The Dirichlet problem for elliptic and degenerate elliptic equations, and related results

A Dissertation

presented to

the Faculty of the Graduate School

University of Missouri

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

by

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July 2016

The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

The Dirichlet problem for elliptic and degenerate elliptic equations, and related results

presented by Phi Long Le, a candidate for the degree of Doctor of Philosophy of

Mathematics, and hereby certify that in their opinion it is worthy of acceptance.

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Professor Igor Verbitsky

Professor Marius Mitrea

Professor Steven Neal

ACKNOWLEDGEMENTS

This dissertation could not be done without the guidance of my advisor Steve Hofmann. From the bottom of my heart, I would like to thank Steve Hofmann for all of his enormous support to me in learning mathematics and also his financial support during five years I am at Columbia. I also want to thank him for his careful guiding and teaching Harmonic Analysis and its Applications to Partial Differential Equations to me during the time I have been studying at University of Missouri - Columbia.

I want to have a special thank to my wife who has been supporting, encouraging me since the beginning of my graduate studies and also she has helped me take care of our daughter which is, to me, it is the harder job than my studies. I also want to thank my daughter since she is the motivation for me to study harder and to finish my studies on time.

Finally, I want to thank all of my friends at Mizzou. Thanks for your discussions in mathematics, telling jokes and other fun things. We have such a good time at Columbia. Special thanks to my friends who helped me when I need during the time my family stayed in Columbia.

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The Dirichlet problem for elliptic and degenerate elliptic equations, and related results

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ABSTRACT

The purpose of this thesis is to understand three different types of problems related to weighted elliptic operators and some results related to solvability of non-degenerate elliptic equations in rough domains. First, we prove that the Dirichlet problem for degenerate elliptic equations $\operatorname{div}(A\nabla u) = 0$ in the upper-half space $(x,t) \in \mathbb{R}^{n+1}_+$ is solvable when $n \geqslant 2$ and the boundary data is in $L^p(\mathbb{R}^n)$ for some $0 < p < \infty$. The coefficient matrix A is only assumed to be measurable, real-valued and t-independent with a degenerate bound and ellipticity controlled by a t-independent A_2-weight μ. It is not required to be symmetric. The result is achieved by proving a Carleson measure estimate for all bounded solutions in order to deduce that harmonic measure is in the A_∞-class with respect to the μ-weighted Lebesgue measure on $\mathbb{R}^n$. The Carleson measure estimate allows us to avoid applying the method of ϵ-approximability, which simplifies the proof obtained recently in the case of uniformly elliptic coefficients. The results have natural extensions to Lipschitz graph domains. Second, We obtain Hodge-decomposition, L^p bounds semi-groups and their gradients, and then we get L^p bounds for Riesz transforms and square functions associated to a degenerate elliptic operator in divergence form, with degeneracy controlled by a weight in the Muckenhoupt class A_2. Finally, we show that for a uniformly elliptic divergence form operator L, defined in an open set Ω with Ahlfors-David regular boundary, BMO-solvability implies scale invariant quantitative absolute continuity (the weak-A_∞ property)

of elliptic-harmonic measure with respect to surface measure on $\partial\Omega$. We do not impose any connectivity hypothesis, qualitative or quantitative; in particular, we do not assume the Harnack Chain condition, even within individual connected components of Ω. In this generality, our results are new even for the Laplacian. Moreover, we obtain a converse, under the additional assumption that Ω satisfies an interior Corkscrew condition, in the special case that L is the Laplacian.

Chapter 1

Introduction

In this doctoral thesis, we investigated three problems as followings:

1. Carleson measure estimates and the Dirichlet problem for degenerate elliptic equations.

2. L^p bounds of Riesz transform and Vertical square functions for degenerate elliptic operators.

3. BMO solvability and absolute continuity of harmonic measure.

The following is a brief introduction to the results in my thesis.

Given a domain $\Omega \subset \mathbb{R}^n$, and a function $g : \partial\Omega \to R$, the Dirichlet problem for Laplace's equation is to find a function u satisfying

$$\begin{cases} \Delta u = 0 \text{ in } \Omega \\ u = g \text{ on } \partial\Omega \end{cases}$$

The Dirichlet problem is fundamental in many area of mathematics and physics. The efforts to solve the problems led to many revolutionary ideas in mathematics. The first serious study of the Dirichlet problem on a general domain with general boundary condition was done by George Green in his Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism, published in 1828. He reduced the problem to the problem of constructing what we now call Green functions, and he argued that Green's function exists for any domain. Since then, there are many results that have been derived. Since the original Dirichlet problem cannot describe some complicated situations in physics, we need to understand the Dirichlet problems in more general forms, and one of the most well-known versions is to replace the equation $\Delta u = 0$ by a divergence form equation $-\operatorname{div} A\nabla u = 0$, where A is an $n \times n$ real valued matrix and A satisfies some elliptic conditions. For nice domains, and nice coefficients, the Dirichlet problem has been studied well. For example, we may know the smoothness of the solution, properties of solutions near the boundary, and solvability of the problem with continuous data on the boundary. An interesting problem is to understand the solution when the boundary data belongs to L^p on the boundary. For L^2 data on the boundary, and radially independent symmetric coefficients in a star-like Lipschitz domain (which case may be reduced, after localizing and pulling back, to the case of t-independent coefficients in the half-space), the problem was treated by Jerison and Kenig. For the 2-dimensional upper half-plane, the solvability of Dirichlet problem was confirmed by Kenig, Koch, Pipher and Toro in 2000 for some p, again for t-independent coefficients, albeit without an assumption of symmetry. In higher dimension (the domain is still upper-half space), the question remained open until 2012, when it was resolved in the affirmative by the work of Hofmann, Kenig, Mayboroda and Pipher, using techniques and results from the solution of Kato problem. My first problem was to understand the L^p solvability of Dirichlet problem in a weighted, degenerate elliptic setting. In our work, we focus on the domain $\mathbb{R}^{n+1}_+ := \{(x,t) : x \in RR^n, t \in \mathbb{R}_+\}$. here, A is an $(n+1)\times(n+1)$ real-valued matrix, it is t-independent, and satisfies the ellipticity conditions $|\langle A(x)\xi, \zeta\rangle| \leqslant \Lambda\mu(x)|\xi||\zeta|$, $|\langle A(x)\xi, \xi\rangle| \geqslant \Lambda^{-1}\mu(x)|\xi|^2$ for some $\Lambda > 1$. We established

Carleson Measure Estimates, i.e., we obtain that for a bounded solution u,

$$\sup_Q \left(\frac{1}{\mu(Q)} \int_0^{\ell(Q)} \int_Q |t\nabla u(x,t)|^2 \, d\mu(x) \frac{dt}{t} \right) \leqslant C \|u\|_\infty^2$$

Base on the above result, we are able to show that the harmonic measure belongs to A_∞ with respect to surface measure – the latter is a quantitative version of absolute continuity with respect to surface measure. From that, we may conclude the Dirichlet problem is solvable for some p large enough.

For our next problem, we start with the assumption that A is an $n \times n$ complex valued matrix and let A satisfy

$$\begin{aligned} \lambda \, \mu(x) \, |\xi|^2 &\leqslant \operatorname{Re}\langle A(x)\xi, \xi \rangle, \\ |\langle A(x)\xi, \psi \rangle| &\leqslant \Lambda \, \mu(x) \, |\xi| \, |\psi|, \end{aligned}$$

for all $\xi, \psi \in \mathbb{C}^n$ and for some uniform λ and Λ, with $0 < \lambda < \Lambda < \infty$. Then we are able to define the square root operator $\sqrt{L}$, where $Lu := -\operatorname{div} A\nabla u$. The well-known Kato problem asked whether we have $\|\sqrt{L}f\|_{L^2(\mathbb{R}^n)} \simeq \|\nabla f\|_{L^2(\mathbb{R}^n)}$, when f is in $\dot{W}^{1,2}(\mathbb{R}^n)$ – the homogeneous Sobolev space. The question has a long history, and a number of famous mathematicians have contributed to its solution. The complete solution was obtained by Auscher, Hofmann, Lacey, Lewis, McIntosh and Tchamitchian in 2001. The resolution of the L^2 Kato problem opened new questions about the problem in L^p. The problem for $p \neq 2$ was treated by many people, and the complete solution was shown by Auscher in a publication in 2007. In 2012, the weighted version (i.e., for degenerate elliptic L) of the L^2 Kato problem was obtained by Cruz-Uribe and Rios. My project in this direction was to verify the Kato problem for degenerate operators (with A_2 degenracy) in weighted L^p levels. We obtained that, for some range $2 - \epsilon < p < 2 + \epsilon$, $\|\sqrt{L}f\|_{L^p} \simeq C\|\nabla f\|_{L^p}$ and also weighted L^p bounds for square functions defined below

$$\begin{aligned} g_{\mathcal{L}_\mu}(f)(x) &= \left(\int_0^\infty \left|\mathcal{L}_\mu^{1/2} e^{-t\mathcal{L}_\mu} f(x)\right|^2 dt\right)^{\frac{1}{2}}, \\ G_{\mathcal{L}_\mu}(f)(x) &= \left(\int_0^\infty \left|\nabla e^{-t\mathcal{L}_\mu} f(x)\right|^2 dt\right)^{\frac{1}{2}}. \end{aligned}$$

We remark that in this context the restriction on the range of p is natural.

The last problem in my thesis concerns solvability of the Dirichlet problem in rough domains. Recently, harmonic measure and its connection with the geometry of the corresponding domain has become an active area with work by Hofmann, Martell, Tolsa, Nazarov, Toro, Pipher, Kenig, ... just name a few. One of the directions of this type of problem is to understand what we can conclude about the boundary of the domain once we know some quantitative properties of the harmonic measure and its converse questions, i.e., what we can say about the quantitative property of harmonic measure when the boundary satisfies some conditions. Another direction is to find the connection between the solvability of the Dirichlet problem, and Square function and Non-tangential maximal function estimates in L^p. In our project, we focused on a different but related problem. We wanted to understand the relation between solvability of the Dirichlet problem with BMO data on the boundary and properties of harmonic measure over domains with Ahlfors – David regular boundaries. Let us consider the operator

$$L := -\operatorname{div} A(X)\nabla,$$

defined in an open set $\Omega \subset \mathbb{R}^{n+1}$, where A is $(n+1)\times(n+1)$, real, L^∞, and satisfies the uniform ellipticity condition

$$\lambda|\xi|^2 \leqslant \langle A(X)\xi, \xi\rangle := \sum_{i,j=1}^{n+1} A_{ij}(X)\xi_j\xi_i, \quad \|A\|_{L^\infty(\mathbb{R}^n)} \leqslant \lambda^{-1}, \tag{1.1}$$

for some $\lambda > 0$, and for all $\xi \in \mathbb{R}^{n+1}$, $X \in \Omega$. We proved that for a domain *Omega* with Ahlfors–David regular boundary, if the BMO-Dirichlet problem is solvable, then its L-harmonic measure belongs to weak -A_∞ with respect to surface measure on the boundary. For the converse result, we were also able to prove that under an additional Corkscrew condition, we have solvability of the Dirichlet problem for the Laplace operator with boundary data in L^p and also in BMO, provided that the harmonic measure belongs to weak- A_∞ with respect to surface measure.

This dissertation is organized as followings. In the Chapter 2, we present some definitions and well-known results that we need for our further arguments. Our results above will be given in details from Chapter 3 to Chapter 5 in the order that we discussed.

Chapter 2

Definitions and basic results

Definition 2.1. Let $1 < p < \infty$ and suppose μ be a locally integrable and positive function. We say μ satisfies Muckenhoupt $A_p(\mathbb{R}^n)$ weight if

$$\sup_B \left(\frac{1}{|B|}\int_B \mu(x)dxdx\right)\left(\frac{1}{|B|}\int_B |\mu(x)|^{\frac{-1}{p-1}}\,dx\right)^{p-1} \leqslant K < \infty\,,$$

where the supremum runs over all balls $B \subset \mathbb{R}^n$ and $|B|$ denotes the Lebesgue measure of the set B.

Now, we would like to recall some function spaces that we need for further arguments.

Definition 2.2. For any open set Ω and $0 < p < \infty$, we define

$$L^p_\mu(\Omega) = \left\{f : \int_\Omega |f(z)|^p\mu(z)dz < \infty\right\},$$

with the norm $\|f\|_{L^p_\mu(\Omega)} := \left(\int_\Omega |f(z)|^p\mu(z)dz\right)^{\frac{1}{p}}$.

We also define

$$L^p_{\mu,\mathrm{loc}}(\Omega) := \left\{f : \int_{\Omega'} |f(z)|^p\mu(z)dz < \infty, \forall\Omega' \subset\subset \Omega\right\}.$$

and the space $L^{p,\infty}_\mu$ for any $p \in (0,\infty)$ is defined to be the set of all measurable functions f such that

$$\begin{aligned}\|f\|_{L^{p,\infty}_\mu} &:= \inf\left\{C > 0 : \mu\left(\{x : |f(x)| > \alpha\}\|\right) \leqslant \frac{C^p}{\alpha^p},\ \text{for all } \alpha > 0\right\}\\ &= \sup\left\{\gamma\left(\mu\left(\{f(x) > \gamma\}\right)\right)^{\frac{1}{p}} : \gamma > 0\right\}.\end{aligned}$$

Next, we give the definition of weak-type (p,p) bounds and strong type (p, q) bounds of an operator T

Definition 2.3. We say an operator T satisfies strong type (p, q) bounds if we have the following

$$\|Tf\|_{L^q_\mu} \leqslant C\|f\|_{L^p_\mu}$$

and we say it satisfies weak-type (p, q) bounds if

$$\|Tf\|_{L^{q,\infty}_\mu} \leqslant C\|f\|_{L^p_\mu}$$

We also would like to recall the definition of some version of Hardy-Littlewood maximal functions and also state some properties of them.

Definition 2.4. For any locally integrable function f, we define the centered Hardy-Littlewood maximal function $M(f)$ as following

$$M(f)(x) = \sup_{r>0} \frac{1}{|B(x, r)|} \int_{B(x,r)} |f(z)|dz,$$

where $|B(x, r)|$ is the Lebesgue measure of the ball $B(x, r)$.

Definition 2.5. For any locally integrable function $f \in L^1_{\mu,\text{loc}}$, we define the weighted centered Hardy-Littlewood maximal function $M_\mu(f)$ as following

$$M^c_\mu(f)(x) = \sup_{r>0} \frac{1}{\mu(B(x, r))} \int_{B(x,r)} |f(z)|\mu(z)dz,$$

where $\mu(B) = \int_B \mu(z)dz$ - the μ measure of the set B.

Definition 2.6. For any locally integrable function $f \in L^1_{\mu,\text{loc}}$, we define the weighted uncentered Hardy-Littlewood maximal function $M_\mu(f)$ as following

$$M_\mu(f)(x) = \sup_{x\in B} \frac{1}{\mu(B)} \int_B |f(z)|\mu(z)dz.$$

Let us recall the following proposition of maximal functions. See more details in the book by Loukas Grafakos [G] for proofs of those results.

Proposition 2.7. *All the maximal functions $M_\mu(f), M(f), M_\mu^c(f)$ satisfy the following properties*

$$\left\|M_\mu(f)\right\|_{L^p_\mu} \lesssim \|f\|_{L^p_\mu}, \quad \left\|M^c_\mu(f)\right\|_{L^p_\mu} \lesssim \|f\|_{L^p_\mu}, \text{and} \left\|M_\mu(f)\right\|_{L^p_\mu} \lesssim \|f\|_{L^p_\mu}$$

for all $1 < p < \infty$.

The following theorem is the solution of Kato problem for weighted operator which was proved in [C-UR2].

Theorem 2.8. *(weighted Kato problem, [C-UR2]) Let $\mathcal{L}_\mu$ be defined below, we have the following result*

$$\left\|\mathcal{L}_\mu^{1/2} f\right\|_{L^2_\mu(\mathbb{R}^n)} \simeq \|\nabla_x f\|_{L^2_\mu(\mathbb{R}^n)},$$

for all $f \in H^1_\mu(\mathbb{R}^n)$.

Let us state here some essential inequalities for weighted setting. The first result is the weighted Poincaré inequality for which the proof was given in Theorem 1.5 of [FKS].

Theorem 2.9. *Let $1 < p < \infty$ and $\mu \in A_p$ Muckenhoupt class. Then there are positive constants $c = c(p, \mu, n)$ and $\delta = \delta(p, \mu, n)$ such that for all function $u \in W^{1,p}_{\mu,loc}(\mathbb{R}^n)$ and for all $1 \leqslant k \leqslant \frac{n}{n-1} + \delta$,*

$$\left(\frac{1}{\mu(B_R)}\int_{B_R} |u(x) - A_{B_R}|^{pk}\mu dx\right)^{\frac{1}{kp}} \leqslant cR\left(\frac{1}{\mu(B_R)}\int_{B_R} |\nabla u|^p \mu dx\right)^{\frac{1}{p}},$$

where either $A_{B_R} = \frac{1}{\mu(B_R)}\int_{B_R} u(x)\mu(x)dx$ or $A_{B_R} = \frac{1}{|B_R|}\int_{B_R} u(x)dx$ and $|B|$ denotes the Lebesgue measure of the set B.

The next theorem we need is the following weighted Sobolev inequality.

Theorem 2.10. *(Theorem 1.2, [FKS]) Take $1 < p < \infty$ and a function $\mu \in A_p$. There exists a positive number δ such that for all $u \in C_0^\infty(B_R)$ and all k satisfying $1 \leqslant k \leqslant \frac{n}{n-1} + \delta$,*

$$\left(\frac{1}{\mu(B_R)} \int_{B_R} |u|^{kp} \mu dx\right)^{\frac{1}{kp}} \leqslant CR \left(\frac{1}{\mu(B_R)} \int_{B_R} |\nabla u|^p \mu dx\right)^{\frac{1}{p}},$$

where C may be taken to depend only on n, the A_p constant of μ, p.

Lemma 2.11. *(Kolmogorov's inequality) Let $f \in L^1_\mu(\mathbb{R}^n)$ and $0 < q < 1$. Then we have*

$$\int_E \left[M_\mu(f)(x)\right]^q \mu \leqslant C_p \mu(E)^{1-q} \|f\|^q_{L^1_\mu},$$

whenever $\mu(E)$ is finite.

Proof. For a proof, see page 100 in [G]. ∎

Now we want to give some properties of A_p weight condition in which we will need them later for our arguments. For a reference, the reader can find proofs in the book by Jose Garcia-Cuerva and J-.L.Rubio De Francia [GF].

Lemma 2.12. *If the measure $\mu \in A_p$ then there is $\epsilon = \epsilon(\mu) > 0$ such that $\mu \in A_{p-\epsilon}$.*

Lemma 2.13 (Doubling property)**.** *If the measure $\mu \in A_p$ then for any $\lambda > 1$ and for any cube Q, we have*

$$\mu(\lambda Q) \leqslant C\lambda^{np} \mu(Q)$$

where λQ is the dilation of Q at the same center with side length $\lambda l(Q)$.

Moreover, we also have the lower bound, i.e., there is some $\delta' > 0$ such that

$$\frac{\mu(E)}{\mu(Q)} \leqslant C \left(\frac{|E|}{|Q|}\right)^{\delta'}$$

where $|E|$ denotes the Lebesgue measure of the set E.

Proof. See Lemma 2.2 and Theorem 2.9 in [GF] for the proofs. ∎

Definition 2.14. Let $\mathcal{T} = (T_t)_{t>0}$ be a family of operators, we say that $\mathcal{T}$ satisfies L^2_μ off-diagonal estimates if for some constants $c > 0$ and $\alpha > 0$ for all closed sets E, F, all $h \in L^2_\mu$ with support in E and for all $t > 0$ we have

$$\|T_t h\|_{L^2_\mu(F)} \leqslant c e^{\frac{-cd(E,F)^2}{t}} \|h\|_{L^2_\mu(E)}.$$

Here and subsequently, $d(E, F)$ denotes the usual Euclidean distance between the two sets.

Proposition 2.15. *For all $z \in \Sigma_\beta$ where $\beta \leqslant \frac{\pi}{4}$, the family $(e^{-z\mathcal{L}_\mu})$, $(z\mathcal{L}_\mu e^{-z\mathcal{L}_\mu})$ and $(\sqrt{|z|}\nabla e^{-z\mathcal{L}_\mu})$ satisfy L^2_μ off diagonal estimates*

Proof. This is a consequence of Lemma 2.10 in [C-UR2] ∎

Next, we give some definitions and notations of domains we will work on.

- We use the letters c, C to denote harmless positive constants, not necessarily the same at each occurrence, which depend only on dimension and the constants appearing in the hypotheses of the theorems (which we refer to as the "allowable parameters"). We shall also sometimes write $a \lesssim b$ and $a \approx b$ to mean, respectively, that $a \leqslant Cb$ and $0 < c \leqslant a/b \leqslant C$, where the constants c and C are as above, unless explicitly noted to the contrary.

- Given a closed set $E \subset \mathbb{R}^{n+1}$, we shall use lower case letters x, y, z, etc., to denote points on E, and capital letters X, Y, Z, etc., to denote generic points in $\mathbb{R}^{n+1}$ (especially those in $\mathbb{R}^{n+1} \setminus E$).

- The open $(n+1)$-dimensional Euclidean ball of radius r will be denoted $B(x,r)$ when the center x lies on E, or $B(X,r)$ when the center $X \in \mathbb{R}^{n+1} \setminus E$. A "surface ball" is denoted $\Delta(x,r) := B(x,r) \cap \partial\Omega$.

- Given a Euclidean ball B or surface ball Δ, its radius will be denoted r_B or r_Δ, respectively.

- Given a Euclidean or surface ball $B = B(X,r)$ or $\Delta = \Delta(x,r)$, its concentric dilate by a factor of $\kappa > 0$ will be denoted $\kappa B := B(X, \kappa r)$ or $\kappa\Delta := \Delta(x, \kappa r)$.

- Given a (fixed) closed set $E \subset \mathbb{R}^{n+1}$, for $X \in \mathbb{R}^{n+1}$, we set $\delta(X) := \operatorname{dist}(X, E)$.

- We let H^n denote n-dimensional Hausdorff measure, and let $\sigma := H^n\big|_E$ denote the "surface measure" on a closed set E of co-dimension 1.

- For a Borel set $A \subset \mathbb{R}^{n+1}$, we let 1_A denote the usual indicator function of A, i.e. $1_A(x) = 1$ if $x \in A$, and $1_A(x) = 0$ if $x \notin A$.

- For a Borel set $A \subset \mathbb{R}^{n+1}$, we let $\operatorname{int}(A)$ denote the interior of A.

- Given a Borel measure μ, and a Borel set A, with positive and finite μ measure, we set $⨍_A f d\mu := \mu(A)^{-1} \int_A f d\mu$.

- We shall use the letter I (and sometimes J) to denote a closed $(n+1)$-dimensional Euclidean dyadic cube with sides parallel to the co-ordinate axes, and we let $\ell(I)$ denote the side length of I. If $\ell(I) = 2^{-k}$, then we set $k_I := k$.

Definition 2.16. (**ADR**) (aka *Ahlfors-David regular*). We say that a set $E \subset \mathbb{R}^{n+1}$, of Hausdorff dimension n, is ADR if it is closed, and if there is some uniform constant C such that

$$\frac{1}{C}\, r^n \leqslant \sigma\big(\Delta(x,r)\big) \leqslant C\, r^n, \quad \forall r \in (0, \operatorname{diam}(E)),\ x \in E, \tag{2.1}$$

where diam(E) may be infinite.

Definition 2.17. (**UR**) (aka *uniformly rectifiable*). An n-dimensional ADR (hence closed) set $E \subset \mathbb{R}^{n+1}$ is UR if and only if it contains "Big Pieces of Lipschitz Images" of $\mathbb{R}^n$ ("BPLI"). This means that there are positive constants θ and M_0, such that for each $x \in E$ and each $r \in (0, \operatorname{diam}(E))$, there is a Lipschitz mapping $\rho = \rho_{x,r} : \mathbb{R}^n \to \mathbb{R}^{n+1}$, with Lipschitz constant no larger than M_0, such that

$$H^n\Big(E \cap B(x,r) \cap \rho\left(\{z \in \mathbb{R}^n : |z| < r\}\right)\Big) \geqslant \theta\, r^n .$$

We recall that n-dimensional rectifiable sets are characterized by the property that they can be covered, up to a set of H^n measure 0, by a countable union of Lipschitz images of $\mathbb{R}^n$; we observe that BPLI is a quantitative version of this fact.

We remark that, at least among the class of ADR sets, the UR sets are precisely those for which all "sufficiently nice" singular integrals are L^2-bounded [DS1]. In fact, for n-dimensional ADR sets in $\mathbb{R}^{n+1}$, the L^2 boundedness of certain special singular integral operators (the "Riesz Transforms"), suffices to characterize uniform rectifiability (see [MMV] for the case $n = 1$, and [NToV] in general). We further remark that there exist sets that are ADR (and that even form the boundary of a domain satisfying interior Corkscrew and Harnack Chain conditions), but that are totally non-rectifiable (e.g., see the construction of Garnett's "4-corners Cantor set" in [DS2, Chapter1]). Finally, we mention that there are numerous other characterizations of UR sets (many of which remain valid in higher co-dimensions); see [DS1, DS2].

Definition 2.18. (**Corkscrew condition**). Following [JK], we say that an open set $\Omega \subset \mathbb{R}^{n+1}$ satisfies the *Corkscrew condition* (more precisely, the *interior* Corkscrew condition) if for some uniform constant $c > 0$ and for every surface ball $\Delta := \Delta(x,r)$, with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, there is a ball $B(X_\Delta, cr) \subset B(x,r) \cap \Omega$. The point $X_\Delta \subset \Omega$ is called a "Corkscrew point" relative to Δ.

Definition 2.19. (**Harnack Chain condition**). Again following [JK], we say that Ω satisfies the *Harnack Chain condition* if there is a uniform constant C such that for every $\rho > 0$, $\Lambda \geqslant 1$, and every pair of points $X, X' \in \Omega$ with $\delta(X)$, $\delta(X') \geqslant \rho$ and $|X - X'| < \Lambda\,\rho$, there is a chain of open balls $B_1, \dots, B_N \subset \Omega$, $N \leqslant C(\Lambda)$, with $X \in B_1$, $X' \in B_N$, $B_k \cap B_{k+1} \neq \emptyset$ and $C^{-1} \operatorname{diam}(B_k) \leqslant \operatorname{dist}(B_k, \partial\Omega) \leqslant C \operatorname{diam}(B_k)$. The chain of balls is called a "Harnack Chain".

Definition 2.20. (**NTA and uniform domains**). Again following [JK], we say that a domain $\Omega \subset \mathbb{R}^{n+1}$ is NTA ("Non-tangentially accessible") if it satisfies the Harnack Chain condition, and if both Ω and $\Omega_{\text{ext}} := \mathbb{R}^{n+1} \setminus \overline{\Omega}$ satisfy the Corkscrew condition. If Ω merely satisfies the Harnack Chain condition and the interior (but not exterior) Corkscrew condition, then it is said to be a *uniform* (aka *1-sided NTA*) domain.

Definition 2.21. (**Chord-arc and 1-sided Chord-arc**). A domain $\Omega \subset \mathbb{R}^{n+1}$ is *Chord-arc* if it is an NTA domain with an ADR boundary; it is *1-sided Chord-arc* if it is a uniform (i.e., 1-sided NTA) domain with ADR boundary.

Definition 2.22. (A_∞, weak-A_∞, and weak-RH_q). Given an ADR set $E \subset \mathbb{R}^{n+1}$, and a surface ball $\Delta_0 := B_0 \cap E$, we say that a Borel measure μ defined on E belongs to $A_\infty(\Delta_0)$ if there are positive constants C and θ such that for each surface ball $\Delta = B \cap E$, with $B \subseteq B_0$, we have

$$\mu(F) \leqslant C \left(\frac{\sigma(F)}{\sigma(\Delta)} \right)^{\theta} \mu(\Delta) \,, \qquad \text{for every Borel set } F \subset \Delta \,. \tag{2.2}$$

Similarly, we say that $\mu \in \text{weak-}A_\infty(\Delta_0)$ if for each surface ball $\Delta = B \cap E$, with $2B \subseteq B_0$,

$$\mu(F) \leqslant C \left(\frac{\sigma(F)}{\sigma(\Delta)} \right)^{\theta} \mu(2\Delta) \,, \qquad \text{for every Borel set } F \subset \Delta \,. \tag{2.3}$$

We recall that, as is well known, the condition $\mu \in \text{weak-}A_\infty(\Delta_0)$ is equivalent to the property that $\mu \ll \sigma$ in Δ_0, and that for some $q > 1$, the Radon-Nikodym derivative $k := d\mu/d\sigma$ satisfies the weak reverse Hölder estimate

$$\left(\fint_\Delta k^q d\sigma \right)^{1/q} \lesssim \fint_{2\Delta} k \, d\sigma \approx \frac{\mu(2\Delta)}{\sigma(\Delta)} \,, \quad \forall \, \Delta = B \cap E, \text{ with } 2B \subseteq B_0 \,. \tag{2.4}$$

We shall refer to the inequality in (2.4) as an "RH_q" estimate, and we shall say that $k \in RH_q(\Delta_0)$ if k satisfies (2.4).

Chapter 3

Carleson measure estimates and the Dirichlet problem for degenerate elliptic equations

3.1 Introduction

We consider the Dirichlet boundary value problem for the degenerate elliptic equation $\operatorname{div}(A\nabla u) = 0$ in the upper-half space $\mathbb{R}^{n+1}_+$ when $n \geqslant 2$ and which we make precise below. The boundary $\mathbb{R}^n \times \{0\}$ is identified with $\mathbb{R}^n$ and we adopt the notation $X = (x,t)$ for points $X \in \mathbb{R}^{n+1}_+$ with coordinates $x \in \mathbb{R}^n$ and $t \in (0,\infty)$. The gradient $\nabla := (\nabla_x, \partial_t)$ and divergence $\operatorname{div} := \operatorname{div}_x + \partial_t$ are with respect to all $(n+1)$-coordinates. The coefficient A denotes an $(n+1)\times(n+1)$-matrix of measurable, real-valued and t-independent functions on $\mathbb{R}^{n+1}_+$. The matrix $A(x) := A(x,t)$ is not required to be symmetric. We suppose that there exist constants $0 < \lambda \leqslant \Lambda < \infty$ and an A_2-weight μ on $\mathbb{R}^n$ such that the degenerate bound and ellipticity

$$|\langle A(x)\xi, \zeta\rangle| \leqslant \Lambda\mu(x)|\xi||\zeta| \quad \text{and} \quad \langle A(x)\xi, \xi\rangle \geqslant \lambda\mu(x)|\xi|^2 \tag{3.1}$$

hold for all $\xi, \zeta \in \mathbb{R}^{n+1}$ and almost every $x \in \mathbb{R}^n$. We use $\langle \cdot, \cdot \rangle$ and $|\cdot|$ to denote the Euclidean inner-product and norm. An A_2-weight μ on $\mathbb{R}^n$ refers to a non-negative locally integrable function $\mu : \mathbb{R}^n \to [0,\infty]$ such that

$$[\mu]_{A_2} := \sup_Q \left(\frac{1}{|Q|} \int_Q \mu(x)\, dx \right) \left(\frac{1}{|Q|} \int_Q \frac{1}{\mu(x)}\, dx \right) < \infty,$$

where $\sup_Q$ denotes the supremum over all dyadic cubes Q in $\mathbb{R}^n$ with volume $|Q|$. We also use μ to denote the measure $\mu(Q) := \int_Q \mu(x)\, dx$ and consider the Lebesgue space $L^p_\mu(\mathbb{R}^n)$ with the norm $\|f\|_{L^p_\mu(\mathbb{R}^n)} := (\int_{\mathbb{R}^n} |f|^p \, d\mu)^{1/p}$ for all $p \in [1,\infty)$. There is also the notation $⨍_Q f\, d\mu := \mu(Q)^{-1} \int_Q f\, d\mu$ whilst $⨍_Q f := |Q|^{-1} \int_Q f(x)\, dx$.

If μ is identically 1, then A is called uniformly elliptic. The solvability of the Dirichlet problem for general nonsymmetric coefficients in that case was obtained only recently by Hofmann, Kenig, Mayboroda and Pipher in [HKMP1]. The result in dimension $n = 1$ had been obtained previously by Kenig, Koch, Pipher and Toro in [KKoPT]. These results assert that for each uniformly elliptic coefficient matrix A, there exists some $p < \infty$ for which the Dirichlet problem is solvable for L^p-boundary data. Conversely, counterexamples in [KKoPT] show that for each $p < \infty$, there exists a uniformly elliptic coefficient matrix A for which the Dirichlet problem is not solvable for L^p-boundary data. In contrast, solvability of the Dirichlet problem for symmetric coefficients in the uniformly elliptic case is well-understood, and we mention only that it was obtained by Jerison and Kenig in [JK] for L^p-boundary data when $2 \leqslant p < \infty$.

The solvability of the Dirichlet problem in the uniformly elliptic case has also been established for a variety of complex coefficient structures (see, for instance, [AS, HKMP2, HMM]). A significant portion of that theory was recently extended to the degenerate elliptic case by Auscher, Rosén and Rule in [ARR] for L^2-boundary data. This extension did not include, however, the results for general nonsymmetric coefficients in [HKMP1, HKMP2]. This paper begins to fill that gap by extending the solvability result for the Dirichlet problem obtained in [HKMP1] to the degenerate elliptic case.

The notion of solvability we consider requires that the A_2-weight μ on $\mathbb{R}^n$ is extended to the t-independent A_2-weight $\mu_+(x,t) := \mu(x)$ on $\mathbb{R}^{n+1}_+$ with $[\mu_+]_{A_2} = [\mu]_{A_2}$. We say that u is a solution of the equation $\operatorname{div} A\nabla u = 0$ in an open set $\Omega \subseteq \mathbb{R}^{n+1}_+$ when $u \in W^{1,2}_{\mu_+,\mathrm{loc}}(\Omega)$ and $\int_{\mathbb{R}^{n+1}_+} \langle A\nabla u, \nabla\Phi\rangle = 0$ for all smooth compactly supported functions $\Phi \in C_c^\infty(\Omega)$. The solution space is the local μ_+-weighted Sobolev space $W^{1,2}_{\mu_+,\mathrm{loc}}$ defined in Section 3.2. The convergence of solutions to boundary data is afforded by estimates for the nontangential maximal operator N_* defined by

$$N_*(u)(x) := \sup_{(y,t)\in\Gamma(x)} |u(y,t)|$$

for all $x \in \mathbb{R}^n$, where the cone $\Gamma(x) := \{(y,t) \in \mathbb{R}^{n+1}_+ : |y-x| < t\}$. If $p \in (1,\infty)$, then the Dirichlet problem for L^p_μ-boundary data, or simply $(D)_{p,\mu}$, is said to be solvable when for each $f \in L^p_\mu(\mathbb{R}^n)$, there exists a solution u such that

$$\begin{cases} \operatorname{div} A\nabla u = 0 \text{ in } \mathbb{R}^{n+1}_+, \\ N_*(u) \in L^p_\mu(\mathbb{R}^n), \\ \lim_{t\to 0} u(\cdot,t) = f, \end{cases} \tag{$(D)_{p,\mu}$}$$

where the limit is required to converge in $L^p_\mu(\mathbb{R}^n)$-norm and in the nontangential sense that $\lim_{\Gamma(x)\ni(y,t)\to(x,0)} u(y,t) = f(x)$ for almost every $x \in \mathbb{R}^n$.

A nonnegative Borel measure ω on a dyadic cube Q_0 in $\mathbb{R}^n$ is said to be in the $A_\infty(Q_0)$-class with respect to μ when there exist constants $C, \theta > 0$, which we call the $A_\infty(Q_0)$-constants, such that

$$\omega(F) \leqslant C \left(\frac{\mu(F)}{\mu(Q)}\right)^\theta \omega(Q) \tag{3.2}$$

for every dyadic cube $Q \subseteq Q_0$ and every Borel set $F \subseteq Q$. This is a scale-invariant version of the absolute continuity of ω with respect to μ. It is well-known, at least in the uniformly elliptic case, that solvability of the Dirichlet problem for L^p-boundary data for some $p < \infty$ is equivalent to the property that an adapted harmonic measure belongs to the A_∞-class with respect to the Lebesgue measure on $\mathbb{R}^n$ (see Theorem 1.7.3 in [K]). In the degenerate case, an adapted harmonic measure ω^X, which we call the the solution measure, can also be defined at each $X \in \mathbb{R}^{n+1}_+$ (see Section 3.5). We prove that solution measure is in the A_∞-class with respect to μ and then deduce the solvability of $(D)_{p,\mu}$ stated in the theorem below. This requires the notation for dyadic cubes Q in $\mathbb{R}^n$ whereby x_Q and $\ell(Q)$ denote the centre and side length of Q, respectively, and $X_Q := (x_Q, \ell(Q))$ denotes the corkscrew point in $\mathbb{R}^{n+1}_+$ relative to Q.

Theorem 3.1. *Let $n \geqslant 2$ and suppose that A is an $(n+1) \times (n+1)$-matrix of measurable, real-valued and t-independent functions on $\mathbb{R}^{n+1}_+$. If A satisfies the degenerate bound and ellipticity in (3.1) for some constants $0 < \lambda \leqslant \Lambda < \infty$ and an A_2-weight μ on $\mathbb{R}^n$, then there exists $p \in (1, \infty)$ such that $(D)_{p,\mu}$, the Dirichlet problem for the equation* $\operatorname{div} A\nabla u = 0$ *in $\mathbb{R}^{n+1}_+$ with $L^p_\mu(\mathbb{R}^n)$-boundary data, is solvable. Moreover, for each dyadic cube Q in $\mathbb{R}^n$, the solution measure w^{X_Q} is in the $A_\infty(Q)$-class with respect to μ and the $A_\infty(Q)$-constant depends only on n, λ, Λ and $[\mu]_{A_2}$.*

Theorem 3.2 (Carleson measure estimate (CME))**.** *Let $n \geqslant 2$ and suppose that A is an $(n+1) \times (n+1)$-matrix of measurable, real-valued and t-independent functions on $\mathbb{R}^{n+1}_+$. If A satisfies the degenerate bound and ellipticity in (3.1) for some constants $0 < \lambda \leqslant \Lambda < \infty$ and an A_2-weight μ on $\mathbb{R}^n$, then any bounded solution $u \in L^\infty(\mathbb{R}^{n+1}_+)$ of the equation* $\operatorname{div} A\nabla u = 0$ *in $\mathbb{R}^{n+1}_+$ satisfies*

$$\sup_Q \frac{1}{\mu(Q)} \int_0^{\ell(Q)} \int_Q |t\nabla u(x,t)|^2 \, d\mu(x) \frac{dt}{t} \leqslant C \|u\|_\infty^2, \tag{3.3}$$

where C depends only on n, λ, Λ and $[\mu]_{A_2}$.

The convention is adopted whereby C denotes a finite positive constant that may change from one line to the next. For $a, b \in \mathbb{R}$, the notation $a \lesssim b$ means that $a \leqslant Cb$ whilst $a \eqsim b$ means that $a \lesssim b \lesssim a$. We write $a \lesssim_p b$ when $a \leqslant Cb$ and we wish to emphasize that C depends on a specified parameter $p > 0$.

3.2 Preliminaries

We dispense with some technical preliminaries concerning A_2-weights μ and degenerate elliptic operators on $\mathbb{R}^n$ for $n \in \mathbb{N}$. The standard dyadic cubes Q and all balls B in $\mathbb{R}^n$ are assumed to be open. For $\alpha > 0$, let αQ and αB denote the concentric dilates of Q and B respectively. For $x \in \mathbb{R}^n$ and $r > 0$, define the ball $B(x, r) := \{y \in \mathbb{R}^n : |y - x| < r\}$. If μ is an A_2-weight on $\mathbb{R}^n$, then the associated measure satisfies the doubling property $\mu(2B) \leqslant [\mu]_{A_2} 2^{2n} \mu(B)$, which implies that

$$\mu(B(x, \alpha r)) \leqslant [\mu]_{A_2} 2^{2n} \alpha^{\log_2 [\mu]_{A_2} + 2n} \mu(B(x, r))$$

for all $x \in \mathbb{R}^n$, $r > 0$ and $\alpha \geqslant 1$ (see, for instance, Section 1.5 in Chapter V of [S2]).

For an open set $\Omega \subseteq \mathbb{R}^n$, the Sobolev space $W^{1,2}_\mu(\Omega)$ is defined as the completion, in the ambient space $L^2_\mu(\Omega)$, of the normed space of all $f \in C^\infty(\Omega)$ with finite norm

$$\|f\|^2_{W^{1,2}_\mu(\Omega)} := \int_\Omega |f|^2 \, d\mu + \int_\Omega |\nabla f|^2 \, d\mu < \infty. \tag{3.4}$$

The embedding of the completion $W^{1,2}_\mu(\Omega)$ in $L^2_\mu(\Omega)$ relies on the A_2-property, which ensures that if $(f_j)_j$ is a $W^{1,2}_\mu(\Omega)$-Cauchy sequence in $C^\infty(\Omega)$ converging to 0 in $L^2_\mu(\Omega)$, then $(f_j)_j$ converges to 0 in $W^{1,2}_\mu(\Omega)$-norm (see Section 2.1 in [FKS]). This also implies that $C^\infty(\Omega)$ is dense in $W^{1,2}_\mu(\Omega)$ and so the gradient extends to a bounded operator $\nabla : W^{1,2}_\mu(\Omega) \to L^2_\mu(\Omega, \mathbb{R}^n)$ which extends (3.4) to all $f \in W^{1,2}_\mu(\Omega)$. The Sobolev space $W^{1,2}_{0,\mu}(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ in $W_\mu^{1,2}(\Omega)$. It can be shown that $W_{0,\mu}^{1,2}(\mathbb{R}^n) = W_\mu^{1,2}(\mathbb{R}^n)$ by following the proof in the unweighted case given in Proposition 1 of Chapter V in [St] but instead using Lemma 2.2 in [ARR] to deduce the convergence of the regularization in $L_\mu^2(\mathbb{R}^n)$. The weighted Sobolev and Poincaré inequalities obtained for continuous functions in Theorems 1.2 and 1.5 in [FKS] also have the following immediate extensions.

Theorem 3.3. *Let $n \geqslant 2$ and $B \subset \mathbb{R}^n$ denote a ball with radius $r(B) > 0$. If μ is an A_2-weight on $\mathbb{R}^n$, then there exists $\delta > 0$, depending only on n and $[\mu]_{A_2}$, such that*

$$\left(\fint_B |f|^{2(\frac{n}{n-1}+\delta)}\, d\mu\right)^{1/(2(\frac{n}{n-1}+\delta))} \lesssim r(B) \left(\fint_B |\nabla f|^2\, d\mu\right)^{1/2} \tag{3.5}$$

for all $f \in W_{0,\mu}^{1,2}(B)$, and

$$\left(\fint_B |f(x) - c_B|^2\, d\mu\right)^{1/2} \lesssim r(B) \left(\fint_B |\nabla f|^2\, d\mu\right)^{1/2} \tag{3.6}$$

for all $f \in W_\mu^{1,2}(B)$ and $c_B \in \left\{\fint_B f\, d\mu, \fint_B f\right\}$, where the implicit constants depend only on n and $[\mu]_{A_2}$.

For $n \in \mathbb{N}$, constants $0 < \lambda \leqslant \Lambda < \infty$ and an A_2-weight μ on $\mathbb{R}^n$, let $\mathcal{E}(n, \lambda, \Lambda, \mu)$ denote the set of all $n \times n$-matrices $\mathcal{A}$ of measurable real-valued functions on $\mathbb{R}^n$ satisfying the degenerate bound and ellipticity

$$|\langle \mathcal{A}(x)\xi, \zeta\rangle| \leqslant \Lambda\mu(x)|\xi||\zeta| \quad \text{and} \quad \langle \mathcal{A}(x)\xi, \xi\rangle \geqslant \lambda\mu(x)|\xi|^2 \tag{3.7}$$

for all $\xi, \zeta \in \mathbb{R}^n$ and almost every $x \in \mathbb{R}^n$. These properties allow us to define $\mathcal{L}_{\mu,\Omega} : \mathrm{Dom}(\mathcal{L}_{\mu,\Omega}) \subseteq L_\mu^2(\Omega) \to$ as the maximal accretive operator in $L_\mu^2(\Omega)$ associated with the bilinear form defined by

$$\mathfrak{a}_\Omega(f, g) := \int_\Omega \langle \mathcal{A}\nabla f, \nabla g\rangle = \int_\Omega \langle \tfrac{1}{\mu}\mathcal{A}\nabla f, \nabla g\rangle\, d\mu \tag{3.8}$$

for all $f, g \in W_{0,\mu}^{1,2}(\Omega)$. The domain of $\mathcal{L}_{\mu,\Omega}$ is dense in $W_{0,\mu}^{1,2}(\Omega)$, and in particular

$$\mathrm{Dom}(\mathcal{L}_{\mu,\Omega}) = \{f \in W_{0,\mu}^{1,2}(\Omega) : \sup_{g \in C_c^\infty(\Omega)} |\mathfrak{a}_\Omega(f, g)|/\|g\|_{L_\mu^2(\Omega)} < \infty\},$$

with

$$\int_\Omega \langle \mathcal{L}_{\mu,\Omega} f, g\rangle \, d\mu = \mathfrak{a}_\Omega(f,g) \tag{3.9}$$

for all $f \in \mathrm{Dom}(\mathcal{L}_{\mu,\Omega})$ and $g \in W^{1,2}_{0,\mu}(\Omega)$. It is equivalent to define $\mathcal{L}_{\mu,\Omega}$ as the composition $-\operatorname{div}_{\mu,\Omega}(\frac{1}{\mu}\mathcal{A}\nabla)$ of unbounded operators, where $-\operatorname{div}_{\mu,\Omega}$ is the adjoint ∇^* of the closed densely-defined operator $\nabla : W^{1,2}_{0,\mu}(\Omega) \subseteq L^2_\mu(\Omega) \to L^2_\mu(\Omega,\mathbb{R}^n)$, that is

$$\int_\Omega \langle -\operatorname{div}_{\mu,\Omega} \mathbf{f}, g\rangle \, d\mu = \int_\Omega \langle \mathbf{f}, \nabla g\rangle \, d\mu \tag{3.10}$$

for all $\mathbf{f} \in \mathrm{Dom}(\operatorname{div}_{\mu,\Omega}) := \mathrm{Dom}(\nabla^*)$ and $g \in W^{1,2}_{0,\mu}(\Omega)$. In view of (3.8) and (3.9), we have the formal identities $\operatorname{div}_{\mu,\Omega} = \frac{1}{\mu}\operatorname{div}_\Omega \mu$ and $\mathcal{L}_{\mu,\Omega} = -\frac{1}{\mu}\operatorname{div}_\Omega(\mathcal{A}\nabla)$.

The space of bounded linear functionals on $W^{1,2}_{0,\mu}(\Omega)$ is denoted by $W^{-1,2}_{0,\mu}(\Omega)$. We interpret the inclusions $W^{1,2}_{0,\mu}(\Omega) \subseteq L^2_\mu(\Omega) \subseteq W^{-1,2}_{0,\mu}(\Omega)$ by identifying $f \in L^2_\mu(\Omega)$ with the functional defined by $\ell_f(g) := \int_\Omega fg \, d\mu$ for all $g \in W^{1,2}_{0,\mu}(\Omega)$. Thus, setting

$$\mathcal{L}_{\mu,\Omega} f(g) := \mathfrak{a}_\Omega(f,g) \qquad \text{and} \qquad -\operatorname{div}_{\mu,\Omega} \mathbf{f}(g) := \int_\Omega \langle \mathbf{f}, \nabla g\rangle \, d\mu$$

for all $f, g \in W^{1,2}_{0,\mu}(\Omega)$ and $\mathbf{f} \in L^2(\Omega,\mathbb{R}^n)$, we obtain an extension of $\mathcal{L}_{\mu,\Omega}$ from (3.9) to a bounded invertible operator from $W^{1,2}_{0,\mu}(\Omega)$ onto $W^{-1,2}_{0,\mu}(\Omega)$, and an extension of $\operatorname{div}_{\mu,\Omega}$ from (3.10) to a bounded operator from $L^2_\mu(\Omega)$ into $W^{-1,2}_{0,\mu}(\Omega)$. The surjectivity of $\mathcal{L}_{\mu,\Omega}$ follows from the Lax–Milgram Theorem. These definitions imply that

$$\|\nabla \mathcal{L}^{-1}_{\mu,\Omega} \operatorname{div}_{\mu,\Omega} \mathbf{f}\|_{L^2_\mu(\Omega,\mathbb{R}^n)} \lesssim \|\mathbf{f}\|_{L^2_\mu(\Omega,\mathbb{R}^n)}$$

for all $\mathbf{f} \in L^2_\mu(\Omega,\mathbb{R}^n)$. The topological direct sum or $W^{1,2}_{0,\mu}(\Omega)$-Hodge decomposition

$$L^2_\mu(\Omega,\mathbb{R}^n) = \{\tfrac{1}{\mu}\mathcal{A}\nabla g : g \in W^{1,2}_{0,\mu}(\Omega)\} \oplus \{\mathbf{h} \in L^2_\mu(\Omega,\mathbb{R}^n) : \operatorname{div}_{\mu,\Omega} \mathbf{h} = 0\} \tag{3.11}$$

follows by writing $\mathbf{f} = \frac{1}{\mu}\mathcal{A}\nabla\mathcal{L}^{-1}_{\mu,\Omega}\operatorname{div}_{\mu,\Omega}\mathbf{f} + (\mathbf{f} - \frac{1}{\mu}\mathcal{A}\nabla\mathcal{L}^{-1}_{\mu,\Omega}\operatorname{div}_{\mu,\Omega}\mathbf{f}) =: \frac{1}{\mu}\mathcal{A}\nabla g + \mathbf{h}$. This decomposition also extends to $L^p_\mu(\Omega,\mathbb{R}^n)$ for all $p \in [2, 2+\epsilon)$ and some $\epsilon > 0$ by recent work of Le in [L], although we do not need it here.

Now let $\Omega = \mathbb{R}^n$ and consider $\operatorname{div}_\mu := \operatorname{div}_{\mu,\mathbb{R}^n}$ as in (3.10) so $\mathcal{L}_\mu := -\operatorname{div}_\mu(\frac{1}{\mu}\mathcal{A}\nabla)$ is maximal accretive, thus having a maximal accretive square root $\mathcal{L}_\mu^{1/2}$, in $L^2_\mu(\mathbb{R}^n)$. The solution of the Kato square root problem in [AHLMT] was recently extended to degenerate elliptic eqations by Cruz-Uribe and Rios in [C-UR2]. This shows that $\|\mathcal{L}_\mu^{1/2} f\|_{L^2_\mu(\mathbb{R}^n)} \approx \|\nabla f\|_{L^2_\mu(\mathbb{R}^n,\mathbb{R}^n)}$ for all $f \in W^{1,2}_\mu(\mathbb{R}^n)$, hence $\operatorname{Dom}(\mathcal{L}_\mu^{1/2}) = W^{1,2}_\mu(\mathbb{R}^n)$.

The operator $\mathcal{L}_\mu$ is also injective and type-$S_{\omega+}$ in $L^2_\mu(\mathbb{R}^n)$ for some $\omega \in (0, \pi/2)$, so it has a bounded $H^\infty(S^o_{\theta+})$-functional calculus in $L^2_\mu(\mathbb{R}^n)$ for each $\theta \in (\omega, \pi)$, where $S^o_{\theta+} := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \theta\}$. See Section 2.2 of [A] for the uniformly elliptic case and Theorems F and G in [ADM] for the general theory. An equivalent property is the validity of the quadratic estimate

$$\int_0^\infty \|\psi(t\mathcal{L}_\mu)f\|^2_{L^2_\mu(\mathbb{R}^n)} \frac{dt}{t} \approx \|f\|^2_{L^2_\mu(\mathbb{R}^n)} \qquad \forall f \in L^2_\mu(\mathbb{R}^n) \tag{3.12}$$

for each holomorphic ψ on $S^o_{\theta+}$ satisfying $|\psi(z)| \lesssim \min\{|z|^\alpha, |z|^{-\beta}\}$ for some $\alpha, \beta > 0$, where the bounded operator $\psi(t\mathcal{L}_\mu)$ on $L^2_\mu(\mathbb{R}^n)$ is defined by a Cauchy integral.

The functional calculus then defines a bounded operator $\varphi(\mathcal{L}_\mu)$ on $L^2_\mu(\mathbb{R}^n)$ for each bounded holomorphic function φ on $S^o_{\theta+}$ and $\|\varphi(\mathcal{L}_\mu)\|_{L^2_\mu(\mathbb{R}^n)\to L^2_\mu(\mathbb{R}^n)} \lesssim_\theta \|\varphi\|_\infty$. Another consequence is that $-\mathcal{L}_\mu$ generates a holomorphic contraction semigroup $(e^{-\zeta\mathcal{L}_\mu})_{\zeta\in S^o_{\pi/2-\omega}\cup\{0\}}$ on $L^2_\mu(\mathbb{R}^n)$, thus $e^{-t\mathcal{L}_\mu} f \in \operatorname{Dom}(\mathcal{L}_\mu)$ and $\partial_t(e^{-t\mathcal{L}_\mu} f) = \mathcal{L}_\mu e^{-t\mathcal{L}_\mu} f$ for all $f \in L^2_\mu(\mathbb{R}^n)$ and $t > 0$. The functional calculus also extends to define an unbounded operator $\phi(\mathcal{L}_\mu)$ on $L^2_\mu(\mathbb{R}^n)$ for each holomorphic function ϕ on $S^o_{\theta+}$ satisfying $|\phi(z)| \lesssim \max\{|z|^\alpha, |z|^{-\beta}\}$ for some $\alpha, \beta > 0$, but the algebra homomorphism property of the functional calculus $(\phi_1(\mathcal{L}_\mu)\phi_2(\mathcal{L}_\mu) = (\phi_1\phi_2)(\mathcal{L}_\mu))$ must then be interpreted in the sense of unbounded linear operators. This allows us to interpret both the semigroup and the square root of $\mathcal{L}_\mu$ in terms of the functional calculus in order to justify some otherwise formal manipulations, beginning with (3.15) in the proof of the following corollary of the solution of the Kato problem in [C-UR2].

Theorem 3.4. *Let $n \geqslant 1$ and suppose that $\mathcal{A} \in \mathcal{E}(n, \lambda, \Lambda, \mu)$ for some constants $0 < \lambda \leqslant \Lambda < \infty$ and an A_2-weight μ on $\mathbb{R}^n$. The operator $\mathcal{L}_\mu := -\operatorname{div}_\mu(\frac{1}{\mu}\mathcal{A}\nabla)$ satisfies*

$$\int_0^\infty \|t\mathcal{L}_\mu e^{-t^2\mathcal{L}_\mu} f\|^2_{L^2_\mu(\mathbb{R}^n)} \frac{dt}{t} \eqsim \|\nabla f\|^2_{L^2_\mu(\mathbb{R}^n,\mathbb{R}^n)} \tag{3.13}$$

and

$$\int_0^\infty \|t^2\nabla_{x,t}\mathcal{L}_\mu e^{-t^2\mathcal{L}_\mu} f\|^2_{L^2_\mu(\mathbb{R}^n,\mathbb{R}^{n+1})} \frac{dt}{t} \lesssim \|\nabla f\|^2_{L^2_\mu(\mathbb{R}^n,\mathbb{R}^n)} \tag{3.14}$$

for all $f \in W^{1,2}_\mu(\mathbb{R}^n)$, where the implicit constants depends only on n, λ, Λ and $[\mu]_{A_2}$.

Proof. The functional calculus of $\mathcal{L}_\mu$ justifies the identity

$$\mathcal{L}_\mu e^{-t^2\mathcal{L}_\mu} f = \mathcal{L}_\mu^{1/2} e^{-t^2\mathcal{L}_\mu} \mathcal{L}_\mu^{1/2} f = e^{-(t^2/2)\mathcal{L}_\mu}\mathcal{L}_\mu e^{-(t^2/2)\mathcal{L}_\mu} f \tag{3.15}$$

for all $f \in \operatorname{Dom}(\mathcal{L}_\mu^{1/2})$ and $t > 0$. The first equality in (3.15), the quadratic estimate in (3.12) and the solution of the Kato problem in [C-UR2] imply that

$$\begin{aligned}
\int_0^\infty \|t\mathcal{L}_\mu e^{-t^2\mathcal{L}_\mu} f\|^2_{L^2_\mu(\mathbb{R}^n)} \frac{dt}{t} &= \int_0^\infty \|(\tau\mathcal{L}_\mu)^{1/2} e^{-\tau\mathcal{L}_\mu}\mathcal{L}_\mu^{1/2} f\|^2_{L^2_\mu(\mathbb{R}^n)} \frac{d\tau}{\tau} \\
&\eqsim \|\mathcal{L}_\mu^{1/2} f\|^2_{L^2_\mu(\mathbb{R}^n)} \\
&\eqsim \|\nabla f\|^2_{L^2_\mu(\mathbb{R}^n,\mathbb{R}^n)}
\end{aligned}$$

for all $f \in \operatorname{Dom}(\mathcal{L}_\mu^{1/2}) = W^{1,2}_\mu(\mathbb{R}^n)$, which proves (3.13).

The bounded $H^\infty(S^o_{\theta+})$-functional calculus of $\mathcal{L}_\mu$ implies the uniform estimate

$$\begin{aligned}
\|t\nabla_{x,t} e^{-t^2\mathcal{L}_\mu} g\|^2_{L^2_\mu(\mathbb{R}^n,\mathbb{R}^{n+1})} &= \|t\partial_t e^{-t^2\mathcal{L}_\mu} g\|^2_{L^2_\mu(\mathbb{R}^n)} + \|t\nabla_x e^{-t^2\mathcal{L}_\mu}\|^2_{L^2_\mu(\mathbb{R}^n,\mathbb{R}^n)} \\
&\lesssim \|t^2\mathcal{L}_\mu e^{-t^2\mathcal{L}_\mu} g\|^2_{L^2_\mu(\mathbb{R}^n)} + \int_{\mathbb{R}^n} t^2\langle A\nabla_x e^{-t^2\mathcal{L}_\mu} g, \nabla_x e^{-t^2\mathcal{L}_\mu} g\rangle \\
&\lesssim \|g\|^2_{L^2_\mu(\mathbb{R}^n)} + \|t^2\mathcal{L}_\mu e^{-t^2\mathcal{L}_\mu} g\|^2_{L^2_\mu(\mathbb{R}^n)}\|e^{-t^2\mathcal{L}_\mu} g\|^2_{L^2_\mu(\mathbb{R}^n)}
\end{aligned}$$

$$\lesssim \|g\|^2_{L^2_\mu(\mathbb{R}^n)}$$

for all $g \in L^2_\mu(\mathbb{R}^n)$ and $t > 0$. Thus, the second equality in (3.15) and the vertical square function estimate in (3.13), which we have already proved, imply that

$$\int_0^\infty \|t^2 \nabla_{x,t} \mathcal{L}_\mu e^{-t^2 \mathcal{L}_\mu} f\|^2_{L^2_\mu(\mathbb{R}^n,\mathbb{R}^{n+1})} \frac{dt}{t} \lesssim \int_0^\infty \|t \mathcal{L}_\mu e^{-(t^2/2)\mathcal{L}_\mu} f\|^2_{L^2_\mu(\mathbb{R}^n)} \frac{dt}{t} \lesssim \|\nabla f\|^2_{L^2_\mu(\mathbb{R}^n,\mathbb{R}^n)}$$

for all $f \in W^{1,2}_\mu(\mathbb{R}^n)$, which proves (3.14). ∎

Now suppose that $\mathbf{f} : \Omega \to \mathbb{R}^n$ is a measurable function for which $\frac{1}{\mu}\mathbf{f} \in L^\infty(\Omega)$. A solution of the inhomogeneous equation $\operatorname{div} \mathcal{A}\nabla u = \operatorname{div} \mathbf{f}$ in $\Omega \subseteq \mathbb{R}^n$ refers to any function $u \in W^{1,2}_{\mu,\mathrm{loc}}(\Omega)$ such that $\int_{\mathbb{R}^n} \langle \mathcal{A}\nabla u - \mathbf{f}, \nabla\Phi \rangle = 0$ for all $\Phi \in W^{1,2}_{0,\mu}(\Omega)$. All solutions of the homogeneous equation $\operatorname{div} \mathcal{A}\nabla u = 0$ in Ω are locally Hölder continuous in the sense that there exist constants $C_0, \alpha > 0$, depending only on n, λ, Λ and $[\mu]_{A_2}$, such that

$$|u(x) - u(y)| \leqslant C_0 \left(\frac{|x-y|}{r(B)} \right)^\alpha \left(\fint_{2B} |u|^2 \, d\mu \right)^{1/2} \tag{3.16}$$

for all $x, y \in B$ and balls B of radius $r(B)$ such that $2B \subseteq \Omega$. This is deduced from Theorem 2.3.12 in [FKS], since the proof does not require that $\mathcal{A}$ is symmetric. The following local boundedness estimate for solutions of the inhomogeneous equation is also needed. The result is a simpler version of Theorem 8.17 in [GT], which we have adapted to degenerate elliptic equations.

Theorem 3.5. *Let $n \geqslant 2$ and suppose that $\mathcal{A} \in \mathcal{E}(n, \lambda, \Lambda, \mu)$ for some constants $0 < \lambda \leqslant \Lambda < \infty$ and an A_2-weight μ on $\mathbb{R}^n$. Let $\Omega \subseteq \mathbb{R}^n$ denote an open set and suppose that $\mathbf{f} : \Omega \to \mathbb{R}^n$ is a measurable function such that $\frac{1}{\mu}\mathbf{f} \in L^\infty(\Omega)$. If $p \in (1, \infty)$ and $\operatorname{div} \mathcal{A}\nabla u = \operatorname{div} \mathbf{f}$ in Ω, then*

$$\|u\|_{L^\infty(B)} \lesssim \left(\frac{1}{\mu(2B)} \int_{2B} |u|^p \, d\mu \right)^{1/p} + r(B) \|\tfrac{1}{\mu}\mathbf{f}\|_{L^\infty(\Omega)} \tag{3.17}$$

for all balls B of radius $r(B) > 0$ such that $2B \subseteq \Omega$, where the implicit constant depends only on p, n, λ, Λ and $[\mu]_{A_2}$.

Proof. Suppose that $\operatorname{div} \mathcal{A}\nabla u = \operatorname{div} \mathbf{f}$ in Ω and consider a ball B such that $2B \subseteq \Omega$. First, assume that u is nonnegative and in $L^\infty(2B)$. Let $\epsilon > 0$, set $k := r(B)\|\frac{1}{\mu}\mathbf{f}\|_{L^\infty(\Omega)}$ and $\bar{u}_\epsilon := u + k + \epsilon$. Let B_r denote the ball concentric to B with radius $r > 0$ and recall the index $\delta > 0$ from the Sobolev inequality in Theorem 3.3. We claim that if $\gamma \in [p, \infty)$ and $r(B) \leqslant r_1 < r_2 \leqslant 2r(B)$, then

$$\left(\fint_{B_{r_1}} \bar{u}_\epsilon^{\gamma(\frac{n}{n-1}+\delta)} \, d\mu \right)^{1/(\gamma(\frac{n}{n-1}+\delta))} \lesssim \left(\gamma \frac{r_1}{r_2 - r_1} \right)^{2/\gamma} \left(\fint_{B_{r_2}} \bar{u}_\epsilon^{\gamma} \, d\mu \right)^{1/\gamma}, \tag{3.18}$$

where the implicit constant depends only on p, n, λ, Λ and $[\mu]_{A_2}$. To prove (3.18), fix $\eta \in C_c^\infty(\Omega)$ such that $\eta : \Omega \to [0,1]$, $\eta \equiv 1$ on B_{r_1}, $\eta \equiv 0$ on $\Omega \setminus B_{r_2}$ and $\|\nabla \eta\|_\infty \leqslant 2/(r_2 - r_1)$. Set $\beta := \gamma - 1$ and $v := \eta^2 \bar{u}_\epsilon^\beta$. Note that $v \in W^{1,2}_{0,\mu}(\Omega)$ with

$$\nabla v = 2\eta \nabla \eta \bar{u}_\epsilon^\beta + \beta \eta^2 \bar{u}_\epsilon^{\beta-1} \nabla u,$$

since $0 < \epsilon \leqslant \bar{u}_\epsilon(x) \leqslant \|u\|_{L^\infty(2B)} + k + \epsilon < \infty$ for almost every $x \in 2B$, thus

$$\int_{\mathbb{R}^n} \langle \mathcal{A}\nabla u - \mathbf{f}, 2\eta \nabla \eta \bar{u}_\epsilon^\beta \rangle = - \int_{\mathbb{R}^n} \langle \mathcal{A}\nabla u - \mathbf{f}, \beta \eta^2 \bar{u}_\epsilon^{\beta-1} \nabla u \rangle.$$

Use this identity and Cauchy's inequality with $\varepsilon > 0$ to obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} \eta^2 \bar{u}_\epsilon^{\beta-1} |\nabla u|^2 \, d\mu &\lesssim_\lambda \int_{\mathbb{R}^n} \eta^2 \bar{u}_\epsilon^{\beta-1} \langle \mathcal{A}\nabla u, \nabla u \rangle \\
&= -2\beta^{-1} \int_{\mathbb{R}^n} \eta \bar{u}_\epsilon^\beta \langle \mathcal{A}\nabla u - \mathbf{f}, \nabla \eta \rangle + \int_{\mathbb{R}^n} \eta^2 \bar{u}_\epsilon^{\beta-1} \langle \mathbf{f}, \nabla u \rangle \\
&\lesssim_\Lambda (p-1)^{-1} \int_{\mathbb{R}^n} \eta \bar{u}_\epsilon^\beta (|\nabla u| + |\tfrac{1}{\mu}\mathbf{f}|) |\nabla \eta| \, d\mu + \int_{\mathbb{R}^n} \eta^2 \bar{u}_\epsilon^{\beta-1} |\tfrac{1}{\mu}\mathbf{f}| |\nabla u| \, d\mu \\
&\lesssim_p \varepsilon \int_{\mathbb{R}^n} \eta^2 \bar{u}_\epsilon^{\beta-1} |\nabla u|^2 \, d\mu + \varepsilon^{-1} \int_{\mathbb{R}^n} \bar{u}_\epsilon^{\beta+1} |\nabla \eta|^2 \, d\mu \\
&\qquad + \int_{\mathbb{R}^n} \bar{u}_\epsilon^{\beta+1} |\nabla \eta|^2 \, d\mu + \int_{\mathbb{R}^n} (\eta / r(B))^2 \bar{u}_\epsilon^{\beta+1} \, d\mu
\end{aligned}$$

$$+ \varepsilon \int_{\mathbb{R}^n} \eta^2 \bar{u}_\epsilon^{\beta-1} |\nabla u|^2 \, d\mu + \varepsilon^{-1} \int_{\mathbb{R}^n} (\eta/r(B))^2 \bar{u}_\epsilon^{\beta+1} \, d\mu,$$

where in the second inequality we used the assumption that $\beta := \gamma - 1 \geqslant p - 1$ and in the final inequality we used the fact that $|\frac{1}{\mu}\mathbf{f}| \leqslant k/r(B) \leqslant \bar{u}_\epsilon/r(B)$ on Ω. Next, choose $\varepsilon > 0$ small enough, depending only on p, λ and Λ, to deduce that

$$\int_{B_{r_1}} \bar{u}_\epsilon^{\beta-1} |\nabla u|^2 \, d\mu \lesssim_{p,\lambda,\Lambda} \int_{\mathbb{R}^n} \bar{u}_\epsilon^{\beta+1} (|\nabla \eta|^2 \, + (\eta/r(B))^2) \, d\mu \lesssim \frac{1}{(r_2 - r_1)^2} \int_{B_{r_2}} \bar{u}_\epsilon^{\beta+1} \, d\mu,$$

where in the final inequality we used the fact that $r(B) \geqslant r_2 - r_1$. Now combine this estimate with the Sobolev inequality (4.13) and recall that $\beta := \gamma - 1$ to obtain

$$\begin{aligned}
\left(\fint_{B_{r_1}} \bar{u}_\epsilon^{\gamma(\frac{n}{n-1}+\delta)} \, d\mu \right)^{1/(\frac{n}{n-1}+\delta)} &\lesssim r_1^2 \fint_{B_{r_1}} |\nabla (\bar{u}_\epsilon^{(\beta+1)/2})|^2 \, d\mu \\
&\lesssim ((\beta+1) r_1)^2 \fint_{B_{r_1}} \bar{u}_\epsilon^{\beta-1} |\nabla u|^2 \, d\mu \\
&\lesssim \left(\gamma \frac{r_1}{r_2 - r_1} \right)^2 \fint_{B_{r_2}} \bar{u}_\epsilon^{\gamma} \, d\mu,
\end{aligned}$$

where the implicit constants depends only on p, n, λ, Λ and $[\mu]_{A_2}$. This proves (3.18).

We now apply the Moser iteration technique to prove (3.17). Set $\chi := \frac{n}{n-1} + \delta$ and define $\Phi(q, r) := \left(\fint_{B_r} \bar{u}_\epsilon^q \, d\mu \right)^{1/q}$ for $q, r > 0$. Estimate (3.18) implies that

$$\Phi(\gamma\chi, r_1) \leqslant \left(C\gamma \frac{r_1}{r_2 - r_1} \right)^{2/\gamma} \Phi(\gamma, r_2)$$

where C depends only on p, n, λ, Λ and $[\mu]_{A_2}$, and it follows by induction that

$$\Phi(p\chi^m, (1 + 2^{-m}) r(B)) \leqslant (4Cp)^{\frac{2}{p} \sum_{k=0}^{m-1} \chi^{-k}} (2\chi)^{\frac{2}{p} \sum_{k=0}^{m-1} k\chi^{-k}} \Phi(p, 2r(B)) \lesssim \Phi(p, 2r(B))$$

for all $m \in \mathbb{N}$. This shows that

$$\|\bar{u}_\epsilon\|_{L^\infty(B)} = \lim_{m \to \infty} \Phi(p\chi^m, r(B)) \lesssim \Phi(p, 2r(B)) = \left(\fint_{2B} \bar{u}_\epsilon^p \, d\mu \right)^{1/p}$$

and therefore

$$\|u\|_{L^\infty(B)} \leqslant \|\bar{u}_\epsilon\|_{L^\infty(B)} \lesssim \left(\fint_{2B} \bar{u}_\epsilon^p \, d\mu \right)^{1/p} \lesssim \left(\fint_{2B} u^p \, d\mu \right)^{1/p} + r(B)\|\tfrac{1}{\mu}\mathbf{f}\|_{L^\infty(\Omega)} + \epsilon$$

for all $\epsilon > 0$, which implies (3.17).

Finally, it remains to remove the assumption that u is nonnegative and in $L^\infty(2B)$. This is achieved by setting $\bar{u}_\epsilon := \max\{u, 0\} + k + \epsilon$ and $\bar{u}_\epsilon := -\min\{u, 0\} + k + \epsilon$ respectively and in each case adjusting the proof above to incorporate the truncated test function $v := \eta^2 \left(h_N(\bar{u}_\epsilon)\bar{u}_\epsilon - (k+\epsilon)^\beta \right)$, where

$$h_N(x) := \begin{cases} x^{\beta-1}, & x \leqslant N + k + \epsilon, \\ (N+k+\epsilon)^{\beta-1}, & x > N + k + \epsilon. \end{cases}$$

We leave the standard details to the reader. ∎

The following self-improvement property for Carleson measures will be used in conjunction with the local Hölder continuity estimate for solutions in (3.16). The result is proved in the unweighted case in Lemma 2.14 in [AHLT]. In that proof, the Lebesgue measure on $\mathbb{R}^n$ can in fact be replaced by any doubling measure, since the Whitney decomposition of open sets can be adapted to any such measure (see, for instance, Lemma 2 in Chapter I of [S2]). The result below then follows.

Lemma 3.6. *Let $n \geqslant 1$ and $Q \subset \mathbb{R}^n$ denote a dyadic cube. Suppose that $\alpha, \beta_0 > 0$ and let $(v_t)_{t>0}$ denote a collection of Hölder continuous functions on Q satisfying*

$$0 \leqslant v_t(x) \leqslant \beta_0 \quad \textit{and} \quad |v_t(x) - v_t(y)| \leqslant \beta_0 \left(\frac{|x-y|}{t} \right)^\alpha$$

for all $x, y \in Q$. If, for an A_2-weight μ on $\mathbb{R}^n$, there exists $\eta \in (0, 1]$, $\beta > 0$ and, for each dyadic cube $Q' \subseteq Q$, a measurable set $F' \subset Q'$ such that

$$\mu(F') \geqslant \eta\mu(Q') \quad \textit{and} \quad \frac{1}{\mu(Q')} \int_0^{l(Q')} \int_{F'} v_t(x) \, d\mu(x) \frac{dt}{t} \leqslant \beta,$$

then

$$\frac{1}{\mu(Q)}\int_0^{l(Q)}\int_Q v_t(x)\,d\mu(x)\frac{dt}{t} \lesssim_{\alpha,\eta} \beta+\beta_0,$$

where the implicit constant depends only on n and $[\mu]_{A_2}$.

3.3 Estimates for maximal operators

For an A_2-weight μ on $\mathbb{R}^n$, define the maximal operators M_μ and $D_{*,\mu}$ by

$$\begin{aligned} M_\mu f(x) &:= \sup_{r>0} \fint_{B(x,r)} |f(y)|\,d\mu(y) \\ D_{\mu,*}f(x) &:= \sup_{r>0}\left(\fint_{B(x,r)}\left(\frac{|f(x)-f(y)|}{|x-y|}\right)^2 d\mu(y)\right)^{1/2} \end{aligned}$$

for all suitable functions f on $\mathbb{R}^n$ and $x\in\mathbb{R}^n$. The (unweighted) Hardy–Littlewood maximal operator is denoted by M. The maximal operator M_μ satisfies the weak-type estimate

$$\mu\big(\{x\in\mathbb{R}^n : |M_\mu f(x)|>\kappa\}\big) \lesssim \kappa^{-1}\|f\|_{L^1_\mu(\mathbb{R}^n)} \tag{3.19}$$

for all $\kappa>0$ and $f\in L^1_\mu(\mathbb{R}^n)$ (see, for instance, Theorem 1 in Chapter I of [S2]).

Lemma 3.7. *Let $n\geqslant 2$. If μ is an A_2-weight on $\mathbb{R}^n$, then*

$$\mu\big(\{x\in\mathbb{R}^n : |D_{\mu,*}f(x)|>\kappa\}\big) \lesssim \kappa^{-2}\|\nabla f\|^2_{L^2_\mu(\mathbb{R}^n,\mathbb{R}^n)} \tag{3.20}$$

for all $\kappa>0$ and $f\in W^{1,2}_\mu(\mathbb{R}^n)$, where the implicit constant depends only on n and $[\mu]_{A_2}$.

Proof. If $f\in C_c^\infty(\mathbb{R}^n)$, then a version of Morrey's inequality (see, for instance, Theorem 3.5.2 in [M]) implies that

$$\frac{|f(x)-f(y)|}{|x-y|} \lesssim M(\nabla f)(x)+M(\nabla f)(y)$$

for almost every $x,y\in\mathbb{R}^n$, hence

$$D_{\mu,*}f(x) \lesssim M(\nabla f)(x)+\big(M_\mu[M(\nabla f)]^2(x)\big)^{1/2}.$$

Estimate (3.20) then follows from (3.19) and the fact that M is a bounded operator on $L^2_\mu(\mathbb{R}^n)$ (see, for instance, Theorem 1 in Chapter V of [S2].) ∎

For $\eta > 0$, define the nontangential maximal operators N_*^η and $\widetilde{N}_*^\eta$ by

$$N_*^\eta u(x) := \sup_{(y,t)\in\Gamma_\eta(x)} |u(y,t)| \quad \text{and} \quad \widetilde{N}_*^\eta u(x) := \sup_{(y,t)\in\Gamma_\eta(x)} \left(\fint_{B(y,\eta t)} |u(z,t)|^2 \, d\mu(z) \right)^{1/2}$$

for all suitable functions u on $\mathbb{R}^{n+1}_+$ and $x \in \mathbb{R}^n$, where the nontangential approach region $\Gamma_\eta(x) := \{(y,t) \in \mathbb{R}^{n+1}_+ : |y-x| < \eta t\}$ denotes the cone in $\mathbb{R}^{n+1}$ with vertex at x and aperture η.

If $\mathcal{A} \in \mathcal{E}(n,\lambda,\Lambda,\mu)$, then since $\mathcal{A}$ has real-valued coefficients, there exists an integral kernel $W_t(x,y)$ such that

$$e^{-t\mathcal{L}_\mu} f(x) = \int_{\mathbb{R}^n} W_t(x,y) f(y) \, d\mu(y) \tag{3.21}$$

for all $f \in C_c^\infty(\mathbb{R}^n)$ and there exists constants $C_1, C_2 > 0$ such that

$$|W_t(x,y)| \leqslant \frac{C_1}{\mu(B(x,\sqrt{t}))} \exp\left(-C_2 \frac{|x-y|^2}{t} \right) \tag{3.22}$$

for all $t > 0$ and $x, y \in \mathbb{R}^n$. This was proved by Cruz-Uribe and Rios under the additional assumption that $\mathcal{A}$ is symmetric (see Theorem 1 and Remark 3 in [C-UR1]). The symmetry assumption can be removed, however, by following their proof and applying the Harnack inequality for degenerate parabolic equations obtained by Ishige in Theorem A of [I], which does not require symmetric coefficients, instead of the version recorded in Proposition 3.8 of [C-UR1].

It will be convenient to consider the semigroup generated by $\mathcal{L}_\mu$ with elliptic homogeneity (t replaced by t^2) and we set $\mathcal{P}_t := e^{-t^2\mathcal{L}_\mu}$ for this purpose.

Lemma 3.8. *Let $n \geqslant 2$ and suppose that $\mathcal{A} \in \mathcal{E}(n,\lambda,\Lambda,\mu)$ for some constants $0 < \lambda \leqslant \Lambda < \infty$ and an A_2-weight μ on $\mathbb{R}^n$. If $x \in \mathbb{R}^n$, $\eta > 0$ and $\alpha \geqslant 1$, then*

$$\sup_{(y,t)\in\Gamma^\eta(x)} |(\eta t)^{-1}[\mathcal{P}_{\eta t}(f - c_{B(x,\alpha\eta t)})](y)|^2 \lesssim_\alpha M_\mu(|\nabla f|^2)(x) \tag{3.23}$$

for all $f \in C_c^\infty(\mathbb{R}^n)$ *and* $c_{B(x,\alpha\eta t)} \in \left\{ \fint_{B(x,\alpha\eta t)} f \, d\mu, \fint_{B(x,\alpha\eta t)} f \right\}$, *and*

$$|N_*^\eta(\partial_t \mathcal{P}_t f)(x)|^2 \lesssim_\eta M_\mu(|\nabla f|^2)(x), \tag{3.24}$$

$$|\eta^{-1} N_*^\eta(\partial_t \mathcal{P}_{\eta t} f)(x)|^2 \lesssim M_\mu(|\nabla f|^2)(x), \tag{3.25}$$

$$|\widetilde{N}_*^\eta(\nabla_\| \mathcal{P}_{\eta t} f)(x)|^2 \lesssim M_\mu(|\eta^{-1} N_*^\eta(\partial_t \mathcal{P}_{\eta t} f)|^2 + |N_*(\partial_t \mathcal{P}_t f)|^2 + |\nabla f|^2)(x), \tag{3.26}$$

for all $f \in C_c^\infty(\mathbb{R}^n)$, *where the implicit constants depend only on* n, λ, Λ *and* $[\mu]_{A_2}$.

Proof. Suppose throughout the proof that $x \in \mathbb{R}^n$, $(y,t) \in \Gamma^\eta(x)$ and $f \in C_c^\infty(\mathbb{R}^n)$. We set $f_{B(x,t)} := \fint_{B(x,t)} f$ and $\tilde{f}_{B(x,t)} := \fint_{B(x,t)} f \, d\mu$.

To prove (3.23), assume without loss of generality that $\eta = 1$ and $\alpha \geqslant 1$. We set $C_0(t) := B(x, \alpha t)$ and define the dyadic annulus $C_j(t) := B(x, 2^j \alpha t) \setminus B(x, 2^{j-1}\alpha t)$ for all $j \in \mathbb{N}$. The Gaussian kernel estimates in (3.21) and (3.22) imply that

$$\begin{aligned} |t^{-1}[\mathcal{P}_t(f - f_{B(x,\alpha t)})](y)| &= t^{-1} \left| \int_{\mathbb{R}^n} W_{t^2}(y,z)[f(z) - f_{B(x,\alpha t)}] \, d\mu(z) \right| \\ &\leqslant \sum_{j=0}^\infty t^{-1} \frac{C_1}{\mu(B(y,t))} \int_{C_j(t)} \exp\left(-C_2 \frac{|y-z|^2}{t^2} \right) |f(z) - f_{B(x,\alpha t)}| \, d\mu(z) =: \sum_{j=0}^\infty I_j. \end{aligned}$$

To estimate I_0, note that $B(x,\alpha t) \subseteq B(y, (1+\alpha)t)$ and apply the doubling property of μ, followed by the Poincaré inequality in (4.13) with $c_B = \fint_{B(x,\alpha t)} f$, to obtain

$$I_0 \lesssim_\alpha t^{-1} \fint_{B(x,\alpha t)} |f(z) - f_{B(x,\alpha t)}| \, d\mu(z) \lesssim \left(\fint_{B(x,\alpha t)} |\nabla f|^2 \, d\mu \right)^{1/2} \lesssim [M_\mu(|\nabla f|^2)(x)]^{1/2}.$$

To estimate I_j, for each $j \in \mathbb{N}$, expand $f(z) - f_{B(x,\alpha t)}$ as a telescoping sum to write

$$\begin{aligned} I_j &\leqslant C_1 e^{-C_2(2^{j-1}\alpha - 1)^2} \frac{\mu(B(x, 2^j \alpha t))}{\mu(B(y,t))} t^{-1} \Bigg(\fint_{B(x,2^j\alpha t)} |f - \tilde{f}_{B(x,2^j\alpha t)}| \, d\mu \\ &\qquad + \sum_{i=1}^j |\tilde{f}_{B(x,2^i\alpha t)} - \tilde{f}_{B(x,2^{i-1}\alpha t)}| + |\tilde{f}_{B(x,\alpha t)} - f_{B(x,\alpha t)}| \Bigg) \\ &\lesssim e^{-C_2(2^{j-1}\alpha - 1)^2} \frac{\mu(B(y,(1+2^j\alpha)t))}{\mu(B(y,t))} \sum_{i=0}^j t^{-1} \fint_{B(x,2^i\alpha t)} |f - \tilde{f}_{B(x,2^i\alpha t)}| \, d\mu \end{aligned}$$

$$\lesssim e^{-C_2(2^{j-1}\alpha-1)^2}(1+2^j\alpha)^{2n}\sum_{i=0}^{j}2^i\alpha\left(\fint_{B(x,2^i\alpha t)}|\nabla f|^2\,d\mu\right)^{1/2}$$
$$\lesssim_\alpha e^{-C_2(2^{j-1}-1)^2}2^{2j(n+1)}[M_\mu(|\nabla f|^2)(x)]^{1/2},$$

where the second inequality relies on the inclusion $B(x,2^j\alpha t)\subseteq B(y,(1+2^j\alpha)t)$, the third inequality uses the doubling property of μ, and the fourth inequality uses the Poincaré inequality in (4.13) with $c_B=\fint_{B(x,2^i\alpha t)}f\,d\mu$. Altogether, we have

$$|t^{-1}[\mathcal{P}_t(f-f_{B(x,\alpha t)})](y)|^2\lesssim_\alpha\left(\sum_{j=0}^{\infty}e^{-C4^j}4^{j(n+1)}\right)^2M_\mu(|\nabla f|^2)(x)\lesssim M_\mu(|\nabla f|^2)(x),$$

which proves (3.23) when $c_{B(x,\alpha t)}=\fint_{B(x,\alpha t)}f$. The proof when $c_{B(x,\alpha t)}=\fint_{B(x,\alpha t)}f\,d\mu$ follows as above by replacing $f_{B(x,\alpha t)}$ with $\tilde{f}_{B(x,\alpha t)}$, since (4.13) can still be applied.

To prove (3.24) and (3.25), suppose that $\eta>0$. The Gaussian kernel estimate for $e^{-t\mathcal{L}_\mu}$ in (3.22) implies that $t\partial_t\mathcal{P}_tf(y)$ has an integral kernel $\widetilde{W}_{t^2}(y,z)$ satisfying

$$|\widetilde{W}_{t^2}(y,z)|\leqslant\frac{C_1}{\mu(B(y,t))}\exp\left(-C_2\frac{|y-z|^2}{t^2}\right)$$

and the conservation property $\int_{\mathbb{R}^n}\widetilde{W}_{t^2}(y,z)\,d\mu(y)=0$ for all $z\in\mathbb{R}^n$ and $t>0$. This follows from Theorem 5 in [C-UR1], where the assumption that $\mathcal{A}$ is symmetric can be removed as per the remarks preceding this lemma. Therefore, we may write

$$|\partial_t\mathcal{P}_tf(y)|=t^{-1}\left|\int_{\mathbb{R}^n}\widetilde{W}_{t^2}(y,z)[f(z)-f_{B(x,\eta t)}]\,d\mu(z)\right|$$

and a change of variables implies that

$$\sup_{(y,t)\in\Gamma^\eta(x)}|\partial_t\mathcal{P}_tf(y)|=\sup_{(y,t)\in\Gamma(x)}t^{-1}\left|\int_{\mathbb{R}^n}\eta\widetilde{W}_{(t/\eta)^2}(y,z)[f(z)-f_{B(x,t)}]\,d\mu(z)\right|.$$

We can then obtain (3.24) by following the proof of (3.23) with $\alpha=1$ in order to show that this is bounded by $M_\mu(|\nabla f|^2)(x)$, since the doubling property of μ ensures that

$$|\eta\widetilde{W}_{(t/\eta)^2}(y,z)|\leqslant\frac{C_{1,\eta}}{\mu(B(y,t))}\exp\left(-C_{2,\eta}\frac{|y-z|^2}{t^2}\right)$$

for some positive constants $C_{1,\eta}$ and $C_{2,\eta}$ that depend on η. We obtain (3.25) as an immediate consequence of (3.24) and the fact that $\eta^{-1}\partial_t\mathcal{P}_{\eta t} = (\partial_s\mathcal{P}_s)|_{s=\eta t}$.

To prove (3.26), let $\eta > 0$, set $u_{\eta t} := \mathcal{P}_{\eta t}f$ and choose a non-negative function $\Phi \in C_c^\infty(B(y, 2\eta t))$ such that $\Phi \equiv 1$ on $B(y, \eta t)$ and $|\nabla_x\Phi| \lesssim (\eta t)^{-1}$. Let $c > 0$ denote a constant that will be chosen later. The definition of $\mathcal{L}_\mu$ implies that

$$\begin{aligned}
\fint_{B(y,\eta t)} |\nabla_x\mathcal{P}_{\eta t}f|^2\, d\mu &\leqslant \fint_{\mathbb{R}^n} |\nabla_x u_{\eta t}|^2\Phi^2\, d\mu \\
&\lesssim \frac{1}{\mu(B(y,\eta t))}\int_{\mathbb{R}^n} \langle A\nabla_x u_{\eta t}, \nabla_x(u_{\eta t} - c)\rangle\Phi^2 \\
&= \frac{1}{\mu(B(y,\eta t))}\int_{\mathbb{R}^n} \left\{\langle A\nabla_x u_{\eta t}, \nabla_x[(u_{\eta t} - c)\Phi^2]\rangle - 2\langle A\nabla_x u_{\eta t}, \nabla_x\Phi(u_{\eta t} - c)\rangle\Phi\right\} \\
&\lesssim \frac{1}{\mu(B(y,\eta t))}\int_{\mathbb{R}^n} \left\{(\mathcal{L}_\mu u_{\eta t})(u_{\eta t} - c)\Phi^2 + |\nabla_x u_{\eta t}||\nabla_x\Phi||(u_{\eta t} - c)\Phi|\right\} d\mu \\
&\leqslant \frac{1}{\mu(B(y,\eta t))}\int_{B(y,2\eta t)} \left(\frac{1}{2\eta^2 t}|\partial_t u_{\eta t}||u_{\eta t} - c|\Phi^2 + |\nabla_x u_{\eta t}||\nabla_x\Phi||u_{\eta t} - c|\Phi\right) d\mu \\
&=: I + II.
\end{aligned}$$

Now fix $c := \tilde{f}_{B(x,3\eta t)}$. To estimate I, we use Cauchy's inequality and the doubling property of μ, combined with the fact that $B(x,\eta t) \subseteq B(y, 2\eta t) \subseteq B(x, 3\eta t)$, to obtain

$$I \lesssim \fint_{B(x,3\eta t)} \left(|\eta^{-1}\partial_t u_{\eta t}|^2 + (\eta t)^{-2}|u_{\eta t} - f|^2 + (\eta t)^{-2}|f - \tilde{f}_{B(x,3\eta t)}|^2\right) d\mu =: I_1 + I_2 + I_3.$$

It is immediate that $I_1 \leqslant M_\mu(|\eta^{-1}N_*^\eta(\partial_t\mathcal{P}_{\eta t}f)|^2)(x)$, whilst the semigroup property

$$|u_{\eta t}(z) - f(z)| = \left|\int_0^{\eta t} \partial_s u_s(z)\, ds\right| \leqslant \eta t N_*(\partial_s u_s)(z)$$

implies that $I_2 \lesssim M_\mu(|N_*(\partial_s u_s)|^2)(x)$, and the Poincaré inequality in (4.12) ensures that $I_3 \lesssim M_\mu(|\nabla f|^2)(x)$, hence

$$I \leqslant M_\mu(|\eta^{-1}N_*^\eta(\partial_t\mathcal{P}_{\eta t}f)|^2)(x) + M_\mu(|N_*(\partial_s u_s)|^2)(x) + M_\mu(|\nabla f|^2)(x).$$

To estimate II, we use Cauchy's inequality with $\epsilon > 0$ to obtain

$$II \lesssim \frac{\epsilon}{\mu(B(y,\eta t))}\int_{\mathbb{R}^n} |\nabla_x u_{\eta t}|^2\Phi^2\, d\mu + \epsilon^{-1}(I_2 + I_3).$$

A sufficiently small choice of $\epsilon > 0$ allows the ϵ-term to be subtracted, yielding

$$\fint_{B(y,\eta t)} |\nabla_x \mathcal{P}_{\eta t} f|^2 \, d\mu \lesssim I + II \lesssim M_\mu(|\eta^{-1} N_*^\eta(\partial_t \mathcal{P}_{\eta t} f)|^2 + |N_*(\partial_t \mathcal{P}_t f)|^2 + |\nabla f|^2)(x),$$

which implies (3.26). ∎

The pointwise estimates in Lemma 3.8 have the following corollary, since the maximal operator M_μ satisfies the weak-type estimate

$$\mu\big(\{x \in \mathbb{R}^n : |M_\mu f(x)| > \kappa\}\big) \lesssim \kappa^{-1} \|f\|_{L^1_\mu(\mathbb{R}^n)}$$

for all $\kappa > 0$ and $f \in L^1_\mu(\mathbb{R}^n)$ (see, for instance, Theorem 1 in Chapter I of [S2]).

Corollary 3.9. *Let $n \geqslant 2$ and suppose that $\mathcal{A} \in \mathcal{E}(n, \lambda, \Lambda, \mu)$ for some constants $0 < \lambda \leqslant \Lambda < \infty$ and an A_2-weight μ on $\mathbb{R}^n$. If $\eta > 0$, then*

$$\mu\big(\{x \in \mathbb{R}^n : |N_*^\eta(\partial_t \mathcal{P}_t f)(x)| > \kappa\}\big) \lesssim_\eta \kappa^{-2} \|\nabla f\|^2_{L^2_\mu(\mathbb{R}^n,\mathbb{R}^n)}, \tag{3.27}$$

$$\mu\big(\{x \in \mathbb{R}^n : |\eta^{-1} N_*^\eta(\partial_t \mathcal{P}_{\eta t} f)(x)| > \kappa\}\big) \lesssim \kappa^{-2} \|\nabla f\|^2_{L^2_\mu(\mathbb{R}^n,\mathbb{R}^n)}, \tag{3.28}$$

$$\mu\big(\{x \in \mathbb{R}^n : |\widetilde{N}_*^\eta(\nabla_\| \mathcal{P}_{\eta t} f)(x)| > \kappa\}\big) \lesssim \kappa^{-2} \|\nabla f\|^2_{L^2_\mu(\mathbb{R}^n,\mathbb{R}^n)}, \tag{3.29}$$

for all $\kappa > 0$ and $f \in W^{1,2}_\mu(\mathbb{R}^n)$, where the implicit constants depend only on n, λ, Λ and $[\mu]_{A_2}$.

3.4 Carleson Measure Estimate

In this section, we give a proof of Theorem 3.2 which is our main theorem. To make the proof easily for readers to follow, we first want to give a brief strategy how we prove Theorem 3.2 then we will give the details of our proof of the theorem.

Lemma 3.10. *Suppose that the hypotheses of Theorem 3.2 hold. If A satisfies the degenerate bound and ellipticity in (3.1) for some constants $0 < \lambda \leqslant \Lambda < \infty$ and an A_2-weight μ*

on $\mathbb{R}^n$, then any bounded solution $u \in L^\infty(4Q \times (0, 4\ell(Q)))$ of the equation $\operatorname{div} A\nabla u = 0$ *in* $4Q \times (0, 4\ell(Q))$ *satisfies*

$$\int_0^{\ell(Q)} \int_Q |t\nabla u(x,t)|^2 \, d\mu(x) \frac{dt}{t} \lesssim \int_0^{4\ell(Q)} \int_{4Q} |t\partial_t u(x,t)|^2 \, d\mu(x) \frac{dt}{t} + \mu(Q)\|u\|_\infty^2,$$

where the implicit constant depends only on n, λ, Λ and $[\mu]_{A_2}$.

Proof. Let $\Phi_Q(t) := \Phi\left(t/l(Q)\right)$, where $\Phi : (0, \infty) \to [0, 1]$ denotes a smooth cut-off function such that $\Phi(t) = 1$ when $t \leqslant 1$ and $\Phi(t) = 0$ when $t \geqslant 2$. Integrating by parts with respect to the t-variable and noting that $\|\partial_t \Phi_Q\|_\infty \lesssim 1/\ell(Q)$, we obtain

$$\begin{aligned}
\mathbf{I} :&= \int_Q \int_0^{2l(Q)} |\nabla u(x,t)|^2 \, \Phi_Q(t) t \, dt d\mu(x) \\
&\approx \int_Q \int_0^{2l(Q)} \partial_t \left(|\nabla u(x,t)|^2 \, \Phi_Q(t) \right) t^2 dt d\mu(x) \\
&\lesssim \int_Q \int_0^{2l(Q)} \langle \nabla \partial_t u(x,t), \nabla u(x,t) \rangle \, \Phi_Q(t) t^2 dt d\mu(x) \\
&\quad + \int_Q \fint_{l(Q)}^{2l(Q)} |\nabla u(x,t)|^2 \, t^2 dt d\mu(x) \\
&=: \mathbf{I}' + \mathbf{I}''.
\end{aligned}$$

For the term $\mathbf{I}'$, we apply Cauchy's inequality with an arbitrary $\epsilon > 0$, to obtain

$$\mathbf{I}' \leqslant \epsilon \mathbf{I} + \frac{1}{\epsilon} \int_Q \int_0^{2l(Q)} |\nabla \partial_t u(x,t)|^2 \, t^3 dt d\mu(x).$$

For the term $\mathbf{I}''$, we apply Caccioppoli's inequality, the doubling property of μ and the fact that $t \approx l(Q)$ in the domain of the integration, to obtain

$$\begin{aligned}
\mathbf{I}'' &\approx l(Q) \int_Q \int_{l(Q)}^{2l(Q)} |\nabla u(x,t)|^2 dt d\mu(x) \\
&\lesssim \frac{1}{l(Q)} \int_{2Q} \int_{l(Q)/2}^{5l(Q)/2} |u(x,t)|^2 dt d\mu(x) \\
&\lesssim \mu(Q) \|u\|_\infty^2.
\end{aligned}$$

We now fix $\epsilon > 0$, depending only on allowable constants, such that altogether

$$\mathbf{I} \lesssim \int_Q \int_0^{2l(Q)} |\nabla \partial_t u(x,t)|^2 \, t^3 dt d\mu(x) + \mu(Q) \|u\|_\infty^2.$$

To complete the estimate, we let $\{W_j : j \in J\}$ denote the collection of Whitney boxes (from a standard Whitney decomposition of $\mathbb{R}^{n+1}_+$) with the property that $W_j \cap (Q \times (0, 2l(Q))) \neq \emptyset$. The coefficient matrix A is t-independent, so $\partial_t u$ is also a solution of $\operatorname{div} A\nabla u = 0$ in each set W_j, hence we may apply Caccioppoli's inequality in combination with the fact that $t \eqsim l(W_j)$ in W_j, to obtain

$$\begin{aligned}
\int_0^{l(Q)} \int_Q |t\nabla u(x,t)|^2 \, d\mu(x) \frac{dt}{t} &\lesssim \sum_{j \in J} \iint_{W_j} |\nabla \partial_t u(x,t)|^2 \, t^3 dt d\mu(x) + \mu(Q)\|u\|_\infty^2 \\
&\lesssim \sum_{j \in J} l(W_j) \iint_{2W_j} |\partial_t u(x,t)|^2 \, dt d\mu(x) + \mu(Q)\|u\|_\infty^2 \\
&\lesssim \int_0^{4l(Q)} \int_{4Q} |t\partial_t u(x,t)|^2 \, d\mu(x) \frac{dt}{t} + \mu(Q)\|u\|_\infty^2,
\end{aligned}$$

as required. ∎

Proof of Theorem 3.2. Let $u \in L^\infty(\mathbb{R}^{n+1}_+)$ denote a bounded solution of the equation $\operatorname{div} A\nabla u = 0$ in $\mathbb{R}^{n+1}_+$. Let $Q \subset \mathbb{R}^n$ denote a dyadic cube. It follows *a fortiori* from Theorem 3.12 that there exist constants $C, c_0 > 0$ and, for each dyadic cube $Q' \subseteq Q$, a measurable set $F' \subset Q'$ such that $\mu(F') \geqslant c_0 \mu(Q')$ and

$$\frac{1}{\mu(Q')} \int_0^{l(Q')} \int_{F'} |t\partial_t u(x,t)|^2 \, d\mu(x) \frac{dt}{t} \leqslant C\|u\|_\infty^2,$$

where C and c_0 depend only on n, λ, Λ and $[\mu]_{A_2}$.

The coefficient matrix A is t-independent, so $\partial_t u$ is also a solution and thus the degenerate version of Moser's estimate in (3.17), followed by Caccioppoli's inequality, shows that

$\|t\partial_t u\|_\infty \lesssim \|u\|_\infty$. Moreover, the degenerate version of the de Giorgi–Nash Hölder regularity for solutions in (3.16) shows that

$$|t\partial_t u(x,t) - t\partial_t u(y,t)| \lesssim \left(\frac{|x-y|}{t}\right)^{\alpha} \|t\partial_t u\|_\infty \lesssim \|u\|_\infty \left(\frac{|x-y|}{t}\right)^{\alpha}$$

for all $x, y \in Q$ and $t > 0$, where all of the implicit constants and the exponent $\alpha > 0$ depend only on n, λ, Λ and $[\mu]_{A_2}$. Therefore, we may apply Lemma 3.6 with $\{v_t, \alpha, \beta_0, \eta, \beta\} := \{(t\partial_t u)^2, \alpha, C\|u\|_\infty^2, c_0, C\|u\|_\infty^2\}$ to obtain

$$\frac{1}{\mu(Q)} \int_0^{l(Q)} \int_Q |t\partial_t u(x,t)|^2 \, d\mu(x) \frac{dt}{t} \lesssim \|u\|_\infty^2,$$

where the implicit constant depends only on n, λ, Λ and $[\mu]_{A_2}$. This estimate holds for all dyadic cubes Q, so by Lemma 3.10, we conclude that (3.3) holds. ∎

First, let us fix some notations. For any given matrix $B = (B_{i,j})$ (no matter its dimensions), we denote $B^* = (B_{j,i})$ as its adjoint (i.e transpose, since our coefficients are real). Since A is an $(n+1) \times (n+1)$ matrix that is t-independent on $\mathbb{R}^{n+1}_+$, we want to write the matrix A in the following form

$$A = \left[\begin{array}{c|c} A_{\|} & \mathbf{b} \\ \hline \mathbf{c} & d \end{array}\right],$$

where $d := A_{n+1,n+1}$, $\mathbf{b} := (A_{i,n+1})_{1 \leqslant i \leqslant n}$, $\mathbf{c} := (A_{n+1,j})_{1 \leqslant j \leqslant n+1}$ and $A_{\|}$ denotes the $n \times n$ submatrix of A with entries $(A_{\|}) := A_{i,j}$, $1 \leqslant i, j \leqslant n$. By rewriting the matrix A in a new form, we will be able to apply some techniques, tools and results from the solution of Kato problem and its related results to get an estimate for the LHS of (3.3) by changing variable method.

Now, we have that the adjoint of the matrix A will be

$$A^* = \left[\begin{array}{c|c} A_{\|}^* & \mathbf{c} \\ \hline \mathbf{b} & d \end{array}\right].$$

To prove the Theorem 3.2, it is enough to prove that for any dyadic cube Q, the LHS of (3.3) is bounded uniformly by the RHS for some constant C that does not depend on Q. So, we fix Q from now on. In order to apply the technique from Kato problem, we also want to write the vectors $\mathbf{b}$ and $\mathbf{c}$ in the following ways

$$\mathbf{c}\mathbb{1}_{5Q} = -A_{\|}^{*}\nabla\varphi + \mathbf{h}, \qquad \mathbf{b}\mathbb{1}_{5Q} = -A_{\|}\nabla\widetilde{\varphi} + \widetilde{\mathbf{h}}, \tag{3.30}$$

where $\varphi, \widetilde{\varphi} \in W^{1,2}_{\mu,0}(5Q)$- the weighted Sobolev space with compact support where the weight $\mu \in A_2$ weight condition, and $\mathbf{h}, \widetilde{\mathbf{h}}$ are divergence free and supported in $5Q$, and where

$$\frac{1}{\mu(5Q)}\int_{5Q}\left(|\nabla\varphi(x)|^2 + \left|\frac{\mathbf{h}(x)}{\mu}\right|^2\right)\mu(x)dx \leqslant C\frac{1}{\mu(5Q)}\int_{5Q}\left|\frac{\mathbf{c}(\mathbf{x})}{\mu}\right|^2\mu(x)dx \leqslant C. \tag{3.31}$$

$$\frac{1}{\mu(5Q)}\int_{5Q}\left(|\nabla\widetilde{\varphi}(x)|^2| + \left|\frac{\widetilde{\mathbf{h}}(x)}{\mu}\right|^2\right)\mu(x)dx \leqslant C\frac{1}{\mu(5Q)}\int_{5Q}\left|\frac{\mathbf{b}(\mathbf{x})}{\mu}\right|^2\mu(x)dx \leqslant C. \tag{3.32}$$

Note that we may have the above decompositions for $\mathbf{b}$ and $\mathbf{c}$ because of the Hodge decomposition proved in (3.11).

Recall that we defined $\mathcal{P}_t := e^{-t^2\mathcal{L}_\mu}$ and $\mathcal{P}_t^* := e^{-t^2\mathcal{L}_\mu^*}$, the heat semigroup associated to $\mathcal{L}_\mu$ and its adjoint $\mathcal{L}_\mu^*$, but enowed with ellipticity homogeneity (thus, t has been squared). As we mentioned above, we will do some changing variables. We will do that many times with different variable in different estimates in our proof and the first time we do is the following

We make a pull-back of the equation $-\operatorname{div} A\nabla u = 0$ in $\mathbb{R}^{n+1}_+$ under the mapping (of course, we will need to justify we can do that)

$$\rho(x,t) := (x, \tau(x,t)) := (x, t - \varphi(x) + \mathcal{P}^*_{\eta t}\varphi(x)),$$

where $\eta > 0$ is small but a fixed number to be chosen, and φ is as in (3.30), and has been extended to all $\mathbb{R}^n$ by setting $\varphi \equiv 0$ in $\mathbb{R}^n \setminus 5Q$. By a little computation, we can get the fact that if u is a solution of $-\operatorname{div} A\nabla u = 0$ in $\mathbb{R}^{n+1}_+$, then we also have that $u_1 := u \circ \rho$ is a solution of the new equation $-\operatorname{div} A_1 \nabla u_1 = 0$ (at least formally now, we will give more details latter), and for $J, \mathbf{p}$ to be defined below,

$$A_1 = \left[\begin{array}{c|c} JA_{\|} & \mathbf{b} + A_{\|}\nabla_x\varphi - A_{\|}\nabla_x\mathcal{P}^*_{\eta t}\varphi \\ \hline \mathbf{h} - A^*_{\|}\nabla_x\mathcal{P}^*_{\eta t}\varphi & \frac{\langle A\mathbf{p},\mathbf{p}\rangle}{J} \end{array}\right],$$

where $\mathbf{h}$ is the divergence free vector in the Hodge decomposition in (3.30), and $J, \mathbf{p}$ are defined as follows:

$$J(x,t) := 1 + \partial\mathcal{P}^*_{\eta t}\varphi, \tag{3.33}$$

is the Jacobian of the change variable $t \to \tau(x,t)$, with x fixed, and

$$\mathbf{p}(x,t) := (\nabla_x\tau(x,t), -1) = (\nabla_x\mathcal{P}^*_{\eta t}\varphi(x) - \nabla_x\varphi(x), -1). \tag{3.34}$$

Let us make precise the meaning of $-\operatorname{div} A_1\nabla u_1 = 0$, i.e. we are showing that the changing variable is valid in some sense. In fact, in the sequel, we will consider u_1 in a certain sawtooth domain Ω_0 in which the mapping $(x,t) \to \rho(x,t)$ is one to one, with range contained in $\mathbb{R}^{n+1}_+$, and in which $J(x,t) \approx 1$ (uniformly). The fact that $-\operatorname{div} A_1\nabla u_1 = 0$ in the sawtooth region then follows from the pointwise identity

$$A\left((\nabla u)\circ\rho\right)\cdot\left((\nabla v)\circ\rho\right)J = A_1\nabla u_1\cdot\nabla v_1, \tag{3.35}$$

for $v \in W^{1,2}(\Omega_0)$, where $v_1 = v \circ \rho$.

Let us explain what we will do next to get the proof. By Lemma 3.6, it suffices to estimate the CME for the new solution u_1 under a smaller subset of Q. Then we extend the LHS of new CME to the whole $\mathbb{R}^{n+1}_+$ by a special cut-off function, then we do integration by parts in time variable t. After integration by parts, there will be some terms having u_1^2, and some term having $\nabla(u_1^2)$. For some terms having u_1^2, the most difficult case which is also the main part of our estimate is when we also have the term $\operatorname{div} \Lambda_1 \nabla(t)$. To estimate that part, we need to use tools from Kato problem, Moser's interior estimate for degenerate elliptic equations as well as the Moser type estimate for degenerate parabolic equations together the properties from our chosen cut-off functions and the geometry of the domain after we changed variable. For the other terms having u_1^2, we bound u_1^2 by $\|u\|_\infty^2$ and use some estimates of the cut-off function and its gradient. To treat the last case which has $\nabla(u_1^2)$, it is easier since we can hide the similar part of LHS of new CME by Holder's inequality with epsilon and bound u_1^2 by $\|u\|_\infty^2$.

Lemma 3.11. *Let $n \geqslant 2$ and suppose that $\mathcal{A} \in \mathcal{E}(n, \lambda, \Lambda, \mu)$ for some constants $0 < \lambda \leqslant \Lambda < \infty$ and an A_2-weight μ on $\mathbb{R}^n$. Let $Q \subseteq \mathbb{R}^n$ denote a dyadic cube and suppose that $\mathbf{f} : 5Q \to \mathbb{R}^n$ is a measurable function such that $\frac{1}{\mu}\mathbf{f} \in L^\infty(5Q)$. Let $\phi \in W^{1,2}_{0,\mu}(5Q)$ and suppose that $\operatorname{div} \mathcal{A}\nabla\phi = \operatorname{div} \mathbf{f}$ in $5Q$. If $\kappa_0 > 0$, $0 < \eta < 1/2$ and $x_0 \in Q$ satisfy $\Lambda(\eta, \phi, \mathcal{A})(x_0) \leqslant \kappa_0$, where*

$$\Lambda(\eta, \phi, \mathcal{A}) := \eta^{-1} N_*^\eta(\partial_t \mathcal{P}_{\eta t}\phi) + N_*(\partial_t \mathcal{P}_t \phi) + [M_\mu(|\nabla_x \phi|^2)]^{1/2} + D_{\mu,*}\phi, \tag{3.36}$$

then

$$|\partial_t \mathcal{P}_{\eta t}\phi(x)| \leqslant \eta\kappa_0 \tag{3.37}$$

and

$$|(I - \mathcal{P}_{\eta t})\phi(x)| \lesssim \eta(\kappa_0 + \|\tfrac{1}{\mu}\mathbf{f}\|_\infty)t \tag{3.38}$$

for all $(x, t) \in \Gamma_\eta(x_0) \cap (2Q \times (0, 4l(Q)))$, where the implicit constant depends only on n, λ, Λ and $[\mu]_{A_2}$.

Proof. Suppose that $\kappa_0 > 0$, $0 < \eta < 1/2$ and $x_0 \in Q$ satisfy $\Lambda(\eta, \phi, \mathcal{A})(x_0) \leqslant \kappa_0$. It follows *a fortiori* that $\eta^{-1} N_*^\eta(\partial_t \mathcal{P}_{\eta t}\phi)(x_0) \leqslant \kappa_0$, so (3.37) holds for all $(x,t) \in \Gamma_\eta(x_0)$.

To prove (3.38), first note that the properties of the semigroup imply that

$$|(I - \mathcal{P}_{\eta t})\phi(x_0)| = \left| \int_0^{\eta t} \partial_s \mathcal{P}_s \phi(x_0) ds \right| \leqslant \eta t \kappa_0 \tag{3.39}$$

for all $t > 0$, since $N_*(\partial_s \mathcal{P}_s \phi)(x_0) \leqslant \kappa_0$. Now let $(x,t) \in \Gamma_\eta(x_0) \cap (2Q \times (0, 4l(Q)))$. We set $\phi_{x_0,\eta t} := \fint_{B(x_0,2\eta t)} \phi(y) dy$ and apply estimate (3.23) with $\alpha = 2$ to obtain

$$|\mathcal{P}_{\eta t}(\phi - \phi_{x_0,\eta t})(x)| \lesssim \eta t [M_\mu(|\nabla_x \phi|^2)(x_0)]^{1/2} \leqslant \eta t \kappa_0. \tag{3.40}$$

Next, since $\operatorname{div} \mathcal{A}\nabla(\phi - \phi(x_0)) = \operatorname{div} \mathcal{A}\nabla\phi = \operatorname{div} \mathbf{f}$ in $5Q$, and since $0 < \eta < 1/2$ ensures that $B(x_0, 2\eta t) \subseteq 5Q$, we may apply the degenerate version of Moser's estimate in (3.17) to obtain

$$\begin{aligned} |\phi(x) - \phi(x_0)| &\lesssim \left(\fint_{B(x_0,2\eta t)} |\phi(y) - \phi(x_0)|^2 d\mu(y) \right)^{1/2} + \eta t \|\tfrac{1}{\mu}\mathbf{f}\|_\infty \\ &\lesssim \eta t (D_{\mu,*}\phi(x_0) + \|\tfrac{1}{\mu}\mathbf{f}\|_\infty) \\ &\lesssim \eta t (\kappa_0 + \|\tfrac{1}{\mu}\mathbf{f}\|_\infty). \end{aligned} \tag{3.41}$$

Combining estimates (3.44), (3.46) and (3.49), we obtain

$$\begin{aligned} |(I - \mathcal{P}_{\eta t})\phi(x)| &\leqslant |\phi(x) - \phi(x_0)| + |(I - \mathcal{P}_{\eta t})\phi(x_0)| \\ &\qquad + |\mathcal{P}_{\eta t}(\phi - \phi_{x_0,\eta t})(x_0)| + |\mathcal{P}_{\eta t}(\phi - \phi_{x_0,\eta t})(x)| \\ &\lesssim \eta(\kappa_0 + \|\tfrac{1}{\mu}\mathbf{f}\|_\infty) t, \end{aligned}$$

which proves (3.38), as the implicit constant depends only on n, λ, Λ and $[\mu]_{A_2}$. ∎

Theorem 3.12. *Suppose that the hypotheses of Theorem 3.2 hold. If A satisfies the degenerate bound and ellipticity in (3.1) for some constants $0 < \lambda \leqslant \Lambda < \infty$ and an A_2-weight μ on $\mathbb{R}^n$, then for any bounded solution $u \in L^\infty(Q \times (0, \ell(Q)))$ of the equation* $\operatorname{div} A\nabla u = 0$

in $Q \times (0, \ell(Q))$*, there exist constants* $C, c_0 > 0$ *and a measurable set* $F \subset Q$ *such that* $\mu(F) \geqslant c_0 \mu(Q)$ *and*

$$\frac{1}{\mu(Q)} \int_0^{l(Q)} \int_F |t \nabla u(x,t)|^2 \, d\mu(x) \frac{dt}{t} \leqslant C \|u\|_\infty^2,$$

where C *and* c_0 *depend only on* n*,* λ*,* Λ *and* $[\mu]_{A_2}$*.*

Proof. Next, we will give details about choosing the set F as we mentioned in the claim above. Now define the set

$$F := \left\{ x \in Q : \Lambda(\eta, \varphi, A_\|^*)(x) + \Lambda(\eta, \widetilde{\varphi}, A_\|)(x) + \widetilde{N}_*^\eta(\nabla \mathcal{P}_{\eta t}^* \varphi)(x) + \widetilde{N}_*^\eta(\nabla \mathcal{P}_{\eta t} \widetilde{\varphi})(x) \leqslant \kappa_0 \right\}, \tag{3.42}$$

where κ_0 is a large number at our disposal. Using the bounds in (3.20), the bounds for $\nabla\varphi$, $\nabla\widetilde{\varphi}$ that come from Hodge decomposition and the bounds for the operators in Λ_1, Λ_2 that were proved in Corollary 3.9, we have

$$\mu(Q \setminus F) \lesssim \kappa_0^{-2} \mu(Q),$$

uniformly in η.

To prove the CME, from the claim above, it is enough to prove that for the set F defined above, we have

$$\int_F \int_0^{l(Q)} A \nabla u(x,t) \cdot \nabla u(x,t) t dt dx \leqslant C_{\eta,\kappa_0} \|u\|_\infty^2 \mu(Q).$$

As we mentioned in the strategy how we prove the theorem, we first make a change variable in time. Here is the detail, let $t \to t - \varphi(x) + \mathcal{P}_{\eta t}^* \varphi(x)$, this gives us

$$\int_F \int_0^{l(Q)} A \nabla u(x,t) \cdot \nabla u(x,t) t dt dx \lesssim \int_F \int_0^{2l(Q)} A_1 \nabla u_1 \cdot \nabla u_1 t dt dx. \tag{3.43}$$

We are allowed to do change variable here because of the followings

$$\left|(I-\mathcal{P}_{\eta t}^{*})\varphi(x)\right| = \left|\int_0^{\eta t} \partial_s \mathcal{P}_s^{*}\varphi(x)ds\right| \leqslant \eta t\kappa_0 \ll \eta^{\frac{1}{2}} t \ll \frac{t}{8}, \quad \forall x \in F, \tag{3.44}$$

since $N_*(\partial_s\mathcal{P}_s\varphi)(x) \leqslant \kappa_0$ and here we choose η small enough, i.e. $\eta \ll \kappa_0^{-2}$. We note at this point that the analogue of (3.44) holds also for $(I-\mathcal{P}_{\eta t})\widetilde{\varphi}$ and more over, by our definition of Λ_1, Λ_2 and the definition of F in (3.42), we have

$$\max\left(|\partial_t\mathcal{P}_{\eta t}\widetilde{\varphi}(x)|, |\partial_t\mathcal{P}_{\eta t}^{*}\varphi(x)|\right) \leqslant \eta\kappa_0 \ll \eta^{\frac{1}{2}}, \quad \forall (x,t) \in \Omega_0, \tag{3.45}$$

where Ω_0 is the sawtooth domain

$$\Omega_0 := \bigcup_{x\in F} \Gamma_0(x),$$

and $\Gamma_0(x)$ denotes the cone with vertex at x and aperture η. Thus, if $(x,t) \in \Omega_0$, then $|x-x_0| < \eta t$ for some $x_0 \in F$, so that, setting

$$\varphi_{x_0,\eta t} := \frac{1}{|B(x_0, 2\eta t)|}\int_{|x_0-y|\leqslant 2\eta t} \varphi(y)dy,$$

Then from the inequality (3.23) with $\alpha = 2$, we obtain

$$|\mathcal{P}_{\eta t}^{*}(\varphi - \varphi_{x_0,\eta t})(x)| \lesssim \eta t\left(M_\mu(|\nabla\varphi|^2)(x_0)\right)^{\frac{1}{2}} \lesssim \eta t\kappa_0 \ll \eta^{\frac{1}{2}} t \quad \forall (x,t) \in \Omega_0. \tag{3.46}$$

Next, we define our smooth cut-off function adapted to Ω_0, as follows. Set

$$\Omega_1 := \bigcup_{x\in F} \Gamma_1(x),$$

where $\Gamma_1(x)$ has aperture $\frac{\eta}{8}$. Let $\delta(x)$:=dist(x,F) and let $\Phi \in C^\infty(\mathbb{R})$, with $\Phi(r) \equiv 1$ if $r \leqslant \frac{1}{16}$, and $\Phi(r) \equiv 0$, if $r > \frac{1}{8}$, and $0 \leqslant \Phi \leqslant 1$. We then set

$$\Psi(x,t) := \Phi\left(\frac{\delta(x)}{\eta t}\right)\Phi\left(\frac{t}{32 l(Q)}\right).$$

Let us make some observations for the cut off function Ψ, and the related sawtooth regions. First, we see that

$$\Psi(x,t) \equiv 1, \qquad \forall (x,t) \in F \times (0, 2l(Q)), \tag{3.47}$$

and also that

$$\operatorname{supp}(\Psi) \subset \Omega_{1,Q} := \Omega_1 \cap (2Q \times (0, 4l(Q)))\,,$$

since η is small.

Next, we claim that

$$|(I - \mathcal{P}^*_{\eta t})\varphi(x)| \ll \eta^{\frac{1}{2}} t, \quad \forall (x,t) \in \Omega_{0,Q} := \Omega_0 \cap (2Q \times (0, 4l(Q))), \tag{3.48}$$

and the similar bound holds for $(I - \mathcal{P}_{\eta t})\widetilde{\varphi}(x)$.

To get the claim, we observe that for $(x,t) \in \Omega_0$, there is $x_0 \in F$ such that

$$x \in \Delta := \Delta(x_0, \eta t) := \{x \in F : |x - x_0| < \eta t\},$$

and $2\Delta \subset 5Q$, since $t < 4l(Q)$ and η is small. Since φ is a $W^{1,2}$ weak solution of the inhomogeneous elliptic equation

$$L^*_{\|}\varphi = \operatorname{div}(\mathbf{c}) \text{ in } 5Q.$$

The equation is also true for $\varphi - k$ for any constant k. From the Theorem 3.5 and the definition of the set F as in (3.42), we have that

$$\begin{aligned} \sup_{\Delta} |\varphi - \varphi(x_0)| &\lesssim \left(\frac{1}{\mu(2\Delta)} \int_{2\Delta} |\varphi(z) - \varphi(x_0)|^2 \mu dz \right)^{\frac{1}{2}} + \eta t \left\| \frac{\mathbf{c}}{\mu} \right\|_{L^\infty} \\ &\lesssim \eta t \left(D_{\mu,*}\varphi(x_0) + \left\| \frac{\mathbf{c}}{\mu} \right\|_{\infty} \right) \lesssim \eta^{1/2} t. \end{aligned} \tag{3.49}$$

Consequently, for $y \in \Delta$,

$$|(I - \mathcal{P}^*_{\eta t}\varphi(y))| \leqslant |\varphi(y) - \varphi(x_0)| + |(I - \mathcal{P}^*_{\eta t})\varphi(x_0)| + |\mathcal{P}^*_{\eta t}(\varphi - \varphi_{x_0,\eta t})(x_0)| +$$
$$+ |\mathcal{P}^*_{\eta t}(\varphi - \varphi_{x_0,\eta t})(y)| \ll \eta^{\frac{1}{2}} t,$$

where we have used the estimates in (3.44), (3.46) and (3.49). In particular, since $x \in \Delta$ we get the claim (3.48). We may also get the same bound for $(I - \mathcal{P}_{\eta t}\widetilde{\varphi})$ by the identical argument.

Moreover, from the change variable, we have the following properties for the Jacobian matrix, for any pair $(x, t) \in \Omega_0$, by (3.45)

$$\begin{aligned} J(x,t) &= \partial_t \left(t - \varphi(x) + \mathcal{P}^*_{\eta t}\varphi(x)\right) \approx 1. \\ \widetilde{J}(x,t) &= \partial_t \left(t - \widetilde{\varphi}(x) + \mathcal{P}_{\eta t}\widetilde{\varphi}(x)\right) \approx 1. \end{aligned}$$

We then have that the mapping

$$\rho(x,t) := (x, \tau(x,t)) := \left(x, t + \mathcal{P}^*_{\eta t}\varphi(x) - \varphi(x)\right),$$

is one-one on $\mathrm{supp}(\Psi)$ with

$$\frac{7t}{8} < \tau(x,t) < \frac{9t}{8}, \quad \forall (x,t) \in \mathrm{supp}(\Psi). \tag{3.50}$$

Consequently, if $\Omega_\beta := \cup_{x \in F}\Gamma_\beta(x)$ is the sawtooth domain with respect to F with cone of aperture β, we have that

$$\Omega_{\frac{8\beta}{9}} \subset \rho(\Omega_\beta) \subset \Omega_{\frac{8\beta}{7}}, \quad \forall \beta \leqslant \eta. \tag{3.51}$$

Now, we want to bound $|\nabla_{x,t}\Psi|$ by the following ways. We set

$$\begin{aligned} E_1 &:= \left\{(x,t) \in Q \times (0, 4l(Q)) : \frac{\eta t}{16} \leqslant \delta(x) \leqslant \frac{\eta t}{8}\right\}. \\ E_2 &:= 2Q \times (2l(Q), 4l(Q)). \end{aligned} \tag{3.52}$$

then from the definition of Ψ, we obtain

$$\left|\nabla_{x,t}\Psi(x,t)\right| \lesssim \frac{1}{\eta t}\mathbb{1}_{E_1}(x,t) + \frac{1}{l(Q)}\mathbb{1}_{E_2}(x,t), \tag{3.53}$$

Now, we give details how we handle the LHS of the (3.43). By (3.47), we have that

$$\begin{aligned}
&\int_F \int_0^{2l(Q)} A_1\nabla u_1 \cdot \nabla u_1 t dt dx \qquad (3.54)\\
\leqslant\ & \iint_{\mathbb{R}^{n+1}_+} A_1\nabla u_1 \cdot \nabla u_1\, \Psi^2\, t dt dx\\
=\ & -\frac{1}{2}\iint_{\mathbb{R}^{n+1}_+} L_1(u_1^2)\Psi^2 t dt dx\\
=\ & -\frac{1}{2}\iint_{\mathbb{R}^{n+1}_+} u_1^2 L^*(t)\Psi^2 dt dx - \frac{1}{2}\iint_{\mathbb{R}^{n+1}_+} A_1\nabla(u_1^2)\cdot\nabla(\Psi^2) t dt dx\\
+\ & \frac{1}{2}\iint_{\mathbb{R}^{n+1}_+} u_1^2\, e_{n+1}\cdot A_1\nabla(\Psi^2) dt dx + \frac{1}{2}\int_F u^2 A_{n+1,n+1} dx\\
=:\ & \mathbf{S} + \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{B},
\end{aligned}$$

where $e_{n+1} = (0, ..., 0, 1)$ and for the boundary term $\mathbf{B}$, we have used that $(A_1^*)_{n+1,n+1}(x,0) = A_{n+1,n+1}(x)$, that $u_1(x,0) = u(x,0)$ on F and that $\Psi(x,0) = \mathbb{1}_F(x)$. Let us give the estimate for the above terms in two steps:

Step 1: In this step, we want to estimate some easy terms first which are $\mathbf{B}, \mathbf{E_1}, \mathbf{E_2}$.

The easiest one is $\mathbf{B}$, so we would like to give the estimate for $\mathbf{B}$ first. Since $u \in L^\infty$, we get the following bound for term $\mathbf{B}$

$$|\mathbf{B}| \leqslant \Lambda \int_Q u^2\, d\mu(x) \leqslant \Lambda\mu(Q)\|u\|_\infty^2.$$

Now we estimate the terms $\mathbf{E}_1$. By Hölder's inequality with small σ to be chosen latter, we bound $|\mathbf{E}_1|$ by two terms as below

$$\begin{aligned}
|\mathbf{E}_1| &= \left|\frac{1}{2}\iint_{\mathbb{R}^{n+1}_+} A_1\nabla(u_1^2)\cdot\nabla(\Psi^2) t dt dx\right|\\
&= 2\left|\iint_{\mathbb{R}^{n+1}_+} A_1\nabla u_1 \cdot \nabla\Psi u_1 \Psi t dt dx\right|
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \quad \sigma \iint_{\mathbb{R}^{n+1}_{+}} A_1 \nabla u_1 \cdot \nabla u_1 \Psi^2 t dt dx + \frac{1}{\sigma} \iint_{\mathbb{R}^{n+1}_{+}} u_1^2 A_1 \nabla \Psi \cdot \nabla \Psi t dt dx \\
&=: \quad \mathbf{E}_1^{'} + \mathbf{E}_1^{''}.
\end{aligned}
$$

We see that $\mathbf{E}_1^{'}$ is similar to the first term of (3.54) except it is really small when we choose σ small enough. Then we can hide the term $\mathbf{E}_1^{'}$ to the LHS of (3.54). So, to finish our estimate for $|E_1|$, we only have to give a bound for the second term $\mathbf{E}_1^{''}$. To do that, we want to bounds $E_1^{''}$ by two terms due to the Hodge decomposition $\mathbf{h} = \mathbf{c}\mathbb{1}_{5Q} + A_{\|}^{*}\nabla\varphi$ and using the fact that $\frac{A}{\mu} \in L^{\infty}$ and also from (3.53). Here is the bound that we have

$$
\mathbf{E}_1^{''} \leqslant \mathbf{E}_{11}^{''} + \mathbf{E}_{12}^{''}
$$

where

$$
\mathbf{E}_{11}^{''} = \frac{C_\eta}{\sigma} \iint_{E_1} u_1^2 \left[1 + \left|\nabla_x (I - \mathcal{P}_{\eta t}^{*})\varphi(x)\right|^2\right] \mu \frac{dt}{t} dx.
$$

and

$$
\begin{aligned}
\mathbf{E}_{12}^{''} &= \frac{C_\eta}{\sigma} \iint_{E_2} u_1^2 \left[1 + \left|\nabla_x (I - \mathcal{P}_{\eta t}^{*})\varphi(x)\right|^2\right] \mu \frac{t dt}{l(Q)^2} dx \\
&\approx \frac{C_\eta}{\sigma} \iint_{E_2} u_1^2 \left[1 + \left|\nabla_x (I - \mathcal{P}_{\eta t}^{*})\varphi(x)\right|^2\right] \mu \frac{dt}{t} dx.
\end{aligned}
$$

We want to take care of $\mathbf{E}_{11}^{''}$ first, to get an estimate for $\mathbf{E}_1^{''}$, we decompose the domain of the integral by dyadic cubes as below

$$
\mathbf{E}_{11}^{''} = \frac{C_\eta}{\sigma} \sum_k \sum_{Q' \in \mathbb{D}_k^{\eta}} \int_{Q'} \int_{2^{-k}}^{2^{-k+1}} u_1^2 \left[1 + \left|\nabla_x (I - \mathcal{P}_{\eta t}^{*})\varphi(x)\right|^2\right] \mathbb{1}_{E_1} \mu \frac{dt}{t} dx, \tag{3.55}
$$

where $\mathbb{D}_k^{\eta}$ denote the grid of dyadic cubes such that

$$
\frac{1}{64}\eta 2^{-k} \leqslant \operatorname{diam} Q^{'} < \frac{1}{32}\eta 2^{-k}, \quad Q^{'} \in \mathbb{D}_k^{\eta}. \tag{3.56}
$$

Let us give an estimate for each element that is non-zero of the summation above. Consider now any fixed k and $Q^{'} \in \mathbb{D}_k^\eta$ for which the double integral in (3.55) is non-zero, thus for which there is a point

$$(x_1, t_1) \in E_1 \bigcap \left(Q^{'} \times [2^{-k}, 2^{-k+1}]\right). \tag{3.57}$$

We now fix such a point (x_1, t_1). By definition of E_1,

$$\frac{\eta t_1}{16} \leqslant \delta(x_1) \leqslant \frac{\eta t_1}{8}. \tag{3.58}$$

In particular, we will have a point $x_0 \in F$ such that

$$|x_1 - x_0| < \frac{\eta t_1}{8}.$$

Note that

$$Q^{'} \subset \Delta^{'} := \Delta(x_0, \eta 2^{-k}) := \{z : |z - x_0| < \eta 2^{-k}\}. \tag{3.59}$$

Consequently,

$$Q^{'} \times [2^{-k}, 2^{-k+1}] \subset \Omega_{0,Q}, \tag{3.60}$$

(we recall that $\Omega_{0,Q} := \Omega_0 \cap (2Q \times (0, 4l(Q)))$. Furthermore, since δ is Lipschitz with norm 1. Using (3.56) and (3.58), we obtain there is a uniform constant C such that

$$Q^{'} \times [2^{-k}, 2^{-k+1}] \subset \widetilde{E}_1 := \left\{(y, s) \in 2Q \times (0, 4l(Q)) : \frac{\eta s}{C}\delta(y) \leqslant C\eta s\right\}. \tag{3.61}$$

It then follows that

$$\mu(Q^{'}) \lesssim \int_{Q^{'}} \int_{2^{-k}}^{2^{-k+1}} \mathbb{1}_{\widetilde{E}_1}(y, s)\mu(y)\frac{ds}{s}dy, \tag{3.62}$$

Now, by (3.36) and definition of F, (3.42), and definition of $\widetilde{N}_*^\eta(v)$

which we recall here

$$\widetilde{N}_*^\eta(v)(x) := \sup_{(y,t):|x-y|\leqslant\eta t}\left(\frac{1}{\mu(B(y,\eta t))}\int_{|y-z|<\eta t}|v(z,t)|^2\mu(z)dz\right)^{\frac{1}{2}},$$

we have, for any $t \in [2^{-k}, 2^{-k+1}]$,

$$\begin{aligned} & \frac{1}{\mu(Q')}\int_{Q'}\left|\nabla_x(I-\mathcal{P}_{\eta t}^*)\varphi(x)\right|^2\mu(x)dx \\ \lesssim\ & \frac{1}{\mu(Q')}\int_{\Delta'}\left|\nabla_x\mathcal{P}_{\eta t}^*\varphi(x)\right|^2\mu(x)dx + \frac{1}{\mu(Q')}\int_{\Delta'}|\nabla_x\varphi(x)|^2\mu(x)dx \\ \lesssim\ & \left(\widetilde{N}_*^\eta(\nabla_x\mathcal{P}_{\eta t}^*)(x_0)\right)^2 + M_\mu\left(|\nabla_x\varphi|^2\right)(x_0) \lesssim \kappa_0^2. \end{aligned} \tag{3.63}$$

Moreover, by definition of u_1, we have $\|u_1\|_\infty \leqslant \|u\|_\infty$. Thus,

$$\begin{aligned} & \int_{Q'}\int_{2^{-k}}^{2^{-k+1}} u_1^2\left(1+\left|\nabla_x(I-\mathcal{P}_{\eta t}^*)\varphi(x)\right|^2\right)\mathbb{1}_{E_1}\mu(x)\frac{dt}{t}dx \\ \leqslant\ & \|u\|_{L^\infty}^2(1+\kappa_0^2)\mu(Q') \\ \leqslant\ & \|u\|_\infty^2(1+\kappa_0^2)\int_{Q'}\int_{2^{-k}}^{2^{-k+1}}\mathbb{1}_{\widetilde{E}_1}\mu(x)\frac{dt}{t}dx. \end{aligned}$$

where we have used (3.62) and (3.63) and the set $\widetilde{E}_1$ was defined in (3.61). Returning to (3.55), we then have

$$\begin{aligned} \mathbf{E}_{11}'' &\leqslant C_{\eta,\kappa_0,\sigma}\sum_k\sum_{Q'\in\mathbb{D}_k^\eta}\|u\|_\infty^2(1+\kappa_0^2)\int_{Q'}\int_{2^{-k}}^{2^{-k+1}}\mathbb{1}_{\widetilde{E}_1}\mu(x)\frac{dt}{t}dx \\ &\leqslant C_{\eta,\kappa_0,\sigma}\|u\|_\infty^2\int_{2Q}\int_{\frac{\delta(y)}{C\eta}}^{\frac{C\delta(y)}{\eta}}\frac{ds}{s}\mu(y)dy \\ &\leqslant C_{\eta,\kappa_0,\sigma}\|u\|_\infty^2\mu(Q). \end{aligned}$$

The term $\mathbf{E}_{12}''$ is easier since there is no singular term for t but we may treat exactly the same way as for $\mathbf{E}_{11}''$, so we also have

$$\mathbf{E}_{12}'' \leqslant C_{\eta,\kappa_0,\sigma}\|u\|_\infty^2\mu(Q).$$

The term $\mathbf{E_2}$ has the same bounds as $\mathbf{E_1''}$. This finishes our **Step 1**. Now, we move to next step which we will give an estimate for the term $\mathbf{S}$. This is also the most difficulty part of our proof for CME theorem.

Step 2: In order to estimate $\mathbf{S}$, let us observe that

$$\begin{aligned} L_1^*(t) &= \operatorname{div}_x A_\parallel^* \nabla_x \mathcal{P}_{\eta t}^* \varphi - \partial_t \left(\frac{1}{J} \langle A\mathbf{p}, \mathbf{p} \rangle \right) \\ &= -L_\parallel^* \mathcal{P}_{\eta t}^* \varphi - \partial_t \left(\frac{1}{J} \langle A\mathbf{p}, \mathbf{p} \rangle \right), \end{aligned}$$

since $\operatorname{div}_x \mathbf{h} = 0$. We then split $\mathbf{S}$ into the summation of two terms

$$\mathbf{S} = \frac{1}{2} \iint_{\mathbb{R}_+^{n+1}} u_1^2 (\mu \mathcal{L}_\mu^* \mathcal{P}_{\eta t}^* \varphi) \Psi^2 dtdx + \frac{1}{2} \iint_{\mathbb{R}_+^{n+1}} u_1^2 \partial_t \left(\frac{1}{J} \langle A\mathbf{p}, \mathbf{p} \rangle \right) \Psi^2 dtdx =: \mathbf{S}_1 + \mathbf{S}_2. \quad (3.64)$$

We treat $\mathbf{S}_1$ first. Note that by definition of $\mathcal{P}_{\eta t}^*$ and also the commutative property of semi-group and the operator $\mathcal{L}_\mu$,

$$\partial_t \mathcal{P}_{\eta t}^* = \partial_t e^{-(\eta t)^2 \mathcal{L}_\mu^*} f = -2\eta^2 t \mathcal{L}_\mu^* \mathcal{P}_{\eta t}^* = -2\eta^2 t \mathcal{P}_{\eta t}^* \mathcal{L}_\mu^*.$$

Integrating by parts in variable t, we have

$$\begin{aligned} \mathbf{S}_1 = &- \frac{1}{2} \iint_{\mathbb{R}_+^{n+1}} u_1^2 \partial_t \left(\mu \mathcal{L}_\mu^* \mathcal{P}_{\eta t}^* \varphi \right) \Psi^2 t dtdx + C_\eta \iint_{\mathbb{R}_+^{n+1}} (u_1 \partial_t u_1) \, \partial_t \mathcal{P}_{\eta t}^* \varphi \Psi^2 \mu dtdx \\ &+ C_\eta \iint_{\mathbb{R}_+^{n+1}} u_1^2 \partial_t \mathcal{P}_{\eta t}^* \varphi \Psi \partial_t \Psi \mu dtdx =: \mathbf{S}_1^{'} + \mathbf{S}_1^{''} + \mathbf{S}_1^{'''}. \end{aligned}$$

Let us handle the latest term $\mathbf{S}_1^{'''}$ with the note that $\partial_t \mathcal{P}_{\eta t}^* \varphi$ is bounded in the support of Ψ (see (3.36)- (3.42)), then

$$\begin{aligned} |\mathbf{S}_1^{'''}| \leqslant C_\eta \iint_{\mathbb{R}_+^{n+1}} \left| u_1^2 \partial_t \mathcal{P}_{\eta t}^* \varphi \Psi \partial_t \Psi \right| \mu dtdx &\leqslant C_\eta \|u\|_\infty^2 \iint_{\mathbb{R}_+^{n+1}} \left| \partial_t \mathcal{P}_{\eta t}^* \varphi \partial_t \Psi \right| \mu dtdx \\ &\leqslant C_\eta \kappa_0 \|u\|_\infty^2 \mu(Q), \end{aligned}$$

by using (3.53) and the definition of the set E_1 and E_2.

To treat the term $\mathbf{S}_1^{'}$, recall that $\partial_t \mathcal{P}_{\eta t}^* = -2\eta^2 t \mathcal{P}_{\eta t}^* \mathcal{L}_\mu^*$, get

$$
\begin{aligned}
|\mathbf{S}_1^{'}| &= \left| \iint_{\mathbb{R}^{n+1}_+} u_1^2 \partial_t \left(L_\|^* \mathcal{P}_{\eta t}^* \varphi \right) \Psi^2 t dt dx \right| \\
&= \left| \iint_{\mathbb{R}^{n+1}_+} u_1^2 \partial_t \left[\mathrm{div}_x A_\|^* \nabla_x (\mathcal{P}_{\eta t}^* \varphi) \right] \Psi^2 t dt dx \right| \\
&= \left| \iint_{\mathbb{R}^{n+1}_+} u_1^2 \left[\mathrm{div}_x A_\|^* \nabla_x (\partial_t \mathcal{P}_{\eta t}^* \varphi) \right] \Psi^2 t dt dx \right| \\
&\lesssim \left| \iint_{\mathbb{R}^{n+1}_+} A_\|^* \nabla_x (\partial_t \mathcal{P}_{\eta t}^* \varphi) \cdot u_1 \nabla_x (u_1) \Psi^2 t dt dx \right| \\
&\quad + \left| \iint_{\mathbb{R}^{n+1}_+} u_1^2 \Psi \nabla_x \Psi \cdot A_\|^* \nabla_x (\partial_t \mathcal{P}_{\eta t}^* \varphi) t dt dx \right| =: \mathbf{J} + \mathbf{K} \\
&\lesssim \sigma \iint_{\mathbb{R}^{n+1}_+} |\nabla_x u_1|^2 \Psi^2 t \mu dt dx + \left(\frac{1}{\sigma} + 1 \right) \left| \iint_{\mathbb{R}^{n+1}_+} u_1^2 \left| \nabla_x \partial_t \mathcal{P}_{\eta t}^* \varphi \right|^2 \Psi^2 t \mu dt dx \right| \\
&\quad + \left| \iint_{\mathbb{R}^{n+1}_+} u_1^2 |\nabla_x \Psi|^2 t \mu dt dx \right| =: \mathbf{S}_{11}^{'} + \mathbf{S}_{12}^{'} + \mathbf{S}_{13}^{'},
\end{aligned}
\tag{3.65}
$$

where we had applied integration by parts in x variable for the second to last line and then the Cauchy–Schwarz inequality with small σ at our disposal.

The term $\mathbf{S}_{13}^{'}$ is essentially like $\mathbf{E}_1^{''}$, and may be handled by a similar argument.

Next, we want to give an estimate for the contribution of $\mathbf{S}_{12}^{'}$, by using $\nabla_x \partial_t \mathcal{P}_{\eta t}^* = \nabla_x \left[(-2\eta^2) t \ \mathcal{P}_{\eta t}^* \mathcal{L}_\mu^* \right]$, and $\mathcal{L}_\mu^* = \frac{1}{\mu} L_\|^*$,

$$
\begin{aligned}
\mathbf{S}_{12}^{'} &\approx \iint_{\mathbb{R}^{n+1}_+} u_1^2 \left| \nabla_x \partial_t \mathcal{P}_{\eta t}^* \varphi \right|^2 \Psi^2 \mu t dt dx \\
&= \iint_{\mathbb{R}^{n+1}_+} 2\eta^2 u_1^2 \left| \nabla_x \left[\mathcal{P}_{\eta t}^* (\mathcal{L}_\mu^* \varphi) \right] \right|^2 \Psi^2 \mu t^3 dt dx \\
&\leqslant C_\eta \|u\|_\infty^2 \iint_{\mathbb{R}^{n+1}_+} \left| t^2 \nabla_x \left[\mathcal{P}_{\eta t}^* (\mathcal{L}_\mu^* \varphi) \right] \right|^2 \Psi^2 \mu \frac{dt}{t} dx \\
&\leqslant C_\eta \|\nabla \varphi\|_{L^2_\mu(\mathbb{R}^n)}^2 \|u\|_\infty^2 \leqslant C_\eta \|u\|_\infty^2 \mu(Q),
\end{aligned}
$$

where the vertical square function estimate in (3.14) was applied to obtain the penultimate inequality.

To see the contribution of the term $\mathbf{S}_{11}^{'}$, we first use the Cauchy–Schwarz inequality with small σ, we get that

$$\begin{aligned}\left|\mathbf{S}_1''\right| &= \left|C_\eta \iint_{\mathbb{R}^{n+1}_+} (u_1\partial_t u_1)\,\partial_t\left(\mathcal{P}^*_{\eta t}\varphi\right)\Psi^2\mu dtdx\right| \\ &\leqslant \sigma \iint_{\mathbb{R}^{n+1}_+} |\partial_t u_1|^2\,\Psi^2 t\mu dtdx + C_{\eta,\sigma}\iint_{\mathbb{R}^{n+1}_+} |u_1|^2\left|\partial_t\mathcal{P}^*_{\eta t}\varphi\right|^2\Psi^2\mu\frac{dt}{t}dx \\ &=: \mathbf{S}_{11}'' + \mathbf{S}_{12}''.\end{aligned}$$

It is easy to see that

$$\mathbf{S}_{12}'' \leqslant C_{\eta,\sigma}\mu(Q).$$

by using the vertical square function estimate in (3.13) to estimate $\mathbf{S}_{12}''$ in the same way as we did for $\mathbf{S}_{12}'$ above.

We observe that, we may combine the terms $\mathbf{S}_{11}''$ and $\mathbf{S}_{11}'$ to get the following fact

$$\mathbf{S}_{11}' + \mathbf{S}_{11}'' = \sigma\int_{\mathbb{R}^{n+1}_+} |\nabla u_1|^2\,\Psi^2 t\mu dtdx, \tag{3.66}$$

where, as above, we used the notation $\nabla \equiv \nabla_{x,t}$.

To get the estimate for the above inequality, we need to get some bound for $|\nabla u_1|$. To make it easier to follow, let us recall that $u_1 = u\circ\rho$ with $\rho(x,t) = (x, t+\mathcal{P}^*_{\eta t}\varphi(x) - \varphi(x)) =: (x,\tau(x,t))$. Thus, we get the following facts

$$\begin{aligned}\partial_t u_1(x,t) &= J(x,t)(\partial_\tau u)(x,\tau(x,t)),\ \text{and} \\ \nabla_x u_1(x,t) &= (\nabla_x u)(x,\tau(x,t)) + (\partial_\tau u)(x,\tau(x,t))\left(\nabla_x\tau(x,t)\right),\end{aligned}$$

where, again, $J(x,t) := \partial_t\tau(x,t) = 1 + \partial_t\mathcal{P}^*_{\eta t}\varphi(x)$. Hence,

$$(\nabla u)\circ\rho = \left(\nabla_x u_1 - \frac{\partial_t u_1}{J}\left(\nabla_x\tau\right), \frac{\partial_t u_1}{J}\right).$$

Since $J \approx 1$ in Ω_0, we have the following bound for $|\nabla u_1|$

$$|\nabla u_1| \lesssim \left|\left(\nabla_x u_1, \frac{\partial_t u_1}{J}\right)\right|$$

$$\begin{aligned} &\lesssim \left| \left(\nabla_x u_1 - \frac{\partial_t u_1}{J}(\nabla_x \tau), \frac{\partial_t u_1}{J} \right) \right| + |\nabla_x \tau||\partial_t u_1| \\ &= |(\nabla u) \circ \rho| + |\nabla_x \tau|\, |\partial_t u_1|\,. \end{aligned}$$

We also note that, by (3.35), the ellipticity of A and the fact that $J \approx 1$, we get the estimate

$$\mu\, |(\nabla u) \circ \rho|^2 \lesssim A_1 \nabla u_1 \cdot \nabla u_1.$$

This implies that we can hide the first term of (3.66) corresponding to the first bound of $|\nabla u_1|$ above on the LHS of (3.54) for small σ. To finish the the bound for (3.66), it remains to treat the term having $|\nabla_x \tau||\partial_t u_1|$, we write the integral in summation of dyadic decomposition as we already did in (3.55) - (3.56), i.e.

$$\iint_{\mathbb{R}^{n+1}_+} |\nabla_x \tau|^2 |\partial_t u_1|^2 \Psi^2 t d\mu(x) dt = \sum_k \sum_{Q' \in \mathbb{D}^\eta_k} \int_{2^{-k}}^{2^{-k+1}} \int_{Q'} |\nabla_x \tau|^2 |\partial_t u_1|^2 \Psi^2 t d\mu(x) dt. \tag{3.67}$$

Now, we want to point out some geometry observations as following.

Take some $t_1 \in [2^{-k}, 2^{-k+1}]$ and a cube $Q' \in \mathbb{D}^\eta_k$ for which $Q' \times t_1$ meets support of Ψ, say at a point (x_1, t_1). Then by construction of Ψ, we have $\delta(x_1) < \frac{\eta t_1}{8}$, when by (3.56) we have $\delta(x) < \frac{\eta t_1}{4}$ for every $x \in Q'$. Thus for each Q', and t_1 above, there is a point $x_0 \in F$ and an n-disk Δ' such that (3.59) and thus also (3.60) and (3.63) hold. In particular, we obtain the following

$$\frac{7}{8} t < \tau(x, t) < \frac{9}{8} t, \quad \forall (x, t) \in I(Q') := Q' \times [2^{-k}, 2^{-k+1}],$$

by (3.48) and definition of $\tau(x, t)$. It then follows that $t \in [2^{-k}, 2^{-k+1}]$ and

$$\begin{aligned} \sup_{x \in Q'} |\partial_t u_1(x, t)| &\approx \sup_{x \in Q'} |(\partial_\tau u)(x, \tau(x, t))| \\ &\leqslant \left(\frac{1}{(\mu \times dt)\left(2Q' \times (\frac{t}{2}, 2t)\right)} \int_{2Q'} \int_{\frac{t}{2}}^{2t} |\partial_s u(y, s)|^2 \mu(y) ds dy \right)^{\frac{1}{2}} \\ &\lesssim \left(\frac{1}{(\mu(2Q')\, t} \int_{2Q'} \int_{\frac{t}{2}}^{2t} |\partial_s u(y, s)|^2 \mu(y) ds dy \right)^{\frac{1}{2}}, \end{aligned} \tag{3.68}$$

where we have applied a Moser's interior estimate for degenerate elliptic equation and t-independent of A (see Corollary 2.3.4 in [FKS]).

Now, with the estimate in (3.68), we return to treat the RHS of (3.67). Since we only want to give bound for the non-zero term in the summation of RHS of (3.67), we look at the set $Q^{'}$ such that $Q^{'} \in \mathbb{D}_k^{\eta}$ for which $I(Q^{'}) := Q^{'} \times [2^{-k}, 2^{-k+1}]$ meets support of Ψ and denote $\mathbb{D}_k^{\eta}(\Psi)$ - the collection of all such $Q^{'}$. Thus there is a point $(x,t) \in I(Q^{'})$ such that $\delta(x) < \frac{\eta t}{8}$ by construction of Ψ. Consequently, for any such $Q^{'}$, by (3.56) we have that

$$\delta(y) < \operatorname{diam}(2Q^{'}) + \frac{1}{8}\eta t \leqslant \frac{3}{16}\eta t \leqslant \frac{3}{8}\eta s, \quad \forall y \in 2Q^{'} \text{ and } s > \frac{t}{2}.$$

Besides that, we also have $t < 4l(Q)$ in $\operatorname{supp}(\Psi)$, so that $s \leqslant 2t$ implies $s < 8l(Q)$. Set

$$\Omega^* := \left\{(y,s) \in \mathbb{R}^{n+1}_+ : \delta(y) < \frac{3\eta s}{8},\ \ 0 < s < 8l(Q)\right\}.$$

As noted above, (3.63) holds in the present context, so that with the help of (3.68)the term (3.67) is comparable to

$$\begin{aligned}
&\sum_k \sum_{Q^{'} \in \mathbb{D}_k^{\eta}(\Psi)} \int_{2^{-k}}^{2^{-k+1}} \int_{Q^{'}} |\nabla_x \tau|^2 \left(\frac{1}{\mu(2Q^{'})}\frac{1}{t}\int_{2Q^{'}}\int_{\frac{t}{2}}^{2t} |\partial_s u(y,s)|^2 \mu ds dy\right) \mu \Psi^2 t dx dt \\
\lesssim &\sum_k \sum_{Q^{'} \in \mathbb{D}_k^{\eta}(\Psi)} \int_{2^{-k}}^{2^{-k+1}} \left(\frac{1}{\mu(Q^{'})}\int_{Q^{'}} |\nabla_x \tau|^2 d\mu(x)\right) \int_{2Q^{'}}\int_{\frac{t}{2}}^{2t} |\partial_s u(y,s)|^2\, \mathbb{1}_{\Omega^*(y,s)} \mu ds dy dt \\
\lesssim & C_{\kappa_0,\eta} \sum_k \sum_{Q^{'} \in \mathbb{D}_k^{\eta}(\Psi)} \int_{2^{-k}}^{2^{-k+1}} \int_{2Q^{'}} |\partial_s u(y,s)|^2\, \mathbb{1}_{\Omega^*(y,s)} s \mu dy ds \\
= & C_{\kappa_0,\eta}\left(\iint_{\Omega^{**}} |\partial_s u|^2\, s\mu dy ds + \iint_{\Omega^* \setminus \Omega^{**}} |\partial_s u|^2\, s\mu dy ds\right) := \mathbf{M} + \mathbf{E},
\end{aligned} \tag{3.69}$$

where we have used (3.63) and the definition of τ for the last inequality and

$$\Omega^{**} := \left\{(y,s) \in \mathbb{R}^{n+1}_+ : \delta(y) < \frac{\eta s}{18}, 0 < s < l(Q)\right\}.$$

for the last equality.

Notice that, by (3.50) - (3.51), we have

$$\rho^{-1}(\Omega^{**}) \subset \Omega_{\frac{\eta}{16}} \cap (2Q \times (0, 2l(Q))),$$

and $\Psi \equiv 1$ on the latter set. Therefore, making change variable $s = \tau(y,t)$, we get

$$\mathbf{M} \leqslant C_{\kappa_0,\eta} \iint_{\mathbb{R}^{n+1}_+} |(\partial_\tau u) \circ \rho|^2 \Psi^2 t dt \mu dy,$$

since $J(y,t) \approx 1$. Again, if σ is chosen small enough we can hide the RHS term of the above inequality to the second term of (3.54).

Next, we want to handle the term $\mathbf{E}$. In order to get an estimate for $\mathbf{E}$, we apply a Moser-type interior estimates for degenerate elliptic equations (see Corollary 2.3.4 in [FKS]), and t-independence of A, we have

$$|\partial_s u(y,s)| \lesssim \frac{1}{s} \|u\|_\infty,$$

Indeed, for any s, by Caccioppoli's inequality and bound the t-derivative by full gradient then for any $(y,s) \in \Omega^* \setminus \Omega^{**}$, take Q_0 be a a cube centered at y with side length s, we obtain

$$\begin{aligned}
|\partial_s u(y,s)| &\leqslant \max_{(x,t)\in Q_0\times[s/2,3s/2]} |\partial_s u(x,t)| \\
&\leqslant \left(\frac{1}{(\mu \times ds)(Q_0 \times (s/2, 3s/2))} \iint_{Q_0\times(s/2,3s/2)} |\partial_t u(y,t)|^2 \mu dy dt \right)^{\frac{1}{2}} \\
&\lesssim \frac{1}{s} \left(\frac{1}{(\mu \times ds)(2Q_0 \times (s, 3s))} \iint_{2Q_0\times(s,3s)} |u(y,t)|^2 \mu dy dt \right)^{\frac{1}{2}} \\
&\lesssim \frac{1}{s} \|u\|_\infty.
\end{aligned}$$

where we used Corollary 2.3.4 in [FKS] for the second inequality and Caccioppoli's inequality for degenerate elliptic equation to get the second to last inequality.

With the above estimate, we now have enough tool to handle $\mathbf{E}$. By definition of $\Omega^* \setminus \Omega^{**}$ and doubling property of μ

$$\mathbf{E} \leqslant C_{\kappa,\eta} \iint_{\Omega^*\setminus\Omega^{**}} \|u\|_\infty^2 \frac{1}{s} s ds \mu dy$$

$$
\begin{aligned}
&\leqslant C_{\kappa,\eta}\int_{2Q}\|u\|_\infty^2\left(\int_{\frac{8\delta(y)}{3\eta}}^{\frac{18\delta(y)}{\eta}}\frac{ds}{s}+\int_{l(Q)}^{8l(Q)}\frac{ds}{s}\right)\mu dy\\
&\leqslant C_{\kappa_0,\eta}\|u\|_\infty^2\mu(2Q)\lesssim C_{\kappa_0,\eta}\|u\|_\infty^2\mu(Q).
\end{aligned}
$$

Thus, we have done for the estimate of $\mathbf{S}_1$ in (3.64). Next, we make an estimate for term $\mathbf{S}_2$, we observe that since A is t-independent

$$
\begin{aligned}
2\mathbf{S}_2 &= \iint_{\mathbb{R}^{n+1}_+} u_1^2\partial_t\left(\frac{1}{J}\langle A\mathbf{p},\mathbf{p}\rangle\right)\Psi^2 dtdx\\
&= \iint_{\mathbb{R}^{n+1}_+} u_1^2\partial_t\left(\frac{1}{J}\right)\langle A\mathbf{p},\mathbf{p}\rangle\Psi^2 dtdx+\iint_{\mathbb{R}^{n+1}_+} u_1^2\frac{1}{J}\langle\partial_t\mathbf{p},A^*\mathbf{p}\rangle\Psi^2 dtdx\\
&+ \iint_{\mathbb{R}^{n+1}_+} u_1^2\frac{1}{J}\langle A\mathbf{p},\partial_t\mathbf{p}\rangle\Psi^2 dtdx =: \mathbf{I}+\mathbf{II}+\mathbf{III},
\end{aligned}
$$

We will estimate the above in order. Recall that $J(x,t)=1+\partial_t\mathcal{P}^*_{\eta t}\varphi(x)$, hence,

$$
\begin{aligned}
\mathbf{I} &= \iint_{\mathbb{R}^{n+1}_+} u_1^2\partial_t\left(\frac{1}{J}\right)\langle A\mathbf{p},\mathbf{p}\rangle\Psi^2 dtdx = -\iint_{\mathbb{R}^{n+1}_+} u_1^2\left(\frac{\partial_t^2\mathcal{P}^*_{\eta t}\varphi(x)}{J^2}\right)\langle A\mathbf{p},\mathbf{p}\rangle\Psi^2 dtdx\\
&= \iint_{\mathbb{R}^{n+1}_+}\partial_t\left(u_1^2\right)\frac{\partial_t\mathcal{P}^*_{\eta t}\varphi(x)}{J^2}\langle A\mathbf{p},\mathbf{p}\rangle\Psi^2 dtdx+\iint_{\mathbb{R}^{n+1}_+} u_1^2\frac{\partial_t\mathcal{P}^*_{\eta t}\varphi(x)}{J^2}\partial_t\langle A\mathbf{p},\mathbf{p}\rangle\Psi^2 dtdx\\
&+\iint_{\mathbb{R}^{n+1}_+} u_1^2\partial_t\mathcal{P}^*_{\eta t}\varphi(x)\partial_t\left(\frac{1}{J^2}\right)\langle A\mathbf{p},\mathbf{p}\rangle\Psi^2 dtdx\\
&+\iint_{\mathbb{R}^{n+1}_+} u_1^2\frac{\partial_t\mathcal{P}^*_{\eta t}\varphi(x)}{J^2}\langle A\mathbf{p},\mathbf{p}\rangle\partial_t\Psi^2 dtdx\\
&=:\mathbf{I}_1+\mathbf{I}_2+\mathbf{I}_3+\mathbf{I}_4,
\end{aligned}
\tag{3.70}
$$

where we did integration by parts in t-variable for the third equality and also used the fact that the boundary terms vanish since $\partial_t^2\mathcal{P}^*_{\eta t}\varphi|_{t=0}=0$ as may be seen by first considering the case that φ is in the domain of -div$A_\|^*\nabla$, and then using a density argument.

Let us recall that,

$$
\mathbf{p} := \left(\nabla_x(\mathcal{P}^*_{\eta t}-I)\varphi,-1\right) = \left(\nabla_x\tau(x,t),-1\right).
$$

Since $\partial_t\mathcal{P}_{\eta t}\varphi$ is bounded, and $J\approx 1$, in Ω_0, the term $\mathbf{I}_4$ may then be handled exactly like $\mathbf{E}''_{11},\mathbf{E}''_{12}$ above.

For the other terms, we need to use other way. Let us start with $\mathbf{I}_1$. By the Cauchy–Schwarz inequality with small σ

$$|\mathbf{I}_1| \leqslant \sigma \iint_{\mathbb{R}^{n+1}_+} |\partial_t u_1|^2 |\mathbf{p}|^2 \Psi^2 t\mu dt dx + \frac{C}{\sigma} \iint_{\mathbb{R}^{n+1}_+} u_1^2 |\partial_t \mathcal{P}_{\eta t}\varphi|^2 |\mathbf{p}|^2 \Psi^2 \mu \frac{dt}{t} dx. \tag{3.71}$$

The first of these terms is similar to (3.67) and we may hide it on the LHS of (3.54), if σ is chosen small enough. For the second term, we use a dyadic decomposition as above:

$$\begin{aligned} &\iint_{\mathbb{R}^{n+1}_+} u_1^2 |\partial_t \mathcal{P}_{\eta t}\varphi|^2 |\mathbf{p}|^2 \Psi^2 \mu \frac{dt}{t} dx \\ =&\sum_k \sum_{Q' \in \mathbb{D}_k^\eta} \int_{Q'} \int_{2^{-k}}^{2^{-k+1}} u_1^2 |\partial_t \mathcal{P}^*_{\eta t}\varphi|^2 |\mathbf{p}|^2 \Psi^2 \mu \frac{dt}{t} dx. \end{aligned} \tag{3.72}$$

By Moser-type parabolic local interior estimates for degenerate parabolic equations (see Theorem B in [F]) with the rescaling $t \to t^2$

$$\left|\partial_t \mathcal{P}^*_{\eta t}\varphi\right| \leqslant \left(\frac{1}{\mu(2Q')t} \iint_{2Q' \times (\frac{t}{2}, 2t)} \left|\partial_s \mathcal{P}^*_{\eta s}\varphi\right|^2 \mu dy ds \right)^{\frac{1}{2}}$$

Then, for each k and $Q' \in \mathbb{D}_k^\eta$, note that $t \approx s$, we get

$$\begin{aligned} &\int_{Q'} \int_{2^{-k}}^{2^{-k+1}} u_1^2 |\partial_t \mathcal{P}^*_{\eta t}\varphi|^2 |\mathbf{p}|^2 \Psi^2 \mu \frac{dt}{t} dx \\ \lesssim &\int_{2^{-k}}^{2^{-k+1}} \|u\|_\infty^2 \left(\int_{2Q'} \int_{2^{-k-1}}^{2^{-k+2}} \left|\partial_s \mathcal{P}^*_{\eta s}\varphi\right|^2 \frac{ds}{s} \mu dy \right) \left(\frac{1}{\mu(Q')} \int_{Q'} |\mathbf{p}|^2 \Psi^2 d\mu(x) \frac{dt}{t} \right) \\ \lesssim &C_{\kappa_0} \|u\|_\infty^2 \left(\int_{2Q'} \int_{2^{-k-1}}^{2^{-k+2}} \left|\partial_s \mathcal{P}^*_{\eta s}\varphi\right|^2 \mathbb{1}_{\Omega_0} \mu \frac{ds}{s} dy \right), \end{aligned} \tag{3.73}$$

since

$$\frac{1}{\mu(Q')} \int_{Q'} |\mathbf{p}|^2 \Psi^2 d\mu(x) \lesssim (\kappa_0 + 1) \qquad \text{by (3.63) and } \Psi^2 \leqslant 1.$$

Now, we can take summation in Q' and in k by using equation (3.72) and the inequality (3.73) and then use square function bound, i.e.

$$\iint_{\mathbb{R}^{n+1}_+} \|u\|_\infty^2 \left|\partial_t \mathcal{P}^*_{\eta t}\varphi\right|^2 |\mathbf{p}|^2 \Psi^2 \mu \frac{dt}{t} dx$$

$$
\begin{aligned}
&\leqslant C_{\kappa_0}\|u\|_\infty^2 \sum_k \sum_{Q'\in\mathbb{D}_k^\eta} \left(\int_{2Q'} \int_{2^{-k-1}}^{2^{-k+2}} \left|\partial_s \mathcal{P}_{\eta s}^* \varphi\right|^2 \mathbb{1}_{\Omega_0} \mu \frac{ds}{s} dy \right) \\
&\leqslant C_{\kappa_0}\|u\|_\infty^2 \iint_{\mathbb{R}_+^{n+1}} \left|\partial_t \mathcal{P}_{\eta t}^* \varphi\right|^2 \mathbb{1}_{\Omega_0} \mu \frac{dt}{t} dx \\
&\leqslant C_{\kappa_0}\|u\|_\infty^2 \int_{\mathbb{R}^n} \left(\int_0^\infty \left|\partial_t \mathcal{P}_{\eta t}^* \varphi\right|^2 \frac{dt}{t} \right) d\mu(x) \\
&= C_{\kappa_0}\|u\|_\infty^2 \int_{\mathbb{R}^n} \int_0^\infty \left|\eta^2 t \mathcal{P}_{\eta t}^* \left(\mathcal{L}_\mu^* \varphi\right)\right|^2 \frac{dt}{t} d\mu(x) \\
&\leqslant C_{\kappa_0}\|u\|_\infty^2 \|\nabla\varphi\|_{L_\mu^2}^2 \leqslant C_{\kappa_0}\|u\|_\infty^2 \mu(Q),
\end{aligned}
$$

where we have used the fact that $\left|\partial_t \mathcal{P}_{\eta t}^* \varphi\right| = \left|\eta^2 t \mathcal{P}_{\eta t}^* \left(\mathcal{L}_\mu^* \varphi\right)\right|$, and L_μ^2 bound for vertical square function $g_{\mathcal{L}_\mu}$ together the definition of φ and the solution of Kato problem.

So, we still have the bound for $\mathbf{I}_1$. Next, we consider the term $\mathbf{I}_2$. By definition of $\mathbf{p}$, we get

$$
|\mathbf{I}_2| \lesssim \iint_{\mathbb{R}_+^{n+1}} u_1^2 \left|\nabla_x \partial_t \mathcal{P}_{\eta t}^* \varphi\right|^2 \Psi^2 t\mu dt dx + \iint_{\mathbb{R}_+^{n+1}} u_1^2 \left|\partial_t \mathcal{P}_{\eta t}^* \varphi\right|^2 |\mathbf{p}|^2 \frac{dt}{t} \mu dx. \tag{3.74}
$$

We observe that the first term above is the same as $\mathbf{S}_{12}'$ in (3.65) and the second term is the same as the second term on RHS of (3.71). We therefore obtain

$$
|\mathbf{I}_2| \lesssim \|u\|_\infty^2 \mu(Q).
$$

To finish our estimate for term $\mathbf{I}$, we only have to give estimate for the term $\mathbf{I}_3$ in (3.70). We observe that from the definition of J, we have

$$
|\mathbf{I}_3| \lesssim \iint_{\mathbb{R}_+^{n+1}} |\mathbf{p}|^2 u_1^2 \left|\partial_t^2 \mathcal{P}_{\eta t}^* \varphi\right|^2 \Psi^2 \mu t dt dx + \iint_{\mathbb{R}_+^{n+1}} |\mathbf{p}|^2 u_1^2 \left|\partial_t \mathcal{P}_{\eta t}^* \varphi\right|^2 \Psi^2 \frac{dt}{t} d\mu(x)
$$

The second term is the same as the second term in $\mathbf{I}_2$. For the first term, once again, decompose $\mathbb{R}^n$ into dyadic cubes

$$
\iint_{\mathbb{R}_+^{n+1}} |\mathbf{p}|^2 u_1^2 \left|\partial_t^2 \mathcal{P}_{\eta t}^* \varphi\right|^2 \Psi^2 \mu t dt dx
$$

$$= \sum_k \sum_{Q' \in \mathbb{D}_k^\eta} \int_{2^{-k}}^{2^{-k+1}} \int_{Q'} u_1^2 \left|\partial_t^2 \mathcal{P}_{\eta t}^* \varphi\right|^2 |\mathbf{p}|^2 \Psi^2 \mu t dt dx.$$

For any $Q^{'}$ such that the integral over $Q^{'}$ of the above term is non-zero, notice that $\partial_t^2 \mathcal{P}_{\eta t} \varphi$ is a solution of a weighted parabolic equation, we apply a Moser interior estimate for weighted parabolic equations to get the following estimate

$$\begin{aligned}
&\int_{2^{-k}}^{2^{-k+1}} \int_{Q'} u_1^2 \left|\partial_t^2 \mathcal{P}_{\eta t}^* \varphi\right|^2 |\mathbf{p}|^2 \Psi^2 t dt \, d\mu(x) \\
&\lesssim \|u\|_\infty^2 \int_{2^{-k}}^{2^{-k+1}} \left(\frac{1}{\mu(Q')2^{-k}} \int_{2Q'} \int_{2^{-k-1}}^{2^{-k+2}} \left|\partial_s^2 \mathcal{P}_{\eta s}^* \varphi\right|^2 ds \, \mu dy \right) \int_{Q'} |\mathbf{p}|^2 \Psi^2 t dt \, d\mu(x) \\
&\approx \|u\|_\infty^2 \int_{2^{-k}}^{2^{-k+1}} 2^k \left(\int_{2Q'} \int_{2^{-k-1}}^{2^{-k+2}} \left|\partial_s^2 \mathcal{P}_{\eta s}^* \varphi\right|^2 ds \, \mu dy \right) \left(\frac{1}{\mu(Q')} \int_{Q'} |\mathbf{p}|^2 \Psi^2 d\mu(x) \right) dt \\
&\lesssim \|u\|_\infty^2 \int_{2^{-k}}^{2^{-k+1}} 2^k \left(\int_{2Q'} \int_{2^{-k-1}}^{2^{-k+2}} \left|\partial_s^2 \mathcal{P}_{\eta s}^* \varphi\right|^2 ds \, \mu dy \right) dt \\
&\approx \|u\|_\infty^2 \int_{2Q'} \int_{2^{-k-1}}^{2^{-k+2}} \left|\partial_s^2 \mathcal{P}_{\eta s}^* \varphi\right|^2 s ds \, \mu dy,
\end{aligned}$$

where we have used (3.63), more precisely, we applied the following estimate

$$\frac{1}{\mu(Q')} \int_{Q'} \left|\nabla_x (I - \mathcal{P}_{\eta t}^*) \varphi(x)\right|^2 \mu(x) dx \lesssim \kappa_0^2,$$

and along with the definition of $\mathbf{p} := \left(\nabla_x (\mathcal{P}_{\eta t}^* - I)\varphi, -1\right)$ and κ_0 is a fixed large number to get the estimate for the last inequality.

From the estimate above, we can give a bound for the first term of the bound for $\mathbf{I}_3$ by taking summation over $Q^{'}$ and k, then using $\partial_t \mathcal{P}_{\eta t}^* = \partial_t e^{-(\eta t)^2 \mathcal{L}_\mu^*} f = -2\eta^2 t \mathcal{L}_\mu^* \mathcal{P}_{\eta t}^* = -2\eta^2 t \mathcal{P}_{\eta t}^* \mathcal{L}_\mu^*$ and bound the t-derivative by full gradient for $\mathcal{P}_{\eta s}^*$, we get

$$\begin{aligned}
&\sum_k \sum_{Q' \in \mathbb{D}_k^\eta} \int_{2^{-k}}^{2^{-k+1}} \int_{Q'} u_1^2 \left|\partial_t^2 \mathcal{P}_{\eta t}^* \varphi\right|^2 |\mathbf{p}|^2 \Psi^2 \mu t dt dx. \\
&= \iint_{\mathbb{R}_+^{n+1}} u_1^2 \left|\partial_t^2 \mathcal{P}_{\eta t}^* \varphi\right|^2 |\mathbf{p}|^2 \Psi^2 t dt d\mu(x) \\
&\lesssim \|u\|_\infty^2 \iint_{\mathbb{R}_+^{n+1}} \left|\partial_s^2 \mathcal{P}_{\eta s}^* \varphi\right|^2 s ds \mu dy
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \|u\|_\infty^2 \int_{\mathbb{R}^n} \left(\int_0^\infty \left| s\partial_s^2 \mathcal{P}_{\eta s}^* \varphi \right|^2 \frac{ds}{s} \right) \mu dy \\
&\lesssim C_\eta \|u\|_\infty^2 \int_{\mathbb{R}^n} \left(\int_0^\infty \left| s^2 \nabla_{x,s} \mathcal{P}_{\eta s}^* \mathcal{L}_\mu^* \varphi \right|^2 \frac{ds}{s} \right) \mu dy \\
&\quad + C_\eta \|u\|_\infty^2 \int_{\mathbb{R}^n} \left(\int_0^\infty \left| s \mathcal{P}_{\eta s}^* \mathcal{L}_\mu^* \varphi \right|^2 \frac{ds}{s} \right) \mu dy \\
&\lesssim C_\eta \|u\|_\infty^2 \|\nabla \varphi\|_{L^2_\mu} \leqslant C_\eta \|u\|_\infty^2 \mu(Q),
\end{aligned}
$$

where the vertical square function estimate in (3.13) and (3.14) were applied in the penultimate inequality.

Hence, we have proved $|\mathbf{I}_3| \lesssim \|u\|_\infty^2 \mu(Q)$. And therefore, we get $|\mathbf{I}| \lesssim \|u\|_\infty^2 \mu(Q)$.

Next, we want to give some estimate for the term $\mathbf{II}$. By definition of $\mathbf{p}$, we have $\partial_t \mathbf{p} = (\nabla_x \partial_t \mathcal{P}_{\eta t}^* \varphi, 0)$, whence it follows from the Hodge decomposition (3.30) that for $x \in 5Q$,

$$
\begin{aligned}
\langle \partial_t \mathbf{p}, A^* \mathbf{p} \rangle &= \left\langle \nabla_x \partial_t \mathcal{P}_{\eta t}^* \varphi, A_\parallel^* \nabla_x \mathcal{P}_{\eta t}^* \varphi \right\rangle - \left\langle \nabla_x \partial_t \mathcal{P}_{\eta t}^* \varphi, A_\parallel^* \nabla_x \varphi \right\rangle - \left\langle \nabla_x \partial_t \mathcal{P}_{\eta t}^* \varphi, \mathbf{c} \right\rangle \\
&= \left\langle \nabla_x \partial_t \mathcal{P}_{\eta t}^* \varphi, A_\parallel^* \nabla_x \mathcal{P}_{\eta t}^* \varphi \right\rangle - \left\langle \nabla_x \partial_t \mathcal{P}_{\eta t}^* \varphi, \mathbf{h} \right\rangle .
\end{aligned} \tag{3.75}
$$

Thus, we can rewrite $\mathbf{II}$ as following

$$
\begin{aligned}
\mathbf{II} &= \iint_{\mathbb{R}^{n+1}_+} u_1^2 \frac{1}{J} \left\langle \nabla_x \partial_t \mathcal{P}_{\eta t}^* \varphi, A_\parallel^* \nabla_x \mathcal{P}_{\eta t}^* \varphi \right\rangle \Psi^2 dtdx \\
&\quad - \iint_{\mathbb{R}^{n+1}_+} u_1^2 \frac{1}{J} \left\langle \nabla_x \partial_t \mathcal{P}_{\eta t}^* \varphi, \mathbf{h} \right\rangle \Psi^2 dtdx \\
&=: \mathbf{II}_1 + \mathbf{II}_2.
\end{aligned}
$$

To estimate $\mathbf{II}_1$, we take integration by part in x variable and obtain

$$
\begin{aligned}
\mathbf{II}_1 &= \iint_{\mathbb{R}^{n+1}_+} u_1^2 \frac{1}{J} \left(\partial_t \mathcal{P}_{\eta t}^* \varphi \right) \left(\mathcal{L}_\mu^* \mathcal{P}_{\eta t}^* \varphi \right) \Psi^2 \mu dtdx \\
&\quad - \iint_{\mathbb{R}^{n+1}_+} \partial_t \mathcal{P}_{\eta t}^* \varphi \left\langle \nabla_x \left(u_1^2 \frac{1}{J} \right), A_\parallel^* \nabla_x \mathcal{P}_{\eta t}^* \varphi \right\rangle \Psi^2 dtdx \\
&\quad - \iint_{\mathbb{R}^{n+1}_+} u_1^2 \frac{1}{J} \partial_t \mathcal{P}_{\eta t}^* \varphi \left\langle \nabla_x \left(\Psi^2 \right), A_\parallel^* \nabla_x \mathcal{P}_{\eta t}^* \varphi \right\rangle dtdx \\
&=: \mathbf{II}_1^{'} + \mathbf{II}_1^{''} + \mathbf{II}_1^{'''}.
\end{aligned}
$$

Since $\mathcal{L}_\mu^* \mathcal{P}_{\eta t}^* = -(2\eta^2 t)^{-1}\partial_t \mathcal{P}_{\eta t}^*$, the term $\mathbf{II}_1^{'}$ is like the second term of RHS of (3.71) but easier since we have a constant instead of $\mathbf{p}$.

To treat the term $\mathbf{II}_1^{''}$, we first apply product rule for $\nabla_x \left(u_1^2 \frac{1}{J}\right)$, and then we use the fact that $J \approx 1$, and $\nabla_x J = \nabla_x \partial_t \mathcal{P}_{\eta t}^* \varphi$ together the Cauchy–Schwarz inequality with small σ, then we have the following bound for $\mathbf{II}_1^{''}$

$$\begin{aligned} \left|\mathbf{II}_1^{''}\right| &\leqslant \sigma \iint_{\mathbb{R}_+^{n+1}} |\nabla_x u_1|^2 \Psi^2 t\mu dt dx + C \iint_{\mathbb{R}_+^{n+1}} u_1^2 |\nabla_x \partial_t \mathcal{P}_{\eta t}^* \varphi|\Psi^2 \mu t dt dx \\ &\quad + C(\frac{1}{\sigma}+1) \iint_{\mathbb{R}_+^{n+1}} u_1^2 |\partial_t \mathcal{P}_{\eta t}^* \varphi|^2 |\nabla_x \mathcal{P}_{\eta t}^* \varphi|^2 \Psi^2 \mu \frac{dt}{t} dx. \end{aligned} \tag{3.76}$$

The first term is bounded by (3.66) and therefore we can handle it in the same way. The second term and the third term are essentially like the two terms bounding $\mathbf{I}_2$ in (3.74); in the last term, the factor $\left|\nabla_x \mathcal{P}_{\eta t}^* \varphi^2\right|$ may be handled like $|\mathbf{p}|^2$, by using (3.63).

To complete our estimate for $\mathbf{II}_1$, we need to take care of $\mathbf{II}_1^{'''}$. By Cauchy-Schwarz inequality and $J \approx 1$, we observe that

$$|\mathbf{II}_1^{'''}| \lesssim \iint_{\mathbb{R}_+^{n+1}} u_1^2 |\nabla_x \Psi|^2 t dt d\mu(x) + \iint_{\mathbb{R}_+^{n+1}} \left|\partial_t \mathcal{P}_{\eta t}^* \varphi\right|^2 \left|\nabla_x \mathcal{P}_{\eta t}^* \varphi\right|^2 \Psi^2 \frac{dt}{t} d\mu(x)$$

The first term is the same $\mathbf{S}_{13}^{'}$ in (3.65) and the second is the same as in the last term in (3.76). This complete the estimate for $\mathbf{II}_1$.

Next, we take care of term $\mathbf{II}_2$. From the Hodge decomposition, $\mathbf{h}$ is divergence free, so

$$\begin{aligned} \mathbf{II}_2 &= \iint_{\mathbb{R}_+^{n+1}} \partial_t \mathcal{P}_{\eta t}^* \varphi \left\langle \nabla_x \left(\frac{u_1^2}{J}\right), \mathbf{h} \right\rangle \Psi^2 dt dx \\ &\quad + \iint_{\mathbb{R}_+^{n+1}} \frac{u_1^2}{J} \partial_t \mathcal{P}_{\eta t}^* \varphi \left\langle \nabla_x \left(\Psi^2\right), \mathbf{h} \right\rangle dt dx \\ &=: \mathbf{II}_2^{'} + \mathbf{II}_2^{''}. \end{aligned}$$

We see that $\mathbf{II}_2^{'}$ can be treated exactly like $\mathbf{II}_1^{''}$ above, and the second term is exactly like

$\mathbf{I}_1'''$ since $\mathbf{h} = \mathbf{c}\mathbb{1}_{5Q} + A_\|^*\nabla_x\varphi$ and

$$\frac{1}{\mu(Q)}\int_{5Q}\left|\frac{\mathbf{h}}{\mu}\right|^2\mu\,dx \lesssim \frac{1}{\mu(Q)}\int_{5Q}\left(|\nabla_x\varphi|^2 + \left|\frac{\mathbf{c}}{\mu}\right|^2\right)\mu\,dx \leqslant C.$$

Therefore, to handle $\mathbf{h}$, we may do via (3.63) just like the factor $\nabla_x\mathcal{P}_{\eta t}^*\varphi$.

Hence, we also have the estimate $\mathbf{II}_2 \lesssim \|u\|_\infty^2\mu(Q)$.

Last, to finish the estimate for $\mathbf{S}$, we treat the term $\mathbf{III}$. By an identity analogous to (3.75), we have

$$\begin{aligned}
\mathbf{III} =& \iint_{\mathbb{R}_+^{n+1}} u_1^2\frac{1}{J}\left\langle A_\|\nabla_x\mathcal{P}_{\eta t}^*\varphi, \nabla_x\nabla_t\mathcal{P}_{\eta t}^*\varphi\right\rangle\Psi^2dtdx\\
&-\iint_{\mathbb{R}_+^{n+1}} u_1^2\frac{1}{J}\left\langle \mathbf{b} + A_\|\nabla_x\varphi, \nabla_x\nabla_t\mathcal{P}_{\eta t}^*\varphi\right\rangle\Psi^2dtdx\\
=&\iint_{\mathbb{R}_+^{n+1}} u_1^2\frac{1}{J}\left\langle \nabla_x\left(\mathcal{P}_{\eta t}^* - I\right)\varphi, A_\|^*\nabla_x\nabla_t\mathcal{P}_{\eta t}^*\varphi\right\rangle\Psi^2dtdx\\
&-\iint_{\mathbb{R}_+^{n+1}} u_1^2\frac{1}{J}\left\langle \mathbf{b}, \nabla_x\nabla_t\mathcal{P}_{\eta t}^*\varphi\right\rangle\Psi^2dtdx\\
=:&\mathbf{III}_1 + \mathbf{III}_2.
\end{aligned}$$

For the first term above, we make integration by parts in x-variable. This implies,

$$\begin{aligned}
\mathbf{III}_1 \;=\;& -\iint_{\mathbb{R}_+^{n+1}}\left(\mathcal{P}_{\eta t}^*\varphi - \varphi\right)\left\langle\nabla_x\left(u_1^2\frac{1}{J}\Psi^2\right), A_\|^*\nabla_x\partial_t\mathcal{P}_{\eta t}^*\varphi\right\rangle dtdx\\
&-\iint_{\mathbb{R}_+^{n+1}} u_1^2\frac{1}{J}\left(\mathcal{P}_{\eta t}^*\varphi - \varphi\right)\left(\mathcal{L}_\mu^*\partial_t\mathcal{P}_{\eta t}^*\varphi\right)\Psi^2\mu dtdx\\
=:&\;\mathbf{III}_1' + \mathbf{III}_1''.
\end{aligned}$$

By (3.48) we have $|\mathcal{P}_{\eta t}^*\varphi - \varphi| \leqslant \eta^{\frac{1}{2}}t < t$ in the support of Ψ. Thus $\mathbf{III}_1'$ upon distributing ∇_x over u_1^2, $\frac{1}{J}$ and Ψ^2 yields integrals that may be handled just like the terms $\mathbf{J}, \mathbf{S}_{12}'$ and $\mathbf{K}$, respectively, in (3.65). So, it is left to find an estimate for $\mathbf{III}_1''$.

In order to take care of $\mathbf{III}_1''$, we first note that for any F such that $\nabla F \in L_\mu^2(\mathbb{R}^n)$, we get the following

$$\int_{\mathbb{R}^n}\left(\int_0^\infty\left|\left(\mathcal{P}_{\eta t}^* - I\right)F\right|^2\frac{dt}{t^3}\right)\mu dx \lesssim \|\nabla F\|_{L_\mu^2(\mathbb{R}^n)}^2, \tag{3.77}$$

as may be seen by using the element identity

$$\int_0^{\eta t} \partial_s \mathcal{P}_{\eta s} ds = \mathcal{P}^*_{\eta t} - \mathcal{P}^*_0$$

then we will use Hardy's inequality to reduce matter to L^2_μ bounds for vertical square functions and then use Kato problem.

Now, we estimate

$$\begin{aligned}
\left|\mathbf{III}_1''\right| &= \left|\iint_{\mathbb{R}^{n+1}_+} u_1^2 \frac{1}{J}\left(\mathcal{P}^*_{\eta t}\varphi - \varphi\right)\left(\mathcal{L}^*_\mu \partial_t \mathcal{P}^*_{\eta t}\varphi\right)\Psi^2 \mu dt dx\right| \\
&\leqslant \int_{\mathbb{R}^n}\int_0^\infty \|u\|^2_\infty \left|\left(\mathcal{P}^*_{\eta t}\varphi - \varphi\right)\left(\mathcal{L}^*_\mu \partial_t \mathcal{P}^*_{\eta t}\varphi\right)\right|\Psi^2 \mu dt dx \\
&\leqslant \|u\|^2_\infty \int_{2Q}\left(\int_0^\infty \left|\left(\mathcal{P}^*_{\eta t}\varphi - \varphi\right)\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}}\left(\int_0^\infty \left|t^2 \partial_t \mathcal{P}^*_{\eta t}\mathcal{L}^*_\mu \varphi\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}} d\mu(x) \\
&\leqslant \|u\|^2_\infty \left(\int_{2Q}\int_0^\infty \left|\left(\mathcal{P}^*_{\eta t}\varphi - \varphi\right)\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}}\left(\int_{2Q}\int_0^\infty \left|t^2 \partial_t \mathcal{P}^*_{\eta t}\mathcal{L}^*_\mu \varphi\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}} d\mu(x) \\
&\leqslant \|u\|^2_\infty \|\nabla\varphi\|^2_{L^2_\mu} \lesssim \|u\|^2_\infty \mu(Q),
\end{aligned}$$

where we have used Hölder's inequality and support of Ψ for the third line, and we applied (3.77) and L^2_μ bound for vertical square function for the last inequality.

To complete the proof of our theorem, it remains to estimate the last term $\mathbf{III}_2$. To this ends, we use the Hodge decomposition to write

$$\begin{aligned}
\mathbf{b}\mathbb{1}_{5Q} &= A_\| \nabla \widetilde{\varphi} + \widetilde{\mathbf{h}} \\
&= (A_\| \nabla_x \tilde{\varphi} - A_\| \nabla_x \mathcal{P}_{\eta t}\widetilde{\varphi}) + A_\| \nabla_x \mathcal{P}_{\eta t}\widetilde{\varphi} + \widetilde{\mathbf{h}},
\end{aligned}$$

where $\mathcal{P}_{\eta t} := e^{-(\eta t)^2 \mathcal{L}_\mu}$ and where $\widetilde{\mathbf{h}}$ is divergence free.

We recall that by construction, the various estimates that we have used for $\varphi, \mathcal{P}_{\eta t}\varphi$ hold also for $\widetilde{\varphi}, \mathcal{P}_{\eta t}\widetilde{\varphi}$. The contribute of $\widetilde{\mathbf{h}}$ may then be handled exactly like $\mathbf{II}_2$ above, while the contribution of $A_\| \nabla_x \mathcal{P}_{\eta t}\widetilde{\varphi}$ maybe handled like $\mathbf{II}_1$ above, i.e., by integration by parts in x to move ∇_x away from $\partial_t \mathcal{P}^*_\eta \varphi$. Finally, the contribution of $(A_\| \nabla_x \widetilde{\varphi} - A_\| \nabla_x \mathcal{P}_{\eta t}\widetilde{\varphi})$ in term $\mathbf{III}_2$

equals

$$\iint_{\mathbb{R}^{n+1}_+} u_1^2 \frac{1}{J} \left\langle \left(\nabla_x \tilde{\varphi} - \nabla_x \mathcal{P}_{\eta t} \widetilde{\varphi}\right), A_\parallel^* \nabla_x \partial_t \mathcal{P}_{\eta t}^* \varphi \right\rangle \Psi dt dx,$$

which can be handled like $\mathbf{III}_1$. ∎

3.5 Solvability of Dirichlet problems for degenerate elliptic equations with L^p boundary data

In this section, we will prove that the harmonic measure for the degenerate elliptic operator $-\mathrm{div} A \nabla$ over the domain domain $\mathbb{R}^{n+1}_+$, where A is the degenerate matrix as above, is A_∞ with respect to the surface measure on the boundary. Then we prove that the Dirichlet problem with L^p_μ data on the boundary is solvable in the sense of nontangential limit for some p.

Let us fix some convention, for any $R \gg 1$, we define $\Omega_R := B(0, R) \cap \mathbb{R}^{n+1}_+$. We see that Ω_R is a Lipschitz domain with Lipschitz constant is bounded by N_0 for all $R > R_0$ where R_0 is large enough. A point $(x, 0) \in \partial\mathbb{R}^{n+1}_+$ is denoted by small letter x and we denote capital letter X if $X = (x, t) \in \mathbb{R}^{n+1}_+$. We use " $\approx$ " to mention that the ratio of the two sides is bounded above and below by positive constants depending only on the allowable parameters such as $n, \mu, \Lambda, \lambda, N_0$.

Recall that we still used the notation μ as an A_2 weight over $\mathbb{R}^n \equiv \partial\mathbb{R}^{n+1}_+$ like we discussed in the previous sections.

Definition 3.13. We say a domain D satisfies "Corkscrew condition" if there are some fix number M_0 and r_0 such that for any $Q \in \partial D$, and for any $r < r_0$ then there exist a Corkscrew point $A = A_r(Q) \in D$ such that $M_0^{-1} < |A - Q| < r$ and $\mathrm{dist}(A, \partial D) > M^{-1} r$

For convenience of our proof later, let us recall a version of Harnack's inequality for degenerate elliptic equations that was proved in [FKS], see Lemma 2.3.5 page 97.

Theorem 3.14. *(Harnack's inequality) Let u be a non-negative solution of $Lu = 0$ belonging to the Sobolev space $W^{1,2}_{\mu}(2B)$ for a ball $B \subset \mathbb{R}^{n+1}_+$. Then we have*

$$\sup_B u \leqslant C \inf_B u, \tag{3.78}$$

with the constant $C = C(\Lambda, \lambda)$ is independent of u and B.

Lemma 3.15. *Fix $x_0 \in \partial\mathbb{R}^{n+1}_+$ and $r_0 > 0$, write $B_0 = B(x_0, r_0), \Delta_0 := B_0 \cap \partial\mathbb{R}^{n+1}_+$ and $B_0' := B(x_0, r_0/2)$. Let $0 < u \in W^{1,2}_{\mu,\mathrm{loc}}(B_0 \cap \mathbb{R}^{n+1}_+) \cap C(\overline{B_0 \cap \mathbb{R}^{n+1}_+})$ verifies $Lu = 0$ in $B_0 \cap \mathbb{R}^{n+1}_+$ in the weak sense and $u \equiv 0$ on Δ_0. Then there exists C, $0 < \alpha \leqslant 1$. depending only on ellipticity, the dimension n and Lipschitz constant N_0 such that*

$$u(X) \leqslant C \left(\frac{|X - x_0|}{r_0}\right)^{\alpha} \sup_{Y \in \overline{B_0 \cap \mathbb{R}^{n+1}_+}} u(Y), \quad \forall X \in B_0' \cap \mathbb{R}^{n+1}_+.$$

Proof. This is a consequence of locally Hölder continuity of the solution u, (see Theorem 2.3.12 in [FKS]). ∎

Theorem 3.16. *Let $u \geqslant 0$, $Lu = 0$ in $R_{2x} := \Delta(x, 2r) \times (0, 2r)$ and also $u(y) = 0$ on $\Delta(x, 2r)$, then for all $X \in R_{2x}$ we have*

$$u(X) \leqslant C(n, \lambda, \Lambda, N_0)\, u(A_r(x)), \tag{3.79}$$

where $A_r(x)$ is the Corkscrew point relative to some surface ball $\Delta(x, 2r) := B(x, 2r) \cap \partial(\mathbb{R}^{n+1}_+)$.

Proof. This Theorem is a consequence of Harnack's inequality, Moser's estimates and Maximal principle. For more details, see Theorem 1.1 in [CFMS]. Notice that, even in [CFMS] the authors proved the Theorem in non-degenerate equations but the proof is still valid once we have all the ingredients above. ∎

Theorem 3.17. *(Comparison Theorem) Let $u, v \geqslant 0$ and a point $x \in \partial\mathbb{R}^{n+1}_+$. Suppose that $Lu = Lv = 0$, $u, v \in W^{1,2}(T_{2r}(x))$, $u, v \in C(\overline{T_{2r}(x)})$, $u = v = 0$ on $\Delta_{2r}(x)$ and $u, v \not\equiv 0$. Then for all $X \in T_{2r}(x)$*

$$\frac{u(A_r(x))}{v(A_r(x))} \approx \frac{u(X)}{v(X)},$$

here $T_{2r}(x) := \Delta(x, 2r) \times (0, 2r)$..

Proof. The proof is the same non-degenerate elliptic equations. See the proof of Theorem 1.4 in [CFMS] for more details. ∎

By a results in [FJK], we have the following definition of Green function and harmonic measure over the domain Ω_R.

Definition 3.18. There is a Green function $g_R : \overline{\Omega}_R \times \overline{\Omega}_R \mapsto [0, \infty]$ satisfying $g(X, y) = 0$ for $y \in \partial\Omega_R$, $-\operatorname{div} A\nabla g(\cdot, Y) = \delta_Y$ where δ_Y is the unit mass at Y. Roughly speaking, we can say the Green function is the week solution of the equation $-\operatorname{div} A\nabla g = \delta_Y$ as a function of X.

Remark 3.19. In the uniform elliptic equations, we have rick theory for the Green function. For examples, we understood the size of Green function, connection between Green function and harmonic measure, etc., for more details about the Green function in this situation,the reader may take a look at [K]. We want to point out that, in degenerate setting, we still have Green function and its size as well as the its connection with the harmonic measure, but only for bounded domains thanks to the work by [FJK], [FJK1] and [FKS]. Due to the structure of the A_2 weight, we may not have the Green function for unbounded domain in general.

Next is the Theorem 3.3 in [FJK1] in which it gave an estimate the size of the green function.

Theorem 3.20. *Let X, Y belong to the set $\Sigma = \{Z : |Z| < \frac{1}{4}\Upsilon\} \subset \mathbb{R}^{n+1}_+$, then*

$$g_R(X,Y) \approx \int_{|X-Y|}^{\Upsilon} \frac{s^2}{\nu(B(X,s))} \frac{ds}{s}.$$

where $\nu(B(X,s))$ is a weighted measure of the ball $B(X,s) \subset \mathbb{R}^{n+1}_+$ and $\nu(B(X,s)) := \iint_{B(X,s)} \mu(x)dxdt$ as we defined above and "$\approx$" means the quotient depends only on the μ measure.

Remark 3.21. The theorem says that the size of the green function over the domain Ω_R does not depend on R nor Ω_R, provided that $R \gg \Upsilon$.

Definition 3.22 (Harmonic measure over domain Ω_R)**.** For any $X \in \Omega_R$, the harmonic measure at the pole X is the measure ω^X on the boundary $\partial\Omega_R$ satisfying that $u(x) = \int_{\partial\Omega_R} hd\Omega^X$ is the solution of the Dirichlet problem $-\operatorname{div} u = 0$ and $u(y) = h(y)$ for all $y \in \partial\Omega_R$ where $h \in C_c(\Omega_R)$.

The following lemma gives us the connection between the Green function and the harmonic measure for degenerate elliptic operators on bounded domain Ω_R.

Lemma 3.23. *Let $r > 0$ and $\Delta := B(x,r) \cap \partial\Omega_R$ for some $x \in \partial\Omega_R$. Suppose that $A_r(x) \in \Omega_R$ be a Corkscrew point corresponding to the surface Δ. If $Y \in \Omega_R \setminus B(x, 4Mr)$, then*

$$g_R(A_r(x), Y) \approx \omega_R^Y(\Delta)\theta(A_r(x), r),$$

where $\theta(A_r(x), r) = \frac{r^2}{\nu(B(A_r(x),r))}$ and $\nu(B(X,r)) := \iint_{B(X,r)} \mu(x)dxdt$ for all $B(X,r) \subset \mathbb{R}^{n+1}_+$ and μ is as in the elliptic condition.

Corollary 3.24. *Let* $\Delta = \Delta_r(x), \Delta^{'} = \Delta_s(x_0) \subset \Delta, Y \in \mathbb{R}^n \setminus T_{2r}(x)$*, then*

$$\omega^{A_r(x)}(\Delta^{'}) \approx \frac{\omega^Y(\Delta^{'})}{\omega^Y(\Delta)}. \tag{3.80}$$

Proof. By Lemma 3.23 above, for R large, we get

$$\begin{aligned} g_R(A_r(x), Y) &\approx \omega_R^Y(\Delta)\theta(A_r(x), r), \\ g_R(A_s(x_0), Y) &\approx \omega_R^Y(\Delta^{'})\theta(A_s(x_0), s), \\ g_R(A_s(x_0), A_r(x)) &\approx \omega_R^{A_r(x)}(\Delta^{'})\theta(A_s(x_0), s). \end{aligned}$$

Set $u(Z) = g_R(Z, Y)$ and $v(Z) = g_R(Z, A_{3r}(x))$ for some Y fixed. Note that $Lu = Lv = 0$ in $T_{2r}(x)$ and also $u = v = 0$ on $\Delta(x, 2r)$. So, by the Comparison Theorem,

$$\frac{u(Z)}{v(Z)} \approx \frac{u(A_r(x))}{v(A_r(x))}, \qquad \forall Z \in T_{7/4r}(x).$$

In particular, we can take $Z = A_s(Q_0)$,

$$\frac{g_R(A_s(x_0), Y)}{g_R(A_s(x_0), A_{3r}(x))} = \frac{u(A_s(x_0))}{v(A_s(x_0))} \approx \frac{u(A_r(x))}{v(A_r(x))} = \frac{g_R(A_r(x), Y)}{g_R(A_r(x), A_{3r}(x))}.$$

Hence,

$$\begin{aligned} \frac{\omega_R^Y(\Delta^{'})}{\omega_R^Y(\Delta)} &\approx \frac{g_R(A_s(x_0), A_{3r}(x))}{g_R(A_r(x), A_{3r}(x))} \cdot \frac{\theta(A_r(x), r)}{\theta(A_s(x_0), s)} \\ &\approx \omega_R^{A_r(x)}(\Delta^{'}), \end{aligned}$$

by the fact that $g_R(A_r(x), A_{3r}(x)) \approx \theta(A_r(x), r)$ (see the proof of Lemma 3 in [FJK] for details). Let $R \to \infty$, we get the desired inequality. ∎

Remark 3.25. In the proof of Lemma 3.23, the authors only used the local properties of solution u, so the constants for ratio are really independent of the size of domain Ω_R or we may say the constant is uniform in R.

Lemma 3.26. *The harmonic measure ω_R^X is doubling and $\omega_R^X(2\Delta) \leqslant C\omega_R^X(\Delta)$ where the constant C is independent on X, R, Δ.*

Proof. This is a consequence of Lemma 3.23 and Harnack's inequality and doubling property of the measure μ. ∎

Definition 3.27 (Harmonic measure for unbounded domain)**.** Let $f \in C_c(\mathbb{R}^n)$, the space of all continuous function with compact support on $\mathbb{R}^n$, there is R large enough so that $\text{supp}(f) \subset \partial\Omega_R$, extend f to be 0 on the rest of $\partial\Omega_R$ and call it f_R. Define

$$u_R(X) := \int_{\partial\Omega_R} f_R d\omega_R^X, \qquad \text{where } X \in \mathbb{R}_+^{n+1}. \tag{3.81}$$

By the maximal principle, u_R is an increasing sequence as R is increasing to $+\infty$ and also that $\|u_R\|_\infty \leqslant \|f\|_\infty$ uniform in R. Define $u(x,t)$ be the limit of $u_R(x,t)$ as R goes to infinity, then we also have $\|u\|_\infty \leqslant \|f\|_\infty$ and also $u_R \to u$ uniformly in any compact subset $K \subset \mathbb{R}_+^{n+1}$. Combine uniform convergence of u_R on a compact set with Caccioppoli's inequality, we deduce $u_R \to u$ in $W_\mu^{1,2}(K)$.

Next, we claim that u is a solution of equation $Lu = 0$. Indeed, for any test function $\varphi \in C_c^\infty(\mathbb{R}^n)$ then $\text{supp}\varphi \subset \Omega_R$ for some large R. The convergences above give

$$\int_{\mathbb{R}^n} A\nabla(u - u_R)\nabla\varphi dx = 0.$$

This prove the claim.

Let us fix $X \in \mathbb{R}_+^{n+1}$. For any positive function $f \in C_c(\mathbb{R}^n)$ define the functional $u(X)$ as above. Then by Riesz representation theorem, there exists a harmonic measure ω^X so that $u(X) = \int_{\mathbb{R}^n} f d\omega^X$.

Lemma 3.28. *The harmonic measure ω^X is a doubling measure for any $X = (x,t) \in \mathbb{R}_+^{n+1}$.*

Proof. For any surface ball $\Delta := B(x,r) \cap \partial\Omega_R$ for any radius $r > 0$ and $x \in \mathbb{R}^n$,

$$\begin{aligned}\omega^X(2\Delta) &= \lim_{R\to\infty} \omega_R^X(2\Delta) \\ &\leqslant C \lim_{R\to\infty} \omega_R^X(\Delta) \\ &= C\omega^X(\Delta),\end{aligned}$$

where we have used the Theorem 1 on page 54 of [EG] for the convergence in the first and last equalities above and the constant C is not depending on Ω_R, X, r as in the Lemma 3.26.

∎

Fix $X_0 = (x_0, t_0) \in \mathbb{R}^{n+1}_+$ and let $\omega := \omega^{X_0}$. Harnack's inequality implies that harmonic measure ω^X differs from ω by a bounded factor. The kernel function is defined for almost every ω as the Radon – Nikodym derivative $K(X,y) := \frac{d\omega^X}{d\omega}(y)$. We see that $K(X,y)$ is well-defined because of the following.

Lemma 3.29. *For any $X \in \mathbb{R}^{n+1}_+$, the harmonic measure ω^X is absolutely continuous with respect to harmonic measure ω.*

Proof. Let O be a Borel subset of $\partial\mathbb{R}^{n+1}_+$ such that $\omega(O) = 0$, we will prove that $\omega^X(O) = 0$. First observe that is is enough to treat the case that O is a bounded Borel set, since in general O is a countable union of bounded Borel sets. Then there is some $R_0 > 0$ large enough such that $O \subset B(0, R_0) \cap \partial\mathbb{R}^{n+1}_+$ and also $X, X_0 \in B(0, R_0)$. Thus by Harnack's Inequality, there is some constant C depending on X_0, X such that

$$\omega^X(O) \leqslant C\omega(O) = 0.$$

This means ω^X is absolute continuous with respect to ω and therefore, $K(X,y)$ is well-defined. ∎

By using Lebesgue Differentiation Theorem and the fact that the harmonic measure is doubling, we get the following corollary

Corollary 3.30. *The Radon – Nikodym derivative $K(X, y)$ satisfies*

$$K(X, y) = \lim_{s\to 0} \frac{\omega^X(\Delta(y, s))}{\omega(\Delta(y, s))},$$

for all X, y.

Now, we have the following results for the kernel $K(X, y)$.

Lemma 3.31. *If $x \in \partial\mathbb{R}^{n+1}_+$ and $A_r(x)$ is the Corkscrew point corresponding to $\Delta := B(x, r)\cap \partial\mathbb{R}^{n+1}_+$, then*

$$K(A_r(x), y) := \frac{d\omega^{A_r(x)}}{d\omega}(y) \leqslant \frac{C}{\omega(\Delta)}\left(\max\left\{\left(\frac{|y-x|}{r}\right)^{\alpha}, 1\right\}\right),$$

for some fixed $0 < \alpha < 1$.

Proof. Let us recall that for any $y \in \partial\mathbb{R}^{n+1}_+$ and for $s > 0$ if $X \in \Omega_R \setminus B(y, 4As)$ then $g_R(A_s(y), X) \approx \omega^X_R(\Delta(y, s))\theta(A_s(y), s)$ and also recall that $\omega := \omega^{X_0}$ for some fixed pole X_0.

Thus for any given $y \in \partial\mathbb{R}^{n+1}_+$ and $s > 0$, there is some positive number $0 < \Upsilon = \Upsilon(X_0, s)$ such that $X_0, A_s(y), A_r(x) \in \{Z : |Z| \leqslant \frac{1}{2}\Upsilon\}$. Choosing $R \gg \Upsilon$, by Theorem 3.20, we have the following estimates

$$\begin{aligned} K_R(A_r(x), y) &= \lim_{s\to 0} \frac{\omega_R^{A_r(x)}(\Delta(y, s))}{\omega_R(\Delta(y, s))} \\ &\approx \lim_{s\to 0} \frac{g_R(A_s(y), A_r(x))}{g_R(A_s(y), X_0)} \\ &\approx \lim_{s\to 0} \left(\int_{|A_s(y)-A_r(x)|}^{\Upsilon} \frac{s^2}{\nu(B(y, s))}\frac{ds}{s})\right)\left(\int_{|A_s(y)-X_0|}^{\Upsilon} \frac{s^2}{\nu(B(y, s))}\frac{ds}{s}\right)^{-1}. \end{aligned}$$

Since the last limit does not depend on $R \gg \Upsilon$, this implies

$$
\begin{aligned}
&\limsup_{R\to\infty} K_R(A_r(x),y)\\
&\approx \lim_{R\to\infty}\lim_{s\to 0}\left(\int_{|A_s(y)-A_r(x)|}^{\Upsilon}\frac{s^2}{\nu(B(y,s))}\frac{ds}{s}\right)\left(\int_{|A_s(y)-X_0|}^{\Upsilon}\frac{s^2}{\nu(B(y,s))}\frac{ds}{s}\right)^{-1}\\
&= \lim_{s\to 0}\lim_{R\to\infty}\left(\int_{|A_s(y)-A_r(x)|}^{\Upsilon}\frac{s^2}{\nu(B(y,s))}\frac{ds}{s}\right)\left(\int_{|A_s(y)-X_0|}^{\Upsilon}\frac{s^2}{\nu(B(y,s))}\frac{ds}{s}\right)\\
&\approx \lim_{s\to 0}\limsup_{R\to\infty}\frac{g_R(A_s(y),A_r(x))}{g_R(A_s(y),X_0)}\\
&\approx \lim_{s\to 0}\lim_{R\to\infty}\frac{\omega_R^{A_r(x)}(\Delta(y,s))}{\omega_R(\Delta(y,s))}\\
&= \lim_{s\to 0}\frac{\omega^{A_r(x)}(\Delta(y,s))}{\omega(\Delta(y,s))} = K(A_r(x),y)\,,
\end{aligned}
$$

where in the next to last step we have used that $\omega_R \to \omega$ in the sense of Radon measures.

Hence, by Lemma 2 in [FJK] we get

$$
\begin{aligned}
K(A_r(x),y) \approx \limsup_{R\to\infty} K_R(A_r(x),y) &\lesssim \lim_{R\to\infty}\frac{1}{\omega_R(\Delta)}\left(\max\left\{\left(\frac{|y-x|}{r}\right)^{\alpha},1\right\}\right)\\
&= \frac{1}{\omega(\Delta)}\left(\max\left\{\left(\frac{|y-x|}{r}\right)^{\alpha},1\right\}\right).
\end{aligned}
$$

∎

Lemma 3.32. *Let $r < 1$ and $n \geqslant 2$. If $x \in \partial\mathbb{R}^{n+1}_+$ and if $A_r(x)$ is the Corkscrew point corresponding to $\Delta(x,r)$ then*

$$
\omega_R^Y(\Delta(x,r)) \geqslant C_0, \quad \forall Y \in B\left(A_r(x), \frac{1}{4}r\right),
$$

where again ω_R is the harmonic measure corresponding to the elliptic operator $L = -\operatorname{div} A\nabla$ over the domain Ω_R and the constant C_0 is not dependent on r, x.

Proof. See the proof of Lemma 3 in [FJK] for more details. ∎

Notice that the estimate $\omega_R^Y(\Delta(x,r)) \geqslant C_0$ is independent on R, so by passing to the limit as $R \to \infty$ we get the following corollary for the harmonic measure ω.

Corollary 3.33. *Let $r < 1$ and $R > 2$. If $x \in \partial\mathbb{R}^{n+1}_+$ and if $A_r(x)$ is the Corkscrew point corresponding to $\Delta(x, r)$ then*

$$\omega^Y(\Delta(x, r)) \geqslant C_0, \quad \forall Y \in B\left(A_r(x), \frac{1}{4}r\right).$$

Lemma 3.34. *Let $A = A_r(x_0)$, $x_0 \in \partial\mathbb{R}^{n+1}_+$, $\Delta_j = \Delta(x_0, 2^j r), R_j = \Delta_j \setminus \Delta_{j-1}$ for $j \geqslant 1$. Then,*

$$\operatorname*{ess\,sup}_{x \in R_j} K(A, x) \leqslant M 2^{-\alpha j} \frac{1}{\omega(\Delta_j)},$$

for some uniform constant M depending on allowable parameters.

Proof. Let Δ' be a small surface. We will estimate $\omega^A(\Delta')/\omega(\Delta')$. Let $A_j = A_{2^j r}(x_0)$. By (3.80), $\frac{\omega_R(\Delta')}{\omega_R(\Delta_j)} \approx \omega_R^{A_j}(\Delta')$. On the other hand, Lemma 3.15 and from (3.79) as well as (3.80) show that

$$\omega_R^A(\Delta') \leqslant M \omega_R^{A_j}(\Delta') \left(\frac{|A - x_0|}{2^j r}\right)^\alpha \leqslant M \frac{\omega_R(\Delta')}{\omega_R(\Delta_j)} 2^{-j\alpha}.$$

By the limit argument as in the Lemma 3.31, we get the desired inequality. ∎

Next, we give the following definition, which was first introduced in [KKoPT]

Definition 3.35. Let $Q_0 \in \mathbb{D}(\mathbb{R}^n)$, where $\mathbb{D}(\mathbb{R}^n)$ is the collection of all dyadic grids of $\mathbb{R}^n$ with side length 2^{-k} for $k \in \mathbb{Z}$. We define a good-ϵ_0 cover of E, of length k, is a collection $\{O_l\}_{l=1}^k$ of nested (relatively) open subsets of Q_0, together with corresponding collection $\mathcal{F} = \{Q_i^l\}_i \subset \mathbb{D}_{Q_0}$, such that

$$O_1 \supset O_2 \supset .... \supset O_{k-1} \supset O_k \supset E, \tag{3.82}$$

$$O_l = \bigcup_{\mathcal{F}_l} Q_i^l, \tag{3.83}$$

and

$$\omega\left(O_l \bigcap Q_i^{l-1}\right) \leqslant \epsilon_0 \ \omega(Q_i^{l-1}), \quad \forall Q_i^{l-1} \in \mathcal{F}_{l-1}. \tag{3.84}$$

We note that by iteration argument, (3.84) implies that

$$\omega(O_l \cap Q_i^m) \leqslant \epsilon_0^{l-m}\omega(Q_i^m), \quad m \leqslant l \leqslant k. \tag{3.85}$$

We omit the details.

Give a dyadic $Q \in \mathbb{D}(\partial\mathbb{R}_+^{n+1})$, the **discretized Carleson region** $\mathbb{D}_Q$ is defined to be

$$\mathbb{D}_Q := \left\{Q' \in \mathbb{D} : Q' \subset Q\right\}. \tag{3.86}$$

Now, for each $Q \in \mathbb{D}(\partial\mathbb{R}_+^{n+1})$, and for a small $\gamma > 0$, we define the "Whitney region"

$$U_Q := Q \times \left(\frac{1}{2}l(Q), l(Q)\right). \tag{3.87}$$

We may then define the *Carleson box* associated to Q, by

$$T_Q := \operatorname{int}\left(\bigcup_{Q' \in \mathbb{D}_Q} U_{Q'}\right),$$

Given $x \in \partial\mathbb{R}_+^{n+1}$, we define the cone with vertex at x by

$$\Gamma(x) := \bigcup_{Q \in \mathbb{D}(\partial\mathbb{R}_+^{n+1}): x \in Q} U_Q.$$

We recall the following result from [KKiPT], see the proof in [KKiPT] for more details with notice that in their proof, they only need the doubling property and Lemma 3.34

Lemma 3.36. *Let $E \subset Q_0$. Given $\epsilon_0 > 0$ sufficient small, there exists $\delta_0 > 0$ such that if $\omega(E) \leqslant \delta_0$, then E has a good-ϵ_0 cover of length k, with*

$$k \approx \frac{log(1/\omega(E))}{\log(1/\epsilon_0)}.$$

Remark 3.37. We fix a small dyadic number $\eta := 2^{-k_0}$ to be chosen, and given $Q \in \mathbb{D}(\mathbb{R}^n)$, we consider the "$k_0$-**grandchildren**" of Q, i.e., the sub-cubes $Q' \subset Q$ with length $l(Q') = \eta l(Q)$. We let $\widetilde{Q}$ denote the particular such grandchild that contains the center x_Q of the cube Q.

Consider the special case that $Q = Q_i^l \in \mathcal{F}_l$, arising in some good-$\epsilon_0$ cover. We then set $\widetilde{Q}_i^l := \widetilde{Q}$, defined as in the previous paragraph, and we further define

$$\widetilde{O}_l := \bigcup_{Q_i^l \in \mathcal{F}_l} \widetilde{Q}_i^l.$$

Let us state the following Lemma which is stated in [KKiPT] whose proof will be given here based on their proof.

Lemma 3.38. *Let $M \gg 1$, and let $\delta_0 > 0$. Suppose that $E \subset Q_0$, and that $\omega(E) \leqslant \delta_0$. If δ_0 is sufficient small, then there is a Borel set $\mathcal{B}$, such that for every $x \in E$, and $Y = (y, t)$the bounded harmonic function defined by $u(Y) := \omega^Y(\mathbb{1}_{\mathcal{B}})$ satisfies*

$$M \leqslant S_T(u)(x).$$

Here, we recall that the truncated square function S_T is defined as following

$$S_T(u)(x) = \left(\int_0^{\ell(Q_0)} \int_{|x-y|<t} |\nabla u(y,t)|^2 \mu(y) \frac{t}{\mu(\Delta(y,t))} dydt \right)^{\frac{1}{2}}. \tag{3.88}$$

Proof. We fix a positive number ϵ_0 to be specific below, and by Lemma 3.36, we may choose $\delta_0 = \delta_0(M)$ (also to be specific below) so that E has a good-ϵ_0 cover length

$$k \approx k(M) \approx \frac{\log(\omega(E))}{\log \epsilon_0}.$$

We set

$$F := \sum_{j=2}^{k} \mathbb{1}_{\widetilde{O}_{j-1} \backslash O_j}. \tag{3.89}$$

Claim 1: F only takes values 0 and 1. Hence $F = \mathbb{1}_S$ for some Borel set $\mathcal{B}$.

Proof of Claim 1: We fix a point x for which $F(x) \neq 0$, so that necessarily, there is an $l \in [2, k]$ such that $\mathbb{1}_{\widetilde{O}_{l-1} \setminus O_l}(x) = 1$. Let l_0 denote the least such index l. Then,

$$\mathbb{1}_{\widetilde{O}_{l_0-1} \setminus O_{l_0}} = 1,$$

implies $x \in \widetilde{O}_{l_0-1} \setminus O_{l_0}$, so $x \notin O_{l_0}$, hence $x \notin O_l$ for all $l > l_0$. Thus $\mathbb{1}_{\widetilde{O}_{l-1} \setminus O_l}(x) = 0$ for all $l > l_0$ and the claim follows.

Remark 3.39. We note that for every $x \in E$, one also has that $x \in O_l, l = 1, 2, ..., k$ and therefore, for each l, there is a cube $Q_i^l \in \mathcal{F}_l$ that contains x. With x fixed, we let $\widehat{Q}_i^l$ denote the particular k_0-**grandchild** (as defined in Remark 3.37 above) of Q_i^l that contains x.

Remark 3.40. Given $Q \subset Q_0$, as above we let X_Q denote a fixed Corkscrew point relative to Q, and we let $\widetilde{X}_Q$ be a fixed Corkscrew point relative to the sub-cube $\widetilde{Q}$ defined in Remark 3.37; equivalently $\widetilde{X}_Q$ is a Corkscrew point relative to the ball $B(x_q, c\eta l(Q))$, where η is the small number fixed in Remark 3.37, and where x_Q is the center of Q.

Set $u(X) := \omega^X(\mathcal{B})$, where $\mathcal{B}$ is the Borel set whose characteristic function equals the function F defined in (3.89). Thus

$$u(X) = \int_{\mathbb{R}^n} F(y) d\omega^X(y) = \int_{\mathbb{R}^n} F(y) K(X, y) d\omega(y), \tag{3.90}$$

where $K(X, y) = \frac{d\omega^X}{d\omega}$.

Claim 2: There is a positive number a, depending only on the allowable parameters, such that if ϵ_0 and η are chosen sufficient small, then

$$\left| u\left(\widetilde{X}_{Q_i^l}\right) - u\left(\widetilde{X}_{\widehat{Q}_i^l}\right) \right| \geqslant a. \tag{3.91}$$

where we have fixed a point $x \in E$, an index $l \in [1, k]$, and the particular Q_i^l that contains x, and where $\widehat{Q}_i^l$ is defined as in Remark 3.39.

Proof of Claim 2: We split the domain $\mathbb{R}^n$ into two domains as below

$$u\left(\widetilde{X}_{Q_i^l}\right) = \int_{\mathbb{R}^n\setminus Q_i^l} F(y)d\omega^{\widetilde{X}_{Q_i^l}}(y) + \int_{Q_i^l} F(y)d\omega^{\widetilde{X}_{Q_i^l}}(y) =: I + II.$$

We note that, by Hölder continuity at the boundary, i.e., Lemma 3.15, we obtain $|I| \lesssim \eta^\alpha$. To estimate the second term, by the definition of F,

$$\begin{aligned} II &= \sum_{j=2}^{l}\int_{Q_i^l}\mathbb{1}_{\widetilde{O}_{j-1}\setminus O_j}d\omega^{\widetilde{X}_{Q_i^l}} + \sum_{j=l+2}^{k}\int_{Q_i^l}\mathbb{1}_{\widetilde{O}_{j-1}\setminus O_j}d\omega^{\widetilde{X}_{Q_i^l}} + \int_{Q_i^l}\mathbb{1}_{\widetilde{O}_l\setminus O_{l+1}}d\omega^{\widetilde{X}_{Q_i^l}} \\ &=: II_1 + II_2 + II_3. \end{aligned}$$

We treat these terms in order. First, we observe that $II_1 = 0$. Indeed, for $j \leqslant l$, we have $Q_i^l \subset O_l \subset O_j$, and therefore, $(\widetilde{O}_{j-1}\setminus O_j)\cap Q_i^l = \emptyset$.

Next, by Harnack's inequality (3.78) and Lemma 3.31 together the representation of u in the last equation of (3.90),

$$\begin{aligned} |II_2| &\leqslant \frac{C_\eta}{\omega(Q_i^l)}\sum_{j=2}^{l}\omega\left(\left(\widetilde{O}_{j-1}\setminus O_j\right)\cap Q_i^l\right) \\ &\leqslant \frac{C_\eta}{\omega(Q_i^l)}\sum_{j=2}^{l}\omega\left(O_{j-1}\cap Q_i^l\right) \\ &\leqslant \frac{C_\eta}{\omega(Q_i^l)}\sum_{j=2}^{l}\epsilon_0^{j-1-l}\omega(Q_i^l) \leqslant C_\eta\epsilon_0, \end{aligned}$$

where the last inequality we have used (3.85).

Now, we turn to term II_3. Observe that $Q_i^l \cap \widetilde{O}_l = \widetilde{Q}_i^l$ by the definition of $\widetilde{O}_l$. Therefore,

$$II_3 = \int_{\widetilde{Q}_i^l} d\omega^{\widetilde{X}_{Q_i^l}} - \int_{\widetilde{Q}_i^l\cap O_{l+1}} d\omega^{\widetilde{X}_{Q_i^l}} =: II_3' - II_3''.$$

By the similar argument as for estimate of the term II, from Harnack's inequality (3.78), Lemma 3.31 and (3.84),

$$II_3'' \leqslant \frac{C_\eta}{\omega(Q_i^l)}\omega\left(O_{l+1}\cap\widetilde{Q}_i^l\right) \leqslant \frac{C_\eta}{\omega(Q_i^l)}\omega\left(O_{l+1}\cap Q_i^l\right) \leqslant C_\eta\epsilon_0.$$

Note that, $II_3' = \widetilde{u}(\widetilde{X}_{Q_i^l})$, where $\widetilde{u}(X) := \omega^X(\widetilde{Q}_i^l)$. Therefore, by (3.33), $II_3' \geqslant C_0$. Choose $Q' \in \mathbb{D}(\partial\mathbb{R}^{n+1}_+)$, distinct from $\widetilde{Q}_i^l$, such that $l(Q') = l(\widetilde{Q}_i^l) \geqslant \operatorname{dist}(Q', \widetilde{Q}_i^l)$. Then by (3.33) and Harnack's inequality, for some $c_1 \approx C_0$, we have $\omega^{\widetilde{X}_{Q_i^l}}(Q') \approx \omega^X(Q') \geqslant c_1$. Consequently,

$$\widetilde{u}(X) \leqslant 1 - \omega^X(\mathbb{R}^n \setminus \widetilde{Q}_i^l) \leqslant 1 - \omega^X(Q') \leqslant 1 - c_1.$$

Without loss of generality, we may suppose that $c_1 \leqslant C_0$, so that $c_1 \leqslant II_3' \leqslant 1 - c_1$. By our previous estimates,

$$u(\widetilde{X}_{Q_i^l}) = II_3' + O(C_\eta \epsilon_0) + O(\eta^\alpha),$$

whence it follows that

$$\frac{3}{4}c_1 \leqslant u(\widetilde{X}_{Q_i^l}) \leqslant 1 - \frac{3}{4}c_1, \tag{3.92}$$

provided that first η, and then ϵ_0 (depending on η), are chosen small enough.

Consider now

$$u\left(\widetilde{X}_{\widehat{Q}_i^l}\right) = \int_{\mathbb{R}^n \setminus \widehat{Q}_i^l} F(y)\, d\omega^{\widetilde{X}_{\widehat{Q}_i^l}}(y) + \int_{\widehat{Q}_i^l} F(y)\, d\omega^{\widetilde{X}_{\widehat{Q}_i^l}}(y) =: \widehat{I} + \widehat{II}.$$

As we did with the term I above, by Lemma 3.15, we obtain $\widehat{I} \lesssim \eta^\alpha$. We then also split $\widehat{II}$ in the same way as we did term II, i.e.,

$$\begin{aligned} \widehat{II} &= \sum_{j=2}^{l} \int_{\widehat{Q}_i^l} \mathbb{1}_{\widetilde{O}_{j-1} \setminus O_j}\, d\omega^{\widetilde{X}_{\widehat{Q}_i^l}} + \sum_{j=l+2}^{k} \int_{\widehat{Q}_i^l} \mathbb{1}_{\widetilde{O}_{j-1} \setminus O_j}\, d\omega^{\widetilde{X}_{\widehat{Q}_i^l}} + \int_{\widehat{Q}_i^l} \mathbb{1}_{\widetilde{O}_l \setminus O_{l+1}}\, d\omega^{\widetilde{X}_{\widehat{Q}_i^l}} \\ &=: \widehat{II}_1 + \widehat{II}_2 + \widehat{II}_3. \end{aligned}$$

Exactly as with terms II_1 and II_2, terms $\widehat{II}_1$ and $\widehat{II}_2$ equal 0 and $O(C_\eta \epsilon_0)$ respectively.

The main term is $\widehat{II}_3$. Observe that

$$\widehat{Q}_i^l \bigcap \left(\widetilde{O}_l \setminus O_{l+1}\right) = \left(\widehat{Q}_i^l \cap \widetilde{Q}_i^l\right) \setminus O_{l+1},$$

and either $\widehat{Q}_i^l \cap \widetilde{Q}_i^l = \emptyset$ or $\widehat{Q}_i^l = \widetilde{Q}_i^l$. In the former case, $\widehat{II}_3 = 0$. Otherwise,

$$\widehat{II}_3 = \int_{\widehat{Q}_i^l} d\omega^{\widetilde{X}_{\widehat{Q}_i^l}} - \int_{\widehat{Q}_i^l \cap O_{l+1}} d\omega^{\widetilde{X}_{\widehat{Q}_i^l}} =: \widehat{II}_3^{'} - \widehat{II}_3^{''}.$$

$\widehat{II}_3^{''}$ is treated identical to the term $II_3^{''}$ by using Lemma 3.31, (3.84) and Harnack's inequality

$$\widehat{II}_3^{''} \leqslant \frac{C_\eta}{\omega(\widehat{Q}_i^l)} \omega(Q_i^l \cap O_{l+1}) \leqslant C_\eta \epsilon_0.$$

Finally, we get

$$\widehat{II}_3^{'} = \omega^{\widetilde{X}_{\widehat{Q}_i^l}} \left(\widehat{Q}_i^l\right) \geqslant 1 - C\eta^\alpha,$$

by Lemma 3.15 applied to $\omega^{\widetilde{X}_{\widehat{Q}_i^l}} \left(\partial \mathbb{R}_+^{n+1} \setminus \widehat{Q}_i^l\right)$.

Combine our estimates, we find that either

$$0 \leqslant u\left(\widetilde{X}_{\widehat{Q}_i^l}\right) \leqslant C\eta^\alpha + C_\eta \epsilon_0, \quad \text{or} \quad u\left(\widetilde{X}_{\widehat{Q}_i^l}\right) \geqslant 1 - \left(C\eta^\alpha + C_\eta \epsilon_0\right). \tag{3.93}$$

In either case, choosing η first, and then ϵ_0, sufficient small, and combining (3.92) and (3.93), we establish Claim 2, with say $a = c_1/2$.

We now return to the proof of Lemma 3.38. Recalling the definition of the Whitney region U_Q (3.87), we define

$$U_Q^\eta := \bigcup_{Q' \subset Q: l(Q') \geqslant \eta^3 l(Q)} U_{Q'}.$$

By the weighted Poincaré inequality (4.12), we have

$$\iint_{U_Q^\eta} \left| f - f_{U_Q^\eta} \right|^2 \mu(x) dx dt \leqslant C_\eta l(Q)^2 \iint_{U_Q^\eta} |\nabla f|^2 \mu(x) dx dt,$$

for all $f \in W^{1,2}(U_Q^\eta)$, where in general $f_U := \frac{1}{l(U)\mu(U)} \iint_U f(x)\mu(x) dx dt$ is the mean value of f on U with the weight μ. Combining the above Poincaré estimate with (3.91), we find that for $\widetilde{\mu}(Q \times I) = \mu(Q) \times l(I)$

$$
\begin{aligned}
a^2 &\leqslant C_\eta \frac{1}{\widetilde{\mu}(U^\eta_{Q^l_i})} \iint_{U^\eta_{Q^l_i}} \left| u\left(\widetilde{X}_{Q^l_i}\right) - u\left(\widetilde{X}_{\widehat{Q}^l_i}\right) \right|^2 \widetilde{\mu}(y) dy dt \\
&\leqslant C_\eta \frac{\left(l(U^\eta_{Q^l_i}) \right)^2}{\widetilde{\mu}(U^\eta_{Q^l_i})} \iint_{U^\eta_{Q^l_i}} |\nabla u|^2 \widetilde{\mu}(y) dy dt \\
&\leqslant C_\eta \iint_{U^\eta_{Q^l_i}} |\nabla u|^2 \frac{l(Q^l_i)}{\mu(Q^l_i)} \mu(x) dx dt,
\end{aligned}
$$

provided that Q^l_i contains some $x \in E$.

Consequently, if we fix x and sum over all indicies i, l such that $x \in Q^l_i$, we find that for all $x \in E$

$$
\frac{a^2}{C_\eta} k \leqslant \sum_{i,l} \iint_{U^\eta_{Q^l_i}} |\nabla u|^2 \frac{l(Q^l_i)}{\mu(Q^l_i)} \mu(x) dx dt \lesssim S_T(u)(x).
$$

because the Whitney region $U^\eta_{Q^l_i}$ has bounded overlaps.

Since $k \approx \frac{\log(1/\omega(E))}{\log(1/\epsilon_0)} \geqslant \frac{\log(1/\delta_0)}{\log(1/\epsilon_0)}$ where η, ϵ_0 have been now fixed, and δ_0 is at our disposal, we obtain the conclusion of Lemma 3.38 by specifying δ_0 small enough. ∎

Theorem 3.41. *Let* $-\operatorname{div} A\nabla$ *be a degenerate elliptic operator defined as before. Then the harmonic measure corresponding to* $-\operatorname{div} A\nabla$ *belongs to Muckenhoupt* A_∞ *class with respect to surface measure on the boundary.*

Proof. With Lemma 3.38 in hand, we may now prove our theorem. By Lemma 3.38, we have that

$$
M^2 \mu(E) \leqslant \int_E |S_T(u)|^2 \mu \leqslant \int_{Q_0} S_T(u)^2 \mu \lesssim \iint_{T^{\text{fat}}_{Q_0}} |\nabla u(y,s)|^2 \mu(y) dy ds \lesssim \mu(Q_0).
$$

where we have applied the Fubini's Theorem to get the third inequality and the last step is the CME result with the fact that the normalized solution u satisfying $\|u\|_\infty = 1$. Since

$M \to \infty$, as $\delta_0 \to 0$, we obtain that for every $\gamma > 0$, there is a positive δ_0 such that

$$\omega(E) \leqslant \delta_0 \approx \delta_0 \omega(Q_0) \Longrightarrow \mu(E) \leqslant \gamma \mu(Q_0).$$

This implies that $\omega \in A_\infty(\mu)$. By transitive property of A_∞ and note that $\mu \in A_2(\sigma)$, we get $\omega \in A_\infty(\sigma)$, where σ denote the surface measure on the boundary. ∎

Since the harmonic measure ω belongs to $A_\infty(\sigma)$ weight condition, we may invoke the result by Dahlberg, Jerison and Kenig in [DJK] to get that for u is a solution of the equation $-\operatorname{div} A\nabla u = 0$ then the Square Function $S(u)$ is bounded by Non-tangential Maximal function $N_*(u)$ in L^p_μ space ("$S < N$" problem), where the Square Function $S(u)$ is defined as follow

$$S(u)(x) := \left(\iint_{\Gamma(x)} |\nabla u(y,t)|^2 \mu(y) \frac{t}{\mu(\Delta(y,t))} dydt \right)^{\frac{1}{2}}.$$

Actually we have a little more stronger results than the L^p_μ bounds. Here is the corollary that we get from [DJK].

Corollary 3.42. *Let $\Phi : [0,\infty) \to [0,\infty)$ be an unbounded, non-decreasing, continuous function satisfying $\Phi(0) = 0$ and $\Phi(2t) \leqslant C\Phi(t)$. There is positive constant c such that if u is a solution of $-\operatorname{div} A\nabla u = 0$ then*

$$\int_{\partial\mathbb{R}^{n+1}_+} \Phi(N_*(u)) d\mu(x) \leqslant c \int_{\partial\mathbb{R}^{n+1}_+} \Phi(S(u)) d\mu(x).$$

Moreover, if the solution u is vanishing at some fix point X_0, we also have the reverse result

$$\int_{\partial\mathbb{R}^{n+1}_+} \Phi(S(u)) d\mu(x) \leqslant c_1 \int_{\partial\mathbb{R}^{n+1}_+} \Phi(N_*(u)) d\mu(x).$$

for some constant c_1.

Next, we would like to give estimate for the Radon – Nikodym derivative with respect to μ measure and surface measure σ.

Lemma 3.43. *For any $Z \in \mathbb{R}^{n+1}_+$. Let $k^Z(x) = \frac{d\omega^Z}{d\sigma}(x)$. There is some $1 < p < \infty$ such that for all $y \in \partial\mathbb{R}^{n+1}_+$, k^Z satisfies*

$$\left(\frac{1}{\sigma(\Delta(y,s))}\int_{\Delta(y,s)} \left|k^Z\right|^p d\sigma\right)^{\frac{1}{p}} \leqslant \frac{1}{\sigma(\Delta(y,s))}\int_{\Delta(y,s)} k^Z d\sigma \leqslant \frac{1}{\sigma(\Delta(y,s))}.$$

Proof. This is a consequence of the result $\omega \in A_\infty(d\sigma)$. ∎

Since we also have $\omega \in A_\infty(d\mu)$, this implies the following lemma.

Lemma 3.44. *For any $Z \in \mathbb{R}^{n+1}_+$. Let $\tilde{k}^Z(x) = \frac{d\omega^Z}{d\mu}(x)$. There is some $1 < p_0 < \infty$ such that for all $y \in \partial\mathbb{R}^{n+1}_+$, $\tilde{k}^Z$ satisfies*

$$\left(\frac{1}{\mu(\Delta(y,s))}\int_{\Delta(y,s)} \left|\tilde{k}^Z\right|^{p_0} d\mu\right)^{\frac{1}{p_0}} \leqslant \frac{1}{\mu(\Delta(y,s))}\int_{\Delta(y,s)} \tilde{k}^Z d\mu \leqslant \frac{1}{\mu(\Delta(y,s))}.$$

Theorem 3.45. *There exists solution of $-\operatorname{div} A\nabla u = 0$ with data $f \in C_c(\mathbb{R}^n)$ that $u(|X|) \to 0$ uniformly as $X \to \infty$. Moreover, that such solution is unique.*

Proof. By our construction, $u(X) = \int_{\mathbb{R}^n} f(y)d\omega^X$ is a solution of $Lu = 0$. Suppose that $\operatorname{supp} f \subset Q_0 \subset \mathbb{R}^n$. For any $X = (x,t) \in \mathbb{R}^{n+1}_+$and for $|X|$ large enough, we have $\operatorname{supp}(f) \subset \Delta(0, 2|X|)$ and so for $\frac{1}{q_0} + \frac{1}{p_0} = 1$ get

$$\begin{aligned} u(X) &= \int_{\mathbb{R}^n} f(y)d\omega^X \\ &= \int_{\mathbb{R}^n} f(y)\tilde{k}^X(y)d\mu \\ &\leqslant \|f\|_\infty \mu(Q_0))^{\frac{1}{q_0}} \left(\int_{\Delta(0,2|X|)} |\tilde{k}^X|^{p_0}(y)d\mu\right)^{\frac{1}{p_0}} \\ &\leqslant \|f\|_\infty \mu(Q_0))^{\frac{1}{q_0}} \mu(\Delta(0,2|X|))^{\frac{1}{p_0}-1} \end{aligned}$$

by the Lemma 3.44 for the last inequality.

So, we have $\lim_{|X|\to\infty} u(X) = 0$ or more precisely that given any $\epsilon > 0$, there is N large such that $u(X) < \epsilon$ for all $|X| > N$. Together with the maximal principle, we have the uniqueness of the solution. ∎

Lemma 3.46. *For q_0 is the conjugate of p_0 as in the Lemma 3.44 above. Suppose $f \in L^{q_0}_\mu(\mathbb{R}^n)$ and*

$$u(X) = \int_{\mathbb{R}^n} f(y)d\omega^X,$$

then

$$\|N_* u\|_{L^{q_0}_\mu(\mathbb{R}^n)} \leqslant C\|f\|_{L^{q_0}_\mu(\mathbb{R}^n)}.$$

Proof. Fix $x_0 \in \mathbb{R}^n$, let $(x,t) \in \Gamma(x_0)$. Write

$$f = \sum_{k=0}^{\infty} f_k,$$

where $f_0 = f\mathbb{1}_{\Delta(x_0,4t)}$ and $f_i = f\mathbb{1}_{\Delta_i}, \Delta_i = \Delta(x_0, 2^i t) \setminus \Delta(x_0, 2^{i-1}t)$ for all $i \geqslant 1$.

WLOG, we may assume that $f \geqslant 0$, otherwise, decompose $f = f^+ - f^-$. Set,

$$u_i(X) := \int_{\mathbb{R}^n} f_i(y)d\omega^X = \int_{\mathbb{R}^n} f_i(y)\tilde{k}^X d\mu.$$

We consider u_0 first, by Harnack's inequality and Hölder's inequality with pairing $\tilde{p}, \tilde{q}$ satisfying $\tilde{q} < q_0$ where $\tilde{q}$ will be chosen later

$$\begin{aligned} u_0(x,t) \approx u_0(x,4t) &= \int_{\mathbb{R}^n} f_0(y)\tilde{k}^{(x,4t)}d\mu \\ &\leqslant \left(\int_{\Delta(P,s)} \left|\tilde{k}^{(x,4t)}\right|^{\tilde{p}} d\mu\right)^{1/\tilde{p}} \left(\int_{\mathbb{R}^n} |f_0|^{\tilde{q}} d\mu(x)\right)^{1/\tilde{q}} \\ &\leqslant \left[\mu(\Delta(x_0,4t))\right]^{-1+1/\tilde{p}} \left(\int_{\Delta(x_0,4t)} |f_0|^{\tilde{q}} d\mu(x)\right)^{1/\tilde{q}} \\ &\leqslant \left(M_\mu |f|^{\tilde{q}}\right)^{1/\tilde{q}}(x_0). \end{aligned}$$

Denote $R_i = \Delta(x_0, 2^i t) \times (0, 2^i t)$, we claim that for any $i > 0$

$$u_i(x,t) \lesssim 2^{-i\alpha}\left(M_\mu |f|^{\tilde{q}}\right)^{1/\tilde{q}}(x_0),$$

where α is the same in the exponent of Hölder continuity of solution of $Lu = 0$.

Indeed, by Hölder continuity at the boundary and (3.79)

$$u_i(x,t) \leqslant C\left(\frac{t}{2^i t}\right)^{\alpha} \sup_{R_i} u_i \leqslant C2^{-i\alpha} u_i(x_0, 2^i t).$$

Hence,

$$\begin{aligned}
u_i(x,t) &\leqslant C2^{-i\alpha}\int_{\Delta_i} f_i \tilde{k}^{(x_0,2^i t)} d\mu \\
&\leqslant C2^{-i\alpha}\left(\int_{\Delta(x_0,2^i t)} \left|\tilde{k}^{(x_0,4t)}\right|^{\tilde{p}} d\mu\right)^{1/\tilde{p}} \left(\int_{\Delta(x_0,2^i t)} |f_i|^{\tilde{q}} d\mu(x)\right)^{1/\tilde{q}} \\
&\leqslant C2^{-i\alpha}\left(\frac{1}{\mu(\Delta(x_0,2^i t))}\int_{\Delta(x_0,2^i t)} |f_i|^{\tilde{q}} d\mu(x)\right)^{1/\tilde{q}} \\
&\leqslant C2^{-i\alpha}\left(M_\mu |f|^{\tilde{q}}\right)^{1/\tilde{q}}(x_0).
\end{aligned}$$

Hence, we just proved that $N_* u \leqslant C\left(M_\mu |f|^{\tilde{q}}\right)^{1/\tilde{q}}$. Notice that since $\tilde{k}^X \in RH_{p_0}$, so there is some $\tilde{p} > p_0$ such that $\tilde{k}^X \in RH_{\tilde{p}}$ and also $\tilde{q} < q_0$ where $\tilde{q}$ (q_0, respectively) the conjugate of $\tilde{p}$ (p_0, respectively).

Therefore, $\|N_* u\|_{L^{q_0}_\mu} \leqslant C\|f\|_{L^{q_0}_\mu}$. ∎

Theorem 3.47. *The Dirichlet problem*

$$(D)_{q_0}\begin{cases} -\operatorname{div} A\nabla u = 0 \quad in \quad \mathbb{R}^{n+1}_+, \\ u|_{\mathbb{R}^n} = f \quad \textit{non-tangential for} \quad f \in L^{q_0}(\mathbb{R}^n), \\ \|N_* u\|_{L^{q_0}_\mu} \leqslant C\|f\|_{L^{q_0}_\mu}, \end{cases}$$

is solvable where q_0 is as in Lemma 3.46.

Proof. Enough to prove the non-tangential convergence. Fix $\epsilon > 0$. Recall that the set of continuous function with compact support $C_0(\mathbb{R}^n)$ is dense in $L^{q_0}_\mu$, let $g \in C_0(\mathbb{R}^n)$ such that $\|f-g\|_{L^{q_0}_\mu} < \epsilon$. Let

$$\begin{aligned}
u_f(X) &= \int_{\mathbb{R}^n} f(y)\, d\omega^X. \\
u_g(X) &= \int_{\mathbb{R}^n} g(y)\, d\omega^X.
\end{aligned}$$

Then for any $x \in \mathbb{R}^n$, let $(y,t) \in \Gamma(x)$, we obtain

$$\begin{aligned} I(x) &= \lim_{(y,t) \xrightarrow{\text{n.t.}} x} |u_f(y,t) - f(x)| \\ &\leqslant \lim_{(y,t) \xrightarrow{\text{n.t.}} x} |u_f(y,t) - u_g(y,t)| + \lim_{(y,t) \xrightarrow{\text{n.t.}} x} |u_g(y,t) - g(x)| + |g(x) - f(x)|. \end{aligned}$$

We see that

$$\lim_{(y,t) \xrightarrow{\text{n.t.}} x} |u_g(y,t) - g(x)| = 0,$$

so it is enough to estimate

$$\lim_{(y,t) \xrightarrow{\text{n.t.}} x} |u_f(y,t) - u_g(y,t)| = \lim_{(y,t) \xrightarrow{\text{n.t.}} x} |u_{(f-g)}(y,t)|.$$

By Lemma 3.46, we get

$$\|N_*(u_{(f-g)})\|_{L^{q_0}_\mu} \leqslant C\|f-g\|_{L^{q_0}_\mu}.$$

Hence,

$$\|I\|_{L^{q_0}_\mu} \leqslant \|N_*(u_{(f-g)})\|_{L^{q_0}_\mu} + \|f-g\|_{L^{q_0}_\mu} \leqslant C\|f-g\|_{L^{q_0}_\mu} = C\epsilon.$$

For any $\eta > 0$,

$$\begin{aligned} \mu(E_\eta) &= \mu\left(\left\{x \in \mathbb{R}^n : \overline{\lim}_{(y,t) \xrightarrow{\text{n.t.}} x} |u_f(y,t) - f(x)| > \eta\right\}\right) \\ &\leqslant \mu\left(\{x \in \mathbb{R}^n : |I(x)| > \eta\}\right) \\ &\leqslant C\frac{1}{q_0}\|I\|_{L^{q_0}_\mu} \leqslant C\frac{\epsilon}{\eta^{q_0}}. \end{aligned}$$

Since ϵ is arbitrary, we have that $\mu\left(E_\eta\right) = 0$. This implies the non-tangential convergence.

∎

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Chapter 4

L^p bounds of Riesz transform and Vertical square functions for degenerate elliptic operators

4.1 Introduction

In this article, we present L^p bounds for the semi-groups $e^{-z\mathcal{L}_\mu}$ and their gradients $\sqrt{z}\nabla e^{-z\mathcal{L}_\mu}$, where $\mathcal{L}_\mu$ is a degenerate elliptic operator, in divergence form, with degeneracy controlled by a weight in the Muckenhoupt class A_2. From those results, we then get L^p bounds for Riesz transforms and square functions associated to $\mathcal{L}_\mu$. The A_2 degeneracy is natural in this context (in particular, the A_2 condition allows for appropriate weighted Sobolev inequalities), and has been widely studied, see, e.g., [FKS], [FJK1], [FJK2], and [C-UR2]. More precisely, suppose that $\mu \in A_2(\mathbb{R}^n)$ (see Definition 4.4 below), and let A be a $n \times n$ complex coefficient matrix-valued function defined on $\mathbb{R}^n$, satisfying the degenerate ellipticity condition

$$\begin{aligned} \lambda\,\mu(x)\,|\xi|^2 &\leqslant \operatorname{Re}\langle A(x)\xi, \xi\rangle, \\ |\langle A(x)\xi, \psi\rangle| &\leqslant \Lambda\,\mu(x)\,|\xi|\,|\psi|, \end{aligned}$$

for all $\xi, \psi \in \mathbb{C}^n$ and for some uniform λ and Λ, with $0 < \lambda < \Lambda < \infty$. We then set

$$Lu := -\operatorname{div} A\nabla u\,,$$

interpreted in the usual weak sense via a sesquilinear form on the weighted homogeneous Sobolev space $\dot{W}^{1,2}_{\mu}(\mathbb{R}^n) =: H^1_{\mu}(\mathbb{R}^n)$, which we define as the completion of $C_0^{\infty}(\mathbb{R}^n)$ with respect to the norm

$$\|f\|_{H^1_\mu} := \left(\int_{\mathbb{R}^n} |\nabla f(x)|^2\, \mu(x)\, dx\right)^{1/2}.$$

We also set

$$\mathcal{L}_\mu := \frac{1}{\mu} L = -\frac{1}{\mu} \operatorname{div} A\nabla.$$

From now on, we say that $x \lesssim y$ if there is a constant C depending only on the fixed parameters such that $x \leqslant Cy$, and $x \simeq y$ if there are constants $c < C$ depending on fixed parameters so that $cy \leqslant x \leqslant Cy$.

The main results in this paper are the following.

Theorem 4.1. *Consider an operator $\mathcal{L}_\mu$ as above. There exists $\epsilon > 0$ depending on dimension, the A_2 constant of μ, and the accretivity constants λ and Λ, such that for all p with $\left|\frac{1}{2} - \frac{1}{p}\right| < \epsilon$, we have*

$$\|e^{-t\mathcal{L}_\mu} f\|_{L^p_\mu(\mathbb{R}^n)} + \|\sqrt{t}\nabla e^{-t\mathcal{L}_\mu} f\|_{L^p_\mu(\mathbb{R}^n)} \leqslant C_p \|f\|_{L^p_\mu(\mathbb{R}^n)}.$$

We define vertical square functions adapted to $\mathcal{L}_\mu$ as follows:

$$g_{\mathcal{L}_\mu}(f)(x) = \left(\int_0^\infty \left|\mathcal{L}_\mu^{1/2} e^{-t\mathcal{L}_\mu} f(x)\right|^2 dt\right)^{\frac{1}{2}}.$$

$$G_{\mathcal{L}_\mu}(f)(x) = \left(\int_0^\infty \left|\nabla e^{-t\mathcal{L}_\mu} f(x)\right|^2 dt\right)^{\frac{1}{2}}.$$

We obtain L^p bounds for the vertical square functions.

Theorem 4.2. *There is an $\epsilon > 0$ depending on n, λ, Λ and $\|\mu\|_{A_2}$ such that for all $p \in [2, 2+\epsilon)$,*

$$\|g_{\mathcal{L}_\mu}(f)\|_{L^p_\mu} + \|G_{\mathcal{L}_\mu}(f)\|_{L^p_\mu} \leqslant C_p \|f\|_{L^p_\mu},$$

where C_p depends also on n, λ, Λ and μ.

We also consider the associated Riesz transforms, and Kato-type square roots.

Theorem 4.3. *There is an $\epsilon > 0$ depending on n, λ, Λ and $\|\mu\|_{A_2}$ such that*

$$\|\nabla f\|_{L^p_\mu} \simeq \|\mathcal{L}_\mu^{1/2} f\|_{L^p_\mu}, \quad 2 - \epsilon < p < 2 + \epsilon.$$

The case $p = 2$ has been treated previously in [C-UR2]. The new contribution in this paper is to show that there is some interval of p near 2 for which the results of [C-UR2] continue to hold. We present the proof of Theorem 4.2 in Section 5.3: Proposition 4.44 treats $g_{\mathcal{L}_\mu}$, and Proposition 4.46 treats $G_{\mathcal{L}_\mu}$. The proof of Theorem 4.3 is given in Section 4.5. These results were originally motivated by their use in the treatment of the Dirichlet problem for certain degenerate elliptic equations [HLM], but are perhaps of intrinsic interest as well. We note that the methods in the present paper are largely based on those of Auscher [A], adapted here to the degenerate setting.

As regards our restrictions on the range of p, let us recall that by an example of Kenig (see [AT, p. 119]), the upper limit $2 + \epsilon$ is optimal for the Riesz transform bound

$$\|\nabla f\|_{L^p_\mu} \lesssim \|\mathcal{L}_\mu^{1/2} f\|_{L^p_\mu},$$

even in the uniformly elliptic case with $\mu \equiv 1$. Moreover, for $\mu \in A_2$, one does not expect to get L^p bounds for $p < 2 - \epsilon$.

We note that while this manuscript was in preparation, we learned that our results, and some additional related ones, have been proved independently in the paper [C-UMR], where the authors obtain, in addition to the results presented here, a wider range of $p < 2$, assuming the stronger condition that $\mu \in A_p$. In addition, for $\mu \in A_1$, they obtain the *unweighted* L^2 bound

$$\int_{\mathbb{R}^n} |\nabla f|^2 \, dx \simeq \int_{\mathbb{R}^n} |\mathcal{L}_\mu^{1/2} f|^2 \, dx .$$

We have also learned that the case $2n/(n+1) < p < 2$ of Theorem 4.3, with $\mu \in A_p$, has been treated independently in [YZ].

4.2 Preliminaries

Throughout the paper, L and $\mathcal{L}_\mu$ will be defined as in the introduction, and μ will be a fixed weight in the Muckenhoupt class A_2. The A_p classes are defined as follows.

Definition 4.4. Let $1 < p < \infty$ and suppose μ be a locally integrable and positive function. We say μ satisfies Muckenhoupt $A_p(\mathbb{R}^n)$ weight if

$$\sup_B \left(\frac{1}{|B|} \int_B \mu(x) dx dx \right) \left(\frac{1}{|B|} \int_B |\mu(x)|^{\frac{-1}{p-1}} \, dx \right)^{p-1} \leqslant K < \infty ,$$

where the supremum runs over all balls $B \subset \mathbb{R}^n$ and $|B|$ denotes the Lebesgue measure of the set B.

Now, we recall some function spaces that we need for future arguments.

Definition 4.5. For any open set Ω and $0 < p < \infty$, we define

$$L^p_\mu(\Omega) = \left\{ f : \int_\Omega |f(z)|^p \mu(z) dz < \infty \right\},$$

with the norm $\|f\|_{L^p_\mu(\Omega)} := \left(\int_\Omega |f(z)|^p \mu(z) dz \right)^{\frac{1}{p}}$.

We also define

$$L^p_{\mu,\mathrm{loc}}(\Omega) := \left\{ f : \int_{\Omega'} |f(z)|^p \mu(z) dz < \infty, \forall \Omega' \subset\subset \Omega \right\}.$$

and the space $L^{p,\infty}_\mu$ for any $p \in (0,\infty)$ is defined to be the set of all measurable functions f such that

$$
\begin{aligned}
\|f\|_{L^{p,\infty}} &:= \inf\left\{C>0 : \mu\left(\{x : |f(x)|>\alpha\}\|\right) \leqslant \frac{C^p}{\alpha^p},\ \text{for all } \alpha>0\right\} \\
&= \sup\left\{\gamma\left(\mu\left(\{f(x)>\gamma\}\right)\right)^{\frac{1}{p}} : \gamma>0\right\}.
\end{aligned}
$$

Next, we give the definition of weak-type (p,p) bounds and strong type (p,q) bounds of an operator T

Definition 4.6. We say an operator T satisfies strong type (p,q) bounds if we have the following

$$\|Tf\|_{L^q_\mu} \leqslant C\|f\|_{L^p_\mu}$$

and we say it satisfies weak-type (p,q) bounds if

$$\|Tf\|_{L^{q,\infty}_\mu} \leqslant C\|f\|_{L^p_\mu}$$

We also would like to recall the definition of some version of Hardy-Littlewood maximal functions and also state some properties of them.

Definition 4.7. For any locally integrable function f, we define the centered Hardy-Littlewood maximal function $M(f)$ as following

$$M(f)(x) = \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(z)|dz,$$

where $|B(x,r)|$ is the Lebesgue measure of the ball $B(x,r)$.

Definition 4.8. For any locally integrable function $f \in L^1_{\mu,\text{loc}}$, we define the weighted centered Hardy- Littlewood maximal function $M_\mu(f)$ as following

$$M^c_\mu(f)(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))}\int_{B(x,r)} |f(z)|\mu(z)dz,$$

where $\mu(B) = \int_B \mu(z)dz$ - the μ measure of the set B.

Definition 4.9. For any locally integrable function $f \in L^1_{\mu,\text{loc}}$, we define the weighted uncentered Hardy-Littlewood maximal function $M_\mu(f)$ as following

$$M_\mu(f)(x) = \sup_{x \in B} \frac{1}{\mu(B)} \int_B |f(z)|\mu(z)dz.$$

Let us recall the following proposition of maximal functions. See more details in the book by L. Grafakos [G] for proofs of those results.

Proposition 4.10. *All the maximal functions $M_\mu(f)$, $M(f)$, $M^c_\mu(f)$ satisfy the following properties*

$$\left\| M_\mu(f) \right\|_{L^p_\mu} \lesssim \|f\|_{L^p_\mu}, \quad \left\| M^c_\mu(f) \right\|_{L^p_\mu} \lesssim \|f\|_{L^p_\mu}, \textit{and} \left\| M_\mu(f) \right\|_{L^p_\mu} \lesssim \|f\|_{L^p_\mu}$$

for all $1 < p < \infty$.

The following theorem is the solution of Kato problem for weighted operator which was proved in [C-UR2].

Theorem 4.11. *(weighted Kato problem, [C-UR2]) Let $\mathcal{L}_\mu$ be defined below, we have the following result*

$$\left\| \mathcal{L}^{1/2}_\mu f \right\|_{L^2_\mu(\mathbb{R}^n)} \simeq \|\nabla_x f\|_{L^2_\mu(\mathbb{R}^n)},$$

for all $f \in H^1_\mu(\mathbb{R}^n)$.

Let us state here some essential inequalities for the weighted setting. The first result is the weighted Poincaré inequality for which the proof was given in Theorem 1.5 of [FKS].

Theorem 4.12. *Let* $1 < p < \infty$ *and* $\mu \in A_p$ *Muckenhoupt class. Then there are positive constants* $c = c(p, \mu, n)$ *and* $\delta = \delta(p, \mu, n)$ *such that for all function* $u \in W^{1,p}_{\mu,loc}(\mathbb{R}^n)$ *and for all* $1 \leqslant k \leqslant \frac{n}{n-1} + \delta$,

$$\left(\frac{1}{\mu(B_R)} \int_{B_R} |u(x) - A_{B_R}|^{pk} \mu dx \right)^{\frac{1}{kp}} \leqslant cR \left(\frac{1}{\mu(B_R)} \int_{B_R} |\nabla u|^p \mu dx \right)^{\frac{1}{p}},$$

where either $A_{B_R} = \frac{1}{\mu(B_R)} \int_{B_R} u(x)\mu(x)dx$ *or* $A_{B_R} = \frac{1}{|B_R|} \int_{B_R} u(x)dx$ *and* $|B|$ *denotes the Lebesgue measure of the set* B.

The next theorem we need is the following weighted Sobolev inequality.

Theorem 4.13. *(Theorem 1.2, [FKS]) Take* $1 < p < \infty$ *and a function* $\mu \in A_p$. *There exists a positive number* δ *such that for all* $u \in C_0^\infty(B_R)$ *and all* k *satisfying* $1 \leqslant k \leqslant \frac{n}{n-1} + \delta$,

$$\left(\frac{1}{\mu(B_R)} \int_{B_R} |u|^{kp} \mu dx \right)^{\frac{1}{kp}} \leqslant CR \left(\frac{1}{\mu(B_R)} \int_{B_R} |\nabla u|^p \mu dx \right)^{\frac{1}{p}},$$

where C *may be taken to depend only on* n, *the* A_p *constant of* μ, p.

Lemma 4.14. *(Kolmogorov's inequality) Let* $f \in L^1_\mu(\mathbb{R}^n)$ *and* $0 < q < 1$. *Then we have*

$$\int_E \left[M_\mu(f)(x) \right]^q \mu \leqslant C_p \mu(E)^{1-q} \|f\|^q_{L^1_\mu},$$

whenever $\mu(E)$ *is finite.*

Proof. For a proof, see page 100 in [G].

∎

Now we want to give some properties of A_p weight condition in which we will need them later for our arguments. For a reference, the reader can find proofs in the book by Jose Garcia-Cuerva and J.-L.Rubio De Francia [GF].

Lemma 4.15. *If the measure $\mu \in A_p$ then there is $\epsilon = \epsilon(\mu) > 0$ such that $\mu \in A_{p-\epsilon}$.*

Lemma 4.16 (Doubling property)**.** *If the measure $\mu \in A_p$ then for any $\lambda > 1$ and for any cube Q, we have*

$$\mu(\lambda Q) \leqslant C\lambda^{np}\mu(Q)$$

where λQ is the dilation of Q at the same center with side length $\lambda l(Q)$.

Moreover, we also have the lower bound, i.e., there is some $\delta' > 0$ such that

$$\frac{\mu(E)}{\mu(Q)} \leqslant C\left(\frac{|E|}{|Q|}\right)^{\delta'}$$

where $|E|$ denotes the Lebesgue measure of the set E.

Proof. See Lemma 2.2 and Theorem 2.9 in [GF] for the proofs. ∎

Definition 4.17. Let $\mathcal{T} = (T_t)_{t>0}$ be a family of operators, we say that $\mathcal{T}$ satisfies L^2_μ off-diagonal estimates if for some constants $c > 0$ and $\alpha > 0$ for all closed sets E, F, all $h \in L^2_\mu$ with support in E and for all $t > 0$ we have

$$||T_t h||_{L^2_\mu(F)} \leqslant ce^{\frac{-cd(E,F)^2}{t}}||h||_{L^2_\mu(E)}.$$

Here and subsequently, $d(E, F)$ denotes the usual Euclidean distance between the two sets.

Proposition 4.18. *For all $z \in \Sigma_\beta$ where $\beta \leqslant \frac{\pi}{4}$, the family $(e^{-z\mathcal{L}_\mu})$, $(z\mathcal{L}_\mu e^{-z\mathcal{L}_\mu})$ and $(\sqrt{|z|}\nabla e^{-z\mathcal{L}_\mu})$ satisfy L^2_μ off diagonal estimates*

Proof. This is a consequence of Lemma 2.10 in [C-UR2] ∎

Now, we state and give a proof of the Hodge decomposition for degenerate elliptic operator. For the case of non-degenerate operators, see [AT]. In the degenerate case, similar results have been proved independently in [C-UMR].

Theorem 4.19. *(Hodge decomposition for $\mathcal{L}_\mu$) For $\frac{\mathbf{f}}{\mu} \in L^p_\mu(\mathbb{R}^n, \mathbb{C}^n)$, we have a decomposition $\mathbf{f} = A\nabla\mathcal{L}_\mu^{-1}\frac{1}{\mu}(\operatorname{div}\mathbf{f}) + \mathbf{h}$ where $\mathbf{h}$ is divergence free, i.e.* $\operatorname{div}\mathbf{h} = 0$ *and also* $\|\nabla\mathcal{L}_\mu^{-1}\frac{1}{\mu}(\operatorname{div}\mathbf{f})\|_{L^p_\mu(\mathbb{R}^n)} \lesssim \|\frac{\mathbf{f}}{\mu}\|_{L^p_\mu(\mathbb{R}^n)}$ *for $p = 2 + \epsilon$, here, $\epsilon > 0$.*

Proof. Observe that $\mathcal{L}_\mu^{-1} = L^{-1}\mu$. Thus

$$
\begin{aligned}
\mathbf{f} &= -A\nabla\mathcal{L}_\mu^{-1}\frac{1}{\mu}\operatorname{div}\mathbf{f} + \left(I + A\nabla\mathcal{L}_\mu^{-1}\frac{1}{\mu}\operatorname{div}\right)\mathbf{f} \\
&= -A\nabla L^{-1}\operatorname{div}\mathbf{f} + \left(I + A\nabla L^{-1}\operatorname{div}\right)\mathbf{f} \\
&= -A\nabla L^{-1}\operatorname{div}\mathbf{f} + \text{divergence free term.}
\end{aligned}
$$

It remains to check that

$$
\|\nabla L^{-1}\operatorname{div}\mathbf{f}\|_{L^p_\mu(\mathbb{R}^n)} \lesssim \left\|\frac{\mathbf{f}}{\mu}\right\|_{L^p_\mu(\mathbb{R}^n)}. \tag{4.1}
$$

Set $v := L^{-1}\operatorname{div}\mathbf{f}$, so $Lv = \operatorname{div}\mathbf{f}$ in the weak sense. Let B be any cube with radius r_B, choose a smooth function η_B satisfying

$$
0 \leqslant \eta_B \leqslant 1, \eta_B \equiv 1 \text{ in } B, \eta_B \equiv 0 \text{ outside } \frac{3}{2}B \text{ and } \|\nabla\eta_B\|_{L^\infty} \leqslant \frac{c}{r_B}.
$$

Since $\nabla v = \nabla(v - C)$ for any constant C, we may replace v by $v - C$, and assume without loss of generality that v has mean value 0 in $(3/2)B$. Then

$$
\frac{1}{\mu(B)}\int_{\frac{3}{2}B}(\nabla v\cdot\nabla v)\eta_B^2\mu \;\lesssim\; \frac{1}{\mu(B)}\int_{\frac{3}{2}B}\operatorname{Re}(A\nabla v\cdot\nabla v)\eta_B^2
$$

$$
\begin{aligned}
&= \frac{1}{\mu(B)}\mathrm{Re}\int_{\frac{3}{2}B}(A\nabla v\cdot\nabla(v\eta_B^2) - \frac{1}{\mu(B)}\mathrm{Re}\int_{\frac{3}{2}B}(A\nabla v\cdot\nabla(\eta_B^2))v \\
&= -\mathrm{Re}\frac{1}{\mu(B)}\int_{\frac{3}{2}B}\mathbf{f}\cdot\nabla(v\eta_B^2) - \frac{1}{\mu(B)}\mathrm{Re}\int_{\frac{3}{2}B}(A\nabla v\cdot\nabla(\eta_B^2))v \\
&= -\mathrm{Re}\frac{1}{\mu(B)}\int_{\frac{3}{2}B}\frac{\mathbf{f}}{\mu}\cdot[(\nabla v)\eta_B^2 + (\nabla\eta_B^2)v)]\mu - \frac{1}{\mu(B)}\mathrm{Re}\int_{\frac{3}{2}B}(A\nabla v\cdot\nabla(\eta_B^2))v \\
&\lesssim \frac{1}{\mu(B)}\int_{\frac{3}{2}B}\left(\frac{|\mathbf{f}|}{\mu}(|\nabla v|\eta_B^2 + |\nabla\eta_B^2||v|)\right)\mu + \left|\frac{C}{\mu(B)}\int_{\frac{3}{2}B}(|\nabla v|\cdot|\nabla(\eta_B^2)|)|v|\mu\right| \\
&\lesssim \frac{1}{\mu(B)}\int_{\frac{3}{2}B}\left(\frac{|\mathbf{f}|}{\mu}\right)^2\mu + \frac{1}{\mu(B)}\int_{\frac{3}{2}B}\frac{|v|^2}{r_B^2}\mu \\
&\qquad \text{(by Cauchy's inequality with epsilons, hiding the small term on LHS)} \\
&\lesssim \frac{1}{\mu(B)}\int_{\frac{3}{2}B}\left(\frac{|\mathbf{f}|}{\mu}\right)^2\mu + \left(\frac{1}{\mu(B)}\int_{\frac{3}{2}B}|\nabla v|^s\mu\right)^{\frac{2}{s}},
\end{aligned}
$$

with $s = 2-\epsilon$, by Lemma 4.15 and Theorem 4.12, since we have reduced to the case that $\int_{(3/2)B} v = 0$. Hence,

$$
\begin{aligned}
\left(\frac{1}{\mu(B)}\int_B|\nabla v|^2\mu\right)^{\frac{1}{2}} &\leqslant \left(\frac{1}{\mu(B)}\int_{\frac{3}{2}B}(\nabla v\cdot\nabla v)\eta_B^2\mu\right)^{\frac{1}{2}} \\
&\lesssim \left(\frac{1}{\mu(\frac{3}{2}B)}\int_{\frac{3}{2}B}\left(\frac{|\mathbf{f}|}{\mu}\right)^2\mu\right)^{\frac{1}{2}} + C\left(\frac{1}{\mu(\frac{3}{2}B)}\int_{\frac{3}{2}B}|\nabla v|^s\mu\right)^{\frac{1}{s}}.
\end{aligned}
$$

Using the following lemma, we get the desired inequality. ∎

Lemma 4.20. *Suppose $p > 1$. Given $g, h \in L^p_\mu(\mathbb{R}^n)$ such that for all dyadic cubes $Q \subset \mathbb{R}^n$*

$$
\left(\frac{1}{\mu(Q)}\int_Q|g|^p\mu\right)^{\frac{1}{p}} \lesssim \left(\frac{1}{\mu(2Q)}\int_{2Q}|g|\mu\right) + \left(\frac{1}{\mu(2Q)}\int_{2Q}|h|^p\mu\right)^{\frac{1}{p}}. \tag{4.2}
$$

Then there is $\epsilon = \epsilon(C,p,n,w,\alpha) > 0$ and $B = B(C,p,n,w,\alpha) > 0$ such that

$$
\int_{\mathbb{R}^n}|g|^{p+\epsilon}\mu \leqslant B\int_{\mathbb{R}^n}|h|^{p+\epsilon}\mu.
$$

We remark that Lemma 4.20 is of course a weighted version of a Gehring-type lemma. The proof is similar to that in the unweighted case, but we include it here for the sake of self-containment. We note that at this point we do not use the full power of the A_2 condition, but only that $\mu(x)dx$ is a doubling measure.

Proof. Given $t > 0$, set $s = s(t) := (3C + 2)t$ for some C to be chosen later. Carry out Calderon-Zygmund decomposition of g^p at the height s^p, i.e., by a stopping time argument we obtain a sequence $\{Q_i\}_{j=1}^{\infty}$ non-overlapping dyadic cubes (see detail in [C-UMP], Proposition A.5) such that

$$s^p \leqslant \frac{1}{\mu(Q_j)} \int_{Q_j} g^p \leqslant C s^p,$$

and $\{g < s\}$ μ a.e. on $\mathbb{R}^n \setminus (\cup Q_j)$.

Set

$$\begin{aligned} Gs &= \{x : g(x) \geqslant s\}. \\ Hs &= \{x : h(x) \geqslant s\}. \\ Gt &= \{x : g(x) \geqslant t\}. \\ Ht &= \{x : h(x) \geqslant t\}. \end{aligned}$$

Then $Gs \subset \cup Q_j$ upto a set of μ-measure 0, therefore, we have the following inequalities

$$\int_{Gs} g^p \mu \leqslant \sum_j \int_{Q_j} g^p \mu \leqslant C_0 s^p \sum_j \mu(Q_j), \tag{4.3}$$

and

$$s \leqslant \left(\frac{1}{\mu(Q_j)} \int_{Q_j} g^p \mu \right)^{\frac{1}{p}} \lesssim \left[\frac{1}{\mu(2Q_j)} \int_{2Q_j} g\mu + \left(\frac{1}{\mu(2Q_j)} \int_{2Q_j} h^p \mu \right)^{\frac{1}{p}} \right], \tag{4.4}$$

$$\frac{1}{\mu(2Q_j)} \int_{2Q_j} g\mu \leqslant t + \frac{1}{\mu(2Q_j)} \int_{Q_j \cap Gt} g\mu, \tag{4.5}$$

$$\left(\frac{1}{\mu(2Q_j)}\int_{2Q_j} h^p\mu\right)^{\frac{1}{p}} \leqslant t + \left(\frac{1}{\mu(2Q_j)}\int_{2Q_j\cap Ht} h^p\mu\right)^{\frac{1}{p}}, \tag{4.6}$$

$$\leqslant 2t + \frac{1}{t^{p-1}\mu(2Q_j)}\int_{2Q_j\cap Ht} h^p\mu, \tag{4.7}$$

where we have used the fact that for any $x > 0$ and for $\frac{1}{p} + \frac{1}{p'} = 1$

$$x^{\frac{1}{p}} = t^{\frac{1}{p'}}(t^{-\frac{1}{p'}}x^{\frac{1}{p}}) \leqslant \frac{t}{p'} + \frac{(t^{-\frac{1}{p'}}x^{\frac{1}{p}})^p}{p} \leqslant t + \frac{x}{t^{p-1}}, \qquad \text{since } p, p' > 1.$$

by Young's inequality for the first inequality.

Apply the above inequality for $x = \frac{1}{\mu(2Q_j)}\int_{2Q_j\cap Ht} h^p\mu$ in (4.6) to get the desired inequality (4.7).

Now, we combine (4.4), (4.5) and (4.7) get

$$\begin{aligned}(3C+2)t = s &\leqslant C\left[t + \frac{1}{\mu(2Q_j)}\int_{2Q_j\cap Gt} g\mu + 2t\frac{1}{t^{p-1}\mu(2Q_j)}\int_{2Q_j\cap Ht} h^p\mu\right] \\ &\leqslant 3Ct + \frac{C}{\mu(2Q_j)}\int_{2Q_j\cap Gt} g\mu + \frac{C}{t^{p-1}\mu(2Q_j)}\int_{2Q_j\cap Ht} h^p\mu.\end{aligned}$$

Hence

$$2t \leqslant \frac{C}{\mu(2Q_j)}\int_{2Q_j\cap Gt} g\mu + \frac{C}{t^{p-1}\mu(2Q_j)}\int_{2Q_j\cap Ht} h^p\mu,$$

or

$$\mu(2Q_j) \leqslant \frac{C}{2t}\int_{2Q_j\cap Gt} g\mu + \frac{C}{t^p}\int_{2Q_j\cap Ht} h^p\mu.$$

By Vitali's covering lemma, there exists a sub-collection $\{Q_j'\}$ such that $2Q_j'$ are pairwise disjoint, and moreover

$$\sum_j \mu(Q_j) \leqslant C\sum_i \mu(2Q_j')$$

$$\leqslant \frac{C}{t}\sum_j\left(\int_{2Q'_j\cap G_t} g\mu + \frac{C}{t^p}\int_{2Q'_j\cap Ht} h^p\mu\right)$$
$$\leqslant \frac{C}{t}\int_{Gt} g\mu + \frac{C}{t^p}\int_{Ht} h^p\mu.$$

Plug the above inequality into (4.3) obtain

$$\int_{Gs} g^p\mu \leqslant C\frac{s^p}{t}\int_{Gt} g\mu + C\frac{s^p}{t^p}\int_{Ht} h^p\mu. \tag{4.8}$$

From the definition of Gs, Gt, for $s > t$ get

$$\int_{Gt\setminus Gs} g^p\mu \leqslant s^{p-1}\int_{Gt\setminus Gs} g\mu. \tag{4.9}$$

Recall that by the choice of s, $s \simeq t$, combine (4.8) and (4.9)

$$\int_{Gt} g^p\mu \leqslant \alpha t^{p-1}\int_{Gt} g\mu + \beta\int_{Ht} h^p\mu, \quad \forall t > 0, \tag{4.10}$$

where α, β depend on n, p, C, μ.

Set

$$\eta(t) = \int_{g>t} g\mu dx.$$

By the theorem of Stieltjes integral

$$\int_{\{g>t\}} g^p\mu = \int_{\{g>t\}} g^{p-1} g\mu dx = -\int_t^\infty s^{p-1} d\eta(s),$$

then (4.11) becomes

$$-\int_t^\infty s^{p-1} d\eta(s) \leqslant \alpha t^{p-1}\eta(t) + \beta\int_{Ht} h^p\mu. \tag{4.11}$$

Fix $0 < T < \infty$, set

$$\eta_T(t) := \begin{cases} \eta(t) & \text{if} \quad 0 < t \leqslant T, \\ 0 & \text{if} \quad t > T. \end{cases}$$

Then (4.11) becomes

$$-\int_t^\infty s^{p-1} d\eta_T(s) \leqslant \alpha t^{p-1}\eta_T(t) + \beta \int_{Ht} h^p \mu. \tag{4.12}$$

Let δ be small enough that $\delta\alpha < 1+\delta$, from the above inequality, we have

$$\begin{aligned}
-\int_0^\infty t^{(p-1)(1+\delta)} d\eta_T(t) &= -\int_0^\infty t^{(p-1)\delta)} t^{p-1} d\eta_T(t) \\
&= \int_0^\infty t^{(p-1)\delta} \left[\int_0^\infty s^{(p-1)d\eta_T(s)}\right]' dt \\
&= \delta(p-1)\int_0^\infty t^{(p-1)\delta-1}\left[-\int_t^\infty s^{p-1} d\eta_T(s)\right] dt \\
&\quad \text{by integration by parts} \\
&\leqslant \delta(p-1)\int_0^\infty \alpha t^{(p-1)\delta-1+p-1}\eta_T(t) dt \\
&\quad +\delta(p-1)\int_0^\infty \beta t^{(p-1)\delta-1}\int_{h>t} h^p \mu dt \\
&=: I + II \\
&= -\frac{\delta\alpha}{1+\delta}\int_0^\infty t^{(p-1)(1+\delta)} d\eta_T(t) + II,
\end{aligned}$$

by integration by parts for part I, then we hide the small part of the RHS to the LHS. Hence,

$$\begin{aligned}
-\int_0^\infty t^{(p-1)(1+\delta)} d\eta_T(t) &\leqslant C\int_0^\infty t^{(p-1)\delta-1}\int_{h>t} h^p \mu dx dt \\
&= C\int_{\mathbb{R}^n} h^p \mu \int_0^h t^{(p-1)\delta-1} dt \\
&= C\int_{\mathbb{R}^n} h^{p+(p-1)\delta}\mu.
\end{aligned}$$

For $\epsilon = (p-1)\delta$, then

$$\begin{aligned}
\int_{\mathbb{R}^n} g^{p+(p-1)\delta}\mu &= \int_{\mathbb{R}^n} g^{(p-1)(1+\delta)} g\mu \\
&= -\int_0^\infty t^{(p-1)(1+\delta)} d\eta_T(t),
\end{aligned}$$

by Stieltjes integral theorem.

Now, we let $T \to \infty$ to get the desired result. ∎

Corollary 4.21. *Suppose $p > 1$. Given $g, h \in L^p_\mu(\mathbb{R}^n)$, suppose that for all dyadic cubes $Q \subset \mathbb{R}^n$*

$$\left(\frac{1}{\mu(Q)}\int_Q |g|^p\mu\right)^{\frac{1}{p}} \lesssim \left(\frac{1}{\mu(2Q)}\int_{2Q} |g|^\theta\mu\right)^{\frac{1}{\theta}} + \left(\frac{1}{\mu(2Q)}\int_{2Q} |h|^p\mu\right)^{\frac{1}{p}},$$

for some $1 \leqslant \theta < p$. Then there is $\epsilon = \epsilon(C, p, n, w, \alpha, \theta) > 0$ and $B = B(C, p, n, w, \alpha) > 0$ such that

$$\int_{\mathbb{R}^n} |g|^{p+\epsilon}\mu \leqslant B\int_{\mathbb{R}^n} |h|^{p+\epsilon}\mu.$$

Proof. This result is a consequence of Lemma 4.2. ∎

4.3 Holomorphic functional calculus on L^2

For this section, we want to recall some facts about holomorphic functional calculus. For more details , we refer the reader to [A, Section 3.2] (in the unweighted case, but here the arguments are similar), and also to [C-UR2].

Let $\mathcal{L}_\mu$ as before, we know that $\mathcal{L}_\mu$ generates a semi-group $(e^{-t\mathcal{L}_\mu})_{t>0}$ which has an analytic extension to a complex semi-group $(e^{-z\mathcal{L}_\mu})_{z\in\Sigma_{\frac{\pi}{2}-\nu}}$ of contractions on L^2 for some $\nu \in [0, \frac{\pi}{2})$ and $\Sigma_\mu := \{z \in \mathbb{C}^* : |\arg z| < \mu.\}$ (see more details in [C-UR2]).

We also have $\mathcal{L}_\mu$ has a bounded holomorphic functional calculus on L^2_μ. In particular, for any $\mu \in (\nu, \pi)$ and any φ holomorphic and bounded in Σ_μ, the operator $\varphi(\mathcal{L}_\mu)$ is bounded on L^2_μ with the estimate

$$\|\varphi(\mathcal{L}_\mu)f\|_{L^2_\mu} \leqslant C\|\varphi\|_\infty\|f\|_{L^2_\mu},$$

the constant C depending only on μ and ν. Moreover, if φ satisfies the technical condition

$$|\varphi(\zeta)| \leqslant C|\zeta|^s(1+|\zeta|)^{-2s}, \tag{4.13}$$

for all $\zeta \in \Sigma_\mu$ for some positive constants c, s, then $\varphi(\mathcal{L}_\mu)$ can be computed using the semi-group. Let $\nu < \theta < \delta < \mu < \frac{\pi}{2}$. One has

$$\varphi(\mathcal{L}_\mu) = \int_{\Gamma_+} e^{-z\mathcal{L}_\mu}\eta_+(z)dz + \int_{\Gamma_-} e^{-z\mathcal{L}_\mu}\eta_-(z)dz, \tag{4.14}$$

where $\Gamma_\pm$ is the half-ray $\mathbb{R}^+e^{\pm i(\frac{\pi}{2}-\theta)}$ and

$$\eta_\pm(z) = \frac{1}{2\pi i}\int_{\gamma_\pm} e^{\zeta z}\varphi(\zeta)d\zeta, \quad z \in \Gamma_\pm,$$

with $\gamma_\pm$ being the half-ray $\mathbb{R}^+e^{\pm i\nu}$.

4.4 L^p boundedness for $(e^{-t\mathcal{L}_\mu})_{t>0}$, $(\sqrt{t}\nabla e^{-t\mathcal{L}_\mu})_{t>0}$ and Vertical Square functions

We recall the vertical square functions defined in the Introduction:

$$\begin{aligned} g_{\mathcal{L}_\mu}(f)(x) &= \left(\int_0^\infty \left|\mathcal{L}_\mu^{1/2}e^{-t\mathcal{L}_\mu}f(x)\right|^2 dt\right)^{\frac{1}{2}}. \\ G_{\mathcal{L}_\mu}(f)(x) &= \left(\int_0^\infty \left|\nabla e^{-t\mathcal{L}_\mu}f(x)\right|^2 dt\right)^{\frac{1}{2}}. \end{aligned}$$

In this section, we will prove the L^p boundedness for the operator $g_{\mathcal{L}_\mu}$ and $G_{\mathcal{L}_\mu}$ for $2 \leqslant p < 2+\epsilon$. The case $p = 2$ is already known. We have the following.

Theorem 4.22. *The vertical square functions $g_{\mathcal{L}_\mu}, G_{\mathcal{L}_\mu}$ above are bounded in $L^2_\mu(\mathbb{R}^n)$.*

Proof. In the case $\mu \equiv 1$, see [A, p. 74]. The same proof also works for degenerate operators with A_2 degeneracy, by the existence of the bounded holomorphic functional calculus in L^2_μ. ∎

4.4.1 Some kind of local $L^2_\mu - L^p_\mu$ off-diagonal estimates for $e^{-t\mathcal{L}_\mu}$

In this section, we will establish some kind of local $L^2_\mu - L^p_\mu$ off-diagonal estimates for $e^{-t\mathcal{L}_\mu}$ over some balls $B = B(x_0, r)$.

We first prove some properties for $e^{-t\mathcal{L}_\mu}$ and then we get similar properties for $\sqrt{t}\nabla e^{-t\mathcal{L}_\mu}$.

Lemma 4.23. *For $2 \leqslant p \leqslant \frac{2n}{n-1} + 2\delta$, where δ is as in Theorem 4.12, then for any ball B with radius r and the function f satisfying supp$f \subset B$, we obtain*

$$\|e^{-t\mathcal{L}_\mu} f\|_{L^p_\mu(B)} \leqslant C\mu(B)^{\frac{1}{p}-\frac{1}{2}} \left(\frac{r}{\sqrt{t}} + 1\right) \|f\|_{L^2_\mu(B)}.$$

Proof. For $2 \leqslant p \leqslant \frac{2n}{n-1} + 2\delta$, for function f satisfying supp$f \subset B$, get

$$\begin{aligned}
\|e^{-t\mathcal{L}_\mu} f\|_{L^p_\mu(B)} &= \mu(B)^{\frac{1}{p}} \left(\frac{1}{\mu(B)} \int_B |e^{-t\mathcal{L}_\mu} f|^p \mu\right)^{\frac{1}{p}} \\
&\leqslant \mu(B)^{\frac{1}{p}} \left[\left(\frac{1}{\mu(B)} \int_B \left|e^{-t\mathcal{L}_\mu} f - \left(\frac{1}{\mu(B)} \int_B e^{-t\mathcal{L}_\mu} f\, \mu\right)\right|^p \mu\right)^{\frac{1}{p}} + \frac{1}{\mu(B)} \int_B \left|e^{-t\mathcal{L}_\mu} f\right| \mu\right] \\
&\leqslant C\mu(B)^{\frac{1}{p}} \left[r\left(\frac{1}{\mu(B)} \int_B |\nabla e^{-t\mathcal{L}_\mu} f|^2 \mu\right)^{\frac{1}{2}} + \left(\frac{1}{\mu(B)} \int_B |e^{-t\mathcal{L}_\mu} f|^2 \mu\right)^{\frac{1}{2}}\right] \\
&\leqslant C\mu(B)^{\frac{1}{p}} \left[\frac{r}{\mu(B)^{\frac{1}{2}}} \frac{\|f\|_{L^2_\mu(B)}}{\sqrt{t}} + \frac{1}{\mu(B)^{\frac{1}{2}}} \|f\|_{L^2_\mu(B)}\right] \\
&\leqslant C\mu(B)^{\frac{1}{p}-\frac{1}{2}} \left(\frac{r}{\sqrt{t}} + 1\right) \|f\|_{L^2_\mu(B)}.
\end{aligned}$$

where we applied weighted Poincaré inequality and Cauchy-Schwarz's inequality to get the second inequality and the assumption supp $f \subset B$ together the L^2_μ bounds for $e^{-t\mathcal{L}_\mu}$ and $\nabla e^{-t\mathcal{L}_\mu}$ to get the third inequality. ∎

Remark 4.24. If we have $r \simeq \sqrt{t}$ then with the hypotheses in the Lemma 4.23, we have

$$\|e^{-t\mathcal{L}_\mu} f\|_{L^p_\mu(B)} \leqslant C\mu(B)^{\frac{1}{p}-\frac{1}{2}} \|f\|_{L^2_\mu(B)}, \quad \forall B = B(x, r).$$

where $2 \leqslant p \leqslant \frac{2n}{n-1} + 2\delta$.

Now, we fix $p_0 = \frac{2n}{n-1} + 2\delta$ and then we interpolate between L^2_μ off-diagonal estimate and $L^2_\mu - L^{p_0}_\mu$ estimate for operator $e^{-t\mathcal{L}_\mu}$ in Lemma 4.23. Let p satisfy $\frac{1}{p} = \frac{\alpha}{p_0} + \frac{1-\alpha}{2}$ for $0 < \alpha < 1$, then for annuli $C_j(B) = 2^{j+1}B \setminus 2^jB$ with $j \geqslant 2$ and with any function $f \in L^2_\mu$ which is supported in a subset of $C_j(B)$ together the condition $t \simeq r^2$, we get the following

$$\begin{aligned}
\|e^{-t\mathcal{L}_\mu} f\|_{L^p_\mu(B)} &\leqslant C\mu(2^jB)^{\alpha\left(\frac{1}{p_0}-\frac{1}{2}\right)} e^{-(1-\alpha)\frac{d(B,C_j)^2}{ct}} \|f\|_{L^2_\mu(C_j(B))} \\
&\leqslant C\mu(2^jB)^{\alpha(\frac{1}{p_0}-\frac{1}{2})} e^{-(1-\alpha)\frac{(2^jr)^2}{ct}} \|f\|_{L^2_\mu(C_j(B))} \\
&\leqslant C\mu(2^jB)^{\left(\frac{1}{p}-\frac{1}{2}\right)} e^{-\frac{(2^jr)^2}{ct}(1-\alpha)} \|f\|_{L^2_\mu(C_j(B))}.
\end{aligned}$$

Thus by doubling property of A_2 condition, $\mu(lB) \leqslant l^{2n}\mu(B)$ and we are in the case $r^2 \simeq t$, we obtain

$$\begin{aligned}
&\left(\frac{1}{\mu(B)}\int_B \left|e^{-t\mathcal{L}_\mu} f\right|^p \mu\right)^{\frac{1}{p}} \leqslant \left(\frac{2^{2jn}}{\mu(2^jB)}\int_B \left|e^{-t\mathcal{L}_\mu} f\right|^p \mu\right)^{\frac{1}{p}} \\
&\leqslant C2^{2jn} e^{\frac{-4^jr^2}{ct}} \left(\frac{1}{\mu(2^jB)}\int_{C_j(B)} |f|^2\mu\right)^{\frac{1}{2}} \leqslant Ce^{\frac{-4^j}{c}} \left(\frac{1}{\mu(2^{j+1}B)}\int_{C_j(B)} |f|^2\mu\right)^{\frac{1}{2}},
\end{aligned}$$

for some constant c depending on previous c and α.

That is the proof of the following lemma.

Lemma 4.25. *For $2 \leqslant p < p_0$ where ϵ as above and if $t \simeq r^2$, suppose that* $\operatorname{supp} f \subset C_j(B)$*, then we have for $\frac{1}{p} = \frac{\alpha}{p_0} + \frac{1-\alpha}{2}$,*

$$\left(\frac{1}{\mu(B)}\int_B \left|e^{-t\mathcal{L}_\mu} f\right|^p \mu\right)^{\frac{1}{p}} \leqslant Ce^{\frac{-4^j}{c}} \left(\frac{1}{\mu(2^{j+1}B)}\int_{C_j(B)} |f|^2\mu\right)^{\frac{1}{2}}, \tag{4.15}$$

for all $j \geqslant 1$, where C could depend on previous C, c and α.

Lemma 4.26. *Let $q_0 := \frac{p_0}{p_0-1} = \frac{2\delta(n-1)+2n}{2\delta(n-1)+n+1}$, the conjugate of p_0, then for $q_0 < q < 2$, and again for any ball B with radius r and if the function f satisfying* $\operatorname{supp} f \subset B$ *then*

$$\|e^{-t\mathcal{L}_\mu} f\|_{L^2_\mu(B)} \leqslant C\mu(B)^{\frac{1}{2}-\frac{1}{q}} \left(\frac{r}{\sqrt{t}} + 1\right) \|f\|_{L^q_\mu(B)}.$$

Proof. By using duality for Lemma 4.23. ∎

Remark 4.27. Notice that $C_j(B) \subset 2^{j+1}B$, so if f is supported in $C_j(B)$, we get

$$\|e^{-t\mathcal{L}_\mu} f\|_{L^2_\mu(B)} \leqslant \|e^{-t\mathcal{L}_\mu} f\|_{L^2_\mu(2^{j+1}B)} \leqslant C\mu(2^{j+1}B)^{\frac{1}{2}-\frac{1}{q}} \left(\frac{2^j r}{\sqrt{t}} + 1\right) \|f\|_{L^q_\mu(C_j(B))}.$$

Now, we fix a number q that satisfy the result of Lemma 4.26 and we still call it q_0. We abuse using the notation for q_0 but it is fine for our purpose. Then we interpolate the above results with the L^2_μ off-diagonal estimate for $e^{-t\mathcal{L}_\mu}$, we get some kind of local $L^q_\mu - L^2_\mu$ off-diagonal estimates, which is the proof of the following lemma.

Lemma 4.28. *For any $\frac{1}{q} = \frac{\alpha}{q_0} + \frac{1-\alpha}{2}$ where $0 < \alpha < 1$, then if $\operatorname{supp} f \subset C_j(B)$, we get*

$$\|e^{-t\mathcal{L}_\mu} f\|_{L^2_\mu(B)} \leqslant C \left[\mu(2^j B)\right]^{\left(\frac{1}{2}-\frac{1}{q}\right)} \left(\frac{2^j r}{\sqrt{t}} + 1\right)^\alpha e^{-\frac{(2^j r)^2(1-\alpha)}{ct}} \|f\|_{L^q_\mu(C_j(B))},$$

for all $j \geqslant 2$, and

$$\|e^{-t\mathcal{L}_\mu} f\|_{L^2_\mu(B)} \leqslant C \left[\mu(4B)\right]^{\left(\frac{1}{2}-\frac{1}{q}\right)} \left(\frac{4r}{\sqrt{t}} + 1\right)^\alpha \|f\|_{L^q_\mu(4B)}$$

for $j = 1$.

Remark 4.29. By the fact that $x^\alpha e^{-\frac{x^2}{c}} \leqslant Ce^{-\frac{x^2}{2c}}$ for all $x \geqslant 1$, we also get

$$\|e^{-t\mathcal{L}_\mu} f\|_{L^2_\mu(B)} \leqslant C \left[\mu(2^j B)\right]^{\frac{1}{2}-\frac{1}{p}} e^{-\frac{(2^j r)^2(1-\alpha)}{ct}} \|f\|_{L^p_\mu(C_j(B))}, \forall j \geqslant 2.$$

By replacing B by the ball $2^j B$ and the fact that $C_{j-1}(B) \subset 2^j B$, we apply Lemma 4.28 for a function f with support in the ball B to get the following lemma:

Lemma 4.30. *Let q_0 be defined as in Lemma 4.26. For $\frac{1}{q} = \frac{\alpha}{q_0} + \frac{1-\alpha}{2}$ where $0 < \alpha < 1$, then if $\operatorname{supp} f \subset B$, we get*

$$\|e^{-t\mathcal{L}_\mu} f\|_{L^2_\mu(C_{j-1}(B))} \leqslant C \left[\mu(2^j B)\right]^{\frac{1}{2}-\frac{1}{q}} \left(\frac{2^j r}{\sqrt{t}} + 1\right)^\alpha e^{-\frac{\operatorname{dist}^2(B, C_{j-1}(B))(1-\alpha)}{ct}} \|f\|_{L^q_\mu(B)}.$$

Remark 4.31. By the identical argument in the previous Remark 4.29, we also get

$$\|e^{-t\mathcal{L}_\mu} f\|_{L^2_\mu(C_{j-1}(B))} \leqslant C\left[\mu(2^j B)\right]^{\frac{1}{2}-\frac{1}{q}} e^{-\frac{(2^j r)^2(1-\alpha)}{ct}} \|f\|_{L^q_\mu(B)}, \forall j \geqslant 2.$$

Remark 4.32. By analytic extension, we also have

$$\|e^{-z\mathcal{L}_\mu} f\|_{L^2_\mu(C_{j-1}(B))} \leqslant C\left[\mu(2^j B)\right]^{\frac{1}{2}-\frac{1}{q}} \left(\frac{2^j r}{\sqrt{|z|}}+1\right)^{\alpha} e^{-\frac{\mathrm{dist}^2(B,C_{j-1}(B))(1-\alpha)}{c|z|}} \|f\|_{L^q_\mu(B)},$$

for all $z \in \Sigma_\beta$ and $\beta \leqslant \frac{\pi}{4}$.

4.4.2 L^p boundedness for $(e^{-t\mathcal{L}_\mu})_{t>0}$ and $(\sqrt{t}\nabla e^{-t\mathcal{L}_\mu})_{t>0}$

In order to get L^p bounds for $(\sqrt{t}\nabla e^{-t\mathcal{L}_\mu})_{t>0}$, first, we want to get the following result.

Lemma 4.33. *For any dyadic cube Q, we have the following*

$$\|\mathcal{L}_\mu u\|_{L^2_\mu(2Q)} \leqslant C\frac{1}{t}\|M_\mu(|u|^{\frac{1}{2}})\|_{L^2_\mu(2Q)}.$$

Moreover, we also have that there is some $\epsilon_1 > 0$ such that

$$\|t\mathcal{L}_\mu u\|_{L^p_\mu} \leqslant C_1\|f\|_{L^p_\mu}$$

for $2-\epsilon_1 < p < 2+\epsilon_1$ where C, C_1 do not depend on t.

Proof. To prove the first estimate, we consider two cases as followings.

Case 1: $t \leqslant l(Q)^2$

For this case, we decompose $f = f_1 + \sum_{i=2}^{\infty} f_i$ where $f_i := f\mathbb{1}_{C_i}$ and C_i is defined as following $C_1 = 4Q$ and $C_i := 2^{i+1}Q \setminus 2^i Q$ for $i > 1$. Then by the off-diagonal estimate for $t\mathcal{L}_\mu e^{-t\mathcal{L}_\mu}$ in Proposition 4.18, we get

$$\|t\mathcal{L}_\mu u\|_{L^2_\mu(Q)} = \|t\mathcal{L}_\mu e^{-t\mathcal{L}_\mu} f\|_{L^2_\mu}(Q) \leqslant \sum_{i=1}^{\infty} \left\|t\mathcal{L}_\mu e^{-t\mathcal{L}_\mu} f_i\right\|_{L^2_\mu(Q)}$$

$$
\begin{aligned}
&\leqslant \sum_{i=2}^{\infty} e^{-\frac{4^i l^2(Q)}{ct}} \|f_i\|_{L^2_\mu(C_i)} + \|f\|_{L^2_\mu(4B)} \\
&\leqslant \sum_{i=2}^{\infty} e^{-\frac{4^i (l(Q))^2}{ct}} \left(\mu(2^i Q)\frac{1}{\mu(Q)} \int_Q \left[M_\mu(f^2)\right]\mu\right)^{\frac{1}{2}} + C \left\| \left[M(|f|^2)\right]^{\frac{1}{2}} \right\|_{L^2_\mu(Q)} \\
&\leqslant \sum_{i=2}^{\infty} e^{-\frac{4^i}{c}} 2^{in} \left\| \left[M_\mu(f^2)\right]^{\frac{1}{2}} \right\|_{L^2_\mu(Q)} + C \left\| \left[M(|f|^2)\right]^{\frac{1}{2}} \right\|_{L^2_\mu(Q)} \\
&\lesssim \left\| \left[M_\mu(f^2)\right]^{\frac{1}{2}} \right\|_{L^2_\mu(Q)},
\end{aligned}
$$

where we have used doubling property of A_2 weight for the second to last inequality.

Case 2: $t \geqslant l(Q)^2$

We decompose the same way as in the Case 1 but for different set $C_i's$. In this situation, we define $C_i := \frac{(2^{i+1}t^2)^{\frac{1}{2}}}{l(Q)^2} Q \setminus \frac{(2^i t^2)^{\frac{1}{2}}}{l(Q)^2} Q$ for $i > 1$ and $C_1 = \frac{(4t^2)^{\frac{1}{2}}}{l(Q)^2} Q$. Thus we have $\mathrm{dist}(C_i, Q) = \left[\left(\frac{(2^i t^2)^{1/2}}{l(Q)^2} - 1\right) l(Q)\right] \simeq \frac{(2^i t^2)^{1/2}}{l(Q)}$. This give us

$$
\begin{aligned}
\|t\mathcal{L}_\mu u\|_{L^2_\mu(Q)} &\leqslant \sum_{i=2}^{\infty} e^{-\frac{2^i t}{c l(Q)^2}} \|f_i\|_{L^2_\mu(C_i)} + \|f\|_{L^2_\mu(4B)} \\
&\leqslant \sum_{i=2}^{\infty} e^{-\frac{2^i t}{c l(Q)^2}} \left(\mu\left(\frac{\left(2^{i+1}t^2\right)^{\frac{1}{2}}}{l(Q)^2} Q\right) \frac{1}{\mu(Q)} \int_Q \left[M_\mu(f^2)\right]\mu\right)^{\frac{1}{2}} \\
&\quad + C \left\| \left[M(|f|^2)\right]^{\frac{1}{2}} \right\|_{L^2_\mu(Q)} \\
&\leqslant \sum_{i=2}^{\infty} e^{-\frac{2^i t}{c l(Q)^2}} \left(\frac{\left(2^{\frac{i}{2}} t\right)}{l(Q)^2}\right)^n \left\| \left[M_\mu(f^2)\right]^{\frac{1}{2}} \right\|_{L^2_\mu(Q)} + C \left\| \left[M(|f|^2)\right]^{\frac{1}{2}} \right\|_{L^2_\mu(Q)} \\
&\lesssim \left\| \left[M_\mu(f^2)\right]^{\frac{1}{2}} \right\|_{L^2_\mu(Q)},
\end{aligned}
$$

where we have used the fact that $e^{-2^i x}(2^i x)^n \leqslant C_0 e^{-\frac{2^i x}{2}} \leqslant C_0 e^{-2^{i-1}}$ for any $x \geqslant 1$ and C_0 is a uniform constant that does not depend on t, Q, i.

Thus, we just proved that $\|\mathcal{L}_\mu u\|_{L^2_\mu(2Q)} \lesssim \frac{1}{t}\|M_\mu(|u|^{\frac{1}{2}})\|_{L^2_\mu(2Q)}$ for any dyadic cube Q. Moreover, we also get immediately that

$$\|t\mathcal{L}_\mu u\|_{L^p_\mu} \lesssim \left\| M_\mu(|u|^{\frac{1}{2}}) \right\|_{L^p_\mu} \lesssim \|f\|_{L^p_\mu}, \forall \quad 2+\epsilon_1 > p > 2,$$

by invoking Corollary 4.21 for some $0 < \epsilon_1$. Then by duality, we have

$$\|t\mathcal{L}_\mu u\|_{L^p_\mu} \lesssim \|f\|_{L^p_\mu}, 2-\epsilon_1 < p < 2$$

Hence,

$$\|t\mathcal{L}_\mu u\|_{L^p_\mu} \lesssim \|f\|_{L^p_\mu},$$

for $2-\epsilon_1 < p < 2+\epsilon_1$. ∎

Now, we would like to give the proof of our first main theorem. For convenience, we restate the Theorem 4.1 here.

Theorem 4.34. *Consider an operator $\mathcal{L}_\mu$ as above. There exists $\epsilon > 0$ depending on dimension, the A_2 constant of μ, and the accretivity constants λ and Λ, such that for all p with $\left|\frac{1}{2} - \frac{1}{p}\right| < \epsilon$, we have*

$$\|e^{-t\mathcal{L}_\mu} f\|_{L^p_\mu(\mathbb{R}^n)} + \|\sqrt{t}\nabla e^{-t\mathcal{L}_\mu} f\|_{L^p_\mu(\mathbb{R}^n)} \leqslant C_p \|f\|_{L^p_\mu(\mathbb{R}^n)}. \tag{4.16}$$

Proof. First, we claim that there is some $\epsilon_2 > 0$ such that $e^{-t\mathcal{L}_\mu}$ is bounded in L^p_μ for all p satisfying $2-\epsilon_2 < p < 2+\epsilon_2$.

Indeed, let $B := B(x, \sqrt{t})$ be the ball centered at some x_0 with radius $\sqrt{t}$. Let us define $C_j := 2^{j+1}B \setminus 2^j B$ for $j > 1$ and $C_1 := 4B$, and also $A_B := \frac{1}{\mu(B)} \int_B e^{-t\mathcal{L}_\mu} f \mu$. Then by the weighted Poincaré Theorem 4.12 and for $f_j := f\mathbb{1}_{C_j}$, we get

$$\left(\frac{1}{\mu(B)} \int_B \left|e^{-t\mathcal{L}_\mu} f - A_B\right|^{2+\delta} \mu\right)^{\frac{1}{2+\delta}} \leqslant C \left(\frac{1}{\mu(B)} \int_B \left|\sqrt{t}\nabla e^{-t\mathcal{L}_\mu} f\right|^2 \mu\right)^{\frac{1}{2}}$$

$$
\begin{aligned}
&\leqslant C\sum_{i=1}^{\infty}\left[\frac{1}{\mu(B)}\right]^{\frac{1}{2}}\left(\int_B\left|\sqrt{t}\nabla e^{-t\mathcal{L}_\mu}f_i\right|^2\mu\right)^{\frac{1}{2}}\\
&\leqslant C\sum_{i=1}^{\infty}\left[\frac{1}{\mu(B)}\right]^{\frac{1}{2}}e^{-\frac{4^j}{c}}\left(\int_{2^jB}|f_i|^2\mu\right)^{\frac{1}{2}}\\
&\leqslant C\sum_{i=1}^{\infty}e^{-\frac{4^j}{c}}2^{jn}\left(\frac{1}{\mu(2^jB)}\int_{2^jB}|f|^2\mu\right)^{\frac{1}{2}}\\
&\leqslant C\frac{1}{\mu(B)}\int_B\left[M_\mu\left(|f|^2\right)\right]^{\frac{1}{2}}\mu,
\end{aligned}
$$

where we applied the L^2_μ off-diagonal estimates for $\sqrt{t}\nabla e^{-t\mathcal{L}_\mu}$ to get the third inequality.

This implies that

$$
\begin{aligned}
&\left(\frac{1}{\mu(B)}\int_B\left|e^{-t\mathcal{L}_\mu}f\right|^{2+\delta}\mu\right)^{\frac{1}{2+\delta}}\\
\leqslant\ &\left(\frac{1}{\mu(B)}\int_B\left|e^{-t\mathcal{L}_\mu}f-A_B\right|^{2+\delta}\mu\right)^{\frac{1}{2+\delta}}+\left(\frac{1}{\mu(B)}\int_B|A_B|^{2+\delta}\mu\right)^{\frac{1}{2+\delta}}\\
\leqslant\ &C\frac{1}{\mu(B)}\int_B\left[M_\mu\left(|f|^2\right)\right]^{\frac{1}{2}}\mu,\\
\leqslant\ &C\left(\frac{1}{\mu(B)}\int_B\left[M_\mu\left(|f|^2\right)\right]^{\frac{2+\delta}{2}}\mu\right)^{\frac{1}{2+\delta}},
\end{aligned}
$$

because $|A_B|\leqslant\left[M_\mu\left(|f|^2\right)\right]^{\frac{1}{2}}$ for the third inequality, or we have

$$
\int_B\left|e^{-t\mathcal{L}_\mu}f\right|^{2+\delta}\mu\leqslant C\int_B\left[M_\mu\left(|f|^2\right)\right]^{\frac{2+\delta}{2}}\mu,
$$

for any ball B with radius $\sqrt{t}$. Hence, by decompose $\mathbb{R}^n$ into dyadic cubes with side length $\sqrt{t}$, we get

$$
\|e^{-t\mathcal{L}_\mu}f\|_{L^{2+\delta}_\mu}\ \leqslant\ C\left\|\left[M_\mu\left(|f|^2\right)\right]^{\frac{1}{2}}\right\|_{L^{2+\delta}_\mu}\leqslant C\|f\|_{L^{2+\delta}_\mu}.
$$

By using duality, we also have

$$
\|e^{-t\mathcal{L}_\mu}f\|_{L^{2-\delta}_\mu}\leqslant C\|f\|_{L^{2-\delta}_\mu}.
$$

Now, we choose $\epsilon_2 = \delta$ to get our claim.

It remains to prove L^p_μ bounds for the gradients. We prove that there is some $\epsilon_2 > 0$ such that $\|\sqrt{t}\nabla e^{-t\mathcal{L}_\mu} f\|_{L^p_\mu(\mathbb{R}^n)} \leqslant C_p \|f\|_{L^p_\mu(\mathbb{R}^n)}$ for all p such that $2 - \epsilon_2 < p < 2 + \epsilon_2$.

First, let us claim that there is some $\epsilon' > 0$ such that $\|\sqrt{t}\nabla e^{-t\mathcal{L}_\mu} f\|_{L^p_\mu(\mathbb{R}^n)} \leqslant C_p \|f\|_{L^p_\mu(\mathbb{R}^n)}$ for all $2 < p < 2 + \epsilon'$. In order to get the claim above, we use Gehring's trick for the proof of reverse Hölder inequality. That means we are proving that there is some $1 > \theta > 0$ such that for any cube Q,

$$\begin{aligned}
&\left(\frac{1}{\mu(Q)}\int_Q \left|\sqrt{t}\nabla u\right|^2 \mu(x)dx\right)^{\frac{1}{2}} \\
&\lesssim \left(\frac{1}{\mu(Q)}\int_{2Q} \left|t\mathcal{L}_\mu u\right|^2 \mu(x)dx\right)^{\frac{1}{2}} + \left(\frac{1}{\mu(Q)}\int_{2Q} \left|M_\mu(|f^2|)\right| \mu(x)dx\right)^{\frac{1}{2}} + \\
&+ \left(\frac{1}{\mu(Q)}\int_{2Q} \left|\sqrt{t}\nabla u\right|^{2-\theta} \mu(x)dx\right)^{\frac{1}{2-\theta}},
\end{aligned} \tag{4.17}$$

where $u := e^{-t\mathcal{L}_\mu} f$.

Then by Corollary 4.21, that implies there is $\epsilon' = \epsilon'(\epsilon_1, \epsilon_2, \theta) > 0$ such that

$$\left(\int \left|\sqrt{t}\nabla u\right|^p\right)^{\frac{1}{p}} \lesssim \left\|(M_\mu(|f^2|))^{\frac{1}{2}}\right\|_{L^p_\mu} + \|t\mathcal{L}_\mu u\|_{L^p_\mu} \lesssim \|f\|_{L^p_\mu},$$

for all $2 < p < 2 + \epsilon'$ where we applied Lemma 4.33, i.e. $\|t\mathcal{L}_\mu u\|_{L^p_\mu} \lesssim \|f\|_{L^p_\mu}$ and L^p_μ boundedness of the Hardy-Littlewood maximal function to get the last inequality.

Now, we return to the proof of our claim $\|\sqrt{t}\nabla e^{-t\mathcal{L}_\mu} f\|_{L^p_\mu(\mathbb{R}^n)} \leqslant C_p \|f\|_{L^p_\mu(\mathbb{R}^n)}$ for all $2 < p < 2 + \epsilon'$ where ϵ' to be chosen. Let η_Q be a smooth function satisfying $\mathbb{1}_Q \leqslant \eta_Q \leqslant \mathbb{1}_{2Q}$ and also $\|\nabla \eta_Q\|_\infty < \frac{1}{l(Q)}$. We want to get the assumption of Corollary 4.21. To get that estimate, we consider two cases like the proof of Lemma 4.33.

Case 1: $l(Q)^2 \lesssim t$.

Let $u := e^{-t\mathcal{L}_\mu} f$, then $\frac{\partial}{\partial t}u = -\mathcal{L}_\mu u$. By ellipticity

$$
\begin{aligned}
&\frac{1}{\mu(Q)}\int_Q |\nabla u|^2\mu \lesssim \frac{1}{\mu(Q)}\int |\nabla u|^2\eta_Q^2\mu \lesssim \frac{1}{\mu(Q)}\mathrm{Re}\int A\nabla u\nabla u\eta_Q^2 dx\\
=&\frac{1}{\mu(Q)}\mathrm{Re}\int A\nabla u\,\nabla\left[(u-c)\eta_Q^2\right]dx - \frac{1}{\mu(Q)}\mathrm{Re}\int A\nabla u\nabla(\eta_Q^2)(u-c)dx\\
=&-\frac{1}{\mu(Q)}\mathrm{Re}\int \frac{\partial u}{\partial t}(u-c)\eta_Q^2\mu(x)dx - \frac{1}{\mu(Q)}\mathrm{Re}\int A\nabla u\nabla\eta_Q^2(u-c)dx\\
\leqslant&\frac{l(Q)^2}{\mu(Q)}\int\left|\frac{\partial u}{\partial t}\right|^2\eta_Q\mu(x)dx + \frac{1}{l(Q)^2}\frac{1}{\mu(Q)}\int_{2Q}|u-c|^2\mu(x)dx\\
&+\frac{1}{\mu(Q)}\int\left|A\nabla u\nabla\eta_Q^2(u-c)\right|dx\\
\lesssim&\frac{l(Q)^2}{\mu(Q)}\int\left|\frac{\partial u}{\partial t}\right|^2\eta_Q\mu(x)dx + \frac{1}{\sigma}\frac{C}{l(Q)^2}\frac{1}{\mu(Q)}\int_{2Q}|u-c|^2\mu(x)dx\\
&+\sigma\frac{1}{\mu(Q)}\int|\nabla u|^2\eta_Q^2\mu.
\end{aligned}
$$

Using $\frac{\partial}{\partial t}u = -\mathcal{L}_\mu u$, the property of A_2 condition, i.e. $\mu \in A_{2-\kappa}$ for some fix small κ, the weighted Sobolev's inequality and hide the σ small part of the right hand side, there is some θ small enough so that

$$
\begin{aligned}
&\left(\frac{1}{\mu(Q)}\int_Q|\nabla u|^2\mu(x)dx\right)^{\frac{1}{2}}\\
\lesssim_\sigma\quad&\left(\frac{l(Q)^2}{\mu(Q)}\int\left|\mathcal{L}_\mu u\right|^2\eta_Q\mu(x)dx\right)^{\frac{1}{2}} + \left(\frac{1}{\mu(2Q)}\int_{2Q}|\nabla u|^{2-\theta}\mu\right)^{\frac{1}{2-\theta}}\\
=\quad&\left(\frac{1}{t}\frac{l(Q)^2}{t}\frac{1}{\mu(Q)}\int\left|t\mathcal{L}_\mu u\right|^2\eta_Q\mu(x)dx\right)^{\frac{1}{2}} + \left(\frac{1}{\mu(2Q)}\int_{2Q}|\nabla u|^{2-\theta}\mu\right)^{\frac{1}{2-\theta}}\\
\leqslant\quad&\left(\frac{1}{t}\frac{1}{\mu(Q)}\int_{2Q}\left|t\mathcal{L}_\mu u\right|^2\eta_Q\mu(x)dx\right)^{\frac{1}{2}} + \left(\frac{1}{\mu(2Q)}\int_{2Q}|\nabla u|^{2-\theta}\mu\right)^{\frac{1}{2-\theta}}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\left(\frac{1}{\mu(Q)}\int_Q|\sqrt{t}\nabla u|^2\mu(x)dx\right)^{\frac{1}{2}} \lesssim_\sigma \left(\frac{1}{\mu(2Q)}\int_{2Q}\left|t\mathcal{L}_\mu u\right|^2\eta_Q\mu(x)dx\right)^{\frac{1}{2}} +&\\
+\left(\frac{1}{\mu(2Q)}\int_{2Q}\left|\sqrt{t}\nabla u\right|^{2-\theta}\mu\right)^{\frac{1}{2-\theta}}&.
\end{aligned}
$$

Case 2: $l(Q)^2 \gtrsim t$.

Let us denote $\overline{u} = \frac{1}{\mu(2Q)} \int_{2Q} u\mu dx$ where u was defined in Case 1 above. By Caccioppoli's inequality for parabolic equations $\partial_t u - \mathcal{L}_\mu u = 0$, as proved in the argument for Case 1 just above, we have

$$\|\nabla(u-\overline{u})\|_{L^2_\mu(Q)} \lesssim \left\|(u-\overline{u})\frac{1}{l(Q)}\right\|_{L^2_\mu(2Q)} + \|u-\overline{u}\|^{\frac{1}{2}}_{L^2_\mu(2Q)} \left\|\frac{\partial u}{\partial t}\right\|^{\frac{1}{2}}_{L^2_\mu(2Q)}.$$

Thus, by weighted Sobolev Theorem 4.12 and triangle inequality as well as $\mu \in A_{2-\kappa}$

$$\begin{aligned}
&\left(\frac{1}{\mu(Q)}\int_Q |\nabla u|^2 \mu\right)^{\frac{1}{2}} \\
&\lesssim \left(\frac{1}{\mu(2Q)}\int_{2Q} |u-\overline{u}|^2\mu\right)^{\frac{1}{2}} \frac{1}{l(Q)} + \frac{1}{\mu(2Q)^{1/2}}\|u-\overline{u}\|^{\frac{1}{2}}_{L^2_\mu(2Q)} \left\|\frac{\partial u}{\partial t}\right\|^{\frac{1}{2}}_{L^2_\mu(2Q)} \\
&\lesssim \left(\frac{1}{\mu(2Q)}\int_{2Q} |\nabla u|^{2-\theta}\mu\right)^{\frac{1}{2-\theta}} + \frac{1}{\mu(2Q)^{1/2}}\left(\|u\|^{\frac{1}{2}}_{L^2_\mu(2Q)} + \|\overline{u}\|^{\frac{1}{2}}_{L^2_\mu(2Q)}\right) \left\|\frac{\partial u}{\partial t}\right\|^{\frac{1}{2}}_{L^2_\mu(2Q)} \\
&\lesssim \left(\frac{1}{\mu(2Q)}\int_{2Q} |\nabla u|^{2-\theta}\mu\right)^{\frac{1}{2-\theta}} + \frac{1}{\mu(2Q)^{1/2}}\|u\|^{\frac{1}{2}}_{L^2_\mu(2Q)}\|\mathcal{L}_\mu u\|^{\frac{1}{2}}_{L^2_\mu(2Q)} \\
&\lesssim \left(\frac{1}{\mu(2Q)}\int_{2Q} |\nabla u|^{2-\theta}\mu\right)^{\frac{1}{2-\theta}} + \left(\frac{1}{\mu(2Q)}\int_{2Q} \left|\frac{u}{\sqrt{t}}\right|^2 \mu\right)^{\frac{1}{2}} + \\
&\quad + \left(\frac{1}{\mu(2Q)}\int_{2Q} |\sqrt{t}\mathcal{L}_\mu u|^2\mu\right)^{\frac{1}{2}}.
\end{aligned}$$

Therefore, from the results of two cases, we conclude that

$$\begin{aligned}
&\left(\frac{1}{\mu(Q)}\int_Q |\sqrt{t}\nabla u|^2 \mu(x)dx\right)^{\frac{1}{2}} \\
&\lesssim \left(\frac{1}{\mu(2Q)}\int_{2Q} \left|t\mathcal{L}_\mu u\right|^2 \eta_Q\mu(x)dx\right)^{\frac{1}{2}} + \left(\frac{1}{\mu(2Q)}\int_{2Q} \left|\sqrt{t}\nabla u\right|^{2-\theta}\mu\right)^{\frac{1}{2-\theta}} \\
&\quad + \left(\frac{1}{\mu(2Q)}\int_{2Q} |u|^2\mu\right)^{\frac{1}{2}}.
\end{aligned}$$

Combine this with (4.17) to get the claim $\|\sqrt{t}\nabla e^{-t\mathcal{L}_\mu} f\|_{L^p_\mu(\mathbb{R}^n)} \leqslant C_p\|f\|_{L^p_\mu(\mathbb{R}^n)}$ for $2 < p < 2+\epsilon'$ by invoking Corollary 4.21, Lemma 4.33 for some $\epsilon' = \epsilon'(\epsilon_1, \epsilon_2, \theta, \mu)$ and the fact that $\left\|\left[M_\mu(u^s)\right]^{\frac{1}{s}}\right\|_{L^p_\mu} \lesssim \|f\|_{L^p_\mu(\mathbb{R}^n)}$ for all $p > s$ and $u = e^{-t\mathcal{L}_\mu} f$.

It remains to prove the result for the case $p < 2$. We have that for any function f, and a cube $B = B(x_0, r)$, define $\left(e^{-\frac{t}{2}\mathcal{L}_\mu} f\right)_i := \left(e^{-\frac{t}{2}\mathcal{L}_\mu} f\right) \mathbb{1}_{C_i}$ and also $f_i := f \mathbb{1}_{C_i(B)}$ where $C_i(B)$'s are annulli corresponding to any ball B defined as following $C_1(B) := 4B$, for $i \geqslant 2$, $C_i(B) = 2^{i+1}B \setminus 2^i B$. Then by L^2 off-diagonal estimates for $\sqrt{t}\nabla e^{-\frac{t}{2}\mathcal{L}_\mu}$ and considering the situation $\sqrt{t} \simeq r$, we obtain

$$\begin{aligned}
\left\| \sqrt{t}\nabla e^{-t\mathcal{L}_\mu} f \right\|_{L^2_\mu(B)} &= \left\| \sqrt{t}\nabla e^{-\frac{t}{2}\mathcal{L}_\mu} e^{-\frac{t}{2}\mathcal{L}_\mu} f \right\|_{L^2_\mu(B)} \\
&\lesssim \sum_{j=2}^{\infty} e^{-\frac{d^2(B, C_j(B))}{ct}} \left\| \left(e^{-\frac{t}{2}\mathcal{L}_\mu} f\right)_j \right\|_{L^2_\mu(C_j(B))} + \left\| e^{-\frac{t}{2}\mathcal{L}_\mu} f \right\|_{L^2_\mu(4B)} \qquad (4.18) \\
&\leqslant \sum_{j\geqslant 1} e^{-\frac{4^j}{c}} \left\| e^{-\frac{t}{2}\mathcal{L}_\mu} f \right\|_{L^2_\mu(2^j B)}.
\end{aligned}$$

Now, we apply $L^q - L^2$ off-diagonal estimates for $e^{-\frac{t}{2}\mathcal{L}_\mu}$ for $q_0 < q < 2$ where $\frac{1}{q} = \frac{\alpha}{q_0} + \frac{1-\alpha}{2}$ and $0 < \alpha < 1$, with the set B is replaced by $2^j B$ and we also note that $r^2 \simeq t$, get

$$\begin{aligned}
&\left\| e^{-\frac{t}{2}\mathcal{L}_\mu} f \right\|_{L^2_\mu(2^j B)} \\
&\lesssim \sum_{k\geqslant 2} \left[\mu(2^k(2^j B))\right]^{\left(\frac{1}{2}-\frac{1}{q}\right)} \left(2^k 2^j\right)^{\alpha} e^{-\frac{\left(2^k 2^j\right)^2(1-\alpha)}{c}} \|f\|_{L^q_\mu(C_k(2^j B))} \\
&+ \left[\mu(2^{j+2}B)\right]^{\left(\frac{1}{2}-\frac{1}{q}\right)} \left(2^{j+2}\right)^{\alpha} \|f\|_{L^q_\mu(2^{j+2}B)} \qquad (4.19) \\
&\lesssim \sum_{k\geqslant 2} \left[\mu(2^{k+j}B))\right]^{\frac{1}{2}-\frac{1}{q}} \left(2^{k+j}\right)^{\alpha} e^{-\frac{\left(4^{k+j}\right)(1-\alpha)}{c}} \|f\|_{L^q_\mu(2^{k+j}B)} \\
&\lesssim \sum_{k\geqslant 2} 4^{n(k+j)} \left[\mu(B))\right]^{\frac{1}{2}-\frac{1}{q_1}} \left(2^{k+j}\right)^{\alpha} e^{-\frac{\left(4^{k+j}\right)(1-\alpha)}{c}} \left\| \left[M(|f|^q)\right]^{\frac{1}{q}} \right\|_{L^{q_1}_\mu(B)},
\end{aligned}$$

for all $j \geqslant 1$, $2 > q_1 > q$ and we have used doubling condition for measure μ and property of maximal function to get the last inequality.

Therefore, from (4.18) and (4.19)

$$\begin{aligned}
&\left\| \sqrt{t}\nabla e^{-t\mathcal{L}_\mu} f \right\|_{L^2_\mu(B)} \\
&\lesssim \sum_{j\geqslant 1} e^{-\frac{4^j}{c}} \sum_{k\geqslant 2} e^{-\frac{\left(4^{k+j}\right)(1-\alpha)}{c}} 4^{n(k+j)} \left(2^{k+j}\right)^{\alpha} \left[\mu(B))\right]^{\frac{1}{2}-\frac{1}{q}} \left\| \left[M(|f|^{q_0})\right]^{\frac{1}{q_0}} \right\|_{L^q_\mu(B)} \\
&\lesssim \left[\mu(B))\right]^{\frac{1}{2}-\frac{1}{q_1}} \left\| \left[M(|f|^q)\right]^{\frac{1}{q}} \right\|_{L^{q_1}_\mu(B)}.
\end{aligned}$$

Hence, for any q_1 such that $q_0 < q_1 < 2$, we may have some q satisfying $q_0 < q < q_1 < 2$ where q_0 was mentioned in paragraph just right above Lemma 4.28, obtain

$$\left(\frac{1}{\mu(B)}\int_B \left|\sqrt{t}e^{-t\mathcal{L}_\mu}f\right|^2\mu\right)^{\frac{1}{2}} \lesssim \left(\frac{1}{\mu(B)}\int_B \left|\left[M_\mu(|f|^q)\right]^{\frac{1}{q}}\right|^{q_1}\mu\right)^{\frac{1}{q_1}}. \tag{4.20}$$

This implies if $\mathbb{D}_{\sqrt{t}}$ is the collections of all dyadic cubes Q of $\mathbb{R}^n$ with side length $l(Q) \simeq \sqrt{t}$, by Hölder inequality, get

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left|\sqrt{t}\nabla e^{-t\mathcal{L}_\mu}f\right|^{q_1}\mu = \sum_{Q\in\mathbb{D}_{\sqrt{t}}}\int_Q\left|\sqrt{t}\nabla e^{-t\mathcal{L}_\mu}f\right|^{q_1}\mu\\
&\leqslant \sum_{Q\in\mathbb{D}_{\sqrt{t}}}\mu(Q)\left(\frac{1}{\mu(Q)}\int_Q\left|\sqrt{t}\nabla e^{-t\mathcal{L}_\mu}f\right|^2\mu\right)^{\frac{q_1}{2}}\\
&\lesssim \sum_{Q\in\mathbb{D}_{\sqrt{t}}}\mu(Q)\left(\frac{1}{\mu(Q)}\int_Q[M_\mu|f|^q]^{\frac{q_1}{q}}\mu\right) = \sum_{Q\in\mathbb{D}_{\sqrt{t}}}\int_Q[M_\mu|f|^q]^{\frac{q_1}{q}}\mu \lesssim \int_{\mathbb{R}^n}|f|^{q_1}\mu,
\end{aligned}$$

by (4.20) and L^q_μ bound of Maximal functions for $q_0 < q < 2$.

Thus, we just proved that there is some ϵ small enough such that for all p with $\left|\frac{1}{2}-\frac{1}{p}\right| < \epsilon$, we have

$$\|e^{-t\mathcal{L}_\mu}f\|_{L^p_\mu(\mathbb{R}^n)} + \|\sqrt{t}\nabla e^{-t\mathcal{L}_\mu}f\|_{L^p_\mu(\mathbb{R}^n)} \leqslant C_p\|f\|_{L^p_\mu(\mathbb{R}^n)}.$$

∎

Remark 4.35. Observe that in the case of real valued operator, by following the proof of Gaussian estimates for semi-group $e^{-t\mathcal{L}_\mu}f$ in [C-UR1] with the note that all the needed results for degenerate parabolic equations still valid in non-symmetric case due to the paper by Kazuhiro Ishige in [I], we also have that the semi-group $e^{-t\mathcal{L}_\mu}$ has Gaussian estimates, so we have L^p-bound for $e^{-t\mathcal{L}_\mu}f$ or $\|e^{-t\mathcal{L}_\mu}f\|_{L^p_\mu} \leqslant C\|f\|_{L^p_\mu}$ for $1 < p < \infty$.

Remark 4.36. By analytic extension (see Section 4.6 in [A]), we have

$$\|e^{-z\mathcal{L}_\mu}f\|_{L^p_\mu(\mathbb{R}^n)} + \|\sqrt{z}\nabla e^{-z\mathcal{L}_\mu}f\|_{L^p_\mu(\mathbb{R}^n)} \leqslant C_p\|f\|_{L^p_\mu(\mathbb{R}^n)}, \tag{4.21}$$

for all $z \in \Sigma_\nu$ for some $0 < \nu < \frac{\pi}{2}$ and $2 - \epsilon < p < 2 + \epsilon$.

By interpolating between L^2_μ off-diagonal estimate and L^p_μ bounds for $\sqrt{t}\nabla e^{-t\mathcal{L}_\mu}$, we get the following L^p_μ off-diagonal estimate corollary.

Corollary 4.37. *Let ϵ be as in Theorem 4.1. The function $\sqrt{t}\nabla e^{-t\mathcal{L}_\mu}$ satisfies $L^p_\mu - L^p_\mu$ off-diagonal estimate, i.e. for any closed set E, F and for any function f supported in E then there is a constant C depending on μ, n, ellipticity and p such that*

$$\left\| \sqrt{t}\nabla e^{-t\mathcal{L}_\mu} f \right\|_{L^p_\mu(F)} \leqslant C e^{-\frac{\mathrm{dist}^2(E,F)}{ct}} \|f\|_{L^p_\mu(E)},$$

where $2 - \epsilon < p < 2 + \epsilon$ and c are uniform constants depending on μ, n and ellipticity conditions.

4.4.3 Some kind of local off-diagonal $L^2_\mu - L^p_\mu$ estimates for $\nabla e^{-t\mathcal{L}_\mu}$

In this section, we will establish some kind of local $L^2_\mu - L^p_\mu$ estimates for $\nabla e^{-t\mathcal{L}_\mu}$ over some balls $B = B(x_0, r)$.

Lemma 4.38. *Let N be a fixed number, for any k such that $1 \leqslant k \leqslant N$, then for any $p \in (2, 2+\epsilon)$ where ϵ was determined in Theorem 4.1, if B is a ball with radius $r = r(B)$, we have*

$$\left(\frac{1}{\mu(B)} \int_B \left|\nabla e^{-t\mathcal{L}_\mu} e^{-kr^2\mathcal{L}_\mu} f\right|^p \mu\right)^{\frac{1}{p}} \leqslant C \sum_{i=1}^{\infty} e^{-\frac{4^{i-1}}{c}} 2^{2n(i+1)} \left(\frac{1}{\mu(2^{i+1}B)} \int_{2^{i+1}B} |\nabla e^{-t\mathcal{L}_\mu} f|^2 \mu\right)^{\frac{1}{2}}. \quad (4.22)$$

where all the constants do not depend on k but depend on N.

Proof. This lemma is a consequence of the L^p_μ off-diagonal estimate (Corollary 4.37) for $\sqrt{t}\nabla e^{-t\mathcal{L}_\mu}$ and weighted Poincaré inequality. Indeed, set $C_i(B) = 2^{i+1}B \setminus 2^i B$ for $i \geqslant 2$ and

$C_1(B) = 4B$. We define

$$A := \frac{1}{\mu(B)} \int_B e^{-t\mathcal{L}_\mu} f\mu,$$

and

$$A_i := \frac{1}{\mu(2^i B)} \int_{2^i B} e^{-t\mathcal{L}_\mu} f\mu.$$

Then,

$$\begin{aligned}
&\left(\frac{1}{\mu(B)} \int_B \left|\nabla e^{-t\mathcal{L}_\mu} e^{-kr^2\mathcal{L}_\mu} f\right|^p \mu\right)^{\frac{1}{p}} = \frac{1}{r}\left(\frac{1}{\mu(B)} \int_B \left|r\nabla e^{-kr^2\mathcal{L}_\mu}\left(e^{-t\mathcal{L}_\mu} f - A\right)\right|^p \mu\right)^{\frac{1}{p}} \\
&\leqslant \sum_{i=1}^{\infty} \frac{1}{r}\left(\frac{1}{\mu(B)} \int_B \left|r\nabla e^{-kr^2\mathcal{L}_\mu}\left[\left(e^{-t\mathcal{L}_\mu} f - A\right)\mathbb{1}_{C_i(B)}\right]\right|^p \mu\right)^{\frac{1}{p}} \\
&\leqslant \sum_{i=1}^{\infty} \frac{1}{r} e^{-\frac{\operatorname{dist}^2(B,C_iB)}{ckr^2}}\left(\frac{1}{\mu(B)} \int_{C_iB} \left|\left(e^{-t\mathcal{L}_\mu} f - A\right)\mathbb{1}_{C_iB}\right|^p \mu\right)^{\frac{1}{p}} \\
&\leqslant \sum_{i=1}^{\infty} \frac{1}{r} e^{-\frac{4^{i-1}}{cN}}\left(\frac{1}{\mu(B)} \int_{C_iB} \left|\left(e^{-t\mathcal{L}_\mu} f - A\right)\mathbb{1}_{C_iB}\right|^p \mu\right)^{\frac{1}{p}}.
\end{aligned} \tag{4.23}$$

where we have used Corollary 4.37 (L^p_μ off-diagonal estimate) to get the second inequality and the fact that

$$e^{-\frac{\operatorname{dist}^2(B,C_iB)}{ckr^2}} \leqslant e^{-\frac{4^{i-1}r^2}{cNr^2}} = e^{-\frac{4^{i-1}}{cN}},$$

for $1 \leqslant k \leqslant N$.

To finish our proof, we need an estimate for the last summation above. Notice that $\mu \in A_2$ Muckenhoupt class, so by weighted Poincaré inequality (Theorem 4.12), we get

$$\begin{aligned}
&\left(\frac{1}{\mu(B)} \int_{C_iB} \left|\left(e^{-t\mathcal{L}_\mu} f - A_{i+1}\right)\mathbb{1}_{C_iB}\right|^p \mu\right)^{\frac{1}{p}} \\
&\leqslant \left(\frac{\mu(2^{i+1}B)}{\mu(B)} \frac{1}{\mu(2^{i+1}B)} \int_{2^{i+1}B} \left|\left(e^{-t\mathcal{L}_\mu} f - A_{i+1}\right)\right|^p \mu\right)^{\frac{1}{p}} \\
&\leqslant C2^{\frac{2n(i+1)}{p}} 2^{i+1} r \left(\frac{1}{\mu(2^{i+1}B)} \int_{2^{i+1}B} \left|\nabla e^{-t\mathcal{L}_\mu} f\right|^2 \mu\right)^{\frac{1}{2}} \\
&\leqslant C2^{2n(i+1)} r \left(\frac{1}{\mu(2^{i+1}B)} \int_{2^{i+1}B} \left|\nabla e^{-t\mathcal{L}_\mu} f\right|^2 \mu\right)^{\frac{1}{2}}.
\end{aligned}$$

By Hölder's inequality and also weighted Poincaré inequality (Theorem 4.12), one obtains

$$\begin{aligned}
\left|A_{j+1}-A_j\right| &= \left|\frac{1}{\mu(2^{j+1}B)}\int_{2^{j+1}B} e^{-t\mathcal{L}_\mu} f\mu - \frac{1}{\mu(2^jB)}\int_{2^jB} e^{-t\mathcal{L}_\mu} f\mu\right| \\
&\leqslant C\frac{1}{\mu(2^{j+1}B)}\int_{2^{j+1}B}\left|e^{-t\mathcal{L}_\mu} f - \frac{1}{\mu(2^{j+1}B)}\int_{2^{j+1}B} e^{-t\mathcal{L}_\mu} f\mu\right|\mu \\
&\leqslant C\left(\frac{1}{\mu(2^{j+1}B)}\int_{2^{j+1}B}\left|e^{-t\mathcal{L}_\mu} f - \frac{1}{\mu(2^{j+1}B)}\int_{2^{j+1}B} e^{-t\mathcal{L}_\mu} f\mu\right|^p\mu\right)^{\frac{1}{p}} \\
&\leqslant C2^{j+1}r\left(\frac{1}{\mu(2^{j+1}B)}\int_{2^{j+1}B}\left|\nabla e^{-t\mathcal{L}_\mu} f\right|^2\mu\right)^{\frac{1}{2}},
\end{aligned}$$

for any $j \leqslant i$.

Hence,

$$\begin{aligned}
&\left(\frac{1}{\mu(B)}\int_{C_iB}\left|A_{j+1}-A_j\right|^p\mu\right)^{\frac{1}{p}} \\
&\leqslant C\left(\frac{\mu(2^{i+1}B)}{\mu(B)}\right)^{\frac{1}{p}} 2^{j+1}r\left(\frac{1}{\mu(2^{j+1}B)}\int_{2^{j+1}B}\left|\nabla e^{-t\mathcal{L}_\mu} f\right|^2\mu\right)^{\frac{1}{2}} \\
&\leqslant C2^{\frac{2n(i+1)}{p}}2^{j+1}r\left(\frac{\mu(2^{i+1}B)}{\mu(2^{j+1}B)}\frac{1}{\mu(2^{i+1}B)}\int_{2^{i+1}B}\left|\nabla e^{-t\mathcal{L}_\mu} f\right|^2\mu\right)^{\frac{1}{2}} \\
&\leqslant C2^{\frac{2n(i+1)}{p}}2^{j+1}2^{n(i-j)}r\left(\frac{1}{\mu(2^{i+1}B)}\int_{2^{i+1}B}\left|\nabla e^{-t\mathcal{L}_\mu} f\right|^2\mu\right)^{\frac{1}{2}}.
\end{aligned}$$

Now, we have enough ingredients to prove Lemma 4.38. Notice that $A = A_1$, then from the estimate (4.23)

$$\begin{aligned}
&\left(\frac{1}{\mu(B)}\int_B\left|\nabla e^{-t\mathcal{L}_\mu}e^{-kr^2\mathcal{L}_\mu} f\right|^p\mu\right)^{\frac{1}{p}} \\
&\leqslant \sum_{i=1}^{\infty}\frac{1}{r}e^{-\frac{4^{i-1}}{cN}}\left(\frac{1}{\mu(B)}\int_{C_iB}\left|\left(e^{-t\mathcal{L}_\mu} f - A\right)\mathbb{1}_{C_iB}\right|^p\mu\right)^{\frac{1}{p}} \\
&\leqslant \sum_{i=1}^{\infty}\frac{1}{r}e^{-\frac{4^{i-1}}{cN}}\left(\frac{1}{\mu(B)}\int_{C_iB}\left|\left(e^{-t\mathcal{L}_\mu} f - A_{i+1}\right)\mathbb{1}_{C_iB}\right|^p\mu\right)^{\frac{1}{p}} \\
&\qquad + \sum_{i=1}^{\infty}\frac{1}{r}e^{-\frac{4^{i-1}}{cN}}\sum_{j=1}^{i}\left(\frac{1}{\mu(B)}\int_{C_iB}\left|A_{j+1}-A_j\right)|^p\mu\right)^{\frac{1}{p}}
\end{aligned}$$

$$\leqslant C\sum_{i=1}^{\infty} e^{-\frac{4^{i-1}}{cN}} 2^{2n(i+1)} \left(\frac{1}{\mu(2^{i+1}B)}\int_{2^{i+1}B} \left|\nabla e^{-t\mathcal{L}_\mu} f\right|^2 \mu\right)^{\frac{1}{2}} +$$

$$+ C\sum_{i=1}^{\infty} e^{-\frac{4^{i-1}}{cN}} \sum_{j=1}^{i} 2^{\frac{2ni}{p}} 2^{j+1} 2^{n(i-j)} \left(\frac{1}{\mu(2^{i+1}B)}\int_{2^{i+1}B} \left|\nabla e^{-t\mathcal{L}_\mu} f\right|^2 \mu\right)^{\frac{1}{2}}$$

$$\leqslant C\sum_{i=1}^{\infty} e^{-\frac{4^{i-1}}{cN}} 2^{2n(i+1)} \left(\frac{1}{\mu(2^{i+1}B)}\int_{2^{i+1}B} \left|\nabla e^{-t\mathcal{L}_\mu} f\right|^2 \mu\right)^{\frac{1}{2}}.$$

since $\sum_{j=1}^{\infty} 2^{j+1-nj} < \infty$.

∎

Lemma 4.39. *There is c depending only on dimension and doubling constant of the weight $\mu \in A_2$ and α below, such that for $\frac{1}{q} = \frac{\alpha}{q_0} + \frac{1-\alpha}{2}$ where $0 < \alpha < 1$, then if $\operatorname{supp} f \subset B$, we get*

$$\left(\frac{1}{\mu(2^jB)}\int_{C_j(B)} \left|\sqrt{z}\nabla e^{-z\mathcal{L}_\mu} f\right|^2 \mu\right)^{\frac{1}{2}}$$

$$\leqslant C(q) j e^{-\frac{\operatorname{dist}^2(C_j(B),B)(1-\alpha)}{c|z|}} \left(\frac{2^j r}{\sqrt{|z|}} + 1\right)^{\alpha} \left(\frac{1}{\mu(B)}\int_B |f|^q \mu\right)^{\frac{1}{q}},$$

for all $z \in \Sigma_\beta$, $\beta \leqslant \frac{\pi}{4}$ and where $C(q)$ is a constant depending on q and fixed parameters. Moreover, we also have the following

$$\left(\frac{1}{\mu(2^jB)}\int_{C_j(B)} \left|\sqrt{z}\nabla e^{-z\mathcal{L}_\mu} f\right|^2 \mu\right)^{\frac{1}{2}} \leqslant Ce^{-\frac{\operatorname{dist}^2(C_j(B),B)(1-\alpha)}{c|z|}} \left(\frac{1}{\mu(B)}\int_B |f|^p \mu\right)^{\frac{1}{p}},$$

for all $2 \leqslant p < 2+\epsilon$, where ϵ is determined in Theorem 4.1.

Proof. The second part of the lemma is a consequence of L^2_μ off-diagonal estimate, [Proposition 4.18] and Hölder's inequality. So we only need to give details for the proof of the first part. It is a consequence of L^2_μ off-diagonal estimate of $\sqrt{z}\nabla e^{-z\mathcal{L}_\mu}$ and local $L^q - L^2$ off-diagonal estimates of $e^{-z\mathcal{L}_\mu}$ as in the Remark 4.32. Indeed, for any function g, set

$g_1 = g\mathbb{1}_{C_1(B)}$ and $g_k = g\mathbb{1}_{C_k(B)}$ for $k \geqslant 2$ where $C_1(B) = 4B$ and $C_k(B) = 2^k B \setminus 2^{k-1}B$ for any ball B as we defined before. First, let us prove for the case $j = 1$. We have

$$\begin{aligned}
&\left(\int_{4B} \left|\sqrt{z}\nabla e^{-z\mathcal{L}_\mu} f\right|^2 \mu\right)^{\frac{1}{2}} \leqslant \sum_{k=1}^{\infty} \left(\int_{4B} \left|\sqrt{z}\nabla e^{-\frac{z}{2}\mathcal{L}_\mu} \left(e^{-\frac{z}{2}\mathcal{L}_\mu} f\right)_k\right|^2 \mu\right)^{\frac{1}{2}} \\
&\lesssim \sum_{k=1}^{\infty} e^{\frac{-\operatorname{dist}^2(C_k(B),4B)}{c|z|}} \left(\int_{C_k(B)} \left|e^{-\frac{z}{2}\mathcal{L}_\mu} f\right|^2 \mu\right)^{\frac{1}{2}} \\
&\lesssim \sum_{k=1}^{\infty} e^{\frac{-\operatorname{dist}^2(C_k(B),4B)}{c|z|}} \left[\mu(2^{k+1}B)\right]^{\frac{1}{2}-\frac{1}{q}} \left(\frac{2^{k+1}r}{\sqrt{|z|}} + 1\right)^{\alpha} e^{-\frac{\operatorname{dist}^2(B,C_k(B))(1-\alpha)}{c|z|}} \|f\|_{L^q_\mu(B)} \\
&\lesssim \sum_{k=2}^{\infty} e^{\frac{-\operatorname{dist}^2(C_k(B),4B)}{c|z|}} \left[\mu(2^{k+1}B)\right]^{\frac{1}{2}-\frac{1}{q}} e^{-\frac{\left(2^{k-1}r\right)^2(1-\alpha)}{c|z|}} \|f\|_{L^q_\mu(B)} \\
&\quad + \left[\mu(2B)\right]^{\frac{1}{2}-\frac{1}{q}} \left(\frac{4r}{\sqrt{|z|}} + 1\right)^{\alpha} \|f\|_{L^q_\mu(B)} \\
&\lesssim \sum_{k=1}^{\infty} \left(2^{k-1}\right)^{\left(\frac{1}{2}-\frac{1}{q}\right)\delta'} \left[\mu(2B)\right]^{\frac{1}{2}-\frac{1}{q}} \left(\frac{4r}{\sqrt{|z|}} + 1\right)^{\alpha} \|f\|_{L^q_\mu(B)} \\
&\lesssim \left[\mu(2B)\right]^{\frac{1}{2}-\frac{1}{q}} \left(\frac{4r}{\sqrt{|z|}} + 1\right)^{\alpha} \|f\|_{L^q_\mu(B)},
\end{aligned}$$

where we have used L^2_μ-offdiagonal estimate to get the second inequality, Remark 4.32 to get the third inequality and the fact that there is some $\delta' > 0$ such that $\frac{\mu(B)}{\mu(2^k B)} \lesssim \left(\frac{|B|}{|2^k B|}\right)^{\delta'}$ since $\mu \in A_2$ and also $\left(\frac{2^{k+1}r}{\sqrt{|z|}} + 1\right)^{\alpha} e^{-\frac{\left(2^{k-1}r\right)^2(1-\alpha)}{c|z|}} \leqslant Ce^{-\frac{\left(2^{k-1}r\right)^2(1-\alpha)}{c|z|}}$ for some new constant c depending on the previous constant c and the constant C is independent of k, r, z, c to get the second to last inequality.

In order to prove for $j \geqslant 2$, we do the same way as for $j = 1$ but we need to do carefully. Here is the details

$$\begin{aligned}
&\left\|\sqrt{z}\nabla e^{-z\mathcal{L}_\mu} f\right\|_{L^2_\mu(C_j B)} \\
&\leqslant C\sum_{k=1}^{j-2} \left\|\sqrt{z}\nabla e^{-\frac{z}{2}\mathcal{L}_\mu} \left(e^{-\frac{z}{2}\mathcal{L}_\mu} f\right)_k\right\|_{L^2_\mu(C_j B)} + \sum_{k=j-1}^{j+1} \left\|\sqrt{z}\nabla e^{-\frac{z}{2}\mathcal{L}_\mu} \left(e^{-\frac{z}{2}\mathcal{L}_\mu} f\right)_k\right\|_{L^2_\mu(C_j B)} \\
&\quad + \sum_{k=j+2}^{\infty} \left\|\sqrt{z}\nabla e^{-\frac{z}{2}\mathcal{L}_\mu} \left(e^{-\frac{z}{2}\mathcal{L}_\mu} f\right)_k\right\|_{L^2_\mu(C_j B)} \\
&=: I + II + III.
\end{aligned}$$

We estimate I, II and III in order. By the L^2 offdiagonal estimate of $\sqrt{z}e^{-z\mathcal{L}_\mu}$ and $L^p - L^2$ offdiagonal estimate of $e^{-z\mathcal{L}_\mu}$ as in Lemma 4.30

$$
\begin{aligned}
I \leqslant & C\sum_{k=1}^{j-1} e^{-\frac{(2^{j-k-1}r)^2}{c|z|}} \left\|e^{-\frac{z}{2}\mathcal{L}_\mu} f\right\|_{L^2_\mu(C_kB)} \\
\leqslant & C\sum_{k=2}^{j-1} e^{-\frac{(2^{j-k-1}r)^2}{c|z|}} e^{-\frac{(2^k r)^2(1-\alpha)}{c|z|}} \left[\mu(2^kB)\right]^{\frac{1}{2}-\frac{1}{q}} \left(\frac{2^k r}{\sqrt{|z|}}+1\right)^\alpha \|f\|_{L^q_\mu(B)} \\
& + e^{-\frac{(2^{j-2}r)^2(1-\alpha)}{c|z|}} \left[\mu(2B)\right]^{\frac{1}{2}-\frac{1}{q}} \left(\frac{2r}{\sqrt{|z|}}+1\right)^\alpha \|f\|_{L^q_\mu(B)}
\end{aligned}
$$

Observe that

$$
e^{-\frac{(2^{j-k-1}r)^2}{c|z|}} e^{-\frac{(2^k r)^2(1-\alpha)}{c|z|}} \leqslant Ce^{-\frac{2^j r^2(1-\alpha)}{c|z|}}, \quad \text{for all } 1 \leqslant k \leqslant j,
$$

and

$$
\left[\mu(2^kB)\right]^{\frac{1}{2}-\frac{1}{q}} \leqslant \left[\mu(B)\right]^{\frac{1}{2}-\frac{1}{q}},
$$

since $q < 2$. So,

$$
I \leqslant C je^{-\frac{2^j r^2(1-\alpha)}{c|z|}} \left[\mu(B)\right]^{\frac{1}{2}-\frac{1}{q}} \left(\frac{2^j r}{\sqrt{|z|}}+1\right)^\alpha \left(\int_B |f|^q\mu\right)^{\frac{1}{q}}. \tag{4.24}
$$

By the similar argument as for I but easier since $k \simeq j$, we have

$$
\begin{aligned}
II \leqslant & C\sum_{k=j-1}^{j+1} \left\|e^{-\frac{z}{2}\mathcal{L}_\mu} f\right\|_{L^2_\mu(C_kB)} \\
\leqslant & C\sum_{k=j-1}^{j+1} e^{-\frac{\left(2^{k-1}r\right)^2(1-\alpha)}{c|z|}} \left(\frac{2^k r}{\sqrt{|z|}}+1\right)^\alpha \left[\mu\left(2^kB\right)\right]^{\frac{1}{2}-\frac{1}{q}} \left(\int_B |f|^q\mu\right)^{\frac{1}{q}} \\
\lesssim & e^{-\frac{\left(2^{j-2}r\right)^2(1-\alpha)}{c|z|}} \left(\frac{2^j r}{\sqrt{|z|}}+1\right)^\alpha \left[\mu\left(2^jB\right)\right]^{\frac{1}{2}-\frac{1}{p}} \left(\int_B |f|^q\mu\right)^{\frac{1}{q}}.
\end{aligned} \tag{4.25}
$$

We need more work for the last term III, again by using offdiagonal estimate of $\sqrt{z}\nabla e^{-z\mathcal{L}_\mu}$ and $L^q - L^2$ offdiagonal estimate as in Lemma 4.30. Using the fact that

$$
\left(\frac{2^k r}{\sqrt{|z|}}+1\right)^\alpha e^{-\frac{\left(2^{k-1}r\right)^2(1-\alpha)}{c|z|}} \leqslant Ce^{-\frac{\left(2^{k-1}r\right)^2(1-\alpha)}{2c|z|}},
$$

where $k, r, |z|$ are positive and doubling property of A_2 condition (see Lemma 4.16), i.e. $\frac{\mu(B)}{\mu(2^k B)} \lesssim \left(\frac{|B|}{|2^k B|}\right)^{\delta'}$, let $\theta_q := \frac{1}{q} - \frac{1}{2} > 0$, we obtain

$$\begin{aligned}
III \leqslant & \sum_{k=j+2}^{\infty} e^{-\frac{\left(2^{k-j-1}r\right)^2}{c|z|}} \left[\mu\left(2^k B\right)\right]^{\frac{1}{2}-\frac{1}{q}} e^{-\frac{\left(2^{k-1}r\right)^2(1-\alpha)}{c|z|}} \left(\int_B |f|^q \mu\right)^{\frac{1}{q}} \\
= & \sum_{m=1}^{\infty} \sum_{k=mj+2}^{(m+1)j+2} e^{-\frac{\left(2^{k-j-1}r\right)^2}{c|z|}} e^{-\frac{\left(2^{k-1}r\right)^2(1-\alpha)}{c|z|}} \left[\mu\left(2^k B\right)\right]^{\frac{1}{2}-\frac{1}{q}} \left(\int_B |f|^q \mu\right)^{\frac{1}{q}} \\
\leqslant & \sum_{m=1}^{\infty} j e^{-\frac{\left(2^{j}r\right)^2(1-\alpha)}{c|z|}} \left[\mu\left(2^j B\right)\right]^{\frac{1}{2}-\frac{1}{q}} 2^{-(m+1)\theta_q \delta'} \left(\int_B |f|^q \mu\right)^{\frac{1}{q}} \\
\leqslant & C(q,\delta') j e^{-\frac{\left(2^{j}r\right)^2(1-\alpha)}{c|z|}} \left[\mu\left(2^j B\right)\right]^{\frac{1}{2}-\frac{1}{q}} \left(\int_B |f|^q \mu\right)^{\frac{1}{q}}.
\end{aligned} \tag{4.26}$$

The estimates in (4.24 – 4.26) imply

$$\begin{aligned}
& \left(\frac{1}{\mu(2^j B)} \int_{C_j B} \left|\sqrt{z}\nabla e^{-z\mathcal{L}_\mu} f\right|^2 \mu\right)^{\frac{1}{2}} \\
\leqslant & C j e^{-\frac{2^j r^2(1-\alpha)}{c|z|}} \left[\frac{1}{\mu(2^j B)}\right]^{\frac{1}{2}} \left[\mu(B)\right]^{\frac{1}{2}} \left(\frac{2^j r}{\sqrt{|z|}}+1\right)^{\alpha} \left(\frac{1}{\mu(B)} \int_B |f|^q \mu\right)^{\frac{1}{q}} \\
& + e^{-\frac{\left(2^{j-2}r\right)^2(1-\alpha)}{c|z|}} \left(\frac{2^j r}{\sqrt{|z|}}+1\right)^{\alpha} \left(\frac{1}{\mu(2^j B)} \int_B |f|^q \mu\right)^{\frac{1}{q}} \\
& + C(q,\delta') j e^{-\frac{\left(2^{j}r\right)^2(1-\alpha)}{c|z|}} \left(\frac{1}{\mu(2^j B)} \int_B |f|^q \mu\right)^{\frac{1}{q}} \\
\leqslant & C j e^{-\frac{2^j r^2(1-\alpha)}{c|z|}} \left(\frac{2^j r}{\sqrt{|z|}}+1\right)^{\alpha} \left(\frac{1}{\mu(B)} \int_B |f|^q \mu\right)^{\frac{1}{q}} \\
& + e^{-\frac{\left(2^{j-2}r\right)^2(1-\alpha)}{c|z|}} \left(\frac{2^j r}{\sqrt{|z|}}+1\right)^{\alpha} \left(\frac{1}{\mu(B)} \int_B |f|^q \mu\right)^{\frac{1}{q}} \\
& + C(q,\delta') j e^{-\frac{\left(2^{j}r\right)^2(1-\alpha)}{c|z|}} \left(\frac{1}{\mu(B)} \int_B |f|^q \mu\right)^{\frac{1}{q}} \\
\leqslant & C(q,\delta') j e^{-\frac{2^j r^2(1-\alpha)}{c|z|}} \left(\frac{2^j r}{\sqrt{|z|}}+1\right)^{\alpha} \left(\frac{1}{\mu(B)} \int_B |f|^q \mu\right)^{\frac{1}{q}}.
\end{aligned}$$

∎

4.4.4 L^p_μ bounds for vertical square functions

Let us recall a simpler version of Theorem 3.1 which was stated and proved in [AM].

Proposition 4.40. *Fix $1 < q < \infty$ and $a > 0$. Let μ belong to A_2 Muckenhoupt class as before. Then there exists $C = C(q, n, a)$ and $K_0' = K_0'(n, a) \geqslant 1$ with the following property. If F, G are non-negative measurable functions such that for every cube Q, there exist non-negative functions G_Q, H_Q with*

$$\begin{aligned} F &\leqslant G_Q + H_Q \quad \text{a.e. on } Q, & (4.27)\\ \left(\frac{1}{\mu(Q)}\int_Q H_Q^q \mu\right)^{\frac{1}{q}} &\leqslant a M_\mu(F)(x) \quad \text{for all } x \in Q, & (4.28)\\ \frac{1}{\mu(Q)}\int_Q G_Q \mu &\leqslant G(x) \quad \text{for all } x \in Q. & (4.29)\end{aligned}$$

Then for all $\lambda > 0$, for all $K > K_0'$ and $\gamma < 1$

$$\mu(\{M_\mu(F) > K\lambda, G \leqslant \gamma\lambda\}) \leqslant C\left(\frac{1}{K^q} + \frac{\gamma}{K}\right)\mu(\{M_\mu(F) > \lambda\}).$$

Remark 4.41. This Proposition 4.40 implies, for $0 < p < q$, for F, G as above then

$$\|M_\mu(F)\|_{L^p_\mu}^p \leqslant CK^p\left(\frac{a^q}{K^q} + \frac{\gamma}{K}\right)\|M_\mu(F)\|_{L^p_\mu}^p + \frac{K^p}{\gamma^p}\|G\|_{L^p_\mu}^p.$$

If we know that $\|M_\mu(F)\|_{L^p_\mu}^p$ is finite then we may choose K large enough and γ small enough so that

$$CK^p\left(\frac{a^q}{K^q} + \frac{\gamma}{K}\right) \leqslant 1 - 2^{-p}.$$

since $p < q$. Then

$$\|M_\mu(F)\|_{L^p_\mu}^p \leqslant C\frac{(2K)^p}{\gamma^p}\|G\|_{L^p_\mu}^p.$$

Remark 4.42. Notice that, in the proof of the the above proposition in [AM], the proof is the same if we replace the collection of cubes Q center at x with sidelength $l(Q)$ by the corresponding collection of balls $B(x, l(Q)/2)$. Indeed, the main ingredient of the proof in

[AM] is that the centered and uncentered maximal functions are comparable, and the fact that those maximal functions satisfy weak-type (p, p) properties. With a little computation, we see that the maximal function defined over balls is still comparable to those defined over cubes. So, it is not an issue if we work with balls instead of cubes.

The following result a the main tool for proving L^p_μ-boundedness for the case $2 < p$.

Theorem 4.43. *Let $p_* \in (2, \infty]$. Suppose that T is sublinear operator acting on L^2_μ and let $A_r, r > 0$ be a family of linear operators acting on L^2_μ. Assume*

$$\left(\frac{1}{\mu(B)}\int_B |T(I - A_{r(B)}f)|^2\mu\right)^{\frac{1}{2}} \leqslant C\left(M_\mu\left(|f|^2\right)\right)^{\frac{1}{2}}(y), \tag{4.30}$$

and

$$\left(\frac{1}{\mu(B)}\int_B |TA_{r(B)}f)|^{p_*}\mu\right)^{\frac{1}{p_*}} \leqslant C\left(M_\mu\left(|Tf|^2\right)\right)^{\frac{1}{2}}(y), \tag{4.31}$$

for all $f \in L^2_\mu$, all ball B and all $y \in B$ where $r(B)$ is the radius of B. If $2 < p < p_*$ and Tf belongs L^p_μ space then T is strong type (p, p). More precisely, for all $f \in L^p \cap L^2$,

$$\|Tf\|_{L^p_\mu} \leqslant c\|f\|_{L^p_\mu},$$

where c depends only on n, p and p_* and C above.

Proof. This theorem is a consequence of the above proposition. With the Remark 4.42 in hand, we will work with balls instead of cubes. Let $f \in L^2_\mu \cap L^p_\mu$. We let $q = \frac{p_*}{2}$ and set $F = |Tf|^2 \in L^{\frac{p}{2}}_\mu$. By the sublinearity of T, for each ball B, recall that $r(B)$ is the radius of the ball B, there are H_B, G_B, more precisely,

$$\begin{aligned} G_B &= 2|T(I - A_{r(B)})f|^2 \quad \text{and} \\ H_B &= 2|TA_{r(B)}f|^2, \end{aligned}$$

so that

$$F \leqslant G_B + H_B \quad \text{for all } x \in B.$$

Hence, applying Proposition 4.40 and Remark 4.41, with $a \equiv 2C^2, G \equiv 2C^2 M_\mu(f^2)$, $M_\mu(F) \in L_\mu^{\frac{p}{2}}$ from the hypotheses, we obtain for $2 < p < p_*$,

$$\|Tf\|_{L_\mu^p}^2 \leqslant \|M_\mu(F)\|_{L_\mu^{p/2}} \leqslant C\|G\|_{L_\mu^{p/2}} \leqslant C\|f\|_{L_\mu^p}^2.$$

∎

Now, we will use the above theorem to prove the L^p bounds of vertical square functions for the case $p > 2$.

Proposition 4.44. *Let ϵ be as in Theorem 4.1. Then for all $p \in [2, 2+\epsilon)$, we have* $\|g_{\mathcal{L}_\mu}(f)\|_{L_\mu^p} \lesssim \|f\|_{L_\mu^p}$.

Proof. We need to verify the condition (4.30) and (4.31) for $p_* = 2+\epsilon$, again we abuse the notation that ϵ be a small number that could be smaller than the previous epsilon and we still call it ϵ. For each ball B with radius $r = r(B)$, we apply Theorem 4.43 to the sublinear operator $T = g_{\mathcal{L}_\mu}$ with $A_r = I - \left(I - e^{-r^2\mathcal{L}_\mu}\right)^m$ for m a large enough integer number. By Minkowski's inequality and for any k - an integer number satisfying $0 \leqslant k \leqslant m-1$

$$\left(\frac{1}{\mu(B)}\int_B \left|g_{\mathcal{L}_\mu}\left(e^{-kr^2\mathcal{L}_\mu} f\right)\right|^{p_*} \mu\right)^{\frac{2}{p_*}} \leqslant \int_0^\infty \left(\frac{1}{\mu(B)}\int_B \left|(t\mathcal{L}_\mu)^{\frac{1}{2}} e^{-t\mathcal{L}_\mu}\left(e^{-kr^2\mathcal{L}_\mu} f\right)\right|^{p_*} \mu\right)^{\frac{2}{p_*}} \frac{dt}{t}.$$

Next, we are using local $L^2 - L^p$ estimates for $e^{-t\mathcal{L}_\mu}$ (see formula (4.15)) and the fact that $\sum_i c_i < \infty$, Cauchy-Schwarz's inequality together taking the integration on RHS over $2^{j+1}B$

instead of $C_j(B)$, we get

$$\left(\frac{1}{\mu(B)}\int_B |e^{-kr^2\mathcal{L}_\mu}h|^{p_*}\mu\right)^{\frac{2}{p_*}} \leqslant \sum_{j\geqslant 1}\frac{c_j}{\mu(2^{j+1}B)}\int_{2^{j+1}B}|h|^2\mu,$$

with $c_j = Ce^{-\frac{4^j}{c}}$ for some constant c, C . By the commutative properties of semi-group, then for each $t>0$ and for $h=(t\mathcal{L}_\mu)^{\frac{1}{2}}e^{-t\mathcal{L}_\mu}f$,

$$\begin{aligned}
&\left(\frac{1}{\mu(B)}\int_B \left|g_{\mathcal{L}_\mu}\left(e^{-kr^2\mathcal{L}_\mu}f\right)\right|^{p_*}\mu\right)^{\frac{2}{p_*}}\\
&\leqslant \int_0^\infty \left[\sum_{j\geqslant 1}\frac{c_j}{\mu(2^{j+1}B)}\int_{2^{j+1}B}\left|\left(t\mathcal{L}_\mu\right)^{\frac{1}{2}}e^{-t\mathcal{L}_\mu}f\right|^2\mu\right]\frac{dt}{t}\\
&\leqslant \sum_{j\geqslant 1}\frac{c_j}{\mu(2^{j+1}B)}\int_{2^{j+1}B}\left|g_{\mathcal{L}_\mu}(f)\right|^2\mu \leqslant CM_\mu\left(\left|g_{\mathcal{L}_\mu}(f)\right|^2\right)(y),
\end{aligned}$$

for all $y\in B$. By power expansion, we get the desired inequality (4.31).

It remains to check (4.30), decompose $f = f_1+f_2+f_3+...$ where $f_i = f\mathbb{1}_{C_i}$ and C_i is defined as before $C_i = C_i(B) = 2^{i+1}B\setminus 2^iB$ for $i\geqslant 2$ and $C_1 := 4B$. We start with $j=1$, we use the L^2_μ boundedness of $g_{\mathcal{L}_\mu}$ and that of $(I-e^{r^2\mathcal{L}_\mu})$ to obtain

$$\left(\frac{1}{\mu(B)}\int_B\left|g_{\mathcal{L}_\mu}\left(\left(I-e^{r^2\mathcal{L}_\mu}\right)^m f_1\right)\right|^2\mu\right)^{\frac{1}{2}} \leqslant C\left(\frac{1}{\mu(4B)}\int_{4B}|f|^2\mu\right)^{\frac{1}{2}}.$$

For $j\geqslant 2$, we write

$$\begin{aligned}
&\frac{1}{\mu(B)}\int_B\left|g_{\mathcal{L}_\mu}\left(\left(I-e^{r^2\mathcal{L}_\mu}\right)^m f_j\right)\right|^2\mu\\
&=\int_0^\infty\frac{1}{\mu(B)}\int_B\left|\left(t\mathcal{L}_\mu\right)^{\frac{1}{2}}e^{-t\mathcal{L}_\mu}\left(I-e^{kr^2\mathcal{L}_\mu}\right)^m f_j\right|^2\mu dx\frac{dt}{t}.
\end{aligned}$$

As in the Section 4.3, one may use the representation (4.14) with the function $\varphi(z) = (tz)^{\frac{1}{2}}e^{-tz}(1-e^{r^2z})^m$. The corresponding functions $\eta_\pm$ satisfy the estimates (see [A], page 79)

$$|\eta_\pm(z)|\leqslant\frac{Ct^{\frac{1}{2}}}{(|z|+t)^{\frac{3}{2}}}\inf\left(1,\frac{r^{2m}}{(|z|+t)^m}\right),\quad z\in\Gamma_\pm.$$

Since for any $0<\beta<\frac{\pi}{2}-w$, $\left(e^{z\mathcal{L}_\mu}\right)_{z\in\Sigma_\beta}$ satisfies L^2 off-diagonal estimates, using the representation (4.14) and the estimate for $\eta_\pm$, $\left\|\left(t\mathcal{L}_\mu\right)^{\frac{1}{2}}e^{-t\mathcal{L}_\mu}\left(I-e^{kr^2\mathcal{L}_\mu}\right)^m f_j\right\|_{L^2_\mu(B)}$ is bounded by

$$C\|f\|_{L^2_\mu(C_j)}\int_{\Gamma_+} e^{-\frac{c4^jr^2}{|z|}}\frac{t^{\frac{1}{2}}}{(|z|+t)^{\frac{3}{2}}}\frac{r^{2m}}{(|z|+t)^m}|dz|,$$

plus the similar term corresponding to integration on Γ_-.

Using the below Lemma 4.45 for the estimate of the above integration, it gives us the bound

$$\frac{C}{4^{jm}}\inf\left(\left(\frac{t}{4^jr^2}\right)^{\frac{1}{2}},\left(\frac{4^jr^2}{t}\right)^m\right)\|f\|_{L^2_\mu(C_j)}.$$

Squaring and integrating with respect to t, get

$$\frac{1}{\mu(B)}\int_B\left|g_{\mathcal{L}_\mu}\left(\left(I-e^{r^2\mathcal{L}_\mu}\right)^m f_j\right)\right|^2\mu\leqslant C2^{jn}4^{-mj}\frac{1}{\mu(2^{j+1}B)}\int_{2^{j+1}B}|f|^2\mu,$$

since

$$\int_0^\infty\frac{C}{4^{jm}}\inf\left(\left(\frac{t}{4^jr^2}\right)^{\frac{1}{2}},\left(\frac{4^jr^2}{t}\right)^m\right)\frac{dt}{t}=\frac{2C}{4^{jm}}+\frac{1}{m4^{jm}}<\frac{2C}{4^{jm}}$$

for m large enough and also $\mu(2^{j+1}B)<2^{jn}\mu(B)$ by doubling property of A_2 weight. This implies hypotheses (4.30) in Theorem 4.43 when we choose m large enough. ∎

Lemma 4.45. *Let $\tau\geqslant 0,\phi\geqslant 0,m>0$ be fixed parameters, and c a positive constant. For some C independent of $j\in\mathbb{N},r>0$ and $t>0$, the integral*

$$I=\int_0^\infty e^{-\frac{4^jr^2}{cs}}\frac{1}{s^{\tau/2}}\frac{t^\phi}{(s+t)^{1+\phi}}\frac{r^{2m}}{(s+t)^m}ds,$$

satisfy the estimate

$$I\leqslant\frac{C}{4^{jm}(2^jr)^\tau}\inf\left(\left(\frac{t}{4^jr^2}\right)^\phi,\left(\frac{4^jr^2}{t}\right)^m\right).$$

Proof. See Lemma 5.4 in [A]. ∎

Proposition 4.46. *Let ϵ be as in Theorem 4.1. Then for all $p \in [2, 2+\epsilon)$, we have* $\|G_{\mathcal{L}_\mu}(f)\|_{L^p_\mu} \lesssim \|f\|_{L^p_\mu}$

Proof. As in the proof for L^p boundedness of $g_{\mathcal{L}_\mu}$, we will check conditions (4.30) and (4.31) then by the Theorem 4.43 we finish our proof. As usual, we choose $A_r = I - (I - e^{-r^2\mathcal{L}_\mu})^m$ for m large enough to be chosen later. Let B be a ball with radius $r = r(B)$ and $k = 1, 2, 3, ..., m$. We will check the condition (4.31) first. By Minkowski integral inequality and for any integer k satisfying $0 \leqslant k \leqslant m-1$ and also take $p_* = 2 + \epsilon$ as in previous Proposition 4.44, we have

$$\begin{aligned}
&\left(\frac{1}{\mu(B)} \int_B \left|G_{\mathcal{L}_\mu}\left(e^{-kr^2\mathcal{L}_\mu} f\right)\right|^{p_*} \mu\right)^{\frac{2}{p_*}} \\
&\leqslant \int_0^\infty \left(\frac{1}{\mu(B)} \int_B \left|\nabla e^{-t\mathcal{L}_\mu}\left(e^{-kr^2\mathcal{L}_\mu} f\right)\right|^{p_*} \mu\right)^{\frac{2}{p_*}} dt.
\end{aligned}$$

Squaring (4.22) both sides then using Cauchy-Schwarz's inequality with the note that $\sum_i g(i) < \infty$ where $g(i) = 2^{2ni} e^{\frac{-4^{i-1}}{c}}$, we have

$$\begin{aligned}
&\int_0^\infty \left(\frac{1}{\mu(B)} \int_B \left|\nabla e^{-t\mathcal{L}_\mu}\left(e^{-kr^2\mathcal{L}_\mu} f\right)\right|^{p_*} \mu\right)^{\frac{2}{p_*}} dt \\
\leqslant &C \int_0^\infty \sum_{j \geqslant 1} g(j) \left(\frac{1}{\mu(2^{j+1}B)} \int_{2^{j+1}B} \left|\nabla e^{-t\mathcal{L}_\mu} f\right|^2 \mu\right) dt.
\end{aligned}$$

Exchange the sum and the integral, the later is equal to

$$C \sum_{j \geqslant 1} \frac{g(j)}{\mu(2^{j+1}B)} \int_{2^{j+1}B} \left|G_{\mathcal{L}_\mu}(f)\right|^2 \mu.$$

which is bounded by the maximal function $CM_\mu\left(\left|G_{\mathcal{L}_\mu}(f)\right|^2\right)(y)$ for $y \in B$. This gives us the assumption (4.31).

It suffices to check the condition (4.30). Again, decompose $f = f_1 + f_2 + f_3 + ...$ where $f_i = f\mathbb{1}_{C_i}$ and C_i is defined as before $C_i = C_i(B) = 2^{i+1}B \setminus 2^i B$ for $i \geqslant 2$ and $C_1 = 4B$. We begin with the following inequality

$$\begin{aligned}
&\left(\frac{1}{\mu(B)}\int_B\left|G_{\mathcal{L}_\mu}\left(\left(I-e^{-r^2\mathcal{L}_\mu}\right)^m f\right)\right|^2\mu\right)^{\frac{1}{2}}\\
\leqslant\ &\sum_{j\leqslant 1}\left(\frac{1}{\mu(B)}\int_B\left|G_{\mathcal{L}_\mu}\left(\left(I-e^{-r^2\mathcal{L}_\mu}\right)^m f_j\right)\right|^2\mu\right)^{\frac{1}{2}}.
\end{aligned}$$

We will give an estimate for the RHS of the above inequality for all $j\geqslant 1$.

For $j=1$, we use the L^2_μ boundedness of $G_{\mathcal{L}_\mu}$ and that of $I-e^{-r^2\mathcal{L}_\mu}$ to obtain

$$\begin{aligned}
&\left(\frac{1}{\mu(B)}\int_B|G_{\mathcal{L}_\mu}\left(\left(I-e^{-r^2\mathcal{L}_\mu}\right)^m f_1\right)|^2\mu\right)^{\frac{1}{2}}\\
&\leqslant C\left(\frac{1}{\mu(B)}\int_B\left|\left(I-e^{-r^2\mathcal{L}_\mu}\right)^m f_1\right|^2\mu\right)^{\frac{1}{2}}\\
&\leqslant C\left(\frac{1}{\mu(4B)}\int_{4B}|f_1|^2\mu\right)^{\frac{1}{2}}.
\end{aligned}$$

For $j\geqslant 2$,

$$\begin{aligned}
&\left(\frac{1}{\mu(B)}\int_B|G_{\mathcal{L}_\mu}\left(\left(I-e^{-r^2\mathcal{L}_\mu}\right)^m f_j\right)|^2\mu\right)^{\frac{1}{2}}\\
&=\int_0^\infty\frac{1}{\mu(B)}\int_B\left|\sqrt{t}\nabla\left(I-e^{-r^2\mathcal{L}_\mu}\right)^m e^{-t\mathcal{L}_\mu}f_j\right|^2\mu\frac{dxdt}{t}.
\end{aligned}$$

We finish our argument by doing the similar argument for function $g_{\mathcal{L}_\mu}$. That is, first, we use the representation (4.14) in Section 4.3 for the function

$$\varphi(z)=\left(1-e^{-r^2z}\right)^m e^{-tz}$$

along with the L^2 off-diagonal estimate together with the estimate on page 45 in [A] for the estimate of $|\eta_\pm(z)|$ for φ above , we have

$$\begin{aligned}
&\left(\int_B\left|\sqrt{t}\nabla\left(I-e^{-r^2\mathcal{L}_\mu}\right)^m e^{-t\mathcal{L}_\mu}f_j\right|^2\mu\right)^{\frac{1}{2}}=\left\|\varphi(\mathcal{L}_\mu)f_j\right\|_{L^2_\mu}\\
\leqslant &C\|f_j\|^2_{L^2_\mu}\int_{\Gamma_+}\frac{\sqrt{t}}{\sqrt{|z|}}\cdot\frac{1}{|z|+t}\cdot\frac{r^{2m}}{(|z|+t)^m}\cdot e^{-\frac{(2^{j-1}r)^2}{ct}}|dz|\\
&+\ \text{ a similar integral over the domain }\Gamma_-.
\end{aligned}$$

then we invoke Lemma 4.45 to get the estimate for the integral above. Finally, we integrate with respect to t to get the condition (4.30) in Theorem 4.43 by the identical argument as in the last part of the proof of Proposition 4.44.

∎

4.5 Riesz transform for degenerate elliptic operators

In this section, we will prove L^p bounds of the Riesz transform for $2-\epsilon < p < 2+\epsilon$ for some small enough ϵ. Moreover, we also will give the proof for L^p bounds of vertical square functions for $p < 2$. Together with the proofs in Section 5.3, we complete the proof for L^p bounds of vertical square functions for all $2-\epsilon < p < 2+\epsilon$.

For future use, we restate a simpler version of the Theorem 8.1 in [AM] where $\mu \in A_2$-condition and note that μ has order doubling $2n$ by Lemma 4.16. This theorem is a tool for us to get L^p bounds of operators for $p < 2$.

Theorem 4.47. *Let $p_* < 2$. Suppose that T is a sublinear operator of strong type $(2,2)$, and let A_r, be a family of linear operators acting on L^2_μ where $\mu \in A_2$. Assume for $j \geqslant 2$*

$$\left(\frac{1}{\mu(2^{j+1}B)}\int_{C_j(B)}\left|T(I-A_{r(B)})f\right|^2\mu\right)^{\frac{1}{2}} \leqslant g(j)\left(\frac{1}{\mu(B)}\int_B |f|^{p_*}\mu\right)^{\frac{1}{p_*}}, \tag{4.32}$$

and for $j \geqslant 1$

$$\left(\frac{1}{\mu(2^{j+1}B)}\int_{C_j(B)}\left|A_{r(B)}f\right|^2\mu\right)^{\frac{1}{2}} \leqslant g(j)\left(\frac{1}{\mu(B)}\int_B |f|^{p_*}\mu\right)^{\frac{1}{p_*}}, \tag{4.33}$$

for all ball B with $r(B)$ the radius of B and all f supported in B. If $\Sigma = \Sigma_j g(j)2^{2nj} < \infty$, then T is of weak type (p_*, p_*), with a bound depending only on the strong type $(2,2)$ bound of T, p_* and Σ, hence bounded on L^p_μ for $p_* < p < 2$.

We are now ready to prove main Theorem 4.3, which we restate here for the reader's convenience.

Theorem 4.48. *There is an $\epsilon' > 0$ such that*

$$\|\nabla f\|_{L^p_\mu} \simeq \|\mathcal{L}_\mu^{1/2} f\|_{L^p_\mu}, \quad 2-\epsilon' < p < 2+\epsilon'.$$

Proof. We prove the case $p < 2$ first by applying Theorem 4.47 to the operator $T = \nabla \mathcal{L}_\mu^{-1/2}$ to obtain a weak type (p_*, p_*) bound for $2 > p_* := \max\left\{\frac{2q_0}{2\alpha+(1-\alpha)q_0}, 2-\epsilon\right\}$ where ϵ is determined in Theorem 4.1 where we recall that $q_0 = \frac{2\delta(n-1)+2n}{2\delta(n-1)+n+1}$, and α is a fixed small number that will be determined in the proof of Lemma 4.49 below. Notice that α will depend only on the allowable parameters. We first introduce the operators $A_r = I - \left(I - e^{-r^2\mathcal{L}_\mu}\right)^m$ where m is some integer to be specified later.

Observe that $A_r = \sum_{k=1}^m c_k e^{-kr^2\mathcal{L}_\mu}$ for some numbers c_k. A direct consequence of the local $L^q - L^2$ off-diagonal estimates (Lemma 4.30 for $q = \frac{2q_0}{2\alpha+(1-\alpha)q_0} \leqslant p_*$ above) and Hölder's inequality is that if B is any ball, with radius $r = r(B)$, and for any function $f \in L^2 \cap L^{p_*}$ that is supported in the ball B, if $j \geqslant 1$ then one gets

$$\left(\frac{1}{\mu(2^jB)}\int_{C_j(B)} |A_r f|^2 \mu\right)^{\frac{1}{2}} \leqslant C e^{-c(2^j)^2(1-\alpha)} 2^{j\alpha} \left(\frac{1}{\mu(B)}\int_B |f|^{p_*}\mu\right)^{\frac{1}{p_*}},$$

where we recall that $C_j(B) = 2^{j+1}B \setminus 2^jB$ for $j \geqslant 2$ and $C_1(B) = 4B$. Hence, assumption (4.33) holds with $g(j) = e^{-c(2^j)^2(1-\alpha)}2^{j\alpha}$.

It remains to check the assumption (4.32) for $j \geqslant 2$. To this end, we note for future reference that by Lemma 4.39 and Hölder's inequality

$$\left(\frac{1}{\mu(2^jB)}\int_{C_jB} \left|\sqrt{t}\nabla e^{-t\mathcal{L}_\mu} f\right|^2 \mu\right)^{\frac{1}{2}}$$

$$\leqslant \; C(p) j e^{-\frac{(2^{j-1}r)^2(1-\alpha)}{ct}} \left(\frac{2^j r}{\sqrt{t}}+1\right)^{\alpha} \left(\frac{1}{\mu(B)}\int_B |f|^{p_*}\mu\right)^{\frac{1}{p_*}}. \quad (4.34)$$

In order to finish our proof, we need the following lemma.

Lemma 4.49. *Let p_* be defined above in the proof of Theorem 4.48, there exists $C > 0$ such that for all balls B with radii $r > 0$ and for any function $f \in L^2_\mu \cap L^{p_*}_\mu$ supported in the ball B. If $j \geqslant 2$ then*

$$\left\{\frac{1}{\mu(2^j B)}\int_{C_j(B)} \left|\nabla \mathcal{L}_\mu^{-1/2}\left(I - e^{r^2\mathcal{L}_\mu}\right)^m f\right|^2 \mu\right\}^{\frac{1}{2}} \leqslant C 2^{jn} 2^{-j(m-1+\alpha)} \left(\frac{1}{\mu(B)}\int_B |f|^{p_*}\mu\right)^{\frac{1}{p_*}},$$

where constant C depends on allowed parameter and also α and m but it does not depend on r, j, B.

Proof. We use the idea in [A]:

$$\begin{aligned}
\nabla \mathcal{L}_\mu^{-1/2}\left(I - e^{r^2\mathcal{L}_\mu}\right)^m f &= \pi^{-\frac{1}{2}} \int_0^\infty \nabla e^{-t\mathcal{L}_\mu}\left(I - e^{-r^2\mathcal{L}_\mu}\right)^m f \frac{dt}{\sqrt{t}} \\
&= \pi^{-\frac{1}{2}} \int_0^\infty g_{r^2}(t) \nabla e^{-t\mathcal{L}_\mu} f dt,
\end{aligned}$$

where using the notation for binomial coefficients,

$$g_s(t) = \sum_{k=0}^m \binom{m}{k} (-1)^k \frac{\chi_{\{t>ks\}}}{\sqrt{t-ks}},$$

and χ is the indicator function of $(0,\infty)$.

By Minkowski's inequality for integral and using the inequality (4.34) above, we get

$$\begin{aligned}
&\left\{\frac{1}{\mu(2^j B)}\int_{C_j B} \left|\nabla \mathcal{L}_\mu^{-1/2}\left(I - e^{r^2\mathcal{L}_\mu}\right)^m f\right|^2 \mu\right\}^{\frac{1}{2}} \\
\leqslant& 2Cj \int_0^\infty |g_{r^2}(t)| \left(\frac{2^j r}{\sqrt{t}}+1\right)^{\alpha} e^{-\frac{2^j r^2(1-\alpha)}{ct}} \frac{dt}{\sqrt{t}} \left(\frac{1}{\mu(B)}\int_B |f|^{p_*}\mu\right)^{\frac{1}{p_*}}.
\end{aligned} \quad (4.35)$$

It remains to estimate the infinite integral, we have

$$\left| g_{r^2}(t) \leqslant \frac{C}{\sqrt{t-kr^2}} \right| \quad \text{if } kr^2 < t \leqslant (k+1)r^2 \leqslant (m+1)r^2,$$

and

$$|g_{r^2}(t)| \leqslant Cr^{2m}t^{-m-\frac{1}{2}} \quad \text{if } t > (m+1)r^2.$$

Observe that

$$\begin{aligned}
&\int_0^\infty |g_{r^2}(t)| \left(\frac{2^j r}{\sqrt{t}} + 1\right)^\alpha e^{-\frac{2^j r^2(1-\alpha)}{ct}} \frac{dt}{\sqrt{t}} \\
=&\int_0^{r^2} |g_{r^2}(t)| \left(\frac{2^j r}{\sqrt{t}} + 1\right)^\alpha e^{-\frac{2^j r^2(1-\alpha)}{ct}} \frac{dt}{\sqrt{t}} \\
&+\sum_{k=1}^m \int_{kr^2}^{(k+1)r^2} |g_{r^2}(t)| \left(\frac{2^j r}{\sqrt{t}} + 1\right)^\alpha e^{-\frac{2^j r^2(1-\alpha)}{ct}} \frac{dt}{\sqrt{t}} + \int_{(m+1)r^2}^\infty |g_{r^2}(t)|\, e^{-\frac{2^j r^2(1-\alpha)}{ct}} \frac{dt}{\sqrt{t}} \\
=:& I + II + III.
\end{aligned}$$

We treat above terms in order,

$$\begin{aligned}
I &\leqslant \int_0^{r^2} \frac{C}{\sqrt{t}} \left(\frac{2^j r}{\sqrt{t}} + 1\right)^\alpha e^{-\frac{2^j r^2(1-\alpha)}{ct}} \frac{dt}{\sqrt{t}} \\
&\leqslant \int_0^{r^2} \frac{C}{t} \left(\frac{2^j r}{\sqrt{t}}\right)^\alpha e^{-\frac{2^j r^2(1-\alpha)}{ct}} dt \\
&= \int_0^{r^2} \frac{C}{t^{1+\alpha/2}} \left(2^j r\right)^\alpha e^{-\frac{2^j r^2(1-\alpha)}{ct}} dt.
\end{aligned}$$

By a simple calculation, we have for α sufficiently small together $t < r^2$ and $j \geqslant 2$

$$\begin{aligned}
\frac{d}{dt}\left(t^{-\frac{\alpha}{2}} e^{-\frac{2^j r^2(1-\alpha)}{ct}}\right) &= t^{-1-\frac{\alpha}{2}} e^{-\frac{2^j r^2(1-\alpha)}{ct}} \left(-\frac{\alpha}{2} + \frac{2^j r^2(1-\alpha)}{ct}\right) \\
&\geqslant t^{-1-\frac{\alpha}{2}} e^{-\frac{2^j r^2(1-\alpha)}{ct}} \left(-\frac{\alpha}{2} + \frac{4(1-\alpha)}{c}\right) \\
&= C_\alpha t^{-1-\frac{\alpha}{2}} e^{-\frac{2^j r^2(1-\alpha)}{ct}} > 0.
\end{aligned}$$

where the constant $C_\alpha > 0$ for sufficiently small α and C_α depends on the constant c of off-diagonal estimates and α. Note that the we determine α by choosing it small enough so that α satisfies $\left(-\frac{\alpha}{2} + \frac{4(1-\alpha)}{c}\right) > 0$, then we fix that value for α.

Hence,

$$I \leqslant \left(2^j r\right)^\alpha C_\alpha r^{-\alpha} e^{-\frac{2^j(1-\alpha)}{c}} = C_\alpha 2^{j\alpha} e^{-\frac{2^j(1-\alpha)}{c}}.$$

For the term II, we have

$$\begin{aligned}
II &\leqslant \sum_{k=1}^{m} \int_{kr^2}^{(k+1)r^2} \frac{C}{\sqrt{t-kr^2}} e^{-\frac{2^j r^2(1-\alpha)}{ct}} \left(\frac{2^j r}{\sqrt{t}}+1\right)^\alpha \frac{dt}{\sqrt{t}} \\
&\leqslant 2\sum_{k=1}^{m} \int_{kr^2}^{(k+1)r^2} \frac{C(m)}{\sqrt{t-kr^2}} e^{-\frac{2^j r^2(1-\alpha)}{ct}} \left(\frac{2^j r}{\sqrt{t}}\right)^\alpha \frac{dt}{\sqrt{t}} \\
&\leqslant 2\sum_{k=1}^{m} \int_{kr^2}^{(k+1)r^2} \frac{C(m)}{\sqrt{t-kr^2}} e^{-\frac{2^j r^2(1-\alpha)}{c(k+1)r^2}} \left(\frac{2^j r}{\sqrt{kr^2}}\right)^\alpha \frac{dt}{\sqrt{kr^2}} \\
&= C(m) \sum_{k=1}^{m} 2^{j\alpha} k^{-\left(\frac{\alpha}{2}+\frac{1}{2}\right)} e^{-\frac{2^j(1-\alpha)}{c(k+1)}} \\
&\leqslant C(m) 2^{j\alpha} e^{-\frac{2^j(1-\alpha)}{c(m+1)}}.
\end{aligned}$$

Now we estimate the last term III,

$$\begin{aligned}
III &\leqslant C \int_{(m+1)r^2}^{\infty} r^{2m} t^{-m-\frac{1}{2}} e^{-\frac{2^j r^2(1-\alpha)}{ct}} \left(\frac{2^j r}{\sqrt{t}}+1\right)^\alpha \frac{dt}{\sqrt{t}} \\
&\leqslant 2C 2^{j\alpha} r^{2m} \int_{(m+1)r^2}^{\infty} t^{-m-1} e^{-\frac{2^j r^2(1-\alpha)}{ct}} dt \\
&\leqslant 2C(m) 2^{j\alpha} r^{2m} \int_{(m+1)r^2}^{\infty} t^{-m-1} \left(\frac{2^j r^2(1-\alpha)}{ct}\right)^{-m+1} dt \\
&= 2C(m) 2^{j\alpha} r^{2m} \left(\frac{2^j r^2(1-\alpha)}{c}\right)^{-m+1} \int_{(m+1)r^2}^{\infty} t^{-2} dt \\
&= 2C(m) 2^{j\alpha} 2^{j(-m+1)} \left(\frac{1-\alpha}{c}\right)^{-m+1}.
\end{aligned}$$

This finishes the proof of our lemma. ∎

We now return to the proof of Theorem 4.48.

Note that (4.32) now follows immediately from Lemma 4.49, provided that m is chosen large enough. By Theorem 4.47 applied to $T = \nabla \mathcal{L}_\mu^{-1/2}$, we therefore get

$$\|\nabla f\|_{L^p_\mu} \lesssim \|\mathcal{L}_\mu^{\frac{1}{2}} f\|_{L^p_\mu}, \quad \text{if} \quad 2-\epsilon < p \leqslant 2.$$

Combining the preceeding estimate with Lemma 4.51 (which will be proved below) and duality, we get

$$\|\nabla f\|_{L^p_\mu} \simeq \|\mathcal{L}_\mu^{\frac{1}{2}} f\|_{L^p_\mu}, \quad \text{if} \quad 2-\epsilon < p < 2+\epsilon.$$

∎

Lemma 4.50. *There exists $\epsilon > 0, C > 0$ such that for all balls B with radii $r > 0$ and $f \in L^2_\mu \cap L^p_\mu$ with support in B and also $j \geqslant 2$, then for all $p > 2-\epsilon$,*

$$\left\{ \frac{1}{\mu(2^j B)} \int_{C_j(B)} \left| \nabla \mathcal{L}_\mu^{-1/2} \left(I - e^{r^2 \mathcal{L}_\mu} \right)^m f \right|^2 \mu \right\}^{\frac{1}{2}} \leqslant C 2^{jn} 2^{-j(m-1)} \left(\frac{1}{\mu(B)} \int_B |f|^p \mu \right)^{\frac{1}{p}},$$

where constant C depends on allowed parameters and m but it does not depend on r, j, B.

In addition, we also obtain the following result and by properties of $\mu \in A_2$, we get for any function $f \in L^2_\mu$ supported in the set $C_j(B)$, the following holds

$$\left\{ \frac{1}{\mu(B)} \int_B \left| \nabla \mathcal{L}_\mu^{-1/2} \left(I - e^{r^2 \mathcal{L}_\mu} \right)^m f \right|^2 \mu \right\}^{\frac{1}{2}} \leqslant C 2^{jn} 2^{-j(m-1)} \left(\frac{1}{\mu(2^j B)} \int_{2^j B} |f|^2 \mu \right)^{\frac{1}{2}}.$$

Proof. The first part is a consequence of the Lemma 4.49 and Hölder's in equality. Next, we give the proof for the second part of the lemma. For our convenience, let us recall that the result of the L^2_μ off-diagonal estimate (Proposition 4.18) implies

$$\left(\int_B \left| \sqrt{t} \nabla e^{-t\mathcal{L}_\mu f} \right|^2 \mu \right)^{\frac{1}{2}} \leqslant C e^{-\frac{\left(2^{j-1} r\right)^2}{ct}} \left(\int_{C_j(B)} |f|^2 \mu \right)^{\frac{1}{2}}.$$

Hence, because $\mu \in A_2$, by doubling properties of measure μ (see Lemma 4.16), we obtain

$$\left(\frac{1}{\mu(B)} \int_B \left| \sqrt{t} \nabla e^{-t\mathcal{L}_\mu f} \right|^2 \mu \right)^{\frac{1}{2}} \leqslant C 2^{(j+1)n} e^{-\frac{\left(2^{j-1} r\right)^2}{ct}} \left(\frac{1}{\mu(2^{j+1} B)} \int_{C_j(B)} |f|^2 \mu \right)^{\frac{1}{2}}. \tag{4.36}$$

Next, as in the proof of Lemma 4.49, from the representation for $\nabla\mathcal{L}^{-1/2}(I-e^{-r^2\mathcal{L}_\mu})^m f$, we have

$$
\begin{aligned}
&\left(\frac{1}{\mu(B)}\int_B \left|\nabla\mathcal{L}^{-1/2}(I-e^{-r^2\mathcal{L}_\mu})^m f\right|^2\mu\right)^{\frac{1}{2}}\\
&\leqslant \left(\frac{1}{\mu(B)}\int_B \left|\pi^{-1/2}\int_0^\infty g_{r^2}(t)\nabla e^{-t\mathcal{L}_\mu} f\,dt\right|^2\mu\right)^{\frac{1}{2}}\\
&\leqslant C\int_0^\infty |g_{r^2}(t)|\left(\frac{1}{\mu(B)}\int_B \left|\sqrt{t}\nabla e^{-t\mathcal{L}_\mu} f\right|^2\mu\right)^{\frac{1}{2}}\frac{dt}{\sqrt{t}}\\
&\leqslant C2^{(j+1)n}\int_0^\infty |g_{r^2}(t)|\,e^{-\frac{\left(2^{j-1}r\right)^2}{ct}}\frac{dt}{\sqrt{t}}\left(\frac{1}{\mu(2^{j+1}B)}\int_{2^{j+1}B}|f|^2\mu\right)^{\frac{1}{2}},
\end{aligned}
$$

where we used (4.36) to get the last inequality.

By the estimate for the integeal $\int_0^\infty |g_{r^2}(t)|\left(\frac{2^j r}{\sqrt{t}}+1\right)^\alpha e^{-\frac{2^j r^2(1-\alpha)}{ct}}\frac{dt}{\sqrt{t}}$ that we computed in the term (4.35), one gets for any $\alpha>0$,

$$
\begin{aligned}
\int_0^\infty |g_{r^2}(t)|\,e^{-\frac{\left(2^{j-1}r\right)^2}{ct}}\frac{dt}{\sqrt{t}} &\leqslant \int_0^\infty |g_{r^2}(t)|\left(\frac{2^j r}{\sqrt{t}}+1\right)^\alpha e^{-\frac{2^j r^2(1-\alpha)}{ct}}\frac{dt}{\sqrt{t}}\\
&\leqslant 2C(m)2^{j\alpha}2^{j(-m+1)}\left(\frac{1-\alpha}{c}\right)^{-m+1}.
\end{aligned}
$$

Hence, by letting $\alpha\to 0$,

$$
2^{(j+1)n}\int_0^\infty |g_{r^2}(t)|\,e^{-\frac{\left(2^{j-1}r\right)^2}{ct}}\frac{dt}{\sqrt{t}}\leqslant 2C(m)2^{(j+1)n}2^{j(-m+1)}.
$$

So, we get the second part of the lemma. ∎

Lemma 4.51. *There is some $\epsilon>0$ such that*

$$
\left\|\nabla\mathcal{L}_\mu^{-\frac{1}{2}} f\right\|_{L^p_\mu}\lesssim \|f\|_{L^p_\mu},
$$

for all $p\in[2,2+\epsilon)$.

Proof. We prove this result by using the Theorem 4.43 for $p_*=2+\epsilon$ where ϵ is determined as in Theorem 4.1. We will check the conditions (4.30) and (4.31) where $r:=r(B)$, the radius of the ball B and $A_{r(B)} = I - \left(I - e^{-r^2\mathcal{L}_\mu}\right)^m$ for some large m to be chosen later and we apply the theorem to the operator $T := \nabla\mathcal{L}_\mu^{-\frac{1}{2}}$.

Now, decompose $f = f_1 + f_2 + ...$ where $f_j = f\mathbb{1}_{C_j(B)}$ and $C_j(B) = 2^{j+1}B \setminus 2^jB$ and $C_1(B) = 4B$. By triangle inequality

$$\left\|\nabla\mathcal{L}_\mu^{-\frac{1}{2}}\left(I - e^{-r^2\mathcal{L}_\mu}\right)^m f\right\|_{L^2_\mu(B)} \leqslant \sum_{j=1}^{\infty}\left\|\nabla\mathcal{L}_\mu^{-\frac{1}{2}}\left(I - e^{-r^2\mathcal{L}_\mu}\right)^m f_j\right\|_{L^2_\mu(B)}.$$

For $j = 1$, by the L^2_μ boundedness of $\nabla\mathcal{L}_\mu^{-\frac{1}{2}}$ and $e^{-t\mathcal{L}_\mu}$, we have

$$\left\|\nabla\mathcal{L}_\mu^{-\frac{1}{2}}\left(I - e^{-r^2\mathcal{L}_\mu}\right)^m f_1\right\|_{L^2_\mu(B)} \lesssim \mu(B)^{\frac{1}{2}}\left(\frac{1}{\mu(4B)}\int_{4B}|f|^2\mu\right)^{\frac{1}{2}}.$$

When $j \geqslant 2$, we apply the second part of Lemma 4.50 to get

$$\left\|\nabla\mathcal{L}_\mu^{-\frac{1}{2}}\left(I - e^{-r^2\mathcal{L}_\mu}\right)^m f_j\right\|_{L^2_\mu(B)} \lesssim C2^{-j(m-n-1)}\mu(B)^{\frac{1}{2}}\left(\frac{1}{\mu(2^jB)}\int_{2^jB}|f|^2\mu\right)^{\frac{1}{2}}.$$

This gives us the condition (4.30) for m large enough.

It remains to check the condition (4.31). Let us recall the result of Lemma 4.38 when $t = 0$, we have

$$\left(\frac{1}{\mu(B)}\int_B\left|\nabla e^{-kr^2\mathcal{L}_\mu}f\right|^{p_*}\mu\right)^{\frac{1}{p_*}} \leqslant C\sum_{i=1}^{\infty}e^{-\frac{4^{i-1}}{c}}2^{2n(i+1)}\left(\frac{1}{\mu(2^{i+1}B)}\int_{2^{i+1}B}|\nabla f|^2\mu\right)^{\frac{1}{2}}.$$

Keep in mind that $A_{r(B)} = \sum_{k=1}^m c_k e^{-kr^2\mathcal{L}_\mu}$, one gets

$$\left(\frac{1}{\mu(B)}\int_B\left|\nabla\mathcal{L}_\mu^{-\frac{1}{2}}e^{-kr^2\mathcal{L}_\mu}f\right|^{p_*}\mu\right)^{\frac{1}{p_*}}$$
$$\leqslant C\sum_{i=1}^{\infty}e^{-\frac{4^{i-1}}{c}}2^{2n(i+1)}\left(\frac{1}{\mu(2^{i+1}B)}\int_{2^{i+1}B}|\nabla\mathcal{L}_\mu^{-\frac{1}{2}}f|^2\mu\right)^{\frac{1}{2}},$$

by replacing f by $\mathcal{L}_\mu^{-\frac{1}{2}}f$. This finishes the condition (4.31).

By the Theorem 4.43, we get the desired inequality. ∎

Proposition 4.52. *For all $q \in (2-\epsilon', 2)$, we have $\|g_{\mathcal{L}_\mu}(f)\|_{L^q_\mu} \lesssim \|f\|_{L^q_\mu}$, where ϵ' is as in the result of Theorem 4.48.*

Proof. This is a consequence of Theorem 4.47, more precisely, we will check the assumption (4.32) and (4.33) for the operator $T = g_{\mathcal{L}_\mu}$ and $A_r = I - (I - e^{-r^2\mathcal{L}_\mu})$ for some $m > 0$ to be chosen later. For any $0 \leqslant k \leqslant m$, from the off-diagonal estimate [Lemma 4.30], we have

$$\begin{aligned}\left(\frac{1}{\mu(2^jB)}\int_{C_jB}\left|e^{-kr^2\mathcal{L}_\mu}f\right|^2\mu\right)^{\frac12} &\leqslant e^{-\frac{(2^jr)^2(1-\alpha)}{cr^2k}}\left(\frac{2^jr}{\sqrt{kr^2}}+1\right)^{\alpha}\left(\frac{1}{\mu(B)}\int_B|f|^q\mu\right)^{\frac1q}\\ &\leqslant e^{-\frac{(2^j)^2(1-\alpha)}{cm}}2^{\alpha j}\left(\frac{1}{\mu(B)}\int_B|f|^q\mu\right)^{\frac1q}.\end{aligned}$$

This give us the assumption (4.33). It remains to check the assumption (4.32). Notice that, by Fubini theorem

$$\begin{aligned}&\frac{1}{\mu(2^jB)}\int_{C_jB}\left|g_L\left(I-e^{-kr^2\mathcal{L}_\mu}\right)^m f\right|^2\mu\\ &=\int_0^\infty\frac{1}{\mu(2^{j+1}B)}\int_{C_jB}\left|\left(t\mathcal{L}_\mu\right)^{\frac12}e^{-t\mathcal{L}_\mu}\left(I-e^{-kr^2\mathcal{L}_\mu}\right)^m f\right|^2\mu dx\frac{dt}{t}.\end{aligned}$$

Apply the representation (4.14) for $\varphi(z) = (tz)^{1/2}e^{-tz}(1-e^{-r^2z})^m$, by the L^2-L^p off-diagonal estimate in Lemma 4.28 and Minkowski's inequality, we have

$$\begin{aligned}&\left(\frac{1}{\mu(2^jB)}\int_{C_jB}\left|\varphi(\mathcal{L}_\mu)f\right|^2\mu\right)^{\frac12}\\ &\leqslant\left(\frac{1}{\mu(2^jB)}\int_{C_jB}\left|\int_{\Gamma_+}e^{-z\mathcal{L}_\mu}f\eta_+(z)dz+\int_{\Gamma_-}e^{-z\mathcal{L}_\mu}f\eta_-(z)dz\right|^2\mu\right)^{\frac12}\\ &\leqslant C\int_{\Gamma_+}e^{-\frac{(2^jr)^2(1-\alpha)}{c|z|}}\left(\frac{2^jr}{\sqrt{z}}+1\right)^{\alpha}\frac{t^{\frac12}}{(|z|+t)^{\frac32}}\frac{r^{2m}}{(|z|+t)^m}|dz|\left(\frac{1}{\mu(B)}\int_B|f|^q\mu\right)^{\frac1q} \qquad (4.37)\\ &\leqslant C\int_{\Gamma_+}e^{-\frac{(2^jr)^2(1-\alpha)}{c|z|}}\left(\frac{2^jr}{\sqrt{|z|}}+1\right)\frac{t^{\frac12}}{(|z|+t)^{\frac32}}\frac{r^{2m}}{(|z|+t)^m}|dz|\left(\frac{1}{\mu(B)}\int_B|f|^q\mu\right)^{\frac1q}\\ &\leqslant C\int_{\Gamma_+}e^{-\frac{(2^jr)^2(1-\alpha)}{c|z|}}2^j\frac{t^{\frac12}}{(|z|+t)^{\frac32}}\frac{r^{2m}}{(|z|+t)^m}|dz|\left(\frac{1}{\mu(B)}\int_B|f|^q\mu\right)^{\frac1q}=:M,\end{aligned}$$

plus the similar term over Γ_-, because $\alpha < 1$ and where $\Gamma_\pm$ are defined in (4.14) above, and for the last inequality we have used that if $r \leqslant |z|$ then $\left(\frac{2^j r}{|z|}+1\right) \leqslant 2\cdot 2^j$ and if $r \geqslant |z|$ then

$$e^{-\frac{(2^j r)^2(1-\alpha)}{c|z|}} \frac{2^j r}{\sqrt{|z|}} \leqslant C_0 e^{-\frac{(2^j r)^2(1-\alpha)}{c_0|z|}},$$

for some C_0, c_0 independent of the ratio $\frac{r}{|z|}$.

Using the Lemma 4.45 for $\phi = \frac{1}{2}, \tau = 0$, get

$$M \leqslant \frac{C2^j}{2^{jm}} \inf\left(\left(\frac{t}{2^j r^2}\right)^{\frac{1}{2}}, \left(\frac{2^j r^2}{t}\right)^m\right)\left(\frac{1}{\mu(B)}\int_B |f|^q \mu\right)^{\frac{1}{q}}.$$

Square and integrate with respect to t, that implies the assumption (4.32) holds for some m large enough. ∎

Proposition 4.53. *For all $q \in (2-\epsilon', 2)$, we have $\|G_{\mathcal{L}_\mu}(f)\|_{L^q_\mu} \lesssim \|f\|_{L^q_\mu}$ where ϵ is as in the result of Theorem 4.48.*

Proof. Like the proof in Proposition 4.52, we will also check the assumptions of Theorem 4.47 for $T = G_{\mathcal{L}_\mu}$ and $A_r = I - \left(I - e^{-r^2\mathcal{L}_\mu}\right)^m$ for some $0 < m \in \mathbb{N}$. We see that assumption (4.33) is a consequence of Lemma 4.30, local $L^q - L^2$ off-diagonal estimates of $e^{-t\mathcal{L}_\mu}$, it remains to check the assumption (4.32). For any ball B with radius r and for function f with supp$f \subset B$, we have the following

$$\begin{aligned}&\frac{1}{\mu(2^{j+1}B)}\int_{C_jB}\left|G_{\mathcal{L}_\mu}\left(\left(I-e^{-r^2\mathcal{L}_\mu}\right)^m f\right)\right|^2 \mu dx\\ &= \int_0^\infty \frac{1}{\mu(2^{j+1})B}\int_{C_jB}\left|\sqrt{t}\nabla e^{-t\mathcal{L}_\mu}\left(I-e^{-kr^2\mathcal{L}_\mu}\right)^m f\right|^2 \mu dx\frac{dt}{t}.\end{aligned}$$

By using the presentation (4.14) for the function $\varphi(z) = e^{-tz}(1-e^{-kr^2 z})^m$ and Lemma 4.39 - the local $L^p - L^2$ off-diagonal estimate for $\sqrt{z}\nabla e^{-z\mathcal{L}_\mu}$ also the following estimate (see [A], page 44-45)

$$|\eta_\pm(z)| \leqslant \frac{C}{|z|+t}\inf\left(1, \frac{r^{2m}}{(|z|+t)^m}\right),$$

we get

$$
\begin{aligned}
&\left(\frac{1}{\mu(2^jB)}\int_{C_jB}\left|\sqrt{t}\nabla\varphi(\mathcal{L}_\mu)f\right|^2\mu dx\right)^{\frac{1}{2}}\\
=&\left(\frac{1}{\mu(2^jB)}\int_{C_jB}\left|\sqrt{t}\nabla e^{-t\mathcal{L}_\mu}\left(I-e^{-kr^2\mathcal{L}_\mu}\right)^m f\right|^2\mu dx\right)^{\frac{1}{2}}\\
\leqslant& C(p)j\int_{\Gamma_+}e^{-\frac{2^jr^2(1-\alpha)}{c|z|}}\left(\frac{2^jr}{\sqrt{|z|}}+1\right)^{\alpha}\frac{t^{\frac{1}{2}}}{(|z|+t)}\frac{1}{|z|^{\frac{1}{2}}}\frac{r^{2m}}{(|z|+t)^m}|dz|\left(\frac{1}{\mu(B)}\int_B|f|^q\mu\right)^{\frac{1}{q}}\\
=:& M^*,
\end{aligned}
$$

plus a similar term corresponding to the integral over Γ_-. Now, doing the same argument as in the proof of previous Proposition 4.52, we get

$$
M^*\leqslant\frac{C2^j}{2^{jm}}\inf\left(\left(\frac{t}{2^jr^2}\right)^{\frac{1}{2}},\left(\frac{2^jr^2}{t}\right)^{m-\frac{1}{2}}\right)\left(\frac{1}{\mu(B)}\int_B|f|^q\mu\right)^{\frac{1}{q}}.
$$

Like before, we square and integrate with respect to t, that gives us the assumption (4.32) for some m large enough. ∎

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Chapter 5

BMO solvability and absolute continuity of harmonic measure

5.1 Introduction

The connection between solvability of the Dirichlet problem with L^p data, and scale-invariant absolute continuity properties of harmonic measure (specifically, that harmonic measure belongs to the Muckenhoupt weight class A_∞ with respect to surface measure on the boundary), is well documented, see the monograph of Kenig [Ke], and the references cited there. Specifically, one obtains that the Dirichlet problem is solvable with data in $L^p(\partial\Omega)$ for some $1 < p < \infty$, if and only if harmonic measure ω with some fixed pole is absolutely continuous with respect to surface measure σ on the boundary, and the Poisson kernel $d\omega/d\sigma$ satisfies a reverse Hölder condition with exponent $p' = p/(p-1)$. The most general class of domains for which such results had previously been known to hold is that of the so-called "1-sided Chord-arc domains" (see Definition 2.21 below).

The connection between solvability of the Dirichlet problem and scale invariant absolute continuity of harmonic measure was sharpened significantly in work of Dindos, Kenig and Pipher [DKP], who showed that harmonic measure satisfies an A_∞ condition with respect to surface measure, if and only if a natural Carleson measure/BMO estimate (to be described in more detail momentarily) holds for solutions of the Dirichlet problem with continuous data. Their proof was nominally carried out in the setting of a Lipschitz domain, but more generally, their arguments apply, essentially verbatim, to Chord-arc domains. The results of [DKP] were recently extended to the setting of a 1-sided Chord-arc domain by Zihui Zhao [Z].

More precisely, consider a divergence form elliptic operator

$$L := -\operatorname{div} A(X)\nabla, \tag{5.1}$$

defined in an open set $\Omega \subset \mathbb{R}^{n+1}$, where A is $(n+1)\times(n+1)$, real, L^∞, and satisfies the uniform ellipticity condition

$$\lambda|\xi|^2 \leqslant \langle A(X)\xi,\xi\rangle := \sum_{i,j=1}^{n+1} A_{ij}(X)\xi_j\xi_i, \qquad \|A\|_{L^\infty(\mathbb{R}^n)} \leqslant \lambda^{-1}, \tag{5.2}$$

for some $\lambda > 0$, and for all $\xi \in \mathbb{R}^{n+1}$, $X \in \Omega$.

Given an open set $\Omega \subset \mathbb{R}^{n+1}$ whose boundary is everywhere regular in the sense of Weiner, and a divergence form operator L as above, we shall say that the Dirichlet problem is *BMO-solvable*[1] for L in Ω if for all continuous f with compact support on $\partial\Omega$, the solution u of the classical Dirichlet problem with data f satisfies the Carleson measure estimate

$$\sup_{x\in\partial\Omega,\, r>0} \frac{1}{\sigma\big(\Delta(x,r)\big)} \iint_{\Omega\cap B(x,r)} |\nabla u(Y)|^2\,\delta(Y)\,dY \leqslant C\|f\|^2_{BMO(\partial\Omega)}\,. \tag{5.3}$$

Here, σ is surface measure on $\partial\Omega$, $\delta(Y) := \operatorname{dist}(Y,\partial\Omega)$, and as usual $B(x,r)$ and $\Delta(x,r) := B(x,r)\cap\partial\Omega$ denote, respectively, the Euclidean ball in $\mathbb{R}^{n+1}$, and the surface ball on $\partial\Omega$, with center x and radius r.

[1] It might be more accurate to refer to this property as “VMO-solvability”, but BMO-solvability seems to be the established terminology in the literature. Under less austere circumstances, e.g., in a Lipschitz or (more generally) a Chord-arc domain, or even in the setting of our Theorem 5.3, where we impose an interior Corkscrew condition, it can be seen that the two notions are ultimately equivalent (see [DKP] for a discussion of this point), but in the more general setting of our Theorem 5.1 this matter is not settled.

For $X \in \Omega$, we let ω_L^X denote elliptic-harmonic measure for L with pole at X, and if the dependence on L is clear in context, we shall simply write ω^X.

The main result of this paper is the following. All terminology used in the statement of the theorem and not discussed already, will be defined precisely in the sequel.

Theorem 5.1. *Suppose that $\Omega \subset \mathbb{R}^{n+1}$ is an open set, not necessarily connected, with Ahlfors-David Regular boundary. Let L be a divergence form elliptic operator defined on Ω. If the Dirichlet problem for L is BMO-solvable in Ω, then harmonic measure belongs to weak-A_∞ in the following sense: for every ball $B = B(x,r)$, with $x \in \partial\Omega$, and $0 < r <$ $\mathrm{diam}(\partial\Omega)$, and for all $Y \in \Omega\setminus 4B$, harmonic measure $\omega_L^Y \in$ weak-$A_\infty(\Delta)$, where $\Delta := B\cap\partial\Omega$, and where the parameters in the weak-A_∞ condition are uniform in Δ, and in $Y \in \Omega \setminus 4B$.*

As mentioned above, this result was established in [DKP], and in [Z], under the more restrictive assumption that Ω is Chord-arc, or 1-sided Chord-arc, respectively. The arguments of [DKP] and [Z] rely both explictly and implicitly on quantitative connectivity of the domain, more precisely, on the Harnack Chain condition (see Definition 2.19 below). The new contribution of the present paper is to dispense with all connectivity assumptions, both qualitative and quantitative. In particular, we do not assume the Harnack Chain condition, even within individual connected components of Ω. In this generality, our results are new even for the Laplacian.

We observe that we draw a slightly weaker conclusion than that of [DKP] (or [Z]), namely, weak-A_∞, as opposed to A_∞, but this is the best that can be hoped for in the absence of connectivity: indeed, clearly, the doubling property of harmonic measure may fail without connectivity. Moreover, even in a connected domain enjoying an "interior big pieces of Lipschitz domains" condition, and having an ADR boundary (and thus, for which harmonic measure belongs to weak-A_∞, by the main result of [BL]), the doubling property may fail in the absence of Harnack Chains; see [BL, Section 4] for a counter-example.

In the particular case that L is the Laplacian, we also obtain the following.

Corollary 5.2. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, not necessarily connected, with Ahlfors-David Regular boundary, and in addition, suppose that Ω satisfies an interior Corkscrew condition (Definition 2.18), and that the Dirichlet problem is BMO-solvable for Laplace's equation in Ω. Then $\partial\Omega$ is uniformly rectifiable (Definition 2.17).*

The proof of the corollary is almost immediate: by Theorem 5.1, harmonic measure belongs to weak-A_∞ (even without the Corkscrew condition), so by the results of [HM][2], in the presence of the interior Corkscrew condition, $\partial\Omega$ is uniformly rectifiable.

We remark that the Corkscrew hypothesis is fairly mild, in the sense that if $\Omega = \mathbb{R}^{n+1} \setminus E$ is the complement of an ADR set, then the Corkscrew condition holds automatically, by a simple pigeon-holing argument.

We also obtain a partial converse to Theorem 5.1.

Theorem 5.3. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, not necessarily connected, with Ahlfors-David Regular (ADR) boundary. Let L be a divergence form elliptic operator defined on Ω, and suppose that elliptic-harmonic measure for L belongs to weak-A_∞ in the sense of the conclusion of Theorem 5.1. Then the Dirichlet problem for L is L^p solvable in Ω, for $p < \infty$ sufficiently large. In the special case that L is the Laplacian, the Dirichlet problem is BMO-solvable, provided also that Ω satisfies an interior Corkscrew condition.*

As noted above, our main new contribution is Theorem 5.1, which establishes the direction BMO-solvability implies $\omega \in$ weak-A_∞; it is in that direction that the lack of connec-

[2]See also [HLMN] and [MT] for more general versions of the result of [HM].

tivity is most problematic. By contrast, our proof of the opposite implication (i.e., Theorem 5.3) is a fairly routine adaptation of the corresponding arguments of [DKP] and of [FN]. On the other hand, let us point out that in Theorem 5.3, we have imposed an extra assumption, namely the Corkscrew condition. At present, we do not know whether the latter hypothesis is necessary to obtain the conclusion of Theorem 5.3, nor do we know whether the conclusion of BMO solvability extends to the case of a general divergence form elliptic operator L.

To provide some further context for our results here, let us mention that recently, Kenig, Kirchheim, Pipher and Toro have shown in [KKiPT] that for a Lipschitz domain Ω, a weaker Carleson measure estimate, namely, a version of (5.3) in which the BMO norm of the boundary data is replaced by $||u||_{L^\infty(\Omega)}$, still suffices to establish that ω_L satisfies an A_∞ condition with respect to surface measure on $\partial\Omega$. Moreover, the argument of [KKiPT] carries over with minor changes to the more general setting of a uniform (i.e., 1-sided NTA) domain with Ahlfors-David regular boundary [HMT]. However, in contrast to our Theorem 5.1, to deduce absolute continuity of harmonic measure under the weaker L^∞ Carleson measure condition seems necessarily to require some sort of connectivity (such as the Harnack Chain condition enjoyed by uniform domains). Indeed, specializing to the case that L is the Laplacian, an example of Bishop and Jones [BiJ] shows that harmonic measure ω need *not* be absolutely continuous with respect to surface measure, even for domains with uniformly rectifiable boundaries, whereas the first named author of this paper, along with J. M. Martell and S. Mayboroda, have shown in [HMM] that uniform rectifiability of $\partial\Omega$ alone suffices to deduce the L^∞ version of (5.3) in the harmonic case (and indeed, for solutions of certain other elliptic equations as well).

The paper is organized as follows. In Section 5.2, we recall some known results from the theory of elliptic PDE. In Sections 5.3 and 5.4, we give the proofs of Theorems 5.1 and 5.3, respectively.

5.2 Preliminaries

In this section, we record some known estimates for elliptic harmonic measure ω_L associated to a divergence form operator L as in (5.1) and (5.2), and for solutions of the equation $Lu = 0$, in an open set Ω with an ADR boundary. We recall that, as a consequence of the ADR property, every point on $\partial\Omega$ is regular in the sense of Wiener (see, e.g., [HLMN, Remark 3.26, Lemma 3.27]).

Lemma 5.4 (Bourgain [Bo]). *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, and suppose that $\partial\Omega$ is n-dimensional ADR. Then there are uniform constants $c \in (0, 1)$ and $C \in (1, \infty)$, depending only on n, ADR, and the ellipticity parameter λ, such that for every $x \in \partial\Omega$, and every $r \in (0, \operatorname{diam}(\partial\Omega))$, if $Y \in \Omega \cap B(x, cr)$, then*

$$\omega_L^Y(\Delta(x, r)) \geqslant 1/C > 0\,. \tag{5.4}$$

We refer the reader to [Bo, Lemma 1] for the proof in the case that L is the Laplacian, but the proof is the same for a general uniformly elliptic divergence form operator.

We note for future reference that in particular, if $\hat{x} \in \partial\Omega$ satisfies $|X - \hat{x}| = \delta(X)$, and $\Delta_X := \partial\Omega \cap B\big(\hat{x}, 10\delta(X)\big)$, then for a slightly different uniform constant $C > 0$,

$$\omega_L^X(\Delta_X) \geqslant 1/C\,. \tag{5.5}$$

Indeed, the latter bound follows immediately from (5.4), and the fact that we can form a Harnack Chain connecting X to a point Y that lies on the line segment from X to $\hat{x}$, and satisfies $|Y - \hat{x}| = c\delta(X)$.

As a consequence of Lemma 5.4, we have the following.

Corollary 5.5. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, and suppose that $\partial\Omega$ is n-dimensional ADR. For $x \in \partial\Omega$, and $0 < r < \operatorname{diam} \partial\Omega$, let u be a non-negative solution of $Lu = 0$ in $\Omega \cap B(x, 2r)$, which vanishes continuously on $\Delta(x, 2r) = B(x, 2r) \cap \partial\Omega$. Then for some $\alpha > 0$,*

$$u(Y) \leqslant C \left(\frac{\delta(Y)}{r}\right)^{\alpha} \frac{1}{|B(x,2r)|} \iint_{B(x,2r)\cap\Omega} u\,, \qquad \forall\, Y \in B(x,r) \cap \Omega\,, \tag{5.6}$$

where the constants C and α depend only on n, ADR and λ.

5.3 Proof of Theorem 5.1: BMO-solvability implies $\omega \in$ weak-A_∞

The basic outline of the proof follows that of [DKP], but the lack of Harnack Chains requires in addition some slightly delicate geometric arguments inspired in part by the work of Bennewitz and Lewis [BL].

We begin by recalling the following deep fact, established in [BL]. Given a point $X \in \Omega$, let $\hat{x} \in \partial\Omega$ be a "touching point" for the ball $B(X, \delta(X))$, i.e., $|X - \hat{x}| = \delta(X)$. Set

$$\Delta_X := \Delta\big(\hat{x}, 10\delta(X)\big)\,. \tag{5.7}$$

Lemma 5.6. *Suppose that $\partial\Omega$ is ADR, and suppose also that there are positive constants c_0 and $\eta < 1$, such that for each $X \in \Omega$, and for every Borel set $F \subset \Delta_X$,*

$$\sigma(F) \geqslant (1-\eta)\sigma(\Delta_X) \quad \Longrightarrow \quad \omega^X(F) \geqslant c_0\,. \tag{5.8}$$

Then $\omega^Y \in$ weak-$A_\infty(\Delta)$, where $\Delta = B \cap \partial\Omega$, for every ball $B = B(x,r)$, with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, and for all $Y \in \Omega \setminus 4B$. Moreover, the parameters in the weak-A_∞

condition depend only on n, ADR, η, c_0, and the ellipticity parameter λ of the divergence form operator L.

Remark 5.7. Lemma 5.6 is not stated explicitly in this form in [BL], but may be gleaned readily from the combination of [BL, Lemma 2.2] and its proof, and [BL, Lemma 3.1]. We mention also that the paper [BL] treats explictly only the case that L is the Laplacian, but the proofs of Lemma 2.2 and Lemma 3.1 in [BL] carry over verbatim to the case of a general uniformly elliptic divergence form operator with real coefficients.

Given the BMO-solvability estimate (5.3), it suffices to verify the hypotheses of Lemma 5.6, with η and c_0 depending only on n, ADR, λ, and the constant C in (5.3). To this end, we fix $X \in \Omega$, and for notational convenience, we set

$$r := \delta(X)\,.$$

We choose $\hat{x} \in \partial\Omega$ so that $|X - \hat{x}| = r$, and let $a \in (0, \pi/10000)$ be a sufficiently small number to be chosen depending only on n and ADR. We then define Δ_X as in (5.7), and set

$$B_X := B(\hat{x}, 10r)\,, \quad B'_X := B(\hat{x}, ar)\,, \quad \Delta'_X := \Delta\big(\hat{x}, ar\big)\,. \tag{5.9}$$

We make the following pair of claims.

Claim 1. For a small enough, depending only on n and ADR, there is a constant $\beta > 0$ depending only on n, a, ADR and λ, and a ball $B_1 := B(x_1, ar) \subset B_X$, with $x_1 \in \partial\Omega$, such that $\operatorname{dist}(B'_X, B_1) \geqslant 5ar$, and

$$\omega_L^X(\Delta_1) \geqslant \beta\, \omega_L^X(\Delta_X)\,, \tag{5.10}$$

where $\Delta_1 := B_1 \cap \partial\Omega$.

Claim 2. Suppose that u is a non-negative solution of $Lu = 0$ in Ω, vanishing continuously on $2\Delta'_X$, with $\|u\|_{L^\infty(\Omega)} \leqslant 1$. Then for every $\varepsilon > 0$,

$$u(X) \leqslant C_\varepsilon \left(\frac{1}{\sigma\left(\Delta_X\right)} \iint_{B_X \cap \Omega} |\nabla u(Y)|^2 \delta(Y)\, dY \right)^{1/2} + C\varepsilon^\alpha\,, \tag{5.11}$$

where $\alpha > 0$ is the Hölder exponent in Corollary 5.5.

Momentarily taking these two claims for granted, we now follow the argument in [DKP], with some minor modifications, in order to establish the hypotheses of Lemma 5.6. Let B_1 and Δ_1 be as in Claim 1. Let $F \subset \Delta_X$ be a Borel set satisfying the first inequality in (5.8), for some small $\eta > 0$. If we choose η small enough, depending only on n, ADR, and the constant a in the definition of B'_X, then

$$\sigma(F_1) \geqslant \left(1 - \sqrt{\eta}\right) \sigma(\Delta_1)\,,$$

where $F_1 := F \cap \Delta_1$. Set $A_1 := \Delta_1 \setminus F_1$, and define

$$f := \max\left(0, 1 + \gamma \log \mu(1_{A_1})\right),$$

where γ is a small number to be chosen, and μ is the usual Hardy-Littlewood maximal operator on $\partial\Omega$. Note that

$$0 \leqslant f \leqslant 1\,, \qquad \|f\|_{BMO(\partial\Omega)} \leqslant C\gamma\,, \qquad 1_{A_1} \leqslant f\,. \tag{5.12}$$

Note also that if $z \in \partial\Omega \setminus 2B_1$, then

$$\mu(1_{A_1})(z) \lesssim \frac{\sigma(A_1)}{\sigma(\Delta_1)} \lesssim \sqrt{\eta}\,,$$

where the implicit constants depend only on n and ADR. Thus, if η is chosen small enough depending on γ, then $1 + \gamma \log \mu(1_{A_1})$ will be negative, hence $f \equiv 0$, on $\partial\Omega \setminus 2B_1$.

In order to work with continuous data, we shall require the following.

Lemma 5.8. *There exists a collection of continuous functions $\{f_s\}_{0<s<ar/1000}$, defined on $\partial\Omega$, with the following properties.*

1. $0 \leqslant f_s \leqslant 1$, *for each* s.

2. $\operatorname{supp}(f_s) \subset 3B_1 \cap \partial\Omega$.

3. $1_{A_1}(z) \leqslant \liminf_{s\to 0} f_s(z)$, *for* ω^X*-a.e.* $z \in \partial\Omega$.

4. $\sup_s \|f_s\|_{BMO(\partial\Omega)} \leqslant C\|f\|_{BMO(\partial\Omega)} \lesssim \gamma$, *where* $C = C(n, ADR)$.

The proof is based on a standard mollification of the function f constructed above. We defer the routine proof to the end of this section.

Let u_s be the solution of the Dirichlet problem for the equation $Lu_s = 0$ in Ω, with data f_s. Then, for a small $\varepsilon > 0$ to be chosen momentarily, by Lemma 5.8, Fatou's lemma, and Claim 2, we have

$$\omega_L^X(A_1) \leqslant \int_{\partial\Omega} \liminf_{s\to 0} f_s \, d\omega^X \leqslant \liminf_{s\to 0} u_s(X) \leqslant C_\varepsilon \gamma + C\varepsilon^\alpha , \tag{5.13}$$

where in the last step we have used (5.11), (5.3), and Lemma 5.8-(4). Combining (5.13) with (5.5), we find that

$$\omega_L^X(A_1) \leqslant \left(C_\varepsilon\gamma + C\varepsilon^\alpha\right) \omega_L^X(\Delta_X) . \tag{5.14}$$

Next, we set $A := \Delta_X \setminus F$, and observe that by definition of A and A_1, along with Claim 1, and (5.14),

$$\omega_L^X(A) \leqslant \omega_L^X(\Delta_X \setminus \Delta_1) + \omega_L^X(A_1) \leqslant \left(1 - \beta + C_\varepsilon\gamma + C\varepsilon^\alpha\right) \omega_L^X(\Delta_X) .$$

We now choose first $\varepsilon > 0$, and then $\gamma > 0$, so that $C_\varepsilon\gamma + C\varepsilon^\alpha < \beta/2$, to obtain that

$$\omega_L^X(F) \geqslant \frac{\beta}{2} \omega_L^X(\Delta_X) \geqslant c\beta ,$$

where in the last step we have used (5.5).

It now remains only to establish the two claims, and to prove Lemma 5.8.

Proof of Claim 1. By translation and rotation, we may suppose without loss of generality that $\hat{x} = 0$, and that the line segment joining $\hat{x}$ to X is purely vertical, thus, $X = re_{n+1}$, where as usual $e_{n+1} := (0, ..., 0, 1)$. Let $\Gamma, \Gamma', \Gamma''$ denote, respectively, the open inverted vertical cones with vertex at X having angular apertures $200a$, $100a$, and $20a$, respectively (recall that $a < \pi/10000$). Then $B'_X \subset \Gamma''$ (where B'_X is defined in (5.9)). Let $B_0 := B(X, r)$ denote the open "touching ball", so that $B_0 \cap \partial\Omega = \emptyset$, and define a closed annular region $R_0 := \overline{5B_0} \setminus B_0$. We now consider two cases:

Case 1. $\partial\Omega \cap (R_0 \setminus \Gamma)$ is non-empty. In this case, we let x_1 be the point in $\partial\Omega \cap (R_0 \setminus \Gamma)$ that is closest to X (if there is more than one such point, we just pick one). Then by construction $r \leqslant |X - x_1| \leqslant 5r$, and the ball $B_1 = B(x_1, ar)$ misses Γ', hence $\operatorname{dist}(B_1, B'_X) \geqslant \operatorname{dist}(B_1, \Gamma'') > 5ar$. Moreover, since x_1 is the closest point to X, setting $\rho := |X - x_1|$, we have that $\Omega' \cap \partial\Omega = \emptyset$, where

$$\Omega' := \big(B(X, \rho) \setminus \overline{\Gamma}\big) \cup B_0 \,.$$

Consequently, we may construct a Harnack Chain within the subdomain $\Omega' \subset \Omega$, connecting X to a point $Y \in B(x_1, car) \cap \Omega'$, with $\delta(Y) \geqslant cr/2$, where c is the constant in Lemma 5.4. Thus, by Harnack's inequality and Lemma 5.4,

$$\omega_L^X(\Delta_1) \gtrsim \omega_L^Y(\Delta_1) \geqslant 1/C \,.$$

Since $\omega_L^X(\Delta_X) \leqslant 1$, we obtain (5.10), and thus Claim 1 holds in the present case.

Case 2. $\partial\Omega \cap (R_0 \setminus \Gamma) = \emptyset$. By ADR, we have that

$$\sigma\big(\Delta(0, 10ar)\big) \leqslant C(ar)^n \,, \qquad \sigma\big(B(X, 4r) \cap \partial\Omega\big) \geqslant r^n/C \,.$$

Thus, for a chosen small enough, depending only on n and ADR, we see that the set $\partial\Omega \cap \big(B(X, 4r) \setminus B(0, 10ar)\big)$ is non-empty. Consequently, under the scenario of Case 2,

$$\partial\Omega \cap \left(\overline{B(X, 4r)} \setminus B(0, 10ar)\right) \subset \Gamma .$$

Define

$$\theta_0 := \min\left\{\theta \in [0, 200a) : \partial\Omega \cap \left(\overline{B(X, 4r)} \setminus B(0, 10ar)\right) \subset \Gamma_\theta\right\},$$

where Γ_θ is the inverted cone with vertex at X of angular aperture θ (if $n + 1 = 2$, it may happen that $\theta_0 = 0$, in which case $\partial\Omega \cap \big(\overline{B(X, 4r)} \setminus B(0, 10ar)$ is contained in the vertical ray pointing straight downward from 0). Then by construction, there is a point

$$x_1 \in \partial\Gamma_{\theta_0} \cap \partial\Omega \cap \left(\overline{B(X, 4r)} \setminus B(0, 10ar)\right)$$

(or, as noted above, x_1 lies on the downward vertical ray if $n+1 = 2$ and $\theta_0 = 0$). Then $B_1 = B(x_1, ar)$ misses $B(0, 9ar)$, so that in particular, $\operatorname{dist}(B_1, B'_X) > 5ar$. Moreover, $\Omega' \cap \partial\Omega = \emptyset$, where now

$$\Omega' := \left(\big(B(X, 4r) \setminus \overline{\Gamma_{\theta_0}}\big) \cup B_0\right) \setminus \overline{B(0, 10ar)}$$

(with the obvious adjustment if $n + 1 = 2$ and $\theta_0 = 0$). Thus, as in Case 1, there is a point $Y \in B(x_1, car) \cap \Omega'$, with $\delta(Y) > cr/2$, which may be joined to X via a Harnack Chain within the subdomain $\Omega' \subset \Omega$, whence by Harnack's inequality, Lemma 5.4, and the fact that $\omega_L^X(\Delta_X) \leqslant 1$, we again obtain (5.10). Claim 1 therefore holds in all cases. ∎

Proof of Claim 2. As in the proof of Claim 1, we may assume by translation and rotation that $\hat{x} = 0$, and that $X = re_{n+1}$, with $r = \delta(X)$. Let Γ denote the *upward* open vertical cone with vertex at 0, of angular aperture $\pi/100$. We let S denote the spherical cap inside Γ, i.e., $S := S^n \cap \Gamma$ (recall that our ambient dimension is $n + 1$). Then by Harnack's inequality, letting μ denote surface measure on the unit sphere, we have

$$u(X) \lesssim \int_S u(r\xi)\,d\mu(\xi) = \int_S \Big(u(r\xi) - u(\varepsilon r\xi)\Big)\,d\mu(\xi) \;+\; O(\varepsilon^\alpha) \;=:\; I + O(\varepsilon^\alpha)\,,$$

where we have used Corollary 5.5 to estimate the "big-O" term. In turn,

$$|I| = \left| \int_S \int_{\varepsilon r}^{r} \frac{\partial}{\partial t}\big(u(t\xi)\big)\,dt\,d\mu(\xi) \right| \;\leqslant\; (\varepsilon r)^{-n} \iint_{\Gamma\cap R_\varepsilon} |\nabla u(Y)|\,dY\,,$$

where $R_\varepsilon := B(0,r) \setminus B(0,\varepsilon r)$, and we have used polar co-ordinates in $n+1$ dimensions.

We then have

$$|I| \lesssim (\varepsilon r)^{-n} r^{(n+1)/2} \left(\iint_{\Gamma\cap R_\varepsilon} |\nabla u(Y)|^2\,dY \right)^{1/2}$$
$$\lesssim\; (\varepsilon)^{-n-1/2} r^{-n/2} \left(\iint_{B(0,r)\cap\Omega} |\nabla u(Y)|^2\,\delta(Y)\,dY \right)^{1/2}\,,$$

where we have used that by construction, $\Gamma \cap R_\varepsilon \subset B(0,r)\cap\Omega$, with $\delta(Y) \approx |Y| \geqslant \varepsilon r$ in $\Gamma \cap R_\varepsilon$. Estimate (5.11) now follows, by ADR and the definition of B_X. ∎

Proof of Lemma 5.8. Let $\zeta \in C_0^\infty(\mathbb{R}^{n+1})$, with

$$\operatorname{supp}(\zeta) \subset B(0,1)\,, \quad \zeta \equiv 1 \text{ on } B(0,1/2)\,, \quad 0 \leqslant \zeta \leqslant 1\,.$$

Given $s \in (0, ar/1000)$, and $z, y \in \partial\Omega$, set

$$\Lambda_s(z,y) := b(z,s)^{-1}\zeta\big(s^{-1}(z-y)\big)\,,$$

where

$$b(z,s) := \int_{\partial\Omega} \zeta\big(s^{-1}(z-y)\big)\,d\sigma(y) \;\approx\; s^n\,, \tag{5.15}$$

uniformly in $z \in \partial\Omega$, by the ADR property. Furthermore,

$$\int_{\partial\Omega} \Lambda_s(z,y)\,d\sigma(y) \equiv 1\,, \qquad \forall\, z \in \partial\Omega\,.$$

We now define

$$f_s(z) := \int_{\partial\Omega} \Lambda_s(z, y)\, f(y)\, d\sigma(y)\,,$$

so that f_s is continuous, by construction. Let us now verify (1)-(4) of Lemma 5.8. We obtain (1) immediately, by (5.12), and the properties of Λ_s, while (2) follows directly from the smallness of s and the fact that $\mathrm{supp}(f) \subset 2B_1 \cap \partial\Omega$. Next, let $z \in \partial\Omega$ be a Lebesgue point (with respect to the measure ω^X) for the function 1_{A_1}, so that

$$1_{A_1}(z) \;=\; \lim_{s\to 0} \int_{\partial\Omega} \Lambda_s(z, y)\, 1_{A_1}\, d\sigma(y) \;\leqslant\; \liminf_{s\to 0} f_s(z)\,,$$

by the last inequality in (5.12). Since ω^X-a.e. $z \in \partial\Omega$ is a Lebesgue point, we obtain (3).

To prove (4), we observe that the second inequality is simply a re-statement of the second inequality in (5.12), so it suffices to show that

$$\|f_s\|_{BMO(\partial\Omega)} \lesssim \|f\|_{BMO(\partial\Omega)}\,, \quad \text{uniformly in } s\,. \tag{5.16}$$

To this end, we fix a surface ball $\Delta = \Delta(x, r)$, and we consider two cases.

Case 1: $s \geqslant r$. In this case, set $c := \fint_{\Delta(x,2s)} f$, so that by ADR, (5.15) and the construction of Λ_s,

$$\fint_\Delta |f_s - c|\, d\sigma \;\lesssim\; \fint_\Delta \fint_{\Delta(x,2s)} |f - c|\, d\sigma \;\lesssim\; \|f\|_{BMO(\partial\Omega)}\,.$$

Case 2: $s < r$. In this case, set $c := \fint_{2\Delta} f$. Then by Fubini's Theorem,

$$\fint_\Delta |f_s(z) - c|\, d\sigma(z) \;\lesssim\; \fint_{2\Delta} |f(y) - c| \int_{\partial\Omega} \Lambda_s(z, y)\, d\sigma(z)\, d\sigma(y) \;\lesssim\; \|f\|_{BMO(\partial\Omega)}\,,$$

where again we have used ADR, (5.15) and the compact support property of $\Lambda_s(z, y)$.

Since these bounds are uniform over all $x \in \partial\Omega$, and $r \in (0, \mathrm{diam}(\partial\Omega))$, we obtain (5.16). ∎

5.4 Proof of Theorem 5.3: $\omega \in$ weak-A_∞ implies L^p and BMO-solvability

In this section, we suppose that Ω is an open set with ADR boundary $\partial\Omega$, and that for every ball $B_0 = B(x_0, r)$, with $x_0 \in \partial\Omega$, and $0 < r < \operatorname{diam}(\partial\Omega)$, and for all $Y \in \Omega \setminus 4B_0$, elliptic-harmonic measure $\omega_L^Y \in$ weak-$A_\infty(\Delta_0)$, where $\Delta_0 := B_0 \cap \partial\Omega$. Thus, $\omega_L^Y \ll \sigma$ in Δ, and the Poisson kernel $k^Y := d\omega_L/d\sigma$ satisfies the weak reverse Hölder condition (2.4), for some uniform $q > 1$. In our proof of BMO-solvability (but not for L^p solvability), we shall also require, at precisely one point in the argument, that the Corkscrew condition (Definition 2.18) is satisfied in Ω. Even in the absence of the Corkscrew condition, it may happen that there is a Corkscrew point X_Δ relative to some particular Δ (e.g., for every $X \in \Omega$, this is true for the surface ball Δ_X as in (5.7), with X itself serving as a Corkscrew point), and in this case, we have the following consequence of the weak-RH_q estimate:

$$\left(\fint_\Delta \left(k^{X_\Delta}\right)^q d\sigma \right)^{1/q} \leqslant C\,\sigma(\Delta)^{-1}\,. \tag{5.17}$$

Indeed, one may cover Δ by a collection of surface balls $\{\Delta' = B' \cap \partial\Omega\}$, in such a way that $X_\Delta \in \Omega \setminus 4B'$, but each Δ' has radius comparable to that of Δ (hence $\sigma(\Delta') \approx \sigma(\Delta)$, by the ADR property), depending on the constant in the Corkscrew condition, and such that the cardinality of the collection $\{\Delta'\}$ is uniformly bounded; one may then readily derive (5.17) by applying (2.4) in each Δ', and using the crude estimate that $\omega^{X_\Delta}(2\Delta')/\sigma(\Delta') \leqslant \sigma(\Delta')^{-1} \approx \sigma(\Delta)^{-1}$.

Our first step is to establish an L^p solvability result. To this end, we define non-tangential "cones" and maximal functions, as follows. First, we fix a collection of standard

Whitney cubes covering Ω, and we denote this collection by $\mathcal{W}$. Given $x \in \partial\Omega$, set

$$\mathcal{W}(x) := \{I \in \mathcal{W} : \operatorname{dist}(x, I) \leqslant 100 \operatorname{diam}(I)\}, \tag{5.18}$$

and define the (possibly disconnected) non-tangential "cone" with vertex at x by

$$\Upsilon(x) := \cup_{I \in \mathcal{W}(x)}. \tag{5.19}$$

For a continuous u defined on Ω, the non-tangential maximal function of u is defined by

$$N_* u(x) := \sup_{Y \in \Upsilon(x)} |u(Y)|. \tag{5.20}$$

Recall that μ denotes the (non-centered) Hardy-Littlewood maximal operator on $\partial\Omega$. We have the following.

Proposition 5.9. *Suppose that there is a $q > 1$, such that (2.4) holds for the Poisson kernel k^Y, for every surface ball $\Delta = B \cap \partial\Omega$, centered on $\partial\Omega$, provided $Y \in \Omega \setminus 4B$. Given g continuous with compact support on $\partial\Omega$, let u be the solution of the Dirichlet problem for L with data g. Then for $p = q/(q-1)$, and for all $x \in \partial\Omega$*

$$N_* u(x) \lesssim (\mu(|g|^p)(x))^{1/p}. \tag{5.21}$$

Thus, for all $s > p$, the Dirichlet problem is L^s-solvable, i.e.,

$$\|N_* u\|_{L^s(\partial\Omega)} \leqslant C_s \|g\|_{L^s(\partial\Omega)}. \tag{5.22}$$

Remark 5.10. As is well known, the weak-RH_q estimate (2.4) is self-improving, i.e., weak-RH_q implies weak-$RH_{q+\varepsilon}$, for some $\varepsilon > 0$, thus, in particular, one may self-improve (5.22) to the case $s = p$.

Proof of Proposition 5.9. Splitting the data g into its positive and negative parts, we may suppose without loss of generality that $g \geqslant 0$, hence also $u \geqslant 0$. Let $x \in \partial\Omega$, fix $Y \in \Upsilon(x)$, and let $\hat{y} \in \partial\Omega$ be a touching point such that $|Y - \hat{y}| = \delta(Y)$. Set

$$\Delta_Y^* := \Delta\big(\hat{y}, 1000\delta(Y)\big)\,, \qquad B_Y^* := B\big(\hat{y}, 1000\delta(Y)\big)\,,$$

and note that $x \in \Delta_Y^*$. Define a continuous partition of unity $\sum_{k\geqslant 0} \varphi_k \equiv 1$ on $\partial\Omega$, such that $0 \leqslant \varphi_k \leqslant 1$ for all $k \geqslant 0$, with

$$\operatorname{supp}(\varphi_0) \subset 4\Delta_Y^*\,, \quad \operatorname{supp}(\varphi_k) \subset R_k := 2^{k+2}\Delta_Y^* \setminus 2^k\Delta_Y^*\,,\ k \geqslant 1\,, \tag{5.23}$$

set $g_k := g\varphi_k$, and let u_k be the solution of the Dirichlet problem with data g_k. Thus, $u = \sum_{k\geqslant 0} u_k$ in Ω. By construction, Y is a Corkscrew point for $4\Delta_Y^*$, and $x \in 4\Delta_Y^*$, hence

$$u_0(Y) \leqslant \int_{\partial\Omega} g_0\, k^Y\, d\sigma \lesssim \left(\fint_{4\Delta_Y^*} g^p\, d\sigma\right)^{1/p} \lesssim \big(\mu\big(g^p\big)(x)\big)^{1/p}\,,$$

where in the next to last step we have used (5.17).

Next, we claim that

$$u_k(Y) \lesssim 2^{-k\alpha}\,\big(\mu\big(g^p\big)(x)\big)^{1/p}\,. \tag{5.24}$$

Given this claim, we may sum in k to obtain (5.21). Thus, it suffices to verify (5.24). To this end, we set

$$\mathcal{W}_k := \big\{I \in \mathcal{W} : I \text{ meets } 2^{k-1}B_Y^*\big\}\,, \qquad \mathcal{W}_k^j := \big\{I \in \mathcal{W}_k : \ell(I) = 2^{-j}\big\}\,,$$

and for each $I \in \mathcal{W}_k$, we fix a point $X_I \in I \cap 2^{k-1}B_Y^*$, and we define

$$\Delta_I := \Delta_{X_I}\,,$$

as in (5.7), with $X = X_I$. We now choose a collection of balls $\{B_i\}_{1\leqslant i\leqslant N}$, with N depending only on n and ADR, and corresponding surface ball $\Delta_i := B_i \cap \partial\Omega$, such that $R_k \subset \cup_{i=1}^N \Delta_i$, and such that for each $i = 1, 2, ..., N$,

$$r_{B_i} \approx 2^k r \quad \text{and} \quad 2^{k-1}B_Y^* \subset \mathbb{R}^{n+1} \setminus 4B_i\,.$$

Then by definition of R_k (see (5.23)), and the ADR property,

$$u_k(X_I) \leqslant \int_{R_k} g\, d\omega^{X_I} \lesssim (2^k r)^n \left(\fint_{2^{k+2}\Delta_Y^*} g^p\, d\sigma \right)^{1/p} \left(\sum_{i=1}^{N} \fint_{\Delta_i} \left(k^{X_I}\right)^q\, d\sigma \right)^{1/q}$$

$$\lesssim \left(\fint_{2^{k+2}\Delta_Y^*} g^p\, d\sigma \right)^{1/p} \lesssim \left(\mu\left(g^p\right)(x)\right)^{1/p}\,, \quad (5.25)$$

where in the next-to-last step we have used the weak-RH_q estimate (2.4) in each Δ_i, along with the crude bound $\omega^{X_I}(2\Delta_i) \leqslant 1$, and the fact that each Δ_i has radius $r_{\Delta_i} \approx 2^k r$.

Next, by Corollary 5.5,

$$\begin{aligned} u_k(Y) &\lesssim 2^{-k\alpha}\, \frac{1}{|2^{k-1}B_Y^*|} \iint_{2^{k-1}B_Y^* \cap \Omega} u_k(Z)\, dZ \\ &\lesssim 2^{-k\alpha} \frac{1}{(2^k r)^{n+1}} \sum_{j:\, 2^{-j} \lesssim 2^k r}\ \sum_{I \in \mathcal{W}_k^j} \iint_I u_k(Z)\, dZ \\ &\approx 2^{-k\alpha} \frac{1}{(2^k r)^{n+1}} \sum_{j:\, 2^{-j} \lesssim 2^k r}\ \sum_{I \in \mathcal{W}_k^j} \ell(I)^{n+1} u_k(X_I) \\ &\lesssim 2^{-k\alpha} \left(\mu\left(g^p\right)(x)\right)^{1/p} \frac{1}{(2^k r)^{n+1}} \sum_{j:\, 2^{-j} \lesssim 2^k r} 2^{-j} \sum_{I \in \mathcal{W}_k^j} \sigma(\Delta_I)\,, \end{aligned}$$

where in the last two lines we have used Harnack's inequality in the Whitney box I, and then (5.25) and the definitions of $\mathcal{W}_k^j$ and of Δ_I. For each fixed j, the surface balls Δ_I, $I \in \mathcal{W}_k^j$, have bounded overlaps, and are all contained in a surface ball of radius $\approx 2^k r$, thus, we obtain the claimed bound (5.24), by the ADR property of $\partial\Omega$. ∎

With Proposition 5.9 in hand, we turn to the proof of BMO-solvability. Our approach here follows that in [DKP], which in turn is based on that of [FN]. We now suppose that the Corkscrew condition holds in Ω, and that L is the Laplacian. In this case, by the result of [HM] (see also [HLMN] and [MT]), the weak-A_∞ condition for harmonic measure implies that $\partial\Omega$ is uniformly rectifiable, and thus, by a result of [HMM2], we have the following square function/non-tangential maximal function estimate: for u harmonic in Ω,

$$\int_{\partial\Omega} \big(Su\big)^p \, d\sigma \leqslant C_p \int_{\partial\Omega} \big(N_* u\big)^p \, d\sigma \,, \tag{5.26}$$

where C_p depends also on n, and the UR constants for $\partial\Omega$ (and thus on the ADR, Corkscrew and weak-A_∞ constants), and where

$$Su(x) := \left(\iint_{\Upsilon(x)} |\nabla u(Y)|^2 \, \delta(Y)^{1-n} \, dY \right)^{1/2},$$

and $\Upsilon(x)$ and $N_* u$ were defined in (5.19) and (5.20).

Now consider a ball $B = B(x,r)$, with $x \in \partial\Omega$, and $0 < r < \operatorname{diam}(\partial\Omega)$, and corresponding surface ball $\Delta = B \cap \partial\Omega$. Let f be continuous with compact support on $\partial\Omega$, and set $h := f - c_\Delta$, where $c_\Delta := \fint_{40\Delta} f$. We construct a smooth partition of unity $\sum_{k \geqslant 0} \varphi_k \equiv 1$ on $\partial\Omega$ as before, but now with 10Δ in place of Δ_Y^*. Set $h_k := h\varphi_k$, and let u_k be the solution to the Dirichlet problem with data h_k. Set

$$\mathcal{W}_B := \big\{ I \in \mathcal{W} : \, I \text{ meets } B \big\}\,, \qquad \mathcal{W}_B^j := \big\{ I \in \mathcal{W}_B : \, \ell(I) = 2^{-j} \big\}\,,$$

and for each $I \in \mathcal{W}_B$, fix a point $X_I \in I \cap B$. As above, let $\Delta_I := \Delta_{X_I}$ be defined as in (5.7), and note that by construction,

$$z \in \Delta_I \quad \Longrightarrow \quad I \in \mathcal{W}(z)\,,$$

where $\mathcal{W}(z)$ is defined in (5.18). Consequently, given $z \in \partial\Omega$,

$$\sum_{I:\, z \in \Delta_I} \iint_I |\nabla u_0(Y)|^2 \, \delta(Y)^{1-n} \, dY \lesssim \big(Su(z)\big)^2 \,. \tag{5.27}$$

Let us note also that

$$I \in \mathcal{W}_B \quad \Longrightarrow \quad \Delta_I \subset \Delta(x, Cr) =: \Delta^*\,,$$

for C chosen large enough. We then have

$$
\begin{aligned}
\iint_{B\cap\Omega} |\nabla u_0(Y)|^2\,\delta(Y)\,dY &\lesssim \sum_{I\in\mathcal{W}_B} \iint_I |\nabla u_0(Y)|^2\,\delta(Y)\,dY \\
&\approx \sum_{I\in\mathcal{W}_B} \fint_{\Delta_I} \iint_I |\nabla u_0(Y)|^2\,\delta(Y)\,dY\,d\sigma \\
&\lesssim \int_{\Delta^*} \left(S\,u_0(z)\right)^2 d\sigma(z) \\
&\lesssim \sigma(\Delta)^{(p-2)/p} \left(\int_{\Delta^*} \left(S\,u_0(z)\right)^p d\sigma(z)\right)^{2/p},
\end{aligned}
$$

where in the last two steps we have used the ADR property and (5.27), and then ADR again. Therefore, by (5.26), and then Remark 5.10, and the definition of u_0,

$$
\frac{1}{\sigma(\Delta)} \iint_{B\cap\Omega} |\nabla u_0(Y)|^2\,\delta(Y)\,dY \lesssim \sigma(\Delta)^{-2/p} \left(\int_{40\Delta} |f - c_\Delta|^p\right)^{2/p} \lesssim \|f\|^2_{BMO(\partial\Omega)}\,.
$$

For $k \geqslant 1$, we set $g_k := |h_k| = |f - c_\Delta|\varphi_k$, and let v_k be the solution of the Dirichlet problem with data g_k. Thus, $|u_k| \leqslant v_k$. For $k \geqslant 0$, set

$$
\widetilde{B} := 40B = B(x, 40r)\,, \quad B_k := 2^k\,\widetilde{B}\,, \quad \Delta_k := B_k \cap \partial\Omega\,,
$$

and let Δ_k^* be a sufficiently large concentric fattening of Δ_k. Given $I \in \mathcal{W}$, define $I^* = ((1+\tau)I$, with τ chosen small enough that $\operatorname{dist}(I^*, \partial\Omega) \approx \operatorname{dist}(I, \partial\Omega) \approx \operatorname{diam}(I)$. Then for $Y \in I^*$, with $I \in \mathcal{W}_B^j$, by Corollary 5.5,

$$
\begin{aligned}
v_k(Y) &\lesssim \left(\frac{\ell(I)}{2^k r}\right)^\alpha \frac{1}{|B_{k-1}|} \iint_{B_{k-1}\cap\Omega} v_k \lesssim \left(2^j 2^k r\right)^{-\alpha} \fint_{\Delta_k^*} N_* v_k\,d\sigma \\
&\lesssim \left(2^j 2^k r\right)^{-\alpha} \left(\fint_{\Delta_k^*} \left(N_* v_k\right)^p d\sigma\right)^{1/p} \lesssim \left(2^j 2^k r\right)^{-\alpha} \left(\fint_{\Delta_{k+2}} |f - c_\Delta|^p\,d\sigma\right)^{1/p} \\
&\lesssim k \left(2^j 2^k r\right)^{-\alpha} \|f\|_{BMO(\partial\Omega)}\,,
\end{aligned}
$$

where in the last two steps we have used Remark 5.10, and a well known telescoping argument. Consequently,

$$\begin{aligned}
\iint_{B\cap\Omega} |\nabla u_k(Y)|^2\, \delta(Y)\, dY &\lesssim \sum_{I\in\mathcal{W}_B} \ell(I) \iint_I |\nabla u_k(Y)|^2\, dY \\
&\lesssim \sum_{I\in\mathcal{W}_B} \ell(I)^{-1} \iint_{I^*} |u_k(Y)|^2\, dY\, d\sigma \\
&\lesssim k^2\, 2^{-2k\alpha}\, \|f\|^2_{BMO(\partial\Omega)} \sum_{j:\, 2^{-j}\lesssim r} \left(2^j r\right)^{-2\alpha} \sum_{I\in\mathcal{W}_B^j} \sigma(\Delta_I) \\
&\lesssim k^2\, 2^{-2k\alpha}\, \|f\|^2_{BMO(\partial\Omega)}\, \sigma(\Delta^*)\,,
\end{aligned}$$

since for each fixed j, the surface balls Δ_I with $I \in \mathcal{W}_B^j$ have bounded overlaps, and are all contained in $\Delta^* = \Delta(x, Cr)$. Dividing by $\sigma(\Delta^*)$ and using ADR, we may then sum in k to obtain (5.3), thus concluding the proof of Theorem 5.3.

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