APPROXIMATE ISOMETRIES AND DISTORTION ENERGY FUNCTIONALS

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The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

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a candidate for the degree of Doctor of Philosophy in Mathematics and hereby certify that in their opinion it is worthy of acceptance.

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To all the suffering

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ABSTRACT

A fundamental problem in Riemannian Geometry and related areas is to determine whether two diffeomorphic compact Riemannian manifolds (M, g_M) and (N, g_N) are isometric; that is, if there exists a diffeomorphism $h : M \to N$ such that $h^*g_N - g_M = 0$, where h^*g_N denotes the pullback of g_N by h. If no such diffeomorphism exists, it is important to know whether there exists a diffeomorphism that most closely resembles an isometry. This is accomplished by minimization of the deformation energy functional

$\Phi(h) = \int_M \|h^*g_N - g_M\|^2.$

We also propose other measures of the distortion produced by some classes of diffeomorphisms and isotopies between two isotopic Riemannian n-manifolds and, with respect to these classes, prove the existence of minimal distortion morphs and diffeomorphisms. In particular, we prove the existence of minimal diffeomorphisms and morphs with respect to distortion due to change of volume. Also, we consider the class of time-dependent vector fields (on an open subset Ω of $\mathbb{R}^{n+1}$ in which the manifolds are embedded) that generate morphs between two manifolds M and N via an evolution equation, define the bending and the morphing distortion energies for these morphs, and prove the existence of minimizers of the corresponding functionals in the set of time-dependent vector fields that generate morphs between M and N and are L^2 functions from $[0, 1]$ to the Sobolev space $W_0^{k,2}(\Omega; R^{n+1})$.

Chapter 1

INTRODUCTION

A fundamental problem in Riemannian Geometry and related areas is to determine whether two diffeomorphic compact Riemannian manifolds (M, g_M) and (N, g_N) are isometric; that is, if there exists a diffeomorphism $h : M \to N$ such that $h^* g_N - g_M = 0$, where $h^* g_N$ denotes the pull-back of g_N by h. If no such diffeomorphism exists, it is important to know whether there exists a diffeomorphism that most closely resembles an isometry. This is accomplished by minimization of the deformation energy functional

$$\Phi(h) = \int_M \|h^* g_N - g_M\|^2 \, \omega_M \tag{1.1}$$

over the space $\mathrm{Diff}(M, N)$ of diffeomorphisms between M and N. In chapters 4 and 5, we prove the existence of minimizers of the above functional in the restricted admissible set of all diffeomorphisms generated by time-dependent vector fields on the ambient space Ω in which the manifolds M and N are embedded.

The minimization of the functional Φ takes on added significance once the physical interpretation of the tensor $h^* g_N - g_M$ is recognized: it is exactly the (nonlinear) strain tensor corresponding to the deformation caused by h in case g_M and g_N are Riemannian metrics inherited from Euclidean space. Thus, it is clear that this functional and its variants must occur in physical problems. Indeed, the minimal distortion problem arises, for example, in manufacturing [44], computer graphics [38], movie making, and medical imaging [11, 17, 40].

From now on, assume that M and N are compact and orientable smooth isotopic Riemannian n-manifolds isometrically embedded into $\mathbb{R}^{n+1}$. In chapter 4 we show that in the special case where M and N are regular closed curves with the length $L(N)$ of N smaller than the length $L(M)$ of M, the functional Φ has no minimizer. The latter result can be illustrated by the example of "wrapping" the curve M of larger length around N without stretching, possibly covering some parts N several times. The deformation energy of such a map is zero, and it can be approximated by smooth maps whose deformation energies converge to zero. Therefore, the infimum of Φ over the admissible set $C^\infty(M, N)$ is not attained. For the case where the domain of Φ is the space $\mathrm{Diff}(M, N)$, the nonexistence of minimizers of the functional Φ follows from a stronger inequality $L(N) < (1/\sqrt{3})\, L(M)$, which is incompatible with the positive second variation condition for Φ at a local minimizer. Thus, a solution of the general problem must take into account at least some global properties of the metric structures of the manifolds M and N.

The general problem of the existence of minimizers of Φ in the admissible set $\mathrm{Diff}(M, N)$ is open. On the other hand, we have proved the existence of minimizers in several cases, where the admissible set is restricted (see theorem 1.0.1 and corollary 1.0.6).

For the case where M and N are Riemann spheres or compact Riemann surfaces of genus greater than one and the admissible set is $\mathrm{HD}(M, N) = \{h \in \mathrm{Diff}(M, N) :$ h is a holomorphic map$\}$ (see chapter 5), we prove the following result.

Theorem 1.0.1. *(i) Let $h_R : \mathbb{R}^3 \to \mathbb{R}^3$ be the radial map given by $h_R(p) = Rp$ for some number $R > 0$. If $M = \mathbb{S}^2$ is the 2-dimensional unit sphere isometrically embedded into $\mathbb{R}^3$ and $N = h_R(M)$, then $h := f \circ h_R|_M$ is a global minimum of the functional Φ, restricted to the admissible set $\mathrm{HD}(M, N)$, whenever f is an isometry of N.*

(ii) Let M and N be compact Riemann surfaces. If $\mathrm{HD}(M, N)$ is not empty and

the genus of M is at least two, then there exists a minimizer of the functional Φ in $\mathrm{HD}(M, N)$.

A key idea in the proof of this theorem is to reduce the functional Φ to a function defined on the homogeneous space of all Möbius transformations (which represent holomorphic diffeomorphisms of the Riemann sphere) modulo compositions with isometries of the extended complex plane.

One of the difficulties encountered in attempts to minimize Φ over $\mathrm{Diff}(M, N)$ is the lack of a complete understanding of the structure of this infinite-dimensional manifold. The natural new approach is to linearize; that is, replace the manifold $\mathrm{Diff}(M, N)$ with a subset of a linear function space. Using this approach, which already appears in the literature on image deformation (see [11, 17, 40]), the distortion energy functional is redefined on time-dependent vector fields that generate isotopies between the manifolds M and N.

Consider time-dependent vector fields on an open ball $\Omega \subset \mathbb{R}^{n+1}$ that contains the manifolds M and N (see Fig. 5.1). A time-dependent vector field $v : \Omega \times [0, 1] \to \mathbb{R}^{n+1}$ with appropriate regularity properties generates an evolution operator $\eta^v(t; s, x)$, where $t, s \in [0, 1]$ and $x \in \Omega$, via the differential equation $dq/dt = v(q, t)$. More precisely the function $t \mapsto \eta^v(t; s, x)$ solves the differential equation $dq/dt = v(q, t)$ with the initial condition $q(s) = x$. If the vector field v is such that the manifold M is mapped to the manifold N by the time-one map $\phi^v(x) = \eta(1; 0, x)$, then v generates the isotopy $F(x, t) = \eta^v(t; 0, x)$, where $(x, t) \in M \times [0, 1]$, between the manifolds M and N.

We study both diffeomorphisms and isotopies between the manifolds M and N that produce minimal distortion. The isotopies of minimal distortion appear in the problems of computer graphics and animation, and are called morphs (see [43]).

Definition 1.0.2. Let M and N be isotopic compact connected smooth n-manifolds (perhaps with boundary) embedded in $\mathbb{R}^{n+1}$ such that M is oriented. A C^∞ isotopy

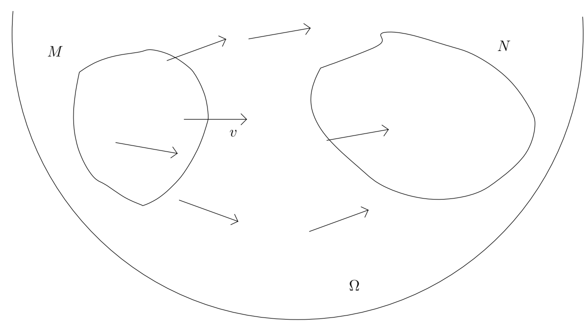

Figure 1.1: The time-dependent vector field $v : \Omega\times[0,1] \to \mathbb{R}^{n+1}$ generates the morph $F^v(p,t)$, which is the solution of the initial value problem $dq/dt = v(q,t),\ q(0) = p$.

$F : M \times [0,1] \to \mathbb{R}^{n+1}$ together with all the intermediate manifolds $M^t := F(M,t)$, equipped with the orientations induced by the maps $f^t = F(\cdot,t) : M \to M^t$ and the Riemannian metrics g_t inherited from $\mathbb{R}^{n+1}$, is called a (smooth) *morph* from M to N.

Let $\mathcal{H}^k := L^2(0,1;W_0^{k,2}(\Omega,\mathbb{R}^{n+1}))$ be the Hilbert space of all L^2 functions from the interval $[0,1]$ to the Sobolev space $W_0^{k,2}(\Omega,\mathbb{R}^{n+1})$ and, by abuse of notation, $v(x,t) = v(t)(x)$ for every $v \in \mathcal{H}^k$ and $(x,t) \in \Omega\times[0,1]$. The inner product on $\mathcal{H}^k$ is defined by

$$\langle v,w\rangle_{\mathcal{H}^k} = \int_0^1 \langle v(t),w(t)\rangle_{k,2}\,dt,$$

where $\langle\cdot,\cdot\rangle_{k,2}$ is the standard inner product on $W_0^{k,2}(\Omega,\mathbb{R}^{n+1})$. Choose $k \in \mathbb{N}$ large enough so that the Sobolev space $W_0^{k,2}(\Omega)$ is embedded into $C^r(\bar{\Omega})$, where $r \geq 2$.

Every vector field $v \in \mathcal{H}^k$ generates a morph $F^v : M \times [0,1] \to \mathbb{R}^{n+1}$ from M to $F^v(M,1)$ via the evolution equation

$$\frac{dq}{dt} = v(q,t). \tag{1.2}$$

More precisely, for every $p \in M$ the function $t \mapsto F^v(p,t)$ is the solution of equation (1.2) with the initial condition $q(0) = p$. By the properties of the evolution operator of equation (1.2), which have been studied by Dupuis, Grenander, and Miller in [11] and by Trouve and Younes in [40], the morph $F^v(p,t)$ is absolutely continuous in the time variable t and C^r in the spatial variable; in symbols, F^v is of class $\mathcal{M}^{\mathrm{ac},r}(M, F^v(M,1))$.

We define the time-one map $\psi^v(p) = F^v(p,1)$, which gives the position of the point $p \in M$ at time one. Let ψ^v denote the restriction to M of the time-one map of (1.2); that is, $\psi^v(p)$ is the time-one position of the point $p \in M$ that evolves along v. The vector fields of interest generate morphs between the manifolds M and N and are bounded in $\mathcal{H}^k$ by a uniform constant P. In symbols, the admissible set of time-dependent vector fields is defined by

$$\mathcal{A}_P^k = \{v \in \mathcal{H}^k : \|v\|_{\mathcal{H}^k} \le P \text{ and } \psi^v \in \mathrm{Diff}^r(M,N)\},$$

where $\mathrm{Diff}^r(M,N)$ is the set of all C^r diffeomorphisms between the manifolds M and N.

Lemma 1.0.3. *For P sufficiently large, the admissible set $\mathcal{A}_P^k$ is nonempty and $\mathcal{A}_P^k$ is weakly closed in $\mathcal{H}^k$.*

For each $t \in [0,1]$ and $v \in \mathcal{A}_P^k$, endow the intermediate manifold $M^{v,t} := F^v(M,t)$ with the Riemannian metric g_t^v inherited from its embedding into $\mathbb{R}^{n+1}$ and let $I\!I_t^v$ denote the corresponding second fundamental form. Distortion energy functionals are defined on the admissible set $\mathcal{A}_P^k$ and measure the distortion energies of the diffeomorphisms and morphs generated by the time-dependent vector fields in $\mathcal{A}_P^k$.

Definition 1.0.4. Let A and B be nonnegative real numbers. The *bending distortion energy* of v is

$$\begin{aligned} E(v) = E(v; A, B) &= A \int_M \|(\psi^v)^* g_N - g_M\|^2 \omega_M \\ &\quad + B \int_M \|(\psi^v)^* I\!I_N - I\!I_M\|^2 \omega_M \end{aligned}$$

and the *morphing distortion energy* of v is

$$\begin{aligned} \mathcal{E}(v) = \mathcal{E}(v; A, B) &= A \int_0^1 \int_M \|F^v(\cdot,t)^* g_t^v - g_M\|^2 \, \omega_M dt \\ &\quad + B \int_0^1 \int_M \|F^v(\cdot,t)^* I\!I_t^v - I\!I_M\|^2 \, \omega_M dt, \end{aligned}$$

where $\|\cdot\|$ is the fiber norm on the tensor bundle of all $(0,2)$ tensor fields on M generated by the fiber inner product $g_M^* \otimes g_M^*$.

Notice that the functionals E and $\mathcal{E}$ compare, in addition to the Riemannian metrics, the embeddings of the manifolds M and N (to avoid, for example, zero distortion energy maps between a square and a round cylinder in $\mathbb{R}^3$).

In chapter 5, we prove that the functionals E and $\mathcal{E}$ have minimizers.

Theorem 1.0.5. *(i) If $P > 0$ and $k \in \mathbb{N}$ are sufficiently large, then the functionals $E : \mathcal{A}_P^k \to \mathbb{R}_+$ and $\mathcal{E} : \mathcal{A}_P^k \to \mathbb{R}_+$ both have minimizers in the admissible set $\mathcal{A}_P^k$.*

In the proof, we show that the functionals E and $\mathcal{E}$ are weakly continuous on the weakly closed subset $\mathcal{A}_P^k$ of the Hilbert space $\mathcal{H}^k$. Assuming the latter, the direct method of calculus of variations implies theorem 1.0.5. The convergence properties of evolution operators generated by weakly convergent vector fields in $\mathcal{H}^k$, which have been studied in [11, 40], also play an important role in the proof.

Corollary 1.0.6. *Let M and N be two isotopic oriented compact connected smooth n-manifolds, perhaps with boundary, isometrically embedded into $\mathbb{R}^{n+1}$. For every $\phi \in \mathrm{Diff}(M)$, let*

$$\mathcal{B}_{P,k}^{\phi} := \{h \in \mathrm{Diff}^2(M,N) : h = \psi^v \circ \phi \text{ for some } v \in \mathcal{H}^k \text{ such that } \|v\|_{\mathcal{H}^k} \leq P\}.$$

If $P > 0$ and $k \in \mathbb{N}$ are sufficiently large, then for every $\phi \in \mathrm{Diff}(M)$ there exists a minimizer of the deformation energy functional Φ in the admissible set $\mathcal{B}_{P,k}^{\phi}$.

Note that every diffeomorphism $h : M \to N \subset \mathbb{R}^{n+1}$ in the admissible set $\mathcal{B}_{P,k}^{\phi}$ is isotopic, as a map from M to $\mathbb{R}^{n+1}$, to a fixed diffeomorphism $\phi : M \to M \subset \mathbb{R}^{n+1}$ via the isotopy $F^v \in \mathcal{M}^{\mathrm{ac},2}(M,N)$ generated by a vector field $v \in \mathcal{A}_P^k$.

In chapter 5, we construct a minimal distortion morph between two circles. The construction involves the solution of a separate constrained optimization problem, and the numerical solution suggests that the constraint $\|v\|_{\mathcal{H}^k} \leq P$ in the definition of the admissible set $\mathcal{A}_P^k$ imposes a restriction on the curvature of the curves $t \mapsto F(t,p)$ of the admissible morphs F between the circles.

Theorem 1.0.5 guarantees the existence of minimizers of the functionals E and $\mathcal{E}$, but the problem of construction of such minimizers in the general case is open. Even in the case where M and N are one-dimensional circles, the construction of a minimal distortion morph between two circles requires delicate analysis (which we have done in chapter 5).

On the other hand, we developed a complete theory (including sufficient conditions) of minimal distortion diffeomorphisms and morphs between the manifolds M and N for the distortion functionals that measure the total infinitesimal relative change of volume.

More precisely, assume that the manifolds M and N are without boundary and define the distortion (due to change of volume) functional $\Lambda : \operatorname{Diff}(M,N) \to \mathbb{R}_+$ by

$$\Lambda(h) = \int_M \Big(\big|J(h)\big| - 1\Big)^2 \omega_M, \tag{1.3}$$

where $J(h)$ denotes the Jacobian determinant of h. Note that $\big|J(h)(p)\big| - 1$ is the infinitesimal relative change of volume at $p \in M$ produced by the diffeomorphisms $h : M \to N$.

We derived the following necessary and sufficient condition for a minimizer of the functional Λ (see chapter 3).

Theorem 1.0.7. *There exists a minimizer of the functional* Λ *over the class* $\operatorname{Diff}(M,N)$ *and the minimum value of* Λ *is* $\Lambda^{\min} = \big(\operatorname{Vol}(M) - \operatorname{Vol}(N)\big)^2 \big/ \operatorname{Vol}(M)$. *A diffeomorphism* $h \in \operatorname{Diff}(M,N)$ *is a minimizer of* Λ *if and only if* $J(h) = \operatorname{Vol}(N)\big/\operatorname{Vol}(M)$.

Because the functional Λ is invariant with respect to left compositions with volume preserving maps, Λ has (infinitely) many minimizers.

The infinitesimal distortion energy of a morph $F \in \mathcal{M}(M,N)$ at $t \in [0,1]$ is the limit of the distortion energy of the transition map $f^{s,t} = f^t \circ (f^s)^{-1}$, where

$f^t = F(t,\cdot) : M \to M^t$, divided by an appropriate power of $(s-t)$ as $s \to t$. The energy of $f^{s,t}$ can be measured using the functional Λ. The total distortion energy of the morph F is then defined as its infinitesimal energy integrated over $t \in [0,1]$.

Definition 1.0.8. The *infinitesimal distortion* of a smooth morph F from M to N at $t \in [0,1]$ is

$$\varepsilon^F(t) = \lim_{s\to t} \frac{\Lambda^{s,t}(f^{s,t})}{(s-t)^2} = \int_M \frac{\big(\frac{d}{dt}J(f^t)\big)^2}{J(f^t)}\omega_M,$$

where the functional $\Lambda^{s,t} : \mathrm{Diff}(M^s, M^t) \to \mathbb{R}_+$ is defined using formula (1.3) by replacing M and N by M^s and M^t respectively.

The *total distortion functional* $\Psi : \mathcal{M}(M,N) \to \mathbb{R}_+$ is given by

$$\Psi(F) = \int_0^1 \varepsilon^F(t)dt = \int_0^1 \Big(\int_M \frac{\big(\frac{d}{dt}J(f^t)\big)^2}{J(f^t)}\omega_M\Big)dt. \tag{1.4}$$

Theorem 1.0.9. *Suppose that the manifolds M and N are connected by a smooth morph.*

(i) There exists a minimizer of Ψ; the minimal value of Ψ is

$$\min_{F\in\mathcal{M}(M,N)} \Psi(F) = 4\big(\sqrt{\mathrm{Vol}(N)} - \sqrt{\mathrm{Vol}(M)}\big)^2.$$

(ii) If the morph $F \in \mathcal{M}(M,N)$ satisfies the equation

$$J(f^t) = \mathrm{Vol}(M^t)/\,\mathrm{Vol}(M)$$

for every $t \in [0,1]$ and the volume of each intermediate manifold M^t is given by

$$\mathrm{Vol}(M^t) = \Big[(\sqrt{\mathrm{Vol}(M)} - \sqrt{\mathrm{Vol}(N)}\,)t - \sqrt{\mathrm{Vol}(M)}\,\Big]^2,$$

then F minimizes the functional Ψ.

The proof of the existence of a distortion minimal morph between every pair of isotopic manifolds M and N uses the concept of a *pairwise minimal morph*: a morph whose transition maps $f^{s,t}$ minimize the corresponding functionals $\Lambda^{s,t}$. The existence of pairwise minimal morphs is a nontrivial fact. It can be proved by rescaling morphs between M and N, which are not necessarily pairwise minimal, to conform to a necessary and sufficient condition for pairwise minimality. Moser's theorem on volume forms (see [30]) plays a crucial role in the proof.

We proved that it suffices to minimize the functional Ψ over the class $\mathcal{PM}(M,N)$ of all pairwise minimal morphs instead of the class $\mathcal{M}(M,N)$ of all smooth morphs. Reduction to a simpler form of the functional Ψ on the admissible set $\mathcal{PM}(M,N)$ allows us to complete the proof of theorem 1.0.9.

Chapter 2

PRELIMINARIES

In this chapter we give the definitions and state known results that will be used in the subsequent chapters. A detailed exposition of Riemannian geometry and analysis on manifolds can be found in [1, 18, 24].

2.1 Manifolds and Tangent Bundles

Definition 2.1.1. A Hausdorff topological space $(M, \mathcal{T})$ is called a topological *manifold* if for every point $p \in M$ there exists an open set $U \in \mathcal{T}$ that contains p and a homeomorphism $\phi : U \to V$, where V is an open subset of a Banach space X. The pair (U, ϕ) is called a *chart* at $p \in M$. If $X = \mathbb{R}^n$, a topological manifold M is called an n-manifold, and we say that the dimension of M, denoted by $\dim(M)$, is n (see figure 2.1 for an illustration of a 2-manifold).

In this thesis, we consider (infinitely) smooth manifolds.

Definition 2.1.2. Let M be a topological manifold. The charts (U_1, ϕ_1) and (U_2, ϕ_2) on M such that $U_1 \cap U_2 \neq \emptyset$ are C^r *compatible* if the *transition functions* $\phi_{12} = \phi_2 \circ (\phi_1)^{-1}$ and $\phi_{21} = \phi_1 \circ (\phi_2)^{-1}$ are C^r diffeomorphisms on their corresponding domains $\phi_1(U_1 \cap U_2)$ and $\phi_2(U_1 \cap U_2)$. Both domains are required to be open subsets of X.

Definition 2.1.3. Let M be a topological manifold. A collection of charts $\mathcal{A} = (U_i, \phi_i)_{i \in I}$, where I is an index set, is called a C^r *differentiable atlas* on M if

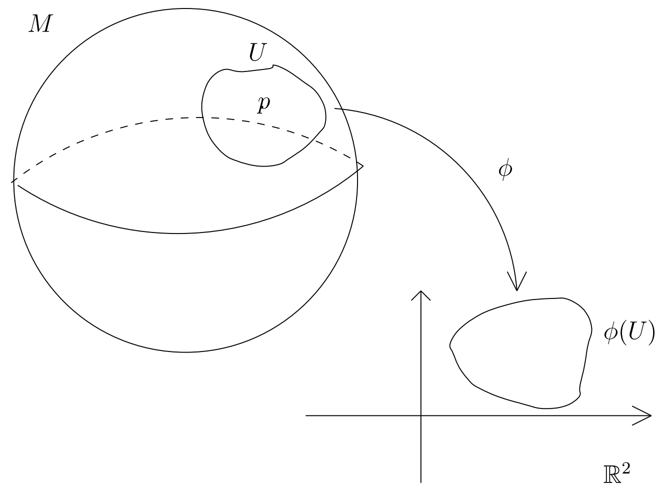

Figure 2.1: A manifold is a topological space locally homeomorphic to a Euclidean space

(i) $\cup_{i \in I} U_i = M$ and

(ii) all the pairs of charts that have nonempty intersection are C^r compatible.

Two C^r atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ of the manifold M are equivalent if and only if their union $\mathcal{A}_1 \cup \mathcal{A}_2$ is also a C^r atlas of M. An equivalence class $\mathcal{D}$ of atlases of the manifold M is called a C^r *differentiable structure* on M.

Definition 2.1.4. A manifold M equipped with a C^r differentiable structure $\mathcal{D}$ is called a C^r *differentiable manifold.* Every chart $(U, \phi) \in \mathcal{A}$, where $\mathcal{A} \in \mathcal{D}$, is called an *admissible* chart. Given $p \in M$, a chart $(U, \phi) \in \mathcal{A} \in \mathcal{D}$ is called an *admissible chart at* $p \in M$ if $p \in U$. For every $p \in M$ there exists an admissible chart at p. A C^∞ differentiable manifold is called a *differentiable manifold* or a *smooth manifold.*

The manifold M in the latter definition has no boundary. The definition of differentiable manifolds with boundaries can be found in [1], chapter 7.2. In the following, we assume that the manifolds under consideration are n-dimensional, smooth, boundaryless, and connected unless it is stated otherwise.

Definition 2.1.5. A *submanifold* of a manifold M is a subset $B \subset M$ such that for each $b \in B$ there is an admissible chart (U, ϕ) at b satisfying the *submanifold property*:

$$\phi : U \to E \times F \text{ and } \phi(U \cap B) = \phi(U) \cap (E \times \{0\}),$$

where E and F are vector spaces.

Definition 2.1.6. Let M and N be manifolds. A map $h : M \to N$ is of class C^1 if for every $p \in M$ and for every chart (W, ψ) of the manifold N at $h(p) \in N$, there exists a chart (U, ϕ) of the manifold M at p such that $h(U) \subset W$ and the *local representative* $\psi \circ h \circ \phi^{-1} : \phi(U) \to \psi \circ h(U)$ is a C^1 function. The functions between manifolds M and N of class C^k are defined analogously.

We denote the set of all C^r maps between manifolds M and N by $C^r(M, N)$, where $r \in \mathbb{N} \cup \{\infty\}$.

Definition 2.1.7. A bijection $h : M \to N$ is called a C^k *diffeomorphism* if both h and h^{-1} are of class C^k, $k \in \mathbb{N} \cup \{\infty\}$. The set of all C^k diffeomorphisms between manifolds M and N is denoted by $\mathrm{Diff}^k(M, N)$. We denote $\mathrm{Diff}(M, N) := \mathrm{Diff}^\infty(M, N)$ and $\mathrm{Diff}^k(M) := \mathrm{Diff}^k(M, M)$. The manifolds M and N are called *diffeomorphic* if $\mathrm{Diff}(M, N)$ is not empty.

Definition 2.1.8. Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $\phi, \psi : \Omega \to \mathbb{R}^n$ be C^r functions. We say that the functions ϕ and ψ are C^r *homotopic* if there exists a C^r function $F : [0, 1] \times \Omega \to \mathbb{R}^n$ such that $F(0, x) = \phi(x)$ and $F(1, x) = \psi(x)$ for all $x \in \Omega$. The function F is called a C^r *homotopy*.

Two C^r diffeomorphisms ϕ and ψ of Ω are C^r *isotopic* if there exists a C^r homotopy $F : [0, 1] \times \Omega \to \Omega$ between them such that the maps $f^t := F(t, \cdot)$ are C^r diffeomorphisms of Ω for all $t \in [0, 1]$.

Definition 2.1.9. A C^1 function $c : (-\varepsilon, \varepsilon) \to M$, where $\varepsilon > 0$, is called a *curve* at $p \in M$ if $c(0) = p$. Let c_1 and c_2 be curves at $p \in M$, and let (U, ϕ) be an admissible chart at p. The curves c_1 and c_2 are *tangent at p with respect to ϕ* if $(\phi \circ c_1)'(0) = (\phi \circ c_2)'(0)$.

It is easy to check that the definition of the tangency of two curves at $p \in M$ does not depend on the choice of the chart at $p \in M$. In fact, the tangency at $p \in M$ defines an equivalence relation on the set of all curves at $p \in M$. We denote the equivalence class of all curves at $p \in M$ tangent to c by $[c]_p$.

Definition 2.1.10. Let M be a smooth manifold. The *tangent space to M at a point $p \in M$* is defined to be the set of equivalence classes of all the curves at $p \in M$:

$$T_pM = \{[c]_p : c \text{ is a curve at } p \in M\}.$$

For $U \subset M$, the disjoint union $TU = \cup_{p\in U} T_pM$ is called the *tangent bundle of M*. The mapping $\tau_M : TM \to M$ defined by $\tau_M([c]_p) = p$ is called the *tangent projection* of M.

Definition 2.1.11. The *tangent $Th : TM \to TN$* of a C^1 function $h : M \to N$ is defined to be $Th([c]_p) = [h \circ c]_{h(p)}$.

Theorem 2.1.12. *Let M be a C^{r+1} n-dimensional manifold with the atlas of admissible charts $\mathcal{A}$. Then the tangent bundle TM is a $2n$-dimensional C^r manifold with the* natural atlas $\{(TU, T\phi) : (U, \phi) \in \mathcal{A}\}$.

The proof of this theorem can be found in [1], section 3.2.

2.2 Immersions and Embeddings

Definition 2.2.1. The closed subspace F of the Banach space E is said to be *split*, or *complemented*, if there is a closed subspace $G \subset E$ such that $E = F \oplus G$.

Let M and N be manifolds.

Definition 2.2.2. A C^r map $f : M \to N$, where $r \geq 1$, is called an *immersion at* $p \in M$ if the derivative map $df_p : T_pM \to T_{f(p)}N$ is injective with closed split range in $T_{f(p)}N$. If the map f is an immersion at each $p \in M$, we say that f is an *immersion.*

Definition 2.2.3. An immersion $f : M \to N$ that is also a homeomorphism onto $f(M)$ with the relative topology induced from N is called an *embedding.*

For example, one may consider n-dimensional manifolds embedded into $\mathbb{R}^{n+1}$.

2.3 Vector Bundles

The tangent bundle is an example of a *vector bundle*, which is defined below.

Definition 2.3.1. Let E and F be vector spaces, and let U be an open subset of E. The Cartesian product $U \times F$ is called a *local vector bundle.* The open set U is called the *base space*, which can be identified with the zero section $U \times \{0\}$. The map $\pi : U \times F \to U$ defined by $\pi(u, f) = u$ is called the *projection* of $U \times F$ onto U. For each $u \in U$, the inverse image $\pi^{-1}(u) = \{u\} \times F =: F_u$ is called the *fiber* over u, which has the vector space structure of F.

To define a vector bundle, we need to introduce the idea of a local vector bundle map.

Definition 2.3.2. Let $U \times F$ and $U' \times F'$ be local vector bundles. A map $\phi : U \times F \to U' \times F'$ is called a C^r *local vector bundle map* if it has the form $\phi(u, f) = (\phi_1(u), \phi_2(u)f)$, where $\phi_1 : U \to U'$ and $\phi_2 : U \to L(F, F')$ are C^r maps. If a map ϕ has an inverse ϕ^{-1} and both the function and its inverse are local vector bundle maps, then ϕ is called a *local vector bundle isomorphism.*

It follows from the definition that every local vector bundle map $\phi : U \times F \to U' \times F'$ satisfies the following properties:

(i) $\phi(F_u) \subset F'_{\phi(u)}$. In other words, ϕ is *fiber preserving*;

(ii) $\phi(u, \cdot) : F_u \to F'_{\phi(u)}$ is a bounded linear map.

Using these local notions, we are now ready to define a vector bundle.

Definition 2.3.3. Let S be a set. A *local vector bundle chart* of S is a pair (W, ϕ), where $W \subset S$ and $\phi : W \to U \times F$ is a bijection onto a local bundle $U \times F$ (the local bundle may depend on ϕ). A *vector bundle atlas* on S is a family $\mathcal{B} = \{(W_i, \phi_i)\}$ of local bundle charts satisfying

(i) $S = \cup_i U_i$

(ii) for every pair (W_i, ϕ_i) and (W_j, ϕ_j) of local bundle charts in $\mathcal{B}$, which have a nonempty intersection $W_i \cap W_j$, the image $\phi_i(W_i \cap W_j)$ is a local vector bundle. Moreover, the transition map $\phi_{ij} = \phi_j \circ \phi_i^{-1}$ restricted to $\phi_i(W_i \cap W_j)$ is a C^∞ local vector bundle isomorphism.

Two vector bundle atlases $\mathcal{B}_1$ and $\mathcal{B}_2$ on S are called equivalent is their union $\mathcal{B}_1 \cup \mathcal{B}_2$ is also a vector bundle atlas of S. A *vector bundle structure* on S is an equivalence class of vector bundle atlases. A *vector bundle* E is a pair $(S, \mathcal{V})$ of a set S equipped with a vector bundle structure $\mathcal{V}$.

Definition 2.3.4. For a vector bundle $E = (S, \mathcal{V})$, the *zero section*, or the *base* is defined to be

$$\begin{aligned} B = \{e \in E : \ & \text{there exists } (W, \phi) \in \mathcal{V}, \text{where } \phi : W \to U \times F, \\ & \text{and } u \in U \text{ such that } e = \phi^{-1}(u, 0)\}. \end{aligned}$$

In other words, B is the union of all the zero sections of the local vector bundles.

The basic properties of vector bundles are summarized in the following theorem.

Theorem 2.3.5. *Let E be a vector bundle. The zero section B of E is a submanifold of E, and there exists a C^∞ surjective map $\pi : E \to B$. Moreover, for each $b \in B$,* *the inverse image $\pi^{-1}(b)$ has a vector space structure induced by an admissible vector bundle chart, with b being the zero element.*

The map $\pi : E \to B$ is called the projection *of the vector bundle E, and the inverse images $\pi^{-1}(b) =: E_b$ are called the* fibers *of the vector bundle E.*

In view of the latter theorem, we will sometimes denote a vector bundle E with the base B and the projection π by $\pi : E \to B$.

Theorem 2.1.12 states that the tangent bundle is a manifold. But we can say more: the tangent bundle is a vector bundle.

Theorem 2.3.6. *Let M be a manifold and $\mathcal{A}$ an atlas of admissible charts. Then*

(i) the natural atlas $T\mathcal{A} = (TU, T\phi)$ is a vector bundle atlas of TM;

(ii) For each $p \in M$, $\tau_M^{-1}(p) = T_pM$ is a fiber of TM. The base B of the vector bundle TM is diffeomorphic to M by the map $\tau_M|_B : B \to M$.

The tangent map Th is fiber preserving: the *derivative map* $\mathbf{d}h_p = \mathbf{d}h(p) = Th|_{T_pM}$ is defined on the fiber T_pM of the tangent bundle TM, and maps it into the fiber $T_{h(p)}N$ of the tangent bundle TN.

The *cotangent bundle* TM^* of a manifold M is the collection $TM^* = \cup_{p\in M}T_pM^*$ of all dual spaces to the tangent spaces T_pM. The cotangent bundle of a smooth manifold is a C^∞ vector bundle with the projection map $\tau_M^* : TM^* \to M$ defined by $\tau_M^*(\alpha_p) = p$ for all $\alpha_p \in T_pM^*$.

Definition 2.3.7. Let E be a vector bundle with the base B and the projection π. A (C^r) function $\theta : B \to E$, where $r \in \mathbb{N} \cup \{\infty\}$, is called a (C^r) *section* of E if $\pi \circ \theta(p) = p$ for all $p \in B$. In other words, to each $p \in M$, the map θ assigns an element $\theta(p)$ of the fiber $\pi^{-1}(p) = E_p$.

The space of all C^r sections of the vector bundle E together with its real (infinite dimensional) vector space structure is denoted by $\Gamma^r(E)$, where $r \in \mathbb{N} \cup \{\infty\}$; $\Gamma(E)$ denotes the space of all sections of E.

Definition 2.3.8. Let M be a smooth manifold. A C^r section of the tangent bundle TM is called a C^r *vector field* on M. A C^r section of the cotangent bundle $TM^\star$ is called a *one-form* on M. We denote the set of all C^r vector fields on M by $\mathfrak{X}^r(M) = \Gamma^r(TM)$, where $r \in \mathbb{N}$. Also, we denote $\mathfrak{X}(M) = \Gamma^\infty(TM)$.

Definition 2.3.9. Let M, N be smooth manifolds, and let $\phi : M \to N$ be a C^r mapping between manifolds M and N.

(i) The *pull-back* of a C^r map $f : N \to N$ is defined to be

$$\phi^* f = f \circ \phi \in C^r(M);$$

(ii) The *push-forward* of a C^r vector field X on M is defined to be:

$$\phi_* X = T\phi \circ X \circ \phi^{-1} \in \Gamma^r(TN).$$

We interchange pull-backs and push-forwards by changing ϕ to ϕ^{-1} according to the rule $\phi_* = (\phi^{-1})^*$ and $\phi^* = (\phi^{-1})_*$.

2.4 Vector Fields and Flows

Let M be a manifold, and $I = (-\varepsilon, \varepsilon) \subset \mathbb{R}$, where $\varepsilon > 0$. Recall that $T_t I = (t, \lambda)$ for all $t \in I$, where $\lambda \in \mathbb{R}$ is the principal part.

Definition 2.4.1. An *integral curve* of a vector field $X \in \mathfrak{X}(M)$ at $p \in M$ is a C^1 curve $c : I \to M$ at p such that $c'(t) = X(c(t))$ for each $t \in I$, where $c'(t) = dc_t(t, 1)$.

Definition 2.4.2. For a vector field X on M, let $\mathcal{D}_X \subset M \times \mathbb{R}$ be the set of all pairs $(p, t) \in M \times \mathbb{R}$ such that there exists an integral curve $c : I \to M$ at p with $t \in I$. The vector field X is *complete* if $\mathcal{D}_X = M \times \mathbb{R}$.

Theorem 2.4.3. *Every C^r vector field with compact support on a manifold M is complete. In particular, every C^r vector field on a compact manifold is complete.*

Proposition 2.4.4. *Let M be a manifold and $X \in \mathfrak{X}^r(M)$, where $r \geq 1$. Then*

(i) $\mathcal{D}_X \supset M \times \{0\}$*;*

(ii) $\mathcal{D}_X$ *is open in* $M \times \mathbb{R}$*;*

(iii) there is a unique C^r mapping $F_X : \mathcal{D}_X \to M$ such that the map $t \mapsto F_X(p,t)$ is an integral curve at p for all $p \in M$;

(iv) for $(p,t) \in \mathcal{D}_X$, the point $(F_X(p,t), s)$ belongs to $\mathcal{D}_X$ if and only if $(p, t+s) \in \mathcal{D}_X$. In this case

$$F_X(p, t+s) = F_X(F_X(p,t), s).$$

Definition 2.4.5. Let M be a manifold and $X \in \mathfrak{X}^r(M)$. The mapping F_X is called the *integral* of X, and the curve $t \mapsto F_X(p,t)$ is called the *maximal integral curve* of X at p for every $p \in M$. If X is complete, then F_X is called the *flow* of X.

We will use the notation $\phi_t(p) = F_X(p,t)$ for the flow of a complete vector field $X \in \mathfrak{X}^r(M)$ of a manifold M. Using statement (iv) of proposition 2.4.4 and the definition of F_X, we see that the flow satisfies the *group law*: $\phi_{t+s} = \phi_t \circ \phi_s$ and ϕ_0 is the identity on M. By definition,

$$\frac{d}{dt}\phi_t(p) = X(\phi_t(p))$$

for all $p \in M$.

Definition 2.4.6. A C^r vector field V on the product manifold $M \times I$ is called a C^r *time-dependent vector field on M.* Note that for each $t \in I$, $V(t) := V(\cdot, t) \in \mathfrak{X}^r(M)$.

Consider the differential equation

$$\frac{dq}{dt} = V(q,t). \tag{2.1}$$

Suppose that $r \geq 1$. For every $p \in M$, there exists a solution $q_s^p : U_s \to M$ of equation (2.1) such that $q_s^p(s) = p$, where $s \in \mathbb{R}$ and U_s is an open neighborhood of

s. Define the *evolution operator* $F : \mathbb{R} \times \mathbb{R} \times M \to M$ by $F(t; s, p) = q_s^p(t)$, where $t \in U_s$ for all $s \in \mathbb{R}$. The evolution operator satisfies the *Chapman-Kolmogorov law*: $F(t; s, F(s, \tau, p)) = F(t; \tau, p)$ and $F(s; s, p) = p$ for all $t, s, \tau \in \mathbb{R}$ and $p \in M$ whenever all the expressions are defined.

2.5 Tensor Fields

The tensor fields on M are smooth sections of tensor bundles, which we define below.

Definition 2.5.1. Let E be a vector space. An *tensor of type* (r, s) (contravariant of order r and covariant of order s) on E is a continuous $(r + s)$ multilinear map from $E^* \times \ldots \times E^* \times E \times \ldots \times E$ to $\mathbb{R}$, where E^* and E in the latter Cartesian product appear r and s times respectively. We denote the set of all tensors of type (r, s) on E by $T^r{}_s(E) = L^{r+s}(E^*, \ldots, E^*, E, \ldots, E; \mathbb{R})$.

Given tensors $t_1 \in T^{r_1}{}_{s_1}(E)$ and $t_2 \in T^{r_2}{}_{s_2}(E)$, the *tensor product* of t_1 and t_2 is the tensor $t_1 \otimes t_2 \in T^{r_1+r_2}{}_{s_1+s_2}(E)$ defined by

$$\begin{aligned}
&(t_1 \otimes t_2)(\beta^1, \ldots \beta^{r_1}, \gamma^1, \ldots, \gamma^{r_2}, f_1, \ldots, f_{s_1}, g_1, \ldots, g_{s_2}) \\
&= t_1(\beta^1, \ldots \beta^{r_1}, f_1, \ldots, f_{s_1}) t_2(\gamma^1, \ldots, \gamma^{r_2}, g_1, \ldots, g_{s_2}),
\end{aligned}$$

where $\beta^j, \gamma^j \in E^*$ and $f_j, g_j \in E$.

Definition 2.5.2. Let $\phi \in L(E, F)$ be an isomorphism. The *push-forward* $T^r{}_s\phi = \phi_* \in L\big(T^r{}_s(E), T^r{}_s(F)\big)$ of ϕ is defined by

$$\phi_* t(\beta^1, \ldots, \beta^r, f_1, \ldots, f_s) = t(\phi^*(\beta^1), \ldots, \phi^*(\beta^r), \phi^{-1}(f_1), \ldots, \phi^{-1}(f_s)),$$

where $t \in T^r{}_s(E)$, $\beta^i \in F^*$, and $f_i \in F$.

Now we are ready to define a tensor bundle.

Definition 2.5.3. Let $\pi : E \to B$ be a vector bundle with $E_b = \pi^{-1}(b)$ denoting the fiber over the point $b \in B$. The *tensor bundle on* E is the collection $T^r{}_s(E) =$

$\cup_{b\in B} T^r{}_s(E_b)$ of tensor spaces and the projection map $\pi^r_s : T^r{}_s(E) \to B$ given by $\pi^r_s(e) = b$ for every $e \in T^r{}_s(E_b)$.

Theorem 2.5.4. *If $\pi : E \to B$ is a vector bundle, so is $\pi^r_s : T^r{}_s(E) \to B$.*

In the special case where $\pi : E \to B$ is the tangent vector bundle of a manifold M, we obtain the vector bundle of *tensor fields* over the manifold M.

Definition 2.5.5. Let M be a manifold with the tangent bundle $\tau_M : TM \to M$. The *vector bundle of tensors of type* (r, s) is defined to be $T^r{}_s(M) := T^r{}_s(TM)$. We identify $T^1{}_0(M)$ with TM and $T^0{}_1(M)$ with the cotangent bundle $\tau^*_M : TM^* \to M$. The zero section of $T^r{}_s(M)$ is identified with M.

Definition 2.5.6. A *tensor field of type* (r, s) on a manifold M is a C^∞ section of the vector bundle of tensor fields $T^r{}_s(M)$. We denote the set of all smooth sections $\Gamma^\infty(T^r{}_s(M))$ together with its (infinite dimensional) real vector space structure by $\mathcal{T}^r{}_s(M)$.

Definition 2.5.7. A *push-forward* of a tensor field $t \in \mathcal{T}^r{}_s(M)$ by a diffeomorphism $\phi : M \to N$ is defined to be $\phi_* t = (T\phi)_* \circ t \circ \phi^{-1}$. The *pull-back* of a tensor field $t \in \mathcal{T}^r{}_s(N)$ by ϕ is given by $\phi^* t = (\phi^{-1})_* t$.

Let us discuss the expression of tensor fields in local coordinates. Let (U, ϕ) be a chart on a given n-manifold M, where $\phi : U \to V \subset \mathbb{R}^n$. Let $\{e_i\}_{i=1}^n$ be the standard basis of $\mathbb{R}^n$. It can be shown that for each $p \in M$, the collection of vectors $\left(\frac{\partial}{\partial x_i}\right)_p := (T\phi)^{-1}(\phi(p), e_i)$, where $1 \le i \le n$, is a basis of the tangent space T_pM. The dual basis $\{(dx^i)_p\}_{i=1}^n$ of the cotangent space T_pM^* is defined by the set of the relations $(dx^i)_p\left(\left(\frac{\partial}{\partial x_j}\right)_p\right) = \delta^i_j$, where δ^i_j is the Kronecker symbol. Let $\{e^i\}_{i=1}^n$ be the dual basis for $\{e_i\}_{i=1}^n$. Then $(dx^i)_p = \phi^*(e^i)(p)$.

Let $t \in \mathcal{T}^r{}_s(M)$ and let (U, ϕ) be a chart on M. Using the coordinates of $\mathbb{R}^n$, the map $\phi : U \to \mathbb{R}^n$ can be expressed in the form

$$\phi(p) = \big(x^1(p), \dots, x^n(p)\big).$$

As usual $\big(x^1(p),\dots,x^n(p)\big)$ are the local coordinates of $p\in M$ and the n-tuple of functions $(x^1,x^2,\dots,x^n)$ is the local coordinate system with respect to (U,ϕ). Because ϕ is a homeomorphism from U onto $\phi(U)$, we identify $p\in U$ and $\phi(p)\in\mathbb{R}^n$ via ϕ.

The components of the tensor t in the chart (U,ϕ) are defined as the following smooth real valued functions on U:

$$p\mapsto t^{i_1\dots i_r}{}_{j_1\dots j_s}(p)=t(p)(dx^{i_1},\dots,dx^{i_r},\frac{\partial}{\partial x^{j_1}},\dots,\frac{\partial}{\partial x^{j_s}}),$$

where all the indices range from 1 to n, and the point $p=\phi^{-1}(x)\in M$ for the bases $\big\{(\frac{\partial}{\partial x_i})_p\big\}_{i=1}^n$ and $\{(dx^i)_p\}_{i=1}^n$ is suppressed.

Using the components of $t\in\mathcal{T}^r{}_s(M)$, for every $p\in U$ the tensor $t(p)\in T^r{}_s(T_pM)$ is given by the expression

$$t(p)=t^{i_1\dots i_r}{}_{j_1\dots j_s}(p)\frac{\partial}{\partial x^{i_1}}\otimes\dots\otimes\frac{\partial}{\partial x^{i_r}}\otimes dx^{j_1}\otimes\dots\otimes dx^{j_s},\tag{2.2}$$

where the Einstein summation convention is used, and the base point p is suppressed. We say that the tensor field $t\in\mathcal{T}^r{}_s(M)$ *is expressed in components by* $t=t^{i_1\dots i_r}{}_{j_1\dots j_s}\frac{\partial}{\partial x^{i_1}}\otimes\dots\otimes\frac{\partial}{\partial x^{i_r}}\otimes dx^{j_1}\otimes\dots\otimes dx^{j_s}$ if equation 2.2 holds at every $p\in M$, where (U,ϕ) is an admissible chart at p. It should be mentioned that such a representation of t *depends on the choice of local coordinates*.

The tensor $(\phi^{-1})^*(t|_U)\in\mathcal{T}^r{}_s(\phi(U))$ is called the local representation of t on U.

It should be mentioned that the tensors in $\mathcal{T}^r_s(M)$ can be viewed as smooth sections of the bundle $TM^*\otimes\dots\otimes TM^*\otimes TM\otimes\dots\otimes TM$, where TM^* and TM appear in the tensor product s and r times respectively, and the tensor product of vector bundles is defined fiberwise.

Definition 2.5.8. A *Riemannian metric* on M is a symmetric tensor field $g\in\mathcal{T}^0{}_2(M)$ such that for every $p\in M$

(i) $g(p)(v,v)>0$ for all $v\in T_pM$ such that $v\neq 0$;

(ii) the map $v \mapsto g(p)(v, \cdot)$ is an isomorphism of T_pM to T_pM^*.

The maps "flat" and "sharp" are defined as follows (see [24]). For every $v \in \Gamma^\infty(TM)$ and $p \in M$

$$v^\flat(p) = g(p)(v(p), \cdot) \in T_pM^*. \tag{2.3}$$

For every $w \in \Gamma^\infty(TM^*)$, the vector field $w^\# \in \Gamma(TM)$ is defined implicitly via the relation

$$w(p) = g(p)(w^\#(p), \cdot), \tag{2.4}$$

where $p \in M$.

Definition 2.5.9. Let $t \in \mathcal{T}^r{}_s(M)$ be a tensor field with components $t^{i_1,\dots,i_r}{}_{j_1,\dots,j_r}$. A *tensor t with its j-th index raised*, where $1 \leq j \leq r$, is defined by

$$\begin{aligned} & t_j^\#(p)(\omega^1, \dots, \omega^r, X_1, \dots, X_{j-1}, \omega^{r+1}, X_{j+1}, \dots, X_s) \\ = \; & t(\omega^1, \dots, \omega^r, X_1, \dots, X_{j-1}, (\omega^{r+1})^\flat, X_{j+1}, \dots, X_s) \end{aligned}$$

for all $\omega^i \in T_pM^*$ and $X_j \in T_pM$.

In the following we will use the Einstein summation convention: we assume the summation with respect to all the repeated indices over the range that is clear from the context.

Definition 2.5.10. The *contraction in lower l and upper k index* of a tensor field $t \in \mathcal{T}^r{}_s(M)$ expressed by $t = t^{i_1,\dots,i_r}{}_{j_1,\dots,j_s} dx^{j_1} \otimes \dots \otimes dx^{j_s} \otimes \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_r}}$ is defined to be

$$\begin{aligned} C_l^k(t) &= t^{i_1,\dots,i_{k-1},p,i_{k+1}\dots,i_r}{}_{j_1,\dots,j_{l-1},p,j_{l+1},\dots,j_s} \\ &\cdot \; dx^{j_1} \otimes \dots \otimes \hat{dx^{j_k}} \otimes dx^{j_s} \otimes \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\hat{\partial}}{\partial x^{i_l}} \otimes \frac{\partial}{\partial x^{i_r}}, \end{aligned}$$

where the "hat" over a vector or a covector means that it is omitted.

Definition 2.5.11. Let $s \in \mathcal{T}^2_s(M)$ and $t \in \mathcal{T}^r_2(M)$. The *contraction* $s : t \in \mathcal{T}^r_s(M)$ is defined to be

$$s : t = s^{km}{}_{j_1,\ldots,j_s} t^{i_1,\ldots,i_r}{}_{km} dx^{j_1} \otimes \ldots \otimes dx^{j_s} \otimes \frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_r}}.$$

Definition 2.5.12. Let $\pi : E \to B$ be a vector bundle, and assume that each fiber $E_p = \pi^{-1}(p)$ is equipped with an inner product $G(p)$. If for all $s_1, s_2 \in \Gamma^\infty(E)$ the correspondence

$$p \mapsto G(p)\big(s_1(p), s_2(p)\big)$$

defines a real valued C^∞ function defined for all $p \in M$, then the collection $\{G(p)\}_{p\in M}$, or simply G, is called a *fiber metric* on M. The *fiber norm* on E is defined analogously.

Definition 2.5.13. Let $h : M \to N$ be a diffeomorphism between Riemannian manifolds (M, g_M) and (N, g_N). The map h is an *isometric diffeomorphism*, or simply an *isometry* if $h^* g_N = g_M$.

2.6 The Lie Derivative

Consider a function $f : M \to \mathbb{R}$ with the tangent $Tf : TM \to T\mathbb{R} = \mathbb{R}\times\mathbb{R}$. A tangent vector in $T\mathbb{R}$ at a base point $\lambda \in \mathbb{R}$ is a pair (λ, μ), where $\mu \in \mathbb{R}$ is the principal part. Therefore, we can write the value of Tf on a tangent vector $v \in T_pM$ as follows: $Tf\, v = (f(p), df(p)v)$. The latter relation defines a functional $df(p) \in T_pM^*$ for each $p \in M$. Therefore, $df \in \Gamma^\infty(TM^*)$ is a *covector field* or a one-form. We call the above defined covector field the *differential of* f.

Definition 2.6.1. Let $f \in C^r(M)$ and $X \in \Gamma^{r-1}(TM)$, $r \geq 1$. The *Lie derivative* of f along the vector field X is defined to be

$$L_X f(p) = X[f](p) = df(p)X(p)$$

for every $p \in M$. We denote the map $p \in M \mapsto df(p)X(p)$ by $df(X)$.

Definition 2.6.2. Let $X, Y \in \mathfrak{X}(M)$. The *Jacobi-Lie bracket* $[X, Y]$ *of* X *and* Y is defined as the unique smooth vector field satisfying the relation $L_{[X,Y]} = [L_X, L_Y] = L_X \circ L_Y - L_Y \circ L_X$. The *Lie derivative* of the vector filed Y in the direction of the vector field X is defined to be the vector field $L_X Y = [X, Y]$.

We have defined the Lie derivative on smooth functions and vector fields. To define the Lie derivative of tensor fields, we will use the concept of a differential operator on the full tensor algebra of a manifold. The Lie derivative is an example of a differential operator.

Definition 2.6.3. A *differential operator* on the full tensor algebra $\mathcal{T}(M)$ of a manifold M is a collection $\{\mathtt{D}^r{}_s(U)\}$ of maps of $\mathcal{T}^r{}_s(U)$ onto itself, where $r, s \geq 0$ and $U \subset M$ is an open set, satisfying the following conditions. For each $\mathtt{D}$ from the collection,

(i) $\mathtt{D}$ is a *tensor derivation*: $\mathtt{D}$ is $\mathbb{R}$-linear and for all $t \in \mathcal{T}^r{}_s(M)$, $\alpha_1, \ldots, \alpha_r \in \Gamma^\infty(TM^*)$ and $X_1, \ldots, X_s \in \Gamma^\infty(TM)$

$$\begin{aligned} \mathtt{D}(t(\alpha_1, \ldots, \alpha_r, X_1, \ldots, X_s)) &= (\mathtt{D}t)(\alpha_1, \ldots, \alpha_r, X_1, \ldots, X_s) \\ &+ \sum_{j=1}^{r} t(\alpha_1, \ldots, \mathtt{D}\alpha_j, \ldots \alpha_r, X_1, \ldots, X_s) \\ &+ \sum_{k=1}^{s} t(\alpha_1, \ldots, \alpha_r, X_1, \ldots, \mathtt{D}X_k, \ldots, X_s); \end{aligned}$$

(ii) $\mathtt{D}$ is *natural with respect to restrictions*: for open sets U, V such that $U \subset V \subset M$ and $t \in \mathcal{T}^r{}_s(V)$

$$(\mathtt{D}t)|_U = \mathtt{D}(t|_U) \in \mathcal{T}^r{}_s(U).$$

Definition 2.6.4. Let $X \in \mathfrak{X}(M)$. The Lie derivative L_X is the unique differential operator on $\mathcal{T}(M)$ such that L_X coincides with the Lie derivative of smooth functions and vector fields defined above.

Proposition 2.6.5. *The Lie derivative is natural with respect to push-forwards. That is, for all* $\phi \in \mathrm{Diff}(M,N)$, $X \in \mathfrak{X}(M)$, *and* $t \in \mathcal{T}^r{}_s(M)$

$$L_{\phi_* X}\phi_* t = \phi_* L_X t.$$

Let $t \in \mathcal{T}^r{}_s(M)$ be a tensor field with components $t^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}$ and let $X \in \mathfrak{X}(M)$, $X = X^i \frac{\partial}{\partial x^i}$. The components of the Lie derivative $L_X t$ of t in the direction X are given by the formula

$$\begin{aligned}
[L_X t]^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s} &= X^k \partial_k t^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s} \\
&- \partial_l X^{i_1} t^{l,i_2,\ldots,i_r}{}_{j_1,\ldots,j_s} - \text{ all upper indices} \\
&+ \partial_{j_1} X^m t^{i_1,i_2,\ldots,i_r}{}_{m,j_2,\ldots,j_s} + \text{ all lower indices,}
\end{aligned} \tag{2.5}$$

where ∂_k denotes differentiation with respect to the variable x_k.

The next theorem states an important property of Lie derivatives.

Theorem 2.6.6. *Let X be a C^k vector field on M with the flow F_λ, and let the tensor field $t \in \mathcal{T}^r{}_s(M)$ be of class C^k. Then on the domain of the flow we have*

$$\frac{d}{d\lambda} F_\lambda^* t = F_\lambda^* L_X t.$$

2.7 Differential Forms

For a vector space E, let $\Lambda^k(E) = L^k{}_a(E;\mathbb{R})$ denote the vector space of skew symmetric real valued multilinear maps, or *exterior k-forms on E*. As with the case of tensors, we can extend this definition fiberwise to a vector bundle over a manifold M.

Definition 2.7.1. Let $\pi : E \to B$ be a vector bundle with fibers E_b. Define the collection $\Lambda^k(E) = \cup_{b\in B}\Lambda^k(E_b)$. The set $\Lambda^k(E)$ is a vector bundle over B with the projection $\pi^k : \Lambda^k(E_b) \to B$ defined by $\pi(t) = b$ for all $t \in \Lambda^k(E_b)$.

The vector bundle of the exterior k-forms on a manifold M is defined to be the vector bundle $\Lambda^k(M) := \Lambda^k(TM)$, where $\tau_M : TM \to M$ is the tangent bundle of the manifold M.

An *exterior k-form* on M is a smooth section of $\Lambda^k(M)$, and we denote $\Omega^k(M) = \Gamma^\infty(\Lambda^k(M))$.

2.8 The Wedge Product, the Exterior Derivative, and the Interior Product

Definition 2.8.1. Let E be a Banach space. The *alternation mapping* $A : {T^0}_k(E) \to {T^0}_k(E)$ is defined by

$$At(e_1, \ldots, e_k) = \frac{1}{k!}\sum_{\sigma \in S_k} (\operatorname{sign}\sigma) t(e_{\sigma(1)}, \ldots, e_{\sigma(k)})$$

for all $t \in {T^0}_k(E)$ and $e_i \in E$, where S_k denotes the set of all permutations of $\{1, 2, \ldots, k\}$.

Definition 2.8.2. Let E be a vector space. We define the *wedge product* of two tensors $\alpha \in {T^0}_k(E)$ and $\beta \in {T^0}_l(E)$ to be the exterior $(k+l)$-form

$$\alpha \wedge \beta = \frac{(k+l)!}{k!l!} A(\alpha \otimes \beta).$$

Example 2.8.3. Let us represent the standard inner product in $\mathbb{R}^n$ as a tensor. Denote the standard basis of $\mathbb{R}^n$ by $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$, and let us denote its dual basis by $\{dx^i\}_{i=1}^n$. Then the standard inner product on $\mathbb{R}^n$ is the tensor $\sum_{i=1}^n dx^i \wedge dx^i \in {T^0}_2(\mathbb{R}^n)$. Indeed, $\sum_{i=1}^n dx^i \wedge dx^i(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}) = \delta_{jk} = \langle \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \rangle_{\mathbb{R}^n}$ as required. For the purpose of brevity, the latter tensor is denoted by $(dx^1)^2 + \ldots + (dx^n)^2$.

Definition 2.8.4. For a given n-dimensional manifold M there is a unique family of mappings $\mathbf{d}^k(U) : \Omega^k(U) \to \Omega^{k+1}(U)$, where $0 \leq k \leq n$ and $U \subset M$ is open, such that each map $\mathbf{d}$ from the family satisfies the following conditions:

(i) $\mathbf{d}$ is a $\wedge$-*antiderivation.* That is, $\mathbf{d}$ is $\mathbb{R}$-linear and for every $\alpha \in \Omega^k(U)$ and $\beta \in \omega^l(U)$

$$\mathbf{d}(\alpha \wedge \beta) = \mathbf{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathbf{d}\beta;$$

(ii) For all $f \in C^\infty(U, \mathbb{R})$, $\mathbf{d}f$ is defined as the differential of f;

(iii) $\mathbf{d}^2 = \mathbf{d} \circ \mathbf{d} = 0$;

(iv) d is *natural with respect to restrictions*. That is, for open sets U, V such that $U \subset V \subset M$ and for every k-form $\alpha \in \Omega^k(V)$, we have $\mathbf{d}(\alpha|_U) = (\mathbf{d}\alpha)|_U$.

Condition (i) is called the *product rule*; condition (iv) means that $\mathbf{d}$ is a *local operator*.

The wedge product of two differential forms on a manifold M is defined pointwise as follows.

Proposition 2.8.5. *The wedge product of two differential forms $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$ is defined as the map $\alpha \wedge \beta : M \to \Lambda^{k+l}(M)$, $(\alpha \wedge \beta)(p) = \alpha(p) \wedge \beta(p)$ for all $p \in M$. The wedge product $\alpha \wedge \beta \in \Omega^{k+l}(M)$ is a $(k+l)$-form, and $\wedge$ is bilinear and associative.*

Definition 2.8.6. Let M be a manifold, $X \in \mathfrak{X}(M)$, and $\omega \in \Omega^{k+1}(M)$. The *interior product* or the *contraction* of X and ω is the contravariant order-k tensor $\mathbf{i}_X\omega$ defined by

$$\mathbf{i}_X\omega(X_1, \ldots, X_k) = \omega(X, X_1, \ldots, X_k).$$

If $\omega \in \Omega^0(M)$, we set $\mathbf{i}_X\omega = 0$.

Theorem 2.8.7. *We have $\mathbf{i}_X : \Omega^k(M) \to \Omega^{k-1}(M)$ for $k = 1, \ldots, m$ and for all $\alpha \in \Omega^k(M)$, $\beta \in \Omega^l(M)$ and $f \in \Omega^0(M)$ the following equalities hold:*

(i) $\mathbf{i}_X$ is a $\wedge-$antiderivation; that is, $\mathbf{i}_X$ is $\mathbb{R}$-linear and

$$\mathbf{i}_X(\alpha \wedge \beta) = \mathbf{i}_X\alpha \wedge \beta + (-1)^k \alpha \wedge \mathbf{i}_X\beta;$$

(ii) $\mathbf{i}_{fX}\alpha = f\mathbf{i}_X\alpha$;

(iii) $L_X\alpha = \mathbf{i}_X\mathbf{d}\alpha + \mathbf{d}\mathbf{i}_X\alpha$ (the "magic" Cartan formula);

(iv) $L_{fX}\alpha = fL_X\alpha + \mathbf{d}f \wedge \mathbf{i}_X\alpha$.

2.9 Volume Forms, Jacobian Determinants, and Divergence

Definition 2.9.1. A *volume form* on an n-manifold M is an n-form $\omega \in \Omega^n(M)$ such that $\omega(p) \neq 0$ for all $p \in M$. The manifold M is called *orientable* if there exists a volume form on M. The pair (M, ω_M) is called a *volume manifold.*

Proposition 2.9.2. *A connected n-manifold M is orientable if and only if there is an n-form $\omega \in \Omega^n(M)$ such that for every other $\nu \in \Omega^n(M)$ there exists $f \in C^\infty(M, \mathbb{R})$ such that $\nu = f\omega$.*

Definition 2.9.3. Let M be an orientable manifold. Two volume forms ω_1 and ω_2 on M are called *equivalent* if there exists a function $f \in C^\infty(M, \mathbb{R})$ such that $f(p) > 0$ for all $p \in M$ and $\omega_1 = f\omega_2$. An *orientation* of M is an equivalence class $[\omega_M]$ of volume forms on M. An *oriented manifold* $(M, [\omega_M])$ is an orientable manifold M together with an orientation $[\omega_M]$ on M.

A chart (U, ϕ) of an oriented n-manifold $(M, [\omega_M])$ is called *positively oriented* if the volume form $\phi^*(\omega_M|_U)$ is equivalent to the standard volume form $dx^1 \wedge \ldots \wedge dx^n \in \Omega^n(\phi(U))$.

Proposition 2.9.4. *Let M and N be n-manifolds equipped with volume forms ω_M and ω_N respectively. For $f \in C^\infty(M, N)$, $f^*\omega_N$ is a volume form on M if and only if f is a local diffeomorphism, i.e. for every $p \in M$ there exists an open neighborhood $V \in M$ such that $f|_V : V \to f(V)$ is a diffeomorphism.*

Definition 2.9.5. Let (M, ω_M) and (N, ω_N) be volume manifolds. A local C^∞ diffeomorphism $f : M \to N$ is called *orientation preserving* if $f^*\omega_N \in [\omega_M]$ and *orientation reversing* if $f^*\omega_N \in [-\omega_M]$.

Proposition 2.9.6. *Let (M, g_M) be a Riemannian manifold. If M is orientable, then there exists a unique volume form ω_M on M such that for every $p \in M$, ω_M*

equals one on all positively oriented orthonormal bases of T_pM. More generally, if $\{v_i\}_{i=1}^n \subset T_pM$ is a positively oriented basis of T_pM, then

$$\omega_M(p)(v_1, \ldots, v_n) = |\det[g_M(p)(v_i, v_j)]|^{1/2}.$$

We say that the volume form ω_M is induced by the Riemannian metric g_M.

Definition 2.9.7. Let M and N be n-manifolds equipped with volume forms ω_M and ω_N respectively. For every $f \in C^\infty(M, N)$, the unique C^∞ function $J(\omega_M, \omega_N)(f) : M \to \mathbb{R}$ such that $f^*\omega_N = J(\omega_M, \omega_N)(f)\omega_M$ is called the *Jacobian determinant* of f with respect to ω_M and ω_N. If the volume forms ω_M and ω_N are understood from the context, we will denote the function $J(\omega_M, \omega_N)(f)$ simply by $J(f)$.

Proposition 2.9.8. *Let (M, ω_M), (N, ω_N) and (S, ω_S) be volume manifolds, and let $f \in \mathrm{Diff}(M, N)$ and $g \in \mathrm{Diff}(N, S)$. Then*

$$J(\omega_M, \omega_S)(g \circ f) = J(\omega_N, \omega_S)(g) \circ f \, J(\omega_M, \omega_N)(f).$$

Definition 2.9.9. Let (M, ω_N) be a volume manifold, and let X be a smooth vector field on M. The unique function $\operatorname{div} X \in C^\infty(M, \mathbb{R})$ such that $L_X\omega_M = \operatorname{div} X\omega_M$ is called the *divergence* of X.

2.10 Riemannian Connection

Definition 2.10.1. Let M be a manifold, and recall that $\mathfrak{X}(M)$ denotes the $C^\infty(M, \mathbb{R})$ module of all C^∞ vector fields on M. Then a bilinear map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

such that for all $X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M, \mathbb{R})$

(i) $\nabla(fX, Y) = f\nabla(X, Y)$,

(ii) $\nabla(X, fY) = (Xf)Y + f\nabla(X, Y)$.

is called a *linear connection*, or an *affine connection* in M. The smooth vector field $\nabla_X Y := \nabla(X, Y)$ is called the *covariant derivative* of Y in the direction X.

Lemma 2.10.2. *Let M be a manifold, and let $U \subset M$ be an open subset. Every linear connection ∇ on M naturally induces a linear connection $\nabla_U : \mathfrak{X}(U) \times \mathfrak{X}(U) \to \mathfrak{X}(U)$.*

Theorem 2.10.3 (Levi-Civita)**.** *Let M be a manifold equipped with a Riemannian metric g_M. There exists a unique linear connection ∇ on M such that for every $X, Y, Z \in \mathfrak{X}(M)$*

(i) $X g_M(Y, Z) = g_M(\nabla_X Y, Z) + g_M(Y, \nabla_X Z)$,

(ii) $\nabla_X Y - \nabla_Y X = [X, Y]$.

The latter connection is called the Levi-Civita*, or the* Riemannian connection *on M.*

Let (M, g_M) be a Riemannian n-manifold with the Riemannian connection ∇. To compute the covariant derivatives of smooth vector fields, it suffices to know the covariant derivatives of the basis vector fields $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$ in coordinate neighborhoods.

Let (U, ϕ) be a coordinate neighborhood of M and the vectors $\{(\frac{\partial}{\partial x^i})_p\}_{i=1}^n$ form a basis of T_pM for every $p \in U$. We define a family of smooth functions $\{\Gamma_{ij}^k\}_{i,j,k=1}^n \subset C^\infty(U, \mathbb{R})$ as follows:

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma_{ij}^k \frac{\partial}{\partial x^k},$$

where, by the Einstein summation convention, we assume summation over all the repeated indices from 1 to n. The functions $\{\Gamma_{ij}^k\}_{i,j,k=1}^n$ are called the *Christoffel symbols* of the Riemannian connection ∇.

The covariant derivative of a smooth function $f \in C^\infty(M, \mathbb{R})$ in the direction $X \in \mathfrak{X}(M)$ is defined to be the C^∞ function $p \mapsto \nabla_X f(p) = df(p)\big(X(p)\big) \in \mathbb{R}$.

We have defined the covariant derivative of smooth functions and vector fields. Similarly to the case of Lie derivatives, we can extend this definition to the full algebra of tensor fields $\mathcal{T}(M)$.

Definition 2.10.4. Let (M, g_M) be a Riemannian manifold with the Riemannian connection ∇, and let $X \in \mathfrak{X}(M)$. The *covariant derivative* ∇_X is the unique differential operator on $\mathcal{T}(M)$ such that ∇_X coincides with the covariant derivative of smooth functions and smooth vector fields defined above.

Let $t \in \mathcal{T}^r{}_s(M)$ be a tensor field on M. The covariant derivative of t is the tensor field $\nabla t \in \mathcal{T}^r_{s+1}(M)$ defined by

$$(\nabla t)(p)(\omega^1, \ldots, \omega^r, X_1, \ldots, X_r, X_{s+1}) = (\nabla_{X_{s+1}} t)(p)(\omega^1, \ldots, \omega^r, X_1, \ldots, X_s)$$

for all $\omega^i \in T_pM^*$ and $X_j \in T_pM$.

Let $t^{b_1 \ldots b_r}{}_{c_1 \ldots c_s}(x)$ be the components of the tensor field $t \in \mathcal{T}^r{}_s(M)$ in the local chart (U, ϕ), where $x \in \phi(U) \subset \mathbb{R}^n$. The following formula for the components of the covariant derivative $\nabla t \in \mathcal{T}^r{}_{s+1}(M)$ is useful in computations:

$$\begin{aligned} (\nabla t)^{b_1 \ldots b_r}{}_{c_1 \ldots c_s a} &= \nabla_a t^{b_1 \ldots b_r}{}_{c_1 \ldots c_s} \\ &= \partial_a t^{b_1 \ldots b_r}{}_{c_1 \ldots c_s} + \sum_i \Gamma^{b_i}_{ap} t^{b_1 \ldots p \ldots b_r}{}_{c_1 \ldots c_s} \\ &- \sum_i \Gamma^p_{ac_i} t^{b_1 \ldots b_r}{}_{c_1 \ldots p \ldots c_s}, \end{aligned} \tag{2.6}$$

where ∂_a denotes the partial differentiation with respect to the variable x_a, and the index p in the last two sums is located at the i-th place. Because ∇ is the Riemannian connection on (M, g_M), we have that $\nabla g_M = 0$.

Definition 2.10.5. The *divergence of a tensor field* $t \in \mathcal{T}^r{}_s(M)$ is defined to be (see [18])

$$\operatorname{div} t = C^r_{s+1}(\nabla t), \tag{2.7}$$

where C^j_i denotes a contraction in lower i and upper j index. Note that $\operatorname{div} T$ is a tensor of type $(r-1, s)$.

The following formula is useful when computing a Lie derivative of the Riemannian metric g_M. The *lowered coordinates* of the vector field $Y = Y^j \frac{\partial}{\partial x^j}$ are given by the formula $Y_m = [Y^\flat]_m = [g_M]_{ml} Y^l$, and the components of the Lie derivative of the Riemannian metric g_M in the direction of the vector field Y are [29]

$$[L_Y g_M]_{km} = \nabla_k Y_m + \nabla_m Y_k. \tag{2.8}$$

In other words, for every $X, Y, Z \in \mathfrak{X}(M)$ we have

$$L_Y g_M(X, Z) = \nabla Y^\flat(X, Z) + \nabla Y^\flat(Z, X).$$

Definition 2.10.6. A submanifold of a Euclidean space of codimension one is called a *hypersurface.*

Definition 2.10.7. Let M be an orientable hypersurface in $\mathbb{R}^{n+1}$ with the smooth unit normal vector field N. That is, for every $p \in M$, $N(p) \in \mathbb{R}^{n+1}$ is such that $\langle N(p), N(p)\rangle = 1$ and $\langle N(p), Y\rangle = 0$ for all $Y \in T_pM$. Denote the natural Riemannian connection (compatible with the standard inner product) on $\mathbb{R}^{n+1}$ by ∇. The *Weintgarten map* or the *shape operator* $W : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is defined by $W(X) = \nabla_X N$. The *second fundamental form* $I\!I \in \mathcal{T}^0{}_2(M)$ on M is defined by

$$I\!I(X, Y) = \langle W(X), Y\rangle$$

for all $X, Y \in \mathfrak{X}(M)$.

2.11 Integration over Manifolds and Stokes' Theorem

In this section we will define integration over volume manifolds. We will use integration over local charts and "patch" the results together over manifolds using partitions of unity.

Definition 2.11.1. A *partition of unity* on a manifold M is a collection $\{(U_i, g_i)_{i\in I}\}$ such that

(i) $\{U_i\}_{i\in I}$ is a locally finite open covering of M. That is, for each $p \in M$, there is an open neighborhood $W \subset M$ of the point p satisfying $W \cap U_i = \emptyset$ except for finitely many indices $i \in I$.

(ii) for each $i \in I$, the C^∞ functions $g_i : M \to \mathbb{R}$ are supported inside U_i and $g_i(p) \geq 0$ for all $p \in M$;

(iii) for each $p \in M$, the finite sum $\sum_{i\in I} g_i(p) = 1$.

If $\mathcal{A} = \{(V_\alpha, \phi_\alpha)\}_{\alpha\in A}$ is an atlas on M, a *partition of unity subordinate to* $\mathcal{A}$ is a partition of unity $\{(U_i, g_i)_{i\in I}\}$ such that every open set U_i is a subset of a chart domain $V_{\alpha(i)}$.

Theorem 2.11.2. *Every second-countable (Hausdorff) n-manifold admits a partition of unity.*

Definition 2.11.3. Let M be an orientable n-manifold with orientation $[\omega_M]$. Suppose that an n-form $\omega \in \Omega^n(M)$ has compact support inside an open set $U \subset M$ such that (U, ϕ) is a positively oriented chart. Then the integral of ω over the chart (U, ϕ) is defined to be

$$\int_{(\phi)} \omega = \int_{\phi(U)} \phi_*(\omega|_U).$$

It can be shown that the latter definition does not depend on the choice of the chart (U, ϕ). Therefore, we can define $\int \omega = \int_{(\phi)} \omega$, where (U, ϕ) is a positively oriented chart containing the compact support of ω.

Definition 2.11.4. Let M be an oriented manifold with an atlas of positively oriented charts $\mathcal{A}$. Let $P = \{(U_\alpha, \phi_\alpha, g_\alpha)\}_{\alpha\in I}$ be a partition of unity subordinate to $\mathcal{A}$. We define the n-forms of compact support $\omega_\alpha = g_\alpha \omega$ and set

$$\int_P \omega = \sum_{\alpha\in I} \int \omega_\alpha. \tag{2.9}$$

Proposition 2.11.5. *The sum in equation 2.9 contains only a finite number of nonzero terms. Moreover, if Q is a partition of unity of M subordinate to an atlas $\mathcal{B}$ of M, which is equivalent to $\mathcal{A}$, then $\int_P \omega = \int_Q \omega$.*

The *integral* of the n-form $\omega \in \Omega^n(M)$ over the manifold M is defined to be $\int_M \omega = \int_P \omega$, where P is a partition of unity on M subordinate to an atlas $\mathcal{A}$ of M.

Theorem 2.11.6 (Stokes)**.** *Let M be an oriented smooth compact n-manifold (without boundary) equipped with the volume form ω_M. For every $\alpha \in \Omega^{n-1}(M)$*

$$\int_M \mathbf{d}\alpha = 0.$$

In particular, for every smooth vector field X on M,

$$\int_M \operatorname{div} X \omega_M = 0.$$

Also we will use (the strong form) of Moser's theorem on volume forms, which we state here for the convenience of the reader (see [30]).

Theorem 2.11.7. *Let τ_t be a family of volume forms defined for $t \in [0,1]$ on a compact manifold M. If*

$$\int_c \tau_t = \int_c \tau_0 \tag{2.10}$$

for every n-cycle c on M, then there exists a one-parameter family of diffeomorphisms $\phi_t : M \to M$ such that

$$\phi_t^* \tau_t = \tau_0 \tag{2.11}$$

and ϕ_0 is the identity mapping. Moreover, the dependence of $\phi_t(m)$ on $m \in M$ and $t \in [0,1]$ is as smooth as in the family τ_t.

2.12 Riemann Surfaces

Let M be a one-dimensional topological manifold modeled on the space of complex numbers (set $X = \mathbb{C}$ in definition 2.1.1). The manifold M endowed with an equivalence class of atlases $[\mathcal{A}]$ is called a *Riemann surface* if all the transition maps between the admissible charts with nonempty intersection are holomorphic (see [34, 23]). Riemann surfaces are orientable.

Let M, N be Riemann surfaces. A map $f : M \to N$ is called *holomorphic* if all its local representations are holomorphic functions wherever they are defined. A holomorphic map $f : M \to N$ with nowhere vanishing derivative is called *conformal*.

In chapter 4.4, we will consider the Riemann sphere. There are several ways of describing the Riemann sphere. It can be viewed as the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$, the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, or the complex projective space $\mathbb{CP}^1$.

Let us check that the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ is a compact Riemann surface (with the topology induced from $\mathbb{R}^3$). Let $N = (0, 0, 1)$ and $S = (1, 0, 0)$ be the north and the south poles of $\mathbb{S}^2$ respectively. Consider the open sets $U_1 = \mathbb{S}^2 \backslash \{N\}$ and $U_2 = \mathbb{S}^2 \backslash \{S\}$ on $\mathbb{S}^2$. Define the maps $z_i : U_i \to \mathbb{C}$ for $i = 1, 2$ by

$$z_1 = \frac{x_1 + i x_2}{1 - x_3} \quad \text{and} \quad z_2 = \frac{x_1 - i x_2}{1 + x_3},$$

where $i \in \mathbb{C}$ is such that $i^2 = -1$. We then have $z_2 = 1/z_1$ on $U_1 \cap U_2$ so that the transition map is indeed holomorphic.

The extension π of the map z_1 maps the sphere $\mathbb{S}^2$ to the extended complex plane $\hat{\mathbb{C}}$ in a bijective manner. The bijective map $\pi : \mathbb{S}^2 \to \hat{\mathbb{C}}$ is a *stereographic projection*. Consider the open sets $V_1 := z_1(U_1) = \mathbb{C}$ and $V_2 := z_2(U_2) = (\mathbb{C} \backslash \{0\}) \cup \{\infty\}$. The extended complex plane is a Riemann surface with coordinate charts

$$\mathrm{id} : V_1 \to \mathbb{C}$$

and

$$\begin{array}{rcl} V_2 & \to & \mathbb{C} \\ z & \mapsto & \dfrac{1}{z}. \end{array}$$

The stereographic projection $\pi : \mathbb{S}^2 \to \hat{\mathbb{C}}$ is a conformal map.

A holomorphic function $f : M \to \hat{\mathbb{C}}$, where M is a Riemann surface, is called *meromorphic* if f is not identically equal to the constant ∞. It can be shown that the meromorphic functions defined on the extended complex plane $\hat{\mathbb{C}}$ are rational functions. Therefore, the automorphisms (bijective meromorphic maps) of the extended complex plane are Möbius transformations

$$f(z) = \frac{az+b}{cz+d},$$

where $ad - bc \neq 0$. The set of all such transformations forms a group under composition. The automorphism group of the Riemann sphere is defined as the group of biholomorphic maps on the Riemann sphere. From the previous remarks, this group $\mathrm{Aut}(\hat{\mathbb{C}})$ coinsides with the group of Möbius transformations.

The Riemann sphere can be identified with the complex projective space $\mathbb{CP}^1$, which is defined as the set of equivalence classes $[z_1, z_2]$ of ordered pairs $(z_1, z_2) \in \mathbb{C} \times \mathbb{C}$ under the equivalence relation $(z_1, z_2) \sim (\lambda z_1, \lambda z_2)$ for all $\lambda \in \mathbb{C}\backslash\{0\}$. The isomorphism $\mathbb{CP}^1 \cong \hat{\mathbb{C}}$ is given by $[z_1, z_2] \mapsto \frac{z_1}{z_2}$, where $z/0 = \infty$.

Recall that the general linear group $\mathrm{GL}(2, \mathbb{C})$ is the group of all invertible linear transformations of $\mathbb{C}^2$, which can be viewed as the group of invertible 2×2 matrices with complex entries. The special linear group $\mathrm{SL}(2, \mathbb{C})$ is the subgroup of $\mathrm{GL}(2, \mathbb{C})$ consisting of all invertible 2×2 matrices with determinant 1. The projective general linear group $\mathrm{PGL}(2, \mathbb{C})$ is the quotient group $\mathrm{GL}(2, \mathbb{C})/\mathcal{S}$, where $\mathcal{S}$ consists of all the scalar multiples of the identity. The group $\mathrm{PGL}(2, \mathbb{C})$ acts on the Riemann sphere $\hat{\mathbb{C}}$ as follows. The action of

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}(2, \mathbb{C})$$

on $[z_1, z_2] \in \mathbb{CP}^1$ is defined to be the equivalence class $[w_1, w_2] \in \mathbb{CP}^1$, where

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

The group of all Möbius transformations is then isomorphic to the projective general linear group $\mathrm{PGL}(2, \mathbb{C})$, and composition in $\mathrm{Aut}(\hat{\mathbb{C}})$ corresponds to matrix multiplication in $\mathrm{GL}(2, \mathbb{C})$. Note that we can always multiply a given matrix in $\mathrm{GL}(2, \mathbb{C})$ by a constant to obtain a matrix with determinant one. Therefore, the group $\mathrm{PGL}(2,\mathbb{C})$ is isomorphic to $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\{\pm I\}$. In summary, we have

$$\mathrm{Aut}\,\hat{\mathbb{C}} = \mathrm{PGL}(2,\mathbb{C}) = \mathrm{PSL}(2,\mathbb{C}). \tag{2.12}$$

Definition 2.12.1. A conformal Riemannian metric on a Riemann surface M is given in local coordinates by

$$\lambda^2(z)dzd\bar{z},$$

where $dzd\bar{z} = (dx + idy) \wedge (dx - idy) = -2idx \wedge dy$ and λ is a positive C^∞ function. If $w \mapsto z(w)$ is a transformation of local coordinates, then the metric transforms to

$$\lambda^2(z)\frac{\partial z}{\partial w}\frac{\partial z}{\partial \bar{w}}dwd\bar{w},$$

where $w = u + iv$, $\frac{\partial}{\partial w} = \frac{1}{2}(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v})$ and $\frac{\partial}{\partial \bar{w}} = \frac{1}{2}(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v})$.

Example 2.12.2. Consider the Riemann sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ with the metric induced on it by the Euclidean metric $g = dx_1^2 + dx_2^2 + dx_3^2$ (see example 2.8.3). Using the stereographic projection $\pi : \mathbb{S}^2 \to \hat{\mathbb{C}}^2$, we induce the conformal metric $\alpha := \pi_* g$ on $\hat{\mathbb{C}}$ given by

$$\alpha(z,\bar{z}) = \frac{4}{(1+|z|^2)^2}dzd\bar{z}.$$

The corresponding isometries of $\hat{\mathbb{C}}$ are the Möbius transformations of the form

$$z \mapsto \frac{az - \bar{c}}{cz + \bar{a}}, \quad |a|^2 + |c|^2 = 1.$$

Recall that $\mathrm{U}(2,\mathbb{C})$ is the unitary group of all unitary complex 2×2 matrices. The group of all isometries of $(\hat{\mathbb{C}}, \alpha)$ is the projective unitary group $\mathrm{PU}(2,\mathbb{C}) = \mathrm{U}(2,\mathbb{C})/\{\pm I\}$.

2.13 The Manifold $\mathrm{Diff}(M,N)$

Let M and N be smooth compact n-manifolds without boundary. The space $C^\infty(M,N)$ is an (infinite dimensional) differentiable manifold (see [19, 32]). The space of all smooth diffeomorphisms $\mathrm{Diff}(M,N)$ is a submanifold of $C^\infty(M,N)$ (see [25], Ch. IX, Sec. 43). The tangent space to $\mathrm{Diff}(M,N)$ at $h \in \mathrm{Diff}(M,N)$ can be identified with the space of all smooth vector fields $\mathfrak{X}(N)$. To explain this identification, let us introduce a convenient notation for the elements of the tangent space $T_h \mathrm{Diff}(M,N)$.

Let $[c]_h \in T_h \mathrm{Diff}(M,N)$ be an equivalence class of curves at $h \in \mathrm{Diff}(M,N)$. The representative $c : I \to \mathrm{Diff}(M,N)$ is a C^1 curve at h, where $I = (-\varepsilon, \varepsilon)$, $\varepsilon > 0$. The open set $I \subset \mathbb{R}$ is a one-dimensional manifold with the natural basis $(\frac{\partial}{\partial t})_{t_0} = (t_0, 1)$, where 1 is the principal part of the tangent space $T_{t_0} I$. Therefore, we can consider the value of the derivative map of $c : I \to \mathrm{Diff}(M,N)$ on $(\frac{\partial}{\partial t})_0$, which we denote by $\frac{d}{dt} c(t)|_{t=0} := dc(0)(\frac{\partial}{\partial t})_0 \in T_h \mathrm{Diff}(M,N)$. By the definition of the derivative map (see the paragraph after theorem 2.3.6), $\frac{d}{dt} c(t)|_{t=0} = [c]_h$. In the following, we will use both notations for the elements of $T_h \mathrm{Diff}(M,N)$, which we call *tangent vectors* at h.

Proposition 2.13.1. *The tangent space $T_h \mathrm{Diff}(M,N)$ can be identified with $\mathfrak{X}(N)$ via a bijective map.*

Proof. An element of $T_h \mathrm{Diff}(M,N)$ is an equivalence class of C^1 curves $[c]_h$ at h, represented by a family of diffeomorphisms $c(t) \in \mathrm{Diff}(M,N)$, where $t \in (-\varepsilon, \varepsilon)$, with $c(0) = h$. For each $q \in N$, this family defines a C^1 curve $s(t) = c(t) \circ (h^{-1}(q))$ in N that passes through q at $t = 0$; hence, it defines a vector $Y(q) \in T_q N$ by $Y(q) := [s]_q = \frac{d}{dt} s(t)|_{t=0}$. The vector field $Y \in \mathfrak{X}(N)$ is thus associated with the equivalence class $[c]_h \in T_h \mathrm{Diff}(M,N)$. In fact, the vector field Y does not depend on the choice of the representative of the equivalence class. On the other hand, to given a vector field $Y \in \mathfrak{X}(N)$ with the flow ϕ_t, we associate the curve $c(t) = \phi_t \circ h$ in $\mathrm{Diff}(M,N)$. The (tangent) equivalence class $[c]_h$ of this curve is an element of $T_h \mathrm{Diff}(M,N)$. $\square$

2.14 Facts from the Calculus of Variations

Let $\mathcal{F}$ be a (perhaps infinite dimensional) differentiable manifold. Consider a functional $E : \mathcal{F} \to \mathbb{R}$. We assume that the functional E is bounded below, that is

$$\inf_{f \in \mathcal{F}} E(f) > -\infty.$$

Let $\mathcal{A} \subset \mathcal{F}$ be a submanifold of $\mathcal{F}$. Consider the problem of minimization of the functional E in the admissible set $\mathcal{A}$.

Definition 2.14.1. We say that there exists a (global) minimum of the functional E in the admissible set $\mathcal{A}$ if $\inf_{f \in \mathcal{A}} E(f) = E(h)$ for some $h \in \mathcal{A}$. The element $h \in \mathcal{A}$ *minimizes* the functional E over the admissible set $\mathcal{A}$, and is called a *minimizer* of E over $\mathcal{A}$.

Let $c : (-\varepsilon, \varepsilon) \to \mathcal{A}$ be a C^1 curve at $h \in \mathcal{A}$. The curve c defines an element of the tangent space $[c]_h \in T_h\mathcal{A}$, which we call a *variational vector field* of h. The *first variation* of the functional E at $h \in \mathcal{A}$ in the direction $[c]_h$ is defined to be

$$\frac{d}{dt} E(c(t))|_{t=0} = dE(h)[c]_h$$

(provided that the derivative exists). We say that the functional E is Gateaux differentiable at $h \in \mathcal{A}$ if the first variation of E exists in every direction $[c]_h \in T_h\mathcal{A}$. (The classical definition of Gateaux differentiability for functions defined on Banach spaces is given in [4], section 1.1)

Definition 2.14.2. We call the equation $dE(h)[c]_h = 0$ for all $[c]_h$ from some chosen subset $\mathcal{Q}_h$ of $T_h\mathcal{A}$ (which will be specified in each particular case), and all the equations with the same set of solutions, an *Euler-Lagrange equation* for the functional E. The equation $dE(h)[c]_h = 0$ for all $[c]_h \in T_h\mathcal{A}$ ($\mathcal{Q}_h = T_h\mathcal{A}$) is called the *complete Euler-Lagrange equation* for E. If h is a minimizer of the functional E, then h satisfies all the Euler-Lagrange equations for E. An element $h \in \mathcal{A}$ is a *critical point* of E if it satisfies some Euler-Lagrange equation for E.

For example, let $\mathcal{F}$ be the Banach space $(C^2([0,l]), \|\cdot\|_{C^2})$, where

$$\|u\|_{C^2} = \sum_{i=0}^{2} \max_{t\in[0,l]} |\frac{d^i}{dt^i}u(t)|.$$

Let $\mathcal{A} = \{u \in \mathcal{F} : u(0) = u(l) = 0\}$, which is a nonempty closed and convex subset of $\mathcal{F}$. Suppose that the functional $E : \mathcal{F} \to \mathbb{R}$ is defined by

$$E(u) = \int_0^l L(u(t), \dot{u}(t), t)\, dt, \tag{2.13}$$

where $\dot{u} = \frac{du}{dt}$ and $L : \mathbb{R}^3 \to \mathbb{R}$ is a C^1 function such that $t \mapsto L(u(t), \dot{u}(t), t)$ is a bounded C^1 function on $(0, l)$ for each $u \in \mathcal{A}$. A function $u \in C^2([0, l])$ is a critical point of E if $\frac{d}{dt}E(c(t))|_{t=0} = 0$ for all C^1 curves $c : (\varepsilon, \varepsilon) \to \mathcal{A}$ at $u \in \mathcal{A}$. If we consider only C^1 curves $c : (-\varepsilon, \varepsilon) \to \mathcal{A}$ at $u \in \mathcal{A}$ that have a special form of our choice, the equation $\frac{d}{dt}E(c(t))|_{t=0} = 0$ still gives a necessary condition for $u \in \mathcal{A}$ to be a minimizer of E. The standard approach in the calculus of variations is to consider the curves of the form $c(t) = u + t\phi$, where $\phi \in C_c^\infty(0, l)$ has compact support on $(0, l)$ [3, 22]. The function ϕ is called a *test function*.

It can be shown that the Euler-Lagrange equation $\frac{d}{dt}E(u + t\phi)|_{t=0} = 0$ for all $\phi \in C_c^\infty(0, l)$ is equivalent to the differential equation

$$\frac{d}{dt}\big(\frac{\partial L}{\partial y_2}(u(t), \dot{u}(t), t)\big) = \frac{\partial L}{\partial y_1}(u(t), \dot{u}(t), t), \tag{2.14}$$

where $\frac{\partial L}{\partial y_i}$ denotes the partial derivative of L with respect to its i^{th} variable, $i = 1, 2$. This equation will be considered in section 4.3, where we investigate the conditions for the existence of minimizers of the deformation energy functional of bending for simple closed curves.

Definition 2.14.3. A sequence $\{v^l\}_{l=1}^\infty \subset \mathcal{F}$ such that

$$\lim_{l\to\infty} E(v^l) = \inf_{f\in\mathcal{F}} E(f) \tag{2.15}$$

for a functional $E : \mathcal{F} \to \mathbb{R}$ is called a *minimizing sequence* of E.

Minimizing sequences exist for E since it is assumed to be bounded from below.

Every critical point $h \in \mathcal{A}$ of the functional $E : \mathcal{F} \to \mathbb{R}$ is a candidate for a minimizer of E. In some cases it is possible to show that h is indeed a minimizer of E over $\mathcal{A}$ by proving the inequality $E(h) \leq E(f)$ for all $f \in \mathcal{A}$. In section 4.3 we use Hölder's inequality to prove that a critical point is a minimizer.

Proposition 2.14.4 (Hölder's inequality). *Suppose that $W \subset \mathbb{R}^n$ is an open set, $u \in L^p(W)$ and $v \in L^q(W)$, where $1 \leq p, q \leq \infty$. If $1/p + 1/q = 1$, then*

$$\int_W |uv| dx \leq \Big(\int_W u^p \, dx \Big)^{1/p} \Big(\int_W v^q \, dx \Big)^{1/q}.$$

Consider the special case when $\mathcal{F}$ is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{F}}$, which induces the norm $\| \cdot \|_{\mathcal{F}}$. Suppose that the admissible set $\mathcal{A} = \mathcal{F}$. As before, E is assumed to be bounded below.

Definition 2.14.5. The functional $E : \mathcal{F} \to \mathbb{R}$ is *coercive* if for every sequence $\{u^l\}_{l=1}^{\infty}$ such that $\|u^l\|_{\mathcal{F}} \to \infty$ as $l \to \infty$, we have $E(u^l) \to \infty$ as $l \to \infty$.

Recall that a sequence $\{u^l\}_{l=1}^{\infty} \subset \mathcal{F}$ converges weakly to $u \in \mathcal{F}$, which we denote by $u^l \rightharpoonup u$, if $\langle v, u^l \rangle_{\mathcal{F}} \to \langle v, u \rangle_{\mathcal{F}}$ as $l \to \infty$ for all $v \in \mathcal{F}$.

Definition 2.14.6. The functional $E : \mathcal{F} \to \mathbb{R}$ is called (sequentially) *weakly lower semicontinuous* if for every weakly convergent sequence $\{u^l\}_{l=1}^{\infty} \subset \mathcal{F}$, with the weak limit $u \in \mathcal{F}$,

$$E(u) \leq \liminf_{l \to \infty} E(u^l).$$

The functional E is (sequentially) *weakly continuous* if under the same assumptions $\lim_{l \to \infty} E(u^l) = E(u)$.

Theorem 2.14.7. *Let $\mathcal{F}$ be a Hilbert space with the norm $\| \cdot \|_{\mathcal{F}}$.*

(i) The functional $u \mapsto \|u\|_{\mathcal{F}}$ is weakly lower semicontinuous;

(ii) Every bounded sequence in $\mathcal{F}$ has a weakly convergent subsequence.

Theorem 2.14.8 (The Direct Method of the Calculus of Variations)**.** *Let $\mathcal{F}$ be a Hilbert space, and suppose that a functional $E : \mathcal{F} \to \mathbb{R}$ is coercive, weakly lower semicontinuous, and bounded below. Then there exists a minimizer of the functional E in $\mathcal{F}$.*

We will formulate regularity results for minimizers of the functional

$$I(u) = \int_a^b F\big(t, u(t), \dot{u}(t)\big)\, dt$$

defined on an appropriate space of functions $u : [a, b] \to \mathbb{R}^d$, where the function $F : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is continuous in its second and third variables and the function $t \mapsto F\big(t, u(t), \dot{u}(t)\big)$ is measurable for all admissible functions $u : [a, b] \to \mathbb{R}^d$. For example, the definition of the functional I is meaningful on the space $\mathrm{AC}[a, b]$ of all absolutely continuous functions on $[a, b]$ (see definition 2.15.7 and theorem 2.15.8).

Let $\delta I(u, \eta)$ denote the Gateaux derivative of the functional I at u in the direction η, i.e. $\delta I(u, \eta) = \frac{d}{ds} I(u + s\eta)|_{s=0}$.

Theorem 2.14.9. *Let the function $F : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be of class C^1 and assume that the derivative $F_p(t, u, p)$ is also of class C^1. Suppose that $u \in C^1([a, b]; \mathbb{R}^d)$ is a solution of the Euler-Lagrange equation*

$$\delta I(u, \eta) = 0$$

for all $\eta \in C^1_c((a, b); \mathbb{R}^d)$. If the Hessian $\big(F_{p_i p_j}(t, u(t), \dot{u}(t))\big)_{1 \le i,j \le d}$ of F with respect to the p variable is nonsingular for all $t \in [a, b]$, then the function u is of class C^2.

Theorem 2.14.10. *Suppose that the function F satisfies the assumptions of the previous theorem. Suppose that $u \in AC([a, b]; \mathbb{R}^d)$ is a solution of the Euler-Lagrange equation*

$$\delta I(u, \eta) = 0$$

for all $\eta \in AC((a, b); \mathbb{R}^d)$ with compact support on (a, b). Assume that $F_u(t, u(t), \dot{u}(t))$ and $F_p(t, u(t), \dot{u}(t))$ are integrable on (a, b) as functions of t. If the Hessian F_{pp} is *positive or negative definite on* $\Omega \times \mathbb{R}^d$*, where* Ω *is an open set containing the graph* $\{(t, u(t)) : t \in [a, b]\}$ *of the function* u*, then* $u \in C^2(I, \mathbb{R}^d)$*.*

The last two theorems are proved in Sec. 1.2 of [22].

We will finish this section with a statement of the Generalized Kuhn-Tucker theorem.

Definition 2.14.11. A subset P of a vector space X is called *a cone with the vertex* $x \in X$, or simply *a cone*, if $P = x + C$, where the set $C \subseteq X$ has the property $\lambda x \in C$ for all nonnegative real numbers λ whenever $x \in C$. The cone P is called a *convex cone* if it is a convex set.

Definition 2.14.12. Let P be a convex cone in a vector space X. For $x, y \in X$ we write $x \geq y$ if $x - y \in P$. The cone P defining this relation is called *the positive cone of* X.

Let $G : X \to Z$ be a Gateaux differentiable map between a vector space X and a normed space Z. We say that the Gateaux differential of G is linear in its increment if the function $h \mapsto \delta G(x, h) = \frac{d}{ds} G(x + sh)|_{s=0}$, defined on X, is linear for every $x \in X$.

Definition 2.14.13. Let X be a vector space and let Z be a normed space with a positive cone P that has a nonempty interior. Let $G : X \to Z$ be a Gateaux differentiable mapping whose Gateaux differential is linear in its increment. A point $x_0 \in X$ is called *a regular point* of the inequality $G(x) \leq 0_Z$, where $0_Z \in Z$ is the zero element, if $G(x_0) \leq 0_Z$ and there exists $h \in X$ such that $G(x_0) + \delta G(x_0, h) < 0_Z$.

We denote the space of all bounded linear functionals on a normed vector space Z by Z^*. The value of a functional $z^* \in Z^*$ at a point $z \in Z$ is denoted by $\langle z, z^* \rangle$.

Theorem 2.14.14 (Generalized Kuhn-Tucker Theorem)**.** *Let* X *be a vector space and let* Z *be a normed space with a positive cone* P*. Assume that* P *has nonempty* *interior. Let $F : X \to \mathbb{R}$ and $G : X \to Z$ be Gateaux differentiable maps whose Gateaux derivatives are linear in their increments. If the point $x_0 \in X$ minimizes the function f subject to the constraint $G(x) \leq 0_Z$, then there exists $z_0^* \in Z^*$ such that $z_0^* \geq 0_{Z^*}$ and the Lagrangian*

$$f(x) + \langle G(x), z_0^* \rangle$$

is stationary at x_0, i.e. its Gateaux derivative at x_0 vanishes in all directions; furthermore, $\langle G(x_0), z_0^* \rangle = 0$.

The proof of this theorem can be found in Sec. 9.4 of [28].

2.15 Facts from Analysis

We will denote the standard Euclidean norm on $\mathbb{R}^r$, where $r \in \mathbb{N}$, by $|\cdot|$. The Euclidean norm $|A|$ of a matrix $A \in \mathbb{R}^r \times \mathbb{R}^l$, where $r, l \in \mathbb{N}$, is the norm of the vector composed of all entries of the matrix A.

Let $\Omega \subset \mathbb{R}^n$ be an open set.

Let $L^1_{loc}(\Omega)$ denote the space of all functions $f : \Omega \to \mathbb{R}$ such that $f \in L^1(K)$ for every compact subset $K \subset \Omega$. We denote the space of all smooth real valued functions with compact support on Ω by $C_c^\infty(\Omega)$.

Definition 2.15.1. We say that a function $f : \Omega \to \mathbb{R}$ is *Hölder continuous* with exponent λ if there exists a constant $C > 0$ such that $|f(x) - f(y)| \leq C|x - y|^\lambda$ for all $x, y \in \Omega$. The λ-th Hölder seminorm of f is defined by

$$[f]_{C^{0,\lambda}(\Omega)} = \sup_{x,y\in\Omega, x\neq y} \Big\{ \frac{|f(x) - f(y)|}{|x - y|^\gamma} \Big\}.$$

The Hölder space $C^{k,\lambda}(\bar{\Omega})$ consists of all functions $f \in C^k(\bar{\Omega})$ for which the norm

$$\|u\|_{C^{j,\lambda}(\bar{\Omega})} = \sum_{|\alpha|\leq j} \|D^\alpha u\|_{C(\bar{\Omega})} + \sum_{|\alpha|=j} [D^\alpha u]_{C^{0,\lambda}(\bar{\Omega})}$$

is finite.

Definition 2.15.2. Suppose that $f, g \in L^1_{loc}(\Omega)$, and $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiindex such that $\alpha_i \in \mathbb{N}$ and $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$. We say that g is the α-th weak partial derivative of f written $D^\alpha f = g$ provided

$$\int_\Omega f D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_\Omega g\phi dx \tag{2.16}$$

for all test functions $\phi \in C_c^\infty(\Omega)$.

Definition 2.15.3. The Sobolev space $W^{k,p}(\Omega)$ is defined as the space of all locally summable functions $f : \Omega \to \mathbb{R}$ such that for each multiindex α with $|\alpha| = k$, $D^\alpha f$ exists in the weak sense and belongs to $L^p(\Omega)$. The Sobolev norm of a function $f \in W^{k,p}(\Omega)$ is defined by

$$\|f\|_{k,p}^p = \sum_{|\alpha| \leq k} \int_\Omega |D^\alpha f|^p dx$$

(see [15], chapter 5).

Theorem 2.15.4. *For each $k = 1, 2, \ldots$ and $p \geq 1$, the Sobolev space $W^{k,p}(\Omega)$ is a Banach space.*

Definition 2.15.5. The Sobolev space $W_0^{k,p}(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$ with respect to the norm $\| \cdot \|_{k,p}$.

The Sobolev space $W^{k,p}(\Omega; \mathbb{R}^n)$ consists of all functions $f : \Omega \to \mathbb{R}^n$ such that their components $f_i \in W^{k,p}(\Omega)$ for all $i = 1, \ldots, n$. The Sobolev norm of a function $f \in W^{k,p}(\Omega; \mathbb{R}^n)$ is defined by $\|f\|_{k,p}^{\mathbb{R}^n} = (\sum_{i=1}^n \|f_i\|_{k,p}^p)^{1/p}$.

For convenience of the reader, we state the Sobolev embedding theorem (see [15], section 5.6).

Theorem 2.15.6 (Sobolev Embedding)**.** *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a C^1 boundary. If $u \in W^{k,p}(\Omega)$ and $k > \frac{n}{p}$, then $u \in C^{j,\lambda}(\bar{\Omega})$ for $j = k - [\frac{n}{p}] - 1$ and some $\lambda \in (0, 1)$. In addition, there is a constant C depending on k, p, n, λ and Ω only, such that*

$$\|u\|_{C^{j,\lambda}(\bar{\Omega})} \leq C\|u\|_{W^{k,2}(\Omega)}. \tag{2.17}$$

Definition 2.15.7. Let I be an interval of the real line $\mathbb{R}$. A function $f : I \to \mathbb{R}^n$ is absolutely continuous on I if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all sequences of pairwise disjoint intervals $\{[x_k, y_k]\}_{k=1}^r \subset I$ satisfying the property

$$\sum_{k=1}^{r} |y_k - x_k| < \delta$$

the values of f at the endpoints of the intervals satisfy the inequality

$$\sum_{k=1}^{r} |f(y_k) - f(x_k)| < \varepsilon.$$

The collection of all absolutely continuous functions from I to $\mathbb{R}^n$ is denoted by $\mathrm{AC}(I, \mathbb{R}^n)$.

The following theorem is proved in [21], chapter 16, section E.

Theorem 2.15.8 (Fundamental Theorem of Calculus for the Lebesgue Integral)**.** *Let $I = [a, b] \subset \mathbb{R}$. A function $f : I \to \mathbb{R}$ is absolutely continuous if and only if there exists a function $g \in L^1(I, \mathbb{R})$ (i.e. $\int_I |g|dx < \infty$) such that*

$$f(x) = f(a) + \int_a^x g(t)\, dt$$

for all $x \in I$. Moreover, every absolutely continuous function $f(x) = f(a) + \int_a^x g(t)\, dt \in \mathrm{AC}(I, \mathbb{R})$ is differentiable almost everywhere on I, and $f' = g$.

Theorem 2.15.9 (Smooth Tietze Extension Theorem)**.** *Let $A \subset \mathbb{R}^n$ be a closed set, and let $g : A \to \mathbb{R}$ be a C^r function, where $r \in \mathbb{N} \cup \{\infty\}$. There exists a C^r function $G : \mathbb{R}^n \to \mathbb{R}$ such that $G|_A = g$. The function G is called a C^r* extension *of g.*

The proof of the general version of this theorem can be found in [1], theorem 5.5.9.

Chapter 3

DISTORTION DUE TO CHANGE OF VOLUME. MINIMAL BENDING AND MORPHING

We consider the problem of distortion minimal morphing of n-dimensional compact connected oriented smooth manifolds without boundary embedded in $\mathbb{R}^{n+1}$. Distortion involves bending and stretching. In this chapter we study the natural cost functional (for change of volume) defined in section 3.1 that measures the total relative change of volume produced by a diffeomorphisms $h \in \mathrm{Diff}(M, N)$. We develop the theory of minimal bending and morphing with respect to this functional. The existence of minimal distortion diffeomorphisms between diffeomorphic manifolds is proved in section 3.1. A definition of minimal distortion morphing between two isotopic manifolds is given, and the existence of minimal distortion morphs between every pair of isotopic embedded manifolds is proved in section 3.2. This functional is invariant under compositions with volume preserving diffeomorphisms; hence, the corresponding minimal maps and morphs are not unique. On the other hand, we prove that the extremals of our functional are (global) minimizers. The main result of this section is theorem 3.2.11, which states the existence of a distortion minimal morph (with respect to change of volume) between every pair of isotopic submanifolds.

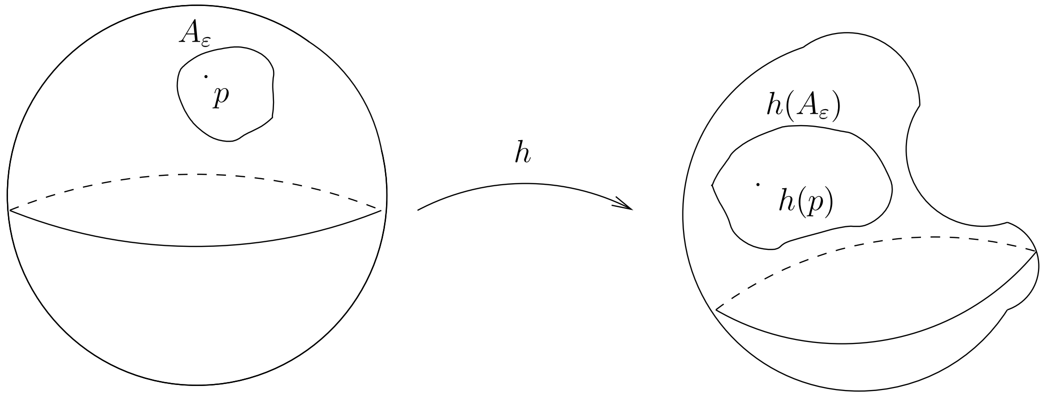

Figure 3.1: The map h changes the volume of the neighborhood A_ε.

3.1 Distortion (due to Change of Volume) Cost Functional. Existence of Minimizers

In this section we prove the existence of distortion minimal diffeomorphisms between diffeomorphic n-dimensional oriented manifolds M and N (which are not necessarily embedded in $\mathbb{R}^{n+1}$) with respective volume forms ω_M and ω_N.

Recall that the Jacobian determinant of a diffeomorphism $h : M \to N$ is defined by the equation $h^*\omega_N = J(\omega_M, \omega_N)(h)\,\omega_M$ (see definition 2.9.7).

The distortion (due to change of volume) $\xi(p)$ at $p \in M$, with respect to a diffeomorphism $h : M \to N$, is defined by

$$\xi(p) = \lim_{\varepsilon \to 0} \frac{\left| \int_{h(A_\varepsilon)} \omega_N \right| - \left| \int_{A_\varepsilon} \omega_M \right|}{\left| \int_{A_\varepsilon} \omega_M \right|} = \big| J(h)(p) \big| - 1, \tag{3.1}$$

where $A_\varepsilon \subset M$, for $\varepsilon > 0$, is a nested family of open neighborhoods of the point $p \in M$ such that $A_\beta \subseteq A_\alpha$ whenever $\alpha > \beta > 0$ and $\cap_{\varepsilon>0} A_\varepsilon = p$ (see figure 3.1).

In other words, the distortion is the infinitesimal relative change of volume with respect to h. It is easy to see that the definition of distortion does not depend on the family of nested sets A_ε.

We denote the set of all smooth diffeomorphisms between manifolds M and N by $\mathrm{Diff}(M, N)$. The *total distortion* functional $\Phi : \mathrm{Diff}(M, N) \to \mathbb{R}$, with respect to the oriented manifolds (M, ω_M) and (N, ω_N), is defined by

$$\Phi(h) = \int_M \Big(\big|J(h)\big| - 1 \Big)^2 \omega_M. \tag{3.2}$$

We will establish necessary and sufficient conditions for a diffeomorphism $h : M \to N$ to be a minimizer of the functional Φ. Also, we will show that a minimizer always exists in $\mathrm{Diff}(M, N)$.

Recall that the tangent space $T_h \mathrm{Diff}(M, N)$ can be identified with $\mathfrak{X}(N)$ (see subsection 2.13). More precisely, $\mathrm{Diff}(M, N)$ *is a Fréchet manifold and its tangent space at* $h \in \mathrm{Diff}(M, N)$ *can be identified with* $\mathfrak{X}(N)$ (see [25]). Indeed, an element of $T_h \mathrm{Diff}(M, N)$ is an equivalence class of curves $[h_\varepsilon]$, represented by a family of diffeomorphisms h_ε with $h_0 = h$, where two curves passing through h are equivalent if they have the same derivative at h. For each $q \in N$, this family defines a curve $\epsilon \mapsto h_\epsilon(h^{-1}(q))$ in N that passes through q at $\varepsilon = 0$; hence, it defines a vector $Y(q) \in T_q N$ by

$$Y(q) := \frac{d}{d\varepsilon} h_\epsilon(h^{-1}(q))\Big|_{\varepsilon=0}.$$

The vector field $Y \in \mathfrak{X}(N)$ is thus associated with the equivalence class $[h_\varepsilon]$. In fact, the vector field Y does not depend on the choice of the representative of the equivalence class. On the other hand, for $Y \in \mathfrak{X}(N)$ with flow ϕ_t, we associate the curve $h_t = \phi_t \circ h$ in $\mathrm{Diff}(M, N)$. The (tangent) equivalence class of this curve is an element in $T_h \mathrm{Diff}(M, N)$.

Proposition 3.1.1 (Euler-Lagrange Equation)**.** *Suppose that* (M, ω_M) *and* (N, ω_N) *are smooth diffeomorphic connected compact oriented* n*-manifolds without boundary. A smooth diffeomorphism* $h : M \to N$ *is a critical point of the total distortion functional* Φ *if and only if* $J(h)$ *is constant.*

Proof. Let $h_\varepsilon : (-1, 1) \to \mathrm{Diff}(M, N)$ be a curve of diffeomorphisms from M to N such that $h_0 = h$. By definition, $h \in \mathrm{Diff}(M, N)$ is a critical point of the functional

$\Phi(h)$, if $\frac{d}{dt}\Phi(h_t)\Big|_{t=0} = 0$. Using the formula

$$\Phi(h) = \int_M J(h)^2 \omega_M - 2\,\mathrm{Vol}(N) + \mathrm{Vol}(M), \tag{3.3}$$

we note that h is a critical point of Φ if and only if

$$2\int_M J(h)\frac{d}{dt}J(h_t)|_{t=0}\,\omega_M = 0.$$

Moreover, using the calculus of differential forms (see [1] and note in particular that L_Y is used to denote the Lie derivative in the direction of the vector field Y), we have that for $h^t = \psi_t \circ h$, where ψ_t is the flow of $Y \in \mathfrak{X}(N)$,

$$\begin{aligned}
\frac{d}{dt}(J(\psi_t \circ h)\omega_M)\Big|_{t=0} &= \frac{d}{dt}\big((\psi_t \circ h)^*\omega_N\big)\Big|_{t=0} \\
&= h^*\frac{d}{dt}(\psi_t^*\omega_N)\Big|_{t=0} \\
&= h^*\psi_t^* L_Y\omega_N\big|_{t=0} \\
&= h^* L_Y \omega_N \\
&= h^*(\mathrm{div}\, Y\omega_N) \\
&= (\mathrm{div}\, Y)\circ hJ(h)\omega_M.
\end{aligned}$$

We will assume that h is orientation preserving. The proof for the orientation reversing case is similar. By Stokes' theorem and the properties of the $\wedge$-antiderivations d and i_Y, we have that

$$\begin{aligned}
\frac{d}{dt}\Phi(h_t)\Big|_{t=0} &= \int_M J(h)^2\,\mathrm{div}\, Y\circ h\,\omega_M = \int_N J(h)\circ h^{-1}\,\mathrm{div}\, Y\omega_N \\
&= \int_N J(h)\circ h^{-1}L_Y\omega_N = \int_N J(h)\circ h^{-1} d\, i_Y\omega_N \\
&= \int_N d(J(h)\circ h^{-1}\wedge i_Y\omega_N) - \int_N d(J(h)\circ h^{-1})\wedge i_Y\omega_N \\
&= \int_N i_Y(d(J(h)\circ h^{-1})\wedge\omega_N) - \int_N i_Y\big(d(J(h)\circ h^{-1})\big)\,\omega_N \\
&= -\int_N d(J(h)\circ h^{-1})(Y)\,\omega_N.
\end{aligned}$$

Hence, $h \in \mathrm{Diff}(M,N)$ is a critical point of the functional $\Phi(h)$ if and only if

$$\int_N d(J(h)\circ h^{-1})(Y)\,\omega_N = 0$$

for all $Y \in \mathfrak{X}(N)$. It follows that if $J(h)$ is constant, then h is a critical point of Φ.

To complete the proof it suffices to show that if

$$\int_N df(Y)\,\omega_N = 0 \tag{3.4}$$

for all $Y \in \mathfrak{X}(N)$, then $df = 0$, where $f := J(h) \circ h^{-1}$.

Suppose, on the contrary, that there exists a continuous vector field $Y \in \mathfrak{X}(N)$ such that (without loss of generality) $df(Y)(q) > 0$ for some point $q \in N$. Because the map $df(Y) : N \to \mathbb{R}$ is continuous, there exists an open neighborhood $U \subset N$ of the point $q \in N$ such that $df(Y)(p) > 0$ for every $p \in U$. After multiplying the vector field Y by an appropriate bump function (see [1]), we obtain a vector field $Z \in \mathfrak{X}(N)$ supported in U such that $\int_N df(Z)\omega_N = \int_U df(Z)\omega_N > 0$, in contradiction to equality (3.4). Hence, $df = 0$. □

Definition 3.1.2. A function $h \in \mathrm{Diff}(M, N)$ is called a *distortion minimal map* if it is a critical point of the total distortion functional Φ.

We will show that every distortion minimal map is a minimizer of the functional Φ (see theorem 3.1.5).

As an immediate corollary of proposition 3.1.1, we have the following theorem.

Theorem 3.1.3. *A function $h \in \mathrm{Diff}(M, N)$ is a distortion minimal map if and only if $|J(h)|$ is the constant function with value $\mathrm{Vol}(N)/\mathrm{Vol}(M)$.*

We will use the elementary properties of distortion minimal maps stated in the following lemma. The proof is left to the reader.

Lemma 3.1.4. *Compositions and inverses of distortion minimal maps are distortion minimal maps.*

Also we will use (the strong form) of Moser's theorem on volume forms (see theorem 2.11.7 and [30]).

Theorem 3.1.5. *If (M, ω_M) and (N, ω_N) are diffeomorphic n-dimensional compact connected oriented manifolds without boundary, then (i) there is a distortion minimal map from M to N, (ii) every distortion minimal map from M to N minimizes the functional Φ, and (iii) the minimum value of Φ is*

$$\Phi_{min} = \frac{\big(\operatorname{Vol}(M) - \operatorname{Vol}(N)\big)^2}{\operatorname{Vol}(M)}. \tag{3.5}$$

Proof. To prove (i), choose a diffeomorphism $h \in \operatorname{Diff}(M, N)$ and note that the differential form $h^*\omega_N$ is a volume on M. Define a new volume on M as follows:

$$\bar{\omega}_M = \frac{\operatorname{Vol}(M)}{\int_M h^*\omega_N} h^*\omega_N.$$

Since

$$\int_M \bar{\omega}_M = \int_M \omega_M$$

and M is compact, by an application of Moser's theorem, there exists a smooth diffeomorphism $f : M \to M$ such that $\omega_M = f^*\bar{\omega}_M$. Hence,

$$\frac{\int_M h^*\omega_N}{\operatorname{Vol}(M)} \omega_M = (h \circ f)^*\omega_N;$$

and $|J(h \circ f)| = \operatorname{Vol}(N)/\operatorname{Vol}(M)$ is constant. Thus, $k = h \circ f$ is a distortion minimal map.

To prove parts (ii) and (iii), note that if k is an arbitrary distortion minimal map from M to N, then

$$\Phi(k) = \big(|J(k)| - 1\big)^2 \operatorname{Vol}(M) = \frac{\big(\operatorname{Vol}(M) - \operatorname{Vol}(N)\big)^2}{\operatorname{Vol}(M)}. \tag{3.6}$$

We claim that this value of Φ is its minimum.

Let $g \in \operatorname{Diff}(M, N)$. By the Cauchy-Schwartz inequality,

$$\begin{aligned} \int_M J(g)^2 \omega_M &\geq \frac{1}{\operatorname{Vol}(M)} \Big(\int_M |J(g)|\, \omega_M \Big)^2 \\ &= \frac{\operatorname{Vol}(N)^2}{\operatorname{Vol}(M)}. \end{aligned}$$

The latter inequality together with formulas (3.3) and (3.6) implies that $\Phi(g) \geq \Phi(k)$ as required. $\square$

Example 3.1.6. Let S_r and S_R be two-dimensional round spheres of radii r and R (respectively) centered at the origin in $\mathbb{R}^3$. Define $h : S_r \to S_R$ by $h(p) = (R/r)\,p$ for $p = (x, y, z) \in S_r$. We will show that h is a distortion minimal map.

Let ω_r (respectively, ω_R) be the standard volume forms on S_r (respectively, S_R) generated by the usual volume form on $\mathbb{R}^{n+1}$.

Using the parametrizations of S_r and S_R by spherical coordinates, it is easy to show that the Jacobian determinant of h is given by

$$J(\omega_r, \omega_R)(h)(m) = \frac{R^2}{r^2} = \frac{\operatorname{Vol}(S_R)}{\operatorname{Vol}(S_r)}$$

for all $m \in S_r$; hence, by theorem 3.1.5, h is a distortion minimal map.

Remark 3.1.7 (Harmonic maps). For $h \in \operatorname{Diff}(M, N)$, the distortion functional (3.2) has value

$$\Phi(h) = \int_M |J(h)|^2 \omega_M - 2\operatorname{Vol}(N) + \operatorname{Vol}(M).$$

Thus, it suffices to consider the minimization problem for the reduced functional Ψ given by

$$\Psi(h) = \int_M |J(h)|^2 \omega_M.$$

We note that if M and N are one-dimensional, then Ψ is the same as

$$\Psi(h) = \int_M |Dh|^2 \omega_M.$$

An extremal of this functional is called a harmonic map (see [12, 13, 14]). Thus, for the one-dimensional case, distortion minimal maps and harmonic maps coincide. On the other hand, there seems to be no obvious relationship in the general case.

3.2 Morphs of embedded manifolds

We will discuss a minimization problem for morphs of compact connected oriented n-manifolds without boundary embedded in $\mathbb{R}^{n+1}$.

3.2.1 Pairwise minimal morphs

Definition 3.2.1. Let M and N be isotopic compact connected smooth n-manifolds without boundary embedded in $\mathbb{R}^{n+1}$ such that M is oriented. A C^∞ isotopy $H : [0,1] \times M \to \mathbb{R}^{n+1}$ together with all the intermediate manifolds $M^t := H(t, M)$, equipped with the orientations induced by the maps $h^t = H(t, \cdot) : M \to M^t$ and the volume forms ω_t generated by the standard volume form on $\mathbb{R}^{n+1}$, is called a (smooth) *morph* from M to N. We denote the set of all morphs between manifolds M and N by $\mathcal{M}(M, N)$.

For simplicity, we will consider only morphs H such that $p \mapsto H(0, p)$ is the identity map. Each manifold $M^t = H(t, M)$ (with $M^0 = M$ and $M^1 = N$) is equipped with the volume form $\omega_t = i_{\eta_t}\Omega$, where

$$\Omega = dx_1 \wedge dx_2 \wedge \ldots \wedge dx_{n+1}$$

is the standard volume form on $\mathbb{R}^{n+1}$ and $\eta_t : M^t \to \mathbb{R}^{n+1}$ is the outer unit normal vector field on M^t with respect to the orientation induced by h^t and the usual metric on $\mathbb{R}^{n+1}$.

Definition 3.2.2. A morph H is *distortion pairwise minimal* (or, for brevity, *pairwise minimal*) if $h^{s,t} = h^t \circ (h^s)^{-1} : M^s \to M^t$ is a distortion minimal map for every s and t in $[0, 1]$. We denote the set of all distortion pairwise minimal morphs between manifolds M and N by $\mathcal{PM}(M, N)$.

By proposition 3.1.1 and theorem 3.1.5, a morph H is pairwise minimal if and only if each Jacobian determinant $J(\omega_s, \omega_t)(h^{s,t})$ is constant.

Proposition 3.2.3. *Let $M = M^0$ and $N = M^1$ be n-dimensional manifolds as in definition 5.1.1. A morph H between M and N is distortion pairwise minimal if and only if*

$$\frac{J(\omega_0, \omega_t)(h^t)(m)}{\mathrm{Vol}(M^t)} = \frac{1}{\mathrm{Vol}(M)} \tag{3.7}$$

for all $t \in [0,1]$ *and* $m \in M$.

Proof. Using lemma 3.1.4 and theorem 3.1.5, it suffices to prove that each map $h^t : M \to M^t$ is minimal if and only if the map (3.7) is constant. An application of theorem 3.1.3 finishes the proof. □

Proposition 3.2.4. *Let* M *and* N *be* n*-dimensional manifolds as in proposition 3.2.3. If there is a morph* G *from* M *to* N*, then there is a distortion pairwise minimal morph between* M *and* N.

Proof. Fix a morph G from M to N with the corresponding family of diffeomorphisms $g^t := G(t,\cdot)$, let $M^t := G(t,M)$, and consider the family of volume forms

$$\bar{\omega}_t = \frac{\mathrm{Vol}(M)}{\mathrm{Vol}(M^t)}(g^t)^*\omega_t$$

defined for $t \in [0,1]$. It is easy to see that

$$\int_M \bar{\omega}_t = \int_M \bar{\omega}_0;$$

hence, by Moser's theorem, there is a family of diffeomorphisms α^t on M such that $\omega_M = (\alpha^t)^*\bar{\omega}_t$. It follows that

$$(g^t \circ \alpha^t)^*\omega_t = \frac{\mathrm{Vol}(M^t)}{\mathrm{Vol}(M)}\omega_M;$$

therefore,

$$J(\omega_M, \omega_t)(g^t \circ \alpha^t)(m) = \frac{\mathrm{Vol}(M^t)}{\mathrm{Vol}(M)}$$

for all $m \in M$. The morph H defined by $H(t,p) := g^t \circ \alpha^t(p)$ for all $t \in [0,1]$ and $p \in M$ is the desired distortion pairwise minimal morph. □

3.2.2 Minimal morphs

We will define distortion minimal morphs of embedded connected oriented smooth n-manifolds without boundary.

For a morph H from M to N, let $E_{s,t}$ denote the total distortion of $h^{s,t}: M^s \to M^t$. We have that

$$
\begin{aligned}
E_{s,t} &= \int_{M^s} \Big(J(h^{s,t}) - 1\Big)^2 \omega_s \\
&= \int_M \Big(\frac{J(h^t)}{J(h^s)} - 1\Big)^2 J(h^s)\, \omega_M .
\end{aligned}
$$

By Taylor's theorem, $E_{s,t}$ has the representation

$$
E_{s,t} = E_{t,t} + \frac{d}{ds}(E_{s,t})\big|_{s=t}(s-t) + \frac{1}{2}\frac{d^2}{ds^2}(E_{s,t})\big|_{s=t}(s-t)^2 + O\big((s-t)^3\big).
$$

Note that $E_{t,t}$ and $\frac{d}{ds}(E_{s,t})\big|_{s=t}$ both vanish, and

$$
\frac{1}{2}\frac{d^2}{ds^2}(E_{s,t})\big|_{s=t} = \int_M \frac{\Big(\frac{d}{dt}J(h^t)\Big)^2}{J(h^t)}\omega_M .
$$

Definition 3.2.5. The *infinitesimal distortion* of a smooth morph H from M to N at $t \in [0,1]$ is

$$
\varepsilon^H(t) = \lim_{s \to t} \frac{E_{s,t}}{(s-t)^2} = \int_M \frac{\Big(\frac{d}{dt}J(h^t)\Big)^2}{J(h^t)}\omega_M .
$$

The *total distortion functional* $\Psi : \mathcal{M}(M,N) \to \mathbb{R}$ is defined by

$$
\Psi(H) = \int_0^1 \varepsilon^H(t)dt = \int_0^1 \left(\int_M \frac{\Big(\frac{d}{dt}J(h^t)\Big)^2}{J(h^t)}\omega_M \right) dt. \tag{3.8}
$$

Definition 3.2.6. A smooth morph is called a *distortion minimal morph* if it minimizes the functional Ψ.

Lemma 3.2.7. *For every morph $H \in \mathcal{M}(M,N)$ there exists a pairwise minimal morph $G \in \mathcal{PM}(M,N)$ such that $\Psi(G) \le \Psi(H)$. In particular, if $H \in \mathcal{M}(M,N)$ is a distortion minimal morph, then there exists a pairwise minimal morph $G \in \mathcal{PM}(M,N)$ such that $\Psi(H) = \Psi(G)$.*

Proof. Let $H \in \mathcal{M}(M,N)$ be a morph with the intermediate states $M^t = H(t,M)$. By proposition 3.2.4, there exists a pairwise minimal morph $G \in \mathcal{PM}(M,N)$ with the same intermediate states. The deformation energy of transition maps satisfies the inequality $E_{s,t}(H) \geq E_{s,t}(G)$ for all $s, t \in [0,1]$ because G is pairwise minimal. Therefore, $\varepsilon^H(t) \geq \varepsilon^G(t)$ for all $t \in [0,1]$, and, consequently,

$$\Psi(H) \geq \Psi(G) \tag{3.9}$$

as required.

If H is distortion minimal, the reverse inequality $\Psi(H) \leq \Psi(G)$ holds and $\Psi(H) = \Psi(G)$ as required. □

Corollary 3.2.8. *(i) The following inequality holds:*

$$\inf_{G \in \mathcal{PM}(M,N)} \Psi(G) \leq \inf_{H \in \mathcal{M}(M,N)} \Psi(H). \tag{3.10}$$

(ii) If there exists a minimizer F of the total distortion functional Ψ over the class $\mathcal{PM}(M,N)$, then F minimizes the functional Ψ over the class $\mathcal{M}(M,N)$ as well:

$$\Psi(F) = \min_{G \in \mathcal{PM}(M,N)} \Psi(G) = \min_{H \in \mathcal{M}(M,N)} \Psi(H). \tag{3.11}$$

Lemma 3.2.9. *The total distortion of a pairwise minimal morph H from M to N is*

$$\Psi(H) = \int_0^1 \frac{\left(\frac{d}{dt}\operatorname{Vol}(M^t)\right)^2}{\operatorname{Vol}(M^t)}\, dt. \tag{3.12}$$

Proof. The proof is an immediate consequence of formula (3.8) and proposition 3.2.3. □

Lemma 3.2.10. *The functional $\bar{\Psi}$ defined by*

$$\bar{\Psi}(\phi) = \int_0^1 \frac{\dot{\phi}^2}{\phi}\, dt \tag{3.13}$$

on the admissible set

$$Q = \left\{\phi \in C^2\big([0,1];(0,\infty)\big) : \phi(0) = \operatorname{Vol}(M), \phi(1) = \operatorname{Vol}(N)\right\}$$

attains its infimum

$$\inf_{\rho \in Q} \bar{\Psi}(\rho) = 4\left(\sqrt{\operatorname{Vol}(N)} - \sqrt{\operatorname{Vol}(M)}\right)^2 \tag{3.14}$$

at the element $\phi \in Q$ given by

$$\phi(t) = \Big[(\sqrt{\mathrm{Vol}(M)} - \sqrt{\mathrm{Vol}(N)}\,)t - \sqrt{\mathrm{Vol}(M)}\,\Big]^2.$$

Proof. The proof is a simple application of the Euler-Lagrange equation and the Cauchy-Schwarz inequality.

The Euler-Lagrange equation for the functional $\bar{\Psi}$ is

$$\frac{2\,\ddot{\phi}\,\phi - \dot{\phi}^2}{\phi^2} = 0.$$

Its solutions have the form

$$\xi(t) = (Ct + D)^2,$$

where the constants C and D must be chosen so that $\xi(0) = \mathrm{Vol}(M)$ and $\xi(1) = \mathrm{Vol}(N)$. Because $\bar{\Psi}(\xi) = 4C^2$, we determine the values $C = \sqrt{\mathrm{Vol}(M)} - \sqrt{\mathrm{Vol}(N)}$ and $D = -\sqrt{\mathrm{Vol}(M)}$ by eliminating the other possible choices of these constants that yield larger values of $\bar{\Psi}$. Hence, the function ϕ in the statement of the theorem is the solution of the Euler-Lagrange equation in Q that yields the smallest value of $\bar{\Psi}$.

By the Cauchy-Schwarz inequality, we have that

$$\begin{aligned} \bar{\Psi}(\eta) = \int_0^1 \frac{\dot{\eta}^2}{\eta}\,dt \quad &\geq \quad \Big(\int_0^1 \frac{\dot{\eta}}{\sqrt{\eta}}\,dt\Big)^2 \\ &= \quad 4(\sqrt{\mathrm{Vol}(M)} - \sqrt{\mathrm{Vol}(N)})^2 = \bar{\Psi}(\phi) \end{aligned}$$

for every $\eta \in Q$. Thus, the critical point ϕ in the statement of the lemma minimizes the functional $\bar{\Psi}$ on Q. $\square$

Using corollary 3.2.8 and lemma 3.2.10, we will minimize the total distortion energy functional Ψ over the set $\mathcal{M}(M, N)$ of all morphs.

Theorem 3.2.11. *Let M and N be two n-dimensional manifolds satisfying the assumptions of definition 5.1.1. If M and N are connected by a smooth morph, then there exists a distortion minimal morph. The minimal value of Ψ is*

$$\min_{H \in \mathcal{M}(M,N)} \Psi(H) = 4\big(\sqrt{\mathrm{Vol}(N)} - \sqrt{\mathrm{Vol}(M)}\big)^2. \tag{3.15}$$

Proof. Let G be a morph between M and N. Without loss of generality, we assume that G is pairwise minimal (see proposition 3.2.4). Set

$$H(t,m) = \lambda(t)G(t,m),$$

where $\lambda : [0,1] \to \mathbb{R}$ is to be determined.

Note that if $M^t = H(t,M)$ and $W^t = G(t,M)$, then

$$\mathrm{Vol}(M^t) = \int_M (h^t)^*\omega_M = \big[\lambda(t)\big]^n \int_M (g^t)^*\omega_M = \big[\lambda(t)\big]^n \,\mathrm{Vol}(W^t).$$

Let $\phi(t)$ be the minimizer of the auxiliary functional Ψ from lemma 3.2.10, and define

$$\lambda(t) = \Big[\frac{\phi(t)}{\mathrm{Vol}(W^t)}\Big]^{\frac{1}{n}}.$$

The volume of the intermediate state M^t is given by $\mathrm{Vol}(M^t) = \phi(t)$; therefore, by corollary 3.2.8 and lemma 3.2.10, the morph H minimizes the total distortion functional Ψ over the class $\mathcal{M}(M,N)$ and

$$\Psi(H) = 4\big(\sqrt{\mathrm{Vol}(N)} - \sqrt{\mathrm{Vol}(M)}\big)^2.$$

□

The next result provides a basic class of distortion minimal morphs.

Proposition 3.2.12. *Suppose that M is an n-dimensional manifold embedded in $\mathbb{R}^{n+1}$ that satisfies the assumptions of definition 5.1.1. If α is a positive real number and*

$$N := \{\alpha m : m \in M\},$$

then the morph given by the family of maps $h^t(m) = \lambda(t)m$, where

$$\lambda(t) = \mathrm{Vol}(M)^{-\frac{1}{n}}\Big[(\sqrt{\mathrm{Vol}(M)} - \sqrt{\mathrm{Vol}(N)}\,)t - \sqrt{\mathrm{Vol}(M)}\Big]^{\frac{2}{n}},$$

is distortion minimal.

Proof. Define $h^t(m) = \lambda(t)m$. It is easy to check that h^t defines a morph from M to N. Also, we have that $J(h^t) := J(\omega_M, \omega_t)(h^t) = \big[\lambda(t)\big]^n$. Since $J(h^t)$ is constant on M, the family h^t defines a pairwise minimal morph H.

We will determine $\lambda(t)$ so that the morph H becomes a minimizer of Ψ over the class $\mathcal{M}(M, N)$. Indeed, by lemma 3.2.10, it suffices to choose λ so that

$$\mathrm{Vol}(M^t) = \big[\lambda(t)\big]^n \,\mathrm{Vol}(M) = \Big[(\sqrt{\mathrm{Vol}(M)} - \sqrt{\mathrm{Vol}(N)}\,)t - \sqrt{\mathrm{Vol}(M)}\,\Big]^2,$$

which yields

$$\lambda(t) = \mathrm{Vol}(M)^{-\frac{1}{n}}\Big[(\sqrt{\mathrm{Vol}(M)} - \sqrt{\mathrm{Vol}(N)}\,)t - \sqrt{\mathrm{Vol}(M)}\,\Big]^{\frac{2}{n}}.$$

The corresponding morph $H(t, m) = \lambda(t)m$ satisfies the equality

$$\Psi(H) = \min_{G \in \mathcal{M}(M,N)} \Psi(G).$$

$\square$

Chapter 4

OPTIMIZATION OF DEFORMATION ENERGY

In this section we address the problem of optimization of deformation energy of diffeomorphisms and homotopies between smooth compact connected Riemannian n-manifolds M and N without boundary embedded in $\mathbb{R}^{n+1}$. As mentioned in chapter 3, the functional $\Phi(h) = \int_M \left(|J(h)| - 1\right)^2 \omega_M$ measures distortion only due to change of volume. In this chapter we consider a different deformation energy functional that measures total deformation of the manifold M produced by a diffeomorphism $h \in \mathrm{Diff}(M, N)$. In section 4.2 we compute the Euler-Lagrange equation for the new functional; in sections 4.3.1 and 4.3.3 we solve the problem of minimal bending and morphing for one-dimensional manifolds embedded in the plane. In section 4.4 we find a minimizer of the total deformation functional among all holomorphic diffeomorphisms between two-dimensional spheres.

4.1 Definition of Total Deformation Energy

Let M and N denote compact, connected and oriented n-manifolds without boundary that are embedded in $\mathbb{R}^{n+1}$ and equip them with the natural Riemannian metrics g_M and g_N inherited from the usual metric of $\mathbb{R}^{n+1}$. These Riemannian manifolds (M, g_M) and (N, g_N) have the volume forms ω_M and ω_N induced by their Riemannian metrics. We assume that M and N are diffeomorphic.

Definition 4.1.1. The strain tensor field $S \in \Gamma(TM^* \otimes TM^*)$ corresponding to $h \in \mathrm{Diff}(M,N)$ is defined to be

$$S = h^* g_N - g_M \tag{4.1}$$

(cf. [38], [29]).

Recall the natural bijection between covectors in T^*M and vectors in TM (see subsection 2.5): To each covector $\alpha_p \in T_pM^*$ assign the vector $\alpha_p^{\#} \in T_pM$ that is implicitly defined by the relation

$$\alpha_p = (g_M)_p(\alpha_p^{\#}, \cdot).$$

Using this correspondence, we introduce the Riemannian metric g_M^* on TM^* by

$$g_M^*(\alpha, \beta) = g_M(\alpha^{\#}, \beta^{\#}),$$

where the base points are suppressed [24].

There is a natural fiber metric G on $TM^* \otimes TM^*$ given by $G = g_M^* \otimes g_M^*$. Consider tensor fields $B, F \in \Gamma^\infty(TM^* \otimes TM^*)$ expressed in components by $B = b_{ij} dx^i \otimes dx^j$ and $F = f_{ij} dx^i \otimes dx^j$ (see the discussion following definition 2.5.7 and formula (2.2)). Then

$$\begin{aligned} G(B,F) &= b_{ij} f_{kl} g_M^*(dx^i, dx^k) g_M^*(dx^j, dx^l) \\ &= b_{ij} f_{kl} [g_M]^{ik} [g_M]^{jl}, \end{aligned} \tag{4.2}$$

where $[g_M]^{ik} = g_M^*(dx^i, dx^k)$ is the (i,k)-th entry of the inverse matrix of $\big([g_M]_{ik}\big)$, and the point $p \in M$ is suppressed.

Definition 4.1.2. The deformation energy functional $\Phi : \mathrm{Diff}(M,N) \to \mathbb{R}_+$ is defined to be

$$\Phi(h) = \int_M \|h^* g_N - g_M\|_G^2 \, \omega_M, \tag{4.3}$$

where $\|\cdot\|_G$ is the fiber norm on $TM^* \otimes TM^*$ induced by the fiber metric G.

The following invariance property of the functional Φ is obvious because the isometries of $\mathbb{R}^{n+1}$ are compositions of translations and rotations, which produce no deformations.

Lemma 4.1.3. *If $k \in \mathrm{Diff}(N)$ is an isometry of N (i.e. $k^* g_N = g_N$), then $\Phi(k \circ h) = \Phi(h)$.*

4.2 The First Variation

In this subsection we will compute the complete Euler-Lagrange equation for the functional $\Phi : \mathrm{Diff}(M, N) \to \mathbb{R}_+$ (see subsection 2.14.2). Let $c : (-\varepsilon, \varepsilon) \to \mathrm{Diff}(M, N)$ be a C^1 curve at $h \in \mathrm{Diff}(M, N)$, which we call a *variation* of h. The equivalence class $[c]_h \in T_h \mathrm{Diff}(M, N)$ can be identified with the smooth vector field $Y \in \mathfrak{X}(N)$ defined by $Y(q) = \frac{d}{dt} c(t) \circ h^{-1}(q)$ for all $q \in N$ (see subsection 2.13). We call the vector field $\mathfrak{X}(N)$ a *variational vector field* of $h \in \mathrm{Diff}(M, N)$. We intend to compute the first variation $d\Phi(h)Y$ for all $Y \in \mathfrak{X}(N)$.

Consider a smooth vector field $X \in \mathfrak{X}(M)$ with flow ϕ_t and a diffeomorphism $h \in \mathrm{Diff}(M, N)$, and suppose that the variation $c(t) = h \circ \phi_t$ induces the variational vector field $Y = h_* X \in \mathfrak{X}(N)$. The diffeomorphism h is a critical point of the functional Φ if and only if

$$\frac{d}{dt} \Phi(h \circ \phi_t)|_{t=0} = D\Phi(h) h_* X = 2 \int_M G(h^* g_N - g_M, L_X h^* g_N) \omega_M = 0 \tag{4.4}$$

for all $X \in \mathfrak{X}(M)$, where $G = g_M^* \otimes g_M^*$.

Let ∇ and $\bar{\nabla}$ be Riemannian connections on M compatible with the Riemannian metrics $\alpha = g_M$ and $\beta = h^* g_N$ respectively, and denote the corresponding Christoffel symbols of ∇ and $\bar{\nabla}$ by Γ^i_{jk} and $\bar{\Gamma}^i_{jk}$.

Let Y be a smooth vector field on M expressed in components by $Y = Y^k \frac{\partial}{\partial x^k}$. The components of the Lie derivative of the Riemannian metric β in the direction of the vector field Y are

$$[L_Y\beta]_{km} = \bar{\nabla}_k Y_m + \bar{\nabla}_m Y_k, \tag{4.5}$$

where $Y_m = [\beta]_{mj} Y^j$ are the lowered coordinates of Y via the Riemannian metric β (see [1]).

Recall that $\mathcal{T}^r{}_s(M)$ is the set of all continuous tensor fields on M contravariant of order r and covariant of order s, or type (r, s).

Definition 4.2.1. Recall that $\alpha = g_M$ and $\beta = h^* g_N$. (i) Define the tensor field $B = (\beta - \alpha)^{\#\#} \in \mathcal{T}^2{}_0(M)$. In other words, B equals the strain tensor field $h^* g_N - g_M$ with both indices raised via the Riemannian metric $\alpha = g_M$. Its components are given by $B^{km} = (\beta_{ij} - \alpha_{ij})\alpha^{ik}\alpha^{jm}$.

(ii) The bilinear form $A : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is defined by

$$A(X, Y) = \bar{\nabla}_X Y - \nabla_X Y \tag{4.6}$$

for all $X, Y \in \mathfrak{X}(M)$.

Remark 4.2.2. The bilinear form A can be viewed as a tensor field of type $(1, 2)$ on M with components

$$A^m{}_{kp} = \bar{\Gamma}^m_{kp} - \Gamma^m_{kp} \tag{4.7}$$

(see [24], proposition 7.10).

Recall that the divergence of a tensor field $\tau \in \mathcal{T}^r{}_s(M)$ is defined to be (see section 2.9)

$$\operatorname{div} \tau = C^r_{s+1}(\nabla \tau), \tag{4.8}$$

where C^j_i denotes the contraction in lower i and upper j index. The divergence of τ, $\operatorname{div} \tau$, is a tensor of type $(r - 1, s)$.

For two tensor fields $\theta \in \mathcal{T}^r{}_2(M)$ and $\tau \in \mathcal{T}^2{}_s(M)$, $\theta : \tau$ denotes the type (r, s) tensor field obtained by the contraction of the two covariant degrees of θ with the two contravariant degrees of τ (see section 2.5).

Lemma 4.2.3. *Let* $\Phi(h) = \int_M \|h^* g_N - g_M\|_G^2 \, \omega_M$ *with domain* $\mathrm{Diff}(M, N)$*. Then*

$$D\Phi(h)(h_* Y) = -4 \int_M g_M(\operatorname{div} B + A : B, Y)\omega_M \tag{4.9}$$

for all vector fields $Y \in \mathfrak{X}(M)$*, where the tensors* A *and* B *are defined above. Moreover,* h *is a critical point of the functional* Φ *if and only if*

$$\operatorname{div} B + A : B = 0.$$

The latter equation can be rewritten in components as follows:

$$\partial_k B^{km} + \Gamma^p_{kp} B^{km} + \bar{\Gamma}^m_{kp} B^{kp} = 0 \tag{4.10}$$

for all $m = 1, 2, \ldots, n$*.*

Proof. For given $Y \in \mathfrak{X}(M)$, consider the vector field

$$X = (\beta_{ij} - \alpha_{ij})\alpha^{ik}\alpha^{jm} Y_m \frac{\partial}{\partial x_k}, \tag{4.11}$$

where $\alpha = g_M$ and $\beta = h^* g_N$. Although we describe X pointwise using local coordinates, X is a well defined smooth vector field on M because it is obtained by various contractions of the tensor fields α, β, and Y. The divergence of the vector field X with respect to the Riemannian metric g_M can be expressed in terms of the components of X as follows:

$$\begin{aligned}
\operatorname{div}_{g_M} X &= \nabla_k X^k \\
&= \nabla_k\big((\beta_{ij} - \alpha_{ij})\alpha^{ik}\alpha^{jm}\big) Y_m + \big((\beta_{ij} - \alpha_{ij})\alpha^{ik}\alpha^{jm}\big)\nabla_k Y_m,
\end{aligned} \tag{4.12}$$

where ∇ is the Riemannian connection on M compatible with the metric g_M.

Using formula (2.6) for the components of the covariant derivative of a tensor, we see that $\nabla_k Y_m = \bar{\nabla}_k Y_m + (\bar{\Gamma}^l_{km} - \Gamma^l_{km}) Y_l$. Taking this into account, we rewrite equality (4.12) in the form

$$\operatorname{div} X = \big(\nabla_k B^{km} + B^{kl}(\bar{\Gamma}^m_{kl} - \Gamma^m_{kl})\big) Y_m + B^{km} \bar{\nabla}_k Y_m. \tag{4.13}$$

On the other hand, from equation (4.4)

$$D\Phi(h)(h_*Y) = 2\int_M G(\beta - \alpha, L_Y\beta)\omega_N. \tag{4.14}$$

The local representation of the integrand in the functional Φ is given by the expression $G(\beta - \alpha, L_Y\beta) = 2B^{km}\bar{\nabla}_k Y_m$. Now we can rewrite the divergence of the vector field X in the form

$$\operatorname{div}_{g_M} X = \frac{1}{2}G(\beta - \alpha, L_Y\beta) + g_M(\operatorname{div} B + S : B, Y). \tag{4.15}$$

By theorem 2.11.6, $\int_M \operatorname{div}_{g_M} X\omega_M = 0$. Using this and equality (4.14), we conclude that

$$D\Phi(h)(h_*Y) = -4\int_M g_M(\operatorname{div} B + S : B, Y)\omega_M$$

as required. □

Lemma 4.2.4. *Let $h \in \operatorname{Diff}(M, N)$. If $h^*g_N = R^2 g_M$ for some $R \in \mathbb{R}$, then h is a critical point of the functional Φ.*

Proof. As before, we denote $\alpha = g_M$, $\beta = h^*g_N$. We need to verify that the equation (4.10) holds. Because

$$\beta_{ij} = R^2\alpha_{ij}, \tag{4.16}$$

we obtain

$$\begin{aligned} \bar{\Gamma}^k_{ij} &= \frac{1}{2}\beta^{ck}[\beta_{ic,j} + \beta_{jc,i} - \beta_{ij,c}] \\ &= \frac{1}{2}\big(\frac{1}{R^2}\alpha^{ck}\big)[R^2\alpha_{ic,j} + R^2\alpha_{jc,i} - R^2\alpha_{ij,c}] = \Gamma^k_{ij}, \end{aligned} \tag{4.17}$$

where $\alpha_{ij,k}$ denotes $\partial_k\alpha_{ij}$. From equations (4.17) and (4.7) we conclude that $A = 0$. Hence, it remains to show that

$$\nabla_k B^{km} = 0 \tag{4.18}$$

for all $m = 1, \ldots, n$. From equation (4.16),

$$B^{km} = (R^2\alpha_{ij} - \alpha_{ij})\alpha^{ik}\alpha^{jm} = (R^2 - 1)\alpha^{km}.$$

Taking into account the identity $\nabla g_M = \nabla \alpha = 0$, we conclude that

$$\nabla_k B^{km} = (R^2 - 1)\nabla_k \alpha^{km} = 0$$

as required. $\square$

Corollary 4.2.5. *Let M be a compact, connected, and oriented smooth n-manifold without boundary embedded into $\mathbb{R}^{n+1}$. The radial map $h_R : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is defined by $h_R(p) = Rp$ for all $p \in \mathbb{R}^{n+1}$, where $R > 0$. Assume that $N = h_R(M)$ is a rescaled version of the manifold M, and the Riemannian metrics g_M and g_N on the manifolds M and N are inherited from $\mathbb{R}^{n+1}$. Then every composition $h = f \circ h_R|_M$ of the radial map $h_R|_M$ with an isometry $f \in \mathrm{Diff}(N)$ is a critical point of the functional Φ.*

Let the tensor field $t \in \mathcal{T}^0{}_2(M)$ be expressed by $t_{ij} dx^i \otimes dx^j$. We will use the following formula for the components of the Lie derivative $L_X t$ of t in the direction of the vector field X (cf. formula 2.5):

$$[L_X t]_{ij} = X^k \frac{\partial t_{ij}}{\partial x^k} + t_{kj} \frac{\partial X^k}{\partial x^i} + t_{ik} \frac{\partial X^k}{\partial x^j}. \tag{4.19}$$

4.3 Minimal Deformation Bending of Simple Closed Curves

4.3.1 First Variation. Minima Among Smooth Maps

In this section M and N are regular smooth simple closed curves in $\mathbb{R}^2$. Their arclengths are denoted by $L(M)$ and $L(N)$ respectively, and they have base points $p \in M$ and $q \in N$. We will determine the minimum of the functional

$$\Phi(h) = \int_M \|h^* g_N - g_M\|_G^2 \omega_M \tag{4.20}$$

over the admissible set

$$\mathcal{A} = \{h \in \mathrm{Diff}(M, N) : h(p) = q\}. \tag{4.21}$$

There exist unique arc length parametrizations $\gamma : [0, L(M)] \to M$ and $\xi : [0, L(N)] \to N$ of M and N respectively, which correspond to the positive orientations of the curves M and N in the plane, and are such that $\gamma(0) = p$ and $\xi(0) = q$. Because M and N are regular smooth curves, the functions $\gamma|_{(0,L(M))}$ and $\xi|_{(0,L(N))}$ are smooth diffeomorphisms. Notice that $[g_M]_{11}(t) = |\dot\gamma(t)|^2 = 1 = [g_M]^{11}(t)$ for $t \in [0, L(M)]$ and $[h^* g_N]_{11}(t) = |Dh\big(\gamma(t)\big)\dot\gamma(t)|^2$. Using formula (4.2) for the metric G, we rewrite functional (4.20) using local coordinates:

$$\Phi(h) = \int_0^{L(M)} \Big(\big|Dh\big(\gamma(t)\big)\dot\gamma(t)\big|^2 - 1\Big)^2 \omega_M. \tag{4.22}$$

Let us denote the local representation of a diffeomorphism $h \in \mathrm{Diff}(M, N)$ by $u = \xi^{-1} \circ h \circ \gamma$. The function u is a diffeomorphism on the open interval $\big(0, L(M)\big)$ and can be continuously extended to the closed interval $[0, L(M)]$ as follows. If h is orientation preserving, we extend u to a continuous function on $[0, L(M)]$ by defining $u(0) = 0$ and $u(L(M)) = L(N)$. In this case $\dot u > 0$. If h is orientation reversing, we define $u(0) = L(N)$ and $u(L(M)) = 0$.

Since

$$\big|\frac{d}{dt}(h \circ \gamma)(t)\big|^2 = \big|\frac{d}{dt}(\xi \circ u)(t)\big|^2 = \dot u^2(t)\big|\dot\xi\big(u(t)\big)\big|^2 = \dot u^2(t)$$

for $t \in \big(0, L(M)\big)$, the original problem of the minimization of functional (4.20) can be reduced to the minimization of the functional

$$\Psi(u) = \int_0^{L(M)} (\dot u^2 - 1)^2 dt \tag{4.23}$$

over the admissible sets

$$\mathcal{B} = \Big\{u \in C^\infty\big([0, L(M)], [0, L(N)]\big) : u(0) = 0, u(L(M)) = L(N)\Big\}$$

and

$$\mathcal{C} = \Big\{u \in C^\infty\big([0, L(M)], [0, L(N)]\big) : u(0) = L(N), u(L(M)) = 0\Big\}.$$

The minima will be shown to correspond to diffeomorphisms in $\mathrm{Diff}(M, N)$.

Lemma 4.3.1. *Suppose that $L(N) \geq L(M)$.*

(i) The function $v(t) = L(N)/L(M)t$, where $t \in [0, L(M)]$, is the unique minimizer of the functional Ψ over the admissible set $\mathcal{B}$.

(ii) The function $w(t) = -L(N)/L(M)t + L(N)$, where $t \in [0, L(M)]$, is the unique minimizer of the functional Ψ over the admissible set $\mathcal{C}$.

Proof. Since the proofs of (i) and (ii) are almost identical, we will only present the proof of the statement (i).

An Euler-Lagrange equation for functional (4.23) can be computed using formula (2.14):

$$4\ddot{u}(3\dot{u}^2 - 1) = 0. \tag{4.24}$$

The only solution of the above equation that belongs to the admissible set $\mathcal{B}$ is $v(t) = \frac{L(N)}{L(M)}t$, where $t \in [0, L(M)]$. Note that v corresponds to a diffeomorphism in $\mathrm{Diff}(M, N)$.

We will show that the critical point v minimizes the functional Ψ; that is,

$$\Psi(u) \geq \Psi(v) = \frac{(L(N)^2 - L(M)^2)^2}{L(M)^3} \tag{4.25}$$

for all $u \in \mathcal{B}$. Using Hölder's inequality

$$L(N) = u(L(M)) = \int_0^{L(M)} \dot{u}(s)\, ds \leq \big[L(M) \int_0^{L(M)} \dot{u}^2(s)\, ds\big]^{1/2},$$

we have that

$$\frac{L(N)^2}{L(M)} \leq \int_0^{L(M)} \dot{u}^2(s)\, ds.$$

Thus, in view of the hypothesis that $L(N) \geq L(M)$,

$$\begin{aligned} \int_0^{L(M)} (\dot{u}^2(s) - 1)\, ds &= \int_0^{L(M)} \dot{u}^2(s)\, ds - L(M) \\ &\geq \frac{L(N)^2 - L(M)^2}{L(M)} \geq 0. \end{aligned} \tag{4.26}$$

After squaring both sides of inequality (4.26), we obtain the inequality

$$\Big(\int_0^{L(M)} (\dot{u}^2(s) - 1)\, ds\Big)^2 \geq \frac{(L(N)^2 - L(M)^2)^2}{L(M)^2}. \tag{4.27}$$

Applying Hölder's inequality to $\Phi(u)$ and taking into account inequality (4.27), we obtain inequality (4.25). Hence, the function $v(t) = L(N)/L(M)t$, where $t \in [0, L(M)]$, minimizes the functional Ψ over the admissible set $\mathcal{B}$. □

Remark 4.3.2. Let us write the Euler-Lagrange equation (4.4) for the one-dimensional case and compare it with equation (4.24).

Recall that

$$[g_M]_{11}(t) = 1, \qquad [h^* g_N]_{11}(t) = \dot{u}(t)^2,$$

and use formula (4.19) to compute

$$[L_Y h^* g_N]_{11}(t) = 2\dot{u}(t)\big(\ddot{u}(t)y(t) + \dot{y}(t)\dot{u}(t)\big) = 2\dot{u}(t)\frac{d}{dt}\big(\dot{u}(t)y(t)\big),$$

where $y(t)$ is the local coordinate of the vector field $Y = y\frac{\partial}{\partial t}$. The function y is smooth on $(0, L(M))$ and satisfies the equation $y(0) = y(L(M))$. We assume that $y \in C_c^\infty\big((0, L(M))\big)$. Using the previous computation and formulas (4.2) and (4.4), we obtain the following Euler-Lagrange equation:

$$\int_0^{L(M)} (\dot{u}^2 - 1)\dot{u}\frac{d}{dt}(\dot{u}y)\, dt = -\int_0^{L(M)} \frac{d}{dt}\big((\dot{u}^2 - 1)\dot{u}\big)\dot{u}y\, dt = 0$$

for all $y \in C_c^\infty\big((0, L(M))\big)$. The latter equation yields

$$\frac{d}{dt}\big((\dot{u}^2 - 1)\dot{u}\big)\dot{u} = \dot{u}\ddot{u}(3\dot{u}^2 - 1) = 0, \tag{4.28}$$

which has the same solutions in the admissible sets $\mathcal{B}$ and $\mathcal{C}$ as equation (4.24).

Proposition 4.3.3. *Suppose that M and N are regular smooth simple closed curves in $\mathbb{R}^2$ with arc lengths $L(M)$ and $L(N)$ and base points $p \in M$ and $q \in N$; γ and ξ are arc length parametrizations of M and N with $\gamma(0) = p$ and $\xi(0) = q$ that induce positive orientations; and, the functions v and w are as in lemma 4.3.1. If*

$L(N) \geq L(M)$, then the functional $\Phi(h)$ defined in display (4.20) has exactly two minimizers in the admissible set

$$\mathcal{A} = \{h \in \mathrm{Diff}(M,N) : h(p) = q\} :$$

the orientation preserving minimizer

$$h_1 = \xi \circ v \circ \gamma^{-1}$$

and the orientation reversing minimizer

$$h_2 = \xi \circ w \circ \gamma^{-1}$$

(where we consider γ as a function defined on $\big[0, L(M)\big)$ so that $\gamma^{-1}(p) = 0$). Moreover, the minimal value of the functional Φ is the admissible set $\mathcal{A}$ is

$$\Phi^{\mathcal{A}}_{min} = \frac{(L(N)^2 - L(M)^2)^2}{L(M)^3}. \tag{4.29}$$

Corollary 4.3.4. *Suppose that M and N are regular smooth simple closed curves in $\mathbb{R}^2$ with arc lengths $L(M)$ and $L(N)$ respectively, where $L(M) \leq L(N)$. Let the functions v and w be defined as in lemma 4.3.1.*

(i) If a diffeomorphism $h \in \mathrm{Diff}(M,N)$ is deformation minimal, i.e. h minimizes the functional Φ in the admissible set $\mathrm{Diff}(M,N)$, then $h = \xi \circ v \circ \gamma^{-1}$ or $h = \xi \circ w \circ \gamma^{-1}$, where γ and ξ are (positive orientation) arc length parametrizations of the curves M and N respectively such that $\xi(0) = h(\gamma(0))$.

(ii) For every arc length parametrizations γ and ξ of M and N respectively, the functions $h_1 = \xi \circ v \circ \gamma^{-1}$ and $h_2 = \xi \circ w \circ \gamma^{-1}$ are deformation minimal. In addition,

$$\min_{f \in \mathrm{Diff}(M,N)} \Phi(f) = \Phi^{\mathcal{A}}_{min} = \frac{(L(N)^2 - L(M)^2)^2}{L(M)^3}.$$

Proof. (i) Suppose that $h \in \mathrm{Diff}(M,N)$ is such that $\Phi(h) \leq \Phi(f)$ for all $f \in \mathrm{Diff}(M,N)$. Fix $p \in M$, and set $q = h(p) \in N$. It is evident that h minimizes the deformation energy functional Φ over the admissible set $\mathcal{A} = \{f \in \mathrm{Diff}(M, N) : f(p) = q\}$. Therefore, by proposition 4.3.3, $h = \xi \circ v \circ \gamma^{-1}$ or $h = \xi \circ w \circ \gamma^{-1}$, where γ and ξ are (positive orientation) arc length parametrizations of M and N respectively such that $\gamma(0) = p$ and $\xi(0) = q$. In addition, $\Phi(h) = (L(N)^2 - L(M)^2)^2/L(M)^3$.

(ii) Let γ and ξ be arc length parametrizations of M and N. We will prove that the functions $h_1 = \xi \circ v \circ \gamma^{-1}$ and $h_2 = \xi \circ w \circ \gamma^{-1}$ are minimizers of Φ over the admissible set $\mathrm{Diff}(M, N)$. Note that

$$\Phi(h_1) = \Phi(h_2) = \frac{(L(N)^2 - L(M)^2)^2}{L(M)^3}. \tag{4.30}$$

Using inequality (4.25), for every diffeomorphism $h \in \mathrm{Diff}(M, N)$ with the local representation $u = \xi^{-1} \circ h \circ \gamma$, we find that

$$\Phi(h) = \Psi(u) \geq \Psi(v) = \frac{(L(N)^2 - L(M)^2)^2}{L(M)^3}. \tag{4.31}$$

Statement (ii) of the theorem follows from this inequality and equality (4.31).

□

Example 4.3.5. For $R > 0$, the radial map $h : \mathbb{R}^2 \to \mathbb{R}^2$ is defined to be $h(z) = Rz$. If M is a regular simple closed curve, $N := h(M)$ and $R > 1$, then $h|_M$ minimizes Φ on $\mathrm{Diff}(M, N)$. To see this fact, let $\gamma(t) = \big(x(t), y(t)\big)$, $t \in [0, L(M)]$, be an arc length parametrization of M. It is easy to see that $\xi(t) = R\big(x(t/R), y(t/R)\big)$, $t \in [0, RL(M)]$ parametrizes $N = h(M)$ by its arc length. By proposition 4.3.3, the minimizer h_1 is

$$\begin{aligned} h_1(z) = \xi\big(v \circ \gamma^{-1}(z)\big) &= \xi\big(R\gamma^{-1}(z)\big) \\ &= \xi(Rt) = R\gamma(t) = Rz \end{aligned}$$

for all $z \in M$. Hence, $h_1 = h|_M$ is the radial map as required.

Lemma 4.3.6. *If $L(N) < L(M)$, then the functional Ψ has no minimum in the admissible set $\mathcal{B}$.*

Proof. Let $\phi : [0, L(M)] \to \mathbb{R}$ be a continuous piecewise linear function such that $\phi(0) = 0$, $\phi(L(M)) = L(N)$, and $\dot{\phi}(t) = \pm 1$ whenever $t \in (0, L(M))$ and the derivative is defined. The graph of ϕ looks like a zig-zag. It is easy to see that ϕ is an element of the Sobolev space $W^{1,4}(0, L(M))$ (one weak derivative in the Lebesgue space L^4). By the standard properties of $W^{1,4}(0, L(M))$, whose usual norm we denote by $\|\cdot\|_{1,4}$, there exists a sequence of smooth functions $\phi_k \in C^\infty[0, L(M)]$ (satisfying the boundary conditions $\phi_k(0) = 0$ and $\phi_k(L(M)) = L(N)$) such that $\|\phi_k - \phi\|_{1,4} \to 0$ as $k \to \infty$. Moreover, there is some constant $C > 0$ such that $\int_0^{L(M)} (\dot{\phi}_k^2 - \dot{\phi}^2)^2 \, dx \le C \|\phi_k - \phi\|_{1,4}^2$. It is easy to see that

$$|\Phi(\phi_k) - \Phi(\phi)| \le C_1 \|\phi_k - \phi\|_{1,4}$$

for some constant $C_1 > 0$. Taking into account the equality $\Psi(\phi) = 0$, we conclude that $\Psi(\phi_k) \to 0$ as $k \to \infty$. Thus, $\{\phi_k\}_{k=1}^\infty$ is a minimizing sequence for the functional Ψ in the admissible set $\mathcal{B}$. On the other hand, there is no function $f \in \mathcal{B}$ such that $\Psi(f) = 0 = \inf_{g \in \mathcal{B}} \Psi(g)$. Therefore, if $L(N) < L(M)$, the functional Φ has no minimum in the admissible set $\mathcal{B}$. $\square$

Corollary 4.3.7. *If $L(N) < L(M)$, then the functional Φ has no minimum in the admissible set*

$$\mathcal{Q} = \{h \in C^\infty(M, N) : h \text{ is orientation preserving and } h(p) = q\}.$$

Let us interpret the result of Lemma 4.3.6. Let $h = \xi \circ \phi \circ \gamma^{-1}$, where $\phi : [0, L(M)] \to \mathbb{R}$ is defined in the proof of Lemma 4.3.6 and γ, ξ are arc length (positive orientation) parametrizations of the curves M and N viewed as periodic functions on $\mathbb{R}$. In case $L(N) < L(M)$, the action of the function h on the curve M can be described as follows. The curve M is cut into segments $\{M_i\}_{i=1}^k$, $k \in \mathbb{N}$, such that $\dot{\phi}$ has a constant value (1 or (-1)) on $\gamma^{-1}(M_i)$. Each segment M_i is wrapped around the curve N counterclockwise or clockwise depending on whether $\dot{\phi}$ equals 1 or (-1) on $\gamma^{-1}(M_i)$ respectively. Since $L(N)$ is less than $L(M)$, some points of N will be covered by segments of M several times. During this process, the segments of the curve M need not be stretched. Hence, as measured by the functional Φ, no strain is produced, i.e. $\Phi(h) = 0$.

The statement of corollary 4.3.7 leaves open an interesting question: Does the functional Φ have a minimum in the admissible set $\mathcal{A}$? Some results in this direction are presented in the next section.

4.3.2 Second variation. Conditions for Nonexistence of Minimum

We will derive a necessary condition for a diffeomorphism $h \in \mathrm{Diff}(M, N)$ to be a minimum of the functional Φ. Let $h_t = h \circ \phi_t$ be a family of diffeomorphisms in $\mathrm{Diff}(M, N)$, where ϕ_t is the flow of a vector field $Y \in \Gamma(TM)$. Using the Lie derivative formula (see [1]), we derive the equations $\frac{d}{dt}(h_t^* g_N) = \phi_t^* L_Y h^* g_N$ and $\frac{d}{dt}(\phi_t^* L_Y h^* g_N) = \phi_t^* L_Y L_Y h^* g_N$. If there exists $\delta > 0$ such that $\Phi(h_t) > \Phi(h)$ for all $|t| < \delta$ and for all variations h_t of h, then h is called a relative minimum of h. If $h \in \mathrm{Diff}(M, N)$ is a relative minimum of Φ, then $\frac{d^2}{dt^2}\Phi(h_t)|_{t=0} > 0$.

Using the previous computations of Lie derivatives, the second variation of Φ is

$$\begin{aligned} \frac{1}{2}\frac{d^2}{dt^2}\Phi(h_t)|_{t=0} &= \int_M G(L_Y h^* g_N, L_Y h^* g_N)\omega_M \\ &+ \int_M G(L_Y L_Y h^* g_N, h^* g_N - g_M)\omega_M. \end{aligned} \tag{4.32}$$

Lemma 4.3.8. *Let M and N be regular simple closed curves parametrized by functions γ and ξ satisfying all the properties stated in lemma 4.3.3. If $h \in \mathrm{Diff}(M, N)$ minimizes the functional Φ in the admissible set $\mathcal{A}$, then the local representation $u = \xi^{-1} \circ h \circ \gamma$ of h satisfies the inequality*

$$\dot{u}^2(t) \geq \frac{1}{3} \tag{4.33}$$

for all $t \in \big(0, L(M)\big)$.

Proof. Using formula (4.19), we compute

$$[L_Y h^* g_N]_{11} = 2(\dot{u}\ddot{u}y + \dot{u}^2\dot{y})$$

and

$$[L_Y L_Y h^* g_N]_{11} = 2(\ddot{u}^2 y^2 + \dot{u}\,\dddot{u}\,y^2 + 5\dot{u}\,\ddot{u}\,\dot{y}\,y + \dot{u}^2\ddot{y}\,y + 2\dot{u}^2\dot{y}^2).$$

Substituting the latter expressions into formula (4.32), we obtain the necessary condition

$$\begin{aligned} W := 4\int_0^{L(M)} \dot{u}^4\,\dot{y}^2\,dt \quad &+ \quad 4\int_0^{L(M)} \dot{u}^2(\dot{u}^2-1)\,\dot{y}^2\,dt \\ &+ \quad 2\int_0^{L(M)} \dot{u}^2(\dot{u}^2-1)\,y\,\ddot{y}\,dt + \ldots \geq 0, \end{aligned}$$

where the integrands of the omitted terms all contain the factor y. After integration by parts, we obtain the inequality

$$\begin{aligned} W = \int_0^{L(M)} \Big(4\dot{u}^4 \quad &+ \quad 4\dot{u}^2(\dot{u}^2-1) \\ &- \quad 2\dot{u}^2(\dot{u}^2-1)\Big)\dot{y}^2\,dt + \ldots \geq 0. \end{aligned} \tag{4.34}$$

Define $y(t) = \varepsilon\rho\big(\frac{t}{\varepsilon}\big)\zeta(t)$, where $\rho(t)$ is a periodic "zig-zag" function defined by the expressions

$$\rho(t) = \begin{cases} t, & \text{if } 0 \leq t < 1/2, \\ 1-t, & \text{if } 1/2 \leq t < 1, \end{cases} \tag{4.35}$$

and $\rho(t+1) = \rho(t)$, $\zeta \in C_c^\infty\big(0, L(M)\big)$. Notice that $\dot{\rho}^2 = 1$ almost everywhere on $\mathbb{R}$ and $\dot{y}^2 = \zeta^2 + O(\varepsilon)$ when $\varepsilon \to 0$. Substitute y into inequality (4.34) and pass to the limit as $\varepsilon \to 0$. All the omitted terms in the expression for W tend to zero because they contain y as a factor. Hence, we have the inequality

$$W \quad = \quad \int_0^{L(M)} \big(4\dot{u}^4 + 2\dot{u}^2(\dot{u}^2-1)\big)\zeta^2\,dt \geq 0,$$

which (after a standard bump function argument) reduces to the inequality

$$\dot{u}^2 \geq 1/3 \tag{4.36}$$

as required. □

Proposition 4.3.9. *If M and N are simple closed curves such that their corresponding arc lengths $L(M)$ and $L(N)$ satisfy the inequality $\frac{L(N)}{L(M)} < \frac{1}{\sqrt{3}}$, then the functional Φ has no minimum in the admissible set $\mathcal{A}$.*

Proof. If $h \in \mathrm{Diff}(M, N)$ is a minimum of the functional Φ, then h satisfies the Euler-Lagrange equation (4.4). Let γ and ξ be parametrizations of the curves M and N with all the properties stated in corollary 4.3.3. By remark 4.3.2, the local representation $u = \xi^{-1} \circ h \circ \gamma$ of h satisfies the ordinary differential equation (4.28) on $(0, L(M))$. In addition, u must satisfy the boundary conditions $u(0) = 0, u(L(M)) = L(N)$ or $u(0) = L(N), u(L(M)) = 0$. Hence, either $u(t) = L(N)/L(M)t$ or $u(t) = -L(N)/L(M)t + L(N)$. Since h minimizes Φ, by lemma 4.3.8 $\dot{u}^2 \geq 1/3$, or, equivalently, $L(N)/L(M) \geq \frac{1}{\sqrt{3}}$. This contradicts the assumption of the theorem. $\square$

4.3.3 Minimal Deformation Morphing of Curves

Recall the definition of a morph between two simple closed curves M and N. In this section, we assume that there is a morph between the curves M and N, and denote the set of all morphs between them by $\mathcal{M}(M, N)$ as before. We assume that each intermediate state $M^t = H(t, M)$ is a regular simple closed curve in $\mathbb{R}^2$ equipped with the Riemannian metric g_t, which is inherited from the standard inner product in $\mathbb{R}^2$. Each intermediate state M^t is equipped with the volume form ω_t induced by the Riemannian metric g_t (see proposition 2.9.6). Recall that the transition maps of a morph $H \in \mathcal{M}(M, N)$ are defined to be $h^{s,t} = h^t \circ (h^s)^{-1} : M^s \to M^t$.

Let us define the deformation energy of morphing. For every $s, t \in [0, 1]$, let $\Phi^{s,t} : \mathrm{Diff}(M^s, M^t) \to \mathbb{R}_+$ be the functional given by

$$\Phi^{s,t}(f) = \int_{M^s} \|f^* g_t - g_s\|^2 \omega_s.$$

Definition 4.3.10. The deformation energy of a transition map $h^{s,t}$ of a morph $H \in \mathcal{M}(M, N)$ is $E^{s,t}(H) := \Phi^{s,t}(h^{s,t})$. The infinitesimal deformation energy of a morph H at a point $s \in [0,1]$ is defined to be

$$\varepsilon_H(s) = \lim_{t \to s} \frac{E^{s,t}(H)}{(s-t)^2}. \tag{4.37}$$

Remark 4.3.11. Note that by the definition $\varepsilon_H(s) \geq 0$ for all $s \in [0,1]$.

We will verify (in lemma 4.3.13) that the above limit exists, and we will compute its value.

Let us construct the arc length local representation of a morph $H(t,p) = h^t(p)$ between curves M and N, where $(t,p) \in [0,1] \times M$. As in definition 5.1.1, we will denote the intermediate states of the morph H by M^t. Let us choose the following parametrizations of the intermediate curves M^t. Let γ^0 parametrize $M = M^0$ by its arc length, induce positive orientation, and satisfy $\gamma^0(0) = p_0$ for a fixed point $p_0 \in M$. Note that γ^0 is unique. For each $t \in [0,1]$, the function $h^t \circ \gamma^0$ parametrizes the curve M^t. Let γ^t be a reparametrization of $h^t \circ \gamma^0$ such that γ^t parametrizes M^t by arc length and gives the curve M^t the same orientation. In addition, we assume that $\gamma^t(0) = h^t \circ \gamma^0(0)$. The function γ^t can be obtained by the Implicit Function Theorem. Indeed, the arc length of the curve M^t is given by the formula $s(t,x) = \int_0^x \left|\frac{d}{d\tau} h^t \circ \gamma^0(\tau)\right| d\tau$. Since $\frac{\partial s}{\partial x} > 0$, the equation $s(t,x) = y$ can be solved for x as a smooth function of t and y. Set $\gamma^t = h^t \circ \gamma^0 \circ x(t,\cdot)$. Notice that the function $\bar{\gamma}(t,x) = \gamma^t(x)$ is smooth in t and compute $\frac{\partial}{\partial y}\gamma^t(y) = \frac{\partial}{\partial x}(h^t \circ \gamma^0)(x(t,y))\frac{\partial x}{\partial y}(t,y)$. But

$$\frac{\partial x}{\partial y}(t,y) = \frac{1}{|\frac{\partial}{\partial x}(h^t \circ \gamma^0)(x(t,y))|} > 0$$

for all $y \in (0, L(M^t))$. Therefore, both γ^t and $h^t \circ \gamma^0$ induce the same orientation on M. In addition, $\gamma^t(0) = h^t \circ \gamma^0(0)$ as required. The local representation G of the morph H is given by the formula $G(t,x) = (\gamma^t)^{-1} \circ H(t, \gamma^0(x))$ for all $x \in (0, L(M))$ and $t \in [0,1]$ or, equivalently, $g^t(x) = (\gamma^t)^{-1} \circ h^t \circ \gamma^0$. The function $G(t,x)$ is smooth jointly in the variables t and x. Note that $g^{s,t} := g^t \circ (g^s)^{-1} = (\gamma^t)^{-1} \circ h^{s,t} \circ \gamma^s$ is the local representation of $h^{s,t}$.

Definition 4.3.12. The above constructed local representation G of the morph $H \in \mathcal{M}(M,N)$ is called the *arc length local representation* of the morph H.

Lemma 4.3.13. *If H is a smooth morph in $\mathcal{M}(M,N)$ with arc length local representation G, then the infinitesimal deformation energy $\varepsilon_H(s)$ exists for all $s \in (0,1)$ and is given by*

$$\varepsilon_H(s) = 4\int_0^{L(M)} \frac{G_{xt}^2(s,x)}{G_x(s,x)}\,dx.$$

Proof. In local coordinates,

$$E^{s,t}(H) = \int_0^{L(M^s)} \left((\dot{g}^{s,t})^2 - 1\right)^2 dx,$$

where the dot denotes differentiation with respect to x. After the change of variables $x = g^s(y)$ we obtain

$$\begin{aligned} E^{s,t}(H) &= \int_0^{L(M)} \left((\dot{g}^{s,t} \circ g^s)^2 - 1\right)^2 \dot{g}^s dx \\ &= \int_0^{L(M)} \left(\Big(\frac{\dot{g}^t}{\dot{g}^s}\Big)^2 - 1\right)^2 \dot{g}^s\,dx \\ &= \int_0^{L(M)} \left(\frac{G_x^2(t,x)}{G_x^2(s,x)} - 1\right)^2 G_x(s,x)\,dx. \end{aligned}$$

It is easy to check that $E^{s,s}(H) = 0$ and

$$\frac{d}{dt}E^{s,t}(H) = 4\int_0^{L(M)} \left(\frac{G_x^2(t,x)}{G_x^2(s,x)} - 1\right)\frac{1}{G_x(s,x)} G_x(t,x)G_{xt}(t,x)\,dx.$$

Therefore, $\frac{d}{dt}E^{s,t}(H)\big|_{t=s} = 0$ and

$$\frac{d^2}{dt^2}E^{s,t}(H)\big|_{t=s} = 8\int_0^{L(M)} \frac{G_{xt}^2(s,x)}{G_x(s,x)}\,dx.$$

Using the Taylor series expansion around the point $t = s$ for the function $t \mapsto E^{s,t}(H)$, it is easy to see that

$$\begin{aligned} \varepsilon_H(s) &= \frac{1}{2}\frac{d^2}{dt^2}E^{s,t}(H)\big|_{t=s} \\ &= 4\int_0^{L(M)} \frac{G_{xt}^2(s,x)}{G_x(s,x)}\,dx. \end{aligned}$$

$\square$

Lemma 4.3.14. *If H is a smooth morph in $\mathcal{M}(M,N)$, then the infinitesimal deformation energy $\varepsilon_H(s)$ exists for all $s \in (0,1)$ and is given by*

$$\varepsilon_H(s) = \lim_{t\to s} \frac{E^{t,s}(H)}{(s-t)^2}. \tag{4.38}$$

Proof. It is easy to check that

$$\begin{aligned} \lim_{t\to s} \frac{E^{t,s}(H)}{(s-t)^2} &= \frac{1}{2}\frac{d^2}{dt^2}E^{t,s}(H)\Big|_{t=s} \\ &= 4\int_0^{L(M)} \frac{G_{xt}^2(s,x)}{G_x(s,x)}\,dx = \varepsilon_H(s). \end{aligned}$$

□

Definition 4.3.15. The quantity

$$\Lambda(H) = \frac{1}{4}\int_0^1 \varepsilon_H(t)\,dt \tag{4.39}$$

is called the *total deformation energy of a morph H* between manifolds M and N.

Remark 4.3.16. If G is the arc length local representation of H, then

$$\Lambda(H) = \int_0^1 \int_0^{L(M)} \frac{G_{xt}^2(t,x)}{G_x(t,x)}\,dx\,dt. \tag{4.40}$$

Example 4.3.17. Let M and N be regular simple closed curves such that $L(M) = L(N)$. We will construct a minimizer of the deformation energy Λ in the admissible set $\mathcal{M}(M,N)$.

Let F be a morph between the curves M and N. Such a morph exists (see [16]). We will rescale F so that the lengths of the intermediate curves remain constant. Define the morph $H(t,x) = \lambda(t)F(t,x)$. Using the notation $W^t = F(t,M)$ and $M^t = H(t,M)$, we find that the length of the intermediate curve M^t is

$$L(M^t) = \lambda(t)L(W^t).$$

Hence, we set $\lambda(t) = \frac{L(M)}{L(W^t)}$ so that $L(M^t) = L(M)$ for all $t \in [0,1]$.

Fix the intermediate curves M^t and their arc length parametrizations γ^t such that $\gamma^t(0) = H(t,\gamma^0(0))$, and define the morph $Q(t,\cdot) = q^t = \gamma^t \circ (\gamma^0)^{-1}$ (this definition is valid because $L(M^t) = L(M)$). The local representation of the functions $q^t = Q(t, \cdot)$ is the identity. Hence, $\Lambda(Q) = 0$. On the other hand, $\Lambda(Y) \geq 0$ for all morphs Y between the curves M and N, which follows from remark 4.3.11. We conclude that Q minimizes the functional Λ over the admissible set $\mathcal{M}(M, N)$ as required.

Definition 4.3.18. A morph H between manifolds M and N is called *orientation preserving (orientation reversing)* if the maps $h^t = H(t, \cdot)$ are orientation preserving (orientation reversing) for all $t \in [0, 1]$ (see definition 2.9.5). We denote the set of all orientation preserving (orientation reversing) morphs between M and N by $\mathcal{M}^+(M, N)$ ($\mathcal{M}^-(M, N)$).

Remark 4.3.19. For a morph $H \in \mathcal{M}(M, N)$, all the diffeomorphisms $h^t : M \to M^t$ are either orientation preserving or orientation reversing. Therefore, every morph $H \in \mathcal{M}(M, N)$ is either orientation preserving or orientation reversing.

Lemma 4.3.20. *For every orientation preserving morph $H^+ \in \mathcal{M}^+(M, N)$ there exists an orientation reversing morph $H^- \in \mathcal{M}^-(M, N)$ with the same intermediate states such that $\Lambda(H^-) = \Lambda(H^+)$ and vice versa.*

Proof. Let $H^+ \in \mathcal{M}^+(M, N)$ with the arc length local representation $G^+(t, x) = (\gamma^t)^{-1} \circ H^+(t, \gamma^0(x))$ for all $t \in [0, 1]$ and $x \in (0, L(M))$ (see definition 4.3.12). Let $\bar{\gamma}^t(x) = \gamma^t(L(M^t) - x)$ for all $x \in (0, L(M^t))$. Define $H^-(t, p) = \bar{\gamma}^t \circ (\gamma^t)^{-1} \circ H^+(t, p)$ for all $p \in M$ and $t \in [0, 1]$. Let us construct the arc length local representation of H^-. It is easy to see that for each $t \in [0, 1]$ the function $\bar{\gamma}^t : (0, L(M^t)) \to M^t$ is the arc length parametrization of M^t that induces the same orientation on M^t as $H^-(t, \cdot)$ and satisfies $\bar{\gamma}^t(0) = H^-(t, \gamma^0(0))$. Therefore, the arc length local representation of H^- is given by $G^-(t, x) = (\bar{\gamma}^t)^{-1} \circ H^-(t, \gamma^0(x)) = G^+(t, x)$ for all $t \in [0, 1]$ and $x \in (0, L(M))$. The statement of the lemma follows from formula (4.40). $\square$

Definition 4.3.21. Let M and N be regular simple closed curves in $\mathbb{R}^2$ such that $L(M) \leq L(N)$. A morph H between M and N satisfying the property $L(M^s) \leq$

$L(M^t)$ for all $s, t \in [0,1]$ such that $s \leq t$ is called a *directional morph.* A directional morph H is called *directional pairwise minimal* if the function $h^{s,t} : M^s \to M^t$ minimizes the functional $\Phi^{s,t}$ over the admissible set $\mathrm{Diff}(M^s, M^t)$ for all $s, t \in [0,1]$ such that $s \leq t$. We denote the set of all directional pairwise minimal morphs between M and N by $\mathcal{PM}_d(M, N)$.

For curves M and N such that $L(M) \leq L(N)$, consider the problem of minimization of the total deformation energy $\Lambda(H)$ over the class

$$\mathcal{Q} \;=\; \{H \in \mathcal{M}(M,N) : H \text{ is an orientation preserving directional morph}\}.$$

Definition 4.3.22. A morph $H \in \mathcal{Q}$ is called *directional deformation minimal* if it minimizes the total deformation functional Λ over the admissible class $\mathcal{Q}$.

Lemma 4.3.23. *Let M and N be regular simple closed curves such that $L(M) \leq L(N)$. There exists a directional pairwise minimal morph $H \in \mathcal{Q}$ between M and N.*

Proof. Let H be a morph between M and N with the intermediate curves W^t and the arc length local parametrization $G(t,x) = (\gamma^t)^{-1} \circ H(t, \gamma^0(x))$, where $t \in [0,1]$ and $x \in (0, L(M))$. Without loss of generality, H is orientation preserving. Indeed, by remark 4.3.19, H is either orientation preserving or orientation reversing. In the latter case, there exists an orientation preserving morph H^+ between M and N by lemma 4.3.20.

We will rescale H in order to obtain a directional morph. Consider the morph $F(t,p) = \lambda(t) H(t,p)$ for $(t,p) \in [0,1] \times M$ with intermediate states $M^t = F(t, M)$ and denote $\phi(t) = L(W^t)$. Because $L(M^t) = \lambda(t)\phi(t)$, we wish to find a smooth function $\lambda : [0,1] \to \mathbb{R}_+$ such that $\lambda(0) = \lambda(1) = 1$ and

$$\frac{d}{dt} L(M^t) = \frac{d}{dt}(\lambda(t)\phi(t)) \geq 0$$

for all $t \in (0,1)$. Let $\xi : [0,1] \to \mathbb{R}_+$ be a smooth nonnegative function such that

$\int_0^1 \xi(s)\, ds = L(N) - L(M)$. It is easy to check that the function

$$\lambda(t) = \frac{1}{\phi(t)}\Big(\int_{\frac{1}{2}}^{t} \xi(s)\, ds + L(M) + \int_0^{\frac{1}{2}} \xi(s)\, ds\Big)$$

satisfies the required conditions. Therefore, the morph $F = \lambda H$ is a directional orientation preserving morph.

We fix the intermediate states $M^t = F(t, M)$ and construct another directional orientation preserving morph $Y \in \mathcal{Q}$, which is directional pairwise minimal. Define the family of functions $z^t(x) = \frac{L(M^t)}{L(M)}x$ for all $x \in (0, L(M))$, where $t \in [0, 1]$. We set $Y(t, p) = \xi^t \circ z^t \circ (\xi^0)^{-1}$, where $\xi^t : (0, L(M)) \to M^t$ are the arc length parametrizations of the intermediate curves M^t associated with the arc length local representation of F (see definition 4.3.12). Recall that $y^t = Y(t, \cdot)$, and notice that the transition functions $y^{s,t} = y^t \circ (y^s)^{-1} : M^s \to M^t$ of the morph Y have the local representation

$$z^{s,t}(x) := (\xi^t)^{-1} \circ y^{s,t} \circ \xi^s = z^t \circ (z^s)^{-1}(x) = \frac{L(M^t)}{L(M^s)}x$$

for all $x \in (0, L(M^s))$. Because F is a directional morph, the inequality $L(M^s) \leq L(M^t)$ holds for all $s, t \in [0, 1]$ such that $s \leq t$. By corollary 4.3.4, each function $y^{s,t}$ minimizes the deformation energy functional $\Phi^{s,t}$ over the set $\mathrm{Diff}(M^s, M^t)$, where $s, t \in [0, 1]$ are such that $s \leq t$. Therefore, the morph Y is directional pairwise minimal. $\square$

Recall that $f \in \mathrm{Diff}(M, N)$ is deformation minimal if f minimizes the deformation energy functional Φ over the admissible set $\mathrm{Diff}(M, N)$.

Lemma 4.3.24. *Let M,N and S be regular smooth simple closed curves in $\mathbb{R}^2$ such that their arc lengths satisfy the inequality $L(M) \leq L(N) \leq L(S)$. If the functions $f : M \to N$ and $g : N \to S$ are deformation minimal, so is the function $g \circ f : M \to S$.*

Proof. The proof follows immediately from corollary 4.3.4. $\square$

Theorem 4.3.25. *For every morph $H \in \mathcal{Q}$ there exists a directional pairwise minimal morph $F \in \mathcal{Q} \cap \mathcal{PM}_d(M, N)$ with the same intermediate states such that $\Lambda(F) \leq \Lambda(H)$.*

Proof. Let $H \in \mathcal{Q}$. By the proof of lemma 4.3.23, there exists a morph $F \in \mathcal{Q} \cap \mathcal{PM}_d(M, N)$ with the same intermediate states. Therefore, $E^{s,t}(H) \geq E^{s,t}(F)$ for all $s, t \in [0, 1]$ such that $s \leq t$, which implies the inequality

$$\lim_{s \to t-} \frac{E^{s,t}(H)}{(s-t)^2} \geq \lim_{s \to t-} \frac{E^{s,t}(F)}{(s-t)^2}$$

for all $t \in (0, 1]$. Hence, $\varepsilon_H(t) \geq \varepsilon_F(t)$ for all $t \in (0, 1]$, and, consequently, $\Lambda(H) \geq \Lambda(F)$ as required. □

Corollary 4.3.26. *The following statements hold.*

(i) $\inf_{H \in \mathcal{Q}} \Lambda(H) \geq \inf_{F \in \mathcal{Q} \cap \mathcal{PM}_d(M,N)} \Lambda(F)$;

(ii) If F minimizes the total deformation functional Λ over the class $\mathcal{Q} \cap \mathcal{PM}_d(M, N)$ of all orientation preserving directional pairwise minimal morphs, then F is deformation minimal, i.e. F minimizes Λ in the admissible set $\mathcal{Q}$.

Lemma 4.3.27. *If $H \in \mathcal{Q}$ is directional pairwise minimal, then*

$$\Lambda(H) = \int_0^1 \frac{(\frac{d}{dt} L(M^t))^2}{L(M^t)} \, dt. \tag{4.41}$$

Proof. If $H \in \mathcal{Q}$ is directional pairwise minimal, then its arc length local representation $G(t, x) = \frac{L(M^t)}{L(M)} x$ for all $t \in [0, 1]$ and $x \in [0, L(M^t)]$. Formula (4.41) follows from equation (4.40). □

Lemma 4.3.28. *Consider the functional*

$$J(\phi) = \int_0^1 \frac{\dot{\phi}^2}{\phi} \, dt$$

defined on the admissible set

$$\mathcal{W} = \{\phi \in \mathbb{C}^\infty([0, 1], \mathbb{R}_+) : \phi(0) = L(M), \phi(1) = L(N), \dot{\phi} \geq 0\}.$$

The functional J attains its minimum at the constant function

$$\phi(t) \equiv L(M) \tag{4.42}$$

if $L(M) = L(N)$ and at

$$\phi(t) = \left[(\sqrt{\mathrm{Vol}(M)} - \sqrt{\mathrm{Vol}(N)}\,)t - \sqrt{\mathrm{Vol}(M)} \right]^2 \tag{4.43}$$

whenever $L(M) \leq L(N)$.

Proof. The proof is identical to the proof of lemma 3.2.10. □

Theorem 4.3.29. *Let M and N be smooth simple closed curves in $\mathbb{R}^2$ such that $L(M) \leq L(N)$. There exists a minimal morph between M and N in the class $\mathcal{Q}$.*

Proof. By corollary 4.3.26, it suffices to minimize Λ over the set $\mathcal{Q} \cap \mathcal{PM}_d(M, N)$. By lemma 4.3.27,

$$\Lambda(H) = \int_0^1 \frac{(\frac{d}{dt} L(M^t))^2}{L(M^t)}\, dt.$$

For directional pairwise minimal morphs, the functional Λ depends only on the length of the intermediate states.

Let $H \in \mathcal{Q}$ be a pairwise minimal morph between M and N. Such a morph exists by lemma 4.3.23. Let

$$F(t,p) = \frac{\phi(t)}{L(H(t,M))} H(t,p),$$

where ϕ minimizes the functional J defined in lemma 4.3.28. The length of the intermediate states $F(t, M)$ coincides with the minimum ϕ of the functional J over the admissible set $\mathcal{W}$. Hence, F is deformation minimal. □

4.4 Minimal Deformation Bending of Two-Dimensional Spheres; Holomorphic Critical Points

In this section we specialize to the manifolds $M = \mathbb{S}^2$ and $N = h_R(\mathbb{S}^2) =: R\,\mathbb{S}^2$ for some $R > 0$, where $\mathbb{S}^2$ is the unit 2-dimensional sphere in $\mathbb{R}^3$, and h_R is the radial map defined by $h_R(y) = R\,y$ for all $y \in \mathbb{R}^3$. As usual, the manifolds M and N are equipped with Riemannian metrics g_M and g_N respectively induced by the Euclidean metric $dx_1^2 + dx_2^2 + dx_3^2$ of $\mathbb{R}^3$ (see example 2.8.3).

The manifolds M and N are Riemann surfaces (see section 2.12). Let us parametrize the spheres $\mathbb{S}^2$ and $R\,\mathbb{S}^2$ on the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$ using stereographic projections. For $(y_1, y_2, y_3) \in \mathbb{S}^2$, the stereographic projection is given by the expression $\pi(y_1, y_2, y_3) = \frac{y_1 + iy_2}{1 - y_3}$.

We will show that maps of the form $h = f \circ h_R|_M$, where f is an isometry on N, minimize the functional Φ in the class of all holomorphic diffeomorphisms between M and N. As we have seen in corollary 4.2.5, maps of this form are critical points of Φ with the domain $\mathrm{Diff}(M, N)$.

The parametrization $\phi : \hat{\mathbb{C}} \to \mathbb{S}^2$ is given by

$$\phi(u + iv) = \left(\frac{2u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2}, \frac{-1 + u^2 + v^2}{1 + u^2 + v^2} \right)^T, \tag{4.44}$$

and the parametrization $\phi_R : \hat{\mathbb{C}} \to R\,\mathbb{S}^2$ of $R\,\mathbb{S}^2$ is given by $\phi_R(u + iv) = R\phi(u + iv)$.

In these coordinates, the Riemannian metrics g_M and g_N are defined by

$$g_M(z, \bar{z}) = \frac{4}{(1 + |z|^2)^2} dz d\bar{z} \tag{4.45}$$

and

$$g_N(z, \bar{z}) = \frac{4R^2}{(1 + |z|^2)^2} dz d\bar{z}. \tag{4.46}$$

Let $h \in \mathrm{Diff}(M, N)$ be a holomorphic map. The local representation $(\phi_R)^{-1} \circ h \circ \phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of h, which (by an abuse of notation) we shall denote by the same letter, is a holomorphic diffeomorphism of the extended complex plane onto itself. We conclude that $h(z)$ has the form $h(z) = M(z)$, where $M(z) = \frac{az+b}{cz+d}$ is a Möbius transformation and $a, b, c, d \in \mathbb{C}$ are such that $ad - bc \neq 0$. For such an h, it is easy to derive the formula

$$h^* g_N(z, \bar{z}) = \frac{4R^2 |bc - ad|^2}{\left(|az + b|^2 + |cz + d|^2 \right)^2} dz d\bar{z}. \tag{4.47}$$

Hence, the problem of minimization of the deformation energy functional Φ defined in display (1.1) over all holomorphic diffeomorphisms from $\mathbb{S}^2$ to $R\,\mathbb{S}^2$ reduces to the problem of minimization of the function

$$\Psi(a,b,c,d) = \int_{\mathbb{R}^2} \Big(\frac{R^2|bc-ad|^2}{\big(|az+b|^2+|cz+d|^2\big)^2} - \frac{1}{(1+|z|^2)^2} \Big)^2 (1+|z|^2)^2 \, dudv, \quad (4.48)$$

where $z = u+iv$, over the group $\mathrm{Aut}(\hat{\mathbb{C}}) = \mathrm{PGL}(2,\mathbb{C})$. Recall that the elements of the projective general linear group $\mathrm{PGL}(2,\mathbb{C})$ are the equivalence classes $[a,b,c,d]$, where $ad-bc \neq 0$ and $(a',b',c',d') \in [a,b,c,d]$ if $(a',b',c',d') = \lambda(a,b,c,d)$ for some $\lambda \in \mathbb{C}\backslash\{0\}$.

Recall that the group of all isometries of the Riemann sphere is the projective unitary group $\mathrm{PU}(2,\mathbb{C})$; that is, every isometry f of $(\mathbb{S}^2, g_M)$ has the local representation (via stereographic projection)

$$f(z) = \frac{az-\bar{c}}{cz+\bar{a}},$$

where $a, c \in \mathbb{C}$ are such that $|a|^2+|c|^2 = 1$.

The functional Φ is invariant with respect to left compositions with isometries; that is, $\Phi(f \circ h) = \Phi(h)$ for every isometry $f \in \mathrm{Diff}(N)$ and $h \in \mathrm{Diff}(M,N)$. Therefore, the reduced function Ψ is well-defined on the quotient of $\mathrm{PGL}(2,\mathbb{C})$ by $\mathrm{PU}(2,\mathbb{C})$, which is the set of all equivalence classes

$$\left[\left[\begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right]\right] = \left\{ \begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix} \in \mathrm{PU}(2,\mathbb{C}) \right\}. \quad (4.49)$$

We note that the equivalence class

$$\left[\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]\right]$$

consists of all the isometries of the unit sphere $(\mathbb{S}^2, g_M)$.

Theorem 4.4.1. *(i) Let $h_R : \mathbb{R}^3 \to \mathbb{R}^3$ be the radial map given by $h_R(p) = Rp$ for some number $R > 0$. If $M = \mathbb{S}^2$ is the 2-dimensional unit sphere isometrically embedded into $\mathbb{R}^3$ and $N = h_R(M)$, then $h := f \circ h_R|_M$ is a global minimum of the* *functional* Φ*, restricted to the admissible set* $\mathrm{HD}(M, N)$*, whenever* f *is an isometry of* N*.*

(ii) Let M *and* N *be compact Riemann surfaces. If* $\mathrm{HD}(M, N)$ *is not empty and the genus of* M *is at least two, then there exists a minimizer of the functional* Φ *in* $\mathrm{HD}(M, N)$*.*

Statement (ii) of theorem 4.4.1 follows immediately from Hurwitz's automorphisms theorem: *The group of automorphisms of a compact Riemann surface of genus greater than one is finite* (see [34]).

Statement (i) is equivalent to the following result.

Theorem 4.4.2. *The equivalence class of the isometries of* $(\mathbb{S}^2, g_M)$ *is the unique minimizer of the function* Ψ *defined on the homogeneous space* $\mathrm{PGL}(2, \mathbb{C})/\,\mathrm{PU}(2, \mathbb{C})$*; that is,*

$$\Psi\left(\left[\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]\right]\right) \leq \Psi\left(\left[\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]\right]\right) \tag{4.50}$$

for all

$$\left[\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]\right] \in \mathrm{PGL}(2, \mathbb{C})/\,\mathrm{PU}(2, \mathbb{C}).$$

Proof. The function Ψ is well-defined on the homogeneous space $\mathrm{PGL}(2)/\,\mathrm{PU}(2)$. Thus, all values of Ψ are obtained by choosing its domain to consist of one representative from each equivalence class.

We claim that each equivalence class

$$\left[\left[\begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array}\right]\right]$$

has a representative of the form

$$\left(\begin{array}{cc} 1 & 0 \\ z & r \end{array}\right),$$

for some $z \in \mathbb{C}$ and $r \in \mathbb{R}_+$.

To prove the claim, note that (without loss of generality) we may assume the determinant of the given representative is unity; that is, $\alpha\delta - \beta\gamma = 1$. We wish to prove the existence of $a, c \in \mathbb{C}$ so that

$$\begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z & r \end{pmatrix} \tag{4.51}$$

for some $z \in \mathbb{C}$ and $r \in \mathbb{R}_+$. In other words, it suffices to solve the system of linear equations

$$\begin{cases} a\alpha - \bar{c}\gamma = 1, \\ a\beta - \bar{c}\delta = 0. \end{cases} \tag{4.52}$$

In view of the equation $\alpha\delta - \beta\gamma = 1$, it follows that $a = \delta$ and $c = \bar{\beta}$. By substitution of a and c into equation (4.51), we find that $z = \bar{\beta}\alpha + \bar{\delta}\gamma$ and $r = |\beta|^2 + |\delta|^2$. This proves the claim.

By the claim, it suffices to consider the value of Ψ only at points of the form $(1, 0, qe^{i\psi}, r)$, where $q \in \mathbb{R}$, $r \in \mathbb{R}_+$, and $\psi \in [0, 2\pi)$. Thus, the theorem is an immediate consequence of the following proposition.

The function $\bar{\Psi} : \mathbb{R} \times [0, 2\pi] \times \mathbb{R}_+ \to \mathbb{R}$ *given by*

$$\bar{\Psi}(q, \psi, r) = \Psi(1, 0, qe^{i\psi}, r) \tag{4.53}$$

attains its global minimum on the set of points $(0, \psi, 1)$.

To prove this result, let us first calculate the integral that represents the function $\bar{\Psi}$.

After passing to polar coordinates ($u = \rho\cos\phi$ and $v = \rho\sin\phi$), we represent $\bar{\Psi}$ in the form

$$\bar{\Psi}(q, \psi, r) = \int_0^\infty \int_0^{2\pi} \Big[\frac{R^2 r^2}{(\xi + \eta\cos(\phi + \psi))^2} - \frac{1}{(1+\rho^2)^2}\Big]^2 (1+\rho^2)^2 \rho \, d\phi d\rho,$$

where $\xi = \rho^2 + \rho^2 q^2 + r^2$ and $\eta = 2\rho q r$. Since the integrand is periodic with respect to ϕ and we are integrating over one period, $\bar{\Psi}(q, \psi, r)$ does not depend on ψ; that is,

$$\bar{\Psi}(q, \psi, r) = \int_0^\infty \int_0^{2\pi} \Big[\frac{R^2 r^2}{(\xi + \eta\cos\phi)^2} - \frac{1}{(1+\rho^2)^2}\Big]^2 (1+\rho^2)^2 \rho \, d\phi d\rho.$$

The inner integral of the equivalent iterated integral is

$$\begin{aligned}
K(\rho): &= \int_0^{2\pi} \Big[\frac{R^2 r^2}{(\xi+\eta\cos(\phi))^2} - \frac{1}{(1+\rho^2)^2}\Big]^2 (1+\rho^2)^2 \rho\, d\phi \\
&= \Big[R^4 r^4 \int_0^{2\pi} \frac{1}{(\xi+\eta\cos(\phi))^4}\, d\phi \\
&\quad - 2R^2 r^2 \frac{1}{(1+\rho^2)^2} \int_0^{2\pi} \frac{1}{(\xi+\eta\cos(\phi))^2}\, d\phi \\
&\quad + 2\pi \frac{1}{(1+\rho^2)^4}\Big] \rho (1+\rho^2)^2. \qquad (4.54)
\end{aligned}$$

Taking into account the inequalities $\xi > |\eta|$ and $\eta > 0$, the integrals in the previous expression are elementary; their values are given by

$$\int_0^{2\pi} \frac{1}{(\xi+\eta\cos(\phi))^2}\, d\phi = \frac{2\pi\xi}{(\xi^2-\eta^2)^{3/2}}$$

and

$$\int_0^{2\pi} \frac{1}{(\xi+\eta\cos(\phi))^4}\, d\phi = \frac{\pi\xi(2\xi^2+3\eta^2)}{(\xi^2-\eta^2)^{7/2}}.$$

By substitution into equation (4.54), we find that

$$\begin{aligned}
\bar{\Psi}(q,\psi,r) &= \int_0^{\infty} K(\rho)\, d\rho \qquad (4.55) \\
&= \pi - 2\pi R^2 + \frac{\pi R^4}{3r^2}\big(1+q^2+(r-1)r\big)(1+q^2+r+r^2).
\end{aligned}$$

The minimum of the function $F(q,r) = \bar{\Psi}(q,\psi,r)$ on $\mathbb{R}\times\mathbb{R}_+$, is easily determined. Indeed, $(0,1)$ is the only critical point of F. Also, the Hessian of F is

$$D^2F(q,r) = \begin{pmatrix} \frac{4\pi R^4(1+3q^2+r^2)}{3r^2} & -\frac{8\pi R^4 q(1+q^2)}{3r^3} \\ -\frac{8\pi R^4 q(1+q^2)}{3r^3} & \frac{2\pi R^4(3+6q^2+3q^4+r^4)}{3r^4} \end{pmatrix}.$$

We note that $\partial^2 F(q,r)/\partial q^2$ and the determinant of the Hessian

$$\det\big(D^2F(q,r)\big) = \frac{8\pi^2 R^8}{9r^6}(3+q^6+3r^2+r^4+r^6+q^4(5+3r^2)+q^2(7+6r^2+3r^4))$$

are both positive by inspection. By Sylvester's criterion, the Hessian is positive definite over the entire domain of F; therefore, F is convex. If follows that $(0,1)$ is the unique global minimizer of F. The minimum of F is

$$F(0,1) = \pi(R^2-1)^2.$$

Clearly, points of the form $(0, \psi, 1)$ are the global minima of $\bar{\Psi}$ on its domain $\mathbb{R} \times [0, 2\pi] \times \mathbb{R}_+$. $\square$

Chapter 5

MINIMAL MORPHS INDUCED BY TIME-DEPENDENT VECTOR FIELDS

As before, let M and N be compact and orientable smooth Riemannian n-manifolds isometrically embedded into $\mathbb{R}^{n+1}$. Recall that a morph between M and N is an isotopy between them together with the set of all intermediate manifolds equipped with the Riemannian metrics inherited from $\mathbb{R}^{n+1}$. Every morph or diffeomorphism between isotopic manifolds produces distortion via stretching and bending.

In the previous chapters we considered distortion energy functionals defined on the infinite dimensional manifolds $\mathrm{Diff}(M, N)$ or $\mathcal{M}(M, N)$. In this chapter, we define distortion energy functionals on time-dependent vector fields, which generate morphs and diffeomorphisms between the manifolds M and N via evolution equations. This approach allows us to treat optimization problems on linear spaces.

Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set containing the manifolds M and N. We define functionals E and $\mathcal{E}$ that measure the distortion of diffeomorphisms and morphs respectively generated by time-dependent vector fields $v : \Omega \times [0, 1] \to \mathbb{R}^{n+1}$ via the evolution equation $dq/dt = v(q, t)$ and prove the existence of minimizers of E and $\mathcal{E}$ in an admissible set $\mathcal{A}_P^k$ of time-dependent vector fields, which is a subset of the closed ball of radius P in the Hilbert space $\mathcal{H}^k$ of all L^2 functions from $[0, 1]$ to the Sobolev space $W_0^{k,2}(\Omega; \mathbb{R}^{n+1})$, where $k \in \mathbb{N}$. We also analyze in detail a concrete example of a minimal morph for the case of circles embedded in the plane.

5.1 Bending and Morphing via Time-Dependent Vector Fields in $\mathbb{R}^{n+1}$

Given a smooth oriented n-manifold S (perhaps with boundary) isometrically embedded into $\mathbb{R}^{n+1}$, we let g_S, ω_S, and $I\!I_S$ denote the Riemannian metric, volume form, and second fundamental form on S associated with this embedding. Also, we let $\operatorname{Int} S$ (respectively, ∂S) denote the interior (respectively, the boundary) of the manifold S.

The definition 5.1.1 of a morph between two embedded manifolds is easily generalized to the case of manifolds with boundary.

Definition 5.1.1. Let M and N be isotopic compact connected smooth n-manifolds (perhaps with boundary) embedded in $\mathbb{R}^{n+1}$ such that M is oriented. A C^∞ isotopy $F : M \times [0,1] \to \mathbb{R}^{n+1}$ together with all the intermediate manifolds $M^t := F(M,t)$, equipped with the orientations induced by the maps $f^t = F(\cdot,t) : M \to M^t$ and the Riemannian metrics g_t inherited from $\mathbb{R}^{n+1}$, is called a (smooth) *morph* from M to N.

Recall that we denote the set of all smooth (respectively, C^r) diffeomorphisms between manifolds M and N by $\operatorname{Diff}(M,N)$ (respectively, $\operatorname{Diff}^r(M,N)$). Similarly, we denote the set of all smooth morphs between M and N by $\mathcal{M}(M,N)$. If F is an isotopy, then each map $F(\cdot,t) : M \to M^t$ induces smooth diffeomorphisms $\operatorname{Int} F(\cdot,t) : \operatorname{Int} M \to \operatorname{Int} M^t$ and $\partial F(\cdot,t) : \partial M \to \partial M^t$ by restriction.

In addition, we consider morphs between manifolds M and N with different regularity properties. For example, we let $\mathcal{M}^{r,\mathrm{ac}}(M,N)$ denote the set of all continuous isotopies $F : M \times [0,1] \to \mathbb{R}^{n+1}$ between M and N such that for each $p \in M$ the map $t \mapsto F(p,t)$ is absolutely continuous on $[0,1]$ and for each $t \in [0,1]$ the function $p \mapsto F(p,t)$ is a C^r diffeomorphism from M onto its image. As in the case of smooth morphs, the diffeomorphism $F(\cdot,t) : M \to M^t$ induces an orientation on the intermediate manifold M^t.

There are several choices for cost functionals that measure the distortion of a diffeomorphism $h \in \text{Diff}(M, N)$ or a morph $F \in \mathcal{M}(M, N)$.

A complete theory of the existence of minimizers of cost functionals that measure distortion of diffeomorphisms and morphs due to change of volume is presented in chapter 3. In this case, the value of the distortion energy functional at a diffeomorphism $h : M \to N$ is defined to be the square of the infinitesimal relative change of volume $|J(h)| - 1$ produced by h integrated over the manifold M, where $J(h)$ is the Jacobian determinant of h. This functional does not take into account the distortion of shape produced by h, which is captured by functionals (1.1) and (5.1), where the fiber metric $\|\cdot\|$ on the bundle of all tensor fields of type $(0, 2)$ is induced by the fiber inner product $g_M^* \otimes g_M^*$ (see section 4.1 and [24]).

The general problem of the existence of minimizers of Φ is open. The special case where M and N are one-dimensional is studied in chapter 4 where, among other results, the functional Φ is shown to have no minimizer in case M and N are circles with the radius of N smaller than the radius of M. Thus, a solution of the general problem must take into account at least some global properties of the metric structures of the manifolds M and N. On the other hand, we proved the existence of minimizers in case M and N are Riemann spheres or compact Riemann surfaces of genus greater than one and the admissible set is $\text{HD}(M, N) = \{h \in \text{Diff}(M, N) :$ h is a holomorphic map$\}$, see theorem 4.4.1 in section 4.4.

If we wish to match, in addition to the Riemannian metrics, the embeddings of the manifolds M and N (to avoid, for example, zero distortion energy maps between a square and a round cylinder in $\mathbb{R}^3$), we arrive at the problem of minimization of the functional

$$\Lambda(h) := \int_M \|h^* g_N - g_M\|^2 \,\omega_M + \int_M \|h^* I\!I_N - I\!I_M\|^2 \,\omega_M \tag{5.1}$$

over the space of diffeomorphisms between M and N, where $I\!I_M$ and $I\!I_N$ are the

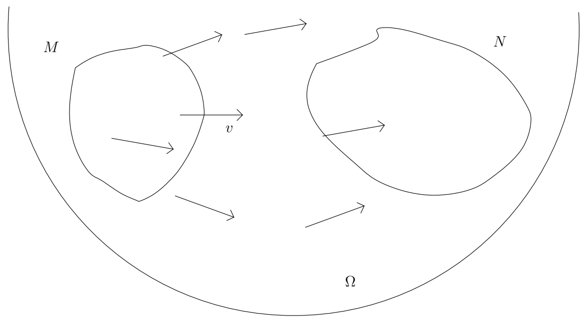

Figure 5.1: The time-dependent vector field $v : \Omega \times [0, 1] \to \mathbb{R}^{n+1}$ generates the morph $F^v(p, t)$, which is the solution of the initial value problem $dq/dt = v(q, t),\ q(0) = p$.

second fundamental forms on the manifolds M and N.

One of the difficulties encountered in attempts to minimize Φ over $\mathrm{Diff}(M, N)$ is the lack of a complete understanding of the structure of this infinite-dimensional space. The natural new approach is to linearize; that is, replace $\mathrm{Diff}(M, N)$ with a subset of a linear function space. Using this approach, which already appears in the literature on image deformation (see [9, 11, 17, 39]), we define our distortion energy functionals on time-dependent vector fields that generate morphs (see Fig. 5.1).

Let us denote the Euclidean norm of an element $A \in \mathbb{R}^m$ by $|A|$ or by $|A|_{\mathbb{R}^m}$. Let $\Omega \subset \mathbb{R}^{n+1}$ be an open ball containing the manifolds M and N, $C_c^\infty(\Omega; \mathbb{R}^{n+1})$ the space of all smooth functions from Ω to $\mathbb{R}^{n+1}$ with compact support, and $V^k := W_0^{k,2}(\Omega; \mathbb{R}^{n+1})$ the closure of $C_c^\infty(\Omega; \mathbb{R}^{n+1})$ in the Sobolev space $W^{k,2}(\Omega; \mathbb{R}^{n+1})$ (see [15]).

The space V^k is a Hilbert space with the inner product

$$\langle f, g\rangle_{V^k} = \sum_{i=1}^{n+1} \sum_{\alpha, |\alpha| \le k} \int_\Omega D^\alpha f_i D^\alpha g_i dx,$$

where $f = (f_1, \ldots, f_{n+1}) : \Omega \to \mathbb{R}^{n+1}$, $\alpha = (\alpha_1, \ldots, \alpha_{n+1})$ is a multi-index with non-negative integer components, $|\alpha| = \alpha_1 + \ldots + \alpha_{n+1}$, and $D^\alpha f_i = \partial^{|\alpha|} f_i / \partial x_1^{\alpha_1} \ldots \partial x_{n+1}^{\alpha_{n+1}}$ is the corresponding weak partial derivative of f_i. We choose $k \in \mathbb{N}$ large enough so that the Sobolev space $W_0^{k,2}(\Omega)$ is embedded into $C^r(\bar{\Omega})$ and $r \ge 1$. By the Sobolev Embedding Theorem (see theorem 2.15.6 and [2, 15]), it suffices to choose $k \ge (n + 1)/2 + r + 1$.

Consider time-dependent vector fields $v : \Omega \times [0, 1] \to \mathbb{R}^{n+1}$ on Ω that belong to the Hilbert space

$$\mathcal{H}^k = L^2(0, 1; V^k) \tag{5.2}$$

(see Fig. 5.1). By an abuse of notation, we will sometimes write $v(x, t) = v(t)(x)$ for $v \in \mathcal{H}^k$ and $(x, t) \in \Omega \times [0, 1]$. A time-dependent vector field $v : \Omega \times [0, 1] \to \mathbb{R}^{n+1}$ belongs to the Hilbert space $\mathcal{H}^k$ if its norm $\|v\|_{\mathcal{H}^k} := (\int_0^1 \|v(\cdot, t)\|_V^2 \, dt)^{\frac{1}{2}}$ is finite. The inner product on $\mathcal{H}^k$ is defined by

$$\langle v, w \rangle_{\mathcal{H}^k} = \int_0^1 \langle v(\cdot, t), w(\cdot, t) \rangle_{V^k} \, dt.$$

Every vector field $v \in \mathcal{H}^k$ generates a morph $F^v : M \times [0, 1] \to \mathbb{R}^{n+1}$ from M to $F^v(M, 1)$ via the evolution equation

$$\frac{dq}{dt} = v(q, t). \tag{5.3}$$

More precisely, let $\eta^v(t; t_0, x)$ be the evolution operator of equation (5.3); that is, for every $t_0 \in [0, 1]$ and $x \in \Omega$ the function $t \mapsto \eta^v(t; t_0, x)$ solves equation (5.3) and satisfies the initial condition $\eta^v(t_0; t_0, x) = x$. The morph F^v is defined by $F^v(p, t) = \eta^v(t; 0, p)$ for all $(p, t) \in M \times [0, 1]$. By the properties of the evolution operator η^v, which have been studied in [11] and [40], the morph $F^v(p, t)$ is of class $\mathcal{M}^{r,\text{ac}}(M, F^v(M, 1))$ (see lemmas 5.4.1 and 5.4.2). The time-one map of the evolution operator η^v is defined to be $\phi^v(x) := \eta^v(1; 0, x)$ for all $x \in \Omega$, and we define $\psi^v = \phi^v|_M$.

Let $\mathcal{A}_P^k$ be the admissible set of all time-dependent vector fields in $\mathcal{H}^k$ that generate morphs between the manifolds M and N and are bounded by a uniform positive constant P. In symbols,

$$\mathcal{A}_P^k = \{v \in \mathcal{H}^k : \psi^v \in \text{Diff}^r(M, N) \text{ and } \|v\|_{\mathcal{H}^k} \leq P\}. \tag{5.4}$$

We will prove that for P sufficiently large, the admissible set $\mathcal{A}_P^k$ is nonempty and $\mathcal{A}_P^k$ is weakly closed in $\mathcal{H}^k$ (see lemma 5.2.10).

Recall that $\mathcal{T}^r{}_s(M)$ denotes the set of all continuous tensor fields on M contravariant of order r and covariant of order s (also called type (r,s)). For a tensor field $\tau_N \in \mathcal{T}^0{}_s(N)$ and a diffeomorphism $h: M \to N$, $h^*\tau_N$ denotes the pull-back of τ_N to M.

For each $t \in [0,1]$ and $v \in \mathcal{A}_P^k$, the manifold $M^{v,t} := F^v(M,t)$ is called an intermediate state of the morph F^v between manifolds M and N generated by the time-dependent vector field v. We endow this intermediate state with the Riemannian metric g_t^v inherited from its embedding in $\mathbb{R}^{n+1}$ and let $I\!I_t^v$ denote the corresponding second fundamental form.

Definition 5.1.2. Let B_1 and B_2 be nonnegative real numbers, F^v the morph, and ϕ^v the time-one map generated by the time-dependent vector field $v \in \mathcal{A}_P^k \subset \mathcal{H}^k$ via the evolution equation (5.3). Recall that $\psi^v := \phi^v|_M$. The *bending distortion energy* of v is

$$\begin{aligned} E(v) = E(v; B_1, B_2) &= B_1 \int_M \|(\psi^v)^* g_N - g_M\|^2 \omega_M \\ &\quad + B_2 \int_M \|(\psi^v)^* I\!I_N - I\!I_M\|^2 \omega_M \end{aligned}$$

and the *morphing distortion energy* of v is

$$\begin{aligned} \mathcal{E}(v) = \mathcal{E}(v; B_1, B_2) &= B_1 \int_0^1 \int_M \|F^v(\cdot,t)^* g_t^v - g_M\|^2 \omega_M dt \\ &\quad + B_2 \int_0^1 \int_M \|F^v(\cdot,t)^* I\!I_t^v - I\!I_M\|^2 \omega_M dt, \end{aligned}$$

where $\|\cdot\|$ is the fiber norm on the tensor bundle $\mathcal{T}^0{}_2(M)$ generated by the fiber inner product $g_M^* \otimes g_M^*$. (Note: We will use the same notation for the fiber norm on the tensor bundle $\mathcal{T}^0{}_s(M)$ generated by the inner product $\otimes_{i=1}^s g_M^*$.)

We will prove that the functionals E and $\mathcal{E}$ have minimizers in $\mathcal{A}_P^k$.

Theorem 5.1.3. *(i) If $P > 0$ and $k \in \mathbb{N}$ are sufficiently large, then each of the functionals $E : \mathcal{A}_P^k \to \mathbb{R}_+$ and $\mathcal{E} : \mathcal{A}_P^k \to \mathbb{R}_+$ has a minimizer in the admissible set $\mathcal{A}_P^k$.*

The detailed conditions on the constants P and k are formulated in theorem 5.2.12.

We note that each diffeomorphism $\psi^v : M \to N$ generated by a time-dependent vector field $v \in \mathcal{A}_P^k$ is isotopic, as a map from M to $\mathbb{R}^{n+1}$, to the inclusion map $i : M \to \mathbb{R}^{n+1}$ via the isotopy $F^v \in \mathcal{M}^{r,\mathrm{ac}}(M, N)$. To minimize the distortion energy of diffeomorphisms from other isotopy classes, we replace the map ψ^v in the definition of the functional E by the diffeomorphism $\psi^v \circ \phi : M \to N$, where ϕ is a fixed diffeomorphism on M. The existence of minimizers of the functional E with the above adjustment guarantees the existence of minimizers of the functionals Φ and Λ defined in displays (1.1) and (5.1) in a restricted admissible set of all C^2 diffeomorphisms between the manifolds M and N, which, considered as maps from M to $\mathbb{R}^{n+1}$, are isotopic to a given map $\phi : M \to \mathbb{R}^{n+1}$.

Theorem 5.1.4. *If $P > 0$ and $k \in \mathbb{N}$ are sufficiently large, then for every $\phi \in \mathrm{Diff}(M)$ both functionals Φ and Λ defined in displays (1.1) and (5.1) respectively have minimizers in the admissible set*

$$\mathcal{B}_{P,\phi}^k := \{h \in \mathrm{Diff}^2(M, N) : h = \psi^v \circ \phi \textit{ for some } v \in \mathcal{A}_P^k\}.$$

In section 5.3 we construct an example of a minimal distortion diffeomorphism and morph between the unit circle $\mathbb{S}^1$ and the circle $\mathbb{S}_R^1$, with radius $R > 1$, in $\mathbb{R}^2$.

While the construction of a minimizer of the functional E does not cause significant difficulties, finding a minimizer of the functional $\mathcal{E}$ is a much more intricate process. Even after we restrict our attention to the family of morphs whose intermediate states are concentric circles, finding a minimal distortion morph requires delicate analysis, which is done in section 5.3.

To find a morph $H(p, t) = \psi(t)p$ with $\psi \in Q_+ := \{\phi \in C^2(0, 1) \cap C[0, 1] : \phi(0) = 1,\ \phi(1) = R,\ \text{and } \phi \text{ is increasing}\}$, which has minimal distortion among the morphs $F \in \mathcal{M}^{3,\mathrm{ac}}(M, N)$ whose intermediate states are circles with increasing radii, we solve the optimization problem

$$\begin{array}{ll} \text{minimize } J(\psi) = \int_0^1 (\psi^2-1)^2\,dt + \int_0^1 (\psi-1)^2\,dt \\ \text{for } \psi \in Q_+ \\ \text{subject to } \int_0^1 \left(\frac{\psi'}{\psi}\right)^2 dt \le A, \end{array} \tag{5.5}$$

where $A > 0$. The inequality constraint in optimization problem (5.5) is derived from the requirement that the vector fields on the set $\Omega \subset \mathbb{R}^2$ generated by the morph H must be bounded by a uniform constant.

More precisely, let Ω be the open ball in $\mathbb{R}^2$ of radius $R+2$ and let $\rho : \mathbb{R}^2 \to \mathbb{R}^2$ be a bump function such that $\rho \equiv 1$ on the open ball $B(0, R+1)$, $\rho \equiv 0$ on Ω^c, and $0 \le \rho \le 1$.

Given $P > 0$, define

$$A(P) := \|\rho\,\mathrm{id}_\Omega\|^{-2}_{W_0^{5,2}(\Omega;\mathbb{R}^2)} P^2.$$

Theorem 5.1.5. *If the constant $A = A(P) > \log^2 R$, then there exists a unique minimal distortion morph $H(p,t) = \psi(t)p$, where $\psi \in Q_+$, between the unit circle $\mathbb{S}^1$ and the circle of radius $R > 1$ in $\mathbb{R}^2$, among all the morphs $F \in \mathcal{M}^{3,\mathrm{ac}}(M,N)$ of the form*

$$F(p,t) = \phi(t)p, \quad \phi \in Q_+$$

that generate the time-dependent vector field

$$v(x,t) = \frac{\phi'(t)}{\phi(t)}\rho(x)x, \quad (x,t) \in \Omega \times [0,1]$$

such that $\|v\|_{\mathcal{H}^5} \le P$.

Moreover, the radial function $\psi \in Q_+$ of the distortion minimal morph H is the unique solution of the optimization problem (5.5) *and solves the initial value problem*

$$\begin{cases} \psi' = \frac{1}{\sqrt{\lambda}}\psi\sqrt{\mu + (\psi^2-1)^2 + (\psi-1)^2}, \\ \psi(0) = 1, \end{cases} \tag{5.6}$$

where the pair of positive constants λ and μ is the unique solution of the system of equations

$$\int_1^R \frac{ds}{s\sqrt{\mu + (s^2-1)^2 + (s-1)^2}} = \frac{1}{\sqrt{\lambda}} \tag{5.7}$$

and

$$\frac{1}{\sqrt{\lambda}}\int_1^R \frac{\sqrt{\mu+(s^2-1)^2+(s-1)^2}}{s}\,ds = A. \tag{5.8}$$

5.2 Existence of Time-Dependent Vector Fields that Generate Minimal Distortion Diffeomorphisms and Morphs

In this section we prove theorem 5.1.3.

We will show that the admissible set $\mathcal{A}_P^k$ is nonempty if P is sufficiently large.

Lemma 5.2.1. *Let M and N be manifolds as in definition 5.1.1. Let F be a smooth morph between the manifolds M and N and assume that $\Omega \subset \mathbb{R}^{n+1}$ is an open ball in $\mathbb{R}^{n+1}$ containing the image $F(M\times[0,1])$ of the morph F. There exists $P_0 > 0$ such that the admissible set $\mathcal{A}_P^k$ is nonempty whenever $P \geq P_0$ and $k \geq \frac{n+1}{2}+2$.*

Proof. The morph $F \in \mathcal{M}(M,N)$ defines the $\mathbb{R}^{n+1}$ valued function

$$v(y,t) = \frac{\partial F}{\partial t}([F(\cdot,t)]^{-1}(y),t)$$

on the compact subset $Q = \{(F(x,t),t) : (x,t) \in M\times[0,1]\}$ of $\mathbb{R}^{n+1}\times\mathbb{R}$.

We will extend the function v to a smooth vector field $w \in \mathcal{H}^k$ such that $\psi^w(M) = N$.

First, notice that the smooth map $G : M\times[0,1] \to \mathbb{R}^{n+1}\times\mathbb{R}$ defined by $G(x,t) = (F(x,t),t)$ is a proper map ($M\times[0,1]$ is compact) and an injective immersion, hence an embedding (see [1]). Therefore, $Q = G(M\times[0,1])$ is a submanifold (with boundary) of $\mathbb{R}^{n+1}\times\mathbb{R}$ (see [19]).

Next, notice that the map $G_1 : Q \to M\times[0,1]$ defined by $G_1(y,t) = ([F(\cdot,t)]^{-1}(y),t)$ is the inverse of G. Because G is an immersion, hence a local diffeomorphism, the map G_1 is smooth. Therefore, the map $v : Q \to \mathbb{R}^{n+1}$ is smooth because it is the composition of two smooth maps $G_1 : Q \to M\times[0,1]$ and $\frac{\partial F}{\partial t} : M\times[0,1] \to \mathbb{R}^{n+1}$.

The smooth function $v : Q \to \mathbb{R}^{n+1}$ can be extended locally. That is, for every $(y,t) \in Q$ there exists an open set $U \subset \mathbb{R}^{n+1} \times \mathbb{R}$ such that $(y,t) \in U$ and a smooth function $v_1 : U \to \mathbb{R}^{n+1}$ such that $v_1|_{U \cap Q} = v|_{U \cap Q}$. This local extension property follows from a more general fact about smooth functions defined on submanifolds: Let S be an s-dimensional smooth submanifold (perhaps with boundary) of $\mathbb{R}^m$ and let $f : S \to \mathbb{R}$ be a smooth function. Then for every $x \in S$ there exists an open set $W \subset \mathbb{R}^m$ containing x and a smooth function $f_1 : W \to \mathbb{R}$ such that $f|_{W \cap S} = f_1|_{W \cap S}$. It is easy to construct a local extension of the function f using submanifold charts on S and the definition of a smooth function whose domain is a submanifold with boundary. The details are left to the reader.

Therefore, the function $v : Q \to \mathbb{R}^{n+1}$ satisfies the conditions of the smooth Tietze extension theorem (see [1]) and can be extended from the closed set $Q \subset \mathbb{R}^{n+1} \times \mathbb{R}$ by a smooth map $\bar{v} : \mathbb{R}^{n+1} \times \mathbb{R} \to \mathbb{R}^{n+1}$.

Finally, define $w(x,t) = \rho(x)\bar{v}(x,t)$, where $\rho : \mathbb{R}^{n+1} \to \mathbb{R}$ is a smooth bump function such that $\rho \equiv 1$ on Q and $\rho \equiv 0$ on $\partial\Omega$, and set $P_0 := \|w\|_{\mathcal{H}^k}$. $\square$

From now on, we assume that the open set Ω in $\mathbb{R}^{n+1}$ is chosen as in the last lemma and the constant $P > 0$ is large enough so that the set $\mathcal{A}_P^k$ is not empty; the number k of weak derivatives satisfies the inequality $k \geq (n+1)/2 + r + 1$, where $r \geq 1$.

For each $v \in \mathcal{A}_P^k$, the time-one map $\psi^v : M \to N$ transforms the interior (respectively, the boundary) of the manifold M to the interior (respectively, the boundary) of the manifold N. The existence and the convergence properties of the evolution operators generated by vector fields $v \in \mathcal{H}^k$ via the evolution equation (5.3) have been studied in [11, 40]. For convenience of the reader, we state some of these properties (which will be useful in our proofs) in Appendix 5.4.

Every time-dependent vector field $v \in \mathcal{A}_P^k \subset \mathcal{H}^k$ generates a morph F^v between the manifolds M and N of class $\mathcal{M}^{r,\mathrm{ac}}(M,N)$.

Let us recall the distortion energy functionals $E : \mathcal{H}^k \to \mathbb{R}_+$ and $\mathcal{E} : \mathcal{H}^k \to \mathbb{R}_+$ (see definition 5.1.2).

One of the main ingredients in the proof of theorem 5.1.3 is the weak continuity of the functionals E and $\mathcal{E}$. We will prove the weak continuity of more general auxiliary functionals, where the tensor fields τ_M and τ_N in the following definition will later be replaced by the first and the second fundamental forms on the manifolds M and N respectively.

Definition 5.2.2. Let M and N be manifolds as in definition 5.1.1. For given continuous tensor fields τ_M and τ_N of type $(0, s)$ on M and N respectively, the functional $J : \mathcal{A}_P^k \to \mathbb{R}_+$ is defined by

$$J(v) = \int_M \|(\psi^v)^* \tau_N - \tau_M\|^2 \, \omega_M.$$

Let $v \in \mathcal{A}_P^k$ be a time-dependent vector field that generates a morph $F^v \in \mathcal{M}^{r,\mathrm{ac}}(M, N)$ from the manifold M to N. Recall that the intermediate state at the time $t \in [0, 1]$ of the morph F^v is denoted by $M^{v,t}$. The Riemannian metric and the second fundamental form on $M^{v,t}$, which are associated with the embedding of $M^{v,t}$ into $\mathbb{R}^{n+1}$, are denoted by g_t^v and $I\!I_t^v$ respectively. The functionals $I_1 : \mathcal{A}_P^k \to \mathbb{R}_+$ and $I_2 : \mathcal{A}_P^k \to \mathbb{R}_+$ are given by

$$I_1(v) = \int_0^1 \int_M \|F^v(\cdot, t)^* g_t^v - g_M\|^2 \, \omega_M \, dt \tag{5.9}$$

and

$$I_2(v) = \int_0^1 \int_M \|F^v(\cdot, t)^* I\!I_t^v - I\!I_M\|^2 \, \omega_M \, dt. \tag{5.10}$$

Definition 5.2.3. Let X be an n-dimensional vector space equipped with the inner product g_X. Let $T^0{}_s(X) = \otimes_{i=1}^s X^*$. For every $v \in X$, we denote the norm of v with respect to the inner product g_X by $|v|_{g_X} = g_X(v, v)^{1/2}$ and the unit sphere by $S_{g_X} = \{v \in X : |v|_{g_X} = 1\}$. Define the norm on $T^0{}_s(X)$ by

$$\|b\|_{g_X} = \max_{v_i \in S_{g_X}} |b(v_1, \ldots, v_s)|.$$

Another norm on $T^0{}_s(X)$ is defined by (see [24])

$$\|b\| = \otimes_{i=1}^{s} g_X^*(b).$$

Let $\mathbb{M}(X) \subset T^0{}_2(X)$ denote the metric space of all inner products on X with the metric $d(g, g') = \|g - g'\|_{g_X}$.

Note that if $\{e_1, \ldots, e_n\}$ is an orthonormal basis of (X, g_X), then

$$\|b\| = \sum_{i_1,\ldots,i_s=1}^{n} b(e_{i_1}, \ldots, e_{i_s})^2. \tag{5.11}$$

Theorem 5.2.4. *The function $\eta : T^0{}_s(X) \times \mathbb{M}(X) \to \mathbb{R}$ defined by $\eta(\beta, g) = \|\beta\|_g$ is continuous on its domain.*

To prove this intuitively obvious lemma, we use some auxiliary facts about the space $\mathbb{M}(X)$ of all metrics on X.

Lemma 5.2.5. *Let $g \in \mathbb{M}(X)$ be a fixed metric. There exists a positive constant $M(g) > 0$ and a positive number $\delta = \delta(g) > 0$ such that for every metric $g' \in \mathbb{M}(X)$ in the δ-neighborhood of g all the vectors v in the unit sphere $S_{g'}$ are bounded, in the standard norm $|\cdot|_{g_X}$, by $M(g)$.*

Remark 5.2.6. It is tempting to question whether the last lemma holds for all $\delta > 0$. To see that it is not so, consider this example.

Let $g = g_X$ and take $g' = \varepsilon g$, where $\varepsilon > 0$. Every vector $v \in S_{g'}$ has the standard norm $|v|_{g_X} = g_X(v, v)^{1/2} = 1/\sqrt{\varepsilon}\, g'(v, v)^{1/2} = 1/\sqrt{\varepsilon}$, which can be as large as desired unless we impose the restriction $|1 - \varepsilon| = \|g' - g\|_{g_X} < 1$.

Proof. Let δ be a positive number to be chosen later and let g' be a metric on X in the δ-neighborhood of g, i.e. $\|g - g'\|_{g_X} < \delta$.

The standard norm of every vector $v \in S_{g'}$ can be estimated as follows:

$$\begin{aligned}
|v|_{g_X}^2 &\leq \|g_X\|_g |v|_g^2 \\
&\leq \|g_X\|_g \|g - g'\|_{g_X} |v|_{g_X}^2 + \|g_X\|_g \\
&< \delta \|g_X\|_g |v|_{g_X}^2 + \|g_X\|_g.
\end{aligned}$$

By choosing $\delta < 1/(2\|g_X\|_g)$, we obtain the inequality $|v|^2_{g_X} \leq 2\|g_X\|_g$ for all $v \in X$. $\square$

Lemma 5.2.7. *Let $g \in \mathbb{M}(X)$. Then for every $\varepsilon > 0$ there exists $\delta = \delta(g) > 0$ such that for all vectors $v \in S_g$ and metrics $g' \in \mathbb{M}(X)$ satisfying $\|g' - g\|_{g_X} < \delta$ we have $\big|v - v/|v|_{g'}\big|_g < \varepsilon$.*

In other words, if $g' \in \mathbb{M}(X)$ is in a sufficiently small neighborhood of g, then every vector v in the unit sphere S_g is close, in the $|\cdot|_g$-norm, to the vector $v/|v|_{g'} \in S_{g'}$.

Proof. Fix $\varepsilon > 0$ and let $v \in S_g$. Without loss of generality, $\varepsilon < 1$. Let $C(g)$ be a positive constant such that $\|\eta\|_g \leq C(g)\|\eta\|_{g_X}$ for all $\eta \in T^0{}_2(X)$.

Choose $\delta < \varepsilon/(2C(g)) < 1/(2C(g))$ and let $g' \in \mathbb{M}(X)$ be in the δ-neighborhood of g, i.e. $\|g - g'\|_{g_X} < \delta$. For every $v \in S_g$ we have

$$\begin{aligned} \big||v|_{g'} - |v|_g\big| &\leq \big||v|_{g'} - |v|_g\big| \, \big||v|_{g'} + |v|_g\big| \\ &= |g'(v,v) - g(v,v)| \\ &\leq \|g' - g\|_g < C(g)\delta < \frac{\varepsilon}{2} \end{aligned}$$

and

$$|v|_{g'} > |v|_g - 1/2 = 1/2.$$

Hence,

$$\begin{aligned} \Big|v - \frac{v}{|v|_{g'}}\Big| &= g\Big(v - \frac{v}{|v|_{g'}}, v - \frac{v}{|v|_{g'}}\Big)^{1/2} \\ &= \frac{\big||v|_{g'} - |v|_g\big|}{|v|_{g'}} < \varepsilon \end{aligned}$$

as required. $\square$

Lemma 5.2.8. *Let $g \in \mathbb{M}(X)$. Then for every $\varepsilon > 0$ there exists $\delta = \delta(g) > 0$ such that for each $g' \in \mathbb{M}(X)$ satisfying $\|g' - g\|_{g_X} < \delta$ and for every $w \in S_{g'}$ we have $\big|w - w/|w|_g\big|_g < \varepsilon$.*

In other words, if $g' \in \mathbb{M}(X)$ is in a small enough neighborhood of g, then every vector w in the unit sphere $S_{g'}$ is close, in the $|\cdot|_g$-norm, to the vector $w/|w|_g \in S_g$.

Proof. By lemma 5.2.5, there exists $M(g) > 0$ and $\delta_1 = \delta_1(g) > 0$ such that for each $g' \in \mathbb{M}(X)$ satisfying $\|g - g'\|_{gX} < \delta_1$ for all $w \in S_{g'}$, we have $|w|_{gX} \leq M(g)$.

Fix $\varepsilon > 0$ and choose $\delta < \min\{\varepsilon/M(g)^2, \delta_1\}$. Let $g' \in \mathbb{M}(X)$ be such that $\|g - g'\|_{gX} < \delta$ and take $w \in S_{g'}$.

Then

$$\begin{aligned}\big|w - 1/|w|_g\, w\big|_g &= \big||w|_{g'} - |w|_g\big| \\ &\leq \big||w|^2_{g'} - |w|^2_g\big| < \delta|w|^2_{gX} < \delta M(g)^2 < \varepsilon\end{aligned}$$

as required. □

Proof of theorem 5.2.4

Proof. Fix $(\beta, g) \in {T^0}_s(X) \times \mathbb{M}(X)$ and $\varepsilon > 0$. Let $\delta < 1$ be a positive number to be specified later, and let $(\beta', g') \in {T^0}_s(X) \times \mathbb{M}(X)$ be such that $\|\beta - \beta'\|_{gX} + \|g - g'\|_{gX} < \delta$.

Let $C(g)$ and $K(g)$ be positive constants such that $\|\alpha\|_g \leq C(g)\|\alpha\|_{gX}$ and $\|\alpha\|_{gX} \leq K(g)\|\alpha\|_g$ for all $\alpha \in {T^0}_s(X)$.

Then

$$\begin{aligned}\big|\|\beta\|_g - \|\beta'\|_{g'}\big| &\leq \big|\|\beta\|_g - \|\beta'\|_g\big| + \big|\|\beta'\|_g - \|\beta'\|_{g'}\big| \\ &< C(g)\delta + \big|\|\beta'\|_g - \|\beta'\|_{g'}\big|.\end{aligned}$$

We will estimate the expression $\big|\|\beta'\|_g - \|\beta'\|_{g'}\big|$.

Let $v_1, \ldots, v_s \in S_g$ be vectors such that $\|\beta'\|_g = |\beta'(v_1, \ldots, v_s)|$. Then

$$\begin{aligned}
\|\beta'\|_g &= |\beta'(v_1 - v_1/|v_1|_{g'} + v_1/|v_1|_{g'}, v_2, \ldots, v_s)| \\
&\leq |\beta'(v_1 - v_1/|v_1|_{g'}, v_2, \ldots, v_s)| \\
&+ |\beta'(v_1/|v_1|_{g'}, v_2 - v_2/|v_2|_{g'}, \ldots, v_s)| \\
&+ \ldots + |\beta'(v_1/|v_1|_{g'}, v_2/|v_2|_{g'}, \ldots, v_s - v_s/|v_s|_{g'})| \\
&+ \|\beta'\|_{g'}. \qquad (5.12)
\end{aligned}$$

By lemma 5.2.5 and the equivalence of the norms $|\cdot|_g$ and $|\cdot|_{g_X}$ on X, there exist $M(g) > 1$ and $\delta_1 = \delta_1(g) > 0$ such that $|w|_g \leq M(g)$ for all $w \in S_{g'}$ whenever $\delta < \delta_1$. In particular, $v_i/|v_i|_{g'} \leq M(g)$ for all $i \in \{1, \ldots, s\}$.

Therefore, for all $\delta < \delta_1$, inequality (5.12) implies

$$\begin{aligned}
\|\beta'\|_g &\leq \|\beta'\|_{g'} + \|\beta'\|_g M(g)^{s-1} \sum_{i=1}^{s} \left|v_i - v_i/|v_i|_{g'}\right|_g \qquad (5.13) \\
&\leq \|\beta'\|_{g'} + C(g)(\|\beta\|_{g_X} + 1) M(g)^{s-1} \sum_{i=1}^{s} \left|v_i - v_i/|v_i|_{g'}\right|_g. \qquad (5.14)
\end{aligned}$$

Let $\delta_2 = \delta_2(g) > 0$ be the δ from lemma 5.2.7 with ε replaced by $C(g)^{-1}s^{-1}(\|\beta\|_{g_X} + 1)^{-1}M(g)^{1-s}\varepsilon$. Then for every $\delta < \min\{\delta_1, \delta_2, 1\}$

$$\|\beta'\|_g < \|\beta'\|_{g'} + \varepsilon.$$

The inequality $\|\beta'\|_{g'} < \|\beta'\|_g + \varepsilon$ is proved using a similar together with lemmas 5.2.8 and 5.2.5. $\square$

For a C^1 Riemannian manifold (S, g_S), let $\|\cdot\|$ be the fiber norm on the bundle of all continuous tensor fields on S of type $(0, s)$ generated by the fiber inner product $\otimes_{i=1}^{s} g_S^*$.

Lemma 5.2.9. *If b is a continuous tensor field of type $(0, s)$ on a C^1 Riemannian n-manifold (S, g_S), then the function*

$$z \mapsto \|b(z)\|_{g_S(z)} \qquad (5.15)$$

is continuous on S. Moreover, the norms $\|\cdot\|$ and $\|\cdot\|_{g_{S(z)}}$ are uniformly equivalent; in fact,

$$\frac{1}{n^{s/2}}\|b(z)\| \leq \|b(z)\|_{g_{S(z)}} \leq \|b(z)\| \tag{5.16}$$

for all $b \in \mathcal{T}^0{}_s(S)$ and $z \in S$.

Proof. The continuity of the function defined in display (5.15) follows immediately from theorem 5.2.4; and, the inequalities in display (5.16) can be easily derived from the definitions of the norms $\|\cdot\|$ and $\|\cdot\|_{g_{S(z)}}$ and formula (5.11). □

Recall that a sequence $\{v^l\}_{l=1}^{\infty} \subset \mathcal{H}^k$ converges weakly to $v \in \mathcal{H}^k$ (in symbols, $v^l \rightharpoonup v$) as $l \to \infty$ if $\langle v^l - v, w\rangle \to 0$ as $l \to \infty$ for every $w \in \mathcal{H}^k$. We call v the weak limit of $\{v^l\}_{l=1}^{\infty}$. A set $\mathcal{Q} \subset \mathcal{H}^k$ is sequentially weakly closed if it contains the weak limit of every weakly convergent sequence $\{v^l\}_{l=1}^{\infty} \subset \mathcal{Q}$.

Lemma 5.2.10. *(i) The admissible set $\mathcal{A}_P^k$ is sequentially weakly closed in $\mathcal{H}^k$.*
(ii) Let b be a continuous tensor field of type $(0, s)$ on the manifold N; and, for every $w \in \mathcal{A}_P^k$, let ψ^w denote the restriction to M of the time-one map of the evolution equation $dq/dt = w(q,t)$. If a sequence $\{v^l\}_{l=1}^{\infty} \subset \mathcal{A}_P^k$ converges weakly to $v \in \mathcal{A}_P^k$ in $\mathcal{H}^k$, then

$$\lim_{l\to\infty} \|(\psi^{v^l})^* b - (\psi^v)^* b\|(p_0) = 0$$

for every $p_0 \in M$.

Proof. (i) Let $\{v^l\}_{l=1}^{\infty} \subset \mathcal{A}_P^k$ and suppose that v^l converges weakly to some $v \in \mathcal{H}^k$ as $l \to \infty$. We will show that $v \in \mathcal{A}_P^k$.

By lemma 5.4.3, $\eta^{v^l}(t; t_0, x) \to \eta^v(t; t_0, x)$ as $l \to \infty$ (in the Euclidean norm) for all $t, t_0 \in [0,1]$ and $x \in \Omega$. Thus, the time-one maps generated by v^l and their inverses converge pointwise: $\phi^{v^l}(x) \to \phi^v(x)$ and $(\phi^{v^l})^{-1}(x) \to (\phi^v)^{-1}(x)$ as $l \to \infty$ for all $x \in \Omega$. Because the manifolds M and N are compact, $\phi^v(M) \subset N$ and

$(\phi^v)^{-1}(N) \subset M$. In view of these inclusions, the C^r diffeomorphism ϕ^v of Ω restricted to M is a diffeomorphism, that is, $\psi^v = \phi^v|_M \in \mathrm{Diff}^r(M, N)$.

By passing to the limit as $l \to \infty$ in the inequality $\|v\|^2_{\mathcal{H}^k} \leq \langle v - v^l, v\rangle + P\|v\|_{\mathcal{H}^k}$, it follows that $\|v\|_{\mathcal{H}^k} \leq P$. Therefore $v \in \mathcal{A}^k_P$, as required.

(ii) For simplicity, let us assume that $s = 2$. Let (U, ξ) be a chart on M at p_0. It suffices to show that

$$B^l := (\phi^{v^l})^* b(X, Y)(p_0) - (\phi^v)^* b(X, Y)(p_0) \to 0 \tag{5.17}$$

as $l \to \infty$ for all smooth vector fields X, Y on U.

Using the notation

$$\begin{aligned}
q^l &= \phi^{v^l}(p_0), \\
q &= \phi^v(p_0), \\
Z^l_y &= D\phi^{v^l} X \circ (\phi^{v^l})^{-1}(y), \\
Z_y &= D\phi^v X \circ (\phi^v)^{-1}(y), \\
Q^l_y &= D\phi^{v^l} Y \circ (\phi^{v^l})^{-1}(y), \text{ and} \\
Q_y &= D\phi^v Y \circ (\phi^v)^{-1}(y)
\end{aligned}$$

for all $y \in N$, the quantity B^l in expression (5.17) is recast in the form

$$\begin{aligned}
B^l &= b(q^l)(Z^l_{q^l}, Q^l_{q^l}) - b(q)(Z_q, Q_q) \\
&= b(q^l)(Z^l_{q^l}, Q^l_{q^l}) - b(q^l)(Z_{q^l}, Q^l_{q^l}) && (5.18) \\
&\quad + b(q^l)(Z_{q^l}, Q^l_{q^l}) - b(q^l)(Z_{q^l}, Q_{q^l}) && (5.19) \\
&\quad + b(q^l)(Z_{q^l}, Q_{q^l}) - b(q)(Z_q, Q_q). && (5.20)
\end{aligned}$$

Using definition 5.2.3 and noting that the Riemannian metric g_N is inherited from $\mathbb{R}^{n+1}$, we estimate difference (5.18) as follows:

$$b(q^l)(Z^l_{q^l}, Q^l_{q^l}) - b(q^l)(Z_{q^l}, Q^l_{q^l}) \leq \|b(q^l)\|_{g_N(q_l)} |Z^l_{q^l} - Z_{q^l}|_{\mathbb{R}^{n+1}} |Q^l_{q^l}|_{\mathbb{R}^{n+1}}.$$

By lemma 5.4.3, $|Z^l_{q^l} - Z_{q^l}|_{\mathbb{R}^{n+1}} \to 0$ as $l \to \infty$, and $|Q^l_{q^l}|_{\mathbb{R}^{n+1}}$ is uniformly bounded in $l \in \mathbb{N}$. By lemma 5.2.9, there exists a constant $C > 0$ such that $\|b(q^l)\|_{g_N(q^l)} \leq C$ for all $l \in \mathbb{N}$. Therefore, difference (5.18) converges to zero as $l \to \infty$. Similarly, it can be shown that difference (5.19) converges to zero as $l \to \infty$. Difference (5.20) converges to zero as $l \to \infty$ because $z \mapsto b(z)(Z_z, Q_z)$ is a continuous function on U. Hence, $B^l \to 0$ as $l \to \infty$. $\square$

We say that a functional $I : \mathcal{H}^k \to \mathbb{R}$ is weakly continuous on $\mathcal{H}^k$ if $I(v^l) \to I(v)$ whenever the sequence $\{v^l\}_{l=1}^{\infty} \subset \mathcal{H}^k$ converges weakly to v in $\mathcal{H}^k$.

Recall that the inequality $k \geq (n+1)/2 + r + 1$ guarantees the embedding of the Sobolev space $W_0^{k,2}(\Omega, \mathbb{R}^{n+1})$ into $C^r(\bar{\Omega}, \mathbb{R}^{n+1})$, where $r \geq 1$.

Lemma 5.2.11. *Assume that the constant $P > 0$ is large enough so that the set $\mathcal{A}^k_P$ is not empty. Let the functionals J, I_1, I_2 be defined as in definition 5.2.2.*
(i) If $k \geq (n+1)/2 + 3$, then the functionals $J : \mathcal{A}^k_P \to \mathbb{R}_+$ and $I_1 : \mathcal{A}^k_P \to \mathbb{R}_+$ are weakly continuous.
(ii) If $k \geq (n+1)/2 + 4$, then the functional $I_2 : \mathcal{A}^k_P \to \mathbb{R}_+$ is weakly continuous.

Proof. Let $\{v^l\}_{l=1}^{\infty} \subset \mathcal{A}^k_P$ and suppose that v^l converges weakly to some $v \in \mathcal{H}^k$ as $l \to \infty$ (in symbols $v^l \rightharpoonup v \in \mathcal{H}^k$). By lemma 5.2.10, $v \in \mathcal{A}^k_P$ and $J(v)$, $I_1(v)$, and $I_2(v)$ are well-defined.
(i) We will show that $\lim_{l\to\infty} J(v^l) = J(v)$.

Let $G := g^*_M \otimes g^*_M$. For tensor fields $a, b \in \mathcal{T}^0{}_2(M)$ and every $p \in M$, we have the equality

$$\left| \|a\|^2(p) - \|b\|^2(p) \right| = |G(a+b, a-b)(p)| \leq \|a+b\|(p)\|a-b\|(p).$$

By applying the Cauchy-Schwarz inequality, we obtain the inequality

$$\begin{aligned}|J(v^l) - J(v)| &\leq \int_M \|(\psi^{v^l})^*\tau_N + (\psi^v)^*\tau_N - 2\tau_M\| \|(\psi^{v^l})^*\tau_N - (\psi^v)^*\tau_N\|\omega_M \\ &\leq \Big(\int_M \|(\psi^{v^l})^*\tau_N + (\psi^v)^*\tau_N - 2\tau_M\|^2\omega_M\Big)^{1/2} \\ &\quad \times \Big(\int_M \|(\psi^{v^l})^*\tau_N - (\psi^v)^*\tau_N\|^2\omega_M\Big)^{1/2}. \end{aligned} \tag{5.21}$$

By lemma 5.2.10,

$$\lim_{l\to\infty} \|(\psi^{v^l})^*\tau_N - (\psi^v)^*\tau_N\|^2(p) = 0 \tag{5.22}$$

for all $p \in M$.

Let $K > 0$ be the constant in display (5.49) of lemma 5.4.3. By lemma 5.2.9 and because the manifold N is compact, there exists a constant $C > 0$ such that $\|\tau_N(z)\|_{g_N(z)} \leq C$ for all $z \in N$. Using the equivalence of norms (5.16), we estimate

$$\begin{aligned}\|(\psi^{v^l})^*\tau_N\|(p) &\leq n^{s/2}\|(\psi^{v^l})^*\tau_N(p)\|_{g_M(p)} \\ &\leq n^{s/2}\|\tau_N(\psi^{v^l}(p))\|_{g_N(\psi^{v^l}(p))}|D\psi^{v^l}(p)|^s \\ &\leq n^{s/2}CK^s. \end{aligned} \tag{5.23}$$

Using inequalities (5.21) and (5.23), limit (5.22), and the Dominated Convergence Theorem, we conclude that $J(v^l) \to J(v)$ as $l \to \infty$.

Let us show that the functional I_1 is weakly continuous. By an estimate analogous to (5.21), it suffices to prove the following statements.

(I) If $p \in M$ and $t \in [0,1]$, then

$$\lim_{l\to\infty} \|F^{v^l}(\cdot,t)^*g_t^{v^l} - F^v(\cdot,t)^*g_t^v\|^2(p) = 0.$$

(II) There exists $S_1 > 0$ such that

$$\|F^{v^l}(\cdot,t)^*g_t^{v^l}\|^2(p) \leq S_1$$

for all $p \in M$, $t \in [0,1]$, and $l \in \mathbb{N}$.

Because all the Riemannian metrics are inherited from $\mathbb{R}^{n+1}$, whose standard inner product is denoted by $\langle \cdot, \cdot \rangle$, we have

$$\begin{aligned} F^{v^l}(\cdot,t)^* g_t^{v^l}(p)(X,Y) - F^v(\cdot,t)^* g_t^v(p)(X,Y) &= \langle D_x F^{v^l}(p,t)X, D_x F^{v^l}(p,t)Y\rangle \\ &\quad - \langle D_x F^v(p,t)X, D_x F^v(p,t)Y\rangle, \end{aligned}$$

for all $p \in M$, $X, Y \in T_pM$, and $t \in [0,1]$, where D_x denotes the derivative with respect to the spatial variable. The right-hand side of this equation converges to zero as $l \to \infty$ by lemma 5.4.3. This completes the proof of statement (I).

By the same lemma and inequality (5.16), for every $p \in M$ we have

$$\begin{aligned} \|F^{v^l}(\cdot,t)^* g_t^{v^l}\|(p) &\leq n\|F^{v^l}(\cdot,t)^* g_t^{v^l}(p)\|_{g_M(p)} \\ &\leq n|D_x F^{v^l}(p,t)|^2 \\ &\leq nK^2. \end{aligned}$$

This inequality implies statement (II).

(ii) We will show the weak continuity of the functional I_2. By an estimate analogous to (5.21), it suffices to show two facts:

(III) If $p \in M$ and $t \in [0,1]$, then

$$\lim_{l\to\infty} \|F^{v^l}(\cdot,t)^* I\!I_t^{v^l} - F^v(\cdot,t)^* I\!I_t^v\|^2(p) = 0.$$

(IV) There exists $S_2 > 0$ such that

$$\|F^{v^l}(\cdot,t)^* I\!I_t^{v^l}\|^2(p) \leq S_2$$

for all $p \in M$, $t \in [0,1]$ and $l \in \mathbb{N}$.

We will first prove statement (IV).

Consider a morph F^w generated by a time-dependent vector field $w \in \mathcal{A}_P^k$. By definition of morphs of class $\mathcal{M}^{r,\mathrm{ac}}(M,N)$ in section **??**, the orientation of each intermediate manifold $M^{w,t}$, where $t \in [0,1]$, is induced by the C^2 diffeomorphism

$F(\cdot,t): M \to M^{w,t}$. Let $\mathcal{N}^{w,t}(z)$ denote the unit normal to the manifold $M^{w,t}$ at the point $z \in M^{w,t}$. We assume that for every positively oriented basis $\{X_i\}_{i=1}^n$ of $T_z M^{w,t}$, the set of vectors $\{X_1, \dots, X_n, \mathcal{N}^{w,t}(z)\}$ is positively oriented in $\mathbb{R}^{n+1}$.

For $p \in M$, let (U,ξ) be a chart at p, choose two smooth vector fields X and Y on U, and let $\gamma : [0,1] \to U$ be a C^1 curve at p such that $\dot\gamma(0) = X_p$. It is evident that the inner product

$$\langle \mathcal{N}^{w,t}(F^w(\gamma(s),t)), D_x F^w(\gamma(s),t) Y_{\gamma(s)} \rangle = 0 \tag{5.24}$$

for every $t, s \in [0,1]$. Let us recall that for every $t \in [0,1]$ the function $x \mapsto F^{w,t}(x,t)$ is defined for all $x \in \Omega$ and denote its second derivative at $x \in \Omega$ by $D_x^2 F(x,t)$. By differentiating expression (5.24) with respect to s at $s = 0$, we obtain the equality

$$\begin{aligned} F^w(\cdot,t)^* I\!I_t^w(p)(X_p,Y_p) &= \langle \bar\nabla_{D_x F^w(p,t) X_p} \mathcal{N}^{w,t}(F^w(p,t)), D_x F^w(p,t) Y_p \rangle \\ &= -\langle \mathcal{N}^{w,t}(F^w(p,t)), D_x^2 F^w(p,t)[X_p, Y_p] \rangle, \end{aligned} \tag{5.25}$$

where $\bar\nabla$ denotes the standard Riemannian connection on $\mathbb{R}^{n+1}$ (see [18]).

For every $p \in M$, let $W_p, Q_p \in T_p M$ be unit length vectors such that

$$\|F^{v^l}(\cdot,t)^* I\!I_t^{v^l}(p)\|_{g_M(p)} = |F^{v^l}(\cdot,t)^* I\!I_t^{v^l}(p)(W_p,Q_p)|.$$

Using inequality (5.16) and equation (5.25), we have the estimates

$$\begin{aligned} \|F^{v^l}(\cdot,t)^* I\!I_t^{v^l}\|(p) &\le n\|F^{v^l}(\cdot,t)^* I\!I_t^{v^l}(p)\|_{g_M(p)} \\ &= n|F^{v^l}(\cdot,t)^* I\!I_t^{v^l}(p)(W_p,Q_p)| \\ &= n|\langle \mathcal{N}^{v^l,t}(F^{v^l}(p,t)), D_x^2 F^{v^l}(p,t)[W_p,Q_p] \rangle| \\ &\le nK. \end{aligned}$$

This completes the proof of statement (IV).

By lemma 5.4.3, if $\alpha \in \{0,1,2\}$, then the derivative $D_x^\alpha F^{v^l}(p,t)$ converges to $D_x^\alpha F^v(p,t)$ as $l \to \infty$ in the Euclidean norm for every $p \in M$ and $t \in [0,1]$. Taking into account equation (5.25), we see that statement (III) follows from the convergence

$$\mathcal{N}^{v^l,t}(F^{v^l}(p,t)) \to \mathcal{N}^{v,t}(F^v(p,t)) \tag{5.26}$$

as $l \to \infty$ in $\mathbb{R}^{n+1}$ for every $p \in M$ and $t \in [0,1]$. $\square$

Theorem 5.2.12. *Assume that the constant $P > 0$ is large enough so that the set $\mathcal{A}_P^k$ is not empty.*
(i) If $k \geq (n+1)/2+3$, then there exists a minimizer of the bending distortion energy functional E in the admissible set $\mathcal{A}_P^k$.
(ii) If $k \geq (n+1)/2+4$, then there exists a minimizer of the morphing distortion energy functional $\mathcal{E}$ in the admissible set $\mathcal{A}_P^k$.

Proof. Let $\{v^l\}_{l=1}^\infty \subset \mathcal{A}_P^k$ be a minimizing sequence of E, that is

$$\lim_{l\to\infty} E(v^l) = \inf_{w\in\mathcal{A}_P^k} E(w) \geq 0.$$

By lemma 5.2.10, the set $\mathcal{A}_P^k$ is sequentially weakly closed and bounded. Therefore, there exists a weakly convergent subsequence $\{v^{l_k}\}_{k=1}^\infty$ with the weak limit $v \in \mathcal{A}_P^k$. The functional E is weakly continuous by lemma 5.2.11. Therefore, $E(v) = \inf_{w\in\mathcal{A}_P^k} E(w)$ and v is a minimizer of E.

The existence of minimizers for the functional $\mathcal{E}$ is proved in the same fashion. $\square$

Remark 5.2.13. Theorem 5.2.12 implies the existence of minimizers of the functional Λ defined in display (5.1) in the admissible set

$$\mathcal{B}_P^k := \{h \in \mathrm{Diff}^2(M,N) : h = \psi^v \text{ for some } v \in \mathcal{A}_P^k\}.$$

The set $\mathcal{B}_P^k$, among other maps, contains smooth diffeomorphisms $f : M \to N \subset \mathbb{R}^{n+1}$ that are homotopic to the inclusion map $i : M \to \mathbb{R}^{n+1}$ and generate time-dependent vector fields in $\mathcal{A}_P^k$.

To minimize the distortion energy of diffeomorphisms from other isotopy classes, we consider the family of maps $\{\psi^v \circ \phi \in \mathrm{Diff}^r(M,N) : v \in \mathcal{A}_P^k\}$, where ϕ is a fixed diffeomorphism of M. Similarly, given a smooth isotopy $G : [0,1]\times M \to M$, we consider the family of morphs $\{F_G^v \in \mathcal{M}^{r,\mathrm{ac}}(M,N) : v \in \mathcal{A}_P^k\}$, where $F_G^v(p,t) = F^v(G(p,t),t)$

for all $(p,t) \in M \times [0,1]$, as candidates for minimal distortion morphs. The most interesting example of this generalization is, perhaps, the case where $G(p,t) = \phi(p)$ for some fixed diffeomorphism $\phi : M \to N$, so that the admissible isotopies are from the class of morphs $F^v(\phi(p),t)$ generated by time-dependent vector fields in $\mathcal{A}_P^k$, where $p \in M$ and $t \in [0,1]$.

The latter idea leads to the definition of the functionals

$$\begin{aligned} E_\phi(v) = E_\phi(v; B_1, B_2) &= B_1 \int_M \|(\psi^v \circ \phi)^* g_N - g_M\|^2 \omega_M \\ &\quad + B_2 \int_M \|(\psi^v \circ \phi)^* I\!I_N - I\!I_M\|^2 \omega_M \end{aligned}$$

and

$$\begin{aligned} \mathcal{E}_G(v) = \mathcal{E}_G(v; B_1, B_2) &= B_1 \int_0^1 \int_M \|F_G^v(\cdot,t)^* g_t^v - g_M\|^2 \, \omega_M dt \\ &\quad + B_2 \int_0^1 \int_M \|F_G^v(\cdot,t)^* I\!I_t^v - I\!I_M\|^2 \, \omega_M dt, \end{aligned}$$

where B_1 and B_2 are nonnegative real numbers (cf. definition 5.1.2), $\phi \in \mathrm{Diff}(M)$ and $G : M \times [0,1] \to M$ is an isotopy. .

Theorem 5.2.12 can be easily generalized to show that for $P > 0$ and $k \in \mathbb{N}$ sufficiently large, both functionals E_ϕ and $\mathcal{E}_G$ have minimizers in $\mathcal{A}_P^k$ for every diffeomorphism $\phi : M \to M$ and isotopy $G : M \times [0,1] \to M$.

Theorem 5.2.14. *Assume that the constant $P > 0$ is large enough so that the set $\mathcal{A}_P^k$ is not empty. Let $\phi \in \mathrm{Diff}(M)$ and let $G : M \times [0,1] \to M$ be an isotopy.*
(i) If $k \geq (n+1)/2+3$, then there exists a minimizer of the bending distortion energy functional E_ϕ in the admissible set $\mathcal{A}_P^k$.
(ii) If $k \geq (n+1)/2+4$, then there exists a minimizer of the morphing distortion energy functional $\mathcal{E}_G$ in the admissible set $\mathcal{A}_P^k$.
(iii) If $k \geq (n+1)/2+3$, then both functionals Φ and Λ defined in displays (1.1) *and* (5.1) *respectively have minimizers in the admissible set*

$$\mathcal{B}_{P,\phi}^k := \{h \in \mathrm{Diff}^2(M,N) : h = \psi^v \circ \phi \text{ for some } v \in \mathcal{A}_P^k\}.$$

The latter theorem is an easy generalization of theorem 5.2.12. More precisely, let $\{b^l\}_{l=1}^{\infty}$ be a sequence of tensor fields in $\mathcal{T}^0{}_s(M)$ such that $\lim_{l\to\infty}\|b^l\|(p) = 0$ and $\|b^l(p)\| \le K$ for all $p \in M$ and $l \in \mathbb{N}$, where K is a positive constant and let $\phi \in \mathrm{Diff}(M)$. Then $\lim_{l\to\infty}\|\phi^* b^l\|(p) = 0$ and there exists a constant $K_1 > 0$ such that $\|\phi^* b^l(p)\| \le K_1$ for all $p \in M$ and $l \in \mathbb{N}$. Using the above observation, lemma 5.2.11 is easily generalized to the case where ψ^v and F^v are replaced with $\psi^v \circ \phi$ and F_G^v respectively, and the proof of theorem 5.2.12 remains the same.

In theorem 5.2.14, the statement (iii), which is equivalent to theorem 5.1.4, follows from the statement (i).

5.3 A Minimal Distortion Morph

We have proved the existence of minimizers of the functionals E and $\mathcal{E}$, which produce minimal distortion diffeomorphisms and morphs between manifolds M and N. In this section, we consider the special case where $M = \mathbb{S}^1$ is the unit circle in the plane and $N = \mathbb{S}^1_R$ is the concentric circle of radius $R > 1$ and construct a minimal distortion diffeomorphism and morph between them.

Our example of a minimal distortion morph in subsection 5.3.2 demonstrates the importance of the bound $\|v\|_{\mathcal{H}^k} \le P$ in the definition of the admissible set $\mathcal{A}^k_P$. If this bound is not imposed, there is a minimizing sequence of morphs $\{F_n\}_{n=1}^{\infty}$ such that the distortion energy

$$\begin{aligned}\Psi(F_n) \quad := \quad & \int_0^1 \int_M \|F_n(\cdot,t)^* g_t^n - g_M\|^2\, \omega_M\, dt \\ & + \int_0^1 \int_M \|F_n(\cdot,t)^* I\!I_t^n - I\!I_M\|^2\, \omega_M\, dt \end{aligned} \tag{5.27}$$

tends to zero, where g_t^n and $I\!I_t^n$ are the first and the second fundamental forms of the intermediate manifold $F_n(M,t)$ induced by its embedding into $\mathbb{R}^2$. An example of such a sequence is $F_n(p,t) = \phi_n(t)p$ for all $t \in [0,1]$ and $p \in M$, where $\phi_n \in C^\infty(0,1) \cap C[0,1]$ is a function whose values remain in the segment $[1,R]$ and such that $\phi_n(t) = 1$ for all $t \in [0, 1-1/n]$ and $\phi_n(1) = R$. From the representation of (5.27) in local coordinates (see (5.30)) we derive

$$\begin{aligned}\Psi(F_n) &= 2\pi \int_0^1 \left[(\phi_n^2 - 1)^2 + (\phi_n - 1)^2\right] dt \\ &\leq 2\pi[(R^2-1)^2 + (R-1)^2]\frac{1}{n};\end{aligned} \tag{5.28}$$

hence, $\lim_{n\to\infty} \Psi(F_n) = 0$. On the other hand, there is no morph H in the space $\mathcal{M}^{r,\mathrm{ac}}(M, N)$ with $r > 1$ such that $\Psi(H) = 0$: otherwise, $H(\cdot, 1)$ would be an isometry between M and N. The sequence $\{F_n\}_{n=1}^{\infty}$ converges pointwise to the discontinuous morph

$$F(p,t) = \begin{cases} p, & \text{if } 0 \leq t < 1, \\ Rp, & \text{if } t = 1 \end{cases}$$

whose distortion energy $\Psi(F)$ vanishes.

Theorem 5.2.12 implies that every sequence of time-dependent vector fields $\{v^n\}_{n=1}^{\infty} \subset \mathcal{H}^k$ such that each $v^n \in \mathcal{H}^k$ generates the morph F_n must be unbounded in $\mathcal{H}^k$.

In our example of a minimal distortion morph, we solve the optimization problem for the minimal distortion morph between $\mathbb{S}^1$ and $\mathbb{S}^1_R$ in the class of morphs, whose intermediate states are circles of increasing radii, that are generated by time-dependent vector fields whose norms are uniformly bounded by a positive constant P. Numerical solutions suggest that the second time-derivative $\partial^2 F/\partial t^2$ of the minimal morph F increases as P increases. In effect, the choice of the constant P in the definition of the admissible set $\mathcal{A}^k_P$ sets a restriction on the magnitude of the curvature of the curves $t \mapsto F(p,t)$, where $p \in M$.

We begin the construction of the minimal distortion morph with the example of a minimal distortion diffeomorphism between $\mathbb{S}^1$ and $\mathbb{S}^1_R$. This example is based on the theory of minimal deformation (as measured by the functional Φ) bending of regular simple closed curves developed in [5].

5.3.1 A Minimal Distortion Diffeomorphism between two Circles

We will construct a minimal distortion diffeomorphism between the circles $M = \mathbb{S}^1$ and $N = \mathbb{S}^1_R$.

For $r \geq 1$, we consider the functional $\Lambda : \mathrm{Diff}^r(M, N) \to \mathbb{R}_+$ defined in display (5.1). Also, using the radius $R > 1$ of $\mathbb{S}^1_R$, we define the radial map $h_R : \mathbb{R}^2 \to \mathbb{R}^2$ by $h_R(p) = Rp$.

Lemma 5.3.1. *The restriction of the radial map h_R to $\mathbb{S}^1$ minimizes the functional $\Lambda : \mathrm{Diff}^2(M, N) \to \mathbb{R}_+$ defined in display* (5.1)*.*

Proof. Fix $p \in M$ and $q \in N$, and let $\gamma : [0, L(M)) \to M$ and $\xi : [0, L(N)) \to N$ be the (positive orientation) arc length parametrizations of M and N respectively such that $\gamma(0) = p$ and $\xi(0) = q$. The distortion energy functional Λ can be recast in the form

$$\begin{aligned} \Lambda(u) &= \int_0^{L(M)} (\dot{u}^2 - 1)^2\, dt + \int_0^{L(M)} (\frac{1}{R}\dot{u}^2 - 1)^2\, dt \\ &=: J_1(u) + J_2(u), \end{aligned} \tag{5.29}$$

where $u = \xi^{-1} \circ h \circ \gamma : [0, L(M)) \to [0, L(N))$ is a local coordinate representation of $h \in \mathrm{Diff}(M, N)$ with $h(p) = q$. By lemma 4.1 in [5], the functions $u_1(t) = L(N)/L(M)t$ and $u_2(t) = -L(N)/L(M)t + L(N)$ minimize the functional J_1 in the admissible set

$$\mathcal{B} = \big\{u \in C^2(0, L(M)) \cap C([0, L(M)]) : u \text{ is a bijection onto } [0, L(N)]\big\}.$$

The proof of the statement that u_1 and u_2 minimize the functional J_2 in $\mathcal{B}$ follows along the same lines.

Therefore, the map $h_R|_{\mathbb{S}^1}$ minimizes the functional (5.1) over the set of all maps $h \in \mathrm{Diff}^2(M, N)$ such that, for our fixed $p \in M$, $h(p) = Rp$.

If $h \in \mathrm{Diff}^2(M, N)$ is such that $h(p) = q \neq Rp$, consider an isometry $f : N \to N$ such that $f(q) = Rp$. Because $f^* g_N = g_N$ and $f^* I\!I_N = I\!I_N$, we obtain $\Lambda(h) = \Lambda(f \circ h) \geq \Lambda(h_R|_{\mathbb{S}^1})$, which proves the lemma. $\square$

As before, let ψ^v denote the time-one map of the vector field $v \in \mathcal{A}_P^k$ restricted to M. Using lemma 5.3.1, it is easy to construct a time-dependent vector field that minimizes the functional $E(v) = \Lambda(\psi^v)$ in the admissible set $\mathcal{A}_P^k$. In fact, every vector field $v^0 \in \mathcal{A}_P^k$ that generates the time-one map ϕ^v such that its restriction to M is $\psi^v = h_R|_M$, minimizes the functional E. An example of such a vector field is

$$v(x,t) = \rho(x)w(x,t)$$

for all x in the open ball $\Omega := B(0, R+2) \subset \mathbb{R}^2$ and $t \in [0,1]$, where

$$w(x,t) = \frac{R-1}{1+(R-1)t}x$$

and $\rho : \mathbb{R}^2 \to \mathbb{R}$ is a bump function such that $0 \leq \rho \leq 1$, $\rho \equiv 1$ in the open ball $B(0, R+1) \subset \mathbb{R}^2$, and $\rho \equiv 0$ on Ω^c. The vector field v generates the morph $F^v(p,t) = (1+(R-1)t)p$, whose time-one map restricted to M is $\psi^v(p) = Rp$.

5.3.2 A Minimal Distortion Morph between two Circles

Let us assume, as before, that $M = \mathbb{S}^1$, $N = \mathbb{S}_R^1$, and $R > 1$.

In the previous subsection, we have constructed a minimizer of the functional E; the construction was quite straight-forward. The time-integral involved in the definition of the functional $\mathcal{E}$ makes the construction of its minimizer a much more intricate process. We will restrict our attention to morphs that operate through images that are concentric circles, while leaving open the question whether a minimizer must be purely radial, as the problem of constructing a minimal morph within this family is difficult enough. Note that although our functional is formally defined in terms of time-dependent vector fields, it is the resulting morphs we will be working with directly.

We will construct a minimal distortion morph between the circles $M = \mathbb{S}^1$ and $N = \mathbb{S}_R^1$ in case $R > 1$. As before, let $\mathcal{M}^{3,\mathrm{ac}}(M,N)$ be the class of morphs between the manifolds M and N that are absolutely continuous in time and class C^3 in the spatial variable. Recall that for a morph $F \in \mathcal{M}^{3,\mathrm{ac}}(M,N)$ we define $f^t = F(\cdot,t) \in \mathrm{Diff}^3(M,M^t)$. We assume that the morph F is generated by a time-dependent vector field $v \in \mathcal{A}_P^k$.

Consider the functional $\Psi : \mathcal{M}^{r,\mathrm{ac}}(M,N) \to \mathbb{R}_+$, where $r \geq 1$, defined by

$$\Psi(F) = \int_0^1 \int_M \|(f^t)^* g_t - g_0\|^2 \, \omega_M \, dt + \int_0^1 \int_M \|(f^t)^* I\!I_t - I\!I_0\|^2 \, \omega_M \, dt,$$

where g_t and $I\!I_t$ are the first and the second fundamental forms on the intermediate state M^t induced by its isometric embedding into $\mathbb{R}^2$. We notice that $\mathcal{E}(v;1,1) = \Psi(F^v)$ for all $v \in \mathcal{A}_P^k$ (see definition 5.1.2).

Fix a point $p \in M$. Let γ be an arc-length parametrization of M that induces the positive orientation on M with $\gamma(0) = p$. Let ξ^t be the arc length reparametrization of M^t obtained from the parametrization $f^t \circ \gamma$ such that $\xi^t(0) = f^t \circ \gamma(p)$ and both ξ^t and $f^t \circ \gamma$ induce the same orientation of M^t. Such a parametrization can be obtained by solving the equation $s(t,x) = y$ for x, where $s(t,x) = \int_0^x |f^t \circ \gamma(\tau)| \, d\tau$ is the arc length function of the curve M^t. Using the implicit solution $x(t,y)$ of $s(t,x) = y$, we define $\xi^t(y) = f^t \circ \gamma \circ x(t,y)$. Because the morph F is generated by a time-dependent vector field $v \in \mathcal{A}_P^k$, lemma 5.4.2 implies that the function $t \mapsto Df^t(p)$, where $p \in M$, is absolutely continuous. It follows that the function $t \mapsto \xi^t(s)$ is continuous for every $s \in [0, L(M^t))$.

The local representation of f^t is given by $u^t(s) = (\xi^t)^{-1} \circ f^t \circ \gamma(s)$, where $s \in [0, 2\pi]$, and the energy $\Psi(F)$ of the morph F is

$$\begin{aligned} \Psi(F) &= \int_0^1 \int_0^{2\pi} \Big(\Big(\frac{du^t}{ds} \Big)^2 - 1 \Big)^2 \, ds \, dt \\ &\quad + \int_0^1 \int_0^{2\pi} \Big(\kappa_t(u^t) \Big(\frac{du^t}{ds} \Big)^2 - 1 \Big)^2 \, ds \, dt, \end{aligned} \tag{5.30}$$

where $\kappa_t : [0, L(M^t)] \to \mathbb{R}$ is the curvature function of the intermediate state M^t.

Let us restrict our attention to the morphs whose intermediate states are circles of increasing radii such that each intermediate state M^t of such a morph F is a circle of radius $\psi(t)$ with $\psi \in C^2(0,1) \cap C[0,1]$ a (strictly) increasing function. In symbols,

$$\psi \in Q_+ := \{\phi \in C^2(0,1) \cap C[0,1] : \phi(0) = 1, \phi(1) = R, \text{ and } \phi \text{ is increasing}\}.$$

The curvature function of M^t is given by $\kappa_t \equiv 1/\psi(t)$. By lemma 5.3.1, the radial map between the circles M and M^t minimizes the functional

$$f^t \mapsto \int_M \|(f^t)^* g_t - g_0\|^2 \, \omega_M + \int_M \|(f^t)^* I\!I_t - I\!I_0\|^2 \, \omega_M.$$

Therefore,

$$\Psi(F) \geq \Psi(H) = 2\pi \int_0^1 (\psi^2 - 1)^2 \, dt + 2\pi \int_0^1 (\psi - 1)^2 \, dt,$$

where the morph $H \in \mathcal{M}^{\infty,2}(M,N)$ is given by $H(p,t) = \psi(t)p$.

To determine the morph $H(p,t) = \psi(t)p$ of smallest distortion energy $\Psi(H)$, we will minimize the functional $J : L^4(0,1) \to \mathbb{R}_+$ defined by

$$J(\psi) := \int_0^1 (\psi^2 - 1)^2 \, dt + \int_0^1 (\psi - 1)^2 \, dt \tag{5.31}$$

over all admissible radial functions ψ. To define the admissible set for the functional J, let us put this example into the context of time-dependent vector fields.

Let Ω be the open ball of radius $R + 2$ in $\mathbb{R}^2$. Given a morph $H(p,t) = \psi(t)p$ (where $\psi \in Q_+$, $t \in [0,1]$, and $p \in M$), let us construct a time-dependent vector field $v \in \mathcal{H}^5 = L^2(0,1; W_0^{5,2}(\Omega; \mathbb{R}^2))$ that generates H, where the number of weak derivatives $k = 5$ is chosen in view of condition (ii) of theorem 5.2.12.

Consider the class of morphs of the plane $\mathbb{R}^2$ that have the form $F(x,t) = \psi(t)x$, where $\psi \in Q_+$. Define a time-dependent vector field $\bar{v} : \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2$ by

$$\bar{v}(F(x,t),t) = \frac{\partial F}{\partial t}(t,x)$$

or, equivalently,

$$\bar{v}(x,t) = \frac{\psi'(t)}{\psi(t)} x.$$

Clearly, the morph F satisfies the differential equation $dq/dt = \bar{v}(q,t)$. To obtain the required vector field v, multiply $\bar{v}$ by a bump function $\rho : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\rho \equiv 1$ on the ball $B(0, R+1)$, $\rho \equiv 0$ on Ω^c, and $0 \leq \rho \leq 1$. The vector field

$$v(x,t) = \frac{\psi'(t)}{\psi(t)}\rho(x)x$$

belongs to the Hilbert space $\mathcal{H}^k$ and generates the morph

$$H(p,t) := \psi(t)p = F|_{M\times[0,1]}(p,t)$$

for all $(p,t) \in M \times [0,1]$.

In theorem 5.2.12, we require the admissible set $\mathcal{A}_P^k$, for some fixed $P > 0$, to contain all vector fields $v \in \mathcal{H}^k$ such that the norm of v is bounded by P and v generates a morph between the manifolds M and N.

Therefore, in addition to the assumption that $\psi \in Q_+$, we must assume that the time-dependent vector fields of the form $v(x,t) = \frac{\psi'(t)}{\psi(t)}\rho(x)x$ are bounded in $\mathcal{H}^k$ by a fixed constant $P > 0$. In symbols, the required bound is

$$\|v\|^2_{\mathcal{H}^k} = \|\rho \cdot \mathrm{id}_\Omega\|^2_{W_0^{5,2}(\Omega;\mathbb{R}^2)} \int_0^1 \left(\frac{\psi'}{\psi}\right)^2 dt \leq P^2.$$

After introducing the constant

$$A := \frac{P^2}{\|\rho \cdot \mathrm{id}_\Omega\|^2_{W_0^{5,2}(\Omega;\mathbb{R}^2)}}, \tag{5.32}$$

we obtain the constraint

$$G(\psi) := \int_0^1 \left(\frac{\psi'}{\psi}\right)^2 dt - A \leq 0. \tag{5.33}$$

Note that the functional J can be written in the form

$$J(\psi) = \int_0^1 u(\psi)\,dt,$$

where the smooth function $u(s) = (s^2-1)^2 + (s-1)^2$ is strictly increasing on $(1,\infty)$.

To find a morph $H(p,t) = \psi(t)p$ with $\psi \in Q_+$, which has minimal distortion among the morphs $F \in \mathcal{M}^{3,\mathrm{ac}}(M,N)$ whose intermediate states are circles with increasing radii, we must solve the optimization problem

$$\begin{aligned}&\text{minimize } J(\psi)\\&\text{for } \psi \in Q_+ = \{\phi \in C^2(0,1) \cap C[0,1] :\\&\phi(0) = 1,\ \phi(1) = R,\ \text{and } \phi \text{ is increasing}\}\\&\text{subject to } G(\phi) \le 0.\end{aligned} \tag{5.34}$$

The solution of problem (5.34) is obtained using the following outline: We will consider the related optimization problem

$$\begin{aligned}&\text{minimize } J(\psi),\\&\psi \in Q^{1,2} := \{\phi \in W^{1,2}(0,1) : \phi(0) = 1, \phi(1) = R\}\\&\text{subject to } G(\phi) \le 0,\end{aligned} \tag{5.35}$$

where (because every function $\psi \in Q^{1,2}$ is absolutely continuous on $[0,1]$) the boundary conditions in the definition of the set $Q^{1,2}$ are to be understood in the classical sense. We will determine the unique minimizer ψ of the optimization problem (5.35) and show that ψ is an increasing C^2 function. Because $Q_+ \subset Q^{1,2}$, the same function ψ is the unique solution of optimization problem (5.34).

Lemma 5.3.2. *There exists a unique solution $\psi \in Q^{1,2}$ of the optimization problem* (5.35)*. Moreover, $1 \le \psi(t) \le R$ for all $t \in [0,1]$.*

Lemma 5.3.2 is proved using the direct method of the calculus of variations. First, we prove the existence of a minimizer of the functional J subject to the constraint $G(\psi) \le 0$ in the admissible set $W^{1,4/3}(0,1)$ with the appropriate boundary conditions, and then we show that the minimizer is, in fact, in class $W^{1,2}(0,1)$. The inequalities $1 \le \psi$ and $\psi \le R$ are proved by contradiction using the cut-off functions $h_1(t) = \max\{1, \psi(t)\}$ and $h_2(t) = \min\{R, \psi(t)\}$, which would yield smaller values of the functional J than the minimizer.

Proof. The proof consists of two main steps: (1) Using the direct method of the calculus of variations, we will prove the existence of a minimizer for the auxiliary optimization problem

$$\begin{array}{l}\text{minimize } J(\psi),\\ \psi \in Q^{1,4/3} := \{\phi \in W^{1,\frac{4}{3}}(0,1) : \psi(0)=1, \psi(1)=R\}\\ \text{subject to } G(\psi) \leq 0.\end{array} \tag{5.36}$$

(2) We will show that the minimizer for problem (5.36) is in $W^{1,2}(0,1)$.

If follows that this minimizer is a minimizer of the optimization problem (5.35).

Let $\{\psi_n\}_{n=1}^{\infty} \subset Q^{1,4/3}$ be a minimizing sequence for the optimization problem (5.36). In particular, $G(\psi_n) \leq 0$ for every positive integer n. In symbols,

$$J(\psi_n) \to \inf_{\psi \in Q^{1,4/3}, G(\psi)\leq 0} J(\psi).$$

We claim that the minimizing sequence is bounded in $W^{1,4/3}(0,1)$. To prove this fact, we use the triangle inequality for the $L^2(0,1)$ norm to make the estimate

$$\begin{aligned}(\int_0^1 \psi_n^4 \, dt)^{1/2} &= (\int_0^1 (\psi_n^2 - 1 + 1)^2 \, dt)^{1/2}\\ &\leq J(\psi_n)^{1/2} + 1\\ &\leq \sqrt{M} + 1,\end{aligned}$$

where $M > 0$ is a uniform bound for the convergent sequence $\{J(\psi_n)\}_{n=1}^{\infty}$. By Hölder's inequality with the conjugate constants 3 and 3/2,

$$\begin{aligned}\int_0^1 |\psi_n'|^{4/3} \, dt &= \int_0^1 |\psi_n|^{4/3} \Big(\frac{|\psi_n'|}{|\psi_n|}\Big)^{4/3} dt\\ &\leq \Big(\int_0^1 |\psi_n|^4 \, dt\Big)^{1/3} \Big(\int_0^1 \Big(\frac{|\psi_n'|}{|\psi_n|}\Big)^2 dt\Big)^{2/3}\\ &\leq \big((\sqrt{M}+1)A\big)^{2/3},\end{aligned}$$

as required.

Because the Banach space $W^{1,4/3}(0,1)$ is reflexive, $\psi_n \rightharpoonup \psi$ weakly in $W^{1,4/3}(0,1)$ for some $\psi \in W^{1,4/3}(0,1)$, up to a subsequence. We have $\psi \in Q^{1,4/3}$ because the subspace $W_0^{1,4/3}(0,1)$ is weakly closed in $W^{1,4/3}(0,1)$.

The integrands $(\psi^2 - 1)^2 + (\psi - 1)^2$ and $(\psi')^2/\psi^2$ of J and G respectively are both convex functions of ψ'. Therefore, the functionals J and G are weakly lower semicontinuous in $W^{1,4/3}(0,1)$ (see theorem 1, Sec. 8.2 in [15]). But then $G(\psi) \leq \liminf_{n\to\infty} G(\psi_n) \leq 0$ and $\psi \in Q^{1,4/3}$ solves optimization problem (5.36).

To prove that $\psi \geq 1$, let us assume, on the contrary, that there exists (in the usual topology of $[0,1]$) an open set W of positive measure such that $\psi(t) < 1$ for all $t \in W$. Define the cut-off function $h_1 \in Q^{1,4/3}$ by $h_1(t) = \max\{1, \psi(t)\}$. It is easy to check that $G(h_1) \leq 0$ and that $J(h_1) < J(\psi)$, which contradicts the minimizing property of ψ. The inequality $\psi(t) \leq R$ for all $t \in [0,1]$ can be verified in a similar fashion, using the cut-off function $h_2(t) = \min\{R, \psi(t)\}$.

Using the inequality $\psi \leq R$, we have the estimate

$$\int_0^1 (\psi')^2\, dt = \int_0^1 \psi^2 \Big(\frac{\psi'}{\psi}\Big)^2\, dt \leq R^2\, A.$$

Therefore, ψ belongs to the space $W^{1,2}(0,1)$.

Finally, the uniqueness of ψ follows from the fact that the equality $J(\psi_1) = J(\psi_2)$, where $\psi_1, \psi_2 \in Q^{1,2}$ are such that $1 \leq \psi_{1,2} \leq R$, implies $u \circ \psi_1(t) = u \circ \psi_2(t)$ for all $t \in [0,1]$, where the function $u(s) = (s^2-1)^2 + (s-1)^2$ is strictly increasing on $(1, +\infty)$.

$\square$

Lemma 5.3.3. *If the constant A in definition* (5.33) *satisfies the inequality $A > (\log R)^2$ (see also equation* (5.32)*) and $\psi \in Q^{1,2}$ is the solution of the optimization problem* (5.35)*, then there exists a constant $\lambda > 0$ such that*

(i) ψ is a critical point of the functional $J + \lambda G$ over the space of variations $W_0^{1,2}(0,1)$, and

(ii) $G(\psi) = 0$.

Moreover, the solution ψ of the optimization problem (5.35) *is in class $C^2(0,1)$.*

Lemma 5.3.3 follows from the Generalized Kuhn-Tucker theorem (see theorem 2.14.14) and a regularity result for weak solutions of Euler-Lagrange equations (see theorem 2.14.10).

Proof. Statements (i) and (ii) follow from the generalized Kuhn-Tucker theorem (see theorem 2.14.14).

We will verify that the minimizer ψ is a regular point of the inequality $G(\psi) \leq 0$. We leave it to the reader to verify that the functionals J and G are Gateaux differentiable at $\psi \geq 1$.

It suffices to show that there exists $h \in W_0^{1,2}(0,1)$ such that

$$\delta G(\psi, h) = \int_0^1 \frac{\psi'}{\psi^3}(h'\psi - \psi' h)\, dt < 0,$$

where $\delta G(\psi, h)$ is the Gateaux derivative of G in the direction h.

Assume, on the contrary, that

$$\int_0^1 \frac{\psi'}{\psi^3}(h'\psi - \psi' h)\, dt = 0 \tag{5.37}$$

for all $h \in W_0^{1,2}(0,1)$. Then ψ satisfies the Euler-Lagrange equation for the functional G whose associated Lagrangian $(\psi'/\psi)^2$ has a positive second derivative with respect to ψ'. By a regularity result for weak solutions of Euler-Lagrange equations (see theorem 2.14.10), ψ is of class $C^2(0,1)$. Therefore, we can integrate by parts in equation (5.37) to obtain the differential equation $\psi'' = (\psi')^2/\psi$.

The function $t \mapsto R^t$ is the unique solution of the latter differential equation satisfying the boundary conditions $\psi(0) = 1$ and $\psi(1) = R$. Therefore, the solution ψ of the optimization problem (5.35) must be $\psi(t) = R^t$. But, there is a function $h_\beta \in Q^{1,2}$ such that $G(h_\beta) \leq 0$ and $J(h_\beta) < J(\psi)$, in contradiction to the minimizing property of ψ. In fact, a family of such functions is given by

$$h_\beta(t) := \begin{cases} 1, & \text{if } t \in [0, \beta]; \\ \frac{R^{2\beta}-1}{\beta}(t-\beta) + 1, & \text{if } t \in (\beta, 2\beta]; \\ R^t, & \text{if } t \in (2\beta, 1] \end{cases}$$

for $\beta > 0$ sufficiently small.

The C^2 regularity of the solution ψ of the optimization problem (5.35) follows from (i) and the special form of the Lagrangian

$$L(q,p) = (q^2-1)^2 + (q-1)^2 + \lambda\Big(\frac{p}{q}\Big)^2$$

associated with the functional $J + \lambda G$ for $\lambda > 0$: it has positive second derivative with respect to p on a neighborhood U of the set $\{(\psi(t), \psi'(t)) : t \in [0, 1]\}$ (see theorem 2.14.10). $\square$

Theorem 5.3.4. *If the constant A in definition* (5.33) *satisfies the inequality $A > (\log R)^2$, then there exists a unique function $\psi \in C^2(0,1) \cap Q^{1,2}$ satisfying conditions (i) and (ii) of lemma 5.3.3 and the following properties.*

(iii) The function ψ is strictly increasing and solves the initial value problem

$$\begin{cases} \psi' = \frac{1}{\sqrt{\lambda}}\psi\sqrt{\mu + (\psi^2 - 1)^2 + (\psi - 1)^2}, \\ \psi(0) = 1, \end{cases} \tag{5.38}$$

where the pair of positive constants λ and μ is the unique solution of the system of equations

$$\int_1^R \frac{ds}{s\sqrt{\mu + (s^2 - 1)^2 + (s - 1)^2}} = \frac{1}{\sqrt{\lambda}} \tag{5.39}$$

and

$$\frac{1}{\sqrt{\lambda}} \int_1^R \frac{\sqrt{\mu + (s^2 - 1)^2 + (s - 1)^2}}{s}\, ds = A. \tag{5.40}$$

(iv) The function ψ is the unique solution of the optimization problem (5.34).

Proof. If $\psi \in Z := C^2(0,1) \cap Q^{1,2}$ is a critical point of the functional $J_\lambda := J + \lambda G : W^{1,2}(0,1) \to \mathbb{R}_+$, then ψ satisfies the Euler-Lagrange equation for J_λ, which is equivalent to the Hamiltonian system

$$\begin{cases} \psi' = \frac{\partial H}{\partial p}(\psi, p), \\ p' = -\frac{\partial H}{\partial \psi}(\psi, p) \end{cases}$$

with the Hamiltonian $H(\psi, p) = p\psi' - L(\psi, \psi')$, where

$$L(\psi, \psi') = (\psi^2 - 1)^2 + (\psi - 1)^2 + \lambda\Big(\frac{\psi'}{\psi}\Big)^2$$

is the integrand of J_λ and $p := \frac{\partial L}{\partial \psi'}(\psi, \psi')$ (see, for example, [10]). Moreover, the Hamiltonian $H(\psi, p)$ is constant along the solutions of the Euler-Lagrange equation for J_λ. Let us denote this constant by μ.

It is easy to see that

$$p = 2\lambda \frac{\psi'}{\psi^2}$$

and the Hamiltonian is given by

$$H(\psi, p) = \frac{1}{4\lambda} p^2 \psi^2 - (\psi^2 - 1)^2 - (\psi - 1)^2.$$

Note that the equation $\psi' = \frac{\partial H}{\partial p}(\psi, p)$ yields $\psi' = \frac{1}{2\lambda} p \psi^2$. By solving the Hamiltonian energy equation

$$\frac{1}{4\lambda} p^2 \psi^2 - (\psi^2 - 1)^2 - (\psi - 1)^2 = \mu \tag{5.41}$$

for p and substituting, we obtain a first-order differential equation for ψ:

$$\psi' = \frac{1}{\sqrt{\lambda}} \psi \sqrt{\mu + (\psi^2 - 1)^2 + (\psi - 1)^2}\,. \tag{5.42}$$

The case with the negative square root is eliminated because the conditions $\psi(0) = 1$ and $\psi(1) = R > 1$ can be used to show that the derivative of ψ is non negative on $(0, 1)$.

In view of equation (5.41), it is easy to see that $\mu + (\psi^2 - 1)^2 + (\psi - 1)^2 \geq 0$ for all $\psi \in Z$. Because $\psi(0) = 1$, we have $\mu \geq 0$. Also, it follows immediately from equation (5.42) that ψ is an increasing function.

Let us use the notation $u(s) = (s^2 - 1)^2 + (s - 1)^2$ and recall that u is a strictly increasing function on $(1, \infty)$. After integrating both sides of equation (5.42) over the interval $0 \leq t \leq 1$ and making the substitution $s = \psi(t)$, we obtain the relation

$$\int_1^R \frac{ds}{s\sqrt{\mu + u(s)}} = \frac{1}{\sqrt{\lambda}}. \tag{5.43}$$

Another relation of λ and μ is obtained from condition (ii) in lemma 5.3.3 (see equation (5.33) for the definition of G). The integrand in the definition of G contains the quantity $(\psi')^2$, which we view as $\psi'\psi'$. We substitute the right-hand side of equation (5.42) for one factor ψ' of this square and leave the other factor ψ' in the resulting integrand. After making the change of variables $s = \psi(t)$, we obtain the equivalent relation

$$\frac{1}{\sqrt{\lambda}} \int_1^R \frac{\sqrt{\mu + u(s)}}{s}\, ds = A. \tag{5.44}$$

We claim that there exists a unique solution (μ, λ) of the equations (5.43) and (5.44). To prove this, substitute for $1/\sqrt{\lambda}$ from equation (5.43) into equation (5.44) to obtain the equation

$$A = f(\mu) \quad := \quad \int_1^R \frac{\sqrt{\mu + u(s)}}{s}\, ds \int_1^R \frac{1}{s\sqrt{\mu + u(s)}}\, ds. \tag{5.45}$$

Make the change of variables $t = u(s)$ in both integrals in display (5.45) and then write $f(\mu)$ as a double integral to obtain the formula

$$f(\mu) = \int_0^{u(R)} \int_0^{u(R)} \frac{\sqrt{\mu + t}}{\sqrt{\mu + s}} \frac{1}{H(t)H(s)} ds\, dt, \tag{5.46}$$

where $H(t) := u^{-1}(t)u'(u^{-1}(t)) \geq 0$ for all $t \in [0, u(R)]$.

By inspection of equation (5.45), it is easy to see that $\lim_{\mu \to 0+} f(\mu) = +\infty$ and $\lim_{\mu \to \infty} f(\mu) = \log^2(R)$. We will show that f is a decreasing function, which guarantees the existence of a unique solution of the equation $f(\mu) = A$ for all $A > \log^2(R)$.

Using formula (5.46), we compute

$$f'(\mu) = \frac{1}{2} \int_0^{u(R)} \int_0^{u(R)} \frac{s - t}{(\mu + s)^{3/2}(\mu + t)^{1/2}} \frac{1}{H(t)H(s)} ds\, dt.$$

Let $D_+ = \{(s, t) \in [0, u(R)]^2 : s > t\}$ and $D_- = \{(s, t) \in [0, u(R)]^2 : s < t\}$. After making a change of variables $\gamma(s, t) = (t, s)$, we see that

$$\iint_{D_+} \frac{s - t}{(\mu + s)^{3/2}(\mu + t)^{1/2}} \frac{1}{H(t)H(s)} ds\, dt = \iint_{D_-} \frac{t - s}{(\mu + t)^{3/2}(\mu + s)^{1/2}} \frac{1}{H(t)H(s)} ds\, dt.$$

Therefore,

$$\begin{aligned} 2f'(\mu) &= \iint_{D_+ \cup D_-} \frac{s - t}{(\mu + s)^{3/2}(\mu + t)^{1/2}} \frac{1}{H(t)H(s)} ds\, dt \\ &= -\iint_{D_-} \frac{(s - t)^2}{(\mu + s)^{3/2}(\mu + t)^{3/2}} \frac{1}{H(t)H(s)} ds\, dt < 0. \end{aligned}$$

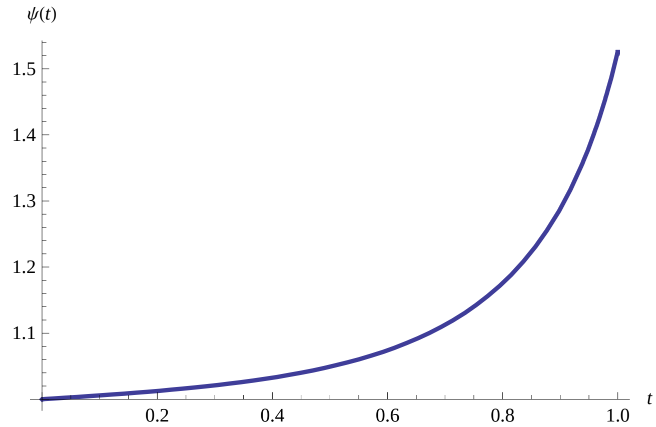

Figure 5.2: Graph of the radius function ψ with $R = 2$, $\mu = 0.001$, $\lambda = 0.306067$, and $A = 1.56296$.

This completes the proof that f is a decreasing function.

There exists a unique solution μ of the equation $f(\mu) = A$ provided that $A > \log^2(R)$. The constant λ is then easily found from equation (5.43).

Having found the unique solution (μ, λ) of the system (5.43) and (5.44), we solve the initial value problem (5.38). In fact, this initial value problem is equivalent to the integral equation

$$\int_1^{\psi(t)} \frac{ds}{s\sqrt{\mu + u(s)}} = \frac{1}{\sqrt{\lambda}} t. \tag{5.47}$$

It follows that the unique solution ψ of the initial value problem (5.38) exists for all $t \in [0, 1]$ and, because of condition (5.43), satisfies $\psi(1) = R$. $\square$

Figs. 5.2 and 5.3 depict graphs of the minimizer ψ of the optimization problem (5.34) with $R = 2$ and $A = f(\mu)$ in case $\mu = 0.001$ for Fig. 5.2 and $\mu = 500$ for Fig. 5.3. Because f is a decreasing function of μ, Fig. 5.2 corresponds to a larger constant A. These plots illustrate that the second derivative of the radius function ψ corresponding to the minimal morph increases as the constant A in definition (5.33) increases.

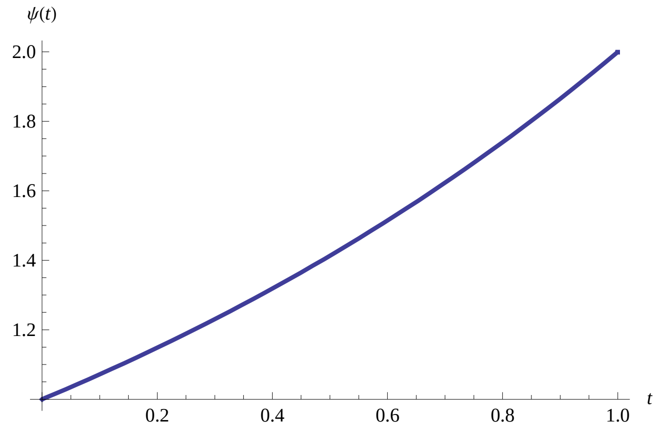

Figure 5.3: Graph of the radius function ψ with $R = 2$, $\mu = 500$, $\lambda = 1045.58$, and $A = 0.480456$.

5.4 Existence and Convergence Results for Evolution Operators

In this section we state results on existence and convergence of certain evolution operators.

We denote the Euclidean norm of an element $A \in \mathbb{R}^m$, where $m \in \mathbb{N}$, by $|A|$ and the Hilbert space $L^2(0,1;V^k)$ by $\mathcal{H}^k$, where the Sobolev space $V^k = W_0^{k,2}(\Omega;\mathbb{R}^{n+1})$ is embedded into $C^r(\bar{\Omega};\mathbb{R}^{n+1})$ and $r \geq 2$. Recall that Sobolev's theorem guarantees the latter embedding if $k \geq (n+1)/2 + r + 1$. The following lemma is proved in [11].

Lemma 5.4.1 (Dupuis, Grenander, Miller)**.** *For every time-dependent vector field $v \in \mathcal{H}^k$ and $t_0 \in [0,1]$, there exists a function $\phi : [0,1] \times \Omega \to \mathbb{R}^{n+1}$ such that $t \mapsto \phi(t,x)$ is the unique absolutely continuous solution of the initial value problem*

$$\begin{cases} \frac{dq}{dt} = v(q,t), \\ q(t_0) = x \end{cases} \tag{5.48}$$

for all $t \in [0,1]$. Moreover, the function $x \mapsto \phi(t,x)$ is a homeomorphism of Ω.

For every $v \in \mathcal{H}^k$ and $x \in \Omega$, let $F^v(x,t)$ be the solution of the evolution equation $dq/dt = v(q,t)$ with the initial condition $F^v(x,0) = x$. For a function $f \in C^r(\Omega)$, denote

$$\|f\|_{r,\infty} = \sum_{\alpha, |\alpha| \leq r} \sup_{x \in \Omega} |D^\alpha f(x)|,$$

where $\alpha = (\alpha_1, \ldots, \alpha_{n+1})$ is a multi-index with nonnegative integer components, $|\alpha| = \alpha_1 + \ldots + \alpha_{n+1}$, and $D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots x_{n+1}^{\alpha_{n+1}}}$.

More general versions of the following two lemmas are proved in [40, Appendix C].

Lemma 5.4.2 (Trouve, Younes). *If $v \in \mathcal{H}^k$ and $F^v : \Omega \times [0,1] \to \mathbb{R}^n$ is defined as above, then the function $x \mapsto F^v(x,t)$ is in class $C^r(\Omega)$ and, for all $q \leq r$,*

$$\frac{\partial}{\partial t} D_x^q F^v(x,t) = D_x^q \big(v(F^v(x,t), t) \big),$$

where D_x^q denotes the derivative with respect to x of order q. Moreover, there exist positive constants C and C' such that

$$\sup_{t \in [0,1]} \|F^v(\cdot, t)\|_{r,\infty} \leq C e^{C' \|v\|_{\mathcal{H}^k}}$$

for all $v \in \mathcal{H}^k$.

Recall that we say $v^l \rightharpoonup v$ weakly in $\mathcal{H}^k$ as $l \to \infty$ if $\langle v^l - v, w \rangle \to 0$ as $l \to \infty$ for all $w \in \mathcal{H}^k$.

Lemma 5.4.3 (Trouve, Younes). *If the sequence $\{v^l\}_{l=1}^\infty \subset \mathcal{H}^k$ converges weakly to $v \in \mathcal{H}^k$ as $l \to \infty$, then*

$$\sup_{t \in [0,1]} \|F^{v^l}(\cdot, t) - F^v(\cdot, t)\|_{r-1,\infty} \to 0$$

as $l \to \infty$. Moreover, there exists a constant $K > 0$ such that

$$\sup_{t \in [0,1]} \|F^{v^l}(\cdot, t)\|_{r,\infty} \leq K \tag{5.49}$$

for all $l \in \mathbb{N}$.

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VITA

Oksana Bihun was born on February 29, 1980, in Lviv, Ukraine, into the family Hryhoriy and Mariya Bihun, and their son Bohdan. She attended Secondary School No. 40 and took classes of piano and bandura at Kos-Anatolsky Secondary Music School No. 1 in Lviv. After completing her secondary education with honors, she entered the Department of Applied Mathematics and Informatics of Ivan Franko National University of Lviv, from which she graduated with an honor diploma of Specialist of Applied Mathematics (equivalent of Masters) in 2001. In 2001–2003 she worked as a teaching and research assistant at the same university. During these years she wrote two books of poetry and sang in Galich Chamber Choir "Yevshan". In 2003 Oksana Bihun was accepted to the graduate school of the University of Missouri, Columbia, USA. In 2005 she received the degree of Master of Applied Mathematics from the same university. During the years she spent in Columbia, Oksana Bihun mastered her English, continued to perform piano music, sing, and write poetry, and has volunteered in Sacred Heart Catholic Church in different roles. Oksana Bihun is presently a graduate student and a teaching assistant at the Mathematics Department of the University of Missouri, Columbia. She is engaged to Damiano Fulghesu and has accepted an Assistant Professor position at Concordia College, Moorhead, Minnesota, USA.