest that the growth of both groups of students was similar between Round 1 and

Round 3 based on the results of the *z*-tests. A comparison of the results of the chi-squared tests in Round 1 and Round 3 presents an apparent decrease in the difference between the MD and TMA groups. This data seems to suggest that the difference between the responses of these groups decrease over the span of the Algebra 1 course, which in turn suggests that the MD group has a higher growth rate than the TMA group.

Limitations

There are several limitations to this study. One limitation stems from the use of the learning trajectory (LT) created by Blanton et al. (2015) as the basis of the theoretical framework of this study. This learning trajectory was developed with early elementary students who were experiencing instruction based on a functional approach to algebra. While this framework was powerful in interpreting the results of this study, because learning trajectories are tied so closely to the context in which they are developed (Blanton et al., 2015), the interpretation of the results of this study may have been unduly impacted by the functional context of the LT.

Furthermore, the differences in types of items examined in this study and those used in the development of the LT may have contributed to the absence of responses consistent with LT levels 1, 2, and 6 in the current study. Because of the exposure to algebra and variable that students in high schools have most likely experienced, it was not expected to see responses consistent with LT level 1 in which there is no variation perceived by the student. While there has been documentation of high school algebra students treating variables as objects (e.g., MacGregor & Stacey, 1997), which is consistent with LT level 2, none of the items selected for this study specifically were designed to illicit this conception of variable (i.e., no multiple-choice options were consistent with this LT level). Because the LT levelv6 is so closely tied to the functional context of the study by Blanton et al. (2015) and because none of the items analyzed referred to functions, it is not surprising that no student responses were consistent with LT level 6.

Another limitation of this study is the result of the use of existing data. Because of the prior existence of this data, the design of this study was restrained. Specifically, because of the percentile information that was available for the IEOC, the cut off scores used to group students were limited to specific scores. Without this limitation, groups could have been created based on similar responses rather than comparing the differences among the pre-determined cut-off points. Because the data were collected in 2013–2014, the students were not available for further data collection or interviews. Thus, the insight that was available into the students' conceptions of variable was limited to inferences made from the responses that they provided to the items analyzed.

Furthermore, the prior existence of the data impacted the statistical analyses that could be performed. Specifically, in considering the growth of the students from Round 1 to Round 3, the data available was not ideally suited to certain tests to determine if the change was significant. Because the data was a pre- and post-test format, with the same students taking each Round, a *t*-test was not appropriate because it assumes independence of the samples. A common fix for this is to perform a paired-samples *t*-test, but this also was not suitable for this data because of the high number of students who only took the Round 1 or Round 3 measures, but not both. A 2 proportions *z*-test provided the best fit of the options considered, but this was still not ideal. A *z*-test assumes some knowledge about the population as a whole. Since the only data that was available was that from this project, by using the *z*-test, the population becomes the sample, so generalizations that can be made to the larger population are limited

A third limitation of this study comes from the placement of the items on the timed ASPM measures. In comparing response patterns across items, differences that were not due to the conceptions of variable of the students may have impacted the results. For example, on item CoV_A17, significant growth was measured from Round 1 to Round 3. Some of this growth will be accounted for because on the Round 3 administration of the measure, students were able to complete more of the items and thus there were fewer students who did not attempt this item on Round 3. This was not because they were better able to respond to this item, but because they were better at answering the previous questions which may or may not have reflected an increase in the sophisticated conception of variable.

A final limitation of this study is the low response rate of the students on the items. While this is related to the previous limitation in that the items that came later on the assessment were sometimes not attempted because the students ran out of time, this does not explain the low response rates on other items place towards the beginning of the measures. Implications of these slow response rates will be addressed in the following section, but they do have a limiting effect upon the interpretation of the results. Because a substantial number of students did not attempt each item, it is difficult to determine how the addition of these students' responses would influence the results. Without performing individual student interviews, it is difficult to ensure that all students respond to each item, but this is not feasible for large groups of students. It is likely that the group of students who did not respond to each item was not a representative group, meaning that without their responses, the response patterns of the group were likely different than they would have been with all students responding. Unfortunately, it is not possible to predict in what ways or magnitudes these impacts would be.

Discussion of Findings and Implications

The findings from this study demonstrate that the differences between Algebra 1 students' responses were limited. While understanding these differences can provide a better understanding of how to support low-achieving algebra students, the lack of more substantial differences is also very important. This section will discuss the implications of these results. Specifically, implications of the differences and the lack of differences will be discussed.

Differences

The primary differences identified by this study between the responses of MD and TMA students were a higher proportion of TMA students' responses that are consistent with LT level 5 on items with formats that involved procedural, algorithmic and properties-based structures. Because of the method of identification of the MD and TMA student groups based on the students' scores on a procedurally-based standardized assessment of mathematical achievement, the IEOC, these difference are not surprising. These differences also could be expected based on the characteristics of MD students identified in Chapter 2. Specifically, Hecht and Vagi's (2010) report of attributes of MD students being specific to a mathematical concept, skill, or domain suggests that achievement in one area of mathematics may not be a good indicator for performance in other mathematical areas. Furthermore, results reported by Malisani and Spagnolo (2009) suggested that even the conception of variable used by a student can vary based on the context and formulation of a problem. Even if these differences were expected, they are important nonetheless.

Items with expressions of the structure $x \pm a$ and $b \pm x$ (for a and b integers and x is a variable), did not show significant results, while those not of this structure did show significant results. For example, in CoV_A09, students compare the expressions $t + 3$ and $5 + t$. The structure of these types expressions are fairly simple, a constant plus or minus a variable. With these types of expressions, there are fewer factors to consider which may provide students with an opportunity to focus more on what the variable means and be less distracted by the structure of the item. On these simpler items, MD and TMA students responded similarly.

On the other hand, expressions with more complex structures may inhibit the students' ability to navigate both the operations, procedure, or properties that form the context of the item and also account for the meaning of the variable. For example, in CoV_B05, students are asked to compare the expressions $2t$ and $t + 2$. In this case, students must engage with what the variable t means and with the relationship between multiplication and addition. This classification of items as more or less procedurally complex may show a transition from the less to more complicated items in which there is little difference between students' responses among groups for the less complex items but the differences increase as the complexity of the item increases by relying more upon procedures, algorithms and properties. While it is possible that this result is due to the lower working memory and attention of MLD students (Geary et al., 2012), this is beyond the scope of this study to investigate.

Whatever the cause, these students with low-achievement seem to be less successful in reasoning about variable quantities when the expressions that contain those variables include more complex operations, procedures, algorithms, and are properties-based. These results support the findings reported in Chapter 2 that MD students perform lower on more advanced problems (Jones et al., 1997), are less likely to make sense of operations (Hecht and Vagi, 2010), or provide a valid explanation of why procedures work and how to use them to solve problems (Mazzocco & Devlin, 2008).

By encasing variables in more complex structures, these students are doubly disadvantaged as not only is the conceptual nature of the item difficult, but also now the procedural aspects of the item complicate it. Thus MD students do not have the same access to a conceptual understanding of variables as their TMA peers do. In other words, the procedural complexity of some items limit the ways in which MD students can develop generalizations about generalized quantities and abstract relationships.

Similarities

While the implications that can be drawn from the analysis of the differences between the responses of the MD and TMA groups are informative and helpful, the result that there are more instances of the lack of differences, which suggest similarities, is more impactful. This similarity between students' responses on conceptually-focused items when they scored differently on a procedurally-based assessment has significant implications.

First, this conceptual similarity is an indication that all students do not have the conception of variable that they need. This lack of differences on these conceptually-focused items are the result of the low achievement of all students on the items analyzed rather than from an unexpected high achievement of the MD students. This finding is different than those reported in Chapter 2 that MD students typically have a more limited conceptual understanding than TMA students (Butler et al., 2005; Geary, 2004; Hecht & Vagi, 2010; Jones et al., 1997). This implication should not be taken to mean that there are no differences in the mathematical understandings of all students or even that different categorizations of students are not useful. It does suggest that all students need help developing more sophisticated conceptions of variable. With the exception of one item, the highest proportion of both groups of students who provided a response that was coded as indicative of providing evidence of a conception of variable that allows for multiple values was 39% as reported in Tables 25 and 28. This means that for more than 60% of all students, at best, variables are viewed as a single unknown value at both the beginning and end of an algebra 1 course. That over 60% of our algebra 1 students at the end of that course may not be able to think about variables as having more than one value is concerning. If this is true, then it is no wonder why these same students struggle to develop a conceptual understanding of a two-variable equation like $3x + 2y = 6$ as representing the relationship between two quantities or similar inequality relationships. Without this foundational understanding, students are not well prepared for any future algebra work. These findings support the results reported in Chapter 2 that most algebra students have conceptions of variable with a low level of sophistication (Küchemann, 1978; Rosnick, 1980; Booth, 1984; Stacey & MacGregor, 1999).

In addition to the high proportion of students who have a conception of variable that only allows for one value, there were an alarming number of students who did not even attempt an explanation. Considering only items that were placed on the first half of the measure to reduce the effect of students running out of time and looking at the Round

3 data, about 40–60% of students did not attempt an explanation on the open-ended items (about 40–50% of TMA and about 60% of MD students). This lack of engagement with the mathematics may be typical of these students' classroom experiences and habits. Without engaging with the mathematics, students will not be able to develop their understanding of the concepts of algebra.

Second, in addition to the overall low proportion of students with a sophisticated conception of variable, the lack of growth by all students over the duration of the algebra 1 course suggests that the current instruction in algebra is insufficient to help students develop a conception that allows variables to represent multiple values simultaneously. In general, traditional mathematics instruction has focused primarily on procedural proficiency with little time allotted for conceptual development. This is also true specifically for a conceptual understanding of variables (Usiskin, 1988). In classroom observations, the ASPM project team verified this when reporting that instruction in project teacher classrooms was almost exclusively procedurally-based (B. Dougherty, personal communication, January 27, 2015). Dede (2004) suggested that the lack of increase in the sophistication of students' conceptions of variable may be attributable to the ways in which variables are initially introduced to students as an unknown value in equations with subsequent steps taken to ascertain its value. Thus many students begin with a single-value conception of variable and struggle to develop more sophisticated concepts of variable later.

Finally, these results showing more similarities than differences between the MD and TMA groups of students suggest that procedurally-based assessments provide an incomplete picture of a student's mathematical understanding and ability. When this incomplete picture is used as a basis for decision-making, lower-achieving students are disproportionally disadvantaged. These disadvantages can take many shapes. From being told that they are low-achieving, to being tracked with the low-ability group students may view themselves as incapable. Students who think of themselves as incapable are less likely to persevere in mathematics and are more likely to give up and stop trying. Boaler (2008) reports that ability grouping impacts not only the opportunities that students have in school mathematics, but also career options.

Conflicting Results

In addition to the cases discussed above in which the results showed either differences or the lack of differences, there were also some conflicting results. These conflicts arose when the results of the chi-squared tests were compared with the results from the *z*-tests. The first instance of conflicting results is with respect to the format of the items. The chi-squared tests for independence used to address Research Questions 2 and 3 provided no evidence that differences (or the lack of differences) were specific to the format of the item for either round. For example, one multiple-choice item (CoV_B05 in Round 3) showed significant differences between the MD and TMA groups of students while another similar multiple-choice item (CoV_A01 in Round 3) did not. Similarly, one open-ended item (CoV_B07 in Round 3) showed significant differences between the MD and TMA groups of students while another similar open-ended item (CoV_A09 in Round 3) did not. But, the data from the *z*-tests provides some evidence that the growth is related to the format of the item. Table 31 reports that no open-ended items provided evidence of growth in the sophistication of conception of variable, while some of the multiple-choice items did. Thus, it is possible that the open-ended format of some items interferes with

This conflict prevents conclusions from being drawn about the differences in MD and TMA students' responses on these formats of items.

The second instance of conflicting results is with respect to the relative growth rates of the MD and TMA groups. In comparing the number of chi-squared tests that produced results suggesting significant differences between these groups in Round 1 with Round 3, these data seem to suggest that the difference between the responses of these groups decrease over the course of the Algebra 1 course. This, in turn, suggests that the MD group had a higher growth rate than the TMA group. In Round 1 there were five out of the 10 items tested that produced showed differences between the MD and TMA groups' responses with a p-value less than 0.0838. But in Round 3, there were only four of the 12 items tested that showed differences with p-value less than 0.0838. The decrease in the number of differences between the responses of the MD and TMA students with p-values in this range suggests that the growth rate of the MD students might have been higher than that of the TMA students because these two groups became more similar over time. But, the z-test data provides little evidence of a difference in growth rate, as discussed in the Chapter 4 section for Research Question 4. These conflicting results mirror the different results obtained by other researchers, some of which have shown MD students with lower (Geary et al., 2012) or higher (Murphy et al., 2007) growth rates than their TMA peers. Furthermore, these differences in the number significant items could have been from chance or been largely due to the small number of responses of the two items that were not analyzed. Factors that could impact these conflicts include the content as well as possible differences in MLD and LMA students' growth rates (Murphy et al., 2007). Because of these mixed results and the lack of definitive evidence as to the growth of students over the course of the algebra course, further research needs to be done to investigate this question

Implications for Practice

The finding that the procedural complexity of some items limit the ways in which MD students can develop generalizations about generalized quantities and abstract relationships has implications in the classroom. All students need to have access to conceptual understanding of variables that is not obstructed by unnecessarily complicated contexts. This is not to say that these students should not be given opportunities to develop this procedural understanding and to integrate it with their understandings of variables and comparing variable quantities. Indeed, both of these and important outcomes for all algebra students to achieve. By unnecessarily complicating the variable-quantity comparisons with procedural, algorithmic, and properties-based context, the opportunities to improve in both the procedural and conceptual areas are interfered with. These results suggest that simple-structured expressions are similarly accessible to all students and could therefore be used as entry point to a classroom discussion or activity around this topic. Once students have established an understanding of the relationships between these expressions with variable quantities in this structure, more complexity could be added step-by-step to scaffold the students to applying a more sophisticated conception of variable in other situations.

The finding that all students do not a have a sophisticated conception of variable and that this level of sophistication does not increase over the course of an Algebra 1 course also has implications for the classroom. Again, this does not mean that there are no differences in the mathematical understandings of all students but that all students need help developing more sophisticated conceptions of variable. Blanton et al. (2015) have shown that young elementary students can develop these sophisticated conceptions of variable. Blanton and Kaput (2011) have advocated for exposing children to a variety of conceptions of variable to help them develop a more sophisticated conception of variable through tasks and activities that engage students with variables with multiple values as a way to help all students develop more sophisticated conceptions of variable.

Summary

Both the similarities and the differences identified in this study between MD and TMA student responses to items designed to assess their ability to compare expressions involving variables support the claim that most students in an algebra 1 course do not have and are not developing a conception of variable that allows for the variable to represent more than one value at a time. The few differences identified suggest that items with more complex operational, procedural, or property-based contexts are less accessible to MD students. The similarities identified support the claim that procedurally focused assessments provide incomplete descriptions of students. In both cases, MD students are disproportionately disadvantaged and these disadvantages impact not only their school experiences, but can have long-lasting impacts on career opportunities as well. These findings have implications for the classroom.

Future Research

Future research is needed to further understand the differences and similarities between the low- and typically-achieving groups of students and on how to implement this understanding to better meet the needs of our low-achieving students. The need of some of this research is illuminated by the limitations identified above. For example, research to develop and empirically verify a learning trajectory that describes the sophistication of algebra 1 students in a CCSSM based instructional setting is needed to provide a strong framework for describing the level and progression of conceptions of variable in that setting.

Further studies should also investigate the conceptions of variables of algebra 1 students through the use of interviews. This is necessary to develop a strong understanding of how students are actually thinking about variable. With this type of an understanding, more effective instruction and interventions can be designed to facilitate the development of more sophisticated conceptions of variable in students. Further studies should also study the differences in low- and typically-achieving students' conceptions of variables in different contexts, such as solving linear equations, multiple variable equations, and geometry.

With the implementation of CCSSM and the occurrence of variables as early as third grade, research also needs to be conducted to investigate the differences in low- and typically-achieving students' conceptions of variables in elementary school. Blanton et al.'s (2015) establishes that young children can develop these sophisticated conceptions of variable and provides a framework with which to describe these conceptions, but differences between low- and typically-achieving students are still unknown.

Closing Statement

In closing, this investigation revealed that the differences between MLD, LMA, and TMA students are limited and seem to be related to the level of complexity of the operations, procedures, and properties that are incorporated into an item. More than these differences, the similarities that were illuminated by this study impact how we perceive what successful and struggling students may look like, and what they need. The overall low proportion of students who have a sophisticated conception of variable suggests that all students need access to better conceptual instruction and experiences in algebra. The context and content of this instruction need to be appropriate for all students, which may mean that MD students required additional support than their TMA peers in developing these sophisticated conceptions of variables.

Despite the similarities that were identified in this study, students who are identified as MLD or MD may be disproportionally disadvantaged by traditional instructional practices and data-based decision making based on procedurally focused assessments and instruction these disadvantages need to be further investigated. These are the students most at risk to be “innumerate” (Geary et al., 2012, p. 206) because of the long lasting impact that being labeled as low-achieving can have on school mathematics opportunities and on future career options. These are the students who need our help. Through experiences with and exposure to variables that represent more than just one value from a younger age, these students can begin to develop a solid foundation upon which to build their future successes, both mathematically and as a contributing member of society.

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Appendix A

Sample of Concept of Variable Measure and Corresponding Materials

Concept of Variable Measure

Jon said, "$m - 1$ is always greater than $1 - m$." Do you agree with Jon?

A. Yes, Jon is correct because m is a positive number.
B. Yes, Jon is correct because you cannot substitute a negative number for m.
C. No, Jon is not correct because $1 - m$ is greater than $m - 1$ when m is negative integer.
D. No, Jon is not correct because these expressions are equivalent.

Answer ___________

If $x = d + 2$ and $d + 2 + x = y$, then which of the following statements is true?

A. $y = 2d + 4 + x$
B. $4d + 2 = y$
C. $y = 2x$
D. $6d = y$

Answer ___________

The equation below is true when $x = 6$:

$$x^3 + x = 222$$

What value of x will make the following equation true?

$$(2x)^3 + 2x = 222$$

$x =$ _____________

If $d - 243 = 542$, what does $d - 245$ equal?

Answer ___________

The solution to the equation $11 - x = 2(13 - 2x)$ is 5. What is the solution to the equation?

$$11 - \frac{x}{3} = 2(13 - 2(\frac{x}{3}))$$

Answer ____________

If $b - 878 = 915$, what does $b - 880$ equal?

Answer ____________

Dan challenged Amy to write an equation that has a solution of 3. Which equation could Amy have written?

A. $4 - x = 10 - 3x$
B. $3 + x = -(x + 3)$
C. $-2x = 6$
D. $x + 2 = 3$

Answer ___________

Carl simplified $9h - h$. He said an equivalent expression was $8h$. Do you agree with Carl?

A. Yes, because the distributive property can be applied to $9h$ and h to simplify it.
B. No, because h is a common factor in both so $h - h$ is 0, that leaves 9.
C. It not possible to determine if Carl is correct because you do not know the value of h.
D. Carl is only correct if h is a positive number.

Answer __________

Bart said, "$t + 3$ is less than $5 + t$."
Circle one: Always true Sometimes true Never true
Explain your answer.

If $h + m = 7$, what does $h + m + 7$ equal?

Answer ___________

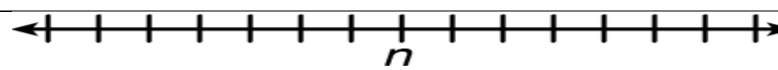
$$y - x = -(-x - y)$$

Circle one: Always true Sometimes true Never true
Explain your answer.

Sharon wrote the equation $c = 3h + 1$. If the value of h is increased by 2, the value of c will …

A. Increase by 1.
B. Increase by 2.
C. Increase by 3.
D. Increase by 6.

Answer ___________



Show on the number line where you would expect to find $n - 3$.

Cathy added $7x + 8y$ and got the sum $15xy$. What would you say to Cathy?

A. You are correct because $8 + 7$ is 15 and $x + y$ is xy.
B. You are correct because 7 and x are being multiplied and 8 and y are being multiplied. The Associative Property lets you combine these terms.
C. You are not correct because unlike terms cannot be combined.
D. It is not possible to know if Cathy is correct because you do not know the value of x and y.

Answer ___________

$$B + E + M = B + R + S$$

Circle one: Always true Sometimes true Never true
Explain your answer.

Jess said, "g^3 is equivalent to $3g$." What would you say to Jess?

A. Yes, it's true because the exponent means to multiply by 3.
B. Yes, it's true because the exponent means to use g as an addend 3 times.
C. No, it's not true because the exponent means to use g as a factor 3 times.
D. No, it's not true if g is a fraction. It is only true for whole numbers.

Answer ___________

Sam said, "$2n + 1$ represents a greater quantity than $2(n + 1)$." Do you agree with Sam?

A. Disagree, the two expressions are equivalent because they have the same numbers and variable.
B. Disagree, the distributive property produces a constant of 2 in the second expression as opposed to 1 in the first expression, and the variable has a coefficient of 2 in both expressions.
C. Agree, $2n + 1$ is greater than $2(n + 1)$ because the 2 is multiplying the n in the first expression, not the 1. The value of n could be greater than 1.
D. You cannot tell which is larger because it depends on the value of n.

Answer ___________

Write a quadratic equation whose solutions are –3 and 7.

Cupcakes cost c cents each and doughnuts cost d cents each. I buy 4 cupcakes and 3 doughnuts. What does $4c + 3d$ stand for or represent?

A. The cupcakes and doughnuts
B. The number of cupcakes and the number of doughnuts I buy
C. The weight of the cupcakes and doughnuts
D. The total cost of the cupcakes and doughnuts

Answer __________

Concept of Variable Standardized Administration Protocol

Concept of Variable

Hand out assessment, keeping the assessments face down. Ask students to keep the assessments face down and write their name, your (teacher's) name, period/block, and the date on the back of the assessment.

Give the standard directions:

The **FIRST** time you administer the CONCEPT OF VARIABLE assessments, say:

The problems on this measure are related to the information you are learning this year in algebra, but they are not in any special order. For example, a mathematical idea from the beginning of the course could be the last problem on the measure. Please look at each problem and decide if you know how to do it. If you know how, solve the problem. If you aren't certain, or think you can't solve the problem, skip it and move to the next one. Don't spend too much time on any one problem. The object of the measure is to answer as many problems correctly in the time available. Once you get to the end, go back and work on the problems you skipped. Remember that you may earn partial credit by giving an explanation, when you are asked to do so.

Some of the problems on this assessment are multiple choice. Other problems may ask you to explain your thinking or describe how you would solve a problem. Be specific when you write your explanations.

Now we'll do the first Concept of Variable measure. You will have 7 minutes to work on this three-page assessment. Remember, your job is to answer as many problems correctly as you can in 7 minutes. Please look at each problem, but if you do not know how to do it, skip it and move on. If you get to the end of the assessment before the time is up, go back and work on the more difficult problems.

When I say begin, please turn turn your paper over and start working. You will have 7 minutes. Please do your best work

Time for 7 minutes. When the timer goes off, say *Stop. Please put your pencils down,* and collect student papers.

Concept of Variable Scoring Guide

Jon said, "m – 1 is always greater than 1 – m." Do you agree with Jon?

A. Yes, Jon is correct because m is a positive number.
B. Yes, Jon is correct because you cannot substitute a negative number for m.
C. No, Jon is not correct because 1 – m is greater than m – 1 when m is negative integer.
D. No, Jon is not correct because these expressions are equivalent.

Score 2 points for C.

If $x = d + 2$ and $d + 2 + x = y$, then which of the following statements is true?

A. $y = 2d + 4 + x$
B. $y = 2x$
C. $4d + 2 = y$
D. $6d = y$

Score 2 points for B
Score 1 point for A

The equation below is true when $x = 6$:

$$x^3 + x = 222$$

What value of x will make the following equation true?

$$(2x)^3 + 2x = 222$$

Score 2 points for $x = 3$

If $d - 243 = 542$, what does $d - 245$ equal?

Score 2 points for 540
Score 1 point for an answer of 785 in the work or as the answer
OR for 544 as the answer

The solution to the equation $11 - x = 2(13 - 2x)$ is 5. What is the solution to the equation?

$$11 - \frac{x}{3} = 2(13 - 2(\frac{x}{3}))$$

Score 3 points for 15
Score 2 points for $\frac{x}{3} = 5$ in the work or as the solution.

If $b - 878 = 915$, what does $b - 880$ equal?

Score 2 points for 913
Score 1 point for an answer of 1793 in the work or as the answer
OR for 917 as the answer

Dan challenged Amy to write an equation that has a solution of 3. Which equation could Amy have written?

A. $4 - x = 10 - 3x$
B. $3 + x = -(x + 3)$
C. $-2x = 6$
D. $x + 2 = 3$

Score 2 points A
Score 1 point for B or C

Carl simplified $9h - h$. He said an equivalent expression was $8h$. Do you agree with Carl?

A. Yes, because the distributive property can be applied to $9h$ and h to simplify it.
B. No, because h is a common factor in both so $h - h$ is 0, that leaves 9.
C. It is not possible to determine if Carl is correct because you do not know the value of h.
D. Carl is only correct if h is a positive number.

Score 2 points for A

Bart said, "$t + 3$ is less than $5 + t$."

Circle one: **Always true** Sometimes true Never true Explain your answer.

Score 1 point for circling Always True.

<u>Score 2 more points</u> if response indicates that *t* is the same quantity in both AND 3 is less than 5.

<u>Score 1 more point</u> if response indicates 3 is less than 5;
OR for substituting a specific value for *t* in both expressions.

If $h + m = 7$, what does $h + m + 7$ equal?

Score 1 for 14 or 7 + 7

$$y - x = -(-x - y)$$

Circle one: Always true Sometimes true Never true

Explain your answer.

Score 2 points for circling Sometimes true
Score 1 point for circling Never true

<u>Score 3 more points</u> for $x = y = 0$ or $x = 0$ if circling Sometimes true.
<u>Score 2 more points</u> if there is mention of multiplying by –1 on the right side of the equation;
OR show $y - x \neq y + x$ if circling Never true.
<u>Score 1 more point</u> for substituting specific values to support Sometimes true or Never true

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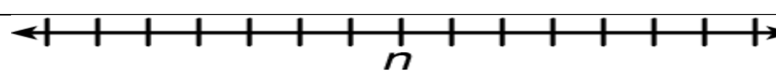
Sharon wrote the equation $c = 3h + 1$. If the value of h is increased by 2, the value of c will …

A. Increase by 1.
B. Increase by 2.
C. Increase by 3.
D. Increase by 6.

Score 1 point for D



Show on the number line where you would expect to find $n - 3$.

Score 2 points for a mark three units to left of *n* and labeled *n* – 3 OR no label
Score 1 point for a mark three units to the left of *n* with a label of –3

Cathy added $7x + 8y$ and got the sum $15xy$. What would you say to Cathy?

A. You are correct because 8 + 7 is 15 and $x + y$ is xy.

B. You are correct because 7 and x are being multiplied and 8 and y are being multiplied. The Associative Property lets you combine these terms. C. You are not correct because unlike terms cannot be combined. D. It is not possible to know if Cathy is correct because you do not know the value of x and y. **Score 1 point for C**
$B + E + M = B + R + S$ Circle one: Always true **Sometimes true** Never true Explain your answer. **Score 1 point for circling Sometimes true** **<u>Score 2 more points</u> for indicating $E + M = R + S$** **<u>Score 1 more point</u> for indicating that it depends on the substitution for E, M, R, S or substitution of the same numbers so that $E + M = R + S$ or for indicating that all of the variables could have the same value ($B = E = M = R = S$).**
Jess said, "g^3 is equivalent to $3g$." What would you say to Jess? A. Yes, it's true because the exponent means to multiply by 3. B. Yes, it's true because the exponent means to use g as an addend 3 times. C. No, it's not true because the exponent means to use g as a factor 3 times. D. No, it's not true if g is a fraction. It is true for whole numbers. **Score 1 point for C**
Sam said, "$2n + 1$ represents a greater quantity than $2(n + 1)$." Do you agree with Sam? A. Disagree, the two expressions are equivalent because they have the same numbers and variable. B. Disagree, the distributive property produces a constant of 2 in the second expression as opposed to 1 in the first expression, and the variable has a coefficient of 2 in both expressions. C. Agree, $2n + 1$ is greater than $2(n + 1)$ because the 2 is multiplying the n in the first expression, not the 1. The value of n could be greater than 1. D. You cannot tell which is larger because it depends on the value of n. **Score 2 points for B**
Write a quadratic equation whose solutions are –3 and 7. **Score 3 points for any quadratic equation equivalent to $(x + 3)(x - 7) = 0$ (It must be an equation.)** **Score 2 point for an expression equivalent to $(x + 3)(x - 7)$** **Score 1 point for an equation equivalent to $x + y = 4$ OR $x - y = \pm 10$.**
Cupcakes cost c cents each and doughnuts cost d cents each. I buy 4 cupcakes and 3 doughnuts. What does $4c + 3d$ stand for or represent? A. The cupcakes and doughnuts B. The number of cupcakes and the number of doughnuts I buy

C. The weight of the cupcakes and doughnuts
D. The total cost of the cupcakes and doughnuts

Score 1 point for D

Appendix B

Sample of Proportional Reasoning Measure and Corresponding Materials

Proportional Reasoning Measure

For every foot of fence built (t), a carpenter needs a consistent number of nails (n). What does the equation tell you?

$$n = 12t + 24$$

A. You need 36 nails and boards.
B. The number of nails increases by 12 for every foot of fence.
C. The number of nails increases 12 times for every foot of fence.
D. The number of nails increases by 24 for every foot of fence.

Answer ___________

Do these ratios represent the same relationship? 3:4 and $\frac{18}{24}$

A. No, because one is written like a fraction.
B. No, because they are different numbers.
C. Yes, because they are both in the ratio of 3 to 4.
D. Yes, because they are written like a ratio.

Answer ___________

In the expression $2j - 8$, the value of j increases by $\frac{1}{2}$. What is the effect on the value of the expression?

A. The value of $2j - 8$ increases by $\frac{1}{2}$.
B. The value of $2j - 8$ increases by 1.
C. The value of $2j - 8$ decreases by $\frac{1}{2}$.
D. The value of $2j - 8$ decreases by 8.

Answer ___________

Given the ratio of 6 miles every 4 hours, which of the following is an equivalent ratio?

A. 12 miles every 8 hours
B. 8 miles every 6 hours
C. 4 miles every 6 hours
D. 2 miles every 3 hours

Answer ___________

Which of the following is closest to the value of n in the equation $\frac{11.8}{5.6} = \frac{n}{3.9}$?

A. 2.8
B. 6.2
C. 8.2
D. 10.1

Answer ___________

Are these ratios equivalent? 3:5 and 6:8 Yes No Explain your answer.
Kerry made lemonade by adding 2 cups of lemon juice for every 3 cups of water. Jon made lemonade by adding 4 cups of lemon juice for every 7 cups of water. Whose lemonade will have a stronger lemon taste? A. Kerry's lemonade will have a stronger lemon taste. B. Jon's lemonade will have a stronger lemon taste. C. Both lemonades will taste the same. D. You cannot tell from the information given. Answer ___________
Are these equivalent equations? $\frac{1}{w} = \frac{1}{a} + \frac{1}{b}$ $w = a + b$ Yes No Explain your answer.
The market value of a house (m) increases steadily each year you own your home. The formula below shows the relationship between the market value and the number of years (y) that you own your home. What does the $3000y$ mean? $m = 3000y + 120{,}000$ A. Your house is worth \$123,000. B. The value of your home increases by \$3000 each year. C. The value of your home increases 3000 times each year. D. The value of your home increases by \$120,000 each year. Answer ___________
In the expression $4b + 1$, the value of b increases by 5. What is the effect on the value of the expression? A. The value of the expression increases by 1. B. The value of the expression increases by 5. C. The value of the expression increases by 10. D. The value of the expression increases by 20. Answer ___________
The y-intercept of a line is at the point (0, –3). As the value of x increases by 1, the y value increases by 4. Which of the following is the slope of the line? A. 1 C. 5 B. 4 D. You cannot tell. Answer ___________
Serena and Jeff have the same number of candies. Serena has 2 green candies for every 3 red candies. Jeff has 3 green candies for every 4 red candies. Which of the following describes the colors of their candies?

A. Serena and Jeff both have equal numbers of red candies.
B. Serena has more red candies than Jeff.
C. Jeff has more red candies than Serena.
D. It is not possible to determine because the total number of candies each of them has is not known.

Answer ___________

Find the value of p: $\frac{2}{25} = \frac{p}{500}$

A. $p = 2$
B. $p = 25$
C. $p = 40$
D. $p = 1000$

Answer ___________

Using the ratios $\frac{5}{7}$, 3:4, and $\frac{2}{3}$, which of the following describes their relationship?

A. $\frac{5}{7} > 3:4 > \frac{2}{3}$
B. $3:4 > \frac{5}{7} > \frac{2}{3}$
C. $\frac{2}{3} > \frac{5}{7} > 3:4$
D. $\frac{2}{3} > 3:4 > \frac{5}{7}$

Answer ___________

Line h has a slope of –7. Line t has a slope of 4. Which of the following is true?

A. Line h has a steeper slope than line t.
B. Line t has a steeper slope than line h.
C. Both lines have the same steepness.
D. It is not possible to tell with this information.

Answer ___________

Are these ratios equivalent? $\frac{3m}{4u}$ $\frac{9m^2}{16u^2}$

Yes No

Explain your answer.

Sam made juice by adding 6 cups of orange juice for every 53 cups of water. Tom made juice by adding 5 cups of orange juice for every 42 cups of water. Which one had the stronger juice?

A. Sam's juice was stronger.
B. Tom's juice was stronger.
C. Their juices were the same.
D. You cannot tell by the information given.

Answer ___________

The ratio of each term to the next term is the same in the following sequence of numbers.

14   42   126

Using the same pattern, what is the next term after 126?

Answer __________

Proportional Reasoning Standardized Administration Protocol

Proportional Reasoning

Hand out assessment, keeping the assessments face down. Ask students to keep the assessments face down and write their name, your (teacher's) name, period/block, and the date on the back of the assessment.

Give the standard directions:

The **FIRST** time you administer the PROPORTIONAL REASONING assessments, say:

The problems on this measure are related to the information you are learning this year in algebra, but they are not in any special order. For example, a mathematical idea from the beginning of the course could be the last problem on the measure. Please look at each problem and decide if you know how to do it. If you know how, solve the problem. If you aren't certain, or think you can't solve the problem, skip it and move to the next one. Don't spend too much time on any one problem. The object of the measure is to answer as many problems correctly in the time available. Once you get to the end, go back and work on the problems you skipped. Remember that you may earn partial credit by giving an explanation, when you are asked to do so.

Some of the problems on this assessment are multiple choice. Other problems may ask you to explain your thinking or describe how you would solve a problem. Be specific when you write your explanations.

Now we'll do the first Proportional Reasoning measure. You will have 7 minutes to work on this three-page assessment. Remember, your job is to answer as many problems correctly as you can in 7 minutes. Please look at each problem, but if you do not know how to do it, skip it and move on. If you get to the end of the assessment before the time is up, go back and work on the more difficult problems.

When I say begin, turn your paper over and start working. You will have 7 minutes. Please do your best work

Time for 7 minutes. When the timer goes off, say *Stop. Please put your pencils down,* and collect student papers.

Proportional Reasoning Scoring Guide

For every foot of fence built (t), a carpenter needs a consistent number of nails (n). What does the equation tell you?

$$n = 12t + 24$$

A. You need 36 nails and boards.
B. The number of nails increases by 12 for every foot of fence.
C. The number of nails increases 12 times for every foot of fence.
D. The number of nails increases by 24 for every foot of fence. **Score 1 point for B**

Do these ratios represent the same relationship? 3:4 and $\frac{18}{24}$

A. No, because one is written like a fraction.
B. No, because they are different numbers.
C. Yes, because they are both in the ratio of 3 to 4.
D. Yes, because they are written like a ratio. **Score 1 point for C**

In the expression $2j - 8$, the value of j increases by $\frac{1}{2}$. What is the effect on the value of the expression?

A. The value of $2j - 8$ increases by $\frac{1}{2}$.
C. The value of $2j - 8$ decreases by $\frac{1}{2}$.

B. The value of $2j - 8$ increases by 1.
D. The value of $2j - 8$ decreases by 8.

Score 1 point for B

Given the ratio of 6 miles every 4 hours, which of the following is an equivalent ratio?

A. 12 miles every 8 hours
B. 8 miles every 6 hours
C. 4 miles every 6 hours
D. 2 miles every 3 hours **Score 1 point for A**

Which of the following is closest to the value of n in the equation $\frac{11.8}{5.6} = \frac{n}{3.9}$?

A. 2.8
C. 8.2
B. **6.2**
D. 10.1

Score 2 points for C
Score 1 point for B

3:5 and 6:8

Are these ratios equivalent? Yes **No** Explain your answer.

Score 1 point for NO

<u>Score 3 more points</u> for explanation using multiplicative thinking (3 x 2 = 6 but 5 x 2 ≠ 8; OR 6 ÷ 3 = 2 but 8 ÷ 2 ≠ 5)
<u>Score 2 more points</u> for simplifying 6:8 as 3:4 and comparing to 3:5
<u>Score 1 more point</u> for dividing 5 by 3 and 8 by 6 and comparing the quotients as

justification <u>OR</u> for indicating 'you can't divide evenly' **<u>Score 0 more</u> points for additive thinking (3 + 2 = 5 and 6 + 2 = 8 OR 3 + 3 = 6 and 3 + 5 = 8)**
Kerry made lemonade by adding 2 cups of lemon juice for every 3 cups of water. Jon made lemonade by adding 4 cups of lemon juice for every 7 cups of water. Whose lemonade will have a stronger lemon taste? A. **Kerry's lemonade will have a stronger lemon taste**. B. Jon's lemonade will have a stronger lemon taste. C. Both lemonades will taste the same. D. You cannot tell by the information given. **Score 1 point for A**
Are these equivalent equations? $\frac{1}{w}=\frac{1}{a}+\frac{1}{b}$ $w = a + b$ Yes No Explain your answer. **Score 1 point for No** **<u>Score 3 more points</u> for transforming to an equivalent equation from one form to another** **<u>Score 2 more points</u> if disproved by substituting variables to show an example** **<u>Score 1 more point</u> if they discuss reciprocals explicitly**
The market value of a house (*m*) increases steadily each year you own your home. The formula below shows the relationship between the market value and the number of years (*y*) that you own your home. What does the 3000*y* mean? $m = 3000y + 120{,}000$ A. Your house is worth $123,000. **B. The value of your home increases by $3000 each year.** C. The value of your home increases 3000 times each year. D. The value of your home increases by $120,000 each year. **Score 1 point for B**
In the expression $4b + 1$, the value of *b* increases by 5. What is the effect on the value of the expression? A. The value of the expression increases by 1. B. The value of the expression increases by 5. C. The value of the expression increases by 10. **D. The value of the expression increases by 20.** **Score 1 point for D**
The *y*-intercept of a line is at the point (0, –3). As the value of *x* increases by 1, the *y* value increases by 4. Which of the following is the slope of the line? A. 1 C. 5 B. **4** D. You cannot tell. **Score 1 point for B**
Serena and Jeff have the same number of candies. Serena has 2 green candies for every 3 red candies. Jeff has 3 green candies for every 4 red candies. Which of the following describes

the colors of their candies? A. Serena and Jeff both have equal numbers of red candies. B. Serena has more red candies than Jeff. C. **Jeff has more red candies than Serena.** D. It is not possible to determine because the total number of candies each of them has is not known. **Score 2 points for C**
Find the value of *p*: $\frac{2}{25} = \frac{p}{500}$ A. $p = 2$ C. $\boldsymbol{p = 40}$ B. $p = 25$ D. $p = 1000$ **Score 1 point for C**
Using the ratios $\frac{5}{7}$, 3:4, and $\frac{2}{3}$, which of the following describes their relationship? A. $\frac{5}{7} > 3{:}4 > \frac{2}{3}$ C. $\frac{2}{3} > \frac{5}{7} > 3{:}4$ B. $\boldsymbol{3{:}4 > \frac{5}{7} > \frac{2}{3}}$ D. $\frac{2}{3} > 3{:}4 > \frac{5}{7}$ **Score 3 points for B** **Score 2 points for A or D** **Score 1 point for C**
Line *h* has a slope of –7. Line *t* has a slope of 4. Which of the following is true? **A. Line *h* has a steeper slope than line *t*.** B. Line *t* has a steeper slope than line *h*. C. Both lines have the same steepness. D. It is not possible to tell with this information. **Score 1 point for A**
Are these ratios equivalent? $\frac{3m}{4u}$ $\frac{9m^2}{16u^2}$ Yes **No** Explain your answer. **Score 1 point for No** **<u>Score 3 more points</u> for explanation indicating squaring a term may not create an equivalent expression** **<u>Score 2 more points</u> if there are values substituted for *m* and *u* to 'prove' it** **<u>Score 1 more point</u> if there is mention about the exponents**
Sam made juice by adding 6 cups of orange juice for every 53 cups of water. Tom made juice by adding 5 cups of orange juice for every 42 cups of water. Which one had the stronger juice? A. Sam's juice was stronger. B. **Tom's juice was stronger**. C. Their juices were the same.

D. You cannot tell by the information given.

Score 1 point for B

The ratio of each term to the next term is the same.

14 42 126

What is the term after 126?

Score 2 points for 378

Score 1 point if they show work where there is multiplication by 3

Appendix C

Sample of Translations, Functions, and Graphs Measure and Corresponding Materials

Translations, Functions, and Graphs Standardized Administration Protocol

Hand out assessment (with the sample page), keeping the assessments face down. Ask students to keep the assessments face down and write their name, their teacher's name, period/block, and the date on the back of the assessment.

You will need to project a copy of the sample page while giving these directions. Give the standard directions:

The **FIRST** time you administer TRANSLATIONS, FUNCTIONS, AND GRAPHING assessments, say:

In algebra, it is important to be able to use information about relationships between and among quantities and translate between different ways of showing that relationship, like equations, graphs, and data tables. The Translations, Functions and Graphing measure doesn't require that you remember specific steps for solving equations. Instead, it lets you show how well you understand the different ways for representing relationships.

Please turn your paper over. This sample page shows some examples of the types of problems on the Translations, Functions, and Graphing measure. At the top of the page, you will see a row of equations. Each one is labeled with a letter (A, B, C, or D). Below this row, you will see a row of graphs. Your first task is to match the letter for each equation in the top row to one of the graphs in the next row. Let's look at an example. The first equation, A, is $y = x + 4$. Which graph does this equation match?

Pause and wait for students to identify the third graph. If they do not, say: *In this equation, if x were 0, what would y be? [4] Do you see a graph that has the point (0,4) on it? Yes, that's right; the third graph has the point (0,4) as part of the line. So you would write the letter A on the line next to this graph.* (Demonstrate writing the letter in the blank.) *You can use your knowledge about equations and graphs to match the graphs in this row with the equations marked A, B, C, and D.*

NOTE to TEACHERS: Feel free to ask students to share other ways to solve this problem (i.e., noting slope and intercept from the equation). Students should understand that there are multiple solution methods, but <u>do not</u> spend a great deal of time 'teaching' students how to do this task.

Now let's look at the data tables in the third row. Notice that there are 5 data tables, but only 4 equations, so you will have to use at least one letter more than once. Look at the third table with the number 7 in the upper left corner. Can you decide which equation goes with this set of values for x and y? (Pause, wait for students to respond with equation C). *Yes, that's right. Equation C is $y = 0$ and in the third data table, the y value is always 0 no matter what value x is. So you would write a C next to this data table* (model writing answer). *Did anyone else solve this problem another way?* Pause and allow comments. *Yes, that's right, there are several ways to solve these problems.*

Some of the problems on this measure are multiple choice and require you to answer with a letter (A, B, C, or D) or a number next to an equation, graph, or table. Problem 10 asks you to identify a graph, so you will circle 1, 2, 3, or 4 to show which graph matches this description.

Other problems may ask you to select a graph, table, or equation and explain your thinking or describe how you would solve a problem. Be specific when you write your explanations. Number 11 is an example of this type of problem.

Now we'll take a minute so <u>you</u> can practice solving the remaining questions on this sample page. If you finish before I say 'Stop', please do NOT turn to the next page. Any questions? Ready, begin. [time for 1 minute] Stop, pencils down.

Now that you've had a chance to try out this type of assessment, do you have any questions? [Only answer procedural questions—do not suggest ways to solve the problems.]

Now we'll do the first Translations, Functions, and Graphing measure. You will have 7 minutes to work on this four-page assessment. Remember, your job is to answer as many problems correctly as you can in 7 minutes. Please look at each problem, but if you do not know how to do it, skip it and move on. If you get to the end of the assessment before the time is up, go back and work on the problems you skipped.

When I say begin, please turn past the sample page and start working. You will have 7 minutes. Please do your best work

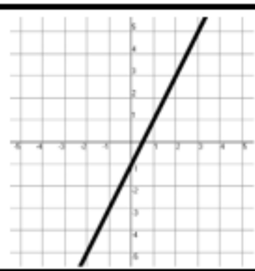
Time for 7 minutes. When the timer goes off, say Stop. Please put your pencils down, and collect student papers.

Translations, Functions, and Graphs Measure

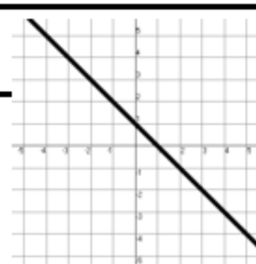
Match each equation to a corresponding graph or data table.

A $y = x$	B $2x = y + 1$	C $y = 1.5$	D $y = -x + 1$

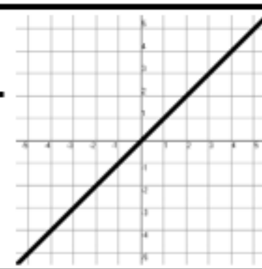
1. ____



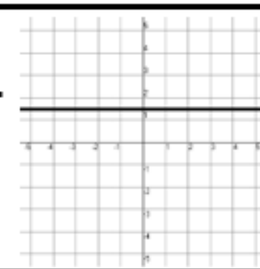
2. ____



3. ____



4. ____



5. ____

x	y
2	1.5
1	1.5
0	1.5
–1	1.5
–2	1.5

6. ____

x	y
2	–1
1	0
0	1
–1	2
–2	3

7. ____

x	y
2	3
1	1
0	–1
–1	–3
–2	–5

8. ____

x	y
4	4
2	2
0	0
–2	–2
–4	–4

9. ____

x	y
4	–3
2	–1
0	1
–2	3
–4	5

10. Which **graph(s)** represents: The sum of a variable and one is equal to two multiplied by a second variable?

11. Which table(s) represents a rate of change of 0?

12. Jake graphed equation B. Jamie graphed the equation $y = 4x - 3$. How does Jamie's graph compare to Jake's graph?

A) Jamie's graph is translated down 3 units from Jake's graph.
B) Jamie's graph is translated up 3 units from Jake's graph.
C) Jamie's graph is steeper than Jake's graph.
D) Jamie's graph is less steep than Jake's graph.

____

Match each data table to a corresponding graph or equation.

A

x	y
4	16
2	4
0	0
–2	4
–4	16

B

x	y
2	4
1	2
0	1
–1	$\frac{1}{2}$
–2	$\frac{1}{4}$

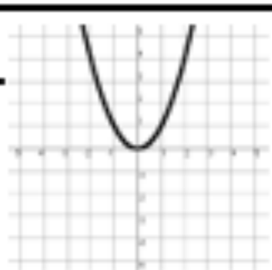
C

x	y
2	5
1	3
0	1
–1	–1
–2	–3

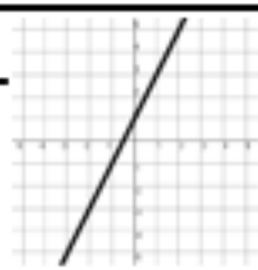
D

x	y
4	3
2	1
0	–1
–2	–3
–4	–5

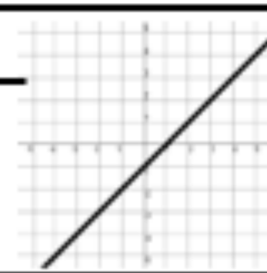
13. ____



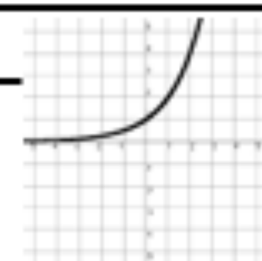
14. ____



15. ____



16. ____



17. $y = 2x + 1$ ____

18. $y = 2^x$ ____

19. $y = x - 1$ ____

20. $y = x^2$ ____

21. $3y = 6x + 3$ ____

22. Which **table**(s) shows a constant rate of change? Explain your answer.

23. Which **equation**(s) shows a non-constant rate of change? Explain your answer.

24. Kate graphed the equation in problem 20. Jim graphed the equation $y = x^2 - 4$. How does Jim's graph compare to Kate's graph?

A) Jim's graph is translated down 4 units from Kate's graph.
B) Jim's graph is translated up 4 units from Kate's graph.
C) Jim's graph is translated left 4 units from Kate's graph.
D) Jim's graph is narrower than Kate's graph.

____

25. Tiles are arranged in a pattern as shown in the drawing below.

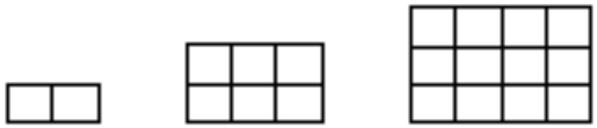


Figure 1 Figure 2 Figure 3

Let y represent the number of tiles in the figure and n represent the figure number. Which of the following functions represents the pattern?

A) $y = n^2 + 2$
B) $y = 2n + 1$
C) $y = n(n + 1)$
D) It is not possible to represent the pattern with a function.

26. The graph of which of the following equations would be perpendicular to the graph of $3y = 6x - 9$?

A) $y = 2x + 3$, because its slope is the same as the given equation.
B) $y = -2x + 3$, because its slope is opposite in sign as the given equation.
C) $y = \frac{1}{2}x + 3$, because its slope is the reciprocal of the slope in the given equation.
D) $y = -\frac{1}{2}x + 3$, because its slope is opposite in sign and the reciprocal of the slope in the given equation

27. Asa drew a line with a slope = 0. Emily drew a line with an undefined slope. Which statement is true?

A) Asa's graph was vertical and crossed the y–axis.
B) Asa's graph was horizontal and crossed the x–axis.
C) Emily's graph was vertical and crossed the x–axis.
D) Emily's graph was horizontal and crossed the y–axis.

28. David graphed the equation $y = 2x^2$. Sarah graphed the equation $y = \frac{1}{2}x^2 + 1$. How does Sarah's graph compare to David's graph?

A) Sarah's graph is wider and is translated up 1 unit from David's graph.
B) Sarah's graph is narrower and is translated up 1 unit from David's graph.
C) Sarah's graph is wider and is translated left 1 unit from David's graph.
D) Sarah's graph is translated up 1 unit and right $\frac{1}{2}$ of a unit from David's graph.

29. Select the pair of equations and explanation resulting in parallel lines.

A) $2x + y = 8$ and $2y = 4x + 16$, because they have the same slope.
B) $2y = -4x - 6$ and $-2y = -4x - 4$, because they have opposite slopes.
C) $y = x + 1$ and $-x + y = 1$, because they have opposite slopes.
D) $y = 3x + 9$ and $2y = 6x - 2$, because they have the same slope.

______

30. Shawn graphed $y = x^2$. Now he wants to make his graph narrower and translated right. What might the equation of his new graph be?

A) $y = 2(x + 1)^2 + 3$
B) $y = \frac{1}{2}(x + 1)^2 - 3$
C) $y = 2(x - 1)^2 - 1$
D) $y = \frac{1}{2}(x - 1)^2 + 1$

______

31. Jill said that $y > -2x + 1$ would have solutions in all quadrants except quadrant III. Do you agree or disagree with Jill? Explain your answer.

32. Which of the following predicts a line that slopes down from left to right and goes through Quadrants II, III, and IV?

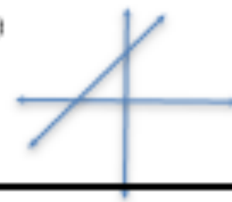
A) $y = 2x - 3$
B) $y = -2x - 3$
C) $y = x + 1$
D) $y = -x + 1$

______

33. Which of the following graphs could represent the equation $y = -0.999 + 1.00001x$?

A) 

B) 

C) 

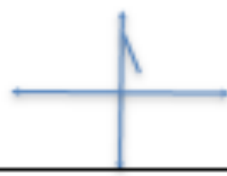
D) 

______

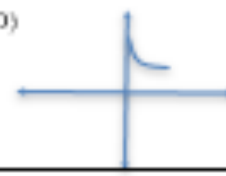
34. Which of the following graphs could represent the table:

x	0	1	2	3	4	5
y	600	562.5	525	487.5	450	412.5

A) 

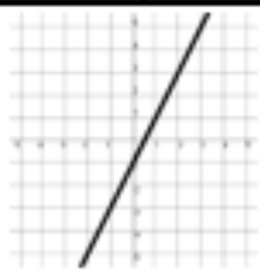
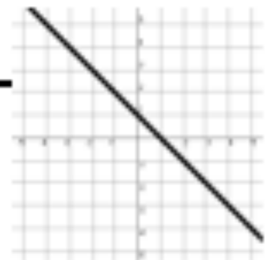
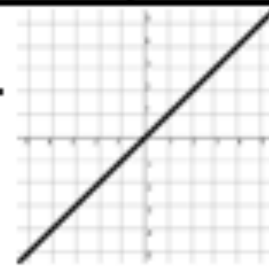
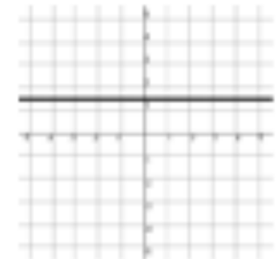
B) 

C) 

D) 

______

Translations, Functions, and Graphs Scoring Guide
Match each equation to a corresponding graph or data table.
Score 1 point for each correct answer for items 1 – 9.

A $y = x$	B $2x = y + 1$	C $y = 1.5$	D $y = -x + 1$
1. B 	2. D 	3. A 	4. C 

5. C

x	y
2	1.5
1	1.5
0	1.5
–1	1.5
–2	1.5

6. D

x	y
2	–1
1	0
0	1
–1	2
–2	3

7. B

x	y
2	3
1	1
0	–1
–1	–3
–2	–5

8. A

x	y
4	4
2	2
0	0
–2	–2
–4	–4

9. D

x	y
4	–3
2	–1
0	1
–2	3
–4	5

10. Which **graph**(s) represents: The sum of a variable and one is equal to two multiplied by a second variable?

Score 1 point for 1

11. Which table(s) represents a rate of change of 0?

Score 1 point for 5
If student gives 2 tables (even if 1 is table 5), it is scored 0.

12. Jake graphed equation B. Jamie graphed the equation $y = 4x - 3$. How does Jamie's graph compare to Jake's graph?

A) Jamie's graph is translated down 3 units from Jake's graph.
B) Jamie's graph is translated up 3 units from Jake's graph.
C) Jamie's graph is steeper than Jake's graph.
D) Jamie's graph is less steep than Jake's graph.

Score 1 point for C.

Match each data table to a corresponding graph or equation.
Score 1 point for correct answers 13–21.

A

x	y
4	16
2	4
0	0
–2	4
–4	16

B

x	y
2	4
1	2
0	1
–1	$\frac{1}{2}$
–2	$\frac{1}{4}$

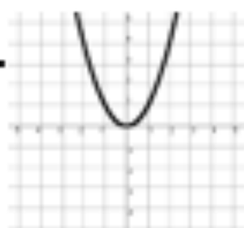
C

x	y
2	5
1	3
0	1
–1	–1
–2	–3

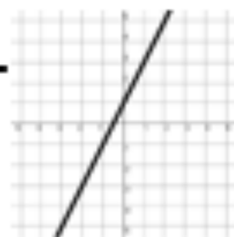
D

x	y
4	3
2	1
0	–1
–2	–3
–4	–5

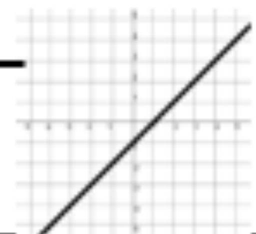
13. A



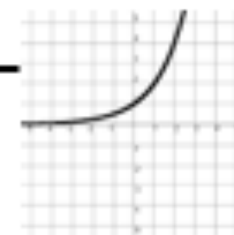
14. C



15. D



16. B



17. $y = 2x + 1$ C

18. $y = 2^x$ B

19. $y = x - 1$ D

20. $y = x^2$ A

21. $3y = 6x + 3$ C

22. Which **table(s)** shows a constant rate of change? Explain your answer. **2 points for C AND D/1 point for C OR D**

<u>2 more points</u> explaining that as x increases by 1, y increases by 2 (C) and as x increases by 1, y increases by 1 (D

<u>1 more point</u> <u>explaining that y increases by 2 in C and by 1 in D</u>

23. Which **equation(s)** shows a non-constant rate of change? Explain your answer.**2 point for 18 AND 20/1 point for 18 OR 20**

<u>2 more points</u> for indicating exponential or quadratic functions are not linear and thus are not constant

<u>1 more point</u> for indicating that the associated table doesn't have consistent differences

24. Kate graphed the equation in problem 20. Jim graphed the equation $y = x^2 - 4$. How does Jim's graph compare to Kate's graph?

A) Jim's graph is translated down 4 units from Kate's graph.
B) Jim's graph is translated up 4 units from Kate's graph.
C) Jim's graph is translated left 4 units from Kate's graph.
D) Jim's graph is narrower than Kate's graph.

Score 1 point for A.

25. Tiles are arranged in a pattern as shown in the drawing below.

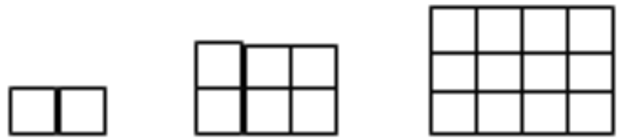


Figure 1 Figure 2 Figure 3

Let y represent the number of tiles in the figure and n represent the figure number. Which of the following functions represents the pattern?

A) $y = n^2 + 2$
B) $y = 2n + 1$
C) $y = n(n + 1)$
D) It is not possible to represent the pattern with a function.

Score 2 points for C
Score 1 point for A

26. The graph of which of the following equations would be perpendicular to the graph of $3y = 6x - 9$?

A) $y = 2x + 3$, because its slope is the same as the given equation.
B) $y = -2x + 3$, because its slope is opposite in sign as the given equation.
C) $y = \frac{1}{2}x + 3$, because its slope is the reciprocal of the slope in the given equation.
D) $y = -\frac{1}{2}x + 3$, because its slope is opposite in sign and the reciprocal of the slope in the given equation

Score 2 points for D
Score 1 point for B or C

27. Asa drew a line with a slope = 0. Emily drew a line with an undefined slope. Which statement is true?

A) Asa's graph was vertical and crossed the y–axis.
B) Asa's graph was horizontal and crossed the x–axis.
C) Emily's graph was vertical and crossed the x–axis.
D) Emily's graph was horizontal and crossed the y–axis.

Score 1 point for C

28. David graphed the equation $y = 2x^2$. Sarah graphed the equation $y = \frac{1}{2}x^2 + 1$. How does Sarah's graph compare to David's graph?

A) Sarah's graph is wider and is translated up 1 unit from David's graph.
B) Sarah's graph is narrower and is translated up 1 unit from David's graph.
C) Sarah's graph is wider and is translated left 1 unit from David's graph.
D) Sarah's graph is translated up 1 unit and right $\frac{1}{2}$ of a unit from David's graph.

Score 1 point for A

29. Select the equations and explanation resulting in parallel lines. A) $2x + y = 8$ and $2y = 4x + 16$, because they have the same slope. B) $2y = -4x - 6$ and $-2y = -4x - 4$, because they have opposite slopes. C) $y = x + 1$ and $-x + y = 1$, because they have opposite slopes. **D)** $y = 3x + 9$ and $2y = 6x - 2$, because they have the same slope. **2 points for D** **1 Point for A**	30. Shawn graphed $y = x^2$. Now he wants to make his graph narrower and translated right. What might the equation of his new graph be? A) $y = 2(x + 1)^2 + 3$ B) $y = \frac{1}{2}(x + 1)^2 - 3$ C) $y = 2(x - 1)^2 - 1$ **2 point for C** D) $y = \frac{1}{2}(x - 1)^2 + 1$ **1 point for A or D**
31. Jill said that $y > -2x + 1$ would have solutions in all quadrants except quadrant III. Do you agree or disagree with Jill? Explain your answer. **1 point for agree** **<u>2 more points</u> indicating that the line y = 3x – 2 passes through 3 quadrants <u>AND</u> all values below the line** **<u>1 more point</u> for substituting values from the appropriate quadrants <u>OR</u> showing a graph**	32. Which of the following predicts a line that slopes down from left to right and goes through Quadrants II, III, and IV? A) $y = 2x - 3$ B) $y = -2x - 3$ C) $y = x + 1$ D) $y = -x + 1$ **2 points for B** **1 point for D**

33. Which of the following graphs could represent the equation $y = -0.999 + 1.00001x$?

A)  B)  C)  D) 

Score 2 points for B

34. Which of the following graphs could represent the table:

x	0	1	2	3	4	5
y	600	562.5	525	487.5	450	412.5

A)  B)  C)  D) 

Score 2 points for B

Appendix D

Sample of Algebra Basic Skills Measure and Corresponding Materials

Algebra Basic Skills Measure

Solve: $9 + a = 15$	$a =$	Solve: $10 - 6 = g$	$g =$
Evaluate: $12 + (-8) + 3$		Simplify: $9 - 4d + 2 + 7d$	
Simplify: $2x + 4 + 3x + 5$		Simplify: $5(3 - n) - n$	
Solve: $4 = 12 - k$	$k =$	Solve: $(q)(5) = 30$	$q =$
Simplify: $4(3 + u) - 7$		Simplify: $2 + w(w - 5)$	
Simplify: $b + b + 2b$		Simplify: $4 - 7b + 5(b - 1)$	
Solve: $\frac{r}{6} = \frac{12}{18}$	$r =$	Solve: 1 foot = 12 inches 5 feet = ____ inches	
Simplify: $7 - 3(x - 2)$		Simplify: $u + 2u - 4u$	
Evaluate: $-5 + (-4) - 1$		Solve: $7 = x + 4$	$x =$
Solve: $9 = 63 \div c$	$c =$	Simplify: $-5(q + 3) + 9$	
Simplify: $2(k - 1) + 4 + 5k$		Evaluate: $(3)(-6)$	
Simplify: $8m - 9(m + 2)$		Simplify: $y^2 + y - 4y + 3y^2$	
Solve: 3 feet = 1 yard ____ feet = 9 yards		Solve: $\frac{12}{2} = \frac{48}{m}$	$m =$

Evaluate: $4 - (-2) + 8$	
Simplify: $2k + 3 - 5(k + 7)$	
Solve: $3 \cdot 8 = m$	$m =$
Solve: $d - 5 = 8$	$d =$
Simplify: $v(6 + v) - 3v$	
Simplify: $-3w^2 + 5w^2 - 5 + 12$	
Solve: 4 quarts = 1 gallon ____ quarts = $3\frac{1}{2}$ gallons	
Simplify: $9 - 4(v + 2)$	
Solve: $28 = 4r$	$r =$
Solve: $\frac{36}{6} = n$	$n =$
Simplify: $16 + 3(p - 4) - 2p$	
Simplify: $3z - 8z + 2 + 9$	
Simplify: $c - 3(c + 2) + 8$	

Solve: $\frac{1.5}{3} = \frac{h}{9}$	$k =$
Simplify: $7b - 4 - 3 - 2b$	
Simplify: $2w - 3(w - 4)$	
Solve: $6 + 7 = v$	$v =$
Evaluate: $-5 + 6 - 6$	
Simplify: $3(c + 2) - 2c$	
Solve: 1 pound = 16 ounces ____ pound = 8 ounces	
Simplify: $6a + 2a - 9 + 3a^2$	
Simplify: $-3(3 + u) - 2u + 5$	
Solve: $\frac{500}{j} = \frac{10}{2}$	$j =$
Simplify: $2c - 3c - cf$	
Solve: $k \div 6 = 8$	$k =$
Evaluate: $(-1)(-5)$	

Algebra Basic Skills Standardized Administration Protocol

ALGEBRA BASIC SKILLS PROBES:

Hand out probe ABS-1 (with the sample page attached), keeping the probes face down. Ask students to keep the probes face down and write their name, my (the teacher's) name, period/block, and the date on the back of the probe.

Give the detailed directions:

The **FIRST** time you administer ALGEBRA BASIC SKILLS algebra probes, say:

Please turn your paper over. This sample page shows some examples of the types of problems on the Algebra Basic Skills measures. The questions include solving simple algebra equations, simplifying expressions by combining like terms, using the distributive property to simplify expressions, and solving proportion, or ratio problems. Please write your answers on the RIGHT side of the dashed vertical line [point to appropriate area to demonstrate]. *Now we'll take a minute so* <u>*you*</u> *can practice doing an Algebra Basic Skills measure. If you finish the sample page before I say 'Stop', please do NOT turn to the next page. Any questions? Ready, begin.* [time for 1 minute and circulate to see that students are writing their answers in the correct area] *Stop, pencils down.*

Now that you've had a chance to try out this type of assessment, do you have any questions? [Only answer procedural questions—do not suggest ways to work the problems.]

Now we'll do the first Algebra Basic Skills measure. You will have 5 minutes to work on this two-page assessment. Write your answers on the right side of the dashed line. Remember, your job is to answer as many problems correctly as you can in 5 minutes. Please look at each problem, but if you do not know how to do it, skip it and move on. If you get to the end of the measure before the time is up, go back and work on the more difficult problems. When you do the simplifying questions, be sure to fully simplify.

When I say begin, please turn past the sample page and start working. You will have 5 minutes. Please do your best work. Ready? Begin.

Time for 5 minutes. When the timer goes off, say *Stop. Please put your pencils down,* and collect student papers.

Algebra Basic Skills Scoring Guide

Solve: $9 + a = 15$	$a =$ **6**
Evaluate: $12 + (-8) + 3$	**7**
Simplify: $2x + 4 + 3x + 5$	$\mathbf{5x + 9}$
Solve: $4 = 12 - k$	$k =$ **8**
Simplify: $4(3 + u) - 7$	$\mathbf{4u + 5}$
Simplify: $b + b + 2b$	$\mathbf{4b}$
Solve: $\frac{r}{6} = \frac{12}{18}$	$r =$ **4**
Simplify: $7 - 3(x - 2)$	$\mathbf{-3x + 13}$
Evaluate: $-5 + (-4) - 1$	**−10**
Solve: $9 = 63 + c$	$c =$ **7**
Simplify: $2(k - 1) + 4 + 5k$	$\mathbf{7k + 2}$
Simplify: $8m - 9(m + 2)$	$\mathbf{-m - 18}$
Solve: 3 feet = 1 yard ____ feet = 9 yards	**27**

Solve: $10 - 6 = g$	$g =$ **4**
Simplify: $9 - 4d + 2 + 7d$	$\mathbf{3d + 11}$
Simplify: $5(3 - n) - n$	$\mathbf{-6n + 15}$
Solve: $(q)(5) = 30$	$q =$ **6**
Simplify: $2 + w(w - 5)$	$\mathbf{w^2 - 5w + 2}$
Simplify: $4 - 7b + 5(b - 1)$	$\mathbf{-2b - 1}$
Solve: 1 feet = 12 inches 5 feet = ____ inches	**60**
Simplify: $u + 2u - 4u$	$\mathbf{-u}$
Solve: $7 = x + 4$	$x =$ **3**
Simplify: $-5(q + 3) + 9$	$\mathbf{-5q - 6}$
Evaluate: $(3)(-6)$	**−18**
Simplify: $y^2 + y - 4y + 3y^2$	$\mathbf{4y^2 - 3y}$
Solve: $\frac{12}{2} = \frac{48}{m}$	$m =$ **8**

Evaluate: $4-(-2)+8$		**14**
Simplify: $2k+3-5(k+7)$		$\mathbf{-3k-32}$
Solve: $3\cdot 8=m$	$m=$	**24**
Solve: $d-5=8$	$d=$	**13**
Simplify: $v(6+v)-3v$		$\mathbf{v^2+3v}$
Simplify: $-3w^2+5w^2-5+12$		$\mathbf{2w^2+7}$
Solve: 4 quarts = 1 gallon ____ quarts = 3 ¼ gallons		**13**
Simplify: $9-4(v+2)$		$\mathbf{-4v+1}$
Solve: $28=4r$	$r=$	**7**
Solve: $\frac{36}{6}=n$	$n=$	**6**
Simplify: $16+2(p-4)-3p$		$\mathbf{-p+8}$
Simplify: $3z-8z+2+9$		$\mathbf{-5z+11}$
Simplify: $c-3(c+2)+8$		$\mathbf{-2c+2}$

Solve: $\frac{1.5}{3}=\frac{h}{9}$	$h=$	**4.5**
Simplify: $7b-4-3-2b$		$\mathbf{5b-7}$
Simplify: $2w-3(w-4)$		$\mathbf{-w+12}$
Solve: $6+7=v$	$v=$	**13**
Evaluate: $-5+6-6$		$\mathbf{-5}$
Simplify: $3(c+2)-2c$		$\mathbf{c+6}$
Solve: 1 pound = 16 ounces ____ pound = 8 ounces		**.5**
Simplify: $6a+2a-9+3a^2$		$\mathbf{3a^2+8a-9}$
Simplify: $-3(3+u)-2u+5$		$\mathbf{-5u-4}$
Solve: $\frac{500}{j}=\frac{10}{2}$	$j=$	**100**
Simplify: $2c-3c-c$		$\mathbf{-2c}$
Solve: $h+6=8$	$h=$	**48**
Evaluate: $(-1)(-5)$		**5**

Appendix E

Sample of Algebra Foundations Measure and Corresponding Materials

Algebra Foundations Standardized Administration Protocol

Hand out the Algebra Foundations measure (with the sample page attached), keeping the probes face down. Ask students to keep the probes face down and write their name, my (the teacher's) name, period/block, and the date on the back of the probe.

Give the standard directions:

The **FIRST** time you administer ALGEBRA FOUNDATIONS probes, say:

Please turn your paper over. This sample page shows some examples of the types of problems on the Algebra Foundations measures. The questions include translating words into expressions, solving simple equations, interpreting line graphs, and completing function or pattern tables. Now we'll take a minute so <u>*you*</u> *can practice doing an Algebra Foundations measure. If you finish the sample page before I say 'Stop', please do NOT turn to the next page. Any questions? Ready, begin.* [time for 1 minute] *Stop, pencils down.*

Now that you've had a chance to try out this type of assessment, do you have any questions? [Only answer procedural questions—do not suggest ways to work the problems.]

Now we'll do the first Algebra Foundations measure. You will have 5 minutes to work on this two-page assessment. Remember, your job is to answer as many problems correctly as you can in 5 minutes. Please look at each problem, but if you do not know how to do it, skip it and move on. If you get to the end of the measure before the time is up, go back and work on the more difficult problems. When you do the simplifying questions, be sure to fully simplify.

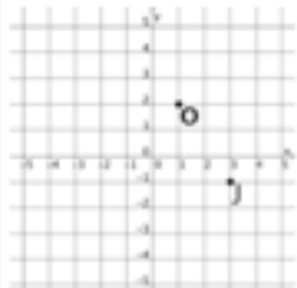
When I say begin, please turn past the sample page and start working. You will have 5 minutes. Please do your best work. Ready? Begin.

Time for 5 minutes. When the timer goes off, say *Stop. Please put your pencils down,* and collect student papers.

Algebra Foundations Measure

List the ordered pair for each point:

J(,   ) O(,   )



Complete the table:

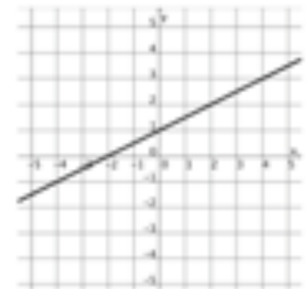
u	$3u$
6	18
7	21
8	
9	27

Complete the table:

n	$4n + 7$
-1	3
-2	
-3	-5
-4	-9

Complete the table:

b	
-3	-6
0	-3
3	0
6	3



Calculate the slope _____

What is the y-intercept? _____

If $y > 9$, two possible values for y are _____ and _____.

Evaluate:
$9 \cdot 4 - 6$
_____

Simplify:
$7a + (2a + a)$
_____

Solve:
$8 = n + 3$
$n =$

Evaluate $4b + 2$ when

$b = 1$ _______

$b = 3$ _______

Write an expression for this phrase:
The difference of a number and 6

Evaluate:
$(-2) \cdot (-4)$
_____

Graph the inequality $m > -4$

Write a word phrase for this expression:
$n + 9$

Evaluate:
$4 + (9 \div 3) - 2^2$
_____

Evaluate:
$(-2)^3$
_____

Write an expression for this phrase:
The product of nine and a number

Evaluate $2x + 4y$ when
$x = 2$ and $y = -3$
_____

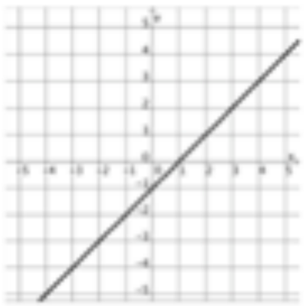
Write a word phrase for this expression:
$7 - b$

Evaluate $8g - 4$ when

$g = 2$ _______

$g = -2$ _______

Simplify:
$6 - 2(b - 4)$
_____



Calculate the slope _____

What is the y-intercept? _____

Complete the table:

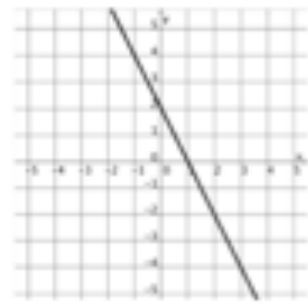
n	
0	0
3	2
6	4
9	6

Complete the table:

k	$2k-7$
-2	-11
2	-3
6	
10	13

Complete the table:

h	
-3	4
1	8
5	12
9	16



Calculate the slope _____

What is the y-intercept? _____

Write a word phrase for this expression:
$x + 4$

Evaluate:
$(-12 + 4) + 5$

_____

Write an expression for this phrase:
The sum of 8 and a number

Solve:
$b = 15 - 8$

_____

Evaluate:
4^2

_____

Graph the inequality $p \leq 3$

Simplify:
$9x - 3 + 4(x + 9)$

_____

Write an expression for this phrase:
The quotient of 10 and a number

If $2a + 4 < 20$, two possible values for a are _____ and _____.

Solve:
$24 + x = 6$
$x =$

_____

Evaluate:
$10 - 3 + 8 + 2$

_____

Simplify:
$12n - 5 + 3 - 7n$

_____

Write a word phrase for this expression:
$h - 5$

Evaluate:
$(-3)(9 - 7)$

_____

Evaluate:
$\sqrt{81}$

_____

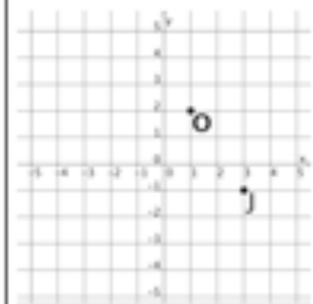
Solve:
$6p = 36$
$p =$

_____

Algebra Foundations Scoring Guide

Find the ordered pair for each point:

J(**3 , –1**) O(**1 , 2**)



Complete the table:

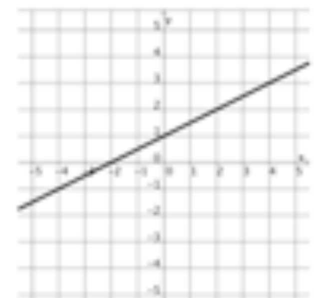
u	$3u$
6	18
7	21
8	**24**
9	27

Complete the table:

n	$4n + 7$
– 1	3
– 2	**-1**
– 3	– 5
– 4	– 9

Complete the table:

b	$b - 3$
– 3	– 6
0	– 3
3	0
6	3



Calculate the slope $\frac{1}{2}$

What is the y-intercept? **1**

If $y > 9$, two possible values for y are _____ and _____

Any number greater than 9

Evaluate:
$9 \cdot 4 - 6$

30

Simplify:
$7a + (2a + a)$

10a

Solve:
$8 = n + 3$
$n =$

5

Evaluate $4b + 2$ when

$b = 1$ **6**

$b = 3$ **14**

Write an expression for this phrase:
The difference of a number and 6

$n - 6$

Evaluate:
$(-2) \cdot (-4)$

8

Graph the inequality $m > -4$

Write a word phrase for this expression:
$n + 9$

The sum of a number and nine

Evaluate:
$4 + (9 + 3) - 2^2$

3

Evaluate:
$(-2)^3$

– 8

Write an expression for this phrase:
The product of nine and a number

9n

Evaluate $2x + 4y$ when
$x = 2$ and $y = -3$

– 8

Write a word phrase for this expression:
$7 - b$

The difference of seven and a number

Evaluate $8g - 4$ when

$g = 2$ **12**

$g = -2$ **– 20**

Simplify:
$6 - 2(b - 4)$

$-2b + 14$

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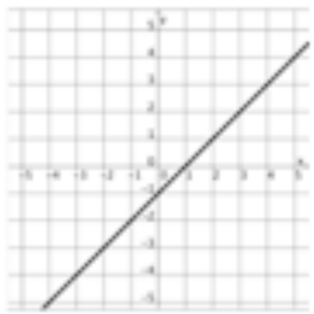


Calculate the slope 1

What is the y-intercept? -1

Complete the table:

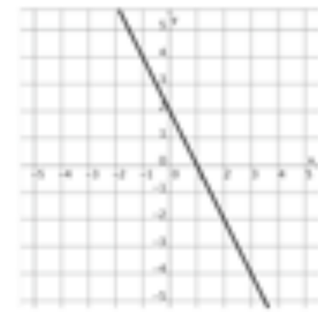
n	$2n + 3$
0	0
3	2
6	4
9	6

Complete the table:

k	$2k - 7$
-2	-11
2	-3
6	**5**
10	13

Complete the table:

h	$h + 7$
-3	4
1	8
5	12
9	16



Calculate the slope -2

What is the y-intercept? 2

Write a word phrase for this expression:
$x + 4$

The quotient of a number and four

Evaluate:
$(-12 + 4) + 5$

2

Write an expression for this phrase:
The sum of 8 and a number

$8 + n$

Solve:
$b = 15 - 8$

7

Evaluate:
4^2

16

Graph the inequality $p \le 3$

Simplify:
$9x - 3 + 4(x + 9)$

$13x + 33$

Write an expression for this phrase:
The quotient of 10 and a number

$\frac{10}{n}$

If $2a + 4 < 20$, two possible values for a are ________ and ________

Any number less than 8

Solve:
$24 + x = 6$
$x =$

4

Evaluate:
$10 - 3 + 8 + 2$

11

Simplify:
$12n - 5 + 3 - 7n$

$5n - 2$

Write a word phrase for this expression:
$h \cdot 5$

The product of a number and five

Evaluate:
$(-3)(9 - 7)$

-6

Evaluate:
$\sqrt{81}$

± 9

Solve:
$6p = 36$
$p =$

6

Appendix F

Sample of Algebra Content Analysis and Corresponding Materials

Algebra Content Analysis Standardized Administration Protocol

Hand out probe ACA-1 (with the sample page attached), keeping them face down. Ask students to keep the probes face down and write their name, my (the teacher's) name, period/block, and the date on the back of the probe.

Give the standard directions:

The **FIRST** time you administer the ALGEBRA CONTENT ANALYSIS algebra probes, say:

The problems on this assessment come from the information you are learning this year in algebra, but they are not in any special order. For example, a problem from the beginning of the course could be the last problem on the measure. Please look at each problem and decide if you know how to do it. If you do, go ahead and answer the problem. If you aren't certain, or think you don't know how to do the problem, skip it and move to the next one. Don't spend too much time on any one problem. The object of the assessment is to answer as many problems correctly in the time available. Once you get to the end, go back and work on the problems you skipped. Remember that you may earn partial credit by showing your work even if you can't do the entire problem. Do NOT make wild guesses because this will cause you to lose points on the assessment.

Please turn your paper over. This sample page shows some examples of the types of problems on the Algebra Content Analysis measures. The problems on this assessment are drawn from the Algebra skills you are learning in this class. The questions are multiple choice. Each problem is worth 3 points, but you can earn partial credit by showing your work. Unless you are completely certain of the correct answer, the best strategy is to show your work. If you do not know the answer, you should NOT make wild guesses. You will lose points from your total score on the measure when you make wild guesses.

Look at the three boxes in the first row labeled A, B, and C. You'll notice that all three have answers and that the problem is the same for all three. Look at the box for Student A. She thought she knew the correct answer, so she just circled her choice at the bottom. Unfortunately, she was incorrect, so she will lose a point for this problem. Student B showed some work and circled an incorrect answer. Because he did part of the problem correctly, Student B will earn 1 out of 3 points on this problem. Student C worked the problem, but did not circle a response. Because she showed some correct work, she will earn 1 out of 3 points on the problem. As you can see from these examples, it is important to show your work on these assessments.

Let's take a minute so you can practice doing an Algebra Content Analysis measure. If you finish the sample page before I say 'Stop', please do NOT turn to the next page. Any questions? Ready, begin. [time for 1 minute] *Stop, pencils down.*

Now that you've had a chance to try out this type of assessment, do you have any questions? [Only answer procedural questions—do not suggest ways to work the problems.]

Now we'll do the first Algebra Content Analysis measure. You will have 7 minutes to work on this two-page assessment. Remember, your job is to answer as many problems correctly as you can in 7 minutes. Please look at each problem. If you do not know how to do it, skip it and move on. If you get to the end of the measure before the time is up, go back and work on the problems you skipped. Remember that you may earn partial credit by showing your work even if you can't do the entire problem. Do NOT make wild guesses because this will cause you to lose points on the assessment.

When I say begin, please turn past the sample page and start working. You will have 7 minutes. Please do your best work. Ready? Begin.

Time for 7 minutes. When the timer goes off, say *Stop. Please put your pencils down*, and collect student papers.

Algebra Content Analysis Measure

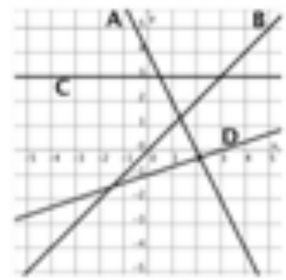
Solve:
$19 = 3x + 4$
$x =$ ______

A) 5
B) $7\frac{2}{3}$
C) 15
D) 45

Evaluate $a^2 - c + 2$ when
$a = 4$ and $c = 6$

A) 1
B) 5
C) 13
D) 34

Which line on the graph represents the equation $y + 2x = 4$?



A) Line A
B) Line B
C) Line C
D) Line D

Simplify:
$3(m + 2) + 2(m - 1)$

A) $5m + 1$
B) $5m + 4$
C) $5m + 8$
D) $6m + 4$

Evaluate the expression:

6^{-2}

A) $\frac{1}{12}$ B) $\frac{1}{36}$
C) -12 D) -36

Solve the system of linear equations:
$x - y = 4$
$x + 2y = 19$

A) $(-2, 19)$
B) $(-1, -5)$
C) $(5, 8)$
D) $(9, 5)$

This graph shows the solution for which inequality?

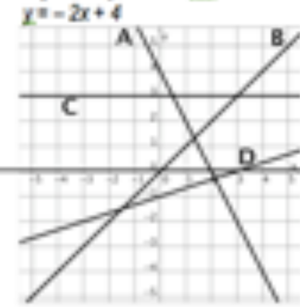
A) $-3x > 9$
B) $x > -3$
C) $2x \leq -6$
D) $3x \geq 9$

Write the equation of a line in slope-intercept form if
slope $= \frac{1}{2}$ and y-intercept $= (0, 3)$.

A) $x = \frac{1}{2}y - 3$ B) $y = \frac{1}{2}x + 3$
C) $y = 3x + \frac{1}{2}$ D) $3y = \frac{1}{2}x + 3$

Evaluate $d + 3c^2$ when $d = -5$ and $c = 1$ A) -2 B) 1 C) 4 D) 8	Solve: $6n + 4 = -3n - 14$ $n =$ _____ A) 6 B) $-\frac{10}{9}$ C) $-\frac{10}{3}$ D) -2	Find the slope of a line that goes through the points $(1, -1)$ and $(5, 2)$. A) $\frac{3}{4}$ B) $\frac{1}{5}$ C) $-\frac{4}{3}$ D) -6	Simplify: $6(2k - 3) - 3(2 - k)$ A) $-18k - 12$ B) $9k - 24$ C) $11k - 9$ D) $15k - 24$
Simplify the expression: $\frac{a^2}{ab^3} \bullet \frac{b^4}{a^3}$ A) $\frac{a^4}{a^3b^3}$ B) $\frac{ab^4}{a^4b^3}$ C) $\frac{b}{a^2}$ D) $\frac{b}{a}$	Solve the system of linear equations: $-6x + 3y = -6$ $2x + 6y = 30$ A) $(2, 6)$ B) $(3, 4)$ C) $(4, -3)$ D) $(6, 3)$	Simplify: $r^2 - 4r + 2r^2 + 7 - 5$ A) $-r^2 - 4r + 12$ B) $3r^2 - 4r + 2$ C) $3r^2 - 4r + 12$ D) $2r + 2$	Write the equation of a line that goes through the points $(5, 3)$ and $(4, 9)$. Use slope-intercept form. A) $y = 33x - 6$ B) $y = 6x + 30$ C) $y = -\frac{1}{6}x + 3\frac{5}{6}$ D) $y = -6x + 33$

Algebra Content Analysis Scoring Guide

Solve: $19 = 3x + 4 =$ $x =$ *Isolates variable by subtracting 4 from each side* $3x = 15$	Evaluate $a^2 - c + 2$ when $a = 4$ and $c = 6$ *Substitutes values for variables* $4^2 - 6 + 2$ $16 - 6 + 2$ $16 - 6 = 10$ $10 + 2$	Which line on the graph represents the equation $y + 2x = 4$? $y = -2x + 4$  	Simplify: $3(m + 2) + 2(m - 1)$ *Distributes either term so that response includes* $3m + 6$ OR $2m - 2$ OR $5m$
Isolates variable and divides by 3 to solve for x **A) 5** B) $7\frac{2}{3}$ C) 15 C) 45	*Further simplifies the expression* $16 - 3$ A) 1 B) 5 **C) 13** D) 34	**A) Line A** B) Line B C) Line C D) Line D	*Distributes both terms correctly* $3m + 6 + 2m - 2$ *OR Distributes either term correctly and includes 5m or + 4 in response* A) $5m + 1$ **B) $5m + 4$** C) $5m + 8$ D) $6m + 4$
Evaluate the expression: 6^{-2} *Applies the negative exponent* $\frac{1}{6^2}$	Solve the system of linear equations: $x - y = 4$ $x + 2y = 19$ *Chooses a correct multiplier for combination (i.e. multiply 1st equation by 2 or -1 or 2nd eq. by -1)* *OR Isolates one variable (x = , y =) for substitution*	This graph shows the solution for which inequality?  $x \le -3$	Write the equation of a line in slope-intercept form if slope $= \frac{1}{2}$ and y-intercept $= (0, 3)$ $y = mx + b$ *OR an expression that includes* $\frac{1}{2}x$
A) $\frac{1}{12}$ **B) $\frac{1}{36}$** C) -12 d) -36	*Solves correctly for 1 variable using either method OR follows steps to solve, but makes computational errors OR substitutes solution for 1 variable if using substitution* A) $(-2, 19)$ B) $(-1, -5)$ C) $(5, 8)$ **D) $(9, 5)$**	*Shows evidence of understanding simplifying of inequalities by dividing answer options to isolate variables* A) $-3x > 9$ B) $x > -3$ **C) $2x \le -6$** D) $3x \ge 9$	$\frac{1}{2}x + 3$ *(but no 'y =')* A) $x = \frac{1}{2}y - 3$ **B) $y = \frac{1}{2}x + 3$** C) $y = 3x + \frac{1}{2}$ D) $3y = \frac{1}{2}x + 3$

Evaluate $d + 3c^2$ when $d = -5$ and $c = 1$ *Substitutes values for variables* $-5 + 3 \cdot 1^2$	Solve: $6n + 4 = -3n - 14$ $n =$ *Isolates variables on 1 side (+ 3n or – 6n to both sides)* *OR isolates constants on 1 side (– 4 or + 14 to both sides)*	Find the slope of a line that goes through the points (1, – 1) and (5, 2). *Draws a graph to determine slope OR uses slope formula* $\frac{2-^-1}{5-1}$ *or* $\frac{^-1-2}{1-5}$	Simplify: $6(2k - 3) - 3(2 - k)$ *Distributes either term so that response includes* $12k - 18$ *OR* $-6 + 3k$ *OR* $15k$
Further simplifies the expression $-5 + 3 \cdot 1$ $-5 + 3$ **A) – 2** B) 1 C) 4 D) 8	*Isolates constants and variables OR isolates variables or constants and divides to solve for n* A) 6 B) $-\frac{10}{9}$ C) $-\frac{10}{3}$ **D) – 2**	*Finds slope, but has negative values in numerator and denominator* $\frac{^-3}{^-4}$ **A)** $\frac{3}{4}$ B) $\frac{1}{5}$ C) $-\frac{4}{3}$ D) – 6	*Distributes both terms correctly* $12k - 18 - 6 + 3k$ *OR Distributes either term correctly and includes 15k or – 24 in response* A) $-18k - 12$ B) $9k - 24$ C) $11k - 9$ **D) $15k - 24$**
Simplify the expression: $\frac{a^2}{ab^3} \cdot \frac{b^4}{a^3}$ *Multiplies numerators and denominators to combine terms OR simplifies either the a or b terms* *EX:* $\frac{a^2b^4}{a^4b^3}$ *or* $\frac{a}{b^3} \cdot \frac{b^4}{a^3}$	Solve the system of linear equations: $-6x + 3y = -6$ $2x + 6y = 30$ *Chooses a correct multiplier for combination (i.e., multiplies 1st equation by – 2 or 2nd eq. by 3)* *OR isolates one variable (x =, y =) for substitution*	Simplify: $r^2 - 4r + 2r^2 + 7 - 5$ *Combines any 2 like terms (i.e., r^2, constants)* $3r^2$ *OR* 2	Write the equation of a line that goes through the points (5, 3) and (4, 9). Use slope-intercept form. *Determines correct slope OR applies slope formula* $\frac{9-3}{4-5}$ *or* $\frac{3-9}{5-4}$ *or* $m = -6$
Simplifies both the a and b terms, but expression is not fully simplified *EX:* $a \cdot \frac{b}{a^3}$ A) $\frac{a^6}{a^{-1}b^3}$ B) $\frac{ab^8}{a^4b^3}$ **C)** $\frac{b}{a^2}$ D) $\frac{b}{a}$	*Solves correctly for 1 variable using either method OR follows steps to solve, but makes computational errors OR substitutes solution for 1 variable if using substitution* A) (2, 6) **B) (3, 4)** C) (4, – 3) d) (6, 3)	*Combines both like terms, but makes an error in computation* *EX:* $3r^2 - 4r + 12$ A) $-r^2 - 4r + 12$ **B) $3r^2 - 4r + 2$** C) $3r^2 - 4r + 12$ D) $2r + 2$	*Attempts to determine y-intercept* $9 = -6(4) + b$ A) $y = 33x - 6$ B) $y = 6x + 30$ C) $y = -\frac{1}{6}x + 3\frac{5}{6}$ **D) $y = -6x + 33$**

VITA

William (Bill) W. DeLeeuw was born in Logan, Utah. He is the oldest of five children, the other four being girls. After graduating from Tooele High School in Tooele, Utah, he attended Snow College in Ephraim, Utah for a year as a mechanical engineering major before serving a two-year mission for the Church of Jesus Christ of Latter Day Saints in British Columbia, Canada. Upon his return, he transferred to Weber State University in Ogden, Utah where he graduated with a Bachelor's Degree in secondary mathematics and physics teaching. While at Weber State University he married his wife, Jessica Salas.

After receiving his Bachelor's Degree, he spent the next 5 years teaching mathematics, physics and coaching soccer at Syracuse High School in Syracuse, Utah. During this time Bill and Jessica welcomed four boys to their family, and Bill earned a Master's Degree in teacher leadership from National University. Bill's teaching assignment at Syracuse High was primarily to teach Algebra 1 to students in grades 10–12. Most of these students having failed a mathematics course at some point in their educational experience, Bill came to feel very strongly that these students were not receiving the education that they needed to be successful in his Algebra 1 classes and in the world at large.

It was at this point that Bill decided to return to school to get his PhD in mathematics education so that he could better meet the needs of those students who were being set up for failure in high school mathematics. He and his family moved from Utah to Missouri so that he could attend the University of Missouri at Columbia as a full time doctoral student. The courses he took during this program, internships in a variety of mathematics methods courses, his research assistantships, and his experiences as a 10-12th grade mathematics teacher guided his path in studying the understandings and conceptions about variables of students who have demonstrated difficulties in mathematics. Specifically, he is interested in supporting the population of students who are struggling with mathematics, but are not diagnosed with a learning disability. He will continue to conduct research and instruct pre-service mathematics teachers as he begins his career as an Assistant Professor at Arizona State University.