Stability of planar fronts for a class of reaction diffusion systems

A Dissertation

presented to

the Faculty of the Graduate School

University of Missouri

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

by

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JULY 2016
The undersigned, appointed by the Dean of the Graduate School, have examined the
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Stability of planar fronts for a class of reaction diffusion systems

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ACKNOWLEDGEMENTS

First of all, I wish to express sincere gratitude to my advisor, Professor Yuri Latushkin, for his vision and direction. His priceless gift to me was not just knowledge, but also an unwavering moral support through every stage of my doctoral studies. His teachings and encouragement will continue to instruct and inspire me.

I would also like to thank my committee members Dr. Carmen Chicone, Dr. Fritz Gesztesy, Dr. Michael Pang, Dr. David Retzloff and Dr. Samuel Walsh for their support and encouragement.

I sincerely thank Dr. Anna Gazaryan whose excellent articles has motivated and helped current project in many ways. I also thank Dr. Roland Schnaubelt for his crucial suggestions in the proof of certain inequality in the first section of this thesis.

Finally, I am very grateful to MU Mathematics department’s academic support staff Amy Crews and Gwen Gilpin. Their timely help and advices are deeply appreciated.
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The purpose of this thesis is to study stability of one-dimensional traveling waves and multidimensional planar fronts as well as space-independent steady states for a class of reaction diffusion systems that arise in combustion theory and chemical-reaction models.

We begin by extending the recent one-dimensional stability results for reaction diffusion equations of this type. Using spaces with exponential weights, we shift the spectrum of the differential operator obtained by linearizing the equation about the front into the stable half-plane, and study the nonlinear equation on the intersection of the unweighted and weighted spaces. In this space, we prove the existence of a stable foliation in vicinity of the traveling front solution, that is, we show that each translation of the front has a stable manifold formed by solutions converging to the translation. The results provide a better understanding of the dynamics near the front, and improve the known stability theorems. In addition, we prove stability of the end states of the front.

We then turn to stability of the planar front solutions and their end states for a class of reaction diffusion equations in multidimensional space. We study the case when the spectrum of the linearization in the direction of the front touches the imaginary axis. This setting is a generalization of the relevant multidimensional stability results known in the literature. Passing to the exponential weights, we obtain appro-
appropriate estimates for the nonlinear terms of the equation governing the evolution of the perturbations of the front in the intersection of the unweighted and weighted spaces. We then show that the unweighted norm of the solutions of the equation with small initial data remains bounded while the weighted norm algebraically decays for large times. Also, we prove stability of the end states of the front.
Chapter 1

Introduction

Planar (traveling) fronts are solutions to partial differential equations that move in a given direction with constant speed without changing their shape and are asymptotic to spatially constant steady-state solutions, the end states. Stability theory of the steady-states and planar (traveling) wave solutions of reaction diffusion equations is an important area in contemporary applied analysis, see, e.g. [H, KP, Sa, VVV] and the vast literature cited in these books, as well as [BGHL, BKSS, K2, KV, LX, LW, R1, R2, R3, R4, TZKS, X1] and the bibliography therein. One of the major methods in the analysis of these solutions is the use of exponentially weighted function spaces [PW, S] which allows one to shift the spectrum of the operator obtained by linearizing the equation about the front into the stable half-plane, and then use decay of the related linear semigroup. In a recent series of papers this method was used for an important class of one-dimensional reaction diffusion equations originating in combustion theory, see [G, GLSS, GLS, GLS1] and a review paper [GLSR] that contains further references. In these papers orbital stability of the one-dimensional traveling fronts was proved for perturbations that belong to the space obtained by intersecting the unweighted and weighted Sobolev spaces. This choice of the space is quite natural as, due to the specific form of the nonlinearity in the equation, its local
Lipschitz property needed to establish well posedness could be indeed obtained in this space. Some additional information was obtained for certain components of the solutions in the unweighted space. Significantly less is known for the multidimensional equations of this type. We mention here an important paper [K2] by T. Kapitula who proved algebraic decay of solutions for the case when the spectrum of the linearization along the front is located in the stable half plane.

Our objective in this thesis is to study one- and multidimensional stability of the space-independent steady-state solutions and planar front solutions of a certain special class of systems of reaction diffusion equations that frequently occur in combustion theory and in chemical reaction models. From a mathematical viewpoint, these equations have a certain “product-triangular” structure in the reaction term similar to that of the equations studied in [GLSS, GLS, GLS1] for the one-dimensional case. A typical example that we have in mind is a “model” system of equations arising in combustion theory for solid fuels, cf. [BLR, GLSS, MS],

\[
\begin{aligned}
&u_t(t,x) = \Delta_x u(t,x) + v(t,x)g(u(t,x)), \; u,v \in \mathbb{R} , \\
v_t(t,x) = \epsilon \Delta_x v(t,x) - \kappa v(t,x)g(u(t,x)), \; x \in \mathbb{R}^d,
\end{aligned}
\]

where \( g(u) = e^{-\frac{u}{\kappa}} \) if \( u > 0 \) and \( g(u) = 0 \) if \( u \leq 0 \), and the parameters \( \epsilon \) and \( \kappa \) satisfy \( 0 \leq \epsilon < 1 \) and \( \kappa > 0 \). More generally, we consider the system of reaction diffusion equations,

\[
Y_t(t,x) = D \Delta_x Y(t,x) + R(Y(t,x)), \; Y \in \mathbb{R}^n, \; x \in \mathbb{R}^d, \; t > 0,
\]

where \( D = \text{diag}(d_1, \cdots, d_n) \) with all \( d_i \geq 0 \), and the function \( R(\cdot) \) is smooth and satisfies some additional special properties listed in the main part of the thesis. Heuristically, the special properties can be described as follows: First, we assume that the
linearization of (1.2) about the front (in an appropriate decomposition $Y = (U, V)^T$, $U \in \mathbb{R}^{n_1}$, $V \in \mathbb{R}^{n_2}$, $n_1 + n_2 = n$, into components $U$ and $V$) has a block upper-triangular structure where one of the diagonal blocks generates a bounded semigroup, while the other generates an exponentially decaying semigroup. Second, we assume that the $U$- and $V$- components of the nonlinear term of (1.2) are the products of a function that depends on $Y$ times the $V$-component of $Y$.

Equations of the special type often appear in combustion models and chemical-reaction models, see, e.g., [GSM, SMS, SKMS, T, VaV]. The fact that the reaction term of the equations studied in the current work has this “product” structure allows one to deal with the complication that the spectrum of the linearization of the equation at the front is assumed to touch the imaginary axis, cf. [Sa, SS]. To eliminate difficulties related to this complication, one employs exponentially weighted spaces, an idea that goes back to [S] and [PW]. As we just mentioned, in the one-dimensional case, the study of this type of nonlinearities was initiated by A. Ghazaryan in [G] and then continued in [GLSS, GLS, GLS1], see also a review paper [GLSR]. In particular, for $d = 1$, it was proved in [GLS] that, under appropriate assumptions on the nonlinearity, the traveling front is orbitally stable, that is, any solution originated in a small vicinity of the front converges exponentially in the weighted norm to a translation of the front. Our objective in this work is therefore two-fold: First, for the one-dimensional case, we want to obtain a better and more detailed picture of the dynamics of the equation in vicinity of the front. Second, for the multidimensional case, we want to take advantage of the special structure of the equation, and prove the decay of solutions for the case when the spectrum of the linearization touches the
Accordingly, the dissertation consists of two parts. In Chapter 2 we study one-dimensional equations of the type (1.1)-(1.2). Extending the work in [GLSS, GLS, GLS1], we study in detail the dynamics of the equation in the vicinity of the traveling front in both exponential weighted and unweighted spaces. Specifically, we show that each translation of the front has a local stable manifold, such that the solutions on this manifold converge to the translation of the front in the weighted space and remain bounded in the unweighted space. Moreover, we show that a small vicinity of the traveling front is foliated by the stable manifolds. Finally, we prove stability of the end states of the front, the space-independent steady-state solutions of the equation, thus completing stability analysis of the fronts conducted in [GLSS, GLS, GLS1].

In Chapter 3 we study planar fronts of the multidimensional equation of the type (1.1)-(1.2). Stability issues for the multidimensional settings are significantly more complicated and studied much less than for $d = 1$. We impose the same “block-triangular” and “product” assumptions as before in order to generalize methods developed in [GLSS, GLS, GLS1] for the multidimensional situation considered by T. Kapitula in [K2]. We consider the linearization of the equation in the direction of the front but, unlike [K2], we assume that its essential spectrum touches the imaginary axis. Unlike the one-dimensional case, passing to the exponentially weighted space does not shift the essential spectrum away from the imaginary axis. Nevertheless, combining some involved multidimensional estimates with a development of the techniques from [GLS], we managed to prove the algebraic decay of the solutions of the nonlinear equation in the weighted space and their boundedness in the unweighted
space. The proof is a rather technical evolution of the methods from [GLS] and [K2]. Finally, completing the theory for the multidimensional case, we study stability of the end states of the front, that is, the space-independent steady-state solutions of the system.
Chapter 2

One Dimensional Reaction Diffusion System

In this chapter we study one dimensional reaction diffusion systems imposing certain special assumptions of the reaction term. In Section 2.1 we prove the existence of a stable foliation in the vicinity of a traveling front solution for systems of reaction diffusion equations in one dimensional space that arise in the study of chemical reaction models and solid fuel combustion. This extends the orbital stability results obtained earlier in a series of papers by A. Ghazaryan, S. Schecter and Y. Latushkin. The essential spectrum of the differential operator obtained by linearization at the front touches the imaginary axis. Passing to the spaces with exponential weights, one can shift the spectrum to the left. We study the nonlinear equations on the intersection of the unweighted and weighted spaces. For each small translation of the front we prove the existence of a stable manifold containing the translation of the front and show that the stable manifolds foliate a small ball centered at the front. In Section 2.2, we address the issue of stability of the end states of the front, and study Lyapunov stability of the steady-state solutions of the reaction diffusion equations of the type considered in [G, GLSS, GLS, GLS1]. In particular, this covers an important model frequently arising in combustion theory [B, VaV].
2.1 Stable foliation in vicinity of a traveling front

In this subsection we study dynamics in the vicinity of the traveling front of a special type of systems of reaction diffusion equations in one dimensional space. We continue the work in [GLS] utilizing the theory of invariant manifolds, cf. [BJ, CHT, L], and give a more detailed analysis of dynamics by proving in Theorem 2.15 the existence of a stable foliation near the front. Specifically, we observe that the set of all translations of the front serves as a local central unstable manifold consisting of fixed points. Next, using the Lyapunov-Perron method, cf. [LL, LPS1, LPS2], we show the existence of a local stable manifold going through each translation of the front. We also show that these manifolds foliate a small neighborhood of the front and therefore each point in the neighborhood belongs to one of them, cf. [BLZ, CHT]. By the standard invariance and attractivity properties of stable manifolds the orbit of the point converges to the translation of the front along the stable manifold as proved in [GLS].

The section is organized as follows. In Subsection 2.1.1 we formulate our assumptions and prove some preliminary results. In Subsection 2.1.2 we study the Lyapunov-Perron operator whose fixed points are the graphs of the invariant manifolds. In Subsection 2.1.3 we formulate and prove our main result on the existence of the stable manifolds.

**Notation.** Throughout the section, $|\cdot|$ and $\langle\cdot,\cdot\rangle$ are the Euclidean norm and the scalar product in $\mathbb{R}^n$. For a given map $f : \mathbb{R}^m \to \mathbb{R}^k$, its differential with respect to $y$ is written as $\partial_y f : \mathbb{R}^m \to B(\mathbb{R}^m, \mathbb{R}^k)$. We let $B(\mathcal{E}, \mathcal{F})$ be the set of linear bounded operators between Banach spaces $\mathcal{E}$ and $\mathcal{F}$, and abbreviate $B(\mathcal{E}) = B(\mathcal{E}, \mathcal{E})$. We denote by $C$ a generic constant that may change from one estimate to another, and
use $T$ to designate transposition. For a Banach space with norm $\|\cdot\|$, we write $B_\delta(\|\cdot\|)$ for the closed ball of radius $\delta$ centered at 0.

We denote by $\mathcal{E}_0$ with norm $\|\cdot\|_0$ either the Sobolev space $H^1$ or the space $BUC$ of bounded uniformly continuous functions on $\mathbb{R}$ with vector values, and by $\mathcal{E}_\alpha$ with norm $\|\cdot\|_\alpha$ the respective space of (exponentially) weighted functions, see (2.12). Let $\|\cdot\|_\beta$ be the norm on the intersection space $\mathcal{E}_\beta = \mathcal{E}_0 \cap \mathcal{E}_\alpha$; i.e., $|y|_\beta := \max\{|y|_0, |y|_\alpha\}$.

2.1.1 The setting

We consider the system of reaction diffusion equations

$$Y_t = DY_{xx} + R(Y), \quad x \in \mathbb{R}, \ t \geq 0,$$ (2.1)

where $D = \text{diag}(d_1, \ldots, d_n)$, $d_j \geq 0$, $Y(t, x) \in \mathbb{R}^n$, and $R : \mathbb{R}^n \to \mathbb{R}^n$ is a $C^3$ function satisfying additional properties listed below.

Passing in (2.1) to the moving coordinate frame $\xi = x - ct$ and redenoting $\xi$ again by $x$, we arrive at the nonlinear equation

$$Y_t = DY_{xx} + cY_x + R(Y), \quad x \in \mathbb{R}, \ t \geq 0.$$ (2.2)

We discuss the wellposedness of this system in Remark 2.3.

**Hypothesis 2.1.** We assume that for some velocity $c \in \mathbb{R}$ the system (2.2) admits a stationary solution $Y_0 \in C^3(\mathbb{R})$; i.e., (2.1) possesses the traveling front solution $Y(t, x) = Y_0(x - ct)$. It is also required that $Y_0(x)$ converges to the end states $Y_\pm$ as $x \to \pm\infty$ exponentially; i.e.,

$$|Y_0(x) - Y_-| \leq Ce^{-\omega_-x}, \quad x \leq 0,$$

$$|Y_0(x) - Y_+| \leq Ce^{-\omega_+x}, \quad x \geq 0,$$ (2.3)
for some $\omega_- < 0 < \omega_+$ and $C > 0$. Replacing $R$ by $\tilde{R}(Y) := R(Y + Y_-)$, we can and will assume that $Y_- = 0$ (and we then drop the tilde).

We further assume that the nonlinear term $R$ in (2.1) and (2.2) has the following product structure.

**Hypothesis 2.2.** The nonlinear term $R$ belongs to $C^3(\mathbb{R}^n, \mathbb{R}^n)$. In appropriate variables $Y = (U,V)^T$ with $U \in \mathbb{R}^{n_1}$, $V \in \mathbb{R}^{n_2}$ and $n_1 + n_2 = n$, we have

$$R(U,0) = (A_1 U, 0)$$

(2.4)

for a constant $n_1 \times n_1$ matrix $A_1$.

In other words, we suppose that

$$R(U,V) = \begin{pmatrix} A_1 U + R_1(U,V) \\ R_2(U,V) \end{pmatrix},$$

where the maps $R_j$ belong to $C^2(\mathbb{R}^n, \mathbb{R}^{n_j})$ and satisfy $R_j(U,0) = 0$ for $j \in \{1,2\}$ and $U \in \mathbb{R}^{n_1}$. Note that condition (2.4) yields $R(0,0) = R(Y_-) = 0$. We also split

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \quad \text{where} \quad D_1 = \text{diag}(d_1, \ldots, d_{n_1}), \quad D_2 = \text{diag}(d_{n_1+1}, \ldots, d_n).$$

Let $q \in \mathbb{R}$. We write $Y_q(x) = Y_0(x - q)$ for the shifted wave. Since (2.2) is translationally invariant, $Y_q$ is again a steady state solution of (2.2) and thus yields a traveling wave solution for (2.1). Linearizing (2.2) at $Y_q$ (that is, substituting $Y_q + Y$ instead of $Y$ in (2.2)), we arrive at the equation

$$Y_t = L_q Y + F_q(Y), \quad \text{where} \quad L_q Y = D Y_{xx} + c Y_x + \partial_Y R(Y_q)Y.$$  

(2.5)

Here, $\partial_Y$ is the differential with respect to $Y \in \mathbb{R}^n$ and the nonlinear term $F_q : \mathbb{R}^n \to \mathbb{R}^n$ is written as

$$F_q(Y) = \int_0^1 (\partial_Y R(Y_q + tY) - \partial_Y R(Y_q)) Y \, dt.$$  

(2.6)
The linearization of (2.2) at \( Y = (0,0)^T \) is given by

\[
Y_t = L^- Y + G(Y), \quad \text{where} \quad L^- Y = D Y_{xx} + c Y_x + \partial_Y R(0) Y
\]  

(2.7)

and \( G : \mathbb{R}^n \to \mathbb{R}^n; \ G(Y) = R(Y) - \partial_Y R(0) Y \). We remark that

\[
(L_q - L^-) Y = B_q Y \quad \text{with} \quad B_q(x) = \partial_Y R(Y_q(x)) - \partial_Y R(0).
\]  

(2.8)

Below we impose conditions on \( L_q \) at \( q = 0 \); i.e., on the linearization at the original traveling wave \( Y_0 \). We further consider \( L_q \) for \( |q| \leq q_0 \) with some \( q_0 > 0 \), which will be fixed sufficiently small in the final theorem. The shifted wave \( Y_q \) decays as in Hypothesis 2.2 with the same exponents \( \omega_{\pm} \) and constants \( C \) only depending on \( q_0 \).

Assumption (2.4) also yields the formulas

\[
\partial_Y R(0,0) = \begin{pmatrix} A_1 & \partial_Y R_1(0,0) \\ 0 & \partial_Y R_2(0,0) \end{pmatrix}, \quad L^- = \begin{pmatrix} L^{(1)} & \partial_Y R_1(0,0) \\ 0 & L^{(2)} \end{pmatrix}
\]  

(2.9)

with the differential expressions

\[
L^{(1)} U = D_1 U_{xx} + c U_x + A_1 U,
\]

\[
L^{(2)} V = D_2 V_{xx} + c V_x + \partial_Y R_2(0,0) V.
\]  

(2.10)

**Remark 2.3.** We consider the equations (2.2) and (2.5) on the space \( \mathcal{E}_0 \) which is either the Sobolev space \( H^1(\mathbb{R})^n \) or the space of bounded uniformly continuous functions \( BUC(\mathbb{R})^n \). It is straightforward to check that the nonlinearites \( R \) and \( F_q \) are Lipschitz on bounded subsets of \( \mathcal{E}_0 \).

For the differential expressions \( L_q \) and \( L^- \) defined in (2.5) and (2.7), respectively, we denote by \( \mathcal{L}_q \) and \( \mathcal{L}^- \) the differential operators on \( \mathcal{E}_0 \) on their natural domain \( \mathcal{D} \) defined as follows. For \( \mathcal{E}_0 = H^1(\mathbb{R})^n \), the domain \( \mathcal{D} \) of \( \mathcal{L}_q \) and of \( \mathcal{L}^- \) consists of the vector functions \( Y = (Y_j)_{j=1}^n \) whose components \( Y_j \) belong to \( H^3(\mathbb{R}) \) if \( d_j > 0 \) and to \( H^2(\mathbb{R}) \) if \( d_j = 0 \). For \( \mathcal{E}_0 = BUC(\mathbb{R})^n \), we choose the domain analogously with \( H^3(\mathbb{R}) \)
replaced by \( BUC^2(\mathbb{R}) \) and \( H^2(\mathbb{R}) \) replaced by \( BUC^1(\mathbb{R}) \), the spaces of differentiable functions which are bounded and have bounded, uniformly continuous derivatives.

The operators \( \mathcal{L}_q \) and \( \mathcal{L}_- \) generate strongly continuous semigroups \( \{T_q(t)\}_{t \geq 0} \) and \( \{S(t)\}_{t \geq 0} \) on \( \mathcal{E}_0 \), respectively, cf. e.g. [GLSS, §2.2].

Standard results then show the local wellposedness of (2.5) in \( \mathcal{E}_0 \) for initial values \( y_0 \) in the domain of \( \mathcal{L}_q \), where the (classical) solutions belong to \( C^1([0, t_0), \mathcal{E}_0) \) and take values in \( \mathcal{D} \). They are given by Duhamel’s formula

\[
Y(t) = T_q(t)y_0 + \int_0^t T_q(t-\tau)F_q(Y(\tau)) \, d\tau, \quad t \geq 0. \tag{2.11}
\]

See e.g. Theorems 6.1.4 and 6.1.6 in [P]. A function \( Y \in C([0, t_0), \mathcal{E}_0) \) satisfying (2.11) is called a \textit{mild solution} of (2.5). This concept is strictly weaker than that of classical solvability. We mostly work with mild solutions. Similar remarks apply to (2.2) and the differential expression \( D\partial_{xx} + c\partial_x \) equipped the same domain \( \mathcal{D} \). Approximating a given initial value \( y_0 \in \mathcal{E}_0 \) in \( \mathcal{E}_0 \) by functions in \( \mathcal{D} \), we see that all mild solutions of (2.2) are given by \( Y_q + Y(t) \) where \( Y(t) \) solves (2.11).

Let \( \alpha = (\alpha_-, \alpha_+) \in \mathbb{R}^2 \). We say that \( \gamma_\alpha : \mathbb{R} \to \mathbb{R} \) is a weight function of class \( \alpha \) if \( \gamma_\alpha \) is \( C^2 \), \( \gamma_\alpha(x) > 0 \) for all \( x \in \mathbb{R} \), and \( \gamma_\alpha(x) = e^{\alpha_-x} \) for \( x \leq -x_0 \) and \( \gamma_\alpha(x) = e^{\alpha_+x} \) for \( x \geq x_0 \) for some \( x_0 > 0 \). We shall always assume that

\[
0 < \alpha_- < -\omega_- \quad \text{and} \quad 0 \leq \alpha_+ < \omega_+, \tag{2.12}
\]

where \( \omega_\pm \) are the exponents mentioned in (2.3). Given such a pair \( \alpha = (\alpha_-, \alpha_+) \), we introduce the weighted space \( \mathcal{E}_\alpha = \{ u : \mathbb{R} \to \mathbb{R}^n : \gamma_\alpha u \in \mathcal{E}_0 \} \) with the norm \( |u|_\alpha = |\gamma_\alpha u|_0 \). (Recall that \( \mathcal{E}_0 \) with norm \( \cdot \) is either \( H^1(\mathbb{R})^n \) or \( BUC(\mathbb{R})^n \).) The intersection space \( \mathcal{E}_\beta = \mathcal{E}_0 \cap \mathcal{E}_\alpha \) is endowed with the norm \( |u|_\beta = \max\{|u|_0, |u|_\alpha\} \).
The differential expressions $L_q$, $L^-$ etc. equipped with their natural domains define operators in $\mathcal{E}_\alpha$ which are denoted by $\mathcal{L}_{q,\alpha}$, $\mathcal{L}_{\alpha}$ etc. (cf. Remark 2.3). On the spectrum of $\mathcal{L}_{0,\alpha}$, we impose the following assumptions.

**Hypothesis 2.4.** In addition to Hypotheses 2.1 and 2.2, we assume that there exists $\alpha = (\alpha_-, \alpha_+) \in \mathbb{R}^2$ such that (2.12) with $\omega_\pm$ from (2.3) and the following assertions hold.

(a) $\text{sup}\{\text{Re} \, \lambda : \lambda \in \text{Sp}_{\text{ess}}(\mathcal{L}_{0,\alpha})\} < 0$ for the differential expression $L_0$ defined in (2.5).

(b) The only element of $\text{Sp}(\mathcal{L}_{0,\alpha})$ in $\{\lambda \in \mathbb{C} : \text{Re} \, \lambda \geq 0\}$ is a simple eigenvalue at $\lambda = 0$ with $Y'_0$ being the respective eigenfunction.

Here the essential spectrum $\text{Sp}_{\text{ess}}(A)$ of a closed densely defined operator contains all points in the spectrum $\text{Sp}(A)$ which are not isolated eigenvalues of finite algebraic multiplicity. We discuss various consequences of the above hypothesis which are important for our proofs.

**Remark 2.5.** We claim that assertions (a) and (b) in Hypothesis 2.4 are satisfied for $\mathcal{E}_0 = H^1(\mathbb{R})^n$ or $\mathcal{E}_0 = BUC(\mathbb{R})^n$ if and only if they hold when $\mathcal{E}_0$ is replaced by the space $L_2(\mathbb{R})^n$ and $\mathcal{E}_{\alpha}$ by the space $L_{\alpha}^2(\mathbb{R})^n$ of functions $u$ with $\gamma_{\alpha} u \in L^2(\mathbb{R})$ which is endowed with the norm $|u|_\alpha = |\gamma_{\alpha} u|_{L^2}$.

Indeed, the “if” part of the claim above is proved in Lemma 3.8 of [GLS]. So we assume Hypothesis 2.4 for $\mathcal{E}_0 = H^1(\mathbb{R})^n$ or $\mathcal{E}_0 = BUC(\mathbb{R})^n$. Then assertion (a) of this hypothesis for $\mathcal{E}_0 = L^2(\mathbb{R})^n$ is true since the right-hand boundary of the essential spectra of $\mathcal{L}_{0,\alpha}$ is the same for all three spaces by [GLS, Lemma 3.5].
To show assertion (b) for $E_0 = L^2(R)^n$, we assume that $L_{0,\alpha}$ on $L^2_\alpha(R)^n$ has an isolated eigenvalue $\lambda$ of finite algebraic multiplicity with $\text{Re} \lambda \geq 0$. By means of the isomorphism $u(\cdot) \mapsto \gamma(\cdot)u(\cdot)$ between $L^2_\alpha(R)^n$ and $L^2(R)^n$ we obtain a differential operator $\hat{L}$ in $L^2(R)^n$ which is similar to $L_{0,\alpha}$ in $L^2_\alpha(R)^n$, cf. [GLS, Eqn. (3.2)], and hence possesses the unstable isolated eigenvalue $\lambda$, too. Palmer’s Dichotomy Theorem in [Pa] says that the first order system corresponding to the second order eigenvalue problem for $\hat{L}$ admits exponential dichotomies on $\mathbb{R}^-$ and $\mathbb{R}^+$. Arguing as in the proof of Lemma 3.8 of [GLS], we see that the respective eigenfunction $Z$ decays exponentially as $x \to \pm \infty$. It thus belongs to $BUC(R)^n$, and also to $H^1(R)^n$ since $Z_x$ can be bounded by $Z$ itself due to the eigenvalue equation, see (3.3) in [GLS]. As a result, $\hat{L}$ in $H^1(R)^n$ or $BUC(R)^n$ has the unstable eigenvalue $\lambda$ and therefore also $L_{0,\alpha}$ in $E_\alpha$. Hypothesis 2.4 now shows that $\lambda = 0$, completing the proof of the claim.

\[\diamond\]

**Lemma 2.6.** Assume that Hypothesis 2.4 holds. Then assertions (a) and (b) in Hypothesis 2.4 are satisfied by the operator $L_{q,\alpha}$ instead of $L_{0,\alpha}$ and by the function $Y_q'$ instead of $Y_0'$.

**Proof.** The operators $L_{q,\alpha}$ and $L_{0,\alpha}$ are similar via the transformation $Y \mapsto Y(\cdot - q)$ which also maps $Y'$ into $Y_q'$. The assertions then easily follow. \[\square\]

**Remark 2.7.** Assume Hypothesis 2.4. Lemma 2.6 says that $\lambda = 0$ is an isolated simple eigenvalue for $L_{q,\alpha}$. We let $P_q^c$ denote the spectral projection for $L_{q,\alpha}$ in $E_\alpha$ onto $\ker L_{q,\alpha} = \text{span}\{Y_q'\}$. Basic operator theory (see, e.g., [DL, Lemma 2.13]) yields that

$$\text{ran}(I_{E_\alpha} - P_q^c) = \ker P_q^c = \text{ran}(L_{q,\alpha}).$$
Moreover, the one-dimensional projection $P_q^c$ is given by

$$P_q^cY = \zeta_q(Y) Y_q', \quad \zeta_q(Y_q') = 1, \quad (2.13)$$

for an element $\zeta_q$ in ker $L_{q,\alpha}^*$ which is also one dimensional, cf. [K, Theorem IV.5.13]. As in the proof of Lemma 2.6, the operators $L_{q,\alpha}^*$ and $L_{0,\alpha}^*$ are similar and therefore the norms of $\zeta_q \in E_{\alpha}^*$ are bounded uniformly for $|q| \leq q_0$. Also, in view of Lemma 3.3 in [GLS], the first three derivatives of the shifted wave $Y_q$ are bounded by $Ce^{-\omega_-\xi}$ for $\xi \leq 0$ and by $Ce^{-\omega_+\xi}$ for $\xi \geq 0$ with $\omega_\pm$ from Hypothesis 2.2 and constants $C$ only depending on $q_0$. We conclude that

$$|P_q^cY|_\alpha = |\zeta_q(Y)||Y_q'|_\alpha \leq C|Y|_\alpha |Y_q'|_\alpha \leq C|Y|_\beta |Y_q'|_\alpha,$$

$$|P_q^cY|_0 = |\zeta_q(Y)||Y_q'|_0 \leq C|Y|_\alpha |Y_q'|_0 \leq C|Y|_\beta |Y_q'|_0.$$

As a consequence, $P_q^c$ induces maps

$$P_q^c \in \mathcal{B}(E_\alpha) \cap \mathcal{B}(E_\beta, E_\alpha) \cap \mathcal{B}(E_\alpha, E_\beta) \cap \mathcal{B}(E_\beta) \cap \mathcal{B}(E_\alpha, E_0) \cap \mathcal{B}(E_\beta, E_0)$$

The complementary projection $P_q^s = I - P_q^c$ thus satisfies

$$P_q^s \in \mathcal{B}(E_\alpha) \cap \mathcal{B}(E_\beta) \cap \mathcal{B}(E_\beta, E_\alpha) \cap \mathcal{B}(E_\beta, E_0).$$

We use the same notation $P_q^c$ and $P_q^s$ on all these spaces and their norms are uniformly bounded for $|q| \leq q_0$. The projections further satisfy

$$\|P_q^c - P_p^c\|_{\mathcal{B}(E_\beta)} \leq C|q - p|, \quad \|P_q^c - P_p^c\|_{\mathcal{B}(E_\alpha)} \leq C|q - p| \quad (2.14)$$

for $|p|, |q| \leq q_0$ and a constant independent of $p$ and $q$. In fact, (2.5) yields

$$L_q - L_p = \partial Y R(Y_0(\cdot - q)) - \partial Y R(Y_0(\cdot - p))$$
\[
Y_0(x - q) - Y_0(x - p) = - \int_0^1 Y_0'(x - p - s(q - p))(q - p) \, ds.
\]  
(2.15)

For \( \mathcal{E}_0 = BUC \) we deduce

\[
\|L_{q,\alpha} - L_{p,\alpha}\|_{\mathcal{B}(\mathcal{E}_0)} = \sup_{x \in \mathbb{R}} |\partial_Y R(Y_0(x - q)) - \partial_Y R(Y_0(x - p))| \leq C|q - p|,
\]
and similarly for \( \mathcal{E}_0 = H^1 \). These estimates can easily be transferred to the resolvents on a sufficiently small circle around 0 which implies the claim (2.14).

\[\text{Remark 2.8.} \] To provide extra information, we now determine \( \zeta_\alpha \) from (2.13) as a solution of a differential equation. Remark 2.5 yields that Hypothesis 2.4 is also true if we replace \( \mathcal{E}_0 \) by \( L^2(\mathbb{R}) \). We first determine \( \zeta_\alpha \) for the operator \( L_{q,\alpha}^* \) acting on the dual \( L^2_\alpha(\mathbb{R})^* \) of the space \( L^2_\alpha(\mathbb{R}) \) of functions with the exponential weight \( \gamma_\alpha \). We recall that the operator \( \gamma_\alpha : L^2_\alpha(\mathbb{R}) \to L^2(\mathbb{R}); Y(\cdot) \mapsto \gamma_\alpha(\cdot)Y(\cdot) \), is an isometric isomorphism. Moreover, \( L^2_\alpha(\mathbb{R})^* \) can be identified with \( L^2 \)-space with the weight \( 1/\gamma_\alpha \), where the duality map between \( L^2_\alpha(\mathbb{R}) \) and \( L^2_\alpha(\mathbb{R})^* \) is given by the usual (real) \( L^2 \)-scalar product. Hence, the adjoint operator \( \gamma_\alpha^* : L^2(\mathbb{R}) \to L^2_\alpha(\mathbb{R})^* \) coincides with the multiplication operator by \( \gamma_\alpha \)

\[
\langle L^2_\alpha(\mathbb{R})^*, \gamma_\alpha Z, Y \rangle_{L^2_\alpha(\mathbb{R})} := \langle \gamma_\alpha Z, Y \rangle_{L^2(\mathbb{R})} = \langle Z, \gamma_\alpha Y \rangle_{L^2(\mathbb{R})}
\]
\[
= (L^2_\alpha(\mathbb{R})^*, \langle \gamma_\alpha \rangle Z, Y \rangle_{L^2_\alpha(\mathbb{R})} \text{ for } Z \in L^2(\mathbb{R}) \text{ and } Y \in L^2_\alpha(\mathbb{R});
\]
here we temporarily denote by \( X, \langle f, g \rangle_X \) the action of a functional \( f \in X^* \) on \( g \in X \) and by \( \langle \cdot, \cdot \rangle_{L^2(\mathbb{R})} \) the scalar product in \( L^2(\mathbb{R}) \).

The operator \( \gamma_\alpha L_{q,\alpha} \gamma_\alpha^{-1} \) in \( L^2(\mathbb{R}) \) is Fredholm since it is similar to the Fredholm operator \( L_{q,\alpha} \) in \( L^2_\alpha(\mathbb{R}) \). The adjoint of \( \gamma_\alpha L_{q,\alpha} \gamma_\alpha^{-1} \) in \( L^2(\mathbb{R}) \) is also Fredholm, and

\[\text{\Diamond} \]
it is equal to $\gamma^{-1}_a L_{q,a}^* \gamma_a$ since $\gamma^*_a = \gamma_a$. We note that the dimension of the kernels is preserved by similarity and duality. The functional $\zeta_q \in \ker L_{q,a}^*$ from (2.13) is then represented by $\zeta_q = \gamma_a Z_q$ where $Z_q \in L^2(\mathbb{R})$ belongs to $\ker (\gamma^{-1}_a L_{q,a}^* \gamma_a)$. In other words, $Z_q \in L^2(\mathbb{R})$ is the unique (up to a normalization) solution on $\mathbb{R}$ of the differential equation $(\gamma^{-1}_a L_{q,a}^* \gamma_a) Z_q = 0$. Reasoning as in the proof of Lemma 3.8 in [GLS] or in Remark 2.5, that is, invoking Palmer’s Theorem [Pa], we conclude that the first order system corresponding to this differential equation has exponential dichotomies on $\mathbb{R}_+$ and $\mathbb{R}_-$. It follows that the solution $Z_q$ decays exponentially to zero as $x \to \pm \infty$. Moreover, $Z_q$ is the translation $Z_0(\cdot - q)$ of $Z_0$, and the decay of the function $Z_q$ is thus uniform in $q$ for $|q| \leq q_0$. Formula (2.13) now yields

$$P^c_q Y = \pi_q(Y) Y'_q \quad \text{with} \quad \pi_q(Y) = \int_{\mathbb{R}} \langle Z_q(x), \gamma_a(x)Y(x) \rangle \, dx$$

(2.17)

for all $Y \in L^2_\alpha(\mathbb{R})$, where $Z_q$ is the exponentially decaying function normalized such that $\pi_q(Y'_q) = 1$.

Finally, returning to the cases $\mathcal{E}_0 = H^1(\mathbb{R})^n$ or $\mathcal{E}_0 = BUC(\mathbb{R})^n$, we notice that $\pi_q(\cdot)$ is a bounded functional on $\mathcal{E}_\alpha$ in both cases. Using also the decay properties of $Y'_q$ recalled in Remark 2.7, we confirm from (2.17) once again that $P^c_q$ is a bounded operator from both $\mathcal{E}_\beta$ and $\mathcal{E}_\alpha$ into $\mathcal{E}_\beta$, with uniform constants for $q \in [-q_0, q_0]$.

**Remark 2.9.** Let $B_q$ be multiplication operator induced by the matrix valued function $B_q(\cdot)$ from (2.8). Lemma 8.2 of [GLS] says that $B_q$ belongs to $\mathcal{B}(\mathcal{E}_\alpha, \mathcal{E}_0)$. As in assertion (3) of this lemma, one also sees that $\|B_q - B_p\|_{\mathcal{B}(\mathcal{E}_\alpha, \mathcal{E}_0)} \leq C|q - p|$ for $q, p \in [-q_0, q_0]$. Inspecting the proofs, we see that the constants do not depend on $p$ and $q$, but on $q_0$.

The operators $L_q$ and $L_{q,a}$ generate strongly continuous semigroups on $\mathcal{E}_0$ and
\( \mathcal{E}_\alpha \), respectively, which are both denoted by \( \{ T_q(t) \}_{t \geq 0} \), see e.g. [GLSS, §2.2]. By Lemma 2.6, there are numbers

\[
0 > -\nu > \sup \{ \text{Re}\lambda : \lambda \in \text{Sp}(L_{q,\alpha}) \setminus \{0\} \},
\]

Lemma 3.13 of [GLS] then yields the exponential decay

\[
\| T_q(t) P^s_q \|_{\mathcal{B}(\mathcal{E}_0)} \leq C e^{-\nu t}, \quad t \geq 0,
\] (2.18)

see also [GLSS]. The constant \( C \) can be chosen uniform in \( q \) because of the transformation used in the proof of Lemma 2.6.

Also the operators \( L^- \) and \( L^-_{\alpha} \) generate strongly continuous semigroups on \( \mathcal{E}_0 \) and \( \mathcal{E}_\alpha \), designated by \( \{ S(t) \}_{t \geq 0} \). Since the multiplication operator \( B_q \) is bounded on these spaces, formula (2.8) implies the variation of constant formula

\[
T_q(t - \tau) = S(t - \tau) + \int_\tau^t S(t - s) B_q T_q(s - \tau) \, ds, \quad t \geq \tau \geq 0, \quad q \in \mathbb{R}. \] (2.19)

The upper triangular structure of the operator \( L^- \) indicated in (2.9) implies an analogous representation of the semigroup

\[
S(t) = \begin{pmatrix} S_1(t) & Q(t) \\ 0 & S_2(t) \end{pmatrix} \quad \text{and} \quad Q(t) = \int_0^t S_1(t - s) \partial V R_1(0,0) S_2(s) \, ds. \] (2.20)

Here \( \{ S_1(t) \}_{t \geq 0} \) and \( \{ S_2(t) \}_{t \geq 0} \) are the semigroups generated by the operators \( L^{(1)} \) and \( L^{(2)} \) from (2.10), respectively. On these semigroups we impose the following assumptions.

**Hypothesis 2.10.** The strongly continuous semigroup \( \{ S_1(t) \}_{t \geq 0} \) is bounded and the semigroup \( \{ S_2(t) \}_{t \geq 0} \) is uniformly exponentially stable on \( \mathcal{E}_0 \); that is,

\[
\| S_1(t) \|_{\mathcal{B}(\mathcal{E}_0)} \leq C, \quad \| S_2(t) \|_{\mathcal{B}(\mathcal{E}_0)} \leq C e^{-\rho t}
\]

for some \( \rho > 0 \) and all \( t \geq 0 \).
Hypothesis 2.10 and (2.20) imply the boundedness of \( \{S(t)\}_{t \geq 0} \) on \( \mathcal{E}_0 \); i.e.,

\[
\|S(t)\|_{\mathcal{B}(\mathcal{E}_0)} \leq C, \quad \text{for all } t \geq 0.
\] (2.21)

We next show that the semigroup \( \{T_q(t)\}_{t \geq 0} \) is bounded on the space \( \mathcal{E}_\beta \), too.

**Lemma 2.11.** Assume Hypotheses 2.4 and 2.10. Take \( q_0 > 0 \) and let \( \alpha = (\alpha_-, \alpha_+) \) satisfy (2.12). Then we have

\[
\sup_{|q| < q_0} \sup_{t \geq 0} \|T_q(t)\|_{\mathcal{B}(\mathcal{E}_\beta)} < \infty.
\] (2.22)

**Proof.** The variation of constant formula (2.19) yields on \( \mathcal{E}_\beta \)

\[
T_q(t)P_q^s = S(t)P_q^s + \int_0^t S(t-s)B_qT_q(s)P_q^s \, ds.
\] (2.23)

As noted in Remark 2.7 and (2.21), the projection \( P_q^s \) belongs to \( \mathcal{B}(\mathcal{E}_\beta, \mathcal{E}_0) \) and to \( \mathcal{B}(\mathcal{E}_\beta, \mathcal{E}_\alpha) \) while the semigroup \( S(t) \) is uniformly bounded in \( \mathcal{E}_0 \) for \( |q| \leq q_0 \) and \( t \geq 0 \), respectively. Using (2.23), these facts, Remark 2.9 and the exponential decay in (2.18), we can estimate

\[
\|T_q(t)P_q^s\|_{\mathcal{B}(\mathcal{E}_\beta, \mathcal{E}_0)} \leq C \|P_q^s\|_{\mathcal{B}(\mathcal{E}_\beta, \mathcal{E}_0)}
\]

\[
+ C \int_0^t \|B_q\|_{\mathcal{B}(\mathcal{E}_\alpha, \mathcal{E}_0)} \|T_q(s)P_q^s\|_{\mathcal{B}(\mathcal{E}_\alpha)} \|P_q^s\|_{\mathcal{B}(\mathcal{E}_\beta, \mathcal{E}_\alpha)} \, ds
\]

\[
\leq C + C \int_0^t e^{-\nu t} \, ds \leq C
\]

for all \( t \geq 0 \) and \( |q| \leq q_0 \), with uniform constants. In view of the inequality \( \|T_q(t)P_q^s\|_{\mathcal{B}(\mathcal{E}_\alpha)} \leq Ce^{-\nu t} \) from (2.18), we have proved (2.22) with \( T_q(t) \) replaced by \( T_q(t)P_q^s \). Writing the semigroup as \( T_q(t) = T_q(t)P_q^s + T_q(t)P_c^s \) on \( \mathcal{E}_\beta \), it remains to show (2.22) with \( T_q(t) \) replaced by \( T_q(t)P_c^s \). Recall from Remark 2.7 that \( P_c^s = I - P_q^s \in \mathcal{B}(\mathcal{E}_\beta) \) projects \( \mathcal{E}_\beta \) onto the kernel of the generators \( \mathcal{L}_{q,0} \) and \( \mathcal{L}_{q,\alpha} \) of the semigroup \( \{T_q(t)\}_{t \geq 0} \) on \( \mathcal{E}_0 \) and \( \mathcal{E}_\alpha \). We conclude that \( T_q(t)P_c^sY = P_c^sY \) for all \( Y \in \mathcal{E}_\beta \) and \( t \geq 0 \). Therefore, \( \|T_q(t)P_c^s\|_{\mathcal{B}(\mathcal{E}_\beta)} \leq C \) for \( t \geq 0 \), completing the proof of (2.22). \( \blacksquare \)
2.1.2 The Lyapunov-Perron operator

In this subsection we introduce the Lyapunov-Perron operator associated with the nonlinear equation (2.5) and show that it is a contraction of a small ball in a certain space of functions $u : \mathbb{R} \to \mathcal{E}_0 \cap \mathcal{E}_\alpha$. First, we establish the main technical estimates for the nonlinearity $F_q : \mathbb{R}^n \to \mathbb{R}^n$ defined in (2.6).

Lemma 2.12. Assume that $\alpha = (\alpha_-, \alpha_+)$ satisfies (2.12) and that the nonlinearity $R \in C^3(\mathbb{R}^n, \mathbb{R}^n)$ fulfills (2.4). Let $\delta_1 > 0$ and choose a radius $\delta \in (0, \delta_1]$. Then for all functions $y = (u, v)$ and $\bar{y} = (\bar{u}, \bar{v})$ from $\mathcal{E}_\beta$ with $|y|_\beta, |\bar{y}|_\beta \leq \delta$ the estimates

$$|F_q(y)|_0 \leq C|y|_0 (|y|_\alpha + |v|_0), \quad (2.24)$$

$$|F_q(y)|_\alpha \leq C|y|_0 |y|_\alpha, \quad (2.25)$$

$$|F_q(y) - F_q(\bar{y})|_0 \leq C(|y - \bar{y}|_0 (|y|_\alpha + |\bar{y}|_\alpha) + |y - \bar{y}|_0 |v|_0 + |\bar{y}|_0 |v - \bar{v}|_0), \quad (2.26)$$

$$|F_q(y) - F_q(\bar{y})|_\alpha \leq |y - \bar{y}|_\alpha (|y|_0 + |\bar{y}|_0) \quad (2.27)$$

are true, where $C = C(\delta_1, q_0)$ and $|q| \leq q_0$.

Proof. Let $|y|_\beta, |\bar{y}|_\beta \leq \delta \leq \delta_1$. From the proof of Lemma 8.3 in [GLS] we recall the representation

$$F_q(y) = I_1(y) + I_2(y) + I_3(y) + I_4(y) + I_5(y),$$

where $Y_q = (U_q, V_q)$, $y = (u, v)$,

$$I_1(y) = \int_0^1 (\partial_a r(Y_q + ty) - \partial_a r(Y_q)) u V_q \, dt,$$

$$I_2(y) = \int_0^1 (\partial_a r(Y_q + ty)) u v \, dt,$$

$$I_3(y) = \int_0^1 (\partial_a r(Y_q + ty) - \partial_a r(Y_q)) v V_q \, dt.$$
\[ I_4(y) = \int_0^1 (\partial_y r(Y_q + ty)\, v) \, tv \, dt, \]
\[ I_5(y) = \int_0^1 (r(Y_q + ty) - r(Y_q)) \, v \, dt, \]

and the function \( r \in C^2(\mathbb{R}^n, \mathbb{R}^{n \times n}) \) is given by
\[ r(u, v) = \int_0^1 \partial_v R(u, tv) \, dt. \]

We note that \( r \) is only applied to functions which are uniformly bounded by \( C(1 + \delta_1) \). It is then straightforward to check the inequalities \( |I_j(y)|_0 \leq C|y|_0 |v|_0 \) for \( j \in \{2, \ldots, 5\} \) and \( |I_j(y)|_\alpha \leq C|y|_0 |y|_\alpha \) for \( j \in \{1, 2, \ldots, 5\} \). Since \( uV_q = (\gamma_\alpha u)(\gamma^{-1}_\alpha V_q) \) and \((\gamma^{-1}_\alpha V_q) \in BUC^1(\mathbb{R}^n)\) by Lemma 3.7 of [GLS], we can further estimate \( |I_1(y)|_0 \leq C|y|_0 |y|_\alpha \), finishing the proof of (2.24) and (2.25). Here and below the constants only depend on \( \delta_1 \) and \( q_0 \).

To show (2.26) and (2.27), we deal with each integral \( I_j \) separately. The terms \(|y - \bar{y}|_0 \,(|y|_\alpha + |\bar{y}|_\alpha)\) and \(|y - \bar{y}|_\alpha \,(|y|_0 + |\bar{y}|_0)\) come from \( I_1 \) while the remaining ones originate from \( I_2 \) through \( I_5 \). We first represent \( I_1(y) - I_1(\bar{y}) \) as
\[ I_1(y) - I_1(\bar{y}) = \int_0^1 \int_0^1 \partial_y \partial_x r(Y_q + st(y - \bar{y}) + t\bar{y})uV_q t(y - \bar{y}) \, ds \, dt \]
\[ + \int_0^1 \int_0^1 \partial_y \partial_x r(Y_q + st\bar{y})(u - \bar{u})V_q t\bar{y} \, ds \, dt. \]

Using \( uV_q(y - \bar{y}) = (\gamma_\alpha u)(\gamma^{-1}_\alpha V_q)(y - \bar{y}) \) and \((u - \bar{u})V_q \bar{y} = (u - \bar{u}) \gamma^{-1}_\alpha V_q \gamma_\alpha y \) as above, we conclude that \(|I_1(y) - I_1(\bar{y})|_0 \leq C|y - \bar{y}|_0 \,(|y|_\alpha + |\bar{y}|_\alpha)\). If we multiply (2.28) by \( \gamma_\alpha \), we directly estimate \(|I_1(y) - I_1(\bar{y})|_\alpha \leq C(|y|_0 + |\bar{y}|_0) \, |y - \bar{y}|_\alpha \) since \(|u| \leq |y|\). Likewise, we write \( I_5(y) - I_5(\bar{y}) \) as
\[ I_5(y) - I_5(\bar{y}) = \int_0^1 \int_0^1 \left( \partial_y r(Y_q + sty) - \partial_y r(Y_q + st\bar{y}) \right) tyv \, ds \, dt \]
\[ + \int_0^1 \int_0^1 \partial_y r(Y_q + ts\bar{y})tv(y - \bar{y}) \, ds \, dt \]
\[ + \int_0^1 \int_0^1 \partial_y r(Y_q + st\bar{y})t\bar{y}(v - \bar{v}) \, ds \, dt. \]
and obtain the bounds $|I_5(y) - I_5(\bar{y})|_0 \leq C(|y - \bar{y}|_0 |v|_0 + |\bar{y}|_0 |v - \bar{v}|_0)$, recalling that $|y|_0 \leq \delta_1$ by assumption. After multiplying (2.29) by $\gamma$, it also follows that $|I_5(y) - I_5(\bar{y})|_\alpha \leq C(|y|_0 |y - \bar{y}|_\alpha + |\bar{y}|_0 |v - \bar{v}|_\alpha)$ since $|v| \leq |y|$. Similarly, the formulas

$$I_2(y) - I_2(\bar{y}) = \int_0^1 (\partial_u r(Y_q + ty) - \partial_u r(Y_q + t\bar{y})) uv dt \tag{2.30}$$

$$+ \int_0^1 \partial_u r(Y_q + t\bar{y})(u - \bar{u})tv dt + \int_0^1 \partial_u r(Y_q + t\bar{y})\bar{u}t(v - \bar{v}) dt,$$

$$I_4(y) - I_4(\bar{y}) = \int_0^1 (\partial_v r(Y_q + ty) - \partial_v r(Y_q + t\bar{y})) vtv dt \tag{2.31}$$

$$+ \int_0^1 \partial_v r(Y_q + t\bar{y})(v - \bar{v})tv dt + \int_0^1 \partial_v r(Y_q + t\bar{y})\bar{v}t(v - \bar{v}) dt$$

imply the inequalities

$$|I_2(y) - I_2(\bar{y})|_0 \leq C(|y - \bar{y}|_0 |v|_0 + |\bar{y}|_0 |v - \bar{v}|_0),$$

$$|I_4(y) - I_4(\bar{y})|_0 \leq C(|y - \bar{y}|_0 |v|_0 + |\bar{y}|_0 |v - \bar{v}|_0).$$

Multiplying (2.30) and (2.31) by $\gamma$, we also derive

$$|I_2(y) - I_2(\bar{y})|_\alpha \leq C(|y|_0 |y - \bar{y}|_\alpha + |\bar{y}|_0 |v - \bar{v}|_\alpha),$$

$$|I_4(y) - I_4(\bar{y})|_\alpha \leq C(|y|_0 |y - \bar{y}|_\alpha + |\bar{y}|_0 |v - \bar{v}|_\alpha).$$

We finally compute

$$I_3(y) - I_3(\bar{y}) = \int_0^1 \int_0^1 \partial_y \partial_v r(Y_q + st(y - \bar{y}) + t\bar{y}) V_q t(y - \bar{y}) ds dt$$

$$+ \int_0^1 \int_0^1 \partial_y \partial_v r(Y_q + st\bar{y})(v - \bar{v}) V_q t\bar{y} ds dt.$$

Again we infer that

$$|I_3(y) - I_3(\bar{y})|_0 \leq C(|y - \bar{y}|_0 |v|_0 + |\bar{y}|_0 |v - \bar{v}|_0),$$

$$|I_3(y) - I_3(\bar{y})|_\alpha \leq C(|y|_0 |y - \bar{y}|_\alpha + |\bar{y}|_0 |v - \bar{v}|_\alpha).$$

This completes the proof of the lemma. $\blacksquare$
Remark 2.13. It follows from the observations after Remark 2.9 that the realization of $L_q$ in $\mathcal{E}_\beta = \mathcal{E}_0 \cap \mathcal{E}_\alpha$ generates a strongly continuous semigroup. The Lipschitz properties proved in the above lemma thus imply that the semilinear equation (2.5) is locally wellposed also in $\mathcal{E}_\beta$, cf. Remark 2.3.

We next establish basic properties of the Lyapunov-Perron operator $\Phi_q(y, z_0)$ defined by

$$\Phi_q(y, z_0)(t) = T_q(t) P_q^s z_0 + \int_0^t T_q(t-\tau) P_q^s F_q(y(\tau)) \, d\tau - \int_t^\infty P_q^c F_q(y(\tau)) \, d\tau, \quad (2.32)$$

where $|q| \leq q_0$ and $z_0 \in \mathcal{E}_0 \cap \mathcal{E}_\alpha = \mathcal{E}_\beta$ satisfies

$$|z_0|_\beta = \max\{|z_0|_0, |z_0|_\alpha\} \leq \delta_0, \quad (2.33)$$

for some $\delta_0 > 0$. Here we use that $P_c^q$ maps into the kernel of the generator of $\{T_q(t)\}_{t \geq 0}$, see Remark 2.7, so that the semigroup is just the identity on the range of $P_c^q$ and we can omit it in the second integral in (2.32).

For a continuous map $y = (u, v) : \mathbb{R} \to \mathcal{E}_\beta = \mathcal{E}_0 \cap \mathcal{E}_\alpha$ we define the norms

$$\|y\|_{\omega, \alpha} = \sup_{t \geq 0} e^{\omega t} |y(t)|_\alpha, \quad \|y\|_{0, 0} = \sup_{t \geq 0} |y(t)|_0, \quad \|v\|_{\omega, 0} = \sup_{t \geq 0} e^{\omega t} |v(t)|_0,$$

where $\omega > 0$ is specified below and $\alpha = (\alpha_-, \alpha_+)$ is given by (2.12). Let $\delta > 0$. Then $(B_\delta, \| \cdot \|)$ is the set of continuous functions $y : \mathbb{R} \to \mathcal{E}_0 \cap \mathcal{E}_\alpha$ such that

$$\|y\| := \max(\|y\|_{\omega, \alpha}, \|y\|_{0, 0}, \|v\|_{\omega, 0}) \leq \delta. \quad (2.34)$$

We recall from Hypothesis 2.10 and (2.18) the exponential estimates

$$\|S_2(t)\|_{B(\mathcal{E}_0)} \leq C e^{-\rho t}, \quad \|T_q(t) P_q^s\|_{B(\mathcal{E}_\alpha)} \leq C e^{-\nu t} \quad (2.35)$$
for $t \geq 0$. For technical reasons (see the next proof), if necessary we have to modify these exponents such that

$$0 < \omega < \rho < \nu.$$  \hfill (2.36)

This is always possible, though one may lose information here. By Lemma 2.11, the semigroup $\{T_q(t)\}_{t \geq 0}$ is bounded in $E_\beta$. The above constants do not depend on $|q| \leq q_0$.

**Lemma 2.14.** Take $q_0 > 0$. Let $\delta > 0$ and $\delta_0 = \delta_0(\delta) > 0$ be small enough. For each $z_0 \in B_{\delta_0}(|\cdot|_\beta)$ the Lyapunov-Perron operator $y \mapsto \Phi_q(y, z_0)$ leaves $B_\delta(\|\cdot\|)$ invariant and is a strict contraction on this ball for all $|q| \leq q_0$. Moreover, for the norm $\|\cdot\|$ defined in (2.34) one has

$$\|\Phi_q(y, z_0) - \Phi_q(\bar{y}, \bar{z}_0)\| \leq C|z_0 - \bar{z}_0|_\beta + C\delta\|y - \bar{y}\|$$  \hfill (2.37)

for some $C > 0$ and all $z_0, \bar{z}_0 \in B_{\delta_0}(|\cdot|_\beta), y, \bar{y} \in B_\delta(\|\cdot\|), and |q| \leq q_0$.

**Proof.** Let $t \geq 0, \delta, \delta_0 > 0, z_0, \bar{z}_0 \in B_{\delta_0}(|\cdot|_\beta), y, \bar{y} \in B_\delta(\|\cdot\|), and |q| \leq q_0$. Below the constants are uniform for $\delta, \delta_0$ and $q$ in bounded subsets. By $\pi_1 y = u$ and $\pi_2 y = v$, we denote the projection of $y = (u, v)$ onto its first and second components. Formulas (2.19) and (2.20) yield

$$\pi_2 T_q(t - \tau) = S_2(t - \tau)\pi_2 + \int_\tau^t S_2(t - s)\pi_2 B_q T_q(s - \tau)\,ds, \quad 0 \leq \tau \leq t.$$  \hfill (2.38)

1a) Using (2.38), (2.35), and Remarks 2.7 and 2.9, the second component of the first integral in (2.32) can be estimated by

$$e^{\omega t} \left| \pi_2 \int_0^t T_q(t - \tau) P_q F_q(y(\tau))\,d\tau \right|_0$$  \hfill (2.39)
\[ \leq C e^{\omega t} \int_0^t \left( e^{-\rho(t-\tau)} |F_q(y(\tau))|_\beta + \int_\tau^t e^{-\rho(t-s)} e^{-\nu(s-t)} |F_q(y(\tau))|_\alpha \, ds \right) \, d\tau, \]

since

\[ |S_2(t-\tau)\pi_2 P_q^s F_q(y(\tau))|_0 \leq \|S_2(t-\tau)\pi_2\|_{\mathcal{B}(\mathcal{E}_0)} \|P_q^s\|_{\mathcal{B}(\mathcal{E}_\beta, \mathcal{E}_0)} |F_q(y(\tau))|_\beta, \quad (2.40) \]

\[ |S_2(t-s)\pi_2 B_q T_q(s-\tau) P_q^s F_q(y(\tau))|_0 \leq \|S_2(t-s)\pi_2\|_{\mathcal{B}(\mathcal{E}_0)} \|B_q\|_{\mathcal{B}(\mathcal{E}_\alpha, \mathcal{E}_0)} \|T_q(s-\tau) P_q^s\|_{\mathcal{B}(\mathcal{E}_\alpha)} |F_q(y(\tau))|_\alpha. \quad (2.41) \]

Because of (2.24) and (2.25), the formulas (2.39) and (2.36) yield

\[ e^{\omega t} \left| \pi_2 \int_0^t T_q(t-\tau) P_q^s F_q(y(\tau)) \, d\tau \right|_0 \leq C e^{\omega t} \int_0^t \left( e^{-\rho(t-\tau)} e^{-\omega \tau} \left( |y(\tau)|_\alpha + |u(\tau)|_0 \right) |y(\tau)|_0 \\
+ \int_\tau^t e^{-\rho(t-s)} e^{-\nu(s-t)} e^{-\omega \tau} |y(\tau)|_\alpha |y(\tau)|_0 \, ds \right) \, d\tau \leq C (\|y\|_{\omega, \alpha} + \|u\|_{\omega, 0}) \|y\|_{0, 0} \int_0^t e^{(\omega-\rho)(t-\tau)} \, d\tau \\
+ C \|y\|_{\omega, \alpha} \|y\|_{0, 0} \int_0^t e^{\omega(t-\tau)} \left( \int_\tau^t e^{-\rho(t-s)} e^{-\nu(s-\tau)} \, ds \right) \, d\tau \leq C \|y\|^2 \leq C \delta^2. \]

We next employ (2.35), (2.25) and (2.36) to bound

\[ e^{\omega t} \left| \int_0^t T_q(t-\tau) P_q^s F_q(y(\tau)) \, d\tau \right|_\alpha \leq C \int_0^t e^{\omega t} e^{-\nu(t-\tau)} e^{-\omega \tau} e^{\omega \tau} |y(\tau)|_0 |y(\tau)|_\alpha \, d\tau \\
\leq C \|y\|_{0, 0} \|y\|_{\omega, \alpha} \leq C \delta^2. \]

To finish with the first integral in (2.32), it remains to control the \(|\cdot|_0\) norm of its first component. Here (2.19), (2.21), Remark 2.7 (in particular, that \(P_q^s \in \mathcal{B}(\mathcal{E}_\beta, \mathcal{E}_0)\)), Remark 2.9, (2.24), (2.35), (2.25) and (2.36) imply the inequalities

\[ \left| \pi_1 \int_0^t T_q(t-\tau) P_q^s F_q(y(\tau)) \, d\tau \right|_0 \]
\[
\left| \pi_1 \int_0^t S(t - \tau) P_q^{s} F_q(y(\tau)) \, d\tau \right| \\
+ \pi_1 \int_0^t \int_\tau^t S(t - s) B_q T_q(s - \tau) P_q^{s} F_q(y(\tau)) \, ds \, d\tau \bigg|_0^t \\
\leq C \int_0^t |F_q(y(\tau))|_\beta \, d\tau \\
+ C \|B_q\|_{\mathcal{B}(E_\alpha,E_0)} \int_0^t \int_\tau^t |T_q(s - \tau) P_q^{s} F_q(y(\tau))|_\alpha \, ds \, d\tau \\
\leq C \int_0^t e^{-\omega \tau} |y(\tau)|_0 e^{\omega \tau} (|y(\tau)|_\alpha + |v(\tau)|) \, d\tau \\
+ C \int_0^t \int_\tau^t e^{-\nu(s-\tau)} |y(\tau)|_0 |y(\tau)|_\alpha \, ds \, d\tau \\
\leq C \|y\|_0,0 (\|y\|_{\omega,\alpha} + \|v\|_{\omega,0}) \\
\int_0^t e^{-\omega \tau} \, d\tau + C \|y\|_0,0 \|y\|_{\omega,\alpha} \int_0^t \int_\tau^t e^{-\nu(s-\tau)} ds \, e^{-\omega \tau} \, d\tau \\
\leq C \delta^2.
\]

1b) We now treat the term \( T_q(t) P_q^{s} z_0 \) in (2.32). From (2.35) and (2.36) we infer
\[
e^{\omega t}|T_q(t) P_q^{s} z_0|_\alpha \leq C e^{\omega t} e^{-\nu t} |z_0|_\alpha \leq C \delta_0.
\]
By means of (2.19), (2.21), Remark 2.7 (in particular, that \( P_q^s \in \mathcal{B}(E_\beta,E_0) \)) and Remark 2.9, as well as (2.35), we next compute
\[
\left| \pi_1 T_q(t) P_q^{s} z_0 \right|_0 \leq |S(t) P_q^{s} z_0|_0 + \int_0^t |S(t - s) B_q T_q(s) P_q^{s} z_0|_0 \, ds \\
\leq C |z_0|_\beta + C \int_0^t \|B_q\|_{\mathcal{B}(E_\alpha,E_0)} e^{-\nu s} |z_0|_\alpha \, ds \leq C \delta_0.
\]
Finally, formulas (2.38), (2.35), Remarks 2.7 and 2.9, as well as inequality (2.36) imply
\[
e^{\omega t}|\pi_2 T_q(t) P_q^{s} z_0|_0 \leq e^{\omega t}|S_2(t) \pi_2 P_q^{s} z_0|_0 + \int_0^t |S_2(t - s) \pi_2 B_q T_q(s) P_q^{s} z_0|_0 \, ds \\
\leq C e^{(\omega - \rho) t} |z_0|_\beta + C \int_0^t e^{\omega t} e^{-\rho(t-s)} e^{-\nu s} |z_0|_\alpha \, ds.
\]
\[ \leq C\delta_0. \]

1c) To show the invariance, it remains to bound the norms of the last integral in (2.32). Remark 2.7 (in particular, that \( P_q^\varepsilon \in B(\mathcal{E}_\alpha, \mathcal{E}_\beta) \)) and estimate (2.25) yield

\[
e^{\omega t} \left| \int_t^\infty P_q^\varepsilon F_q(y(\tau)) d\tau \right|_\beta \leq C \int_t^\infty e^{\omega \tau} |F_q(y(\tau))|_\alpha d\tau
\]

\[
\leq C \int_t^\infty e^{\omega \tau} e^{-\omega \tau} |y(\tau)|_0 |y(\tau)|_\alpha d\tau
\]

\[
\leq C \|y\|_{0,0} \|y\|_{\omega,\alpha} \leq C\delta^2.
\]

We thus have shown that \( \Phi_q(\cdot, z_0) \) leaves the ball \( \mathcal{B}_\delta(\| \cdot \|) \) invariant if first \( \delta > 0 \) and then \( \delta_0 > 0 \) are chosen small enough.

2) For the contractivity we have to estimate the difference

\[
\Phi_q(y, z_0) - \Phi_q(\bar{y}, z_0) = \int_0^t T_q(t - \tau) P_q^s (F_q(y(\tau)) - F_q(\bar{y}(\tau))) d\tau
\]

(2.42)

\[
- \int_t^\infty P_q^\varepsilon (F_q(y(\tau)) - F_q(\bar{y}(\tau))) d\tau.
\]

2a) Using (2.19), (2.21), (2.35), Remark 2.7 (in particular, that \( P_q^s \in B(\mathcal{E}_\beta, \mathcal{E}_0) \)) and Remark 2.9, we bound the first integral by

\[
\left| \int_0^t T_q(t - \tau) P_q^s (F_q(y(\tau)) - F_q(\bar{y}(\tau))) d\tau \right|_0
\]

\[
\leq \left| \int_0^t S(t - \tau) P_q^s (F_q(y(\tau)) - F_q(\bar{y}(\tau))) d\tau \right|_0
\]

\[
+ \left| \int_0^t \int_\tau^t S(t - s) B_q T_q(s - \tau) P_q^s (F_q(y(\tau)) - F_q(\bar{y}(\tau))) ds d\tau \right|_0
\]

\[
\leq C \int_0^t |F_q(y(\tau)) - F_q(\bar{y}(\tau))|_\beta d\tau
\]

\[
+ C \int_0^t \int_\tau^t \|B_q\|_{B(\mathcal{E}_\alpha, \mathcal{E}_\alpha)} e^{-\nu(s-\tau)} |F_q(y(\tau)) - F_q(\bar{y}(\tau))|_\alpha ds d\tau.
\]

The inequalities (2.26) and (2.27) then lead to

\[
\left| \int_0^t T_q(t - \tau) P_q^s (F_q(y(\tau)) - F_q(\bar{y}(\tau))) d\tau \right|_0
\]
\[ \leq C \int_0^t e^{-\omega \tau} \left[ |y(\tau) - \bar{y}(\tau)|_0 + \omega \tau (|y(\tau)|_0 + |\bar{y}(\tau)|_0) + |\bar{v}(\tau)|_0 \right] \, d\tau \]
\[ + C \int_0^t e^{-\omega \tau} |\bar{v}(\tau) - \bar{v}(\tau)|_0 + e^{\omega \tau} |y(\tau) - \bar{y}(\tau)|_0 (|y(\tau)|_0 + |\bar{y}(\tau)|_0) \, d\tau \]
\[ \leq C \|y(t) - \bar{y}(t)|_{0,0} \left( \|y(t)\|_{\omega,\alpha} + \|\bar{y}(t)\|_{\omega,\alpha} + \|v(t)\|_{\omega,0} \right) + C \|\bar{y}(t)\|_{0,0} \|v(t) - \bar{v}(t)\|_{\omega,0} \]
\[ + C \|y(t) - \bar{y}(t)\|_{\omega,\alpha} (\|y(t)\|_{0,0} + \|\bar{y}(t)\|_{0,0}) \]
\[ \leq C \delta \|y(t) - \bar{y}(t)\|_{\omega,\alpha} \]

The $|.|_\alpha$-norm of the first integral in (2.42) is estimated by

\[ e^{\omega t} \left| \int_0^t T_q(t-\tau) P^q_s(F_q(y(\tau)) - F_q(\bar{y}(\tau))) \, d\tau \right|_\alpha \]
\[ \leq C \int_0^t e^{\omega t} e^{-\nu(t-\tau)} e^{-\omega \tau} |y(\tau) - \bar{y}(\tau)|_0 (|y(\tau)|_0 + |\bar{y}(\tau)|_0) \, d\tau \]
\[ \leq C \int_0^t e^{(\omega - \nu)(t-\tau)} \|y(t) - \bar{y}(t)\|_{\omega,\alpha} (\|y(t)\|_{0,0} + \|\bar{y}(t)\|_{0,0}) \]
\[ \leq C \delta \|y(t) - \bar{y}(t)\|_{\omega,\alpha} \]

employing (2.35), (2.27), and (2.36). As in (2.40) and (2.41), for the second component we use formulas (2.38) and (2.35), Remark 2.7 (in particular, that $P^q_s \in B(\mathcal{E}_{\beta}, \mathcal{E}_0)$) and Remark 2.9, as well as inequalities (2.26), (2.27) and (2.36) to derive the estimates

\[ e^{\omega t} \left| \int_0^t T_q(t-\tau) P^q_s(F_q(y(\tau)) - F_q(\bar{y}(\tau))) \, d\tau \right|_0 \]
\[ \leq e^{\omega t} \left| \int_0^t S_2(t-\tau) \pi_2 P^q_s(F_q(y(\tau)) - F_q(\bar{y}(\tau))) \, d\tau \right|_0 \]
\[ + e^{\omega t} \left| \int_0^t \int_\tau^t S_2(t-s) \pi_2 B_q T_q(s-\tau) P^q_s(F_q(y(\tau)) - F_q(\bar{y}(\tau))) \, ds \, d\tau \right|_0 \]
\[ \leq C \int_0^t e^{(\omega - \nu)(t-\tau)} e^{\omega \tau} \left[ |y(\tau) - \bar{y}(\tau)|_0 (|y(\tau)|_0 + |\bar{y}(\tau)|_0 + |v(\tau)|_0) \right. \]
\[ + |y(\tau)|_0 |v(\tau) - \bar{v}(\tau)|_0 + |y(\tau) - \bar{y}(\tau)|_0 (|y(\tau)|_0 + |\bar{y}(\tau)|_0) \, d\tau \]
\[ \leq C \int_0^t e^{\omega t} \left[ |y(\tau) - \bar{y}(\tau)|_0 (|y(\tau)|_0 + |\bar{y}(\tau)|_0) + |y(\tau) - \bar{y}(\tau)|_0 (|y(\tau)|_0 + |\bar{y}(\tau)|_0) \right] \, d\tau \]
\[ + C \int_0^t \int_\tau^t e^{\omega t} \left[ |y(\tau) - \bar{y}(\tau)|_0 (|y(\tau)|_0 + |\bar{y}(\tau)|_0) + |y(\tau) - \bar{y}(\tau)|_0 (|y(\tau)|_0 + |\bar{y}(\tau)|_0) \right] \, d\tau \]
\[ \leq C \delta \|y(t) - \bar{y}(t)\|_{\omega,\alpha} \]
\[
\begin{align*}
e^{\omega \tau} |y(\tau) - \bar{y}(\tau)|_\alpha (|y(\tau)|_0 + |\bar{y}(\tau)|_0) \, d\tau \\
\leq C \left( \|y - \bar{y}\|_{0,0} (\|y\|_{\omega,\alpha} + \|\bar{y}\|_{\omega,\alpha} + \|v\|_{\omega,0}) + \|\bar{y}\|_{0,0} \|v - \bar{v}\|_{\omega,0} \\
+ \|y - \bar{y}\|_{\omega,\alpha} (\|y\|_{0,0} + \|\bar{y}\|_{0,0}) \right)
\end{align*}
\]

\[\leq C\delta \|y - \bar{y}\|.
\]

As a result, the \(\|\cdot\|\)-norm of the first integral in (2.42) is dominated by \(C\delta \|y - \bar{y}\|\).

2b) For the second integral in (2.42), Remark 2.7 and inequality (2.27) yield

\[
\begin{align*}
e^{\omega t} \left| \int_t^{\infty} P_t^c (F_q(y(\tau)) - F_q(\bar{y}(\tau))) \, d\tau \right|_\beta \\
\leq C \int_t^{\infty} e^{\omega(t-\tau)} e^{\omega \tau} |y(\tau) - \bar{y}(\tau)|_\alpha (|y(\tau)|_0 + |\bar{y}(\tau)|_0) \, d\tau \\
\leq C \|y - \bar{y}\|_{\omega,\alpha} (\|y\|_{0,0} + \|\bar{y}\|_{0,0}) \\
\leq C\delta \|y - \bar{y}\|.
\end{align*}
\]

We have thus established

\[
\|\Phi_q(y, z_0) - \Phi_q(\bar{y}, z_0)\| \leq C\delta \|y - \bar{y}\|,
\]

finishing the proof of the first part of Lemma 2.14.

3) It remains to show the estimate

\[
\|\Phi_q(y, z_0) - \Phi_q(y, \bar{z}_0)\| = \|T_q(\cdot) P^s_q(z_0 - \bar{z}_0)\| \leq C|z - \bar{z}_0|_\beta.
\]

The inequalities (2.35) and (2.36) first imply that

\[
\|T_q(\cdot) P^s_q(z_0 - \bar{z}_0)\|_{\omega,\alpha} \leq C|z - \bar{z}_0|_\alpha.
\]

To treat the norm \(\|\cdot\|_{0,0}\), from (2.19), (2.21), Remark 2.7 (in particular, that \(P^s_q \in \mathcal{B}(\mathcal{E}_\beta, \mathcal{E}_0)\)), Remark 2.9 and (2.35), we conclude

\[
|T_q(t) P^s_q(z_0 - \bar{z}_0)|_0 \leq |S(t) P^s_q(z_0 - \bar{z}_0)|_0 + \int_0^t |S(t-s)B_k T_q(s) P^s_q(z_0 - \bar{z}_0)|_0 \, ds
\]

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\[
\leq C|z_0 - \bar{z}_0|_\beta + C \int_0^t e^{-\nu s} \, ds \, |z_0 - \bar{z}_0|_\alpha \\
\leq C|z_0 - \bar{z}_0|_\beta.
\]

In a similar way, formula (2.38), Remarks 2.7 and 2.9, as well as inequalities (2.35) and (2.36) yield the remaining bound for \( \|\pi_2 T_q(\cdot) P^s_q(z_0 - \bar{z}_0)\|_{\omega,0} \), so that (2.44) follows.

\[ \text{2.1.3 Stable manifolds} \]

For a small \( q_0 > 0 \) and each \( q \in [-q_0, q_0] \), we now construct a function \( \phi_q : \text{ran}(P^s_q) \to P^c_q \) whose graph contains \( Y_q \) and it is a stable manifold \( \mathcal{M}^s_q \) for the system (2.2). We further prove that the sets \( \mathcal{M}^s_q \) satisfy the standard properties of stable manifolds and that they foliate a small neighborhood of \( Y_0 \).

Let \( \delta, \delta_0 > 0 \) be sufficiently small and \( q_0 > 0 \). Take \( |q| \leq q_0 \) and \( z_0 \in \text{ran}(P^s_q) \cap \mathbb{B}_{\delta_0}(| \cdot |_\beta) \). Lemma 2.14 then yields a unique function \( y^q_{z_0} : \mathbb{R}_+ \to \mathcal{E}_\beta \) which belongs to \( \mathbb{B}_\delta(|| \cdot ||) \) and is a fixed point of the Lyapunov-Perron operator \( \Phi_q(\cdot, z_0) \); that is,

\[
y^q_{z_0}(t) = T_q(t)z_0 + \int_0^t T_q(t - \tau) P^s_q F_q(y^q_{z_0}(\tau)) \, d\tau - \int_0^\infty P^c_q F_q(y^q_{z_0}(\tau)) \, d\tau \tag{2.45}
\]

for \( t \geq 0 \). At \( t = 0 \) we obtain the identity

\[
y^q_{z_0}(0) = z_0 - \int_0^\infty P^c_q F_q(y^q_{z_0}(\tau)) \, d\tau
\]

for all \( z_0 \in \text{ran}(P^s_q) \cap \mathbb{B}_{\delta_0}(| \cdot |_\beta) \). We define the function \( \phi_q : \text{ran}(P^s_q) \cap \mathbb{B}_{\delta_0}(| \cdot |_\beta) \to \text{ran}(P^c_q) \) by

\[
\phi_q(z_0) = -\int_0^\infty P^c_q F_q(y^q_{z_0}(\tau)) \, d\tau. \tag{2.46}
\]

In this notation, we have \( y^q_{z_0}(0) = z_0 + \phi_q(z_0) \) so that \( y^q_{z_0}(0) \) belongs to the graph \( \text{graph}_{\delta_0} \phi_q \) of \( \phi_q \) over the small neighborhood \( \text{ran}(P^s_q) \cap \mathbb{B}_{\delta_0}(| \cdot |_\beta) \) of 0. Adding and
substracting the term $\int_0^t P_q F_q(y_q(z_0)(\tau)) d\tau$, we deduce from (2.45) that the fixed point $y = y_{z_0}^q$ of the Lyapunov-Perron operator satisfies the equation

$$y(t) = T_q(t)y(0) + \int_0^t T_q(t-\tau)F_q(y(\tau)) d\tau, \quad t \geq 0. \quad (2.47)$$

Consequently, $y = y_{z_0}^q$ is the mild solution of the nonlinear equation (2.5) in $B_\delta(\|\cdot\|)$, and the function $Y_q + y$ solves (2.2) in the mild sense, cf. Remark 2.3. By uniqueness, $y_0^q$ is the 0 function. Let also $\bar{z}_0$ belong to $\text{ran}(P_\sigma^q) \cap B_{\delta_0}(\|\cdot\|_\beta)$. Taking a sufficiently small $\delta > 0$ in (2.37), we deduce the estimates

$$\|y_{z_0}^q - y_{z_0}^q\| \leq C|z_0 - \bar{z}_0|_{\beta}, \quad \|y_{z_0}^q\| \leq C|z_0|_{\beta}. \quad (2.48)$$

For a number $\eta > 0$ to be fixed below, the stable manifold $M_q^s$ is then defined by

$$M_q^s = \{Y_q + z_0 + \phi_q(z_0) : z_0 \in \text{ran}(P_\sigma^q) \cap B_{\delta_0}(\|\cdot\|_\beta) \cap (Y_0 + B_{\eta}(\|\cdot\|_\beta))\}, \quad (2.49)$$

where $|q| \leq q_0$ and $Y_0 + B_{\eta}(\|\cdot\|_\beta)$ is the closed ball in $E_\beta = E_\alpha \cap E_0$ with radius $\eta$ and centered at the original traveling wave $Y_0$.

**Theorem 2.15.** Assume Hypotheses 2.4 and 2.10. Let $q_0, \delta, \delta_0, \eta > 0$ be sufficiently small, $|q| \leq q_0$, and $\omega > 0$ be given by (2.36). Then the ball $Y_0 + B_{\eta}(\|\cdot\|_\beta)$ is foliated by the stable manifolds $M_q^s$ from (2.49) for the nonlinear equation (2.2) and the following assertions hold.

(i) Each $M_q^s$ is a Lipschitz manifold in $E_\beta$. If $Y(0) \in M_q^s$ and the mild solution $Y(t; Y(0))$ of (2.2) belongs to $Y_0 + B_{\eta}(\|\cdot\|_\beta)$ for some $t \geq 0$, then $Y(t; Y(0))$ is contained in $M_q^s$.

(ii) For each $Y(0) \in M_q^s$ there exists a solution $Y(t; Y(0))$ of (2.2) which exists for all $t \geq 0$ and satisfies $|Y(t; Y(0)) - Y_q|_{\beta} \leq \delta$ as well as
\( (a) \ |Y(t; Y(0)) - Y_q|_\alpha \leq Ce^{-\omega t} |Y(0) - Y_q|_\beta, \)

\( (b) \ |\pi_1(Y(t; Y(0)) - Y_q) - U_q|_0 \leq C |Y(0) - Y_q|_\beta, \)

\( (c) \ |\pi_2(Y(t; Y(0)) - Y_q) - V_q|_0 \leq Ce^{-\omega t} |Y(0) - Y_q|_\beta \)

for all \( t \geq 0. \) Here, \( Y_q = (U_q, V_q) = Y_0(\cdot - q) \) is the shifted traveling wave, \( \pi_1 : Y = (U, V) \to U, \) and \( \pi_2 : Y = (U, V) \to V. \)

(iii) If \( Y(t; Y(0)), t \geq 0, \) is a mild solution of (2.2) with \( Y(0) \in Y_0 + \mathbb{B}_\eta(|\cdot|_\beta) \) that satisfies properties (a)-(c) in item (ii), then \( Y(0) \) belongs to \( \mathcal{M}_q^s. \)

(iv) For \( q \neq \bar{q}, \) we have \( \mathcal{M}_q^s \cap \mathcal{M}_{\bar{q}}^s = \emptyset. \) Moreover, \( Y_0 + \mathbb{B}_\eta(|\cdot|_\beta) = \bigcup_{|q| \leq q_0} \mathcal{M}_q^s. \)

(v) The map \([-q_0, q_0] \to \text{ran}(P^*_q); q \mapsto \phi_q(P^*_q z_0), \) is Lipschitz for each \( z_0 \in \mathbb{B}_{\delta_0}(|\cdot|_\beta). \)

As a result, for each \( Y(0) \in Y_0 + \mathbb{B}_\eta(|\cdot|_\beta) \) there exists exactly one shift \( q \in [-q_0, q_0] \) such that \( Y(0) \in \mathcal{M}_q^s. \)

The following lemma will be used in the proof of Theorem 2.15. Recall the definition of the ball \( \mathbb{B}_\delta(||\cdot||) \) in (2.34).

**Lemma 2.16.** Assume Hypotheses 2.4 and 2.10. Let \( \delta, \delta_0 > 0 \) be chosen small enough, \( q_0 > 0, \) and let \( |q| \leq q_0. \) Take \( y_0 \in \mathcal{E}_\beta = \mathcal{E}_\alpha \cap \mathcal{E}_0. \) Let \( y = Y(\cdot; y_0) \in C([0, t_0), \mathcal{E}_0 \cap \mathcal{E}_\alpha) \) be the mild solution of the nonlinear equation (2.5) with the initial value \( y(0) = y_0, \) where \( t_0 \in (0, \infty]. \) Set \( z_0 = P^*_q y_0 \) and assume that \( |z_0|_\beta \leq \delta_0. \) Then the following assertions are equivalent.

\( (a) \ y_0 = z_0 + \phi_q(z_0) \in \text{graph}_{\delta_0} \phi_q. \)

\( (b) \ y \) can be extended to a global mild solution of (2.5) in \( \mathbb{B}_\delta(||\cdot||), \) and it is the fixed point \( y^*_{q_0} \) of the Lyapunov-Perron operator \( \Phi_q(\cdot, z_0) \) from (2.32).
(c) $y$ can be extended to a global mild solution of (2.5) in $\mathbb{B}_\delta(\|\cdot\|)$.

**Proof.** (a)⇒(b): Assertion (a) and the equations (2.46) and (2.45) yield

$$y_0 = z_0 + \phi_q(z_0) = z_0 - \int_0^\infty P^c_q F_q(y^q_{z_0}(\tau)) \, d\tau = y^q_{z_0}(0),$$

where $y^q_{z_0} \in \mathbb{B}_\delta(\|\cdot\|)$ is the fixed point of $\Phi_q(\cdot, z_0)$. Since their initial values are the same, the mild solutions $y$ and $y^q_{z_0}$ coincide by uniqueness of (2.47); i.e., (b) holds.

(b)⇒(c): This implication is obvious.

(c)⇒(a): In view of (c), Lemma 2.14 shows that the integral

$$z_c := P^c_q y_0 + \int_0^\infty P^c_q F_q(y(\tau)) \, d\tau \in \text{ran}(P^c_q)$$

exists. Since $y$ solves (2.47) and $T_q(t - \tau)$ is the identity on $\text{ran}(P^c_q)$, we can write

$$y(t) = T_q(t)y_0 + \int_0^t T_q(t - \tau)F_q(y(\tau)) \, d\tau$$

$$= T_q(t)P^*_q y_0 + \int_0^t T_q(t - \tau)P^*_q F_q(y(\tau)) \, d\tau - \int_t^\infty P^c_q F_q(y(\tau)) \, d\tau$$

$$+ P^c_q y_0 + \int_t^\infty P^c_q F_q(y(\tau)) \, d\tau + \int_0^t P^c_q F_q(y(\tau)) \, d\tau,$$

using again Lemma 2.14 and (c). The definition of $\Phi_q(y, z_0)$ in (2.32) then yields

$$y(t) = \Phi_q(y, z_0)(t) + z_c, \quad t \geq 0. \tag{2.50}$$

Due to (c) and (2.34), the functions $y$ and $\Phi_q(y, z_0)$ tend to 0 in $E_\alpha$ as $t \to \infty$, and hence $z_c = 0$. Equation (2.50) thus implies $y = \Phi_q(y, z_0)$ so that (a) is a consequence of the observations after (2.46). ■

**Proof of Theorem 2.15.** Recall from Remark 2.3 that all mild solutions of (2.2) are given by $y + Y_q$ for a mild solution $y$ of (2.5).
(i) and (ii). Equations (2.45) and (2.46) show that \( z_0 + \phi_q(z_0) \) is the value of \( \Phi(z_0, \phi_q(z_0)) \) at \( t = 0 \). From (2.48) we then deduce that \( \phi_q \) and hence \( M^s_q \) are Lipschitz in \( \mathcal{E}_\beta = \mathcal{E}_0 \cap \mathcal{E}_\alpha \).

Let \( y_0 + Y_q \) belong to \( M^s_q \), where \( z_0 = P^s_q y_0 \in \text{ran}(P^s_q) \cap \mathbb{B}_{\delta_0}(\| \cdot \|_\beta) \). By Lemma 2.16, the fixed point \( y^s_{z_0} \) is the mild solution \( Y(\cdot; y_0) \) of (2.5) in \( \mathbb{B}_{\delta}(\| \cdot \|) \) of (2.5) with the initial value \( y_0 \). Combined with (2.48) and Remark 2.7, these facts imply (ii).

Take \( t_0 > 0 \) such that \( |y(t_0) + Y_q - Y_0|_\beta \leq \eta \). It is easy to see that \( y(t_0 + \cdot) \) still belongs to \( \mathbb{B}_0(\| \cdot \|) \) and that it is the mild solution of (2.5) with the initial value \( y(t_0) \).

Moreover, Remark 2.7 (in particular, that \( P^s_q \in \mathcal{B}(\mathcal{E}_\beta) \)) and (2.15) yield

\[
|P^s_q y(t_0)|_\beta \leq C \left( |y(t_0) + Y_q - Y_0|_\beta + |Y_0 - Y_q|_\beta \right) \leq C(\eta + q) \leq \delta_0, \tag{2.51}
\]

if we choose \( \eta > 0 \) and \( q_0 \) small enough. (Note that the constants are uniform for \( q \) in compact intervals and independent of \( \eta \).) Therefore, \( y(t_0) + Y_q \) is contained in \( M^s_q \) thanks to Lemma 2.16. So (i) is shown.

(iii). Take \( Y(0) \in Y_0 + \mathbb{B}_\eta(\| \cdot \|_\beta) \) that satisfies properties (a)–(c) in item (ii). The function \( y(t) = Y(t; Y(0)) - Y_q \) is a mild solution of (2.5) with initial value \( Y(0) - Y_q \).

Using again (2.15), we can estimate

\[
|Y(0) - Y_q|_\beta \leq |Y(0) - Y_0|_\beta + |Y_q - Y_0|_q \leq \eta + Cq.
\]

Possibly decreasing \( \eta, q_0 > 0 \), we deduce from conditions (a)–(c) the inequality (2.34) for \( y \) and from Remark 2.7 the estimate \( |P^s_q(Y(0) - Y_q)|_\beta \leq \delta_0 \). Lemma 2.16 now yields that \( y(0) \in \text{graph}_{1_{\delta_0}, \phi_q} \), proving (iii).

(iv). By Theorem 3.14 in [GLS], we can fix a sufficiently small radius \( \eta > 0 \) such that for each point \( Y(0) \) in the ball \( Y_0 + \mathbb{B}_\eta(\| \cdot \|_\beta; Y_0) \) there exists a shift \( q = q(Y(0)) \)
such that the solution \( Y(\cdot; Y(0)) \) of (2.2) satisfies properties (a)-(c) of item (ii). We remark that in Theorem 3.14 we can choose the same number \( \delta > 0 \) as in the current proof and exponents \( \nu, \rho > \omega \) which are different from our exponents \( \nu \) and \( \rho \) in (2.36).

Item (iii) then implies that \( Y(0) \) is contained in \( \mathcal{M}^s_\bar{q} \). If \( Y(0) \) is also an element of \( \mathcal{M}^s_q \) for some \( \bar{q} \in [-q_0, q_0] \), then the corresponding solution \( y \) would converge both to \( Y_q \) and \( Y_\bar{q} \) as \( t \to \infty \), and so \( q = \bar{q} \). Hence, (iv) holds.

(v). Let \( |q|, |ar{q}| \leq q_0 \) and \( z_0 \in \mathbb{B}_{\delta_0}(\| \cdot \|_{\beta}) \). The maps \( q \mapsto P^e_q \in \mathcal{B}(\mathcal{E}_\kappa, \mathcal{E}_\beta) \), \( q \mapsto P^s_q \in \mathcal{B}(\mathcal{E}_\kappa) \) and \( q \mapsto B_q \in \mathcal{B}(\mathcal{E}_\alpha, \mathcal{E}_0) \) are Lipschitz for \( \kappa \in \{\beta, \alpha\} \) due to (2.14) and Remark 2.9. Lemma 3.7 of [GLS] implies that \( \gamma_\alpha Y'_0 \) and \( \gamma_{\alpha}^{-1} Y'_0 \) are bounded. Using (2.6) and (2.15), we then deduce the estimate

\[
|F_q(Y) - F_\bar{q}(Y)|_\beta \leq C |Y|_\kappa |q - \bar{q}|
\]

for all \( Y \in \mathcal{E}_\kappa \) and \( \kappa \in \{0, \alpha\} \). In view of (2.46), for (v) it remains to check that the map \( q \mapsto y^q_{z_0} =: y_q \) is Lipschitz for \( \| \cdot \| \). Since \( y_q \) is the fixed point, we infer from (2.32) the identity

\[
y_q - y_\bar{q} = \Phi_q(y_q, z_0) - \Phi_\bar{q}(y_q, z_0) + \Phi_q(y_\bar{q}, z_0) - \Phi_\bar{q}(y_\bar{q}, z_0).
\]

By (2.43), the second difference on the right-hand side is bounded by \( C\delta \|y_q - y_\bar{q}\| \) and can thus be absorbed by the left-hand side possibly after decreasing \( \delta > 0 \) once more.

To control the other difference, we note that the bounded perturbation theorem and (2.16) imply that \( q \mapsto T_q(t) \in \mathcal{B}(\mathcal{E}_\kappa) \) is Lipschitz for \( \kappa \in \{0, \alpha\} \) and uniformly for \( t \geq 0 \) in compact sets, see Corollary 3.1.3 of [P]. To extend this property to \( \mathbb{R}_+ \), let \( t \in (n, n + 1] \). We write

\[
T_q(t)P^e_q - T_\bar{q}(t)P^e_\bar{q} = (T_q(t - n) - T_\bar{q}(t - n))T_q(n)P^e_q
\]
\[ + T_q(t-n)T_q(n)P^s_q(P^s_q - P^s_{\bar{q}}) \]
\[ + T_q(t-n) \sum_{k=0}^{n-1} T_q(n-k-1)P^s_q(T_q(1) - T_{\bar{q}}(1))T_q(k)P^s_{\bar{q}} \]
\[ + T_q(t-n)(P^s_q - P^s_{\bar{q}})T_q(n)P^s_{\bar{q}}. \]

In the exponential decay estimate (2.18) for \( T_q(t)P^s_q \) we can replace \( \nu \) by a slightly larger number, see Lemma 3.13 of [GLS]. This and the above mentioned facts lead to the inequality
\[
\| T_q(t)P^s_q - T_{\bar{q}}(t)P^s_{\bar{q}} \|_{B(E_\alpha)} \leq Ce^{-\nu t} |q - \bar{q}|, \quad t \geq 0.
\]

As in Lemma 2.14 one can now show that
\[
\| \Phi_q(z_0, y_q) - \Phi_{\bar{q}}(z_0, y_{\bar{q}}) \| \leq C|q - \bar{q}|.
\]

Summing up, (v) is true. ■

To conclude, we briefly mention two motivating examples borrowed from [GLS1] that fit our setting. More details can be found in the papers [GLSS] and [GLS], respectively. We stress, however, that for this type of examples the Hypotheses 2.1, 2.2 and 2.4 (a) can rigorously be verified not in all cases while the absence of the unstable eigenvalues required in Hypothesis 2.4 (b) is usually checked only numerically for certain ranges of the parameter values.

**Example 2.17. Gasless combustion.** A simple combustion model in one space dimension has been mentioned in the Introduction and is given by the system
\[
\partial_t u = \partial_{xx} u + vg(u), \quad \partial_t v = -\beta vg(u),
\]
where \( g(u) = e^{-\frac{u}{\beta}} \) if \( u > 0 \) and \( g(u) = 0 \) if \( u \leq 0 \). In this system, \( u \) is the temperature, \( v \) is the concentration of unburned fuel, \( g \) is the unit reaction rate, and \( \beta > 0 \) is a
constant parameter. This system was a primary guiding example in [G, GLSS, GLS, GLS1, GLSR]. One motivation for looking at this well-studied problem, in which the reactant does not diffuse, was heat-enhanced methods of oil recovery in which the reactant is coke contained in the rock formation, see [AY]. The value \( u = 0 \) represents the ignition temperature and is also taken to be the background temperature, at which the reaction does not take place.

Clearly, Hypothesis 2.2 is satisfied here. One looks for traveling waves \( Y_0 = (u_0, v_0) \) such that \( Y_- = (u_-, 0) \) with \( u_- > 0 \), \( Y_+ = (0, 1) \), and \( (u_0(x), v_0(x)) \) approaches these end states exponentially as \( x \to \pm \infty \). For each \( \beta > 0 \) there is a unique \( c > 0 \) for which such a wave exists, cf. [GLS1, §3.2]. This wave represents a combustion front that leaves behind of it high temperature \( u_- = 1/\beta \) and no fuel, while in front of it temperature is 0 and there is fuel, with concentration normalized to 1. As discussed in Paragraph 3.2 of [GLS1], Hypothesis 2.10 is true and Hypothesis 2.4 can be verified (partly numerically) for small \( \beta > 0 \).

We note the lack of diffusion in the second equation which inspired the linear Lemma 3.13 in [GLS] used to derive the exponential decay (2.18) from the spectral assumptions in Hypothesis (2.4), and the form of the nonlinear term in this and related problems which inspired the triangular and product structure of the nonlinearity in the current paper that follows from Hypothesis 2.2.

Example 2.18. Exothermic-endothermic chemical reactions. A model in which two chemical reactions occur at rates determined by temperature was studied in [SMS, SKMS], see also [GLS]. One reaction is exothermic (produces heat), the other is
endothermic (absorbs heat). The system reads

\begin{align}
\partial_t y_1 &= \partial_{xx} y_1 + y_2 f_2(y_1) - \sigma y_3 f_3(y_1), \\
\partial_t y_2 &= d_2 \partial_{xx} y_2 - y_2 f_2(y_1), \\
\partial_t y_3 &= d_3 \partial_{xx} y_3 - \tau y_3 f_3(y_1).
\end{align}

(2.52)\hfill (2.53)\hfill (2.54)

Here $y_1$ is the temperature, $y_2$ is the quantity of an exothermic reactant, and $y_3$ is the quantity of an endothermic reactant. The parameters $\sigma$ and $\tau$ are positive, and there are positive constants $a_i$ and $b_i$ such that $f_i(u) = a_i e^{-b_i u}$ for $u > 0$ and $f_i(u) = 0$ for $u \leq 0$. In [SMS, SKMS] it is shown numerically that in certain parameter regimes there exist traveling wave solutions $Y_0$ of (2.52)–(2.54) with speed $c > 0$ and the end states $Y_- = (1 - \frac{\sigma}{\tau}, 0, 0)$ and $Y_+ = (0, 1, 1)$. Moreover, both end states are approached at an exponential rate, the zero eigenvalue of the linearization is simple, and there are no other eigenvalues in the right half plane. A rigorous motivation for the existence of such traveling wave is also given in [GLS, Section 9.2]. Assuming the existence of the traveling wave with these properties, the remaining hypotheses of the current paper are easy to verify.

2.2 Stability of the end states

In this section we prove that a steady-state solution of a class of reaction diffusion systems is Lyapunov stable in the intersection of the Sobolev space $H^1(\mathbb{R})$ and an exponentially weighted space $H^1_\alpha(\mathbb{R})$. The steady-state solution considered herein is the end state of the traveling front associated with the system, and thus the current results complement recent papers by A. Ghazaryan, Y. Latushkin and S. Schecter where stability of the traveling fronts was investigated.
We will introduce some basic definitions and study the spectrum of the operator obtained by linearizing the equation about the end state in Section 2.2.1. Section 2.2.2 will mainly focus on the nonlinear term in the system, and some estimates needed for the subsequent proofs. The proof of stability of the steady state is given in Section 2.2.3, see there the main Theorem 2.38. Finally, in Section 2.2.4 we give generalizations for the reaction diffusion systems of the type considered in [G, GLSS, GLS, GLS1].

2.2.1 The definitions and the linearized operator

In this subsection we introduce some definitions, linearize system (1.1) at an end state and study the spectrum of the operator obtained by linearization.

To set the stage, we define

\[
g(u_1) = \begin{cases} 
    e^{-\frac{1}{u_1}} & \text{if } u_1 > 0; \\
    0 & \text{if } u_1 \leq 0,
\end{cases}
\]  

and consider the combustion system of two equations in \( \mathbb{R}^d \),

\[
\begin{align*}
    u_{1t}(t, x) &= \Delta_x u_1(t, x) + u_2(t, x)g(u_1(t, x)), \quad u_1, u_2 \in \mathbb{R}, \\
    u_{2t}(t, x) &= \epsilon \Delta_x u_2(t, x) - \kappa u_2(t, x)g(u_1(t, x)), \quad x \in \mathbb{R}^d,
\end{align*}
\]

where the parameters \( \epsilon \) and \( \kappa \) satisfy \( 0 \leq \epsilon < 1 \) and \( \kappa > 0 \). In what follows we call this a model system. Let

\[
    u(t, x) = \begin{pmatrix} u_1(t, x) \\ u_2(t, x) \end{pmatrix}, \quad f(u(t, x)) = \begin{pmatrix} f_1(u_1, u_2) \\ f_2(u_1, u_2) \end{pmatrix} = \begin{pmatrix} u_2(t, x)g(u_1(t, x)) \\ -\kappa u_2(t, x)g(u_1(t, x)) \end{pmatrix}.
\]

Then system (2.56) can be rewritten in the vector form as

\[
    u_t(t, x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Delta_x u(t, x) + f(u(t, x)).
\]  

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In system (2.57), given a fixed vector \( e \in \mathbb{R}^d \), we pass to the moving in the direction of \( e \) coordinate frame \( x \mapsto x + cte \) via the formula

\[
w(t, x) = u(t, x + cte), \, x \in \mathbb{R}^d, \, u, w \in \mathbb{R}^2, \, t \geq 0.
\]  

(2.58)

It follows that

\[
w_t(t, x) = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \Delta_x w(t, x) + c(e \cdot \nabla_x)w(t, x) + f(w(t, x)).
\]  

(2.59)

Assume that \( u(t, x) \) is a solution of (2.57). We define \( w(t, x) \) as in (2.58) and record (2.57) at the point \( x + cte \). By a direct computation, it follows that \( w(t, x) \) is a solution of (2.59) recorded at the point \( x + cte \). Conversely, assume that \( w(t, x) \) is a solution of (2.59), we then denote \( u(t, x) \) as \( u(t, x) = w(t, x - cte) \) and record (2.59) at the point \( x - cte \). It is clear that \( u(t, x) \) is also a solution of (2.57) recorded at the point \( x \). Therefore the equations (2.57) and (2.59) are equivalent, that is, each solution of (2.57) generates a solution of (2.59) and vice versa.

We are concerned with the traveling waves for (2.57), that is, the \( t \)-independent solutions \( w(x) \) of (2.59) that approach constant states as \( x \to \pm \infty \),

\[
\lim_{x \to -\infty} w(x) = u_-, \quad \lim_{x \to +\infty} w(x) = u_+.
\]

Such solutions are called \textit{pulses} if \( u_- = u_+ \) and \textit{fronts} if \( u_- \neq u_+ \). Often, these solutions approach both end states at an exponential rate, i.e., there exist numbers \( K > 0 \) and \( \omega_- < 0 < \omega_+ \) such that

\[
||w(x) - u_-|| \leq Ke^{\omega_-x} \text{ for } x \leq 0 \quad \text{and} \quad ||w(x) - u_+|| \leq Ke^{\omega_+x} \text{ for } x \geq 0.
\]

As we have mentioned above, in this section we are interested in the end-states of the fronts and pulses, that is, in stability of the equilibrium (that is, \( x \)-independent)
solutions. It is clear that systems (2.56) or (2.59) have equilibrium solutions of two types: one type is when \( u_1(x) \) is equal to a real constant while \( u_2(x) = 0 \), and the other type is when \( u_1(x) = 0 \) while \( u_2(x) \) is equal to a real constant. In particular, we can choose \( u_1 = 1/\kappa, u_2 = 0 \), which is the equilibrium corresponds to the completely burnt reactant and is located behind the front, and \( u_1 = 0, u_2 = 1 \) which corresponding to the unburnt substance. In other words, we choose \( u_\sim = (1/\kappa, 0) \) and \( u_+ = (0, 1) \), see [GLSS] and Remark 2.26 for more explanations why \( u_\sim \) and \( u_+ \) are chosen this way.

To begin the analysis one linearizes (2.59) about the traveling wave \( w(x) \), and obtains a linear differential equation \( \partial_t v = Lv \), with \( v(t) \in \mathcal{X} \), for an appropriate function space \( \mathcal{X} \) of allowable perturbations. The differential expression \( L \) generates a differential operator \( \mathcal{L} \) in \( \mathcal{X} \) with an appropriate domain. Our usual choices of the space \( \mathcal{X} \) in this section is either the Sobolev space \( H^1 \), or a weighted Sobolev space, or the intersection of the two.

**Definition 2.19.** We call a time-independent solution \( w(x) \) of (2.59) *spectrally stable* in \( \mathcal{X} \) if the spectrum \( \text{Sp}(\mathcal{L}) \) of the operator \( \mathcal{L} \) is contained in \( \{ \lambda : \text{Re} \lambda < a \} \cup \{0\} \) for some \( a > 0 \), in addition, when \( w(x) \) is a pulse or a front, 0 is a simple eigenvalue of \( \mathcal{L} \).

An important degenerate (partly parabolic) example of model (2.56) is the following one dimensional gasless combustion model of a solid fuel, studied in [GLSS],

\[
\begin{align*}
\partial_t u_1 &= \partial_{xx} u_1 + u_2 g(u_1), \\
\partial_t u_2 &= -\kappa u_2 g(u_1), \quad x \in \mathbb{R},
\end{align*}
\]

(2.60)

with \( \kappa > 0 \), where \( u_1 \) is the temperature, \( u_2 \) represents the concentration of unburned
fuel, and $g$ is the unit reaction rate. The authors of [GLSS] investigated a traveling wave solution $(u_1, u_2)(\xi)$, $\xi = x - ct$ for $c > 0$ where $c$ is the speed of the front. Furthermore, $(u_1, u_2)(\xi)$ approaches the end states exponentially. However, the traveling wave is not spectrally stable in $H^1(\mathbb{R})$. The authors introduced a weight function $e^{\alpha \xi}$, where $\alpha$ is positive and small, such that the perturbation of the traveling wave solution that belongs to this weighted space will approach 0 exponentially near the right end state, that is, when $\xi \to \infty$. In the weighted space, the nonlinear terms in (2.60) does not yield a locally Lipschitz mapping. In order to prove stability of the traveling wave, the authors used the method in [G] to prove that perturbations of the traveling wave that are small in both the weighted norm and the unweighted norm decay exponentially to the traveling wave in the weighted norm. As we will see in what follows, the study of stability of the end-states of the front encounters similar difficulties.

The problem of stability for a general time-independent solution $w(x)$ of (2.59) is quite hard. Therefore, we now want to study a time independent solution $w(x)$ of (2.59) of the form $w(x) = \phi(x \cdot e)$ where we assume that the function $\phi$ depends on only one scalar variable. Plugging $z = x \cdot e$ in equation (2.59) and assuming that $e \cdot e = 1$ we have an ODE:

$$
\begin{pmatrix}
1 & 0 \\
0 & \epsilon
\end{pmatrix}
\phi_{zz}(z) + c\phi_z(z) + f(\phi(z)) = 0.
$$

We can now perturb the function $\phi$ by either

(i) adding a function that depends only on one space variable $z$, that is, considering the solution $w(t, x)$ of (2.59) with the initial condition

$$
w(0, x) = \phi(x \cdot e) + v(0, x \cdot e)
$$

(2.61)
with some \( \mathbf{v} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2 \) from an appropriate function space; or by

(ii) adding a function that depends on all spatial variables, that is, considering the solution \( \mathbf{w}(t, \mathbf{x}) \) of (2.59) with the initial condition

\[
\mathbf{w}(0, \mathbf{x}) = \phi(\mathbf{x} \cdot \mathbf{e}) + \mathbf{v}(0, \mathbf{x})
\]

with some \( \mathbf{v} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^2 \) from an appropriate function space.

**Definition 2.20.** We call \( \phi \) the 1-dim front if the perturbations of type (i) are considered, and we call \( \phi \) the planar front if the perturbations of type (ii) are considered.

This leads to the spectral analysis of two different operators acting on \( H^1(\mathbb{R})^2 \) or \( H^1(\mathbb{R}^d)^2 \) respectively.

In this section, we will only consider perturbing the steady state solution \( \mathbf{u}_- \) of (2.59) using the perturbation of type (i). Assuming that \( \mathbf{w}(t, \mathbf{x}) = \mathbf{w}(t, z) \) and linearizing (2.59) at \( \mathbf{u}_- \) we have

\[
\mathbf{w}_t(t, z) = \begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix} \partial_{zz} \mathbf{w}(t, z) + c \partial_z \mathbf{w}(t, z) + \partial_u f(\mathbf{u}_-) \mathbf{w}(t, z),
\]

where \( \partial_u f(\mathbf{u}_-) \) is the differential of the map \( f \) with respect to \( \mathbf{u} \) evaluated at \( \mathbf{u}_- \).

Suppose that the linear operator associated with (2.62) has essential spectrum in the half plane \( \text{Re} \lambda \geq 0 \), which means the steady state solution will not be spectrally stable. To fix this problem, one introduces a weight function that shifts the essential spectrum to the left when it is possible. Given a real parameter \( \alpha \), we shall say that \( \gamma_\alpha : \mathbb{R} \mapsto \mathbb{R} \) is a weight function of class \( \alpha \) if \( \gamma_\alpha(z) = e^{\alpha z} \) for \( z \in \mathbb{R} \). The weighted space with the weight function \( \gamma_\alpha \) is defined as

\[
H^1_\alpha(\mathbb{R}) = \{ u \in H^1_{\text{loc}}(\mathbb{R}) : \gamma_\alpha(\cdot)u(\cdot) \in H^1(\mathbb{R}) \}.
\]
We denote the norm of $u$ on the unweighted space $H^1(\mathbb{R})$ by $\| \cdot \|_0$ and the norm on the weighted space $H^1_\alpha(\mathbb{R})$ by $\|u\|_\alpha = \|\gamma_\alpha u\|_0$. A big difficulty is that the nonlinear term in (2.62) will not yield a locally Lipschitz mapping on the weighted space $H^1_\alpha(\mathbb{R})$.

To fix this problem, we introduce a new space

$$ E := H^1(\mathbb{R}) \cap H^1_\alpha(\mathbb{R}) \quad \text{with} \quad \|u\|_E = \max\{\|u\|_0, \|u\|_\alpha\}. \quad (2.64) $$

To describe the property of the perturbation of solutions to be bounded in time (or decaying) with respect to an exponentially weighted norm, we shall use the term *convective stability*.

**Definition 2.21.** We say that the perturbation $v(\cdot, t)$ of the solution $w(\cdot, t)$ of (2.59) is

(i) **convectively Lyapunov stable** provided $\sup_{t \geq 0} \|v(\cdot, t)\|_\alpha < \infty$;

(ii) **convectively exponentially stable** provided $\lim_{t \to \infty} e^{\nu t} \|v(\cdot, t)\|_\alpha = 0$ for some positive $\nu > 0$;

(iii) **convectively algebraically stable** provided $\lim_{t \to \infty} t^n \|v(\cdot, t)\|_\alpha = 0$ for some positive integer $n > 0$.

We say that the equilibrium solution is **equidirectionally stable** if it is stable for any vector $e$.

In this section, we will consider a perturbation depending only on one space variable $z$ of an equilibrium solution $u_-$, that is, we want to fix a function $v(z)$ in weighted space $E^2$ such that the solution of (2.59) satisfies (2.61).

Since we want to use only type (i) perturbations, a solution is needed of the form $w(t, z) = u_- + v(t, z)$ with $v(t, \cdot)$ which belongs to an appropriate space of functions.
on \( \mathbb{R} \). With this notation we will have the following equation for the perturbation \( v(t,z) \):

\[
v_t(t,z) = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} v_{zz}(t,z) + cv_z(t,z) + f(v(t,z) + u_-). \tag{2.65}
\]

Introducing the nonlinear term

\[
H(v(t,z)) = f(u_- + v(t,z)) - f(u_-) - \partial_u f(u_-) v(t,z),
\]

equation (2.65) can be rewritten as follows:

\[
v_t(t,z) = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} v_{zz}(t,z) + cv_z(t,z) + \partial_u f(u_-) v(t,z) + H(v(t,z)). \tag{2.66}
\]

Since

\[
\partial_u f(u_-) = \begin{pmatrix} u_2 e^{-1/u_1} & e^{-1/u_1} \\ -\kappa u_2 e^{-1/u_1} & -\kappa e^{-1/u_1} \end{pmatrix} \bigg|_{u=u_-} = \begin{pmatrix} 0 & e^{-\kappa} \\ 0 & -\kappa e^{-\kappa} \end{pmatrix},
\]

we therefore have

\[
v_t(t,z) = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} v_{zz}(t,z) + cv_z(t,z) + \begin{pmatrix} 0 & e^{-\kappa} \\ 0 & -\kappa e^{-\kappa} \end{pmatrix} v(t,z) + H(v(t,z)).
\]

We now define the constant coefficient linear differential expression \( L \) by

\[
L = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \partial_{zz} + c\partial_z \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & e^{-\kappa} \\ 0 & -\kappa e^{-\kappa} \end{pmatrix}. \tag{2.67}
\]

To determine the stability of the zero solution in (2.66), we need spectral information regarding the linear operator associated with (2.67). Consider the system of differential expressions \( L \) given by (2.67). We will now define several differential operators associated with \( L \).

We define the linear operator \( \mathcal{L} \) on \( H^1(\mathbb{R})^2 \) by the formula \( u \to Lu \) and the domain of \( \mathcal{L} \) as the set of \((u_1,u_2)\) where \( u_1,u_2 \in H^3(\mathbb{R}) \). For the space \( L^2(\mathbb{R}) \), the domain of \( \mathcal{L} \) in \( L^2(\mathbb{R})^2 \) is the set of \((u_1,u_2)\) where \( u_1,u_2 \in H^2(\mathbb{R}) \).
We will show below that the spectrum of $L$ touches the imaginary axis so that the equilibrium solution is not spectrally stable in $H^1(\mathbb{R})^2$. A way out of this problem is to use a weighted space $H^1_\alpha(\mathbb{R})^2$.

We define the operator $L_\alpha$ on $H^1_\alpha(\mathbb{R})^2$ as the linear operator given by the formula $u \rightarrow Lu$, and the domain of $L_\alpha$ is the set \{(u_1, u_2) : \gamma_\alpha(\cdot)u_1(\cdot), \gamma_\alpha(\cdot)u_2(\cdot) \in H^3(\mathbb{R})\},$ see formula (2.63). Similarly, we define the weighted space $L^2_\alpha(\mathbb{R}) := \{u : \gamma_\alpha(\cdot)u(\cdot) \in L^2(\mathbb{R})\}$, the domain of $L_\alpha$ on $L^2_\alpha(\mathbb{R})^2$ is the set \{(u_1, u_2) : \gamma_\alpha(\cdot)u_1(\cdot), \gamma_\alpha(\cdot)u_2(\cdot) \in H^2(\mathbb{R})\}.

We denote by $L_\mathcal{E}$ the linear operator on $\mathcal{E}^2$ given by $u \rightarrow Lu$ where the domain of $L_\mathcal{E}$ is the set of $u$ on $\mathcal{E}^2$ satisfying $u \in \text{dom}(L) \cap \text{dom}(L_\alpha)$, where $\text{dom}(L)$ and $\text{dom}(L_\alpha)$ are respective domains defined above.

We recall that for a general closed density defined operator $T$, the resolvent set $\rho(T)$ is the set of $\lambda \in \mathbb{C}$ such that $T - \lambda I$ has a bounded inverse. The complement of $\rho(T)$ is the spectrum $\text{Sp}(T)$. It is the union of the discrete spectrum $\text{Sp}_d(T)$, which is the set of isolated points in $\text{Sp}(T)$ that are eigenvalues of $T$ of finite algebraic multiplicity, and the essential spectrum $\text{Sp}_{\text{ess}}(T)$, which is the rest.

In the remaining part of this subsection we collect several elementary facts about the constant coefficient differential operator $L$. We will first use Fourier transform to find $\text{Sp}(L)$ on $L^2(\mathbb{R})^2$. The operator $L$ on $L^2(\mathbb{R})^2$ is similar to the operator of multiplication on $L^2(\mathbb{R})^2$ by the matrix-valued function $M(\theta)$, where

$$M(\theta) = -\begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \theta^2 + i\theta c I + \partial_u f(u_-), \theta \in \mathbb{R},$$

(2.68)

see e.g. [EN, Section 6.5]. The spectrum of $L$ on $L^2(\mathbb{R})^2$ is the closure of the union over $\theta \in \mathbb{R}$ of the spectra of the matrices $M(\theta)$. Hence the spectrum of $L$ is equal to
closure of the set of \( \lambda \in \mathbb{C} \) for which there exists \( \theta \in \mathbb{R} \) such that

\[
\det(M(\theta) - \lambda I) = \det\left(-\begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \theta^2 + (i\theta c - \lambda) I + \partial u f(u_\cdot))\right) = 0.
\]

It is a collection of curves of the form \( \lambda = \lambda_k(\theta) \), where \( \lambda_k(\theta) \) are the eigenvalues of the matrices \( M(\theta) \).

Actually, the spectrum of \( \mathcal{L} \) on \( L^2(\mathbb{R})^2 \) is equal to its spectrum on \( H^1(\mathbb{R})^2 \), which is proved in the following lemma.

**Lemma 2.22.** The constant coefficient linear operator \( \mathcal{L} \) associated with the differential expression \( L \) in (2.67) have the same spectrum on \( L^2(\mathbb{R})^2 \) and \( H^1(\mathbb{R})^2 \).

**Proof.** We will denote the operator associated with \( L \) by \( \mathcal{L}_{L^2} \) on \( L^2(\mathbb{R})^2 \) and by \( \mathcal{L}_{H^1} \) on \( H^1(\mathbb{R})^2 \). We recall that the operator \( \partial_{z,L^2} \) has the domain \( H^1(\mathbb{R}) \) and spectrum \( i\mathbb{R} \). Therefore, the operator

\[
\mathcal{D} = \begin{pmatrix} \partial_{z,L^2} + \mathcal{I} & 0 \\ 0 & \partial_{z,L^2} + \mathcal{I} \end{pmatrix} : H^1(\mathbb{R}) \times H^1(\mathbb{R}) \mapsto L^2(\mathbb{R}) \times L^2(\mathbb{R})
\]

is an isomorphism. Using the identity

\[
\mathcal{D} \mathcal{L}_{H^1} \mathbf{v} = \mathcal{L}_{L^2} \mathcal{D} \mathbf{v}, \text{ for all } \mathbf{v} \in \text{dom} \mathcal{L}_{H^1} = H^3(\mathbb{R})^2,
\]

yields \( \mathcal{D} \mathcal{L}_{H^1} \mathcal{D}^{-1} = \mathcal{L}_{L^2} \). Thus, we can conclude that \( \text{Sp}(\mathcal{L}_{H^1}) = \text{Sp}(\mathcal{D} \mathcal{L}_{L^2} \mathcal{D}^{-1}) = \text{Sp}(\mathcal{L}_{L^2}) \) as claimed.

By Lemma 2.22 and the discussion preceding the lemma, for the operator \( \mathcal{L} \) associated with the differential expression

\[
L = \begin{pmatrix} \partial_{zz} + c \partial_z & e^{-\kappa} \\ 0 & \epsilon \partial_{zz} + c \partial_z - \kappa e^{-\kappa} \end{pmatrix}, \quad (2.69)
\]
the spectrum of $L$ on $L^2(\mathbb{R})^2$ and $H^1(\mathbb{R})^2$ is
\[
\text{Sp}(L) = \bigcup_{\theta \in \mathbb{R}} \text{Sp}\left(\begin{pmatrix} -\theta^2 + ci\theta & e^{-\kappa} \\ 0 & -\epsilon\theta^2 + ci\theta - \kappa e^{-\kappa} \end{pmatrix} \right) = \bigcup_{\theta \in \mathbb{R}} (-\theta^2 + ci\theta) \bigcup_{\theta \in \mathbb{R}} (-\epsilon\theta^2 + ci\theta - \kappa e^{-\kappa}).
\] (2.70)

Hence, the spectrum of $L$ on $L^2(\mathbb{R})^2$ is the union of the two curves $\lambda_1 = -\theta^2 + ci\theta$ and $\lambda_2 = -\epsilon\theta^2 + ci\theta - \kappa e^{-\kappa}$ where $\theta \in \mathbb{R}$, therefore $\sup\{\text{Re}\lambda : \lambda \in \text{Sp}(L)\} = 0$. Thus the spectrum of $L$ on $L^2(\mathbb{R})^2$ and $H^1(\mathbb{R})^2$ touches the imaginary axis.

Next, we will tackle $\text{Sp}(L_\alpha)$ on $H^1_\alpha(\mathbb{R})^2$, which can be described as following. Let $\mathcal{E}_0$ be $L^2(\mathbb{R})$ or $H^1(\mathbb{R})$ and $\mathcal{E}_\alpha = \{u : \gamma_\alpha(z)u(z) \in \mathcal{E}_0\}$. The linear operator $\mathcal{M}$ defined by $\mathcal{M}u = \gamma_\alpha u$ is an isomorphism from $\mathcal{E}^2_\alpha$ to $\mathcal{E}^2_0$. Define the linear operator $\hat{\mathcal{L}} = \mathcal{M}L_\alpha \mathcal{M}^{-1}$ on $\mathcal{E}^2_0$, with domain $H^2(\mathbb{R})^2$ if $\mathcal{E}^2_0 = L^2(\mathbb{R})^2$, or domain $H^3(\mathbb{R})^2$ if $\mathcal{E}^2_0 = H^1(\mathbb{R})^2$. It is therefore similar to $L_\alpha$ on $\mathcal{E}^2_\alpha$ and hence has the same spectrum.

Assume that $(v_1, v_2)$ belongs to the weighted space with the weight function $\gamma_\alpha$. It follows that $(v_1, v_2) = \gamma^{-1}_\alpha(\tilde{v}_1, \tilde{v}_2)$ with $\tilde{v} = (\tilde{v}_1, \tilde{v}_2) \in L^2(\mathbb{R})^2$. By substituting into the formula for $L \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and multiplying by $\gamma_\alpha$ and noticing that
\[
\hat{\partial}_z \tilde{v} = \gamma_\alpha \partial_z \gamma^{-1}_\alpha \tilde{v} = \gamma_\alpha (\gamma^{-1}_\alpha) \hat{\partial}_z \tilde{v} + \gamma^{-1}_\alpha \tilde{v} = (\partial_z - \alpha) \tilde{v},
\]
we can rewrite the linear differential expression
\[
\hat{L} \tilde{v} = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} (\partial_z - \alpha)^2 + c(\partial_z - \alpha) + \begin{pmatrix} 0 & e^{-\kappa} \\ 0 & -\kappa e^{-\kappa} \end{pmatrix} \tilde{v} = \begin{pmatrix} L - 2\alpha \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} & \alpha^2 \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} - c\alpha I \end{pmatrix} \tilde{v}.
\]

Via the Fourier transform, the operator $\hat{\mathcal{L}}$ on $L^2(\mathbb{R})^2$ is similar to the operator of multiplication on $L^2(\mathbb{R})^2$ by the matrix-valued function
\[
N(\theta) = -\theta^2 \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} + i\theta(cI - 2\alpha \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}) + \alpha^2 \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} - c\alpha I + \begin{pmatrix} 0 & e^{-\kappa} \\ 0 & -\kappa e^{-\kappa} \end{pmatrix}.
\]
Hence the spectrum of $\hat{L}$ on $L^2(\mathbb{R})^2$ equals to that of multiplication by $N$ on $L^2(\mathbb{R})^2$.

Thus, we find that the spectrum of the operator $\hat{L}$ is the union of the two curves

$$\lambda_1 = -\theta^2 + (c - 2\alpha)\theta i + \alpha^2 - c\alpha$$
$$\lambda_2 = -\varepsilon\theta^2 + (c - 2\alpha\varepsilon)\theta i + \varepsilon\alpha^2 - c\alpha - \kappa e^{-\kappa}$$

for all $\theta \in \mathbb{R}$. Then

$$\sup\{\Re \lambda : \lambda \in \text{Sp}_{\text{ess}}(\mathcal{L}_\alpha)\} = \sup\{\Re \lambda : \lambda \in \text{Sp}_{\text{ess}}(\hat{\mathcal{L}})\}$$
$$= \max\{\alpha^2 - c\alpha, \varepsilon\alpha^2 - c\alpha - \kappa e^{-\kappa}\}$$
$$= \alpha^2 - c\alpha.$$

The linear operator $\mathcal{L}_\alpha$ is constant-coefficient operator, so $\text{Sp}(\mathcal{L}_\alpha) = \text{Sp}_{\text{ess}}(\mathcal{L}_\alpha)$. Also we have the following analogue of Lemma 2.22.

**Lemma 2.23.** The linear operator $\mathcal{L}_\alpha$ associated with the differential expression, that is defined in (2.69), has the same spectrum on $L^2_\alpha(\mathbb{R})^2$ and $H^1_\alpha(\mathbb{R})^2$.

For $\alpha \in (0, c/2)$, we will have $\sup\{\Re \lambda : \lambda \in \text{Sp}(\mathcal{L}_\alpha)\} < 0$ so that the spectrum $\text{Sp}(\mathcal{L}_\alpha)$ has been moved to the left of the imaginary axis. We summarize this result as the following proposition.

**Proposition 2.24.** On the unweighted space $H^1(\mathbb{R})^2$, one has

$$\sup\{\Re \lambda : \lambda \in \text{Sp}(\mathcal{L})\} = 0,$$

the spectrum of $\mathcal{L}$ will touch the imaginary axis. On the weighted space $H^1_\alpha(\mathbb{R})^2$, if $0 < \alpha < c/2$, then the spectrum of $\mathcal{L}_\alpha$ will be bounded away from the imaginary axis and $\sup\{\Re \lambda : \lambda \in \text{Sp}(\mathcal{L}_\alpha)\} < -\nu$ for some $\nu > 0$.

In system (2.66), we have the following triangular structure,

$$\mathbf{v}_t = \begin{pmatrix} \partial_{zz} + c\partial_z & e^{-\kappa} \\ 0 & \varepsilon\partial_{zz} + c\partial_z - \kappa e^{-\kappa} \end{pmatrix} \mathbf{v} + H(\mathbf{v}).$$
Let
\[ L_1 = \partial_{zz} + c\partial_z, \quad (2.71) \]
\[ L_2 = \varepsilon \partial_{zz} + c\partial_z - \kappa e^{-\kappa}, \quad (2.72) \]

and for \( i = 1, 2 \), let \( \mathcal{L}_i \) be the operators on \( H^1(\mathbb{R}) \) defined by \( v_i \mapsto L_i v_i \), the domain of \( \mathcal{L}_i \) on \( H^1(\mathbb{R}) \) is the set of \( v_i \) where \( v_i \in H^3(\mathbb{R}) \).

**Lemma 2.25.** Consider the operators \( \mathcal{L}, \mathcal{L}^{(1)} \) and \( \mathcal{L}^{(2)} \) associated with the differential expressions (2.69), (2.71) and (2.72), respectively,

(1) The operator \( \mathcal{L}_1 \) generates a bounded semigroup on \( H^1(\mathbb{R}) \);

(2) The operator \( \mathcal{L}_2 \) on \( H^1(\mathbb{R}) \) satisfies \( \sup \{ Re \lambda : \lambda \in \text{Sp}(\mathcal{L}_2) \} < 0 \);

(3) The following is true on \( H^1(\mathbb{R}) \):

(a) \( \sup \{ Re \lambda : \lambda \in \text{Sp}(\mathcal{L}_1) \} \leq 0 \);

(b) \( \sup \{ Re \lambda : \lambda \in \text{Sp}(\mathcal{L}) \} \leq 0 \);

(c) There exist \( \rho > 0 \) and \( K > 0 \) such that \( \| e^{t \mathcal{L}_2} \|_{H^1(\mathbb{R}) \to H^1(\mathbb{R})} \leq Ke^{-\rho t} \) for \( t \geq 0 \).

**Proof.** We claim that the semigroups generated by the operators \( \mathcal{L}_i, \ i = 1, 2 \), on \( L^2(\mathbb{R}) \) and \( H^1(\mathbb{R}) \) are similar (This gives yet another way to prove that \( \mathcal{L}_i \) has the same spectrum on \( L^2(\mathbb{R}) \) and \( H^1(\mathbb{R}) \) for \( i = 1, 2 \)). We denote \( \mathcal{L}_i \) on \( L^2(\mathbb{R}) \) as \( \mathcal{L}_i \mathcal{L}^2 \) and \( \mathcal{L}_i \) on \( H^1(\mathbb{R}) \) as \( \mathcal{L}_i \mathcal{H}^1 \).

Recall that the Fourier transform is an isomorphism of \( H^1(\mathbb{R}) \) onto \( L^2_m(\mathbb{R}) \), where the weight function is \( m(\theta) = (1 + |\theta|^2)^{1/2} \) for \( \theta \in \mathbb{R} \). The operator of multiplication
by the function $m(\theta)$ is an isomorphism of $L^2_m(\mathbb{R})$ onto $L^2(\mathbb{R})$. Under the Fourier transform followed by this isomorphism of $H^1(\mathbb{R})$ onto $L^2_m(\mathbb{R})$, the operator of differentiation on $H^1(\mathbb{R})$ is similar to the operator of multiplication by $i\theta$ on $L^2(\mathbb{R})$. Using this, we have $m\mathcal{F}_1\mathcal{L}_{iH^1} = M(\theta)m\mathcal{F}_1$, where $M(\theta)$ is defined in (2.68). The operator of multiplication by $i\theta$ on $L^2(\mathbb{R})$ is similar to the operator of differentiation on $L^2(\mathbb{R})$ via the Fourier transform, and thus we have $\mathcal{F}_2\mathcal{L}_{iL^2} = M(\theta)\mathcal{F}_2$. It follows that

$$\mathcal{L}_{iH^1} = (m\mathcal{F}_1)^{-1}M(\theta)m\mathcal{F}_1 = (m\mathcal{F}_1)^{-1}(\mathcal{F}_2\mathcal{L}_{iL^2}\mathcal{F}_2^{-1})(m\mathcal{F}_1),$$

and thus the operators on $H^1(\mathbb{R})$ and $L^2(\mathbb{R})$ associated with the same constant-coefficient differential expression are similar. Therefore the semigroups they generate are similar, proving the claim.

The operator $\mathcal{L}_1$ generates a bounded semigroup on $L^2(\mathbb{R})$ by Proposition A.1(1) of [GLS]. So (1) is proved because $\mathcal{L}_1$ on $H^1(\mathbb{R})$ is similar to $\mathcal{L}_1$ on $L^2(\mathbb{R})$.

By using the Fourier transform, we can find that the spectrum of $\mathcal{L}_1$ on $L^2(\mathbb{R})$ is the curve $\lambda_1 = -\theta^2 + ci\theta$ and the spectrum of $\mathcal{L}_2$ on $L^2(\mathbb{R})$ is the curve $\lambda_2 = -\epsilon\theta^2 + ci\theta - \kappa e^{-\kappa}$. Thus $\sup\{Re\lambda : \lambda \in \text{Sp}(\mathcal{L}_1)\} \leq 0$ and $\sup\{Re\lambda : \lambda \in \text{Sp}(\mathcal{L}_2)\} < 0$ on $L^2(\mathbb{R})$. It is also true on $H^1(\mathbb{R})$, proving (2) and (3)(a), (b).

Assertion (3)(c) is a direct consequence of (2), see [GLS, Lemma 3.13].

**Remark 2.26.** To conclude this subsection we explain why the end states $\mathbf{u}_-$ and $\mathbf{u}_+$ for the model system

$$\begin{cases}
    u_{1t}(t, \mathbf{x}) = \partial_{zz}u_1(t, \mathbf{x}) + c\partial_z u_1 + u_2(t, \mathbf{x})g(u_1(t, \mathbf{x})), \ u_1, u_2 \in \mathbb{R}, \\
    u_{2t}(t, \mathbf{x}) = \epsilon \partial_{zz}u_2(t, \mathbf{x}) + c\partial_z u_2 - \kappa u_2(t, \mathbf{x})g(u_1(t, \mathbf{x})), \ \mathbf{x} \in \mathbb{R}^d,
\end{cases}$$

where $g$ is defined in (2.55), were chosen as $\mathbf{u}_- = (1/\kappa, 0)$ and $\mathbf{u}_+ = (0, 1)$, see the discussion earlier in this subsection.
Let $\phi = (\phi_1, \phi_2)$ be a time-independent solution of the model system so that $\phi$ satisfies the ODE system

$$\begin{cases}
\partial_{zz}\phi_1(x) + c\partial_z\phi_1 + \phi_2(x)g(\phi_1(x)) = 0, \\
\epsilon\partial_{zz}\phi_2(x) + c\partial_z\phi_2 - \kappa\phi_2(x)g(\phi_1(x)) = 0.
\end{cases} \tag{2.74}$$

We are interested in solutions of (2.74) that satisfy the boundary conditions at $z \to \pm \infty$,

$$(\phi_1, \phi_2)(-\infty) = (\phi_1^*, 0), \quad (\phi_1, \phi_2)(\infty) = (0, 1).$$

Such solutions represent traveling combustion fronts. Here, the left temperature $\phi_1^*$ is an unknown to be determined.

In the ODE system (2.74), we set $\phi_3 = \partial_z\phi_1$ and $\phi_4 = \partial_z\phi_2$, and also use prime to denote the derivative with respect to $z$, to obtain the following first-order system:

$$\phi_1' = \phi_3, \tag{2.75}$$

$$\phi_2' = \phi_4, \tag{2.76}$$

$$\phi_3' = -c\phi_3 + \phi_2g(\phi_1), \tag{2.77}$$

$$\phi_4' = -\frac{1}{\epsilon}[c\phi_4 - \kappa\phi_2g(\phi_1)]. \tag{2.78}$$

By adding (2.77) to (2.78) multiplied by $\epsilon/\kappa$, we obtain the following equation,

$$\phi_1'' + c\phi_1' + \frac{\epsilon}{\kappa}\phi_2'' + \frac{c}{\kappa}\phi_2' = 0. \tag{2.79}$$

This expression can be integrated once to produce a function of $z$ that is constant along any traveling wave. We denote this constant by $k$ so that

$$\phi_3 + c\phi_1 + \frac{\epsilon}{\kappa}\phi_4 + \frac{c}{\kappa}\phi_2 = \text{constant} := k. \tag{2.80}$$
For the solution that approaches \((\phi_1, \phi_2, \phi_3, \phi_4) = (0, 1, 0, 0)\) as \(z \to \infty\), we must have \(k = \frac{\xi}{\kappa}\). Substituting \(k = \frac{\xi}{\kappa}\) into equation (2.80) we have
\[
\phi_3 = -c\phi_1 - \frac{\varepsilon}{\kappa} \phi_4 - \frac{c}{\kappa} \phi_2 + \frac{c}{\kappa} \to 0, \quad \phi_1 \to \phi_1^*, \quad \phi_2 \to 0 \text{ and } \phi_4 \to 0 \text{ as } z \to -\infty \quad (2.81)
\]
since the steady solution of the system (2.75)-(2.78) approaches \((\phi_1, \phi_2, \phi_3, \phi_4) = (\phi_1^*, 0, 0, 0)\) as \(z \to -\infty\). Thus we necessarily have \(\phi_1^* = \frac{1}{\kappa}\).

### 2.2.2 Estimates of the nonlinear term

In this subsection, we will mainly focus on the nonlinear term \(H(v)\) in system (2.66) and will show that the nonlinear term yields a locally Lipschitz mapping on the intersection space \(E\), which is essential in proving the existence of solutions of (2.59).

The exposition is quite elementary and serves as a motivation for the discussion in the next chapter. In particular, we substitute \(u = (1/\kappa, 0)\) to the nonlinear term \(H(v)\), to obtain
\[
H(v) = f \left( \begin{pmatrix} 1/\kappa \\ 0 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) - \begin{pmatrix} 0 & e^{-\kappa} \\ 0 & -\kappa e^{-\kappa} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}
\]
\[
= \begin{pmatrix} v_2 e^{-\frac{1}{v_1+1/\kappa}} \\ -\kappa v_2 e^{-\frac{1}{v_1+1/\kappa}} \end{pmatrix} - \begin{pmatrix} v_2 e^{-\kappa} \\ -\kappa v_2 e^{-\kappa} \end{pmatrix}
\]
\[
= \begin{pmatrix} v_2 \left( e^{-\frac{1}{v_1+1/\kappa}} - e^{-\kappa} \right) \\ -\kappa v_2 \left( e^{-\frac{1}{v_1+1/\kappa}} - e^{-\kappa} \right) \end{pmatrix}.
\]

We introduce the notation \(k = \begin{pmatrix} 1 \\ -\kappa \end{pmatrix}\), then \(H(v)\) can be written as
\[
H(v) = k \left( g \left( \frac{1}{\kappa} + v_1 \right) - g \left( \frac{1}{\kappa} \right) \right) v_2,
\]
(2.82)

where \(g(\cdot)\) is defined as in equation (2.55).

In order to prove that \(H(\cdot)\) is a locally Lipschitz mapping on an appropriate space, we will use below the inclusion \(g(u) \in C^\infty(\mathbb{R})\) when \(u \in \mathbb{R}\). To prove this inclusion,
we first show that \( \lim_{u \to 0^+} g^{(n)}(u) = 0 \) for \( n \in \mathbb{N} \). Indeed, by L’Hospital’s rule for \( u \in \mathbb{N} \),

\[
\lim_{u \to 0^+} u^{-n}e^{-1/u} = \lim_{x \to \infty} x^n e^{-x} = 0. \tag{2.83}
\]

On the other hand, if \( u > 0 \) then \( g(u) = e^{-1/u} \) and it follows that

\[
g'(u) = e^{-1/u}u^{-2}, \quad g''(u) = e^{-1/u}(u^{-4} - 2u^{-3}), \ldots,
\]

\[
g^{(n)}(u) = e^{-1/u}(u^{-2n} + c_{-2n+1}u^{-2n+1} + \cdots + c_{-n-1}u^{-n-1})
\]

are all continuous functions for \( u > 0 \). Using (2.83), we can conclude that \( g^{(n)}(u) \) approaches 0 as \( u \to 0^+ \) for any \( n \in \mathbb{N} \) and thus \( g^{(n)} \) is continuous for all \( u \). The required inclusion \( g(u) \in C^\infty(\mathbb{R}) \) follows.

We recall notation (2.64), that is, \( E = H^1(\mathbb{R}) \cap H^1_\alpha(\mathbb{R}) \) with the norm \( ||u||_E = \max\{||u||_0, ||u||_\alpha\} \). In order to prove the Lipschitz property of \( H(\cdot) \) on \( E \), we will also need the following elementary proposition.

**Proposition 2.27.** (1) If \( u, v \in H^1(\mathbb{R}) \), then \( uv \in H^1(\mathbb{R}) \); also there exists a constant \( C > 0 \) such that \( ||uv||_0 \leq C||u||_0||v||_0 \).

(2) If \( u, v \in E \), then \( uv \in H^1_\alpha(\mathbb{R}) \); also there exists a constant \( C > 0 \) such that \( ||uv||_\alpha \leq C||u||_0||v||_\alpha \).

(3) If \( u, v \in E \), then \( uv \in E \); also there exists a constant \( C > 0 \) such that \( ||uv||_E \leq C||u||_E||v||_E \).

**Proof.** Assertion (1) is a well-know result, see [AF, Theorem 5.23]. Assertion (2) can be proved by

\[
||uv||_\alpha = ||\gamma_\alpha uv||_0 \leq C||u||_0||\gamma_\alpha v||_0 = C||u||_0||v||_\alpha.
\]

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To show assertion (3), let \( u, v \in \mathcal{E} \). Then by (1),
\[
||uv||_0 \leq C||u||_0 ||v||_0 \leq C||u||_\mathcal{E} ||v||_\mathcal{E} ,
\]
and by (2), \( ||uv||_\alpha \leq C||u||_\alpha ||v||_\alpha \leq C||u||_\mathcal{E} ||v||_\mathcal{E} \). Therefore \( uv \in \mathcal{E} \) and \( ||uv||_\mathcal{E} \leq C||u||_\mathcal{E} ||v||_\mathcal{E} \). 

Let \( \mathcal{U} \subset \mathbb{R} \) and let \( C^0(\mathcal{U}) \) denote the space of bounded \( C^0 \) functions \( m : \mathcal{U} \rightarrow \mathbb{R} \) with the sup norm, which we now denote \( || \cdot ||_{C^0} \). More generally, let \( C^k(\mathcal{U}) \) denote the space of \( C^k \) functions \( m : \mathcal{U} \rightarrow \mathbb{R} \) such that \( m, \partial m, \cdots, \partial^k m \) are all bounded functions, with the \( C^k \)-norm:
\[
||m||_{C^k} = ||m||_{C^0} + ||\partial m||_{C^0} + \cdots + ||\partial^k m||_{C^0} .
\]

**Proposition 2.28.** Let \( m(\cdot) \in C^2(\mathbb{R}) \). Then the formula
\[
v(z) \mapsto m(v(z))
\]
defines mappings from \( H^1(\mathbb{R}) \) to \( H^1(\mathbb{R}) \) and from \( \mathcal{E} \) to \( \mathcal{E} \). The first is Lipschitz on any set of the form \( \{ v : ||v||_0 \leq K \} \); the second is Lipschitz on any set of the form \( \{ v : ||v||_\mathcal{E} \leq K \} \).

**Proof.** We have
\[
m(v(z) + \bar{v}(z)) - m(v(z)) = \int_0^1 \partial_v m(v(z) + tv(z))dt\bar{v}(z) .
\]
(2.84)
Therefore,
\[
||m(v + \bar{v}) - m(v)||_{L^2} \leq ||m||_{C^1} ||\bar{v}||_{L^2} ,
\]
and
\[
||\gamma_\alpha (m(v + \bar{v}) - m(v))||_{L^2} \leq ||m||_{C^1} ||\gamma_\alpha \bar{v}||_{L^2} .
\]
Also, differentiating (2.84),
\[
\partial_z \left( m(v(z) + \bar{v}(z)) - m(v(z)) \right) = \int_0^1 \partial_z^2 m(v(z) + t\bar{v}(z)) (\partial_z v + t\partial_z \bar{v}) dt \bar{v}(z) + \int_0^1 \partial_v m(v(z) + t\bar{v}(z)) dt \partial_z \bar{v}.
\]
Therefore
\[
||\partial_z \left( m(v(z) + \bar{v}(z)) - m(v(z)) \right)||_{L^2} \leq ||m||_{C^2} ||\partial_z v||_{L^2} ||\bar{v}||_{L^\infty} + \frac{1}{2} ||m||_{C^2} ||\partial_z \bar{v}||_{L^2} ||\bar{v}||_{L^\infty} + ||m||_{C^1} ||\partial_z \bar{v}||_{L^2},
\]
and, because \( \bar{v} \in H^1(\mathbb{R}) \subset L^\infty(\mathbb{R}) \) by the Sobolev embedding theorem, we have
\[
||\partial_z \left( m(v(z) + \bar{v}(z)) - m(v(z)) \right)||_{L^2} \leq ||m||_{C^2} ||\partial_z v||_{L^2} ||\bar{v}||_{H^1} + \frac{1}{2} ||m||_{C^2} ||\partial_z \bar{v}||_{L^2} ||\bar{v}||_{H^1} + ||m||_{C^1} ||\partial_z \bar{v}||_{L^2},
\]
and similarly
\[
||\gamma_\alpha \partial_z \left( m(v(z) + \bar{v}(z)) - m(v(z)) \right)||_{L^2} \leq ||m||_{C^2} ||\partial_z v||_{L^2} ||\gamma_\alpha \bar{v}||_{H^1} + \frac{1}{2} ||m||_{C^2} ||\partial_z \bar{v}||_{L^2} ||\gamma_\alpha \bar{v}||_{H^1} + ||m||_{C^1} ||\gamma_\alpha \partial_z \bar{v}||_{L^2}.
\]
If \( ||v||_0 \) and \( ||v + \bar{v}||_0 \) are both bounded by the constant \( K \), then \( ||\bar{v}||_0 \leq 2K \), and due to (2.84) there exists a constant \( C_K > 0 \) depending on \( K \), such that
\[
||m(v(z) + \bar{v}(z)) - m(v(z))||_0 \leq C_K ||\bar{v}(z)||_0.
\]
Similarly, if \( ||v||_\varepsilon, ||v + \bar{v}||_\varepsilon \) are both bounded by the constant \( K \), then \( ||\bar{v}||_\varepsilon \leq 2K \),
\[
||m(v(z) + \bar{v}(z)) - m(v(z))||_0 \leq C_K ||\bar{v}||_0
\]
and
\[
||m(v(z) + \bar{v}(z)) - m(v(z))||_\alpha \leq C_K ||\bar{v}||_\alpha,
\]
thus \( ||m(v(z) + \bar{v}(z)) - m(v(z))||_\varepsilon \leq C_K ||\bar{v}||_\varepsilon. \)
Proposition 2.29. Let \( m(\cdot) \in C^2(\mathbb{R}) \). Consider the formula

\[
(u(z), v(z)) \mapsto m(u(z))v(z).
\] (2.85)

(1) Formula (2.85) defines a mapping from \( H^1(\mathbb{R})^2 \) to \( H^1(\mathbb{R}) \) that is locally Lipschitz on any set of the form \( \{(u, v) : \|u\|_0 + \|v\|_0 \leq K\} \).

(2) Formula (2.85) defines a mapping from \( \mathcal{E}^2 \) to \( \mathcal{E} \) that is locally Lipschitz on any set of the form \( \{(u, v) : \|u\|_\mathcal{E} + \|v\|_\mathcal{E} \leq K\} \).

Proof. See Proposition 5.6 in [GLSS]. \( \blacksquare \)

Proposition 2.30. Let \( m(v_1) = g(1/\kappa + v_1) - g(1/\kappa) \) with \( g \) from (2.55).

(1) The formula \( v_1(z) \mapsto m(v_1(z)) \) defines a mapping from \( H^1(\mathbb{R}) \) to \( H^1(\mathbb{R}) \) that is Lipschitz on any set of the form \( \{v_1 : \|v_1\|_0 \leq K\} \) and there is a constant \( C_K > 0 \) depending on \( K \) such that \( \|m(v_1)\|_0 \leq C_K\|v_1\|_0 \).

(2) If \( \mathbf{v} = (v_1, v_2) \in H^1(\mathbb{R})^2 \), and \( H(\mathbf{v}) = k(g(1/\kappa + v_1) - g(1/\kappa)) \) is given by (2.82), then \( H(\mathbf{v}) \) defines a mapping from \( H^1(\mathbb{R})^2 \) to \( H^1(\mathbb{R})^2 \) that is Lipschitz on any set of the form \( \{\mathbf{v} : \|\mathbf{v}\|_0 \leq K\} \) and \( \|H(\mathbf{v})\|_0 \leq C_K\|v_1\|_0\|v_2\|_0 \leq C_K\|\mathbf{v}\|_0^2 \).

Proof. (1) The Lipschitz property follows from Proposition 2.28. Because this mapping is Lipschitz and \( m(0) = g(1/\kappa + 0) - g(1/\kappa) = 0 \), we have \( \|m(v_1)\|_0 \leq C_K\|v_1\|_0 \).

(2) Since \( H(\mathbf{v}) = km(v_1)v_2 \), (2) follows from (1) and Proposition 2.29.(2). \( \blacksquare \)

Proposition 2.31. Consider the formula

\[
\mathbf{v} = (v_1(z), v_2(z)) \mapsto H(v_1(z), v_2(z)),
\] (2.86)

where \( H(\mathbf{v}) \) is given by (2.82).
(1) If \( v \in \mathcal{E}^2 \), then \( H(v) \in H_\alpha^1(\mathbb{R})^2 \), also \( H(\cdot) \) is Lipschitz on any set of the form 
\( \{ v : \|v\|_\mathcal{E} \leq K \} \) and there is a constant \( C_K > 0 \) depending on \( K \) such that 
\( \|H(v)\|_\alpha \leq C_K\|v\|_0\|v\|_\alpha \).

(2) Formula (2.82) for \( H(v_1, v_2) \) defines a mapping from \( \mathcal{E}^2 \) to \( \mathcal{E}^2 \) that is Lipschitz on any set of the form 
\( \{ v : \|v\|_\mathcal{E} \leq K \} \) and \( \|H(v)\|_\mathcal{E} \leq C_K\|v\|_\mathcal{E}^2 \).

Proof. To prove (1), we use the following inequality:
\[
\|H(v)\|_\alpha = \|\gamma_\alpha H(v)\|_0 = \|\gamma_\alpha k v_2(g(v_1 + 1/\kappa) - g(1/\kappa))\|_0 \\
\leq \|k m(v_1(z))\|_0\|\gamma_\alpha v_2\|_0 \\
\leq C_K\|v_1\|_0\|v_2\|_\alpha \text{ (using Proposition 2.28)} \\
\leq C_K\|v\|_0\|v\|_\alpha
\]
for some \( C_K > 0 \) depending on \( K \). Assertion (2) can be proved similarly to Proposition 2.30 (2) by using Proposition 2.29 (2).

2.2.3 Nonlinear stability

In this subsection, we prove the Lyapunov stability of the equilibrium solutions of 
the model system (2.66) on \( \mathcal{E}^2 \). The operator \( \mathcal{L}_\mathcal{E} \) generates a strongly continuous 
semigroup on \( \mathcal{E}^2 \). The nonlinear term yields a locally Lipschitz mapping on \( \mathcal{E}^2 \) by 
Proposition 2.31. Therefore we can apply the following standard result.

**Lemma 2.32.** Let \( \mathcal{X} \) be a Banach space. Consider the system
\[
\mathbf{v}_t = \mathcal{L} \mathbf{v}(t) + H(\mathbf{v}(t)), \quad t \geq 0,
\]
where \( H(\mathbf{v}) \) is locally Lipschitz continuous in \( \mathbf{v} \) and the operator \( \mathcal{L} : \text{dom}(\mathcal{L}) \subset \mathcal{X} \mapsto \mathcal{X} \) generates a \( C_0 \) semigroup \( T(t) \) on \( \mathcal{X} \). For any \( \mathbf{v}^0 \in \mathcal{X} \) the system has a unique
mild solution \( v \) with the initial value \( v^0 \). The solution is defined for the time \( t \) in the maximal interval \( 0 \leq t < t_{\max}(v^0) \) where \( 0 < t_{\max}(v^0) \leq \infty \).

Proof. See [P, Theorem 6.1.4].

Next, we recall yet another standard fact. Consider a system of the form

\[
v_t = Lv + H(v(t)),
\]

where the operator \( L \) is defined by the formula \( v \mapsto Lv \) with the domain \( \text{dom}(L) \subset \mathcal{E} \) and generates a \( C_0 \)-semigroup on \( \mathcal{E} \), and \( H(\cdot) \) is a locally Lipschitz mapping from \( \mathcal{E} \) into \( \mathcal{E} \). Let \( E \) be given by

\[
E = \{(v^0, t) \in \mathcal{E} \times \mathbb{R}^+ : 0 \leq t < t_{\max}(v^0)\};
\]

so that the set \( E \) is open in \( \mathcal{E} \times \mathbb{R}^+ \), and the map \((v^0, t) \mapsto v\) from \( E \) to \( \mathcal{E} \) is continuous. We have the following lemma.

Lemma 2.33. For each \( \delta > 0 \), if \( 0 < \gamma < \delta \), then there exists \( T \) depending on \( \gamma \) and \( \delta \), with \( 0 < T \leq \infty \), such that the following is true: if \( v^0 \in \mathcal{E} \) satisfies

\[
||v^0||_{\mathcal{E}} \leq \gamma \tag{2.87}
\]

and \( 0 \leq t < T \), then the solution \( v(t) \in \mathcal{E} \) of (2.66) is defined and satisfies

\[
||v(t)||_{\mathcal{E}} \leq \delta. \tag{2.88}
\]

Proof. The proof is the same as the proof in [SY, Theorem 46.4].

If \( \delta, \gamma > 0 \) are fixed, let \( T(\gamma, \delta) \) denote the supremum of all \( T \) such that (2.88) holds for all \( 0 \leq t < T \) whenever (2.87) is satisfied.

By Proposition 2.24, we can obtain the following results:
Lemma 2.34. Let \( \mathcal{L}_\alpha : \text{dom}(\mathcal{L}_\alpha) \subset H^1_\alpha(\mathbb{R})^2 \mapsto H^1_\alpha(\mathbb{R})^2 \) be the operator defined in subsection 2.2.1. There exists \( \nu > 0 \) that satisfies

\[
\sup\{\Re \lambda : \lambda \in \text{Sp}(\mathcal{L}_\alpha)\} < -\nu.
\] (2.89)

Furthermore, there exists \( K > 0 \) such that \( \|e^{t\mathcal{L}_\alpha}\|_{B(H^1_\alpha(\mathbb{R})^2)} \leq Ke^{-\nu t} \) for \( t \geq 0 \).

Proof. Recall (2.67). The operator \( \mathcal{L}_\alpha \) generates an analytic semigroup provided \( \epsilon > 0 \) and a strongly continuous semigroup provided \( \epsilon = 0 \). As shown in [GLSS], in either case the differential operator \( \mathcal{L} \) associated with the differential expression in (2.67) enjoys the spectral mapping property, that is, the boundary of the spectrum of the semigroup operator \( e^{t\mathcal{L}_\alpha} \) is controlled by the boundary of the spectrum of the semigroup generator \( \mathcal{L}_\alpha \) for any \( \epsilon \geq 0 \). By Proposition 2.24 we can choose \( \nu > 0 \) such that \( \sup\{\Re \lambda : \lambda \in \text{Sp}(\mathcal{L}_\alpha)\} < -\nu \). Furthermore, by the above mentioned semigroup property, see, e.g. Proposition 4.3 in [GLSS], there exists \( K > 0 \) such that \( \|e^{t\mathcal{L}_\alpha}\|_{H^1_\alpha(\mathbb{R})^2 \rightarrow H^1_\alpha(\mathbb{R})^2} \leq Ke^{-\nu t} \).

We are ready to begin the stability analysis. We will first show that the solution of system (2.66) is exponentially decaying in \( \| \cdot \|_\alpha \) norm.

Proposition 2.35. Let \( \nu > 0 \) satisfies (2.89). Then there exist \( \delta_1 > 0 \) and \( K_1 > 0 \) such that for every \( \delta \in (0, \delta_1) \) and every \( \gamma \) with \( 0 < \gamma < \delta \), the following is true for the mild solution \( v = v(t) \) of (2.66) with the initial value \( v_0 \): If \( v_0 \in \mathcal{E}^2 \) satisfies (2.87) so that \( v(t) \) satisfies (2.88) for \( 0 \leq t < T(\delta, \gamma) \), then

\[
\|v(t)\|_\alpha \leq K_1 e^{-\nu t}\|v_0\|_\alpha, \quad 0 \leq t < T(\delta, \gamma).
\] (2.90)

Proof. Because \( v(t) \) is a mild solution of (2.66) on \( \mathcal{E}^2 \), it satisfies the integral equation

\[
v(t) = e^{t\mathcal{L}_\epsilon}v_0 + \int_0^t e^{(t-s)\mathcal{L}_\epsilon} H(v(s))ds.
\] (2.91)
Since $v^0 \in \mathcal{E}^2$ by assumption, it is clear that $H(v(s))$ is in $H^1_\alpha(\mathbb{R})$ by Proposition 2.31, so we have

$$e^{t\mathcal{L}_\varepsilon} v^0 = e^{t\mathcal{L}_\alpha} v^0 \quad \text{and} \quad e^{(t-s)\mathcal{L}_\varepsilon} H(v(s)) = e^{(t-s)\mathcal{L}_\alpha} H(v(s)).$$

Next, we replace $\mathcal{L}_\varepsilon$ by $\mathcal{L}_\alpha$ in (2.91) and choose $\bar{\nu} > \nu > 0$ such that

$$\sup\{\Re \lambda : \lambda \in \Sp(\mathcal{L}_\alpha)\} < -\bar{\nu} < \nu < 0$$

and define $k = \bar{\nu}/\nu > 1$.

By Lemma 2.34, there exists $K_1 > 0$ such that $||e^{t\mathcal{L}_\alpha}|| \leq K_1 e^{-\bar{\nu}t}$ for all $t \geq 0$. Pick any $\delta' > 0$, for $0 < \gamma < \delta'$, and notice that if $||v^0||_\varepsilon < \gamma$, then $||v(s)||_\varepsilon < \delta'$ for all $s \in (0, T(\delta', \gamma))$ by Lemma 2.33.

With the aid of Proposition 2.31(1), there exists a constant $C_{\delta'} > 0$ depending on $\delta'$ such that for all $t \in [0, T(\delta', \gamma))$, using that $||v(s)||_\varepsilon \leq \delta'$ when $s \in (0, T(\delta', \gamma))$, it follows that

$$||v(t)||_\alpha \leq K_1 e^{-\bar{\nu}t} ||v^0||_\alpha + \int_0^t K_1 e^{-\bar{\nu}(t-s)} C_{\delta'} ||v(s)||_\varepsilon ||v(s)||_\alpha ds.$$  

For each $\delta < \delta'$, and $0 < \gamma < \delta$, if $||v^0||_\varepsilon < \gamma$, then $||v(s)||_\varepsilon < \delta$ for all $s \in (0, T(\delta, \gamma))$ by Lemma 2.33. Then, for all $t \in [0, T(\delta', \gamma))$, we have

$$||v(t)||_\alpha \leq K_1 e^{-\bar{\nu}t} ||v^0||_\alpha + K_1 C_{\delta'} \delta \int_0^t e^{-\bar{\nu}(t-s)} ||v(s)||_\varepsilon ds.$$  

(2.92)

Applying Gronwall’s inequality for the function $e^{\bar{\nu}t} ||v(t)||_\alpha$, we conclude that the inequality

$$e^{\bar{\nu}t} ||v(t)||_\alpha \leq K_1 ||v^0||_\alpha + K_1 C_{\delta'} \delta \int_0^t e^{\bar{\nu}s} ||v(s)||_\varepsilon ds,$$
implies, by Gronwall’s inequality, that
\[ e^{\nu t} \| v(t) \|_\alpha \leq K_1 \| v^0 \|_\alpha e^{K_1 \nu \delta t}, \]
so that
\[ \| v(t) \|_\alpha \leq K_1 \| v^0 \|_\alpha e^{K_1 \nu \delta t - \nu t}. \]

By choosing \( \delta_1 < \min\{ \delta', (k-1) \nu K_1 C_{\delta'} \} \), we can conclude that (2.90) holds for any \( \delta \in (0, \delta_1) \).

We now show that the solution of system (2.66) is bounded in \( \| \cdot \|_0 \) norm and the second component \( v_2(t) \) is exponentially decaying in \( \| \cdot \|_0 \) norm.

**Proposition 2.36.** Let \( \rho > 0 \) be chosen as in Lemma 2.25(3), and \( \delta_1 \) be given by Proposition 2.35. Assume that \( \nu < \rho \) where \( \nu \) satisfies (2.89). Then there exist a \( \delta_2 \in (0, \delta_1) \) and \( C_1 > 0 \) such that for every \( \delta \in (0, \delta_2) \) and every \( \gamma \) with \( 0 < \gamma < \delta \), the following is true: If \( 0 \leq t < T(\delta, \gamma) \), and \( v^0 \in E^2 \) satisfies (2.87), so that \( v(t) \in E^2 \) satisfies (2.88), then the following estimates hold:

\[ \| v_1(t) \|_0 \leq C_1 \| v^0 \|_E, \]  
\[ \| v_2(t) \|_0 \leq C_1 e^{-\rho t} \| v^0 \|_E. \]

**Proof.** We write system (2.66) as a nonautonomous linear system on \( H^1(\mathbb{R})^2 \):

\[ v_{1t} = L_1 v_1 + e^{-\nu} v_2 + H_1(v_1(t), v_2(t)), \]  
\[ v_{2t} = L_2 v_2 + H_2(v_1(t), v_2(t)), \]

where \( L_1, L_2 \) are defined in (2.71) and (2.72), \( v(t) = (v_1, v_2)(t) \) is a fixed solution of (2.66), and

\[ H_1(v) = v_2(g(v_1 + 1/\kappa) - g(v_1)), \]
\( H_2(v) = -\kappa v_2 \left( (g(v_1 + 1/\kappa) - g(v_1)) \right) . \)

We note that \((v_1, v_2)\) is the solution of (2.95)-(2.96) with the value \((v_1^0, v_2^0)\) at \(t = 0\), that is \((v_1, v_2)(t) = (v_1, v_2)(t, v_1^0, v_2^0)\).

With the help of Proposition 2.30(2), we can find a constant \(C_{\delta_1} > 0\) so that

\[
||H_1(v_1, v_2)||_0 \leq C_{\delta_1} ||v_1||_0 ||v_2||_0, \tag{2.97}
\]

and

\[
||H_2(v_1, v_2)||_0 = || -\kappa H_1(v) ||_0 \leq C_{\delta_1} ||v_1||_0 ||v_2||_0. \tag{2.98}
\]

when \(||v||_0 \leq \delta_1\).

The solution of (2.96) in \(H^1(\mathbb{R})\) can be written as

\[
v_2(t) = e^{t\mathcal{L}_2}v_2^0 + \int_0^t e^{(t-s)\mathcal{L}_2}H_2(v_1(s), v_2(s))ds. \tag{2.99}
\]

We then choose some \(\bar{\rho} > \rho > 0\) and \(k = \bar{\rho}/\rho > 1\) such that

\[
\sup\{\text{Re} \lambda : \lambda \in \text{Sp}(\mathcal{L}_2)\} < -\bar{\rho} := -k\rho.
\]

By Lemma 2.25 (3), there exists \(K_2 > 0\) such that \(||e^{t\mathcal{L}_2}||_{\text{B}(H^1(\mathbb{R}))} \leq K_2 e^{-\bar{\rho}t}.\) For each \(\delta \in (0, \delta_1)\) and \(\gamma \in (0, \delta)\), if \(||v^0||_\varepsilon \leq \gamma\) then

\[
||v_1(s)||_0 \leq ||v_1(s)||_\varepsilon \leq ||v(s)||_\varepsilon \leq \delta.
\]

By Lemma 2.33, we can obtain the following estimate for \(v_2(t)\) by using (2.98):

\[
||v_2(t)||_0 \leq K_2 e^{-\bar{\rho}t} ||v_2^0||_0 + \int_0^t K_2 e^{-\bar{\rho}(t-s)}C_{\delta_1} ||v_1(s)||_0 ||v_2(s)||_0 ds
\]

\[
\leq K_2 e^{-\bar{\rho}t} ||v_2^0||_0 + \int_0^t K_2 e^{-\bar{\rho}(t-s)}C_{\delta_1} \delta ||v_2(s)||_0 ds.
\]
We then calculate
\[ e^{\bar{\rho} t} \| v_2(t) \|_0 \leq K_2 \| v_2^0 \|_\varepsilon + K_2 C_{\delta_1} \delta \int_0^t e^{\bar{\rho} s} \| v_2(s) \|_0 ds \]
\[ \leq K_2 \| v^0 \|_\varepsilon + K_2 C_{\delta_1} \delta \int_0^t e^{\bar{\rho} s} \| v_2(s) \|_0 ds. \]

By applying Gronwall’s inequality to \( e^{\bar{\rho} t} \| v_2(t) \|_0 \), we infer that
\[ \| v_2(t) \|_0 \leq K_2 \| v^0 \|_\varepsilon e^{K_2 C_{\delta_1} \delta t - \bar{\rho} t}. \]

Let \( \delta_2 < \min(\delta_1, (\frac{k-1}{K_2 C_{\delta_1}})). \) Then for \( \delta < \delta_2 \) it follows that
\[ \| v_2(t) \|_0 \leq K_2 \| v^0 \|_\varepsilon e^{-\rho t} \text{ for all } t \in [0, T(\delta, \gamma)) \] (2.100)

proving (2.94). Let us show the proof of (2.93). The solution of (2.95) in \( H^1(\mathbb{R}) \) satisfies
\[ v_1(t) = e^{t \mathcal{L}_1} v_1^0 + \int_0^t e^{(t-s) \mathcal{L}_1} (e^{-\kappa v_2(s) + H_1(v_1(s), v_2(s))) ds.} \] (2.101)

First, because \( \mathcal{L}_1 \) generates a bounded semigroup by Lemma 2.25 (1), there exists a constant \( K_3 > 0 \), such that \( \| e^{t \mathcal{L}_1} \|_{B(H^1(\mathbb{R}))} \leq K_3. \) By using (2.97) and the fact that
\[ \| e^{-\kappa v_2(s)} \|_0 \leq \| v_2(s) \|_0 \]
for \( \kappa > 0 \) we infer that
\[ \| v_1(t) \|_0 \leq K_3 \| v_1^0 \|_0 + \int_0^t (K_3 C_{\delta_1} \| v_2(s) \|_0 \| v_1(s) \|_0 + K_3 \| v_2(s) \|_0) ds. \]

Also, using the fact that \( \| v_1(s) \|_0 \leq \| v(s) \|_0 \leq \| v(s) \|_\varepsilon < \delta < \delta_2 \), we have, for a constant \( C_{\delta_1, \delta_2} > 0 \) independent of \( \delta \), that
\[ \| v_1(t) \|_0 \leq K_3 \| v_1^0 \|_\varepsilon + \int_0^t (K_3 C_{\delta_1, \delta_2} \| v_2(s) \|_0 + K_3 \| v_2(s) \|_0) ds \]
\[ K_3 ||v_1(t)||_\mathcal{E} + \int_0^t K_3 C_{\delta_1, \delta_2} ||v_2(s)||_\mathcal{E} ds \]

Then we use (2.100) to obtain

\[
||v_1(t)||_0 \leq K_3 ||v^0||_\mathcal{E} + \int_0^t K_2 K_3 C_{\delta_1, \delta_2} e^{-\rho s} ||v^0||_\mathcal{E} ds \\
\leq K_3 ||v^0||_\mathcal{E} + K_2 K_3 C_{\delta_1, \delta_2} ||v^0||_\mathcal{E} \int_0^t e^{-\rho s} ds \\
\leq C_2 ||v^0||_\mathcal{E}
\]

for some \( C_2 > 0 \). In conclusion, there exists a constant \( C_1 > 0 \) such that for \( \delta \in (0, \delta_2) \) and \( \gamma \in (0, \delta) \), the inequalities (2.93) and (2.94) hold when \( t \in [0, T(\delta, \gamma)) \).

We now complete the proof of nonlinear stability of the end state \( u_- \).

**Remark 2.37.** We claim that the end state \( u_- \) of (2.66) is Lyapunov stable in \( ||\cdot||_\mathcal{E} \).

The proof of the Lyapunov stability of the end state \( u_- \) is, in fact, contained in the next theorem and relies on the following bootstrap argument based on Proposition 2.35 and 2.36. Indeed, in particular, these propositions yield the existence of constants \( \delta_0 > 0 \) and \( C_{\delta_0} > 0 \) such that for every \( \delta \in (0, \delta_0) \) and every \( \gamma \in (0, \delta) \), there exists \( T(\delta, \gamma) \), such that for every \( t \in [0, T(\delta, \gamma)) \) the inequalities

\[
||v(t)||_\mathcal{E} < \delta \quad \text{and} \quad ||v(t)||_\mathcal{E} \leq C_{\delta_0} ||v^0||_\mathcal{E}
\]

hold for the solution \( v(t) \) of (2.66) with the initial value \( v^0 \in \mathcal{E} \) as long as \( ||v^0||_\mathcal{E} < \gamma \).

Let us show that for each \( \delta \in (0, \delta_0) \), there is an \( \eta \) such that if \( ||v^0||_\mathcal{E} < \eta \) then \( ||v(t)||_\mathcal{E} < \delta \) for all \( t \geq 0 \), that is, that the end state \( u_- \) of (2.66) is Lyapunov stable in \( \mathcal{E} \). Indeed, assuming \( C_{\delta_0} > 1 \) with no loss of generality, set \( \eta = \frac{\delta}{2C_{\delta_0}} \) and assume \( ||v^0||_\mathcal{E} < \eta \). Then \( ||v(T(\delta, \gamma))||_\mathcal{E} < \delta/2 \) by (2.102), and thus the solution \( v \) with the initial value \( v(T(\delta, \gamma)) \) satisfies (2.102) again for \( t \in [T(\eta, \gamma), 2T(\eta, \gamma)) \), again by
Proposition 2.35 and 2.36. So, these propositions can be applied for all \( t \geq 0 \), proving the Lyapunov stability. In addition, as long as these propositions are applicable, we obtain a more refined information about the behavior of the solution such as its boundedness in \( \| \cdot \|_0 \)-norm and the exponential decay in \( \| \cdot \|_\alpha \)-norm, see items (3)-(4) of the next theorem. We now proceed with a more formal exposition of the stability statement.

Given an initial value \( v^0 \in \mathcal{E}^2 \), let \( v(t) = v(t, v^0) \) be the solution of (2.66) in \( \mathcal{E}^2 \) with \( v(0) = v^0 \), which we showed exists on \( 0 \leq t < t_{\text{max}}(v^0) \) by Lemma 2.32. We shall show that \( v(t) \in \mathcal{E}^2 \) is defined, and is bounded in \( \| \cdot \|_\mathcal{E} \)-norm and exponentially decaying for all time \( t > 0 \). In fact, the following stability results holds true. We note that the small constant in the next theorem can be chosen as \( \delta_0 = \delta_2 \) where \( \gamma_2 \) is chosen as in Proposition 2.36.

**Theorem 2.38.** There exist constants \( C > 0 \) and \( \nu > 0 \) such that for each \( 0 < \delta < \delta_0 \), we can find \( \eta > 0 \) such that if \( \| v^0 \|_{\mathcal{E}} \leq \eta \), then the following is true for all \( t > 0 \):

1. \( v(t) \) is defined;
2. \( \| v(t) \|_{\mathcal{E}} \leq \delta \);
3. \( \| v(t) \|_\alpha \leq Ce^{-\nu t}\| v^0 \|_\alpha \);
4. \( \| v_1(t) \|_0 \leq C\| v^0 \|_{\mathcal{E}} \);
5. \( \| v_2(t) \|_0 \leq Ce^{-\nu t}\| v^0 \|_{\mathcal{E}} \).

**Proof.** Choose \( \nu \) as in Lemma 2.34. Choose \( \delta_0 = \delta_2 \) as indicated in Proposition 2.36. Let \( C \) be a constant satisfying \( C > \max\{1, K_1, C_1\} \) with \( K_1 \) and \( C_1 \) given as
in Propositions 2.35 and 2.36. Let $0 < \gamma < \delta < \delta_0$ and set $\eta = C^{-1}\gamma$. Assume $v^0 \in \mathcal{E}$ satisfies $\|v^0\|_\mathcal{E} \leq \eta$. Since $\|v^0\|_\mathcal{E} < \eta < \delta$, the solution $v(t)$ exists and satisfies assertion (2)-(5) in the theorem for $t \in [0, T(\delta, \eta))$ by Propositions 2.35 and 2.36.

We claim that $T(\delta, \eta) = \infty$, so that the proof is finished as soon as the claim is justified. For this, for any $T \in (0, T(\delta, \eta))$ we consider the solution with the initial data $v(T)$. Note that (3)-(5) for $t = T$ yield $\|v(T)\|_\mathcal{E} \leq C\|v^0\|_\mathcal{E} \leq C\eta \leq \gamma$ and thus Lemma 2.33 applies and gives $\|v(T + t)\|_\mathcal{E} \leq \delta$ for $t \in (0, T(\delta, \gamma))$. Therefore, we proved that if $\|v^0\|_\mathcal{E} \leq \eta$ then $\|v(t)\|_\mathcal{E} \leq \delta$ for all $t \in [0, T(\delta, \gamma) + T)$. This shows that $T(\delta, \eta) \geq T(\delta, \gamma) + T$ and therefore implies $T(\delta, \eta) \geq T(\delta, \gamma) + T(\delta, \eta)$ and thus $T(\delta, \eta) = \infty$ as claimed. \(\blacksquare\)

### 2.2.4 A more general system

In this subsection we will consider a more general reaction-diffusion system than (2.57) with more general nonlinearity $f(\cdot)$ and coefficients matrix $D$ given by

$$u_t(t,x) = D \Delta_x u(t,x) + f(u(t,x)), \quad (2.103)$$

where $u \in \mathbb{R}^n$, $x \in \mathbb{R}^d$, $t \geq 0$, $D = \text{diag}(d_1, \ldots, d_n)$ with all $d_i \geq 0$, and the function $f : \mathbb{R}^n \to \mathbb{R}^n$ is smooth, see Hypothesis 2.39. We will study an $x$-independent equilibrium solution $u_-$ to (2.103) and its perturbation depending only on $z = x \cdot e - ct$.

By the discussion in Subsection 2.2.1, replacing the spatial variable $x = (x_1, \ldots, x_n)$ by the moving variable $z$ in (2.103), we obtain

$$u_t(t,z) = D u_{zz}(t,z) + cu_z(t,z) + f(u(t,z)), \quad z \in \mathbb{R}. \quad (2.104)$$

Without loss of generality we shall take the equilibrium solution $u_-$ to be 0. Information about the stability of the solution 0 is encoded in the spectrum of the operator
obtained by linearizing (2.104) about 0, 

\[ u_t = Du_{zz} + cu_z + \partial_u f(0)u =: Lu, \quad (2.105) \]

where \( \partial_u \) is the differential with respect to \( u \).

Let \( L \) be the operator defined on \( H^1(\mathbb{R})^n \) given by \( u \mapsto Lu \), with the domain \( u \in H^3(\mathbb{R})^n \). Define the weight function \( \gamma_\alpha(z) = e^{\alpha z} \), spaces \( H^1_\alpha(\mathbb{R}) \) and \( E = H^1(\mathbb{R}) \cap H^1_\alpha(\mathbb{R}) \) as in Subsection 2.2.1. Analogously to \( L \), let \( L_\alpha \) be the operator defined on \( H^1_\alpha(\mathbb{R})^n \) given by \( u \mapsto Lu \), with the domain being the set of \( u \) where \( \gamma_\alpha u \in H^3(\mathbb{R})^n \).

Throughout we impose the following assumptions on \( f(\cdot) \) in (2.103).

**Hypothesis 2.39.** (a) In appropriate variables \( u = (u_1, u_2) \), \( u_1 \in \mathbb{R}^{n_1} \), \( u_2 \in \mathbb{R}^{n_2} \), 

\[ n_1 + n_2 = n, \text{ we assume that for some constant } n_1 \times n_1 \text{ matrix } A_1, \]

\[ f(u_1, 0) = (A_1 u_1, 0)^T. \]

(b) The function \( f \) is \( C^3 \) from \( \mathbb{R}^n \) to \( \mathbb{R}^n \).

If Hypothesis 2.39 holds, then

\[ f(u_1, u_2) = f(u_1, 0) + f(u_1, u_2) - f(u_1, 0) \]

\[ = \left( \begin{array}{c} A_1 u_1 \\ 0 \end{array} \right) + \int_0^1 \partial_{u_2} f(u_1, tu_2)dtu_2 \]

\[ = \left( \begin{array}{c} A_1 u_1 + F_1(u_1, u_2)u_2 \\ F_2(u_1, u_2)u_2 \end{array} \right), \]

where \( F_1 \) and \( F_2 \) are some matrix-valued functions of size \( n_1 \times n_2 \) and \( n_2 \times n_2 \), respectively. We write

\[ D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \quad f(u) = \begin{pmatrix} f_1(u_1, u_2) \\ f_2(u_1, u_2) \end{pmatrix}, \]

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where each $D_i$ is a nonnegative diagonal matrix of size $n_i \times n_i$, and $f_i : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}^{n_i}$ for $i = 1, 2$.

Equation (2.104) now reads

\[
\begin{align*}
\mathbf{u}_{1t} &= D_1 \mathbf{u}_{1zz} + \mathbf{c}\mathbf{u}_{1z} + f_1(\mathbf{u}_1, \mathbf{u}_2), \\
\mathbf{u}_{2t} &= D_2 \mathbf{u}_{2zz} + \mathbf{c}\mathbf{u}_{2z} + f_2(\mathbf{u}_1, \mathbf{u}_2).
\end{align*}
\]  

(2.106)  

(2.107)

If we linearize (2.107) at $(0, 0)$, the constant-coefficient linear equation depends only on $\mathbf{u}_2$ since $f(0, 0) = 0$ by Hypothesis 2.39.(a):

\[
\begin{align*}
\mathbf{u}_{2t} &= D_2 \mathbf{u}_{2zz} + \mathbf{c}\mathbf{u}_{2z} + \partial_{\mathbf{u}_2} f_2(0, 0) \mathbf{u}_1 + \partial_{\mathbf{u}_2} f_2(0, 0) \mathbf{u}_2 \\
&= D_2 \mathbf{u}_{2zz} + \mathbf{c}\mathbf{u}_{2z} + \partial_{\mathbf{u}_2} f_2(0, 0) \mathbf{u}_2.
\end{align*}
\]  

(2.108)

We denote by $L_2 \mathbf{u}_2$ the right-hand side of (2.108), and let $\mathcal{L}_2$ be the operator defined on $H^1(\mathbb{R})^{n_2}$ given by $\mathbf{u} \rightarrow L_2 \mathbf{u}$, with the domain $\mathbf{u} \in H^3(\mathbb{R})^{n_2}$.

In addition, we linearize (2.106) at $(0, 0)$, and by Hypothesis 2.39.(a), the constant-coefficient linear equation reads:

\[
\begin{align*}
\mathbf{u}_{1t} &= D_1 \mathbf{u}_{1zz} + \mathbf{c}\mathbf{u}_{1z} + \partial_{\mathbf{u}_1} f_1(0, 0) \mathbf{u}_1 + \partial_{\mathbf{u}_2} f_1(0, 0) \mathbf{u}_2 \\
&= D_1 \mathbf{u}_{1zz} + \mathbf{c}\mathbf{u}_{1z} + A_1 \mathbf{u}_1 + \partial_{\mathbf{u}_2} f_1(0, 0) \mathbf{u}_2.
\end{align*}
\]  

(2.109)

We denote $L_1 \mathbf{u}_1 = D_1 \mathbf{u}_{1zz} + \mathbf{c}\mathbf{u}_{1z} + A_1 \mathbf{u}_1$, thus $\mathbf{u}_{1t} = L_1 \mathbf{u} + \partial_{\mathbf{u}_2} f_1(0, 0) \mathbf{u}_2$. Let $\mathcal{L}_1$ be the operator defined on $H^1(\mathbb{R})^{n_1}$ given by $\mathbf{u} \rightarrow L_1 \mathbf{u}$, with the domain $\mathbf{u} \in H^3(\mathbb{R})^{n_1}$.

With additional assumptions listed below, we will show that the perturbations of the steady state solution that are initially small in both the unweighted norm and weighted norm stay small in the unweighted norm and decay exponentially in the weighted norm. In addition, the $\mathbf{u}_2$-component of the perturbation decays exponentially in the unweighted norm.

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We will now study the spectrum of $\mathcal{L}$.

**Lemma 2.40.** The linear operator $\mathcal{L}$ associated with $L$ in (2.105) have the same spectrum on $L^2(\mathbb{R})^n$ and $H^1(\mathbb{R})^n$. Also, the linear operators $\mathcal{L}_i$ associated with $L_i$ in (2.108) and (2.109) have the same spectra on $L^2(\mathbb{R})^{n_i}$ and $H^1(\mathbb{R})^{n_i}$ for $i = 1, 2$.

**Proof.** We will consider the operator associated with

$$L = D \partial_{zz} + c \partial_z + \partial_u f(0, 0).$$

Here we denote $\mathcal{L}$ on $L^2(\mathbb{R})^n$ as $\mathcal{L}_{L^2}$ and $\mathcal{L}$ on $H^1(\mathbb{R})^n$ as $\mathcal{L}_{H^1}$.

We recall that the operator $\partial_{z,L^2}$ has the domain $H^1(\mathbb{R})$ and spectrum $i\mathbb{R}$. Therefore, the operator

$$\mathcal{D} = \begin{pmatrix} \partial_{z,L^2} + I_{L^2} & 0 \\ 0 & \partial_{z,L^2} + I_{L^2} \end{pmatrix} : H^1(\mathbb{R})^{n_1} \times H^1(\mathbb{R})^{n_2} \mapsto L^2(\mathbb{R})^{n_1} \times L^2(\mathbb{R})^{n_2}$$

is an isomorphism, which maps dom($\mathcal{L}_{H^1}$) onto dom($\mathcal{L}_{L^2}$). Since $\mathcal{L}$ is a constant coefficient operator, using the identity $\mathcal{D} \mathcal{L}_{H^1} \mathbf{v} = \mathcal{L}_{L^2} \mathcal{D} \mathbf{v}$, for all $\mathbf{v} \in $ dom $\mathcal{L}_{H^1} = H^3(\mathbb{R})$, we can obtain that $\mathcal{D} \mathcal{L}_{H^1} \mathcal{D}^{-1} = \mathcal{L}_{L^2}$. Thus, we can conclude that $\text{Sp}(\mathcal{L}_{H^1}) = \text{Sp}(\mathcal{L}_{L^2})$.

Similarly, we can conclude that the linear operators $\mathcal{L}_1$ and $\mathcal{L}_2$ associated with $L_1$ and $L_2$ in (2.108) and (2.109) have the same spectra on $L^2(\mathbb{R})^{n_i}$ and $H^1(\mathbb{R})^{n_i}$.

Next, we will tackle $\text{Sp}(\mathcal{L}_\alpha)$ on $H^1_\alpha(\mathbb{R})^n$, which can be done as follows.

**Lemma 2.41.** The linear operator $\mathcal{L}_\alpha$ has the same spectrum on both $L^2_\alpha(\mathbb{R})^n$ and $H^1_\alpha(\mathbb{R})^n$.

**Proof.** Let $\mathcal{E}_0$ be $L^2(\mathbb{R})$ or $H^1(\mathbb{R})$ and $\mathcal{E}_\alpha = \{u : \gamma_\alpha(z)u(z) \in \mathcal{E}_0\}$. The linear operator $\mathcal{M}$ defined by $\mathcal{M}u = \gamma_\alpha u$ is an isomorphism from $\mathcal{E}_\alpha^n$ to $\mathcal{E}_0^n$. Define the linear
map \( \mathcal{L} = M \mathcal{L}_\alpha M^{-1} \) on \( E_0^n \), with the domain being \( H^2(\mathbb{R})^n \) if \( E_0 = L^2(\mathbb{R})^n \), or the domain being \( H^3(\mathbb{R})^n \) if \( E_0 = H^1(\mathbb{R})^n \). For future use we record \( \hat{\mathcal{L}} \) corresponding to the operator \( \mathcal{L} \), i.e., \( \hat{\mathcal{L}}W = DW_{zz} + (c - 2\alpha)W_z + (\alpha^2D - c\alpha I + \partial_u f(0))W \) for \( W \in \text{dom}(\hat{\mathcal{L}}) \). The operator \( \hat{\mathcal{L}} \) is therefore similar to \( \mathcal{L}_\alpha \) on \( E_0^n \) and hence has the same spectrum. Using the similar proof as in Lemma 2.40, the operator \( \hat{\mathcal{L}} \) has the same spectrum on \( H^1(\mathbb{R})^n \) and \( L^2(\mathbb{R})^n \). \( \blacksquare \)

In order to discuss the Lyapunov stability of the steady state solution of (2.104), we need some additional hypotheses.

**Hypothesis 2.42.** There exists \( \alpha > 0 \) such that \( \sup \{ \text{Re}\lambda : \lambda \in \text{Sp}(\mathcal{L}_\alpha) \} < 0 \) on \( H^1_\alpha(\mathbb{R})^n \).

We now give a simple sufficient condition when Hypothesis 2.42 is satisfied.

**Lemma 2.43.** Assume that Hypothesis 2.39 holds and \( \partial_u(0,0) \) is non-positive, then Hypothesis 2.42 holds.

**Proof.** Via the Fourier transform, the operator \( \hat{\mathcal{L}} \) on \( L^2(\mathbb{R})^n \) is similar to the operator of multiplication on \( L^2(\mathbb{R})^n \) by the matrix-valued function

\[
M(\theta) = -D\theta^2 + i\theta(cI - 2\alpha D) + \alpha^2D - c\alpha I + \partial_u f(0,0).
\]

Hence the spectrum of \( \hat{\mathcal{L}} \) on \( L^2(\mathbb{R})^n \) equals that of multiplication by \( M \) on \( L^2(\mathbb{R})^n \).

The spectrum of \( \mathcal{L}_\alpha \) is equal to the set of \( \lambda \in \mathbb{C} \) for which there exists \( \theta \in \mathbb{R} \) such that

\[
\det(M(\theta) - \lambda I) = \det \left( -D\theta^2 + i\theta(cI - 2\alpha D) + \alpha^2D - c\alpha I - \lambda I + \partial_u f(0,0) \right) = 0.
\]

Thus we can find \( \alpha > 0 \) such that \( \sup \{ \text{Re}\lambda : \lambda \in \text{Sp}(\mathcal{L}_\alpha) \} < 0 \). \( \blacksquare \)
We need some more hypotheses on $L_1$ and $L_2$ introduced in (2.108) and (2.109).

**Hypothesis 2.44.** In addition to Hypotheses 2.39 and 2.42, we assume the following:

1. The operator $L_1$ generates a bounded semigroup on $L^2(\mathbb{R})^{n_1}$ and $H^1(\mathbb{R})^{n_1}$.

2. The operator $L_2$ satisfies $\sup\{\text{Re} \lambda : \lambda \in \text{Sp}(L_2)\} < 0$ on $L^2(\mathbb{R})^{n_2}$ and $H^1(\mathbb{R})^{n_2}$.

We will now rewrite the equation for the perturbation $v(t, z)$ of the steady state solution $u_+ = 0$ in the form amenable for the subsequence analysis.

We seek a solution to (2.104) of the form $u(t, z) = 0 + v(t, z)$, with this notation, $v(t, z)$ satisfies

$$v_t = Dv_{zz} + cv_z + \partial u f(0)v + f(v) - f(0) - \partial u f(0)v. \quad (2.110)$$

Note that

$$f(v) - f(0) - \partial u f(0)v = \int_0^1 (\partial u f(tv) - \partial u f(0))dtv.$$

We define

$$N(v) = \int_0^1 (\partial u f(tv) - \partial u f(0))dt, \quad (2.111)$$

as an $n \times n$ matrix-valued function of $v$. Using (2.111), we rewrite (2.110) as

$$v_t = Lv + N(v)v. \quad (2.112)$$

This is the main semilinear equation that we will study.

**Proposition 2.45.** Assume that Hypothesis 2.44 hold. Then the following is true:

(a) There exists $\alpha > 0$ such that on the weighted space $H^{1}(\mathbb{R})^n$, the spectrum of $L_\alpha$ will be bounded away from the imaginary axis and $\sup\{\text{Re} \lambda : \lambda \in \text{Sp}(L_\alpha)\} < -\nu$ for some $\nu > 0$. Also there exists $K > 0$ such that for all $t \geq 0$,

$$\|e^{tL_\alpha}\|_{\mathcal{B}(H^{1}(\mathbb{R})^n)} \leq Ke^{-\nu t}. \quad (2.113)$$
(b) On $H^1(\mathbb{R}^n)$, we have $\sup \{ \Re \lambda : \lambda \in \text{Sp}(\mathcal{L}^2) \} < -\rho$ for some $\rho > 0$. Moreover, there exists $K > 0$ such that $\|e^{t\mathcal{L}^2}\|_{C(\mathbb{R}^n)} < K e^{-\rho t}$ for $t \geq 0$.

Proof. Statement (a) holds by Hypothesis 2.42 and Lemma 2.41. Statement (b) follows from Hypothesis 2.44(2) and Lemma 2.40. ■

**Proposition 2.46.** Assume Hypothesis 2.39, consider the nonlinearity $N(\mathbf{v})$ defined in (2.111). We have:

1. If $\mathbf{v} \in E^n$, then $N(\mathbf{v})\mathbf{v} \in H^1(\mathbb{R})^n$, and on any bounded neighborhood of the form $\{ \mathbf{v} : \|\mathbf{v}\|_E \leq K \}$ there is a constant $C_K > 0$ such that $\|N(\mathbf{v})\mathbf{v}\|_\alpha \leq C_K\|\mathbf{v}\|_0\|\mathbf{v}\|_\alpha$.

2. If $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2) \in E^n$, then $N(\mathbf{v})\mathbf{v} \in H^1(\mathbb{R})^n$, and on any bounded neighborhood of the form $\{ \mathbf{v} : \|\mathbf{v}\|_0 \leq K \}$ there is a constant $C_K > 0$ such that $\|N(\mathbf{v})\mathbf{v}\|_0 \leq C_K\|\mathbf{v}_1\|_0\|\mathbf{v}_2\|_0 \leq C_K\|\mathbf{v}\|_0^2$.

3. The formula $\mathbf{v} \mapsto N(\mathbf{v})\mathbf{v}$ defines a mapping from $E^n$ to $E^n$ that is locally Lipschitz on any bounded neighborhood of the form $\{ \mathbf{v} : \|\mathbf{v}\|_E \leq K \}$ in $E^n$.

Proof. We refer to [GLS, Proposition 7.5, Proposition 7.6], while supressing the $q$ dependence and letting $Y_q = 0$ in the formula of $N(Y_q, \tilde{Y})$, see (4.3) in [GLS]. ■

Now, applying Lemma 2.25 we show that (2.112) has a mild solution on $E^n$ for at least a short time period. Moreover, by [SY, Theorem 64.2], we have the following result.

**Proposition 2.47.** Assume Hypothesis 2.44. For each $\delta > 0$, if $0 < \gamma < \delta$, then there exists $T(\delta, \gamma)$, such that if $\|\mathbf{v}^0\|_E < \gamma$, then for $t \in (0, T(\delta, \gamma))$, the solution $\mathbf{v}(t)$ to (2.112) is defined in $E^n$ and satisfies $\|\mathbf{v}(t)\|_E < \delta$. 72
We now establish exponential decay of the solutions of (2.112) on $H^1_\alpha(\mathbb{R})^n$.

**Proposition 2.48.** Assume Hypothesis 2.44. Let $\nu > 0$ satisfy (2.113). Then there exist $\delta_1 > 0$ and $K_1 > 0$ such that for every $\delta \in (0, \delta_1)$ and every $\gamma$ with $0 < \gamma < \delta$, the following is true: Let $v^0 \in \mathcal{E}^n$ satisfies $||v^0||_\mathcal{E} < \gamma$ so that $v(t)$ satisfies $||v(t)||_\mathcal{E} < \delta$ for $0 \leq t < T(\delta, \gamma)$. Then

$$||v(t)||_\alpha \leq K_1 e^{-\nu t} ||v^0||_\alpha \quad \text{for} \quad 0 \leq t < T(\delta, \gamma). \quad (2.114)$$

**Proof.** Because $v(t)$ is a mild solution on $\mathcal{E}^n$, it satisfies the integral equation

$$v(t) = e^{tL_\mathcal{E}} v^0 + \int_0^t e^{(t-s)L_\mathcal{E}} N(v(s))v(s)ds. \quad (2.115)$$

Since $v^0 \in \mathcal{E}^n$ by assumption, it is clear that $N(v)\nu$ is in $H^1_\alpha(\mathbb{R})^n$ by Proposition 2.46.(1), so we have

$$e^{tL_\mathcal{E}} v^0 = e^{tL_\alpha} v^0 \quad \text{and} \quad e^{(t-s)L_\mathcal{E}} N(v(s))v(s) = e^{(t-s)L_\alpha} N(v(s))v(s).$$

Next, we replace $L_\mathcal{E}$ by $L_\alpha$ in (2.115) and choose $\bar{\nu} > \nu > 0$ such that

$$\sup\{\Re \lambda : \lambda \in \text{Sp}(L_\alpha)\} < -\bar{\nu} := -k\nu$$

for some $k = \bar{\nu}/\nu > 1$ and close to 1. There exists $K_1 > 0$ such that $||e^{tL_\alpha}|| \leq K_1 e^{-\bar{\nu} t}$ for all $t \geq 0$ by Proposition 2.45.(a).

Pick any $\delta' > 0$. For any $\gamma$ so that $0 < \gamma < \delta'$, if $||v^0||_\mathcal{E} < \gamma$. Then $||v(s)||_\mathcal{E} < \delta'$ for all $s \in (0, T(\delta', \gamma))$ by Proposition 2.47.

With the aid of Proposition 2.46(1), there exists a constant $C_{\delta'} > 0$ depending on $\delta'$ such that for $||v(s)||_\mathcal{E} \leq \delta'$ when $s \in (0, T(\delta', \gamma))$, it follows that

$$||v(t)||_\alpha \leq K_1 e^{-\beta t} ||v^0||_\alpha + \int_0^t K_1 e^{-\beta(t-s)} C_{\delta'} ||v(s)||_0 ||v(s)||_\alpha ds.$$
For each $\delta < \delta'$, and $0 < \gamma < \delta$, if $||v^0||_E < \gamma$, then $||v(s)||_E < \delta$ for all $s \in (0, T(\delta, \gamma))$ by Proposition 2.47 again, then

$$||v(t)||_\alpha \leq K_1 e^{-\beta t} ||v^0||_\alpha + K_1 C_{\beta'} \delta \int_0^t e^{-\beta(t-s)} ||v(s)||_\alpha ds. \quad (2.116)$$

Applying Gronwall’s inequality for the function $e^{\beta t} ||v(t)||_\alpha$, we can obtain that the inequality

$$e^{\beta t} ||v(t)||_\alpha \leq K_1 ||v^0||_\alpha + K_1 C_{\beta'} \delta \int_0^t e^{\beta s} ||v(s)||_\alpha ds,$$

implies, by Gronwall’s inequality, that

$$||v(t)||_\alpha \leq K_1 ||v^0||_\alpha e^{K_1 C_{\beta'} \delta t - \beta t}.$$

By choosing $\delta_1 < \min\{\delta', (k - 1) \frac{\nu}{K_1 C_{\beta'}}\}$, we can conclude that (2.114) holds for any $\delta \in (0, \delta_1)$. ■

We now show that the $v_1$ component of the solution $v = (v_1, v_2)^T$ of (2.112) is bounded in $H^1(\mathbb{R})^{n_1}$ while $v_2$ component is exponentially decaying on $H^1(\mathbb{R})^{n_2}$.

**Proposition 2.49.** Assume Hypothesis 2.44. Let $\rho > 0$ be chosen as in Proposition 2.45. (b), and $\delta_1$ be given by Proposition 2.48. Then there exist a $\delta_2 \in (0, \delta_1)$ and $C_1 > 0$ such that for every $\delta \in (0, \delta_2)$ and every $\gamma$ with $0 < \gamma < \delta$, the following is true: If $0 \leq t < T(\delta, \gamma)$, and $v^0 \in E^n$ satisfies $||v^0||_E < \gamma$, then $v(t) \in E^n$ satisfies $||v(t)||_E < \delta$ and the following estimates hold:

$$||v_1(t)||_0 \leq C_1 ||v^0||_E, \quad (2.117)$$

$$||v_2(t)||_0 \leq C_1 e^{-\rho t} ||v^0||_E. \quad (2.118)$$
Proof. We write system (2.112) as a nonautonomous linear system in $H^1(\mathbb{R})^n$,

\begin{align*}
v_{1t} &= L_1 v_1 + \partial_{u_2} f_1(0,0) v_2 + H_1(v_1(t), v_2(t)), \\
v_{2t} &= L_2 v_2 + H_2(v_1(t), v_2(t)),
\end{align*}

(2.119) (2.120)

where $v(t) = (v_1, v_2)(t)$ is a fixed solution of (2.112), and we introduce the nonlinear terms $H_i, i = 1, 2$, as follows:

$$
\begin{pmatrix}
H_1(v(t)) \\
H_2(v(t))
\end{pmatrix} = N(v(t)) = \begin{pmatrix}
N_1(v(t))v(t) \\
N_2(v(t))v(t)
\end{pmatrix}.
$$

With the help of Proposition 2.46(2), we can find a constant $C_{\delta_1} > 0$ so that if

$$||v||_0 \leq \delta_1$$

then

$$
||H_1(v_1, v_2)||_0 \leq C_{\delta_1} ||v_1||_0 ||v_2||_0,
$$

(2.121)

$$
||H_2(v_1, v_2)||_0 \leq C_{\delta_1} ||v_1||_0 ||v_2||_0.
$$

(2.122)

The mild solution of (2.120) in $H^1(\mathbb{R})^n$ satisfies the integral equation

$$v_2(t) = e^{t \mathcal{L}_2} v_2^0 + \int_0^t e^{(t-s) \mathcal{L}_2} H_2(v_1(s), v_2(s)) ds.
$$

(2.123)

We then choose some $\bar{\rho} > \rho > 0$ and $k = \bar{\rho}/\rho > 1$ such that

$$
sup\{Re\lambda : \lambda \in Sp(\mathcal{L}_2)\} < -\bar{\rho} := -k\rho.
$$

By Proposition 2.45.(b), there exists $K_2 > 0$ such that

$$
||e^{t \mathcal{L}_2}||_{B(H^1(\mathbb{R})^n)} \leq K_2 e^{-\bar{\rho} t}.
$$

For each $\delta \in (0, \delta_1)$ and $\gamma \in (0, \delta)$, if $||v^0||_\mathcal{E} \leq \gamma$ then $t \in (0, T(\delta, \gamma))$ and $s \in (0, t)$ yield

$$
||v_2(s)||_0 \leq ||v_2(s)||_\mathcal{E} \leq ||v(s)||_\mathcal{E} \leq \delta.
$$
By Proposition 2.47, we can obtain from (2.123) the following estimate for \( \|v_2(t)\|_0 \) by using (2.122):

\[
\|v_2(t)\|_0 \leq K_2 e^{-\beta t} \|v_0\|_0 + \int_0^t K_2 e^{-\rho(t-s)} C_\delta \|v(s)\|_0 \|v_2(s)\|_0 ds
\]

\[
\leq K_2 e^{-\beta t} \|v_0\|_0 + \int_0^t K_2 e^{-\rho(t-s)} C_\delta \|v_2(s)\|_0 ds.
\]

We then calculate

\[
e^\beta \|v_2(t)\|_0 \leq K_2 \|v_0\|_0 + \int_0^t K_2 e^{\rho s} C_\delta \|v_2(s)\|_0 ds
\]

\[
\leq K_2 \|v_0\|_0 + K_2 C_\delta \delta \int_0^t e^{\rho s} \|v_2(s)\|_0 ds.
\]

By applying Gronwall’s inequality for \( e^\beta \|v_2(t)\|_0 \), we infer that

\[
\|v_2(t)\|_0 \leq K_2 \|v_0\|_0 e^{K_2 C_\delta \delta t - \beta t}.
\]

Let \( \delta_2 = \min(\delta_1, \frac{(k-1)\rho}{K_2 C_\delta_1}) \). Then for \( \delta < \delta_2 \), it follows that

\[
\|v_2(t)\|_0 \leq K_2 \|v_0\|_0 e^{-\rho t} \text{ for all } t \in [0, T(\delta, \gamma)).
\] (2.124)

The mild solution of (2.119) in \( H^1(\mathbb{R})^n \) satisfies the integral equation

\[
v_1(t) = e^{t\mathcal{L}_1} v_1^0 + \int_0^t (e^{(t-s)\mathcal{L}_1} (\partial v_2 f_1(0, 0)v_2(s) + H_1(v_1(s), v_2(s)))) ds.
\]

First, because \( \mathcal{L}_1 \) generates a bounded semigroup, there exists a constant \( K_3 > 0 \), such that \( \|e^{t\mathcal{L}_1}\|_{\mathcal{B}(H^1(\mathbb{R})^n)} \leq K_3 \). By using (2.121) and the fact that

\[
\|\partial v_2 f_1(0, 0)v_2(s)\|_0 \leq C \|v_2(s)\|_0
\]

since \( f \in C^3 \), equation (2.124) now reads

\[
\|v_1(t)\|_0 \leq K_3 \|v_1^0\|_0 + \int_0^t (K_3 C_\delta \|v_2(s)\|_0 \|v_1(s)\|_0 + K_3 C \|v_2(s)\|_0) ds. \quad (2.125)
\]
Also, using the facts that

\[ ||v_1^0||_0 \leq ||v_1^0||_{\mathcal{E}} \leq ||v^0||_{\mathcal{E}} \quad \text{and} \quad ||v_1(s)||_0 \leq ||v(s)||_0 \leq ||v(s)||_{\mathcal{E}} < \delta < \delta_2, \]

we infer from the previous inequality (2.125) that

\[ ||v_1(t)||_0 \leq K_2 ||v_1^0||_{\mathcal{E}} + \int_0^t K_2 C_\delta_1 (C + \delta_2) ||v_2(s)||_0 ds. \]

Then, using (2.124), for a constant \( C_{\delta_1, \delta_2} > 0 \), we conclude that

\[ ||v_1(t)||_0 \leq K_2 ||v_1^0||_{\mathcal{E}} + K_2 K_3 C_{\delta_1, \delta_2} ||v^0||_{\mathcal{E}} \int_0^t e^{-\rho s} ds \leq C_2 ||v^0||_{\mathcal{E}} \]

for some \( C_2 > 0 \).

In conclusion, for \( \delta \in (0, \delta_2) \) and \( \gamma \in (0, \delta) \), there exists a constant \( C_1 > 0 \) such that (2.117) and (2.118) hold when \( t \in [0, T(\delta, \gamma)) \).

Given an initial value \( v^0 \in \mathcal{E}^n \), let \( v(t) = v(t, v^0) \) be the mild solution of (2.112) in \( \mathcal{E}^n \) with \( v(0) = v^0 \), which we showed exists for at least a short time period. We shall show that \( v(t) \in \mathcal{E}^n \) is defined and bounded for all time \( t > 0 \), as we are ready to formulate our main result to establish the Lyapunov stability of the steady state solution \( u_- \). The small constant \( \delta_0 \) in the next theorem can be taken as \( \delta_0 = \delta_2 \), where \( \delta_2 \) is chosen as in Proposition 2.49.

**Theorem 2.50.** Assume Hypothesis 2.44. Then there exist constants \( \delta_0 > 0 \), \( C > 0 \) and \( \nu > 0 \) such that for each \( 0 < \delta < \delta_0 \), we can find \( \eta > 0 \) such that if \( ||v^0||_{\mathcal{E}} \leq \eta \), the following is true for all \( t > 0 \):

1. \( v(t) \) is defined;
2. \( ||v(t)||_{\mathcal{E}} \leq \delta; \)
\[(3) \quad \|v(t)\|_{\alpha} \leq Ce^{-\nu t}\|v^0\|_{\alpha};\]

\[(4) \quad \|v_1(t)\|_0 \leq C\|v^0\|_\varepsilon;\]

\[(5) \quad \|v_2(t)\|_0 \leq Ce^{-\nu t}\|v^0\|_\varepsilon.\]

**Proof.** The proof uses the same bootstrap argument as the proof of Theorem 2.38. Indeed, choose \(\nu\) as in Lemma 2.45, and \(C > \max\{1, C_1, K_1\}\) with \(C_1\) and \(K_1\) given as in Propositions 2.48 and 2.49. Take \(\delta_0 = \delta_2\) from Proposition 2.49. Let \(0 < \gamma < \delta < \delta_0\) and \(\eta = C^{-1}\gamma\). If \(\|v^0\|_\varepsilon \leq \eta\) then Propositions 2.48 and 2.49 with \(\gamma\) replaced by \(\eta < \gamma\) imply assertion (2)-(5) for \(t \in [0, T(\delta, \eta))\), and it remains to prove that \(T(\delta, \eta) = \infty\) since then we have (2)-(5) for all \(t > 0\).

To show \(T(\delta, \eta) = \infty\) it is enough to see that \(T(\delta, \eta) \geq T(\delta, \eta) + T(\delta, \gamma)\), or that \(T(\delta, \eta) \geq T(\delta, \gamma) + T\) for any \(T \in (0, T(\delta, \eta))\). But for such a \(T\) the solution satisfies \(\|v(T)\|_\varepsilon \leq C\|v^0\|_\varepsilon \leq C\eta \leq \gamma\), by using (3)-(5) at \(t = T\). Applying Proposition 2.47, we conclude that if \(\|v^0\|_\varepsilon \leq \eta\) then \(\|v(T + t)\|_\varepsilon \leq \delta\) for all \(t \in [0, T(\delta, \gamma))\), thus \(T(\delta, \eta) \geq T(\delta, \gamma) + T\) as needed. \(\blacksquare\)
Chapter 3
Multidimensional Reaction Diffusion System

In this chapter we study stability of the space independent steady state solutions and planar front solutions of a certain special class of systems of reaction diffusion equations that frequently occur in combustion theory. From a mathematical viewpoint, these equations have a certain “product-triangular” structure in the reaction term analogous to that of the equations studied in [GLSS, GLS, GLS1] for the one-dimensional case. A typical example that we have in mind is System (1.1) and a more general system (1.2). Descriptively, we are imposing some assumptions on the nonlinearity that allows one to generalize methods developed in [GLSS, GLS, GLS1] for the multidimensional situation considered by T. Kapitula in [K2].

Specifically, we discuss the following two problems. First, we consider a traveling front $\phi = \phi(x \cdot e - ct)$ for the reaction diffusion equation moving in the direction of a given $d$-dimensional vector $e$. The front has the end states, $u_-$ and $u_+$, which are some space independent steady state solutions of the reaction diffusion equation. In Sections 3.1, we study stability of the end state $u_-$, first for the model problem (1.1), and then for the general case (1.2). Next, in Section 3.2, we consider perturbations of the front itself, and show how to develop the techniques in [K2] to prove that the
perturbations originated in a small vicinity of the front decay to zero algebraically in time. The respective result in Theorem 3.63 is the central result of this chapter.

### 3.1 Stability of the space-independent steady states

In this pilot section we study the stability of the end state of systems of the model form (1.1). The exposition is quite elementary and should help understand a much more involved analysis in Section 3.2. We use the space of functions with exponential weights, and shift the spectrum of the differential operator obtained by linearizing (1.1) about the end state to the stable half of the plane. Next, we study the nonlinear equation on the intersection of the unweighted and weighted spaces. This section is organized as follows: We study the spectrum of the operator generated by linearizing (1.1) about the end state in both unweighted and weighted spaces in Subsection 3.1.1, show the Lipschitz property of the nonlinear term $H(v(t, x))$ in Subsection 3.1.2, and prove stability of the constant steady state solution $u_-$ in Subsection 3.1.3. Finally, we prove stability of a more general reaction diffusion system in Subsection 3.1.4.

#### 3.1.1 The setting in the model case

In this subsection we linearize system (1.1) about the end state of the front and study the spectrum of the operator obtained by the linearization.

We consider the reaction diffusion system of equations for the $(2 \times 1)$ vector $u = (u_1, u_2)^T \in \mathbb{R}^2$,

$$
\begin{align*}
    u_{1t}(t, x) &= \Delta_x u_1(t, x) + u_2(t, x)g(u_1(t, x)), \quad u_1, u_2 \in \mathbb{R}, \\
    u_{2t}(t, x) &= \epsilon \Delta_x u_2(t, x) - \kappa u_2(t, x)g(u_1(t, x)), \quad x \in \mathbb{R}^d,
\end{align*}
$$

where $g(u_1) = e^{-\frac{1}{u_1}}$ if $u_1 > 0$ and $g(u_1) = 0$ if $u_1 \leq 0$, and the parameters $\epsilon$ and $\kappa$
satisfy $0 \leq \epsilon < 1$ and $\kappa > 0$, and we assume here $d \geq 2$. This system often appears in combustion theory, and is a model system studied in the one-dimensional case $d = 1$ in Chapter 2. The cases $d = 2$ and $d = 3$ are of interest in combustion theory.

We will consider a planar front solution $\phi(\cdot)$ moving in the direction of a given vector $e \in \mathbb{R}^d$ with a constant speed $c > 0$. This means the following: Without loss of generality, we assume that $e = (1, 0, \ldots, 0)$. Consider a $t$-dependent change of variables $z = x_1 - ct$, $x_j = x_j$, $j = 2, \ldots, d$ in (3.1). Re-denoting $x = (z, x_2, \ldots, x_d)$ again, the reaction diffusion system (3.1) in these new moving coordinates will become

$$u_t(t, x) = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \Delta_x u(t, x) + c(e \cdot \nabla_x)u(t, x) + f(u(t, x)), \quad (3.2)$$

where $f(u(t, x)) = \begin{pmatrix} f_1(u_1, u_2) \\ f_2(u_1, u_2) \end{pmatrix} = \begin{pmatrix} u_2(t, x)g(u_1(t, x)) \\ -\kappa u_2(t, x)g(u_1(t, x)) \end{pmatrix}$.

A traveling wave solution $\phi$ for (3.1) is a $t$-independent solution $\phi = \phi(z)$ of (3.2), that is, a function that depends only on $z$, the variable along $e$, so that $\phi$ satisfies the ordinary differential equation

$$0 = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \phi''(z) + c\phi'(z) + f(\phi(z)) = 0. \quad (3.3)$$

The traveling wave solution $\phi$ is called a planar front if there exist $x$-independent steady state solution of (3.2), the end states, that satisfies the asymptotic relations

$$u_- = \lim_{z \to -\infty} \phi(z), \quad u_+ = \lim_{z \to +\infty} \phi(z).$$

We consider the left end state for the front $\phi$ given by $u_- = (1/\kappa, 0)$, see, e.g. [GLSS] and Remark 2.26, which is an $x$-independent steady state solution to (3.2) (and (3.3)). Our objective in this section is to study stability of the end state $u_- = (1/\kappa, 0)$.

We will consider perturbations of the constant solution $u_- = (1/\kappa, 0)$ that depend on all spatial variables of the system, that is, we consider the solutions $u(t, x) =$
\[ u_+ + v(t, x) \text{ of } (3.2) \text{ with the initial conditions} \]
\[ u(0, x) = u_- + v(0, x), \]

where \( v : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^2 \) is taken from an appropriate function space. Substituting \( u(t, x) = u_- + v(t, x) \) into system (3.2), we have:

\[ v_t(t, x) = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \Delta_x v(t, x) + c \partial_z v(t, x) + f(u_- + v(t, x)). \quad (3.4) \]

Linearizing the nonlinear term \( f(u_- + v(t, x)) \) at \( u_- = (1/\kappa, 0) \) gives:

\[
\begin{aligned}
f(u_- + v(t, x)) &= f(u_-) + \partial_u f(u_-) v(t, x) + H(v(t, x)) \\
&= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & e^{-\kappa} \\ 0 & -\kappa e^{-\kappa} \end{pmatrix} \begin{pmatrix} v_1(t, x) \\ v_2(t, x) \end{pmatrix} + H(v(t, x)),
\end{aligned}
\]

where we introduced the nonlinear term by

\[ H(v(t, x)) = f(u_- + v(t, x)) - f(u_-) - \partial_u f(u_-) v(t, x). \quad (3.5) \]

We therefore have the following semilinear equation for the perturbations of the end states,

\[ v_t(t, x) = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \Delta_x v(t, x) + c \partial_z v(t, x) + \begin{pmatrix} 0 & e^{-\kappa} \\ 0 & -\kappa e^{-\kappa} \end{pmatrix} v(t, x) + H(v(t, x)). \quad (3.6) \]

We now define the linear differential expression \( L \) by

\[ L = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \Delta_x + c \partial_z + \begin{pmatrix} 0 & e^{-\kappa} \\ 0 & -\kappa e^{-\kappa} \end{pmatrix}. \quad (3.7) \]

We will consider a differential operator \( \mathcal{L} \) associated with the differential expression \( L \) in the Sobolev space \( H^k(\mathbb{R}^d)^2 \) of \((2 \times 1)\)-vector valued functions and throughout assume that \( k \geq \left\lfloor \frac{d+1}{2} \right\rfloor \). As we will see soon, the essential spectrum of \( \mathcal{L} \) will touch the imaginary axis. This prevents \( u_- \) from being stable in the space \( H^k(\mathbb{R}^d)^2 \), and we
will have to replace the space by another space $H^k_\alpha(\mathbb{R}^d)^2$, with an exponential weight with respect to the variable $z$. In this new space the operator $\mathcal{L}$ will become spectrally stable, but then the nonlinearity will lose the local Lipschitz property needed to see the well-posedness of (3.2). To gain it back, as in [GLSS, GLS, GLS1] and Subsection 2.2.2, we will pass to the intersection space $H^k(\mathbb{R}^d)^2 \cap H^k_\alpha(\mathbb{R}^d)^2$, and perform further analysis there.

To study the stability of the solution to (3.6), we first need spectral information regarding the linear operator associated with (3.7). We will define several linear operators associated with $L$ given by (3.7) in different spaces.

For $\mathcal{E}_0$ being the Sobolev spaces $H^k(\mathbb{R}^d)$ ($k = 1, 2, \ldots$ and we often define $H^0(\mathbb{R}^d) = L^2(\mathbb{R}^d)$), which are suited for the study of nonlinear stability because they are closed under multiplication, we denote the norm in $\mathcal{E}_0$ by $\| \cdot \|_0$. In this section, we define the weight function of class $\alpha \in \mathbb{R}$ similarly by $e^{\alpha z}$ for $z \in \mathbb{R}$, as in the one-dimensional case, and now in $H^k(\mathbb{R}^d)$ we will let

$$\gamma_\alpha(z, x_2, \ldots, x_d) = e^{\alpha z}, \text{ for } x = (z, x_2, \ldots, x_d) \in \mathbb{R}^d.$$  

For the fixed weight function $\gamma_\alpha$ we define $\mathcal{E}_\alpha = \{ u : \gamma_\alpha u \in \mathcal{E}_0\}$, with the norm $\| u \|_\alpha = \| \gamma_\alpha u \|_0$. Note that by this definition, $\mathcal{E}_\alpha = H^k_\alpha(\mathbb{R}) \otimes H^k(\mathbb{R}^{d-1})$. Here and below we use the fact that $H^k(\mathbb{R}^d)$ can be written as the tensor product $H^k(\mathbb{R}^d) = H^k(\mathbb{R}) \otimes H^k(\mathbb{R}^{d-1})$. For general results on tensor products and operators on tensor products we refer to [RS1, Section VIII.10]. Thus, in this section $H^k_\alpha(\mathbb{R}^d) = \{ u : e^{\alpha z}u \in H^k(\mathbb{R}^d)\}$

**Remark 3.1.** We will involve several operators associated with each differential expressions considered below. We will use the following notation for these operators. If $B$ is a $(2 \times 2)$ system of $n$ differential expressions as, for instance, in (3.7), then we
shall use notation $B : \mathcal{E}_0^2 \to \mathcal{E}_0^2$ and $B_\alpha : \mathcal{E}_\alpha^2 \to \mathcal{E}_\alpha^2$ to denote the linear operator in $\mathcal{E}_0^2$ and $\mathcal{E}_\alpha^2$, respectively, given by the formula $u \mapsto Bu$, with their “natural” domains, that is, for $k = 0, 1, \cdots$, we use $\mathcal{L} : \mathcal{E}_0 \to \mathcal{E}_0$ to denote the linear operator given by the formula $u \mapsto Lu$, whose domain is $H^{k+2}(\mathbb{R}^d)^2$; and use $\mathcal{L}_\alpha : \mathcal{E}_\alpha \to \mathcal{E}_\alpha$ to denote the operator in $\mathcal{E}_\alpha^2$ given by the formula $u \mapsto Lu$, whose domain is the set of $(u_1, u_2)$ where $\gamma_\alpha u_1, \gamma_\alpha u_2 \in H^{k+2}(\mathbb{R}^d)$.

We also introduce the intersection space:

$$\mathcal{E} := \mathcal{E}_0 \cap \mathcal{E}_\alpha, \text{ with } \|u\|_E = \max\{\|u\|_0, \|u\|_\alpha\}. \quad (3.8)$$

We use notation $\mathcal{L}_E : \mathcal{E}^2 \to \mathcal{E}^2$ to denote the linear operator given by $u \mapsto Lu$ with the domain of $\mathcal{L}_E$ being the set of $(u_1, u_2)$ satisfying $(u_1, u_2) \in \text{dom}(\mathcal{L}) \cap \text{dom}(\mathcal{L}_\alpha)$, where $\text{dom}(\mathcal{L})$ and $\text{dom}(\mathcal{L}_\alpha)$ are respective domains defined above.

First, we will use Fourier transform to explore the spectrum of the constant coefficient differential operator $\mathcal{L}$ on $L^2(\mathbb{R}^d)$, and the spectrum of the constant coefficient differential operator $\mathcal{L}_\alpha$ on $L^2_\alpha(\mathbb{R}) \otimes L^2(\mathbb{R}^{d-1})$, respectively. We will use the following elementary proposition to show that the spectrum of $\mathcal{L}$ on $\mathcal{E}_0^2$ touches the imaginary axis, and the spectrum of $\mathcal{L}_\alpha$ on $\mathcal{E}_\alpha^2$ will be away from the imaginary axis.

**Proposition 3.2.** Assume that $\mathcal{L}$ and $\mathcal{L}_\alpha$ are the constant coefficient linear differential operators associated with the differential expression $L$ in (3.7). On the unweighted space $\mathcal{E}_0^2 = H^k(\mathbb{R}^d)^2$ for all integers $k \geq 0$, one has

$$\sup\{\Re \lambda : \lambda \in \text{Sp}(\mathcal{L})\} = 0,$$

so that the spectrum of $\mathcal{L}$ touches the imaginary axis. By choosing $\alpha \in (0, c/2)$, one
has
\[
\sup\{\text{Re}\lambda : \lambda \in \text{Sp}(L_\alpha)\} < -\nu
\]
for some \(\nu > 0\) so that the spectrum of \(L_\alpha\) is shifted to the left of the imaginary axis on the weighted space \(E_\alpha^2 = H^k_\alpha(\mathbb{R})^2 \otimes H^k(\mathbb{R}^{d-1})^2\).

**Proof.** By Lemma 3.3 proved next, it is enough to consider the case \(k = 0\), that is, to assume that \(E_0 = L^2(\mathbb{R}^d)\). To find \(\text{Sp}(L)\) in the unweighted space \(E^2_0\), we can use Fourier transform. By properties of Fourier transform, see, e.g., [EN, Section 6.5], the operator \(L\) on \(L^2(\mathbb{R}^d)^2\) is similar to the operator on \(L^2(\mathbb{R}^d)^2\) of multiplication by the matrix-valued function
\[
M(\xi) = -(\xi_1^2 + \xi_2^2 + \cdots + \xi_d^2) \begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix} + i\xi_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & e^{-\kappa} \\ 0 & -\kappa e^{-\kappa} \end{pmatrix},
\]
where \(\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d\). Thus the spectrum of \(L\) on \(L^2(\mathbb{R}^d)^2\) is the closure of the union over \(\xi \in \mathbb{R}^d\) of the spectra of the matrices \(M(\xi)\). Hence the spectrum of \(L\) is equal to the closure of the set of \(\lambda \in \mathbb{C}\) for which there exists \(\xi \in \mathbb{R}^d\) such that
\[
\det (M(\xi) - \lambda I) = \det \left( -(\xi_1^2 + \xi_2^2 + \cdots + \xi_d^2) \begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix} + i\xi_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & e^{-\kappa} \\ 0 & -\kappa e^{-\kappa} \end{pmatrix} \right) = 0.
\]
It is a collection of curves \(\lambda = \lambda(\xi)\), where \(\lambda(\xi)\) are the eigenvalues of the matrices \(M(\xi)\). Thus the spectrum of the operator \(L\) is
\[
\text{Sp}(L) = \bigcup_{\xi \in \mathbb{R}^d} \text{Sp}\left( -(\xi_1^2 + \cdots + \xi_d^2) + ci\xi_1 \begin{pmatrix} 1 & 0 \\ 0 & e^{-\kappa} \end{pmatrix} -\epsilon(\xi_1^2 + \cdots + \xi_d^2) + ci\xi_1 - \kappa e^{-\kappa} \right)
\]
\[
= \bigcup_{\xi \in \mathbb{R}^d} \left( -(\xi_1^2 + \cdots + \xi_d^2) + ci\xi_1 \right) \bigcup_{\xi \in \mathbb{R}^d} \left( -\epsilon(\xi_1^2 + \cdots + \xi_d^2) + ci\xi_1 - \kappa e^{-\kappa} \right).
\]
This implies that the spectrum of \(L\) in \(L^2(\mathbb{R}^d)^2\) touches the imaginary axis when \(\xi = (\xi_1, \ldots, \xi_d) = (0, \ldots, 0)\).
We also need \( \operatorname{Sp}(L_{\alpha}) \) on the weighted space \( \mathcal{E}_{\alpha}^{2} \). First define the linear map \( N : \mathcal{E}_{\alpha} \mapsto \mathcal{E}_{0} \) given by \( Nv = \gamma_{\alpha}v \), and notice that by definition \( N \) is an isomorphism of \( \mathcal{E}_{\alpha} \) onto \( \mathcal{E}_{0} \). In particular, we can define a linear operator \( \hat{L} = NL_{\alpha}N^{-1} \) on \( \mathcal{E}_{0}^{2} = L^{2}(\mathbb{R}^{d})^{2} \), with the domain \( \operatorname{dom}(\hat{L}) = H^{2}(\mathbb{R}^{d})^{2} \) since \( N^{-1} \) maps \( \operatorname{dom}(\hat{L}) \) in \( \operatorname{dom}(L_{\alpha}) \). The operator \( \hat{L} \) is similar to \( L_{\alpha} \) on \( \mathcal{E}_{\alpha}^{2} \) and hence has the same spectrum.

In particular, let us consider the operator \( \partial_{z,\alpha} \) on \( \mathcal{E}_{\alpha} \) with

\[
\operatorname{dom}(\partial_{z,\alpha}) = H^{1}_{\alpha}(\mathbb{R}) \otimes H^{1}(\mathbb{R}^{d-1}).
\]

Fix any \( v \in H^{1}(\mathbb{R}^{d}) = \operatorname{dom}(\hat{\partial}_{z}) \) when \( \hat{\partial}_{z} \) is considered in \( L^{2}(\mathbb{R}^{d}) \), and \( \hat{\partial}_{z} = N\partial_{z,\alpha}N^{-1} \).

Then, temporarily redenoting \( \gamma_{\alpha}(z) = e^{\alpha z} \), we have

\[
\partial v = N\partial_{z,\alpha}N^{-1}v = \gamma_{\alpha}\partial_{z}(\gamma_{-\alpha}v) = \gamma_{\alpha}(\gamma'_{-\alpha}v + \gamma_{-\alpha}\partial_{z}v)
\]

\[
= \gamma_{\alpha}(-\alpha\gamma_{-\alpha}v + \gamma_{-\alpha}\partial_{z}v)
\]

\[
= (\partial_{z} - \alpha)v.
\]

Denoting \( y = (x_{2}, \ldots, x_{d}) \), then \( x = (z, y) \in \mathbb{R}^{d} \), a similar computation shows that for each

\[
v = (v_{1}, v_{2})^{T} \in \operatorname{dom}\hat{L} = L^{2}(\mathbb{R}^{d})^{2} \subset L^{2}(\mathbb{R}^{d})^{2},
\]

we have:

\[
\hat{L}v = e^{\alpha z}
\]

\[
\left( \begin{array}{cc}
1 & 0 \\
0 & \epsilon
\end{array} \right)
\left( \begin{array}{c}
\Delta_{y} + \partial_{zz}
\
-\kappa \epsilon
\end{array} \right)
\left( \begin{array}{c}
ev^{-\kappa}
\
0
\end{array} \right)
\left( e^{-\alpha z}v \right)
\]

\[
= \left( \begin{array}{c}
1 \\
\epsilon
\end{array} \right) \Delta_{y}v + \left( \begin{array}{c}
1 \\
\epsilon
\end{array} \right) (\alpha^{2}v - 2\alpha\partial_{z}v + \partial_{zz}v) + c(\partial_{z}v - \alpha v) + \left( \begin{array}{c}
0 \\
0
\end{array} \right) v
\]

\[
= \left( \begin{array}{cc}
1 & 0 \\
\epsilon & 0
\end{array} \right) \Delta_{x}v + (cI - 2\alpha \left( \begin{array}{c}
1 \\
\epsilon
\end{array} \right)) \partial_{z}v + (\alpha^{2} \left( \begin{array}{c}
1 \\
\epsilon
\end{array} \right) - c\alpha I + \left( \begin{array}{c}
0 \\
0 \epsilon e^{-\kappa}
\end{array} \right) v.
\]

Via the Fourier transform, the operator \( \hat{L} \) on \( L^{2}(\mathbb{R}^{d})^{2} \) is similar to the operator of
multiplication on $L^2(\mathbb{R}^d)^2$ by the matrix-valued function

$$N(\xi) = -\|\xi\|^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (i\xi_1 c - \alpha c) I + (\alpha^2 - 2i\xi_1 \alpha) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} e^{-\kappa} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\|\xi\|^2 + (c - 2a)\xi_1 + \alpha^2 - \alpha \xi_1 + \alpha^2 - c\alpha - \kappa e^{-\kappa} \\ -\|\xi\|^2 + (c - 2a)\xi_1 + \alpha^2 - \alpha \xi_1 + \alpha^2 - c\alpha - \kappa e^{-\kappa} \end{pmatrix}$$

where $\|\xi\|^2 = \xi_1^2 + \cdots + \xi_d^2$. Hence,

$$\text{Sp}(L_{\alpha}) = \bigcup_{\xi \in \mathbb{R}^d} (-\xi_1^2 + \cdots + \xi_d^2) + (c - 2\alpha)i\xi_1 + \alpha^2 - c\alpha$$

$$\bigcup_{\xi \in \mathbb{R}^d} (-\xi_1^2 + \cdots + \xi_d^2) + (c - 2\alpha)i\xi_1 + \alpha^2 - c\alpha - \kappa e^{-\kappa}).$$

Then

$$\sup\{\text{Re}\lambda : \lambda \in \text{Sp}(L_{\alpha})\} = \sup\{\text{Re}\lambda : \lambda \in \text{Sp}(L)\}$$

$$= \max(\alpha^2 - c\alpha, \epsilon\alpha^2 - c\alpha - \kappa e^{-\kappa})$$

$$= \alpha^2 - c\alpha.$$ 

Thus we conclude that for $\alpha \in (0, c/2)$, one has $\sup\{\text{Re}\lambda : \lambda \in \text{Sp}(L_{\alpha})\} < 0$ so that the spectrum $\text{Sp}(L_{\alpha})$ on the weighted space $\mathcal{E}_0^2$ is moved to the left of the imaginary axis for $\mathcal{E}_0 = L^2(\mathbb{R}^d) = H^0(\mathbb{R}^d)$. \(\blacksquare\)

**Lemma 3.3.** *The linear constant coefficient differential operator $L$ associated with $L$ defined in (3.7) has the same spectrum on $L^2(\mathbb{R}^d)^2$ and on $H^k(\mathbb{R}^d)^2$ for all integers $k > 0$; similarly, the operator $L_{\alpha}$ associated with $L$ defined in (3.7) has the same spectrum on $L^2_{\alpha}(\mathbb{R}) \otimes L^2(\mathbb{R}^{d-1})$ and on $H^k_{\alpha}(\mathbb{R}) \otimes H^k(\mathbb{R}^{d-1})$, for all integers $k > 0$.***

*Proof.* To show that the spectrum of $L$ in $H^k(\mathbb{R}^d)^2$ is the same as the spectrum of $L^2(\mathbb{R}^d)^2$, we let $F_1$ denote the Fourier transform acting from $H^k(\mathbb{R}^d)^2$ into $L^2_{\alpha}(\mathbb{R}) \otimes L^2(\mathbb{R}^{d-1})$, where $L^2_{\alpha}(\mathbb{R}) \otimes L^2(\mathbb{R}^{d-1})$ is the weighted $L^2$-space with the standard weight $m(\xi) = (1 +
By the standard property of the Fourier transform we have \( F_1 \Delta_x = -|\xi|^2_{\mathbb{R}^d} F_1 \) and \( F_1 \partial_z = -i \xi_1 F_1 \). Thus \( F_1 \mathcal{L} = M F_1 \) for a matrix-valued function \( M = M(\xi) \) obtained from (3.7) by replacing \( \Delta_x \) by \( -|\xi|^2_{\mathbb{R}^d} \) and \( \partial_z \) by \( -i \xi_1 \).

On the other hand, the operator of multiplication by \( m(\cdot) \) is an isomorphism of \( L^2_m(\mathbb{R}^d)^2 \) onto \( L^2(\mathbb{R}^d)^2 \). Let us denote by \( \mathcal{L}_{H^k} \) the operator \( \mathcal{L} \) associated with \( L \) on the space \( H^k(\mathbb{R}^d)^2 \), and by \( \mathcal{L}_{L^2} \) the operator \( \mathcal{L} \) associated with \( L \) on the space \( L^2(\mathbb{R}^d)^2 \). By the previous paragraph we then have \( m F_1 \mathcal{L}_{H^k} = M m F_1 \). (Here and below we allow a slight abbreviation of notation, the proper writing is that \( u \in \text{dom}(\mathcal{L}_{H^k}) \) implies \( m F_1 u \in \text{dom}(M) \) and \( m F_1 \mathcal{L}_{H^k} u = M m F_1 u \) for all \( u \in \text{dom}(\mathcal{L}_{H^k}) \).

We remark that the operator of multiplication by \(-i \xi_j \), \( j = 1, \ldots, d \) on \( L^2(\mathbb{R}^d)^2 \) is similar to the operator of differentiation \( \partial_{x_j} \) on \( L^2(\mathbb{R}^d)^2 \) via the Fourier transform \( F_2 \). This implies that \( F_2 \mathcal{L}_{L^2} = M F_2 \) with the same matrix-valued function \( M \) as above. It follows that

\[
\mathcal{L}_{H^k} = (m F_1)^{-1} M m F_1 = (m F_1)^{-1} (F_2 \mathcal{L}_{L^2} F_2^{-1}) (m F_1),
\]

therefore the spectrum of \( \mathcal{L} \) on \( H^k(\mathbb{R}^d)^2 \) is the same as the spectrum of \( \mathcal{L} \) on \( L^2(\mathbb{R}^d)^2 \) because the operators on \( H^k(\mathbb{R}^d)^2 \) and \( L^2(\mathbb{R}^d)^2 \) are similar.

By analogous argument, the spectrum of \( \mathcal{L}_{\alpha} \) on \( L^2_\alpha(\mathbb{R}) \otimes L^2(\mathbb{R}^{d-1}) \) is the same as the spectrum of \( \mathcal{L}_{\alpha} \) on \( H^k_{\alpha}(\mathbb{R}) \otimes H^k(\mathbb{R}^{d-1}) \).

**Remark 3.4.** Recall that we denote \( y = (x_2, \ldots, x_d) \). Let \( \Delta_y \) be the operator given by the differential expression \( \partial^2_{x_2} + \cdots + \partial^2_{x_d} \), where the domain of \( \Delta_y \) on the Hilbert space \( H^k(\mathbb{R}^{d-1}) \) is the set of \( u \) such that \( u \in H^{k+2}(\mathbb{R}^{d-1}) \). We denote by \( \mathcal{L}_{1,\alpha} : H^k_{\alpha}(\mathbb{R})^2 \to H^k_{\alpha}(\mathbb{R})^2 \) the operator given by the differential expression \( \partial_{x_2} + c \partial_{x} + \begin{pmatrix} 0 & e^{-\kappa} \\ 0 & -\kappa e^{-\kappa} \end{pmatrix} \), and \( \text{dom}(\mathcal{L}_{1,\alpha}) = H^{k+2}_{\alpha}(\mathbb{R})^2 \subset H^k_{\alpha}(\mathbb{R})^2 \). The operator \( \mathcal{L}_{\alpha} \) on \( \mathcal{E}_{\alpha}^2 = H^k_{\alpha}(\mathbb{R})^2 \otimes H^k(\mathbb{R}^{d-1})^2 \)
can be written as $\mathcal{L}_{1,\alpha} \otimes I_{H^k(\mathbb{R}^{d-1})} + I_{H^k_\alpha(\mathbb{R})} \otimes \Delta_y$. We have yet another approach to prove Proposition 3.2 by using Theorem XIII.34, Theorem XIII.35 and Corollary 1 from [RS4]. Indeed, since $\mathcal{L}_{1,\alpha}$ and $\Delta_y$ are the generators of bounded semigroups on Hilbert spaces $H^k_\alpha(\mathbb{R})$ and $H^k(\mathbb{R}^{d-1})$ respectively, we have

$$\text{Sp}(\mathcal{L}_{1,\alpha} \otimes I_{H^k(\mathbb{R}^{d-1})} + I_{H^k_\alpha(\mathbb{R})} \otimes \Delta_y) = \text{Sp}(\mathcal{L}_{1,\alpha}) + \text{Sp}(\Delta_y), \quad (3.13)$$

see [RS4]. Thus, $\text{Sp}(\mathcal{L}_\alpha) = \text{Sp}(\mathcal{L}_{1,\alpha}) + \text{Sp}(\Delta_y)$. It is easy to see that the spectrum of $\Delta_y$ on $H^k(\mathbb{R}^{d-1})$ is the non-negative semiline $(-\infty, 0]$ and we have showed that the spectrum of $\mathcal{L}_{1,\alpha}$ on $H^k_\alpha(\mathbb{R})$ satisfies $\sup\{\text{Re}\lambda : \lambda \in \text{Sp}(\mathcal{L}_{1,\alpha})\} < -\nu$ for some $\nu > 0$, thus Proposition 3.2 is proved. Moreover, the same argument shows that if $\Gamma$ is the curve that bounds the spectrum of $\mathcal{L}_{1,\alpha}$ on the right, then $\text{Sp}(\mathcal{L}_\alpha)$ is the entire solid part of the plane bounded by $\Gamma$. ∎

Now notice that the differential expression in (3.7) has the following triangular structure,

$$L = \begin{pmatrix} \Delta_x + c\partial_z & e^{-\kappa} \\ 0 & \epsilon \Delta_x + c\partial_z - \kappa e^{-\kappa} \end{pmatrix}.$$ 

Let

$$L^{(1)} = \Delta_x + c\partial_z; \quad (3.14)$$

$$L^{(2)} = \epsilon \Delta_x + c\partial_z - \kappa e^{-\kappa}, \quad (3.15)$$

and for $i = 1, 2$, let $\mathcal{L}^{(i)}$ be the operator on $H^k(\mathbb{R}^d)$ defined by $v_i \mapsto L^{(i)}v_i$, with the domain of $\mathcal{L}^{(i)}$ to be $H^{k+2}(\mathbb{R}^d)$, for $k = 0, 1, 2, ...$.

**Lemma 3.5.** Consider the operators $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$ on $H^k(\mathbb{R}^d)$ defined by the differential expressions $L^{(1)}$ and $L^{(2)}$ given in (3.14) and (3.15).
(1) The operator $L^{(1)}$ generates a bounded strongly continuous semigroup on $H^k(\mathbb{R}^d)$;

(2) The operator $L^{(2)}$ satisfies $\sup\{\Re \lambda : \lambda \in \text{Sp}(L^{(2)})\} < 0$ on $H^k(\mathbb{R}^d)$;

(3) The following is true on $H^k(\mathbb{R}^d)$:

(a) $\sup\{\Re \lambda : \lambda \in \text{Sp}(L^{(1)})\} \leq 0$;

(b) There exist $K > 0$ and $\rho > 0$ such that for the strongly continuous semigroup $\{e^{tL^{(2)}}\}_{t \geq 0}$, one has $\|e^{tL^{(2)}}\|_{\mathcal{B}(H^k(\mathbb{R}^d))} \leq Ke^{-\rho t}$ for all $t \geq 0$.

Proof. As in Lemma 3.3, we can prove that the operators $L^{(i)}$, $i = 1, 2$ have the same spectrum on $H^k(\mathbb{R}^d)$ and on $L^2(\mathbb{R}^d)$.

Using the Fourier transform, we find that the spectrum of $L^{(1)}$ on $L^2(\mathbb{R}^d)$ is the union of the curves $\lambda_1(\xi) = -\epsilon(\xi_1^2 + \cdots + \xi_d^2) + c\xi_1$ for all $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$, thus $\sup\{\Re \lambda : \lambda \in \text{Sp}(L^{(1)})\} \leq 0$ on $L^2(\mathbb{R})$ which proves (3)(a). By the proof of Proposition A.1(1) in [GLS], the operator $L^{(1)}$ generates a bounded semigroup on $L^2(\mathbb{R}^d)$. Operators on $H^k(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$ associated with the same constant-coefficient differential expression are similar, see (3.12), therefore the semigroup they generate are similar, so (1) is proved.

The spectrum of $L^{(2)}$ on $L^2(\mathbb{R}^d)$ is the union of the curves $\lambda_2(\xi) = -\epsilon(\xi_1^2 + \cdots + \xi_d^2) + c\xi_1 - \kappa e^{-\kappa}$ for all $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$, and therefore $\sup\{\Re \lambda : \lambda \in \text{Sp}(L^{(2)})\} < 0$ on $L^2(\mathbb{R}^d)$, also on $H^k(\mathbb{R}^d)$ by Lemma 3.3, proving (2).

Assertion (3)(b) is a direct consequence of (2), see [GLS] Lemma 3.13. □
3.1.2 Nonlinear terms in the model case

In this subsection we study the nonlinear terms defined in (3.5) and prove the nonlinearity is locally Lipschitz on the intersection space $E$.

Recall that we introduced the nonlinear term of system (3.6) as in formula (3.5), that is,

$$
H(v(t,x)) = f(u_+ + v(t,x)) - f(u_-) - \partial_u f(u_-) v(t,x)
$$

$$
= f\left(\begin{pmatrix}1/\kappa + v_1 \\ v_2\end{pmatrix}\right) - \begin{pmatrix}0 \\ 0\end{pmatrix} - \begin{pmatrix}0 \\ 0\end{pmatrix} - \begin{pmatrix}e^{-\kappa} \\ -\kappa e^{-\kappa}\end{pmatrix} v(t,x)
$$

$$
= \begin{pmatrix}v_2(e^{-\frac{1}{\kappa} + v_1} - e^{-\kappa}) \\ -\kappa v_2(e^{-\frac{1}{\kappa} + v_1} - e^{-\kappa})\end{pmatrix},
$$

(3.16)

In Subsection 2.2.2, we proved that the nonlinear term $H(\cdot)$ is a locally Lipschitz map in the one-dimensional space if $v_1, v_2 \in H^1(\mathbb{R})$. To obtain the Lipschitz property of the nonlinear term on the multidimensional space $H^k(\mathbb{R}^d)$ and $H^k_0(\mathbb{R}) \otimes H^k(\mathbb{R}^{d-1})$, we need the space $E$ defined in equation (3.8).

It will be convenient to write $H(v)$ as follows:

$$
H(v) = \begin{pmatrix}1 \\ -\kappa\end{pmatrix} \begin{pmatrix}g\left(\frac{1}{\kappa} + v_1\right) - g\left(\frac{1}{\kappa}\right)\end{pmatrix} v_2,
$$

(3.17)

where $v = (v_1, v_2)$ and $g(\cdot)$ is defined as in (1.1).

The proofs below will be based on the fact that the Sobolev embedding yields the inequality

$$
||uv||_{H^k(\mathbb{R}^d)} \leq C ||u||_{H^k(\mathbb{R}^d)} ||v||_{H^k(\mathbb{R}^d)}
$$

(3.18)

for $2k > d$ (see [AF, Theorem 4.39]). We begin with some elementary facts.

**Lemma 3.6.** Assume that $k \geq \left[\frac{d+1}{2}\right]$, and consider $E_0 = H^k(\mathbb{R}^d)$. Then the following assertions hold.
\(1\) If \(u, v \in \mathcal{E}_0\), then \(uv \in \mathcal{E}_0\), and there exists a constant \(C > 0\) such that
\[
\|uv\|_0 \leq C\|u\|_0\|v\|_0.
\]

\(2\) If \(u, v \in \mathcal{E}\), then \(uv \in \mathcal{E}_\alpha\), and there exists a constant \(C > 0\) such that
\[
\|uv\|_\alpha \leq C\|u\|_0\|v\|_\alpha.
\]

\(3\) If \(u, v \in \mathcal{E}\), then \(uv \in \mathcal{E}\), and there exists a constant \(C > 0\) such that
\[
\|uv\|_\mathcal{E} \leq C\|u\|_\mathcal{E}\|v\|_\mathcal{E}.
\]

**Proof.** Assertion (1) is in fact (3.18). Assertion (2) can be proved by using (3.18) since
\[
\|uv\|_\alpha = \|\gamma_\alpha uv\|_0 \leq C\|u\|_0\|\gamma_\alpha v\|_0 = C\|u\|_0\|v\|_\alpha.
\]

To show (3), let \(u, v \in \mathcal{E}\). Then by (1),
\[
\|uv\|_0 \leq C\|u\|_0\|v\|_0 \leq C\|u\|_\mathcal{E}\|v\|_\mathcal{E},
\]
and by (2),
\[
\|uv\|_\alpha \leq C\|u\|_0\|v\|_\alpha \leq C\|u\|_\mathcal{E}\|v\|_\mathcal{E}.
\]
Therefore \(uv \in \mathcal{E}\) and \(\|uv\|_\mathcal{E} \leq C\|u\|_\mathcal{E}\|v\|_\mathcal{E}\). ■

The nonlinearities of type (3.17) is a combination of the Nemytskij-type operator \(v_1 \mapsto g(1/\kappa + v_1)\) and multiplication operator by \(v_2\). In what follows we will need to establish local Lipschitz properties of this and more general operators of the type
\(v \mapsto m(v(\cdot))v(\cdot)\) where \(m(\cdot)\) is a given function and \(v \in H^k(\mathbb{R}^d)\). The one-dimensional results of this type can be found in [GLS, Proposition 7.2]. We present an analogue of the proof of [GLS, Proposition 7.2] in Appendix A. Also, by using the same
technique as in the appendix but dropping \( q \) from Proposition A.3, we record the following corollary that can be used to study the components of the map \( H(\cdot) \) from (3.17).

**Corollary 3.7.** Let \( \mathcal{E}_0 = H^k(\mathbb{R}^d) \), and \( \mathcal{E}_\alpha \) and \( \mathcal{E} \) be defined accordingly. If \( k \geq \left[ \frac{d+1}{2} \right] \) and \( m(\cdot) \in C^\infty(\mathbb{R}) \), then the formula

\[
v(x) \mapsto m(v(x))v(x), \quad x \in \mathbb{R}^d,
\]
defines mappings from \( \mathcal{E}_0 \) to \( \mathcal{E}_0 \), and from \( \mathcal{E} \) to \( \mathcal{E} \). The first is locally Lipschitz on any set of the form \( \{v : \|v\|_0 \leq K\} \); the second is locally Lipschitz on any set of the form \( \{v : \|v\|_\mathcal{E} \leq K\} \).

**Proposition 3.8.** Let \( \mathcal{E}_0 = H^k(\mathbb{R}^d) \), and \( \mathcal{E}_\alpha \) and \( \mathcal{E} \) be defined accordingly, let \( k \geq \left[ \frac{d+1}{2} \right] \) and \( v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \), and consider the formula

\[
H(v) = \begin{pmatrix} 1 \\ -\kappa \end{pmatrix} v_2 \left(g(v_1 + 1/\kappa) - g(1/\kappa)\right)
\]
given in (3.16).

1. \( H(\cdot) \) defines a mapping from \( \mathcal{E}^2_0 \) to \( \mathcal{E}^2_0 \) that is locally Lipschitz on any set of the form \( \{v : \|v\|_0 \leq K\} \).

2. \( H(\cdot) \) defines a mapping from \( \mathcal{E}^2 \) to \( \mathcal{E}^2 \) that is locally Lipschitz on any set of the form \( \{v : \|v\|_\mathcal{E} \leq K\} \).

**Proof.** By Subsection 2.2.2 of the one-dimensional case, we know that \( g(\cdot) \in C^\infty(\mathbb{R}) \) is a smooth bounded function. Let

\[
m(v_1) = g(v_1 + 1/\kappa) - g(1/\kappa),
\]
then $m(\cdot)$ is also a smooth and bounded function. By applying Corollary 3.7 or Proposition A.3 to the components of the vector-valued map $H$, we finish the proof.

Proposition 3.9. Let $\mathcal{E}_0 = H^k(\mathbb{R}^d)$, let $\mathcal{E}_\alpha$ and $\mathcal{E}$ be defined accordingly, let $k \geq \left\lceil \frac{d+1}{2} \right\rceil$ and $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, and consider the formula

$$H(v) = \begin{pmatrix} 1 \\ -\kappa \end{pmatrix} v_2 \left( g(v_1 + 1/\kappa) - g(1/\kappa) \right)$$

given in (3.16).

(1) If $v \in \mathcal{E}_2$, then there exist a constant $C_K > 0$ such that

$$\|H(v)\|_\alpha \leq C_K \|v\|_0 \|v\|_\alpha$$

on any set of the form $\{v : \|v\|_{\mathcal{E}} \leq K\}$.

(2) If $v \in \mathcal{E}_2$, then there exist a constant $C_K > 0$ such that

$$\|H(v)\|_{\mathcal{E}} \leq C_K \|v\|_{\mathcal{E}}^2$$

on any set of the form $\{v : \|v\|_{\mathcal{E}} \leq K\}$.

Proof. Note that

$$g(v_1 + 1/\kappa) - g(1/\kappa) = \left( \int_0^t g'(1/\kappa + tv_1) dt \right) v_1.$$ 

Since $\int_0^t g'(1/\kappa + tv_1) dt$ is also a smooth function, by Corollary 3.7, we have

$$\|g(v_1 + 1/\kappa) - g(1/\kappa)\|_0 \leq C_K \|v_1\|_0.$$ 

Also note that $\|v_1\|_0 \leq \|v\|_0$ and $\|v_2\|_\alpha \leq \|v\|_\alpha$ because $v_1$ and $v_2$ are components of the vector $v$. Then (1) holds since

$$\|H(v)\|_\alpha = \|\gamma_\alpha H(v)\|_0 = \left\| \gamma_\alpha \begin{pmatrix} 1 \\ -\kappa \end{pmatrix} v_2 \left( g(v_1 + 1/\kappa) - g(1/\kappa) \right) \right\|_0$$

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\[ C \| (g(v_1 + 1/\kappa) - g(1/\kappa)) \|_0 \gamma_\alpha v_2 \|_0 \]
\[ \leq C_K \| v_1 \|_0 \| v_2 \|_\alpha \leq C_K \| v \|_0 \| v \|_\alpha. \]

Similarly, by using the fact that \( \| v_2 \|_0 \leq \| v \|_0 \), we have
\[ \| H(v) \|_0 = \left\| \begin{pmatrix} 1 \\ -\kappa \end{pmatrix} v_2 (g(v_1 + 1/\kappa) - g(1/\kappa)) \right\|_0 \]
\[ \leq C \| (g(v_1 + 1/\kappa) - g(1/\kappa)) \|_0 \| v_2 \|_0 \]
\[ \leq C_K \| v_1 \|_0 \| v_2 \|_0 \leq C_K \| v \|_0 \| v \|_0. \]

thus,
\[ \| H(v) \|_{\mathcal{E}} = \max \{ \| H(v) \|_0, \| H(v) \|_\alpha \} \]
\[ \leq \max \{ C_K \| v \|_0 \| v \|_0, C_K \| v \|_0 \| v \|_\alpha \} \]
\[ \leq C_K \| v \|_{\mathcal{E}} \| v \|_{\mathcal{E}}, \]

and (2) is proved. \( \blacksquare \)

### 3.1.3 Stability of the end state in the model case

In this subsection we prove the stability of the zero solution of system (3.6) and thus the stability of the end state \( u_- = (1/\kappa, 0) \) of (3.2). By Proposition 3.2 and Proposition 3.8, we know that, given initial data \( v^0 \in \mathcal{E}^2 \), the system (3.6) has a unique mild solution \( v(t, v^0) \) with \( v(0, v^0) = v^0 \). The solution is defined for \( t \in [0, t_{\max}(v^0)) \) where \( 0 < t_{\max}(v^0) \leq \infty \); see, e.g., [P, Theorem 6.1.4]. The set \( \{(t, v^0) \in \mathbb{R}^+ \times \mathcal{E}^2 : 0 \leq t < t_{\max}(v^0) \} \) is open in \( \mathbb{R}^+ \times \mathcal{E}^2 \), and the map \( (t, v^0) \mapsto v(t, v^0) \) from this set to \( \mathcal{E}^2 \) is continuous; see, e.g., [SY, Theorem 46.4]. We summarize these facts as follows.

**Proposition 3.10.** Let \( \mathcal{E}_0 = H^k(\mathbb{R}^d) \) with \( k \geq \lceil d+1/2 \rceil \). For each \( \delta > 0 \), if \( 0 < \gamma < \delta \), then there exists \( T(\gamma, \delta) \) depending on \( \gamma \) and \( \delta \), with \( 0 < T(\gamma, \delta) \leq \infty \), such that the
following is true: if \( v^0 \in \mathcal{E}^2 \) satisfies
\[
||v^0||_{\mathcal{E}} \leq \gamma \tag{3.19}
\]
and \( 0 \leq t < T \), then the solution \( v(t) \in \mathcal{E}^2 \) of (3.6) is defined and satisfies
\[
||v(t)||_{\mathcal{E}} \leq \delta. \tag{3.20}
\]

Since the differential operator in (3.7) is constant coefficient, the analysis in the multidimensional case is very similar to the discussion in the one-dimensional case conducted in Subsection 2.2.3. For instance, as in that subsection, we can prove the following proposition which shows that \( v(t, v^0) \in \mathcal{E}^2 \) is exponentially decaying in the weighted norm given \( v^0 \) is small in \( \mathcal{E}^2 \).

**Proposition 3.11.** Let \( \mathcal{E}_0 = H^k(\mathbb{R}^d) \) with \( k \geq \lfloor \frac{d+1}{2} \rfloor \). Choose \( \nu > 0 \) as in Proposition 3.2. Then there exist \( \delta_1 > 0 \), and \( K_1 > 0 \) such that for every \( \delta \in (0, \delta_1) \) and every \( \gamma \) with \( 0 < \gamma < \delta \), the following is true: Let \( v^0 \in \mathcal{E}^2 \) satisfies (3.19) so that \( v(t) \) satisfies (3.20) for \( 0 \leq t < T(\delta, \gamma) \). Then
\[
||v(t)||_\alpha \leq K_1 e^{-\nu t} ||v^0||_\alpha \quad \text{for} \quad 0 \leq t < T(\delta, \gamma). \tag{3.21}
\]

**Proof.** The proof is similar to the Proposition 2.35 in the one-dimensional case, thus will be omitted. \( \blacksquare \)

On the unweighted space \( \mathcal{E}^2_0 = H^k(\mathbb{R}^d)^2 \), we rewrite system (3.6) as follows,
\[
v_{1t} = L^{(1)} v_1 + e^{-\kappa} v_2 + H_0(v),
\]
\[
v_{2t} = L^{(2)} v_2 - \kappa H_0(v),
\]
where \( H_0(v) = v_2(g(v_1 + 1/\kappa) - g(v_1)) \). Using Proposition 3.8 and Proposition 3.9, we have that \( H_0(\cdot) \) defines a mapping from \( \mathcal{E}^2_0 \) to \( \mathcal{E}_0 \) that is locally Lipschitz on any
set of the form \( \{ v : \| v \|_0 \leq K \} \) and \( \| H_0(v) \|_0 \leq C_K \| v \|_0^2 \). Therefore, similarly to Proposition 2.36 in the one-dimensional case, we obtain the following estimate.

**Proposition 3.12.** Let \( \mathcal{E}_0 = H^k(\mathbb{R}^d) \) with \( k \geq \left[ \frac{d+1}{2} \right] \). Choose \( \rho > 0 \) as in Lemma 3.5(3b), and \( \delta_1 \) be given by Proposition 3.11. Assume that \( \nu < \rho \) where \( \nu \) are chosen as in Proposition 3.2. Then there exist a \( \delta_2 \in (0, \delta_1) \) and \( C_1 > 0 \) such that for every \( \delta \in (0, \delta_2) \) and every \( \gamma \) with \( 0 < \gamma < \delta \), the following is true: If \( 0 \leq t < T(\delta, \gamma) \), and \( v^0 \in \mathcal{E}^2 \) satisfies (3.19), so that the solution \( v(t) \in \mathcal{E}^2 \) of (3.6) satisfies (3.20), then the following estimates hold:

\[
\| v_1(t) \|_0 \leq C_1 \| v^0 \|_\mathcal{E},
\]

\[
\| v_2(t) \|_0 \leq C_1 e^{-\rho t} \| v^0 \|_\mathcal{E}.
\]

Finally, by using a bootstrap argument as in the proof of Theorems 2.38 and 2.50, we may combine the estimations from Proposition 3.11 and Proposition 3.12 to obtain the main theorem of this subsection given next. The small \( \delta_0 > 0 \) in the next theorem can be chosen as \( \delta_0 = \delta_2 \) where \( \delta_2 \) is given in Proposition 3.12.

**Theorem 3.13.** Let \( \mathcal{E}_0 = H^k(\mathbb{R}^d) \) with \( k \geq \frac{d+1}{2} \) and consider the semilinear system (3.6). There exist constants \( C > 0 \) and \( \nu > 0 \) such that for each \( 0 < \delta < \delta_2 \), we can find a small \( \delta_0 > 0 \) and \( \eta \) satisfying \( 0 < \eta < \delta \) such that if \( \| v^0 \|_\mathcal{E} \leq \eta \), the following is true for the solution \( v(t) \) of (3.6) for all \( t > 0 \):

1. \( v(t) \) is defined in \( \mathcal{E}^2 \);
2. \( \| v(t) \|_\mathcal{E} \leq \delta \);
3. \( \| v(t) \|_\alpha \leq C e^{-\nu t} \| v^0 \|_\alpha \);
\begin{align}
(4) \quad \|v_1(t)\|_0 & \leq C\|v^0\|_\varepsilon; \\
(5) \quad \|v_2(t)\|_0 & \leq Ce^{-\nu t}\|v^0\|_\varepsilon.
\end{align}

**Proof.** The proof is very similar to the proof of Theorem 2.38 in the one-dimensional case, and will be omitted. ■

### 3.1.4 End states for a general system

In this subsection we consider a more general than (3.1) reaction-diffusion system with more general nonlinearity $f(\cdot)$ and coefficients matrix $D$:

\begin{equation}
\frac{u_t}{u(t,x)} = D\Delta_x u(t,x) + f(u(t,x)), \quad (3.22)
\end{equation}

where $u \in \mathbb{R}^n$, $x \in \mathbb{R}^d$, $t \geq 0$, $D = \text{diag}(d_1, ..., d_n)$ with all $d_i \geq 0$, and the function $f(u)$ is smooth and satisfies Hypothesis 3.14 below. We will study an $x$-independent equilibrium solution $u_-$ to (3.22), $f(u_-) = 0$, and its perturbation depending on the spatial variable $x \in \mathbb{R}^d$.

By replacing the spatial variable $x = (x_1, ..., x_n)$ by the moving variable $(x_1 - ct, x_2, ..., x_d)$ with the speed $c$ in (3.22), and letting $z = x_1 - ct$, re-denoting $x = (z, x_2, \cdots, x_d)$, we obtain

\begin{equation}
\frac{u_t}{u(t,x)} = D\Delta_x u(t,x) + c\partial_z u(t,x) + f(u(t,x)). \quad (3.23)
\end{equation}

Without loss of generality we shall take the equilibrium solution $u_-$ to be 0. Information about the stability of the zero solution is encoded in the spectrum of the operator obtained by linearizing (3.23) about zero,

\begin{equation}
u_t = D\Delta_x u + c\partial_z u + \partial_u f(0)u =: Lu, \quad (3.24)
\end{equation}
where \( \partial_u \) is the differential with respect to \( u \).

Let \( \mathcal{E}_0 \) be the Sobolev spaces \( H^k(\mathbb{R}^d) \), and define the weight function

\[
\gamma_\alpha(z,x_2,...,x_d) = e^{\alpha z},
\]

and the spaces \( \mathcal{E}_\alpha \) and \( \mathcal{E} = \mathcal{E}_0 \cap \mathcal{E}_\alpha \) as in Subsection 3.1.1. Analogously to the example that has been discussed in Subsection 3.1.1, we use \( \mathcal{L} \) to denote the operator defined on \( \mathcal{E}_0^n \) given by the map \( u \to Lu \), with the domain \( H^{k+2}(\mathbb{R}^d)^n \), and use \( \mathcal{L}_\alpha \) to denote the operator defined on \( \mathcal{E}_\alpha^n \) given by \( u \to Lu \), with the domain being the set of \( u \) where \( \gamma_\alpha u \in H^{k+2}(\mathbb{R}^d)^n \).

Throughout we impose the following assumptions on \( f(\cdot) \) in (3.22).

**Hypothesis 3.14.** Consider system (3.22).

(a) In appropriate variables \( u = (u_1, u_2) \), \( u_1 \in \mathbb{R}^{n_1}, u_2 \in \mathbb{R}^{n_2}, n_1 + n_2 = n \), we assume that for some constant \( n_1 \times n_1 \) matrix \( A_1 \), one has

\[
f(u_1, 0) = (A_1 u_1, 0)^T.
\]

(b) The function \( f \) is \( C^{k+3} \) from \( \mathbb{R}^n \) to \( \mathbb{R}^n \).

If Hypothesis 3.14 holds, then

\[
f(u_1, u_2) = f(u_1, 0) + f(u_1, u_2) - f(u_1, 0)
= \left( A_1 u_1 \right) + \int_0^1 \partial_{u_2} f(u_1, tu_2) dt u_2
= \left( A_1 u_1 + \tilde{f}_1(u_1, u_2) u_2 \right),
\]

where \( \tilde{f}_1 \) and \( \tilde{f}_2 \) are some matrix-valued functions of size \( n_1 \times n_2 \) and \( n_2 \times n_2 \), respectively.
We write
\[
D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \quad f(u) = \begin{pmatrix} f_1(u_1, u_2) \\ f_2(u_1, u_2) \end{pmatrix},
\]
where each $D_i$ is a nonnegative diagonal matrix of size $n_i \times n_i$, and $f_i : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_i}$ for $i = 1, 2$. Equation (3.23) now reads
\[
\partial_t u_1 = D_1 \Delta_x u_1 + c \partial_z u_1 + f_1(u_1, u_2),
\]
(3.25)
\[
\partial_t u_2 = D_2 \Delta_x u_2 + c \partial_z u_2 + f_2(u_1, u_2).
\]
(3.26)

If we linearize (3.26) at $(0, 0)$, then the constant-coefficient linear equation depends only on $u_2$ since $f(0,0) = 0$ by Hypothesis 3.14.(a) yielding:
\[
\partial_t u_2 = D_2 \Delta_x u_2 + c \partial_z u_2 + \partial_{u_2} f_2(0,0) u_2
\]
\[
= D_2 \Delta_x u_2 + c \partial_z u_2 + \partial_{u_2} f_2(0,0) u_2.
\]
(3.27)

We denote by $L^{(2)} u_2$ the right-hand side of (3.27), and let $L^{(2)}$ be the operator defined on $H^k(\mathbb{R}^d)^{n_2}$ given by $u \to L^{(2)} u$, with the domain $H^{k+2}(\mathbb{R}^d)^{n_2}$.

In addition, we linearize (3.25) at $(0, 0)$, and by Hypothesis 3.14.(a), the respective constant-coefficient linear equation reads:
\[
\partial_t u_1 = D_1 \Delta_x u_1 + c \partial_z u_1 + \partial_{u_1} f_1(0,0) u_1 + \partial_{u_2} f_2(0,0) u_2
\]
\[
= D_1 \Delta_x u_1 + c \partial_z u_1 + A_1 u_1 + \partial_{u_2} f_1(0,0) u_2.
\]
(3.28)

We denote $L^{(1)} u_1 = D_1 \Delta_x u_1 + c \partial_z u_1 + A_1 u_1$, thus $\partial_t u_1 = L^{(1)} u + \partial_{u_2} f_1(0,0) u_2$. Let $L^{(1)}$ be the operator defined on $H^k(\mathbb{R}^d)^{n_1}$ given by $u \to L^{(1)} u$, with the domain $H^{k+2}(\mathbb{R}^d)^{n_1}$.

With some additional assumptions listed below, we will show that the perturbations of the left end state $u_-$ that are initially small in both the unweighted norm
and weighted norm stay small in the unweighted norm and decay exponentially in the weighted norm. In addition, the $u_2$-component of the perturbation decays exponentially in the unweighted norm. We will now study the spectrum of $L$. First, we need the following lemma.

**Lemma 3.15.** The linear operator $L$ associated with $L$ in (3.24) have the same spectrum on $L^2(\mathbb{R}^d)^n$ and $H^k(\mathbb{R}^d)^n$, the linear operators $L^{(i)}$ associated with $L^{(i)}$ in (3.27) and (3.28) have the same spectra on $L^2(\mathbb{R}^d)^{n_i}$ and $H^k(\mathbb{R}^d)^{n_i}$ for $i = 1, 2$; similarly, the linear operator $L_\alpha$ has the same spectrum on both $L^2_\alpha(\mathbb{R})^n \otimes L^2(\mathbb{R}^{d-1})^n$ and $H^k_\alpha(\mathbb{R})^n \otimes H^k(\mathbb{R}^{d-1})^n$.

**Proof.** Because $L$ is associated with the constant-coefficient differential expression $L$, we can use the same proof as in Lemma 3.3. ■

In order to discuss the stability of the perturbation of the end state $u_-$ of (3.23), we need some additional hypotheses.

**Hypothesis 3.16.** In addition to Hypothesis 3.14, we assume that there exists a constant $\alpha > 0$ such that $\sup\{\text{Re}\lambda : \lambda \in \text{Sp}(L_\alpha)\} < 0$ on $L^2_\alpha(\mathbb{R})^n \otimes L^2(\mathbb{R}^{d-1})^n$.

As in Subsection 3.1.1, let $y = (x_2, ..., x_d) \in \mathbb{R}^{d-1}$, so that $x = (z, y) \in \mathbb{R}^d$, and denote $L_{1,\alpha} = D\partial_{x_2} + c\partial_z + \partial_u f(0)u$ and $\Delta_y = \partial^2_{x_2} + \cdots + \partial^2_{x_d}$, and next define the linear operators $L_{1,\alpha} : H^k_\alpha(\mathbb{R})^n \to H^k_\alpha(\mathbb{R})^n$, where $\text{dom}(L_{1,\alpha}) = H^{k+2}_\alpha(\mathbb{R})^n \subset H^k_\alpha(\mathbb{R})^n$, and $\Delta_y : H^k(\mathbb{R}^{d-1})^n \to H^k(\mathbb{R}^{d-1})^n$ where $\text{dom}(\Delta_y) = H^{k+2}(\mathbb{R}^{d-1})^n \subset H^k(\mathbb{R}^{d-1})$. Then the operator $L_\alpha$ on the space $H^k_\alpha(\mathbb{R})^n$ can be represented as

$$L_\alpha = L_{1,\alpha} \otimes I_{H^k(\mathbb{R}^{d-1})} + I_{H^k_\alpha(\mathbb{R})} \otimes \Delta_y.$$
Hypothesis 3.16 holds if there exist a constant $\alpha > 0$ such that $\sup \{ \text{Re}\lambda : \lambda \in \text{Sp}(L_{1,\alpha}) \} < 0$ on $H_{\alpha}^k(\mathbb{R})^n$. Indeed, by using Remark 3.4, we have

$$\text{Sp}(L_{1,\alpha} \otimes I_{H^k(\mathbb{R}^{d-1})} + I_{H^k_{\mathbb{R}^d}} \otimes \Delta_y) = \text{Sp}(L_{1,\alpha}) \cup \text{Sp}(\Delta_y).$$

Note that the spectrum of $L_{1,\alpha}$ on $L_{\alpha}^2(\mathbb{R}^d)$ and $H_{\alpha}^k(\mathbb{R}^d)$ are equal similarly to Lemma 3.15, and thus we can see that if $\sup \{ \text{Re}\lambda : \lambda \in \text{Sp}(L_{1,\alpha}) \} < 0$ on $L_{\alpha}^2(\mathbb{R})^n$, then Hypothesis 3.16 will be satisfied for any $H_{\alpha}^k(\mathbb{R}^d)$.

We need some more hypotheses for the operators $L^{(1)}$ and $L^{(2)}$ introduced in (3.27) and (3.28).

**Hypothesis 3.17.** In addition to Hypothesis 3.16, we assume the following:

1. The operator $L^{(1)}$ generates a bounded semigroup on the spaces $L_{\alpha}^2(\mathbb{R}^d)^{n_1}$ and $H_{\alpha}^k(\mathbb{R}^d)^{n_1}$.

2. The operator $L^{(2)}$ satisfies $\sup \{ \text{Re}\lambda : \lambda \in \text{Sp}(L^{(2)}) \} < 0$ on $L_{\alpha}^2(\mathbb{R}^d)^{n_2}$ and $H_{\alpha}^k(\mathbb{R}^d)^{n_2}$.

We will now rewrite equation (3.23) for the perturbation $v(t, z)$ of the end state $u_- = 0$, in the form amenable for the subsequent analysis. We seek a solution to (3.23) of the form $u(t, z) = u_- + v(t, z)$. With this notation, $v = v(t, z)$ satisfies

$$v_t = D\Delta_x v + c\partial_z v + \partial_u f(0)v + f(v) - f(0) - \partial_u f(0)v.$$  \hspace{1cm} (3.29)

Note that

$$f(v) - f(0) - \partial_u f(0)v = \left( \int_0^1 (\partial_u f(tv) - \partial_u f(0))dt \right) v.$$  \hspace{1cm} (3.30)

We define

$$N(v) = \int_0^1 (\partial_u f(tv) - \partial_u f(0))dt,$$  \hspace{1cm} (3.30)
as an $n \times n$ matrix-valued function of $v$. Note that $N(v)v \in \mathbb{R}^n$ for any $v \in \mathbb{R}^n$. Using (3.30), we rewrite (3.29) as

$$v_t = Lv + N(v)v.$$  \hspace{1cm} (3.31)

This is the semilinear equation for the perturbation that we will study.

**Proposition 3.18.** Assume that Hypothesis 3.17 holds. Then the following is true:

(1) There exists $\alpha > 0$ such that on the weighted space $\mathcal{E}_\alpha^n$, the spectrum of $L_\alpha$ will be bounded away from the imaginary axis so that $\sup\{\Re \lambda : \lambda \in \text{Sp}(L_\alpha)\} < -\nu$ for some $\nu > 0$. Also, there exists $K > 0$ such that

$$||e^{t L_\alpha}||_{\mathcal{B}(\mathcal{E}_\alpha^n)} \leq Ke^{-\nu t} \quad \text{for all} \quad t \geq 0. \hspace{1cm} (3.32)$$

(2) On the unweighted space $\mathcal{E}_0^{n_2}$, we have $\sup\{\Re \lambda : \lambda \in \text{Sp}(L^{(2)})\} < -\rho$ for some $\rho > 0$, and there exists $K > 0$ such that $||e^{t L^{(2)}}||_{\mathcal{B}(\mathcal{E}_0^{n_2} \to \mathcal{E}_0^{n_2})} \leq Ke^{-\rho t}$ for all $t \geq 0$.

**Proof.** Statement (a) holds by Hypothesis 3.16 and Lemma 3.15. Statement (b) follows from Hypothesis 2.44(2) and Lemma 3.15. \qed

Throughout the rest of this section, we will always assume $k \geq \lceil \frac{d+1}{2} \rceil$ for $\mathcal{E}_0 = H^k(\mathbb{R}^d)$ and $\mathcal{E}_\alpha = H^k_\alpha(\mathbb{R}) \otimes H^k(\mathbb{R}^{d-1})$.

**Proposition 3.19.** Assume $k \geq \frac{d+1}{2}$ and let $\mathcal{E}_0 = H^k(\mathbb{R}^d)$. Given $f \in C^{k+3}(\mathbb{R}^n; \mathbb{R}^n)$, consider the nonlinearity $N(v)$ defined in (3.30). Then we have:

(1) If $v \in \mathcal{E}^n$, then $N(v)v \in \mathcal{E}_\alpha^n$, and on any bounded neighborhood of the form $\{v : ||v||_\varepsilon \leq K\}$ there is a constant $C_K > 0$ such that $||N(v)v||_\alpha \leq C_K||v||_0||v||_\alpha$. 

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(2) If \( v \in \mathcal{E}^n \), then \( N(v)v \in \mathcal{E}^n_0 \), and on any bounded neighborhood of the form \( \{ v : ||v||_0 \leq K \} \) there is a constant \( C_K > 0 \) such that \( ||N(v)v||_0 \leq C_K ||v||_0^2 \).

(3) The formula \( v \mapsto N(v)v \) defines a mapping from \( \mathcal{E}^n \) to \( \mathcal{E}^n \) that is locally Lipschitz on any bounded neighborhood of the form \( \{ v : ||v||_\mathcal{E} \leq K \} \) in \( \mathcal{E}^n \).

Proof. We will refer to Corollary 3.7 for the components of \( N(v)v \).

Note that

\[
N(v) = \int_0^1 (\partial_u f(tv) - \partial_u f(0)) \, dt = \int_0^1 \left( \int_0^1 \partial_{u^2} f(stv) \, ds \right) tv \, dt.
\]

By Corollary 3.7 applied to the components of the vector under the integral, the mapping \( v \mapsto N(v) \) is locally Lipschitz on sets of the form \( \{ v : ||v||_0 \leq K \} \) and satisfies

\[
||N(v)||_0 \leq C_K ||v||_0,
\]

(3.33)

Thus by Lemma 3.6(1) and (3.33), we conclude that the nonlinearity \( N(v)v \) satisfies

\[
||N(v)v||_0 \leq ||N(v)||_0 ||v||_0 \leq C_K ||v||_0 ||v||_0,
\]

while by Lemma 3.6(2) and (3.33), it satisfies

\[
||N(v)v||_\alpha = ||\gamma_\alpha N(v)v||_0 \leq ||N(v)||_0 ||\gamma_\alpha v||_0 \leq C_K ||v||_0 ||v||_\alpha,
\]

thus proving (1) and (2). Next, we use the definition of \( |||v||_\mathcal{E} \) and infer

\[
||N(v)v||_\mathcal{E} = \max\{||N(v)v||_0, ||N(v)v||_\alpha\}
\leq \max\{C_K ||v||_0 ||v||_0, C_K ||v||_0 ||v||_\alpha\}
\leq C_K ||v||_\mathcal{E} ||v||_\mathcal{E},
\]

proving (3). \( \blacksquare \)
By the general theory, we now know that (3.31) has a mild solution on $\mathcal{E}^n$ for at least a short time period, see, e.g., [P, Theorem 6.1.4]. Moreover, by [SY, Theorem 64.2], we have the following result.

**Proposition 3.20.** Assume Hypothesis 3.17. For each $\delta > 0$, if $0 < \gamma < \delta$, then there exists $T(\delta, \gamma)$, such that if $\|v^0\|_\varepsilon < \gamma$, then for $t \in (0, T(\delta, \gamma))$, the solution $v(t)$ to (3.31) is defined in $\mathcal{E}^n$ and satisfies $\|v(t)\|_\varepsilon < \delta$.

We now establish the exponential decay of the solutions of (3.31) on $\mathcal{E}_\alpha^n$. The proofs are similar to the proof of one-dimensional case thus will be omitted.

**Proposition 3.21.** Assume Hypothesis 3.17. Let $\nu > 0$ satisfy (3.32). There exist $\delta_1 > 0$ and $K_1 > 0$ such that for every $\delta \in (0, \delta_1)$ and every $\gamma$ with $0 < \gamma < \delta$, the following is true: If $v^0 \in \mathcal{E}^n$ satisfies $\|v^0\|_\varepsilon < \gamma$ so that $v(t)$ satisfies $\|v(t)\|_\varepsilon < \delta$ for $0 \leq t < T(\delta, \gamma)$, then

$$\|v(t)\|_\alpha \leq K_1 e^{-\nu t} \|v^0\|_\alpha, \text{ for } 0 \leq t < T(\delta, \gamma).$$

(3.34)

We now show that $u_1$ component of (3.31) is bounded in $H^k(\mathbb{R}^d)^{n_1}$ while $u_2$-component is exponentially decaying in $H^k(\mathbb{R}^d)^{n_2}$.

**Proposition 3.22.** Assume Hypothesis 3.17. Let $\rho > 0$ be chosen as in Proposition 3.18.(b), and $\delta_1$ be given by Proposition 2.48. Then there exist a $\delta_2 \in (0, \delta_1)$ and $C_1 > 0$ such that for every $\delta \in (0, \delta_2)$ and every $\gamma$ with $0 < \gamma < \delta$, the following is true: If $0 \leq t < T(\delta, \gamma)$, and $v^0 \in \mathcal{E}^n$ satisfies $\|v^0\|_\varepsilon < \gamma$, then $v(t) \in \mathcal{E}^n$ satisfies $\|v(t)\|_\varepsilon < \delta$ and the following estimates hold:

$$\|v_1(t)\|_0 \leq C_1 \|v^0\|_\varepsilon,$$

(3.35)

$$\|v_2(t)\|_0 \leq C_1 e^{-\rho t} \|v^0\|_\varepsilon.$$

(3.36)
Given initial value $v^0 \in \mathcal{E}^n$, let $v(t) = v(t, v^0)$ be the solution of (3.31) in $\mathcal{E}^n$ with $v(0) = v^0$, which we showed exists for at least a short time period. Now by using the same bootstrap argument as in the proofs of Theorems 2.38 and 2.50, we are ready to prove our main result that establish the stability of the left end state $u_-$. The small constant $\delta_0$ in the next theorem can be chosen as $\delta_0 = \delta_2$, where $\delta_2$ is given as in Proposition 3.22.

**Theorem 3.23.** Let $k \geq \frac{d+1}{2}$, $\mathcal{E}_0 = H^k(\mathbb{R}^d)$ and $\mathcal{E}_\alpha = H^k_\alpha(\mathbb{R}) \otimes H^k(\mathbb{R}^{d-1})$. Assume Hypothesis 3.17. Then there exist constants $C > 0$, $\nu > 0$ and a small $\delta_0 > 0$ such that for each $0 < \delta < \delta_0$, we can find $\eta > 0$ such that if $||v^0||_\mathcal{E} \leq \eta$, then the following is true for all $t > 0$:

1. $v(t)$ is defined in $\mathcal{E}^n$;

2. $||v(t)||_\mathcal{E} \leq \delta$;

3. $||v(t)||_\alpha \leq C e^{-\nu t} ||v^0||_\alpha$;

4. $||v_1(t)||_0 \leq C ||v^0||_\mathcal{E}$;

5. $||v_2(t)||_0 \leq C e^{-\nu t} ||v^0||_\mathcal{E}$.

We omit the proof as it is identical to the proof of Theorems 2.38 and 2.50.

### 3.2 Multidimensional stability of a planar front

In this section we study the planar front solutions of a general reaction diffusion system (3.37) in a weighted multi-dimensional space, and prove that the small perturbation of the front decays algebraically to zero as $t \to \infty$. 

In Subsection 3.2.1 we list the assumptions that we impose on system (3.37). We study the spectrum of the operator obtained by linearizing the system at the planar front in Subsection 3.2.2. In Subsection 3.2.3 we obtain a system for the perturbations of the planar front to be studied. We prove the stability of the system in Subsection 3.2.4.

3.2.1 Assumptions

In this subsection we introduce the multidimensional reaction diffusion system, and impose some hypotheses on the system. We will consider a general reaction-diffusion system

$$u_t(t, x) = D \Delta_x u(t, x) + f(u(t, x)), \quad (3.37)$$

where $u \in \mathbb{R}^n$, $x \in \mathbb{R}^d$, $t \geq 0$, $D = (d_1, \ldots, d_n)$ with $d_i > 0$ for $i = 1, \ldots, n$, and the function $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is smooth.

Given the vector $e = (1, 0, \ldots, 0) \in \mathbb{S}^d$, we will make a change of variable $z = e \cdot x - ct$ for some velocity $c > 0$. Redenoting again $x = (z, x_2, \ldots, x_d)$, we arrive to the equation

$$u_t = D \Delta_z u + cu_z + f(u), \quad (3.38)$$

where $\Delta = \partial_z^2 + \partial_{x_2}^2 + \cdots + \partial_{x_n}^2$. A traveling wave solution $\phi = \phi(z)$ for system (3.37) is a smooth function of $z \in \mathbb{R}$ that is a steady state (time independent) solution of (3.38). The solution $\phi$ satisfies

$$0 = D \partial_z^2 \phi + c \partial_z \phi + f(\phi), \quad (3.39)$$

since $\phi$ depends only on $z$. We further assume that there exist time-independent solutions $\phi_\pm \in \mathbb{R}^n$ of (3.37) so that $f(\phi_\pm) = 0$ and there exist constants $K > 0$ and
$\omega_- < 0 < \omega_+$ such that

$$||\phi(z) - \phi_-||_{\mathbb{R}^n} \leq Ke^{-\omega_+ z} \text{ for } z \geq 0 \quad \text{and} \quad ||\phi(z) - \phi_+||_{\mathbb{R}^n} \leq Ke^{-\omega_- z} \text{ for } z \leq 0. \quad (3.40)$$

If the latter assumption holds then the traveling wave is called a planar front. Without loss of generality we may assume that one of the end state, $\phi_-$ or $\phi_+$, is zero. In what follows, we assume that $\phi_- = 0$ and study the stability of the planar front $\phi$ of (3.37).

Linearizing (3.38) about $\phi$, we now define the linear differential variable coefficients expression $L$ by

$$L = D\Delta_x + c\partial_z + df(\phi), \quad (3.41)$$

where $df(\phi)$ is the differential of the function $f$ evaluated at the planar front solution $\phi = \phi(z)$.

To study the stability of the planar front $\phi$, we need to show the spectral information for the linear operator $\mathcal{L}$ associated with the differential expression (3.41) on the desired Sobolev space $H^k(\mathbb{R}^d)^n$, which is suited to the study of nonlinear stability because it is closed under multiplication. We denote $\mathcal{E}_0 = H^k(\mathbb{R}^d)$, the Soblev space, and we set $H^0(\mathbb{R}^d) = L^2(\mathbb{R}^d)$, denoting the norm in $\mathcal{E}_0$ by $\| \cdot \|_0$.

We use the tensor product notation and write $\mathcal{E}_0$ as $H^k(\mathbb{R}) \otimes H^k(\mathbb{R}^{d-1})$, for any $u \in H^k(\mathbb{R})$ and $v \in H^k(\mathbb{R}^{d-1})$, denote $u \otimes v = u(z)v(x_2, \ldots, x_d) \in \mathcal{E}_0$ for $k = 0, 1, 2, \ldots$. For the remainder of this section we will decompose any $x \in \mathbb{R}^d$ as $x = (z, y) \in \mathbb{R} \otimes \mathbb{R}^{d-1}$, where $z = x_1 - ct$ and $y = (x_2, \ldots, x_d)$.

Thus we can use a decomposition of $\mathcal{L}$ on $\mathcal{E}_0$ as follows,

$$\mathcal{L} = \mathcal{L}_1 \otimes I_{H^k(\mathbb{R}^{d-1})^n} + I_{H^k(\mathbb{R})^n} \otimes \Delta_y,$$

where $\mathcal{L}_1$ is associated with the one-dimensional differential variable coefficients ex-
pression

\[ L_1 = D\partial^2_z + c\partial_z + df(\phi), \]  \hspace{1cm} (3.42)

that depends only on \( z \), and

\[ \Delta_y = D(\partial^2_{x_2} + \cdots + \partial^2_{x_d}). \]  \hspace{1cm} (3.43)

We will be concerned with the problem that on \( \mathcal{E}^n_0 \) the essential spectrum of the linear operator associated with \( L_1 \) as in (3.42) may touch the imaginary axis. To fix this, one introduces a class of weight functions of exponential type in order to stabilize the spectrum. Let \( \alpha = (\alpha_-, \alpha_+) \in \mathbb{R}^2 \). We call \( \gamma_\alpha : \mathbb{R} \to \mathbb{R} \) a weight function of class \( \alpha \) if \( 0 < \gamma_\alpha(z) \) for all \( z \in \mathbb{R} \), the function \( \gamma_\alpha \in C^{k+3}(\mathbb{R}) \), and

\[ \gamma_\alpha(z) = \begin{cases} e^{\alpha_- z}, & \text{for large negative } z, \\ e^{\alpha_+ z}, & \text{for large positive } z. \end{cases} \]  \hspace{1cm} (3.44)

Following the setting in [GLS], we will always assume that

\[ 0 < \alpha_- < -\omega_- \quad \text{and} \quad 0 \leq \alpha_+ < \omega_+, \]

where \( \omega_- \) and \( \omega_+ \) were introduced in (3.40) when we defined the planar front. The condition \( \alpha_+ < \omega_+ \) ensures that \( \phi' \) is in the weighted space, and the conditions \( \alpha_- < \omega_- \) and \( 0 \leq \alpha_+ \) ensure that \( \gamma_\alpha^{-1}(z)\phi(z) \) is bounded, which will be used to prove the stability of the front.

For a fixed weight function \( \gamma_\alpha \), let \( \mathcal{E}_\alpha = \{ u : \gamma_\alpha \otimes I_{H^k(\mathbb{R}^{d-1})}u \in \mathcal{E}_0 \} \), with the norm \( \|u\|_\alpha = \|\gamma_\alpha u\|_0 \). Note that by the definition of \( \mathcal{E}_\alpha \), we can represent the weighted space \( \mathcal{E}_\alpha \) by \( H^k_\alpha(\mathbb{R}) \otimes H^k(\mathbb{R}^{d-1}) \). Here, for a function \( u = u(z, y) \) we denote by \( \gamma_\alpha \otimes I_{H^k(\mathbb{R}^{d-1})}u \) is the function of \( (z, y) \) defined by

\[ (\gamma_\alpha \otimes I_{H^k(\mathbb{R}^{d-1})}u)(z, y) = \gamma_\alpha(z)u(z, y), \quad (z, y) \in \mathbb{R}^d \]
Remark 3.24. As in Remark 3.1, for \( k = 0, 1, 2, \ldots \), we use notation \( \mathcal{L} : \mathcal{E}_0^n \to \mathcal{E}_0^n \) to denote the linear operator given by the formula \( u \mapsto Lu \), where \( \text{dom} \mathcal{L} = H^{k+2}(\mathbb{R}^d)_n \subset \mathcal{E}_0^n \); similarly, the linear operator \( \mathcal{L}_1 : H^k(\mathbb{R})^n \to H^k(\mathbb{R})^n \) is defined by the formula \( u \mapsto L_1u \), where \( \text{dom} \mathcal{L}_1 = H^{k+2}(\mathbb{R})^n \subset H^k(\mathbb{R})^n \); the linear operator \( \Delta_y : H^k(\mathbb{R}^{d-1})^n \to H^k(\mathbb{R}^{d-1})^n \) is defined by (3.43), and its domain is \( H^{k+2}(\mathbb{R}^{d-1})^n \); we use notation \( \mathcal{L}_\alpha : \mathcal{E}_\alpha^n \to \mathcal{E}_\alpha^n \) to denote the operator given by the formula \( u \mapsto Lu \), and \( \text{dom} \mathcal{L}_\alpha = H^{k+2}(\mathbb{R}_\alpha)^n \subset H^k(\mathbb{R}_\alpha)^n \); and use notation \( \mathcal{L}_{1,\alpha} : H_{1,\alpha}^k(\mathbb{R})^n \to H_{1,\alpha}^k(\mathbb{R})^n \) to denote the operator given by \( u \mapsto L_1u \), and \( \text{dom} \mathcal{L}_{1,\alpha} = H_{1,\alpha}^{k+2}(\mathbb{R})^n \subset H_{1,\alpha}^k(\mathbb{R})^n \).

We continue to let \( \mathcal{B}(X,Y) \) denote the space of bounded operators from \( X \) to \( Y \), and \( \mathcal{B}(X) \) denote the space of bounded operators from \( X \) to \( X \).

We consider system (3.37) and assume that the following hypotheses are true for (3.37) in this subsection. Recall that \( \mathcal{E}_0 = H^k(\mathbb{R}^d) \).

Hypothesis 3.25. The function \( f \) is in \( C^{k+3}(\mathbb{R}^n; \mathbb{R}^n) \).

Hypothesis 3.26. The system (3.37) has a planar front \( \phi(z) \), \( z = x_1 - ct \), for which there exist numbers \( K > 0 \) and \( \omega_- < 0 < \omega_+ \) such that

\[
||\phi(z)|| \leq Ke^{-\omega_- z} \text{ for } z \leq 0 \quad \text{and} \quad ||\phi(z) - \phi_+|| \leq Ke^{-\omega_+ z} \text{ for } z \geq 0.
\]

Hypotheses 3.25 and 3.26 imply that \( f(\phi_\pm) = 0 \) and the following lemma.

Lemma 3.27. There exists \( K > 0 \) such that the following is true: \( ||\phi^{(m)}(z)|| \leq Ke^{-\omega_- z} \text{ for } z \leq 0 \), and \( ||\phi^{(m)}(z)|| \leq Ke^{-\omega_+ z} \text{ for } z \geq 0 \), here \( m = 1, \ldots, k+1 \).

Remark 3.28. Note that the \( z \)-derivative \( \phi' \) of the planar front \( \phi \) clearly satisfies the equation \( L\phi' = 0 \), this follows by taking \( z \)-derivative of the equation (3.38). ◇
Hypothesis 3.29. In addition to Hypothesis 3.26, we assume that there exists $\alpha = (\alpha_-, \alpha_+) \in \mathbb{R}^2$ such that the following assertions are true:

1. $0 < \alpha_- < -\omega_-$.

2. $0 \leq \alpha_+ < \omega_+.$

3. For the linear operator $\mathcal{L}_{1, \alpha} : H^k_{\alpha}(\mathbb{R})^n \to H^k_{\alpha}(\mathbb{R})^n$, there exists $\nu > 0$ such that

$$\sup\{\text{Re} \lambda : \lambda \in \text{Sp}_{\text{ess}}(\mathcal{L}_{1, \alpha})\} < -\nu,$$

and the only element of $\text{Sp}(\mathcal{L}_{1, \alpha})$ in $\{\lambda : \text{Re} \lambda \geq 0\}$ is a simple eigenvalue 0.

Hypothesis 3.26, Lemma 3.27 and Hypothesis 3.29 imply the following lemma.

Lemma 3.30. Assume Hypotheses 3.26 and 3.29. Then

1. $\gamma^{-1}_\alpha \phi$ is a $C^k(\mathbb{R})^n$ function on $\mathbb{R}$, exponentially approaching zero as $z \to \infty$ or $z \to -\infty$ together with the derivatives $\gamma^{-1}_\alpha(\phi)^{(m)}$ exponentially approach zero as $z \to +\infty$ or $z \to -\infty$ for $m = 0, 1, 2, \ldots, k$.

2. $\gamma_\alpha \phi$ is a $k$ times continuous differentiable function on $\mathbb{R}$, exponentially approaching infinity as $z \to \infty$ and zero as $z \to -\infty$, while $\gamma_\alpha \phi^{(m)}$ exponentially approach zero as $z \to +\infty$ or $z \to -\infty$ for $m = 1, 2, \ldots, k$.

We will also assume that the nonlinearity in system (3.37) has certain “product” structure, see (3.47). To describe this structure, let $u = (u_1, u_2)$, $u_1 \in \mathbb{R}^{n_1}$, $u_2 \in \mathbb{R}^{n_2}$, and $n_1 + n_2 = n$, and write

$$f(u) = \begin{pmatrix} f_1(u_1, u_2) \\ f_2(u_1, u_2) \end{pmatrix}, f_i : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_i}, i = 1, 2, \quad D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}.$$
Equation (3.37) now reads
\[
\partial_t u_1 = D_1 \Delta_x u_1 + f_1(u_1, u_2),
\]
\[
\partial_t u_2 = D_2 \Delta_x u_2 + f_2(u_1, u_2).
\]

(3.45)

Equation (3.38) now reads
\[
\partial_t u_1 = D_1(\partial_{zz} + \Delta_y)u_1 + c\partial_z u_1 + f_1(u_1, u_2),
\]
\[
\partial_t u_2 = D_2(\partial_{zz} + \Delta_y)u_2 + c\partial_z u_2 + f_2(u_1, u_2).
\]

Without loss of generality, we shall take \( u_- \) to be 0, and write
\[
\phi(z) = (\phi_1(z), \phi_2(z)), \quad u_+ = (u_{1,+}, u_{2,+}).
\]

Let \( L_1^- \) and \( L_1^+ \) denote the constant-coefficient linear expressions obtained by linearizing the right-hand side of (3.38) at 0 and \( u_+ \), respectively, thus
\[
L_1^- = D\partial_{zz} + c\partial_z + df(0), \quad L_1^+ = D\partial_{zz} + c\partial_z + df(u_+).
\]

(3.46)

We will assume the following.

**Hypothesis 3.31.** There is a \( n_1 \times n_1 \) constant matrix \( A_1 \) such that \( f(u_1, 0) = (A_1 u_1, 0) \).

With this hypothesis then
\[
f(u_1, u_2) = \left( \begin{array}{c}
A_1 u_1 + \tilde{f}_1(u_1, u_2) u_2 \\
\tilde{f}_2(u_1, u_2) u_2
\end{array} \right),
\]

(3.47)

where \( \tilde{f}_1 \) and \( \tilde{f}_2 \) are matrix-valued functions of size \( n_1 \times n_2 \) and \( n_2 \times n_2 \), respectively.

Let
\[
L_1^{(1)} = D_1 \partial_{zz} + c\partial_z + d_{u_1} f_1(0, 0) = D_1 \partial_{zz} + c\partial_z + A_1,
\]

(3.48)
where \( d_i f \) is the derivative of \( f \) with respect to \( u_i \), \( i = 1, 2 \).

For \( i = 1, 2 \), obviously, \( L_1^{(i)} \) is a constant coefficient linear differential expression on \( \mathbb{R}^{n_i} \). By Hypothesis 3.31, using (A.1), we infer

\[
L_1^{-} = \begin{pmatrix} L_1^{(1)} & d_{u_2} f_1(0, 0) \\ 0 & L_1^{(2)} \end{pmatrix}.
\]

(3.50)

We note that from (3.42) and (3.46) one can conclude that for the expression (3.42), we have

\[
L_1 = L_1^{-} + (df(\phi) - df(0)),
\]

and then from (3.50), we finally have

\[
L_1 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} L_1^{(1)} & d_{u_2} f_1(0, 0) \\ 0 & L_1^{(2)} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + (df(\phi) - df(0)) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.
\]

(3.51)

Define now the operators \( \mathcal{L}_1^{(1)} \) and \( \mathcal{L}_1^{(2)} \) as prescribed in Remark 3.1. We will impose the next hypothesis which, in particular, will imply stability in the unweighted norm of the end state \((0, 0)\), that is, at one end of the planar front.

**Hypothesis 3.32.**

1. The operator \( \mathcal{L}_1^{(1)} \) on \( H^k(\mathbb{R})^{n_1} \) induced by (3.48) generates a bounded semigroup, that is, \( \| e^{t \mathcal{L}_1^{(1)}} \|_{\mathcal{B}(H^k(\mathbb{R}))} \leq K \) for some \( K > 0 \) and all \( t \geq 0 \);

2. The operator \( \mathcal{L}_1^{(2)} \) on \( H^k(\mathbb{R})^{n_2} \) induced by (3.49) satisfies

\[
\sup \{ \text{Re} \lambda : \lambda \in \text{Sp}(\mathcal{L}_1^{(2)}) \} < 0,
\]

so that there exist numbers \( \rho > 0 \) and \( K > 0 \), for which the inequality

\[
\| e^{t \mathcal{L}_1^{(2)}} \|_{\mathcal{B}(H^k(\mathbb{R}))} \leq Ke^{-\rho t}
\]

holds for all \( t \geq 0 \).
Remark 3.33. We note that assertions (1) and (2) in Hypothesis 3.32 imply that on the unweighted space $H^k(\mathbb{R}^d)$, the following assertions are true:

(a) $\sup\{\text{Re}\lambda : \lambda \in \text{Sp}(\mathcal{L}^{(1)}_1)\} \leq 0$;

(b) $\sup\{\text{Re}\lambda : \lambda \in \text{Sp}(\mathcal{L}^-_1)\} \leq 0$.

\[\Box\]

### 3.2.2 Spectrum and projection operators

In this section we will discuss the projection corresponding to the isolated eigenvalue zero of the one-dimensional operator $\mathcal{L}_{1,\alpha}$ associated with (3.42) on the weighted space $H^k_\alpha(\mathbb{R})$ and show how this projection works for the multidimensional operator $\mathcal{L}_\alpha$ in $H^k_\alpha(\mathbb{R}^d)$. The fact that the spectrum of the linearization can touch the imaginary axis is one of the main difficulties in the stability analysis of system (3.37) in multidimensional space. In the one-dimensional case [GLS], the authors have imposed the hypotheses under which the wave is spectrally stable on $H^1_\alpha(\mathbb{R})^n$, i.e., the linear operator associated with the one-dimensional differential expression $L_1 = D\partial_x^2 + c\partial_x + df(\phi)$ has only one simple eigenvalue at 0, with all others points in the spectrum being located to the left of the imaginary axis on the weighted space $H^1_\alpha(\mathbb{R})^n$. One can refer to the discussion in [GLS, Section 3.1]. Indeed, let $L^-_1$ and $L^+_1$ be defined as in (3.50). Then the spectrum of $\mathcal{L}^-_{1,\alpha}$ on $H^k_\alpha(\mathbb{R})^n$, respectively, $\mathcal{L}^+_{1,\alpha}$, is equal to the set of $\lambda \in \mathbb{C}$ for which there exists $\theta \in \mathbb{R}$ such that

$$\det \left( -\theta^2 D + i\theta (c - 2\alpha_+ - \lambda) I + \alpha^2 I - c\alpha_- I + df(0) \right) = 0,$$

respectively, such that

$$\det \left( -\theta^2 D + i\theta (c - 2\alpha_- - \lambda) I + \alpha^2 I - c\alpha_+ I + df(u_+) \right) = 0.$$
By using [GLS, Lemma 3.5], the right-hand boundary of $\text{Sp}_{\text{ess}}(\mathcal{L}_{1,\alpha})$ is exactly the right-hand boundary of the set $\text{Sp}(\mathcal{L}_{1,\alpha}^-) \cup \text{Sp}(\mathcal{L}_{1,\alpha}^+)$. For example, in the discussion of stability at the steady state solution, we have showed for the model system (3.1) that the spectrum of $\mathcal{L}_{1,\alpha}^-$ associated with the differential expression

$$L_{1,\alpha}^- = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \partial_{zz} + c \partial_z + \begin{pmatrix} 0 & e^{-\kappa} \\ 0 & -\kappa e^{-\kappa} \end{pmatrix}$$

touches the imaginary axis when the operator is considered in the unweighted space $H^k(\mathbb{R}^d)^2$, and the spectrum of operator $\mathcal{L}_{1,\alpha}^-$ on $H^k_{\alpha}(\mathbb{R}) \otimes H^k(\mathbb{R}^{d-1})$ is the union of two curves $-\theta^2 + i\theta(c - 2\alpha) + (\alpha^2 - c\alpha)$ and $-\epsilon\theta^2 + i\theta(c - 2\epsilon\alpha) + (\epsilon\alpha^2 - c\alpha) - \kappa e^{-\kappa}$ which are located in the left half plane, and the two curves are also right-hand boundary of $\text{Sp}_{\text{ess}}(\mathcal{L}_{1,\alpha})$, see Figure 3.1.

![Figure 3.1: The spectrum of $\mathcal{L}_{1,\alpha}$](image)

For the multidimensional operator $\mathcal{L}_{\alpha}$, the situation is radically different, see Figure 3.2. One concern about the spectrum of $\mathcal{L}_{\alpha}$ is the influence of the point spectrum at zero of the operator $\mathcal{L}_{1,\alpha}$ in the multidimensional case as opposite to the one-dimensional case. To see this influence, we recall the definition of approximate point spectrum, see [EN, Lemma IV.1.9]. Given an operator $\mathcal{A} : D(\mathcal{A}) \subseteq X \to X$, we call
the set
\[ \text{Sp}_{\text{ap}}(\mathcal{A}) := \{ \lambda \in \mathbb{C} : \lambda - \mathcal{A} \text{ is not injective or range of } \lambda - \mathcal{A} \text{ is not closed in } X \} \]
the approximate point spectrum of $\mathcal{A}$. Equivalently, this is the set of approximate eigenvalues $\lambda$ for which there is a sequence of unit vectors $a_1, a_2, \ldots$ for which
\[ \lim_{N \to \infty} \| \mathcal{A}a_N - \lambda a_N \| = 0. \]

We will use the following proposition to show what kind of influence in Figure 3.1 is caused by the isolated point spectrum $\{0\}$ of the operator $\mathcal{L}_{1,\alpha}$ when we consider the system on the weighted space $\mathcal{E}_n^\alpha = H_k^k(\mathbb{R})^n \otimes H_k^k(\mathbb{R}^{d-1})^n$.

**Proposition 3.34.** Assume Hypothesis 3.29. Given the linear operator $\mathcal{L}_\alpha$ associated with $L$ introduced in (3.41), the isolated eigenvalue 0 of the operator $\mathcal{L}_{1,\alpha}$ will be connected to the essential spectrum of $\mathcal{L}_\alpha$ for $d > 1$ by a semiline contained in $\text{Sp}_{\text{ess}}(\mathcal{L}_\alpha)$.

Thus, the essential spectrum of $\mathcal{L}_\alpha$ is no longer bounded away from the imaginary axis on the weighted space $\mathcal{E}_n^\alpha$ as it was for $d = 1$. Moreover, each point $\eta \in \text{Sp}_{\text{ap}}(\mathcal{L}_{1,\alpha})$ will generate a semiline contained in the spectrum $\text{Sp}(\mathcal{L}_{1,\alpha} \otimes I + I \otimes \Delta_y)$.

**Proof.** We first offer an elementary proof that does not use the formula for the spectra of tensor product of operators, cf. (3.52) below. Assume that $\sup \{ \text{Re} \lambda : \lambda \in \text{Sp}_{\text{ess}}(\mathcal{L}_{1,\alpha}) \} < 0$ and $\{0\}$ is a simple eigenvalue of $\mathcal{L}_{1,\alpha}$ by Hypothesis 3.29. We will use the decomposition $\mathcal{L}_\alpha = \mathcal{L}_{1,\alpha} \otimes I_{H^k(\mathbb{R}^{d-1})^n} + I_{H_k^k(\mathbb{R})^n} \otimes \Delta_y$.

First, we recall that the essential spectrum of $\Delta_y$ on $H^k(\mathbb{R}^{d-1})^n$ or $L^2(\mathbb{R}^{d-1})$ satisfies
\[ \text{Sp}_{\text{ess}}(\Delta_y) = \bigcup_{(\xi_2, \ldots, \xi_d) \in \mathbb{R}^{d-1}} - (\xi_2^2 + \cdots + \xi_d^2) = (-\infty, 0]; \]
therefore each point in the spectrum of $\Delta_y$ belongs to the boundary of the spectrum.

By [EN, Proposition IV.1.10], the boundary of the spectrum $\text{Sp}(\Delta_y)$ is contained in the approximate point spectrum $Sp_{ap}(\Delta_y)$, thus every point on the non-positive axis is contained in $Sp_{ap}(\Delta_y)$.

Using [EN, Lemma IV.1.9], for any $\lambda \in Sp_{ap}(\Delta_y)$ with the approximate eigenvectors sequence $\{a_N\}$, we can form a sequence $b_N = l \otimes a_N = l(z)a_N(x_2, \ldots, x_d)$, where the unit $H^k$-norm functions $a_N \in D(\Delta_y)$, which is $H^{k+2}(\mathbb{R}^{d-1})$, satisfy

$$\lim_{N \to \infty} \| \Delta_y a_N - \lambda a_N \|_{H^k(\mathbb{R}^{d-1})} = 0,$$

and $l \in D(\mathcal{L}_{1,\alpha})$, which is in $H^{k+2}_\alpha(\mathbb{R})^n$, and $l$ is a $H^k_\alpha(\mathbb{R})$-unit eigenvector corresponding to the isolated eigenvalue 0 of the operator $\mathcal{L}_{1,\alpha}$, i.e., $\mathcal{L}_{1,\alpha}l = 0$. Then $\{b_N\}$ is a sequence of unit vectors in $H^k_\alpha(\mathbb{R})^n \otimes H^k(\mathbb{R}^{d-1})^n$, and a calculation shows that

$$\lim_{N \to \infty} \| \mathcal{L}_\alpha b_N - \lambda b_N \|_\alpha = \lim_{N \to \infty} \| (\mathcal{L}_{1,\alpha} \otimes I_{H^k(\mathbb{R}^{d-1})^n} + I_{H^k_\alpha(\mathbb{R})^n} \otimes \Delta_y) b_N - \lambda b_N \|_\alpha$$

$$= \lim_{N \to \infty} \| \mathcal{L}_{1,\alpha} l \otimes a_N + l \otimes \Delta_y a_N - \lambda l \otimes a_N \|_\alpha$$

$$= \lim_{N \to \infty} \| l \otimes (\Delta_y a_N - \lambda a_N) \|_\alpha$$

$$= \lim_{N \to \infty} \| \Delta_y a_N - \lambda a_N \|_{H^k(\mathbb{R}^{d-1})}$$

$$= 0.$$

Thus applying [EN, Lemma IV.1.9] again, $\lambda$ is also an approximate eigenvalue of $\mathcal{L}_\alpha$ with a sequence of approximate eigenvectors $\{b_N\}$.

To prove the last assertion, we pick $l_N \in D(\mathcal{L}_{1,\alpha})$ so that $\|(\mathcal{L}_{1,\alpha} - \eta I) l_N \|_\alpha \to 0$ as $N \to \infty$, $\lambda \in (-\infty, 0]$ and $a_N$ as before. Then, letting $b_N = l_N a_N$, one has

$$\| \mathcal{L}_\alpha b_N - (\eta + \lambda) b_N \|_\alpha = \| (\mathcal{L}_{1,\alpha} \otimes I + I \otimes \Delta_y)(l_N \otimes a_N) - (\eta + \lambda)(l_N \otimes a_N) \|_\alpha$$
\[
= \left\| (\mathcal{L}_{1,\alpha} l_N \otimes a_N) - (\eta l_N \otimes a_N) + (l_N \otimes \Delta_y a_N) - (l_N \otimes \lambda a_N) \right\|_{\alpha}
\leq \left\| (\mathcal{L}_{1,\alpha} - \eta l_N) \otimes a_N \right\|_{\alpha} + \left\| l_N \otimes (\Delta_y a_N - \lambda a_N) \right\|_{\alpha}
\]
which converges to zero as \( N \to \infty \). Thus, \( \eta + \lambda \in \text{Sp}_{ap}(\mathcal{L}_{\alpha}) \).

![Figure 3.2: Approximate eigenvalues of \( \mathcal{L}_{1,\alpha} \) will generate semilines contained in \( \text{Sp}_{\text{ess}}(\mathcal{L}_{1,\alpha}) \)](image)

Alternatively, we can also refer to Remark 3.4, see (3.13) and [RS4], to obtain:

\[
\text{Sp}(\mathcal{L}_{1,\alpha} \otimes I_{H^k(\mathbb{R}^{d-1})^n} + I_{H^k(\mathbb{R})^n} \otimes \Delta_y) = \text{Sp}(\mathcal{L}_{\alpha}) + \text{Sp}(\Delta_y). \tag{3.52}
\]

This gives an alternative proof of the properties listed in the proposition. In particular, we can conclude that every point on the boundary of spectrum of \( \mathcal{L}_{1,\alpha} \) will generate a semi-line to \(-\infty\), so that the essential spectrum of \( \mathcal{L}_{\alpha} \) on \( E^n_{\alpha} \) will now touch the imaginary axis by the semiline generated by this eigenvalue 0, in contrast to the 1-dimensional case, where zero eigenvalue is an isolated point in \( \text{Sp}(\mathcal{L}_{1,\alpha}) \).

We will impose the following standing assumption that will be used from this point on to the end of this section.

**Hypothesis 3.35.** In addition to Hypotheses 3.25, 3.26, 3.29, 3.31 and 3.32, we assume that the diffusion matrix in (3.37) is identity, \( D = I_{n \times n} \).

Since 0 is an isolated point in the spectrum of \( \mathcal{L}_{1,\alpha} \) by Hypothesis 3.29, we can define the Riesz spectral projection \( P_\alpha \) of \( \mathcal{L}_{1,\alpha} \) on \( H^k_{\alpha}(\mathbb{R})^n \) onto the 1-dimensional
space $N(\mathcal{L}_{1,\alpha})$. The projection $P_\alpha$ commutes with $e^{t\mathcal{L}_{1,\alpha}}$ for all $t \geq 0$. Since the operator $\mathcal{L}_{1,\alpha}$ is Fredholm of index zero and 0 is a simple eigenvalue of $\mathcal{L}_{1,\alpha}$, [DL, Lemma 2.13] yields that

$$H^k_\alpha(\mathbb{R})^n = \text{ran} \mathcal{L}_{1,\alpha} \oplus \ker \mathcal{L}_{1,\alpha}; \quad \ker P_\alpha = \text{ran} \mathcal{L}_{1,\alpha}.$$  

Hypothesis 3.29 yields that $\text{ran} P_\alpha = \ker \mathcal{L}_{1,\alpha}$ is spanned by $\phi'$. As in Remark 2.8 it can be shown that there exists a unique $H^k$-smooth function $\tilde{e} : \mathbb{R} \to \mathbb{R}^n$ such that the function $\gamma^{-1}_\alpha(\cdot)\tilde{e}(\cdot)$ is exponentially decaying, $\tilde{e}$ solves the adjoint equation $L^*_{1,\alpha}\tilde{e} = 0$ and satisfies $\int_{\mathbb{R}}(\tilde{e}(s), \phi'(s))_{\mathbb{R}^n} ds = 1$. Then for $V \in H^k_\alpha(\mathbb{R})^n$ the operator $P_\alpha$ can be written as follows,

$$(P_\alpha V)(z) = \left( \int_{\mathbb{R}} (\tilde{e}(s), V(s))_{\mathbb{R}^n} ds \right) \phi'(z), \quad z \in \mathbb{R},$$

cf. (2.17) with $Z_q$ replaced by $\gamma^{-1}_\alpha \tilde{e}$. Let $Q_\alpha = I - P_\alpha$ be the projection in $H^k_\alpha(\mathbb{R})^n$ onto $\text{ran} \mathcal{L}_{1,\alpha}$ with kernel $N(\mathcal{L}_\alpha)$. The operator $Q_\alpha$ also commutes with $e^{t\mathcal{L}_{1,\alpha}}$ for all $t \geq 0$. Using Hypothesis 3.29 we have the following result, see also [GLS, Lemma 3.13].

**Lemma 3.36.** Assume Hypothesis 3.35. We choose $\nu > 0$ such that

$$\sup\{\text{Re}\lambda : \lambda \in \text{Sp}_{\text{ess}}(\mathcal{L}_{1,\alpha})\} < -\nu.$$  

There exists $K > 0$ such that for $t \geq 0$,

$$\|e^{t\mathcal{L}_{1,\alpha}}Q_\alpha\|_{\mathcal{B}(H^k_\alpha(\mathbb{R}))} \leq Ke^{-\nu t}.$$

Now, for convenience, for $U \in H^k_\alpha(\mathbb{R})^n \otimes H^k(\mathbb{R}^{d-1})^n$ we denote

$$(\pi_\alpha U)(y) = \int_{\mathbb{R}} (\tilde{e}(s), U(s, y))_{\mathbb{R}^n} ds, \quad y \in \mathbb{R}^{d-1}. \quad (3.53)$$

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In what follows we will need an operator on $\mathcal{E}_\alpha^n = H_k^1(\mathbb{R})^n \otimes H_k^1(\mathbb{R}^{d-1})$ defined by

$$\mathcal{P}U = (P_\alpha \otimes I_{H_k^1(\mathbb{R}^{d-1})}) U,$$

so that

$$(\mathcal{P}U)(z, y) = \left( \int_{\mathbb{R}} (\tilde{e}(s), U(s, y))_{\mathbb{R}^n} ds \right) \phi'(z)$$

$$= (\pi_\alpha U)(y)\phi'(z), \quad (z, y) \in \mathbb{R}^d.$$ (3.55)

We now show that $\pi_\alpha$ and $\mathcal{P}$ have the following properties.

**Lemma 3.37.** Assume Hypothesis 3.35. The linear operators $\mathcal{P}$ and $\pi_\alpha$ satisfy: $\mathcal{P} \in \mathcal{B}(\mathcal{E}_0 \cap \mathcal{E}_\alpha)$ and $\pi_\alpha \in \mathcal{B}(\mathcal{E}_0 \cap \mathcal{E}_\alpha, H_k^1(\mathbb{R}^{d-1}))$ for $k \geq \left[ \frac{d+1}{2} \right]$ and $\alpha = (\alpha_-, \alpha_+) \in \mathbb{R}_+^2$ from Hypothesis 3.29. Moreover, $\pi_\alpha \in \mathcal{B}(L_1^1(\mathbb{R}) \otimes L^1(\mathbb{R}^{d-1}), L^1(\mathbb{R}^{d-1}))$.

**Proof.** Since $\|\gamma_\alpha^{-1}(z)\tilde{e}(z)\|_{\mathbb{R}^n} \to 0$ exponentially fast as $|z| \to \infty$, there exist $\zeta_- < 0 < \zeta_+$ and $K > 0$ such that $\|\gamma_\alpha^{-1}(z)\tilde{e}(z)\|_{\mathbb{R}^n} \leq Ke^{-\zeta_- z}$ for $z \leq 0$, and $\|\gamma_\alpha^{-1}(z)\tilde{e}(z)\|_{\mathbb{R}^n} \leq Ke^{-\zeta_+ z}$ for $z \geq 0$. We pick $U \in \mathcal{E}_0^n \cap \mathcal{E}_\alpha^n$, and first consider the $L^2$-norm, so that

$$\|\pi_\alpha U\|^2_{L^2(\mathbb{R}^{d-1})} = \int_{\mathbb{R}^{d-1}} |(\pi_\alpha U)(y)|^2 dy$$

$$= \int_{\mathbb{R}^{d-1}} \left| \int_{\mathbb{R}} (\gamma_\alpha^{-1}(s)\tilde{e}(s), \gamma_\alpha(s)U(s, y))_{\mathbb{R}^n} \right|^2 dy$$

$$\leq \int_{\mathbb{R}^{d-1}} \left( \int_{\mathbb{R}} \|\gamma_\alpha^{-1}(s)\tilde{e}(s)\|_{\mathbb{R}^n} \|\gamma_\alpha(s)U(s, y)\|_{\mathbb{R}^n} ds \right)^2 dy$$

$$\leq \int_{\mathbb{R}^{d-1}} \left( \int_{\mathbb{R}} \|\gamma_\alpha^{-1}(s)\tilde{e}(s)\|^2_{\mathbb{R}^n} ds \right) \left( \int_{\mathbb{R}} \|\gamma_\alpha(s)U(s, y)\|^2_{\mathbb{R}^n} ds \right) dy$$

by Hölder’s inequality. Since we have,

$$\|\gamma_\alpha^{-1}(s)\tilde{e}(s)\|_{\mathbb{R}^n} \leq \begin{cases} Ke^{-\zeta_- s}, & \text{for } s \leq 0, \\ Ke^{-\zeta_+ s}, & \text{for } s \geq 0, \end{cases}$$

(3.56)

this clearly shows that

$$\int_{\mathbb{R}} \|\gamma_\alpha^{-1}(s)\tilde{e}(s)\|^2_{\mathbb{R}^n} ds \leq K \left( \int_{-\infty}^0 e^{-2\zeta_- s} ds + \int_0^{\infty} e^{-2\zeta_+ s} ds \right) \leq C$$

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for some \( C > 0 \). Thus,

\[
\|\pi_\alpha U\|_{L^2(R^{d-1})}^2 \leq C \int_{R^{d-1}} \int_R \|\gamma_\alpha(s)U(s, y)\|^2 \, ds \, dy = C \|\gamma_\alpha U\|_{L^2(R^d)}^2 \tag{3.57}
\]

\[
\leq C \max\{\|U\|_{L^2(R^d)}, \|U\|_{L^2(R^d)}^2\} \leq C \|U\|^2_{\bar{E}}. \tag{3.58}
\]

For \( H^k \)-norms, we again employ the equivalent Sobolev norm (see [NS], page 316).

Let \( x = (z, y) \in R^d \) and \( y = (x_2, ..., x_d) \in R^{d-1} \),

\[
\|f\|_{H^k(R^{d-1})} \sim \|f\|_{L^2(R^{d-1})} + \sum_{a_2 + ... + a_d = k} \left\| \frac{\partial^k}{\partial x_2^{a_2} ... \partial x_d^{a_d}} f \right\|_{L^2(R^{d-1})},
\]

where the sum extends over all \((d - 1)\)-tuples \((a_2, ..., a_d)\) of non-negative integers with \(\sum_{t=2}^{d} a_t = k\), and \(\frac{\partial^t}{\partial x_t^{a_t}}\) is the \(a_t\)-th differentiation of functions with respect to \(x_t\), \(t = 2, ..., d\).

We already have the estimates for \(\|\pi_\alpha U\|_{L^2(R^{d-1})}\) for \(U \in \mathcal{E}_{\alpha}^n \cap \mathcal{E}_0^n\). Also by Hölder's inequality it follows as before that

\[
\left\| \frac{\partial^k}{\partial x_2^{a_2} ... \partial x_d^{a_d}} \pi_\alpha U \right\|_{L^2(R^{d-1})}^2 = \int_{R^{d-1}} \left\| \frac{\partial^k}{\partial x_2^{a_2} ... \partial x_d^{a_d}} (\pi_\alpha U)(y) \right\|^2 \, dy
\]

\[
\leq \int_{R^{d-1}} \left( \int_R \|\gamma_\alpha^{-1}(s)\bar{e}(s)\|_{R^n} \left\| \gamma_\alpha(s) \frac{\partial^k}{\partial x_2^{a_2} ... \partial x_d^{a_d}} U(s, y) \right\| \, ds \right)^2 \, dy
\]

\[
\leq \int_{R^{d-1}} \left( \int_R \|\gamma_\alpha^{-1}(s)\bar{e}(s)\|^2_{R^n} \, ds \right) \left( \int_R \left\| \gamma_\alpha(s) \frac{\partial^k U(s, y)}{\partial x_2^{a_2} ... \partial x_d^{a_d}} \right\|^2_{R^n} \, ds \right) \, dy
\]

\[
\leq C \int_{R^{d-1}} \int_R \|\gamma_\alpha(s) \frac{\partial^k}{\partial x_2^{a_2} ... \partial x_d^{a_d}} U(s, y) \|^2_{R^n} \, ds \, dy
\]

\[
\leq C \|U\|^2_{\alpha} \leq C \|U\|^2_{\bar{E}}, \tag{3.59}
\]

thus finishing the proof of \(\pi_\alpha \in \mathcal{B}(\mathcal{E}_{\alpha}^n \cap \mathcal{E}_0^n, H^k(R^{d-1}))\).

For the estimates for the \(L^1\)-norm of \(\pi_\alpha U\), we note that

\[
\|\pi_\alpha U\|_{L^1(R^{d-1})} = \int_{R^{d-1}} |(\pi_\alpha U)(y)| \, dy \tag{3.60}
\]

\[
= \int_{R^{d-1}} \left| \int_R (\bar{e}(s), U(s, y))_{R^n} \, ds \right| \, dy
\]

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\[ \leq \int_{\mathbb{R}^{d-1}} \left( \int_{\mathbb{R}} \| \gamma_\alpha^{-1}(s) e(s) \|_{\mathbb{R}^n} \| \gamma_\alpha(s) U(s, y) \|_{\mathbb{R}^n} ds \right) dy \]
\[ \leq \int_{\mathbb{R}^{d-1}} \left( \int_{\mathbb{R}} C \| \gamma_\alpha(s) U(s, y) \|_{\mathbb{R}^n} ds \right) dy \]
\[ \leq C \| \gamma_\alpha U \|_{L^1(\mathbb{R}^d)} \]
\[ \leq C \| U \|_{L^1(\mathbb{R}^{d}) \otimes L^1(\mathbb{R}^{d-1})}. \quad (3.61) \]

Now we consider \( PU \) for \( U \in \mathcal{E}_\alpha^\alpha \) noting that \( P = P_\alpha \otimes I_{H^k(\mathbb{R}^{d-1})} \). As in Remark 2.7, \( P_\alpha \) is a bounded operator from \( H^k_\alpha(\mathbb{R}) \cap H^k(\mathbb{R}) \) to \( H^k_\alpha(\mathbb{R}) \cap H^k(\mathbb{R}) \). Therefore, by Lemma 3.48 we have

\[ \| P \|_{\mathcal{B}(\mathcal{E})} = \| P_\alpha \|_{\mathcal{B}(H^k_\alpha(\mathbb{R}) \cap H^k(\mathbb{R}))} \| I \|_{\mathcal{B}(H^k(\mathbb{R}^{d-1}))} \leq C, \]

finishing the proof of this lemma. \( \blacksquare \)

**Remark 3.38.** Indeed, Lemma 3.37 and the definition \( PU(z, y) = (\pi_\alpha U)(y) \phi'(z) \), \( z, y \in \mathbb{R}^d \), imply that

\[ \| P U \| = \| \pi_\alpha U \|_{H^k(\mathbb{R}^{d-1})} \| \phi' \|_{H^k_\alpha(\mathbb{R})} \leq C \| U \| \| \phi' \|_{H^k_\alpha(\mathbb{R})}, \]
\[ \| P U \|_0 = \| \pi_\alpha U \|_{H^k(\mathbb{R}^{d-1})} \| \phi' \|_{H^k(\mathbb{R})} \leq C \| U \| \| \phi' \|_{H^k(\mathbb{R})}. \]

The first inequality implies that \( P \) is a bounded operator from \( E_\alpha \) to \( E_\alpha \) and also from \( E \) to \( E_\alpha \) while the second inequality implies that \( P \) is a bounded operator from \( E_\alpha \) to \( E_0 \) and also from \( E \) to \( E_0 \). The two inequalities also yield that \( Q = I - P \) is a bounded operator from \( E_\alpha \) to \( E \) and from \( E \) to \( E \). \( \diamond \)

The projection \( P_\alpha \) is initially defined as the Riesz projection for the operator \( L_{1,\alpha} \), and now we need to verify that \( P_\alpha L_\alpha = L_\alpha P \). Using \( P_\alpha L_{1,\alpha} = L_{1,\alpha} P \) yields

\[ P L_\alpha = (P_\alpha \otimes I_{H^k(\mathbb{R}^{d-1})^n}) (L_{1,\alpha} \otimes I_{H^k(\mathbb{R}^{d-1})^n} + I_{H^k_\alpha(\mathbb{R})^n} \otimes \Delta_y) \]
\[ P_\alpha \mathcal{L}_{1,\alpha} \otimes I_{H^k(\mathbb{R}^{d-1})^n} + P_\alpha \otimes \Delta_y \]

and

\[ \mathcal{L}_\alpha \mathcal{P} = (\mathcal{L}_{1,\alpha} \otimes I_{H^k(\mathbb{R}^{d-1})^n} + I_{H^k(\mathbb{R})^n} \otimes \Delta_y)(P_\alpha \otimes I_{H^k(\mathbb{R}^{d-1})}) \]

\[ = \mathcal{L}_{1,\alpha} P_\alpha \otimes I_{H^k(\mathbb{R}^{d-1})} + P_\alpha \otimes \Delta_y \]

\[ = \mathcal{L}_{1,\alpha} P_\alpha \otimes I_{H^k(\mathbb{R}^{d-1})^n} + P_\alpha \otimes \Delta_y. \]

However, the relation \( P_\alpha \mathcal{L}_{1,\alpha} = \mathcal{L}_\alpha \mathcal{P} \) is not true in the event when the diffusion matrix \( D = \text{diag}(d_1, ..., d_n) \), with different \( d_i \), is a general diffusion matrix, because in this more general case we write

\[ \mathcal{L}_\alpha = DI_{H^k(\mathbb{R})^n} \otimes \Delta_y + \mathcal{L}_{1,\alpha} \otimes I_{H^k(\mathbb{R}^{d-1})^n}, \]

and notice that \( D \) doesn’t commute with \( P_\alpha \). This explains the reason for assuming Hypothesis 3.35 instead of Hypothesis 3.29.

### 3.2.3 The system of evolution equations

In this subsection we derive and prove well-posedness of the system of evolution equations governing the perturbation of the planar front.

Let \( \mathcal{E} := \mathcal{E}_0 \cap \mathcal{E}_\alpha \), with norm \( \|u\|_\mathcal{E} = \max\{\|u\|_0, \|u\|_\alpha\} \). We use Remark 3.1 and let \( \mathcal{L}_\mathcal{E} \) denote the linear operator in \( \mathcal{E}^n \) given by \( u \to Lu \) where the domain of \( \mathcal{L}_\mathcal{E} \) is the set of \( u \) satisfying \( u \in \text{dom}(\mathcal{L}) \cap \text{dom}(\mathcal{L}_\alpha) \), where \( \text{dom}(\mathcal{L}) \) and \( \text{dom}(\mathcal{L}_\alpha) \) are respective domains defined in Remark 3.24. Throughout, we denote \( \mathcal{E}_0 = H^k(\mathbb{R}^d) \) and \( \mathcal{E}_\alpha = H^k(\mathbb{R}) \otimes H^k(\mathbb{R}^{d-1}) \).

We also define the projection operator \( \mathcal{Q} \) by \( \mathcal{Q} U = (I_{\mathcal{E}_\alpha} - \mathcal{P})U \), and denote the spaces \( \text{ran} \mathcal{P} = \{U \in \mathcal{E}_\alpha^n : U = \mathcal{P}U\} \) and \( \text{ran} \mathcal{Q} = \{U \in \mathcal{E}_\alpha^n : U = \mathcal{Q}U\} \). In fact,
if $U \in \text{ran} \mathcal{Q}$, then $\pi_\alpha U = 0$ because $\mathcal{P}U = 0$. Here, $\mathcal{P}$ is the projection defined in (3.55).

Lemmas 3.27 and 3.30 imply that $\phi' \in \mathcal{E}^n$, therefore if $v \in \mathcal{E}^n \hookrightarrow \mathcal{E}_\alpha^n$, then $\mathcal{P}v \in \mathcal{E}^n$, and then $\mathcal{Q}v = (I - \mathcal{P})v \in \mathcal{E}^n$. Hence we may define $\mathcal{P}_\mathcal{E}$ and $\mathcal{Q}_\mathcal{E}$ to be the operators acting in $\mathcal{E}^n$ and given by restricting $\mathcal{P}$ and $\mathcal{Q}$ to $\mathcal{E}^n$. Since $\mathcal{E}^n \hookrightarrow \mathcal{E}_\alpha^n$, the operators $\mathcal{P}_\mathcal{E}$ and $\mathcal{Q}_\mathcal{E}$ are also bounded. It follows that $\mathcal{E}^n = \text{ran} \mathcal{P}_\mathcal{E} \oplus \text{ran} \mathcal{Q}_\mathcal{E}$ where $\text{ran} \mathcal{P}_\mathcal{E} = \text{ran} \mathcal{L}_\alpha \cap \mathcal{E}^n$, see Lemma 3.37 and Remark 3.38 above.

In the space $\mathcal{E}^n$ we consider a solution of equation (3.38) of the form

$$u(z, y, t) = \phi(z - q(y, t)) + v(z, y, t), \quad (z, y) \in \mathbb{R}^d,$$

(3.62)

where $(v, q) \in \text{ran} \mathcal{Q}_\mathcal{E} \otimes H^k(\mathbb{R}^{d-1})$. Here, we decompose the perturbation $u$ of the front $\phi$ into a spatial translation component, i.e., the component in the direction of the front, $\phi(z - q(y, t))$, and a normal component $v$, so that $v = v(\cdot, y, t)$ belongs to the range of $\mathcal{L}_{1, \alpha}$, $\text{ran} \mathcal{Q}_\alpha = \text{ran} \mathcal{L}_{1, \alpha}$, for each $(y, t) \in \mathbb{R}^{d-1} \times \mathbb{R}^+$. We need the following lemma to show that the coordinate system $(v, q)$ is well defined, that is, given a perturbation $\tilde{v} \in \mathcal{E}^n$ of the original front $\phi$ suitably small, in order to use this system, we need to prove that there exists a unique pair $(v, q) \in \text{ran} \mathcal{Q}_\mathcal{E} \times H^k(\mathbb{R}^{d-1})$ such that $\phi + \tilde{v} = \phi(\cdot - q) + v$.

**Lemma 3.39.** Assume Hypothesis 3.35 and $k \geq \lceil \frac{d+1}{2} \rceil$. If $\tilde{v} \in \mathcal{E}^n$ is small enough, then there exists $(v, q) \in \text{ran} \mathcal{Q}_\mathcal{E} \times H^k(\mathbb{R}^{d-1})$ depending on $\tilde{v}$ such that

$$\phi(z) + \tilde{v}(z, y) = \phi(z - q(y)) + v(z, y), \quad (z, y) \in \mathbb{R}^d.$$

(3.63)

**Proof.** Following the proof of [K2, Lemma 2.2], we write

$$\phi(\cdot - q) - \phi(\cdot) = -\int_0^1 \phi'((\cdot - sq)q ds,$$
where $\phi'$ is the derivative of $\phi$ with respect to $z$.

We claim that the right-hand side of the last equation is in $\mathcal{E}^n$ since $\phi'$ and its derivatives decay to zero as $z \to \pm\infty$ faster than $\gamma_\alpha$ by Hypothesis 3.29. Indeed, since $\omega_+ > \alpha_+$ and $-\omega_- > \alpha_-$ we have

$$
\|\phi'(-sq(\cdot))q(\cdot)\|_{L^2_d(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} |q(y)|^2 \int_{\mathbb{R}} |\phi'(z - sq(y))|^2 \gamma_\alpha^2(z) \, dz \, dy
\leq c\|q\|_{L^2(\mathbb{R}^d)} \left( \int_0^\infty e^{-2\omega_+ z} e^{2\alpha_+ z} \, dz + \int_0^0 e^{-2\omega_- z} e^{2\alpha_- z} \, dz \right)
\leq C\|q\|_{H^k(\mathbb{R}^d)},
$$

where we used the estimate

$$
|\phi'(z - sq(y))| \leq K e^{-\omega_\pm (z - sq(y))} = Ke^{sq(y)} e^{-\omega_\pm z} \leq Ce^{-\omega_\pm z}
$$

and the fact that $q \in H^k(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$. Therefore, $\phi(\cdot - q) - \phi(\cdot)$ is in $\mathcal{E}^n$ provided $q \in H^k(\mathbb{R}^d)$. We then write (3.63) as

$$
\tilde{v} = v - q \int_0^1 \phi'(\cdot - sq) \, ds
$$

(3.64)

and apply $\pi_\alpha$, see (3.53), to both sides of (3.64). Since $v \in \text{ran \, } Q \mathcal{E} = \text{ker \, } P \mathcal{E}$, we have that $\pi_\alpha(v) = 0$, and equation (3.64) becomes

$$
\pi_\alpha(\tilde{v}) = -q \left( \int_0^1 \pi_\alpha(\phi'(\cdot - sq)) \, ds \right).
$$

Define a mapping $G$ from $H^k(\mathbb{R}^d) \times \mathcal{E}^n$ to $H^k(\mathbb{R}^d)$ by

$$
G(q, \tilde{v}) = \pi_\alpha(\tilde{v}) + q \left( \int_0^1 \pi_\alpha(\phi'(\cdot - sq)) \, ds \right).
$$

(3.65)

We note that $G(0, 0) = 0$ and $\frac{\partial G}{\partial q}(0, 0) = I$. Now the Implicit Function Theorem yields the existence of $q(\tilde{v})$ near $\tilde{v} = 0$ with $G(q(\tilde{v}), \tilde{v}) = 0$. Applying $Q \mathcal{E}$ to (3.64)
with this \( q \), we have

\[
v = Q_\varepsilon v = Q_\varepsilon \tilde{v} + Q_\varepsilon \left( q \int_0^1 \phi'(\cdot - sq) \, ds \right).
\]

Since \( q(\tilde{v}) \) has been found by solving (3.65), the desired coordinate system is well-defined. ■

Using the result of Lemma 3.39, the desired pair \( (v, q) \in \text{ran} \, Q_\varepsilon \otimes H^k(\mathbb{R}^{d-1}) \) is well defined. We will now substitute (3.62) into (3.38). (The computations will be similar to the discussion of [K2, Section 2], it will be stated for readers’ conveniences). It follows that

\[
\partial_t u = \partial_t(\phi(z - q)) + \partial_t v = \partial_t v - \phi'(z - q) \partial_t q.
\]

Let \( \phi_q(z) = \phi(z - q) \). With this notation we will calculate the right-hand side of equation (3.38) by using the fact that \( \phi_q(z - q) \) is also a planar front that satisfies the equation

\[
\partial_{zz} \phi_q + c \partial_z \phi_q + f(\phi_q) = 0.
\]

Indeed, we have for \( u = \phi_q + v \) from (3.38):

\[
\Delta_x u + c \partial_z u + f(u) = \Delta_x (\phi_q + v) + c \partial_z (\phi_q + v) + f(\phi_q + v)
\]

\[
= \Delta_x \phi_q + c \partial_z \phi_q + f(\phi_q + v) + \Delta_x v + c \partial_z v
\]

\[
= \partial_{zz} \phi_q + c \partial_z \phi_q + f(\phi_q) + df(\phi_q) v + f(\phi_q + v) - f(\phi_q) - df(\phi_q) v + \Delta_y \phi_q
\]

\[
+ \Delta_x v + c \partial_z v
\]

\[
= (\Delta_x v + c \partial_z v + df(\phi_q) v) + f(\phi_q + v) - f(\phi_q) - df(\phi_q) v + \Delta_y \phi_q.
\]

\[ (3.67) \]
Note that

\[ f(u + v) - f(u) - df(u)v = \left( \int_0^1 df(u + sv) - df(u)ds \right) v. \]

We define

\[ N(u, v) = \int_0^1 df(u + sv) - df(u)ds, \quad (3.68) \]
as an \( n \times n \) matrix-valued function of \( (u, v) \). Using (3.68), we combine (3.66) and (3.67) to write (3.38) as follows,

\[ \partial_t v - \phi'_q \partial_t q = L v + (df(\phi_q) - df(\phi))v + N(\phi_q, v)v + \Delta_y \phi_q, \quad (3.69) \]

where \( L \) is the differential expression defined in (3.41).

We now tackle the term \( \Delta_y \phi_q \). Recall that \( q = q(y) \in H^k(\mathbb{R}^{d-1}) \), and thus

\[
\Delta_y \phi_q = \partial_{x_2}^2 \phi_q + \cdots + \partial_{x_d}^2 \phi_q \\
= \partial_{x_2} (-\phi'_q \partial_{x_2} q) + \cdots + \partial_{x_d} (-\phi'_q \partial_{x_d} q) \\
= (\phi''_q \partial_{x_2} q)(\partial_{x_2} q) - \phi'_q \partial_{x_2}^2 q + \cdots + (\phi''_q \partial_{x_d} q)(\partial_{x_d} q) - \phi'_q \partial_{x_d}^2 q \\
= \phi''_q (\nabla_y q \cdot \nabla_y q) - \phi'_q \Delta_y q,
\]

where \( \nabla_y q = (\partial_{x_2} q, \cdots, \partial_{x_d} q) \). Combining this with (3.69), we see that \( v \) satisfies the equation

\[ \partial_t v - \phi'_q \partial_t q = L v + (df(\phi_q) - df(\phi))v + N(\phi_q, v)v + \phi''_q (\nabla_y q \cdot \nabla_y q) - \phi'_q \Delta_y q, \]
or, in other words, the equation

\[ \partial_t v = L v + (df(\phi_q) - df(\phi))v + N(\phi_q, v)v + (\partial_t q - \Delta_y q) \phi'_q + (\nabla_y q \cdot \nabla_y q) \phi''_q. \quad (3.70) \]
Let us assume that $v(\cdot, \cdot, t)$ is in ran $Q_{\mathcal{E}} \cap \mathcal{E}^n$ for every $t \geq 0$, that it, $P_{\mathcal{E}} v = 0$. We apply the projection $P_{\mathcal{E}}$ in (3.70) and conclude that $q$ satisfies the equation

$$
(-\pi_\alpha \phi') \partial_t q = (\pi_\alpha \phi'') (\nabla_y q \cdot \nabla_y q) - (\pi_\alpha \phi') \Delta_y q
+ \pi_\alpha ((df(\phi) - df(\phi)) v + N(\phi, v) v).
$$

(3.71)

It would be convenient to divide both sides of (3.71) by $-\pi_\alpha (\phi'')(y)$. For this we use the following result from [K2, Lemma 2.3], whose proof in one-dimensional case can also be found in [GLS] Lemma 4.1.

**Lemma 3.40.** Assume Hypothesis 3.35. There are constants $\delta_0 > 0$ and $C > 0$ such that if $\|q\|_{L^\infty(\mathbb{R}^{d-1})} < \delta_0$, then

$$
1 - C\delta_0 \leq 1 - C\|q\|_{L^\infty(\mathbb{R}^{d-1})} \leq |\pi_\alpha (\phi')(y)| \leq 1 + C\|q\|_{L^\infty(\mathbb{R}^{d-1})} \leq 1 + C\delta_0,
$$

$$
C(1 - \delta_0) \leq C(1 - \|q\|_{L^\infty(\mathbb{R}^{d-1})}) \leq |\pi_\alpha (\phi')(y)| \leq C(1 + \|q\|_{L^\infty(\mathbb{R}^{d-1})})
\leq C(1 + \delta_0),
$$

for all $y \in \mathbb{R}^{d-1}$, where $\phi_q = \phi(\cdot - q)$.

**Proof.** It follows from Lemma 3.30 that $\gamma_\alpha(z)\phi''(z) \to 0$ exponentially as $z \to \pm \infty$, therefore the mapping $q \to \phi'(z - q)$ is differentiable from $H^k(\mathbb{R}^{d-1})$ into $\mathcal{E}^n$, and

$$
\phi'(z - q) = \phi' - q \int_0^1 \phi''(z - sq) ds.
$$

(3.72)

Since $\pi_\alpha \phi' = 1$, applying $\pi_\alpha$ from (3.53) on both sides of (3.72) we have

$$
\pi_\alpha (\phi'(z - q))(y) = 1 - q(y) \pi_\alpha \int_0^1 \phi''(z - sq)(y) ds
$$

(3.73)

for all $y \in \mathbb{R}^{d-1}$. We note that $\pi_\alpha \in \mathcal{B}(\mathcal{E}, H^k(\mathbb{R}^{d-1}))$ by Lemma 3.37, so that by Lemma 3.27 and (3.56) we have, due to $H^k(\mathbb{R}^{d-1}) \hookrightarrow L^\infty(\mathbb{R}^{d-1})$:

$$
\left\| \pi_\alpha \int_0^1 \phi''(\cdot - sq(\cdot)) ds \right\|_{L^\infty(\mathbb{R}^{d-1})}
$$
\[ = \left\| \int_{\mathbb{R}} \left( \tilde{e}(z), \int_0^1 \phi''(z - sq(s))ds \right)_{\mathbb{R}^n} \right\|_{L^\infty(\mathbb{R}^{d-1})} \leq C. \]

Therefore
\[
\|q(\cdot)\pi_{\alpha} \int_0^1 \phi''(\cdot - sq(s))ds\|_{L^\infty(\mathbb{R}^{d-1})} \leq C \|q\|_{L^\infty(\mathbb{R}^{d-1})} \left\| \pi_{\alpha} \int_0^1 \phi''(\cdot - sq(s))ds \right\|_{L^\infty(\mathbb{R}^{d-1})} \leq C \|q\|_{L^\infty(\mathbb{R}^{d-1})}.
\]

Thus for \( \|q\|_{L^\infty(\mathbb{R}^{d-1})} < \delta_0 \) and all \( y \in \mathbb{R}^{d-1} \) we have from (3.73) the first inequality required in the lemma. By writing \( \phi''(z - q) = \phi'' - q \int_0^1 \phi'''(z - sq)ds \) and arguing analogously, we have the second inequality in Lemma 3.40. \( \blacksquare \)

Assuming \( \|q\|_{L^\infty(\mathbb{R}^{d-1})} \leq \delta_0 \) with \( \delta_0 \) from Lemma 3.40, we introduce the notation, for any \((v,q) \in \mathcal{E} \times H^k(\mathbb{R}^{d-1})\):

\[
G(v,q) = (df(\phi_q) - df(\phi))v + N(\phi_q, v),
\]

\[
K_1(q) = -\frac{\pi_{\alpha} \phi_q''}{\pi_{\alpha} \phi_q'},
\]

\[
K_2(q) = -\frac{1}{\pi_{\alpha} \phi_q'}. \tag{3.74}
\]

We now divide both sides of (3.71) by \( \pi_{\alpha} \phi_q' \) to obtain the equation

\[
\partial_t q = \Delta_y q + K_1(q)(\nabla_y q) \cdot (\nabla_y q) + K_2(q) \pi_{\alpha}(G(v, q)). \tag{3.75}
\]

The following fact is a modification of the argument leading to [K2, eq(2.23)].

It will be used to derive various estimates for nonlinearities in evolution equations studied below.
Lemma 3.41. Assume Hypothesis 3.35. Let the functions

\[ K_1 = K_1(q)(y) \quad \text{and} \quad K_2 = K_2(q)(y) \]

for \( q \in H^k(\mathbb{R}^{d-1}) \) be defined as in (3.74). There exist constants \( \delta_0 > 0 \) and \( C > 0 \) such that for \( \|q\|_{H^k(\mathbb{R}^{d-1})} \leq \delta_0 \) we have

\[ \|K_i(q)\|_{L^\infty(\mathbb{R}^{d-1})} \leq C(1 + \|q\|_{H^k(\mathbb{R}^{d-1})}), \quad i = 1, 2. \quad (3.76) \]

Moreover, the formulas for \( K_i, \quad i = 1, 2 \), define locally Lipschitz mappings \( q \mapsto K_i(q) \)

from \( H^k(\mathbb{R}^{d-1}) \) to \( L^\infty(\mathbb{R}^{d-1}) \).

Proof. The Sobolev imbedding gives \( \|q\|_{L^\infty(\mathbb{R}^{d-1})} \leq C\|q\|_{H^k(\mathbb{R}^{d-1})} \) for \( 2k \geq d + 1 > d \), so as long as \( \|q\|_{H^k(\mathbb{R}^{d-1})} \) stays small, the functions \( K_1 = K_1(q)(y) \) and \( K_2 = K_2(q)(y) \)

will remain uniformly bounded for all \( y \in \mathbb{R}^{d-1} \).

Indeed, since \( \phi \) is smooth, we use Lemma 3.40 to conclude that there exist \( \delta_0 > 0 \) and \( C > 0 \) such that for \( \|q\|_{H^k(\mathbb{R}^{d-1})} \leq \delta_0 \) one has

\[
\begin{align*}
\|K_1(q)\|_{L^\infty(\mathbb{R}^{d-1})} &\leq \frac{C + C\|q\|_{L^\infty(\mathbb{R}^{d-1})}}{1 - C\|q\|_{L^\infty(\mathbb{R}^{d-1})}} \leq \frac{C + C\delta_0}{1 - C\delta_0}, \\
\|K_2(q)\|_{L^\infty(\mathbb{R}^{d-1})} &\leq \frac{1}{1 - C\|q\|_{L^\infty(\mathbb{R}^{d-1})}} \leq \frac{1}{1 - C\delta_0}.
\end{align*}
\]

This implies (3.76).

To show the Lipschitz property, we let \( q, \tilde{q} \) belong to the bounded set

\[ \{ q : \|q\|_{H^k(\mathbb{R}^{d-1})} \leq K \}, \]

and then use Lemma 3.40 to see that

\[
\begin{align*}
1 - CK &\leq |\pi_\alpha \phi_q'(y)| \leq 1 + CK, \quad 1 - CK \leq |\pi_\alpha \phi_{\tilde{q}}'(y)| \leq 1 + CK, \\
C - CK &\leq |\pi_\alpha \phi_q''(y)| \leq C + CK, \quad C - CK \leq |\pi_\alpha \phi_{\tilde{q}}''(y)| \leq C + CK.
\end{align*}
\]
We then return to the task of deriving the evolution equation for the perturbation \( \phi_q + v \). Applying the projection operator \( Q_\mathcal{E} \) to (3.70) yields the equation

\[
\partial_t v = Lv + Q_\mathcal{E}(G(v, q) + (\partial_t q - \Delta yq)\phi'_q + (\nabla_y q)^2\phi''_q),
\]

(3.77)
where \( G(v, q) \) is defined as in (3.74). Combining (3.77) and (3.75) we have the system

\[
\begin{align*}
\partial_t v &= L v + Q \varepsilon \left( G(v, q) + (\partial_t q - \Delta_y q) \phi'_q + (\nabla_y q) \cdot (\nabla_y q) \phi''_q \right), \\
\partial_t q &= \Delta_y q + K_1(q)(\nabla_y q) \cdot (\nabla_y q) + K_2(q) \pi \alpha G(v, q). 
\end{align*}
\] (3.78)

To make the system look nicer, define

\[
w(y) = \nabla_y q(y), \quad y \in \mathbb{R}^{d-1},
\]

\[
F_1(v, q, w) = G(v, q) + (\partial_t q - \Delta_y q) \phi'_q + (w \cdot w) \phi''_q,
\] (3.79)

and

\[
F_2(v, q, w) = K_1(q)(w \cdot w) + K_2(q) \pi \alpha G(v, q).
\] (3.80)

Because \( \partial_t q - \Delta_y q = F_2(v, q, w) \) by (3.75), plugging this in (3.80) we have the following relation between \( F_1 \) and \( F_2 \),

\[
F_1(v, q, w) = G(v, q) + F_2(v, q, w) \phi'_q + (w \cdot w) \phi''_q.
\] (3.81)

Also notice that from (3.70) we have, using \( v \in \mathrm{ran} \mathcal{Q} \varepsilon \) and \( \phi'_q \in \ker P \varepsilon \), that

\[
\mathcal{P} \varepsilon \left( G(v, q) + (\partial_t q - \Delta_y q) \phi'_q + (w \cdot w) \phi''_q \right) = 0,
\]

which implies that \( F_1(v, q, w) = \mathcal{Q} \varepsilon F_1(v, q, w) \) and

\[
\mathcal{Q} \varepsilon F_1(v, q, w) = \mathcal{Q} \varepsilon \left( G(v, q) + (\partial_t q - \Delta_y q) \phi'_q + (w \cdot w) \phi''_q \right)
\]

\[
= G(v, q) + (\partial_t q - \Delta_y q) \phi'_q + (w \cdot w) \phi''_q.
\]

Thus applying \( \nabla_y \) to the second equation in (3.78) we finally arrive at the following system for \((v, q, w) \in \mathrm{ran} \mathcal{Q} \varepsilon \times H^k(\mathbb{R}^{d-1}) \times H^k(\mathbb{R}^{d-1})^{d-1}\):

\[
\begin{align*}
\partial_t v &= Lv + F_1(v, q, w), \\
\partial_t q &= \Delta_y q + F_2(v, q, w), \\
\partial_t w &= \Delta_y w + \nabla_y \cdot F_2(\sigma, w, v),
\end{align*}
\] (3.82)
where the nonlinear terms $F_1$ is defined in (3.79) and (3.81), and $F_2$ is defined in (3.80). This is the main system of equations which we will study.

First, we will study the behavior of the nonlinear terms in (3.82). For the rest of this section we will write $\| \cdot \|_{H^k(\mathbb{R}^{d-1})}$ as $\| \cdot \|_{H^k}$, and continue to use notation $\| \cdot \|_0 = \| \cdot \|_{H^k(\mathbb{R}^d)}$ and $\| \cdot \|_\alpha = \| \cdot \|_{H^k(\mathbb{R}) \otimes H^k(\mathbb{R}^{d-1})}$.

**Lemma 3.42.** Assume $k \geq \frac{[d+1]}{2}$. If $q_1, q_2 \in H^k(\mathbb{R}^{d-1})$ and $\psi : \mathbb{R} \to \mathbb{R}$ is a smooth function such that $\psi'(z)$ is exponentially decaying to 0 as $z \to \pm \infty$, then the function $\sigma(z, y) = \psi'(z - q_1(y))q_2(y)$, $(z, y) \in \mathbb{R}^d$, satisfies

$$\| \sigma \|_{H^k(\mathbb{R}^d)} = \| \psi'(\cdot - q_1(\cdot))q_2(\cdot) \|_{H^k(\mathbb{R}^d)} \leq C \| q_2 \|_{H^k(\mathbb{R}^{d-1})},$$

where $C = C(\| \psi \|_{H^k(\mathbb{R})}, \| q_1 \|_{H^k(\mathbb{R}^{d-1})})$ is bounded in each ball of the form

$$\{ q_1 : \| q_1 \|_{H^k(\mathbb{R}^{d-1})} \leq K \}.$$

**Proof.** The $L^2$-norm in the left-hand side of (3.83) is given by

$$\| \sigma \|_{L^2(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^{d-1}} \left( \int_{\mathbb{R}} |\psi'(z - q_1(y))|^2 \, dz \right) |q_2(y)|^2 \, dy$$

$$= \int_{\mathbb{R}^{d-1}} \left( \int_{\mathbb{R}} |\psi'(z)|^2 \, dz \right) |q_2(y)|^2 \, dy$$

$$\leq C \| q_2 \|_{H^k(\mathbb{R}^{d-1})},$$

since $\psi'$ is exponentially decaying to 0. Taking derivatives,

$$\frac{\partial \sigma}{\partial z} = \psi''(z - q_1(y))q_2(y),$$

$$\frac{\partial \sigma}{\partial x_j} = \psi''(z - q(y))q_2(y) \frac{\partial q_1}{\partial x_j} + \psi'(z - q_1(y)) \frac{\partial q_2}{\partial x_j}, \ j = 2, \ldots, d,$$

and arguing analogously, one has the assertion in the lemma for $k = 1$. Indeed,

$$\| \psi''(\cdot - q(\cdot))q_2(\cdot) \frac{\partial}{\partial x_j} q_1(\cdot) \|_{L^2(\mathbb{R}^d)} \leq C \left\| q_2 \frac{\partial}{\partial x_j} q_1(\cdot) \right\|_{L^2(\mathbb{R}^{d-1})}.$$
\[ \leq C \| q_2 \|_{L^\infty(\mathbb{R}^{d-1})} \left\| \frac{\partial}{\partial x_j} q_1 \right\|_{L^2(\mathbb{R}^{d-1})} \leq C \| q_2 \|_{H^k(\mathbb{R}^d)} \| q_1 \|_{H^k(\mathbb{R}^d)}. \]

A calculation similar to the proof of Proposition A.3 concludes the proof of the lemma. Indeed, instead of (A.7) and (A.8) in the proof of Proposition A.3 we will use

\[ k - \frac{d - 1}{2} = k - \frac{d}{2} + \frac{1}{2} > n_i - \frac{d}{p_i} + \frac{1}{p_i} = n_i - \frac{d - 1}{p_i}, \]

which proves the embedding \( H^k(\mathbb{R}^{d-1}) \hookrightarrow W^{n_i,p_i}(\mathbb{R}^{d-1}) \) by Lemma A.1.

Using Lemma 3.42 we now prove the following propositions to estimate the 0-norm and \( \alpha \)-norm of the nonlinear term \( G(v,q) \) introduced in (3.74).

**Proposition 3.43.** Assume Hypothesis 3.35 and \( k \geq \left\lceil \frac{d+1}{2} \right\rceil \), let \( \mathcal{E}_0 = H^k(\mathbb{R}^d) \), and let \( \mathcal{E}_\alpha \) and \( \mathcal{E} \) be defined accordingly. The following assertions hold:

1. Formula \( (v,q) \mapsto (df(\phi_q) - df(\phi))v \) defines a mapping from \( \mathcal{E}_0^n \times H^k(\mathbb{R}^{d-1}) \) to \( \mathcal{E}_0^n \) that is locally Lipschitz on any set of the form \( \{(v,q) : \|v\|_0 + \|q\|_{H^k} \leq K\} \).

On such a set there is a constant \( C_K \) depending on \( K \) such that

\[ \|(df(\phi_q) - df(\phi))v\|_0 \leq C_K \|q\|_{H^k} \|v\|_0. \]

2. Formula \( (v,q) \mapsto (df(\phi_q) - df(\phi))v \) defines a mapping from \( \mathcal{E}^n \times H^k(\mathbb{R}^{d-1}) \) to \( \mathcal{E}^n \) that is locally Lipschitz on any set of the form \( \{(v,q) : \|v\|_\mathcal{E} + \|q\|_{H^k} \leq K\} \).

On such a set there is a constant \( C_K \) depending on \( K \) such that

\[ \|(df(\phi_q) - df(\phi))v\|_\mathcal{E} \leq C_K \|q\|_{H^k} \|v\|_\mathcal{E} \]

and therefore

\[ \|(df(\phi_q) - df(\phi))v\|_\alpha \leq C_K \|q\|_{H^k} \|v\|_\alpha. \]
Proof. We define \( p(q, v) \in H^k(\mathbb{R}^d) \) for \( q \in H^k(\mathbb{R}^{d-1}) \) and \( v \in H^k(\mathbb{R}^d) \) by the formula

\[
p(q, v)(z, y) = \left( df(\phi(z - q(y))) - df(\phi(z)) \right) v(x),
\]
and write

\[
p(q, v) = \int_0^1 \frac{d}{ds}df(\phi(\cdot - sq))vds = -\int_0^1 d^2f(\phi(\cdot - sq))(\phi'(\cdot - sq)q, v) ds.
\]

(3.87)

Since \( x \mapsto d^2f(\phi(z - sq(y))) \) is a smooth function with bounded derivatives, using Lemma 3.42 we conclude that \( p(q, v) \in H^k(\mathbb{R}^d) \) and satisfies

\[
\|p(q, v)\|_{H^k(\mathbb{R}^d)} \leq C_K\|q\|_{H^k(\mathbb{R}^{d-1})}\|v\|_{H^k(\mathbb{R}^d)};
\]

also, multiplying (3.87) by \( \gamma_\alpha \), we have

\[
\|p(q, v)\|_\alpha \leq C_K\|q\|_{H^k(\mathbb{R}^{d-1})}\|v\|_\alpha.
\]

This yields the required inequalities in the proposition.

To show the locally Lipschitz estimates for \( p(q, v) \) and \( \gamma_\alpha p(q, v) \), we may pass to components in the vector equation (3.87). Thus, it is enough to show that the following map, \( \tilde{l} : H^k(\mathbb{R}^{d-1}) \times H^k(\mathbb{R}^{d-1}) \times H^k(\mathbb{R}^d) \rightarrow H^k(\mathbb{R}^d) \) is locally Lipschitz, where

\[
\tilde{l}(q_1, q_2, v)(z, y) = l(\psi(z - q_1(y)))\psi'(z - q_1(y))q_2(y)v(x), x = (z, y) \in \mathbb{R}^d,
\]

where \( \psi : \mathbb{R} \rightarrow \mathbb{R} \) is exponentially decaying to some constants \( \psi_\pm \) as \( z \rightarrow \pm\infty \), and the derivatives \( \psi^{(m)}(z) \rightarrow 0 \) as \( z \rightarrow \pm\infty \) exponentially, and \( l : \mathbb{R} \rightarrow \mathbb{R} \) is a \( C^{k+3} \) smooth function with bounded derivatives. Recall that \( k \geq \left\lceil \frac{d+1}{2} \right\rceil \) and thus \( H^k(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d) \). Since the derivatives of \( l \) are bounded, using Lemma 3.42 we have

\[
\|\tilde{l}(q_1, q_2, v)\|_{H^k(\mathbb{R}^d)} \leq C\|l\|_{C^{k+3}}\|q_2\|_{H^k(\mathbb{R}^{d-1})}\|v\|_{H^k(\mathbb{R}^d)},
\]

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and thus the map $\tilde{l}$ is well defined.

We will now proceed with the locally Lipschitz estimates for $\tilde{l}$. To show the estimate for the variation in $q_1$. We fix $q_2$, $v$ and write:

$$\tilde{l}(q_1, q_2, v) - \tilde{l}(\bar{q}_1, q_2, v) = \left( l(\psi(\cdot - q_1)) - l(\psi(\cdot - \bar{q}_1)) \right) \psi'(\cdot - q_1) q_2 v$$

$$+ l(\psi(\cdot - \bar{q}_1))(\psi'(\cdot - q_1) - \psi'(\cdot - \bar{q}_1)) q_2 v.$$ 

We write

$$l(\psi(z - q_1(y))) - l(\psi(z - \bar{q}_1(y)))$$

$$= \int_0^1 \frac{d}{ds} l(\psi(z - q_1(y) - (s - 1)(q_1(y) - \bar{q}_1(y)))) ds$$

$$= \int_0^1 l'(\psi(\cdot))(\psi'(\cdot)(q_1(y) - \bar{q}_1(y))) ds.$$ 

(3.88)

Since $l$ has bounded derivatives, applying Lemma 3.42 again shows

$$\|l(\psi(\cdot - q_1)) - l(\psi(\cdot - \bar{q}_1))\|_{H^k(\mathbb{R}^d)} \leq C_K \|q_1 - \bar{q}_1\|_{H^k(\mathbb{R}^{d-1})}.$$ 

On the other hand,

$$\psi'(z - q_1(y)) - \psi'(z - \bar{q}_1(y)) = \int_0^1 \frac{d}{ds} \psi'(z - q_1(y) - (s - 1)(q_1(y) - \bar{q}_1(y))) ds$$

$$= - \int_0^1 \psi''(\cdot)(q_1(y) - \bar{q}_1(y)) ds.$$ 

Another application of Lemma 3.42 yields

$$\|\tilde{l}(q_1, q_2, v) - \tilde{l}(\bar{q}_1, q_2, v)\|_{H^k(\mathbb{R}^d)} \leq C_K \|q_1 - \bar{q}_1\|_{H^k(\mathbb{R}^{d-1})}.$$ 

The estimate for the variation in $q_2$ are similar. The estimate for $v$ follows from Proposition A.3 by fixing $q_1$ and $q_2$.

Multiplying $\tilde{l}$ by $\gamma_\alpha$ and working with $l(\psi(\cdot - q_1))\psi'(\cdot - q_1) q_2 \gamma_\alpha v$ gives the local Lipschitz estimate of $p(q, v)$ in the $\alpha$-norm. □
Proposition 3.44. Assume Hypothesis 3.35 and \( k \geq \left\lceil \frac{d+1}{2} \right\rceil \), let \( E_0 = H^k(\mathbb{R}^d) \), and let \( E_\alpha \), and \( E \) be defined accordingly. We recall notation (3.68).

(1) The formula \((v, q) \mapsto N(\phi_q, v)\) defines a mapping from \( E_0^a \times H^k(\mathbb{R}^{d-1}) \) to \( E_0^{a_2} \) that is locally Lipschitz and \( O(||v||_0) \) as \( ||v||_0 \to 0 \) uniformly on any bounded neighborhood of \((0, 0)\) in \( E_0^a \times H^k(\mathbb{R}^{d-1}) \).

(2) The formula \((v, q) \mapsto N(\phi_q) v\) defines a mapping from \( E_0^a \times H^k(\mathbb{R}^{d-1}) \) to \( E_0^a \) that is locally Lipschitz on any bounded neighborhood of \((0, 0)\) in \( E_0^a \times H^k(\mathbb{R}^{d-1}) \).

Proof. (1) We write
\[
N(\phi_q, v) = \int_0^1 (df(\phi_q + sv) - df(\phi_q)) \, ds
= \int_0^1 \int_0^1 \frac{d}{d\tau} (df(\phi_q + s\tau v)) \, d\tau \, ds
= \int_0^1 \int_0^1 d^2 f(\phi_q + s\tau v) sv \, d\tau \, ds.
\]

Passing to components in the vector \( d^2 f(\cdot) v \), it is enough to show that the following map, \( \tilde{l} : H^k(\mathbb{R}^{d-1}) \times H^k(\mathbb{R}^d) \times H^k(\mathbb{R}^d) \to H^k(\mathbb{R}^d) \), defined by
\[
\tilde{l}(q, u, v)(z, y) = l(\psi(z - q(y)), u(x)) \, v(x), \quad x = (z, y) \in \mathbb{R}^d,
\]
is locally Lipschitz, where \( \psi : \mathbb{R} \to \mathbb{R} \) is a function exponentially decaying to some constants \( \psi_{\pm} \) as \( z \to \pm \infty \) and \( \psi^{(m)}(z) \to 0 \) exponentially, \( m = 1, 2, ... \) and \( l : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is a \( C^{k+3} \)-smooth bounded function with bounded derivatives.

Recall that \( k \geq \left\lceil \frac{d+1}{2} \right\rceil \) and thus \( H^k(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d) \). Then
\[
||\tilde{l}(q, u, v)||_{L^2(\mathbb{R}^d)} \leq ||l||_{C^{k+3}} ||v||_{L^2(\mathbb{R}^d)}.
\]

Taking derivatives, and denoting by \( l'_j \) the derivative with respect to the \( j \)-th...
variable, we have
\[
\frac{\partial \tilde{l}}{\partial z} = l'_1(\cdot)\psi'(z-q(y))v(x) + l'_2(\cdot)\frac{\partial u}{\partial z}v(x) + l(\cdot)\frac{\partial v}{\partial z},
\]
\[
\frac{\partial \tilde{l}}{\partial x_j} = l'_1(\cdot)\psi'(z-q(y))\frac{\partial q}{\partial x_j}v(x) + l'_2(\cdot)\frac{\partial u}{\partial x_j}v(x) + l(\cdot)\frac{\partial v}{\partial x_j}, \quad j = 2, \ldots, d.
\]

Since \(l'_1, l'_2, \psi'\) and \(\frac{\partial q}{\partial x_j}\) are bounded, we conclude that \(\tilde{l}(q,u,v) \in H^1(\mathbb{R}^d)\). A similar calculation (cf. the proof of Proposition A.3 with \(q(z,y)\) in the proof replaced by \(\phi'(z - q(y))\)) shows that \(\tilde{l}(q,u,v) \in H^k(\mathbb{R}^d)\). Thus, the map \(\tilde{l}\) is well defined.

Let us proceed with the proof of the local Lipschitz property.

1) Variation in \(q\) gives
\[
\tilde{l}(q,u,v) - \tilde{l}(\bar{q},u,v) = \left( l(\psi(z - q), u) - l(\psi(z - \bar{q}), u) \right) v
\]
\[
= \int_0^1 \frac{d}{ds} l(\psi(z - q - (s-1)(q - \bar{q})), u) v \, ds
\]
\[
= -\int_0^1 l'_1(\psi(\cdot), u)\psi'(\cdot)(q - \bar{q})v \, ds.
\]

Since \(l'_1\) and its derivatives are bounded, the main part of the estimate
\[
\|\tilde{l}(q,u,v) - \tilde{l}(\bar{q},u,v)\|_{H^k(\mathbb{R}^d)} \leq C_K \|q - \bar{q}\|_{H^k(\mathbb{R}^{d-1})}
\]
on sets of the form \(\{(q,u,v) : \|q\|_{H^k(\mathbb{R}^{d-1})} + \|u\|_{H^k(\mathbb{R}^d)} + \|v\|_{H^k(\mathbb{R}^d)} \leq K\}\) is therefore reduced to Lemma 3.42.

2) Variation in \(u\). The estimate
\[
\|\tilde{l}(q,u,v) - \tilde{l}(q,\bar{u},v)\|_{H^k(\mathbb{R}^d)} = \|(l(\psi(z - q), u) - l(\psi(z - q), \bar{u})) v\|_{H^k(\mathbb{R}^d)}
\]
\[
\leq C_K \|u - \bar{u}\|_{H^k(\mathbb{R}^d)}
\]
follows from Proposition A.3 by considering a \(q\)-independent function \(m(u) := l(\psi(z - q), u)\) in that proposition (we fix \(q\) first and use Proposition A.3 for \(u \mapsto l(\psi_q, u)\) mapping \(H^k(\mathbb{R}^d)\) into \(H^k(\mathbb{R}^d)\)).
3) The estimate for the variation in $v$ also follows from Proposition A.3 for fixed $q$ and $u$.

This concludes the proof the first assertion in part (1) of Proposition 3.44.

Using the Lipschitz property and that $N(\phi_q, 0) = 0$ we conclude that

$$\|N(\phi_q, v)\|_{H^k(\mathbb{R}^d)} = \|N(\phi_q, v) - N(\phi_q, 0)\|_{H^k(\mathbb{R}^d)} \leq C_K \|v\|_{H^k(\mathbb{R}^d)}$$

on any set of the form $\{(v, q) : \|q\|_{H^k(\mathbb{R}^{d-1})} + \|v\|_{H^k(\mathbb{R}^d)} \leq K\}$ as required.

The proof of part (2) follows from part (1) since $H^k(\mathbb{R}^d)$ is an algebra, see (3.18), for instance, the estimate of the variation in $q$ is

$$\|N(\phi_q, v)v - N(\phi_{\bar{q}}, v)v\| = \|\tilde{l}(q, u, v) - \tilde{l}(\bar{q}, u, v)\|_{H^k(\mathbb{R}^d)} \leq C_K \|q - \bar{q}\|_{H^k(\mathbb{R}^{d-1})} \|v\|_{H^k(\mathbb{R}^d)};$$

the estimate of variation in $v$ follows by fixing $q$ and considering the map $v \rightarrow l(\psi_q, v)v$ in Proposition A.3. 

**Proposition 3.45.** Assume Hypothesis 3.35 and $k \geq \frac{d+1}{2}$, let $\mathcal{E}_0 = H^k(\mathbb{R}^d)$, and $\mathcal{E}_\alpha$ and $\mathcal{E}$ be defined accordingly. We consider $N(\phi_q, v)$ defined as in (3.68).

1. If $v \in \mathcal{E}_\alpha$, then $N(\phi_q, v)v \in \mathcal{E}_\alpha$, and for any ball of radius $K$ centered at $(0, 0)$ in $\mathcal{E}_\alpha \times H^k(\mathbb{R}^{d-1})$ there is a constant $C_K > 0$ depending on $K$ such that for any $(v, q)$ in the ball one has

$$\|N(\phi_q, v)v\|_\alpha \leq C_K \|v\|_0 \|v\|_\alpha.$$

2. The formula $(v, q) \mapsto N(\phi_q, v)$ defines a mapping from $\mathcal{E}_\alpha \times H^k(\mathbb{R}^{d-1})$ to $\mathcal{E}_\alpha$ that is locally Lipschitz on any bounded neighborhood of $(0, 0)$ in $\mathcal{E}_\alpha \times H^k(\mathbb{R}^{d-1})$. 

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(3) The formula \((v, q) \mapsto N(\phi, v)\) defines a mapping from \(E^n \times H^k(\mathbb{R}^{d-1})\) to \(E^n\) that is locally Lipschitz on any bounded neighborhood of \((0, 0)\) in \(E^n \times H^k(\mathbb{R}^{d-1})\).

**Proof.** (1) Using Proposition 3.44 (1) and (3.18) we infer

\[ ||N(\phi, v)||_\alpha = ||N(\phi, v)\gamma_\alpha v||_0 \leq ||N(\phi, v)||_0 ||\gamma_\alpha v||_0 \leq C_K ||v||_0 ||v||_\alpha.\]

To show the local Lipschitz property in part (2) and (3) of the proposition, we note that

\[ \gamma_\alpha N(\phi, v) = \int_0^1 \int_0^1 d^2 f(\phi + s\tau v) s\gamma_\alpha v d\tau ds.\]

Now the Lipschitz assertion follows by working with \(\tilde{l}(q, u, \gamma_\alpha v)\) similarly to Proposition 3.44(1) and (2).

**Proposition 3.46.** Assume Hypothesis 3.35 and \(k \geq \lceil \frac{d+1}{2} \rceil\). The formula

\[ (q, w) \mapsto (w \cdot w)\phi_q'' \]

defines a locally Lipschitz mapping from \(H^k(\mathbb{R}^{d-1}) \times H^k(\mathbb{R}^{d-1})\) to \(E^n\) on any bounded set of the form \(\{(q, w) : ||q||_{H^k} + ||w||_{H^k} \leq K\}\), and the mapping satisfies

\[ ||(w \cdot w)\phi_q''||_0 \leq C_K ||w||_{H^k}^2, \quad \text{and} \quad ||(w \cdot w)\phi_q''||_\alpha \leq C_K ||w||_{H^k}^2.\]

**Proof.** Recall that by Lemma 3.27, \(\phi''\) and its derivatives are exponentially decaying to 0 as \(z \to \pm \infty\), and by Lemma 3.30, \(\gamma_\alpha \phi^{(m)}\) is exponentially decaying to 0 as \(z \to \pm \infty\) for \(m = 1, 2, ..., k + 1\).

For a fixed \(q \in H^k(\mathbb{R}^{d-1})\), to show the local Lipschitz estimate in \(w\), we use the Sobolev embedding \(H^k(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)\), and inequality (3.18) for \(H^k(\mathbb{R}^{d-1})\) with
\[ k \geq \left[ \frac{d+1}{2} \right] > \frac{(d-1)+1}{2} \] and observe that, using Lemma 3.42 with \( q_2 = w \cdot \bar{w} \cdot \bar{w} \), we have

\[
\| (w \cdot w - \bar{w} \cdot \bar{w}) \phi_\eta'' \|_0 \leq C \| (w - \bar{w}) \cdot (w + \bar{w}) \|_{H^k(\mathbb{R}^{d-1})} \\
\leq C \| w + \bar{w} \|_{H^k(\mathbb{R}^{d-1})} \| w - \bar{w} \|_{H^k(\mathbb{R}^{d-1})} \\
\leq C_K \| w - \bar{w} \|_{H^k(\mathbb{R}^{d-1})},
\]

\[
\| (w \cdot w - \bar{w} \cdot \bar{w}) \phi_\eta'' \|_\alpha \leq C \| (w - \bar{w}) \cdot (w + \bar{w}) \|_{H^k(\mathbb{R}^{d-1})} \\
\leq C \| w + \bar{w} \|_{H^k(\mathbb{R}^{d-1})} \| w - \bar{w} \|_{H^k(\mathbb{R}^{d-1})} \\
\leq C_K \| w - \bar{w} \|_{H^k(\mathbb{R}^{d-1})},
\]

with some \( C_K > 0 \) depending on \( K \). This yields the inequalities required in the proposition.

The local Lipschitz estimate in \( q \) is proved similarly to Proposition 3.44 using Lemma 3.42.

**Proposition 3.47.** Assume Hypothesis 3.35 and \( k \geq \left[ \frac{d+1}{2} \right] \). Let \( \mathcal{E}_0 = H^k(\mathbb{R}^d) \), and let \( \mathcal{E}_\alpha \) and \( \mathcal{E} \) be defined accordingly. Formula (3.74) for \( G(v,q) \), formula (3.79) for \( F_1(v,q,w) \), and formula (3.80) for \( F_2(v,q,w) \) define locally Lipschitz mappings from \( \mathcal{E}^n \times H^k(\mathbb{R}^{d-1}) \times H^k(\mathbb{R}^{d-1}) \) to \( \mathcal{E}^n, \mathcal{E}^n, \) and \( H^k(\mathbb{R}^{d-1}) \) respectively, on any set of the form \( \{ (v,q,w) : \| v \|_\mathcal{E} + \| q \|_{H^k} + \| w \|_{H^k} < K \} \) with the Lipschitz constant denoted by \( C_K \). Moreover, if \( \| v \|_\mathcal{E} + \| q \|_{H^k} + \| w \|_{H^k} < K \), then for some \( C_K > 0 \) depending on \( K \) one has:

(a)

\[
\| G(v,q) \|_\alpha \leq C_K (\| v \|_0 + \| q \|_{H^k}) \| v \|_\alpha.
\]
(b)\[
\|F_1(v, q, w)\|_\alpha \leq C_K (\|v\|_0 \|v\|_\alpha + \|q\|_{H^k} \|v\|_\alpha + \|w\|_{H^k}^2),
\]

(c)\[
\|F_2(v, q, w)\|_{H^k} \leq C_K (\|v\|_0 \|v\|_\alpha + \|q\|_{H^k} \|v\|_\alpha + \|w\|_{H^k}^2),
\]

(d)\[
\|F_2(v, q, w)\|_{L^1(\mathbb{R}^{d-1})} \leq C_K (\|v\|_0 \|v\|_\alpha + \|q\|_{H^k} \|v\|_\alpha + \|w\|_{H^k}^2).
\]

Proof. The local Lipschitz property of \((df(\phi_q) - df(\phi))v\) on \(E^n\) has been proved in Proposition 3.43, and the local Lipschitz property of \(N(\phi_q, v)v\) on \(E^n \times H^k(\mathbb{R}^{d-1})\) has been proved in Proposition 3.45(3). Combining the local Lipschitz properties of the two terms, we have the locally Lipschitz property of \(G(v, q)\) on \(E^n \times H^k(\mathbb{R}^{d-1})\). The proof of (a) follows from (3.74) together with Proposition 3.43 and 3.45.

In formula (3.80) for \(F_2(v, q, w)\), we first consider the term \(K_1(q)(w \cdot w)\). The Lipschitz estimate of the variation in \(q\) follows from the triangular inequality, inequality (3.18) and Lemma 3.41 because

\[
\|K_1(q)(w \cdot w) - K_1(\tilde{q})(\tilde{w} \cdot \tilde{w})\|_{H^k(\mathbb{R}^{d-1})} \leq \|K_1(q) - K_1(\tilde{q})\|_{H^k(\mathbb{R}^{d-1})} \|w \cdot w\|_{H^k(\mathbb{R}^{d-1})} + \|K_1(\tilde{q})\|_{H^k(\mathbb{R}^{d-1})} \|(w \cdot w - \tilde{w} \cdot \tilde{w})\|_{H^k(\mathbb{R}^{d-1})} \leq C_K (\|q - \tilde{q}\|_{H^k(\mathbb{R}^{d-1})} + \|w - \tilde{w}\|_{H^k(\mathbb{R}^{d-1})}).
\]

We then consider the term \(K_2(q)\pi_\alpha G(v, q)\). The Lipschitz estimate of the variation in \(q\) follows from the triangular inequality, (3.18), Lemma 3.37, the Lipschitz property
of \( G(v, q) \) on \( \mathcal{E}^n \times H^k(\mathbb{R}^{d-1}) \), Lemma 3.41 and the estimates in part (a) because

\[
\| K_2(q) \pi_{\alpha} G(v, q) - K_2(\tilde{q}) \pi_{\alpha} G(v, \tilde{q}) \|_{H^k(\mathbb{R}^{d-1})} \\
\leq \| K_2(q) \pi_{\alpha} G(v, q) - K_2(q) \pi_{\alpha} G(v, \tilde{q}) \|_{H^k(\mathbb{R}^{d-1})} \\
+ \| K_2(q) \pi_{\alpha} G(v, \tilde{q}) - K_2(\tilde{q}) \pi_{\alpha} G(v, \tilde{q}) \|_{H^k(\mathbb{R}^{d-1})} \\
\leq \| K_2(q) \|_{L^\infty(\mathbb{R}^{d-1})} \| \pi_{\alpha} \|_{B(\mathcal{E}, H^k(\mathbb{R}^{d-1}))} \| G(v, q) - G(v, \tilde{q}) \|_\varepsilon \\
+ \| K_2(q) - K_2(\tilde{q}) \|_{L^\infty(\mathbb{R}^{d-1})} \| \pi_{\alpha} \|_{B(\mathcal{E}, H^k(\mathbb{R}^{d-1}))} \| G(v, \tilde{q}) \|_\varepsilon \\
\leq C_K \| q - \tilde{q} \|_{H^k(\mathbb{R}^{d-1})}.
\]

The estimates in part (c) follows from (3.18), (3.57), (3.59) and (3.76):

\[
\| K_2(q) \pi_{\alpha} G(v, q) \|_{H^k(\mathbb{R}^{d-1})} \leq \| K_2(q) \|_{L^\infty(\mathbb{R}^{d-1})} \| \pi_{\alpha} \|_{B(\mathcal{E}, H^k(\mathbb{R}^{d-1}))} \| G(v, q) \|_\varepsilon \\
\leq \| K(q) \|_{L^\infty(\mathbb{R}^{d-1})} \| \pi_{\alpha} \|_{B(\mathcal{E}, H^k(\mathbb{R}^{d-1}))} \| G(v, q) \|_\varepsilon \\
\leq C_K \| q \|_{H^k(\mathbb{R}^{d-1})}.
\]  (3.89)

and

\[
\| K_1(q) (w \cdot w) \|_{H^k(\mathbb{R}^{d-1})} \leq \| K_1(q) \|_{L^\infty(\mathbb{R}^{d-1})} \| w \cdot w \|_{H^k(\mathbb{R}^{d-1})} \leq C_K \| w \|_{H^k(\mathbb{R}^{d-1})}^2.  \]  (3.90)

Combining estimates in part (a), (3.89) and (3.90) we have part (c).

For part (b), in formula (3.81) of \( F_1(v, q, w) \), we already have the Lipschitz property of \( G(v, q) \) mapping into \( \mathcal{E}^n \) by part (a) and the Lipschitz property of the term \( (w \cdot w) \varphi''(q) \) mapping into \( \mathcal{E}^n \) by Proposition 3.46. To prove the Lipschitz estimate for \( \phi'_q F_2(v, q, w) \) of the variation in \( v \) and \( w \), we apply the Lipschitz property of the map \( (v, q) \mapsto F_2(v, q, w) \) for a fixed \( q \). To prove the Lipschitz property of the variation in \( q \mapsto \phi'_q F_2(v, q, w) \), we use the fact that \( \phi' \) decays exponentially to 0 and Lemma 3.42.
with $q_2 = F_2(v, q, w)$. We have the inequality

$$\|F_1(v, q, w)\|_\alpha = \|\gamma \alpha F_1(v, q, w)\|_0$$

$$\leq \|G(v, q)\|_\alpha + C\|F_2(v, q, w)\|_{H^k(\mathbb{R}^{d-1})} + \|\gamma \alpha (w \cdot w)\phi''_q\|_0.$$ 

We can now use the result of the estimation of $G(v, q)$ in part (a) to deal with the first term, then use the estimation for $F_2(v, q, w)$ in part (c), while the estimation of the last term is given by Proposition 3.46.

To prove part (d), we use Cauchy-Schwarz inequality, (3.60), the fact that $\|\cdot\|_{L^2} \leq \|\cdot\|_{H^k}$; also invoking Proposition 3.44(1) and Lemma 3.37 we can conclude that

$$\|\pi \alpha G(v, q)\|_{L^1(\mathbb{R}^{d-1})} \leq C\|\gamma \alpha G(v, q)\|_{L^1(\mathbb{R}^{d-1})}$$

$$\leq C\left(\|\gamma \alpha N(\phi_q, v)\|_{L^1(\mathbb{R}^d)} + \|\gamma \alpha (df(\phi_q) - df(\phi))v\|_{L^1(\mathbb{R}^d)}\right)$$

$$\leq C\left(\|N(\phi_q, v)\|_{L^2(\mathbb{R}^d)}\|\gamma \alpha v\|_{L^2(\mathbb{R}^d)}\right)$$

$$\quad + \|df(\phi_q) - df(\phi)\|_{L^2(\mathbb{R}^d)}\|\gamma \alpha v\|_{L^2(\mathbb{R}^d)}$$

$$\leq C_K(\|v\|_0\|v\|_\alpha + \|q\|_{H^k}\|v\|_\alpha).$$

Similarly, using Cauchy-Schwarz inequality, we infer

$$\|w \cdot w\|_{L^1(\mathbb{R}^{d-1})} \leq \|w\|_{L^2(\mathbb{R}^{d-1})}\|w\|_{L^2(\mathbb{R}^{d-1})} \leq \|w\|^2_{H^k(\mathbb{R}^{d-1})},$$

and thus we have

$$\|F_2(v, q, w)\|_{L^1(\mathbb{R}^{d-1})} \leq C\left(\|\pi \alpha G(v, q)\|_{L^1(\mathbb{R}^{d-1})} + \|w \cdot w\|_{L^1(\mathbb{R}^{d-1})}\right)$$

$$\leq C_K(\|v\|_0\|v\|_\alpha + \|q\|_{H^k}\|v\|_\alpha + \|w\|^2_{H^k}).$$

This finishes the proof of the required inequalities in part (d). □

We will frequently use the following lemma from [RS1, page 299].

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Lemma 3.48. Let $A$ and $B$ be bounded operators on Hilbert spaces $H_1$ and $H_2$. Then
$$
\|A \otimes B\|_{\mathcal{B}(H_1 \otimes H_2)} = \|A\|_{\mathcal{B}(H_1)} \|B\|_{\mathcal{B}(H_2)}.
$$

In what follows, we will need estimates for the semigroups generated by the linear operators $L_\alpha$, $L_\xi$, $\Delta_y$ and $L^{(i)}$ for $i = 1, 2$, cf. (3.41), (3.48) and (3.49), see Lemmas 3.49, 3.50 and 3.52 below.

Lemma 3.49. Assume Hypothesis 3.29, and choose $\nu$ such that
$$
\sup \{\Re \lambda : \lambda \in \text{Sp}(L_{1,\alpha}) \text{ and } \lambda \neq 0 \} < -\nu.
$$

There exists $K > 0$ such that for all $t \geq 0$ one has
$$
\|e^{tL_\alpha} Q\|_{\mathcal{B}(E_\alpha)} \leq Ke^{-\nu t}.
$$

Proof. Recall that $Q = Q_\alpha \otimes I_{H^k(\mathbb{R}^{d-1})}$, and $L_\alpha = L_{1,\alpha} \otimes I_{H^k(\mathbb{R}^{d-1})} + I_{H^k(\mathbb{R})} \otimes \Delta_y$. Thus from the proof of [RS4, Theorem XIII.35], we have $e^{tL_\alpha} Q = e^{tL_{1,\alpha} Q_\alpha} \otimes e^{t\Delta_y} I_{H^k(\mathbb{R}^{d-1})}$. The operators $L_{1,\alpha}$ and $\Delta_y$ both generate bounded semigroups on $\text{ran} \ Q_\alpha = \text{ran} \ L_{1,\alpha}$ and $H^k(\mathbb{R}^{d-1})$, cf. Lemma 3.36 and Lemma 3.52.a), thus by Lemma 3.48, we infer
$$
\|e^{tL_{1,\alpha}} Q_\alpha \otimes e^{t\Delta_y} I_{H^k(\mathbb{R}^{d-1})}\|_{\mathcal{B}(E_\alpha)} = \|e^{tL_{1,\alpha}} Q_\alpha\|_{\mathcal{B}(H^k(\mathbb{R}))} \|e^{t\Delta_y}\|_{\mathcal{B}(H^k(\mathbb{R}^{d-1}))},
$$
finishing the proof. □

We consider the operator $L^-$ on $E_0$ associated with the differential expression
$$
L^- = L^-_1 \otimes I_{H^k(\mathbb{R}^{d-1})} + I_{H^k(\mathbb{R})} \otimes \Delta_y,
$$
where $L^-_1$ is defined in (3.50), and let
$$
L^{(1)} = \Delta_x + c\partial_z + A_1 = L^{(1)}_1 \otimes I_{H^k(\mathbb{R}^{d-1})} + I_{H^k(\mathbb{R})} \otimes \Delta_y,
$$
(3.92)
\[ L^{(2)} = \Delta x + c \partial_z + d_{u_2} f_2(0) = L_1^{(2)} \otimes I_{H^k(\mathbb{R}^{d-1})} + I_{H^k(\mathbb{R})} \otimes \Delta_y. \]

where \( A_1 \) is introduced in Hypothesis 3.31, and \( L_1^{(i)} \), \( i = 1, 2 \) are defined in (3.48) and (3.49). Thus

\[ L^- = \begin{pmatrix} L_1^{(1)} & d_{u_2} f_1(0,0) \\ 0 & L^{(2)} \end{pmatrix}, \]

and the linearization (3.41) about the front is given by the formula

\[ L = L^- + (df(\phi) - df(0)) \otimes I_{H^k(\mathbb{R}^{d-1})}. \]  

(3.94)

As in [GLS, Lemma 8.2(1)], the operator \( df(\phi) - df(0) \) is a bounded operator from \( H^k_\alpha(\mathbb{R}) \) into \( H^k(\mathbb{R}) \). We therefore have

\[ (df(\phi) - df(0)) \otimes I_{H^k(\mathbb{R}^{d-1})} \in \mathcal{B}(E_0, E_0). \]  

(3.95)

**Lemma 3.50.** Assume Hypothesis 3.35. Let \( \mathcal{L}^{(i)}, i = 1, 2 \) be the operators given by the differential expressions introduced in (3.92) on \( E_0 \). We choose \( \rho > 0 \) to satisfy

\[ \sup \{ \text{Re} \lambda : \lambda \in \text{Sp}(L_1^{(2)}) \text{ and } \lambda \neq 0 \} < -\rho. \]

There exists \( K > 0 \) such that, for all \( t \geq 0 \),

\[ \| e^{t \mathcal{L}^{(1)}} \|_{\mathcal{B}(E_0)} \leq K, \]

(3.96)

\[ \| e^{t \mathcal{L}^{(2)}} \|_{\mathcal{B}(E_0)} \leq K e^{-\rho t}, \]

moreover, the operator \( \mathcal{L}^- \) given by the differential expression (3.91) generates a bounded semigroup on \( E_0 \), that is,

\[ \| e^{t \mathcal{L}^-} \|_{\mathcal{B}(E_0)} \leq K \text{ for all } t \geq 0. \]  

(3.97)
Proof. We will use the fact that
\[ e^{t \mathcal{L}^{(i)}} = e^{t (L_1^{(i)} \otimes I_{H^k(\mathbb{R}^{d-1})} + I_{H^k(\mathbb{R})} \otimes \Delta_y})} = e^{t L_1^{(i)}} \otimes e^{t \Delta_y}, \]
for \( i = 1, 2 \), see [RS4, Theorem XIII.35].

By Hypothesis 3.32(1), the operator \( L_1^{(1)} \) generates a bounded semigroup on \( H^k(\mathbb{R}) \), thus we may use Lemma 3.48 to have \( \| e^{t L_1^{(1)}} \otimes e^{t \Delta_y} \| = \| e^{t L_1^{(1)}} \| \| e^{t \Delta_y} \| < K \) for some \( K > 0 \) and all \( t \geq 0 \). Similarly, by using Hypothesis 3.32(2) and Lemma 3.48 we have \( \| e^{t L_1^{(2)}} \otimes e^{t \Delta_y} \| = \| e^{t L_1^{(2)}} \| \| e^{t \Delta_y} \| < Ke^{-\rho t} \) for some \( K > 0 \) and all \( t \geq 0 \).

Let \( \{ S(t) \}_{t \geq 0} \) be the semigroup generated by the operator \( \mathcal{L}^- \) and let \( \{ S_i(t) \}_{t \geq 0}, i = 1, 2 \) be the semigroups generated by the operators \( L_i^{(i)}, i = 1, 2 \). The triangular structure of the operator \( \mathcal{L}^- \) as in (3.93) yields the triangular structure of the semigroup \( \{ S(t) \}_{t \geq 0} \), that is
\[ S(t) = \begin{pmatrix} S_1(t) & Q(t) \\ 0 & S_2(t) \end{pmatrix}, \]
where \( Q(t) = \int_0^t S_1(t-s) \partial u_2 f_1(0) S_2(s) ds \). \( (3.98) \)

Equation (3.98) and inequalities (3.96) imply that
\[ \| S(t) \|_{B(\mathcal{E})} = \| e^{t \mathcal{L}^-} \|_{B(\mathcal{E})} \leq K \]
for all \( t \geq 0 \) as required in (3.97).

We will now derive from Lemma 3.49 and Lemma 3.50 that the semigroup generated by the operator \( \mathcal{L} \) on \( \mathcal{E} \) is also bounded.

Lemma 3.51. Assume Hypothesis 3.35. Let \( \mathcal{L}_\mathcal{E} \) be the operator on \( \mathcal{E} \) given by the differential expressions introduced in (3.41). There exists \( K > 0 \) such that
\[ \| e^{t \mathcal{L}_\mathcal{E}} \|_{B(\mathcal{E})} \leq K \] for all \( t \geq 0 \).
Proof. The operator $L_\mathcal{E}$ is given by the same differential expression (3.41) on $\mathcal{E} = \mathcal{E}_0 \cap \mathcal{E}_\alpha$, and the operator $Q_\mathcal{E}$ is given by restricting $Q$ to $\mathcal{E}$, thus

$$\|e^{t L_\mathcal{E}} Q_\mathcal{E}\|_{B(\mathcal{E}, \mathcal{E}_0)} \leq Ke^{-\nu t}$$

by Lemma 3.49; therefore, it remains to estimate $\|e^{t L_\mathcal{E}} Q_\mathcal{E}\|_{B(\mathcal{E}, \mathcal{E}_0)}$ and $\|e^{t L_\mathcal{E}} P_\mathcal{E}\|_{B(\mathcal{E})}$.

We recall that $\text{ran } Q_\mathcal{E} = \text{ran } L_\mathcal{E} \cap \mathcal{E}_n$, and $Q_\mathcal{E}$ commutes with $L_\mathcal{E}$ and $e^{t L_\mathcal{E}}$. Then the variation of constant formula and (3.94) yield

$$e^{t L_\mathcal{E}} = e^{t L_-} + \int_0^t e^{(t-s) L_-} \left( (df(\phi) - df(0)) \otimes I_{H_k(\mathbb{R}^{d-1})} \right) s L_\mathcal{E} ds.$$

Multiplying by $Q_\mathcal{E}$, estimating the norm in $B(\mathcal{E}, \mathcal{E}_0)$ and using (3.95) and Remark 3.38 yields

$$\|e^{t L_\mathcal{E}} Q_\mathcal{E}\|_{B(\mathcal{E}, \mathcal{E}_0)} \leq \|e^{t L_-}\|_{B(\mathcal{E}_0)} \|Q_\mathcal{E}\|_{B(\mathcal{E}, \mathcal{E}_0)}$$

$$+ \int_0^t \|e^{(t-s) L_-}\|_{B(\mathcal{E}_0)} \|df(\phi) - df(0)\|_{B(\mathcal{E}_0)} \|e^{s L_\mathcal{E}} Q_\mathcal{E}\|_{B(\mathcal{E}_0)} \|Q_\mathcal{E}\|_{B(\mathcal{E}, \mathcal{E}_0)} ds.$$

By using (3.97) and (3.99), we conclude that

$$\|e^{t L_\mathcal{E}} Q_\mathcal{E}\|_{B(\mathcal{E}, \mathcal{E}_0)} \leq K + \int_0^t Ke^{-\nu s} ds \leq K,$$

for some $K > 0$ and all $t \geq 0$. Combined with (3.99) this shows that the semigroup $\{e^{t L_\mathcal{E}} Q_\mathcal{E}\}_{t \geq 0}$ is bounded in $\text{ran } Q_\mathcal{E}$.

We note that $\mathcal{E} = \text{ran } P_\mathcal{E} \oplus \text{ran } Q_\mathcal{E}$ and $e^{t L_\mathcal{E}} = e^{t L_- P_\mathcal{E} \oplus e^{t L_\mathcal{E}} Q_\mathcal{E}}$. In order to finish the proof of Lemma 3.51, we will need to show that the semigroup $\{e^{t L_\mathcal{E}} P_\mathcal{E}\}$ is bounded in $\text{ran } P_\mathcal{E}$, that is,

$$\|e^{t L_\mathcal{E}} P_\mathcal{E}\|_{B(\mathcal{E}_0)} \leq K, \|e^{t L_\mathcal{E}} P_\mathcal{E}\|_{B(\mathcal{E}, \mathcal{E}_0)} \leq K$$

for all $t \geq 0$. 

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Recall that $P_E$ projects onto the kernel of $L$ and $L_\alpha$ on $E_0$ and $E_\alpha$ of the generators of the semigroups $e^{tL}$ and $e^{tL_\alpha}$; thus we have

$$e^{tL_\alpha}P_E = P_E, \quad e^{tL}P_E = P_E,$$

where $P_E \in B(E, E_0)$ and $P_E \in B(E_\alpha)$ as mentioned in Remark 3.38. This implies that

$$\|e^{tL}P_E\|_{B(E, E_0)} = \|P_E\|_{B(E, E_0)} \leq K \quad \text{and} \quad \|e^{tL_\alpha}P_E\|_{B(E_\alpha)} = \|P_E\|_{B(E_\alpha)} \leq K$$

for all $t \geq 0$ thus finishing the proof. 

We will need the following standard estimates, see, e.g., [K2, Lemma 3.2].

**Lemma 3.52.** The semigroup $S_{\Delta y}(t)$ generated by the linear operator $\Delta_y$ for all $t > 0$ satisfies the following decay estimates with some $\beta > 0$:

(a) $\|S_{\Delta y}(t)u\|_{H^k(\mathbb{R}^{d-1})} \leq C\|u\|_{H^k(\mathbb{R}^{d-1})}$,

(b) $\|S_{\Delta y}(t)u\|_{H^k(\mathbb{R}^{d-1})} \leq C(1 + t)^{-(d-1)/4}\|u\|_{L^1(\mathbb{R}^{d-1})} + Ce^{-\beta t}\|u\|_{H^k(\mathbb{R}^{d-1})}$,

(c) $\|\nabla_y S_{\Delta y}(t)u\|_{H^k(\mathbb{R}^{d-1})} \leq Ct^{-1/2}\|u\|_{H^k(\mathbb{R}^{d-1})}$,

(d) $\|\nabla_y S_{\Delta y}(t)u\|_{H^k(\mathbb{R}^{d-1})} \leq C(1 + t)^{-(d+1)/4}\|u\|_{L^1(\mathbb{R}^{d-1})} + Ct^{-1/2}e^{-\beta t}\|u\|_{H^k(\mathbb{R}^{d-1})}$.

After these preliminaries, we are ready to begin the analysis of the system (3.82). We recall formula (3.80) and (3.81) for the nonlinear term $F_1$, $F_2$ and the notation $S_{L_E}(t) = e^{tL_E}$ for the semigroup generated by the operator $L_E$ associated with linearization (3.41) about the front. By the variation of constants formula, the mild solution to (3.82) on ran $Q_E \times H^k(\mathbb{R}^{d-1}) \times H^k(\mathbb{R}^{d-1})$ satisfy the equations

$$v(t) = S_{L_E}(t)v^0 + \int_0^t S_{L_E}(t-s)F_1(v(s), q(s), w(s))ds,$$

(3.100)
\[ q(t) = S_{\Delta_y}(t)q^0 + \int_0^t S_{\Delta_y}(t-s)F_2(v(s), q(s), w(s))ds, \]
\[ w(t) = S_{\Delta_y}(t)w^0 + \int_0^t \nabla_y S_{\Delta_y}(t-s)F_2(v(s), q(s), w(s))ds, \]

where \((v^0, q^0, w^0)\) is the initial condition of (3.82). Note that we used the fact that
\[ S_{\Delta_y}(t) \nabla F_2 = \nabla S_{\Delta_y}(t) F_2 \]
in (3.100).

Next, supplying the proof of [K2, Lemma 3.4], we will show the existence and
uniqueness of the mild solutions of (3.100) in the following proposition, which is an
application of a standard semigroup theory (see [P, Theorem 6.1.4]). In the proof
we will only use the fact that \(L_E\) generates a strongly continuous semigroup, even
though we have proved that \(L_E\) generates a bounded strongly continuous semigroup.

Indeed, since the operator \(L_E\) generates a strongly continuous semigroup, and the
nonlinearities \(F_1\) and \(F_2\) are locally Lipschitz with Lipschitz constant \(C_K\) on the set
of the form \(\{(v, q, w) : \|v\|_E + \|q\|_{H^k} + \|w\|_{H^k} < K\}\), the integrable at \(t = 0\) estimate
in Lemma 3.52 (c) yields the following fact.

**Proposition 3.53.** Assume Hypothesis 3.35 and \(k \geq \frac{d+1}{2}\). For any initial data
\((v^0, q^0, w^0) \in \text{ran } Q_E \times H^k(\mathbb{R}^{d-1}) \times H^k(\mathbb{R}^{d-1})\),

system (3.82) has a unique mild solution (that is, a solution of (3.100)) so that
\[ (v(t), q(t), w(t)) \in \text{ran } Q_E \times H^k(\mathbb{R}^{d-1}) \times H^k(\mathbb{R}^{d-1})d^{-1} \]
in the maximal interval \(0 \leq t < t_{\text{max}}\), where \(0 < t_{\text{max}} \leq \infty\).

**Proof.** For any \(t_0 \geq 0\), let \(V^0 \in \mathcal{X} := \text{ran } Q_E \times H^k(\mathbb{R}^{d-1}) \times H^k(\mathbb{R}^{d-1})d^{-1}\), and define
a mapping \(\mathcal{J} : C([t_0, t_1]; \mathcal{X}) \to C([t_0, t_1]; \mathcal{X})\) by the formula
\[ (\mathcal{J}V)(t) = T(t-t_0)V^0 + \int_{t_0}^t BT(t-s)F(V(s))ds , \ t_0 \leq t \leq t_1, \] (3.101)
where \( t_1 \) will be defined in a minute,

\[
V(t) = (v(t), q(t), w(t)) \in \mathcal{X}, \quad T(t) = S_{L\ell}(t) \oplus S_{\Delta y}(t) \oplus S_{\Delta y}(t), \quad B = I \oplus I \oplus \nabla_y,
\]

with \( \text{ran} \ T(t) \subset \text{dom}(B) \) and

\[
F(V(s)) = (F_1(v, q, w), F_2(v, q, w), F_2(v, q, w)) \quad (3.102)
\]

with \( F_1 \) and \( F_2 \) defined as in (3.82), cf., (3.74), (3.79) and (3.80)

The operator \( L\ell \) generates a strongly continuous semigroup on \( E := E_0 \cap E_\alpha \). By Lemma 3.52 (a) we know that the semigroup \( S_{\Delta y} \) generated by \( \Delta y \) is bounded on \( H^k(\mathbb{R}^{d-1}) \). We then define \( C(t_0) > 0 \) such that \( \| BT(t) \|_{B(\mathcal{X})} \leq C(t_0)t^{-1/2} \) for all \( t \in [t_0, t_0 + 1) \), using Lemma 3.52(c). We also define

\[
M(t_0) = \max\{\sup\{\|T(t)\|_{B(\mathcal{X})} : t_0 \leq t \leq t_0 + 1\}, C(t_0)\}, \quad K(t_0) = 2\|V^0\|_\mathcal{X} M(t_0),
\]

and

\[
\delta(t_0, \|u^0\|_\mathcal{X}) = \min\left\{1, \left(\frac{\|V^0\|_\mathcal{X}}{2(K(t_0)C_{K(t_0)})}\right)^2\right\}, \quad (3.103)
\]

where \( C_{K(t_0)} \) is the local Lipschitz constant of \( F_i, \ i = 1, 2 \) as in Proposition 3.47.

Let \( t_1 = t_0 + \delta(t_0, \|u^0\|) \) where \( \delta(t_0, \|u^0\|) \) is defined by (3.103). We claim that the mapping \( \mathcal{J} \) maps the ball of radius \( K(t_0) \) centered at \( 0 \) of \( C([t_0, t_1]; \mathcal{X}) \) into itself. Indeed, for all \( t \in [t_0, t_1] \)

\[
\| (\mathcal{J} V)(t) \|_\mathcal{X} \leq M(t_0)\|V^0\|_\mathcal{X} + \int_{t_0}^{t} M(t_0)(t - s)^{-1/2}\|F(V(s))\|_\mathcal{X} ds
\]

\[
\leq M(t_0)\|V^0\|_\mathcal{X} + M(t_0)K(t_0)\int_{t_0}^{t} K(t_0)(t - s)^{-1/2} ds
\]

\[
\leq M(t_0) \left(\|V^0\|_\mathcal{X} + 2K(t_0)C_{K(t_0)}(t - t_0)^{1/2}\right)
\]

\[
\leq 2M(t_0)\|V^0\|_\mathcal{X}
\]

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and the ball is invariant under $\mathcal{J}$ as claimed. In this ball, $\mathcal{J}$ satisfies a uniform Lipschitz condition with the Lipschitz constant $L = L(K(t_0), t_0)$. That is, for all $t \in [t_0, t_1]$ we have

$$
||((\mathcal{J}V_1)(t) - (\mathcal{J}V_2)(t))||_X \leq \int_{t_0}^{t} ||BT(t - s)(F(V_1(s)) - F(V_2(s)))||_X ds \quad (3.104)
$$

$$
\leq M(t_0)C_K(t_0)||V_1 - V_2||_{C([t_0, t_1]; X)}(t - t_0)^{1/2}.
$$

Using (3.101) and (3.104) and induction on $n$ it follows that on $C([t_0, t_1]; X)$ for all $t \in [t_0, t_1]$ we have

$$
||((\mathcal{J}^nV_1)(t) - (\mathcal{J}^nV_2)(t))||_X \leq \frac{(2M(t_0)C_K(t_0)(t - t_0)^{1/2})^n}{n!}||V_1 - V_2||_{C([t_0, t_1]; X)}
$$

$$
\leq \frac{(2M(t_0)C_K(t_0))^n}{n!}||V_1 - V_2||_{C([t_0, t_1]; X)}.
$$

For $n$ large enough $\frac{(2M(t_0)C_K(t_0))^n}{n!} < 1$ and by applying the contraction principle we conclude that $\mathcal{J}$ has a unique fixed point $u$ in the ball of radius $K(t_0)$ in $C([t_0, t_1]; X)$. This is the desired solution of the system on the interval $[t_0, t_1]$, and we conclude that the solution to (3.82) exists for at least a short time period.

Let $[0, t_{\text{max}})$ be the maximal interval of existence of the mild solution $V(t)$ of (3.82). We need to prove the uniqueness of the local mild solution $V = V_1(t)$ of (3.82). We note that if $V_2(t)$ is also a mild solution of (3.82) on an interval $[0, t_0]$, then letting

$$
M = \max\{C(t_{\text{max}} - 1), \sup\{||T(t)||_{\mathcal{B}(X)} : 0 < t < t_{\text{max}}\}\},
$$

and

$$
K = \max\{||V_1(t)||_X, ||V_2(t)||_X : t \in (0, t_0)\},
$$

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we infer, for \( t \in (0, t_0] \):

\[
||V_1(t) - V_2(t)||_X \leq ||T(t)V_1^0 - T(t)V_2^0||_X \\
+ \int_0^t ||BT(t-s)(F(V_1(s)) - F(V_2(s)))||_X ds \\
\leq M||V_1^0 - V_2^0||_X + \int_0^t MC_K(t-s)^{-1/2}||V_1(s) - V_2(s)||_X ds.
\]

This implies, by Gronwall’s inequality, that

\[
||V_1(t) - V_2(t)||_X \leq M||V_1^0 - V_2^0||_X e^{\int_0^t MC_K(t-s)^{-1/2} ds} \\
\leq M||V_1^0 - V_2^0||_X e^{2t/4MC_K} \\
\leq M||V_1^0 - V_2^0||_X e^{2t_{\max}/4MC_K},
\]

whenever \( 0 \leq t < t_{\max} \). This yields both the uniqueness of \( V(t) \) and the Lipschitz continuity of the map \( V^0 \to V(t) \) for \( 0 \leq t < t_{\max} \). □

We then consider (3.82) on \( \mathcal{Q}_E \times H^k(\mathbb{R}^{d-1}) \times H^k(\mathbb{R}^{d-1}) \) and, combining Proposition 3.53 and [SY, Theorem 64.2], conclude the following.

**Lemma 3.54.** Assume Hypothesis 3.35. For each \( \delta > 0 \), if \( 0 < \gamma < \delta \), then there exists \( T \), with \( 0 < T \leq \infty \), such that the following is true: if \( (v^0, q^0, w^0) \in \mathcal{Q}_E \times H^k(\mathbb{R}^{d-1}) \times H^k(\mathbb{R}^{d-1}) \) satisfies

\[
||v^0||_{\mathcal{E}} + ||q^0||_{H^k} + ||w^0||_{H^k} \leq \gamma;
\]

and \( 0 \leq t < T \), then the solution \( (v(t), q(t), w(t)) \in \mathcal{Q}_E \times H^k(\mathbb{R}^{d-1}) \times H^k(\mathbb{R}^{d-1}) \) of (3.100) with the initial data \( (v^0, q^0, w^0) \) is defined and satisfies

\[
||v(t)||_{\mathcal{E}} + ||q(t)||_{H^k} + ||w(t)||_{H^k} \leq \delta.
\]
**Definition 3.55.** Let $T(\delta, \gamma)$ denote the supremum of all $T$ such that (3.106) holds for all $0 \leq t < T$ whenever (3.105) is satisfied.

Having established the local in time existence of the solution of (3.82), we will now proceed with the estimates on the algebraic decay and boundness of the solution, see Propositions 3.58 and 3.62. We begin by proving that it suffices to verify the estimates in Proposition 3.58 and 3.62 only for large values of $t$, that is, we will now show the simple fact that, given an initial condition $V^0 = (v^0, q^0, w^0)$, in a small time period the mild solution

$$V = (v, q, w)(t, v^0, q^0, w^0) \in \mathcal{X} := \text{ran} \mathcal{Q}_\mathcal{E} \times H^k(\mathbb{R}^d - 1) \times H^k(\mathbb{R}^{d-1})^{d-1}$$

satisfies estimate (3.107) in the next corollary. Given a $K > 0$, we continue to denote by $C_K$ the Lipschitz constant from Proposition 3.47 for the nonlinearity $F$ from (3.102) on the ball

$$\{V(t) \in \mathcal{X} : \|V(t)\|_X = \|v(t)\|_\mathcal{E} + \|q(t)\|_{H^k(\mathbb{R}^{d-1})} + \|w(t)\|_{H^k(\mathbb{R}^{d-1})} \leq K\}.$$

**Corollary 3.56. Assume Hypothesis 3.35. Given a $K > 0$, there exists a small enough $\delta_0 < K$ such that for any $\gamma$, $\delta$ satisfying $0 < \gamma < \delta < \delta_0$, the mild solution $V(t) = (v(t), q(t), w(t))$ of (3.82) satisfying $\|V(t)\|_X \leq \delta$ on the interval $t \in [0, T(\delta, \gamma))$ is continuous with respect to the initial data $V^0 = (v^0, q^0, w^0)$ satisfying $\|V^0\|_X \leq \gamma$. Moreover, if $T(\delta, \gamma) \leq 1$, then

$$\|V(t)\|_X \leq C(K)\|V^0\|_X \text{ for all } t \in [0, T(\delta, \gamma)), \quad (3.107)$$

where $C(K)$ is a constant that depends on $K$ but is independent of $\delta$ and $\gamma$. 

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Proof. Since the estimate in Lemma 3.52 is integrable at \( t = 0 \), the continuity with respect to initial data is indeed a simple modification of the standard argument, see [SY, Theorem 64.2], as in the proof of Proposition 3.53 above.

We will now show estimate (3.107). Recall that in Proposition 3.53 we let
\[
T(t) = S_{L_{\xi}}(t) \oplus S_{\Delta_y}(t) \oplus S_{\Delta_y}(t), \quad B = I \oplus I \oplus \nabla_y
\]
and
\[
F(V(s)) = (F_1(v,q,w), F_2(v,q,w), F_2(v,q,w)).
\]
We also recall that the semigroup \( \{T(t)\}_{t \geq 0} \) is bounded by Lemma 3.51 and Lemma 3.52, and define \( M = \max\{\sup\{\|T(t)\|_{B(X)} : t \geq 0\}, C\} \), where \( C \) is the constant from Lemma 3.52(c). The variation of constant formula (3.100) and Proposition 3.47 and Lemma 3.54 yield
\[
\|V(t)\|_X \leq M\|V^0\|_X + MC_K\delta \int_0^t (t-s)^{-1/2}\|V(s)\|_X ds
\]
\[
\leq M\|V^0\|_X + 2MC_K\delta \sup_{0 \leq t < T(\delta, \gamma)} \|V(t)\|_X t^{1/2} \quad (\text{noting that } 0 \leq t < T(\delta, \gamma) \leq 1)
\]
\[
\leq M\|V^0\|_X + 2MC_K\delta_0 \sup_{0 \leq t < T(\delta, \gamma)} \|V(t)\|_X,
\]
for all \( t \in [0, T(\delta, \gamma)) \). Thus by choosing \( \delta_0 \leq \min\{K, \frac{1}{4MC_K}\} \), we conclude that for any \( 0 < \delta < \delta_0 \) and \( 0 < \gamma < \delta \), if \( T(\delta, \gamma) \leq 1 \), then \( \|V(t)\|_X \leq C(K)\|V^0\|_X \) for some \( C(K) \) depending on \( K \) for all \( t \in [0, T(\delta, \gamma)) \).

3.2.4 The algebraic decay of solutions

In this subsection we show that the \( \| \cdot \|_\alpha \)-norm of the solution \( v(t) \) of (3.82) decays algebraically as \( t \to \infty \), the \( \| \cdot \|_0 \)-norm of the second component \( v_2(t) \) of \( v(t) \) also decays algebraically as \( t \to \infty \), while the \( \| \cdot \|_0 \)-norm of the first component \( v_1(t) \) of \( v(t) \) is bounded provided the initial value of the solution is chosen close to zero.
Define the quantity $E_k$ for the initial data $(v^0, q^0, w^0) \in (\mathcal{E}, H^k(\mathbb{R}^{d-1}), H^k(\mathbb{R}^{d-1}))$ by

$$E_k = \|v^0\|_{\mathcal{E}} + \|q^0\|_{H^{k+1}(\mathbb{R}^{d-1})} + \|q^0\|_{W^{1,1}(\mathbb{R}^{d-1})}. \quad (3.108)$$

Note that here we assume $q^0 \in H^{k+1}(\mathbb{R}^{d-1})$ and $q^0 \in W^{1,1}(\mathbb{R}^{d-1})$ so that in a short time period for which (3.82) has a solution, $w(t)$ satisfies $w(t) = \nabla_y q(t)$ and $w(t) \in H^k(\mathbb{R}^{d-1})$ and $w(t) \in L^1(\mathbb{R}^{d-1})$.

We will need the following lemma to prove various estimates.

**Lemma 3.57.** Suppose $a, b, c > 0$, then

(1) $\int_0^{t/2} (1 + t - s)^{-b}(1 + s)^{-c} ds \leq (1 + t)^{-a}$, if $a \leq b$, $a \leq b + c - 1$, $c \neq 1$; or if $a < b$, $a \leq b + c - 1$, $c = 1$,

(2) $\int_{t/2}^t (1 + t - s)^{-b}(1 + s)^{-c} ds \leq (1 + t)^{-a}$, if $a \leq c$, $a \leq b + c - 1$, $b \neq 1$; or if $a < c$, $a \leq b + c - 1$, $b = 1$,

(3) $\int_0^t e^{-b(t-s)}(1 + s)^{-c} ds \leq (1 + t)^{-c}$.

The proof by a direct computation can be found in [X1].

We now show that the $\|\cdot\|_{\alpha}$-norm of $v(t)$ and the $\|\cdot\|_{H^k(\mathbb{R}^{d-1})}$-norms of $q(t)$ and $w(t)$, in fact, decay to zero algebraically as long as $t$ grows but the $\|\cdot\|_{\mathcal{E}}$-norm of $v(t)$ and the $\|\cdot\|_{H^k(\mathbb{R}^{d-1})}$ norms of $q(t)$ and $w(t)$ remain small. Recall notation (3.108) and the definition of $T(\delta, \gamma)$ given in Definition 3.55.

**Proposition 3.58.** Assume Hypothesis 3.35 and $k \geq \left[\frac{d+1}{2}\right]$. Choose $\nu > 0$ as in Lemma 3.49. There exist $\delta_1 > 0$ and $K_1 > 0$ such that for every $\delta \in (0, \delta_1)$ and every
\( \gamma \) with 0 < \( \gamma < \delta \), the following is true: If \( E_k < \gamma \), then the solution \((v(t), q(t), w(t))\) of (3.82) with the initial data \((v^0, q^0, w^0)\) for \( t \in [0, T(\delta, \gamma)) \), satisfies the estimates

\[
\|v(t)\|_\alpha \leq K_1 (1 + t)^{-(d+1)/2} E_k,
\]

\[
\|q(t)\|_{H^k} \leq K_1 (1 + t)^{-(d-1)/4} E_k,
\]

\[
\|w(t)\|_{H^k} \leq K_1 (1 + t)^{-(d+1)/4} E_k.
\]

**Proof.** In Corollary 3.56 we have discussed the behavior of the solution of (3.82) in a small time period \( t \in [0, T(\delta, \gamma)) \) when \( T(\delta, \gamma) \leq 1 \), therefore, without loss of generality, we may assume that \( T(\delta, \gamma) > 1 \) and \( t > 1 \) is large in this proof.

We recall that \( \text{ran} \, Q_\varepsilon = \text{ran} \, L_\alpha \cap \mathcal{E}^\alpha \), thus for \( v \in \text{ran} \, Q_\varepsilon \) we can replace \( L_\varepsilon \) by \( L_\alpha \) in the first equation in (3.100). Applying the semigroup estimates given in Lemma 3.52 and Lemma 3.49 to equations (3.100) yields

\[
\|v(t)\|_\alpha \leq C e^{-\nu t} \|v^0\|_\alpha + C \int_0^t e^{-\nu (t-s)} \|F_1(v(s), q(s), w(s))\|_\alpha ds,
\]

(3.109)

\[
\|q(t)\|_{H^k} \leq C \left((1 + t)^{-(d-1)/4} \|q^0\|_{L^1(\mathbb{R}^{d-1})} + e^{-\beta t} \|q^0\|_{H^k(\mathbb{R}^{d-1})}\right)
+ C \int_0^t e^{-\beta (t-s)} \|F_2(v(s), q(s), w(s))\|_{H^k(\mathbb{R}^{d-1})} ds
+ C \int_0^t (1 + t - s)^{-(d-1)/4} \|F_2(v(s), q(s), w(s))\|_{L^1(\mathbb{R}^{d-1})} ds,
\]

\[
\|w(t)\|_{H^k} \leq C \left((1 + t)^{-(d+1)/4} \|q^0\|_{L^1(\mathbb{R}^{d-1})} + t^{-1/2} e^{-\beta t} \|q^0\|_{H^k(\mathbb{R}^{d-1})}\right)
+ C \int_0^t (t-s)^{-1/2} e^{-\beta (t-s)} \|F_2(v(s), q(s), w(s))\|_{H^k(\mathbb{R}^{d-1})} ds
+ C \int_0^t (1 + t - s)^{-(d+1)/4} \|F_2(v(s), q(s), w(s))\|_{L^1(\mathbb{R}^{d-1})} ds.
\]

For \( t > 1 \), there exist some \( C > 0 \) such that

\[
e^{-\nu t} \|v^0\|_\alpha \leq C e^{-\nu t} E_k,
\]

\[
(1 + t)^{-(d-1)/4} \|q^0\|_{L^1(\mathbb{R}^{d-1})} + e^{-\beta t} \|q^0\|_{H^k(\mathbb{R}^{d-1})} \leq C (1 + t)^{-(d-1)/4} E_k,
\]

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Pick any $\delta' > 0$. For $0 < \gamma < \delta'$, if $E_k < \gamma$ then, by Lemma 3.54,

$$||v(s)||_E + ||q(s)||_{H^k} + ||w(s)||_{H^k} < \delta'$$

for all $s \in [0, T(\delta', \gamma))$.

In this bounded set Proposition 3.47 (b) and (c) state that the norms of the nonlinear terms $F_1((v(s), q(s), w(s)))$ and $F_2((v(s), q(s), w(s)))$ can be estimated as follows:

$$\|F_1(v(s), q(s), w(s))\|_\alpha$$

$$\leq C\delta'(||v(s)||_0||v(s)||_\alpha + ||q(s)||_{H^k}||v(s)||_\alpha + ||w(s)||_{H^k}^2),$$

$$\|F_2(v(s), q(s), w(s))\|_{H^k}$$

$$\leq C\delta'(||v(s)||_0||v(s)||_\alpha + ||q(s)||_{H^k}||v(s)||_\alpha + ||w(s)||_{H^k}^2).$$

Then the inequalities (3.109) can rewritten as follows:

$$||v(t)||_\alpha \leq Ce^{-\nu t}E_k$$

$$+ CC\delta' \int_0^t e^{-\nu(t-s)}(||v(s)||_0||v(s)||_\alpha + ||q(s)||_{H^k}||v(s)||_\alpha + ||w(s)||_{H^k}^2)ds, $$

$$||q(t)||_{H^k} \leq C(1 + t)^{-(d-1)/4}E_k$$

$$+ CC\delta' \int_0^t e^{-\beta(t-s)}(||v(s)||_0||v(s)||_\alpha + ||q(s)||_{H^k}||v(s)||_\alpha + ||w(s)||_{H^k}^2)ds$$

$$+ CC\delta' \int_0^t (1 + t - s)^{-(d-1)/4}(||v(s)||_0||v(s)||_\alpha + ||q(s)||_{H^k}||v(s)||_\alpha$$

$$+ ||w(s)||_{H^k}^2)ds,$$

$$||w(t)||_{H^k} \leq C(1 + t)^{-(d+1)/4}E_k$$

$$+ CC\delta' \int_0^t (t - s)^{-1/2}e^{-\beta(t-s)}(||v(s)||_0||v(s)||_\alpha + ||q(s)||_{H^k}||v(s)||_\alpha$$

$$+ ||w(s)||_{H^k}^2)ds + CC\delta' \int_0^t (1 + t - s)^{-(d+1)/4}(||v(s)||_0||v(s)||_\alpha$$

$$+ ||q(s)||_{H^k}||v(s)||_\alpha + ||w(s)||_{H^k}^2)ds.$$
For each $\delta < \delta'$, and $0 < \gamma < \delta$, if $E_k < \gamma$, by Lemma 3.54 we have

$$||v(s)||_\alpha \leq ||v(s)||_E < \delta, \text{ for all } s \in (0, T(\delta, \gamma)),$$

thus

$$||v(t)||_\alpha \leq Ce^{-\nu t} E_k$$

$$(3.110)$$

$$+ CC_{\delta'} \int_0^t \frac{e^{-\nu(t-s)}(\delta||v(s)||_\alpha + ||v(s)||_\alpha ||q(s)||_{H^{k}} + ||w(s)||_{H^{k}}^2)ds,}$$

$$||q(t)||_{H^{k}} \leq C(1+t)^{-(d-1)/4} E_k + CC_{\delta'} \int_0^t e^{-\beta(t-s)}(\delta||v(s)||_\alpha$$

$$+ ||v(s)||_\alpha ||q(s)||_{H^{k}} + ||w(s)||_{H^{k}}^2)ds + CC_{\delta'} \int_0^t (1 + t - s)^{-(d-1)/4}$$

$$\left(\delta||v(s)||_\alpha + ||v(s)||_\alpha ||q(s)||_{H^{k}} + ||w(s)||_{H^{k}}^2\right)ds,$$

$$||w(t)||_{H^{k}} \leq C(1+t)^{-(d+1)/4} E_k + CC_{\delta'} \int_0^t (t - s)^{-1/2} e^{-\beta(t-s)}(\delta||v(s)||_\alpha$$

$$+ ||v(s)||_\alpha ||q(s)||_{H^{k}} + ||w(s)||_{H^{k}}^2)ds + CC_{\delta'} \int_0^t (1 + t - s)^{-(d+1)/4}$$

$$\left(\delta||v(s)||_\alpha + ||v(s)||_\alpha ||q(s)||_{H^{k}} + ||w(s)||_{H^{k}}^2\right)ds.$$

We now define

$$M_v(t) = \sup_{0<s\leq t} (1+s)^{(d+1)/2}||v(s)||_\alpha,$$

$$M_q(t) = \sup_{0<s\leq t} (1+s)^{(d-1)/4}||q(s)||_{H^{k}},$$

$$M_w(t) = \sup_{0<s\leq t} (1+s)^{(d+1)/4}||w(s)||_{H^{k}}.$$

Using the above definitions, (3.110) can be rewritten as follows:

$$||v(t)||_\alpha \leq Ce^{-\nu t} E_k + CC_{\delta'} M_v(t) \int_0^t e^{-\nu(t-s)}(1 + s)^{-(d+1)/2}ds$$

$$+ CC_{\delta'} M_v(t) M_q(t) \int_0^t e^{-\nu(t-s)}(1 + s)^{-3(d+1)/4}ds$$

$$+ CC_{\delta'} M^2_w(t) \int_0^t e^{-\nu(t-s)}(1 + s)^{-(d+1)/2}ds.$$
\[ \|q(t)\|_{H^k} \leq C (1 + t)^{-(d-1)/4} E_k + CC_{\delta'} \delta M_v(t) \int_0^t e^{-\beta(t-s)} (1 + s)^{-(d+1)/2} ds \\
+ CC_{\delta'} M_v(t) M_q(t) \int_0^t e^{-\beta(t-s)} (1 + s)^{-(3d+1)/4} ds \\
+ CC_{\delta'} M^2_v(t) \int_0^t e^{-\beta(t-s)} (1 + s)^{-(d+1)/2} ds \\
+ CC_{\delta'} \delta M_v(t) \int_0^t (1 + t - s)^{-(d-1)/4} (1 + s)^{-(d+1)/2} ds \\
+ CC_{\delta'} M_v(t) M_q(t) \int_0^t (1 + t - s)^{-(d-1)/4} (1 + s)^{-(3d+1)/4} ds \\
+ CC_{\delta'} M^2_v(t) \int_0^t (1 + t - s)^{-(d-1)/4} (1 + s)^{-(d+1)/2} ds, \]

\[ \|w(t)\|_{H^k} \leq C (1 + t)^{-(d-1)/4} E_k \\
+ CC_{\delta'} \delta M_v(t) \int_0^t (t - s)^{-1/2} e^{-\beta(t-s)} (1 + s)^{-(d+1)/2} ds \\
+ CC_{\delta'} M_v(t) M_q(t) \int_0^t (t - s)^{-1/2} e^{-\beta(t-s)} (1 + s)^{-(3d+1)/4} ds \\
+ CC_{\delta'} M^2_v(t) \int_0^t (t - s)^{-1/2} e^{-\beta(t-s)} (1 + s)^{-(d+1)/2} ds \\
+ CC_{\delta'} \delta M_v(t) \int_0^t (1 + t - s)^{-(d+1)/4} (1 + s)^{-(d+1)/2} ds \\
+ CC_{\delta'} M_v(t) M_q(t) \int_0^t (1 + t - s)^{-(d+1)/4} (1 + s)^{-(3d+1)/4} ds \\
+ CC_{\delta'} M^2_v(t) \int_0^t (1 + t - s)^{-(d+1)/4} (1 + s)^{-(d+1)/2} ds. \]

Estimating the integrals by Lemma 3.57 one can get

\[ \|v(t)\|_{\alpha} \leq Ce^{-\nu t} E_k + CC_{\delta'} \delta M_v(t) (1 + t)^{-(d+1)/2} \\
+ CC_{\delta'} M_v(t) M_q(t) (1 + t)^{-(3d+1)/4} + CC_{\delta'} M^2_v(t) (1 + t)^{-(d+1)/2}, \]

\[ \|q(t)\|_{H^k} \leq C (1 + t)^{-(d-1)/4} E_k + CC_{\delta'} \delta M_v(t) (1 + t)^{-(d-1)/4} \\
+ CC_{\delta'} M_v(t) M_q(t) (1 + t)^{-(d-1)/4} + CC_{\delta'} M^2_v(t) (1 + t)^{-(d-1)/4}, \]

\[ \|w(t)\|_{H^k} \leq C (1 + t)^{-(d+1)/4} E_k + CC_{\delta'} \delta M_v(t) (1 + t)^{-(d+1)/4} \\
+ CC_{\delta'} M_v(t) M_q(t) (1 + t)^{-(d+1)/4} + CC_{\delta'} M^2_v(t) (1 + t)^{-(d+1)/4}. \]
Multiplying the above inequalities by \((1 + t)^{(d+1)/2}\), \((1 + t)^{(d-1)/4}\) and \((1 + t)^{(d+1)/4}\), respectively, one has for some \(C > 0\)

\[
(1 + t)^{(d+1)/2}\|v(t)\|_\alpha \leq C(1 + t)^{(d+1)/2}e^{-\nu t}E_k + C\delta M_v(t) + CM_v(t)M_q(t)(1 + t)^{-(d-1)/4} + CM^2_v(t),
\]

\[
(1 + t)^{(d-1)/4}\|q(t)\|_{H^k} \leq CE_k + C\delta M_v(t) + CM_v(t)M_q(t) + CM^2_v(t),
\]

\[
(1 + t)^{(d+1)/4}\|w(t)\|_{H^k} \leq CE_k + C\delta M_v(t) + CM_v(t)M_q(t) + CM^2_v(t).
\]

Since each of the functions \(M_v(t), M_q(t), M_w(t)\) is an increasing function of \(t\), it can be concluded that for \(t \in [1, T(\gamma, \delta))\),

\[
M_v(t) \leq CE_k + C(\delta M_v(t) + M_v(t)M_q(t) + M^2_w(t)),
\]

\[
M_q(t) \leq CE_k + C(\delta M_v(t) + M_v(t)M_q(t) + M^2_w(t)),
\]

\[
M_w(t) \leq CE_k + C(\delta M_v(t) + M_v(t)M_q(t) + M^2_w(t)). \tag{3.111}
\]

Now let \(M(t) = M_v(t) + M_q(t) + M_w(t)\). Then by (3.111), we infer that for all \(t \in [1, T(\gamma, \delta))\), we have the quadratic estimate:

\[
M(t) \leq CE_k + C\delta M(t) + CM^2(t),
\]

for some \(C > 0\) that depends on \(\delta'\). Note that by Corollary 3.56 \(M(t) \leq C(\delta')E_k\) for \(0 \leq t \leq 1\) and some constant \(C(\delta') > 0\) which depends on \(\delta'\). Choose \(\delta_1 \leq \min\{1/2C, \delta'\}\) and \(0 < \gamma < \delta < \delta_1\), then absorbing the term \(1/2M(t)\) into the left-hand side, we have the final quadratic inequality for \(M(t)\)

\[
M(t) \leq 2CE_k + 2CM^2(t) \quad \text{for all } t \in [0, T(\delta, \gamma)).
\]

Since this inequality holds for all \(t \in [0, T(\delta, \gamma))\), by continuity of \(M(\cdot)\), the expression \(M(t)\) can not “jump” over the first root of the respective quadratic equation. This
root, in turn, can be controlled by $K_1E_k$ as long as $E_k$ is kept sufficiently small. Indeed, let $M_1 = \frac{1 - \sqrt{1 - 16C^2E_k}}{4C}$ denote the first root of the equation $2CM^2 - M + 2CE_k = 0$. If $E_k < 1/16C^2$ then

$$M_1 = \frac{16C^2E_k}{4C(1 + \sqrt{1 - 16C^2E_k})} < 4CE_k.$$  

Since $M(t)$ is continuous in $t$ and $M(0) = E_k$, it follows that if

$$\delta_1 \leq \min\{\delta', 1/2C, 1/16C^2\},$$

then for all $\delta \in (0, \delta_1)$ and $0 < E_k \leq \gamma < \delta$, see Lemma 3.54, we have the inequality $M(t) \leq M_1 \leq K_1E_k$ for some some $K_1 > 0$ and all $t \in [0, T(\delta, \gamma))$. ■

Next, we will show that the 0-norm of the solution $v(t)$ remains bounded for large $t$. Together with the decay of the $\alpha$-norm for large $t$ this implies smallness of $\mathcal{E}$-norm of the solution provided the initial conditions are small. But as long as this happens, we can again prove decay, leading to a bootstrap argument used in our main Theorem 3.63 proved below.

The following heuristic comments is in ordr. Since $\alpha_+ \geq 0$, the weight function $\gamma_\alpha(z)$ is bounded away from 0 for large $z$. Thus, to establish the decay of $\|v(t, \cdot, \cdot)\|_\alpha$, it is enough to prove the decay of the solution near $z = \infty$, but it is not enough to prove the decay at the left end, i.e., it could be possible that $\|v(t, \cdot, \cdot)\|_{0, -\infty} \to \infty$ near $z = -\infty$ even though $\|v(t, \cdot, \cdot)\|_{\alpha, -\infty}$ is algebraically decaying since $\gamma_\alpha(z) \to 0$ as $z \to -\infty$. It is because of this, we need to use the “product-triangular” structure of the nonlinearity to show the boundedness in the unweighted norm.

We start by formulating the following three lemmas, whose proofs resemble the proofs in [GLS, Lemma 8.1, Lemma 8.2 and Lemma 8.3].

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Lemma 3.59. Assume Hypothesis 3.63 and $k \geq \frac{[d+1]}{2}$. Then the entries of the matrix-valued function $(df(\phi) - df(0))\gamma^{-1}_\alpha$ belong to $H^k(\mathbb{R})$.

Proof. This follows from

$$(df(\phi) - df(0))\gamma^{-1}_\alpha = \left( \int_0^1 d^2 f(s\phi)ds \right) \phi\gamma^{-1}_\alpha$$

where $f(\cdot)$ is a $C^{k+3}$ smooth function by Hypothesis 3.25 and $\phi\gamma^{-1}_\alpha \in H^k(\mathbb{R})$ using Lemma 3.30(1).

We will now use Lemma 3.42 to prove an analogue of Proposition 3.43(1) with $\|v\|_0$ in the right-hand side replaced by $\|v\|_\alpha$ and Proposition 3.45(1) with $\| \cdot \|_\alpha$ in the left-hand side replaced by $\| \cdot \|_0$. We recall that $\phi = \phi(z)$ and that the function $(z, y) \mapsto (df(\phi(z) - q(y)) - df(0))v(y)$ is in $\mathcal{E}_0 = H^k(\mathbb{R}^d)$.

Lemma 3.60. Assume Hypothesis 3.35 and $k \geq \frac{d+1}{2}$. For each $k > 0$, there is a constant $C_K > 0$ such that if $q \in H^k(\mathbb{R}^d)$ and $v \in \mathcal{E}_\alpha$ satisfy $\|v\|_\mathcal{E} + \|q\|_{H^k} \leq K$, then

1. $\|(df(\phi) - df(0))v\|_0 \leq C_K \|v\|_\alpha$;
2. $\|(df(\phi_q) - df(\phi))v\|_0 \leq C_K \|q\|_{H^k} \|v\|_\alpha$.
3. For $(v, q)$ in a bounded neighborhood of $(0, 0)$ in $\mathcal{E}^n \times H^k(\mathbb{R}^{d-1})$, and $v = (v_1, v_2)^T$ with $v_i \in \mathcal{E}^{n_i}$, $i = 1, 2$, one has

$$\|N(\phi_q, v)v\|_0 \leq C_K \|v\|_0(\|v\|_\alpha + \|v_2\|_0),$$

where $N(\cdot, \cdot)$ is defined in equation (3.68).

Proof. Lemma 3.59 and (3.18) yield (1) since

$$\|(df(\phi) - df(0))v\|_0 \leq \|(df(\phi) - df(0))\gamma^{-1}_\alpha\|_{H^k(\mathbb{R})} \|\gamma v\|_0 \leq C_K \|v\|_\alpha.$$
To see (2), we write, as in (3.87),
\[
(df(\phi_q) - df(\phi)) v = - \int_0^1 \frac{d^2 f(\phi(z - sq(y)))}{ds} (\phi'(z - sq(y))q, v) \, ds 
= - \int_0^1 \frac{d^2 f(\phi(z - sq(y)))}{ds} (\gamma_{\alpha}^{-1} \phi'(z - sq(y))q, \gamma_{\alpha}v) \, ds.
\] (3.112)

We will be using an argument similar to Lemma 3.42 to prove that
\[
\left\| (df(\phi_q) - df(\phi)) v \right\|_0 \leq C_K ||q||_{H^k} ||v||_\alpha.
\] (3.113)

Indeed, let us indicate the main steps in proving (3.113).

Passing to the components in the right-hand side of the vector equation (3.112), the problem of proving (3.113) is reduced to proving the inequality
\[
\left\| \sigma \right\|_{H^k(\mathbb{R}^d)} \leq C ||q||_{H^k(\mathbb{R}^{d-1})}.
\] (3.114)

where \(\sigma(z, y) = \gamma^{-1}_{\alpha}(z) \psi'(z - q(y))q(y), x = (z, y) \in \mathbb{R}^d,\) and \(\psi\) is the function as in Lemma 3.42 with exponentially decaying derivatives. Indeed, as soon as (3.114) is proved, the inequality \(\left\| \phi'(\cdot - sq(\cdot))q(\cdot) v(\cdot) \right\|_{H^k(\mathbb{R}^d)} \leq \left\| \sigma \right\|_{H^k(\mathbb{R}^d)} \left\| v \right\|_{H^k(\mathbb{R}^d)}\) yields (3.113) by (3.112).

To begin the proof of (3.114), we denote \(m(x) = \gamma_{\alpha}(z - q(y))\) so that \(\sigma(x) = m(x)(\gamma_{\alpha}^{-1} \psi')(z - q(y))q(y).\) We note that \(\psi_1(z) = \gamma^{-1}_{\alpha}(z) \psi'(z)\) exponentially decays at \(z \to \pm \infty.\) Using \(q \in H^k(\mathbb{R}^{d-1}) \hookrightarrow L^{\infty}(\mathbb{R}^{d-1}),\) and formula (3.44) for \(\gamma_{\alpha}(z),\) we conclude that \(m(x) = e^{-\alpha - q(y)}\) for \(z \leq -r\) and \(m(x) = e^{-\alpha + q(y)}\) for \(z \geq r\) for some large \(r > 0\) uniformly in \(y \in \mathbb{R}^{d-1};\) moreover, \(m \in L^{\infty}(\mathbb{R}^d).\) Similarly to the calculation in (3.84), the \(L^2(\mathbb{R}^d)\)-norm of \(\sigma\) can be estimated as
\[
\left\| \sigma \right\|_{L^2(\mathbb{R}^d)} \leq \left\| m \right\|_{L^{\infty}(\mathbb{R}^d)}, \quad \left\| (\gamma_{\alpha}^{-1} \psi')(\cdot - q(\cdot))q(\cdot) \right\|_{L^2(\mathbb{R}^d)} \leq C \left\| q \right\|_{H^k(\mathbb{R}^d)}.
\] (3.115)
We will now show how to estimate the \(L^2(\mathbb{R}^d)\)-norm of the derivatives of \(\sigma\). The \(z\)-derivative,
\[
\frac{\partial \sigma}{\partial z} = (\gamma^{-1}_\alpha)'(z)\psi'(z - q(y))q(y) + \gamma^{-1}_\alpha(z)\psi''(z - q(y))q(y),
\]
(3.116)
is the sum of two terms that can handled similarly to (3.115). Taking derivatives with respect to \(x_j, j = 2, \ldots, d\), yields, as in (3.85),
\[
\frac{\partial \sigma}{\partial x_j} = \gamma^{-1}_\alpha \psi''(z - q(y)) \frac{\partial q}{\partial x_j} q(y) + \gamma^{-1}_\alpha(z)\psi'(z - q(y)) \frac{\partial q}{\partial x_j} q(y),
\]
(3.117)
The \(L^2(\mathbb{R}^d)\)-norm of the first term can be estimated as in (3.86) and (3.115), that is,
\[
\|\gamma^{-1}_\alpha(\cdot)\psi''(\cdot - q(\cdot)) \frac{\partial q}{\partial x_j} q\|_{L^2(\mathbb{R}^d)} \leq m \|q\|_{L^\infty(\mathbb{R}^d)} \|\gamma^{-1}_\alpha(\cdot)\psi''(\cdot - q(\cdot)) \frac{\partial q}{\partial x_j} q\|_{L^2(\mathbb{R}^d)}
\]
\[
\leq C \|\frac{\partial q}{\partial x_j} q\|_{L^2(\mathbb{R}^{d-1})}
\]
\[
\leq C \|q\|_{L^\infty(\mathbb{R}^{d-1})} \|\frac{\partial q}{\partial x_j} q\|_{L^2(\mathbb{R}^{d-1})}
\]
\[
\leq C \|q\|_{H^k(\mathbb{R}^{d-1})} \|q\|_{H^k(\mathbb{R}^{d-1})}.
\]
A similar (easier) calculation works for the second term in (3.117). This proves assertion (3.114) for \(k = 1\). Higher order derivatives are handled similarly as in the proof of Proposition A.3.

This concludes the proof of assertion (2) in the lemma.

To prove (3), we will recall the following convenient representation of the nonlinearity \(v = (v_1, v_2)^T \mapsto N(\phi_q, v) v\) borrowed from the proof of [GLS, Lemma 8.3],
\[
N(\phi_q, v) v = I_1(v) + I_2(v) + I_3(v) + I_4(v) + I_5(v),
\]
where \(\phi_q = (\phi_1(z - q), \phi_2(z - q))^T = (\phi_{1,q}, \phi_{2,q})^T, v = (v_1, v_2)^T\),
\[
I_1(v) = \int_0^1 (\partial_{u_1} r(\phi_q + tv) - \partial_{u_1} r(\phi_q)) v_1 \phi_{2,q} dt,
\]
(3.118)
\[ I_2(v) = \int_0^1 (\partial_{u_1} r(\phi_q + tv)v_1 tv_2 dt, \]
\[ I_3(v) = \int_0^1 (\partial_{u_2} r(\phi_q + tv) - \partial_{u_2} r(\phi_q)) v_2 \phi_{2,q} dt, \]
\[ I_4(v) = \int_0^1 (\partial_{u_2} r(\phi_q + tv)v_2) tv_2 dt, \]
\[ I_5(v) = \int_0^1 (r(\phi_q + tv) - r(\phi_q)) v_2 dt, \]

and the \( n \times n \) matrix valued \( C^k \) function \( r = r(u_1, u_2) \) is given by

\[
r(u_1, u_2) = \int_0^1 \partial_{u_2} f(u_1, su_2) ds.
\]

The proof of the required estimates for each \( I_j, j = 1, 2, ..., 5 \) is similar to the proof of assertion (2) above and use Lemma 3.30. For instance, for \( j = 1 \), passing in the integral to the third derivative of \( f \) (which is a \( C^k \)-bounded function by Hypothesis 3.25), we will reduce the problem to making an estimate for the \( H^k(\mathbb{R}^d) \)-norm of the function \( vv_1 \phi_{2,q} \). Writing \( v_1 \phi_{2,q} = (\gamma_\alpha v_1)(\gamma_\alpha^{-1} \phi_{2,q}) \) and using that \( H^k(\mathbb{R}^d) \) is an algebra, in order to prove that

\[
\|I_1(v)\|_0 \leq C\|v\|_0\|v_1\|_\alpha,
\] (3.119)

it is enough to show that the \( H^k(\mathbb{R}^d) \)-norm of \( \sigma(z, y) = \gamma_\alpha^{-1}(z)\phi_2(z - q(y))w(x) \) with \( w = \gamma_\alpha v_1 \) is bounded by \( C\|w\|_{H^k(\mathbb{R}^d)} \). This follows because \( q \in H^k(\mathbb{R}^{d-1}) \hookrightarrow L^\infty(\mathbb{R}^{d-1}) \) yields the existence of a large \( r > 0 \) such that, uniformly for \( y \in \mathbb{R}^{d-1} \), we have

\[
|\gamma_\alpha^{-1}(z)\phi_2(z - q(y))| \leq \begin{cases} K e^{-\alpha z} e^{-\omega_+(z-q(y))}, & z \leq -r, \\ K e^{-\alpha z} (|\phi_2| + e^{-\omega_+(z-q(y))}), & z \leq -r. \end{cases}
\]

Using \( e^q \in L^\infty(\mathbb{R}^{d-1}) \) and Hypothesis 3.29 we conclude that \( \gamma_\alpha^{-1}(\cdot)\phi_2(\cdot - q(\cdot)) \) is bounded. A similar argument, as in the proof of (2) above, applies for the derivatives
of $\sigma$. This completes the proof of 3.119. For $j = 2, \ldots, 5$ the estimates $\| I_j(v) \|_0 \leq C \| v \|_0 \| v_2 \|_0$ are straightforward since each integral has a factor $v_2$ and both derivatives of $f$ and $\phi_{2,q}$ are $k$-smooth with bounded derivatives. Combining the estimates for $j = 1, \ldots, k$ yields assertion (3).

The next lemma gives an analogue of the estimate in Proposition 3.47(b) with $\| \cdot \|_\alpha$ in the left-hand side replaced by $\| \cdot \|_0$.

**Lemma 3.61.** Assume Hypothesis 3.35 and $k \geq d+1$ \textsuperscript{2}. For each $K > 0$ there is a constant $C_K > 0$ such that if $(v, q, w) \in E_\alpha \times H^k(\mathbb{R}^d) \times H^k(\mathbb{R}^d)$ is the solution of (3.38) and $\| v \|_E + \| q \|_{H^k} + \| w \|_{H^k} \leq K$, then

$$\| F_1(v, q, w) \|_0 \leq C_K(\| v \|_0 \| v \|_\alpha + \| v \|_0 \| v_2 \|_0 + \| q \|_{H^k} \| v \|_\alpha + \| w \|_{H^k}^2).$$

**Proof.** We recall formula (3.81) for $F_1(v, q, w)$, and use Lemma 3.42 for the estimate of the second term in this formula,

$$\| \phi'_q F_2(v, q, w) \|_0 \leq C(\| v \|_0 \| v \|_\alpha + \| q \|_{H^k} \| v \|_\alpha + \| w \|_{H^k}^2).$$

Analogously, the third term in (3.81) is estimated as

$$\| \phi''_q (w \cdot w) \|_0 \leq C \| w \|_{H^k}^2,$$

the proofs are presented in the proof of Proposition 3.47(b). It remains to show the $\| \cdot \|_0$-norm estimate of $G(v, q) = (df(\phi_q) - df(\phi))v + N(\phi_q, v)v$. Indeed, this estimate follows from Lemma 3.60 (2) and (3):

$$\| G(v, q) \|_0 \leq \| (df(\phi_q) - df(\phi))v \|_0 + \| N(\phi_q, v)v \|_0 \leq K(\| q \|_{H^k} \| v \|_\alpha + \| v \|_0 \| v \|_\alpha + \| v \|_0 \| v_2 \|_0).$$
Adding the above estimates of the three terms of (3.81) finishes the proof. We recall notation (3.108) and the definition of $T(\delta, \gamma)$ given in Definition 3.55.

We are ready to prove the analogue of Proposition 3.58 for $\| \cdot \|_0$-norm.

**Proposition 3.62.** Assume Hypothesis 3.35 and $k \geq \lceil \frac{d+1}{2} \rceil$. Choose $\rho > 0$ to satisfy
\[
\sup \{ \text{Re} \lambda : \lambda \in \text{Sp}(L^{(2)}_1) \} < -\rho,
\]
and $\delta_1$ as indicated in Proposition 3.58. There exist $\delta_2 \in (0, \delta_1)$ and $K_2 > 0$ such that for every $\delta \in (0, \delta_2)$ and every $\gamma$ with $0 < \gamma < \delta$, the following is true: if $E_k \leq \gamma$, then the solution to (3.82) for $t \in [0, T(\delta, \gamma))$ satisfies the estimates
\[
\| v_1(t) \|_0 \leq K_2 E_k; \quad (3.120)
\]
\[
\| v_2(t) \|_0 \leq K_2 (1 + t)^{-(d+1)/2} E_k. \quad (3.121)
\]

**Proof.** Using (3.51), we write the first equation in (3.82) as follows,
\[
\partial_t v_1 = L^{(1)} v_1 + du_2 f(0, 0) v_2 + H_1(q,v_1,v_2), \quad (3.122)
\]
\[
\partial_t v_2 = L^{(2)} v_2 + H_2(q,v_1,v_2), \quad (3.123)
\]
where we split $f = (f_1, f_2)^T$ and introduce the nonlinearity $H = (H_1, H_2)^T$ by the formula
\[
\begin{pmatrix}
H_1(v_1, v_2, q, w) \\
H_2(v_1, v_2, q, w)
\end{pmatrix} = F_1(v, q, w) + (df(\phi) - df(0)) v.
\]
Since $(v_1, v_2, q, w)(t)$ is a fixed solution of (3.38) in $\mathcal{E}_n \times H^k(\mathbb{R}^{d-1}) \times H^k(\mathbb{R}^{d-1})$, we may regard (3.122)-(3.123) as a nonautonomous linear system on $\mathcal{E}_{0}^{n}$. The mild solutions of (3.122) and (3.123) satisfy the following system of integral equations,
\[
v_1(t) = e^{t L^{(1)}_1} v_1^0 + \int_0^t e^{(t-s) L^{(1)}_1} (du_2 f(0, 0) v_2(s) + H_1(v(s), q(s), w(s))) \, ds,
\]
\[\text{(3.124)}\]
\[ v_2(t) = e^{tL_2} v_2^0 + \int_0^t e^{(t-s)L_2} H_2(v(s), q(s), w(s)) \, ds. \]  

(3.125)

As in the proof of Proposition 3.58, we may assume that \( t \in [1, T(\delta, \gamma)) \) and large.

By Lemma 3.50, we know that

\[ \|e^{tL_2}\|_{\mathcal{B}(E_0)} \leq K e^{-\rho t}. \]

By the definition of \( T(\delta, \gamma) \), for \( 0 < \delta < \delta_1 \), if \( 0 < \gamma < \delta \) and \( E_k < \gamma \), then

\[ \|v(s)\|_\mathcal{E} + \|q(s)\|_{H^k} + \|w(s)\|_{H^k} < \delta < \delta_1, \text{ for all } s \in [1, T(\gamma, \delta)). \]

It follows from Lemmas 3.60 and 3.61 that there exists a constant \( C_{\delta_1} > 0 \) such that

\[ \|H_i(v_1(s), v_2(s), q(s), w(s))\| \leq C_{\delta_1}(\|v(s)\|_{\alpha} + \|v(s)\|_0 \|v_2(s)\|_0 \]

\[ + \|v(s)\|_0 \|v(s)\|_{\alpha} + \|q(s)\|_{H^k} \|v(s)\|_{\alpha} + \|w(s)\|_{H^k}^2) \]

for \( i = 1, 2 \). Thus by Proposition 3.58, Lemma 3.57 and (3.126), and also by the fact that \( \|v(t)\|_0 < \delta \), formula (3.125) yields the estimates for \( v_2(t) \) in \( E_0^{n_2} \) as follows:

\[ \|v_2(t)\|_0 \leq K e^{-\rho t} E_k + K \int_0^t e^{-\rho(t-s)} C_{\delta_1}(\|v(s)\|_{\alpha} + \|v(s)\|_0 \|v_2(s)\|_0 \]

\[ + \|v(s)\|_0 \|v(s)\|_{\alpha} + \|q(s)\|_{H^k} \|v(s)\|_{\alpha} + \|w(s)\|_{H^k}^2) ds \]

\[ \leq K e^{-\rho t} E_k + K C_{\delta_1} \int_0^t e^{-\rho(t-s)}(\delta \|v_2(s)\|_0 + (1 + \delta) \|v(s)\|_{\alpha} \]

\[ + \|q(s)\|_{H^k} \|v(s)\|_{\alpha} + \|w(s)\|_{H^k}^2) ds \]

\[ \leq K e^{-\rho t} E_k + K C_{\delta_1} \delta \int_0^t e^{-\rho(t-s)} \|v_2(s)\|_0 ds \]

\[ + K C_{\delta_1} E_k (1 + \delta) \int_0^t e^{-\rho(t-s)}(1 + s)^{-(d+1)/2} ds \]

\[ + K C_{\delta_1} E_k \int_0^t e^{-\rho(t-s)}(1 + s)^{-(3d+1)/4} ds \]

\[ + K C_{\delta_1} E_k \int_0^t e^{-\rho(t-s)}(1 + s)^{-(d+1)/2} ds \]

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\[
K e^{-\rho t} E_k + KC_\delta_1 \int_0^t e^{-\rho(t-s)} \|v_2(s)\|_0 ds \\
+ KC_\delta_1 E_k (1 + \delta)(1 + t)^{-(d+1)/2} + KC_\delta_1 E_k (1 + t)^{-(3d+1)/4} \\
+ KC_\delta_1 E_k (1 + t)^{-(d+1)/2} \\
\leq KE_k (e^{-\rho t} + (1 + t)^{-(d+1)/2} + (1 + t)^{-(3d+1)/4}) \\
+ KC_\delta_1 \int_0^t e^{-\rho(t-s)} \|v_2(s)\|_0 ds \\
\leq K(1 + t)^{-(d+1)/2} E_k + K \delta \int_0^t e^{-\rho(t-s)} \|v_2(s)\|_0 ds,
\]
for some $K > 0$ and all $t \in [1, T(\gamma, \delta))$. We now define

\[
M_{v_2}(t) = \sup_{0 < s \leq t} (1 + s)^{(d+1)/2} \|v_2(s)\|_0.
\]

Applying Lemma 3.57 yields,

\[
\|v_2(t)\|_0 \leq K (1 + t)^{-(d+1)/2} E_k + K \delta M_{v_2}(t) \int_0^t e^{-\rho(t-s)} (1 + s)^{-(d+1)/2} ds \quad (3.128)
\]

Multiplying (3.128) by $(1 + t)^{(d+1)/2}$ implies:

\[
(1 + t)^{(d+1)/2} v_2(t) \leq KE_k + K \delta M_{v_2}(t).
\]

Because the function $M_{v_2}(t)$ is an increasing function of $t$, we therefore conclude that

\[
M_{v_2}(t) \leq KE_k + K \delta M_{v_2}(t)
\]
for some $C > 0$. By choosing $\delta_2 < \min\{\delta_1, \frac{1}{2K}\}$, we obtain that if $0 < \delta < \delta_2$ and $0 < \gamma < \delta$ then

\[
\|v_2(t)\|_0 \leq K_2 (1 + t)^{-(d+1)/2} E_k, \quad (3.129)
\]
for some $K_2 > 0$ on the time interval $t \in [1, T(\gamma, \delta))$, thus finishes the proof of (3.121).
To prove (3.120), we first use Lemma 3.50 in (3.124) to infer
\[ \| e^{tL(E)} \|_{\mathcal{B}(e_0)} \leq K. \]
Then the estimation of the solution to (3.124) satisfies the following estimate based on (3.126):
\[
\| v_1(t) \|_0 \leq KE_k + KC_{\delta_1} \int_0^t \left[ C \| v_2(s) \|_0 + \| v(s) \|_\alpha + \| v(s) \|_0 \| v_2(s) \|_0 \right. \\
\left. + \| v(s) \|_0 \| v(s) \|_\alpha + \| q(s) \|_{H^k} \| v(s) \|_\alpha + \| w(s) \|^2_{H_k} \right] ds.
\]
Since \( 0 < E_k < \gamma < \delta < \delta_2 \), for \( t \in [1, T(\gamma, \delta)] \) we have \( \| v(t) \|_0 < \delta \) by Lemma 3.54 and the definition 3.55 of \( T(\delta, \gamma) \). Therefore,
\[
\| v_1(t) \|_0 \leq KE_k + KC_{\delta_1} \int_0^t \left( C \| v_2(s) \|_0 + \| v(s) \|_\alpha + \delta \| v_2(s) \|_0 + \delta \| v(s) \|_\alpha \right. \\
\left. + \| q(s) \|_{H^k} \| v(s) \|_\alpha + \| w(s) \|^2_{H_k} \right) ds \]
\[
\leq KE_k + K(C + \delta) \int_0^t \| v_2(s) \|_0 ds + K(1 + \delta) \int_0^t \| v(s) \|_\alpha ds \\
+ K \int_0^t \| q(s) \|_{H^k} \| v(s) \|_\alpha ds + K \int_0^t \| w(s) \|^2_{H_k} ds.
\]
Finally, we may apply Proposition 3.58, equation (3.121) and Lemma 3.57 to obtain that
\[
\| v_1(t) \|_0 \leq KE_k + KC_{\delta_1} (C + \delta) E_k \int_0^t (1 + s)^{-(d+1)/2} ds \\
+ KC_{\delta_1} (1 + \delta) E_k \int_0^t (1 + s)^{-(d+1)/2} ds \\
+ KC_{\delta_1} E_k \int_0^t (1 + s)^{-(3d+1)/4} ds \\
+ KC_{\delta_1} E_k \int_0^t (1 + s)^{-(d+1)/2} ds \\
\leq K_2 E_k,
\]
for some \( K_2 > 0 \).

Now we are ready to present the main theorem of this section by using the bootstrap argument similar to in the proofs of Theorems 2.38 and 2.50. The constant \( \delta_0 \)
in the next theorem can be taken to be $\delta_0 = \delta_2$, where $\delta_2$ is chosen as in Proposition 3.62.

**Theorem 3.63.** Assume Hypothesis 3.35 and $k \geq \lceil \frac{d+1}{2} \rceil$. There exist a small $\delta_0 > 0$ and a constant $C > 0$ such that for each $0 < \delta < \delta_0$ there exists $0 < \eta < \delta$ such that the following is true. Let $(v^0, q^0, w^0) \in \mathcal{E}^n \times H^k(\mathbb{R}^{d-1}) \times H^k(\mathbb{R}^{d-1})^{d-1}$ be the initial condition satisfying $E_k = \|v^0\| + \|q^0\|_{H^{k+1}(\mathbb{R}^{d-1})} + \|q^0\|_{W^{1,1}(\mathbb{R}^{d-1})} \leq \eta$ and let $(v(t), q(t), w(t)) \in \mathcal{E}^n \times H^k(\mathbb{R}^{d-1}) \times H^k(\mathbb{R}^{d-1})^{d-1}$ be the solution of the evolution equation (3.82) with the initial condition $(v^0, q^0, w^0)$. Then for all $t > 0$,

1. $(v(t), q(t), w(t))$ is defined in $\mathcal{E}^n \times H^k(\mathbb{R}^{d-1})$;

2. $\|v(t)\| + \|q(t)\|_{H^k} + \|w(t)\|_{H^k} \leq \delta$;

3. $\|v(t)\|_\alpha \leq C(1 + t)^{-(d+1)/2} E_k$;

4. $\|q(t)\|_{H^k} \leq C(1 + t)^{-(d-1)/4} E_k$;

5. $\|w(t)\|_{H^k} \leq C(1 + t)^{-(d+1)/4} E_k$;

6. $\|v_1(t)\|_0 \leq CE_k$;

7. $\|v_2(t)\|_0 \leq C(1 + t)^{-(d+1)/2} E_k$.

**Proof.** Choose $\delta_0 = \delta_2$ as indicated in Proposition 3.62, and fix $C > \max\{1, K_1, K_2\}$ with $K_1$ and $K_2$ given in Propositions 3.58 and 3.62 respectively. Let $0 < \gamma < \delta < \delta_0$ and set $\eta = C^{-1}\gamma/3$. Let $(v^0, q^0, w^0) \in \mathcal{Q}_{\mathcal{E}} \times H^k(\mathbb{R}^{d-1}) \times H^k(\mathbb{R}^{d-1})^{d-1}$ be the initial value of the solution $(v(t), q(t), w(t)) \in \mathcal{Q}_{\mathcal{E}} \times H^k(\mathbb{R}^{d-1}) \times H^k(\mathbb{R}^{d-1})^{d-1}$ of equation (3.38) such that $E_k \leq \eta$. Since $\eta < \gamma < \delta$, we can apply Propositions 3.58
and 3.62 with $\gamma$ replaced by $\eta$ and conclude that for all $t \in [0, T(\gamma, \eta))$ assertions (1)- (7) of the theorem hold.

We claim that $T(\delta, \eta) = \infty$; thus the theorem holds as soon as the claim is proved.

To prove the claim, fix any $T \in (0, T(\delta, \eta))$ and consider the solution at the point $t = T$, we conclude that

$$\|v(T, v^0, q^0, w^0)\|_E + \|q(T, v^0, q^0, w^0)\|_{H^k} + \|w(T, v^0, q^0, w^0)\|_{H^k} \leq 3CE_k \leq 3C\eta = \gamma.$$ 

We now apply Lemma 3.54 for the solution with the initial data $(v(T), q(T), w(T))$. This lemma says that for all $t \in [0, T(\delta, \gamma))$ we have the inequality

$$\|v(t + T)\|_E + \|q(t + T)\|_{H^k} + \|w(t + T)\|_{H^k} \leq \delta.$$ 

So we conclude that if $E_k \leq \eta$ then $\|v(t)\|_E + \|q(t)\|_{H^k} + \|w(t)\|_{H^k} \leq \delta$ for all $t \in [0, T + T(\delta, \gamma))$. But this means, using Definition 3.55, that $T(\delta, \eta) \geq T + T(\delta, \gamma)$ for each $T \in (0, T(\delta, \gamma))$. Taking the sup over such $T$ yields $T(\delta, \eta) \geq T(\delta, \eta) + T(\delta, \gamma)$ and therefore $T(\delta, \gamma) = \infty$ as claimed, completing the proof of the theorem. \hfill \blacksquare
Appendix A

Lipschitz Properties of the Nemytskij Operator

In this appendix we prove the Lipschitz properties of the Nemytskij operator (A.1) induced by the nonlinear term. In order to do so, we need the following lemma from [RS, Remark 2] (see page 31), and a generalized Hölder’s inequality (see, e.g., [WZ, Exercise 8.6]):

**Lemma A.1.** Given Sobolev spaces \( W^{k,p}(\mathbb{R}^d) \) and \( W^{k_0,p_0}(\mathbb{R}^d) \), if \( k > k_0 \) and

\[ k - \frac{d}{p} > k_0 - \frac{d}{p_0} \]

then the Sobolev embedding \( W^{k,p}(\mathbb{R}^d) \hookrightarrow W^{k_0,p_0}(\mathbb{R}^d) \) holds.

**Lemma A.2.** Assume that \( r \in (0, \infty) \) and \( p_1, ..., p_n \in (0, \infty] \) are such that

\[ \sum_{k=1}^{n} \frac{1}{p_k} = \frac{1}{r}. \]

Then for all \( \mu \)-measurable real- or complex-valued functions \( f_1, ..., f_n \),

\[ \left\| \prod_{k=1}^{n} f_k \right\|_{L^r(\mu)} \leq \prod_{k=1}^{n} \| f_k \|_{L^{p_k}(\mu)}. \]

In particular, \( f_k \in L^{p_k}(\mu) \) for all \( k \in \{1, \ldots, n\} \) implies that \( \prod_{k=1}^{n} f_k \in L^r(\mu) \).
Recall notation $\mathcal{E}_0 = H^k(\mathbb{R}^d)$ and $\mathcal{E}_\alpha = H^k_\alpha(\mathbb{R}^d) = H^k_\alpha(\mathbb{R}) \times H^k(\mathbb{R}^{d-1})$ and $\mathcal{E} = \mathcal{E}_0 \cap \mathcal{E}_\alpha$, where $\gamma_\alpha$ is defined in (3.44). Next we will show an analogue of [GLS, Proposition 7.2].

**Proposition A.3.** Assume $k \geq \left[ \frac{d+1}{2} \right]$, and let $m: (q, u) \mapsto m(q, u) \in \mathbb{R}$ be a function from $C^{k+1}(\mathbb{R}^2)$. Consider the formula

$$
(q(x), u(x), v(x)) \mapsto m(q(x), u(x))v(x), \tag{A.1}
$$

where $q(\cdot), u(\cdot), v(\cdot): \mathbb{R}^d \mapsto \mathbb{R}$, and the variable $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$.

(1) Formula (A.1) defines a mapping from $H^k(\mathbb{R}^d) \times \mathcal{E}_0^2$ to $\mathcal{E}_0$ that is locally Lipschitz on any set of the form $\{(q, u, v) : \|q\|_0 + \|u\|_0 + \|v\|_0 \leq K\}$.

(2) Formula (A.1) defines a mapping from $H^k(\mathbb{R}^d) \times \mathcal{E}_0^2$ to $\mathcal{E}$ that is locally Lipschitz on any set of the form $\{(q, u, v) : \|q\|_0 + \|u\|_0 + \|v\|_0 \leq K\}$.

**Proof.** We shall employ the equivalent Sobolev norm (see, e.g., [NS], page 316):

$$
\|f\|_{H^k} \sim \|f\|_{L^2} + \sum_{a_1+\cdots+a_d=k} \left\| \frac{\partial^k}{\partial x_1^{a_1} \cdots \partial x_d^{a_d}} f \right\|_{L^2}, \tag{A.2}
$$

where the sum extends over all $d$-tuples $(a_1, \ldots, a_d)$ of non-negative integers with $\sum_{t=1}^d a_t = k$, and $\frac{\partial^{a_t}}{\partial x_t^{a_t}}$ is the $a_t$-th differentiation of functions with respect to $x_t$, $t = 1, \ldots, d$.

First we consider variation in $q$. We have

$$
m(q + \bar{q}, u)v - m(q, u)v = \left( \int_0^1 m_q(q + t\bar{q}, u)dt \right) \bar{q}v.
$$

Therefore the estimate of $m(q + \bar{q}, u)v - m(q, u)v$ on $L^2(\mathbb{R}^d)$ follows from

$$
\|m(q + \bar{q}, u)v - m(q, u)v\|_{L^2} \leq \|m\|_{C^1} \|\bar{q}\|_{L^\infty} \|v\|_{L^2}.
$$

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by using the fact that \( H^k(\mathbb{R}^d) \) can be embedded into \( L^\infty(\mathbb{R}^d) \); and the estimate of \( m(q + \bar{q}, u)v - m(q, u)v \) on \( L^2(\mathbb{R}) \otimes L^2(\mathbb{R}^{d-1}) \) follows from

\[
\|\gamma_\alpha(m(q + \bar{q}, u)v - m(q, u)v)\|_{L^2} \leq \|m\|_{C^1} \|\bar{q}\|_{H^k}\|v\|_{L^2},
\]

Let us denote \( m_1(q, \bar{q}, u) = \int_0^1 m_q(q + t\bar{q}, u)dt \). In order to estimate the terms

\[
\frac{\partial^k}{\partial x_1^{a_1} \cdots \partial x_d^{a_d}} (m_1(q, \bar{q}, u)\bar{q}v)
\]

on \( L^2(\mathbb{R}^d) \), we will need the General Leibniz Rule (see p.318 of [O]): if \( f_1, \ldots, f_m \) are all \( n \)-times differentiable functions, then their product \( f_1 \cdots f_m \) is also \( n \)-times differentiable and its \( n \)th derivative is given by

\[
(f_1 f_2 \cdots f_m)^{(n)} = \sum_{k_1 + k_2 + \cdots + k_m = n} \binom{n}{k_1, k_2, \ldots, k_m} \prod_{1 \leq t \leq m} f_t^{(k_t)},
\]

where the sum extends over all m-tuples \((k_1, \ldots, k_m)\) of non-negative integers with \( \sum_{i=1}^m k_i = n \) and \( \binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1! k_2! \cdots k_m!} \) are the multinomial coefficients.

Fix any tuple \((a_1, \ldots, a_d)\) and use the General Leibniz Rule. Then we have

\[
\frac{\partial^k}{\partial x_1^{a_1} \cdots \partial x_d^{a_d}} (m_1(q, \bar{q}, u)\bar{q}v) = \sum_{b_1 + c_1 + e_1 = a_1} \binom{a_1}{b_1, c_1, e_1} \frac{\partial^{a_d}}{\partial x_d^{a_d}} m_1(q, \bar{q}, u) \cdot \frac{\partial^{e_d}}{\partial x_d^{e_d}} \bar{q} \cdot \frac{\partial^{c_d}}{\partial x_d^{c_d}} v,
\]

\[
= \sum_{b_d + c_d + e_d = a_d} \binom{a_d}{b_d, c_d, e_d} \sum_{b_{d-1} + c_{d-1} + e_{d-1} = a_{d-1}} \binom{a_{d-1}}{b_{d-1}, c_{d-1}, e_{d-1}} \frac{\partial^{b_{d-1}}}{\partial x_{d-1}^{b_{d-1}}} m_1(q, \bar{q}, u) \cdot \frac{\partial^{e_{d-1}+c_{d-1}}}{\partial x_{d-1}^{e_{d-1}+c_{d-1}}} \bar{q} \cdot \frac{\partial^{c_{d-1}+e_{d-1}}}{\partial x_{d-1}^{c_{d-1}+e_{d-1}}} v
\]

\[
= \cdots \sum_{b_1 + c_1 + e_1 = a_1} \binom{a_1}{b_1, c_1, e_1} \frac{\partial^{b_1}}{\partial x_1^{b_1}} m_1(q, \bar{q}, u) \cdot \frac{\partial^{c_1+e_1}}{\partial x_1^{c_1} \cdots x_d^{e_1}} \bar{q} \cdot \frac{\partial^{e_1+e_1}}{\partial x_1^{e_1} \cdots x_d^{e_1}} v,
\]

\[
\leq \|m\|_{C^1} \|\bar{q}\|_{H^k}\|v\|_{L^2},
\]

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where $a_1 + \cdots + a_d = k$.

We now refer to the Higher Chain Formula (see [Ts, Lemma 1]): Consider a mapping $M : x \in X \subset \mathbb{R}^d \rightarrow (q(x), \bar{q}(x), u(x)) \in G \subset \mathbb{R}^3 \rightarrow m_1 \in \mathbb{R}$, where $X, G$ are open subsets of $\mathbb{R}^d, \mathbb{R}^3$ respectively and $g, h$ are sufficiently smooth functions, denote 

$$(g_1(x), g_2(x), g_3(x)) = (q(x), \bar{q}(x), u(x)).$$

For each $i$ in the set $J_s$ of integers $1, 2, \ldots, s$, where $s = b_1 + \cdots + b_d$, let $t_i$ denote one of the independent variables $x_1, \ldots, x_d$. A partition of $J_s$ is a family of pairwise disjoint nonempty subsets of $J_s$ whose union is $J_s$. Sets in a partition are called blocks. A block function is to assign a label to each block of a partition. The set of all block functions from a partition $P$ of $J_s$ into $J_3$ is denoted by $P_3$. The set of all partitions of $J_s$ is denoted by $P_s$, then

$$\frac{\partial^s m_2(g(x))}{\partial t_1 \cdots \partial t_s} = \sum_{P \in P_s} \sum_{\lambda \in P_3} \left\{ \left( \prod_{B \in P} \left( \frac{\partial}{\partial g_{\lambda}(B)} \right) \right)^{m_1} \left\{ \prod_{B \in P} \left[ \prod_{b \in B} \left( \frac{\partial}{\partial t_b} \right) g_{\lambda}(B) \right] \right\} \right\}. \quad (A.4)$$

Note that the phrase "$B \in P$" means that $B$ runs through the list of all of the blocks of the partition $P$, let us denote by $|P|$ the number of blocks in the partition $P$, then write the partition $P$ as $P = \{B_1, \ldots, B_{|P|}\}$, and let $|B_i|$ be the size of the block $B_i$, $i = 1, 2, \ldots, |P|$. Now for fixed multinomial coefficients $(a_i, b_i, c_i, e_i)$ $(i = 1, \ldots, d)$, fixed $P \in P_s$ and $\lambda \in P_3$ we need to estimate the following term in both $L^2(\mathbb{R}^d)$ and $L^2_\alpha(\mathbb{R}) \otimes L^2(\mathbb{R})^{d-1}$:

$$\left\{ \left( \prod_{B \in P} \frac{\partial}{\partial g_{\lambda}(B)} \right)^{m_1} \left\{ \prod_{B \in P} \left[ \prod_{b \in B} \left( \frac{\partial}{\partial t_b} \right) g_{\lambda}(B) \right] \right\} \frac{\partial^{c_1 + \cdots + c_d}}{\partial x_1^{c_1} \cdots \partial x_d^{c_d}} \frac{\partial^{e_1 + \cdots + e_d}}{\partial x_1^{e_1} \cdots \partial x_d^{e_d}}, \right.$$
Lemma A.2 according to
\[
\frac{1}{2} = \sum_{i=1}^{\left| P \right|+2} \frac{1}{p_i},
\]
where \( p_i \) will be chosen in a minute. Denoting \( P = \{B_1, B_2, \ldots, B_{\left| P \right|}\} \) and \( l = \left| P \right| + 2 \) (note that \( 3 \leq l \leq k \)), also introduce the \( l \)-tuple
\[
(n_1, n_2, \ldots, n_l) = (\left| B_1 \right|, \left| B_2 \right|, \ldots, \left| B_{l-2} \right|, c_1 + \cdots + c_d, e_1 + \cdots + e_d)
\]
we obtain
\[
\left\| \left\{ \left( \prod_{B \in P} \left( \frac{\partial}{\partial g_{\lambda(B)}} \right) \right)^{m_1} \left( \prod_{B \in P} \left( \frac{\partial}{\partial t_b} \right) \right) \right\} \frac{\partial^{c_1+\cdots+c_d} \bar{q}}{\partial x_1^{c_1} \cdots \partial x_d^{c_d}} \frac{\partial^{e_1+\cdots+e_d} v}{\partial x_1^{e_1} \cdots \partial x_d^{e_d}} \right\|_{L^2} 
\]
\[
\leq \left\| m_1^{(\left| P \right|)} \right\|_{L^\infty} \prod_{b \in B_1} \left\| \left( \frac{\partial}{\partial t_b} \right) \right\|_{L^{p_1}} \cdots \prod_{b \in B_{l-2}} \left\| \left( \frac{\partial}{\partial t_b} \right) \right\|_{L^{p_l}} 
\]
\[
\left\| \frac{\partial^{c_1+\cdots+c_d} \bar{q}}{\partial x_1^{c_1} \cdots \partial x_d^{c_d}} \right\|_{L^{p_{l-1}}} \left\| \frac{\partial^{e_1+\cdots+e_d} v}{\partial x_1^{e_1} \cdots \partial x_d^{e_d}} \right\|_{L^{p_l}} 
\]
\[
\leq \left\| m_1^{(\left| P \right|)} \right\|_{L^\infty} \left\| g_{\lambda(B_1)} \right\|_{W^{n_1,p_1}} \cdots \left\| g_{\lambda(B_{l-2})} \right\|_{W^{n_{l-2},p_{l-2}}} \left\| \bar{q} \right\|_{W^{n_{l-1},p_{l-1}}} \left\| v \right\|_{W^{n_l,p_l}},
\]
where \( W^{k,p} \) are the Sobolev spaces of \( k \) times differentiable functions from \( L^p \). Note that in equation (A.5), \( \sum_{i=1}^{\left| P \right|} \left| B_i \right| = \sum_{j=1}^{d} b_j \), this is because of the fact that \( P \) is one of the partitions of the \( b_1 + \cdots + b_d \) indices of \( (x_1, \ldots, x_1, \ldots, x_d, \ldots, x_d) \), and \( \{B_1, \ldots, B_{\left| P \right|}\} \) are all blocks in the partition \( P \).

Then by Lemma A.1, in order to prove that \( W^{k,2}(\mathbb{R}^d) = H^k(\mathbb{R}^d) \hookrightarrow W^{n_i,p_i}(\mathbb{R}^d) \), since all \( b_1 + \cdots + b_d, c_1 + \cdots + c_d \) and \( e_1 + \cdots + e_d \) are nonzero, it is obvious that \( k > n_i \), we will need to show that \( k - \frac{d}{2} > n_i - \frac{d}{p_i} \). By choosing \( \frac{1}{p_i} = (\frac{1}{2} - \frac{k}{d}) \frac{1}{l} + \frac{n_i}{d} \), it follows that
\[
\sum_{i=1}^{l} \frac{1}{p_i} = (\frac{1}{2} - \frac{k}{d}) + \sum_{i=1}^{l} \frac{n_i}{d} = \frac{1}{2} - \frac{k}{d} + \frac{k}{d} = \frac{1}{2},
\]
(A.6)
\[ n_i - \frac{d}{p_i} = n_i - d\left(\frac{1}{2} - \frac{k}{d}\right)\frac{1}{l} + \frac{n_i}{d} \]
\[ = n_i - \left(\frac{d}{2l} - \frac{k}{l} + \frac{n_i}{d}\right) \]
\[ = \frac{k}{l} - \frac{d}{2l} \]
\[ = \frac{1}{l}(k - \frac{d}{2}). \tag{A.7} \]

Since \( k \geq \frac{d+1}{2} \) and \( l > 2 \), we can conclude that
\[ k - \frac{d}{2} > \frac{1}{l}(k - \frac{d}{2}) = n_i - \frac{d}{p_i}, \quad i = 1, \ldots, l, \tag{A.8} \]

therefore \( H^k(\mathbb{R}^d) \) can be embedded into each \( W^{n_i, p_i}(\mathbb{R}^d), \quad i = 1, \ldots, l \), thus following (A.5), we have
\[ \| m_{\{P\}} \| L_1 \cdot \prod_{B \in \mathcal{P}} \left[ \prod_{b \in B} \left( \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \cdot \frac{\partial^{c_1 + \cdots + c_d}}{\partial x_1^{c_1} \cdots \partial x_d^{c_d}} \tilde{q} \cdot \frac{\partial^{e_1 + \cdots + e_d}}{\partial x_1^{e_1} \cdots \partial x_d^{e_d}} v \quad \text{L}^2 \]
\[ \leq \| m_{\{P\}} \| L_\infty \| g_{\lambda(B_1)} \| W^{n_1, p_1} \cdots \| g_{\lambda(B_{l-2})} \| W^{n_{l-2}, p_{l-2}} \| \tilde{q} \| W^{n_{l-1}, p_{l-1}} \| v \| W^{n_l, p_l} \]
\[ \leq C \| m_{\{P\}} \| L_\infty \| g_{\lambda(B_1)} \| H^k \cdots \| g_{\lambda(B_{l-2})} \| H^k \| \tilde{q} \| H^k \| v \| H^k. \tag{A.9} \]

This proof required \( |P| \neq 0, c_1 + \cdots + c_d \) and \( e_1 + \cdots + e_d \) are all non-zero such that \( l > 1 \) which will be needed in inequality (A.8), if not, see following discussions:

**Case 1.2:** If \( c_1 + \cdots + c_d = e_1 + \cdots + e_d = 0 \) and \( |P| \neq 0 \), we use the Sobolev embedding \( H^k(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d) \) and Lemma 3.6(1), so that
\[ \| m_{\{P\}} \| L_\infty \cdot \prod_{B \in \mathcal{P}} \left[ \prod_{b \in B} \left( \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \cdot \frac{\partial^{c_1 + \cdots + c_d}}{\partial x_1^{c_1} \cdots \partial x_d^{c_d}} \tilde{q} \cdot \frac{\partial^{e_1 + \cdots + e_d}}{\partial x_1^{e_1} \cdots \partial x_d^{e_d}} v \quad \text{L}^2 \]
\[ = \| m_{\{P\}} \| L_\infty \cdot \prod_{B \in \mathcal{P}} \left[ \prod_{b \in B} \left( \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \cdot \| \tilde{q} v \| L_\infty \]
\[ \leq \| m_{\{P\}} \| L_\infty \cdot \prod_{B \in \mathcal{P}} \left[ \prod_{b \in B} \left( \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \| \tilde{q} v \| L_\infty. \]
\[\begin{align*}
&\leq C\|m_1^{(P')}\|_{L^\infty} \left\| \prod_{B \in P} \left( \prod_{b \in B} \left( \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right) \right\|_{L^2} \|\bar{q}v\|_{H^k} \\
&\leq C\|m_1^{(P')}\|_{L^\infty} \left\| \prod_{B \in P} \left( \prod_{b \in B} \left( \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right) \right\|_{L^2} \|\bar{q}\|_{H^k} \|v\|_{H^k},
\end{align*}\]

let \(l = |P|\), when \(l = 1\), we use the inequality \(\left\| \frac{\partial^k}{\partial x_1^{a_1} \cdots \partial x_d^{a_d}} g_i \right\|_{L^2} \leq \|u\|_{H^k}\), where \(g_i\) is one of \((g_1, g_2, g_3) = (q, \bar{q}, u)\), thus

\[\begin{align*}
&\left\| m_1^{(P')} \cdot \frac{\partial^k}{\partial x_1^{a_1} \cdots \partial x_d^{a_d}} g_i \cdot \bar{q}v \right\|_{L^2} \leq \|m_1^{(1)}\|_{L^\infty} \|g_i\|_{H^k} \|\bar{q}v\|_{H^k} \\
&\quad \leq C\|m\|_{C^2} \|g_i\|_{H^k} \|\bar{q}\|_{H^k} \|v\|_{H^k};
\end{align*}\]

when \(l \geq 2\), let \(P = \{B_1, B_2, \ldots, B_l\}\), the \(l\)-tuple \((n_1, \ldots, n_l) = (|B_1|, \ldots, |B_l|)\). Then use Lemma A.1 and Lemma A.2 according to \(\frac{1}{p_i} = (\frac{1}{2} - \frac{b}{d})\frac{1}{4} + \frac{n_i}{d}, i = 1, \ldots, l\) we obtain that

\[\begin{align*}
&\left\| \prod_{B \in P} \left( \prod_{b \in B} \left( \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right) \right\|_{L^2} \leq \prod_{b \in B_1} \left( \frac{\partial}{\partial t_b} \right) g_{\lambda(B_1)} \cdots \prod_{b \in B_l} \left( \frac{\partial}{\partial t_b} \right) g_{\lambda(B_l)} \\
&\quad \leq \|g_{\lambda(B_1)}\|_{W^{n_1, p_1}} \cdots \|g_{\lambda(B_l)}\|_{W^{n_l, p_l}} \\
&\quad \leq \|g_{\lambda(B_1)}\|_{H^k} \cdots \|g_{\lambda(B_l)}\|_{H^k},
\end{align*}\]

from which we conclude that

\[\begin{align*}
&\left\| m_1^{(l)} \cdot \prod_{B \in P} \left( \prod_{b \in B} \left( \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right) \cdot \bar{q}v \right\|_{L^2} \leq \|m_1^{(l)}\|_{L^\infty} \|g_{\lambda(B_1)}\|_{H^k} \cdots \|g_{\lambda(B_l)}\|_{H^k} \|\bar{q}v\|_{H^k} \\
&\quad \leq C\|m\|_{C^{l+1}} \|g_{\lambda(B_1)}\|_{H^k} \cdots \|g_{\lambda(B_l)}\|_{H^k} \|\bar{q}\|_{H^k} \|v\|_{H^k}.
\end{align*}\]

**Case 1.3:** If \(b_1 + \cdots + b_d = c_1 + \cdots + c_d = 0\) and \(e_1 + \cdots + e_d \neq 0\), we are evaluating the term \(m_1(q, \bar{q}, u)\bar{q} \frac{\partial^k}{\partial x_1^{a_1} \cdots \partial x_d^{a_d}} v\) on \(L^2(\mathbb{R}^d)\), which will be given by the Sobolev embedding \(H^k(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)\):

\[\begin{align*}
&\left\| m_1(q, \bar{q}, u)\bar{q} \frac{\partial^k}{\partial x_1^{a_1} \cdots \partial x_d^{a_d}} v \right\|_{L^2} \leq \|m_1\|_{L^\infty} \|\bar{q}\|_{L^\infty} \left\| \frac{\partial^k}{\partial x_1^{a_1} \cdots \partial x_d^{a_d}} v \right\|_{L^2}.
\end{align*}\]
Similarly, if \( b_1 + \cdots + b_d = e_1 + \cdots + e_d = 0 \) and \( c_1 + \cdots + c_d \neq 0 \), we have

\[
\left\| m_1(q, \bar{q}, u) \frac{\partial^k}{\partial x_1^{a_1} \cdots \partial x_d^{a_d}} \bar{q} \right\|_{L^2} \leq \left\| m_1 \right\|_{L^\infty} \left\| v \right\|_{L^\infty} \left\| \frac{\partial^k}{\partial x_1^{a_1} \cdots \partial x_d^{a_d}} \bar{q} \right\|_{L^2} \leq C \left\| m \right\|_{C^1} \| \bar{q} \|_{H^k} \| v \|_{H^k}.
\]

**Case 1.4:** If \( b_1 + \cdots + b_d = 0, \ c_1 + \cdots + c_d \neq 0 \) and \( e_1 + \cdots + e_d \neq 0 \) we let \( \frac{1}{p_i} = \left(\frac{1}{2} - \frac{k}{d}\right) \frac{1}{\nu} + \frac{n_i}{d}, \ i = 1, 2, \)

\[
\left\| m_1(q, \bar{q}, u) \cdot \frac{\partial^{e_1 + \cdots + e_d}}{\partial x_1^{e_1} \cdots \partial x_d^{e_d}} \bar{q} \cdot \frac{\partial^{e_1 + \cdots + e_d}}{\partial x_1^{e_1} \cdots \partial x_d^{e_d}} v \right\|_{L^2} \leq \left\| m_1 \right\|_{L^\infty} \left\| \frac{\partial^{e_1 + \cdots + e_d}}{\partial x_1^{e_1} \cdots \partial x_d^{e_d}} \bar{q} \right\|_{L^{p_1}} \left\| \frac{\partial^{e_1 + \cdots + e_d}}{\partial x_1^{e_1} \cdots \partial x_d^{e_d}} v \right\|_{L^{p_2}} \leq C \left\| m \right\|_{C^1} \| \bar{q} \|_{W^{\nu, p_1}} \| v \|_{W^{\nu, p_2}} \leq C \left\| m \right\|_{C^1} \| \bar{q} \|_{H^k} \| v \|_{H^k}.
\]

**Case 1.5:** If \( c_1 + \cdots + c_d = 0, \ |P| \neq 0 \) and \( e_1 + \cdots + e_d \neq 0 \), we are evaluating

\[
m_1^{(P)}(q, \bar{q}, u) \cdot \prod_{B \in P} \left[ \prod_{b \in B} \left( \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \cdot \bar{q} \cdot \frac{\partial^{e_1 + \cdots + e_d}}{\partial x_1^{e_1} \cdots \partial x_d^{e_d}} v
\]
on \( L^2(\mathbb{R}^d) \), this could be done by letting \( l = |P| + 1, \ P = \{B_1, B_2, ..., B_{l-1}\} \), the \( l \)-tuple \( (n_1, ..., n_l) = (|B_1|, ..., |B_{l-1}|, e_1 + \cdots + e_d) \) and applying Lemma A.1 and Lemma A.2 according to \( \frac{1}{p_i} = \left(\frac{1}{2} - \frac{k}{d}\right) \frac{1}{\nu} + \frac{n_i}{d}, \ i = 1, ..., l \) and the Sobolev Embedding \( H^k(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d) \), we obtain that

\[
\| m_1^{(P)}(q, \bar{q}, u) \cdot \prod_{B \in P} \left[ \prod_{b \in B} \left( \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \cdot \bar{q} \cdot \frac{\partial^{e_1 + \cdots + e_d}}{\partial x_1^{e_1} \cdots \partial x_d^{e_d}} v \|_{L^2} \leq C \left\| m_1^{(l-1)} \right\|_{L^\infty} \| \bar{q} \|_{L^\infty} \left\| \prod_{b \in B_1} \left( \frac{\partial}{\partial t_b} \right) g_{\lambda(B_1)} \right\|_{L^{p_1}} \cdots
\]

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Combining the inequalities (A.9)-(A.12), and the fact that the Sobolev Embedding $H^k \hookrightarrow L^{\infty}$, applying Lemma A.1 and Lemma A.2 according to $1 \leq \|B\|_q, u_1 \leq \|B\|_{W^{n, p_1}} \cdots \|B\|_{W^{n, p_1-1}} \|v\|_{W^{n, p_1-1}}$

$$\leq C\|m\|C^1\|\bar{q}\|_{H^k} \|B\|_{W^{n, p_1}} \cdots \|B\|_{W^{n, p_1}} \|v\|_{H^k}$$

Similarly, if $e_1 + \cdots + e_d = 0$, $|P| \neq 0$ and $e_1 + \cdots + e_d \neq 0$, we let $l = |P| + 1$, $P = \{B_1, B_2, \ldots, B_{l-1}\}$, the $l$-tuple $(n_1, \ldots, n_l) = (|B_1|, \ldots, |B_{l-1}|, e_1 + \cdots + e_d)$ and applying Lemma A.1 and Lemma A.2 according to $\frac{1}{p_i} = (\frac{1}{2} - \frac{k}{d}) \frac{1}{l} + \frac{n}{d}$, $i = 1, \ldots, l$ and the Sobolev Embedding $H^k(\mathbb{R}^d) \hookrightarrow L^{\infty}(\mathbb{R}^d)$

$$\|m^{(l)}_1(q, \bar{q}, u) \cdot \prod_{B \in P} \prod_{B \in B} \left( \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \| \frac{\partial^{e_1 + \cdots + e_d}}{\partial x_1^{e_1} \cdots \partial x_d^{e_d}} \bar{q} \cdot v \|_{L^2}$$

\begin{equation}
\leq \|m^{(l-1)}_1\|_{L^\infty} \|v\|_{L^\infty} \prod_{B \in B} \left( \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \| \frac{\partial^{e_1 + \cdots + e_d}}{\partial x_1^{e_1} \cdots \partial x_d^{e_d}} \bar{q} \cdot v \|_{L^p_1}
\end{equation}

$$\leq C\|m\|C^1\|\bar{q}\|_{H^k} \|B\|_{W^{n, p_1}} \cdots \|B\|_{W^{n, p_1}} \|v\|_{W^{n, p_1}}$$

Combining the inequalities (A.9)-(A.12), and the fact that $|P| \leq k$, we have proved that

\begin{equation}
\|m^{(l)}_1(q, \bar{q}, u) \cdot \prod_{B \in P} \prod_{B \in B} \left( \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \| \frac{\partial^{e_1 + \cdots + e_d}}{\partial x_1^{e_1} \cdots \partial x_d^{e_d}} \bar{q} \cdot v \|_{L^2}
\end{equation}

\begin{equation}
\leq \|m\|C^{k+1} \|\lambda(B)\|_{H^k} \cdots \|\lambda(B_{l-1})\|_{H^k} \|\bar{q}\|_{H^k} \|v\|_{H^k}
\end{equation}

Similarly, let $l = |P| + 2$, $P = \{B_1, \ldots, B_{l-2}\}$, the $l$-tuple $(n_1, \ldots, n_l) = (|B_1|, \ldots,
Let $|B_{l-2}|, c_1 + \cdots + c_d, e_1 + \cdots + e_d$ and let $\gamma = e^{\alpha x_1},$

\[
\left\| \gamma \left( m_1(P) (q, \bar{q}, u) \cdot \prod_{B \in P} \left( \prod_{b \in B} \left( \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right) \right) \cdot \frac{\partial^{e_1 + \cdots + e_d} \bar{q}}{\partial x_1^{e_1} \cdots \partial x_d^{e_d}} \right\|_{L^2}
\]

\[
(A.14)
\]

\[
\leq m_1(P) \left\| \prod_{b \in B_1} \left( \frac{\partial}{\partial t_b} \right) g_{\lambda(B_1)} \right\|_{L^p_1} \cdots \left\| \prod_{b \in B_{l-2}} \left( \frac{\partial}{\partial t_b} \right) g_{\lambda(B_{l-2})} \right\|_{L^p_{l-2}} \\
\left\| \frac{\partial^{c_1 + \cdots + c_d} q}{\partial x_1^{c_1} \cdots \partial x_d^{c_d}} \right\|_{L^{p_1}_{l-1}} \left\| \gamma \alpha \frac{\partial^{e_1 + \cdots + e_d} \bar{q}}{\partial x_1^{e_1} \cdots \partial x_d^{e_d}} \right\|_{L^p_l}
\]

\[
\leq m_1 \left\| g_{\lambda(B_1)} \right\|_{W^{n_1, p_1}} \cdots \left\| g_{\lambda(B_{l-2})} \right\|_{W^{n_{l-2}, p_{l-2}}} \left\| q \right\|_{W^{n_{l-1}, p_{l-1}}} \left\| \gamma \alpha \right\|_{W^{n_l, p_l}}
\]

\[
\leq m \left\| g_{\lambda(B_1)} \right\|_{H^k} \cdots \left\| g_{\lambda(B_{l-2})} \right\|_{H^k} \left\| q \right\|_{H^k} \left\| \gamma \alpha \right\|_{H^k};
\]

the similar discussion when $|P| = 0, c_1 + \cdots + c_d = 0$ or $c_1 + \cdots + c_d = 0$ also follows.

Next we consider variations in $u$. We have

\[
m(q, u + \bar{u})v - m(q, u)v = \int_0^1 m_u(q, u + t\bar{u}) dt \bar{u} v.
\]

It yields

\[
\left\| m(q, u + \bar{u})v - m(q, u)v \right\|_{L^2} \leq \left\| m \right\|_{C^1} \left\| \bar{u} \right\|_{L^\infty} \left\| v \right\|_{L^2},
\]

by Sobolev embedding $H^k(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$

\[
\left\| m(q, u + \bar{u})v - m(q, u)v \right\|_{L^2} \leq \left\| m \right\|_{C^1} \left\| \bar{u} \right\|_{H^k} \left\| v \right\|_{L^2},
\]

and

\[
\left\| \gamma (m(q, u + \bar{u})v - m(q, u)v) \right\|_{L^2} \leq \left\| m \right\|_{C^1} \left\| \bar{u} \right\|_{H^k} \left\| \gamma \alpha \right\|_{L^2}.
\]

We denote $m_2(q, u, \bar{u}) = \int_0^1 m_u(q, u + t\bar{u}) dt, g_1(x) = q(x), g_2(x) = u(x)$ and $g_3(x) = \bar{u}(x)$, by generalizing the Leibniz rule again for $\sum_{t=1}^d a_t = k$ we obtain that:

\[
\frac{\partial^k}{\partial x_1^{a_1} \cdots \partial x_d^{a_d}} m_2(g_1, g_2, g_3) \bar{u} v
\]

(A.15)
Let $s = b_1 + \cdots + b_d$, we will also employ the Higher Chain Formula, for each $i$ in the set $J_s$ of integers $1, 2, \ldots, s$, let $t_i$ denote one of the independent variables $x_1, \ldots, x_d$. A partition of $J_s$ is a family of pairwise disjoint nonempty subsets of $J_s$ whose union is $J_s$. Sets in a partition are called blocks. A block function is to assign a label to each block of a partition. The set of all block functions from a partition $P$ of $J_s$ into $J_3$ is denoted by $P_3$. The set of all partitions of $J_s$ is denoted by $P_s$, then

\[
\frac{\partial^s m_2(g(x))}{\partial t_1 \cdots \partial t_s} = \sum_{P \in P_s} \sum_{\lambda \in P_3} \left\{ \left( \prod_{B \in P} \frac{\partial}{\partial g_\lambda(B)} \right) m_2 \right\} \left\{ \prod_{B \in P} \left[ \prod_{b \in B} \left( \frac{\partial}{\partial t_b} \right) g_\lambda(B) \right] \right\}. \quad \text{(A.16)}
\]

Now for fixed multinomial coefficients $(\alpha_i)_{b_i, c_i, e_i}$ ($i = 1, \ldots, d$), fixed $P \in P_s$ and $\lambda \in P_3$. We need to estimate the following term in both $L^2(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d \otimes L^2(\mathbb{R}^{d-1}))$:

\[
\left\{ \left( \prod_{B \in P} \frac{\partial}{\partial g_\lambda(B)} \right) m_2 \right\} \left\{ \prod_{B \in P} \left[ \prod_{b \in B} \left( \frac{\partial}{\partial t_b} \right) g_\lambda(B) \right] \right\} \frac{\partial^{c_1 + \cdots + c_d} u}{\partial x_1^{c_1} \cdots \partial x_d^{c_d}} \frac{\partial^{e_1 + \cdots + e_d} v}{\partial x_1^{e_1} \cdots \partial x_d^{e_d}}.
\]

**Case 2.1:** If all $|P|$, $c_1 + \cdots + c_d$, $e_1 + \cdots + e_d > 0$, let $l = |P| + 2$ $(2 < l \leq k)$ and $P = \{B_1, \ldots, B_{l-2}\}$, we will use Lemma A.1 and Lemma A.2 according to $(n_1, \ldots, n_l) = \{|B_1|, \ldots, |B_{l-2}|, c_1 + \cdots + c_d, e_1 + \cdots + e_d\}$ and $\frac{1}{p_i} = (\frac{1}{2} - \frac{1}{d}) \frac{1}{2} + \frac{n_i}{d}$ for $i = 1, 2, \ldots, l$ to obtain the inequality:

\[
\left\| \left\{ \left( \prod_{B \in P} \frac{\partial}{\partial g_\lambda(B)} \right) m_2 \right\} \left\{ \prod_{B \in P} \left[ \prod_{b \in B} \left( \frac{\partial}{\partial t_b} \right) g_\lambda(B) \right] \right\} \frac{\partial^{c_1 + \cdots + c_d} u}{\partial x_1^{c_1} \cdots \partial x_d^{c_d}} \frac{\partial^{e_1 + \cdots + e_d} v}{\partial x_1^{e_1} \cdots \partial x_d^{e_d}} \right\|_{L^2} \leq \quad \text{(A.17)}
\]

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When it is not the case that not all \(|P|, c_1 + \cdots + c_d, e_1 + \cdots + e_d\) are positive, we have the following discussions:

**Case 2.2:** If \(|P| = 0\) and \(c_1 + \cdots + c_d, e_1 + \cdots + e_d > 0\), we apply Lemma A.1 and Lemma A.2 according to \(l = 2\), \((n_1, n_2) = (c_1 + \cdots + c_d, e_1 + \cdots + e_d)\) and \(\frac{1}{p_i} = (1 - \frac{k}{d}) \frac{1}{l} + \frac{n_i}{d}\) so that \(H^k(\mathbb{R}^d) \hookrightarrow W^{n_i, p_i}(\mathbb{R}^d)\) for \(i = 1, 2\) and obtain:

\[
\left\| m_2(g_1, g_2, g_3) \left( \frac{\partial^{c_1 + \cdots + c_d}}{\partial x_1^{c_1} \cdots \partial x_d^{c_d}} \bar{u}, \frac{\partial^{e_1 + \cdots + e_d}}{\partial x_1^{e_1} \cdots \partial x_d^{e_d}} v \right) \right\|_{L^2} \\
\leq \left\| m_2 \right\|_{L^\infty} \left\| \frac{\partial^{c_1 + \cdots + c_d}}{\partial x_1^{c_1} \cdots \partial x_d^{c_d}} \bar{u}, \frac{\partial^{e_1 + \cdots + e_d}}{\partial x_1^{e_1} \cdots \partial x_d^{e_d}} v \right\|_{L^2} \\
\leq \left\| m_2 \right\|_{C_1} \left\| \frac{\partial^{c_1 + \cdots + c_d}}{\partial x_1^{c_1} \cdots \partial x_d^{c_d}} \bar{u} \right\|_{L^p} \left\| \frac{\partial^{e_1 + \cdots + e_d}}{\partial x_1^{e_1} \cdots \partial x_d^{e_d}} v \right\|_{L^p} \\
\leq \left\| m_2 \right\|_{C_1} \| \bar{u} \|_{W^{n_1, p_1}} \| v \|_{W^{n_2, p_2}} \\
\leq \left\| m_2 \right\|_{C_1} \| \bar{u} \|_{H^k} \| v \|_{H^k}.
\]

**Case 2.3:** If \(e_1 + \cdots + e_d = 0\), and \(|P|, c_1 + \cdots + c_d > 0\), let \(l = |P| + 1\) (\(2 \leq l \leq k\)) and \(P = \{B_1, ..., B_{l-1}\}\), then we apply Lemma A.1 and Lemma A.2 according to \((n_1, ..., n_l) = (|B_1|, ..., |B_{l-1}|, c_1 + \cdots + c_d)\) and \(\frac{1}{p_i} = (1 - \frac{k}{d}) \frac{1}{l} + \frac{n_i}{d}\), \(i = 1, ..., l\) and the Sobolev embedding \(H^k(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)\) to obtain

\[
\left\| \left( \prod_{B \in P} \frac{\partial}{\partial y(B)} \right) m_2 \right\| \left\{ \prod_{B \in P} \left[ \prod_{b \in B} \frac{\partial}{\partial t_b} \right] g_\lambda(B) \right\} \left( \frac{\partial^{c_1 + \cdots + c_d}}{\partial x_1^{c_1} \cdots \partial x_d^{c_d}} \bar{u} \right) v \right\|_{L^2} \\
(A.19)
\]

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or if
\[ \| \prod_{b \in B} \left( \frac{\partial}{\partial t_b} \right) g_\lambda(B) \| L^1 \leq C \| m \|_{C^{l-1}} \| v \|_{H^k} \]

Similarly, if \( c_1 + \cdots + c_d = 0 \) and \( \| P \|, e_1 + \cdots + e_d > 0 \), we can have
\[ \left\| \prod_{b \in B} \left( \frac{\partial}{\partial t_b} \right) m_2 \right\|_{L^1} \leq \| m \|_{C^{l-1}} \| v \|_{H^k} \]

Case 2.4: If \( \| P \| = e_1 + \cdots + e_d = 0 \) and \( c_1 + \cdots + c_d \neq 0 \), we use the Sobolev embedding \( H^k(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d) \), then
\[ \left\| \frac{\partial}{\partial t} \left( \prod_{b \in B} \left( \frac{\partial}{\partial t_b} \right) \right) g_\lambda(B) \right\|_{L^1} \leq C \| m \|_{C^{l-1}} \| v \|_{H^k} \]

or if \( \| P \| = c_1 + \cdots + c_d = 0 \) and \( e_1 + \cdots + e_d \neq 0 \), we have
\[ \left\| \prod_{b \in B} \left( \frac{\partial}{\partial t_b} \right) m_2 \right\|_{L^1} \leq \| m \|_{C^{l-1}} \| v \|_{H^k} \]
\[ \leq C||m||_{C^1}||\vec{u}||_{H^k}||v||_{H^k}. \]

**Case 2.5:** If \( c_1 + \cdots + c_d = e_1 + \cdots + e_d = 0 \) and \( |P| \neq 0 \), let \( l = |P| \) (1 \( \leq l \leq k \)) and \( P = \{ B_1, \ldots, B_l \} \): when \( l = 1 \), \( g_{\lambda(B)} = g_i \), \( i = 1, 2, 3 \) we can use the inequality

\[
\left\| \frac{\partial^k}{\partial x_1^{a_1} \cdots \partial x_d^{a_d}} g_i \right\|_{L^2} \leq ||g_i||_{H^k},
\]

then use Sobolev embedding \( H^k(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d) \) and Lemma 3.6(1) we obtain for \( i = 1, 2, 3 \)

\[
\left\| m^{(1)}_2 \left( \frac{\partial^k}{\partial x_1^{a_1} \cdots \partial x_d^{a_d}} g_i \right) \vec{u} \right\|_{L^2} \leq ||m||_{C^1} ||g_i||_{H^k} ||\vec{u}||_{L^\infty} \tag{A.23}
\]

\[
\leq ||m||_{C^1} ||g_i||_{H^k} ||\vec{u}||_{H^k}
\]

\[
\leq ||m||_{C^1} ||g_i||_{H^k} ||\vec{u}||_{H^k} ||v||_{H^k};
\]

when \( l \geq 2 \), we use the Sobolev embedding \( H^k(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R}) \), Lemma A.1 and Lemma A.2 according to \( (n_1, \ldots, n_l) = (|B_1|, \ldots, |B_l|) \) and \( \frac{1}{p_i} = \left( \frac{1}{2} - \frac{k}{d} \right) + \frac{n_i}{d} \) for \( i = 1, \ldots, l \) and Lemma 3.6(1) to obtain

\[
\left\| \left\{ \prod_{B \in P} \left( \frac{\partial}{\partial g_{\lambda(B)}} \right) \right\} \left\{ \prod_{B \in P} \left\{ \prod_{b \in B} \left( \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right\} \right\} \vec{u} \right\|_{L^2} \tag{A.24}
\]

\[
\leq ||m_2||_{C^l} \left\| \prod_{B \in P} \left\{ \prod_{b \in B} \left( \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right\} \right\|_{L^2} ||\vec{u}||_{L^\infty}
\]

\[
\leq C||m_2||_{C^l} \left\| \prod_{b \in B_1} \left( \frac{\partial}{\partial t_b} \right) g_{\lambda(B_1)} \right\|_{L^{p_1}} \cdots \left\| \prod_{b \in B_1} \left( \frac{\partial}{\partial t_b} \right) g_{\lambda(B_1)} \right\|_{L^{p_l}} ||\vec{u}||_{H^k}
\]

\[
\leq C||m||_{C^{l+1}} ||g_{\lambda(B_1)}||_{H^k} \cdots ||g_{\lambda(B_l)}||_{H^k} ||\vec{u}||_{H^k} ||v||_{H^k}
\]

Combining above inequalities (A.17)-(A.24), we can conclude that:

\[
\left\| \left\{ \prod_{B \in P} \left( \frac{\partial}{\partial g_{\lambda(B)}} \right) \right\} \left\{ \prod_{B \in P} \left\{ \prod_{b \in B} \left( \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right\} \right\} \frac{\partial^{c_1+\cdots+c_d}\vec{u}}{\partial x_1^{c_1} \cdots \partial x_d^{c_d}} \frac{\partial^{e_1+\cdots+e_d}v}{\partial x_1^{e_1} \cdots \partial x_d^{e_d}} \right\|_{L^2} \tag{A.25}
\]
Similarly, we let $l = |P| + 2$ (2 < $l$ ≤ $k$), $P = \{B_1, ..., B_{l-2}\}$ and $\gamma_\alpha = \epsilon^{\alpha x_1}$, we will use Lemma A.1 and Lemma A.2 according to $(n_1, ..., n_l) = \{|B_1|, ..., |B_{l-2}|, c_1 + \cdots + c_d, e_1 + \cdots + e_d\}$ and $\frac{1}{p_i} = (\frac{1}{2} - \frac{k}{d})\frac{1}{l} + \frac{m_a}{d}$ for $i = 1, 2, ..., l$ to obtain that

$$\|m\|_{C^{k+1}} \|g_{\lambda(B_1)}\|_{H^k} \cdots \|g_{\lambda(B_{l-2})}\|_{H^k} \|\bar{u}\|_{H^k} \|v\|_{H^k}.$$  

We have

$$m(q, u)(v + \bar{v}) - m(q, u)v = m(q, u)\bar{v}.$$  

Therefore

$$\|m(q, u)\bar{v}\|_{L^2} \leq \|m\|_{L^\infty} \|\bar{v}\|_{L^2},$$

and

$$\|\gamma_\alpha m(q, u)\bar{v}\|_{L^2} \leq \|m\|_{L^\infty} \|\gamma_\alpha \bar{v}\|_{L^2},$$

Also by using the general Leibniz rule we have:

$$\frac{\partial^k}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}(m(q, u)\bar{v}) = \tag{A.27}$$
Thus for fixed binomial coefficients \( \binom{a_j}{b_j} \) are the binomial coefficients, and let \( s = b_1 + \cdots + b_d \), 

\((g_1(x), g_2(x)) = (q(x), u(x))\) we will also employ the Higher Chain Formula, for each \( i \) in the set \( J_s \) of integers 1, 2, ..., \( s \), let \( t_i \) denote one of the independent variables \( x_1, \ldots, x_d \). A partition of \( J_s \) is a family of pairwise disjoint nonempty subsets of \( J_s \) whose union is \( J_s \). Sets in a partition are called blocks. A block function is to assign a label to each block of a partition. The set of all block functions from a partition \( P \) of \( J_s \) into \( J_2 \) is denoted by \( P_2 \). The set of all partitions of \( J_s \) is denoted by \( P_s \), then

\[
\frac{\partial^s m(q, u)}{\partial t_1 \cdots \partial t_s} = \sum_{\lambda \in P_s} \sum_{\lambda' \in P_2} \left\{ \left( \prod_{B \in P} \frac{\partial}{\partial g_{\lambda(B)}} \right)^m \left( \prod_{B \in P} \left( \frac{\partial}{\partial t_b} \right)^{g_{\lambda(B)}} \right) \right\},
\]

thus for fixed binomial coefficients \( \binom{a_j}{b_j} \), \( j = 1, \ldots, d \), fixed partition \( P \in P_s \) and fixed block function \( \lambda \in P_2 \) we need to estimate

\[
\left\{ \left( \prod_{B \in P} \frac{\partial}{\partial g_{\lambda(B)}} \right)^m \left( \prod_{B \in P} \left( \frac{\partial}{\partial t_b} \right)^{g_{\lambda(B)}} \right) \right\} \frac{\partial^{a_1-b_1+\cdots+a_d-b_d}}{\partial x_1^{a_1-b_1} \cdots \partial x_d^{a_d-b_d} \vec{v}},
\]

in both \( L^2(\mathbb{R}^d) \) and \( L^2(\mathbb{R}) \otimes L^2(\mathbb{R}^{d-1}) \), we discuss the estimation in the following cases:

**Case 3.1:** If \( b_1 + \cdots + b_d = 0 \), then

\[
\left\| \left( \prod_{B \in P} \frac{\partial}{\partial g_{\lambda(B)}} \right)^m \left( \prod_{B \in P} \left( \frac{\partial}{\partial t_b} \right)^{g_{\lambda(B)}} \right) \right\|_{L^2} \leq \left\| m(q, u) \frac{\partial^{a_1+\cdots+a_d}}{\partial x_1^{a_1} \cdots \partial x_d^{a_d} \vec{v}} \right\|_{L^2}
\]

\[
\leq \left\| m \right\|_{L^\infty} \left\| \frac{\partial^{a_1+\cdots+a_d}}{\partial x_1^{a_1} \cdots \partial x_d^{a_d} \vec{v}} \right\|_{L^2} \leq \left\| m \right\|_{L^\infty} \left\| \vec{v} \right\|_{H^k}.
\]
Case 3.2: If $0 < b_1 + \cdots + b_d < a_1 + \cdots + a_d$, let $P = \{B_1, B_2, \ldots, B_{l-1}\}$, the $l$-tuple $(n_1, \ldots, n_l) = (|B_1|, \ldots, |B_{l-1}|, a_1 - b_1 + \cdots + a_d - b_d)$. Then we use Lemma A.1 and Lemma A.2 according to Case 3.2:

\[
\left\| \left( \prod_{b \in B} \left( \frac{\partial}{\partial g_{\lambda(B)}} \right) \right)^m \left\{ \prod_{b \in B} \left( \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right\} \left\| \frac{\partial^{a_1-b_1+\cdots+a_d-b_d}}{\partial x_1^{a_1-b_1} \cdots \partial x_d^{a_d-b_d}} \bar{v} \right\|_{L^2}^2 \right. \\
\left. \leq \| m^{(l-1)} \|_{L^\infty} \left\| \left\| \prod_{b \in B} \left( \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right\|_{L^{p_1}} \cdots \left\| \prod_{b \in B_{l-1}} \left( \frac{\partial}{\partial t_b} \right) g_{\lambda(B_{l-1})} \right\|_{L^{p_{l-1}}} \right. \\
\left. \left. \left\| \frac{\partial^{a_1-b_1+\cdots+a_d-b_d}}{\partial x_1^{a_1-b_1} \cdots \partial x_d^{a_d-b_d}} \bar{v} \right\|_{L^{p_l}} \right. \\
\left. \leq \| m \|_{C^{l-1}} \| g_{\lambda(B_1)} \|_{W^{n_1,p_1}} \cdots \| g_{\lambda(B_{l-1})} \|_{W^{n_{l-1},p_{l-1}}} \| \bar{v} \|_{W^{n_l,p_l}} \right. \\
\left. \leq \| m \|_{C^{l-1}} \| g_{\lambda(B_1)} \|_{H^k} \cdots \| g_{\lambda(B_{l-1})} \|_{H^k} \| \bar{v} \|_{H^k}. \right. 
\]

Case 3.3: If $b_1 + \cdots + b_d = a_1 + \cdots + a_d$, when $|P| = 1$, $g_{\lambda(B)} = g_i$, $i = 1, 2$ or 3, we use Sobolev embedding $H^k(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ then

\[
\left\| \left( \prod_{b \in P} \left( \frac{\partial}{\partial g_{\lambda(B)}} \right) \right)^m \left\{ \prod_{b \in B} \left( \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right\} \left\| \frac{\partial^{a_1-b_1+\cdots+a_d-b_d}}{\partial x_1^{a_1-b_1} \cdots \partial x_d^{a_d-b_d}} \bar{v} \right\|_{L^2}^2 \right. \\
\left. \leq \| m^{(1)} \|_{L^\infty} \| \bar{v} \|_{L^\infty} \left\| \frac{\partial^{a_1+\cdots+a_d}}{\partial x_1^{a_1} \cdots \partial x_d^{a_d}} g_i \right\|_{L^2} \\
\left. \leq \| m \|_{C^1} \| \bar{v} \|_{H^k} \| g_i \|_{H^k}; \right. 
\]

when $|P| = l > 1$, let $P = \{B_1, B_2, \ldots, B_l\}$, the $l$-tuple $(n_1, \ldots, n_l) = (|B_1|, \ldots, |B_l|)$. Then we use Lemma A.1 and Lemma A.2 according to $\frac{1}{p_i} = \left( \frac{1}{2} - \frac{k}{d} \right)_l + \frac{n_i}{d}$, $i = 1, \ldots, l$ and the Sobolev embedding $H^k(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ to obtain that

\[
\left\| \left( \prod_{b \in P} \left( \frac{\partial}{\partial g_{\lambda(B)}} \right) \right)^m \left\{ \prod_{b \in B} \left( \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right\} \left\| \frac{\partial^{a_1-b_1+\cdots+a_d-b_d}}{\partial x_1^{a_1-b_1} \cdots \partial x_d^{a_d-b_d}} \bar{v} \right\|_{L^2}^2 \right. \\
\left. \leq \| m \|_{C^{l-1}} \| \bar{v} \|_{H^k} \| g_i \|_{H^k}. \right. 
\]
achieved, then the mappings are also locally Lipschitz on the given sets on $E$ by using the fact that $\|\cdot\|_E$

show that the mappings are locally Lipschitz on the given sets on $E$

Similarly, let $l = |P| + 1$, $P = \{B_1, \ldots, B_{l-1}\}$, the $l$-tuple $(n_1, \ldots, n_l) = (|B_1|, \ldots, |B_{l-1}|, a_1 - b_1 + \cdots + a_d - b_d)$ and let $\gamma_\alpha = e^{\alpha x_1}$,

$$
\| \gamma_\alpha \left( \left( \prod_{B \in P} \frac{\partial}{\partial g_\lambda(B)} \right)^m \left( \prod_{B \in P} \frac{\partial}{\partial B} g_\lambda(B) \right) \right) \|_{L^2} \\
\leq \|m^{(|P|)}\|_{L^\infty} \prod_{b \in B_1} \left( \frac{\partial}{\partial B} g_\lambda(B_1) \right) \cdots \prod_{b \in B_{l-1}} \left( \frac{\partial}{\partial B} g_\lambda(B_{l-1}) \right) \\
\leq \|m\|_{C^l} \|g_\lambda(B_1)\|_{W^{n,1,p_1}} \cdots \|g_\lambda(B_{l-1})\|_{W^{n_{l-1},p_{l-1}}} \|\gamma_\alpha \bar{v}\|_{W^{n_1,p_1}} \\
\leq \|m\|_{C^l-1} \|g_\lambda(B_1)\|_{H^k} \cdots \|g_\lambda(B_{l-1})\|_{H^k} \|\gamma_\alpha \bar{v}\|_{H^k};$

the similar discussion in the weighted norm when $b_1 + \cdots + b_d = 0$, or $b_1 + \cdots + b_d = a_1 + \cdots + a_d$ also follows.

Using the separate Lipschitz estimates for variations in $q$, $u$, and $v$, one can easily show that the mappings are locally Lipschitz on the given sets on $E_\alpha$. On $E = E_\alpha \cap E_\alpha$, by using the fact that $\|\cdot\|_0 \leq \|\cdot\|_E$ and $\|\cdot\|_\alpha \leq \|\cdot\|_E$, the estimates can also be achieved, then the mappings are also locally Lipschitz on the given sets on $E$.
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