SURFACE TO SURFACE CHANGES OF VARIABLES AND APPLICATIONS

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by

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The undersigned, appointed by the Dean of the Graduate School, have examined the thesis entitled

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a candidate for the degree of Master of Science for Teachers and hereby certify that in their opinion it is worthy of acceptance.

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ABSTRACT

The present thesis addresses a number of basic problems in relation to integration over surfaces in the Euclidean space, such as

- how the surface measure and unit normal changes under a smooth diffeomorphism

- how the integration process is affected by a surface to surface change of variables.

We provide precise answers to these and other related issues, and discuss a number of applications, such as the invariance of Lebesgue and Sobolev spaces on surfaces.

Chapter 1

Introduction

The main aim of this thesis can be heuristically described as follows. Consider a sufficiently regular surface $\Sigma \subseteq \mathbb{R}^n$ and assume that a C^∞-diffeomorphism F of the ambient space $\mathbb{R}^n$ has been given. Set $\widetilde{\Sigma} := F(\Sigma)$. Does it follow that $\widetilde{\Sigma}$ is also a regular surface, and if so, then how does the geometry of $\widetilde{\Sigma}$ relate to that of Σ? A concrete aspect of the latter issue is: how is $\widetilde{\nu}$, the unit normal to $\widetilde{\Sigma}$, related to ν, the unit normal to Σ?

Going futher, it is natural to ask how the integration process on $\widetilde{\Sigma}$ is related to that on Σ. More concretely, given a reasonable function $f : \Sigma \longrightarrow \mathbb{R}$, what is the relationship between the integral of this function on Σ and that of $f \circ F^{-1}$ on $\widetilde{\Sigma}$? In essence, we would like to generalize the celebrated *Classical Change of Variables Formula* which gives the relationship of the integration process between open subsets of $\mathbb{R}^{n-1}$. The latter then becomes a particular case of our theory, corresponding to the situation when the surfaces involved are flat.

We shall address all the aforementioned issues and, in fact, go on to consider finer aspects of the intergration theories on Σ and $\widetilde{\Sigma}$. Specifically, we shall identify how the Lebesgue and Sobolev spaces transform under the surface-to-surface change of variables

$$\Sigma \ni x \mapsto F(x) \in \widetilde{\Sigma}. \tag{1.1}$$

The first order of business is to define the integral over a surface Σ. Recall that if $\Sigma \subseteq \mathbb{R}^n$ is the graph of a C^1 function $\phi : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ and if $f : \Sigma \longrightarrow \mathbb{R}$ is measurable, then by definition

$$\int_\Sigma f \, d\sigma := \int_{\mathbb{R}^{n-1}} f(x', \phi(x')) \sqrt{1 + \|\nabla \phi(x')\|^2} \, dx'.$$

A set $E \subseteq \Sigma$ is called measurable if $\{x' \in \mathbb{R}^{n-1} : (x', \phi(x')) \in E\}$ is a *Borel* measurable set in $\mathbb{R}^{n-1}$. Moreover, a function $f : \Sigma \longrightarrow \mathbb{R}$ is measurable if $f^{-1}(I)$ is a measurable set of Σ for all $I \subseteq \mathbb{R}$ such that I is an open interval.

In practical applications, it is often the case that Σ has a "nice" (local) parametrization. By this, we mean that for all $x \in \Sigma$ there exists $r > 0$ such that $\Sigma \cap B(x, r) = P(\mathcal{O})$ where

(i) $\mathcal{O}$ is an open subset of $\mathbb{R}^{n-1}$;

(ii) $P : \mathcal{O} \longrightarrow \mathbb{R}^n$ is injective;

(iii) $P : \mathcal{O} \longrightarrow \mathbb{R}^n$ is a C^1 map;

(iv) $\text{rank}[DP(u)] = n - 1$, for all $u = (u_1, \ldots, u_{n-1}) \in \mathcal{O}$, where

$$\left(\frac{D(P_1, \ldots, P_n)}{D(u_1, \ldots, u_{n-1})} \right)(u) = DP(u)$$

is the *Jacobian* matrix of P.

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A basic issue is then finding a way of expressing this when a parametrization P for Σ is available. In the three-dimensional setting, a classical formula to this effect is

$$\int_\Sigma f \, d\sigma = \int_\mathcal{O} f \circ P \, \|\partial_1 P \times \partial_2 P\| \, du_1 du_2.$$

Efforts of extending this to more general situations run into two immediate difficulties. First, generally speaking, surfaces can only be parametrized *locally* and typically lack a global parametrization. We overcome this problem by making appeal to the so-called *Partition of Unity*. Informally speaking, this allows us to piece together into a global fashion, local results, which is a very useful feature.

The second difficulty is finding an appropriate substitute for the cross-poduct $\partial_1 P \times \partial_2 P$ when $n \neq 3$. When $n = 3$, it is well-known that, given any two vectors $v_1, v_2 \in \mathbb{R}^3$, one has

$$v_1 \times v_2 = \det \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ e_1 & e_2 & e_3 \end{pmatrix},$$

where e_1, e_2, e_3 is the standard orthonormal basis in $\mathbb{R}^3$. In spite of the fact that this seems an intrinsic three-diemsnional operation with vectors, here we are able to generalize the concept of cross-product to Euclidean spaces of arbitrary dimension. The key feature of our extension is the observation that, in $\mathbb{R}^n$, the cross-product should actually involve $n-1$ vectors (so that, when $n = 3$, we are back to considering two vectors). More specifically, given $(v_1, v_2, \ldots, v_{n-1})$, $n-1$ vectors in $\mathbb{R}^n$, we define their *Cross Product in $\mathbb{R}^n$* as

$$v_1 \times v_2 \times \cdots \times v_{n-1} = \det \begin{pmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ v_{n-1\,1} & v_{n-1\,2} & \cdots & v_{n-1\,n} \\ e_1 & e_2 & \cdots & e_n \end{pmatrix},$$

where $e_1, e_2, \ldots, e_n$ are the vectors of the standard orthonormal basis in $\mathbb{R}^n$. The above, is to be understood as the vector which is obtained by formally expanding the determinant with respect to the last line. Some of the properties of the cross product which we establish are as follows:

1. $\langle v_1 \times v_2 \times \cdots \times v_{n-1}, v_n \rangle$ is the (oriented) volume of the parallelopiped spanned by the vectors $v_1, \ldots, v_n$ in $\mathbb{R}^n$.

2. The vector $v_1 \times v_2 \times \cdots \times v_{n-1}$ is perpendicular to each of the vectors $v_1, \ldots, v_{n-1}$;

3. If A is an $n \times n$ invertible matrix and $v_1, \ldots, v_{n-1}$ are $n-1$ vectors in $\mathbb{R}^n$, then

$$Av_1 \times \cdots \times Av_{n-1} = (\det A)(A^{-1})^\top (v_1 \times \cdots \times v_{n-1}),$$

4. If $\mathcal{R}$ is a rotation of $\mathbb{R}^n$ about the origin, then

$$\|\mathcal{R}v_1 \times \cdots \times \mathcal{R}v_{n-1}\| = \|v_1 \times \cdots \times v_{n-1}\|.$$

Having introduced this new concept of multidimensional cross-product, it is natural to speculate that the following is true:

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Theorem 1. *Assume that $\Sigma \subset \mathbb{R}^n$ is a surface that has a global canonical parametrization $P : \mathcal{O} \to \Sigma \hookrightarrow \mathbb{R}^n$, where $\mathcal{O}$ is an open subset of $\mathbb{R}^{n-1}$. Then for every absolutely integrable function $f : \Sigma \longrightarrow \mathbb{R}$, there holds*

$$\int_\Sigma f \, d\sigma = \int_\mathcal{O} f \circ P \, \|\partial_1 P \times \ldots \times \partial_{n-1} P\| \, du_1 \ldots du_{n-1}.$$

In Chapter 5, starting from the definition of integration on Σ and making use of the properties of the cross product in $\mathbb{R}^n$, we show that this is indeed the case. It is worth mentioning that the above formula is a key ingredient in the proof of many of the subsequent results we establish in this thesis. In particular, this plays a paramount role in the generalization of the *Classical Change of Variables Formula.*

For practical applications, it is also of interest to derive a formula similar to the one given above which makes no direct reference to the multi-dimensional cross product. This is indeed possible, as we prove the following:

Theorem 2. *Assume that $\Sigma \subset \mathbb{R}^n$ is a surface that has a global parametrization $P : \mathcal{O} \to \Sigma \hookrightarrow \mathbb{R}^n$, where $\mathcal{O}$ is an open subset of $\mathbb{R}^{n-1}$. Then for every absolutely integrable function $f : \Sigma \longrightarrow \mathbb{R}$, there holds*

$$\int_\Sigma f \, d\sigma = \int_\mathcal{O} (f \circ P) \left(\sum_{j=1}^n \left[\det \left(\frac{D(P_1 \ldots \widehat{P}_j \ldots P_n)}{D(u_1 \ldots u_{n-1})} \right) \right]^2 \right)^{\frac{1}{2}} du_1 \ldots du_{n-1},$$

where hat indicates omission.

Having successfully linked the integration process on Σ to the cross product on $\mathbb{R}^n$, we next aim to produce a formula for the unit normal on Σ. Specifically, we show:

Theorem 3. *Assume that $\Sigma \subset \mathbb{R}^n$ is a surface which has a global canonical parametrization $P : \mathcal{O} \to \Sigma \hookrightarrow \mathbb{R}^n$, where $\mathcal{O}$ is an open subset of $\mathbb{R}^{n-1}$. If ν is the unit normal to the surface Σ, then*

$$\nu \circ P = \frac{\partial_1 P \times \partial_2 P \times \ldots \times \partial_{n-1} P}{\|\partial_1 P \times \partial_2 P \times \ldots \times \partial_{n-1} P\|} \qquad \text{on } \mathcal{O}.$$

Again, for various practical considerations it is useful to derive a formula for ν independent of the multidimensional cross product. That formula reads as follows:

Theorem 4. *Assume that $\Sigma \subset \mathbb{R}^n$ is a surface which has a global canonical parametrization $P : \mathcal{O} \to \Sigma \hookrightarrow \mathbb{R}^n$, where $\mathcal{O}$ is an open subset of $\mathbb{R}^{n-1}$. If ν is the unit normal to the surface Σ, then for every $j \in \{1, 2, \ldots, n\}$ there holds*

$$\nu_j \circ P = \frac{(-1)^{j+1} \det(\mathrm{A_j})}{\left(\sum_{k=1}^n \left[\det \left(\frac{\mathrm{D(P_1 \ldots \hat{P}_j \ldots P_n)}}{\mathrm{D(u_1 \ldots u_{n-1})}} \right) \right]^2 \right)^{\frac{1}{2}}},$$

where

$$A_j = \begin{pmatrix} \partial_1 P_1 & \ldots & \partial_1 P_{j-1} & \partial_1 P_{j+1} & \ldots & \partial_1 P_n \\ \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\ \partial_{n-1} P_1 & \ldots & \partial_{n-1} P_{j-1} & \partial_{n-1} P_{j+1} & \ldots & \partial_{n-1} P_n \end{pmatrix}$$

and $\hat{P}_j$ means that P_j is omitted for $1 \le j \le n$.

Granted the tools mentioned above, we are then well-positioned to start exploring the relationship between the integration processes on Σ and $\widetilde{\Sigma}$. At the same time, another aspect we are concerned with is understanding how the geometries of Σ and $\widetilde{\Sigma}$ (manifested through their respective unit normals) are related. Below we list a

number of motivational questions that we will consider. To set the stage, recall that we are assuming that Σ is a nice surface in $\mathbb{R}^n$ and that

$$F : \mathbb{R}^n \longrightarrow \mathbb{R}^n \qquad C^\infty\text{-diffeomorphism}$$

is a diffeomorphism (i.e., F is of class C^∞, F is bijective, and F^{-1} is of class C^∞).

Question 1

Under the above hypotheses, does it follow that

$$\widetilde{\Sigma} = F(\Sigma)$$

is also a smooth surface in $\mathbb{R}^n$?

If the answer to this question is "yes", then we may also consider:

Question 2

How does the unit normal $\tilde{\nu}$ to $\widetilde{\Sigma}$ relate to the unit normal ν to Σ ?

Question 3

How does the surface area element $d\tilde{\sigma}$ for $\widetilde{\Sigma}$ relate to the surface area element $d\sigma$ for Σ ?

Question 4

How does the integration process on $\widetilde{\Sigma}$ relate to the integration process on Σ ? (i.e. is there a change of variables formula from Σ to $\widetilde{\Sigma}$)

As mentioned above, the main tools needed to elucidate these issues are diligently addressed by Theorems 1-4 above. Based on these, in Chapter 7 we then establish

the following:

Theorem 5. *Let $\Sigma = P(\mathcal{O})$ where $P : \mathcal{O} \longrightarrow \mathbb{R}^n$ is a global parametrization of Σ. Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a C^∞-diffeomorphism. Then $\widetilde{\Sigma} := F(\Sigma)$ is a smooth surface in $\mathbb{R}^n$.*

Furthermore, if $f : \widetilde{\Sigma} \longrightarrow \mathbb{R}$ is an arbitrary absolutely integrable function, then

$$\int_{\widetilde{\Sigma}} f \, d\widetilde{\sigma} = \int_{\Sigma} (f \circ F) \, |\det(DF)| \, \left\| ((DF)^{-1})^\top \nu \right\| \, d\sigma.$$

A remarkable feature of the above surface-to-surface change of variables formula is that the *Classical Change of Variables Formula* becomes a particular case of it. Before proving that this is indeed the case, let us recall the actual statement of the latter.

The Classical Change of Variables Formula. *Let $D \subseteq \mathbb{R}^n$ be such that D is open, and let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be an arbitrary absolutely integrable function. Consider next a function g such that:*

1. *$g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a C^∞-diffeomorphism;*

2. *$g(\mathcal{O}) = D$;*

3. *$\mathcal{O} \subseteq \mathbb{R}^n$, $\mathcal{O}$ open.*

Then

$$\int_D f(x) \, dx = \int_{\mathcal{O}} f(g(y)) \, |\det(Dg)(y)| \, dy.$$

In order to show this is a particular case of Theorem 5, we shall regard $\mathcal{O}$ as a flat surface in $\mathbb{R}^{n+1}$ by making the following identifications:

(i) $\widetilde{\Sigma} := D \times \{0\}$;

(ii) $\widetilde{f}(x, 0) := f(x)$, $x \in D$;

(iii) $d\widetilde{\sigma} = dx_1 dx_2 \ldots dx_n = dx$.

Next, we define

1. $\Sigma := \mathcal{O} \times \{0\}$;

2. $F(x, x_{n+1}) := (g(x), x_{n+1})$,

so that $F : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ becomes a C^∞-diffeomorphism. Note that

$$(\widetilde{f} \circ F)(x, 0) = \widetilde{f}(F(x, 0)) = \widetilde{f}(g(x), 0) = f(g(x))$$

and, given that the surface is flat,

$$d\sigma = dx_1 dx_2 \ldots dx_n = dx.$$

Moreover, it can be shown that the following identities hold:

$$|\det(DF)(x, x_{n+1})| = |\det(Dg)(x)|,$$

and

$$\left\| (((DF)(x, x_{n+1}))^{-1})^\top \nu(x, x_{n+1}) \right\| = 1.$$

Using the above substitutions and identities, it is a straightforward matter to show that the *Classical Change of Variables* is simply a particular case of our formula in Theorem 5.

Moving on, we produce an explicit formula for $\widetilde{\nu}$, the unit normal on $\widetilde{\Sigma}$ which reads as follows:

Theorem 6. *Let $\Sigma \subseteq \mathbb{R}^n$ be a surface with unit normal ν and let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a C^∞-diffeomorphism. Denote $\widetilde{\Sigma} := F(\Sigma)$ and let $\widetilde{\nu}$ be the unit normal to $\widetilde{\Sigma}$. Then*

$$\widetilde{\nu} = \frac{(DF^{-1})^\top (\nu \circ F^{-1})}{\|(DF^{-1})^\top (\nu \circ F^{-1})\|} \quad \text{on } \widetilde{\Sigma}.$$

In turn, having established a *Surface-to-Surface Change of Variables Formula* along with a formula for the unit normal, in Chapter 11 we turn our attention to showing the invariance of Lebesgue and Sobolev spaces defined on surfaces. Our first result in this regard is the following:

Theorem 7. *Assume that $\Sigma \subset \mathbb{R}^n$ is a C^1 surface, $\mathcal{O} \subset \mathbb{R}^n$ is an open neighborhood of Σ, and $F : \mathcal{O} \to \mathbb{R}^n$ be an orientation preserving C^1-diffeomorphism onto its image. Set $\widetilde{\Sigma} := F(\Sigma)$. Then for each $1 \le p < \infty$, the operator*

$$T : L^p(\Sigma) \longrightarrow L^p(\widetilde{\Sigma})$$

defined by

$$T(f) := f \circ F^{-1}, \qquad f \in L^p(\Sigma),$$

is well-defined, linear, and bounded. In fact, T is an isomorphism.

Once this theorem is proved, we then define and establish basic formulas for the tangential gradient of a function $f : \Sigma \longrightarrow \mathbb{R}$. The definition of the tangential gradient is as follows. Given a C^1 surface $\Sigma \subset \mathbb{R}^n$ with unit normal ν, we define the tangential gradient of a function $f : \Sigma \longrightarrow \mathbb{R}$ by

$$\nabla_{tan} f := \nabla f - \langle \nabla f, \nu \rangle \nu.$$

We can think of the tangential gradient coordinate-wise in the following manner:

$$\left(\nabla_{tan} f \right)_j = \sum_{k=1}^{n} \nu_k \partial_{\tau_{kj}} f, \qquad 1 \le j \le n,$$

where we have set

$$\partial_{\tau_{jk}} f := \left(\nu_j \partial_k - \nu_k \partial_j \right) f, \qquad 1 \le j, k \le n.$$

As a consequence, there exist dimensional constants $C_1, C_2 > 0$ such that

$$C_1 \| \nabla_{tan} f \| \le \sum_{1 \le j, k \le n} | \partial_{\tau_{jk}} f | \le C_2 \| \nabla_{tan} f \|,$$

pointwise on Σ.

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The next order of business is to define a Sobolev space of order one on Σ. We do so as follows. Let $1 \leq p < \infty$, and set

$$W^{1,p}(\Sigma) := \left\{ f \in L^p(\Sigma) : (\nabla_{tan} f)_j \in L^p(\Sigma), \ 1 \leq j \leq n \right\}.$$

This becomes a Banach space when equipped with the norm

$$\|f\|_{W^{1,p}(\Sigma)} := \|f\|_{L^p(\Sigma)} + \sum_{j=1}^{n} \|(\nabla_{tan} f)_j\|_{L^p(\Sigma)}.$$

To make matters simpler in the calculations we will subsequently attempt, we will need an equivalent norm to the one given above. We show that an equivalent norm on $W^{1,p}(\Sigma)$ is given by

$$\|f\|_{W^{1,p}(\Sigma)} = \|f\|_{L^p(\Sigma)} + \sum_{1 \leq j,\, k \leq n} \|\partial_{\tau_{jk}} f\|_{L^p(\Sigma)}.$$

Before stating the main invariance result for Sobolev spaces, we define and discuss the properties of the tensor product between two vectors in $\mathbb{R}^n$. We define the tensor product between $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$ to be

$$a \otimes b := (a_j b_k)_{1 \leq j,k \leq n}.$$

In other words, $a \otimes b$ is the $n \times n$ matrix whose jk-th entry is $a_j b_k$. Having defined the tensor product, we then discuss properties of the tensor product. They are the following:

$$(a \otimes b)^\top = b \otimes a, \qquad \forall\, a, b \in \mathbb{R}^n,$$

and

$$\langle a \otimes b, c \rangle = \langle b, c \rangle\, a, \qquad \forall\, a, b, c \in \mathbb{R}^n.$$

Also,

$$a \otimes b - b \otimes a = a_b \otimes b - b \otimes a_b, \qquad \text{where} \quad a_b := a - \langle a, b \rangle b.$$

After establishing the above tools, we are finally ready to state and prove the following theorem:

Theorem 8. *Assume that $\Sigma \subset \mathbb{R}^n$ is a C^1 surface, $\mathcal{O} \subset \mathbb{R}^n$ is an open neighborhood of Σ, and $F : \mathcal{O} \to \mathbb{R}^n$ be an orientation preserving C^1-diffeomorphism onto its image. Set $\widetilde{\Sigma} := F(\Sigma)$. Then for each $1 \le p < \infty$, the operator*

$$T : W^{1,p}(\Sigma) \longrightarrow W^{1,p}(\widetilde{\Sigma})$$

defined by

$$T(f) := f \circ F^{-1}, \qquad f \in W^{1,p}(\Sigma),$$

is well-defined, linear, and bounded. In fact, T is an isomorphism.

In closing, we wish to mention that the proof of the above result is subtle and makes essential use of Theorem 7, concerning the invariance of Lebesgue spaces under a surface-to-surface change of variables, as well as properties of the tensor product of vectors, as recalled above.

Chapter 2

Properties of the Cross Product in $\mathbb{R}^n$

The first order of business is to define the cross product $v_1 \times v_2 \times \cdots \times v_{n-1}$ of $n-1$ vectors $v_1, \ldots, v_{n-1} \in \mathbb{R}^n$. Below and elsewhere, $e_1, \ldots, e_n$ are the vectors of the standard orthonormal basis of $\mathbb{R}^n$.

Definition 2.0.1. *If $v_1 = (v_{11}, \ldots, v_{1n}), \ldots, v_{n-1} = (v_{n-1\,1}, \ldots, v_{n-1\,n})$ are $n-1$ vectors in $\mathbb{R}^n$ then*

$$v_1 \times v_2 \times \cdots \times v_{n-1} = \det \begin{pmatrix} v_{11} & v_{12} & \ldots & v_{1n} \\ v_{21} & v_{22} & \ldots & v_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ v_{n-1\,1} & v_{n-1\,2} & \ldots & v_{n-1\,n} \\ e_1 & e_2 & \ldots & e_n \end{pmatrix}, \tag{2.1}$$

where the determinant is understood as computed by formally expanding it with respect to the last row, the result being a vector in $\mathbb{R}^n$.

Below we summarize some of the main properties of the cross product.

Proposition 2.0.1. *The cross product introduced in Definition 2.0.1 enjoys the following properties:*

(i) $\langle v_1 \times v_2 \times \cdots \times v_{n-1}, v_n \rangle$ is the (oriented) volume of the parallelopiped spanned by the vectors $v_1, \ldots, v_n$ in $\mathbb{R}^n$.

(ii) The vector $v_1 \times v_2 \times \cdots \times v_{n-1}$ is perpendicular to each of the vectors $v_1, \ldots, v_{n-1}$;

(iii) If A is an $n \times n$ invertible matrix and $v_1, \ldots, v_{n-1}$ are $n-1$ vectors in $\mathbb{R}^n$, then

$$Av_1 \times \cdots \times Av_{n-1} = (\det A)(A^{-1})^{\top}(v_1 \times \cdots \times v_{n-1}), \tag{2.2}$$

where "$\top$" stands for transposition of matrices.

(iv) If $\mathcal{R}$ is a rotation of $\mathbb{R}^n$ about the origin, then

$$\|\mathcal{R}v_1 \times \cdots \times \mathcal{R}v_{n-1}\| = \|v_1 \times \cdots \times v_{n-1}\|. \tag{2.3}$$

Proof: From (2.1),

$$v_1 \times \ldots \times v_{n-1} = \sum_{j=1}^{n} (-1)^{j+1} e_j \det A_j \tag{2.4}$$

where

$$A_j = \begin{pmatrix} v_{11} & \cdots & v_{j-1} & v_{j+1} & \cdots & v_{1n} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ v_{n-1\,1} & \cdots & v_{n-1\,j-1} & v_{n-1\,j+1} & \cdots & v_{n-1\,n} \end{pmatrix} \tag{2.5}$$

Then by (2.4) and (2.1),

$$\langle v_1 \times \ldots \times v_{n-1}, v_n \rangle = \left[\sum_{j=1}^{n} (-1)^{j+1} (\det A_j) e_j \right] \cdot v_n = \sum_{j=1}^{n} (-1)^{j+1} v_{nj} \det A_j$$

$$= \det \begin{pmatrix} v_{11} & \ldots & v_{1n} \\ \vdots & \ldots & \vdots \\ v_{n-1\,1} & \ldots & v_{n-1\,n} \\ v_{n1} & \ldots & v_{nn} \end{pmatrix}. \tag{2.6}$$

As in (2.1), this determinant is to be expanded by the last row. Moreover, this determinant is known to be the volume. This finishes (i) of (2.0.1).

For (ii), observe that for every fixed $j \in \{1, 2, \ldots, n-1\}$, we have by (2.6)

$$\langle v_1 \times \ldots \times v_{n-1}, v_j \rangle = \det \begin{pmatrix} v_{11} & \ldots & v_{1n} \\ \vdots & & \vdots \\ v_{j1} & \ldots & v_{jn} \\ \vdots & & \vdots \\ v_{n-1\,1} & \ldots & v_{n-1\,n} \\ v_{j1} & \ldots & v_{jn} \end{pmatrix} = 0,$$

since this is the determinant of a matrix which has two identical rows. The proof of this statement can be found in the Appendix. Now, since generally speaking,

$$v \perp w \iff \langle v, w \rangle = 0 \tag{2.7}$$

(i.e., two vectors are perpendicular if and only if their dot product is zero). This finishes (ii) of Proposition 2.0.1.

Before proving (iii), we will make use of the following facts whose proofs can be found in the Appendix. Let A be an $n \times n$ matrix and $v, w \in \mathbb{R}^n$, then

$$\langle A^\top v, w \rangle = \langle v, Aw \rangle, \tag{2.8}$$

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and

$$\text{if } \langle u, w \rangle = \langle v, w \rangle \quad \text{for all } w \in \mathbb{R}^n, \text{ then } u = v. \tag{2.9}$$

Moreover, if B is an $n \times n$ matrix and $\lambda \in \mathbb{R}$, then

$$\lambda \langle u, v \rangle = \langle \lambda u, v \rangle = \langle u, \lambda v \rangle, \tag{2.10}$$

$$\det B = \det(B^\top), \tag{2.11}$$

and

$$\det(AB) = \det A \cdot \det B. \tag{2.12}$$

In particular if B is an $n \times n$ invertible matrix, then

$$(B^{-1})^\top = (B^\top)^{-1}. \tag{2.13}$$

Returning to the mainstream discussion, let $v_n \in \mathbb{R}^n$. Using (2.8) gives

$$\left\langle A^\top (Av_1 \times \ldots \times Av_{n-1}), v_n \right\rangle = \left\langle Av_1 \times \ldots \times Av_{n-1}, Av_n \right\rangle.$$

By (2.6), we have

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$$\langle Av_1 \times \ldots \times Av_{n-1}, Av_n \rangle \;=\; \det \begin{pmatrix} (Av_1)_1 & \ldots & (Av_1)_n \\ \vdots & & \vdots \\ (Av_n)_1 & \ldots & (Av_n)_n \end{pmatrix}$$

$$= \det \begin{pmatrix} \sum_{i=1}^{n} a_{1i}v_{1i} & \ldots & \sum_{i=1}^{n} a_{ni}v_{1i} \\ \vdots & & \vdots \\ \sum_{i=1}^{n} a_{1i}v_{ni} & \ldots & \sum_{i=1}^{n} a_{ni}v_{ni} \end{pmatrix}$$

$$= \det \begin{pmatrix} \sum_{i=1}^{n} v_{1i}a_{1i} & \ldots & \sum_{i=1}^{n} v_{1i}a_{ni} \\ \vdots & & \vdots \\ \sum_{i=1}^{n} v_{ni}a_{1i} & \ldots & \sum_{i=1}^{n} v_{ni}a_{ni} \end{pmatrix}$$

$$= \det \left[\begin{pmatrix} v_{11} & \ldots & v_{1n} \\ \vdots & & \vdots \\ v_{n1} & \ldots & v_{nn} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & \ldots & a_{n1} \\ \vdots & & \vdots \\ a_{1n} & \ldots & a_{nn} \end{pmatrix} \right] \quad (2.14)$$

Using (2.12), (2.11), (2.6),and (2.10) in (2.14) yield

$$\det \left[\begin{pmatrix} v_{11} & \ldots & v_{1n} \\ \vdots & & \vdots \\ v_{n1} & \ldots & v_{nn} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & \ldots & a_{n1} \\ \vdots & & \vdots \\ a_{1n} & \ldots & a_{nn} \end{pmatrix} \right] \;=\; \det \begin{pmatrix} v_{11} & \ldots & v_{1n} \\ \vdots & & \vdots \\ v_{n1} & \ldots & v_{nn} \end{pmatrix} \cdot \det(A^{\top})$$

$$= (\det A) \langle v_1 \times \ldots \times v_{n-1}, v_n \rangle$$

$$= \langle (\det A)(v_1 \times \ldots \times v_{n-1}), v_n \rangle .$$

Hence,

$$\left\langle A^{\top}(Av_1 \times \ldots \times Av_{n-1}), v_n \right\rangle = \langle (\det A)(v_1 \times \ldots \times v_{n-1}), v_n \rangle \qquad \text{for all } v_n \in \mathbb{R}^n \quad (2.15)$$

Using (2.9) on (2.15), we have

$$A^{\top}(Av_1 \times \ldots \times Av_{n-1}) = (\det A)(v_1 \times \ldots \times v_{n-1}) \qquad \text{for all } v_1, \ldots, v_{n-1} \in \mathbb{R}^n \quad (2.16)$$

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Multiplying by $(A^\top)^{-1}$ on both sides of (2.16) and using (2.13) gives

$$Av_1 \times \ldots \times Av_{n-1} = (\det A)(A^{-1})^\top (v_1 \times \ldots \times v_{n-1}).$$

This concludes (iii) of (2.0.1).

For (iv), let us make the following observation:

$$\mathcal{R} : \mathbb{R}^n \longrightarrow \mathbb{R}^n \quad \text{is a } \textit{rotation} \Longleftrightarrow \begin{cases} (1) \exists\, A, \text{ an } n \times n \text{ matrix, such that } A^{-1} = A^\top, \\ (2)\, \mathcal{R}(x) = A \cdot x, \ \forall x \in \mathbb{R}^n. \end{cases} \qquad (2.17)$$

In other words, we will not distinguish between the rotation $\mathcal{R}$ itself and its associated matrix A. We will also make use of the following facts that if $\lambda \in \mathbb{R}$, $v \in \mathbb{R}^n$, and A is an invertible $n \times n$ matrix, then

$$\|\lambda v\| = |\lambda| \, \|v\|\,, \qquad (2.18)$$

and

$$\det(A^{-1}) = \frac{1}{\det A}. \qquad (2.19)$$

Returning to the proof of (iv) we find, by (iii),

$$\mathcal{R}v_1 \times \ldots \times \mathcal{R}v_{n-1} = (\det \mathcal{R})(\mathcal{R}^{-1})^\top (v_1 \times \ldots \times v_{n-1}).$$

Taking the norm and using (2.17) and (2.18) yields

$$\|\mathcal{R}v_1 \times \ldots \times \mathcal{R}v_{n-1}\| = |\det \mathcal{R}|\, \|\mathcal{R}(v_1 \times \ldots \times v_{n-1})\|. \tag{2.20}$$

<u>Claim 1</u> For every rotation $\mathcal{R}$, one has

$$|\det \mathcal{R}| = 1. \tag{2.21}$$

To prove **<u>Claim 1</u>**, we will apply the determinant to the identity $\mathcal{R}^{-1} = \mathcal{R}^\top$. This gives $\det(\mathcal{R}^{-1}) = \det(\mathcal{R}^\top)$. Using (2.19) and (2.11), we have

$$\frac{1}{\det \mathcal{R}} = \det \mathcal{R} \quad \Rightarrow \quad 1 = (\det \mathcal{R})^2$$

$$\Rightarrow \quad \det \mathcal{R} = \pm 1$$

$$\Rightarrow \quad |\det \mathcal{R}| = 1.$$

This finishes **<u>Claim 1</u>**.

<u>Claim 2</u> For every rotation $\mathcal{R}$, one has

$$\|\mathcal{R}u\| = \|u\|, \qquad \forall\, u \in \mathbb{R}^n. \tag{2.22}$$

Note that both sides of the equation in **<u>Claim 2</u>** are nonnegative numbers, thus it is enough to show that $\|\mathcal{R}u\|^2 = \|u\|^2$. To this end, we write

$$\|\mathcal{R}u\|^2 = \langle \mathcal{R}u, \mathcal{R}u \rangle. \tag{2.23}$$

Using (2.8) and (2.17) in (2.23) we get

$$\langle \mathcal{R}u, \mathcal{R}u \rangle = \langle \mathcal{R}^\top \mathcal{R}u, u \rangle = \langle \mathcal{R}^{-1} \mathcal{R}u, u \rangle = \langle u, u \rangle = \|u\|^2.$$

This finishes **Claim 2**.

We now return to the mainstream discussion. Using **Claim 1** and **Claim 2** in (2.20), we get

$$\|\mathcal{R}v_1 \times \ldots \times \mathcal{R}v_{n-1}\| = \|v_1 \times \ldots \times v_{n-1}\|.$$

This finishes (iv) of Proposition 2.0.1. The proof of this result is therefore completed.

$\square$

Chapter 3

Partition of Unity

Many of the results we prove in this thesis have a global nature although the proof requires working at a local level, at least in a first stage. A convenient mathematical tool that permits us to patch together local results is the *Partition of Unity Theorem*. Heuristically, this means that the constant function 1 can be broken up in a number of (smooth) bump functions whose supports have an a priori specified location.

To facilitate further considerations, here we state and prove (in a self-contained fashion) a version of the *Partition of Unity* which suits our purposes.

Theorem 3.0.2. *Let $K \subset \mathbb{R}^n$ be compact, $K \subset \bigcup_{j=1}^{J} U_j$ where U_j is open for $j \in \{1, \ldots, J\}$. Then there exists a finite collection of C^∞ functions $\{\varphi_j\}_{j=1}^{J}$ such that*

1. *For every $1 \le j \le J$, $\text{supp}(\varphi_j)$ is compact and contained in U_j;*

2. *For every $1 \le j \le J$, $0 \le \varphi_j \le 1$;*

3. *$\sum_{j=1}^{J} \varphi_j(x) = 1$, for every $x \in K$.*

Before proceeding with the proof of Theorem 3.0.2, we state and prove the following two lemmas which will be utilized in the proof.

Lemma 3.0.3. *If C is compact, and U is an open set such that $C \subset U$, then there exists a compact set D such that $C \subset \overset{\circ}{D} \subset D \subset U$.*

Proof: Let $V = U^c \cap \overline{B_R(0)}$, where $R > 0$ is large enough so that $\overline{U} \subset B_R(0)$. Then V is compact, so there exists $a > 0$ such that

$$a = \text{dist}(V, C) = \inf_{\substack{x \in C \\ y \in V}} |x - y|. \tag{3.1}$$

Note that $\bigcup_{x \in C} B_{\frac{a}{4}}(x)$ is an open cover of the compact set C. Then by the *Heine-Borel Theorem*, there exists $J \subseteq C$ finite such that $C \subseteq \bigcup_{x \in J} B_{\frac{a}{4}}(x)$. Let

$$D = \bigcup_{x \in J} \overline{B_{\frac{a}{4}}(x)}. \tag{3.2}$$

Then D is compact, and $C \subset \overset{\circ}{D} \subset D \subset U$. This completes the proof of Lemma 3.0.3.

Lemma 3.0.4. *If D is a compact set, and U is an open set such that $D \subset U$, then there exists $\psi \in C^\infty$ such that $\psi > 0$ on D, and $\psi = 0$ outside some open set contained in U.*

Proof: Let

$$f(y) = \begin{cases} e^{-\frac{1}{(y-1)^2}} \cdot e^{-\frac{1}{(y+1)^2}}, & y \in (-1, 1), \\ 0, & y \notin (-1, 1). \end{cases} \tag{3.3}$$

Then $f \in C^\infty(\mathbb{R})$, and $f > 0$ on $(-1, 1)$. Let $\epsilon > 0$. For each $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, let $g_a(x) := f\left(\frac{x_1 - a_1}{\epsilon}\right) \times \cdots \times f\left(\frac{x_n - a_n}{\epsilon}\right)$, where f is as defined in (3.3) and $x \in \mathbb{R}^n$. Then $g_a \in C^\infty(\mathbb{R}^n)$, and

$$\begin{cases} g_a > 0 & \text{on} \quad (a_1 - \epsilon, a_1 + \epsilon) \times \cdots \times (a_n - \epsilon, a_n + \epsilon), \\ g_a = 0 & \text{elsewhere.} \end{cases} \tag{3.4}$$

Set $\alpha = \text{dist}(D, \overline{U}) > 0$. For every $x \in D$, $B_{\frac{\alpha}{4}}(x) \subset U$. Moreover, there exists $\epsilon' > 0$ such that $O_{x_l} = (x_1^l - \epsilon', x_1^l + \epsilon') \times \cdots \times (x_n^l - \epsilon', x_n^l + \epsilon') \subset U$ for each $x_l = (x_1^l, x_2^l, \ldots, x_n^l) \in D$. Hence, $D \subset \bigcup_{x_l \in D} O_{x_l}$. Since D is compact, we can extract a finite subcover such that $D \subset \bigcup_{l=1}^{M} O_{x_l}$. Let

$$\psi(x) = \sum_{l=1}^{M} g_{x_l}(x). \tag{3.5}$$

Then $\psi \in C^\infty$, $\psi > 0$ on $\bigcup_{l=1}^{M} O_{x_l}$, and $\psi = 0$ outside $\bigcup_{l=1}^{M} O_{x_l}$. This proves Lemma 3.0.4.

Now we are ready to give the *Proof of Theorem 3.0.2*.

Let $C_1 := K \backslash \bigcup_{j=2}^{J} U_j$. Then C_1 is compact, and $C_1 \subset U_1$. By Lemma 3.0.3, there exists a compact set D_1 such that $C_1 \subset \overset{\circ}{D_1} \subset D_1 \subset U_1$. Consequently $K \subset D_1 \cup \bigcup_{j=2}^{J} U_j$. By induction, we can construct the sets $D_1, D_2, \ldots, D_J$ such that $K \subset \bigcup_{j=1}^{J} \overset{\circ}{D_j}$, where each D_j is compact and $D_j \subset U_j$ for every $1 \leq j \leq J$. Indeed, if $K \subset \overset{\circ}{D_1} \cup \overset{\circ}{D_2} \cup \ldots \cup \overset{\circ}{D_k} \cup U_{k+1} \cup \ldots \cup U_J$, we let

$$C_{k+1} := K \backslash \left[\bigcup_{j=1}^{k} \overset{\circ}{D_j} \cup \bigcup_{j=k+2}^{J} U_j \right]. \tag{3.6}$$

and by Lemma 3.0.3 we find D_{k+1} with the desired properties.

We now define the functions φ_j by setting

$$\varphi_j(x) := \frac{\psi_j(x)}{\sum_{j=1}^{J} \psi_j(x)}. \tag{3.7}$$

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Then $\varphi_j \in C^\infty$, supp $\varphi_j \subset U_j$, $0 \leq \varphi_j \leq 1$ for every $j \in \{1, 2, \ldots, J\}$, and if $x \in K$, then $x \in \bigcup_{j=1}^{J} \overset{\circ}{D}_j$ and $\sum_{j=1}^{J} \varphi_j(x) = 1$. This completes the proof of Theorem 3.0.2. $\qquad\square$

Chapter 4

Definition of Surfaces and Integrals in Surfaces

Recall that if $\Sigma \subseteq \mathbb{R}^n$ is the graph of a C^1 function $\phi : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ and if $f : \Sigma \longrightarrow \mathbb{R}$ is measurable, then by definition

$$\int_{\Sigma} f \, d\sigma := \int_{\mathbb{R}^{n-1}} f(x', \phi(x')) \sqrt{1 + \|\nabla\phi(x')\|^2} \, dx'. \tag{4.1}$$

Recall that a set $E \subseteq \Sigma$ is called measurable if $\{x' \in \mathbb{R}^{n-1} : (x', \phi(x')) \in E\}$ is a *Borel* measurable set in $\mathbb{R}^{n-1}$. Moreover, a function $f : \Sigma \longrightarrow \mathbb{R}$ is measurable if $f^{-1}(I)$ is a measurable set of Σ for all $I \subseteq \mathbb{R}$ such that I is an open interval.
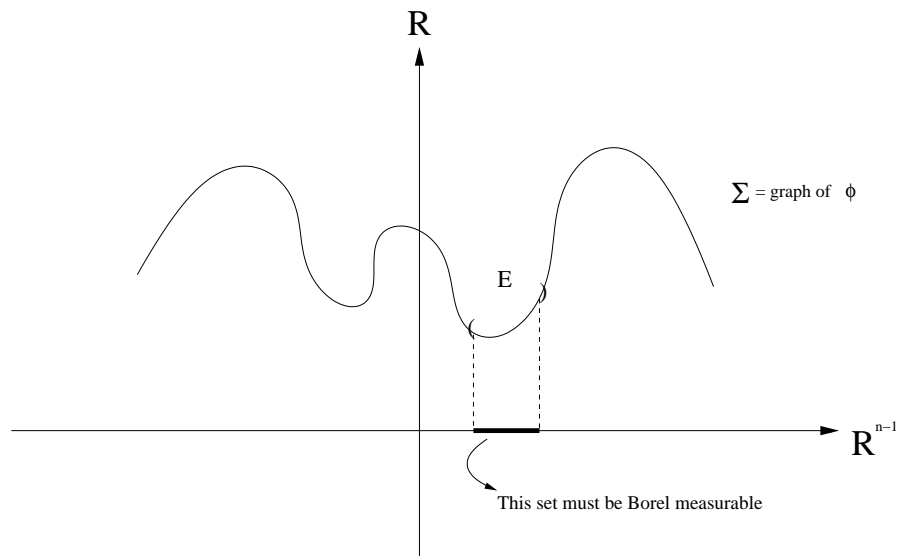
A few comments about (4.1) being a natural definition for integration are in order here:

1. If Σ is flat, i.e. $\Sigma = \mathbb{R}^{n-1} \times \{0\}$, then $\phi = 0$ and (4.1) becomes the equality between $\int_\Sigma f \, d\sigma$ and $\int_{\mathbb{R}^{n-1}} f(x', 0) \, dx'$.

2. If Σ is the graph of $\phi : \mathbb{R} \longrightarrow \mathbb{R}$, i.e. the length of a curve in $\mathbb{R}^2$, then the length of Σ is given by

$$\text{length}(\Sigma_{a,b}) = \int_a^b \sqrt{1 + (\phi'(x))^2} \, dx.$$
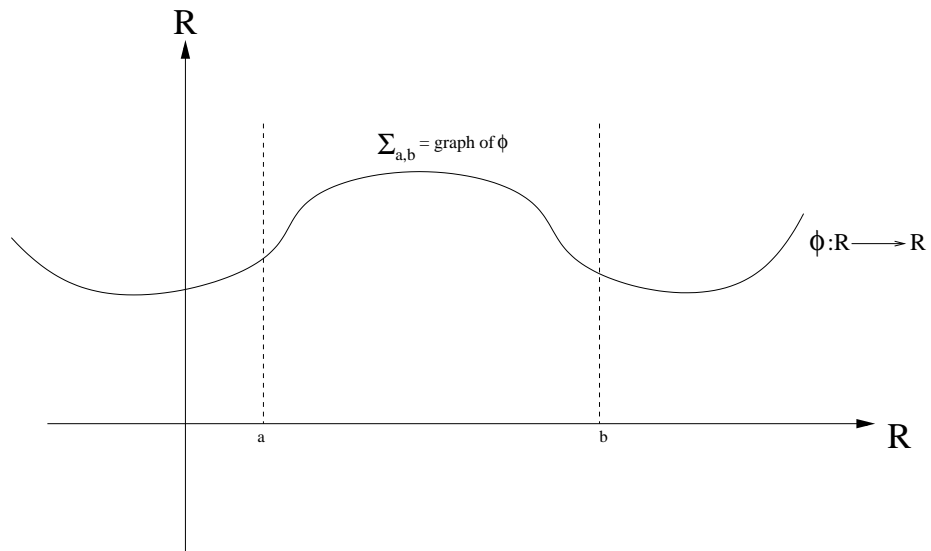
This is in agreement with (4.1) when $n = 2$, and $f = \chi_{\Sigma_{a,b}}$. I.e. $\int_\Sigma f \, d\sigma = \text{length}(\Sigma_{a,b})$ and $\int_{\mathbb{R}^{n-1}} f(x', \phi(x')) \sqrt{1 + \|\nabla\phi(x')\|^2} \, dx' = \int_a^b \sqrt{1 + (\phi'(x))^2} \, dx.$

Let Σ be a C^1 smooth surface in $\mathbb{R}^n$. By definition, this means that locally, Σ has a C^1 parametrization. That is, for every $x \in \Sigma$ there exists $r > 0$ such that

$$\Sigma \cap B(x,r) = P(\mathcal{O}) \tag{4.2}$$

where

$$P : \mathcal{O} \longrightarrow \mathbb{R}^n \quad \text{is a } C^1 \text{ parametrization.} \tag{4.3}$$

By definition, the latter condition means that:

(i) $\mathcal{O}$ is an open subset of $\mathbb{R}^{n-1}$;

(ii) $P : \mathcal{O} \longrightarrow \mathbb{R}^n$ is injective;

(iii) $P : \mathcal{O} \longrightarrow \mathbb{R}^n$ is a C^1 map;

(iv) $\text{rank}[DP(u)] = n-1$, for all $u = (u_1, \dots, u_{n-1}) \in \mathcal{O}$, $\tag{4.4}$

where

$$\left(\frac{D(P_1, \dots, P_n)}{D(u_1, \dots, u_{n-1})} \right)(u) = DP(u) \tag{4.5}$$

is the *Jacobian* matrix of P. It is useful to remark that the last condition above means that for all $u \in \mathcal{O}$, there exists $j \in \{1, 2, \dots, n\}$ such that

$$\det\left(\frac{D(P_1, \dots, P_{j-1}, P_{j+1}, \dots, P_n)}{D(u_1, \dots, u_{n-1})} \right)(u) \neq 0, \tag{4.6}$$

where $P_1, P_2, \dots, P_n$ are the components of P. Or equivalently

$$\nabla P_1, \ldots, \nabla P_{j-1}, \nabla P_{j+1}, \ldots, \nabla P_n \quad \text{are } n-1 \text{ linearly independent vectors in } \mathbb{R}^{n-1} (4.7)$$

When (4.7) holds with $j = n$ at every $u \in \mathcal{O}$, we shall refer to P as being a <u>canonical</u> parametrization. When instead of (4.2) we have $\Sigma = P(\mathcal{O})$, we shall refer to Σ as having a <u>global</u> parametrization.
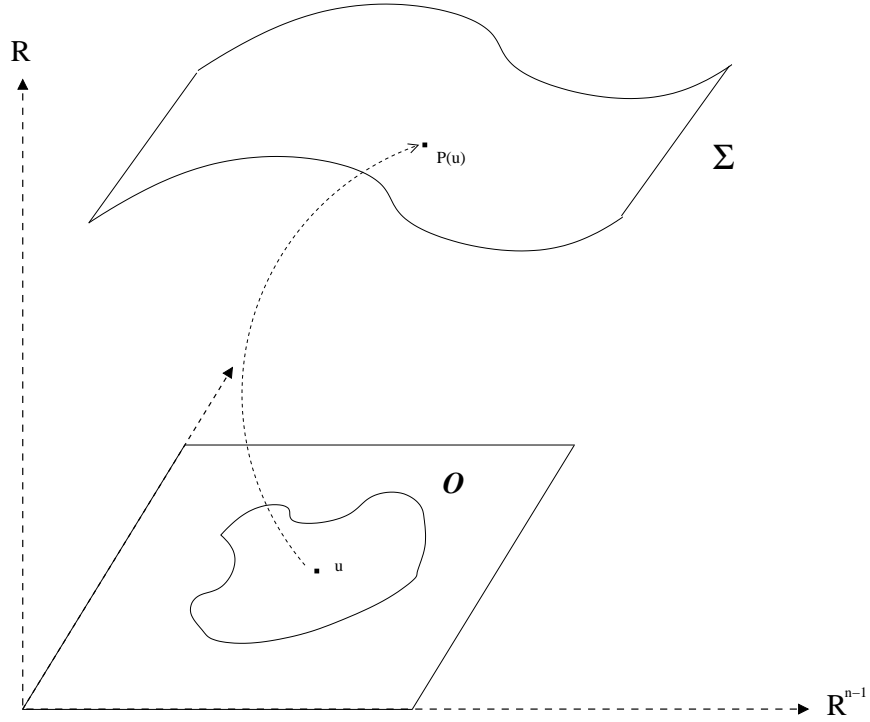
Definition 4.0.2. *If $\Sigma \subseteq \mathbb{R}^n$ is a surface which is locally given as the graph of C^1 functions (in an appropriate system of coordinates), then we define*

$$\int_\Sigma f \, d\sigma := \sum_{j \in J} \int_\Sigma \psi_j f \, d\sigma$$

where $\{\psi_j\}_{j \in J}$ form a Partition of Unity with the property that $(\mathrm{supp}\psi_j) \cap \Sigma$ is contained in the graph of a C^1 function.

Chapter 5

Applications of the Cross Product to Parametrizations

Recall (4.1). We next turn our attention to deriving an alternative formula for the integral of a function on a surface, which emphasizes the role played by the parametrization of the surface.

Theorem 5.0.5. *Assume that* $\Sigma \subset \mathbb{R}^n$ *is a surface that has a global canonical parametrization* $P : \mathcal{O} \to \Sigma \hookrightarrow \mathbb{R}^n$, *where* $\mathcal{O}$ *is an open subset of* $\mathbb{R}^{n-1}$. *Then for every absolutely integrable function* $f : \Sigma \longrightarrow \mathbb{R}$, *there holds*

$$\int_\Sigma f \, d\sigma = \int_\mathcal{O} f \circ P \, \|\partial_1 P \times \ldots \times \partial_{n-1}P\| \, du_1 \ldots du_{n-1}. \tag{5.1}$$

Proof: We divide the proof into several steps.

Step I:

Let us define $P' : \mathcal{O} \longrightarrow \mathbb{R}^{n-1}$, by

$$P'(u) := \Big(P_1(u), \ldots, P_{n-1}(u) \Big), \quad u \in \mathcal{O}. \tag{5.2}$$

We will prove (5.1) under the additional assumptions that $P'(\mathcal{O}) \in \mathbb{R}^{n-1}$ is open, and

$$P' : \mathcal{O} \longrightarrow P'(\mathcal{O}) \quad \text{has a } C^1 \text{ inverse.} \tag{5.3}$$

We need a function ϕ such that $P(u) = (x', \phi(x'))$, so that we can think of Σ as the graph of ϕ. In other words, we need

$$x_j = P_j(u), \quad 1 \leq j \leq n - 1, \tag{5.4}$$

and

$$\phi(x') = P_n(u). \tag{5.5}$$

Thus, condition (5.4) amounts to

$$P'(u) = x'. \tag{5.6}$$

Recalling that P' has a C^1 inverse, (5.6) can be written as

$$u = (P')^{-1}(x'). \tag{5.7}$$

Using this and (5.5), we can take

$$\phi(x') := P_n((P')^{-1}(x')). \tag{5.8}$$

Note that for ϕ chosen as above, we have

$$(x', \phi(x')) = P(u), \quad \text{if } u = (P')^{-1}(x'). \tag{5.9}$$

Indeed

$$P(u) = \Big(P'(u), P_n(u)\Big) = \Big(P'((P')^{-1}(x')), P_n((P')^{-1}(x'))\Big) = \Big(x', \phi(x')\Big). \tag{5.10}$$

Since Σ is globally the graph of ϕ, we can invoke (4.1) to write

$$\int_\Sigma f \, d\sigma = \int_{\mathbb{R}^{n-1}} f(x', \phi(x'))\sqrt{1 + \|\nabla\phi(x')\|^2} dx'.$$

In the second integral, make the change of variables

$$\mathbb{R}^{n-1} \ni x' = P'(u), \quad u \in \mathcal{O}. \tag{5.11}$$

Note that the corresponding *Jacobian* is

$$dx' = \left| \det\left(\frac{D(P_1 \dots P_{n-1})}{D(u_1 \dots u_{n-1})}\right)(u) \right| du_1 \dots du_{n-1}. \tag{5.12}$$

This integral now becomes

$$\int_\mathcal{O} f\Big(P'(u), \phi(P'(u))\Big)\sqrt{1 + \|(\nabla\phi)(P'(u))\|^2} \times \tag{5.13}$$

$$\times \left| \det\left(\frac{D(P_1 \dots P_{n-1})}{D(u_1 \dots u_{n-1})}\right)(u) \right| du_1 \dots du_{n-1}.$$

Since $\phi(P'(u)) = P_n\Big((P')^{-1}(P'(u))\Big) = P_n(u)$, we have

$$f\Big(P'(u), \phi(P'(u))\Big) = f(P'(u), P_n(u)) = f(P(u)) = (f \circ P)(u). \tag{5.14}$$

Hence, matters have been reduced to showing that

$$\sqrt{1 + \|(\nabla\phi)(P'(u))\|^2} \cdot \left| \det\left(\frac{D(P_1 \dots P_{n-1})}{D(u_1 \dots u_{n-1})}\right)(u) \right|$$

$$= \sqrt{1 + \|(\nabla\phi)(P'(u))\|^2} \cdot |\det(DP')(u)|$$

$$= \|(\partial_1 P)(u) \times \dots \times (\partial_{n-1}P)(u)\|. \tag{5.15}$$

Taking the derivative of (5.8) and using the *Chain Rule* we get

$$\nabla\phi(x') = (D\phi)(x') = D\Big(P_n[(P')^{-1}(x')]\Big) \tag{5.16}$$

$$= (\nabla P_n)[(P')^{-1}(x')] \cdot D[(P')^{-1}](x')$$

$$= \nabla P_n(u) \cdot D[(P')^{-1}](x').$$

Note that $P' \circ [(P')^{-1}] = I$, the identity function in $\mathbb{R}^{n-1}$. Taking the derivative and evaluating at x' we obtain

$$D\Big(P' \circ [(P')^{-1}]\Big)(x') = I_{(n-1)\times(n-1)} \tag{5.17}$$

$$\Rightarrow (DP')((P')^{-1}(x')) \cdot D[(P')^{-1}](x') = I_{(n-1)\times(n-1)}.$$

Thus $D[(P')^{-1}](x') = [(DP')(u)]^{-1}$. Hence, using this in (5.16) we arrive at

$$\nabla\phi(x') = \nabla P_n(u) \cdot [(DP')(u)]^{-1}. \tag{5.18}$$

Taking the transpose of (5.18) gives

$$(\nabla\phi(x'))^\top = (\nabla P_n(u)[(DP')(u)]^{-1})^\top = ([(DP')(u)]^{-1})^\top(\nabla P_n(u))^\top,$$

whose norm yields

$$
\begin{aligned}
\|\nabla\phi(x')\| &= \|(\nabla\phi(x'))^\top\| = \|([(DP')(u)]^{-1})^\top(\nabla P_n(u))^\top\| && (5.19)\\
&= \|([(DP')(u)]^{-1})^\top(\nabla P_n)(u)\|.
\end{aligned}
$$

Hence, (5.19) implies

$$
\sqrt{1 + \|\nabla\phi(x')\|^2} \cdot |\det(DP')(u)| \tag{5.20}
$$
$$
= \sqrt{1 + \|([(DP')(u)]^{-1})^\top(\nabla P_n)(u)\|^2} \cdot |\det(DP')(u)|.
$$

<u>Claim</u> Consider $v_1, \ldots, v_{n-1} \in \mathbb{R}^n$ arbitrary vectors and arrange their components vertically as columns in the $n \times (n-1)$ matrix

$$
\begin{pmatrix}
v_{11} & v_{21} & \cdots & v_{n-1\,1} \\
v_{12} & v_{22} & \cdots & v_{n-1\,2} \\
\vdots & \vdots & \cdots & \vdots \\
v_{1\,n-1} & v_{2\,n-1} & \cdots & v_{n-1\,n-1} \\
\hline
v_{1n} & v_{2n} & \cdots & v_{n-1\,n}
\end{pmatrix}. \tag{5.21}
$$

Let M be the $(n-1) \times (n-1)$ matrix obtained by eliminating the last row in this matrix. Also, let w be the last row in (5.21), viewed as a vector in $\mathbb{R}^{n-1}$. Then, if M is invertible, we have

$$| \det M | \sqrt{1 + \| M^{-1} w \|^2} = \| v_1 \times \ldots \times v_{n-1} \| . \tag{5.22}$$

Proof of Claim.

Set $A := \left(\begin{array}{c|c} M^{-1} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array} \right) \in M_{n \times n}$. Then by (2.19), we have

$$| \det A | = \left| \det(M^{-1}) \right| = | \det M |^{-1} , \tag{5.23}$$

where the determinant is found by expanding along the last row.

Letting $A = (b_{jk})_{1 \le j, k \le n-1}$, we find

$$
\begin{aligned}
Av_j &= \left(\begin{array}{ccc|c} b_{11} & \ldots & b_{1\,n-1} & 0 \\ \vdots & \ldots & \vdots & \vdots \\ b_{n-1\,1} & \ldots & b_{n-1\,n-1} & 0 \\ \hline 0 & \ldots & 0 & 1 \end{array} \right) \cdot \begin{pmatrix} v_{j1} \\ v_{j2} \\ \vdots \\ v_{jn} \end{pmatrix} \\
&= \left(\sum_{k=1}^{n} b_{1k} v_{jk}, \sum_{k=1}^{n} b_{2k} v_{jk}, \ldots, \sum_{k=1}^{n} b_{jk} v_{jk}, \ldots, \sum_{k=1}^{n} b_{n-1\,k} v_{jk}, v_{jn} \right) \\
&= (0, \ldots, 0, 1, 0, \ldots, 0, v_{jn}), \tag{5.24}
\end{aligned}
$$

where 1 represents the *j-th* place.

Thus,

$$Av_j = (0, \ldots, 0, 1, 0, \ldots, 0, v_{jn}) \qquad \text{for } 1 \le j \le n - 1. \tag{5.25}$$

Recall that (2.2) is the following:

$$Av_1 \times \cdots \times Av_{n-1} = (\det A)(A^{-1})^{\top}(v_1 \times \cdots \times v_{n-1}).$$

Multiplying by $|\det A|^{-1} = |\det M|$ and $A^\top$ in (2.2), and then taking the norm yields

$$\|v_1 \times \ldots \times v_{n-1}\| = |\det M| \left\|A^\top(Av_1 \times \ldots \times Av_{n-1})\right\|. \tag{5.26}$$

Notice,

$$
\begin{aligned}
Av_1 \times \ldots \times Av_{n-1} &= \det \left(\begin{array}{cccc|c} 1 & 0 & \ldots & 0 & v_{1n} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & \ldots & 0 & 1 & v_{n-1\,n} \\ \hline e_1 & e_2 & \ldots & e_{n-1} & e_n \end{array}\right) \\[2mm]
&= \det \left(\begin{array}{ccc|c} \multicolumn{3}{c|}{I_{(n-1)\times(n-1)}} & w \\ \hline e_1 & \ldots & & e_n \end{array}\right) \\[2mm]
&= w_1 e_1 + w_2 e_2 + \ldots + w_{n-1} e_{n-1} + e_n, \tag{5.27}
\end{aligned}
$$

where $(w_1, w_2, \ldots, w_n) = (v_{1n}, v_{2n}, \ldots, v_{n-1\,n}) = w$. So,

$$
\begin{aligned}
A^\top &(Av_1 \times \ldots \times Av_{n-1}) \\[2mm]
&= \left(\begin{array}{c|c} (M^{-1})^\top & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right) \cdot \left[w_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \ldots + w_{n-1} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right] \\[2mm]
&= A^\top w_1 e_1 + \ldots + A^\top w_{n-1} e_{n-1} + A^\top e_n \\[2mm]
&= w_1 A^\top e_1 + \ldots + w_{n-1} A^\top e_{n-1} + A^\top e_n \\[2mm]
&= \left(\sum_{j=1}^{n-1} w_j A^\top e_j\right) + A^\top e_n. \tag{5.28}
\end{aligned}
$$

Focusing on $A^\top e_j$ yields

$$A^\top e_j \;=\; \left(\begin{array}{ccc|c} b_{11} & \ldots & b_{n-1\,1} & 0 \\ \vdots & \ldots & \vdots & \vdots \\ b_{1\,n-1} & \ldots & b_{n-1\,n-1} & 0 \\ \hline 0 & \ldots & 0 & 1 \end{array}\right) \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{5.29}$$

$$= \begin{cases} \begin{pmatrix} b_{1j} \\ \vdots \\ b_{n-1\,j} \\ 0 \end{pmatrix} & \text{if } 1 \le j \le n-1, \\[3em] \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} & \text{if } j = n. \end{cases}$$

Hence,

$$\left(\sum_{j=1}^{n-1} w_j A^\top e_j\right) + A^\top e_n = \left[\sum_{j=1}^{n-1} w_j \begin{pmatrix} b_{1j} \\ \vdots \\ b_{n-1\,j} \\ 0 \end{pmatrix}\right] + e_n$$

$$= \begin{pmatrix} w_1 b_{11} \\ \vdots \\ w_1 b_{n-1\,1} \\ 0 \end{pmatrix} + \begin{pmatrix} w_2 b_{12} \\ \vdots \\ w_2 b_{n-1\,2} \\ 0 \end{pmatrix} + \ldots + \begin{pmatrix} w_{n-1} b_{1\,n-1} \\ \vdots \\ w_{n-1} b_{n-1\,n-1} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$= \left(\sum_{j=1}^{n-1} w_j b_{1j}, \sum_{j=1}^{n-1} w_j b_{2j}, \ldots, \sum_{j=1}^{n-1} w_j b_{n-1\,j}, 1\right)^\top$$

$$= \left(\sum_{j=1}^{n-1} b_{1j} w_j, \sum_{j=1}^{n-1} b_{2j} w_j, \ldots, \sum_{j=1}^{n-1} b_{n-1\,j} w_j, 1\right)^\top$$

$$= \begin{pmatrix} M^{-1} w \\ \hline 1 \end{pmatrix}. \tag{5.30}$$

Thus,

$$A^\top(Av_1 \times \ldots \times Av_{n-1}) = \begin{pmatrix} M^{-1}w \\ \hline 1 \end{pmatrix}. \tag{5.31}$$

Taking the norm of (5.31) gives

$$\left\| A^\top(Av_1 \times \ldots \times Av_{n-1}) \right\| = \sqrt{1 + \|M^{-1}w\|^2}. \tag{5.32}$$

Using (5.26) in (5.32) gives

$$\|v_1 \times \ldots \times v_{n-1}\| = |\det M| \sqrt{1 + \|M^{-1}w\|^2}. \tag{5.33}$$

This finishes the proof of the **<u>Claim</u>**.

To continue, we specialize the **<u>Claim</u>** to the following situation:

Set

$$M := (DP')^\top \in M_{(n-1)\times(n-1)} \tag{5.34}$$

and

$$w := \nabla P_n \in \mathbb{R}^{n-1}. \tag{5.35}$$

Let $v_1, \ldots, v_{n-1} \in \mathbb{R}^n$ be such that

$$v_j := \partial_j P \qquad \text{for } 1 \leq j \leq n-1, \tag{5.36}$$

then $(v_{jn})_{1 \le j \le n-1} = w$.

Thus,

$$
M = \begin{pmatrix} \partial_1 P_1 & \dots & \partial_{n-1} P_1 \\ \partial_1 P_2 & \dots & \partial_{n-1} P_2 \\ \vdots & \dots & \vdots \\ \partial_1 P_{n-1} & \dots & \partial_{n-1} P_{n-1} \end{pmatrix} = \begin{pmatrix} v_{11} & \dots & v_{n-1\,1} \\ v_{12} & \dots & v_{n-1\,2} \\ \vdots & \dots & \vdots \\ v_{1\,n-1} & \dots & v_{n-1\,n-1} \end{pmatrix}, \tag{5.37}
$$

Putting (5.20), (2.13), (5.34), (5.35), (2.11), (5.33), and (5.36) together, we get

$$
\sqrt{1 + \|\nabla\phi(P'(u))\|^2} \cdot |\det[(DP')(u)]|
$$

$$
= \sqrt{1 + \|([(DP')(u)]^{-1})^\top (\nabla P_n)(u)\|^2} \cdot |\det[(DP')(u)]|
$$

$$
= \sqrt{1 + \|([(DP')(u)]^\top)^{-1} (\nabla P_n)(u)\|^2} \cdot |\det[(DP')(u)]|
$$

$$
= \sqrt{1 + \|M^{-1}w\|^2} \cdot \left| \det\left(M^\top\right) \right|
$$

$$
= \sqrt{1 + \|M^{-1}w\|^2} \cdot |\det M|
$$

$$
= \|v_1 \times \dots \times v_{n-1}\|
$$

$$
= \|(\partial_1 P)(u) \times \dots \times (\partial_{n-1} P)(u)\|. \tag{5.38}
$$

In particular,

$$
\int_\Sigma f \, d\sigma = \int_{\mathcal{O}} f \circ P \, \|\partial_1 P \times \dots \times \partial_{n-1} P\| \, du, \tag{5.39}
$$

finishing the proof of **Step I**.

Step II:

For all $j \in \mathbb{N}$ define

$$\mathcal{O}_j := \left\{ x \in \mathcal{O} : \ \mathrm{dist}\,(x, \partial\mathcal{O}) > \frac{1}{j} \right\} \cap B(0, j). \tag{5.40}$$

Then $\mathcal{O}_j$ is open, bounded, and in particular

1. $\overline{\mathcal{O}_j} \subseteq \mathcal{O}$, for all $j \in \mathbb{N}$; $\hspace{3cm}$ (5.41)

2. $\mathcal{O}_j \subseteq \mathcal{O}_{j+1} \subseteq \mathcal{O}$, for all $j \in \mathbb{N}$; $\hspace{2.5cm}$ (5.42)

3. $\displaystyle\bigcup_{j=1}^{\infty} \mathcal{O}_j = \mathcal{O}$.

Set

$$\Sigma_j := P(\mathcal{O}_j). \tag{5.43}$$

Then it follows from the above properties of the $\mathcal{O}_j$'s that

1. $\Sigma_j \subseteq \Sigma$, for all $j \in \mathbb{N}$;

2. $\Sigma_j \subseteq \Sigma_{j+1}$, for all $j \in \mathbb{N}$; $\hspace{3cm}$ (5.44)

3. $\displaystyle\bigcup_{j=1}^{\infty} \Sigma_j = \Sigma$.

If

$$\int_{\Sigma_j} f \, d\sigma = \int_{\mathcal{O}_j} f \circ P \, \|\partial_1 P \times \ldots \times \partial_{n-1} P\| \, du, \quad \text{for every } j \in \mathbb{N}, \tag{5.45}$$

then, by the *Lebesgue Dominated Convergence Theorem* and (5.45), we have

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$$\int_\Sigma f \, d\sigma = \int_\Sigma \lim_{j \to \infty} (f \chi_{\Sigma_j}) \, d\sigma = \lim_{j \to \infty} \int_\Sigma (f \chi_{\Sigma_j}) \, d\sigma$$

$$= \lim_{j \to \infty} \int_{\Sigma_j} f \, d\sigma$$

$$= \lim_{j \to \infty} \int_{\mathcal{O}_j} (f \circ P) \, \| \partial_1 P \times \ldots \times \partial_{n-1} P \| \, du$$

$$= \lim_{j \to \infty} \int_{\mathcal{O}} (f \circ P) \, \| \partial_1 P \times \ldots \times \partial_{n-1} P \| \, (\chi_{\mathcal{O}_j}) du$$

$$= \int_{\mathcal{O}} (f \circ P) \, \| \partial_1 P \times \ldots \times \partial_{n-1} P \| \, du. \tag{5.46}$$

Above, the second and sixth equalities are by the *Lebesgue Dominated Convergence Theorem.* (The reader may verify in the Appendix why the *Lebesgue Dominated Convergence Theorem* applies in the above situation.) The third and fifth equalities follow from properties of the indicator function.

This finishes the proof of **Step II**.

Step III:

We need to prove (5.45) for each fixed j. So, fix $j \in \mathbb{N}$. Then,

$$\det \left(\frac{D(P_1 \ldots P_{n-1})}{D(u_1 \ldots u_{n-1})} \right) (u) = \det \left(\frac{DP'}{Du} \right) (u) \neq 0, \quad \forall \, u = (u_1, \ldots, u_{n-1}) \in \mathcal{O}. \tag{5.47}$$

The *Inverse Function Theorem* implies there exists $V_u \subseteq \mathcal{O}$, an open neighborhood of u, such that $P'(V_u)$ is open in $\mathbb{R}^{n-1}$ and $P'|_{V_u} : V_u \longrightarrow P'(V_u)$ has a C^1-inverse. Recall (5.41). Note that $\overline{\mathcal{O}_j}$ is compact for all $j \in \mathbb{N}$. It follows that we can find an open cover of $\overline{\mathcal{O}_j}$ for each $j \in \mathbb{N}$. In other words, we have

$$\overline{\mathcal{O}_j} \subseteq \mathcal{O} = \bigcup_{u \in \mathcal{O}} V_u, \tag{5.48}$$

which implies there exists $I_j \subseteq \mathcal{O}$, a finite set of points, such that

$$\overline{\mathcal{O}_j} \subseteq \bigcup_{u \in I_j} V_u. \tag{5.49}$$

Hence, $\mathcal{O}_j \subseteq \bigcup_{u \in I_j} V_u$ and, thus,

$$\mathcal{O}_j = \bigcup_{u \in I_j} (V_u \cap \mathcal{O}_j), \tag{5.50}$$

as one can check without difficulty. Let $\{\psi_u\}_{u \in I_j}$ be a finite *Partition of Unity* subordinated to $\{P(V_u \cap \mathcal{O}_j)\}_{u \in I_j}$. I.e.,

1. $\operatorname{supp}(\psi_u) \subseteq P(V_u \cap \mathcal{O}_j) \quad$ for all $u \in I_j$; $\tag{5.51}$

2. $0 \leq \psi_u \leq 1 \quad$ for all $u \in I_j$;

3. $\displaystyle\sum_{u \in I_j} \psi_u = 1. \tag{5.52}$

Next, write

$$\int_{\Sigma_j} f \, d\sigma = \int_{\Sigma_j} \left(\sum_{u \in I_j} \psi_u \right) f \, d\sigma = \sum_{u \in I_j} \int_{\Sigma_j} \psi_u f \, d\sigma. \tag{5.53}$$

To continue, we need the following fact whose proof is in the Appendix:

$$\operatorname{supp}(AC) \subseteq \operatorname{supp}(A) \cap \operatorname{supp}(C). \tag{5.54}$$

Specializing (5.54) to $\psi_u f$ and using (5.51), we have

$$\sum_{u \in I_j} \int_{\Sigma_j} \psi_u f \, d\sigma = \sum_{u \in I_j} \int_{P(V_u \cap \mathcal{O}_j)} \psi_u f \, d\sigma. \tag{5.55}$$

For each $u \in I_j$, $P(V_u \cap \mathcal{O}_j)$ is a surface with a global C^1 parametrization, and having the additional property that $P' : V_u \cap \mathcal{O}_j \longrightarrow P'(V_u \cap \mathcal{O}_j)$ is globally invertible, with a C^1 inverse. Hence, this is a setting in which the result in **Step I** applies. Consequently, for all $u \in I_j$, by **Step I**,

$$\int_{\Sigma_j} \psi_u f \, d\sigma = \int_{P(V_u \cap \mathcal{O}_j)} \psi_u f \, d\sigma = \int_{V_u \cap \mathcal{O}_j} (\psi_u f) \circ P \, \|\partial_1 P \times \ldots \times \partial_{n-1} P\| \, du \tag{5.56}$$

Summing up over $u \in I_j$ then gives

$$\sum_{u \in I_j} \int_{P(V_u \cap \mathcal{O}_j)} \psi_u f \, d\sigma = \sum_{u \in I_j} \int_{V_u \cap \mathcal{O}_j} (\psi_u f) \circ P \, \|\partial_1 P \times \ldots \times \partial_{n-1} P\| \, du. \tag{5.57}$$

In order to proceed, we need the following facts whose proofs can be found in the Appendix:

$$(AB) \circ C = (A \circ C)(B \circ C), \tag{5.58}$$

and if A, C are continuous functions, then

$$\operatorname{supp}(A \circ C) \subseteq C^{-1}(\operatorname{supp}(A)). \tag{5.59}$$

Using (5.58) in (5.57) yields

$$\sum_{u \in I_j} \int_{V_u \cap \mathcal{O}_j} (\psi_u f) \circ P \, \|\partial_1 P \times \ldots \times \partial_{n-1} P\| \, du \tag{5.60}$$

$$= \sum_{u \in I_j} \int_{V_u \cap \mathcal{O}_j} (\psi_u \circ P)(f \circ P) \, \|\partial_1 P \times \ldots \times \partial_{n-1} P\| \, du.$$

Let us look a little closer at $\psi_u \circ P$. From (5.59), (5.51), and the fact that P is a *bijection* in this setting, we have

$$
\begin{aligned}
\text{supp}(\psi_u \circ P) \quad &\subseteq \quad P^{-1}(\text{supp } \psi_u) \\[2mm]
&\subseteq \quad P^{-1}\Big(P(V_u \cap \mathcal{O}_j)\Big) \\[2mm]
&= \quad V_u \cap \mathcal{O}_j \\[2mm]
&\subseteq \quad \mathcal{O}_j.
\end{aligned}
\tag{5.61}
$$

Utilizing (5.61), bringing the summation inside the integral on (5.60), and using (5.52) give

$$\sum_{u \in I_j} \int_{V_u \cap \mathcal{O}_j} (\psi_u \circ P)(f \circ P) \, \|\partial_1 P \times \ldots \times \partial_{n-1} P\| \, du$$

$$= \int_{\mathcal{O}_j} \sum_{u \in I_j} \left[(\psi_u \circ P)(f \circ P) \right] \|\partial_1 P \times \ldots \times \partial_{n-1} P\| \, du$$

$$= \int_{\mathcal{O}_j} \left(\sum_{u \in I_j} \psi_u \circ P \right) (f \circ P) \, \|\partial_1 P \times \ldots \times \partial_{n-1} P\| \, du$$

$$= \int_{\mathcal{O}_j} \left[\left(\sum_{u \in I_j} \psi_u \right) \circ P \right] (f \circ P) \, \|\partial_1 P \times \ldots \times \partial_{n-1} P\| \, du$$

$$= \int_{\mathcal{O}_j} (1 \circ P)(f \circ P) \, \|\partial_1 P \times \ldots \times \partial_{n-1} P\| \, du$$

$$= \int_{\mathcal{O}_j} (f \circ P) \, \|\partial_1 P \times \ldots \times \partial_{n-1} P\| \, du. \tag{5.62}$$

This concludes the proof of **Step III**, and consequently the proof of Theorem (5.0.5).

$\square$

Below we single out a useful special case of the previous theorem.

Corollary 5.0.6. *Assume that $\Sigma \subseteq \mathbb{R}^n$ is a surface, $x_0 \in \Sigma$, and let $r_0 > 0$ be such that $P : \mathcal{O} \to \Sigma \cap B(x_0, r_0) \hookrightarrow \mathbb{R}^n$, where $\mathcal{O} \subseteq \mathbb{R}^{n-1}$ is a local parametrization near x_0. Then,*

$$\int_{\Sigma \cap B(x_0, r_0)} f \, d\sigma = \int_{\mathcal{O}} f \circ P \, \|\partial_1 P \times \ldots \times \partial_{n-1} P\| \, du_1 \ldots du_{n-1}, \tag{5.63}$$

for every absolutely integrable function $f : \Sigma \longrightarrow \mathbb{R}$ with support contained in $\Sigma \cap B(x_0, r_0)$.

Proof. Use Theorem (5.0.5) with Σ replaced by $\Sigma \cap B(x_0, r_0)$ (which becomes a surface with a global parametrization). $\square$

For practical applications it is of interest to express the norm of the cross product $\partial_1 P \times \ldots \times \partial_{n-1} P$ (appearing in the right-hand side of (5.63)) in a way which aviods using the multi-dimensional cross product. The scope of the theorem below is to do just that.

Theorem 5.0.7. *Assume that $\Sigma \subset \mathbb{R}^n$ is a surface that has a global canonical parametrization $P : \mathcal{O} \to \Sigma \hookrightarrow \mathbb{R}^n$, where $\mathcal{O}$ is an open subset of $\mathbb{R}^{n-1}$. Then for every absolutely integrable function $f : \Sigma \longrightarrow \mathbb{R}$, there holds*

$$\int_\Sigma f \, d\sigma = \int_{\mathcal{O}} (f \circ P) \left(\sum_{j=1}^n \left[\det \left(\frac{D(P_1 \ldots \widehat{P}_j \ldots P_n)}{D(u_1 \ldots u_{n-1})} \right) \right]^2 \right)^{\frac{1}{2}} du_1 \ldots du_{n-1}, \quad (5.64)$$

where $\widehat{P}_j$ is omitted for $1 \le j \le n$.

Proof. Observe that (2.1) implies

$$\partial_1 P \times \ldots \times \partial_{n-1} P = \det \begin{pmatrix} \partial_1 P_1 & \partial_1 P_2 & \ldots & \partial_1 P_n \\ \partial_2 P_1 & \partial_2 P_2 & \ldots & \partial_2 P_n \\ \vdots & \vdots & \ldots & \vdots \\ \partial_{n-1} P_1 & \partial_{n-1} P_2 & \ldots & \partial_{n-1} P_n \\ e_1 & e_2 & \ldots & e_n \end{pmatrix}$$

$$= \sum_{j=1}^n (-1)^{j+1} e_j \det A_j, \quad (5.65)$$

where as before

$$A_j = \begin{pmatrix} \partial_1 P_1 & \ldots & \partial_1 P_{j-1} & \partial_1 P_{j+1} & \ldots & \partial_1 P_n \\ \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\ \partial_{n-1} P_1 & \ldots & \partial_{n-1} P_{j-1} & \partial_{n-1} P_{j+1} & \ldots & \partial_{n-1} P_n \end{pmatrix}. \quad (5.66)$$

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So,

$$\partial_1 P \times \ldots \times \partial_{n-1} P = \sum_{j=1}^{n} (-1)^{j+1} e_j \det A_j$$

$$= \Big((-1)^2 \det A_1, \ (-1)^3 \det A_2, \ldots, \ (-1)^{n+1} \det A_n \Big),$$

and, hence,

$$\|\partial_1 P \times \ldots \times \partial_{n-1} P\| = \Big([(-1)^2 \det A_1]^2 + [(-1)^3 \det A_2]^2 + \ldots + [(-1)^{n+1} \det A_n]^2 \Big)^{\frac{1}{2}}$$

$$= \Big([\det A_1]^2 + [\det A_2]^2 + \ldots + [\det A_n]^2 \Big)^{\frac{1}{2}}$$

$$= \Big(\sum_{j=1}^{n} [\det A_j]^2 \Big)^{\frac{1}{2}}. \tag{5.67}$$

Next, we expand the last expression above as

$$\left(\left[\det \begin{pmatrix} \partial_1 P_2 & \ldots & \partial_1 P_n \\ \vdots & \ldots & \vdots \\ \partial_{n-1} P_2 & \ldots & \partial_{n-1} P_n \end{pmatrix} \right]^2 + \ldots + \left[\det \begin{pmatrix} \partial_1 P_1 & \ldots & \partial_1 P_{n-1} \\ \vdots & \ldots & \vdots \\ \partial_{n-1} P_1 & \ldots & \partial_{n-1} P_{n-1} \end{pmatrix} \right]^2 \right)^{\frac{1}{2}}$$

$$= \left(\sum_{j=1}^{n} \left[\det \left(\frac{D(P_1 \ldots \widehat{P}_j \ldots P_n)}{D(u_1 \ldots u_{n-1})} \right) \right]^2 \right)^{\frac{1}{2}}. \tag{5.68}$$

This finishes the proof of Theorem (5.0.7). □

A few comments are in order here:

1. $\int_\Sigma f \, d\sigma$ is independent of the cross product;

2. P does not necessarily have to be the canonical parametrization.

Moving on, we now discuss how the unit normal to the surface can be expressed in terms of a parametrization.

Theorem 5.0.8. *Assume that $\Sigma \subset \mathbb{R}^n$ is a surface which has a global canonical parametrization $P : \mathcal{O} \to \Sigma \hookrightarrow \mathbb{R}^n$, where $\mathcal{O}$ is an open subset of $\mathbb{R}^{n-1}$. If ν is the <u>unit normal</u> to the surface Σ, then*

$$\nu \circ P = \frac{\partial_1 P \times \partial_2 P \times \ldots \times \partial_{n-1} P}{\|\partial_1 P \times \partial_2 P \times \ldots \times \partial_{n-1} P\|} \qquad on \; \mathcal{O}. \tag{5.69}$$

Proof: Since $P : \mathcal{O} \longrightarrow \mathbb{R}^n$ is a parametrization of the surface Σ, we can deduce that $(\partial_1 P)(u), \ldots, (\partial_{n-1} P)(u)$ are tangent vectors to Σ at the point $P(u) \in \Sigma$, for every $u \in \mathcal{O}$. Furthermore, by (4.7), these tangent vectors are linearly independent. Hence $\partial_1 P(u), \ldots, \partial_{n-1} P(u)$ form a basis for the tangent plane to Σ at $P(u)$. By (ii) in (2.0.1), $\partial_1 P(u) \times \ldots \times \partial_{n-1} P(u)$ is perpendicular on the tangent plance for Σ at $P(u)$. Hence, $\nu(P(u))$, i.e. the unit normal to Σ at $P(u)$ is given by

$$\nu(P(u)) = \frac{\partial_1 P(u) \times \ldots \times \partial_{n-1} P(u)}{\|\partial_1 P(u) \times \ldots \times \partial_{n-1} P(u)\|} \qquad \text{for every } u \in \mathcal{O}.$$

This finishes the proof of (5.69). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

As we have done for Theorem (5.0.7) above, we would like to express the unit normal ν to the surface Σ which avoids using the multi-dimensional cross product. We do so in the corollary below.

Corollary 5.0.9. *Assume that $\Sigma \subset \mathbb{R}^n$ is a surface which has a global canonical parametrization $P : \mathcal{O} \to \Sigma \hookrightarrow \mathbb{R}^n$, where $\mathcal{O}$ is an open subset of $\mathbb{R}^{n-1}$. If ν is the <u>unit normal</u> to the surface Σ, then for every $j \in \{1, 2, \ldots, n\}$ there holds*

$$\nu_j \circ P = \frac{(-1)^{j+1} \det(A_j)}{\left(\sum_{k=1}^n \left[\det \left(\frac{D(P_1...\hat{P}_j...P_n)}{D(u_1...u_{n-1})} \right) \right]^2 \right)^{\frac{1}{2}}}, \tag{5.70}$$

where

$$A_j = \begin{pmatrix} \partial_1 P_1 & \ldots & \partial_1 P_{j-1} & \partial_1 P_{j+1} & \ldots & \partial_1 P_n \\ \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\ \partial_{n-1} P_1 & \ldots & \partial_{n-1} P_{j-1} & \partial_{n-1} P_{j+1} & \ldots & \partial_{n-1} P_n \end{pmatrix}$$

and $\hat{P}_j$ is omitted for $1 \le j \le n$.

Proof: Use (5.69) with the numerator replaced by (5.65) and the denominator replaced by (5.68). Since we are only concerned with the *j-th* component of ν, (5.65) becomes $(-1)^{j+1} \det(A_j)$. $\square$
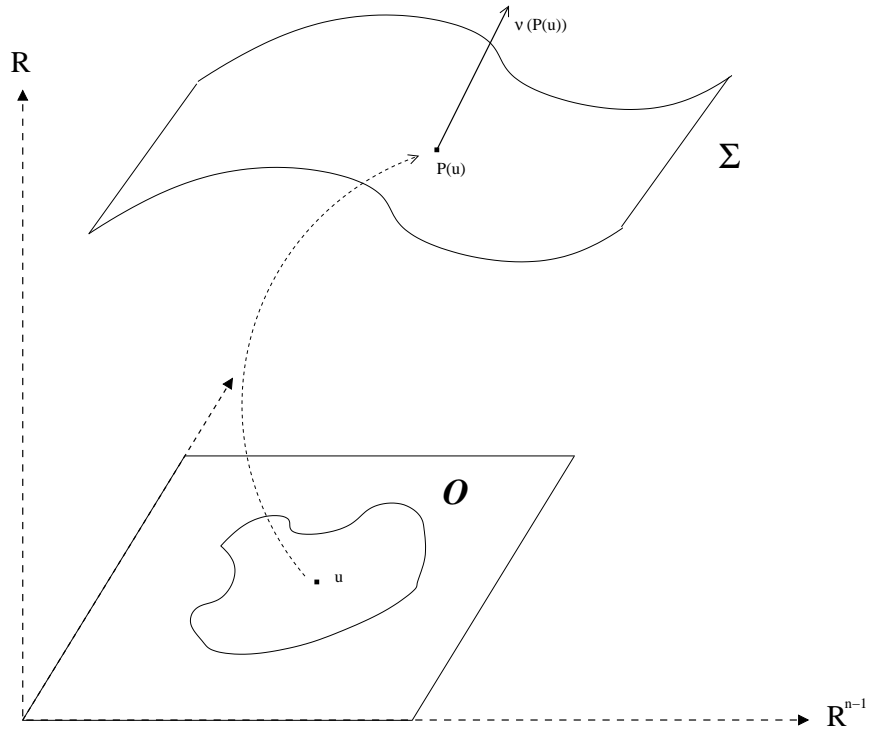
Chapter 6

Motivational Questions

In (5.1) above, $d\sigma$ stands for the surface area element (sometimes denoted dS).

Consider next

$$F : \mathbb{R}^n \longrightarrow \mathbb{R}^n \qquad C^\infty\text{-diffeomorphism} \tag{6.1}$$

i.e., F is of class C^∞, F is bijective, and F^{-1} is of class C^∞.

The basic issues which this project is addressing are as follows:

Question 1

Under the above hypotheses, does it follow that

$$\widetilde{\Sigma} = F(\Sigma) \tag{6.2}$$

is also a smooth surface in $\mathbb{R}^n$?

If the answer to this question is "yes", then we may also consider:

Question 2

How does the unit normal $\tilde{\nu}$ to $\widetilde{\Sigma}$ relate to the unit normal ν to Σ?

Question 3

How does the surface area element $d\widetilde{\sigma}$ for $\widetilde{\Sigma}$ relate to the surface area element $d\sigma$ for Σ?

Question 4

How does the integration process on $\widetilde{\Sigma}$ relate to the integration process on Σ? (i.e. is there a change of variables formula from Σ to $\widetilde{\Sigma}$)

Chapter 7

Surface to Surface Change of Variables

Question 4 is the main source of motivation for the subsequent work.

Theorem 7.0.10. *Let $\Sigma = P(\mathcal{O})$ where $P : \mathcal{O} \longrightarrow \mathbb{R}^n$ is a global parametrization of Σ. Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, a C^∞-diffeomorphism, be such that $F(\Sigma) = \widetilde{\Sigma}$, and let $f : \widetilde{\Sigma} \longrightarrow \mathbb{R}$ be an arbitrary absolutely integrable function. Then*

$$\int_{\widetilde{\Sigma}} f \, d\widetilde{\sigma} = \int_{\Sigma} (f \circ F) \, | \, det(DF)| \, \big\|((DF)^{-1})^{\top} \nu\big\| \, d\sigma. \tag{7.1}$$

Proof: The key ingredient for Question 4 is that we need a parametrization for $\widetilde{\Sigma}$. Specifically, the parametrization we are looking for is the following:

Claim

$$F \circ P : \mathcal{O} \longrightarrow \mathbb{R}^n \quad \text{is a parametriztion for } \widetilde{\Sigma}. \tag{7.2}$$

To show $F \circ P$ is a parametrization we need to verify:

(i) $\mathcal{O}$ is an open subset of $\mathbb{R}^{n-1}$;

(ii) $F \circ P : \mathcal{O} \longrightarrow \mathbb{R}^n$ is injective;

(iii) $F \circ P : \mathcal{O} \longrightarrow \mathbb{R}^n$ is a C^1 map;

(iv) rank $[D(F \circ P)(u)] = n - 1$, for all $u = (u_1, \ldots, u_{n-1}) \in \mathcal{O}$, where

$$\left(\frac{DF(P_1, \ldots, P_n)}{D(u_1, \ldots, u_{n-1})} \right)(u) = D(F \circ P)(u).$$

Note that $\mathcal{O} \subseteq \mathbb{R}^{n-1}$ is open by assumption. Since F is a diffeomorphism and P is injective, then $F \circ P$ is injective. Also, F is a C^∞ map and P is a C^1 map, thus $F \circ P$ is a C^1 map. Hence, (i-iii) above have been verified. To show (iv), we will make use of the following definitions from Linear Algebra.

Let A be a $n \times m$ matrix, then

- row-rank of $A :=$ the maximal number of linearly independent rows of A;

- column-rank of $A :=$ the maximal number of linearly independent columns of A.

From these definitions, one can deduce the following Theorems:

1. row-rank of $A =$ column-rank of A

$$= \text{a number, from now on referred to as } \text{rank}(A);$$

2. $\text{rank}(A) = \text{rank}(A^\top)$;

3. If $B \in M_{m \times l}$ with $\text{rank}(B) = m$, then $\text{rank}(AB) = \text{rank}(A)$;

4. If $C \in M_{l \times n}$ with $\text{rank}(C) = n$, then $\text{rank}(CA) = \text{rank}(A)$; \hfill (7.3)

5. If $A \in M_{n \times n}$ is an invertible matrix, then $\text{rank}(A) = n$. \hfill (7.4)

Returning to the verification of (iv), we have by the *Chain Rule*

$$D(F \circ P)(u) = (DF)(P(u)) \cdot DP(u).$$

Hence,

$$\text{rank}[D(F \circ P)(u)] = \text{rank}[(DF)(P(u)) \cdot DP(u)]. \tag{7.5}$$

Since F is a C^∞-diffeomorphism, then $(DF)(P(u))$ is an invertibe $n \times n$ matrix. Thus, (7.4) implies that the $\text{rank}[(DF)(P(u))] = n$. In addition, we note that (4.4) implies that the rank $(DP(u)) = n - 1$. What we have shown thus far is that the matrices $(DF)(P(u))$ and $DP(u)$ fit the hypothesis for (7.3). Making use of (7.3) in (7.5) yields

$$
\begin{aligned}
\text{rank}[D(F \circ P)(u)] &= \text{rank}[(DF)(P(u)) \cdot DP(u)] \\
&= \text{rank}[DP(u)] \\
&= n - 1.
\end{aligned}
$$

This finishes (iv) and consequently the proof of the **Claim**.

Having established a parametrization for $\widetilde{\Sigma}$, we now focus on $(DF) \circ P$. Set

$$A := (DF) \circ P. \tag{7.6}$$

In particular, A is a $n \times n$ matrix-valued function defined in $\mathcal{O}$. Note that, by the *Chain Rule*, for every $i, j \in \{1, \ldots, n\}$,

$$\partial_j(F_i \circ P) = \sum_{k=1}^{n}[(\partial_k F_i) \circ P]\partial_j P_k \tag{7.7}$$

$$= \text{the } j\text{-th component of } A\,\partial_j P,$$

where A is the $n \times n$ matrix from (7.6) and $\partial_j P$ is a vector in $\mathbb{R}^n$. Thus,

$$\partial_j(F \circ P) = A\partial_j P, \qquad \forall j \in \{1, \dots, n\}\,. \tag{7.8}$$

Recall formula (2.2); taking the norm and using (2.18) gives

$$\|Av_1 \times \dots \times Av_{n-1}\| = |\det A| \left\|(A^{-1})^\top(v_1 \times \dots \times v_{n-1})\right\|. \tag{7.9}$$

Then we can write for any absolutely integrable function

$$f : \widetilde{\Sigma} \longrightarrow \mathbb{R}$$

the following sequence of identities:

$$
\begin{aligned}
\int_{\widetilde{\Sigma}} f \, d\widetilde{\sigma} &= \int_{\mathcal{O}} f \circ (F \circ P) \left\|\partial_1(F \circ P) \times \dots \times \partial_{n-1}(F \circ P)\right\| dx_1 \dots dx_{n-1} \\
&= \int_{\mathcal{O}} (f \circ F) \circ P \left\|A\,\partial_1 P \times A\partial_2 P \times \dots \times A\,\partial_{n-1}P\right\| dx_1 \dots dx_{n-1} \tag{7.10} \\
&= \int_{\mathcal{O}} (f \circ F) \circ P \,|\det A| \left\|(A^{-1})^\top(\partial P_1 \times \dots \times \partial_{n-1}P)\right\| dx_1 \dots dx_{n-1},
\end{aligned}
$$

where A is as in (7.6). Replacing A with its actual formula then gives

$$
\begin{aligned}
\int_{\widetilde{\Sigma}} f \, d\widetilde{\sigma} &= \int_{\mathcal{O}} (f \circ F) \circ P \,|\det[(DF) \circ P]| \times \\
&\qquad\qquad \times \left\|[((DF)^{-1})^\top \circ P](\partial_1 P \times \dots \times \partial_{n-1}P)\right\| dx_1 \dots dx_{n-1}. \tag{7.11}
\end{aligned}
$$

Next, write

$$[((DF)^{-1})^\top \circ P](\partial_1 P \times \ldots \times \partial_{n-1} P)$$

$$= [((DF)^{-1})^\top \circ P] \left(\frac{\partial_1 P \times \ldots \times \partial_{n-1} P}{\|\partial_1 P \times \ldots \times \partial_{n-1} P\|} \right) \|\partial_1 P \times \ldots \times \partial_{n-1} P\| \quad (7.12)$$

and recall formula (5.69). Together, these give

$$[((DF)^{-1})^\top \circ P](\partial_1 P \times \ldots \times \partial_{n-1} P)$$

$$= [((DF)^{-1})^\top \circ P](\nu \circ P) \|\partial_1 P \times \ldots \times \partial_{n-1} P\| . \quad (7.13)$$

Replacing (7.13) back in (7.11) gives

$$\int_{\widetilde{\Sigma}} f \, d\widetilde{\sigma} = \int_{\mathcal{O}} (f \circ F) \circ P \, |[\det(DF)] \circ P| \, \|[((DF)^{-1})^\top \circ P](\nu \circ P)\| \, \cdot$$

$$\cdot \|\partial_1 P \times \ldots \times \partial_{n-1} P\| \, dx_1 \ldots dx_{n-1}. \quad (7.14)$$

Finally, by (5.1) applied to the function $(f \circ F) \, |\det(DF)| \, \|(DF^{-1})^\top \nu\|$ in place of f, we can write (7.14) as

$$\int_{\widetilde{\Sigma}} d\widetilde{\sigma} = \int_{\Sigma} (f \circ F) \, |\det(DF)| \, \|((DF)^{-1})^\top \nu\| \, d\sigma. \quad (7.15)$$

This finishes the proof of Theorem (7.0.10). □

A few comments are in order here.

Corollary 7.0.11. *With the usual assumptions and notational inventions, one has:*

$$\int_{\widetilde{\Sigma}} f \, d\widetilde{\sigma} = \int_{\Sigma} (f \circ F) \, | \, det(DF)| \, \big\| [(DF^{-1})^{\top} \circ F] \nu \big\| \, d\sigma. \tag{7.16}$$

Proof: By the *Inverse Function Theorem,*

$$(DF)^{-1} = (DF^{-1}) \circ F. \tag{7.17}$$

So, (7.1) can be also written as (7.16). $\qquad\square$

Remark

The above considerations are local in nature (i.e., it was assumed that the surface Σ has a global parametriztion P). In general, once formulas (7.1) and (7.16) have been established, one can then use a **Partition of Unity** to "glue" together these local remarks. We do so in the following section.

Remark

Remarkable particular cases are obtained in the following cases:

1. F is a rotation in $\mathbb{R}^n$;

2. F is a translation in $\mathbb{R}^n$.

Let us consider these cases in more detail.

Consider $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ as a rotation in $\mathbb{R}^n$. Then by (2.17) we may regard F as an $n \times n$ unitary matrix $\mathcal{R}$. Let Σ be a surface in $\mathbb{R}^n$ and define

$$F(x) := \mathcal{R}x, \quad \text{for } x \in \mathbb{R}^n. \tag{7.18}$$

With this definition, one can immediately deduce that

$$(f \circ F)(x) = f(F(x)) = f(\mathcal{R}x). \tag{7.19}$$

Since F is a rotation, then F is a C^∞ diffeomorphism. Thus, it is meaningful to discuss F^{-1}. The formula for F^{-1} is as follows:

$$F^{-1}(x) = \mathcal{R}^{-1}x = \mathcal{R}^\top x. \tag{7.20}$$

By definition $\widetilde{\Sigma} := F(\Sigma)$. Using (7.18), we have

$$\begin{aligned}
\widetilde{\Sigma} &= F(\Sigma) = \text{the rotated version of } \Sigma \text{ by } \mathcal{R} \\
&= \mathcal{R}(\Sigma). \tag{7.21}
\end{aligned}$$

Letting $\mathcal{R} = (r_{ij})_{1 \le i, j \le n}$ and $x = (x_i)_{1 \le i \le n}$, we have

$$\mathcal{R}x = \begin{pmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & & \vdots \\ r_{n1} & \cdots & r_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \left(\sum_{i=1}^{n} r_{1i}x_i, \ldots, \sum_{i=1}^{n} r_{ni}x_i \right). \tag{7.22}$$

Taking the derivative of (7.18) and using (7.22) yield

$$\begin{aligned}
(DF)(x) &= D(\mathcal{R}x) = \begin{pmatrix} \partial_1 \left(\sum_{i=1}^{n} r_{1i}x_i \right) & \cdots & \partial_n \left(\sum_{i=1}^{n} r_{1i}x_i \right) \\ & \vdots & \vdots \\ \partial_1 \left(\sum_{i=1}^{n} r_{ni}x_i \right) & \cdots & \partial_n \left(\sum_{i=1}^{n} r_{ni}x_i \right) \end{pmatrix} \\
&= \begin{pmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & & \vdots \\ r_{n1} & \cdots & r_{nn} \end{pmatrix} \\
&= \mathcal{R}. \tag{7.23}
\end{aligned}$$

Hence, taking the determinant and the absolute value of (7.23) and using (2.21) give

$$|\det(DF)(x)| = |\det(\mathcal{R})| = 1. \tag{7.24}$$

Taking the transpose of (7.17) and using (7.23), we have

$$\left([(DF^{-1}) \circ F](x) \right)^{\top} = \left([(DF)(x)]^{-1} \right)^{\top} = (\mathcal{R}^{-1})^{\top} = \mathcal{R}. \tag{7.25}$$

Finally, by (2.22), we have

$$\|\mathcal{R}\nu\| = \|\nu\| = 1. \tag{7.26}$$

Putting (7.19), (7.21), (7.24), and (7.26) in (7.16) we can write

$$\int_{\Sigma} f(\mathcal{R}x) \, d\sigma(x) = \int_{\mathcal{R}(\Sigma)} f \, d\sigma_{\mathcal{R}(\Sigma)}, \tag{7.27}$$

where $d\sigma_{\mathcal{R}(\Sigma)}$ represents the surface measure on $\mathcal{R}(\Sigma)$, i.e. the rotated version of Σ by $\mathcal{R}$. In particular, if Σ is a sphere centered at 0, then

$$\int_{\Sigma} f(\mathcal{R}x) \, d\sigma(x) = \int_{\Sigma} f \, d\sigma. \tag{7.28}$$

This finishes the case when F is a rotation.

Consider $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ as a translation in $\mathbb{R}^n$. Recall

$$\int_{\mathbb{R}^n} f(x+y) \, dy = \int_{\mathbb{R}^n} f(y) dy, \quad \forall x \in \mathbb{R}^n.$$

We would like to know what happens if we have a general surface in place of $\mathbb{R}^n$. This statement gives rise to the following work. Given Σ, a surface in $\mathbb{R}^n$ and $x \in \mathbb{R}^n$, define $x + \Sigma = \{x + y : y \in \Sigma\}$. Then for any $x \in \mathbb{R}^n$ and any absolutely integrable function $f : x + \Sigma \longrightarrow \mathbb{R}$, we have

$$\int_\Sigma f(x+y) \, d\sigma(y) = \int_{x+\Sigma} f(z) \, d\sigma_x(z), \tag{7.29}$$

where σ_x is the surface measure of $x + \Sigma$.

Proof of (7.29): Fix x and assume $f : x + \Sigma \longrightarrow \mathbb{R}$ is absolutely integrable. Choose

$$F(y) := x + y \text{ for all } y \in \mathbb{R}^n. \tag{7.30}$$

This definition for F implies that F is a C^∞ diffeomorphism of $\mathbb{R}^n$. Moreover

$$(f \circ F)(y) = f(F(y)) = f(x + y). \tag{7.31}$$

By definition $\widetilde{\Sigma} := F(\Sigma)$, thus by (7.30)

$$\widetilde{\Sigma} = x + \Sigma. \tag{7.32}$$

Taking the derivative of (7.30), we have $(DF)(y) = I_{n \times n}$ which implies

$$|\det(DF)| = 1. \tag{7.33}$$

Since F is a C^∞ diffeomorphism, it is meaningful to discuss F^{-1}. One can easily deduce the following formula: $F^{-1}(y) = y - x$. Hence, by the *Inverse Function Theorem*

$$\left([(DF^{-1}) \circ F](y) \right)^\top = \left([(DF)(y)]^{-1} \right)^\top = \left((I_{n\times n})^{-1} \right)^\top = I_{n\times n}. \qquad (7.34)$$

Thus,

$$\|I_{n\times n}\, \nu\| = \|\nu\| = 1. \qquad (7.35)$$

Putting (7.31), (7.32), (7.33), and (7.35) in (7.16) give the desired result. $\qquad \square$

Chapter 8

Proof of (7.1) for Surfaces That Do Not Have a Global Parametrization (i.e., Σ a compact set.)

Step 1

For all $x \in \Sigma$, choose $R_x > 0$ such that

$$\Sigma \cap B(x, R_x) = P_x(\mathcal{O}_x) \tag{8.1}$$

where $P_x : \mathcal{O}_x \longrightarrow \mathbb{R}^n$ is a parametrization satisfying:

1. $\mathcal{O}_x \in \mathbb{R}^{n-1}$ is open;

2. $P_x : \mathcal{O}_x \longrightarrow \mathbb{R}^n$ is a C^1 map;

3. $P_x : \mathcal{O}_x \longrightarrow \mathbb{R}^n$ is injective;

4. $\text{rank}[DP_x(u)] = n - 1$, for all $u = (u_1, \ldots, u_{n-1}) \in \mathcal{O}_x$, where

$$\left(\frac{D(P_{x_1}, \ldots, P_{x_n})}{D(u_1, \ldots, u_{n-1})} \right)(u) = DP_x(u).$$

Step 2

Note that $\Sigma \subseteq \bigcup_{x \in \Sigma} B(x, R_x)$, and since Σ is compact, there exists $x_1, \ldots, x_J \in \Sigma$ such that

$$\Sigma \subseteq \bigcup_{j=1}^{J} B(x_j, R_{x_j}). \tag{8.2}$$

Invoking Theorem (3.0.2) gives there exists $\{\phi_j\}_{j=1}^{J} \in C^\infty(\mathbb{R}^n)$, $\phi_j : \mathbb{R}^n \longrightarrow \mathbb{R}$, such that

1. $\mathrm{supp}(\phi_j)$ is a compact subset of $B(x_j, R_{x_j})$ for all $j \in \{1, \ldots, J\}$; (8.3)

2. $0 \le \phi_j \le 1$ for all $j \in \{1, \ldots, J\}$;

3. $\sum_{j=1}^{J} \phi_j(x) = 1$ for all $x \in \Sigma$. (8.4)

Step 3

Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be such that F is a C^∞-diffeomorphism. Set $\widetilde{\Sigma} = F(\Sigma)$ and let $f : \widetilde{\Sigma} \longrightarrow \mathbb{R}$ be an arbitrary absolutely integrable function. With $\{\phi_j\}_{j=1}^{J}$ as in **Step 2**, define $\widetilde{\phi}_j : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\widetilde{\phi}_j := \phi_j \circ F^{-1}. \tag{8.5}$$

Notice, for all $y \in \widetilde{\Sigma}$, there exists $x \in \Sigma$ such that $F(x) = y$. Using this fact, (8.5), and (8.4) give

$$\sum_{j=1}^{J} \widetilde{\phi}_j(y) = \sum_{j=1}^{J} \widetilde{\phi}_j(F(x)) = \sum_{j=1}^{J} \phi_j(x) = 1.$$

Thus,

$$\sum_{j=1}^{J} \widetilde{\phi}_j = 1 \quad \text{on } \widetilde{\Sigma}. \tag{8.6}$$

For each $j \in \{1, \dots, J\}$, consider the function

$$\widetilde{\phi}_j f : \widetilde{\Sigma} \longrightarrow \mathbb{R}$$

Claim

$$\int_{\widetilde{\Sigma}} \widetilde{\phi}_j f \, d\widetilde{\sigma} = \int_{\Sigma} (\widetilde{\phi}_j f) \circ F \, |\det(DF)| \, \|((DF)^{-1})^{\top} \nu\| \, d\sigma. \tag{8.7}$$

To prove this **Claim**, we will make use of the following facts (already used earlier in a different context), and whose proofs can be found in the Appendix:

$$(AB) \circ C = (A \circ C)(B \circ C),$$

$$\text{supp}(AC) \subseteq \text{supp}(A) \cap \text{supp}(C).$$

$$\text{supp}(A \circ C) \subseteq C^{-1}(\text{supp}(A)),$$

and (if F is injective), then

$$F(A) \cap F(B) = F(A \cap B). \tag{8.8}$$

Returning to the proof of the **Claim**, we find by (5.58) and (8.5)

$$\operatorname{supp} [(\widetilde{\phi}_j f) \circ F] = \operatorname{supp} [(\widetilde{\phi}_j \circ F)(f \circ F)] = \operatorname{supp} [\phi_j (f \circ F)].$$

Thus, by (5.54) and (8.3)

$$\operatorname{supp} [\phi_j (f \circ F)] \subseteq \operatorname{supp} \phi_j \cap \operatorname{supp} (f \circ F) \subseteq \operatorname{supp} \phi_j \subseteq B(x_j, R_{x_j}).$$

Hence,

$$\operatorname{supp} [(\widetilde{\phi}_j f) \circ F] \subseteq B(x_j, R_{x_j}). \tag{8.9}$$

Utilizing (8.9) in the **Claim** yields

$$\int_\Sigma (\widetilde{\phi}_j f) \circ F \, |\det(DF)| \, \left\| ((DF)^{-1})^\top \nu \right\| \, d\sigma$$

$$= \int_{\Sigma \cap B(x_j, R_{x_j})} (\widetilde{\phi}_j f) \circ F \, |\det(DF)| \, \left\| ((DF)^{-1})^\top \nu \right\| \, d\sigma \tag{8.10}$$

and recalling (8.1) gives

$$\int_{\Sigma \cap B(x_j, R_{x_j})} (\widetilde{\phi}_j f) \circ F \, |\det(DF)| \, \left\| ((DF)^{-1})^\top \nu \right\| \, d\sigma$$

$$= \int_{P_{x_j}(\mathcal{O}_{x_j})} (\widetilde{\phi}_j f) \circ F \, |\det(DF)| \, \left\| ((DF)^{-1})^\top \nu \right\| \, d\sigma.$$

Thus, (7.1) applies to $\underline{\Sigma \cap B(x_j, R_{x_j})}$ providing

$$\int_{\Sigma \cap B(x_j, R_{x_j})} (\widetilde{\phi}_j f) \circ F \,|\det(DF)| \,\|((DF)^{-1})^\top \nu\| \, d\sigma = \int_{F(\Sigma \cap B(x_j, R_{x_j}))} (\widetilde{\phi}_j f) \, d\widetilde{\sigma} \quad (8.11)$$

By (5.54), (8.5), and recalling $f : \widetilde{\Sigma} \longrightarrow \mathbb{R}$, we have

$$\operatorname{supp}(\widetilde{\phi}f) \subseteq \operatorname{supp} \widetilde{\phi}_j \cap \operatorname{supp} f \subseteq \operatorname{supp}(\phi_j \circ F^{-1}) \cap \widetilde{\Sigma}.$$

Using (5.59), (8.3), and (8.8) give

$$\operatorname{supp}(\phi_j \circ F^{-1}) \cap \widetilde{\Sigma} \;\subseteq\; F(\operatorname{supp}\phi_j) \cap F(\Sigma)$$

$$\subseteq\; F(B(x_j, R_{x_j})) \cap F(\Sigma)$$

$$=\; F(\Sigma \cap B(x_j, R_{x_j})).$$

Thus,

$$\operatorname{supp}(\widetilde{\phi}_j f) \subseteq F(\Sigma \cap B(x_j, R_{x_j})) \subseteq \widetilde{\Sigma}, \quad (8.12)$$

and

$$\int_{F(\Sigma \cap B(x_j, R_{x_j}))} (\widetilde{\phi}_j f) \, d\widetilde{\sigma} = \int_{\widetilde{\Sigma}} \widetilde{\phi}_j f \, d\widetilde{\sigma}. \quad (8.13)$$

Putting (8.10), (8.11), and (8.13) together gives

$$\int_\Sigma (\widetilde{\phi}_j f) \circ F \,|\det(DF)| \, \big\|((DF)^{-1})^\top \nu\big\| \, d\sigma = \int_{\widetilde{\Sigma}} \widetilde{\phi}_j f \, d\widetilde{\sigma}$$

which finishes the **Claim**.

Having established the **Claim** for every $j \in \{1, \ldots, J\}$, let us sum up formulas like the **Claim** over all j's and obtain

$$\sum_{j=1}^J \left(\int_{\widetilde{\Sigma}} \widetilde{\phi}_j f \, d\widetilde{\sigma} \right) = \sum_{j=1}^J \left(\int_\Sigma (\widetilde{\phi}_j f) \circ F \,|\det(DF)| \, \big\|((DF)^{-1})^\top \nu\big\| \, d\sigma \right).$$

Hence, bringing the sums inside the integrals and using (8.6) on

$$\sum_{j=1}^J (\widetilde{\phi}_j f) = \left(\sum_{j=1}^J \widetilde{\phi}_j \right) f = f \quad \text{on } \widetilde{\Sigma}, \tag{8.14}$$

and noting that

$$\sum_{j=1}^J \left[(\widetilde{\phi}_j f) \circ F \,|\det(DF)| \, \big\|((DF)^{-1})^\top \nu\big\| \right] =$$

$$= \sum_{j=1}^J \left[(\widetilde{\phi}_j \circ F)(f \circ F) \,|\det(DF)| \, \big\|((DF)^{-1})^\top \nu\big\| \right]$$

$$= \left(\sum_{j=1}^J \phi_j \right) (f \circ F) \,|\det(DF)| \, \big\|((DF)^{-1})^\top \nu\big\|$$

$$= (f \circ F) \,|\det(DF)| \, \big\|((DF)^{-1})^\top \nu\big\| \quad \text{on } \Sigma, \tag{8.15}$$

we obtain

$$\int_{\widetilde{\Sigma}} f \, d\widetilde{\sigma} = \int_\Sigma (f \circ F) \,|\det(DF)| \, \big\|((DF)^{-1})^\top \nu\big\| \, d\sigma$$

which is (7.1) in full generality! $\qquad\qquad \square$

Chapter 9

Formula for the Unit Normal

The goal of this section is to explain how the unit normal to a surface changes as the surface Σ is mapped by a diffeomorphism into another surface $\widetilde{\Sigma}$. Recall formula (2.2):

$$Av_1 \times \ldots \times Av_n = |\det A| \, (A^{-1})^\top (v_1 \times \ldots \times v_{n-1}). \tag{9.1}$$

Consider next Question 2.

Theorem 9.0.12. *Let $\Sigma \subseteq \mathbb{R}^n$ be a surface with unit normal ν and let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a C^∞-diffeomorphism. Denote $\widetilde{\Sigma} := F(\Sigma)$ and let $\widetilde{\nu}$ be the unit normal to $\widetilde{\Sigma}$. Then*

$$\tilde{\nu} = \frac{(DF^{-1})^\top (\nu \circ F^{-1})}{\|(DF^{-1})^\top (\nu \circ F^{-1})\|} \quad on \ \widetilde{\Sigma}. \tag{9.2}$$

Proof: Given that ν and $\widetilde{\nu}$ are defined locally. There is no loss of generality in assuming that Σ has a global parametrization P. Thus, $\widetilde{\Sigma}$ has $F \circ P$ as a global parametrization. Therefore, granted (7.2), formula (5.69) gives

$$\tilde{\nu} \circ (F \circ P) = \frac{\partial_1(F \circ P) \times \ldots \times \partial_{n-1}(F \circ P)}{\|\partial_1(F \circ P) \times \ldots \times \partial_{n-1}(F \circ P)\|} \tag{9.3}$$

$$= \frac{A\,\partial_1 P \times \ldots \times A\,\partial_{n-1}P}{\|A\,\partial_1 P \times \ldots \times A\,\partial_{n-1}P\|}$$

$$= \frac{|\det A|\,(A^{-1})^\top(\partial_1 P \times \ldots \times \partial_{n-1}P)}{|\det A|\,\|(A^{-1})^\top(\partial_1 P \times \ldots \times \partial_{n-1}P)\|}$$

$$= \frac{(A^{-1})^\top \left(\frac{\partial_1 P \times \ldots \times \partial_{n-1}P}{\|\partial_1 P \times \ldots \times \partial_{n-1}P\|}\right)}{\left\|(A^{-1})^\top \left(\frac{\partial_1 P \times \ldots \times \partial_{n-1}P}{\|\partial_1 P \times \ldots \times \partial_{n-1}P\|}\right)\right\|}$$

$$= \frac{[((DF)^{-1})^\top \circ P](\nu \circ P)}{\|[((DF)^{-1})^\top \circ P](\nu \circ P)\|} \qquad \text{on } \mathcal{O}.$$

"Dropping" P in (9.3) gives

$$\tilde{\nu} \circ F = \frac{((DF)^{-1})^\top \nu}{\|((DF)^{-1})^\top \nu\|} \quad \text{on } \Sigma. \tag{9.4}$$

Composing with F^{-1} on the right and recalling (7.17), allows us to re-write (9.4) in the form of (9.2). This finishes Theorem (9.0.12). $\qquad \square$

Chapter 10

Relationship with the *Classical Change of Variables Formula*

Let $D \subseteq \mathbb{R}^n$ be such that D is open, and let $f : D \longrightarrow \mathbb{R}$ be an arbitrary absolutely integrable function. We want to make a change of variables, say $x = g(y)$ where:

1. $g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a C^∞-diffeomorphism;

2. $g(\mathcal{O}) = D$;

3. $\mathcal{O} \subseteq \mathbb{R}^n$, $\mathcal{O}$ open.

The *Classical Change of Variables* yields:

$$\int_D f(x) \, dx = \int_{\mathcal{O}} f(g(y)) \, |\det(Dg)(y)| \, dy. \tag{10.1}$$

Our goal is to show (10.1) is a particular case of (7.1). In order to do this, we regard $\mathcal{O}$ as a flat surface in $\mathbb{R}^{n+1}$. Let $F : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ and define

$$F(x, x_{n+1}) := (g(x), x_{n+1}). \tag{10.2}$$

Moreover, define

$$\Sigma := \mathcal{O} \times \{0\} \tag{10.3}$$

and

$$\widetilde{\Sigma} := F(\Sigma) = \{F(x, x_{n+1}) : (x, x_{n+1}) \in \Sigma\}$$

$$= \{(g(x), x_{n+1}) : x \in \mathcal{O} \text{ and } x_{n+1} = 0\} = D \times \{0\} \,. \tag{10.4}$$

Let $\widetilde{f} : \widetilde{\Sigma} \longrightarrow \mathbb{R}$ and define

$$\widetilde{f}(x, 0) := f(x), \quad x \in \mathbb{R}^n. \tag{10.5}$$

From (7.1), we know

$$\int_{\widetilde{\Sigma}} \widetilde{f}(x, x_{n+1}) \, d\widetilde{\sigma} = \int_{\Sigma} (\widetilde{f} \circ F)(x, x_{n+1}) \, |\det(DF)(x, x_{n+1})| \times$$

$$\times \left\| (((DF)(x, x_{n+1}))^{-1})^{\top} \nu(x, x_{n+1}) \right\| \, d\sigma. \tag{10.6}$$

Summarizing what we have thus far gives:

1. $\widetilde{\Sigma} = D \times \{0\}$;

2. $\widetilde{f}(x, 0) = f(x)$;

3. $d\widetilde{\sigma} = dx_1 dx_2 \dots dx_n = dx$ \tag{10.7}

and

1. $\Sigma = \mathcal{O} \times \{0\}$;

2. $(\widetilde{f} \circ F)(x, 0) = \widetilde{f}(F(x, 0)) = \widetilde{f}(g(x), 0) = f(g(x))$; $\qquad$ (10.8)

3. $d\sigma = dx_1 dx_2 \ldots dx_n = dx$. $\qquad$ (10.9)

Claim 1

$$|\det(DF)(x, x_{n+1})| = |\det(Dg)(x)|. \qquad (10.10)$$

Recall

$$F(x, x_{n+1}) = (g(x), x_{n+1}) = (g_1(x), g_2(x), \ldots, g_n(x), x_{n+1}); \qquad (10.11)$$

thus,

$$
(DF)(x, x_{n+1}) = \begin{pmatrix} \frac{\partial g_1(x)}{\partial x_1} & \frac{\partial g_1(x)}{\partial x_2} & \cdots & \frac{\partial g_1(x)}{\partial x_n} & \frac{\partial g_1(x)}{\partial x_{n+1}} \\[2mm] \frac{\partial g_2(x)}{\partial x_1} & \frac{\partial g_2(x)}{\partial x_2} & \cdots & \frac{\partial g_2(x)}{\partial x_n} & \frac{\partial g_2(x)}{\partial x_{n+1}} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \frac{\partial g_n(x)}{\partial x_1} & \frac{\partial g_n(x)}{\partial x_2} & \cdots & \frac{\partial g_n(x)}{\partial x_n} & \frac{\partial g_n(x)}{\partial x_{n+1}} \\[2mm] \frac{\partial x_{n+1}}{\partial x_1} & \frac{\partial x_{n+1}}{\partial x_2} & \cdots & \frac{\partial x_{n+1}}{\partial x_n} & \frac{\partial x_{n+1}}{\partial x_{n+1}} \end{pmatrix}
$$

$$
= \begin{pmatrix} \partial_1 g_1(x) & \partial_2 g_1(x) & \cdots & \partial_n g_1(x) & 0 \\ \partial_1 g_2(x) & \partial_2 g_2(x) & \cdots & \partial_n g_2(x) & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \partial_1 g_n(x) & \partial_2 g_n(x) & \cdots & \partial_n g_n(x) & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}
$$

$$
= \left(\begin{array}{c|c} Dg(x) & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array} \right). \qquad (10.12)
$$

Hence,

$$\det(DF)(x, x_{n+1}) = \det\left(\begin{array}{c|c} Dg(x) & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right) = \det(Dg)(x),$$

by expanding the determinant with respect to the last row, and

$$|\det(DF)(x, x_{n+1})| = |\det(Dg)(x)|.$$

This proves **Claim 1**.

Claim 2

$$\left\|\left(((DF)(x, x_{n+1}))^{-1}\right)^\top \nu(x, x_{n+1})\right\| = 1. \tag{10.13}$$

From (10.12),

$$(DF)(x, x_{n+1}) = \left(\begin{array}{c|c} Dg(x) & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right).$$

Note also that

$$(DF)(x, x_{n+1}) \cdot ((DF)(x, x_{n+1}))^{-1} = I_{(n+1)\times(n+1)}.$$

Thus,

$$\left(\begin{array}{c|c} Dg(x) & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right) \cdot \left(\begin{array}{c|c} Dg(x) & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right)^{-1} = I_{(n+1)\times(n+1)}.$$

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Hence,

$$((DF)(x, x_{n+1}))^{-1} = \left(\begin{array}{c|c} Dg(x) & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array} \right)^{-1} = \left(\begin{array}{c|c} (Dg(x))^{-1} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array} \right) \qquad (10.14)$$

(the proof of this step can be found in the Appendix) and

$$(((DF)(x, x_{n+1}))^{-1})^{\top} = \left(\begin{array}{c|c} (Dg(x))^{-1} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array} \right)^{\top} = \left(\begin{array}{c|c} ((Dg(x))^{-1})^{\top} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array} \right) \quad (10.15)$$

In this context

$$\nu(x, x_{n+1}) = (0, 0, \ldots, 0, 1), \qquad (10.16)$$

therefore,

$$(((DF)(x, x_{n+1}))^{-1})^{\top} \cdot \nu(x, x_{n+1}) = \left(\begin{array}{c|c} ((Dg(x))^{-1})^{\top} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array} \right) \cdot \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = (0, 0, \ldots, 1).$$

Hence,

$$\left\| (((DF)(x, x_{n+1}))^{-1})^{\top} \cdot \nu(x, x_{n+1}) \right\| = \|0, 0, \ldots, 0, 1\| = 1;$$

this proves **Claim 2**.

Putting (10.4), (10.5), (10.7), (10.3), (10.8), (10.10), (10.13), (10.9) into (10.6) give

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$$\int_{D\times\{0\}} f(x)\,dx = \int_{\mathcal{O}\times\{0\}} f(g(x)) \cdot |\det Dg(x)| \cdot 1\,dx.$$

Dropping $(\times\,\{0\})$ gives (10.1) as desired.

Chapter 11

Invariance of Lebesgue and Sobolev Spaces on Surfaces

In this section, the goal is to analyze how Lebesgue and Sobolev spaces defined on surfaces in $\mathbb{R}^n$ transform under the operator of composition with a smooth diffeomorphism. Our first result in this regard is the following.

Theorem 11.0.13. *Assume that $\Sigma \subset \mathbb{R}^n$ is a C^1 surface, $\mathcal{O} \subset \mathbb{R}^n$ is an open neighborhood of Σ, and $F : \mathcal{O} \to \mathbb{R}^n$ be an orientation preserving C^1-diffeomorphism onto its image. Set $\widetilde{\Sigma} := F(\Sigma)$. Then for each $1 \leq p < \infty$, the operator*

$$T : L^p(\Sigma) \longrightarrow L^p(\widetilde{\Sigma}) \tag{11.1}$$

defined by

$$T(f) := f \circ F^{-1}, \qquad f \in L^p(\Sigma), \tag{11.2}$$

is well-defined, linear, and bounded. In fact, T is an isomorphism.

Proof: To show T is <u>well-defined</u> and <u>bounded</u>, we need to show that there exists $c \in \mathbb{R}$ such that $\|Tf\|_{L^p(\widetilde{\Sigma})} \leq c \|f\|_{L^p(\Sigma)} \ \forall f \in L^p(\Sigma)$. Let $f \in L^p(\Sigma)$. Then f is measurable and

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$$\|f\|_{L^p(\Sigma)} = \left(\int_\Sigma |f|^p \, d\sigma \right)^{\frac{1}{p}} < \infty. \tag{11.3}$$

By (7.16), written for $|f|^p \circ F^{-1}$ in place of f, we have

$$
\begin{aligned}
\int_{\widetilde{\Sigma}} (|f|^p \circ F^{-1}) \, d\widetilde{\sigma} &= \int_\Sigma (|f|^p \circ F^{-1} \circ F) \, |\det(DF)| \, \left\| [(DF^{-1})^\top \circ F]\nu \right\| \, d\sigma \\
&= \int_\Sigma |f|^p \, |\det(DF)| \, \left\| [(DF^{-1})^\top \circ F]\nu \right\| \, d\sigma.
\end{aligned} \tag{11.4}
$$

Furthermore, by (5.58),

$$\int_{\widetilde{\Sigma}} (|f|^p \circ F^{-1}) \, d\widetilde{\sigma} = \int_{\widetilde{\Sigma}} \left| f \circ F^{-1} \right|^p \, d\sigma. \tag{11.5}$$

Let us make use of the following facts whose proofs can be found in the Appendix. Given an $n \times n$ invertible matrix $A = (a_{jk})_{1 \leq j,k \leq n}$ and $x \in \mathbb{R}^n$, then

$$|\det(A)| \leq n! \left(\max_{1 \leq j,k \leq n} |a_{jk}| \right)^n, \tag{11.6}$$

and

$$\|Ax\| \leq n^{\frac{3}{2}} \left(\max_{1 \leq j,k \leq n} |a_{jk}| \right) \|x\|. \tag{11.7}$$

Utilizing (11.6) and (11.7) in (11.4), we have

$$\int_{\Sigma} |f|^p \, |\det(DF)| \, \left\| [(DF^{-1})^\top \circ F]\nu \right\| \, d\sigma \tag{11.8}$$

$$\leq \int_{\Sigma} |f|^p \, n! \left(\max_{1 \leq j,k \leq n} \|\partial_j F_k\|_{L^\infty} \right)^n n^{\frac{3}{2}} \left(\max_{1 \leq j,k \leq n} \left\| [\partial_j F_k^{-1}]^\top \circ F \right\|_{L^\infty} \right) \|\nu\| \, d\sigma$$

$$= c' \int_{\Sigma} |f|^p \, d\sigma.$$

where in the above calculation we have used the fact that $\|\nu\| = 1$ and have defined

$$c' := n^{\frac{3}{2}} n! \left(\max_{1 \leq j,k \leq n} \|\partial_j F_k\|_{L^\infty} \right)^n \cdot \left(\max_{1 \leq j,k \leq n} \left\| [\partial_j F_k^{-1}]^\top \circ F \right\|_{L^\infty} \right).$$

Combining (11.5) and (11.8) give

$$\int_{\widetilde{\Sigma}} |f \circ F^{-1}|^p \, d\widetilde{\sigma} \leq c' \int_{\Sigma} |f|^p \, d\sigma. \tag{11.9}$$

Thus, raising both sides of the equation to the $\frac{1}{p}$ power in (11.9) and using (11.3) yields

$$\left(\int_{\widetilde{\Sigma}} |f \circ F^{-1}|^p \, d\widetilde{\sigma} \right)^{\frac{1}{p}} \leq c \left(\int_{\Sigma} |f|^p \, d\sigma \right)^{\frac{1}{p}} < \infty, \tag{11.10}$$

where $c := (c')^{\frac{1}{p}}$. Hence, T is well-defined and bounded.

To show T is <u>linear</u>, we need to verify that

1. $T(f + g) = Tf + Tg \quad \forall f, g \in L^p(\Sigma)$;

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2. $T(\lambda f) = \lambda(Tf) \quad \forall f \in L^p(\Sigma), \, c \in \mathbb{R}.$

Let $f, g \in L^p(\Sigma)$, then

$$T(f + g) = (f + g) \circ F^{-1} = (f \circ F^{-1}) + (g \circ F^{-1}) = Tf + Tg,$$

and

$$T(\lambda f) = \lambda f \circ F^{-1} = \lambda(f \circ F^{-1}) = \lambda(Tf).$$

To show T is an <u>isomorphism</u>, we need to show that $\exists R : L^p(\widetilde{\Sigma}) \longrightarrow L^p(\Sigma)$ such that $T(Rf) = R(Tf) = f$. Defining $Rf := f \circ F$ satisfies the above condition. (Note that R is well-defined since it is of the same type as T itself, but with F, Σ, and $\widetilde{\Sigma}$ replaced by F^{-1}, $\widetilde{\Sigma}$ and Σ). $\qquad\square$

Given a C^1 surface $\Sigma \subset \mathbb{R}^n$ with unit normal ν, define the <u>tangential gradient</u> of a function $f : \Sigma \to \mathbb{R}$ by

$$\nabla_{tan} f := \nabla f - \langle \nabla f, \nu \rangle \nu. \tag{11.11}$$

Thus, coordinate-wise,

$$\left(\nabla_{tan} f\right)_j = \partial_j f - \sum_{k=1}^n (\partial_k f) \nu_k \nu_j, \qquad 1 \le j \le n, \tag{11.12}$$

or

$$\left(\nabla_{tan}f\right)_j = \sum_{k=1}^{n} \nu_k \partial_{\tau_{kj}} f, \qquad 1 \le j \le n, \tag{11.13}$$

where we have set

$$\partial_{\tau_{jk}} f := (\nu_j \partial_k - \nu_k \partial_j) f, \qquad 1 \le j, k \le n. \tag{11.14}$$

Let us verify that (11.13) holds by using the definition of $\partial_{\tau_{kj}} f$,

$$
\begin{aligned}
\left(\nabla_{tan}f\right)_j &= \nu_1(\nu_1\partial_j - \nu_j\partial_1)f + \ldots + \nu_j(\nu_j\partial_j - \nu_j\partial_j)f + \ldots + \nu_n(\nu_n\partial_j - \nu_j\partial_n)f \\[2mm]
&= \nu_1^2 \partial_j f - \nu_1\nu_j\partial_1 f + \ldots + \nu_j^2 \partial_j f - \nu_j\nu_j\partial_j f + \ldots + \nu_n^2 \partial_j f - \nu_n\nu_j\partial_n f \\[2mm]
&= \partial_j f(\nu_1^2 + \ldots + \nu_n^2) - \nu_j(\nu_1\partial_1 f + \ldots + \nu_n\partial_n f) \\[2mm]
&= \partial_j f \|\nu\|^2 - \nu_j \langle \nabla f, \nu \rangle \\[2mm]
&= \partial_j f - \langle \nabla f, \nu \rangle \nu_j \\[2mm]
&= \partial_j f - \sum_{k=1}^{n} [(\partial_k f)\nu_k]\nu_j.
\end{aligned}
$$

This calculation concludes the verification of (11.13). We shall refer to these as being tangential derivatives on Σ.

It follows that

$$\nu_j\left(\nabla_{tan}f\right)_k - \nu_k\left(\nabla_{tan}f\right)_j = \partial_{\tau_{jk}} f, \qquad 1 \le j, k \le n. \tag{11.15}$$

As a consequence, there exist dimensional constants $C_1, C_2 > 0$ such that

$$C_1\|\nabla_{tan}f\| \leq \sum_{1\leq j,\,k\leq n} |\partial_{\tau_{jk}}f| \leq C_2\|\nabla_{tan}f\|, \tag{11.16}$$

pointwise on Σ. Let us show that (11.16) holds. First, we need to check that there exists a constant $C_0 > 0$ such that $\|\nabla_{tan}f\| \leq C_0 \sum_{1\leq j,\,k\leq n} |\partial_{\tau_{jk}}f|$. In order to show this, we will make use of the fact that all norms are equivalent and use (11.13). That is

$$
\begin{aligned}
\|\nabla_{tan}f\|_2 &\leq C_0\|\nabla_{tan}f\|_1 = C_0 \sum_{j=1}^{n} |(\nabla_{tan}f)_j| \\
&= C_0 \sum_{j=1}^{n} \left| \sum_{k=1}^{n} \nu_k \partial_{\tau_{kj}}f \right| \\
&\leq C_0 \sum_{1\leq j,\,k\leq n} \left| \nu_k \partial_{\tau_{kj}}f \right| \\
&\leq C_0 \sum_{1\leq j,\,k\leq n} \left| \partial_{\tau_{kj}}f \right|,
\end{aligned}
$$

for some $C_0 > 0$. Letting $C_1 := \frac{1}{C_0}$ gives the first inequality in (11.16). To show the second inequality of (11.16), we will use (11.15) for fixed $j \in \{1, 2, \ldots, n\}$. Doing so, we have

$$
\begin{aligned}
\sum_{k=1}^{n} \left| \partial_{\tau_{jk}}f \right| &\leq |(\nabla_{tan}f)_j| \Big(|\nu_1| + \ldots + |\nu_n| \Big) + |\nu_j| \Big(|(\nabla_{tan}f)_1| + \ldots + |(\nabla_{tan}f)_n| \Big) \\
&= \sum_{k=1}^{n} \Big(|(\nabla_{tan}f)_j|\,|\nu_k| + |\nu_j|\,|(\nabla_{tan}f)_k| \Big). \tag{11.17}
\end{aligned}
$$

Summing over all $j \in \{1, 2, \ldots, n\}$ in (11.17) and using eqivalence of norms again yields

$$\sum_{1 \leq j, k \leq n} |\partial_{\tau_{jk}} f| \ \leq \ \sum_{j=1}^{n} \left(\sum_{k=1}^{n} \Big(|(\nabla_{tan} f)_j| \, |\nu_k| + |\nu_j| \, |(\nabla_{tan} f)_k| \Big) \right)$$

$$\leq \ 2 \Big(|\nu_1| + \ldots + |\nu_n| \Big) \Big(|(\nabla_{tan} f)_1| + \ldots + |(\nabla_{tan} f)_n| \Big)$$

$$= \ 2 \|\nu\|_2 \|\nabla_{tan} f\|_1$$

$$\leq \ C_2 \|\nabla_{tan} f\|_2,$$

where $C_2 := 2C$ for some $C > 0$. This concludes the verification of (11.16).

Next, if $1 \leq p < \infty$, we define a Sobolev space of order one on Σ by setting

$$W^{1,p}(\Sigma) := \big\{ f \in L^p(\Sigma) : (\nabla_{tan} f)_j \in L^p(\Sigma), \ 1 \leq j \leq n \big\}. \tag{11.18}$$

This becomes a Banach space when equipped with the norm

$$\|f\|_{W^{1,p}(\Sigma)} := \|f\|_{L^p(\Sigma)} + \sum_{j=1}^{n} \|(\nabla_{tan} f)_j\|_{L^p(\Sigma)}. \tag{11.19}$$

<u>**Claim**</u> An equivalent norm on $W^{1,p}(\Sigma)$ is given by

$$\|f\|_{W^{1,p}(\Sigma)} = \|f\|_{L^p(\Sigma)} + \sum_{1 \leq j, k \leq n} \|\partial_{\tau_{jk}} f\|_{L^p(\Sigma)}. \tag{11.20}$$

Proof of Claim. In order to prove this claim we will show

$$\|f\|_{L^p(\Sigma)} \leq C_1 \left(\|f\|_{L^p(\Sigma)} + \sum_{1 \leq j, k \leq n} \|\partial_{\tau_{jk}} f\|_{L^p(\Sigma)} \right), \tag{11.21}$$

and

$$\sum_{j=1}^{n} \|(\nabla_{tan} f)_j\|_{L^p(\Sigma)} \le C_2 \left(\|f\|_{L^p(\Sigma)} + \sum_{1 \le j,\, k \le n} \|\partial_{\tau_{jk}} f\|_{L^p(\Sigma)} \right), \tag{11.22}$$

where $C_1,\, C_2 > 0$. After we have done this, we must also show that (11.20) is bounded by (11.19). Specifically, we must show

$$\|f\|_{L^p(\Sigma)} \le C_3 \left(\|f\|_{L^p(\Sigma)} + \sum_{j=1}^{n} \|(\nabla_{tan} f)_j\|_{L^p(\Sigma)} \right), \tag{11.23}$$

and

$$\sum_{1 \le j,\, k \le n} \|\partial_{\tau_{jk}} f\|_{L^p(\Sigma)} \le C_4 \left(\|f\|_{L^p(\Sigma)} + \sum_{j=1}^{n} \|(\nabla_{tan} f)_j\|_{L^p(\Sigma)} \right), \tag{11.24}$$

where $C_3,\, C_4 > 0$.

To show (11.21) holds, observe that clearly

$$\|f\|_{L^p(\Sigma)} \le \|f\|_{L^p(\Sigma)} + \sum_{1 \le j,\, k \le n} \|\partial_{\tau_{jk}} f\|_{L^p(\Sigma)}.$$

Taking $C_1 := 1$ gives (11.21) as desired.

To show (11.22) holds, we will make use of the following facts:

$$|(\nabla_{tan} f)_j| \le \|\nabla_{tan} f\| \qquad \text{for all } j \in \{1, 2, \ldots, n\}. \tag{11.25}$$

and if $a_i \ge 0$, $0 < p < \infty$, then

$$(a_1 + a_2 + \ldots + a_n)^p \le C_{n,p}(a_1^p + a_2^p + \ldots + a_n^p), \tag{11.26}$$

where $C_{n,p} > 0$ is a constant that depends on n and p. (The proof of this fact can be found in the Appendix.) Making use of (11.25) and (11.16) in (11.22), we arrive at

$$
\begin{aligned}
\sum_{j=1}^{n} \|(\nabla_{tan} f)_j\|_{L^p(\Sigma)} &= \sum_{j=1}^{n} \left(\int_\Sigma |(\nabla_{tan} f)_j|^p \, d\sigma \right)^{\frac{1}{p}} \\
&\leq \sum_{j=1}^{n} \left(\int_\Sigma \|\nabla_{tan} f\|^p \, d\sigma \right)^{\frac{1}{p}} \\
&\leq \sum_{j=1}^{n} \left(\int_\Sigma C_p \left(\sum_{1 \leq k, l \leq n} |\partial_{\tau_{kl}} f| \right)^p d\sigma \right)^{\frac{1}{p}},
\end{aligned}
\tag{11.27}
$$

where $C_p := \left(\frac{1}{c_1} \right)^p$ for some $c_1 > 0$. Specializing (11.26) to $\sum\limits_{1 \leq k, l \leq n} |\partial_{\tau_{kl}} f|$, we have

$$
\begin{aligned}
\sum_{j=1}^{n} \|(\nabla_{tan} f)_j\|_{L^p(\Sigma)} &\leq \sum_{j=1}^{n} \left(\int_\Sigma C_{n,p} \left(\sum_{1 \leq k, l \leq n} |\partial_{\tau_{kl}} f|^p \right) d\sigma \right)^{\frac{1}{p}} \\
&= n \left(\int_\Sigma C_{n,p} \left(\sum_{1 \leq k, l \leq n} |\partial_{\tau_{kl}} f|^p \right) d\sigma \right)^{\frac{1}{p}} \\
&= C_0 \left(\sum_{1 \leq k, l \leq n} \int_\Sigma |\partial_{\tau_{kl}} f|^p \, d\sigma \right)^{\frac{1}{p}},
\end{aligned}
\tag{11.28}
$$

where $C_{n,p} := c_2 C_p$ for some $c_2 > 0$, and $C_0 := n(C_{n,p})^{\frac{1}{p}}$. Using (11.26) again with $\sum\limits_{1 \leq k, l \leq n} \int_\Sigma |\partial_{\tau_{kl}} f|^p \, d\sigma$ gives

$$
\begin{aligned}
\sum_{j=1}^{n} \|(\nabla_{tan} f)_j\|_{L^p(\Sigma)} &\leq C_2 \sum_{1 \leq k, l \leq n} \left(\int_\Sigma |\partial_{\tau_{kl}} f|^p \, d\sigma \right)^{\frac{1}{p}} \\
&= C_2 \sum_{1 \leq k, l \leq n} \|\partial_{\tau_{kl}} f\|_{L^p(\Sigma)} \\
&\leq C_2 \left(\|f\|_{L^p(\Sigma)} + \sum_{1 \leq k, l \leq n} \|\partial_{\tau_{kl}} f\|_{L^p(\Sigma)} \right),
\end{aligned}
\tag{11.29}
$$

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where $C_2 := c_3 C_0$ for some $c_3 > 0$. Combining the constants, we see that $C_2 = \frac{n c_3}{c_1} c_2^{\frac{1}{p}}$.

To show (11.23) holds, observe that clearly

$$\|f\|_{L^p(\Sigma)} \leq \|f\|_{L^p(\Sigma)} + \sum_{1 \leq j,\, k \leq n} \|(\nabla_{tan} f)_j\|_{L^p(\Sigma)}.$$

Taking $C_3 := 1$ gives (11.23) as desired.

To show (11.24), observe that from (11.16) we have

$$
\begin{aligned}
\sum_{1 \leq j,\, k \leq n} \| \partial_{\tau_{jk}} f \|_{L^p(\Sigma)} &= \sum_{1 \leq j,\, k \leq n} \left(\int_\Sigma | \partial_{\tau_{jk}} f |^p \, d\sigma \right)^{\frac{1}{p}} \\
&\leq \sum_{1 \leq j,\, k \leq n} \left(\int_\Sigma \left(\sum_{1 \leq j,\, k \leq n} | \partial_{\tau_{jk}} f | \right)^p d\sigma \right)^{\frac{1}{p}} \\
&\leq C_0 \sum_{1 \leq j,\, k \leq n} \left(\int_\Sigma \|(\nabla_{tan} f)\|_2^p \, d\sigma \right)^{\frac{1}{p}} \\
&= C_1 \left(\int_\Sigma \|(\nabla_{tan} f)\|_2^p \, d\sigma \right)^{\frac{1}{p}},
\end{aligned}
\tag{11.30}
$$

where $C_1 := C_0 n^2$ for some $C_0 > 0$. Using the fact that all norms are equivalent, we arrive at

$$
\begin{aligned}
\sum_{1 \leq j,\, k \leq n} \| \partial_{\tau_{jk}} f \|_{L^p(\Sigma)} &\leq C_2 \left(\int_\Sigma \|(\nabla_{tan} f)\|_1^p \, d\sigma \right)^{\frac{1}{p}} \\
&= C_2 \left(\int_\Sigma \left(\sum_{j=1}^n |(\nabla_{tan} f)_j| \right)^p d\sigma \right)^{\frac{1}{p}},
\end{aligned}
\tag{11.31}
$$

where $C_2 := C_1 c$ for some $c > 0$. Using (11.26) in (11.31) yields

$$\sum_{1 \le j, k \le n} \| \partial_{\tau_{jk}} f \|_{L^p(\Sigma)} \le C_3 \left(\sum_{j=1}^n \int_\Sigma |(\nabla_{tan} f)_j|^p \, d\sigma \right)^{\frac{1}{p}},$$

where $C_3 := C_2 a^{\frac{1}{p}}$ for some $a > 0$. Utilizing (11.26) again gives

$$
\begin{aligned}
\sum_{1 \le j, k \le n} \| \partial_{\tau_{jk}} f \|_{L^p(\Sigma)} &\le C_4 \sum_{j=1}^n \left(\int_\Sigma |(\nabla_{tan} f)_j|^p \, d\sigma \right)^{\frac{1}{p}} \\
&= C_4 \sum_{j=1}^n \| (\nabla_{tan} f)_j \|_{L^p(\Sigma)} \\
&\le C_4 \left(\| f \|_{L^p(\Sigma)} + \sum_{j=1}^n \| (\nabla_{tan} f)_j \|_{L^p(\Sigma)} \right), \quad (11.32)
\end{aligned}
$$

where $C_4 := C_3 b$ for some $b > 0$. Combining the constants, we see that $C_4 = n^2 C_0 c b a^{\frac{1}{p}}$. This finishes the proof of the **<u>Claim</u>**.

Before stating our main invariance result for Sobolev spaces, we digress for the purpose of briefly discussing the tensor product of two vectors in $\mathbb{R}^n$. Specifically, if $a = (a_1, ..., a_n) \in \mathbb{R}^n$ and $b = (b_1, ..., b_n) \in \mathbb{R}^n$ are given, then we set

$$a \otimes b := (a_j b_k)_{1 \le j, k \le n}. \quad (11.33)$$

That is, $a \otimes b$ is the $n \times n$ matrix whose jk-entry is $a_j b_k$. Some of the most basic properties of this operation are summarized in the proposition below.

Proposition 11.0.14. *One has:*

$$(a \otimes b)^\top = b \otimes a, \qquad \forall \, a, b \in \mathbb{R}^n, \quad (11.34)$$

and

$$\langle a \otimes b, c \rangle = \langle b, c \rangle \, a, \qquad \forall \, a, b, c \in \mathbb{R}^n. \tag{11.35}$$

Also,

$$a \otimes b - b \otimes a = a_b \otimes b - b \otimes a_b, \qquad where \;\; a_b := a - \langle a, b \rangle b. \tag{11.36}$$

Proof. As far as (11.34) is concerned, we have

$$(a \otimes b) = \begin{pmatrix} a_1 b_1 & \dots & a_1 b_n \\ \vdots & & \vdots \\ a_n b_1 & \dots & a_n b_n \end{pmatrix}, \tag{11.37}$$

and, hence,

$$\begin{aligned} (a \otimes b)^\top &= \begin{pmatrix} a_1 b_1 & \dots & a_n b_1 \\ \vdots & & \vdots \\ a_1 b_n & \dots & a_n b_n \end{pmatrix} = \begin{pmatrix} b_1 a_1 & \dots & b_1 a_n \\ \vdots & & \vdots \\ b_n a_1 & \dots & n_n a_n \end{pmatrix} \\ &= b \otimes a, \end{aligned} \tag{11.38}$$

as desired. Next, write

$$\begin{aligned} \langle a \otimes b, c \rangle &= \begin{pmatrix} a_1 b_1 & \dots & a_1 b_n \\ \vdots & & \vdots \\ a_n b_1 & \dots & a_n b_n \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \left(\sum_{j=1}^n a_1 b_j c_j, \sum_{j=1}^n a_2 b_j c_j, \dots, \sum_{j=1}^n a_n b_j c_j \right) \\ &= \left(\sum_{j=1}^n b_j c_j a_1, \sum_{j=1}^n b_j c_j a_2, \dots, \sum_{j=1}^n b_j c_j a_n \right) = \left(\langle b, c \rangle \, a_1, \langle b, c \rangle \, a_2, \dots, \langle b, c \rangle \, a_n \right) \\ &= \langle b, c \rangle \, a, \end{aligned} \tag{11.39}$$

which proves (11.35).

For the proof of (11.36), note that

$$a \otimes b - b \otimes a = \begin{pmatrix} a_1b_1 & \cdots & a_1b_n \\ \vdots & & \vdots \\ a_nb_1 & \cdots & a_nb_n \end{pmatrix} - \begin{pmatrix} b_1a_1 & \cdots & b_1a_n \\ \vdots & & \vdots \\ b_na_1 & \cdots & b_na_n \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \cdots & a_1b_n - b_1a_n \\ & \vdots & \vdots \\ a_nb_1 - b_na_1 & \cdots & 0 \end{pmatrix}. \tag{11.40}$$

Also,

$$(a - \langle a, b \rangle b) \otimes b = \left[(a_1, a_2, \ldots, a_n) - \left(\sum_{i=1}^n a_i b_i b_1, \sum_{i=1}^n a_i b_i b_2, \ldots, \sum_{i=1}^n a_i b_i b_n \right) \right] \otimes b$$

$$= \left(a_1 - \sum_{i=1}^n a_i b_i b_1, a_2 - \sum_{i=1}^n a_i b_i b_2, \ldots, a_n - \sum_{i=1}^n a_i b_i b_n \right) \otimes b$$

$$= \begin{pmatrix} c_1b_1 & \cdots & c_1b_n \\ \vdots & & \vdots \\ c_nb_1 & \cdots & c_nb_n \end{pmatrix} \tag{11.41}$$

where

$$c_j := a_j - \sum_{i=1}^n a_i b_i b_j \qquad \forall j \in \{1, 2, \ldots, n\}. \tag{11.42}$$

Using the same argument given above, we find

$$b \otimes (a - \langle a, b \rangle b) = \begin{pmatrix} b_1c_1 & \cdots & b_1c_n \\ \vdots & & \vdots \\ b_nc_1 & \cdots & b_nc_n \end{pmatrix}. \tag{11.43}$$

Thus,

$$[(a - \langle a, b \rangle \, b) \otimes b] - [b \otimes (a - \langle a, b \rangle \, b)] = \begin{pmatrix} c_1 b_1 & \cdots & c_1 b_n \\ \vdots & & \vdots \\ c_n b_1 & \cdots & c_n b_n \end{pmatrix} - \begin{pmatrix} b_1 c_1 & \cdots & b_1 c_n \\ \vdots & & \vdots \\ b_n c_1 & \cdots & b_n c_n \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \cdots & c_1 b_n - b_1 c_n \\ \vdots & & \vdots \\ c_n b_1 - b_n c_1 & \cdots & 0 \end{pmatrix}. \tag{11.44}$$

For the goal we have in mind, it is enough to show that $a_j b_k - b_j a_k = c_j b_k - b_j c_k$ for all $j \in \{1, 2, \ldots, n\}$. From (11.42) we have

$$\begin{aligned} c_j b_k - b_j c_k &= \left(a_j - \sum_{i=1}^n a_i b_i b_j \right) b_k - b_j \left(a_k - \sum_{i=1}^n a_i b_i b_k \right) \\ &= a_j b_k - b_j a_k - \sum_{i=1}^n a_i b_i b_j b_k + \sum_{i=1}^n a_i b_i b_j b_k \\ &= a_j b_k - b_j a_k. \end{aligned} \tag{11.45}$$

Thus, (11.44) can be rewritten as

$$\begin{pmatrix} 0 & \cdots & c_1 b_n - b_1 c_n \\ \vdots & & \vdots \\ c_n b_1 - b_n c_1 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 0 & \cdots & a_1 b_n - b_1 a_n \\ \vdots & & \vdots \\ a_n b_1 - b_n a_1 & \cdots & 0 \end{pmatrix} \tag{11.46}$$

Thus, (11.40) and (11.46) are identical. This finishes Proposition (11.0.14). $\qquad \square$

After this preamble, we are ready to state and prove the following theorem:

Theorem 11.0.15. *Assume that $\Sigma \subset \mathbb{R}^n$ is a C^1 surface, $\mathcal{O} \subset \mathbb{R}^n$ is an open neighborhood of Σ, and $F : \mathcal{O} \to \mathbb{R}^n$ be an orientation preserving C^1-diffeomorphism onto its image. Set $\widetilde{\Sigma} := F(\Sigma)$. Then for each $1 \leq p < \infty$, the operator*

$$T : W^{1,p}(\Sigma) \longrightarrow W^{1,p}(\widetilde{\Sigma}) \tag{11.47}$$

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defined by

$$T(f) := f \circ F^{-1}, \qquad f \in W^{1,p}(\Sigma), \tag{11.48}$$

is well-defined, linear, and bounded. In fact, T is an isomorphism.

Proof. To show T is <u>well-defined</u> and <u>bounded</u>, we need to show that there exists $c \in \mathbb{R}$ such that $\|Tf\|_{W^{1,p}(\widetilde{\Sigma})} \leq \|f\|_{W^{1,p}(\Sigma)}$ for all $f \in W^{1,p}(\Sigma)$.

Fix $f \in W^{1,p}(\Sigma)$. By (11.20), we have

$$
\begin{aligned}
\|T(f)\|_{W^{1,p}(\widetilde{\Sigma})} &\approx \|T(f)\|_{L^p(\widetilde{\Sigma})} + \sum_{1 \leq j,\, k \leq n} \|\partial_{\widetilde{\tau}_{jk}}(Tf)\|_{L^p(\widetilde{\Sigma})} \\
&\approx \|f \circ F^{-1}\|_{L^p(\widetilde{\Sigma})} + \sum_{1 \leq j,\, k \leq n} \|\partial_{\widetilde{\tau}_{jk}}(Tf)\|_{L^p(\widetilde{\Sigma})}.
\end{aligned}
\tag{11.49}
$$

Using Theorem (11.0.13) and (11.23) in (11.49) yields

$$
\begin{aligned}
\|f \circ F^{-1}\|_{L^p(\widetilde{\Sigma})} &\leq C_1 \|f\|_{L^p(\Sigma)} \\
&\leq C_1 \|f\|_{W^{1,p}(\Sigma)} \\
&= C_1 \left(\|f\|_{L^p(\Sigma)} + \sum_{j=1}^{n} \|(\nabla_{tan} f)_j\|_{L^p(\Sigma)} \right).
\end{aligned}
\tag{11.50}
$$

The next order of business is to get an upper bound for $\sum_{1 \leq j,\, k \leq n} \|\partial_{\widetilde{\tau}_{jk}}(Tf)\|_{L^p(\widetilde{\Sigma})}$. For each $j, k \in \{1, \ldots, n\}$, denote by $\partial_{\widetilde{\tau}_{jk}}$ the tangential derivative on $\widetilde{\Sigma}$ given by $\widetilde{\nu}_j \partial_k - \widetilde{\nu}_k \partial_j$, we have

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$$\partial_{\tilde{\tau}_{jk}}(f \circ F^{-1}) = \tilde{\nu}_j \partial_k(f \circ F^{-1}) - \tilde{\nu}_k \partial_j(f \circ F^{-1})$$

$$= \tilde{\nu}_j \sum_{\ell=1}^{n} \big((\partial_\ell f) \circ F^{-1}\big) \partial_k F_\ell^{-1} - \tilde{\nu}_k \sum_{r=1}^{n} \big((\partial_r f) \circ F^{-1}\big) \partial_j F_r^{-1} \quad (11.51)$$

Employing Theorem 9.0.12 we further write

$$\tilde{\nu}_j \sum_{\ell=1}^{n} \big((\partial_\ell f) \circ F^{-1}\big) \partial_k F_\ell^{-1}$$

$$= \frac{\big((DF^{-1})^\top (\nu \circ F^{-1})\big)_j \sum_{\ell=1}^{n} \big((\nabla f) \circ F^{-1}\big)_\ell (DF^{-1})_{\ell k}}{\|(DF^{-1})^\top (\nu \circ F^{-1})\|}. \quad (11.52)$$

In order to proceed, we will use the fact that the *jk-th* entry of the product of three matrices is as follows:

$$[(a_{\alpha\beta})_{\alpha,\beta}(b_{\gamma\delta})_{\gamma,\delta}(c_{\eta\zeta})_{\eta,\zeta}]_{jk} = \sum_{i,l=1}^{n} a_{ji} b_{il} c_{lk}. \quad (11.53)$$

Letting

1. $(DF^{-1})^\top =: A = (a_{\alpha\beta})_{1\le\alpha,\beta\le n}$;

2. $(\nu \circ F^{-1}) =: v = (v_i)_{1\le i\le n}$;

3. $((\nabla f) \circ F^{-1}) =: w = (w_i)_{1\le i\le n}$;

4. $DF^{-1} =: B = (b_{\gamma\delta})_{1\le\gamma,\delta\le n}$;

the numerator of (11.52) can be written in the form

$$(Av)_j \left(\sum_{l=1}^{n} w_l b_{lk} \right) = \left(\sum_{q=1}^{n} a_{jq} v_q \right) \left(\sum_{l=1}^{n} w_l b_{lk} \right)$$

$$= \sum_{q,\,l=1}^{n} a_{jq} v_q w_l b_{lk}$$

$$= \sum_{q,\,l=1}^{n} a_{jq}(v \otimes w)_{ql} b_{lk}. \tag{11.54}$$

Using (11.53) in (11.54), we have

$$\sum_{q,\,l=1}^{n} a_{jq}(v \otimes w)_{ql} b_{lk} = [A(v \otimes w)B]_{jk}$$

$$= \left[(DF^{-1})^\top \big((\nu \circ F^{-1}) \otimes ((\nabla f) \circ F^{-1}) \big)(DF^{-1}) \right]_{jk} \tag{11.55}$$

Putting the numerator of (11.52), (11.54) and (11.55) together we see

$$\big((DF^{-1})^\top (\nu \circ F^{-1}) \big)_j \sum_{\ell=1}^{n} \big((\nabla f) \circ F^{-1} \big)_\ell (DF^{-1})_{\ell k}$$

$$= \left[(DF^{-1})^\top \big((\nu \circ F^{-1}) \otimes (\nabla f \circ F^{-1}) \big)(DF^{-1}) \right]_{jk}. \tag{11.56}$$

Using (11.56) in (11.52) gives

$$\widetilde{\nu}_j \sum_{\ell=1}^{n} \big((\partial_\ell f) \circ F^{-1} \big) \partial_k F_\ell^{-1} = \frac{\left[(DF^{-1})^\top \big((\nu \circ F^{-1}) \otimes (\nabla f \circ F^{-1}) \big)(DF^{-1}) \right]_{jk}}{\| (DF^{-1})^\top (\nu \circ F^{-1}) \|} \tag{11.57}$$

Recall that given matrices A, B, and C, $(ABC)^\top = C^\top B^\top A^\top$. Also, the *jk-th* entry of $A^\top$ is the *kj-th* entry of A. Using these facts along with (11.34), we can write

$$\left[(DF^{-1})^\top \big((\nu \circ F^{-1}) \otimes (\nabla f \circ F^{-1}) \big)(DF^{-1}) \right]_{jk}$$

$$= \left[(DF^{-1})^\top \big((\nabla f \circ F^{-1}) \otimes (\nu \circ F^{-1}) \big)(DF^{-1}) \right]_{kj}. \tag{11.58}$$

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Thus, based on (11.51) and (11.58),

$$\partial_{\tilde{\tau}_{jk}}(f \circ F^{-1}) = \frac{\left[(DF^{-1})^{\top}\big((\nabla f \circ F^{-1}) \otimes (\nu \circ F^{-1})\big)(DF^{-1})\right]_{kj}}{\|(DF^{-1})^{\top}(\nu \circ F^{-1})\|}$$
$$-\frac{\left[(DF^{-1})^{\top}\big((\nabla f \circ F^{-1}) \otimes (\nu \circ F^{-1})\big)(DF^{-1})\right]_{jk}}{\|(DF^{-1})^{\top}(\nu \circ F^{-1})\|}. \qquad (11.59)$$

Using the same reasoning as in (11.58), this further gives

$$\partial_{\tilde{\tau}_{jk}}(f \circ F^{-1})$$
$$= \frac{\left[(DF^{-1})^{\top}\big((\nabla f \circ F^{-1}) \otimes (\nu \circ F^{-1})\big) - \big((\nu \circ F^{-1}) \otimes (\nabla f \circ F^{-1})\big)(DF^{-1})\right]_{kj}}{\|(DF^{-1})^{\top}(\nu \circ F^{-1})\|}$$
$$= \frac{\left[(DF^{-1})^{\top}\big(a \otimes b - b \otimes a\big)(DF^{-1})\right]_{kj}}{\|(DF^{-1})^{\top}(\nu \circ F^{-1})\|}, \qquad (11.60)$$

where

$$a := \nabla f \circ F^{-1} \qquad \text{and} \qquad b := \nu \circ F^{-1}. \qquad (11.61)$$

From this and (11.36) we may finally conclude that, for every j, k,

$$\partial_{\tilde{\tau}_{jk}}(f \circ F^{-1}) = \frac{\left[(DF^{-1})^{\top}\big[(\nabla_{tan} f \otimes \nu - \nu \otimes \nabla_{tan} f) \circ F^{-1}\big](DF^{-1})\right]_{kj}}{\|(DF^{-1})^{\top}(\nu \circ F^{-1})\|}. (11.62)$$

In order to proceed, we will again use the fact that the *kj-th* entry of the product of three matrices is as follows:

$$[(a_{\alpha\beta})_{\alpha, \beta}(b_{\gamma\delta})_{\gamma, \delta}(c_{\eta\varsigma})_{\eta, \varsigma}]_{kj} = \sum_{i, l=1}^{n} a_{ki} b_{il} c_{lj}. \qquad (11.63)$$

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Let us denote $(DF^{-1})^\top =: A = (a_{\alpha\beta})_{\alpha,\,\beta}$. Using (11.63) in the numerator of (11.62), we have

$$
\left[A\Big((\nabla_{tan} f \otimes \nu) \circ F^{-1} \Big) A^\top \right]_{kj} - \left[A\Big((\nu \otimes \nabla_{tan} f) \circ F^{-1} \Big) A^\top \right]_{kj}
$$

$$
= \sum_{i,\,l=1}^{n} a_{ki}\Big((\nabla_{tan} f)_i \circ F^{-1} \Big)(\nu_l \circ F^{-1}) a_{jl}
$$

$$
- \sum_{i,\,l=1}^{n} a_{ki}(\nu_i \circ F^{-1})\Big((\nabla_{tan} f)_l \circ F^{-1} \Big) a_{jl}. \tag{11.64}
$$

Taking the absolute value of (11.62) and using the *Triangle Inequality*, (11.64) and (11.25) yield

$$
\begin{aligned}
\left| \partial_{\tilde{\tau}_{jk}} (f \circ F^{-1}) \right| &\leq \sum_{i,\,l=1}^{n} |a_{ki}| \left| (\nabla_{tan} f)_i \circ F^{-1} \right| \left| \nu_l \circ F^{-1} \right| |a_{jl}| \\
&\quad + \sum_{i,\,l=1}^{n} |a_{ki}| \left| \nu_i \circ F^{-1} \right| \left| (\nabla_{tan} f)_l \circ F^{-1} \right| |a_{jl}| \\
&\leq \sum_{i,\,l=1}^{n} \left| \max_{1\leq i,l\leq n} a_{il} \right| \| (\nabla_{tan} f) \circ F^{-1} \| \| \nu \circ F^{-1} \| \left| \max_{1\leq i,l\leq n} a_{il} \right| \\
&\quad + \sum_{i,\,l=1}^{n} \left| \max_{1\leq i,l\leq n} a_{il} \right| \| \nu \circ F^{-1} \| \| (\nabla_{tan} f) \circ F^{-1} \| \left| \max_{1\leq i,l\leq n} a_{il} \right| \\
&\leq 2 \sum_{i,\,l=1}^{n} \left(\max_{1\leq i,l\leq n} |a_{i,l}| \right)^2 \| (\nabla_{tan} f) \circ F^{-1} \| \\
&= C_2 \| (\nabla_{tan} f) \circ F^{-1} \|, \tag{11.65}
\end{aligned}
$$

where $C_2 := 2n^2 \left(\max_{1\leq i,l\leq n} |a_{i,l}| \right)^2$. In summary, we have

$$
|\partial_{\tilde{\tau}_{jk}} (f \circ F^{-1})| \leq C \, \| (\nabla_{tan} f) \circ F^{-1} \|, \tag{11.66}
$$

pointwise on $\widetilde{\Sigma}$, where $C = C(\Sigma, F, n) > 0$ is a constant which depends only on F, Σ, and n.

With this in hand, we return to the mainstream discussion of finding an upper bound for $\sum_{1 \leq j, k \leq n} \|\partial_{\widetilde{\tau}_{jk}}(Tf)\|_{L^p(\widetilde{\Sigma})}$. Using (11.66), we now deduce that

$$
\begin{aligned}
\|\partial_{\widetilde{\tau}_{jk}}(f \circ F^{-1})\|_{L^p(\widetilde{\Sigma})} &= \left(\int_{\widetilde{\Sigma}} |\partial_{\widetilde{\tau}_{jk}}(f \circ F^{-1})|^p \, d\widetilde{\sigma} \right)^{\frac{1}{p}} \\
&\leq C_0 \left(\int_{\widetilde{\Sigma}} \|(\nabla_{tan} f) \circ F^{-1}\|^p \, d\widetilde{\sigma} \right)^{\frac{1}{p}} \\
&= C_0 \left(\int_{\widetilde{\Sigma}} \left(\sum_{j=1}^{n} \left[(\nabla_{tan} f \circ F^{-1})_j \right]^2 \right)^{\frac{p}{2}} d\widetilde{\sigma} \right)^{\frac{1}{p}},
\end{aligned}
$$

where $C_0 > 0$. Using (11.26) and interchanging the summation and integral yields

$$
\begin{aligned}
\|\partial_{\widetilde{\tau}_{jk}}(f \circ F^{-1})\|_{L^p(\widetilde{\Sigma})} &\leq C_1 \left(\int_{\widetilde{\Sigma}} \sum_{j=1}^{n} \left[(\nabla_{tan} f \circ F^{-1})_j \right]^p \, d\widetilde{\sigma} \right)^{\frac{1}{p}} \\
&= C_1 \left(\sum_{j=1}^{n} \int_{\widetilde{\Sigma}} \left[(\nabla_{tan} f \circ F^{-1})_j \right]^p \, d\widetilde{\sigma} \right)^{\frac{1}{p}},
\end{aligned}
$$

where $C_1 > 0$. Utilizing (11.26) again gives

$$
\begin{aligned}
\|\partial_{\widetilde{\tau}_{jk}}(f \circ F^{-1})\|_{L^p(\widetilde{\Sigma})} &\leq C_2 \sum_{j=1}^{n} \left(\int_{\widetilde{\Sigma}} \left[(\nabla_{tan} f \circ F^{-1})_j \right]^p \, d\widetilde{\sigma} \right)^{\frac{1}{p}} \\
&= C_2 \sum_{j=1}^{n} \|(\nabla_{tan} f \circ F^{-1})_j\|_{L^p(\widetilde{\Sigma})},
\end{aligned}
$$

where $C_2 > 0$. The key step now is to use Theorem (11.0.13) in order to obtain

$$
\|\partial_{\widetilde{\tau}_{jk}}(f \circ F^{-1})\|_{L^p(\widetilde{\Sigma})} \leq C_3 \sum_{j=1}^{n} \|(\nabla_{tan} f)_j\|_{L^p(\Sigma)}, \tag{11.67}
$$

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for some constant $C_3 := C(\Sigma, F) > 0$ which depends only on F and Σ. Hence, using (11.67), we finally have

$$
\begin{aligned}
\sum_{1 \le j, k \le n} \|\partial_{\widetilde{\tau}_{jk}}(Tf)\|_{L^p(\widetilde{\Sigma})} &\le \sum_{1 \le j, k \le n} \left(C_3 \sum_{j=1}^n \|(\nabla_{tan}f)_j\|_{L^p(\Sigma)} \right) \\
&= C_4 \sum_{j=1}^n \|(\nabla_{tan}f)_j\|_{L^p(\Sigma)} \\
&\le C_4 \left(\|f\|_{L^p(\Sigma)} + \sum_{j=1}^n \|(\nabla_{tan}f)_j\|_{L^p(\Sigma)} \right), \qquad (11.68)
\end{aligned}
$$

where $C_4 := n^2 C_3 > 0$.

Based on (11.50), (11.68) and Theorem 11.0.13, we may now deduce that

$$
\begin{aligned}
\|f \circ F^{-1}\|_{W^{1,p}(\widetilde{\Sigma})} &\approx \|f \circ F^{-1}\|_{L^p(\widetilde{\Sigma})} + \sum_{j,k=1}^n \|\partial_{\widetilde{\tau}_{jk}}(f \circ F^{-1})\|_{L^p(\widetilde{\Sigma})} \\
&\le C \left(\|f\|_{L^p(\Sigma)} + \sum_{j=1}^n \|(\nabla_{tan}f)_j\|_{L^p(\Sigma)} \right) \\
&= C\|f\|_{W^{1,p}(\Sigma)}, \qquad (11.69)
\end{aligned}
$$

where the constant $C := C_1 + C_4 = C(\Sigma, F, n) > 0$ depends only on F, Σ, and n. In other words,

$$
\|T(f)\|_{W^{1,p}(\widetilde{\Sigma})} \le C\|f\|_{W^{1,p}(\Sigma)}, \qquad (11.70)
$$

with C independent of f. This shows that the operator T in (11.47)-(11.48) is well-defined and bounded. That T is linear, is already contained in Theorem 11.0.13. Much as there, the inverse of T in (11.47)-(11.48) is given by the operator of composition with F, so T in (11.47)-(11.48) is in fact an isomorphism. $\qquad \square$

Appendix A

Function Theory

1. (5.58) is as follows: $(AB) \circ C = (A \circ C)(B \circ C)$.

   Proof:

   $$[(AB) \circ C](x) = (AB)(C(x)) = A(C(x)) \cdot B(C(x)) = [(A \circ C)(x)][(B \circ C)(x)].$$

   This is true for all x; thus dropping x will give the desired result. $\square$

2. We will need to prove certain critera about the suppport of a function. In order to do these proofs, we will make use of the following proof:

   $$x \notin \ \text{supp}(A) \iff \exists\, r > 0 \text{ such that } A = 0 \text{ on } B(x, r). \tag{1.1}$$

   Proof:

   By definition the support of a function A is as follows:

   $$\text{supp}(A) = \overline{\{x : \ A(x) \neq 0\}};$$

specifically, note that $\mathrm{supp}(A)$ is a closed set.

" $\Rightarrow$ "

Assume $x_0 \notin \mathrm{supp}(A)$.

This implies there exists $r > 0$ such that $B(x_0, r) \cap \overline{\{x : A(x) \neq 0\}} = \emptyset$.

Thus, $B(x_0, r) \cap \{x : A(x) \neq 0\} = \emptyset$. Hence $A = 0$ on $B(x_0, r)$.

" $\Leftarrow$ "

Assume there exists $r > 0$ such that $A = 0$ on $B(x, r)$.

This implies $B(x, r) \cap \overline{\{x : A(x) \neq 0\}} = \emptyset$. Thus, $B(x, r) \cap \mathrm{supp}(A) = \emptyset$.

Hence, $x \notin \mathrm{supp}(A)$. $\qquad\qquad\square$

3. (5.54) is as follows: $\mathrm{supp}(AB) \subseteq \mathrm{supp}(A) \cap \mathrm{supp}(B)$.

Proof:

I will prove the contrapositive for this proof; that is, if $x \notin \mathrm{supp}(A) \cap \mathrm{supp}(B)$, then $x \notin \mathrm{supp}(AB)$. So,

$$
\begin{aligned}
x \notin \mathrm{supp}(A) \cap \mathrm{supp}(B) \;&\Rightarrow\; x \in (\mathrm{supp}(A) \cap \mathrm{supp}(B))^c \\
&\Rightarrow\; x \in (\mathrm{supp}(A))^c \cup (\mathrm{supp}(B))^c \\
&\Rightarrow\; x \notin \mathrm{supp}(A) \text{ or } x \notin \mathrm{supp}(B).
\end{aligned}
$$

Case 1: $x \notin \mathrm{supp}(A)$

Since $\mathrm{supp}(A)$ is a closed set, then $\exists\, r > 0$ such that $B(x, r) \cap \mathrm{supp}(A) = \emptyset$.

Thus, $A = 0$ on $B(x, r)$ which implies $A \cdot B = 0$ on $B(x, r)$.

Hence, (1.1) implies $x \notin \text{supp}(AB)$.

Case 2: $x \notin \text{supp}B$

Follow pf. of Case 1 with A interchanged with B.

Both cases yield the same result, thus proving (5.54). $\qquad\square$

4. (5.59) is as follows: if A, C are continuous, then $\text{supp}(A \circ C) \subseteq C^{-1}(\text{supp}(A))$.

   Proof: Recall that the definiton of the support of function f is as follows:

   $$\text{supp} f := \overline{\{x : f(x) \neq 0\}}.$$

   We shall establish the contrapositive for this proof. That is

   $$C^{-1}\left((\text{supp}(A))^c\right) \subseteq \left(\text{supp}(A \circ C)\right)^c.$$

   Let $x_0 \in C^{-1}\left((\text{supp}(A))^c\right)$. Then $C(x_0) \notin \text{supp}A$.

   This implies there exists $r_0 > 0$ such that $A = 0$ on $B(C(x_0), r_0)$.

   Since C is a continuous function, there exists $r > 0$ such that $x \in B(x_0, r) \Rightarrow$ $C(x) \in B(C(x_0), r_0)$. Thus $(A \circ C)(x) = A(C(x)) = 0$ if $x \in B(x_0, r)$. By (1.1), we have $x \notin \text{supp}(A \circ C)$ for all $x \in B(x_0, r)$. Hence, $x_0 \notin \text{supp}(A \circ C)$. $\qquad\square$

5. (8.8) is as follows: If F is injective, then $F(A) \cap F(B) = F(A \cap B)$.

Proof: We will use double inclusion for this proof.

"$\subseteq$"

Let $y \in F(A) \cap F(B)$.

Then there exists $x \in A$ such that $F(x) = y$ and there exists $x' \in B$ such that $F(x') = y$. Since F is injective and $F(x) = F(x')$, then $x = x'$. Thus, $x \in A \cap B$. Hence, $F(x) = y \in F(A \cap B)$.

"$\supseteq$"

Let $y \in F(A \cap B)$.

Then there exists $x \in A \cap B$ such that $F(x) = y$. Thus, $x \in A$ and $x \in B$. This implies $F(x) \in F(A)$ and $F(x) \in F(B)$. Hence, $y \in F(A) \cap F(B)$. $\qquad\square$

6. Justification as to why the *Lebesgue Dominated Convergence Theorem* applies to (5.46).

Proof:

For $j \in \mathbb{N}$, write

$$\int_{\Sigma_j} f \, d\sigma = \int_{\Sigma} (f \chi_{\Sigma_j}) \, d\sigma,$$

where $f \chi_{\Sigma_j}$ is the sequence of functions to which we will apply the *Lebesgue Dominated Convergence Theorem.* In order to do this, we need

(a) $(f \chi_{\Sigma_j})(x) \to f(x)$ as $j \to \infty$, for σ-a.e. $x \in \Sigma$;

(b) $\left|(f \chi_{\Sigma_j})(x)\right| \leq |f(x)|$, for all $j \in \mathbb{N}$, σ-a.e. $x \in \Sigma$.

To show (a), let $x \in \Sigma$. Then there exists $j_x \in \mathbb{N}$ such that $x \in \Sigma_{j_x}$. Using (5.44), we have $x \in \Sigma_j$ for all $j \geq j_x$. This implies $(f\chi_{\Sigma_j})(x) = f(x)\chi_{\Sigma_j}(x) = f(x)$, for all $j \geq j_x$. Hence (a) has been verified.

To show (b), note that $\left|(f\chi_{\Sigma_j})(x)\right| = |f(x)| \left|\chi_{\Sigma_j}(x)\right|$. Furthermore, note that

$$\left|\chi_{\Sigma_j}(x)\right| = \begin{cases} 0 & \text{if } x \notin \Sigma_j \\ 1 & \text{if } x \in \Sigma_j \end{cases} .$$

Thus, $\left|\chi_{\Sigma_j}(x)\right| \leq 1$ for all $x \in \Sigma_j$. Hence, $\left|(f\chi_{\Sigma_j})(x)\right| \leq |f(x)|$ for all $x \in \Sigma$.

Note that the same proof applies with Σ, f, and (5.44) replaced by $\mathcal{O}$, $f \circ P$, and (5.42). $\square$

7. (11.26) is as follows: If $a_i \geq 0$, and $0 < p < \infty$, then

$$(a_1 + a_2 + \ldots + a_n)^p \leq C_{n,p}(a_1^p + a_2^p + \ldots + a_n^p),$$

where $C_{n,p} > 0$ is a constant that depends on n and p.

Proof:

We will use *Math Induction* on n. Fix $p \in (0, \infty)$. For $n = 1$, we have $a_1^p \leq a_1^p$. Therefore, taking $C_{n,p} := 1$, we find the statement is true for $n = 1$. Let us assume the statement is true for the *n-th* term. We need to verify that the statement is true for the *(n+1)-th* term. That is, we need to verify that

$$(a_1 + a_2 + \ldots + a_{n+1})^p \leq C_{n,p}(a_1^p + a_2^p + \ldots + a_{n+1}^p). \tag{1.2}$$

Before we verify (1.2), it is to our advantage to show the statement holds when $n = 2$. Specifically for $a_1, a_2 > 0$, we need to show that

$$(a_1 + a_2)^p \leq C_{n,p}(a_1^p + a_2^p). \tag{1.3}$$

Let us define $x := \frac{a_1}{a_2} > 0$. Dividing (1.3) by a_2 we now need to verify $(1+x)^p \leq C_{n,p}(x^p + 1)$, or equivalently

$$\frac{(1+x)^p}{1+x^p} \leq C_{n,p}.$$

To this end, let us consider the function

$$f(x) = \frac{(1+x)^p}{1+x^p} \qquad x \in [0, \infty).$$

It suffices to show f is bounded on the interval $[0, \infty)$. Clearly, f is continuous on $[0, \infty)$, and note that $\lim_{x \to \infty} f(x) = 1$. This implies for all $\epsilon > 0$ there exists $r > 0$ such that $|f(x) - 1| < \epsilon$ if $x > r$. Take $\epsilon = \frac{1}{2}$ and denote by r_0 the corresponding r. Hence, $|f(x) - 1| < \frac{1}{2}$ if $x \in (r_0, \infty)$. That is, $f(x) \in (\frac{1}{2}, \frac{3}{2})$ if $x \in (r_0, \infty)$. In other words f is bounded on (r_0, ∞). Notice that the complement of (r_0, ∞) (restricted to the non-negative numbers) is $[0, r_0]$. In particular this is a compact set. Recall that f is continuous on $[0, \infty)$. Thus, it is continuous on this compact set. Hence, the *Boundedness Theorem* implies

f is bounded on $[0, r_0]$. Putting everything together we have shown that f is bounded on $[0, \infty)$, and consequently we have shown (1.3) holds.

Returning to the mainstream discussion of verifying (1.2), write

$$(a_1 + \ldots + a_{n+1})^p = \Big((a_1 + \ldots + a_n) + a_{n+1} \Big)^p. \tag{1.4}$$

Using (1.3) and the induction hypothesis in (1.4) yields

$$
\begin{aligned}
(a_1 + \ldots + a_{n+1})^p &\leq C_{n,p}(a_1 + \ldots + a_n)^p + C_{n,p}a_{n+1}^p \\
&\leq C_{n,p}(a_1^p + \ldots + a_n^p) + C_{n,p}a_{n+1}^p \\
&= C_{n,p}(a_1^p + \ldots + a_n^p + a_{n+1}^p).
\end{aligned}
$$

This finishes the proof. $\qquad\square$

Appendix B

Linear Algebra Proofs

The following are facts from Linear Algebra whose proofs will be left to the reader.

Let A be an $n \times n$ matrix, $u, v \in \mathbb{R}^n$, and $\lambda \in \mathbb{R}$. Then

(a) $A(\lambda v) = \lambda(Av)$;

(b) $\|\lambda v\| = |\lambda| \, \|v\|$;

(c) $\det(A) = \det(A^\top)$;

(d) $\det(AB) = \det(A)\det(B)$;

(e) $\det(A^{-1}) = \left(\det(A)\right)^{-1}$;

(f) $\lambda \langle u, v \rangle = \langle \lambda u, v \rangle = \langle u, \lambda v \rangle$.

8. Let A, B be $n \times n$ invertible matrices with the property that

$$A \cdot B = I_{n \times n}. \tag{2.1}$$

Then, $B = A^{-1}$.

Proof:

Taking the determinant on both sides of (2.1) yields: $\det(AB) = \det(I) = 1$. This implies $\det(A) \cdot \det(B) = 1$. Thus, $\det(A) \neq 0$. Hence, A is invertible (i.e. A^{-1} exists). Multiplying (2.1) by A^{-1} on the left yields:

$$A^{-1}(AB) = A^{-1}I \Rightarrow (A^{-1}A)B = A^{-1} \Rightarrow B = A^{-1}.$$

$\square$

9. Let $\mathbb{A} = \left(\begin{array}{c|c} A & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array} \right) \in M_{(n+1)\times(n+1)}$ where A is a $n \times n$ invertible matrix.
Then, $\mathbb{A}^{-1} = \left(\begin{array}{c|c} A^{-1} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array} \right)$.

Proof:

Denote

$$
\begin{aligned}
A &= (a_{ij})_{1 \leq i,j \leq n}, \\
A^{-1} &= (b_{ij})_{1 \leq i,j \leq n}, \\
I_{n\times n} &= (\delta_{ij}) \quad \text{where} \quad \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}
\end{aligned} \tag{2.2}
$$

Since A is an invertible $n \times n$ matrix, $I_{n\times n} = A \cdot A^{-1}$. Specifically, the ij^{th} entry of $I_{n\times n}$ will match the ij^{th} entry of $A \cdot A^{-1}$. That is

$$\delta_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj} \qquad \forall i, j \in \{1, \ldots, n\}. \tag{2.3}$$

We must show that $\left(\begin{array}{c|c} A & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right) \cdot \left(\begin{array}{c|c} A^{-1} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right) = I_{(n+1)\times(n+1)}.$

So,

$$\left(\begin{array}{c|c} A & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right) \cdot \left(\begin{array}{c|c} A^{-1} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right) =$$

$$= \left(\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & 0 \\ a_{21} & a_{22} & \ldots & a_{2n} & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} & 0 \\ \hline 0 & 0 & \ldots & 0 & 1 \end{array}\right) \cdot \left(\begin{array}{cccc|c} b_{11} & b_{12} & \ldots & b_{1n} & 0 \\ b_{21} & b_{22} & \ldots & b_{2n} & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ b_{n1} & b_{n2} & \ldots & b_{nn} & 0 \\ \hline 0 & 0 & \ldots & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{cccc|c} \sum\limits_{k=1}^{n} a_{1k}b_{k1} & \sum\limits_{k=1}^{n} a_{1k}b_{k2} & \ldots & \sum\limits_{k=1}^{n} a_{1k}b_{kn} & 0 \\ \sum\limits_{k=1}^{n} a_{2k}b_{k1} & \sum\limits_{k=1}^{n} a_{2k}b_{k2} & \ldots & \sum\limits_{k=1}^{n} a_{2k}b_{kn} & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ \sum\limits_{k=1}^{n} a_{nk}b_{k1} & \sum\limits_{k=1}^{n} a_{nk}b_{k2} & \ldots & \sum\limits_{k=1}^{n} a_{nk}b_{kn} & 0 \\ \hline 0 & 0 & \ldots & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{cccc|c} \delta_{11} & \delta_{12} & \ldots & \delta_{1n} & 0 \\ \delta_{21} & \delta_{22} & \ldots & \delta_{2n} & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ \delta_{n1} & \delta_{n2} & \ldots & \delta_{nn} & 0 \\ \hline 0 & 0 & \ldots & 0 & 1 \end{array}\right). \tag{2.4}$$

Utilizing (2.2) in (2.4) gives

$$\left(\begin{array}{c|c} A & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right) \cdot \left(\begin{array}{c|c} A^{-1} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right) = \left(\begin{array}{ccccc|c} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ \hline 0 & 0 & 0 & \ldots & 0 & 1 \end{array}\right) = I_{(n+1)\times(n+1)}.$$

By (2.1), $\left(\begin{array}{c|c} A^{-1} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right) = \mathbb{A}^{-1}.$ $\qquad\square$

10. Given an $n \times n$ invertible matrix $A = (a_{jk})_{1 \leq j,k \leq n}$, then

$$|\det(A)| \leq n! \left(\max_{1 \leq j,k \leq n} |a_{jk}| \right)^n.$$

Proof:

Recall that if $A = (a_{jk})_{1 \leq j,k \leq n}$, then

$$\det A = \sum_{\sigma \in S_n} (-1)^{\text{sign}\,\sigma} a_{1\,\sigma(1)} a_{2\,\sigma(2)} \cdots a_{n\,\sigma(n)}, \tag{2.5}$$

where S_n = the group of permutations of the set $\{1, 2, \ldots, n\}$, and $(\text{sign}\,\sigma)$ is the sign of σ (i.e., the number of transpositions in σ). Taking the absolute value of (2.5) and using the *Triangle Inequality* gives

$$
\begin{aligned}
|\det A| &\leq \sum_{\sigma \in S_n} \left| (-1)^{\text{sign}\,\sigma} \right| \left| a_{1\sigma(1)} \right| \left| a_{2\sigma(2)} \right| \cdots \left| a_{n\sigma(n)} \right| \\
&\leq \sum_{\sigma \in S_n} \left(\max_{1 \leq j,k \leq n} |a_{jk}| \right) \left(\max_{1 \leq j,k \leq n} |a_{jk}| \right) \cdots \left(\max_{1 \leq j,k \leq n} |a_{jk}| \right) \\
&= n! \left(\max_{1 \leq j,k \leq n} |a_{jk}| \right)^n.
\end{aligned}
$$

This concludes the proof. $\square$

11. Given an $n \times n$ invertible matrix $A = (a_{jk})_{1 \leq j,k \leq n}$ and $x \in \mathbb{R}^n$, then

$$\|Ax\|_2 \le n^{\frac{3}{2}} \left(\max_{1 \le j,k \le n} |a_{jk}| \right) \|x\|_2 \, .$$

Proof:

Recall that for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, and $p \in [1, \infty)$ we have

$$\|x\|_p = (x_1^p + x_2^p + \ldots + x_n^p)^{\frac{1}{p}} \, .$$

Let $A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix}$, and $v = (v_1, \ldots, v_n)$. Then,

$$\begin{aligned} Ax &= \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &= \left(\sum_{k=1}^n a_{1k}x_k, \sum_{k=1}^n a_{2k}x_k, \ldots, \sum_{k=1}^n a_{nk}x_k \right) . \end{aligned} \tag{2.6}$$

Taking the 2-norm of (2.6) gives

$$\begin{aligned} \|Ax\|_2 &= \left(\left[\sum_{k=1}^n a_{1k}x_k \right]^2 + \left[\sum_{k=1}^n a_{2k}x_k \right]^2 + \ldots + \left[\sum_{k=1}^n a_{nk}x_k \right]^2 \right)^{\frac{1}{2}} \\ &\le \left(\left[\sum_{k=1}^n \left(\max_{1 \le j,k \le n} |a_{jk}| \right) |x_k| \right]^2 + \ldots + \left[\sum_{k=1}^n \left(\max_{1 \le j,k \le n} |a_{jk}| \right) |x_k| \right]^2 \right)^{\frac{1}{2}} \\ &= \left(n \left[\left(\max_{1 \le j,k \le n} |a_{jk}| \right) \sum_{k=1}^n |x_k| \right]^2 \right)^{\frac{1}{2}} \\ &= \sqrt{n} \left(\max_{1 \le j,k \le n} |a_{jk}| \right) \sum_{k=1}^n |x_k| \, . \end{aligned} \tag{2.7}$$

Note that $|x_k|$ above has the following property:

$$|x_k| \leq \|x\|_2, \quad \forall k \in \{1, 2, \ldots, n\}. \tag{2.8}$$

Using (2.8) in (2.7) yields

$$
\begin{aligned}
\|Ax\|_2 &\leq \sqrt{n} \left(\max_{1 \leq j,\, k \leq n} |a_{jk}| \right) \sum_{k=1}^{n} \|x\|_2 \\
&= n^{\frac{3}{2}} \left(\max_{1 \leq j, k \leq n} |a_{jk}| \right) \|x\|_2.
\end{aligned}
$$

This finishes the proof. □

12. Assume $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ has two identical rows. Then $\det(A) = 0$.

Proof:

Assume the *j-th* row is identical to the *k-th* row; that is

$$
A = \begin{pmatrix}
a_{11} & \cdots & a_{1n} \\
\vdots & & \vdots \\
a_{j1} & \cdots & a_{jn} \\
a_{j1} & \cdots & a_{jn} \\
\vdots & & \vdots \\
a_{n1} & \cdots & a_{nn}
\end{pmatrix}.
$$

Subtracting the *j-th* row from the *k-th* row yields a matrix with the same determinant as A and the *k-th* row equal to the zero vector in $\mathbb{R}^n$; that is

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{j1} & \dots & a_{jn} \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}.$$

Expanding along the *k-th* row gives the desired result. □

13. Let A, B be $n \times n$ matrices, then

$$(AB)^\top = B^\top A^\top. \tag{2.9}$$

Proof:

Let $A = (a_{ij})_{1 \leq i,j \leq n}$ and $B = (b_{ij})_{1 \leq i,j \leq n}$. Then

$$AB = \begin{pmatrix} \sum_{i=1}^{n} a_{1i}b_{i1} & \dots & \sum_{i=1}^{n} a_{1i}b_{in} \\ \vdots & & \vdots \\ \sum_{i=1}^{n} a_{ni}b_{i1} & \dots & \sum_{i=1}^{n} a_{ni}b_{in} \end{pmatrix},$$

and

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$$(AB)^\top = \begin{pmatrix} \sum\limits_{i=1}^{n} a_{1i}b_{i1} & \dots & \sum\limits_{i=1}^{n} a_{ni}b_{i1} \\ \vdots & & \vdots \\ \sum\limits_{i=1}^{n} a_{1i}b_{in} & \dots & \sum\limits_{i=1}^{n} a_{ni}b_{in} \end{pmatrix} = \begin{pmatrix} \sum\limits_{i=1}^{n} b_{i1}a_{1i} & \dots & \sum\limits_{i=1}^{n} b_{i1}a_{ni} \\ \vdots & & \vdots \\ \sum\limits_{i=1}^{n} b_{in}a_{1i} & \dots & \sum\limits_{i=1}^{n} b_{in}a_{ni} \end{pmatrix}$$

$$= \begin{pmatrix} b_{11} & b_{21} & \dots & b_{n1} \\ b_{12} & b_{22} & \dots & b_{n2} \\ \vdots & \vdots & & \vdots \\ b_{1n} & b_{2n} & \dots & b_{nn} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{pmatrix}$$

$$= B^\top A^\top.$$

This concludes the proof. □

14. Assume A is an $n \times n$ invertible matrix, then

$$(A^{-1})^\top = (A^\top)^{-1}.$$

Proof:

We need to show $(A^{-1})^\top A^\top = I_{n\times n}$ and $A^\top (A^{-1})^\top = I_{n\times n}$. Using (2.9) yields

$$(A^{-1})^\top A^\top = (AA^{-1})^\top = (I_{n\times n})^\top = I_{n\times n}.$$

The same argument shows $A^\top (A^{-1})^\top = I_{n\times n}$. □

15. Let A be an $n \times n$ matrix and $v, w \in \mathbb{R}^n$, then

$$\langle A^\top v, w \rangle = \langle v, Aw \rangle. \tag{2.10}$$

Proof:

Let $A = (a_{jk})_{1 \leq j,k \leq n}$, then $A^\top = (a_{kj})_{1 \leq j,k \leq n}$.

Let $v = (v_1, v_2, \ldots, v_n)$ and $w = (w_1, w_2, \ldots, w_n)$. So,

$$A^\top v = \begin{pmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \left(\sum_{i=1}^{n} a_{i1} v_i, \sum_{i=1}^{n} a_{i2} v_i, \ldots, \sum_{i=1}^{n} a_{in} v_i \right).$$

Thus,

$$
\begin{aligned}
\langle A^\top v, w \rangle &= \left(\sum_{i=1}^{n} a_{i1} v_i, \ldots, \sum_{i=1}^{n} a_{in} v_i \right) \cdot (w_1, \ldots, w_n) \\[2mm]
&= \sum_{i=1}^{n} a_{i1} v_i w_1 + \sum_{i=1}^{n} a_{i2} v_i w_2 + \ldots + \sum_{i=1}^{n} a_{in} v_i w_n \\[2mm]
&= \sum_{i=1}^{n} v_i a_{i1} w_1 + \sum_{i=1}^{n} v_i a_{i2} w_2 + \ldots + \sum_{i=1}^{n} v_i a_{in} w_n \\[2mm]
&= (v_1 a_{11} w_1 + \ldots + v_n a_{n1} w_1) + (v_1 a_{12} w_2 + \ldots + v_n a_{n2} w_2) + \ldots \\[2mm]
&\quad + (v_1 a_{1n} w_n + \ldots + v_n a_{nn} w_n) \\[2mm]
&= (v_1 a_{11} w_1 + v_1 a_{12} w_2 + \ldots + v_1 a_{1n} w_n) \\[2mm]
&\quad + (v_2 a_{21} w_1 + v_2 a_{22} w_2 + \ldots + v_2 a_{2n} w_n) + \ldots \\[2mm]
&\quad + (v_n a_{n1} w_1 + v_n a_{n2} w_2 + \ldots + v_n a_{nn} w_n) \\[2mm]
&= \sum_{i=1}^{n} v_1 a_{1i} w_i + \sum_{i=1}^{n} v_2 a_{2i} w_i + \ldots + \sum_{i=1}^{n} v_n a_{ni} w_i \\[2mm]
&= (v_1, v_2, \ldots, v_n) \cdot \left(\sum_{i=1}^{n} a_{1i} w_i, \sum_{i=1}^{n} a_{2i} w_i, \ldots, \sum_{i=1}^{n} a_{ni} w_i \right). \quad (2.11)
\end{aligned}
$$

Focusing on $\left(\sum\limits_{i=1}^{n} a_{1i}w_i, \sum\limits_{i=1}^{n} a_{2i}w_i, \ldots, \sum\limits_{i=1}^{n} a_{ni}w_i \right)$ gives

$$\left(\sum_{i=1}^{n} a_{1i}w_i, \sum_{i=1}^{n} a_{2i}w_i, \ldots, \sum_{i=1}^{n} a_{ni}w_i \right) = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$
$$= Aw. \tag{2.12}$$

Putting (2.11) and (2.12) together, we find

$$\langle A^{\top}v, w \rangle = \langle v, Aw \rangle.$$

This finishes the proof. $\qquad\square$

16. If $u, v \in \mathbb{R}^n$ are such that $\langle u, w \rangle = \langle v, w \rangle$ for all $w \in \mathbb{R}^n$, then $u = v$.

Proof:

$\langle u, w \rangle = \langle v, w \rangle$ implies $\langle u - v, w \rangle = 0$ for all $w \in \mathbb{R}^n$. Since this expression is true for all $w \in \mathbb{R}^n$, let $w = u - v$. In doing so we obtain $\langle u - v, u - v \rangle = 0$. This implies $\|u - v\|^2 = 0$. Thus $u_i - v_i = 0$ for all $i \in \{1, 2, \ldots, n\}$. Hence $u = v$. $\qquad\square$

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