

**MAXIMAL FOURIER INTEGRALS AND
MULTILINEAR MULTIPLIER OPERATORS**

A Dissertation presented to
the Faculty of the Graduate School
at the University of Missouri

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

by

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July 2016

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MAXIMAL FOURIER INTEGRALS AND
MULTILINEAR MULTIPLIER OPERATORS

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ACKNOWLEDGMENTS

I owe my deepest gratitude to Dr. Loukas Grafakos for introducing me to Harmonic Analysis and for tirelessly guiding me and inspiring me to push my research forward. I had a hard time in the summer 2011 when my mother unexpectedly passed away after an unsuccessful brain tumor surgery. At that time, it was really hard for me to keep my concentration on research and to be productive. Luckily, Dr. Grafakos was enormously sympathetic and encouraged me to return to Columbia and to continue my research. I am indebted to him for responsively answering all my questions and for spending hours to explore my ideas. I am also grateful to him for being my supervisor during my graduate study at Mizzou.

I would like to express my gratitude to the faculty and staff of the Department of Mathematics at the University of Missouri, Columbia for running interesting courses, and for constantly helping international students with their special needs. Many thanks, in particular, to Dr. Peter Casazza for introducing me Frame Theory and letting me be a part of his research group, to Dr. Steven Hofmann, Dr. David G. Retzloff, and Dr. Igor E. Verbitsky for being members of my committee.

I owe thanks to the Vietnamese Ministry of Education and Training and the Vietnam International Education Development which supported me for the first two years of my doctoral program. I am so thankful to the Department of Mathematics at the University of Missouri Columbia and to the Department of Mathematics of College of Education at Hue University in Vietnam for their vital support.

Finally, I cannot thank enough my family for understanding my choice to leave Vietnam and to pursue a doctoral degree in the USA. Especially, I would like to express my deep gratitude to my wife Phuong Truong, who has endlessly supported my research work and has assisted me to complete this dissertation timely, and to my son Hayden and my little daughter Hannah who made this process more enjoyable.

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ABSTRACT

The first topic of this dissertation is concerned with the L^2 boundedness of a maximal Fourier integral operator which arises by transferring the spherical maximal operator on the sphere \mathbb{S}^n to a Euclidean space of the same dimension. Thus, we obtain a new proof of the boundedness of the spherical maximal function on \mathbb{S}^n .

In the second part, we obtain boundedness for m -linear multiplier operators from a product of Lebesgue (or Hardy spaces) on \mathbb{R}^n to a Lebesgue space on \mathbb{R}^n , with indices ranging from zero to infinity. The multipliers lie in an L^2 -based Sobolev space on \mathbb{R}^{mn} uniformly over all annuli, just as in Hörmander's classical multiplier condition. Moreover, via proofs or counterexamples, we find the optimal range of indices for which the boundedness holds within this class of multilinear Fourier multipliers.

Chapter 1

Introduction

Fourier singular integral operators naturally appear in the study of linear partial differential equations and have proved to be useful tools in the solution of linear partial differential equations with variable coefficients. These operators generalize the Fourier transform, or general Fourier multiplier operators, and pseudo-differential operators, such as those that arise in the study of the wave equation. The fundamental L^2 -estimate for Fourier integral operators was established by Hörmander [21]; this was extended by Seeger, Sogge, and Stein [34] to L^p for $1 < p < \infty$. In this dissertation we consider maximal Fourier singular integral operators on \mathbb{R}^n that arise from natural process of a pullback of a maximal operator on the unit sphere \mathbb{S}^n in \mathbb{R}^{n+1} . Such maximal Fourier integral operators are associated with symbols of three variable $\sigma(t, \xi, x)$, where the supremum is taken over $t > 0$, x is the space variable, and ξ is the frequency variable. Under certain nondegeneracy conditions on σ and on the phase, we establish the L^2 boundedness of such maximal Fourier integral operators via an extension of the Rubio de Francia's technique [6]. As an application we obtain an alternative direct proof of Sogge's result [35] about establishing the boundedness of the spherical maximal operator on the unit sphere \mathbb{S}^n for $n \geq 3$.

In the second part of this dissertation, we address the following question: What

is the minimal regularity of a multilinear multiplier so that the associated multilinear operator is bounded from a product of Hardy spaces to another Lebesgue space? Inspired by the work of Calderón and Torchinsky [3], Grafakos and Kalton [13], and Miyachi and Tomita [27], we found an answer to the question. We provide necessary and sufficient conditions on multipliers which lie uniformly over all annuli in an L^2 -based Sobolev space for the associated operators to be bounded from a product of Lebesgue (or Hardy) spaces to another Lebesgue space for the entire range of indices possible.

The essential results in this work are organized as follows: Chapter 2 contains the L^2 boundedness of a maximal Fourier integral and a short proof of the boundedness of the spherical maximal function on the unit sphere \mathbb{S}^n . The study of minimal regularity of multilinear multipliers is given in Chapter 3.

Chapter 2

Maximal Fourier Integrals

2.1 Historical overview

The spherical maximal operator on \mathbb{R}^n was introduced by Stein in [39] as the supremum of spherical averages of a function. Precisely, this is defined as the operator

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{\sigma_r(\mathbb{S}(x, r))} \left| \int_{\mathbb{S}(x, r)} f(y) d\sigma_r(y) \right|, \quad x \in \mathbb{R}^n \quad (2.1)$$

where σ_r is the induced measure on the sphere $\mathbb{S}(x, r) = \{y \in \mathbb{R}^n : |y - x| = r\}$, and f is any function on \mathbb{R}^n , initially taken to be in the Schwartz class. The study of this operator has provided the impetus for the subsequent development of Fourier integral operators and their maximal counterparts. Stein [39] obtained the $L^p(\mathbb{R}^n)$ boundedness of \mathcal{M} whenever for $p > \frac{n}{n-1}$ and $n \geq 3$. The two-dimensional case $n = 2$ turned out to be more complicated and was completed about a decade later by Bourgain [2]. An alternative proof of these results was given by Mockenhaupt, Seeger and Sogge [28].

The study of the operator \mathcal{M} in (2.1) has stimulated significant new research. Greenleaf [19] has considered the case where the sphere is replaced by a surface with

non-zero principal curvature. Sogge and Stein [36] have studied the case where the Gaussian curvature of the surface is nonvanishing to infinite order at any point and later extended this result to variable coefficient maximal operators associated with a family of surfaces of finite order [37]; the same authors have also considered the case of non-vanishing rotational curvature in [38]. In this work, Sogge and Stein studied the maximal operator with respect to dilations of $(n - 1)$ -dimensional compact surfaces embedded in an n -dimensional manifold. Using intricate estimates for oscillatory integrals, Iosevich and Sawyer [23] provided a necessary and sufficient condition for the maximal function associated with surface measure on a smooth convex surface having finite order of contact with its tangent lines, to be bounded on $L^p(\mathbb{R}^n)$ for $p > 2$.

The spherical maximal function on non-Euclidean ambient spaces has also been considered. In this situation, this operator is defined as the supremum of all averages over geodesic spheres. Kohn [24] has obtained the boundedness for this spherical maximal operator on hyperboloids. Ionescu [22] extended this result for noncompact symmetric spaces of real rank one. In [30] Narayanan and Thangavelu showed the L^p -boundedness of the spherical maximal operator on the Heisenberg group \mathbb{H}^n for $p > 2n/(2n - 1)$ and $n \geq 2$. Müller and Seeger [29] have proved the boundedness of the spherical maximal function on 2-step nilpotent Lie groups; since \mathbb{S}^3 is a 2-step nilpotent Lie group, this result covers the spherical maximal function on \mathbb{S}^3 . Fischer [8] has considered the case of free 2-step nilpotent Lie groups. Nevo and Ratnakumar [31] have proved endpoint restricted weak type $(\frac{n}{n-1}, \frac{n}{n-1})$ estimates for the spherical maximal operator acting on radial functions defined on n dimensional symmetric spaces of constant curvature; these fail for general functions. Sogge [35] has employed the powerful machinery of Fourier integral operators to deduce the boundedness of the spherical maximal operator on compact manifolds without boundary and positive injectivity radius.

Rubio de Francia [6] studied the $L^p(\mathbb{R}^n)$ boundedness of maximal operators formed by dilations of a fixed singular multiplier transformation. In the present chapter, we provide an extension of the result in [6], obtaining the boundedness for a maximal (over $t > 0$) Fourier integral operator containing a symbol of three variable $\sigma(t, \xi, x)$, under certain nondegeneracy conditions on σ and on the phase. As an application we obtain an alternative proof of the boundedness of the spherical maximal operator on the n -dimensional sphere \mathbb{S}^n for $n \geq 3$, which is transferred to \mathbb{R}^n via the stereographic projection.

2.2 Maximal Fourier integral operator

The main result of this section is the boundedness of a maximal Fourier integral operator on a domain possessing a special geometrical property, called the cone property; this property is crucial in proving the Sobolev embedding theorem; see [1]. An open set Ω in \mathbb{R}^n has the cone property if there exists a finite truncated cone C whose vertex is the origin such that for each point $x \in \Omega$ there exists a rigid motion R of \mathbb{R}^n such that $x + R[C]$ is contained in Ω . We have the following theorem concerning maximal Fourier integral operators on such domains.

Theorem 2.1. *Let Ω be an open set in \mathbb{R}^n having the cone property. Fix $0 < \lambda \leq \infty$ and let $\sigma : [0, \lambda) \times \mathbb{R}^n \times \Omega \rightarrow \mathbb{C}$ be a C^k function, for some $k \geq n$. Let $\varphi : [0, \lambda) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a C^1 -function. For a Schwartz function f supported inside Ω , define the maximal Fourier integral operator*

$$T^* f(x) = \sup_{0 < t < \lambda} \left| \int_{\mathbb{R}^n} \widehat{f}(\xi) \sigma(t, \xi, x) e^{2\pi i \varphi(t, x) \cdot \xi} d\xi \right|, \quad \forall x \in \Omega.$$

Suppose that there exist constants $A_1, A_2, r > 0$ and $a > \frac{1}{2}$ such that $\sigma(t, \xi, x)$ is supported in the set $\{(t, \xi, x) \in [0, \lambda) \times \mathbb{R}^n \times \Omega : |t\xi| \geq r\}$ and such that

1. $|\partial_x^\alpha \sigma(t, \xi, x)| + \frac{|\partial_t \partial_x^\alpha \sigma(t, \xi, x)|}{|\xi|} \leq |t\xi|^{-a} \psi_\alpha(x)$ with $\psi_\alpha \in L^1(\Omega)$ for all $|\alpha| \leq k, \xi \neq 0$;
2. $A_1 |\det((J_x \varphi)(t, x))| \geq 1$ and $|\partial_t \varphi(t, x)| \leq A_2$ for all $x \in \Omega$ and $0 \leq t < \lambda$.

Then, there exists a finite constant $C(\Omega, a, n, k, A_1, A_2, \psi)$ such that

$$\|T^* f\|_{L^2(\Omega)} \leq C(\Omega, a, n, k, A_1, A_2, \psi) r^{\frac{1}{2}-a} \|f\|_{L^2(\Omega)}$$

for all Schwartz functions f . Thus T^* extends to a bounded map from $L^2(\Omega)$ to $L^2(\Omega)$ with constant depending only on $a, n, k, A_1, A_2, r, \psi$, and on Ω .

Proof. We suppose first that $\sigma(t, \xi, x)$ is supported in the set $s \leq |t\xi| \leq 4s$ for some $s > 0$. For $0 \leq t < \lambda$, denote

$$T^t f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) \sigma(t, \xi, x) e^{2\pi i \varphi(t, x) \cdot \xi} d\xi.$$

Then

$$|T^t f(x)|^2 \leq 2 \left(\int_0^\lambda |T^t f(x)|^2 \frac{dt}{t} \right)^{1/2} \left(\int_0^\lambda \left| t \frac{d}{dt} (T^t f(x)) \right|^2 \frac{dt}{t} \right)^{1/2}.$$

We have

$$\begin{aligned} \frac{d}{dt} (T^t f(x)) &= \int_{\mathbb{R}^n} \widehat{f}(\xi) \partial_t \sigma(t, \xi, x) e^{2\pi i \varphi(t, x) \cdot \xi} d\xi \\ &\quad + \sum_{\ell=1}^n \int_{\mathbb{R}^n} \widehat{f}(\xi) \sigma(t, \xi, x) \partial_t \varphi_\ell(t, x) \xi_\ell e^{2\pi i \varphi(t, x) \cdot \xi} d\xi, \end{aligned}$$

in which φ_ℓ is the ℓ th component function of φ . Denote

$$G_\sigma f(x) = \left(\int_0^\lambda |T^t f(x)|^2 \frac{dt}{t} \right)^{1/2} = \left(\int_0^\lambda \left| \int_{\mathbb{R}^n} \widehat{f}(\xi) \sigma(t, \xi, x) e^{2\pi i \varphi(t, x) \cdot \xi} d\xi \right|^2 \frac{dt}{t} \right)^{1/2}.$$

Since Ω has the cone property, the Sobolev embedding theorem for Ω ([1]) says that

for $k \geq n$ there is a constant $C = C(\Omega, n, k)$ such that

$$\sup_{z \in \Omega} \left| \int_{\mathbb{R}^n} \widehat{f}(\xi) \sigma(t, \xi, z) e^{2\pi i \varphi(t, x) \cdot \xi} d\xi \right| \leq C \sum_{|\alpha| \leq k} \int_{\Omega} \left| \int_{\mathbb{R}^n} \widehat{f}(\xi) \partial_z^\alpha \sigma(t, \xi, z) e^{2\pi i \varphi(t, x) \cdot \xi} d\xi \right| dz.$$

It follows that

$$G_\sigma f(x) \leq C \sum_{|\alpha| \leq k} \int_{\Omega} \left(\int_0^\lambda \left| \int_{\mathbb{R}^n} \widehat{f}(\xi) \partial_z^\alpha \sigma(t, \xi, z) e^{2\pi i \varphi(t, x) \cdot \xi} d\xi \right|^2 \frac{dt}{t} \right)^{1/2} dz.$$

Taking the L^2 -norm, the above inequality gives

$$\|G_\sigma f\|_{L^2(\Omega)} \leq C \sum_{|\alpha| \leq k} \int_{\Omega} \left(\int_0^\lambda \int_{\Omega} \left| \int_{\mathbb{R}^n} \widehat{f}(\xi) \partial_z^\alpha \sigma(t, \xi, z) e^{2\pi i \varphi(t, x) \cdot \xi} d\xi \right|^2 dx \frac{dt}{t} \right)^{1/2} dz.$$

Now letting $y = \varphi(t, x)$ and noting that $A_1 |\det(J_x \varphi)(t, x)| \geq 1$, we have

$$\begin{aligned} \|G_\sigma f\|_{L^2(\Omega)} &\leq A_1 C \sum_{|\alpha| \leq k} \int_{\Omega} \left(\int_0^\lambda \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \widehat{f}(\xi) \partial_z^\alpha \sigma(t, \xi, z) e^{2\pi i y \cdot \xi} d\xi \right|^2 dy \frac{dt}{t} \right)^{1/2} dz \\ &\leq A_1 C \sum_{|\alpha| \leq k} \int_{\Omega} \left(\int_0^\lambda \int_{\mathbb{R}^n} \left| \widehat{f}(\xi) \partial_z^\alpha \sigma(t, \xi, z) \right|^2 d\xi \frac{dt}{t} \right)^{1/2} dz. \end{aligned}$$

By the hypothesis $|\partial_x^\alpha \sigma(t, \xi, x)| \leq |t\xi|^{-a} \psi_\alpha(x)$, we deduce

$$\|G_\sigma f\|_{L^2(\Omega)} \leq \sqrt{\ln(4)} A_1 C s^{-a} \|f\|_{L^2(\Omega)} \sum_{|\alpha| \leq k} \|\psi_\alpha\|_{L^1(\Omega)}. \quad (2.2)$$

Denote $\sigma^b(t, \xi, x) = t \partial_t \sigma(t, \xi, x)$ and $\sigma_\ell^\sharp(t, \xi, x) = t \xi_\ell \sigma(t, \xi, x)$ ($\ell = 1, \dots, n$). Then, it is easy to see that

$$|T^* f(x)|^2 \leq G_\sigma f(x) \left(G_{\sigma^b} f(x) + A_2 \sum_{\ell=1}^n G_{\sigma_\ell^\sharp} f(x) \right), \quad \forall x \in \Omega.$$

Taking the integral of both sides of the preceding inequality, we obtain

$$\|T^* f\|_{L^2(\Omega)}^2 \leq \|G_\sigma f\|_{L^2(\Omega)} \left(\|G_{\sigma^b} f\|_{L^2(\Omega)} + A_2 \sum_{\ell=1}^n \|G_{\sigma_\ell^\#} f\|_{L^2(\Omega)} \right).$$

Applying estimate (2.2) to G_σ , G_{σ^b} , and $G_{\sigma_\ell^\#}$ we conclude that

$$\|T^* f\|_{L^2(\Omega)} \leq C(\Omega, n, k, A_1, A_2) s^{\frac{1}{2}-a} \|f\|_{L^2(\Omega)} \sum_{|\alpha| \leq k} \|\psi_\alpha\|_{L^1(\Omega)}. \quad (2.3)$$

Assume now that σ is supported in the set $|t\xi| \geq r$. Fix functions $\gamma_0, \gamma \in C_0^\infty$ such that γ is supported in $\frac{r}{2} \leq |\xi| \leq 2r$ and

$$\gamma_0(\xi) + \sum_{j \geq 0} \gamma(2^{-j}\xi) = 1, \quad \forall \xi \neq 0,$$

where γ_0 is supported in $|\xi| \leq r$. Since σ vanishes on $|t\xi| < r$, we have

$$\sigma(t, \xi, x) = \sum_{j \geq 0} \sigma(t, \xi, x) \gamma(2^{-j}t\xi)$$

for all $t \in [0, \lambda)$, $\xi \neq 0$ and $x \in \Omega$. Set $\sigma_j(t, \xi, x) = \sigma(t, \xi, x) \gamma(2^{-j}t\xi)$, $j \geq 0$ and

$$T_j^* f(x) = \sup_{0 < t < \lambda} \left| \int_{\mathbb{R}^n} \widehat{f}(\xi) \sigma_j(t, \xi, x) e^{2\pi i \varphi(t, x) \cdot \xi} d\xi \right|.$$

Notice that σ_j enjoys the same properties as σ and is supported in $2^{j-1}r \leq |t\xi| \leq 2^{j+1}r$. It is easy to see that

$$T^* f(x) \leq \sum_{j \geq 0} T_j^* f(x), \quad \forall x \in \Omega.$$

Combining the above inequality with estimate (2.3) for each σ_j we obtain

$$\|T^* f\|_{L^2(\Omega)} \leq C(\Omega, n, A_1, A_2) \sum_{j \geq 0} 2^{(j-1)(\frac{1}{2}-a)} r^{\frac{1}{2}-a} \|f\|_{L^2(\Omega)} \sum_{|\alpha| \leq k} \|\psi_\alpha\|_{L^1(\Omega)}.$$

Thus, we have just derived the inequality

$$\|T^* f\|_{L^2(\Omega)} \leq C(\Omega, a, n, A_1, A_2) r^{\frac{1}{2}-a} \|f\|_{L^2(\Omega)} \sum_{|\alpha| \leq k} \|\psi_\alpha\|_{L^1(\Omega)}$$

for all Schwartz functions on \mathbb{R}^n supported in Ω . This result can be extended to $L^p(\Omega)$ functions via the following lemma whose easy proof is omitted. \square

Lemma 2.2. *Let X be a normed space and let $T : Y \rightarrow \mathbb{R}$ be a sublinear map initially defined on a dense subspace Y of X such that*

$$\|T(y)\|_X \leq C \|y\|_X$$

for all $y \in Y$. Then T can be extended to a bounded map on X with the same norm.

2.3 The spherical maximal operator on the sphere

We now apply our theorem to derive the boundedness of the spherical maximal operator on the sphere. Let \mathbb{S}^n be the n -sphere on \mathbb{R}^{n+1} . For a point $x \in \mathbb{S}^n$ and $r > 0$ we define the $(n-1)$ -sphere

$$\Gamma(x, r) = \{y \in \mathbb{S}^n : |x - y| = 2r\}$$

and the spherical cap

$$\Delta(x, r) = \{y \in \mathbb{S}^n : |x - y| < 2r\}.$$

For brevity, we set $\Delta(x) = \Delta(x, 1/4)$ for all $x \in \mathbb{S}^n$.

In analogy with the Euclidean setting, the spherical maximal operator is defined in the spherical-geometry setting as follows: Given a function $g \in L^1(\mathbb{S}^n)$, we define

$$\mathcal{M}_S(g)(x) = \sup_{0 < r < 1/8} \frac{1}{r^{n-1}} \left| \int_{\Gamma(x,r)} g(y) d\mathcal{H}^{n-1}(y) \right|, \quad (2.4)$$

where \mathcal{H}^{n-1} is the induced $(n-1)$ -Hausdorff measure of \mathbb{R}^{n+1} on $\Gamma(x, r)$. The main result of this section is to show an alternative proof of the following [35]:

Theorem 2.3 ([35]). *For $n \geq 3$ and $p > \frac{n}{n-1}$, there is a constant $C = C(n, p)$ such that*

$$\|\mathcal{M}_S(g)\|_{L^p(\mathbb{S}^n, \sigma_n)} \leq C \|g\|_{L^p(\mathbb{S}^n, \sigma_n)}, \quad (2.5)$$

holds for all smooth functions g on \mathbb{S}^n , where $\sigma_n = \mathcal{H}^n$ is the spherical measure on \mathbb{S}^n and the constant C depending only on p and n . Therefore, \mathcal{M}_S can be uniquely extended to a bounded operator on $L^p(\mathbb{S}^n, \sigma_n)$ with the same constant.

The main idea of the proof is to apply the stereographic projection to transfer \mathcal{M}_S to the Euclidean space where one has many available tools, such as the Fourier transform, maximal Fourier integral operators and the Hardy-Littlewood maximal operator.

Now estimate (2.5) can be localized because of the compactness of the sphere. In fact, we can find a positive integer N_0 such that $\mathbb{S}^n \subset \cup_{j=1}^{N_0} \Delta(x_j, 1/8)$. For $g \in L^1(\mathbb{S}^n)$, using partition of unity ρ_j subordinated to $\Delta(x_j, 1/8)$ to split the function g into several pieces, $g = \sum_{j=1}^{N_0} g\rho_j$, where $0 \leq \rho_j \leq 1$ and $\text{supp}(\rho_j) \subset \Delta(x_j, 1/8)$. The sublinearity of the operator in (2.4) yields immediately the inequality

$$\mathcal{M}_S(g) \leq \sum_{j=1}^{N_0} \mathcal{M}_S(g_j),$$

with $g_j = g\rho_j$, $1 \leq j \leq N_0$. Note that if function g is supported in $\Delta(x_0, \frac{1}{8})$, then

$\mathcal{M}_S(g)$ is supported in $\Delta(x_0) = \Delta(x_0, \frac{1}{4})$. Thus, to prove the estimate (2.5), it is enough to show that

$$\int_{\Delta(x_j)} \mathcal{M}_S(g_j)^p d\sigma_n \leq C(n, p) \int_{\Delta(x_j)} |g_j|^p d\sigma_n,$$

where g_j is supported in $\Delta(x_j, \frac{1}{8})$. Let $e_{n+1} = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$. Since

$$\mathcal{M}_S(g \circ R^{-1})(Rx) = \mathcal{M}_S(g)(x)$$

for all rotations $R \in O(n+1)$. A suitable choice of rotations, it suffices to prove the inequality

$$\int_{\Delta(-e_{n+1})} \mathcal{M}_S(g)^p(x) d\sigma_n(x) \leq C(n, p) \int_{\Delta(-e_{n+1})} |g(x)|^p d\sigma_n(x) \quad (2.6)$$

for every integrable function g supported in $\Delta(-e_{n+1}, \frac{1}{8})$.

Now the stereographic projection $\Pi : \mathbb{R}^n \rightarrow \mathbb{S}^n$, given by

$$\Pi(x) = \frac{2}{1+|x|^2}x + \frac{1-|x|^2}{1+|x|^2}e_{n+1},$$

will allow us to transfer the problem to the Euclidean setting. Fix $x \in B(0, 1) \subset \mathbb{R}^n$ and denote $\Sigma(x, r) = \Pi^{-1}(\Gamma(\Pi(x), r))$, ($0 < r < 1/8$). Note that $\Sigma(x, r)$ is an $(n-1)$ -sphere $\mathbb{S}^{n-1}(c_x, \rho_x) \subset \mathbb{R}^n$. Let $\mu_{n-1} = (\Pi^{-1})_*(\mathcal{H}^{n-1})$ be the pushforward measure of \mathcal{H}^{n-1} associated with the map Π^{-1} . Then

$$\begin{aligned} \mathcal{M}_S(g)(\Pi(x)) &= \sup_{0 < r < 1/8} \frac{1}{r^{n-1}} \left| \int_{\Gamma(\Pi(x), r)} g(y) d\mathcal{H}^{n-1}(y) \right| \\ &= \sup_{0 < r < 1/8} \frac{1}{r^{n-1}} \left| \int_{\Sigma(x, r)} (g \circ \Pi)(z) d\mu_{n-1}(z) \right|. \end{aligned}$$

Since the measure μ_{n-1} is comparable with the normal surface measure on $\Sigma(x, r) =$

$\mathbb{S}^{n-1}(c_x, \rho_x)$, we have

$$\mathcal{M}_S(g)(\Pi(x)) \approx \sup_{0 < r < 1/8} \frac{1}{r^{n-1}} \left| \int_{\mathbb{S}^{n-1}(c_x, \rho_x)} (g \circ \Pi)(y) d\sigma_{n-1}(y) \right|.$$

(The notation \approx means that the quantities are comparable.)

Now change variables

$$y = \frac{x}{1 - r^2(1 + |x|^2)} - \frac{r\sqrt{1 - r^2}(1 + |x|^2)}{1 - r^2(1 + |x|^2)} z,$$

then $\mathcal{M}_S(g)(\Pi(x))$ is comparable to

$$\sup_{0 < r < 1/8} a(r, x)^{n-1} \left| \int_{\mathbb{S}^{n-1}} (g \circ \Pi) \left(\frac{x - r\sqrt{1 - r^2}(1 + |x|^2)z}{1 - r^2(1 + |x|^2)} \right) d\sigma_{n-1}(z) \right|,$$

in which the quantity $a(r, x) = \frac{\sqrt{1 - r^2}(1 + |x|^2)}{1 - r^2(1 + |x|^2)}$ is comparable to a positive constant for all $0 < r < 1/8$ and $|x| < 1$. Set $s = r^2(1 + |x|^2)$, ($0 < s < 1/32$). Then $\mathcal{M}_S(g)(\Pi(x))$ can be controlled by

$$\sup_{0 < s < \frac{1}{32}} \left| \int_{\mathbb{S}^{n-1}} (g \circ \Pi) \left(\frac{x}{1 - s} - \frac{\sqrt{s}\sqrt{1 + |x|^2 - s}}{1 - s} z \right) d\sigma_{n-1}(z) \right|.$$

This sequence of calculations leads us to define an operator acting on every locally integrable function h on \mathbb{R}^n by setting

$$\mathcal{N}(h)(x) = \sup_{0 < s < \frac{1}{32}} \left| \int_{\mathbb{S}^{n-1}} h \left(\frac{x}{1 - s} - \frac{\sqrt{s}\sqrt{1 + |x|^2 - s}}{1 - s} z \right) d\sigma_{n-1}(z) \right|.$$

The required inequality (2.6) will be established if the operator \mathcal{N} is bounded from L^p to L^p when acting on functions supported in the unit ball. The following result is the main ingredient needed to prove Theorem 2.3.

Lemma 2.4. For $n \geq 3$ and $p > \frac{n}{n-1}$, the operator \mathcal{N} satisfies the estimate

$$\|\mathcal{N}(h)\|_{L^p(B(0,2))} \leq C \|h\|_{L^p(B(0,2))} \quad (2.7)$$

for all smooth functions h supported in the ball $B(0, 2)$.

The proof of Lemma 2.4 will be provided in the next section. Now we examine certain examples that provide us information about the behavior of this operator.

Fix $x_0 \in \mathbb{R}^n$ and $\delta > 0$. Acting \mathcal{N} on $\chi_{B(x_0, \delta)}$ yields

$$\mathcal{N}(\chi_{B(x_0, \delta)})(x) = \sup_{0 < s < \frac{1}{32}} \int_{\mathbb{S}^{n-1}} \chi_{B\left(0, \frac{(1-s)\delta}{\sqrt{s}\sqrt{1+|x|^2-s}}\right)}\left(\frac{x-(1-s)x_0}{\sqrt{s}\sqrt{1+|x|^2-s}} - z\right) d\sigma_{n-1}(z).$$

Since $\frac{1-s}{\sqrt{1+|x|^2-s}} > 1/2$ for all $0 < s < 1/32$ and $|x| < 1$,

$$\mathcal{N}(\chi_{B(x_0, \delta)})(x) \geq \sup_{0 < s < \frac{1}{32}} \int_{\mathbb{S}^{n-1}} \chi_{B\left(\frac{(1-s)x_0-x}{\sqrt{s}\sqrt{1+|x|^2-s}}, \frac{\delta}{\sqrt{s}}\right)}(z) d\sigma_{n-1}(z).$$

Denote

$$G_{x_0} = \bigcup_{0 < s < \frac{1}{32}} \{x \in \mathbb{R}^n : |x - x_0|^2 = 1 - s + s|x_0|^2\}.$$

For $0 < s < \frac{1}{32}$, let $x_s = \frac{(1-s)x_0-x}{\sqrt{s}\sqrt{1+|x|^2-s}} \in \mathbb{S}^{n-1}$. Then we have

$$\mathcal{N}(\chi_{B(x_0, \delta)})(x) \geq \int_{\mathbb{S}^{n-1}} \chi_{B(x_s, \frac{\delta}{\sqrt{s}})}(z) d\sigma_{n-1}(z) \geq b_n \left(\frac{\delta}{\sqrt{s}}\right)^{n-1} \geq c_n \delta^{n-1}.$$

Thus, we have just established the lower bound

$$\mathcal{N}(\chi_{B(x_0, \delta)})(x) \geq c_n \delta^{n-1} \chi_{G_{x_0}}.$$

This inequality tells us that \mathcal{N} does not map L^p to L^p for $p < \frac{n}{n-1}$.

More interestingly, note that $|G_{x_0}|$ is arbitrarily large whenever x_0 near infinity;

consequently, the operator \mathcal{N} fails to map $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ globally for all $0 < p < \infty$. However, for $p > \frac{n}{n-1}$ the local boundedness of the operator will be established in the next section.

To exclude the case $p = \frac{n}{n-1}$, we use the inequality

$$\mathcal{N}(\chi_{B(0,1)})(x) \geq c_n \left(\frac{1-|x|^2}{|x|} \right)^{n-1}, \quad \forall \quad 0 < |x| < \frac{1}{4\sqrt{2}}.$$

2.4 The proof of Theorem 2.3

To complete the proof of Theorem 2.3, we fix a smooth function h on \mathbb{R}^n supported in some ball and we prove (2.7) for this h . The passage to general h is achieved via Lemma 2.2.

Before turning to the proof of Theorem 2.3, let us recall the operator

$$\mathcal{N}(h)(x) = \sup_{0 < s < \frac{1}{32}} \left| \int_{\mathbb{S}^{n-1}} h \left(\frac{x}{1-s} - \frac{\sqrt{s}\sqrt{1+|x|^2-s}}{1-s} z \right) d\sigma_{n-1}(z) \right|,$$

for which we need to obtain L^p bounds. Expressing the smooth function h in terms of its Fourier transform \widehat{h} , we may write:

$$\mathcal{N}(h)(x) = \sup_{0 < s < \frac{1}{32}} \left| \int_{\mathbb{R}^n} \widehat{h}(\xi) m \left(\sqrt{1-s+|x|^2} \frac{\sqrt{s}}{1-s} \xi \right) e^{\frac{2\pi i}{1-s} x \cdot \xi} d\xi \right|,$$

where $m(\xi) = \widehat{d\sigma}(\xi)$ is the Fourier transform of the spherical measure. It is easy to see that if h is supported in the unit ball then $\mathcal{N}(h)$ is supported in $B(0,2)$. This operator is similar to the one that was introduced by Rubio de Francia [6]. However, the method that was used in [6] is based on Plancherel's theorem and cannot be applied directly because the multiplier here is a function of two variables x and ξ . To avoid this issue, we need to modify slightly the parameter of the operator.

In fact, the substitution $t = \frac{\sqrt{s}}{1-s} \sqrt{1-s+|x|^2}$ yields a bigger maximal operator

$$T^*h(x) = \sup_{0 < t < 1/8} \left| \int_{\mathbb{R}^n} \widehat{h}(\xi) m(t\xi) e^{2\pi i \alpha(t, |x|^2) x \cdot \xi} d\xi \right|,$$

in which $\alpha : (0, \infty) \times [0, \infty) \rightarrow (0, \infty)$ is determined by

$$\alpha(t, s) = \frac{2(1+t^2)}{\sqrt{s^2 + 2s(1+2t^2) + 1} + 1 - s}.$$

First, we have the following elementary properties of the function α :

1. $1 \leq \alpha(t, s) \leq 2$ for all $(t, s) \in (0, 1) \times [0, \infty)$.
2. $2s \frac{\partial \alpha}{\partial s}(t, s) + \alpha(t, s) \geq \frac{1}{2}$ for all $(t, s) \in (0, \frac{1}{2}) \times [0, \infty)$.
3. For fixed $t \in (0, \frac{1}{2})$, the map $x \mapsto \varphi_t(x) = \alpha(t, |x|^2)x$ is one-to-one on \mathbb{R}^n .
4. The Jacobian matrix of $\varphi_t(x)$ is precisely represented as the sum of two matrices

$$J(\varphi_t(x)) = \alpha(t, |x|^2)I_n + 2 \frac{\partial \alpha}{\partial s}(t, |x|^2)B,$$

when B is the matrix with entries $b_{ij} = x_i x_j$.

5. The determinant of the Jacobian matrix is bounded below by a uniform constant

$$\begin{aligned} \det(J(\varphi_t(x))) &= \alpha(t, |x|^2)^n \left(1 + \frac{2 \frac{\partial \alpha}{\partial s}(t, |x|^2)}{\alpha(t, |x|^2)} |x|^2 \right) \\ &= \alpha(t, |x|^2)^{n-1} \left(2|x|^2 \frac{\partial \alpha}{\partial s}(t, |x|^2) + \alpha(t, |x|^2) \right) \geq \frac{1}{2}, \quad \forall t \in (0, \frac{1}{2}). \end{aligned}$$

6. $|\frac{\partial \alpha}{\partial t}(t, s)| \leq 6$ for all $t \in (0, 1)$ and $s \geq 0$.

Our goal is to establish the L^p local boundedness for the operator T^* . To do this, we first fix functions $\psi_0, \psi \in C^\infty(\mathbb{R}^n)$ such that $\widehat{\psi}$ is smooth and supported in

$\frac{1}{2} \leq |\xi| \leq 2$ and that

$$\widehat{\psi}_0(\xi) + \sum_{j \geq 1} \widehat{\psi}(2^{-j}\xi) = 1,$$

where $\widehat{\psi}_0$ is a smooth function supported inside the ball $|\xi| \leq 2$. Now using the idea of the Littlewood-Paley decomposition to break the function h into frequencies, i.e.

$$h = (\psi_0)_t * h + \sum_{j \geq 1} \psi_{2^{-j}t} * h,$$

where for a function g , define the dilation $g_t(x) = t^{-n}g(x/t)$, $t > 0$, $x \in \mathbb{R}^n$. We have the following simple estimate

$$T^*h(x) \leq \sum_{j=0}^{\infty} T_j^*h(x), \quad x \in \mathbb{R}^n,$$

with

$$T_0^*h(x) = \sup_{0 < t < 1/8} \left| \int_{\mathbb{R}^n} \widehat{h}(\xi) m(t\xi) \widehat{\psi}_0(t\xi) e^{2\pi i \alpha(t, |x|^2) x \cdot \xi} d\xi \right|,$$

and

$$T_j^*h(x) = \sup_{0 < t < 1/8} \left| \int_{\mathbb{R}^n} \widehat{h}(\xi) m(t\xi) \widehat{\psi}(2^{-j}t\xi) e^{2\pi i \alpha(t, |x|^2) x \cdot \xi} d\xi \right|, \quad j \geq 1.$$

Next, we will show that T_0^* is dominated by the Hardy-Littlewood maximal operator for all $|x| \leq 2$; consequently,

$$\|T_0^*\|_{L^p(B(0,2)) \rightarrow L^p(B(0,2))} < \infty, \quad \forall 1 < p < \infty.$$

Indeed, we have

$$\begin{aligned} & \int_{\mathbb{R}^n} \widehat{h}(\xi) m(t\xi) \widehat{\psi}_0(t\xi) e^{2\pi i \alpha(t, |x|^2) x \cdot \xi} d\xi \\ &= \int_{\mathbb{R}^n} h(y) \int_{\mathbb{R}^n} m(t\xi) \widehat{\psi}_0(t\xi) e^{2\pi i (\alpha(t, |x|^2) x - y) \cdot \xi} d\xi dy \\ &= \frac{1}{t^n} \int_{\mathbb{R}^n} h(y) \int_{\mathbb{R}^n} m(\xi) \widehat{\psi}_0(\xi) e^{2\pi i \frac{(\alpha(t, |x|^2) x - y) \cdot \xi}{t}} d\xi dy \end{aligned}$$

$$= \frac{1}{t^n} \int_{\mathbb{R}^n} h(y) (\psi_0 * d\sigma_{n-1}) \left(\beta(t, |x|^2)x + \frac{x-y}{t} \right) dy,$$

where

$$\beta(t, s) = \frac{2t\alpha(t, s)}{\sqrt{s^2 + 2s(1 + 2t^2) + 1} + s + 1 + 2t^2}.$$

Notice that $0 < \beta(t, s) \leq 1$ for all $(t, s) \in (0, \frac{1}{2}) \times [0, \infty)$.

Since the smooth function $\widehat{\psi_0}$ has compact support, ψ_0 is a Schwartz function.

Applying Lemma 2.5 to ψ_0 with $j = 0$ deduces

$$\begin{aligned} \left| (\psi_0 * d\sigma_{n-1}) \left(\beta(t, |x|^2)x + \frac{x-y}{t} \right) \right| &\leq C(n) \left(3 + \left| \beta(t, |x|^2)x + \frac{x-y}{t} \right| \right)^{-n-1} \\ &\leq C(n) \left(1 + \left| \frac{x-y}{t} \right| \right)^{-n-1}. \end{aligned}$$

Thus, T_0^* is majorized by Hardy-Littlewood maximal function on $B(0, 2)$.

It remains to show that

$$\sum_{j \geq 1} \|T_j^* h\|_{L^p(B(0,2))} \leq C(n, p) \|h\|_{L^p(B(0,2))}, \quad p > n/(n-1). \quad (2.8)$$

For $j \geq 1$, in view of the fact that

$$|m(t\xi)| + |\partial_t(m(t\xi))| / |\xi| \leq C |t\xi|^{-\frac{n-1}{2}}, \quad \forall \xi \neq 0,$$

Theorem 2.1 implies the estimate

$$\|T_j^* h\|_{L^2(B(0,2))} \leq C 2^{(1-\frac{n}{2})j} \|h\|_{L^2 B(0,2)}. \quad (2.9)$$

Next, we show that $T_j^* h$ is pointwise controlled by a multiple of the Hardy-Littlewood maximal function with a constant that grows like a multiple of 2^j . Indeed,

we will show that

$$T_j^* h(x) \leq C(n)2^j Mh(x), \quad \forall x \in B(0, 2). \quad (2.10)$$

Using the weak type $(1, 1)$ boundedness of the Hardy-Littlewood maximal function and (2.10), we deduce

$$\|T_j^* h\|_{L^{1,\infty}(B(0,2))} \leq C(n)2^j \|h\|_{L^1(B(0,2))} \quad (2.11)$$

for our fixed function h supported inside the ball $B(0, 2)$.

Now interpolating between inequalities (2.11) and (2.9) yields:

$$\|T_j^* h\|_{L^p(B(0,2))} \leq C(n, p)2^{j(\frac{n}{p}+1-n)} \|h\|_{L^p(B(0,2))}, \quad (1 < p < 2). \quad (2.12)$$

Summing up all inequalities from (2.12) and noting that $p > \frac{n}{n-1}$, the inequality (2.8) is established for $\frac{n}{n-1} < p < 2$; hence, Lemma 2.4 is proved for $\frac{n}{n-1} < p < 2$. Since the operator \mathcal{N} maps L^∞ to L^∞ , estimate (2.7) is also true for $p \geq 2$ via an interpolation between L^p (for some $\frac{n}{n-1} < p < 2$) and L^∞ .

Thus we are only left with establishing inequality (2.10). The following lemma, whose proof can be found in [10, p. 339–340], can be used to prove (2.10).

Lemma 2.5. *Support ϕ is a Schwartz function in \mathbb{R}^n . For every $M > n$, the convolution of $\phi_{2^{-j}}$ with the spherical measure $d\sigma_{n-1}$ can be estimated pointwise by*

$$|\phi_{2^{-j}} * d\sigma_{n-1}(x)| \leq \frac{C(M, n)2^j}{(3 + |x|)^M}, \quad j \geq 0,$$

where $C(M, n)$ is a finite constant depending only on M and n .

Now, the same argument as in proving the pointwise estimate for $T_0^* h(x)$ can be used together with the above lemma to deduce the inequality (2.10).

So far we have been working with a smooth function g with compact support. To extend the boundedness for \mathcal{M}_S to all functions in $L^p(\mathbb{S}^n, \sigma_n)$, we use Lemma [2.2](#).

Chapter 3

Multilinear Multiplier Operators

3.1 Historical overview

Let σ be a bounded function on \mathbb{R}^n . We denote by T_σ the linear Fourier multiplier operator, whose action on Schwartz functions is given by

$$T_\sigma(f)(x) = \int_{\mathbb{R}^n} \sigma(\xi) \widehat{f}(\xi) e^{2\pi i x \xi} d\xi. \quad (3.1)$$

Mikhlin's classical result [26] states that the T_σ admits an L^p -bounded extension for $1 < p < \infty$, whenever

$$|\partial_\xi^\alpha \sigma(\xi)| \leq C_\alpha |\xi|^{-|\alpha|}, \quad \xi \neq 0 \quad (3.2)$$

for all multi-indices α with $|\alpha| \leq [\frac{n}{2}] + 1$. This result was refined by Hörmander [20] who proved that (3.2) can be replaced by the Sobolev-norm condition

$$\sup_{j \in \mathbb{Z}} \|\sigma(2^j(\cdot)) \widehat{\psi}\|_{W^s} < \infty, \quad (3.3)$$

for some $s > \frac{n}{2}$, where $\widehat{\psi}$ is a smooth function supported in $\frac{1}{2} \leq |\xi| \leq 2$ that satisfies

$$\sum_{j \in \mathbb{Z}} \widehat{\psi}(2^{-j}\xi) = 1$$

for all $0 \neq \xi \in \mathbb{R}^n$. Here $\|g\|_{W^s} = \|(I - \Delta)^{s/2}g\|_{L^2}$, where I is the identity operator and $\Delta = \sum_{j=1}^n \partial_j^2$ is the Laplacian on \mathbb{R}^n .

Calderón and Torchinsky [3] showed that the Fourier multiplier operator in (3.1) admits a bounded extension from the Hardy space H^p to H^p with $0 < p \leq 1$ if

$$\sup_{t>0} \|\sigma(t \cdot) \widehat{\psi}\|_{W^s} < \infty$$

and $s > \frac{n}{p} - \frac{n}{2}$. Here the index $s = \frac{n}{p} - \frac{n}{2}$ is critical in the sense that the boundedness of T_σ on H^p does not hold if $s \leq \frac{n}{p} - \frac{n}{2}$. This was pointed out later by Miyachi and Tomita [27].

The multilinear counterpart of the Fourier multiplier theory has been rather similar in the formulation of results, but substantially more complicated in its proofs. The theory of multilinear operators, and in particular that of multilinear multiplier operators, originated in the work of Coifman and Meyer [4], [5], [25] and resurfaced in the work of Grafakos and Torres [18]. Multilinear Fourier multipliers are bounded functions σ on $\mathbb{R}^{mn} = \mathbb{R}^n \times \dots \times \mathbb{R}^n$ associated with the m -linear Fourier multiplier operator in the following way

$$T_\sigma(f_1, \dots, f_m)(x) = \int_{\mathbb{R}^{mn}} e^{2\pi i x \cdot (\xi_1 + \dots + \xi_m)} \sigma(\xi_1, \dots, \xi_m) \widehat{f}_1(\xi_1) \cdots \widehat{f}_m(\xi_m) d\vec{\xi}, \quad (3.4)$$

where f_j are in the Schwartz space of \mathbb{R}^n and $d\vec{\xi} = d\xi_1 \cdots d\xi_m$.

Tomita [40] obtained $L^{p_1} \times \dots \times L^{p_m} \rightarrow L^p$ boundedness ($1 < p_1, \dots, p_m, p < \infty$) for multilinear multiplier operators under a condition analogous to (3.3). Grafakos and Si [17] extended Tomita's results to the case $p \leq 1$ by using L^r -based Sobolev

norms for σ with $1 < r \leq 2$. Fujita and Tomita [9] provided weighted extensions of these results but also noticed that the Sobolev space W^s in (3.3) can be replaced by a product-type Sobolev space $W^{(s_1, \dots, s_m)}$ when $p > 2$. Grafakos, Miyachi, and Tomita [15] extended the range of p in [9] to $p > 1$ and obtained boundedness even in the endpoint case where all but one indices p_j are equal to infinity. Miyachi and Tomita [27] provided extensions of the Calderón and Torchinsky results [3] for Hardy spaces in the bilinear case; note that in [27] it was pointed out that the conditions on the indices are sharp, even in the linear case, i.e., in the Calderón and Torchinsky theorem.

Following this stream of work, we provide extensions of the result of Calderón and Torchinsky [3] ($m = 1$) and of Miyachi and Tomita [27] ($m = 2$) to the case $m \geq 3$. Our work is inspired by that of Calderón and Torchinsky [3], Grafakos and Kalton [13], and certainly of Miyachi and Tomita [27]. As in [27], we find necessary and sufficient conditions, which coincide with those in [27] when $m = 2$, that imply boundedness for multilinear multiplier operators on a products of Hardy or Lebesgue spaces on the entire range of indices $0 < p \leq \infty$. One important aspect of this work is an appropriate regularization of the multilinear multiplier operator which allows the interchange of its action with infinite sums of H^{p_j} atoms (see Section 3.3). Also, we point out that the complexity of the problem increases significantly as m increases. In fact, the main difficulty concerns the case where $1 < p_j < 2$, in which the boundedness holds exactly in the interior of a convex simplex in \mathbb{R}^m . This simplex has $m2^{m-1} + 1$ vertices but it is not enough to obtain the corresponding estimates for the vertices of the simplex, since interpolation between the vertices does not yield minimal smoothness in the interior. We overcome this difficulty by establishing estimates for all the points inside the simplex being arbitrarily close to those $m2^{m-1} + 1$ points without losing smoothness.

We introduce the Sobolev spaces that will be used throughout this chapter. First,

for $x \in \mathbb{R}^n$ we set $\langle x \rangle = \sqrt{1 + |x|^2}$. For $s_1, \dots, s_m > 0$, we denote by $W^{(s_1, \dots, s_m)}$ the Sobolev space (of product type) consisting all functions $f \in L^2(\mathbb{R}^{mn})$ such that

$$\|f\|_{W^{(s_1, \dots, s_m)}} := \left(\int_{\mathbb{R}^{mn}} |\widehat{f}(y_1, \dots, y_m) \langle y_1 \rangle^{s_1} \cdots \langle y_m \rangle^{s_m}|^2 dy_1 \cdots dy_m \right)^{\frac{1}{2}} < \infty.$$

Throughout this work, let ψ be a smooth function on \mathbb{R}^{mn} whose Fourier transform $\widehat{\psi}$ is supported in $\frac{1}{2} \leq |\xi| \leq 2$ and satisfies

$$\sum_{j \in \mathbb{Z}} \widehat{\psi}(2^{-j}\xi) = 1, \quad \xi \neq 0.$$

Also, $H^p(\mathbb{R}^n)$ denotes the real-variable Hardy space of Fefferman and Stein [7], for $0 < p \leq \infty$. This space coincides with the Lebesgue space $L^p(\mathbb{R}^n)$ when $1 < p \leq \infty$. The following is the main result of this chapter.

Theorem 3.1. *Let $0 < p_1, \dots, p_m \leq \infty$, $0 < p \leq \infty$, $\frac{1}{p_1} + \dots + \frac{1}{p_m} = \frac{1}{p}$, $s_1, \dots, s_m > n/2$, and suppose*

$$\sum_{k \in J} \left(\frac{s_k}{n} - \frac{1}{p_k} \right) > -\frac{1}{2} \tag{3.5}$$

for every nonempty subset $J \subset \{1, 2, \dots, m\}$. If σ satisfies

$$A := \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \widehat{\psi}\|_{W^{(s_1, \dots, s_m)}} < \infty, \tag{3.6}$$

then we have

$$\|T_\sigma\|_{H^{p_1} \times \dots \times H^{p_m} \rightarrow L^p} \lesssim A, \tag{3.7}$$

where L^p should be replaced by BMO if $p = \infty$. Moreover, this result is optimal in the sense that if (3.6) and (3.7) are valid then we must necessarily have $s_j \geq n/2$ for all $1 \leq j \leq m$, and

$$\sum_{k \in J} \left(\frac{s_k}{n} - \frac{1}{p_k} \right) \geq -\frac{1}{2}$$

for every nonempty subset J of $\{1, 2, \dots, m\}$.

Remark 3.2. When $0 < p_j \leq 1$ for all $1 \leq j \leq m$, conditions (3.5) imply that $s_j > \frac{n}{2}$. Moreover, the condition in (3.6) is sufficient to guarantee that σ lies in $L^\infty(\mathbb{R}^{mn})$. Indeed, suppose that σ is a function on \mathbb{R}^{mn} that satisfies (3.6). It is easy to see that $\widehat{\psi}(\frac{1}{2}x) + \widehat{\psi}(x) + \widehat{\psi}(2x) = 1$ for all $1 \leq x \leq 2$. Now we want to verify that $|\sigma(2^{j_0}x)|$ is uniformly bounded in $j_0 \in \mathbb{Z}$ for a.e. $1 \leq |x| \leq 2$. Applying the Cauchy-Schwarz inequality and using the conditions $s_k > \frac{n}{2}$, we write

$$\begin{aligned}
& |\sigma(2^{j_0}x)| \\
&= \left| \sum_{|l| \leq 1} \sigma(2^{j_0}x) \widehat{\psi}(2^l x) \right| \leq \sum_{|l| \leq 1} \left| \int_{\mathbb{R}^{mn}} (\sigma(2^{j_0-l}\cdot) \widehat{\psi})^\vee(\xi) e^{2^{l+1}\pi i x \xi} d\xi \right| \\
&\leq \sum_{|l| \leq 1} \int_{\mathbb{R}^{mn}} \prod_{k=1}^m (1 + |\xi_k|^2)^{-\frac{s_k}{2}} \left| \prod_{k=1}^m (1 + |t\xi_k|^2)^{\frac{s_k}{2}} (\sigma(2^{j_0-l}\cdot) \widehat{\psi})^\vee(\xi_1, \dots, \xi_m) \right| d\xi_1 \cdots d\xi_m \\
&\leq \sum_{|l| \leq 1} C(s_1, \dots, s_m, n) \|\sigma_{j_0-l} \widehat{\psi}\|_{W(s_1, \dots, s_m)} \leq 3C(s_1, \dots, s_m, n) \sup_{j \in \mathbb{Z}} \|\sigma_j \widehat{\psi}\|_{W(s_1, \dots, s_m)},
\end{aligned}$$

for almost all x satisfying $1 \leq |x| \leq 2$. Here we set $\sigma_j(\vec{\xi}) = \sigma(2^j \vec{\xi})$. Thus

$$\|\sigma\|_{L^\infty(\mathbb{R}^{mn})} \leq 3C(s_1, \dots, s_m, n) \sup_{j \in \mathbb{Z}} \|\sigma_j \widehat{\psi}\|_{W(s_1, \dots, s_m)} < \infty.$$

Throughout this chapter, we use the notation $A \lesssim B$ to indicate that $A \leq CB$, where the constant C is independent of any essential parameters, and $A \approx B$ if both $A \lesssim B$ and $B \lesssim A$ hold simultaneously.

3.2 Preliminaries and known results

Now fix $0 < p \leq \infty$ and a Schwartz function Φ with $\widehat{\Phi}(0) \neq 0$. Then the Hardy space H^p contains all tempered distributions f on \mathbb{R}^n such that

$$\|f\|_{H^p} := \left\| \sup_{0 < t < \infty} |\Phi_t * f| \right\|_{L^p} < \infty.$$

It is well known that the definition of the Hardy space does not depend on the choice of the function Φ . Note that $H^p = L^p$ for all $p > 1$. When $0 < p \leq 1$, one of nice features of Hardy spaces is the atomic decomposition. More precisely, any function $f \in H^p$ ($0 < p \leq 1$) can be decomposed as $f = \sum_k \lambda_k a_k$, where a_k 's are L^∞ -atoms for H^p supported in cubes Q_k such that $\|a_k\|_{L^\infty} \leq |Q_k|^{-\frac{1}{p}}$ and $\int x^\gamma a_k(x) dx = 0$ for all $|\gamma| < N$, and the coefficients λ_k satisfy $\sum_k |\lambda_k|^p \leq 2^p \|f\|_{H^p}^p$. The order N of the moment condition can be taken arbitrarily large.

A fundamental L^2 estimate for T_σ is given in the following theorem.

Theorem 3.3 ([15]). *If $s_1, \dots, s_m > n/2$, then*

$$\|T_\sigma\|_{L^2 \times L^\infty \times \dots \times L^\infty \rightarrow L^2} \leq C \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \widehat{\psi}\|_{W^{(s_1, \dots, s_m)}}.$$

The following two lemmas are essentially contained in [27], modulo a few minor modifications.

Lemma 3.4 ([27]). *Let m be a positive integer, σ be a function defined on \mathbb{R}^{mn} , and $K = \sigma^\vee$, the inverse Fourier transform of σ . Suppose that σ is supported in the ball $\{y \in \mathbb{R}^{mn} : |y| \leq 2\}$ and suppose $1 \leq l \leq n$, $s_i \geq 0$ for $1 \leq i \leq m$ and $1 \leq p \leq q \leq \infty$. Then for each multi-index α there exists a constant C_α such that*

$$\|\langle y_1 \rangle^{s_1} \cdots \langle y_l \rangle^{s_l} \partial_y^\alpha K(y)\|_{L^q(\mathbb{R}^{ml}, dy_1 \cdots dy_l)} \leq C_\alpha \|\langle y_1 \rangle^{s_1} \cdots \langle y_l \rangle^{s_l} K(y)\|_{L^p(\mathbb{R}^{ml}, dy_1 \cdots dy_l)},$$

where $y = (y_1, \dots, y_m)$ with $y_j \in \mathbb{R}^n$.

Proof. Denote $y = (y', y'')$, where $y' = (y_1, \dots, y_l)$ and $y'' = (y_{l+1}, \dots, y_m)$. Take φ a Schwartz function on \mathbb{R}^{ln} such that $\widehat{\varphi}(y') = 1$ for all $y' \in \mathbb{R}^{ln}$, $|y'| \leq 2$. It is easy to see that $\sigma(y', y'') = \sigma(y', y'')\widehat{\varphi}(y')$ for all $y' \in \mathbb{R}^{ln}$ and $y'' \in \mathbb{R}^{(m-l)n}$. Using the inverse Fourier transform we have

$$\begin{aligned} K(y', y'') &= \left(K * (\varphi \otimes \delta_0) \right)(y', y'') \\ &= \int_{\mathbb{R}^{ln} \times \mathbb{R}^{(m-l)n}} K(y' - u', y'' - u'') \varphi(u') \delta_0(u'') du' du'' \\ &= \int_{\mathbb{R}^{ln}} K(y' - u', y'') \varphi(u') du', \end{aligned}$$

where δ_0 is the Dirac distribution. Therefore,

$$\begin{aligned} \langle y_1 \rangle^{s_1} \cdots \langle y_l \rangle^{s_l} |K(y', y'')| &= \langle y_1 \rangle^{s_1} \cdots \langle y_l \rangle^{s_l} \left| \int_{\mathbb{R}^{ln}} K(y' - u', y'') \varphi(u') du' \right| \\ &\lesssim \int_{\mathbb{R}^{ln}} \left(\prod_{j=1}^l \langle y_j - u_j \rangle^{s_j} \right) |K(y' - u', y'')| \langle u_1 \rangle^{s_1} \cdots \langle u_l \rangle^{s_l} |\varphi(u')| du' \\ &\leq C_1 \|\langle y_1 \rangle^{s_1} \cdots \langle y_l \rangle^{s_l} K(y', y'')\|_{L^p(\mathbb{R}^{ln}, dy')} \|\langle u_1 \rangle^{s_1} \cdots \langle u_l \rangle^{s_l} \varphi(u')\|_{L^{p'}(\mathbb{R}^{ln}, du')} \\ &\leq C_2 \|\langle y_1 \rangle^{s_1} \cdots \langle y_l \rangle^{s_l} K(y', y'')\|_{L^p(\mathbb{R}^{ln}, dy')}. \end{aligned}$$

Here we used Hölder's inequality in the second to last line. Thus, we have proved

$$\|\langle y_1 \rangle^{s_1} \cdots \langle y_l \rangle^{s_l} \partial_y^\alpha K(y)\|_{L^\infty(\mathbb{R}^{ml}, dy_1 \cdots dy_l)} \lesssim \|\langle y_1 \rangle^{s_1} \cdots \langle y_l \rangle^{s_l} K(y)\|_{L^p(\mathbb{R}^{ml}, dy_1 \cdots dy_l)}.$$

Now interpolate between the last inequality above with the trivial identity

$$\|\langle y_1 \rangle^{s_1} \cdots \langle y_l \rangle^{s_l} \partial_y^\alpha K(y)\|_{L^p(\mathbb{R}^{ml}, dy_1 \cdots dy_l)} = \|\langle y_1 \rangle^{s_1} \cdots \langle y_l \rangle^{s_l} K(y)\|_{L^p(\mathbb{R}^{ml}, dy_1 \cdots dy_l)},$$

we obtain the required inequality in the lemma. \square

Lemma 3.5 ([27]). *Let $s_i > \frac{n}{2}$ for $1 \leq i \leq m$, and let $\widehat{\zeta}$ be a smooth function which is*

supported in an annulus centered at zero. Suppose that Φ is a smooth function away from zero that satisfies the estimates

$$|\partial_\xi^\alpha \Phi(\xi)| \leq C_\alpha |\xi|^{-|\alpha|}$$

for all $\xi \in \mathbb{R}^{mn}$, $\xi \neq 0$, and for all multi-indices α . Then there exists a constant C such that

$$\sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \Phi(2^j \cdot) \widehat{\zeta}\|_{W^{(s_1, \dots, s_m)}} \leq C \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \widehat{\psi}\|_{W^{(s_1, \dots, s_m)}}.$$

Adapting the Calderón and Torchinsky interpolation techniques in the multilinear setting (for details on this we refer to [15, p. 318]) allows us to interpolate between two estimates for multilinear multiplier operators from a product of some Hardy spaces or Lebesgue spaces to Lebesgue spaces.

Theorem 3.6 ([15]). *Let $0 < p_i, p_{i,k} \leq \infty$ and $s_{i,k} > 0$ for $i = 0, 1$ and $1 \leq k \leq m$. For $0 < \theta < 1$, set $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{p_k} = \frac{1-\theta}{p_{0,k}} + \frac{\theta}{p_{1,k}}$, and $s_k = (1-\theta)s_{0,k} + \theta s_{1,k}$. Assume that the multilinear operator T_σ defined in (3.4) satisfies the estimates*

$$\|T_\sigma\|_{H^{p_{i,1}} \times \dots \times H^{p_{i,m}} \rightarrow L^{p_i}} \leq C_i \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \widehat{\psi}\|_{W^{(s_{i,1}, \dots, s_{i,m})}}, \quad i = 0, 1,$$

where L^{p_i} should be replaced by BMO if $p_i = \infty$. Then

$$\|T_\sigma\|_{H^{p_1} \times \dots \times H^{p_m} \rightarrow L^p} \leq C \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \widehat{\psi}\|_{W^{(s_1, \dots, s_m)}},$$

where L^p should be replaced by BMO if $p = \infty$.

The following result is due to Fujita and Tomita [9] for $2 < p < \infty$, while the extension to $p > 1$ and the endpoint case where all but one indices are equal to infinity is due to Grafakos, Miyachi and Tomita [15].

Theorem 3.7 ([9, 15]). *Let $1 < p_1, \dots, p_m \leq \infty$, $1 < p < \infty$ and $\frac{1}{p_1} + \dots + \frac{1}{p_m} = \frac{1}{p}$. If σ satisfies (3.6), then T_σ is bounded from $L^{p_1} \times \dots \times L^{p_m}$ to L^p with constant at most a multiple of A .*

We also use the following lemmas.

Lemma 3.8 ([15, Lemma 3.3]). *Let $s > n/2$, $\max\{1, n/s\} < q < 2$, and*

$$\zeta_j(x) = 2^{jn}(1 + |2^j x|)^{-sq}, \quad j \in \mathbb{Z}, \quad x \in \mathbb{R}^n.$$

Suppose $\sigma \in W^{(s, \dots, s)}(\mathbb{R}^{mn})$ and $\text{supp } \sigma \subset \{|\xi| \leq 2^{j+1}\}$ for some $j \in \mathbb{Z}$. Then there exists a constant $C > 0$ depending only on m, n, s , and q such that

$$|T_\sigma(f_1, \dots, f_m)(x)| \leq C \|\sigma(2^j \cdot)\|_{W^{(s, \dots, s)}} (\zeta_j * |f_1|^q)(x)^{1/q} \dots (\zeta_j * |f_m|^q)(x)^{1/q}$$

for all $x \in \mathbb{R}^n$.

Lemma 3.9 ([15, Lemma 3.2]). *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\widehat{\varphi}(0) = 0$, and set*

$$\Delta_j f(x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \widehat{\varphi}(2^{-j} \xi) \widehat{f}(\xi) d\xi, \quad j \in \mathbb{Z}.$$

Let $\varepsilon > 0$ and $\zeta_j(x) = 2^{jn}(1 + |2^j x|)^{-n-\varepsilon}$, $j \in \mathbb{Z}$, $x \in \mathbb{R}^n$. Then the following inequalities hold for each $0 < q < 2$:

$$\sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} |\Delta_j f(x)|^2 dx \leq C \|f\|_{L^2}^2, \quad (3.8)$$

$$\sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} (\zeta_j * |f|^q)(x)^{2/q} (\zeta_j * |\Delta_j g|^q)(x)^{2/q} dx \leq C_q \|f\|_{L^2}^2 \|g\|_{BMO}^2. \quad (3.9)$$

Lemma 3.10. *Suppose $\{F_j\} \subset \mathcal{S}'(\mathbb{R}^n)$ and suppose there exists a constant $B > 1$ such that $\text{supp } \widehat{F}_j \subset \{\zeta \in \mathbb{R}^n \mid B^{-1}2^j \leq |\zeta| \leq B2^j\}$ for all $j \in \mathbb{Z}$. Then, for each*

$0 < p < \infty$,

$$\left\| \sum_j F_j \right\|_{H^p} \lesssim \left\| \left(\sum_j |F_j|^2 \right)^{1/2} \right\|_{L^p}.$$

The preceding lemma is well known in the Littlewood-Paley theory, see for example [41, 5.2.4] and [11, Lemma 7.5.2].

3.3 Regularization of the multiplier

In this section, we show that the operator defined in (3.1) with enough smoothness of the multiplier can be approximated by a family of very nice operators.

For a Schwartz function K , we denote the multilinear operator of convolution type associated with the kernel K by

$$T^K(f_1, \dots, f_m)(x) = \int_{\mathbb{R}^{mn}} K(x - y_1, \dots, x - y_m) f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m.$$

Theorem 3.11. *Let σ be a function on \mathbb{R}^{mn} and $s_k > \frac{n}{2}$ for $1 \leq k \leq m$ satisfying (3.6). Then there exists a family of functions $(\sigma^\varepsilon)_{0 < \varepsilon < \frac{1}{2}}$ such that $K^\varepsilon := (\sigma^\varepsilon)^\vee$ is smooth and compactly supported for every $0 < \varepsilon < \frac{1}{2}$; also*

$$\sup_{0 < \varepsilon < \frac{1}{2}} \sup_{j \in \mathbb{Z}} \|\sigma^\varepsilon(2^j \cdot) \widehat{\psi}\|_{W^{(s_1, \dots, s_m)}} \lesssim \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \widehat{\psi}\|_{W^{(s_1, \dots, s_m)}}, \quad (3.10)$$

and

$$\lim_{\varepsilon \rightarrow 0} \|T_\varepsilon(f_1, \dots, f_m) - T_\sigma(f_1, \dots, f_m)\|_{L^2} = 0 \quad (3.11)$$

for all functions $f_k \in L^{2m}$, $1 \leq k \leq m$, where T_ε are multilinear singular integral operators of convolution type associated to K^ε .

The following lemma is the first step in constructing such a family of functions σ^ε as stated in Theorem 3.11.

Lemma 3.12. *Let φ be a Schwartz function. Suppose σ is a function on \mathbb{R}^{mn} satisfying (3.6) for $s_k > \frac{n}{2}$. Then we have*

$$\sup_{\varepsilon > 0} \sup_{j \in \mathbb{Z}} \|[(\varphi_\varepsilon * \sigma)(2^j \cdot)] \widehat{\psi}\|_{W^{(s_1, \dots, s_m)}} \lesssim \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \widehat{\psi}\|_{W^{(s_1, \dots, s_m)}},$$

where $\varphi_\varepsilon(x_1, \dots, x_m) = \varepsilon^{-mn} \varphi(\varepsilon^{-1}x_1, \dots, \varepsilon^{-1}x_m)$ for all $x_k \in \mathbb{R}^n$, $1 \leq k \leq m$.

Proof. For $\rho \in \mathbb{Z}$, $\rho \geq 2$ denote

$$F_\rho = \{y \in \mathbb{R}^{mn} : 2^{\rho-1} - 2 \leq |y| \leq 2^{\rho+1} + 2\}.$$

Fix $x = (x_1, \dots, x_m) \in \mathbb{R}^{mn}$. Then we have

$$\begin{aligned} (\varphi_\varepsilon * \sigma)(2^j x) \widehat{\psi}(x) &= \left\{ \int \varepsilon^{-mn} \varphi(\varepsilon^{-1}y) \sigma(2^j x - y) dy \right\} \widehat{\psi}(x) \\ &= \left\{ \int \varepsilon^{-mn} 2^{jmn} \varphi(\varepsilon^{-1} 2^j y) \sigma(2^j(x - y)) dy \right\} \widehat{\psi}(x) \\ &= \sum_{\rho \in \mathbb{Z}} \left\{ \int \varphi_{\varepsilon 2^{-j}}(y) \sigma(2^j(x - y)) \widehat{\psi}(2^{-\rho}(x - y)) dy \right\} \widehat{\psi}(x) \\ &= \sum_{\rho \leq -3} \left\{ \int \varphi_{\varepsilon 2^{-j}}(x - y) \sigma(2^j y) \widehat{\psi}(2^{-\rho} y) dy \right\} \widehat{\psi}(x) \end{aligned} \quad (3.12)$$

$$+ \sum_{|\rho| \leq 2} \int \varphi_{\varepsilon 2^{-j}}(y) \sigma(2^j(x - y)) \widehat{\psi}(2^{-\rho}(x - y)) \widehat{\psi}(x) dy \quad (3.13)$$

$$+ \sum_{\rho \geq 3} \left\{ \int \varphi_{\varepsilon 2^{-j}}(x - y) \sigma(2^j y) \widehat{\psi}(2^{-\rho} y) dy \right\} \widehat{\psi}(x). \quad (3.14)$$

The $W^{(s_1, \dots, s_m)}$ norm of term (3.13) can be estimated easily by

$$\begin{aligned} &\sum_{|\rho| \leq 2} \int |\varphi_{\varepsilon 2^{-j}}(y)| \|\sigma(2^j(\cdot - y)) \widehat{\psi}(2^{-\rho}(\cdot - y)) \widehat{\psi}\|_{W^{(s_1, \dots, s_m)}} dy \\ &\leq \sum_{|\rho| \leq 2} \int |\varphi_{\varepsilon 2^{-j}}(y)| \|\sigma(2^j(\cdot - y)) \widehat{\psi}(2^{-\rho}(\cdot - y))\|_{W^{(s_1, \dots, s_m)}} \|\widehat{\psi}\|_{W^{(s_1, \dots, s_m)}} dy \\ &\lesssim \sum_{|\rho| \leq 2} \|\sigma(2^{j+\rho} \cdot) \widehat{\psi}\|_{W^{(s_1, \dots, s_m)}} \int |\varphi_{\varepsilon 2^{-j}}(y)| dy \lesssim \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \widehat{\psi}\|_{W^{(s_1, \dots, s_m)}}, \end{aligned}$$

in which the second last inequality follows from the fact [9, Proposition A.2] that

$$\|fg\|_{W^{(s_1, \dots, s_m)}} \lesssim \|f\|_{W^{(s_1, \dots, s_m)}} \|g\|_{W^{(s_1, \dots, s_m)}},$$

when $f, g \in W^{(s_1, \dots, s_m)}$ for $s_1, \dots, s_m > \frac{n}{2}$.

Now fix integer numbers $N_k \geq s_k$, ($1 \leq k \leq m$) and set $N = N_1 + \dots + N_m$. Since $\|f\|_{W^{(s_1, \dots, s_m)}} \leq \|f\|_{W^N}$, the Sobolev norm of the term in (3.12) is bounded by

$$\begin{aligned} & \sum_{\rho \leq -3} \left\| \left\{ \int \varphi_{\varepsilon 2^{-j}}(\cdot - y) \sigma(2^j y) \widehat{\psi}(2^{-\rho} y) dy \right\} \widehat{\psi} \right\|_{W^N} \\ & \lesssim \sum_{\rho \leq -3} \sum_{|\alpha| + |\beta| \leq N} \left\| \left\{ \int (\varepsilon 2^{-j})^{-|\alpha|} (\partial^\alpha \varphi)_{\varepsilon 2^{-j}}(\cdot - y) \sigma(2^j y) \widehat{\psi}(2^{-\rho} y) dy \right\} \partial^\beta \widehat{\psi} \right\|_{L^2} \\ & = \sum_{\rho \leq -3} \sum_{|\alpha| + |\beta| \leq N} \left\| \left\{ \int_{\frac{1}{4} \leq |y| \leq \frac{9}{4}} \frac{(\partial^\alpha \varphi)_{\varepsilon 2^{-j}}(y)}{(\varepsilon 2^{-j})^{|\alpha|}} \sigma(2^j(\cdot - y)) \widehat{\psi}(2^{-\rho}(\cdot - y)) dy \right\} \partial^\beta \widehat{\psi} \right\|_{L^2} \\ & \lesssim \sum_{\rho \leq -3} \sum_{|\alpha| \leq N} \int \left(\frac{|y|}{\varepsilon 2^{-j}} \right)^{|\alpha|} |(\partial^\alpha \varphi)_{\varepsilon 2^{-j}}(y)| \left\| \sigma(2^j(\cdot - y)) \widehat{\psi}(2^{-\rho}(\cdot - y)) \right\|_{L^2} dy \\ & \lesssim \sum_{\rho \leq -3} 2^{\frac{\rho n}{2}} \left\| \sigma(2^{j+\rho} \cdot) \widehat{\psi} \right\|_{L^2} \sum_{|\alpha| \leq N} \int \left(\frac{|y|}{\varepsilon 2^{-j}} \right)^{|\alpha|} |(\partial^\alpha \varphi)_{\varepsilon 2^{-j}}(y)| dy \\ & \lesssim \sum_{\rho \leq -3} 2^{\frac{\rho n}{2}} \left\| \sigma(2^{j+\rho} \cdot) \widehat{\psi} \right\|_{W^s} \sum_{|\alpha| \leq N} \int |y|^{|\alpha|} |(\partial^\alpha \varphi)(y)| dy \\ & \lesssim \sup_{j \in \mathbb{Z}} \left\| \sigma(2^j \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}}. \end{aligned}$$

Finally, we deal with term (3.14). We have

$$\begin{aligned} & \left\| \sum_{\rho \geq 3} \left\{ \int \varphi_{\varepsilon 2^{-j}}(\cdot - y) \sigma(2^j y) \widehat{\psi}(2^{-\rho} y) dy \right\} \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}} \\ & \leq \sum_{\rho \geq 3} \left\| \left\{ \int \varphi_{\varepsilon 2^{-j}}(\cdot - y) \sigma(2^j y) \widehat{\psi}(2^{-\rho} y) dy \right\} \widehat{\psi} \right\|_{W^N} \\ & \lesssim \sum_{\rho \geq 3} \sum_{|\alpha| + |\beta| \leq N} \left\| \left\{ \int (\varepsilon 2^{-j})^{-|\alpha|} (\partial^\alpha \varphi)_{\varepsilon 2^{-j}}(\cdot - y) \sigma(2^j y) \widehat{\psi}(2^{-\rho} y) dy \right\} \partial^\beta \widehat{\psi} \right\|_{L^2} \\ & = \sum_{\rho \geq 3} \sum_{|\alpha| + |\beta| \leq N} \left\| \left\{ \int_{F_\rho} (\varepsilon 2^{-j})^{-|\alpha|} (\partial^\alpha \varphi)_{\varepsilon 2^{-j}}(y) \sigma(2^j(\cdot - y)) \widehat{\psi}(2^{-\rho}(\cdot - y)) dy \right\} \partial^\beta \widehat{\psi} \right\|_{L^2} \end{aligned}$$

$$\begin{aligned}
&\lesssim \sum_{\rho \geq 3} \sum_{|\alpha|+|\beta| \leq N} \int_{F_\rho} \left(\frac{|y|}{2^\rho \varepsilon 2^{-j}} \right)^{|\alpha|} |(\partial^\alpha \varphi)_{\varepsilon 2^{-j}}(y)| \left\| \sigma(2^j(\cdot - y)) \widehat{\psi}(2^{-\rho}(\cdot - y)) \partial^\beta \widehat{\psi} \right\|_{L^2} dy \\
&= \sum_{\rho \geq 3} \sum_{|\alpha|+|\beta| \leq N} \int_{F_\rho} \left(\frac{|y|}{2^\rho \varepsilon 2^{-j}} \right)^{|\alpha|} |(\partial^\alpha \varphi)_{\varepsilon 2^{-j}}(y)| \left\| \sigma(2^j \cdot) \widehat{\psi}(2^{-\rho} \cdot) (\partial^\beta \widehat{\psi})(\cdot + y) \right\|_{L^2} dy \\
&\leq \sum_{\rho \geq 3} \sum_{|\alpha|+|\beta| \leq N} \int_{F_\rho} 2^{\frac{\rho n}{2}} \left(\frac{|y|}{\varepsilon 2^{-j}} \right)^{|\alpha|} |(\partial^\alpha \varphi)_{\varepsilon 2^{-j}}(y)| \left\| \sigma(2^{j+\rho} \cdot) \widehat{\psi} (\partial^\beta \widehat{\psi})(2^\rho \cdot + y) \right\|_{L^2} dy \\
&\leq \sum_{\rho \geq 3} \sum_{|\alpha| \leq N} \int_{F_\rho} 2^{\frac{\rho n}{2}} \left(\frac{|y|}{\varepsilon 2^{-j}} \right)^{|\alpha|} |(\partial^\alpha \varphi)_{\varepsilon 2^{-j}}(y)| \left\| \sigma(2^{j+\rho} \cdot) \widehat{\psi} \right\|_{L^2(B(-2^{-\rho}y, 2^{1-\rho}))} dy \\
&\lesssim \sum_{|\alpha| \leq N} \sum_{\rho \geq 3} \left\| \sigma(2^{j+\rho} \cdot) \widehat{\psi} \right\|_{L^\infty} \int_{F_\rho} \left(\frac{|y|}{\varepsilon 2^{-j}} \right)^{|\alpha|} |(\partial^\alpha \varphi)_{\varepsilon 2^{-j}}(y)| dy \\
&\leq \sum_{|\alpha| \leq N} \sum_{\rho \geq 3} \left\| \sigma(2^{j+\rho} \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}} \int_{F_\rho} \left(\frac{|y|}{\varepsilon 2^{-j}} \right)^{|\alpha|} |(\partial^\alpha \varphi)_{\varepsilon 2^{-j}}(y)| dy \\
&\lesssim \sup_{\rho \in \mathbb{Z}} \left\| \sigma(2^{j+\rho} \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}} \sum_{|\alpha| \leq N} \sum_{\rho \geq 3} \int_{F_\rho} \left(\frac{|y|}{\varepsilon 2^{-j}} \right)^{|\alpha|} |(\partial^\alpha \varphi)_{\varepsilon 2^{-j}}(y)| dy \\
&\lesssim \sup_{j \in \mathbb{Z}} \left\| \sigma(2^j \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}}.
\end{aligned}$$

The proof of the lemma is now complete. \square

Proof of Theorem 3.11. Fix $0 < \varepsilon < \frac{1}{2}$. Choose a smooth function φ such that $\widehat{\varphi}$ is supported in the unit ball and $\widehat{\varphi}(0) = 1$. Denote by $\sigma^\varepsilon = \varphi_\varepsilon * (\sigma \phi^\varepsilon)$, where $\phi^\varepsilon = \theta(\varepsilon^{-1} \cdot) - \theta(\varepsilon \cdot)$, and θ is a smooth function satisfying $\theta(x) = 0$ for all $|x| \leq 1$ and $\theta(x) = 1$ for all $|x| \geq 2$. We note that these functions are suitable regularized versions of the multiplier in Theorem 3.11. Indeed, let $K^\varepsilon = (\sigma^\varepsilon)^\vee = (\sigma \phi^\varepsilon)^\vee \widehat{\varphi}(\varepsilon(\cdot))$; then, K^ε are smooth functions with compact support for all $0 < \varepsilon < \frac{1}{2}$.

Using the fact that

$$|\partial^\alpha \phi^\varepsilon(\xi)| \leq C_{\alpha, \theta} |\xi|^{-\alpha}, \quad \xi \neq 0, \quad 0 < \varepsilon < \frac{1}{2},$$

Lemma 3.12 applied to the function $\sigma \phi^\varepsilon$ combined with Lemma 3.5 gives

$$\sup_{0 < \varepsilon < \frac{1}{2}} \sup_{j \in \mathbb{Z}} \left\| \sigma^\varepsilon(2^j \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}} \lesssim \sup_{0 < \varepsilon < \frac{1}{2}} \sup_{j \in \mathbb{Z}} \left\| \sigma(2^j \cdot) \phi^\varepsilon(2^j \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}}$$

$$\lesssim \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \widehat{\psi}\|_{W^{(s_1, \dots, s_m)}},$$

which yields (3.10). Thus, we are left with establishing (3.11). For $\varepsilon > 0$, now recall

$$\begin{aligned} T_\varepsilon(f_1, \dots, f_m)(x) &= \int K^\varepsilon(x - y_1, \dots, x - y_m) f_1(y_1) \cdots f_m(y_m) dy \\ &= \int \sigma^\varepsilon(\xi_1, \dots, \xi_m) \widehat{f_1}(\xi_1) \cdots \widehat{f_m}(\xi_m) e^{2\pi i x(\xi_1 + \cdots + \xi_m)} d\xi. \end{aligned}$$

Involve estimate (3.10) with Theorem 3.7, we can see that T_σ and T_ε are uniformly bounded from $L^{2m} \times \cdots \times L^{2m} \rightarrow L^2$ for all $0 < \varepsilon < \frac{1}{2}$. By density, it suffices to verify (3.11) for all functions in the Schwartz class.

Now fix Schwartz functions f_k , for $1 \leq k \leq m$. The Fourier transform of $T_\sigma(f_1, \dots, f_m)$ can be written by

$$\int_{\mathbb{R}^{n(m-1)}} \sigma(\xi_1, \dots, \xi_{m-1}, \xi - \sum_{l=1}^{m-1} \xi_l) \widehat{f_1}(\xi_1) \cdots \widehat{f_{m-1}}(\xi_{m-1}) \widehat{f_m}(\xi - \sum_{l=1}^{m-1} \xi_l) d\xi_1 \cdots d\xi_{m-1}.$$

Similarly, the Fourier transform of $T_\varepsilon(f_1, \dots, f_m)$ is

$$\int_{\mathbb{R}^{n(m-1)}} \sigma^\varepsilon(\xi_1, \dots, \xi_{m-1}, \xi - \sum_{l=1}^{m-1} \xi_l) \widehat{f_1}(\xi_1) \cdots \widehat{f_{m-1}}(\xi_{m-1}) \widehat{f_m}(\xi - \sum_{l=1}^{m-1} \xi_l) d\xi_1 \cdots d\xi_{m-1}.$$

We now claim that σ^ε converges pointwise to σ . Take this claim for granted, we have

$$\left(T_\varepsilon(f_1, \dots, f_m) \right)^\wedge(\xi) \rightarrow \left(T_\sigma(f_1, \dots, f_m) \right)^\wedge(\xi), \quad \varepsilon \rightarrow 0 \quad (3.15)$$

for a.e. $\xi \in \mathbb{R}^n$. Notice that

$$\|T_\varepsilon(f_1, \dots, f_m) - T_\sigma(f_1, \dots, f_m)\|_{L^2} = \left\| \left(T_\varepsilon(f_1, \dots, f_m) \right)^\wedge - \left(T_\sigma(f_1, \dots, f_m) \right)^\wedge \right\|_{L^2}.$$

Since $\|\sigma^\varepsilon\|_{L^\infty} \lesssim \|\sigma\|_{L^\infty} < \infty$ for all $\varepsilon > 0$, Lebesgue's dominated convergence theorem

implies that

$$(T_\varepsilon(f_1, \dots, f_m))^\wedge \longrightarrow (T_\sigma(f_1, \dots, f_m))^\wedge \quad \text{as } \varepsilon \rightarrow 0$$

in L^2 , and this establishes (3.11).

It remains to prove (3.15) about the pointwise convergence of σ^ε as $\varepsilon \rightarrow 0$. Now fix $j_0 \in \mathbb{Z}$, we want to show that $\sigma^\varepsilon(x) \rightarrow \sigma(x)$ for a.e. a.e. $2^{j_0} \leq |x| \leq 2^{j_0+1}$. Indeed, let $0 < \varepsilon < \min \{2^{2j_0-2}, 2^{-|j_0|-2}\}$ be an arbitrarily small positive number. Then we have

$$\begin{aligned} |\sigma^\varepsilon(x) - \sigma(x)| &\leq \int_{|y| \leq \sqrt{\varepsilon}} |\varphi_\varepsilon(y)| |\sigma(x-y)| \sup_{2^{j_0} \leq |x| \leq 2^{j_0+1}} |\phi^\varepsilon(x-y) - 1| dy \\ &\quad + \int_{|y| \leq \sqrt{\varepsilon}} |\varphi_\varepsilon(y)| |\sigma(x-y) - \sigma(x)| dy \\ &\quad + \int_{|y| > \sqrt{\varepsilon}} |\varphi_\varepsilon(y)| |\sigma(x-y)\phi^\varepsilon(x-y) - \sigma(x)| dy. \end{aligned}$$

The first integral vanishes since $\phi^\varepsilon(x) = 1$ for all $2\varepsilon \leq |x| \leq \frac{1}{\varepsilon}$. To estimate the second integral, we denote

$$\widehat{\Psi}(x) = \sum_{|l| \leq 2} \widehat{\psi}(2^{-l}x).$$

Then $\widehat{\Psi}(x) = 1$ for all $\frac{1}{4} \leq |x| \leq 4$. Therefore we have

$$\widehat{\Psi}(2^{-j_0}(x-y)) = \widehat{\Psi}(2^{-j_0}x) = 1$$

for all $2^{j_0} \leq |x| \leq 2^{j_0+1}$ and $|y| \leq 2^{j_0-1}$. Now recall $\sigma_j(x) = \sigma(2^jx)$ and estimate

$$\begin{aligned} &\int_{|y| \leq \sqrt{\varepsilon}} |\varphi_\varepsilon(y)| |\sigma(x-y) - \sigma(x)| dy \\ &= \int_{|y| \leq \sqrt{\varepsilon}} |\varphi_\varepsilon(y)| |\sigma(x-y)\widehat{\Psi}(2^{-j_0}(x-y)) - \sigma(x)\widehat{\Psi}(2^{-j_0}x)| dy \\ &\leq \|\varphi\|_{L^1} \sup_{|y| \leq \sqrt{\varepsilon}} \|\sigma(\cdot - y)\widehat{\Psi}(2^{-j_0}(\cdot - y)) - \sigma\widehat{\Psi}(2^{-j_0}\cdot)\|_{L^\infty} \end{aligned}$$

$$\begin{aligned}
&\leq \|\varphi\|_{L^1} \sum_{j=j_0-2}^{j_0+2} \sup_{|y| \leq \sqrt{\varepsilon}} \|(\sigma_j \widehat{\psi})(\cdot - 2^{-j}y) - (\sigma_j \widehat{\psi})\|_{L^\infty} \\
&= \|\varphi\|_{L^1} \sum_{j=j_0-2}^{j_0+2} \sup_{|y| \leq 2^{-j}\sqrt{\varepsilon}} \|(\sigma_j \widehat{\psi})(\cdot - y) - (\sigma_j \widehat{\psi})\|_{L^\infty}.
\end{aligned}$$

We would like to show

$$\lim_{\varepsilon \rightarrow 0} \sup_{|y| \leq 2^{-j}\sqrt{\varepsilon}} \|(\sigma_j \widehat{\psi})(\cdot - y) - (\sigma_j \widehat{\psi})\|_{L^\infty} = 0.$$

The preceding limit apparently converges to 0 as $\varepsilon \rightarrow 0$ because $\sigma_j \widehat{\psi} \in W^{(s_1, \dots, s_m)}$ for $s_k > \frac{n}{2}$, $1 \leq k \leq m$. The last term is majorized by

$$C \|\sigma\|_{L^\infty} \int_{|y| \geq \frac{1}{\sqrt{\varepsilon}}} |\varphi(y)| dy,$$

which tends to 0 when $\varepsilon \rightarrow 0$.

Thus $\sigma^\varepsilon(x) \rightarrow \sigma(x)$ as $\varepsilon \rightarrow 0$ for a.e. $2^{j_0} \leq |x| \leq 2^{j_0+1}$. Hence, σ^ε converges to σ pointwise on \mathbb{R}^{mn} . Also $\|\sigma^\varepsilon\|_{L^\infty(\mathbb{R}^{mn})} \lesssim \|\sigma\|_{L^\infty(\mathbb{R}^{mn})}$ uniformly for all $\varepsilon > 0$. The proof of Theorem 3.11 is now complete. \square

Proposition 3.13. *Let K be a smooth function on \mathbb{R}^{mn} with compact support. Then we have*

$$\|T^K\|_{H^{p_1} \times \dots \times H^{p_m} \rightarrow L^p} \leq C_K < \infty$$

for all $0 < p_1, \dots, p_m, p < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$, where T^K is the multilinear singular integral operator of convolution type associated with the kernel K .

Proof. The boundedness of the operator T^K can be deduced from [12, Lemma 4.2], which provides the estimate (for some sufficiently large integer N)

$$|T^K(f_1, \dots, f_m)(x)| \lesssim \prod_{k=1}^m \mathcal{M}_N(f_k)(x), \quad (3.16)$$

for all $f_k \in L^2 \cap H^{p_k}$, in which

$$\mathcal{M}_N(f)(x) = \sup_{\varphi \in \mathfrak{F}_N} \sup_{t>0} \sup_{y \in B(x,t)} |(\varphi_t * f)(y)|$$

is the grand maximal function with respect to N , and

$$\mathfrak{F}_N := \left\{ \varphi \in \mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + |x|)^N \sum_{|\alpha| \leq N+1} |\partial^\alpha \varphi(x)| dx \leq 1 \right\}.$$

Taking the L^p quasinorm, applying Holder's inequality to (3.16), and using the quasinorm equivalence of some maximal functions [11, Theorem 6.4.4] yields

$$\|T^K(f_1, \dots, f_m)\|_{L^p} \lesssim \prod_{k=1}^m \|\mathcal{M}_N(f_k)\|_{L^{p_k}} \leq C_K \prod_{k=1}^m \|f_k\|_{H^{p_k}}.$$

□

Working with smooth kernels K with compact support comes handy when dealing with infinite sums of atoms, since we are able to freely interchange the action of T^K with infinite sums of atoms. Precisely, a consequence of the boundedness of T^K , given in Proposition 3.13, is the following result.

Proposition 3.14. *Let $0 < p_1, \dots, p_m \leq 1$ and let K be a smooth function with compact support. Then for every $f_k \in H^{p_k}$ with atomic representation $f_k = \sum_{j_k} \lambda_{k,j_k} a_{k,j_k}$, where a_{k,j_k} are L^∞ -atoms for H^{p_k} and $\sum_{j_k} |\lambda_{k,j_k}|^{p_k} \leq 2^{p_k} \|f_k\|_{H^{p_k}}^{p_k}$ for $1 \leq k \leq m$. Then*

$$T^K(f_1, \dots, f_m)(x) = \sum_{j_1} \cdots \sum_{j_m} \lambda_{1,j_1} \cdots \lambda_{m,j_m} T^K(a_{1,j_1}, \dots, a_{m,j_m})(x)$$

for a.e. $x \in \mathbb{R}^n$.

Proof. Let $0 < p < \infty$ be number such that $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. For any positive

integers N_1, \dots, N_m , Proposition 3.13 gives the estimate

$$\begin{aligned} & \|T^K(f_1, \dots, f_m) - \sum_{j_1=1}^{N_1} \cdots \sum_{j_m=1}^{N_m} \lambda_{1,j_1} \cdots \lambda_{m,j_m} T^K(a_{1,j_1}, \dots, a_{m,j_m})\|_{L^p} \\ & \leq C_K \sum_{k=1}^m \|f_k - \sum_{j_k=1}^{N_k} \lambda_{k,j_k} a_{k,j_k}\|_{H^{p_k}} \prod_{l \neq k} \|f_l\|_{H^{p_l}}. \end{aligned}$$

Now passing to the limit, we obtain

$$T^K(f_1, \dots, f_m)(x) = \sum_{j_1=1}^{\infty} \cdots \sum_{j_m=1}^{\infty} \lambda_{1,j_1} \cdots \lambda_{m,j_m} T^K(a_{1,j_1}, \dots, a_{m,j_m})(x)$$

for a.e. $x \in \mathbb{R}^n$. □

Similarly, we can obtain the following result.

Proposition 3.15. *Let $0 < p_1, \dots, p_l \leq 1$ and $1 < p_{l+1}, \dots, p_m \leq \infty$. Let K be a smooth function with compact support. Suppose $f_i \in H^{p_i}$, $1 \leq i \leq l$, has atomic representation $f_i = \sum_{k_i} \lambda_{i,k_i} a_{i,k_i}$, where a_{i,k_i} are L^∞ -atoms for H^{p_i} and $\sum_{k_i} |\lambda_{i,k_i}|^{p_i} \leq 2^{p_i} \|f_i\|_{H^{p_i}}^{p_i}$. Suppose $f_i \in L^{p_i}$ for $l+1 \leq i \leq m$. Then*

$$T^K(f_1, \dots, f_m)(x) = \sum_{k_1} \cdots \sum_{k_l} \lambda_{1,k_1} \cdots \lambda_{l,k_l} T^K(a_{1,k_1}, \dots, a_{l,k_l}, f_{l+1}, \dots, f_m)(x)$$

for almost all $x \in \mathbb{R}^n$.

3.4 Minimality of conditions

In this section we will show that conditions (3.5) and $s_k > \frac{n}{2}$ are minimal in general that guarantee boundedness for multilinear multiplier operators. We fix a smooth function ψ whose Fourier transform is supported in $\{2^{-\frac{3}{4}} \leq |\xi| \leq 2^{\frac{3}{4}}\}$, it satisfies

$\widehat{\psi}(\xi) = 1$ for all $2^{-\frac{1}{4}} \leq |\xi| \leq 2^{\frac{1}{4}}$, and for some nonzero constant c

$$\sum_{j \in \mathbb{Z}} \widehat{\psi}(2^{-j}\xi) = c, \quad \xi \neq 0.$$

Now we have the following theorem:

Theorem 3.16. *Let $0 < p_k \leq \infty$, $0 < p < \infty$, and $s_k > 0$ for $1 \leq k \leq m$. Suppose that the estimate*

$$\|T_\sigma(f_1, \dots, f_m)\|_{L^p} \lesssim \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \widehat{\psi}\|_{W^{(s_1, \dots, s_m)}} \prod_{k=1}^m \|f_k\|_{H^{p_k}}$$

holds for all $f_k \in H^{p_k}$ and $\sigma \in L^\infty$ such that $\sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \widehat{\psi}\|_{W^{(s_1, \dots, s_m)}} < \infty$.

The following conditions are then necessary:

$$s_k \geq \frac{n}{2}, \quad \forall 1 \leq k \leq m, \quad (3.17)$$

and

$$\sum_{k \in J} \left(\frac{s_k}{n} - \frac{1}{p_k} \right) \geq -\frac{1}{2} \quad (3.18)$$

for every nonempty subset $J \subset \{1, \dots, m\}$.

The following lemma is obvious by changing variables, so its proof is omitted.

Lemma 3.17. *Let φ be a nontrivial Schwartz function and $s > 0$. Then*

$$\left(\int |\varphi(\varepsilon y)|^2 (1 + |y|^2)^s dy \right)^{\frac{1}{2}} \approx \varepsilon^{-\frac{n}{2} - s}$$

for all $0 < \varepsilon \leq 1$.

Proof of Theorem 3.16. We show first the necessary conditions (3.17) for $1 \leq k \leq m$.

Without loss of generality, we will show $s_1 \geq \frac{n}{2}$. To establish this inequality, we need

to construct some functions σ^ε , ($0 < \varepsilon \ll 1$), and $f_k \in H^{p_k}$ such that $\|f_k\|_{H^{p_k}} = 1$ for all $1 \leq k \leq m$, and $\|T_{\sigma^\varepsilon}(f_1, \dots, f_m)\|_{L^p} \approx 1$, and further that

$$\sup_{j \in \mathbb{Z}} \|\sigma^\varepsilon(2^j \cdot) \widehat{\psi}\|_{W^{(s_1, \dots, s_m)}} \lesssim \varepsilon^{\frac{n}{2} - s_1}.$$

Once these functions are constructed, one observes that

$$1 \approx \|T_{\sigma^\varepsilon}(f_1, \dots, f_m)\|_{L^p} \lesssim \sup_{j \in \mathbb{Z}} \|\sigma^\varepsilon(2^j \cdot) \widehat{\psi}\|_{W^{(s_1, \dots, s_m)}} \prod_{k=1}^m \|f_k\|_{H^{p_k}} \lesssim \varepsilon^{\frac{n}{2} - s_1}$$

for all $0 < \varepsilon \ll 1$. Therefore we must have $s_1 \geq \frac{n}{2}$.

Let φ be a nontrivial Schwartz function such that $\widehat{\varphi}$ is supported in the unit ball, and let $\phi_2 = \dots = \phi_{m-1}$ be Schwartz functions whose Fourier transforms, $\widehat{\phi}_2$, is supported in an annulus $\frac{1}{17m} \leq |\xi| \leq \frac{1}{13m}$, and identical to 1 on $\frac{1}{16m} \leq |\xi| \leq \frac{1}{14m}$. Similarly, fix a Schwartz function ϕ_m with $\widehat{\phi}_m \subset \{\xi \in \mathbb{R}^n : \frac{12}{13} \leq |\xi| \leq \frac{14}{13}\}$ and $\widehat{\phi}_m \equiv 1$ on an annulus $\frac{25}{26} \leq |\xi| \leq \frac{27}{26}$. Take $a, b \in \mathbb{R}^n$ with $|a| = \frac{1}{15m}$ and $|b| = 1$.

For $0 < \varepsilon < \frac{1}{240m}$, set

$$\sigma^\varepsilon(\xi_1, \dots, \xi_m) = \widehat{\varphi}\left(\frac{\xi_1 - a}{\varepsilon}\right) \widehat{\phi}_2(\xi_2) \cdots \widehat{\phi}_m(\xi_m).$$

It is easy to check that $\text{supp } \sigma^\varepsilon \subset \{2^{-\frac{1}{4}} \leq |\xi| \leq 2^{\frac{1}{4}}\}$; hence, $\sigma^\varepsilon(2^j \cdot) \widehat{\psi} = \sigma^\varepsilon$ for $j = 0$ and $\sigma^\varepsilon(2^j \cdot) \widehat{\psi} = 0$ for $j \neq 0$. This directly implies that

$$\sup_{j \in \mathbb{Z}} \|\sigma^\varepsilon(2^j \cdot) \widehat{\psi}\|_{W^{(s_1, \dots, s_m)}} = \|\sigma^\varepsilon\|_{W^{(s_1, \dots, s_m)}}.$$

Taking the inverse Fourier transform of σ^ε gives

$$(\sigma^\varepsilon)^\vee(x_1, \dots, x_m) = \varepsilon^n e^{2\pi i a \cdot x_1} \varphi(\varepsilon x_1) \phi_2(x_2) \cdots \phi_m(x_m).$$

Now apply Lemma 3.17 to have

$$\|\sigma^\varepsilon\|_{W(s_1, \dots, s_m)} \lesssim \varepsilon^{\frac{n}{2} - s_1}.$$

Thus

$$\sup_{j \in \mathbb{Z}} \|\sigma^\varepsilon(2^j \cdot) \widehat{\psi}\|_{W(s_1, \dots, s_m)} \lesssim \varepsilon^{\frac{n}{2} - s_1}.$$

Now choose $\widehat{f}_k(\xi) = \varepsilon^{\frac{n}{p_k} - n} \widehat{\varphi}(\frac{\xi - a}{\varepsilon})$ for $1 \leq k \leq m - 1$, and $\widehat{f}_m(\xi) = \varepsilon^{\frac{n}{p_m} - n} \widehat{\varphi}(\frac{\xi - b}{\varepsilon})$. Then we will show that these functions are what we needed to construct.

In the following estimates, we will use the fact, its proof can be done by using the Littlewood-Paley characterization for Hardy spaces, that if f is a function whose Fourier transform is supported in a fixed annulus centered at the origin, then $\|f\|_{H^p} \approx \|f\|_{L^p}$ for $0 < p < \infty$, (cf. [9, Remark 7.1]).

Indeed, using the above fact and checking that each \widehat{f}_k is supported in an annulus centered at zero and not depending on ε allow us to estimate H^p -norms via L^p -norms, namely

$$\|f_k\|_{H^{p_k}} \approx \|f_k\|_{L^{p_k}} = 1, \quad (1 \leq k \leq m).$$

Thus, we are left with showing that $\|T_\sigma(f_1, \dots, f_m)\|_{L^p} \approx 1$. Notice that $\widehat{\phi}_k(\xi) = 1$ on the support of \widehat{f}_k for $2 \leq k \leq m$. Therefore,

$$\begin{aligned} T_{\sigma^\varepsilon}(f_1, \dots, f_m)(x) &= \left(\widehat{\varphi}\left(\frac{\cdot - a}{\varepsilon}\right) \varepsilon^{\frac{n}{p_1} - n} \widehat{\varphi}\left(\frac{\cdot - a}{\varepsilon}\right) \right)^\vee(x) \left(\widehat{\phi}_2 \widehat{f}_2 \right)^\vee(x) \cdots \left(\widehat{\phi}_m \widehat{f}_m \right)^\vee(x) \\ &= \left(\widehat{\varphi}\left(\frac{\cdot - a}{\varepsilon}\right) \varepsilon^{\frac{n}{p_1} - n} \widehat{\varphi}\left(\frac{\cdot - a}{\varepsilon}\right) \right)^\vee(x) \left(\widehat{f}_2 \right)^\vee(x) \cdots \left(\widehat{f}_m \right)^\vee(x) \\ &= \varepsilon^{\frac{n}{p_1} + \dots + \frac{n}{p_m}} e^{2\pi i[(m-1)a+b] \cdot x} (\varphi * \varphi)(\varepsilon x) [\varphi(\varepsilon x)]^{m-1} \\ &= \varepsilon^{\frac{n}{p}} e^{2\pi i[(m-1)a+b] \cdot x} (\varphi * \varphi)(\varepsilon x) [\varphi(\varepsilon x)]^{m-1}, \end{aligned}$$

which obviously gives $\|T_{\sigma^\varepsilon}(f_1, \dots, f_m)\|_{L^p} \approx 1$. So far, we have proved that $s_1 \geq \frac{n}{2}$; hence, by symmetry, we have $s_k \geq \frac{n}{2}$ for all $1 \leq k \leq m$.

It now remains to show (3.18). By symmetry, we just only need to prove that

$$\sum_{k=1}^r \left(\frac{s_k}{n} - \frac{1}{p_k} \right) \geq -\frac{1}{2} \quad (3.19)$$

for some fixed $1 \leq r \leq m$. To achieve our goal, we construct a multiplier σ^ε such that

$$\sup_{j \in \mathbb{Z}} \|\sigma^\varepsilon(2^j \cdot) \widehat{\psi}\|_{W^{(s_1, \dots, s_m)}} \lesssim \varepsilon^{\frac{n}{2} - s_1 - \dots - s_r}$$

for $0 < \varepsilon \ll 1$ and functions f_k satisfying $\|f_k\|_{H^{p_k}} \approx 1$ for $1 \leq k \leq m$ and

$$\|T_{\sigma^\varepsilon}(f_1, \dots, f_m)\|_{L^p} \approx \varepsilon^{n - \frac{n}{p_1} - \dots - \frac{n}{p_r}}.$$

Then inequalities

$$\begin{aligned} \varepsilon^{n - \frac{n}{p_1} - \dots - \frac{n}{p_r}} &\approx \|T_{\sigma^\varepsilon}(f_1, \dots, f_m)\|_{L^p} \\ &\leq \sup_{j \in \mathbb{Z}} \|\sigma^\varepsilon(2^j \cdot) \widehat{\psi}\|_{W^{(s_1, \dots, s_m)}} \prod_{k=1}^m \|f_k\|_{H^{p_k}} \\ &\lesssim \varepsilon^{\frac{n}{2} - s_1 - \dots - s_r}, \end{aligned}$$

for all small positive numbers ε yield (3.19).

We construct functions that give us enough ingredients to establish the multiplier σ^ε and functions f_k , ($1 \leq k \leq m$), as mentioned above. Take two smooth functions φ, ϕ such that $\varphi(0) \neq 0$, $\widehat{\varphi}$ is supported in $\{\xi \in \mathbb{R}^n : |\xi| \leq \frac{1}{19mr}\}$ and $\widehat{\varphi}(\xi) = 1$ for all $|\xi| \leq \frac{1}{30mr}$, and that $\widehat{\phi}$ is supported in an annulus $\frac{1}{23m} \leq |\xi| \leq \frac{1}{19m}$ and $\widehat{\phi}(\xi) = 1$ for all $\frac{1}{22m} \leq |\xi| \leq \frac{1}{20m}$. Fix $a, b \in \mathbb{R}^n$ such that $|a| = r^{-\frac{1}{2}}$, $|b| = \frac{1}{21m}$.

For $0 < \varepsilon < \frac{1}{462m}$, for $r > 1$, define

$$\begin{aligned} &\sigma^\varepsilon(\xi_1, \dots, \xi_m) \\ &= \widehat{\varphi}\left(\frac{1}{r\varepsilon} \sum_{l=1}^r (\xi_l - a)\right) \widehat{\varphi}\left(\frac{1}{r} \sum_{l=1}^r (\xi_l - \xi_2)\right) \cdots \widehat{\varphi}\left(\frac{1}{r} \sum_{l=1}^r (\xi_l - \xi_r)\right) \widehat{\phi}(\xi_{r+1}) \cdots \widehat{\phi}(\xi_m). \end{aligned}$$

Note that in the case $r = 1$ we will replace the above by the following function

$$\sigma^\varepsilon(\xi_1, \dots, \xi_m) = \widehat{\varphi}\left(\frac{1}{\varepsilon}(\xi_1 - a)\right)\widehat{\phi}(\xi_2) \cdots \widehat{\phi}(\xi_m).$$

Once again, we have $\text{supp } \sigma^\varepsilon \subset \{\xi \in \mathbb{R}^n : 2^{-\frac{1}{4}} \leq |\xi| \leq 2^{\frac{1}{4}}\}$, which, as in the previous case, implies that

$$\sup_{j \in \mathbb{Z}} \|\sigma^\varepsilon(2^j \cdot) \widehat{\psi}\|_{W(s_1, \dots, s_m)} = \|\sigma^\varepsilon\|_{W(s_1, \dots, s_m)}.$$

Taking the inverse Fourier transform of σ^ε , we have

$$\begin{aligned} (\sigma^\varepsilon)^\vee(x_1, \dots, x_m) &= \int \widehat{\varphi}\left(\frac{1}{r\varepsilon} \sum_{l=1}^r (\xi_l - a)\right) \widehat{\varphi}\left(\frac{1}{r} \sum_{l=1}^r (\xi_l - \xi_2)\right) \cdots \widehat{\varphi}\left(\frac{1}{r} \sum_{l=1}^r (\xi_l - \xi_r)\right) \times \\ &\quad \times \widehat{\phi}(\xi_{r+1}) \cdots \widehat{\phi}(\xi_m) e^{2\pi i(x_1 \xi_1 + \cdots + x_r \xi_r + x_{r+1} \xi_{r+1} + \cdots + x_m \xi_m)} d\xi_1 \cdots d\xi_m. \end{aligned}$$

Now we set

$$y_1 = \frac{1}{r} \sum_{l=1}^r (\xi_l - a), y_k = \frac{1}{r} \sum_{l=1}^r (\xi_l - \xi_k), \quad 2 \leq k \leq r, y_j = \xi_j, \quad r+1 \leq j \leq m.$$

The above change of variables yields

$$(\sigma^\varepsilon)^\vee(x_1, \dots, x_m) = r e^{2\pi i a \cdot \sum_{l=1}^r x_l} \varepsilon^n \varphi\left(\varepsilon \sum_{l=1}^r x_l\right) \varphi(x_1 - x_2) \cdots \varphi(x_1 - x_r) \phi(x_{r+1}) \cdots \phi(x_m).$$

Taking Sobolev norm deduces

$$\|\sigma^\varepsilon\|_{W(s_1, \dots, s_m)} = C \varepsilon^n \left(\int_{\mathbb{R}^{nr}} \left| \varphi\left(\varepsilon \sum_{l=1}^r x_l\right) \prod_{l=2}^r \varphi(x_1 - x_l) \right|^2 \prod_{l=1}^r (1 + |x_l|^2)^{s_l} dx_1 \cdots dx_r \right)^{\frac{1}{2}},$$

where $C = r \|\phi\|_{W^{s_{r+1}}} \cdots \|\phi\|_{W^{s_m}}$. Next, we show that

$$\begin{aligned} & \int_{\mathbb{R}^{nr}} \left| \varphi\left(\varepsilon \sum_{l=1}^r x_l\right) \varphi(x_1 - x_2) \cdots \varphi(x_1 - x_r) \right|^2 \prod_{l=1}^r (1 + |x_l|^2)^{s_l} dx_1 \cdots dx_r \\ & \lesssim \varepsilon^{-n-2(s_1+\cdots+s_r)}. \end{aligned} \quad (3.20)$$

In fact, a suitable change variables together with Lemma 3.17 yields

$$\begin{aligned} & \int_{\mathbb{R}^{nr}} \left| \varphi\left(\varepsilon \sum_{l=1}^r x_l\right) \varphi(x_1 - x_2) \cdots \varphi(x_1 - x_r) \right|^2 \prod_{l=1}^r (1 + |x_l|^2)^{s_l} dx_1 \cdots dx_r \\ & = \frac{1}{r} \int_{\mathbb{R}^{nr}} |\varphi(\varepsilon y_1) \varphi(y_2) \cdots \varphi(y_r)|^2 \left(1 + \frac{1}{r^2} \left|\sum_{l=1}^r y_l\right|^2\right)^{s_1} \\ & \quad \prod_{l=2}^r \left(1 + \left| -y_l + \frac{1}{r} \sum_{l=1}^r y_l \right|^2\right)^{s_l} dy_1 \cdots dy_r \\ & \lesssim \int_{\mathbb{R}^{nr}} |\varphi(\varepsilon y_1) \varphi(y_2) \cdots \varphi(y_r)|^2 \prod_{l=1}^r (1 + |y_l|^2)^{s_1+\cdots+s_r} dy_1 \cdots dy_r \\ & \lesssim \int_{\mathbb{R}^{nr}} |\varphi(\varepsilon y_1)|^2 (1 + |y_1|^2)^{s_1+\cdots+s_r} dy_1 \\ & \lesssim \varepsilon^{-n-2s_1-\cdots-2s_r}, \end{aligned}$$

where the implicit constants do not depend on ε . Inequality (3.20) gives us

$$\sup_{j \in \mathbb{Z}} \|\sigma^\varepsilon(2^j \cdot) \widehat{\psi}\|_{W^{(s_1, \dots, s_m)}} = \|\sigma^\varepsilon\|_{W^{(s_1, \dots, s_m)}} \lesssim \varepsilon^{\frac{n}{2}-s_1-\cdots-s_r}.$$

To construct functions f_k , we fix a smooth function ζ such that $\widehat{\zeta}$ is supported in the ball $\{\xi \in \mathbb{R}^n : |\xi - a| \leq \frac{1}{3m}\}$ and is identical to 1 on $\{\xi \in \mathbb{R}^n : |\xi - a| \leq \frac{3}{19m}\}$. Now set $f_1 = \cdots = f_r = \zeta$ and $\widehat{f}_k(\xi) = \varepsilon^{\frac{n}{p_k}-n} \widehat{\varphi}\left(\frac{\xi-b}{\varepsilon}\right)$ for $r+1 \leq k \leq m$. It is clear that

$$\|f_k\|_{H^{p_k}} \approx \|f_k\|_{L^{p_k}} \approx 1, \quad 1 \leq k \leq m.$$

Moreover, $\widehat{f}_1(\xi_1) \cdots \widehat{f}_r(\xi_r) = 1$ on the support of the function

$$\widehat{\varphi}\left(\frac{1}{r\varepsilon} \sum_{l=1}^r (\xi_l - a)\right) \widehat{\varphi}\left(\frac{1}{r} \sum_{l=1}^r (\xi_l - \xi_2)\right) \cdots \widehat{\varphi}\left(\frac{1}{r} \sum_{l=1}^r (\xi_l - \xi_r)\right)$$

and also $\widehat{\phi}(\xi) = 1$ on the support of the functions \widehat{f}_k for all $r+1 \leq k \leq m$. Therefore we have

$$T_{\sigma^\varepsilon}(f_1, \dots, f_m)(x) = r e^{2\pi i(ra+(m-r)b) \cdot x} \varepsilon^n \varphi(\varepsilon r x) [\varphi(0)]^{r-1} \varepsilon^{\frac{n}{p_{r+1}} + \dots + \frac{n}{p_m}} [\varphi(\varepsilon x)]^{m-r}.$$

Taking L^p -norm, we deduce

$$\|T_{\sigma^\varepsilon}(f_1, \dots, f_m)\|_{L^p} \approx \varepsilon^{n - \frac{n}{p_1} - \dots - \frac{n}{p_r}},$$

which is the last thing we needed to obtain for our construction. Notice that the above argument also works for $p_k = \infty$. \square

3.5 Endpoint estimate

In this section we consider two endpoint estimates for multilinear singular integral operators. In the first case all indices are equal to infinity and in the second case one index is 1 and the others are equal to infinity.

For $x \in \mathbb{R}^n$ and $1 \leq k \leq m$, define

$$\Gamma_x^k = \{(y_1, \dots, y_m) \in \mathbb{R}^{mn} : |y_k| > 2|x|\}.$$

We say that a locally integrable function $K(y_1, \dots, y_m)$ on $\mathbb{R}^{mn} \setminus \{0\}$ satisfies a

coordinate-type Hörmander condition if for some finite constant A we have

$$\sum_{k=1}^m \int_{\Gamma_x^k} |K(y_1, \dots, y_{k-1}, x - y_k, y_{k+1}, \dots, y_m) - K(y_1, \dots, y_m)| d\vec{y} \leq A \quad (3.21)$$

for all $0 \neq x \in \mathbb{R}^n$. Another type of (bi)-linear Hörmander condition of geometric nature appeared in Pérez and Torres [33].

Denote by $\Lambda_p = \{(p, \infty, \dots, \infty), (\infty, p, \infty, \dots, \infty), \dots, (\infty, \dots, \infty, p)\}$ the set of all m -tuples with $(m - 1)$ entries equal to infinity and only one entry equal to p . The following result provides a version of the classical multilinear Calderón-Zygmund theorem in which the kernel satisfies a coordinate-type Hörmander condition under the initial assumption that the operator is bounded on Lebesgue spaces with indices in Λ_2 . We denote by L_c^∞ the space of all compactly supported bounded functions.

Theorem 3.18. *Suppose that an m -linear singular integral operator of convolution type T with kernel K is bounded from $L^{q_1} \times \dots \times L^{q_m}$ to L^2 with norm at most B for all $(q_1, \dots, q_m) \in \Lambda_2$. If K satisfies the coordinate-type Hörmander condition (3.21), then*

$$\|T(f_1, \dots, f_m)\|_{BMO} \lesssim (A + B) \|f_1\|_{L^\infty} \cdots \|f_m\|_{L^\infty} \quad (3.22)$$

for all f_j in L_c^∞ . Moreover, T has a bounded extension which satisfies

$$\|T(f_1, \dots, f_m)\|_{L^{1,\infty}} \lesssim (A + B) \|f_l\|_{L^1} \prod_{\substack{l=1 \\ k \neq l}}^m \|f_k\|_{L^\infty} \quad (3.23)$$

for $f_l \in L^1$, and $f_k \in L_c^\infty$ for all $1 \leq k \leq m$, $k \neq l$.

Proof. Fix a cube Q . To prove (3.22) we show that there exists a constant C_Q such that

$$\frac{1}{|Q|} \int_Q |T(f_1, \dots, f_m)(x) - C_Q| dx \lesssim (A + B) \|f_1\|_{L^\infty} \cdots \|f_m\|_{L^\infty}. \quad (3.24)$$

We decompose each function $f_l = f_l^0 + f_l^1$, where $f_l^0 = f_l \chi_{Q^*}$ and $f_l^1 = f_l \chi_{(Q^*)^c}$. Let F be the set of the 2^m sequences of length m consisting of zeros and ones. We claim that for each sequence $\vec{k} = (k_1, \dots, k_m)$ in F there is a constant $C_{\vec{k}}$ such that

$$\frac{1}{|Q|} \int_Q |T(f_1^{k_1}, \dots, f_m^{k_m})(x) - C_{\vec{k}}| dx \lesssim (A + B) \|f_1\|_{L^\infty} \cdots \|f_m\|_{L^\infty}. \quad (3.25)$$

Assuming the validity of the preceding claim we obtain (3.24) with $C_Q = \sum_{\vec{k} \in F} C_{\vec{k}}$.

Next, we want to establish (3.25) for each $\vec{k} \in F$. If $\vec{k} = (k_1, \dots, k_m)$ has at least one zero entry we pick $C_{\vec{k}} = 0$. Without loss of generality, we may assume that $k_1 = 0$. Since T maps $L^2 \times L^\infty \times \cdots \times L^\infty$ to L^2 , we have

$$\begin{aligned} \frac{1}{|Q|} \int_Q |T(f_1^{k_1}, \dots, f_m^{k_m})(x)| dx &\leq \left(\frac{1}{|Q|} \int_Q |T(f_1^{k_1}, \dots, f_m^{k_m})(x)|^2 dx \right)^{\frac{1}{2}} \\ &\leq \left(\frac{1}{|Q|} \int_{\mathbb{R}^n} |T(f_1^{k_1}, \dots, f_m^{k_m})(x)|^2 dx \right)^{\frac{1}{2}} \\ &\leq B |Q|^{-\frac{1}{2}} \|f_1^0\|_{L^2} \|f_2^{k_2}\|_{L^\infty} \cdots \|f_m^{k_m}\|_{L^\infty} \\ &\leq B |Q|^{-\frac{1}{2}} |Q^*|^{\frac{1}{2}} \|f_1\|_{L^\infty} \cdots \|f_m\|_{L^\infty} \\ &\lesssim B \|f_1\|_{L^\infty} \cdots \|f_m\|_{L^\infty}. \end{aligned}$$

Now suppose that $\vec{k} = (1, \dots, 1)$. Set $C_{\vec{k}} = T(f_1^{k_1}, \dots, f_m^{k_m})(x_Q)$, where x_Q is the center of the cube Q . Then, by the coordinate-type Hörmander condition (3.21), we have

$$\begin{aligned} &\frac{1}{|Q|} \int_Q |T(f_1^1, \dots, f_m^1)(x) - C_{\vec{k}}| dx \\ &\leq \frac{1}{|Q|} \int_Q \int_{\mathbb{R}^{nm}} |K(x - y_1, \dots, x - y_m) - K(x_Q - y_1, \dots, x_Q - y_m)| \prod_{k=1}^m |f_k^1(y_k)| d\vec{y} dx \\ &\leq \frac{\prod_{k=1}^m \|f_k\|_{L^\infty}}{|Q|} \int_Q \sum_{k=1}^m \int_{\Gamma_{x-x_Q}^k} |K(y_1, \dots, (x-x_Q) - y_k, \dots, y_m) - K(y_1, \dots, y_m)| d\vec{y} dx \\ &\lesssim A \|f_1\|_{L^\infty} \cdots \|f_m\|_{L^\infty}. \end{aligned}$$

This completes the proof of (3.22) and we are left with establishing (3.23). Fix $\lambda > 0$. It is enough to show that

$$|\{x \in \mathbb{R}^n : |T(f_1, \dots, f_m)(x)| > 2\lambda\}| \lesssim (A + B) \frac{1}{\lambda} \|f_1\|_{L^1} \|f_2\|_{L^\infty} \cdots \|f_m\|_{L^\infty}.$$

By scaling, we may assume that $\|f_1\|_{L^1} = \|f_2\|_{L^\infty} = \cdots = \|f_m\|_{L^\infty} = 1$. Let δ be a positive number chosen later and let $f_1 = g_1 + b_1$ be the Calderón-Zygmund decomposition at height $\delta\lambda$, and $b_1 = \sum_j b_{1,j}$, where $b_{1,j}$ are functions supported in the (pairwise disjoint) cubes Q_j such that

$$\begin{aligned} \text{supp}(b_{1,j}) &\subset Q_j, & \int b_{1,j}(x) dx &= 0, \\ \|b_{1,j}\|_{L^1} &\leq 2^{n+1} \delta \lambda |Q_j|, & \sum_j |Q_j| &\leq \frac{1}{\delta \lambda}, \\ \|g_1\|_{L^\infty} &\leq 2^n \delta \lambda, & \|g_1\|_{L^1} &\leq 1. \end{aligned}$$

Now we can estimate

$$\begin{aligned} |\{x \in \mathbb{R}^n : |T(f_1, \dots, f_m)(x)| > 2\lambda\}| &\leq |\{x \in \mathbb{R}^n : |T(g_1, \dots, f_m)(x)| > \lambda\}| \\ &\quad + |\{x \in \mathbb{R}^n : |T(b_1, \dots, f_m)(x)| > \lambda\}|. \end{aligned}$$

Since T maps $L^2 \times L^\infty \times \cdots \times L^\infty$ to L^2 , the first part can be controlled by

$$\begin{aligned} |\{x \in \mathbb{R}^n : |T(g_1, \dots, f_m)(x)| > \lambda\}| &\leq \frac{1}{\lambda^2} \int_{\mathbb{R}^n} |T(g_1, f_2, \dots, f_m)(x)|^2 dx \\ &\leq \frac{B^2}{\lambda^2} \|g_1\|_{L^2}^2 \|f_2\|_{L^\infty}^2 \cdots \|f_m\|_{L^\infty}^2 \\ &\leq \frac{2^n B^2 \delta}{\lambda}. \end{aligned}$$

To estimate the second part, we set $G = \cup_j Q_j^*$. Then we have

$$\begin{aligned} |\{x \in \mathbb{R}^n : |T(b_1, \dots, f_m)(x)| > \lambda\}| &\leq |G| + |\{x \in G^c : |T(b_1, \dots, f_m)(x)| > \lambda\}| \\ &\leq |G| + \frac{1}{\lambda} \sum_j \int_{(Q_j^*)^c} |T(b_{1,j}, \dots, f_m)(x)| dx. \end{aligned}$$

Notice that

$$|G| \leq \sum_j |Q_j^*| \lesssim \sum_j |Q_j| \leq \frac{1}{\delta \lambda}.$$

Denote by c_j the center of the cube Q_j . Invoking condition (3.21) yields

$$\begin{aligned} &\int_{(Q_j^*)^c} |T(b_{1,j}, \dots, f_m)(x)| dx \\ &\leq \int_{(Q_j^*)^c} \left| \int K(x - y_1, \dots, x - y_m) b_{1,j}(y_1) f_2(y_2) \cdots f_m(y_m) d\vec{y} \right| dx \\ &\leq \int_{(Q_j^*)^c} \left| \int [K(x - y_1, y_2, \dots, y_m) - K(x - c_j, y_2, \dots, y_m)] b_{1,j}(y_1) \prod_{k=2}^m f_k(x - y_k) d\vec{y} \right| dx \\ &\leq \prod_{k=2}^m \|f_k\|_{L^\infty} \int_{(Q_j^*)^c} \int |K(x - y_1, y_2, \dots, y_m) - K(x - c_j, y_2, \dots, y_m)| |b_{1,j}(y_1)| d\vec{y} dx \\ &\leq \int_{Q_j} \left\{ \int_{\Gamma_{y_1 - c_j}^1} |K(y_1 - z_1, z_2, \dots, z_m) - K(z_1, z_2, \dots, z_m)| d\vec{z} \right\} |b_{1,j}(y_1)| dy_1 \\ &\leq A \|b_{1,j}\|_{L^1}. \end{aligned}$$

Therefore

$$\frac{1}{\lambda} \sum_j \int_{(Q_j^*)^c} |T(b_{1,j}, \dots, f_m)(x)| dx \leq \frac{A}{\lambda} \sum_j \|b_{1,j}\|_{L^1} \leq \frac{2^{n+1} A}{\lambda}.$$

Choosing $\delta = B^{-1}$ and combining the preceding inequalities we obtain

$$|\{x \in \mathbb{R}^n : |T(f_1, \dots, f_m)(x)| > 2\lambda\}| \leq \frac{1}{\lambda} (2^n B + B + 2^{n+1} A) \leq 2^{n+1} (A + B) \frac{1}{\lambda},$$

which yields (3.23). □

This result allows us to obtain intermediate estimates between the results in [9] (in which $2 < p_k < \infty$ and $2 < p < \infty$) and the results in [15] (in which $1 < p_k \leq \infty$ and $1 < p \leq 2$).

Corollary 3.19. *Let $1 < p_k \leq \infty$ and $1 < p < \infty$ satisfy $1/p_1 + \dots + 1/p_m = 1/p$. Assume that (3.6) holds for a function σ on \mathbb{R}^{mn} where $s_k > n/2$ for all k . Then the multilinear Fourier multiplier operator T_σ maps $L^{p_1} \times \dots \times L^{p_m}$ to L^p .*

Proof. Note that Sobolev condition (3.6) for σ implies Hörmander condition (3.21) for $K = \sigma^\vee$. The proof of this implication is standard in the linear case and similarly, in the m -linear case, it follows by freezing all but one variable (in the bilinear case it is contained in [27]). We are now able to apply Theorem 3.18 to T_σ , and hence Corollary 3.19 follows. Interpolating between (3.22) and (3.23) yields that T_σ maps $L^p \times L^\infty \times \dots \times L^\infty$ to L^p for all $1 < p < \infty$. By symmetry, we deduce that T_σ is bounded from $L^{q_1} \times \dots \times L^{q_m}$ to L^p for all $(q_1, \dots, q_m) \in \Lambda_p$ and $1 < p < \infty$. Once again, by interpolation, we have that T_σ maps from $L^{p_1} \times \dots \times L^{p_m}$ to L^p for all $1 < p_1, \dots, p_m \leq \infty$ and $1 < p < \infty$ such that $\frac{1}{p_1} + \dots + \frac{1}{p_m} = \frac{1}{p}$ with norm at most a multiple of A . \square

Corollary 3.20. *Let σ be a bounded function on $\mathbb{R}^{mn} \setminus \{0\}$ which satisfies (3.6) with $s_k > n/2$ for all $k = 1, \dots, m$. Then we have the estimate*

$$\|T_\sigma(f_1, \dots, f_m)\|_{BMO} \lesssim A \|f_1\|_{L^\infty} \cdots \|f_m\|_{L^\infty}$$

for all functions $f_k \in L_c^\infty$.

Proof. As before condition (3.6) for σ implies (3.21) for $K = \sigma^\vee$. Applying Theorem 3.18 to T_σ , Corollary 3.20 follows. \square

3.6 Discretization of the multiplier

For $1 \leq \kappa \neq \ell \leq m$, we introduce sets

$$U_{\kappa,\ell} = \{(\xi_1, \dots, \xi_m) \in (\mathbb{R}^n)^m : \max_{j \neq \kappa, \ell} |\xi_j| \leq \frac{11}{10} |\xi_\kappa| \leq \frac{11}{50m} |\xi_\ell|\}, \quad (3.26)$$

$$W_{\kappa,\ell} = \{(\xi_1, \dots, \xi_m) \in (\mathbb{R}^n)^m : \max_{j \neq \kappa, \ell} |\xi_j| \leq \frac{11}{10} |\xi_\kappa|, \frac{1}{10m} |\xi_\ell| \leq |\xi_\kappa| \leq 2|\xi_\ell|\}. \quad (3.27)$$

Now we can construct smooth homogeneous of degree zero functions $\phi_{\kappa,\ell}$ and $\psi_{\kappa,\ell}$ supported in $U_{\kappa,\ell}$ and $W_{\kappa,\ell}$, respectively such that

$$\sum_{1 \leq \kappa \neq \ell \leq m} \left(\phi_{\kappa,\ell}(\xi_1, \dots, \xi_m) + \psi_{\kappa,\ell}(\xi_1, \dots, \xi_m) \right) = 1$$

for every $(\xi_1, \dots, \xi_m) \in \mathbb{R}^{mn} \setminus \{0\}$; see [11, Exercise 7.5.4].

This decomposition of unity leads to a discretization of σ as follows

$$\sigma = \sigma \sum_{1 \leq \kappa \neq \ell \leq m} (\phi_{\kappa,\ell} + \psi_{\kappa,\ell}) = \sum_{1 \leq \kappa \neq \ell \leq m} \phi_{\kappa,\ell} \sigma + \sum_{1 \leq \kappa \neq \ell \leq m} \psi_{\kappa,\ell} \sigma. \quad (3.28)$$

Notice that, by Lemma 3.5, if σ satisfies (3.6), then so do $\phi_{\kappa,\ell} \sigma$ and $\psi_{\kappa,\ell} \sigma$. In the sequel, to obtain the boundedness of T_σ from $H^{p_1} \times \dots \times H^{p_m}$ to L^p , we reduce the matter to establishing the boundedness of $T_{\sigma \phi_{\kappa,\ell}}$ and $T_{\sigma \psi_{\kappa,\ell}}$ for all $1 \leq \kappa \neq \ell \leq m$.

We also use the following dyadic decomposition of the multiplier

$$\sigma = \sum_{j \in \mathbb{Z}} \sigma_j, \quad \sigma_j(\xi) = \sigma(\xi) \widehat{\psi}(2^{-j} \xi). \quad (3.29)$$

From now on, we denote

$$G_\sigma^1(f_1, \dots, f_m) = \sum_{j \in \mathbb{Z}} |T_{\sigma_j}(f_1, \dots, f_m)|, \quad (3.30)$$

$$G_\sigma^2(f_1, \dots, f_m) = \left(\sum_{j \in \mathbb{Z}} |T_{\sigma_j}(f_1, \dots, f_m)|^2 \right)^{1/2}. \quad (3.31)$$

It is easy to see that $|T_\sigma(f_1, \dots, f_m)| \leq G_\sigma^1(f_1, \dots, f_m)$.

3.7 The proof of the main result

In this section, we prove Theorem 3.1 by considering four cases.

3.7.1 The first case: $0 < p_i \leq 1, 1 \leq i \leq m$

In this case, we prove the following result.

Theorem 3.21. *Let $\frac{n}{2} < s_1, \dots, s_m < \infty$, $0 < p_1, \dots, p_m \leq 1$, and $\frac{1}{p_1} + \dots + \frac{1}{p_m} = \frac{1}{p}$. Suppose (3.5) holds for every nonempty subset J of $\{1, 2, \dots, m\}$. Then (3.7) holds.*

Proof. By regularization, we may assume that the inverse Fourier transform of σ is smooth and compactly supported. If this case is established, then Theorem 3.11 yields the existence of a family of multilinear multiplier operators $(T_\varepsilon)_{0 < \varepsilon < \frac{1}{2}}$ associated with a family of multipliers $(\sigma^\varepsilon)_{0 < \varepsilon < \frac{1}{2}}$ such that $K^\varepsilon = (\sigma^\varepsilon)^\vee$ are smooth functions with compact supports for all $0 < \varepsilon < \frac{1}{2}$, and that (3.10), (3.11) hold. Fix $f_k \in H^{p_k} \cap L^{2m}$, ($1 \leq k \leq m$). The L^2 convergence in (3.11) implies that we can find a sequence of positive numbers $(\varepsilon_j)_j$ convergent to 0 such that

$$\lim_{j \rightarrow \infty} T_{\varepsilon_j}(f_1, \dots, f_m)(x) = T_\sigma(f_1, \dots, f_m)(x)$$

for a.e. $x \in \mathbb{R}^n$. Fatou's lemma connecting with (3.10) gives us

$$\begin{aligned} \|T_\sigma(f_1, \dots, f_m)\|_{L^p} &\leq \liminf_{j \rightarrow \infty} \|T_{\varepsilon_j}(f_1, \dots, f_m)\|_{L^p} \\ &\lesssim \sup_{0 < \varepsilon < \frac{1}{2}} \|T_\varepsilon(f_1, \dots, f_m)\|_{L^p} \\ &\lesssim \sup_{0 < \varepsilon < \frac{1}{2}} \sup_{j \in \mathbb{Z}} \|\sigma^\varepsilon(2^j \cdot) \widehat{\psi}\|_{W^{(s_1, \dots, s_m)}} \|f_1\|_{H^{p_1}} \cdots \|f_m\|_{H^{p_m}} \end{aligned}$$

$$\lesssim \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \widehat{\psi}\|_{W^{(s_1, \dots, s_m)}} \|f_1\|_{H^{p_1}} \cdots \|f_m\|_{H^{p_m}},$$

thus establishing the claimed estimate for a general multiplier σ .

In view of this deduction, we suppose σ^\vee is smooth and compactly supported. The aim is to show that

$$\|T_\sigma(f_1, \dots, f_m)\|_{L^p} \lesssim \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \widehat{\psi}\|_{W^{(s_1, \dots, s_m)}} \|f_1\|_{H^{p_1}} \cdots \|f_m\|_{H^{p_m}}. \quad (3.32)$$

By the decomposition of σ in (3.28), we only need to prove (3.32) for $\phi_{\kappa, \ell} \sigma$ or $\phi_{\kappa, \ell} \sigma^\vee$ in place of σ and for all $1 \leq \kappa \neq \ell \leq m$.

Fix functions $f_k \in H^{p_k}$. Using atomic representations for H^{p_k} -functions, write

$$f_k = \sum_{j_k \in \mathbb{Z}} \lambda_{k, j_k} a_{k, j_k}, \quad (1 \leq k \leq m),$$

where a_{k, j_k} are L^∞ -atoms for H^{p_k} satisfying

$$\text{supp}(a_{k, j_k}) \subset Q_{k, j_k}, \quad \|a_{k, j_k}\|_{L^\infty} \leq |Q_{k, j_k}|^{-\frac{1}{p_k}}, \quad \int_{Q_{k, j_k}} x^\alpha a_{k, j_k}(x) dx = 0$$

for all $|\alpha|$ large enough, and $\sum_{j_k} |\lambda_{k, j_k}|^{p_k} \leq 2^{p_k} \|f_k\|_{H^{p_k}}^{p_k}$.

Denote $\sigma^0 = \phi_{\kappa, \ell} \sigma$ and $\sigma^1 = \psi_{\kappa, \ell} \sigma$. We first prove (3.32) for T_{σ^0} . For (ξ_1, \dots, ξ_m) in the support $U_{\kappa, \ell}$ of σ^0 , we always have $|\xi_1 + \dots + \xi_m| \approx |\xi_\ell|$. Now we further decompose σ^0 as in (3.29) with $\sigma_j^0(\xi) = \sigma^0(\xi) \widehat{\psi}(2^{-j} \xi)$. In the support of σ_j^0 , we have

$$|\xi_1 + \dots + \xi_m| \approx |\xi_\ell| \approx 2^j.$$

Therefore, the Fourier transform of $T_{\sigma_j^0}(f_1, \dots, f_m)$ is supported in $B^{-1}2^j \leq |\xi| \leq B2^j$

for some $B > 1$. By Lemma 3.10,

$$\begin{aligned}
\|T_{\sigma^0}(f_1, \dots, f_m)\|_{L^p} &\leq \|T_{\sigma^0}(f_1, \dots, f_m)\|_{H^p} \\
&\leq \left\| \left(\sum_{j \in \mathbb{Z}} |T_{\sigma_j^0}(f_1, \dots, f_m)|^2 \right)^{1/2} \right\|_{L^p} \\
&= \|G_{\sigma^0}^2(f_1, \dots, f_m)\|_{L^p}.
\end{aligned} \tag{3.33}$$

For now we can assume that $K = (\sigma^0)^\vee$ is smooth and compactly supported, Proposition 3.14 yields that

$$T_{\sigma_j^0}(f_1, \dots, f_m)(x) = \sum_{j_1} \cdots \sum_{j_m} \lambda_{1,j_1} \cdots \lambda_{m,j_m} T_{\sigma_j^0}(a_{1,j_1}, \dots, a_{m,j_m})(x) \tag{3.34}$$

for a.e. $x \in \mathbb{R}^n$. Then (3.34) yields

$$\begin{aligned}
G_{\sigma^0}^2(f_1, \dots, f_m) &= \left(\sum_{j \in \mathbb{Z}} |T_{\sigma_j^0}(f_1, \dots, f_m)|^2 \right)^{1/2} \\
&\leq \sum_{j_1} \cdots \sum_{j_m} |\lambda_{1,j_1}| \cdots |\lambda_{m,j_m}| G_{\sigma^0}^2(a_{1,j_1}, \dots, a_{m,j_m}).
\end{aligned} \tag{3.35}$$

To control the pointwise estimate of $G_{\sigma^0}^2(a_{1,j_1}, \dots, a_{m,j_m})$, we need the following lemma whose proof will be given in Section 3.8.

Lemma 3.22. *Let $n/2 < s_1, \dots, s_m < \infty$, $0 < p_1, \dots, p_m \leq 1$ be numbers and let σ be a function satisfying (3.5) and (3.6). Suppose a_k are atoms supported in the cube Q_k , ($k = 1, \dots, m$) such that*

$$\|a_k\|_{L^\infty} \leq |Q_k|^{-\frac{1}{p_k}}, \quad \int_{Q_k} x^\alpha a_k(x) dx = 0,$$

for all $|\alpha| \leq N_k$ with $N_k = \lceil n(\frac{1}{p_k} - 1) \rceil + 1$. Then there exist positive functions b_1, \dots, b_m depending only on $m, n, \sigma, s_i, p_i, N_i$, and Q_i , $1 \leq i \leq m$, such that

1. If σ is supported in some $U_{\kappa,\ell}$, then

$$|G_\sigma^2(a_1, \dots, a_m)| \lesssim A b_1 \cdots b_m; \quad (3.36)$$

2. If σ is supported in some $W_{\kappa,\ell}$, then

$$G_\sigma^1(a_1, \dots, a_m) \lesssim A b_1 \cdots b_m, \quad (3.37)$$

and such that $\|b_k\|_{L^{p_k}} \lesssim 1$ for all $1 \leq k \leq m$, where G_σ^0 and G_σ^1 are defined in (3.30) and (3.30); $U_{\kappa,\ell}$ and $W_{\kappa,\ell}$ are sets introduced in (3.26) and (3.27) respectively.

Since σ^0 is supported in $U_{\kappa,\ell}$, Lemma 3.22 combined with (3.35) deduces

$$\begin{aligned} G_{\sigma^0}^2(f_1, \dots, f_m) &\lesssim A \sum_{j_1} \cdots \sum_{j_m} |\lambda_{1,j_1}| \cdots |\lambda_{m,j_m}| b_{1,k_1} \cdots b_{m,k_m} \\ &= A \prod_{k=1}^m \left(\sum_{j_k} |\lambda_{k,j_k}| b_{k,j_k} \right), \end{aligned}$$

where b_{k,j_k} are positive functions with uniformly bounded L^{p_k} -norms. Apply Hölder's inequality to obtain

$$\|G_{\sigma^0}^2(f_1, \dots, f_m)\|_{L^p} \lesssim A \|f_1\|_{H^{p_1}} \cdots \|f_m\|_{H^{p_m}},$$

combined with (3.33) to yield (3.32) for σ^0 . Here we used the fact that $p_k \leq 1$ and

$$\left\| \sum_{j_k} |\lambda_{k,j_k}| b_{k,j_k} \right\|_{L^{p_k}}^{p_k} \leq \sum_{j_k} |\lambda_{k,j_k}|^{p_k} \|b_{k,j_k}\|_{L^{p_k}}^{p_k} \lesssim \sum_{j_k} |\lambda_{k,j_k}|^{p_k} \lesssim \|f_k\|_{H^{p_k}}^{p_k}.$$

We now turn into σ^1 . We also decompose $\sigma^1 = \sum_{j \in \mathbb{Z}} \sigma_j^1$, where $\sigma_j^1(\xi) = \sigma^1(\xi) \widehat{\psi}(2^{-j}\xi)$.

By a similar argument as in the case for σ^0 , we have

$$T_{\sigma^1}(f_1, \dots, f_m) = \sum_{j \in \mathbb{Z}} T_{\sigma_j^1}(f_1, \dots, f_m),$$

and

$$T_{\sigma_j^1}(f_1, \dots, f_m) = \sum_{j_1} \cdots \sum_{j_m} \lambda_{1,j_1} \cdots \lambda_{m,j_m} T_{\sigma_j^1}(a_{1,j_1}, \dots, a_{m,j_m}).$$

Therefore

$$\begin{aligned} |T_{\sigma^1}(f_1, \dots, f_m)| &\leq \sum_{j \in \mathbb{Z}} |T_{\sigma_j^1}(f_1, \dots, f_m)| \\ &\leq \sum_{j \in \mathbb{Z}} \sum_{j_1} \cdots \sum_{j_m} |\lambda_{1,j_1}| \cdots |\lambda_{m,j_m}| |T_{\sigma_j^1}(a_{1,j_1}, \dots, a_{m,j_m})| \\ &= \sum_{j_1} \cdots \sum_{j_m} |\lambda_{1,j_1}| \cdots |\lambda_{m,j_m}| \sum_{j \in \mathbb{Z}} |T_{\sigma_j^1}(a_{1,j_1}, \dots, a_{m,j_m})| \\ &= \sum_{j_1} \cdots \sum_{j_m} |\lambda_{1,j_1}| \cdots |\lambda_{m,j_m}| G_{\sigma^1}^1(a_{1,j_1}, \dots, a_{m,j_m}). \end{aligned}$$

Since σ^1 is supported in $W_{\kappa, \ell}$, Lemma 3.22 allows us to replace $G_{\sigma^1}^1(a_{1,j_1}, \dots, a_{m,j_m})$ by $A b_{1,j_1} \cdots b_{m,j_m}$ in the above inequality to obtain

$$|T_{\sigma^1}(f_1, \dots, f_m)| \lesssim A \prod_{k=1}^m \left(\sum_{j_k} |\lambda_{k,j_k}| b_{k,j_k} \right),$$

which again combines with Hölder's inequality to establish (3.32) for σ^1 . This completes the proof of Theorem 3.21. \square

3.7.2 The second case: $0 < p_i \leq 1$ or $p_i = \infty$

Theorem 3.23. *Let $\frac{n}{2} < s_1, \dots, s_m < \infty$, $0 < p_1, \dots, p_l \leq 1$, $1 \leq l < m$, and $\frac{1}{p_1} + \cdots + \frac{1}{p_l} = \frac{1}{p}$. Suppose (3.5) holds for every nonempty subset J of $\{1, 2, \dots, l\}$.*

Then

$$\|T_\sigma\|_{H^{p_1} \times \dots \times H^{p_l} \times L^\infty \times \dots \times L^\infty \rightarrow L^p} \lesssim \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \widehat{\psi}\|_{W^{(s_1, \dots, s_m)}}. \quad (3.38)$$

Proof. The proof of this theorem is similar to that of Theorem 3.21. We just only need to sketch the key differences. The aim is to show that

$$\|T_\sigma(f_1, \dots, f_m)\|_{L^p} \lesssim A \|f_1\|_{H^{p_1}} \cdots \|f_l\|_{H^{p_l}} \prod_{i=l+1}^m \|f_i\|_{L^\infty}.$$

Fix functions $f_i \in H^{p_i}$. Using atomic representations for H^{p_i} -functions, write

$$f_i = \sum_{k_i \in \mathbb{Z}} \lambda_{i, k_i} a_{i, k_i}, \quad 1 \leq i \leq l,$$

where a_{i, k_i} are L^∞ -atoms for H^{p_i} satisfying

$$\text{supp}(a_{i, k_i}) \subset Q_{i, k_i}, \quad \|a_{i, k_i}\|_{L^\infty} \leq |Q_{i, k_i}|^{-\frac{1}{p_i}}, \quad \int_{Q_{i, k_i}} x^\alpha a_{i, k_i}(x) dx = 0$$

for $|\alpha| < N_i$ with N_i large enough, and $\sum_{k_i} |\lambda_{i, k_i}|^{p_i} \leq 2^{p_i} \|f_i\|_{H^{p_i}}^{p_i}$. To complete the proof of the theorem, we repeat the argument in the proof of Theorem 3.21 with noting that Lemma 3.22 is replaced by the following.

Lemma 3.24. *Let $\frac{n}{2} < s_1, \dots, s_m < \infty$, $0 < p_1, \dots, p_l \leq 1$, $1 \leq l < m$, and suppose (3.5) holds for all $J \subset \{1, \dots, l\}$. Let σ be a function satisfying (3.6). Suppose a_i , $1 \leq i \leq l$, are atoms supported in the cube Q_i such that*

$$\|a_i\|_{L^\infty} \leq |Q_i|^{-\frac{1}{p_i}}, \quad \int_{Q_i} x^\alpha a_i(x) dx = 0$$

for all $|\alpha| \leq N_k$ with $N_k = \lceil n(\frac{1}{p_k} - 1) \rceil + 1$. Then there exist positive functions b_1, \dots, b_l as stated in Lemma 3.22 such that

1. If σ is supported in some $U_{\kappa,\ell}$, then

$$|G_\sigma^2(a_1, \dots, a_l, f_{l+1}, \dots, f_m)| \lesssim A b_1 \cdots b_l \|f_{l+1}\|_{L^\infty} \cdots \|f_m\|_{L^\infty};$$

2. If σ is supported in some $W_{\kappa,\ell}$, then

$$G_\sigma^1(a_1, \dots, a_l, f_{l+1}, \dots, f_m) \lesssim A b_1 \cdots b_l \|f_{l+1}\|_{L^\infty} \cdots \|f_m\|_{L^\infty}.$$

□

3.7.3 The third case: $0 < p_i \leq 1$ or $2 \leq p_i \leq \infty$

Theorem 3.25. *Let $\frac{n}{2} < s_1, \dots, s_m < \infty$, $p_1, \dots, p_m \in (0, 1] \cup [2, \infty]$, $0 < p < \infty$, and $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}$. Assume there exists at least one index i such that $p_i \in (0, 1]$ and also assume the condition (3.5) holds for every nonempty subset J of $\{1, 2, \dots, m\}$. Then (3.7) holds.*

Proof. In addition to the assumptions of the theorem, we also assume there exists at least one i such that $p_i \in [2, \infty)$, since otherwise the claim is already covered by Theorems 3.21 or 3.23. Thus without loss of generality, we may assume that $0 < p_1, \dots, p_l \leq 1$, $2 \leq p_{l+1}, \dots, p_\rho < \infty$, $p_{\rho+1} = \cdots = p_m = \infty$, $1 \leq l < \rho \leq m$, and $\frac{1}{p_1} + \cdots + \frac{1}{p_\rho} = \frac{1}{p}$. Our goal is to establish the estimate

$$\|T_\sigma\|_{H^{p_1} \times \cdots \times H^{p_l} \times L^{p_{l+1}} \times \cdots \times L^{p_\rho} \times L^\infty \times \cdots \times L^\infty \rightarrow L^p} \lesssim \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \widehat{\psi}\|_{W^{(s_1, \dots, s_m)}}. \quad (3.39)$$

Assume momentarily the validity of the following estimate

$$\|T_\sigma\|_{H^{p_1} \times \cdots \times H^{p_l} \times \underbrace{L^2 \times \cdots \times L^2}_{(\rho-l)\text{-times}} \times \underbrace{L^\infty \times \cdots \times L^\infty}_{(m-\rho)\text{-times}} \rightarrow L^p} \lesssim \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \widehat{\psi}\|_{W^{(s_1, \dots, s_m)}}. \quad (3.40)$$

Then using Theorem 3.6 to interpolate between (3.40) and (3.38), we obtain the estimate (3.39) as required. (In fact, since the condition (3.5) with $(p_i)_{i=1,\dots,m}$ in the estimates of (3.38), (3.39), and (3.40) gives the same restriction on $(s_i)_{i=1,\dots,m}$, in order to deduce (3.39) from (3.40) and (3.38), we may fix $(s_i)_{i=1,\dots,m}$ and could use the usual real or complex interpolation for linear operators.) Thus it suffices to prove (3.40). In the rest of the proof, we assume $p_{l+1} = \dots = p_\rho = 2$.

Before we proceed to the proof of (3.40), we shall see that it is sufficient to consider σ that has support in some cone. Indeed, using decomposition (3.29), we may assume additionally that σ is supported in some $U_{\kappa,\ell}$ or $W_{\kappa,\ell}$ as indicated in (3.26) or (3.27).

To simplify notation, we also assume

$$\sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \widehat{\psi}\|_{W^{(s_1, \dots, s_m)}} = 1. \quad (3.41)$$

We shall divide the proof into two subcases. First case: σ is supported in $U_{\kappa,\ell}$. Second case: σ is supported in $W_{\kappa,\ell}$. In the first case, we shall use a Littlewood-Paley function. Notice that, in this case, the support of the Fourier transform of $T_{\sigma_j}(f_1, \dots, f_m)$ is included in the annulus $\{\xi \in \mathbb{R}^n : B^{-1}2^j \leq |\xi| \leq B2^j\}$ with some constant $B > 1$. Hence, by Lemma 3.10, we have

$$\|T_\sigma(f_1, \dots, f_m)\|_{H^p} \lesssim \left\| \left(\sum_{j \in \mathbb{Z}} |T_{\sigma_j}(f_1, \dots, f_m)|^2 \right)^{1/2} \right\|_{L^p}. \quad (3.42)$$

Thus, in the first case, we recall the function

$$G_\sigma^2(f_1, \dots, f_m) = \left(\sum_{j \in \mathbb{Z}} |T_{\sigma_j}(f_1, \dots, f_m)|^2 \right)^{1/2}$$

and prove the estimate

$$\|G_\sigma^2(f_1, \dots, f_m)\|_{L^p} \lesssim \prod_{i=1}^m \|f_i\|_{H^{p_i}}, \quad (3.43)$$

which combined with (3.42) implies that

$$\|T_\sigma(f_1, \dots, f_m)\|_{L^p} \lesssim \prod_{i=1}^m \|f_i\|_{H^{p_i}}, \quad (3.44)$$

where G_σ^1 is defined in (3.30).

In the second case, we shall directly prove (3.44). The essential part of the proofs of (3.43) and (3.44) are given in the following lemma.

Lemma 3.26. *Let $\frac{n}{2} < s_1, \dots, s_m < \infty$, $0 < p_1, \dots, p_l \leq 1$, $p_{l+1} = \dots = p_\rho = 2$, $p_{\rho+1} = \dots = p_m = \infty$, $1 \leq l < \rho \leq m$, and suppose (3.5) holds for every nonempty subset J of $\{1, \dots, l\}$. Let σ be a function satisfying (3.41). Assume a_i , $1 \leq i \leq l$, are H^{p_i} atoms such that*

$$\text{supp } a_i \subset Q_i, \quad \|a_i\|_{L^\infty} \leq |Q_i|^{-1/p_i}, \quad \int a_i(x) x^\alpha dx = 0$$

for $|\alpha| < N_i$, where N_i is a sufficiently large positive integer and Q_i is a cube, and that $f_{l+1}, \dots, f_\rho \in L^2$ and $f_{\rho+1}, \dots, f_m \in L^\infty$. Then there exist functions b_1, \dots, b_l and F_{l+1}, \dots, F_ρ such that

1. *If σ is supported in some $U_{\kappa, \ell}$, then*

$$G_\sigma^2(a_1, \dots, a_l, f_{l+1}, \dots, f_m)(x) \lesssim \prod_{i=1}^l b_i(x) \cdot \prod_{i=l+1}^{\rho} F_i(x) \cdot \prod_{i=\rho+1}^m \|f_i\|_{L^\infty}; \quad (3.45)$$

2. *If σ is supported in some $W_{\kappa, \ell}$, then*

$$|G_\sigma^1(a_1, \dots, a_l, f_{l+1}, \dots, f_m)(x)| \lesssim \prod_{i=1}^l b_i(x) \cdot \prod_{i=l+1}^{\rho} F_i(x) \cdot \prod_{i=\rho+1}^m \|f_i\|_{L^\infty}, \quad (3.46)$$

where the function b_i depends only on m , n , $(s_i)_{i=1, \dots, m}$, $(p_i)_{i=1, \dots, m}$, σ , i , a_i , and $(f_i)_{i=\rho+1, \dots, m}$; the function F_i depends only on m , n , $(s_i)_{i=1, \dots, m}$, i , f_i , and $(f_i)_{i=\rho+1, \dots, m}$;

and they satisfy the estimates $\|b_i\|_{L^{p_i}} \lesssim 1$ and $\|F_i\|_{L^2} \lesssim \|f_i\|_{L^2}$.

The proofs of this lemma will be given in Section 3.8. We shall continue the proof of Theorem 3.25. To utilize the above lemmas, we decompose $f_i \in H^{p_i}$, $1 \leq i \leq l$, into atoms as $f_i = \sum_{k_i \in \mathbb{Z}} \lambda_{i,k_i} a_{i,k_i}$ with λ_{i,k_i} , a_{i,k_i} , and the cubes Q_{i,k_i} being the same as in the proof of Theorem 3.23.

Consider the first case when σ is supported in some $U_{\kappa,\ell}$. In this case, Lemma 3.26 yields functions b_{i,k_i} ($1 \leq i \leq l$, $k_i \in \mathbb{Z}$) and F_i ($l+1 \leq i \leq \rho$) such that

$$|G_\sigma^2(a_{1,k_1}, \dots, a_{l,k_l}, f_{l+1}, \dots, f_m)(x)| \lesssim \prod_{i=1}^l b_{i,k_i}(x) \cdot \prod_{i=l+1}^\rho F_i(x) \cdot \prod_{i=\rho+1}^m \|f_i\|_{L^\infty}$$

and $\|b_{i,k_i}\|_{L^{p_i}} \lesssim 1$ and $\|F_i\|_{L^2} \lesssim \|f_i\|_{L^2}$. Notice that b_{i,k_i} do not depend on k_j with $j \neq i$ and F_i do not depend on k_1, \dots, k_l . By the sublinear property of square function, we have

$$\begin{aligned} G_\sigma^2(f_1, \dots, f_m) &\leq \sum_{k_1} \cdots \sum_{k_l} |\lambda_{1,k_1} \cdots \lambda_{l,k_l}| G_\sigma^2(a_{1,k_1}, \dots, a_{l,k_l}, f_{l+1}, \dots, f_m) \\ &\lesssim \sum_{k_1} \cdots \sum_{k_l} |\lambda_{1,k_1} \cdots \lambda_{l,k_l}| \prod_{i=1}^l b_{i,k_i}(x) \cdot \prod_{i=l+1}^\rho F_i(x) \cdot \prod_{i=\rho+1}^m \|f_i\|_{L^\infty} \\ &= \prod_{i=1}^l \left(\sum_{k_i} |\lambda_{i,k_i}| b_{i,k_i}(x) \right) \cdot \prod_{i=l+1}^\rho F_i(x) \cdot \prod_{i=\rho+1}^m \|f_i\|_{L^\infty}. \end{aligned}$$

The above pointwise inequality and Hölder's inequality now give (3.43).

Next consider the second case when σ is supported in some $W_{\kappa,\ell}$. In this case, using (3.45) implies

$$\begin{aligned} |T_\sigma(f_1, \dots, f_m)| &\leq G_\sigma^1(f_1, \dots, f_m) \\ &\lesssim \sum_{k_1} \cdots \sum_{k_l} |\lambda_{1,k_1}| \cdots |\lambda_{l,k_l}| G_\sigma^1(a_{1,k_1}, \dots, a_{m,k_m}) \\ &\lesssim \left(\sum_{k_1} \cdots \sum_{k_l} |\lambda_{1,k_1} \cdots \lambda_{l,k_l}| \prod_{i=1}^l b_{i,k_i} \right) \cdot \prod_{i=l+1}^\rho F_i \cdot \prod_{i=\rho+1}^m \|f_i\|_{L^\infty} \end{aligned}$$

$$= \prod_{i=1}^l \left(\sum_{k_i} |\lambda_{i,k_i}| b_{i,k_i} \right) \cdot \prod_{i=l+1}^{\rho} F_i \cdot \prod_{i=\rho+1}^m \|f_i\|_{L^\infty}.$$

Arguing in the same way as in the first case, we obtain (3.44). Thus the proof of Theorem 3.25 is reduced to Lemmas 3.26, which is verified in Section 3.8. \square

3.7.4 The last case: $0 < p_i \leq \infty$

In this subsection, we shall prove the estimate (3.7) for the entire range $0 < p_i \leq \infty$. To simplify notation, we use the letters \mathbf{s} and \mathbf{p} to denote (s_1, \dots, s_m) and (p_1, \dots, p_m) , respectively.

We shall slightly change the formulation of the claim of Theorem 3.1. We assume $0 < p_1, \dots, p_m \leq \infty$,

$$\infty > s_1, \dots, s_m \geq n/2, \quad (3.47)$$

and assume they satisfy (3.18) for every nonempty subset J of $\{1, \dots, m\}$. We shall prove the estimate

$$\|T_\sigma\|_{H^{p_1} \times \dots \times H^{p_m} \rightarrow L^p} \lesssim \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \widehat{\psi}\|_{W^{(s_1+\varepsilon, \dots, s_m+\varepsilon)}} \quad (3.48)$$

holds for every $\varepsilon > 0$, where $1/p = 1/p_1 + \dots + 1/p_m$ and the space L^p should be replaced by BMO if $p_1 = \dots = p_m = p = \infty$. The inequality (3.48) is equivalent to the estimate given in Theorem 3.1. The proof will be given in two steps.

In the first step, we fix \mathbf{s} satisfying (3.47) and consider the set $\Delta(\mathbf{s})$ that consists of all $(1/p_1, \dots, 1/p_m) \in [0, \infty)^m$ such that the condition (3.18) holds for every nonempty subset J of $\{1, \dots, m\}$. We prove the following lemma.

Lemma 3.27. *If \mathbf{s} satisfies (3.47), then $\Delta(\mathbf{s})$ is the convex hull of the point $(0, \dots, 0)$*

and the points $(1/p_1, \dots, 1/p_m)$ that satisfy

$$1/p_i = 0 \text{ or } 1/p_i = s_i/n \text{ or } 1/p_i = s_i/n + 1/2 \text{ for all } i, \quad (3.49)$$

and

$$1/p_i = s_i/n + 1/2 \text{ for exactly one } i. \quad (3.50)$$

Proof. Fix $\mathbf{s} = (s_1, \dots, s_m)$ such that $s_i \geq \frac{n}{2}$ for all $1 \leq i \leq m$. Condition (3.18) gives a clearer presentation of the set $\Delta(\mathbf{s})$ as

$$\Delta(m, \mathbf{s}) = \left\{ \left(\frac{1}{p_1}, \dots, \frac{1}{p_m} \right) \in \mathbb{R}^m : 0 \leq \frac{1}{p_i} \leq \frac{s_i}{n} + \frac{1}{2}, \sum_{i \in J} \frac{1}{p_i} \leq \sum_{i \in J} \frac{s_i}{n} + \frac{1}{2} \right\},$$

where J runs over all non-empty subsets of $\{1, \dots, m\}$. We let H denote the convex hull of $(0, \dots, 0)$ and of all the points $(1/p_1, \dots, 1/p_m)$ that satisfy (3.49) and (3.50).

We will show that $\Delta(m, \mathbf{s}) = H$ by induction in m .

The case when $m = 2$ is trivial because $\Delta(2, \mathbf{s})$ is the convex hull of the following points $(0, 0)$, $(\frac{s_1}{n} + \frac{1}{2}, 0)$, $(\frac{s_1}{n} + \frac{1}{2}, \frac{s_2}{n})$, $(0, \frac{s_2}{n} + \frac{1}{2})$ and $(\frac{s_1}{n}, \frac{s_2}{n} + \frac{1}{2})$; hence, the statement of Lemma 3.27 holds obviously in this case.

Now fix an $m > 2$ and suppose that the statement of the lemma is true for $m - 1$. For $1 \leq k \leq m$, denote

$$\Delta^k(m, \mathbf{s}) = \left\{ \left(\frac{1}{p_1}, \dots, \frac{1}{p_m} \right) \in \Delta(m, \mathbf{s}) : 0 \leq \frac{1}{p_k} \leq \frac{s_k}{n} \right\},$$

$$F_0^k(m, \mathbf{s}) = \left\{ \left(\frac{1}{p_1}, \dots, \frac{1}{p_m} \right) \in \Delta(m, \mathbf{s}) : \frac{1}{p_k} = 0 \right\},$$

$$F_1^k(m, \mathbf{s}) = \left\{ \left(\frac{1}{p_1}, \dots, \frac{1}{p_m} \right) \in \Delta(m, \mathbf{s}) : \frac{1}{p_k} = \frac{s_k}{n} \right\},$$

and

$$\Delta^0(m, \mathbf{s}) = \left\{ \left(\frac{1}{p_1}, \dots, \frac{1}{p_m} \right) \in \Delta(m, \mathbf{s}) : \frac{s_i}{n} \leq \frac{1}{p_i} \leq \frac{s_i}{n} + \frac{1}{2}, \forall 1 \leq i \leq m \right\}.$$

It is easy to see that $\Delta(m, \mathbf{s}) = \cup_{k=0}^m \Delta^k(m, \mathbf{s})$. We observe that H is a subset of $\Delta(m, \mathbf{s})$, since each vertex of H obviously sits inside the convex set $\Delta(m, \mathbf{s})$. Thus, it suffices to prove that $\Delta^k(m, \mathbf{s})$ is a subset of H for every $0 \leq k \leq m$.

We first consider $\Delta^k(m, \mathbf{s})$ for $1 \leq k \leq m$. By induction, the face $F_0^k(m, \mathbf{s})$ is the convex hull of the following points $(0, \dots, 0)$ and $(\frac{1}{p_1}, \dots, \frac{1}{p_m})$, where $\frac{1}{p_k} = 0$, $\frac{1}{p_i} \in \{0, \frac{s_i}{n}, \frac{s_i}{n} + \frac{1}{2}\}$ for $i \neq k$, and there exists exactly one $i \neq k$ such that $\frac{1}{p_i} = \frac{s_i}{n} + \frac{1}{2}$. Similarly, the face $F_1^k(m, \mathbf{s})$ is determined by the same constraints for all variables $\frac{1}{p_i}, i \neq k$ as those for $F_0^k(m, \mathbf{s})$. Therefore, by induction, we have that $F_1^k(m, \mathbf{s})$ is the convex hull of the points $(0, \dots, 0, \frac{s_k}{n}, 0, \dots, 0)$ and $(\frac{1}{p_1}, \dots, \frac{1}{p_m})$, where $\frac{1}{p_k} = \frac{s_k}{n}, \frac{1}{p_i} \in \{0, \frac{s_i}{n}, \frac{s_i}{n} + \frac{1}{2}\}$ for $i \neq k$, and there exists exactly one $i \neq k$ such that $\frac{1}{p_i} = \frac{s_i}{n} + \frac{1}{2}$. Note that the point $(0, \dots, 0, \frac{s_k}{n}, 0, \dots, 0)$ belongs to the line segment that joins the origin $(0, \dots, 0)$ with $(0, \dots, 0, \frac{s_k}{n} + \frac{1}{2}, 0, \dots, 0)$. Thus $F_0^k(m, \mathbf{s})$ and $F_1^k(m, \mathbf{s})$ are contained in H , and hence, $\Delta^k(m, \mathbf{s})$ is a subset of H since $\Delta^k(m, \mathbf{s})$ is a convex hull of two faces $F_0^k(m, \mathbf{s})$ and $F_1^k(m, \mathbf{s})$.

It remains to check that $\Delta^0(m, \mathbf{s}) \subset H$. In this case, we note that the constraints $0 \leq \frac{1}{p_i} - \frac{s_i}{n} \leq \frac{1}{2}, \forall 1 \leq i \leq m$ and

$$\sum_{i=1}^m \left(\frac{1}{p_i} - \frac{s_i}{n} \right) \leq \frac{1}{2}$$

imply that $\Delta^0(m, \mathbf{s})$ is a standard m -simplex with vertices $(\frac{s_1}{n}, \dots, \frac{s_m}{n})$ and $(\frac{1}{p_1}, \dots, \frac{1}{p_m})$, where $\frac{1}{p_i} \in \{\frac{s_i}{n}, \frac{s_i}{n} + \frac{1}{2}\}$ for $1 \leq i \leq m$, and there exists exactly one i such that $\frac{1}{p_i} = \frac{s_i}{n} + \frac{1}{2}$, which implies $\Delta^0(m, \mathbf{s}) \subset H$ with noting that the point $(\frac{s_1}{n}, \dots, \frac{s_m}{n}) \in F_1^k(m, \mathbf{s}) \subset H$. \square

By virtue of Lemma 3.27 and Theorem 3.6, to prove the estimate (3.48) under the assumptions (3.47) and (3.18), it is sufficient to show it for $\mathbf{p} = (\infty, \dots, \infty)$ and for \mathbf{p} satisfying (3.49) and (3.50). For $\mathbf{p} = (\infty, \dots, \infty)$, the estimate (3.48) with *BMO* in place of L^p is established in Corollary 3.19. Thus it is sufficient to consider the latter points.

In the second step, we shall prove the following lemma, which will complete the proof of Theorem 3.1.

Lemma 3.28. *Estimate (3.48) holds if \mathbf{s} and \mathbf{p} satisfy (3.47), (3.49), and (3.50).*

Proof. For $\mathbf{p} \in (0, \infty]^m$, we define $\ell(\mathbf{p})$ to be the number of the indices $i \in \{1, \dots, m\}$ such that $1 < p_i < 2$. We shall prove the claim by induction on $\ell(\mathbf{p})$.

The conditions (3.47) and (3.50) imply in particular that there exists at least one i such that $p_i \leq 1$. Hence if $\ell(\mathbf{p}) = 0$ then the claim directly follows from Theorem 3.25.

Assume $\ell_0 \geq 1$ and assume the claim holds if $\ell(\mathbf{p}) < \ell_0$. Let

$$(\mathbf{p}^0, \mathbf{s}^0) = (p_1^0, \dots, p_m^0, s_1^0, \dots, s_m^0)$$

be a point that satisfies the conditions (3.47), (3.49), and (3.50), and satisfies $\ell(\mathbf{p}^0) = \ell_0$. There exists an index i such that $1 < p_i^0 < 2$. Notice that $1/p_i^0 = s_i^0/n$ for this index i . Without loss of generality, we assume $1 > 1/p_1^0 = s_1^0/n > 1/2$. Then the condition (3.50) implies that there exists exactly one i such that $2 \leq i \leq m$ and $1/p_i^0 = s_i^0/n + 1/2$. Consider the following two points:

$$(\mathbf{p}', \mathbf{s}') = (1, p_2^0, \dots, p_m^0, n, s_2^0, \dots, s_m^0),$$

$$(\mathbf{p}'', \mathbf{s}'') = (2, p_2^0, \dots, p_m^0, n/2, s_2^0, \dots, s_m^0).$$

Both $(\mathbf{p}', \mathbf{s}')$ and $(\mathbf{p}'', \mathbf{s}'')$ satisfy the conditions (3.47), (3.49), and (3.50), and $\ell(\mathbf{p}') =$

$\ell(\mathbf{p}'') = \ell_0 - 1$. Hence by the induction hypothesis the estimate (3.48) holds for $(\mathbf{p}', \mathbf{s}')$ and $(\mathbf{p}'', \mathbf{s}'')$. Then, by Theorem 3.6, it follows that the estimate (3.48) also holds for $(\mathbf{p}^0, \mathbf{s}^0)$. This completes the proof of Lemma 3.28. \square

3.8 Proofs of technical lemmas

Throughout this section, a_k are H^{p_k} -atoms supported in the cube Q_k such that

$$\|a_k\|_{L^\infty} \leq |Q_k|^{-\frac{1}{p_k}}, \quad \int_{Q_k} x^\alpha a_k(x) dx = 0,$$

for all $|\alpha| \leq N_k$ with $N_k = \lceil n(\frac{1}{p_k} - 1) \rceil + 1$ and for all $1 \leq k \leq m$. Denote by Q^* the dilation of the cube Q with factor $2\sqrt{n}$. Recall functions G_σ^1 and G_σ^2 that are defined in (3.30) and (3.31). Let \mathcal{P}_m be the set of all subsets of $\{1, \dots, m\}$. For real numbers $s_1, \dots, s_m > \frac{n}{2}$, let $s = \min(s_1, \dots, s_m) > \frac{n}{2}$. We can find a real number $1 < q < 2$ such that $sq > n$. Also let M_q be the maximal operator defined by

$$M_q(f)(x) = \sup_{r>0} \left(\frac{1}{r^n} \int_{|x-y|<r} |f(y)|^q dy \right)^{1/q}. \quad (3.51)$$

3.8.1 Proof of Lemma 3.22

We partition \mathbb{R}^n into disjoint subsets as $\mathbb{R}^n = \bigcup_J E_J$, where J runs over \mathcal{P}_m and E_J is defined by

$$E_J = \bigcap_{i \in J} (Q_i^*)^c \cap \bigcap_{i \in J^c} Q_i^*. \quad (3.52)$$

Let first prove (3.36) with assumption that σ is supported in some $U_{\kappa, \ell}$. Assume momentarily that for each $J \in \mathcal{P}_m$ we can construct positive functions b_k^J such that

$$|G_\sigma^2(a_1, \dots, a_m)(x)| \lesssim A b_1^J(x) \cdots b_m^J(x) \quad \forall x \in E_J, \quad (3.53)$$

and that $\|b_k^J\|_{L^{p_k}} \lesssim 1$. Then the desired functions b_k stated in Lemma 3.22 can be constructed by letting $b_k = \sum_{J \in \mathcal{P}_m} b_k^J \chi_{E_J}$. Thus, it is enough to prove (3.53) for each $J \in \mathcal{P}_m$.

Fix $J \in \mathcal{P}_m$ and divide the proof of (3.36) into two subcases: $J \neq \emptyset$ and $J = \emptyset$. For $J \neq \emptyset$, we may assume that $J = \{1, \dots, r\}$ for some $1 \leq r \leq m$. Fix $x \in E_J = (\cap_{k=r+1}^m Q_k^*) \setminus (\cup_{k=1}^r Q_k^*)$ and denote $g_j = T_{\sigma_j}(a_1, \dots, a_m)$. Then

$$g_j(x) = \int_{\mathbb{R}^{mn}} 2^{jmn} K_j(2^j(x - y_1), \dots, 2^j(x - y_m)) a_1(y_1) \cdots a_m(y_m) dy_1 \cdots dy_m$$

with $K_j = (\sigma(2^j \cdot) \widehat{\psi})^\vee$. Let c_k be the center of the cube Q_k ($1 \leq k \leq m$). For $1 \leq k \leq r$, since $x \notin Q_k^*$ and $y_k \in Q_k$, $|x - c_k| \approx |x - y_k|$. Fix $1 \leq l \leq r$. Using Lemma 3.4 with $s_k > \frac{n}{2}$ and applying the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} & \prod_{k=1}^r \langle 2^j(x - c_k) \rangle^{s_k} |g_j(x)| \\ & \lesssim 2^{jmn} \int_{Q_1 \times \cdots \times Q_m} \left(\prod_{k=1}^r \langle 2^j(x - y_k) \rangle^{s_k} \right) |K_j(2^j(x - y_1), \dots, 2^j(x - y_m))| \prod_{k=1}^m \|a_k\|_{L^\infty} d\vec{y} \\ & \leq 2^{jmn} \int_{Q_1 \times \cdots \times Q_m} \left(\prod_{k=1}^r \langle 2^j(x - y_k) \rangle^{s_k} \right) |K_j(2^j(x - y_1), \dots, 2^j(x - y_m))| \prod_{k=1}^m |Q_k|^{-\frac{1}{p_k}} d\vec{y} \\ & \leq 2^{jrn} \prod_{k=1}^m |Q_k|^{-\frac{1}{p_k}} \int_{Q_1 \times \cdots \times Q_r \times \mathbb{R}^{(m-r)n}} \left(\prod_{k=1}^r \langle 2^j(x - y_k) \rangle^{s_k} \right) \times \\ & \quad \times |K_j(2^j(x - y_1), \dots, 2^j(x - y_r), y_{r+1}, \dots, y_m)| dy_1 \cdots dy_r dy_{r+1} \cdots dy_m \\ & \leq 2^{jrn} \prod_{k=1}^r |Q_k|^{1-\frac{1}{p_k}} \prod_{k=r+1}^m |Q_k|^{-\frac{1}{p_k}} \int_{\mathbb{R}^{(m-r)n}} \int_{Q_l} |Q_l|^{-1} \langle 2^j(x - y_l) \rangle^{s_l} \times \\ & \quad \times \left\| \left(\prod_{\substack{k=1 \\ k \neq l}}^r \langle y_k \rangle^{s_k} \right) K_j(y_1, \dots, y_{l-1}, 2^j(x - y_l), y_{l+1}, \dots, y_m) \right\|_{L^\infty(dy_1 \cdots \widehat{dy}_l \cdots dy_r)} dy_l dy_{r+1} \cdots dy_m \\ & \lesssim 2^{jrn} \prod_{k=1}^r |Q_k|^{1-\frac{1}{p_k}} \prod_{k=r+1}^m |Q_k|^{-\frac{1}{p_k}} \int_{\mathbb{R}^{(m-r)n}} \int_{Q_l} |Q_l|^{-1} \langle 2^j(x - y_l) \rangle^{s_l} \times \end{aligned}$$

$$\begin{aligned}
& \times \left\| \left(\prod_{\substack{k=1 \\ k \neq l}}^r \langle y_k \rangle^{s_k} \right) K_j(y_1, \dots, y_{l-1}, 2^j(x - y_l), y_{l+1}, \dots, y_m) \right\|_{L^2(dy_1 \dots \widehat{dy}_l \dots dy_r)} dy_l dy_{r+1} \dots dy_m \\
& \lesssim 2^{jrn} \prod_{k=1}^r |Q_k|^{1-\frac{1}{p_k}} \prod_{k=r+1}^m |Q_k|^{-\frac{1}{p_k}} \int_{Q_l} |Q_l|^{-1} \langle 2^j(x - y_l) \rangle^{s_l} \times \\
& \times \left\| \left(\prod_{\substack{k=1 \\ i \neq k}}^m \langle y_k \rangle^{s_i} \right) K_j(y_1, \dots, y_{l-1}, 2^j(x - y_l), y_{l+1}, \dots, y_m) \right\|_{L^2(dy_1 \dots \widehat{dy}_l \dots dy_m)} dy_l \\
& = 2^{jrn} \left(\prod_{k=1}^r |Q_k|^{1-\frac{1}{p_k}} \right) h_j^{(l,0)}(x) \prod_{k=r+1}^m b_k^J(x) \tag{3.54}
\end{aligned}$$

for all $x \in E_J$, where

$$\begin{aligned}
h_j^{(l,0)}(x) &= \frac{1}{|Q_l|} \int_{Q_l} \langle 2^j(x - y_l) \rangle^{s_l} \\
& \times \left\| \left(\prod_{\substack{k=1 \\ k \neq l}}^m \langle y_k \rangle^{s_k} \right) K_j(y_1, \dots, y_{l-1}, 2^j(x - y_l), y_{l+1}, \dots, y_m) \right\|_{L^2(dy_1 \dots \widehat{dy}_l \dots dy_m)} dy_l
\end{aligned}$$

and $b_k^J(x) = |Q_k|^{-\frac{1}{p_k}} \chi_{Q_k^*}(x)$ for $r+1 \leq k \leq m$. A direct computation gives

$$\|h_j^{(l,0)}\|_{L^2} \leq 2^{-\frac{jn}{2}} \|\sigma(2^j \cdot) \widehat{\psi}\|_{W(s_1, \dots, s_m)} = A 2^{-\frac{jn}{2}}.$$

Using the vanishing moment condition of a_k and Taylor's formula, we write

$$\begin{aligned}
g_j(x) &= 2^{jmn} \sum_{|\alpha|=N_l} C_\alpha \int_{\mathbb{R}^{mn}} \left\{ \int_0^1 (1-t)^{N_l-1} \right. \\
& \times \partial_{y_l}^\alpha K_j \left(2^j(x - y_1), \dots, 2^j(x - c_l - t(y_l - c_l)), \dots, 2^j(x - y_m) \right) \\
& \left. \times (2^j(y_l - c_l))^\alpha a_1(y_1) \dots a_m(y_m) dt \right\} dy_1 \dots dy_m.
\end{aligned}$$

Repeat the preceding argument to obtain

$$\prod_{k=1}^r \langle 2^j(x - c_k) \rangle^{s_k} |g_j(x)| \lesssim 2^{jrn} \left(\prod_{k=1}^r |Q_k|^{1-\frac{1}{p_k}} \right) h_j^{(l,1)}(x) \prod_{k=r+1}^m b_k^J(x) \tag{3.55}$$

for all $x \in E_J$, where $b_k^J(x) = |Q_k|^{-\frac{1}{p_k}} \chi_{Q_k^*}(x)$ for $r+1 \leq k \leq m$ and

$$\begin{aligned} h_j^{(l,1)}(x) &= \sum_{|\alpha|=N_l} \int_{Q_l} \left\{ \int_0^1 \langle 2^j x_{c_l, y^t}^t \rangle^{s_l} \right. \\ &\quad \times \left\| \left(\prod_{\substack{k=1 \\ k \neq l}}^m \langle y_k \rangle^{s_k} \right) \partial_{y_l}^\alpha K_j(y_1, \dots, y_{l-1}, 2^j x_{c_l, y^t}^t, y_{l+1}, \dots, y_m) \right\|_{L^2(dy_1 \cdots \widehat{dy_l} \cdots dy_m)} \\ &\quad \times (2^j \ell(Q_l))^{N_l} |Q_l|^{-1} dt \Big\} dy_l, \end{aligned}$$

with $x_{c_l, y^t}^t = x - c_l - t(y_l - c_l)$. Applying Minkowski's inequality together with Lemmas 3.4 and 3.5 implies that

$$\|h_j^{(l,1)}\|_{L^2} \leq A 2^{-\frac{jn}{2}} (2^j \ell(Q_l))^{N_l}.$$

Combining inequalities (3.54) and (3.55), we get

$$\begin{aligned} &\prod_{k=1}^r \langle 2^j(x - c_k) \rangle^{s_k} |g_j(x)| \\ &\leq 2^{jrn} \left(\prod_{k=1}^r |Q_k|^{1-\frac{1}{p_k}} \right) \min \left\{ h_j^{(l,0)}(x), h_j^{(l,1)}(x) \right\} \prod_{k=r+1}^m b_k^J(x) \end{aligned} \quad (3.56)$$

for all $1 \leq l \leq r$. The inequalities in (3.56) imply that

$$\begin{aligned} &|g_j(x)| \\ &\leq 2^{jrn} \prod_{k=1}^r |Q_k|^{1-\frac{1}{p_k}} \prod_{k=1}^r \langle 2^j(x - c_k) \rangle^{-s_k} \min_{1 \leq l \leq r} \left\{ h_j^{(l,0)}(x), h_j^{(l,1)}(x) \right\} \prod_{k=r+1}^m b_k^J(x) \end{aligned} \quad (3.57)$$

for all $x \in E_J$.

Now we need to construct functions u_j^k ($1 \leq k \leq r$) such that

$$|g_j(x)| \lesssim A \prod_{k=1}^r u_j^k(x) \prod_{k=r+1}^m b_k^J(x) \quad (3.58)$$

for all $x \in E_J$ and that

$$\left\| \sum_j u_j^k \right\|_{L^{p_k}} \lesssim 1, \quad \forall 1 \leq k \leq r. \quad (3.59)$$

Note that

$$\begin{aligned} |G_\sigma^2(a_1, \dots, a_m)(x)| &= \left(\sum_{j \in \mathbb{Z}} |T_{\sigma_j}(a_1, \dots, a_m)(x)|^2 \right)^{1/2} = \left(\sum_{j \in \mathbb{Z}} |g_j(x)|^2 \right)^{1/2} \\ &\leq \sum_{j \in \mathbb{Z}} |g_j(x)| = A \sum_{j \in \mathbb{Z}} \prod_{k=1}^r u_j^k(x) \prod_{k=r+1}^m b_k^J(x). \end{aligned}$$

Then (3.53) follows by taking $b_k^J = \sum_j u_j^k$ when $1 \leq k \leq r$ and $b_k^J = |Q_k|^{-\frac{1}{p_k}} \chi_{Q_k^*}$ when $r+1 \leq k \leq m$.

To proceed (3.58), we can choose $0 < \lambda_k < \min \left\{ \frac{1}{2}, \frac{s_k}{n} - \frac{1}{p_k} + \frac{1}{2} \right\}$ such that

$$\sum_{k=1}^r \lambda_k = \frac{r-1}{2}.$$

This is possible since conditions (3.5) implies that

$$\sum_{k=1}^r \min \left\{ \frac{1}{2}, \frac{s_k}{n} - \frac{1}{p_k} + \frac{1}{2} \right\} > \frac{r-1}{2}.$$

Set $\alpha_k = \frac{1}{p_k} - \frac{1}{2} + \lambda_k$ and $\beta_k = 2(\frac{1}{p_k} - \alpha_k)$. Then we have

$$\sum_{k=1}^r \alpha_k = \sum_{k=1}^r \frac{1}{p_k} - \frac{1}{2},$$

$\beta_k > 0$ and $\beta_1 + \dots + \beta_r = 1$. Now define

$$u_j^k = A^{-\beta_k} 2^{jn} |Q_k|^{1-\frac{1}{p_k}} \langle 2^j(\cdot - c_k) \rangle^{-s_k} \chi_{(Q_k^*)^c} \min \left\{ h_j^{(k,0)}, h_j^{(k,1)} \right\}^{\beta_k}, \quad 1 \leq k \leq r. \quad (3.60)$$

Then, from (3.57), it is easy to see that (3.53) holds for all $x \in E_J$. It remains to

check (3.59).

Since $\frac{1}{p_k} = \alpha_k + \frac{\beta_k}{2}$, setting $\frac{1}{p'_k} = 1 - \frac{1}{p_k}$, Hölder's inequality gives

$$\|u_j^k\|_{L^{p_k}} \leq A^{-\beta_k} 2^{jn} |Q_k|^{\frac{1}{p'_k}} \|\langle 2^j(\cdot - c_k) \rangle^{-s_k} \chi_{(Q_k^*)^c}\|_{L^{\frac{1}{\alpha_k}}} \left\| \min \{h_j^{(k,0)}, h_j^{(k,1)}\}^{\beta_k} \right\|_{L^{\frac{2}{\beta_k}}}$$

for all $1 \leq k \leq r$. Notice that $n < \frac{s_k}{\alpha_k}$, we have

$$\left\| \langle 2^j(\cdot - c_k) \rangle^{-s_k} \chi_{(Q_k^*)^c} \right\|_{L^{1/\alpha_k}} \lesssim 2^{-jn\alpha_k} \min \{1, (2^j \ell(Q_k))^{\alpha_k n - s_k}\}$$

and

$$\begin{aligned} \left\| \left(\min \{h_j^{(k,0)}, h_j^{(k,1)}\} \right)^{\beta_k} \right\|_{L^{2/\beta_k}} &\leq \min \{ \|h_j^{(k,0)}\|_{L^2}^{\beta_k}, \|h_j^{(k,1)}\|_{L^2}^{\beta_k} \} \\ &\lesssim \left(A 2^{-jn/2} \min \{1, (2^j \ell(Q_k))^{N_k}\} \right)^{\beta_k}. \end{aligned}$$

Therefore

$$\begin{aligned} \|u_j^k\|_{L^{p_k}} &\leq 2^{jn} |Q_k|^{1 - \frac{1}{p_k}} 2^{-\frac{jn}{p_k}} \min \{1, (2^j \ell(Q_k))^{\alpha_k n - s_k}\} \min \{1, (2^j \ell(Q_k))^{N_k \beta_k}\} \\ &\lesssim (2^j \ell(Q_k))^{n - \frac{n}{p_k}} \min \{1, (2^j \ell(Q_k))^{\alpha_k n - s_k}\} \min \{1, (2^j \ell(Q_k))^{N_k \beta_k}\}. \end{aligned}$$

This inequality is enough to verify (3.59). Thus (3.53) is established for $J \neq \emptyset$.

Now we turn into the second subcase when $J = \emptyset$. Fix $x \in E_\emptyset = \cap_{k=1}^m Q_k^*$. Since σ is supported in $U_{\kappa, \ell}$, $|\xi_1 + \dots + \xi_m| \approx |\xi_\ell| \approx 2^j$ for all (ξ_1, \dots, ξ_m) in the support of σ_j , where σ_j is the function appeared in the decomposition (3.29) of σ . Choose a smooth function φ such that $\widehat{\varphi}$ is supported in an annulus and

$$\sigma_j(\xi_1, \dots, \xi_m) \widehat{\varphi}(\xi_\ell) = \sigma_j(\xi_1, \dots, \xi_m), \quad \forall (\xi_1, \dots, \xi_m) \neq (0, \dots, 0).$$

Applying Lemma 3.8 to $T_{\sigma_j}(a_1, \dots, a_m)$ with the fact that

$$T_{\sigma_j}(a_1, \dots, a_m)(x) = T_{\sigma_j}(a_1, \dots, a_{\ell-1}, \Delta_j(a_\ell), a_{\ell+1}, \dots, a_m)(x)$$

yields

$$|T_{\sigma_j}(a_1, \dots, a_m)(x)| \leq A d_{1,j}(x) \cdots d_{m,j}(x), \quad \forall x \in E_\emptyset,$$

where $d_{i,j} = M_q(a_i)\chi_{Q_i^*}$ when $i \neq \ell$ and $d_{\ell,j} = M_q(|\Delta_j(a_\ell)|)\chi_{Q_\ell^*}$. Here we used the inequality $(\zeta_j * |f|^q)^{1/q} \lesssim M_q(f)$, which is clear since $sq > n$. Thus

$$G_\sigma^2(a_1, \dots, a_m)(x) = \left(\sum_{j \in \mathbb{Z}} |T_{\sigma_j}(a_1, \dots, a_m)(x)|^2 \right)^{1/2} \leq A b_1^\emptyset(x) \cdots b_m^\emptyset(x), \quad (3.61)$$

for all $x \in E_\emptyset$ with $b_i^\emptyset = M_q(a_i)\chi_{Q_i^*}$ for $i \neq \ell$ and $b_\ell^\emptyset = \left(\sum_{j \in \mathbb{Z}} M_q(|\Delta_j(a_\ell)|)^2 \right)^{1/2} \chi_{Q_\ell^*}$. The L^2 -boundedness of M_q and Hölder's inequality imply that

$$\|b_i^\emptyset\|_{L^{p_i}} \leq \|M_q(a_i)\|_{L^2} |Q_i^*|^{1/p_i - 1/2} \lesssim \|a_i\|_{L^2} |Q_i|^{1/p_i - 1/2} \leq 1, \quad i \neq \ell.$$

Similarly, we can estimate $\|b_\ell^\emptyset\|_{L^{p_\ell}} \leq 1$. Therefore (3.61) yields (3.53) for $J = \emptyset$, which proves the first part of Lemma 3.22.

We now turn into the proof of (3.37) by assuming that σ is supported in some $W_{\kappa,\ell}$. Fix $x \in E_J$ for some $\emptyset \neq J \in \mathcal{P}_m$. Recall $g_j = T_{\sigma_j}(a_1, \dots, a_m)$ as in the previous case. Then the inequality (3.58) still holds for this case, i.e.

$$|g_j(x)| \lesssim A \prod_{k=1}^r u_j^k(x) \prod_{k=r+1}^m b_k^J(x),$$

where u_j^k is defined in (3.60) satisfying (3.59) and $J \neq \emptyset$. Thus

$$\begin{aligned} G_\sigma^1(a_1, \dots, a_m)(x) &= \sum_{j \in \mathbb{Z}} |g_j(x)| \leq A \left(\sum_{j \in \mathbb{Z}} \prod_{k=1}^r u_j^k(x) \right) \prod_{k=r+1}^m b_k^J(x) \\ &\leq A b_1^J(x) \cdots b_m^J(x) \end{aligned} \quad (3.62)$$

where $b_k^J = \sum_j u_j^k$ when $1 \leq k \leq r$ and $b_k^J = |Q_k|^{-\frac{1}{p_k}} \chi_{Q_k^*}$ when $r+1 \leq k \leq m$.

It remains to prove (3.62) for $J = \emptyset$. Since σ is supported in $W_{\kappa,\ell}$, inside the

support of σ_j we have $|\xi_\kappa| \approx |\xi_\ell| \approx 2^j$. Choose another function φ such that such that $\widehat{\varphi}$ is supported in an annulus and

$$\sigma_j(\xi_1, \dots, \xi_m) \widehat{\varphi}(\xi_\kappa) \widehat{\varphi}(\xi_\ell) = \sigma_j(\xi_1, \dots, \xi_m), \quad \forall (\xi_1, \dots, \xi_m) \neq (0, \dots, 0). \quad (3.63)$$

Applying Lemma 3.8 to $T_{\sigma_j}(a_1, \dots, a_m)$ deduces

$$|T_{\sigma_j}(a_1, \dots, a_m)(x)| \leq A d_{1,j}(x) \cdots d_{m,j}(x), \quad \forall x \in E_\emptyset,$$

where $d_{i,j} = M_q(a_i) \chi_{Q_i^*}$ when $i \neq \kappa, \ell$ and $d_{i,j} = M_q(|\Delta_j(a_i)|) \chi_{Q_i^*}$ when $i = \kappa, \ell$. Summing over $j \in \mathbb{Z}$ the above inequality and using the Cauchy-Schwarz inequality yields (3.62) for $J = \emptyset$ with $b_i^\emptyset = M_q(a_i) \chi_{Q_i^*}$ when $i \neq \kappa, \ell$ and

$$b_i^\emptyset = \left(\sum_{j \in \mathbb{Z}} M_q(|\Delta_j(a_i)|)^2 \right)^{1/2} \chi_{Q_i^*}$$

when $i = \kappa, \ell$. The proof of Lemma 3.22 is complete.

3.8.2 Proof of Lemma 3.24

The proof of Lemma 3.24 is very similar to that of 3.22. Indeed, we partition \mathbb{R}^n into disjoint subsets as $\mathbb{R}^n = \bigcup_J E_J$, where J runs over \mathcal{P}_l (the set of all subset of $\{1, \dots, l\}$) and E_J is defined by (3.52).

The argument in the case when $J \neq \emptyset$ for this lemma is nearly identical to the one given in Lemma 3.22. The minor difference is that we use the function

$$g_j(x) = T_{\sigma_j}(a_1, \dots, a_l, f_{l+1}, \dots, f_m)(x).$$

Therefore, we omit the detail computations for the case when $J \neq \emptyset$.

When $J = \emptyset$, the proof is a bit complicated comparing to the previous one. For

all $x \in E_\emptyset$, we will prove that

$$G_\sigma^2(a_1, \dots, a_l, f_{l+1}, \dots, f_m)(x) \leq A b_1^\emptyset(x) \cdots b_l^\emptyset(x) \|f_{l+1}\|_{L^\infty} \cdots \|f_m\|_{L^\infty}, \quad (3.64)$$

if σ is supported in $U_{\kappa, \ell}$ and that

$$G_\sigma^1(a_1, \dots, a_l, f_{l+1}, \dots, f_m)(x) \leq A b_1^\emptyset(x) \cdots b_l^\emptyset(x) \|f_{l+1}\|_{L^\infty} \cdots \|f_m\|_{L^\infty}, \quad (3.65)$$

if σ is supported in $W_{\kappa, \ell}$.

Let first establish (3.64), where the proof is divided into 2 cases by considering whether ℓ belongs to the set $\{1, \dots, l\}$. If $\ell \in \{1, \dots, l\}$, the argument in the proof of Lemma 3.22 works well for this case. Therefore, we just only need to consider the case when $\ell \notin \{1, \dots, l\}$. Without loss of generality, we may assume that $\|f_i\|_{L^\infty} = 1$ for all $l+1 \leq i \leq m$. Applying Lemma 3.8 to $T_{\sigma_j}(a_1, \dots, a_m)$ gives

$$|g_j(x)| = |T_{\sigma_j}(a_1, \dots, a_l, f_{l+1}, \dots, f_m)(x)| \leq A d_{1,j}(x) \cdots d_{l,j}(x), \quad \forall x \in E_\emptyset,$$

where $d_{1,j} = (\zeta_j * |a_1|^q)^{1/q} (\zeta_j * |\Delta_j f_\ell|^q)^{1/q} \chi_{Q_1^*}$ and $d_{i,j} = M_q(a_i) \chi_{Q_i^*}$ when $1 < i \leq l$. Square and sum over j the above inequality to deduce (3.64) with

$$b_1^\emptyset = \left(\sum_{j \in \mathbb{Z}} (\zeta_j * |a_1|^q)^{2/q} (\zeta_j * |\Delta_j f_\ell|^q)^{2/q} \right)^{1/2} \chi_{Q_1^*},$$

and $b_i^\emptyset = M_q(a_i) \chi_{Q_i^*}$ for $1 < i \leq l$. By (3.9), we can see that $\|b_1^\emptyset\|_{L^{p_1}} \lesssim 1$, which completes the proof of (3.64).

To prove (3.65), we consider 3 cases: (A) $\kappa, \ell \in \{1, \dots, l\}$; (B) only one of κ or ℓ belongs to $\{1, \dots, l\}$; (C) neither κ nor ℓ belongs to $\{1, \dots, l\}$. The first two cases (A) and (B) can be proceeded as in the proof for (3.64) above. For the case (C),

(3.65) is also valid for the following choice of functions

$$b_1^\emptyset = \left(\sum_{j \in \mathbb{Z}} (\zeta_j * |a_1|^q)^{2/q} (\zeta_j * |\Delta_j f_\kappa|^q)^{2/q} \right)^{1/2} \chi_{Q_1^*},$$

$$b_2^\emptyset = \left(\sum_{j \in \mathbb{Z}} (\zeta_j * |a_2|^q)^{2/q} (\zeta_j * |\Delta_j f_\ell|^q)^{2/q} \right)^{1/2} \chi_{Q_2^*},$$

and $b_i^\emptyset = M_q(a_i) \chi_{Q_i^*}$ for $2 < i \leq l$. This completes the proof of the lemma.

Remark 3.29. In the proof for (3.65) we actually required $l \geq 2$. However, the above argument still works for $l = 1$ with $b_1^\emptyset = A^{-1} |T_\sigma(a_1, f_2, \dots, f_m)| \chi_{Q_1^*}$.

3.8.3 Proof of Lemma 3.26

We use the following notations:

$$\text{I} = \{1, \dots, l\}, \quad \text{II} = \{l+1, \dots, \rho\}, \quad \text{III} = \{\rho+1, \dots, m\}, \quad \Lambda = \{1, \dots, m\}.$$

Recall that we are assuming $\text{I} \neq \emptyset$ and $\text{II} \neq \emptyset$ (the set III might be empty). For a subset $B = \{i_1, \dots, i_k\}$ of $\{1, \dots, m\}$, we write $y_B = (y_{i_1}, \dots, y_{i_k})$ and $dy_B = dy_{i_1} \cdots dy_{i_k}$.

Let a_i ($i \in \text{I}$) and f_i ($i \in \text{II} \cup \text{III}$) be functions as mentioned in the lemma. Without loss of generality, we may assume $\|f_i\|_{L^\infty} = 1$ for $i \in \text{III}$. We use the decomposition (3.29) and write

$$g = T_\sigma(a_1, \dots, a_l, f_{l+1}, \dots, f_m) = \sum_{j \in \mathbb{Z}} g_j,$$

where $g_j = T_{\sigma_j}(a_1, \dots, a_l, f_{l+1}, \dots, f_m)$.

To prove the pointwise estimate (3.46) with noting that σ is supported in $W_{\kappa, \ell}$, we divide \mathbb{R}^n as $\mathbb{R}^n = \bigcup_{J \subset \text{I}} E_J$, where J runs over all subsets of I and E_J is defined by (3.52). In order to prove (3.46), it is sufficient to construct functions b_i^J ($i \in \text{I}$)

and F_i^J ($i \in \text{II}$), for each $J \subset \text{I}$, such that

$$|g(x)| \chi_{E_J}(x) \lesssim b_1^J(x) \dots b_l^J(x) F_{l+1}^J(x) \dots F_\rho^J(x), \quad (3.66)$$

where the function b_i^J depends only on $m, n, (s_i)_{i \in \Lambda}, (p_i)_{i \in \Lambda}, \sigma, J, i, a_i$, and $(f_i)_{i \in \text{III}}$; the function F_i^J depends only on $m, n, (s_i)_{i \in \Lambda}, J, i, f_i$, and $(f_i)_{i \in \text{III}}$; and they satisfy the estimates

$$\|b_i^J\|_{L^{p_i}} \lesssim 1, \quad (3.67)$$

$$\|F_i^J\|_{L^2} \lesssim \|f_i\|_{L^2}. \quad (3.68)$$

In fact, if this is proved, then the desired functions can be obtained by $b_i = \sum_{J \subset \text{I}} b_i^J$ and $F_i = \sum_{J \subset \text{I}} F_i^J$.

First, we shall prove the estimate (3.66) for $J = \emptyset$, $E_\emptyset = Q_1^* \cap \dots \cap Q_l^*$. The argument to be given below will show the estimate (3.66) with some combination of the following choices of b_i^\emptyset and F_i^\emptyset :

$$b_i^\emptyset(x) = M_q(a_i)(x) \chi_{Q_i^*}(x), \quad (3.69)$$

$$b_i^\emptyset(x) = \left(\sum_{j \in \mathbb{Z}} M_q(\Delta_j a_i)(x)^2 \right)^{1/2} \chi_{Q_i^*}(x), \quad (3.70)$$

$$b_i^\emptyset(x) = \left(\sum_{j \in \mathbb{Z}} (\zeta_j * |a_i|^q)(x)^{2/q} (\zeta_j * |\Delta_j f_k|^q)(x)^{2/q} \right)^{1/2} \chi_{Q_i^*}(x), \quad k \in \text{III}, \quad (3.71)$$

$$F_i^\emptyset(x) = M_q(f_i)(x), \quad (3.72)$$

$$F_i^\emptyset(x) = \left(\sum_{j \in \mathbb{Z}} M_q(\Delta_j f_i)(x)^2 \right)^{1/2}, \quad (3.73)$$

$$F_i^\emptyset(x) = \left(\sum_{j \in \mathbb{Z}} (\zeta_j * |f_i|^q)(x)^{2/q} (\zeta_j * |\Delta_j f_k|^q)(x)^{2/q} \right)^{1/2}, \quad k \in \text{III}, \quad (3.74)$$

where $\zeta_j(x) = 2^{jn}(1 + |2^j x|)^{-sq}$ is the function in Lemma 3.8 and M_q is defined in

(3.51). The above functions b_i^\emptyset and F_i^\emptyset depend on other things as mentioned in the lemma. We shall see that they also satisfy the estimates (3.67) and (3.68). For F_i^\emptyset given by (3.72) or (3.73), the L^2 -boundedness of M_q , $q < 2$, and Lemma 3.9 (3.8) give the L^2 -estimate (3.68). For F_i^\emptyset given by (3.74), Lemma 3.9 (3.9) yields the same L^2 -estimate since $\|f_k\|_{BMO} \lesssim \|f_k\|_{L^\infty} = 1$ for $k \in \text{III}$. For b_i^\emptyset given by (3.69), the L^2 -estimate $\|M_q(a_i)\|_{L^2} \lesssim \|a_i\|_{L^2}$ and Hölder's inequality give the estimate (3.67):

$$\|b_i^\emptyset\|_{L^{p_i}} \leq \|M_q(a_i)\|_{L^2} |Q_i^*|^{1/p_i-1/2} \lesssim \|a_i\|_{L^2} |Q_i|^{1/p_i-1/2} \leq 1.$$

For b_i^\emptyset given by (3.70) or (3.71), the same estimate is proved in a similar way.

We divide the proof of (3.66) for $J = \emptyset$ into the following six cases, (1)–(6), depending on the indices κ and ℓ involved in the definition of $W_{\kappa,\ell}$.

(1) $\kappa, \ell \in \text{I}$. In this case, without loss of generality, we assume $\{\kappa, \ell\} = \{1, 2\} \subset \text{I}$. Choose the function φ satisfying (3.63) on the support of σ_j . We write

$$g_j = T_{\sigma_j}(\Delta_j a_1, \Delta_j a_2, a_3, \dots, a_l, f_{l+1}, \dots, f_\rho, \dots, f_m).$$

By Lemma 3.8, we have the pointwise estimate

$$\begin{aligned} |g_j| &\lesssim (\zeta_j * |\Delta_j a_1|^q)^{1/q} (\zeta_j * |\Delta_j a_2|^q)^{1/q} (\zeta_j * |a_3|^q)^{1/q} \cdots (\zeta_j * |a_l|^q)^{1/q} \\ &\quad \times (\zeta_j * |f_{l+1}|^q)^{1/q} \cdots (\zeta_j * |f_\rho|^q)^{1/q} \cdots (\zeta_j * |f_m|^q)^{1/q} \\ &\lesssim M_q(\Delta_j a_1) M_q(\Delta_j a_2) M_q(a_3) \cdots M_q(a_l) M_q(f_{l+1}) \cdots M_q(f_\rho), \end{aligned}$$

where $M_q(f)$ is the maximal operator in (3.51). Summing over $j \in \mathbb{Z}$ and using the Cauchy-Schwarz inequality, we obtain

$$|g| \lesssim \left(\sum_{j \in \mathbb{Z}} \left\{ M_q(\Delta_j a_1) \right\}^2 \right)^{\frac{1}{2}} \left(\sum_{j \in \mathbb{Z}} \left\{ M_q(\Delta_j a_2) \right\}^2 \right)^{\frac{1}{2}} M_q(a_3) \cdots M_q(a_l) M_q(f_{l+1}) \cdots M_q(f_\rho).$$

This implies (3.66) for $J = \emptyset$ with b_i^\emptyset of (3.70) for $i = 1, 2$, with b_i^\emptyset of (3.69) for $3 \leq i \leq l$, and with F_i^\emptyset of (3.72) for $l + 1 \leq i \leq \rho$.

(2) $\kappa, \ell \in \text{II}$. In this case, we may assume $\{\kappa, \ell\} = \{l + 1, l + 2\} \subset \text{II}$. Then write

$$g_j = T_{\sigma_j}(a_1, \dots, a_l, \Delta_j f_{l+1}, \Delta_j f_{l+2}, f_{l+3}, \dots, f_\rho, \dots, f_m).$$

Hence, by Lemma 3.8,

$$|g_j| \lesssim M_q(a_1) \cdots M_q(a_l) M_q(\Delta_j f_{l+1}) M_q(\Delta_j f_{l+2}) M_q(f_{l+3}) \cdots M_q(f_\rho).$$

Taking sum over $j \in \mathbb{Z}$ and using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |g| &\lesssim M_q(a_1) \cdots M_q(a_l) \left(\sum_{j \in \mathbb{Z}} \left\{ M_q(\Delta_j f_{l+1}) \right\}^2 \right)^{1/2} \left(\sum_{j \in \mathbb{Z}} \left\{ M_q(\Delta_j f_{l+2}) \right\}^2 \right)^{1/2} \\ &\quad \times M_q(f_{l+3}) \cdots M_q(f_\rho). \end{aligned}$$

This implies (3.66) for $J = \emptyset$ with b_i^\emptyset of (3.69) for $1 \leq i \leq l$, with F_i^\emptyset of (3.73) for $i = l + 1, l + 2$, and with F_i^\emptyset of (3.72) for $l + 3 \leq i \leq \rho$.

(3) $\kappa, \ell \in \text{III}$. Without loss of generality, we assume $\{\kappa, \ell\} = \{\rho + 1, \rho + 2\} \subset \text{III}$.

Then g_j can be written as

$$g_j = T_{\sigma_j}(a_1, \dots, a_l, f_{l+1}, \dots, f_\rho, \Delta_j f_{\rho+1}, \Delta_j f_{\rho+2}, f_{\rho+3}, \dots, f_m)$$

and Lemma 3.8 yields

$$\begin{aligned} |g_j| &\lesssim (\zeta_j * |a_1|^q)^{1/q} M_q(a_2) \cdots M_q(a_l) \\ &\quad \times (\zeta_j * |f_{l+1}|^q)^{1/q} M_q(f_{l+2}) \cdots M_q(f_\rho) (\zeta_j * |\Delta_j f_{\rho+1}|^q)^{1/q} (\zeta_j * |\Delta_j f_{\rho+2}|^q)^{1/q}. \end{aligned}$$

Taking sum over $j \in \mathbb{Z}$ and using the Cauchy-Schwarz inequality, we obtain

$$|g| \lesssim \left(\sum_{j \in \mathbb{Z}} (\zeta_j * |a_1|^q)^{2/q} (\zeta_j * |\Delta_j f_{\rho+1}|^q)^{2/q} \right)^{1/2} M_q(a_2) \cdots M_q(a_l) \\ \times \left(\sum_{j \in \mathbb{Z}} (\zeta_j * |f_{l+1}|^q)^{2/q} (\zeta_j * |\Delta_j f_{\rho+2}|^q)^{2/q} \right)^{1/2} M_q(f_{l+2}) \cdots M_q(f_\rho).$$

This implies (3.66) for $J = \emptyset$ with the following functions: b_i^\emptyset is (3.71) with $i = 1$ and $k = \rho + 1$; b_i^\emptyset is (3.69) for $2 \leq i \leq l$; F_{l+1}^\emptyset is (3.74) with $i = l + 1$ and $k = \rho + 2$; and F_i^\emptyset is (3.72) for $l + 2 \leq i \leq \rho$.

(4) $\kappa \in \text{I}$ and $\ell \in \text{II}$. Without loss of generality, we assume $\kappa = 1$ and $\ell = l + 1$.

Then

$$g_j = T_{\sigma_j}(\Delta_j a_1, a_2, \dots, a_l, \Delta_j f_{l+1}, f_{l+2}, \dots, f_\rho, \dots, f_m)$$

and Lemma 3.8 yields

$$|g_j| \lesssim M_q(\Delta_j a_1) M_q(a_2) \cdots M_q(a_l) M_q(\Delta_j f_{l+1}) M_q(f_{l+2}) \cdots M_q(f_\rho).$$

Taking sum over $j \in \mathbb{Z}$ and using the Cauchy-Schwarz inequality, we obtain

$$|g| \lesssim \left(\sum_{j \in \mathbb{Z}} \left\{ M_q(\Delta_j a_1) \right\}^2 \right)^{1/2} M_q(a_2) \cdots M_q(a_l) \\ \times \left(\sum_{j \in \mathbb{Z}} \left\{ M_q(\Delta_j f_{l+1}) \right\}^2 \right)^{1/2} M_q(f_{l+2}) \cdots M_q(f_\rho).$$

This implies (3.66) for $J = \emptyset$ with b_i^\emptyset of (3.70) for $i = 1$, b_i^\emptyset of (3.69) for $2 \leq i \leq l$, with F_i^\emptyset of (3.73) for $i = l + 1$, and with F_i^\emptyset of (3.72) for $l + 2 \leq i \leq \rho$.

(5) $\kappa \in \text{II}$ and $\ell \in \text{III}$. Without loss of generality, we assume $\kappa = l + 1$ and $\ell = \rho + 1$. Then we have

$$g_j = T_{\sigma_j}(a_1, \dots, a_l, \Delta_j f_{l+1}, f_{l+2}, \dots, f_\rho, \Delta_j f_{\rho+1}, f_{\rho+2}, \dots, f_m)$$

and Lemma 3.8 yields

$$|g_j| \lesssim (\zeta_j * |a_1|^q)^{1/q} M_q(a_2) \cdots M_q(a_l) M_q(\Delta_j f_{l+1}) M_q(f_{l+2}) \cdots M_q(f_\rho) (\zeta_j * |\Delta_j f_{\rho+1}|^q)^{1/q}.$$

Taking sum over $j \in \mathbb{Z}$ and using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |g| &\lesssim \left(\sum_{j \in \mathbb{Z}} (\zeta_j * |a_1|^q)^{2/q} (\zeta_j * |\Delta_j f_{\rho+1}|^q)^{2/q} \right)^{1/2} M_q(a_2) \cdots M_q(a_l) \\ &\quad \times \left(\sum_{j \in \mathbb{Z}} \left\{ M_q(\Delta_j f_{l+1}) \right\}^2 \right)^{1/2} M_q(f_{l+2}) \cdots M_q(f_\rho). \end{aligned}$$

This implies (3.66) for $J = \emptyset$ with the following functions: b_i^\emptyset is (3.71) with $i = 1$ and $k = \rho + 1$; b_i^\emptyset is (3.69) for $2 \leq i \leq l$; F_i^\emptyset is (3.73) for $i = l + 1$; and F_i^\emptyset is (3.72) for $l + 2 \leq i \leq \rho$.

(6) $\kappa \in \text{I}$ and $\ell \in \text{III}$. Without loss of generality, we assume $\kappa = 1$ and $\ell = \rho + 1$.

Then g_j can be written as

$$g_j = T_{\sigma_j}(\Delta_j a_1, a_2, \dots, a_l, f_{l+1}, \dots, f_\rho, \Delta_j f_{\rho+1}, f_{\rho+2}, \dots, f_m)$$

and Lemma 3.8 yields

$$\begin{aligned} |g_j| &\lesssim M_q(\Delta_j a_1) M_q(a_2) \cdots M_q(a_l) \\ &\quad \times (\zeta_j * |f_{l+1}|^q)^{1/q} M_q(f_{l+2}) \cdots M_q(f_\rho) (\zeta_j * |\Delta_j f_{\rho+1}|^q)^{1/q}. \end{aligned}$$

Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |g| &\lesssim \left(\sum_{j \in \mathbb{Z}} \left\{ M_q(\Delta_j a_1) \right\}^2 \right)^{1/2} M_q(a_2) \cdots M_q(a_l) \\ &\quad \times \left(\sum_{j \in \mathbb{Z}} (\zeta_j * |f_{l+1}|^q)^{2/q} (\zeta_j * |\Delta_j f_{\rho+1}|^q)^{2/q} \right)^{1/2} M_q(f_{l+2}) \cdots M_q(f_\rho). \end{aligned}$$

This implies (3.66) for $J = \emptyset$ with the following functions: b_i^\emptyset is (3.70) for $i = 1$; b_i^\emptyset is (3.69) for $2 \leq i \leq l$; F_{l+1}^\emptyset is (3.74) with $i = l + 1$ and $k = \rho + 1$; and F_i^\emptyset is (3.72) for $l + 2 \leq i \leq \rho$. Thus we have proved (3.66) for $J = \emptyset$.

Next we shall prove (3.66) for $J \neq \emptyset$. Here we do not need the assumption that σ is supported in $W_{\kappa, \ell}$. We fix a nonempty subset J of I . We shall prove that there exist functions $u_{k,j}^J$, $k \in J$, $j \in \mathbb{Z}$, such that

$$|g_j(x)|\chi_{E_J}(x) \lesssim \prod_{k \in J} u_{k,j}^J(x) \cdot \prod_{i \in I \setminus J} |Q_i|^{-1/p_i} \chi_{Q_i^*}(x) \cdot \prod_{i \in \text{II}} M_q(f_i)(x) \quad (3.75)$$

for all $j \in \mathbb{Z}$ and all $x \in \mathbb{R}^n$; the function $u_{k,j}^J$ depends only on m , n , $(s_i)_{i \in \Lambda}$, $(p_i)_{i \in \Lambda}$, σ , J , k , j , N_k , and Q_k , and satisfies the estimate

$$\|u_{k,j}^J\|_{L^{p_k}} \lesssim \min\{(2^j \ell(Q_k))^{\gamma_k}, (2^j \ell(Q_k))^{-\delta_k}\}, \quad (3.76)$$

where γ_k and δ_k are positive constants that will be given in terms of n , k , J , $(s_i)_{i \in \Lambda}$, $(p_i)_{i \in \Lambda}$, and N_k . If we have these functions $u_{k,j}^J$, then we have (3.66) with the functions

$$\begin{aligned} b_k^J &= \sum_{j \in \mathbb{Z}} u_{k,j}^J \quad \text{for } k \in J, \\ b_i^J &= |Q_i|^{-1/p_i} \chi_{Q_i^*} \quad \text{for } i \in I \setminus J, \\ F_i^J &= M_q(f_i) \quad \text{for } i \in \text{II}. \end{aligned}$$

In fact, b_k^J , $k \in J$, depends only on m , n , $(s_i)_{i \in J}$, $(p_i)_{i \in J}$, σ , J , k , N_k , and Q_k , and the estimate (3.67) follows from (3.76). The estimate (3.67) for b_i^J with $i \in I \setminus J$ is obvious and the estimate (3.68) for F_i^J with $i \in \text{II}$ holds by the L^2 -boundedness of M_q , $q < 2$. Thus it is sufficient to construct the functions $u_{k,j}^J$.

Before we proceed to the construction of $u_{k,j}^J$, we observe that it is sufficient to treat only the case $j = 0$. In fact, if we have (3.75)–(3.76) for $j = 0$, then the case of

general $j \in \mathbb{Z}$ can be derived by the use of the dilation formula

$$T_{\sigma_j}(f_1, \dots, f_m)(x) = T_{\sigma_j(2^j \cdot)}(f_1(2^{-j} \cdot), \dots, f_m(2^{-j} \cdot))(2^j x)$$

and by a simple computation.

Thus we shall consider $g_0(x)$. Using $K_0 = (\sigma_0)^\vee$, we write

$$g_0(x) = \int_{\mathbb{R}^{mn}} K_0(x - y_1, \dots, x - y_m) \prod_{i \in \text{I}} a_i(y_i) \cdot \prod_{i \in \text{II} \cup \text{III}} f_i(y_i) dy_1 \cdots dy_m. \quad (3.77)$$

We write c_i to denote the center of the cube Q_i . Since $|x - y_i| \approx |x - c_i|$ for $x \notin Q_i^*$ and $y_i \in Q_i$, from (3.77) we see that the following inequalities hold for $x \in E_J$:

$$\begin{aligned} & \prod_{i \in J} \langle x - c_i \rangle^{s_i} \cdot |g_0(x)| \\ & \lesssim \int_{\mathbb{R}^{mn}} \prod_{i \in J} \langle x - y_i \rangle^{s_i} \cdot |K_0(x - y_1, \dots, x - y_m)| \prod_{i \in \text{I}} |a_i(y_i)| \cdot \prod_{i \in \text{II} \cup \text{III}} |f_i(y_i)| dy_1 \cdots dy_m \\ & \leq \int_{\mathbb{R}^{mn}} \prod_{i \in J} \langle x - y_i \rangle^{s_i} \cdot |K_0(x - y_1, \dots, x - y_m)| \prod_{i \in \text{I}} |Q_i|^{-1/p_i} \chi_{Q_i}(y_i) \cdot \prod_{i \in \text{II}} |f_i(y_i)| dy_1 \cdots dy_m. \end{aligned}$$

We now fix a $k \in J$ and estimate the last integral as

$$\begin{aligned} & \leq \int_{\mathbb{R}^n} \left\| \prod_{i \in J \cup \text{II}} \langle x - y_i \rangle^{s_i} \cdot K_0(x - y_1, \dots, x - y_m) \right\|_{L^\infty(y_{J \setminus \{k\}}) L^1(y_{\text{I} \setminus J}) L^{q'}(y_{\text{II}}) L^1(y_{\text{III}})} \\ & \quad \times \left\| \prod_{i \in J} |Q_i|^{-1/p_i} \chi_{Q_i}(y_i) \right\|_{L^1(y_{J \setminus \{k\}})} \left\| \prod_{i \in \text{I} \setminus J} |Q_i|^{-1/p_i} \chi_{Q_i}(y_i) \right\|_{L^\infty(y_{\text{I} \setminus J})} \\ & \quad \times \left\| \prod_{i \in \text{II}} \langle x - y_i \rangle^{-s_i} f_i(y_i) \right\|_{L^q(y_{\text{II}})} dy_k, \end{aligned}$$

where we used the following notation for mixed norm and its obvious generalization:

$$\|F(z_1, z_2)\|_{L^p(z_1) L^q(z_2)} = \left[\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |F(z_1, z_2)|^p dz_1 \right)^{q/p} dz_2 \right]^{1/q}.$$

Recall that the mixed norms satisfy

$$\|F(z_1, z_2)\|_{L^p(z_1)L^q(z_2)} \leq \|F(z_1, z_2)\|_{L^q(z_2)L^p(z_1)} \quad \text{if } p < q. \quad (3.78)$$

Since $s_i > n/2$, the Cauchy-Schwarz inequality gives

$$\|F(x - y_1, \dots, x - y_m)\|_{L^1(y_B)} \lesssim \left\| F(x - y_1, \dots, x - y_m) \prod_{i \in B} \langle x - y_i \rangle^{s_i} \right\|_{L^2(y_B)}. \quad (3.79)$$

Now repeated applications of (3.78), (3.79), and Lemma 3.4 yield

$$\begin{aligned} & \left\| \prod_{i \in J \cup \text{II}} \langle x - y_i \rangle^{s_i} \cdot K_0(x - y_1, \dots, x - y_m) \right\|_{L^\infty(y_{J \setminus \{k\}})L^1(y_{\text{I} \setminus J})L^{q'}(y_{\text{II}})L^1(y_{\text{III}})} \\ & \lesssim \left\| \prod_{i \in \Lambda} \langle x - y_i \rangle^{s_i} \cdot K_0(x - y_1, \dots, x - y_m) \right\|_{L^\infty(y_{J \setminus \{k\}})L^2(y_{\text{I} \setminus J})L^{q'}(y_{\text{II}})L^2(y_{\text{III}})} \\ & \lesssim \left\| \prod_{i \in \Lambda} \langle x - y_i \rangle^{s_i} \cdot K_0(x - y_1, \dots, x - y_m) \right\|_{L^2(y_{\Lambda \setminus \{k\}})} \\ & = \left\| \langle x - y_k \rangle^{s_k} \prod_{i \in \Lambda \setminus \{k\}} \langle z_i \rangle^{s_i} \cdot K_0(z_1, \dots, x - y_k, \dots, z_m) \right\|_{L^2(z_{\Lambda \setminus \{k\}})}. \end{aligned}$$

Since $s_i q > n$ by our choice of q , we have

$$\left\| \prod_{i \in \text{II}} \langle x - y_i \rangle^{-s_i} f_i(y_i) \right\|_{L^q(y_{\text{II}})} \lesssim \prod_{i \in \text{II}} M_q(f_i)(x).$$

Combining the above inequalities, we obtain the following estimate for $x \in E_J$:

$$\prod_{i \in J} \langle x - c_i \rangle^{s_i} \cdot |g_0(x)| \lesssim h^{(k,0)}(x) \prod_{i \in J} |Q_i|^{-1/p_i+1} \cdot \prod_{i \in \text{I} \setminus J} |Q_i|^{-1/p_i} \cdot \prod_{i \in \text{II}} M_q(f_i)(x), \quad (3.80)$$

where

$$h^{(k,0)}(x) = |Q_k|^{-1} \int_{Q_k} \left\| \langle x - y_k \rangle^{s_k} \prod_{i \in \Lambda \setminus \{k\}} \langle z_i \rangle^{s_i} \cdot K_0(z_1, \dots, x - y_k, \dots, z_m) \right\|_{L^2(z_{\Lambda \setminus \{k\}})} dy_k.$$

We have

$$\begin{aligned}
& \|h^{(k,0)}\|_{L^2(\mathbb{R}^n)} \\
& \leq |Q_k|^{-1} \int_{Q_k} \left\| \langle x - y_k \rangle^{s_k} \prod_{i \in \Lambda \setminus \{k\}} \langle z_i \rangle^{s_i} \cdot K_0(z_1, \dots, x - y_k, \dots, z_m) \right\|_{L^2(z_{\Lambda \setminus \{k\}}) L^2(x)} dy_k \\
& = \left\| \prod_{i \in \Lambda} \langle z_i \rangle^{s_i} \cdot K_0(z_1, \dots, z_m) \right\|_{L^2(z_\Lambda)} = \|\sigma_0\|_{W^{(s_1, \dots, s_m)}}.
\end{aligned}$$

Thus, by the assumption (3.41),

$$\|h^{(k,0)}\|_{L^2(\mathbb{R}^n)} \leq 1. \quad (3.81)$$

On the other hand, using the vanishing moment condition of a_k and Taylor's formula, we can write $g_0(x)$ as

$$\begin{aligned}
g_0(x) &= \sum_{|\alpha|=N_k} C_\alpha \int_{\mathbb{R}^{mn}} \left\{ \int_0^1 (1-t)^{N_k-1} \right. \\
& \quad \times \partial_k^\alpha K_0(x - y_1, \dots, x_{c_k, y_k}^t, \dots, x - y_m) \\
& \quad \left. \times (y_k - c_k)^\alpha a_1(y_1) \cdots a_l(y_l) f_{l+1}(y_{l+1}) \cdots f_m(y_m) dt \right\} dy_1 \cdots dy_m,
\end{aligned}$$

where $\partial_k^\alpha K_0(z_1, \dots, z_m) = \partial_{z_k}^\alpha K_0(z_1, \dots, z_m)$ and $x_{c_k, y_k}^t = x - c_k - t(y_k - c_k)$. Hence the following inequality holds for $x \in E_J$:

$$\begin{aligned}
\prod_{i \in J} \langle x - c_i \rangle^{s_i} \cdot |g_0(x)| &\lesssim \sum_{|\alpha|=N_k} \int_{\mathbb{R}^{mn}} \left\{ \int_0^1 \langle x_{c_k, y_k}^t \rangle^{s_k} \prod_{i \in J \setminus \{k\}} \langle x - y_i \rangle^{s_i} \right. \\
& \quad \times \left| \partial_k^\alpha K_0(x - y_1, \dots, x_{c_k, y_k}^t, \dots, x - y_m) \right| \\
& \quad \left. \times \ell(Q_k)^{N_k} \prod_{i \in I} |Q_i|^{-1/p_i} \chi_{Q_i}(y_i) \cdot \prod_{i \in \Pi} |f_i(y_i)| dt \right\} dy_1 \cdots dy_m.
\end{aligned}$$

Using this inequality and arguing in the same way as before, we obtain the following

estimate for $x \in E_J$:

$$\prod_{i \in J} \langle x - c_i \rangle^{s_i} \cdot |g_0(x)| \lesssim h^{(k,1)}(x) \prod_{i \in J} |Q_i|^{-1/p_i+1} \cdot \prod_{i \in I \setminus J} |Q_i|^{-1/p_i} \cdot \prod_{i \in \Pi} M_q(f_i)(x), \quad (3.82)$$

where

$$\begin{aligned} h^{(k,1)}(x) &= |Q_k|^{-1+N_k/n} \sum_{|\alpha|=N_k} \int_{\substack{0 < t \leq 1 \\ y_k \in Q_k}} \\ &\times \left\| \langle x_{c_k, y_k}^t \rangle^{s_k} \prod_{i \in \Lambda \setminus \{k\}} \langle z_i \rangle^{s_i} \cdot \partial_k^\alpha K_0(z_1, \dots, x_{c_k, y_k}^t, \dots, z_m) \right\|_{L^2(z_{\Lambda \setminus \{k\}})} dt dy_k. \end{aligned}$$

Using Lemma 3.4, we obtain

$$\|h^{(k,1)}\|_{L^2(\mathbb{R}^n)} \lesssim |Q_k|^{N_k/n}. \quad (3.83)$$

From two estimates (3.80) and (3.82), we have

$$|g_0(x)| \lesssim \prod_{i \in J} \langle x - c_i \rangle^{-s_i} |Q_i|^{-1/p_i+1} \cdot \prod_{i \in I \setminus J} |Q_i|^{-1/p_i} \cdot \prod_{i \in \Pi} M_q(f_i)(x) \times \min\{h^{(k,0)}(x), h^{(k,1)}(x)\}$$

for all $x \in E_J$ and for each $k \in J$. We take positive numbers $(\beta_k)_{k \in J}$ satisfying $\sum_{k \in J} \beta_k = 1$ and take a geometric mean of the above estimates to obtain

$$|g_0(x)| \chi_{E_J}(x) \lesssim \prod_{k \in J} u_k^J(x) \cdot \prod_{i \in I \setminus J} |Q_i|^{-1/p_i} \chi_{Q_i^*}(x) \cdot \prod_{i \in \Pi} M_q(f_i)(x),$$

where

$$u_k^J(x) = \langle x - c_k \rangle^{-s_k} |Q_k|^{-1/p_k+1} \chi_{(Q_k^*)^c}(x) \left(\min\{h^{(k,0)}(x), h^{(k,1)}(x)\} \right)^{\beta_k}.$$

We choose β_k , $k \in J$, so that we have

$$\beta_k > 0, \quad \frac{s_k}{n} > \frac{1}{p_k} - \frac{\beta_k}{2}, \quad \sum_{k \in J} \beta_k = 1.$$

This is possible since $\frac{1}{2} > \sum_{k \in J} \max\{0, 1/p_k - s_k/n\}$ by virtue of our condition (3.5).

If we write $1/p_k - \beta_k/2 = 1/r_k$, then $r_k > 0$ and Hölder's inequality gives

$$\begin{aligned} \|u_k^J\|_{L^{p_k}} &\leq \left\| \langle x - c_k \rangle^{-s_k} |Q_k|^{-1/p_k+1} \chi_{(Q_k^*)^c}(x) \right\|_{L^{r_k}} \\ &\quad \times \left\| (\min\{h^{(k,0)}(x), h^{(k,1)}(x)\})^{\beta_k} \right\|_{L^{2/\beta_k}}. \end{aligned}$$

Since $s_k r_k > n$, we have

$$\left\| \langle x - c_k \rangle^{-s_k} |Q_k|^{-1/p_k+1} \chi_{(Q_k^*)^c}(x) \right\|_{L^{r_k}} \approx \begin{cases} |Q_k|^{-1/p_k+1} & \text{if } |Q_k| \leq 1 \\ |Q_k|^{-1/p_k+1-s_k/n+1/r_k} & \text{if } |Q_k| > 1. \end{cases}$$

By (3.81) and (3.83), we have

$$\begin{aligned} \left\| (\min\{h^{(k,0)}(x), h^{(k,1)}(x)\})^{\beta_k} \right\|_{L^{2/\beta_k}} &\leq \min \left\{ \|h^{(k,0)}\|_{L^2}^{\beta_k}, \|h^{(k,1)}\|_{L^2}^{\beta_k} \right\} \\ &\lesssim \begin{cases} |Q_k|^{N_k \beta_k / n} & \text{if } |Q_k| \leq 1 \\ 1 & \text{if } |Q_k| > 1. \end{cases} \end{aligned}$$

Thus

$$\|u_k^J\|_{L^{p_k}} \lesssim \begin{cases} |Q_k|^{N_k \beta_k / n - 1/p_k + 1} & \text{if } |Q_k| \leq 1 \\ |Q_k|^{-1/p_k + 1 - s_k/n + 1/r_k} & \text{if } |Q_k| > 1, \end{cases}$$

which implies (3.76) for $j = 0$ with $\gamma_k = N_k \beta_k - n/p_k + n$ and $\delta_k = n/p_k - n + s_k - n/r_k$.

We have $\gamma_k > 0$ since N_k is sufficiently large and $\delta_k > 0$ since $\delta_k = n\beta_k/2 - n + s_k \geq n\beta_k/2 - n/p_k + s_k > 0$ by our choice of β_k . This completes the proof (3.65).

Now we turn into (3.64). Since the proof of (3.64) is similar to that of (3.65), we

shall briefly indicate only the key points. We use the same notation as above. Define

$$G(x) := G_\sigma^2(a_1, \dots, a_l, f_{l+1}, \dots, f_m)(x) = \left(\sum_{j \in \mathbb{Z}} |g_j(x)|^2 \right)^{1/2}.$$

It is sufficient to prove the estimate

$$G(x)\chi_{E_J}(x) \lesssim \prod_{i \in I} b_i^J(x) \cdot \prod_{i \in \Pi} F_i(x) \quad (3.84)$$

for each subset J of I , where b_i^J and F_i have the same properties as in (3.66).

First we consider the case $J = \emptyset$, $E_\emptyset = Q_1^* \cap \dots \cap Q_l^*$. We divide the proof into the following three cases, (1)–(3), depending on the index κ, ℓ in the definition of $U_{\kappa, \ell}$.

(1) $\ell \in I$. Without loss of generality, we assume $\ell = 1$. We can write

$$g_j = T_{\sigma_j}(\Delta_j a_1, a_2, \dots, a_l, f_{l+1}, \dots, f_\rho, \dots, f_m).$$

By Lemma 3.8, we have

$$|g_j| \lesssim M_q(\Delta_j a_1) M_q(a_2) \cdots M_q(a_l) M_q(f_{l+1}) \cdots M_q(f_\rho).$$

Hence

$$G \lesssim \left(\sum_{j \in \mathbb{Z}} \left\{ M_q(\Delta_j a_1) \right\}^2 \right)^{1/2} M_q(a_2) \cdots M_q(a_l) M_q(f_{l+1}) \cdots M_q(f_\rho).$$

Thus we obtain (3.84) for $J = \emptyset$ with

$$b_1^\emptyset = \left(\sum_{j \in \mathbb{Z}} \left\{ M_q(\Delta_j a_1) \right\}^2 \right)^{1/2} \chi_{Q_1^*},$$

$$b_i^\emptyset = M_q(a_i) \chi_{Q_i^*} \quad \text{for } 2 \leq i \leq l,$$

$$F_i^\emptyset = M_q(f_i) \quad \text{for } l+1 \leq i \leq \rho.$$

(2) $\ell \in \text{II}$. Without loss of generality, we assume $\ell = l + 1$. We can write

$$g_j = T_{\sigma_j}(a_1, \dots, a_l, \Delta_j f_{l+1}, f_{l+2}, \dots, f_\rho, \dots, f_m).$$

By Lemma 3.8, we have

$$|g_j| \lesssim M_q(a_1) \cdots M_q(a_l) M_q(\Delta_j f_{l+1}) M_q(f_{l+2}) \cdots M_q(f_\rho).$$

Hence

$$G \lesssim M_q(a_1) \cdots M_q(a_l) \left(\sum_{j \in \mathbb{Z}} \left\{ M_q(\Delta_j f_{l+1}) \right\}^2 \right)^{1/2} M_q(f_{l+2}) \cdots M_q(f_\rho).$$

Thus we obtain (3.84) for $J = \emptyset$ with

$$\begin{aligned} b_i^\emptyset &= M_q(a_i) \chi_{Q_i^*} \quad \text{for } 1 \leq i \leq l, \\ F_{l+1}^\emptyset &= \left(\sum_{j \in \mathbb{Z}} \left\{ M_q(\Delta_j f_{l+1}) \right\}^2 \right)^{1/2}, \\ F_i^\emptyset &= M_q(f_i) \quad \text{for } l+2 \leq i \leq \rho. \end{aligned}$$

(3) $\ell \in \text{III}$. Without loss of generality, we assume $\ell = \rho + 1$. We can write

$$g_j = T_{\sigma_j}(a_1, \dots, a_l, f_{l+1}, \dots, f_\rho, \Delta_j f_{\rho+1}, \dots, f_m).$$

Lemma 3.8 yields

$$|g_j| \lesssim (\zeta_j * |a_1|^q)^{1/q} M_q(a_2) \cdots M_q(a_l) M_q(f_{l+1}) \cdots M_q(f_\rho) (\zeta_j * |\Delta_j f_{\rho+1}|^q)^{1/q}.$$

Hence

$$G \lesssim \left(\sum_{j \in \mathbb{Z}} (\zeta_j * |a_1|^q)^{2/q} (\zeta_j * |\Delta_j f_{\rho+1}|^q)^{2/q} \right)^{1/2} M_q(a_2) \cdots M_q(a_l) M_q(f_{l+1}) \cdots M_q(f_\rho).$$

Thus we obtain (3.84) for $J = \emptyset$ with

$$\begin{aligned} b_1^\emptyset &= \left(\sum_{j \in \mathbb{Z}} (\zeta_j * |a_1|^q)^{2/q} (\zeta_j * |\Delta_j f_{\rho+1}|^q)^{2/q} \right)^{1/2} \chi_{Q_1^*}, \\ b_i^\emptyset &= M_q(a_i) \chi_{Q_i^*} \quad \text{for } 2 \leq i \leq l, \\ F_i^\emptyset &= M_q(f_i) \quad \text{for } l+1 \leq i \leq \rho. \end{aligned}$$

Finally we prove (3.84) for $J \neq \emptyset$. The proof is immediate. Observe that the estimate of $g_j(x)$ on E_J , $J \neq \emptyset$, given in the the proof of (3.65) holds in the present case as well, since we did not use the restriction on the support of σ in that argument. Also observe that there we have actually proved the estimate

$$\sum_{j \in \mathbb{Z}} |g_j(x)| \chi_{E_J}(x) \lesssim b_1^J(x) \cdots b_l^J(x) F_{l+1}^J(x) \cdots F_\rho^J(x)$$

for $J \neq \emptyset$. Thus the estimate (3.84) for $J \neq \emptyset$ also holds since

$$G(x) = \left(\sum_{j \in \mathbb{Z}} |g_j(x)|^2 \right)^{1/2} \leq \sum_{j \in \mathbb{Z}} |g_j(x)|.$$

This completes the proof of the lemma.

Appendix A

Interpolation illustration in \mathbb{R}^3

A.1 The smoothness condition in the unit cube

In this section, we will provide some pictures to visualize how the interpolation works in the proof of Theorem 3.1 and the appropriate smoothness needed for each region inside the unit cube in \mathbb{R}^3 . We consider the following unit cube

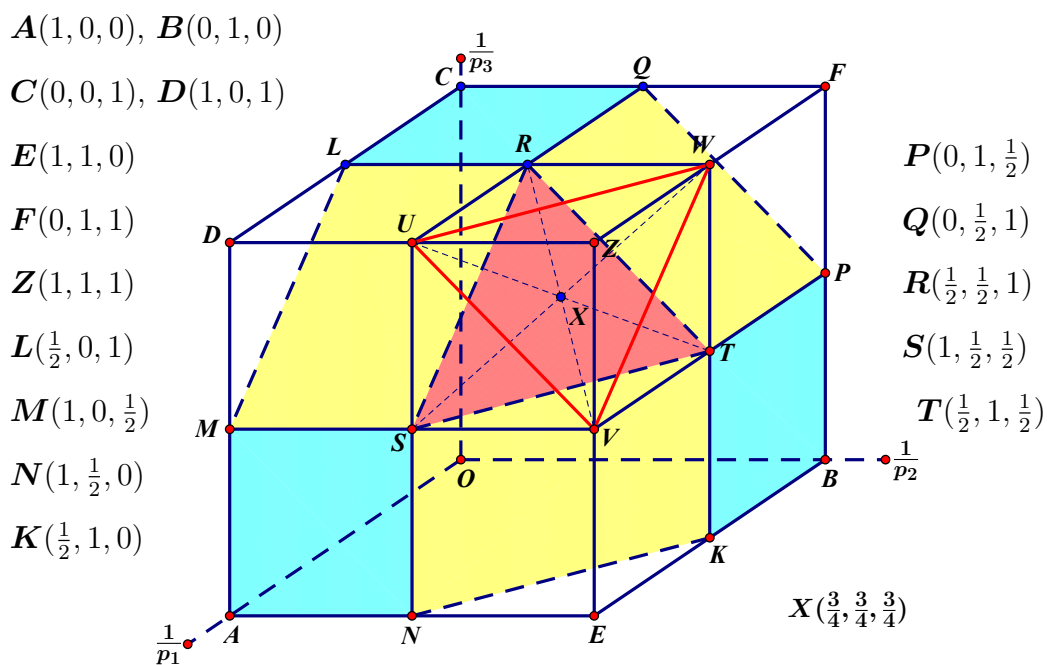


Figure A.1: The unit cube

A.1.1 Near the origin

In this case, we just only need the following smoothness

$$s_1 > \frac{n}{2}, \quad s_2 > \frac{n}{2}, \quad s_3 > \frac{n}{2}.$$

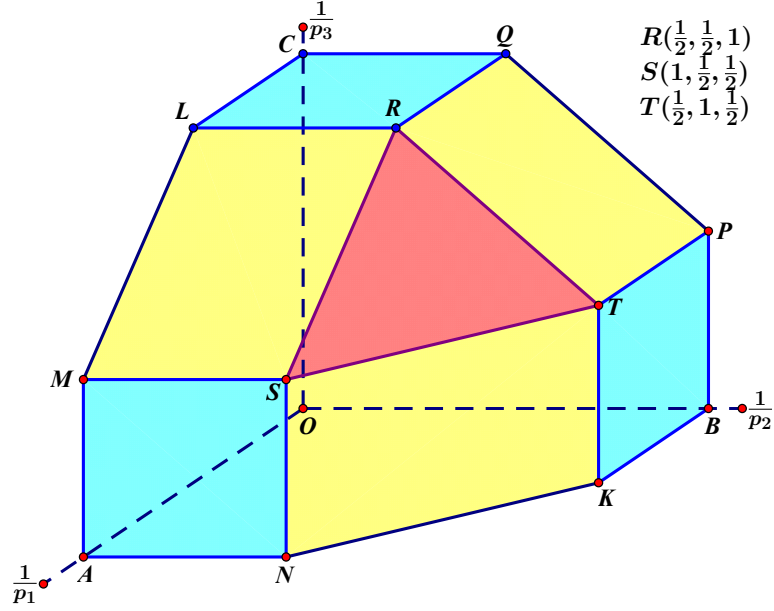


Figure A.2: Region near the origin

A.1.2 Three wedges

In this case, the smoothness depends on each region. For example, in the polyhedron *DMLUSR*, we need the following conditions on the indices

$$s_1 > \frac{n}{2}, \quad s_2 > \frac{n}{2}, \quad s_3 > \frac{n}{2}, \quad \frac{s_1}{n} + \frac{s_3}{n} > \frac{1}{p_1} + \frac{1}{p_3} - \frac{1}{2}.$$

In the prism *WTRFLQ* we require

$$s_1 > \frac{n}{2}, \quad s_2 > \frac{n}{2}, \quad s_3 > \frac{n}{2}, \quad \frac{s_2}{n} + \frac{s_3}{n} > \frac{1}{p_2} + \frac{1}{p_3} - \frac{1}{2}.$$

In the polyhedron $VSTENK$, we need

$$s_1 > \frac{n}{2}, \quad s_2 > \frac{n}{2}, \quad s_3 > \frac{n}{2}, \quad \frac{s_1}{n} + \frac{s_2}{n} > \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{2}.$$

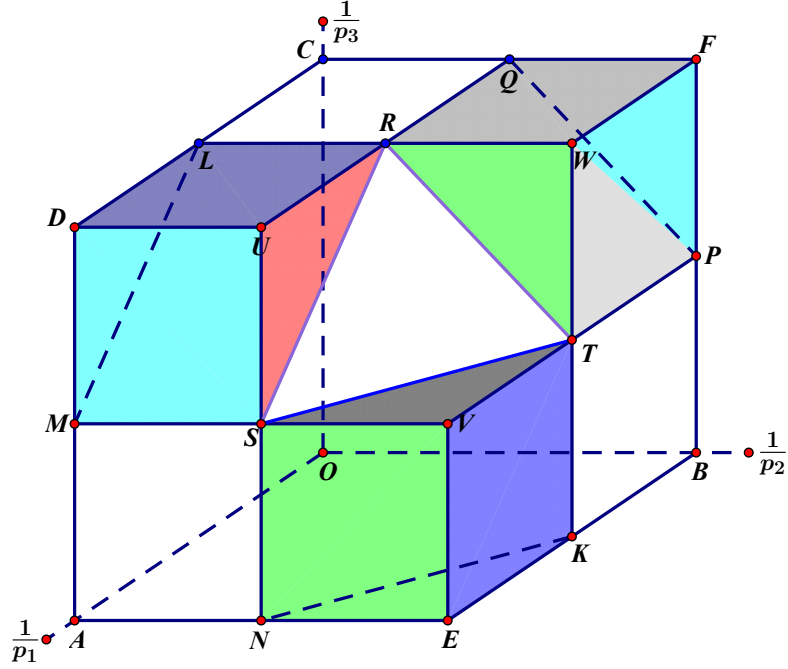


Figure A.3: The wedges

A.1.3 The lower local tetrahedron

Now we are looking for the smooth condition for which the operator T_σ is bounded on the tetrahedron $XSTR$. In this case, we need the following condition on the indices

$$s_1 > \frac{n}{2}, \quad s_2 > \frac{n}{2}, \quad s_3 > \frac{n}{2}, \quad \frac{s_1}{n} + \frac{s_2}{n} + \frac{s_3}{n} > \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} - \frac{1}{2}.$$

Note that the vertex of this tetrahedron is $X(\frac{3}{4}, \frac{3}{4}, \frac{3}{4})$ and its base is the triangle formed by three points $R(\frac{1}{2}, \frac{1}{2}, 1)$, $S(1, \frac{1}{2}, \frac{1}{2})$, and $T(\frac{1}{2}, 1, \frac{1}{2})$.

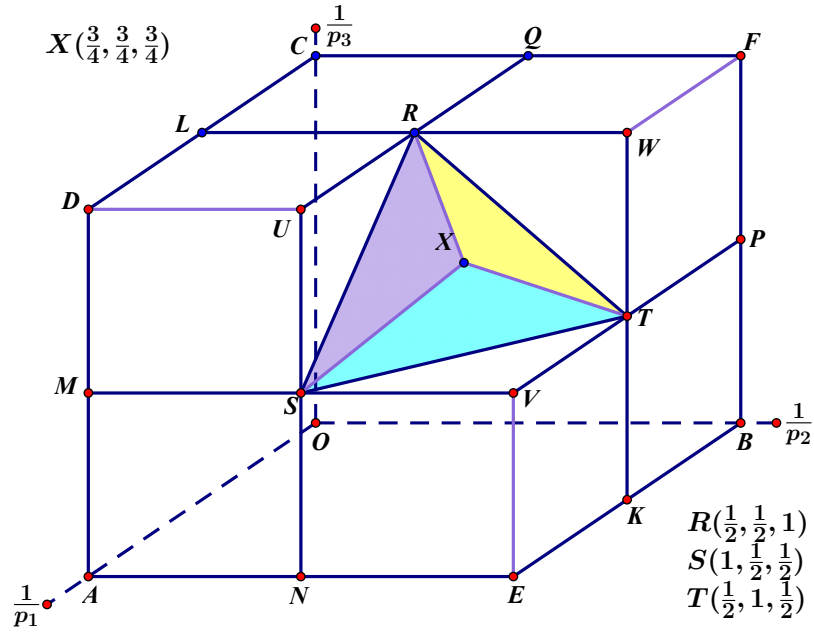


Figure A.4: The lower local tetrahedron

A.1.4 Three side local tetrahedrons

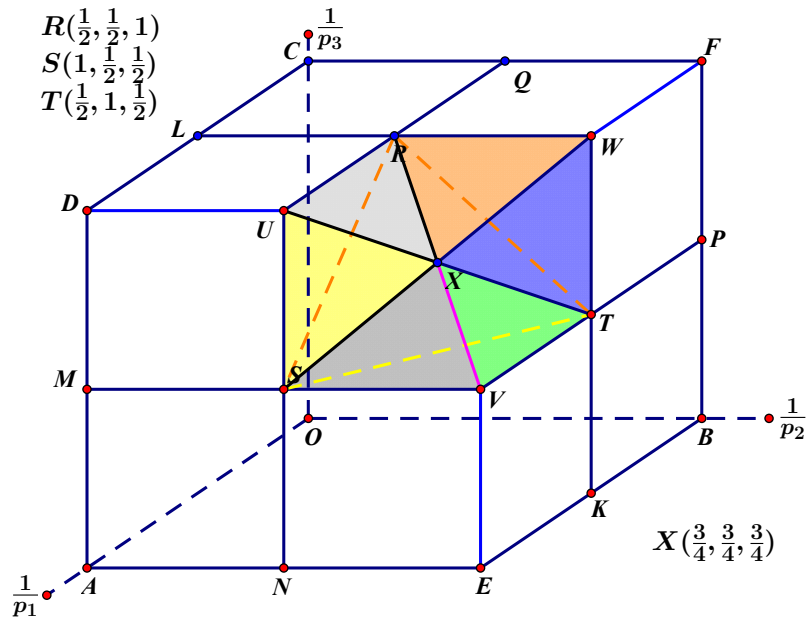


Figure A.5: Three side local tetrahedrons

In this case we have three regions. Inside the tetrahedron $X.STV$, the smoothness

conditions are

$$s_1 > \frac{n}{2}, \quad s_2 > \frac{n}{2}, \quad s_3 > \frac{n}{2}, \quad \frac{s_1}{n} + \frac{s_2}{n} > \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{2}, \quad \frac{s_1}{n} + \frac{s_2}{n} > \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{2}.$$

Inside the tetrahedron $X.SUR$, we require

$$s_1 > \frac{n}{2}, \quad s_2 > \frac{n}{2}, \quad s_3 > \frac{n}{2}, \quad \frac{s_1}{n} + \frac{s_3}{n} > \frac{1}{p_1} + \frac{1}{p_3} - \frac{1}{2}, \quad \frac{s_1}{n} + \frac{s_2}{n} > \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{2}.$$

Inside the tetrahedron $X.TRW$, we need

$$s_1 > \frac{n}{2}, \quad s_2 > \frac{n}{2}, \quad s_3 > \frac{n}{2}, \quad \frac{s_2}{n} + \frac{s_3}{n} > \frac{1}{p_2} + \frac{1}{p_3} - \frac{1}{2}, \quad \frac{s_1}{n} + \frac{s_2}{n} > \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{2}.$$

A.1.5 Three face-out local tetrahedrons

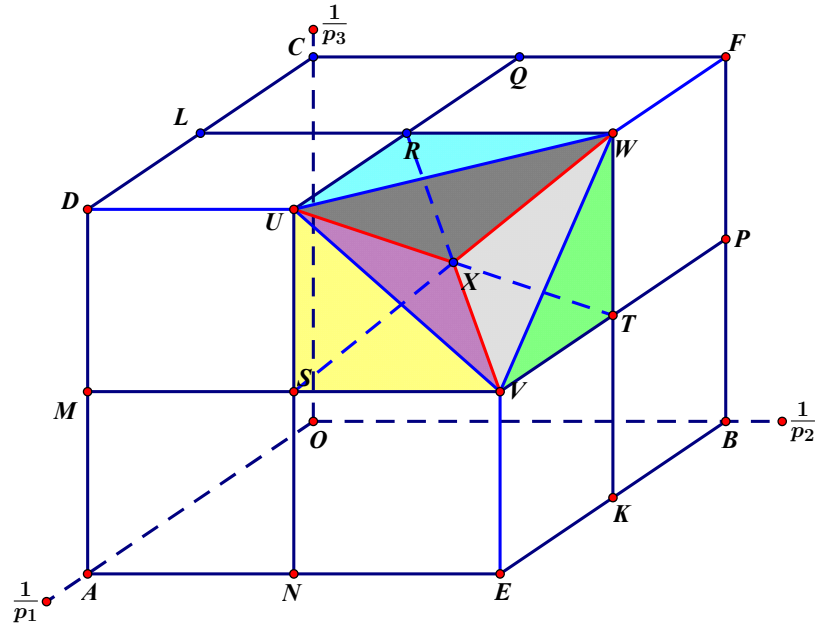


Figure A.6: Three face-out local tetrahedrons

In this case we still have three regions. Inside the tetrahedron $X.USV$, the smooth-

ness conditions are

$$\begin{aligned} s_1 > \frac{n}{2}, \quad s_2 > \frac{n}{2}, \quad s_3 > \frac{n}{2}, \quad \frac{s_1}{n} + \frac{s_2}{n} + \frac{s_3}{n} > \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} - \frac{1}{2}, \\ \frac{s_1}{n} + \frac{s_2}{n} > \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{2}, \quad \frac{s_1}{n} + \frac{s_3}{n} > \frac{1}{p_1} + \frac{1}{p_3} - \frac{1}{2}. \end{aligned}$$

Inside the tetrahedron $X.VTW$, we need

$$\begin{aligned} s_1 > \frac{n}{2}, \quad s_2 > \frac{n}{2}, \quad s_3 > \frac{n}{2}, \quad \frac{s_1}{n} + \frac{s_2}{n} + \frac{s_3}{n} > \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} - \frac{1}{2}, \\ \frac{s_1}{n} + \frac{s_2}{n} > \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{2}, \quad \frac{s_2}{n} + \frac{s_3}{n} > \frac{1}{p_2} + \frac{1}{p_3} - \frac{1}{2}. \end{aligned}$$

Inside the tetrahedron $X.UWR$, we need

$$\begin{aligned} s_1 > \frac{n}{2}, \quad s_2 > \frac{n}{2}, \quad s_3 > \frac{n}{2}, \quad \frac{s_1}{n} + \frac{s_2}{n} + \frac{s_3}{n} > \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} - \frac{1}{2}, \\ \frac{s_1}{n} + \frac{s_3}{n} > \frac{1}{p_1} + \frac{1}{p_3} - \frac{1}{2}, \quad \frac{s_2}{n} + \frac{s_3}{n} > \frac{1}{p_2} + \frac{1}{p_3} - \frac{1}{2}. \end{aligned}$$

A.1.6 Two upper local tetrahedrons

In this case, the region is formed by two tetrahedrons $X.UVW$ and $Z.UVW$, where $X(\frac{3}{4}, \frac{3}{4}, \frac{3}{4})$, $Z(1, 1, 1)$, $U(1, \frac{1}{2}, 1)$, $V(1, 1, \frac{1}{2})$, and $W(\frac{1}{2}, 1, 1)$. In these two tetrahedrons, we need all the inequalities appeared in (3.5), i.e.,

$$\begin{aligned} s_1 > \frac{n}{2}, \quad s_2 > \frac{n}{2}, \quad s_3 > \frac{n}{2}, \quad \frac{s_1}{n} + \frac{s_2}{n} + \frac{s_3}{n} > \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} - \frac{1}{2}, \\ \frac{s_1}{n} + \frac{s_2}{n} > \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{2}, \quad \frac{s_1}{n} + \frac{s_3}{n} > \frac{1}{p_1} + \frac{1}{p_3} - \frac{1}{2}, \quad \frac{s_2}{n} + \frac{s_3}{n} > \frac{1}{p_2} + \frac{1}{p_3} - \frac{1}{2}. \end{aligned}$$

Note that when the indices p_j are greater or equal to 2, we need less smoothness indices s_j . The smoothness indices are getting bigger when the integrability indices

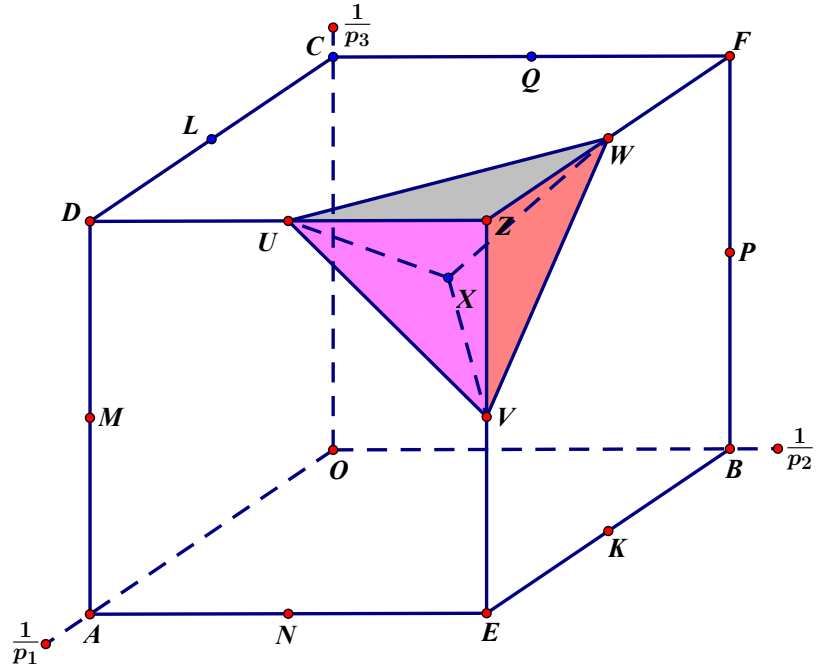


Figure A.7: Two upper local tetrahedrons

become closer to the point $Z(1, 1, 1)$ on the unit cube. In particular, to obtain the boundedness of T_σ from a product of Lebesgue spaces L^{p_j} , $1 < p_j < 2$, into another Lebesgue space, we need more derivatives which translates into the inequalities involving s_j given in the upper local tetrahedrons (see Figure A.7 above).

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VITA

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