

*Approved  
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A SIMPLE PRESENTATION OF FOURIER'S SERIES

by

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## PREFACE

The usual presentations of the development of a function into a sine or cosine series either show the possibility of such a development for special cases, only, or else give an outline of the general proof of Dirichlet.\* The latter is rather complicated, and too difficult for a beginner, while the former is very unsatisfactory since the theory is applied to many functions for which it is not shown to be valid.

It is the chief purpose of this paper to give a relatively simple proof for the possibility of representing a large class of functions by means of convergent sine or cosine series, - generally classed as Fourier's Series. This seems desirable because of the growing importance of trigonometric series in their application to applied mathematics, along with their historical value in the development of the theory of functions and mathematical analysis. Even those who are not specially inter-

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\* The treatment of Bocher (Annals of Math. Vol. VII, Ser. 2.) is an exception to this statement and is perhaps the simplest that has yet appeared.

ested in advanced mathematics may find an elementary treatment of the subject of some value in giving them some knowledge of the subject at a first reading.

In order to make this paper somewhat a unit in itself I have made free use of materials found in standard treatises on the subject. I have consulted a number of discussions but have drawn most freely upon Byerly's "Fourier's Series and Spherical Harmonics", Hobson's "Theory of Functions of a Real Variable", and Bocher's treatment of the subject in the Annals of Mathematics, Vol. VII, Ser. 2. Further references will be given in the foot notes.

I wish to acknowledge here my indebtedness to my adviser, Dr. Louis Ingold, for his untiring efforts in assisting me in the preparation of this paper. I deeply appreciate the assistance and extend my sincerest thanks for the same.

Thos. W. Jackson.

## CHAPTER I

## PRELIMINARY, - DEFINITIONS AND THEOREMS.

1) The following theorems and definitions are merely listed here for convenience and may be omitted by one who is already familiar with them.

2) CONVERGENCE:

An infinite series is said to be convergent when the sum of the first  $n$  terms approaches a limit as  $n$  increases without limit.\*

3) UNIFORM CONVERGENCE:

If we have given a series

$$f(x) = U_0(x) + U_1(x) + U_2(x) + \dots + U_n(x) + \dots$$

it is said to be Uniformly Convergent in the interval  $(ab)$  if it converges for every value of  $x$  between  $a$  and  $b$ , and if, corresponding to any arbitrarily preassigned positive number  $\epsilon$ , a positive number  $N$ , independent of  $x$ , can be found such that the absolute value of the remainder;  $R_n(x)$ ; of the given series after  $n$  terms is less than  $\epsilon$  for every value of  $x$  which lies in the interval  $(ab)$  when  $n$  is greater than  $N$ .\*\*

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\* Woods and Bailey, Anal. Geom. and Cal. Pg. 405.

\*\* This is essentially the definition given in Goursat-Hedrick, Math. Analysis, Vol. I, Pg. 361.

## 4) THEOREM I:

If  $f(x)$  is always less than a fixed number  $M$  then

$$\int_a^b f(x) dx < M(b - a).*$$

## 5) THEOREM II:

If a series whose terms are integrable converges to an integrable function uniformly in an interval  $(ab)$  it may be integrated term by term, provided the limits of integration lie in the interval  $(ab)$ .\*\*

Proof:

Let  $x_1$  and  $x_2$  be any two values of  $x$  which lie between  $a$  and  $b$ , and let  $N$  be a positive integer such that  $|R_n(x)| < \epsilon$  for all values of  $x$ , between  $a$  and  $b$ , when  $n \geq N$ .

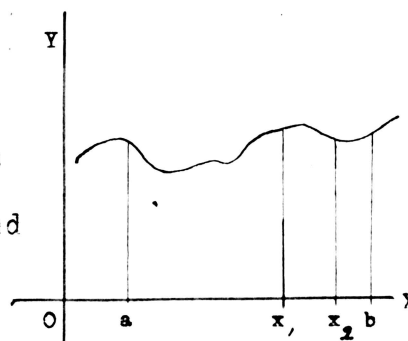


Fig. 1

Let  $f(x) = U_0(x) + U_1(x) + U_2(x) + \dots$

and let us set

$$D_n = \int_{x_1}^{x_2} f(x) dx - \int_{x_1}^{x_2} U_0(x) dx - \int_{x_1}^{x_2} U_1(x) dx - \int_{x_1}^{x_2} U_2(x) dx - \dots$$

$$\dots - \int_{x_1}^{x_2} U_n(x) dx - \dots = \int_{x_1}^{x_2} R_n dx.$$

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\* The proof is easy. It may be obtained by the method used for the First Law of the Mean. Goursat-Hedrick, Vol. I, Pg. 151.

\*\* This proof is based on Goursat-Hedrick. Math. Analy. Vol. I, Pg. 364. The proof there is more general. (?)

The absolute value of  $D_n$  is  $< \epsilon |x_2 - x_1|$  when  $n \geq N$ .<sup>\*</sup>  
Hence  $D_n$  approaches zero as  $n$  increases indefinitely,  
and we have the equation

$$\int_{x_1}^{x_2} f(x) dx = \int_{x_1}^{x_2} U_0(x) dx + \int_{x_1}^{x_2} U_1(x) dx + \int_{x_1}^{x_2} U_2(x) dx + \int_{x_1}^{x_2} U_3(x) dx \\ + \int_{x_1}^{x_2} U_4(x) dx + \dots + \int_{x_1}^{x_2} U_m(x) dx + \dots \quad (A)$$

6) THEOREM III:

If a series whose terms are integrable converges uniformly in every interval contained wholly within the interval  $(ab)$  and if the sum of the series is integrable while the remainder,  $R_n$ , after  $n$  terms never exceeds a fixed number  $R$ , then the series may be integrated term by term.

Proof:

In the interval  $(a+h, b-h)$   $R_n < \eta$  if  $n > N$ . And we may write

$$\int_a^b R_n(x) dx = \int_a^{a+h} R_n(x) dx + \int_{a+h}^{b-h} R_n(x) dx + \int_{b-h}^b R_n(x) dx \\ < R(h) + (b-a-2h)\eta + R(h) \quad \text{or} \\ < 2R(h) + \eta(b-a-2h).^{**}$$

Now choose (1)  $h < \frac{\epsilon}{4R}$  and (2)  $\eta < \frac{\epsilon}{2(b-a-2h)}$ .

Then we have

\* By the Law of the Mean. See Davis, The Cal. Pg. 257.

\*\* By the Law of the Mean. See note \*.

$\int_a^b R_n(x) dx < \epsilon$  and therefore the sum of the integrals

of the first  $n$  terms approaches the integral of the sum of the series.

This result can evidently be extended to apply to cases where there are any finite number of points of non uniform convergence in the interval  $(ab)$ .

## CHAPTER II

## FOURIER'S SERIES.

7) A series of the form

$$\frac{b_0}{2} + b_1 \cos X + b_2 \cos 2X + b_3 \cos 3X + \dots + b_n \cos nX + \dots \\ \dots + a_1 \sin X + a_2 \sin 2X + a_3 \sin 3X + \dots + a_n \sin nX + \dots$$

where the coefficients  $b_0, b_1, b_2, \dots, a_1, a_2, a_3, \dots$  do not involve  $x$ , is called a trigonometric series. If the coefficients are determined in the manner given in paragraphs nine and eleven the series is called a Fourier's Series.

The sum of the double series\* is often expressed as  $C + \sum (b_n \cos nX + a_n \sin nX)$ , where  $C$  is a constant.

In what follows we shall consider the sine and cosine series separately.\*\*

8) NUMERICAL CONSIDERATION OF SINE SERIES:

We may actually determine the coefficients of a sine series so that the curve represented by the series may be made to pass through any given point, or any finite number of points, on the straight line  $y = x$ , between the limits zero and  $\pi$ .

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\* When this series is written as the sum of a sine and cosine series it is called a double Fourier's Series, while either part is referred to as a single Fourier's Series.

\*\* These functions have the period  $2\pi$ . Def:  $f(x)$  is periodic, with period  $k$ , if  $f(x+k) = f(x)$ . See Woods and Bailey, Anal. Geom. and Cal. Pg. 427.



(a) If we select the point  $P_1 = \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ , (see Fig.2), then  $y = a_1 \sin X$ , or  $\frac{\pi}{2} = a_1(1)$ , and  $a_1 = \frac{\pi}{2}$ . Thus we get the equation  $y = \frac{\pi}{2} \sin X$ . This curve will pass through the points  $(0,0)$ ,  $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ , and  $(\pi, 0)$ .

(b) If we select the two points  $P_2 = \left(\frac{\pi}{4}, \frac{\pi}{4}\right)$ , and  $P_3 = \left(\frac{3\pi}{4}, \frac{3\pi}{4}\right)$  we can solve the two resulting simultaneous equations

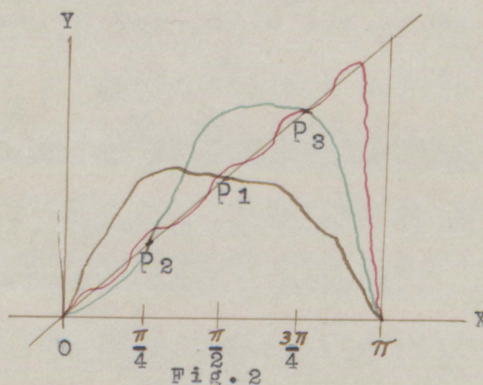
$$\frac{\pi}{4} = a_1 \sin \frac{\pi}{4} + a_2 \sin \frac{\pi}{2}$$

$$\frac{3\pi}{4} = a_1 \sin \frac{3\pi}{4} + a_2 \sin \frac{3\pi}{2}$$

for  $a_1$  and  $a_2$ . Substitute these values in

$$y = a_1 \sin X + a_2 \sin 2X \text{ and}$$

we get  $y = \frac{\pi}{2} \sin X - \frac{\pi}{4} \sin 2X$ . This curve will pass through the points  $(0,0)$ ,  $P_2$ ,  $P_3$ , and  $(\pi, 0)$ .



We may in like manner actually determine the coefficients of a sine series so as to make the curve pass through any finite number of points on the line  $y = f(x) = x$ , between the limits zero and  $\pi$ .

This suggests the possibility of representing the straight line,  $y = x$ , to any degree of accuracy, between  $x = 0$  and  $x = \pi$ , by suitably determining the coefficients of a sufficiently large number of terms of a sine series of the form

$$f(x) = a_1 \sin X + a_2 \sin 2X + a_3 \sin 3X + \dots + a_n \sin NX.$$

In fact it may be possible to represent the

function exactly, between zero and  $\pi$ , by means of a convergent, infinite series of this type.

It will appear in the sequel that not only the function  $x$  but a very large class of functions, including almost all the continuous functions likely to be met with, may be represented by infinite series of the Fourier type.

9) GENERAL METHOD OF DETERMINING THE COEFFICIENTS OF A SINE SERIES:

In case a known function is represented by a series  $f(x) = a_1 \sin X + a_2 \sin 2X + a_3 \sin 3X + a_4 \sin 4X + \dots$  (B) which converges in such a way that when it is multiplied by  $\sin MX$  (where  $M$  is a positive integer, not zero) it may be integrated term by term, then the coefficients are easily determined. This is due to the fact that

$$\int_0^{\pi} \sin MX \sin NX dx = 0, \text{ when } M \neq N^* \text{ and}$$

$$\int_0^{\pi} \sin MX \sin NX dx = \frac{\pi}{2}, \text{ when } M = N^*.$$

Assuming, then, that the series

$$f(x) \sin MX = a_1 \sin X \sin MX + a_2 \sin 2X \sin MX + a_3 \sin 3X \sin MX + \dots + a_m \sin^2 MX + \dots + a_n \sin NX \sin MX + \dots$$

may be integrated term by term we have

$$\int_0^{\pi} f(x) \sin MX dx = 0 + 0 + 0 + \dots + a_m \frac{\pi}{2} + 0 + 0 + \dots$$

\* See formulae 111 and 112, Pg. 44 of Tables, The Calculus, by Davis.

Hence  $\int_0^\pi f(x)\sin Mx dx = a_m \frac{\pi}{2}$  ,

and  $a_m = \frac{2}{\pi} \int_0^\pi f(x)\sin Mx dx$ . (C)

Thus we may obtain the coefficients for any sine series, under the conditions above.

10) EXAMPLES OF SINE DEVELOPMENTS:\*

(a) Let  $f(x) = 1$ .

Then  $a_m = \frac{2}{\pi} \int_0^\pi (1)\sin Mx dx = \frac{2}{\pi} \left[ -\frac{\cos Mx}{M} \right]_0^\pi$   
 $= \frac{2}{\pi M} [1 - (-1)^M] = 0$ , when  $M$  is even and  
 $= \frac{4}{\pi M}$ , when  $M$  is odd.

Therefore

$f(x) = 1 = \frac{4}{\pi} \left( \frac{\sin X}{1} + \frac{\sin 3X}{3} + \frac{\sin 5X}{5} + \dots \right)$ ,

provided this series may be integrated term by term when multiplied by  $\sin Mx$ .

(b) Example (a) gives at once a sine development for any constant,  $k$ . It is

$k = \frac{4k}{\pi} \left( \frac{\sin X}{1} + \frac{\sin 3X}{3} + \frac{\sin 5X}{5} + \dots \right)$ .

(c) Let  $f(x) = x$ .

Then  $a_m = \frac{2}{\pi} \int_0^\pi (x)\sin Mx dx = -\frac{2}{\pi M^2} [\sin Mx - Mx \cos Mx]_0^\pi$   
 $= \left( -\frac{\pi(-1)^M}{M} \right) \left( \frac{2}{\pi} \right) = -\frac{2(-1)^M}{M}$

Therefore

$f(x) = x = 2 \left( \frac{\sin X}{1} - \frac{\sin 2X}{2} + \frac{\sin 3X}{3} - \frac{\sin 4X}{4} + \dots \right)$ ,

provided this series may be integrated term by term when multiplied by  $\sin Mx$ .

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 \*Developed by Byerly, Fourier's Ser. Sp. Har. Pg. 39.

11) GENERAL METHOD OF DETERMINING THE COEFFICIENTS OF A  
COSINE SERIES:

Analogous to the discussion in paragraph nine, let  

$$f(x) = \frac{b_0}{2} + b_1 \cos X + b_2 \cos 2X + b_3 \cos 3X + \dots \quad (D)$$

be a known function represented by a series which  
converges in such a way that when it is multiplied  
by  $\cos MX$ , (where  $M$  is a positive integer), it may  
be integrated term by term, then the coefficients  
may be determined. This is due to the fact that

$$\int_0^{\pi} \cos MX \cos NX dx = 0, \text{ when } M \neq N, \text{ and equal to } \frac{\pi}{2}$$

when  $M = N$ .\*

Multiply the expression (D) by  $\cos MX$  and we get  

$$f(x) \cos MX = \frac{b_0}{2} \cos MX + b_1 \cos X \cos MX + b_2 \cos 2X \cos MX +$$

$$\dots + b_m \cos^2 MX + \dots + b_n \cos NX \cos MX + \dots \quad (E)$$

Integrate both members from zero to  $\pi$  and we get

$$\int_0^{\pi} f(x) \cos MX dx = 0 + 0 + 0 + \dots + b_m \frac{\pi}{2} + 0 + 0 + \dots$$

Or 
$$\int_0^{\pi} f(x) \cos MX dx = b_m \frac{\pi}{2}, \text{ and}$$

$$b_m = \frac{2}{\pi} \int_0^{\pi} f(x) \cos MX dx. \text{ (This is also valid for } M = 0). \quad (F)$$

Thus, under the above conditions, we may obtain  
the coefficients of a cosine series.

In general the coefficients of either a sine or  
cosine series may be determined, provided the series  
can be integrated term by term, between the limits  
zero and  $\pi$ , when multiplied by  $\sin MX$  or  $\cos MX$ .

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\* See foot note page 9

## 12) EXAMPLES OF COSINE DEVELOPMENTS:

(a) Let  $f(x) = 1$ .

$$\begin{aligned} \text{Then } b_m &= \frac{2}{\pi} \int_0^{\pi} f(x) \cos Mx dx = \frac{2}{\pi} \int_0^{\pi} (1) \cos Mx dx \\ &= \frac{2}{\pi} \left[ \frac{\sin Mx}{M} \right]_0^{\pi} = 0, \text{ when } M = 1, 2, 3, \dots \end{aligned}$$

$$\text{But } b_0 = \frac{2}{\pi} \int_0^{\pi} \cos 0x dx = \frac{2}{\pi} \int_0^{\pi} dx = 2, \text{ when } M = 0.$$

In this case each term of the development is zero except the first, which is unity the given function.

(b) Let  $f(x) = x$ .

$$\begin{aligned} \text{Then } b_0 &= \frac{2}{\pi} \int_0^{\pi} (x) dx = \pi, \text{ and } b_m = \frac{2}{\pi} \int_0^{\pi} (x) \cos Mx dx \\ &= \frac{2}{M^2 \pi} (\cos M\pi - 1) = \frac{2}{M^2 \pi} [(-1)^M - 1]. \text{ Therefore} \end{aligned}$$

$f(x) = x = \frac{\pi}{2} - \frac{4}{\pi} \left( \frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right)$ ,  
provided this series may be integrated term by term,  
between zero and  $\pi$ , when multiplied by  $\cos Mx$ .

## 13) REPRESENTABLE FUNCTIONS:

The above developments are only formal for we have proved that they represent a true development of the function, between zero and  $\pi$ , only when the series, (B) or (D), converges in such a way as to be integrable term by term when multiplied by  $\sin Mx$  or  $\cos Mx$ , respectively.

It is natural to inquire just what functions may be represented by a sine or cosine series. This question, however, has never been answered com-

pletely, but it is known that certain large classes of functions may be so represented.

In Chapter IV it will be shown that any function of a real variable can be developed into a sine or cosine series, and that the function and series will be equal for all values of  $x$  between zero and  $\pi$ , provided the function is single-valued, finite, and periodic of period  $2\pi$ , and is the integral of a finite integrable function.

#### 14) THE SUM OF THE FIRST N TERMS OF A SINE SERIES:

We will here consider the sum of the first  $n$  terms of the Fourier development of any function into a sine series. This sum may, of course, be considered whether the series represents the function or not, indeed whether the series converges or not.

Let  $f(x)$  be any function of  $x$  and let the Fourier development of  $f(x)$  be

$$a_1 \sin X + a_2 \sin 2X + a_3 \sin 3X + \dots + a_n \sin nX + \dots, \quad (G)$$

where any coefficient  $a_m = \frac{2}{\pi} \int_0^{\pi} f(v) \sin mV dv$ .

Substitute for the coefficients in series (G)

and we get

$$\begin{aligned} & \frac{2}{\pi} \int_0^{\pi} f(v) \sin v dv \cdot \sin X + \frac{2}{\pi} \int_0^{\pi} f(v) \sin 2v dv \cdot \sin 2X + \dots \\ & + \frac{2}{\pi} \int_0^{\pi} f(v) \sin mV dv \cdot \sin mX + \dots + \frac{2}{\pi} \int_0^{\pi} f(v) \sin nV dv \cdot \sin nX + \dots \quad (H) \end{aligned}$$

Let  $S_n$  be the sum of the first  $n$  terms of series

(H). \* Then

$$\begin{aligned}
 S_n &= \frac{2}{\pi} \int_0^\pi f(v) [\sin v \sin X + \sin 2v \sin 2X + \dots + \sin Nv \sin NX] dv \\
 &= \frac{1}{\pi} \int_0^\pi f(v) [\cos(v-x) - \cos(v+x) + \cos 2(v-x) - \cos 2(v+x) \\
 &\quad + \dots + \cos N(v-x) - \cos N(v+x)] dv \\
 &= \frac{1}{\pi} \int_0^\pi f(v) [\cos(v-x) + \cos 2(v-x) + \dots + \cos N(v-x)] dv \\
 &\quad - \frac{1}{\pi} \int_0^\pi f(v) [\cos(v+x) + \cos 2(v+x) + \dots + \cos N(v+x)] dv \\
 &= \frac{1}{\pi} \int_0^\pi f(v) \left[ -\frac{1}{2} + \frac{1}{2} \frac{\sin(2N+1)\frac{v-x}{2}}{\sin\frac{v-x}{2}} \right] dv \\
 &\quad - \frac{1}{\pi} \int_0^\pi f(v) \left[ -\frac{1}{2} + \frac{1}{2} \frac{\sin(2N+1)\frac{v+x}{2}}{\sin\frac{v+x}{2}} \right] dv \quad ** \\
 &= \frac{1}{2\pi} \int_0^\pi f(v) \frac{\sin(2N+1)\frac{v-x}{2}}{\sin\frac{v-x}{2}} dv - \frac{1}{2\pi} \int_0^\pi f(v) \frac{\sin(2N+1)\frac{v+x}{2}}{\sin\frac{v+x}{2}} dv. \quad (I)
 \end{aligned}$$

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\* Taken from Byerly, Fourier's Ser. & Sph. Har. Pg. 55.

\*\* Lemma: See Byerly, Fourier's Ser. & Sph. Har. Pg. 32.

Let  $S = \cos \theta + \cos 2\theta + \cos 3\theta + \dots + \cos N\theta$ ,  
 and multiply by  $2\cos \theta$ . Then we have  
 $2S\cos \theta = 2\cos^2 \theta + 2\cos \theta \cos 2\theta + 2\cos \theta \cos 3\theta + \dots + 2\cos \theta \cos N\theta$   
 $= 1 + \cos \theta + \cos 2\theta + \cos 3\theta + \dots + \cos(N-1)\theta$   
 $\quad + \cos 2\theta + \cos 3\theta + \cos 4\theta + \dots + \cos(N+1)\theta$   
 $= 2S + 1 + \cos(N+1)\theta - \cos \theta - \cos N\theta.$

Therefore

$$\begin{aligned}
 S &= -\frac{1}{2} + \frac{\cos N\theta - \cos(N+1)\theta}{2(1 - \cos \theta)} \\
 &= -\frac{1}{2} + \frac{1}{2} \frac{\sin(2N+1)\frac{\theta}{2}}{\sin\frac{\theta}{2}}.
 \end{aligned}$$

## CHAPTER III

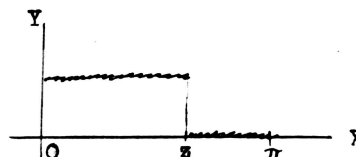
## SPECIAL FOURIER'S SERIES.

## 15) SPECIAL FUNCTION:

Having obtained a formula for the sum of the first  $n$  terms of our sine series we shall in this and the following sections consider the limit approached by this sum in the important special case where  $f(x) = 1$ , when  $0 < x < z$  and  $f(x) = 0$ , when  $z < x < \pi$ .

It can be shown that  $S_n$  really approaches the limit  $f(x)$ , just defined, as  $n$  becomes infinite.

In this case  $S_n$ , (formula (I) page 14), reduces to the follow-



ing:

$$S_n = \frac{1}{2\pi} \int_0^z f(v) \frac{\sin(2N+1)\frac{v-x}{2}}{\sin\frac{v-x}{2}} dv + \frac{1}{2\pi} \int_z^\pi f(v) \frac{\sin(2N+1)\frac{v-x}{2}}{\sin\frac{v-x}{2}} dv - \frac{1}{2\pi} \int_0^z f(v) \frac{\sin(2N+1)\frac{v+x}{2}}{\sin\frac{v+x}{2}} dv - \frac{1}{2\pi} \int_z^\pi f(v) \frac{\sin(2N+1)\frac{v+x}{2}}{\sin\frac{v+x}{2}} dv.$$

Since the function  $f(v)$  is zero for values of  $v$  between  $z$  and  $\pi$  the second and fourth terms of this expression for  $S_n$  are equal to zero and we have

$$S_n = \frac{1}{2\pi} \int_0^z \frac{\sin(2N+1)\frac{v-x}{2}}{\sin\frac{v-x}{2}} dv - \frac{1}{2\pi} \int_0^z \frac{\sin(2N+1)\frac{v+x}{2}}{\sin\frac{v+x}{2}} dv. \quad (J)$$

Now let  $\frac{v-x}{2} = \theta$ . Then when  $v = 0$ ,  $\theta = -\frac{x}{2}$ , and when  $v = z$ ,  $\theta = \frac{z-x}{2}$ , while  $dv = 2d\theta$ .



Also let  $\frac{v+x}{2} = \theta$ . Then when  $v = 0$ ,  $\theta = \frac{x}{2}$ , and when  $v = z$ ,  $\theta = \frac{z+x}{2}$ , while  $dv = 2d\theta$ .

Substitute these values in equation (J) and we

have

$$S_n = \frac{1}{\pi} \int_{-\frac{x}{2}}^{\frac{z-x}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta - \frac{1}{\pi} \int_{\frac{x}{2}}^{\frac{z+x}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta. \quad (K)$$

Or

$$S_n = \frac{1}{\pi} \int_0^{\frac{x}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta + \frac{1}{\pi} \int_0^{\frac{z-x}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta + \frac{1}{\pi} \int_0^{\frac{x}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta - \frac{1}{\pi} \int_0^{\frac{z+x}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta. \quad (L)$$

In order to find the limit of  $S_n$  it will be necessary to obtain the limit of each of the integrals on the right hand side of equation (L). This we shall do in the next section.

16) PROOF THAT  $\lim_{N \rightarrow \infty} \int_0^b \frac{\sin(2N+1)x}{\sin x} dx = \frac{\pi}{2}$ : (Where  $0 < b < \frac{\pi}{2}$ ).

This proof is taken from Byerly's "Fourier's Series and Spherical Harmonics", paragraphs 34 and 35.\*\*

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\* See Byerly, Fourier's Ser. & Sph. Har., foot of Pg. 59.  
Let  $\beta = -\theta$ , then substitute and we get

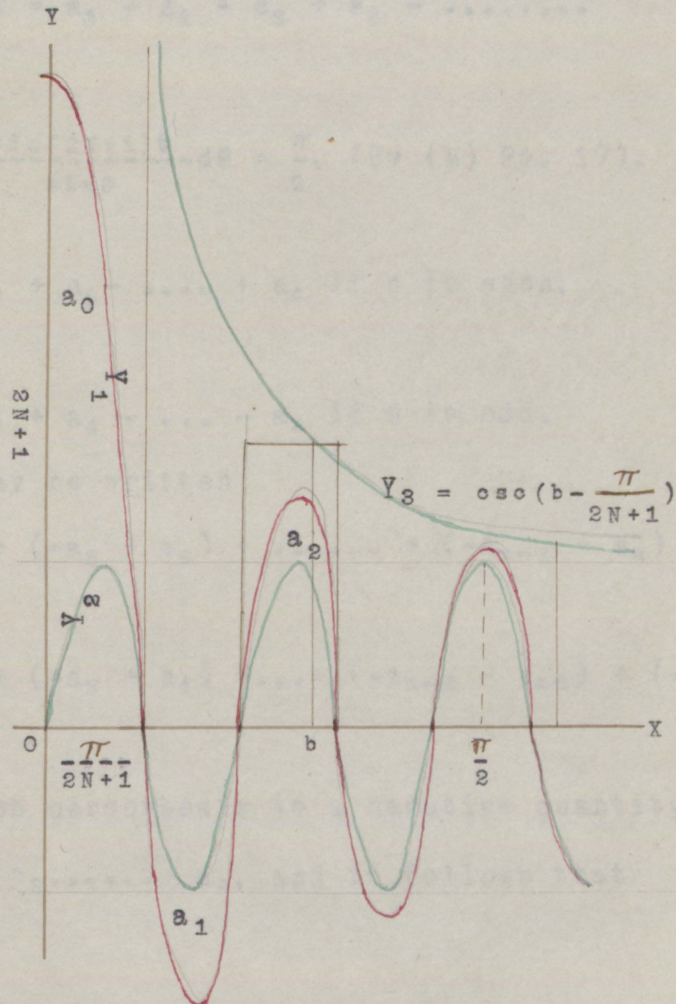
$$\begin{aligned} \int_{-\frac{x}{2}}^0 \frac{\sin(2N+1)\theta}{\sin\theta} d\theta &= - \int_{\frac{x}{2}}^0 \frac{\sin(2N+1)\beta}{\sin\beta} d\beta \\ &= \int_0^{\frac{x}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta. \end{aligned}$$

\*\* Because of the importance of this limit and also to make the paper reasonably complete the discussion of Byerly is reproduced here in full.

Since  $\frac{\sin(2N+1)\theta}{\sin\theta} = \frac{1}{2} + \cos 2\theta + \cos 4\theta + \dots + \cos 2N\theta^*$   
 and  $\int_0^{\pi/2} \cos 2N\theta d\theta = 0$ , we have  $\int_0^{\pi/2} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta = \frac{\pi}{2}$ . (M)

Draw the curve  $Y_1 = \frac{\sin(2N+1)X}{\sin X}$  by first drawing the curve  $Y_2 = \sin(2N+1)X$  and then dividing the length of each ordinate by the value of the sine of the corresponding abscissa. See Figure 3.

The successive arches into which the curve  $Y_2 = \sin(2N+1)X$  is divided by the axis of  $X$  are equal, and consequently their areas are equal. Each arch has unit altitude and  $\frac{\pi}{2N+1}$  for its base and is symmetrical with respect to the ordinate of its highest or lowest point



Since  $\sin X$  increases as  $X$  increases from  $0$  to  $\frac{\pi}{2}$ , the ordinate of any point of the curve  $Y_1$  will be shorter than the

\* See Lemma page 14. (Foot note).

ordinate of the corresponding point in the preceding arch, and consequently the area of each arch of  $Y_1$  will be less than that of the arch before it.

If  $a_0, a_1, a_2, a_3, \dots, a_{n-1}$ , are the areas of the successive arches and  $a_n$  that of the incomplete arch terminated by the ordinate corresponding to  $x = \frac{\pi}{2}$ ,

$$\text{then } \int_0^{\frac{\pi}{2}} \frac{\sin(2N+1)x}{\sin x} dx = a_0 - a_1 + a_2 - a_3 + \dots$$

But

$$\int_0^{\frac{\pi}{2}} \frac{\sin(2N+1)x}{\sin x} dx = \int_0^{\frac{\pi}{2}} \frac{\sin(2N+1)\theta}{\sin \theta} d\theta = \frac{\pi}{2}. \quad (\text{By (M) Pg. 17}).$$

Hence

$$\frac{\pi}{2} = a_0 - a_1 + a_2 - a_3 + a_4 - \dots + a_n \text{ if } n \text{ is even,}$$

or

$$\frac{\pi}{2} = a_0 - a_1 + a_2 - a_3 + a_4 - \dots - a_n \text{ if } n \text{ is odd.}$$

These equations may be written

$$\frac{\pi}{2} = a_0 + (-a_1 + a_2) + (-a_3 + a_4) + \dots + (-a_{n-1} + a_n)$$

and

$$\frac{\pi}{2} = a_0 + (-a_1 + a_2) + (-a_3 + a_4) + \dots + (-a_{n-2} - a_{n-1}) + (-a_n),$$

respectively.

In either case each parenthesis is a negative quantity since  $a_0 > a_1 > a_2 > a_3 \dots > a_n$ , and it follows that  $a_0$  is greater than  $\frac{\pi}{2}$ .

Again

$$\frac{\pi}{2} = (a_0 - a_1) + (a_2 - a_3) + \dots + (a_{n-2} - a_{n-1}) + a_n \text{ if}$$

$n$  is even, and

$$\frac{\pi}{2} = (a_0 - a_1) + (a_2 - a_3) + \dots + (a_{n-1} - a_n) \text{ if } n \text{ is odd.}$$

In either case each parenthesis is positive and it follows that  $a_0 - a_1$  is less  $\frac{\pi}{2}$ .

Since  $a_0 > \frac{\pi}{2} > a_0 - a_1$ ,  $a_0$  and  $a_0 - a_1$  differ from  $\frac{\pi}{2}$  by less than they differ from each other, that is, by less than  $a_1$ .

In like manner we can show that  $a_0 - a_1$  and  $a_0 - a_1 + a_2$  differ from  $\frac{\pi}{2}$  by less than  $a_2$ ; and in general that  $a_0 - a_1 + a_2 - a_3 + a_4 - \dots + a_k$  differs from  $\frac{\pi}{2}$  by less than  $a_k$ ; or even that  $a_0 - a_1 + a_2 - \dots \pm \frac{a_k}{p}$  differs from  $\frac{\pi}{2}$  by less than  $a_k$  no matter what the value of  $p$ , provided  $p$  is greater than unity.

Finally, from what has been proved above, it follows

that  $\int_0^b \frac{\sin(2N+1)x}{\sin x} dx$ , where  $b$  is some value between  $\frac{\pi}{2N+1}$  and  $\frac{\pi}{2}$ , differs from  $\frac{\pi}{2}$  by less than the area of the arch in which the ordinate  $Y_1$  corresponding to  $x = b$  falls if this ordinate divides an arch, or by less than the area of the arch next beyond the point  $(b, 0)$  if the curve crosses the axis of  $X$  at that point.

The area of the arch in question is less than  $\frac{\pi}{2N+1}$ , its base, multiplied by  $\frac{1}{\sin(b - \frac{\pi}{2N+1})}$ , a value greater than the length of its longest ordinate.

Therefore  $\int_0^b \frac{\sin(2N+1)x}{\sin x} dx$  differs from  $\frac{\pi}{2}$  by less than  $\frac{\pi}{2N+1} \cdot \frac{1}{\sin(b - \frac{\pi}{2N+1})} = \frac{\pi}{2N+1} \operatorname{csc}(b - \frac{\pi}{2N+1})$ . (N)

Now if  $n$  is indefinitely increased the expression (N) approaches zero as its limit, and we get the important result

$$\text{Limit}_{N = \infty} \int_0^b \frac{\sin(2N+1)x}{\sin x} dx = \frac{\pi}{2} \quad \text{if } 0 < b < \frac{\pi}{2}. \quad (0)$$

### 17) LIMIT OF $S_n$ IN THE SPECIAL CASE OF § 15:

We are now able to obtain the limit of  $S_n$  as  $n$  becomes infinite in the special case which was taken up in article 15. The discussion is divided into three cases, viz: Case I, where  $0 < x < z$  and  $z = \frac{\pi}{2}$ .

Case II, where  $x = z$  and  $z = \frac{\pi}{2}$ .

Case III, where  $z < x < \pi$  and (a)  $\frac{z+x}{2} = \frac{\pi}{2}$   
(b)  $\frac{z+x}{2} > \frac{\pi}{2}$ .

Case I: (When  $0 < x < z$ , and  $z = \frac{\pi}{2}$ ).

From equation (L), page 16, the

$$\text{Limit}_{N = \infty} S_n = \frac{1}{\pi} \cdot \frac{\pi}{2} + \frac{1}{\pi} \cdot \frac{\pi}{2} + \frac{1}{\pi} \cdot \frac{\pi}{2} - \frac{1}{\pi} \cdot \frac{\pi}{2} = 1 \quad \text{identically.}$$

For the limit of each of the four integrals in (L) is synonymous with the expression (0) above.

Case II: (When  $x = z$ , and  $z = \frac{\pi}{2}$ ).

From equation (L), page 16, the

$$\begin{aligned} \text{Limit}_{N = \infty} S_n &= \text{Limit}_{N = \infty} \left[ \frac{1}{\pi} \int_0^{z/2} \frac{\sin(2N+1)\theta}{\sin \theta} d\theta + \frac{1}{\pi} \int_0^0 \frac{\sin(2N+1)\theta}{\sin \theta} d\theta \right. \\ &\quad \left. + \frac{1}{\pi} \int_0^{z/2} \frac{\sin(2N+1)\theta}{\sin \theta} d\theta - \frac{1}{\pi} \int_0^z \frac{\sin(2N+1)\theta}{\sin \theta} d\theta \right] \\ &= \frac{1}{\pi} \cdot \frac{\pi}{2} + \frac{1}{\pi} \cdot (0) + \frac{1}{\pi} \cdot \frac{\pi}{2} - \frac{1}{\pi} \cdot \frac{\pi}{2} = \frac{1}{2} \quad \text{identically.} \end{aligned}$$

For in this case the limit of the second integral is zero because the limits of integration are each equal to zero, while the limits of the other three integrals each satisfy the conditions in equation (0) above.\*

Case III: (a) (When  $z < x < \pi$ , and  $\frac{z+x}{2} < \frac{\pi}{2}$ ).

From equation (L), page 16,

$$S_n = \frac{1}{\pi} \int_0^{\frac{x}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta - \frac{1}{\pi} \int_0^{\frac{x-z}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta^{**}$$

$$+ \frac{1}{\pi} \int_0^{\frac{x}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta - \frac{1}{\pi} \int_0^{\frac{z+x}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta.$$

And the

$$\lim_{N \rightarrow \infty} S_n = \frac{1}{\pi} \cdot \frac{\pi}{2} - \frac{1}{\pi} \cdot \frac{\pi}{2} + \frac{1}{\pi} \cdot \frac{\pi}{2} - \frac{1}{\pi} \cdot \frac{\pi}{2} = 0 \text{ identically.}$$

For here again the limits of the four integrals each satisfy the conditions of equation (0) above.

Case III: (b) (When  $z < x < \pi$ , and  $\frac{z+x}{2} > \frac{\pi}{2}$ ).

-----  
 \* Bôcher shows that the Fourier development of such a function converges to  $\frac{1}{2}[f(x+0) + f(x-0)]$  at points of discontinuity. Annals of Math. Vol. VII, Ser. 2, Pgs. 113-115. Also see Hobson, Theory of Functions of a Real Variable, Pg. 759. We shall not consider discontinuous functions but these could be treated by precisely the same method as that used by Bôcher.

$$** \int_0^{\frac{z-x}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta = \int_0^{\frac{x-z}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta$$

$$= - \int_{-\frac{x-z}{2}}^0 \frac{\sin(2N+1)\theta}{\sin\theta} d\theta = - \int_0^{\frac{x-z}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta.$$

See foot note page 16.

In this case

$$\begin{aligned}
 S_n &= \frac{1}{\pi} \int_0^{\frac{x}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta + \frac{1}{\pi} \int_0^{\frac{z-x}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta \\
 &+ \frac{1}{\pi} \int_0^{\frac{x}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta - \frac{1}{\pi} \int_0^{\frac{z+x}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta \quad (\text{from (L)}) \\
 &= \frac{1}{\pi} \int_0^{\frac{x}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta - \frac{1}{\pi} \int_0^{\frac{z-x}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta^* \\
 &+ \frac{1}{\pi} \int_0^{\frac{x}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta - \frac{1}{\pi} \int_0^{\frac{z+x}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta \\
 &- \frac{1}{\pi} \int_{\frac{\pi}{2}}^{\frac{z+x}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta. \quad (P)
 \end{aligned}$$

Now let  $\beta = \pi - \theta$ , then when  $\theta = \frac{\pi}{2}$ ,  $\beta = \frac{\pi}{2}$ , and when

$$\theta = \frac{z+x}{2}, \quad \beta = \pi - \frac{z+x}{2}. \quad \text{The } d\theta = -d\beta.$$

Substitute these values in the last term of (P)

and we have

$$\begin{aligned}
 \int_{\frac{\pi}{2}}^{\frac{z+x}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta &= - \int_{\frac{\pi}{2}}^{\pi - \frac{z+x}{2}} \frac{\sin(2N+1)\beta}{\sin\beta} d\beta = \int_{\pi - \frac{z+x}{2}}^{\frac{\pi}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta \\
 &= \int_0^{\frac{\pi}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta - \int_0^{\pi - \frac{z+x}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta. \quad **
 \end{aligned}$$

Substitute these two integrals for the last term of

(P) and we get

---

\* See foot note page 21.

\*\* See foot note page 16.

$$\begin{aligned}
S_n = & \frac{1}{\pi} \int_0^{\frac{x}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta - \frac{1}{\pi} \int_0^{\frac{x-z}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta \\
& + \frac{1}{\pi} \int_0^{\frac{x}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta - \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta \\
& - \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta + \frac{1}{\pi} \int_0^{\frac{\pi - \frac{z+x}{2}}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta.
\end{aligned}$$

And

$$\text{Limit } N = \infty \quad S_n = \frac{1}{\pi} \cdot \frac{\pi}{2} - \frac{1}{\pi} \cdot \frac{\pi}{2} + \frac{1}{\pi} \cdot \frac{\pi}{2} - \frac{1}{\pi} \cdot \frac{\pi}{2} - \frac{1}{\pi} \cdot \frac{\pi}{2} + \frac{1}{\pi} \cdot \frac{\pi}{2} = 0.$$

We have thus shown that the sum of  $n$  terms of  $f(x)$  = 1, when  $0 < x < z$  and  $f(x) = 0$ , when  $z < x < \pi$ , expressed as a sine series, really approaches the limit unity, or zero, as  $n$  becomes infinite. We have therefore the following formula:

$$\left. \begin{aligned}
1 &= \frac{2}{\pi} \sum \sin Mx \frac{1 - \cos Mz}{M}, \text{ when } 0 < x < z. \\
\frac{1}{2} &= \text{ " " " " " , when } x = z. \\
0 &= \text{ " " " " " , when } z < x < \pi.
\end{aligned} \right\} \quad (Q)$$

Clearly the series converges to zero when  $x = 0$ . We may if we like regard  $x$  as a fixed number between zero and  $\pi$  and  $z$  as a variable. Then we have a function of  $z$  defined by the formula:

$$\left. \begin{aligned}
0 &= \frac{2}{\pi} \sum \frac{1 - \cos Mz}{M} \sin Mx, \text{ when } 0 < z < x. \\
\frac{1}{2} &= \text{ " " " " " , when } z = x. \\
1 &= \text{ " " " " " , when } x < z < \pi.
\end{aligned} \right\} \quad (R)$$



## 18) REGION OF UNIFORM CONVERGENCE:

Before reaching our final result it will be necessary to integrate the series just obtained (and also other series obtained from this one by multiplying by certain functions). We must therefore determine whether the series may be integrated term by term. For this purpose we investigate the region of uniform convergence. We first show that

$$\left| \frac{\pi}{2} - \int_0^x \frac{\sin(2N+1)\theta}{\sin\theta} d\theta \right| < \eta \text{ for all values of } x \text{ between}$$

$\frac{h}{2}$  and  $\frac{\pi}{2}$ , provided  $n > N$  (independent of  $x$ ).

Proof:

Choose  $n$  large enough to ensure that the area of the arch in which the ordinate of  $V_1$  (see Fig. 3, page 17) corresponding to  $x = \frac{h}{2}$  falls shall be less than  $\eta$ . Then the above inequality holds for all values of  $x$  between  $\frac{h}{2}$  and  $\frac{\pi}{2}$  for the same value of  $n$ .

Now in Case I, article 17, page 20,

$$S_n = \frac{1}{\pi} \int_0^{\frac{x}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta + \frac{1}{\pi} \int_0^{\frac{z-x}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta \\ + \frac{1}{\pi} \int_0^{\frac{x}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta - \frac{1}{\pi} \int_0^{\frac{z+x}{2}} \frac{\sin(2N+1)\theta}{\sin\theta} d\theta, \text{ by equa-}$$

tion (L), page 16.

If  $\frac{z-x}{2} > \frac{h}{2}$ ,  $n$  can be taken large enough so that

each term differs from  $\frac{1}{\pi} \cdot \frac{\pi}{2}$ , or  $\frac{1}{2}$ , by less than  $\frac{\epsilon}{4}$ .

Therefore  $S_n$  approaches the limit 1 uniformly between  $h$  and  $z-h$ . (By definition, article 3, page 3).

In precisely the same way it can be shown that  $S_n$  approaches zero uniformly between  $x = z+h$  and  $x = \pi-h$ .

It is easily seen that the region of uniform convergence remains the same if the series is multiplied by any bounded integrable function, and that if  $z$  be regarded as variable and  $x$  constant the series converges uniformly between  $0$  and  $x-h$  and also between  $x+h$  and  $\pi-h$ .

These series may therefore (by Theorem III, Pg. 5) be integrated term by term either with respect to  $x$  or  $z$ . They may also be integrated after they are multiplied by any bounded integrable function.

## CHAPTER IV

A CLASS OF FUNCTIONS REPRESENTED  
BY A FOURIER'S SERIES.

## 19) SPECIAL EXAMPLE:

We are now able to obtain the sine or cosine development of any number of functions, for if we multiply the series obtained at the end of article 17, (series (R) page 23), by any bounded integrable function of  $z$  and integrate, the resulting sine series will represent the integral of the left hand side, which will be a function of  $x$ . To illustrate we integrate the series as it stands with respect to  $z$ . Thus

$$\int_0^{\pi} \begin{matrix} [0] \\ [1] \end{matrix} dz = \frac{2}{\pi} \sum \left[ \int_0^{\pi} \frac{1 - \cos MZ}{M} dz \right] \sin MX, \quad \begin{matrix} (0 < z < x) \\ (x < z < \pi) \end{matrix}$$

or

$$\int_0^x (0) dz + \int_x^{\pi} (1) dz = \frac{2}{\pi} \sum \left[ \frac{z - \frac{1}{M} \sin MZ}{M} \right]_0^{\pi} \sin MX;$$

or

$$(0) + \pi - x = \frac{2}{\pi} \sum \left[ \frac{\pi - 0}{M} \right] \sin MX;$$

hence  $\frac{\pi - x}{2} = \sum \frac{\sin MX}{M}.$  \* (S)

Thus we have obtained the sine series which represents the function  $\frac{\pi - x}{2}$  for values of  $x$  between zero and  $\pi$ .

---

\* This series is discussed by Hobson, Theory of functions of a Real Variable, Pg. 648. Also by Bôcher, Annals of Math., Vol. VII, Ser. 2.

## 20) GENERAL RESULT:

Instead of integrating series (R) as it stands we may first multiply by any bounded integrable function  $f(z)$  and then integrate term by term.

Multiply series (R) and we get

$$\left. \begin{aligned} 0 &= \frac{2}{\pi} \sum f(z) \frac{1 - \frac{\cos MZ}{M}}{\sin MX}, \text{ when } 0 < z < x; \\ f(z) &= \frac{2}{\pi} \sum f(z) \frac{1 - \frac{\cos MZ}{M}}{\sin MX}, \text{ when } x < z < \pi. \end{aligned} \right\} \quad (\text{T})$$

Hence

$$\int_x^\pi f(z) dz = \frac{2}{\pi} \sum \left[ \int_0^\pi f(z) \frac{1 - \frac{\cos MZ}{M}}{\sin MX} dz \right] \sin MX. \quad (\text{U})$$

Thus we have obtained a sine series which represents the function  $\int_x^\pi f(z) dz$ . Since  $f(z)$  is an arbitrary bounded integrable function we have an exceedingly general result which we state as follows:

ANY FUNCTION WHICH CAN BE EXPRESSED AS THE INTEGRAL, BETWEEN THE LIMITS  $x$  AND  $\pi$ , OF A BOUNDED INTEGRABLE FUNCTION CAN BE REPRESENTED BY MEANS OF A CONVERGENT SINE SERIES.

## 21) COMPARISON OF COEFFICIENTS WITH FOURIER'S COEFFICIENTS:

It remains to show that the coefficients of this general series are really the Fourier's Coefficients of the function which the series represents. By article 9, page 9, this will follow provided we can show that the series can be integrated term by term when multiplied by  $\sin MX$ .

We conclude from article 18, page 24, that if  $n > N$   $R_n < \eta$ , provided  $z$  and  $x$  are both greater than  $h$  and  $|z-x| < h$ . (The integer  $N$  depends on  $h$  but may be chosen independent of both  $z$  and  $x$  with the above restrictions).

The remainder after  $n$  terms of our general series is

$$\int_0^\pi R_n dz \text{ and we may write}$$

$$\int_0^\pi R_n dz = \int_0^h R_n dz + \int_h^{x-h} R_n dz + \int_{x-h}^{x+h} R_n dz + \int_{x+h}^\pi R_n dz.$$

Between the limits  $z = 0$  and  $z = h$ , and also between  $z = x-h$  and  $z = x+h$ , little is known about  $R_n$  except that it always remains less than some fixed number  $R$ . Elsewhere between  $0$  and  $\pi$   $R_n$  is less than  $\eta$  if  $n$  is sufficiently large.

We have, therefore, by the Law of the Mean

$$\left| \int_0^\pi R_n dz \right| < hR + \eta(x-2h) + 2hR + \eta(\pi-x-h)$$

$$< 3hR + \eta(\pi-3h).$$

So far  $h$  and  $\eta$  are arbitrary but  $N$  depends upon  $h$ .

Let us choose first  $h < \frac{\epsilon}{6R}$ , second  $\eta < \frac{\epsilon}{2(\pi-3h)}$ , and third  $n > N$ , (for  $h$  thus chosen). Substitute these values in the above inequality and we get

$$\left| \int_0^\pi R_n dz \right| < \epsilon, \text{ if } n > N, \text{ for every value of } x \text{ between } h \text{ and } \pi.$$

Hence our general series converges uniformly in the interval  $(h, \pi)$  where  $h$  is arbitrary. Furthermore the region of uniform convergence is not affected by multiplying by  $\sin MX$ . We may therefore conclude, by Theorem III, page 5, that the general series (U), page 27, may be integrated term by term after being multiplied by  $\sin MX$ . And it follows that the coefficients of this series are indeed the Fourier's

Coefficients of the function  $\int_x^\pi f(z)dz$ ; hence we have

$$\int_0^\pi \left[ \sin MX \int_z^\pi f(z)dz \right] dx = \int_0^\pi f(z) \frac{1 - \frac{\cos MZ}{M}}{M} dz. \tag{V}$$

This can be verified by integration by parts whenever  $\int_x^\pi f(z)dz$  can be differentiated with respect to  $x$ .

In conclusion note that series (Q), page 23, can be modified, by using series (S), page 26, into the form

$$\left. \begin{aligned} 0 &= \frac{x}{\pi} + \frac{2}{\pi} \sum \sin MX \frac{\cos MZ}{M}, \text{ when } 0 < x < z; \\ 1 &= \frac{x}{\pi} + \frac{2}{\pi} \sum \sin MX \frac{\cos MZ}{M}, \text{ when } z < x < \pi. \end{aligned} \right\} \tag{W}$$

And from this we obtain

$$\int_z^\pi f(x)dx = \frac{1}{\pi} \int_0^\pi (x)f(x)dx + \frac{2}{\pi} \sum \left[ \int_0^\pi f(x) \sin MX dx \right] \frac{\cos MZ}{M},$$

which gives a cosine development for the function

$$\int_z^\pi f(x)dx.$$

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UNIVERSITY OF MISSOURI  
COLUMBIA

DEPARTMENT OF PHYSICS

December 26 1919

Dean Walter Miller,  
Graduate School,  
University

Dear Doctor Miller:

I have carefully gone through the dissertation submitted by Thomas W. Jackson on A Simple Presentation of Fourier's Series, and I recommend that it be accepted as fulfilling the requirement for the Master's degree.

Yours very truly,

*O. M. Stewart*



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