# ASYMPTOTIC UNCONDITIONALITY IN BANACH SPACES

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The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

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a candidate for the degree of Doctor of Philosophy

and hereby certify that in their opinion it is worthy of acceptance.

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ABSTRACT

We show that a separable real Banach space embeds almost isometrically in a space $Y$ with a shrinking 1-unconditional basis if and only if $\lim_{n\to\infty} \|x^* + x_n^*\| = \lim_{n\to\infty} \|x^* - x_n^*\|$ whenever $x^* \in X^*$, $(x_n^*)_{n=1}^\infty$ is a weak$^*$-null sequence and both limits exist. If $X$ is reflexive then $Y$ can be assumed reflexive. These results provide the isometric counterparts of recent work of Johnson and Zheng.

# Chapter 1

# Introduction

In this thesis we consider only real Banach spaces. Recently, Johnson and Zheng [11] gave an intrinsic characterization of separable reflexive Banach spaces which embed isomorphically into a reflexive Banach space with unconditional basis. Precisely a separable reflexive Banach space $X$ embeds into a (reflexive) Banach space with unconditional basis if and only if $X$ has the unconditional tree property (UTP), i.e. for some $C$, every weakly null tree has a $C$-unconditional branch. The use of tree properties to describe subspaces of certain Banach spaces is a recent development in Banach space theory which originates in [15] and was later developed in [24].

The results of [15] and [24] are both, in a certain sense, isomorphic versions of earlier isometric results from [17]. In the latter paper, for $1 < p < \infty$, it is shown that if $X$ is a separable Banach space containing no copy of $\ell_1$, then $X$ $(1+\delta)-$embeds in an $\ell_p-$sum of finite-dimensional spaces for every $\delta > 0$ if and only if

$$\lim_{n\to\infty}(\|x+x_n\|^p - \|x\|^p - \|x_n\|^p) = 0$$

whenever $x \in X$ and $(x_n)_{n=1}^\infty$ is a weakly null sequence. Similarly, again assuming

$X$ is separable and contains no copy of $\ell_1$, $X$ $(1+\delta)-$embeds into $c_0$ for every $\delta > 0$ if and only if

$$\lim_{n\to\infty}(\|x+x_n\| - \max(\|x\|, \|x_n\|)) = 0$$

whenever $x \in X$ and $(x_n)_{n=1}^{\infty}$ is weakly null.

In [15] it was shown that a separable Banach space $X$, containing no copy of $\ell_1$, embeds isomorphically into $c_0$ if and only if every weakly null tree has a $c_0$-branch; the corresponding result for $1 < p < \infty$ was given in [24] where it was shown that a reflexive Banach space $X$ embeds isomorphically into an $\ell_p-$sum of finite-dimensional spaces if and only if every weakly null tree has an $\ell_p$-branch. We remark that in [15] the proof of the isomorphic result was given by renorming and reducing to a situation very similar to the isometric result.

The aim of this thesis is to prove an isometric analogue of the Johnson-Zheng theorem. We say that a separable Banach space $X$ has *property (au)* if and only if given any $x \in X$ and $\delta > 0$ there is a closed subspace $F$ of finite codimension such that

$$\|x-y\| \leq (1+\delta)\|x+y\|, \qquad y \in F.$$

This could be restated as

$$\lim_{d\in D}(\|x+x_d\| - \|x-x_d\|) = 0$$

whenever $x \in X$ and $(x_d)_{d\in D}$ is a bounded weakly null net. If $X$ has separable dual we may replace nets by sequences in this definition. In general, we call the sequential version of this property, ($\omega$-au). There is also a natural dual notion; a separable Banach space $X$ has *property (au$^*$)* if given any $x^* \in X^*$ and $\delta > 0$ there is a weak$^*$ closed subspace $F$ of finite codimension in $X^*$ such that

$$\|x^* - y^*\| \leq (1+\delta)\|x^* + y^*\|, \qquad y^* \in F.$$

This is equivalent to

$$\lim_{n\to\infty} (\|x^* + x_n^*\| - \|x^* - x_n^*\|) = 0$$

whenever $x^* \in X^*$ and $(x_n^*)_{n=1}^\infty$ is a weak$^*$ null sequence in $X^*$. Both these concepts already exist in the literature under different names (see [28], [18], [22] and [15].) It can be shown that (au$^*$) implies (au) (Proposition 3.0.7 below) but the converse is false (take $X = \ell_1$).

Our main result (Theorem 5.0.16) is that a separable Banach space $X$ has property (au$^*$) if and only if for every $\delta > 0$ there is a Banach space $Y$ with a shrinking 1-unconditional basis and a subspace $X_\delta$ of $Y$ with $d(X, X_\delta) < 1 + \delta$; $Y$ may be assumed reflexive when $X$ is reflexive. A special case of this theorem was already implicit in the literature. Recall that a separable Banach space $X$ has the *unconditional metric approximation property (UMAP)* [5] if there is a sequence of finite rank operators such that $\lim_{n\to\infty} T_n x = x$ for $x \in X$ and $\lim_{n\to\infty} \|I - 2T_n\| = 1$; if additionally $\lim_{n\to\infty} T_n^* x^* = x^*$ for $x^* \in X^*$ we say that $X$ has shrinking (UMAP). Lima [18] showed that if $X$ is a separable Banach space with property (au$^*$) and such that $X^*$ has the approximation property then $X$ has (shrinking) (UMAP). In [7] (Corollary IV.4) it is shown that if $X$ has shrinking (UMAP) then $X$ can be $(1+\delta)-$embedded in a space with a shrinking 1-unconditional basis; unfortunately the proof of this result is inaccurate (as Haskell Rosenthal has pointed out to us) and we give a corrected proof below (contained in Proposition 4.0.13). Thus the novelty in Theorem 5.0.16 is the removal of the approximation property hypothesis. Let us also remark at this point that Johnson and Zheng [12] have informed us that they have extended the methods of [11] to show that a separable Banach space $X$ with separable dual embeds isomorphically into a space with a shrinking unconditional basis if and only if $X^*$ has the weak*-(UTP). This provides a complete isomorphic analogue of Theorem 5.0.16.

If $X$ is reflexive then (au) is equivalent to (au*) and so Theorem 5.0.16 could be restated using property (au). We conjecture that if $X$ contains no copy of $\ell_1$ then (au) and (au*) are equivalent. We are not quite able to prove this, but we do prove a result very close to it. We say that a separable Banach space has *property (WABS) (weak alternating Banach-Saks property)* if given any bounded sequence $(x_n)_{n=1}^{\infty}$ we can find a sequence of convex blocks $(y_n)_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \sup_{r_1<r_2<\cdots<r_n} \|\frac{1}{n}\sum_{j=1}^{n}(-1)^j y_{r_j}\| = 0.$$

This condition is implied by reflexivity or the Alternating Banach-Saks property. Then $X$ has property (au*) if and only if $X$ has property (au) and (WABS). The example of the James space [10] shows then there is a space with separable dual and (UTP) which has no equivalent renorming to have property (au).

# Chapter 2

# Preliminaries

Unless stated otherwise, $X$ will be assumed to be a real, infinite dimensional Banach space.

- Given a subset $E$ of a normed space $Y$, we denote by $\langle E\rangle$ the linear span of $E$, that is, the intersection of all subspaces of $Y$ containing $E$, equivalently, the least subspace on $Y$ containining $E$, equivalently, the space of all linear combinations of elements of $E$.

  We denote by $[E]$ the closed linear span of $E$, that is, the intersection of all closed subspaces of $Y$ containing $E$, equivalently, the least closed subspace of $Y$ containing $E$. Since $Y$ is a normed space, $[E] = \overline{\langle E\rangle}$, although this is not generally true, for topological vector spaces, see Exercise 1.12 of [21].

- A sequence of elements $(e_n)_{n=1}^{\infty}$ in $X$ is said to be a *basis* of $X$ if for each $x \in X$ there is a unique sequence of scalars $(a_n)_{n=1}^{\infty}$ such that $x = \sum_{n=1}^{\infty} a_n e_n$, that is, $\|x - \sum_{i=1}^{n} a_i e_i\| \xrightarrow{n\to\infty} 0$.

  A sequence $(e_n)_{n=1}^{\infty} \subset X$ is called a *basic sequence* if it is a basis for its closed linear span $[e_n : n \in \mathbb{N}]$.

A sequence of finite-rank operators $(T_n)_{n=1}^\infty$, with $T_n : X \to X$, is said to be a *finite dimensional decomposition (FDD)* if for each $x \in X$ there is a unique sequence $(x_n)_{n=1}^\infty$ with $x_n \in T_n(X)$ such that $x = \sum_{n=1}^\infty x_n$. In this case, it can be shown that the operators $T_n$ are bounded linear projections, and that $x_n = T_n x$. See eg Theorem 1.1.3 of [1].

Thus if each operator $T_n$ has rank 1, then we may choose a basis $(e_n)_{n=1}^\infty$ for $X$, with $e_n \in T_n(X)$ for all $n$.

If $(T_n)_{n=1}^\infty$ is an FDD for $X$, then the Uniform Boundedness Principle implies that the partial sum operators $S_n = \sum_{i=1}^n T_i$ are uniformly bounded, although this already follows from the proof of Theorem 1.1.3 of [1].

The quantity $K = \sup_{n\in\mathbb{N}} \|S_n\|$ is called the *FDD constant*, and in the case of a basis, the *basis constant*.

Suppose that $(e_n)_{n=1}^\infty$ is a basis for $X$, that $p_0 = 0 < p_1 < p_2 < \cdots$ and that $(a_n)_{n=1}^\infty$ are scalars. Then a sequence of nonzero vectors $(u_n)_{n=1}^\infty$ in $X$ of the form $u_n = \sum_{j=p_{n-1}+1}^{p_n} a_j e_j$ is called a *block basic sequence* of $(e_n)_{n=1}^\infty$.

In this case, it can be shown that $(u_n)_{n=1}^\infty$ is a basic sequence with basis constant at most that of $(e_n)_{n=1}^\infty$. See eg Lemma 1.3.5 of [1].

A formal series $\sum_{n=1}^\infty x_n$ in $X$ is said to *converge unconditionally* precisely when $\sum_{n=1}^\infty x_{\pi(n)}$ converges for every permutation $\pi$ of $\mathbb{N}$. In this case, it can be shown that all permutations of the series have the same limit. It can also be shown that $\sum_{n=1}^\infty x_n$ converges unconditionally if and only if $\sum_{n=1}^\infty \epsilon_n x_n$ converges for every choice of signs $\epsilon_n = \pm 1$. Moreover, it can be shown that in this case the set $\{\sum_{i=1}^{\infty} \epsilon_i x_n : \epsilon_i = \pm 1\}$ is compact, in particular, $\sup_{\epsilon_i=\pm 1} \| \sum_{i=1}^{\infty} \epsilon_i x_i\| < \infty$. See eg Chapter 10 of [4]. Since

$$\sum_{i=1}^{n} \epsilon_i x_i = \frac{1}{2}\left(\left(\sum_{i=1}^{n} \epsilon_i x_i + \sum_{i=n+1}^{\infty} x_i\right) + \left(\sum_{i=1}^{n} \epsilon_i x_i + \sum_{i=n+1}^{\infty} (-x_i)\right)\right)$$

we have

$$\sup_{n\in\mathbb{N}} \sup_{\epsilon_i=\pm 1} \| \sum_{i=1}^{n} \epsilon_i x_i \| = \sup_{\epsilon_i=\pm 1} \| \sum_{i=1}^{\infty} \epsilon_i x_i \| < \infty. \tag{2.1}$$

An FDD $(T_n)_{n=1}^{\infty}$ for $X$ is called *unconditional* if for each $x \in X$ the series $x = \sum_{n=1}^{\infty} T_n x$ converges unconditionally. Given $(\epsilon_n)_{n=1}^{\infty} \in \{-1, 1\}^{\mathbb{N}}$, let $M_{(\epsilon_n)} : X \to X$ be the isomorphism defined by $M_{(\epsilon_n)}(\sum_{n=1}^{\infty} T_n x) = \sum_{n=1}^{\infty} \epsilon_n T_n x$. Then the Uniform Boundedness Principle implies that

$$K_u = \sup\{\|M_{(\epsilon_n)}\| : (\epsilon_n)_{n=1}^{\infty} \in \{-1, 1\}^{\mathbb{N}}\} < \infty.$$

In this case, the quantity $K_u$ is called the *unconditional constant* of the FDD, and the FDD is said to be $K_u$*-unconditional*.

Note that (2.1) implies that

$$\begin{aligned}
K_u &= \sup_{\epsilon_n=\pm 1} \sup_{\| \sum_{n=1}^{\infty} T_n x\|=1} \| \sum_{n=1}^{\infty} \epsilon_n T_n x \| \\
&= \sup_{\| \sum_{n=1}^{\infty} T_n x\|=1} \sup_{\epsilon_n=\pm 1} \| \sum_{n=1}^{\infty} \epsilon_n T_n x \| \\
&= \sup_{\| \sum_{n=1}^{\infty} T_n x\|=1} \sup_{n\in\mathbb{N}} \sup_{\epsilon_i=\pm 1} \| \sum_{i=1}^{n} \epsilon_i T_i x \| \\
&= \sup_{n\in\mathbb{N}} \sup_{\epsilon_i=\pm 1} \sup_{\| \sum_{n=1}^{\infty} T_n x\|=1} \| \sum_{i=1}^{n} \epsilon_i T_i x \| \\
&= \sup_{n\in\mathbb{N}} \sup_{\epsilon_i=\pm 1} \| \sum_{i=1}^{n} \epsilon_i T_i \|.
\end{aligned}$$

Necessarily $K_u \geq 1$, but as pointed out on page 19 of [19], $X$ can be equivalently renormed by $|||x||| = \sup_{(\epsilon_n)_{n=1}^{\infty} \in \{-1,1\}^{\mathbb{N}}} \|M_{(\epsilon_n)} x\|$, and with respect to this norm, the FDD $(T_n)_{n=1}^{\infty}$ is 1-unconditional.

Note that if an FDD $(T_n)_{n=1}^{\infty}$ is $\lambda$-unconditional, then so is the sequence $(T_n^*)_{n=1}^{\infty}$.

Given an FDD $(T_n)_{n=1}^{\infty}$ for $X$, the sequence $(T_n^*)_{n=1}^{\infty}$ is an FDD for $H = [T_n^*(X^*) : n \in \mathbb{N}]$. But in general $H$ need not equal all of $X^*$. That is, although for all $x^* \in X^*$, $\sum_{i=1}^{n} T_i^* x^* \xrightarrow{n \to \infty} x^*$ weak$^*$, there may be some $x^* \in X^*$ for which the series does not converge in norm.

The FDD is called *shrinking* precisely when $H = X^*$. In case each $T_n$ has rank 1, so we may choose a basis $(e_n)_{n=1}^{\infty}$ for $X$ with $e_n \in T_n(X)$, the operators $T_n$ have the form $T_n(\sum_{i=1}^{\infty} a_n e_n) = a_n e_n$. The bounded linear functional $x \mapsto a_n$ is denoted by $e_n^*$. Thus $T_n x = e_n^*(x) e_n$. The sequence $(e_n^*)_{n=1}^{\infty}$ are called the *biorthogonal functionals* of the basis $(e_n)_{n=1}^{\infty}$. They have the property that $e_n^*(e_m) = \delta_{m,n}$. The dual operators $T_n^*$ have the form $T_n^* x^* = x * (e_n) e_n^*$. Then a basis $(e_n)_{n=1}^{\infty}$ for $X$ is shrinking if and only if $(e_n^*)_{n=1}^{\infty}$ is a basis for $X^*$.

As in Proposition 3.2.6 of [1], an FDD is shrinking if and only if whenever $x^* \in X^*$,

$$\lim_{N \to \infty} \|x^*|_{[T_n(X):n>N]}\| = 0,$$

where $x^*|_{[T_n(X):n>N]}$ denotes the restriction of $x^*$ to the closed linear span of ranges of the $T_n$ such that $n > N$.

- A formal series $\sum_{n=1}^{\infty} x_n$ in $X$ is said to be *weakly unconditionally Cauchy*

*(WUC)* if and only if, for each $x^* \in X^*$ the series $\sum_{n=1}^{\infty} x^*(x_n)$ is unconditionally convergent in $\mathbb{R}$. Recall that for series in $\mathbb{R}$, unconditional convergence is equivalent to absolute convergence, by a theorem of Riemann. In infinite dimensional Banach spaces, absolute convergence is strictly stronger than unconditional convergence. See eg Chapter 8 of [1] and Chapter 1 of [21]. Note that a WUC series need not be weakly convergent. See eg Section 2.4 of [1].

In Lemma 2.4.6 of [1] it is shown that a formal series $\sum_{n=1}^{\infty} x_n$ in $X$ is WUC if and only if there exists a $C > 0$ such that for any finite subset $F$ of $\mathbb{N}$ and all $\epsilon_n = \pm 1$,

$$\|\sum_{n \in F} \epsilon_n x_n\| \leq C.$$

Note that this is equivalent to the condition that for all $n \in \mathbb{N}$ and all $\epsilon_i = \pm 1$,

$$\|\sum_{i=1}^{n} \epsilon_i x_i\| \leq C.$$

One implication is immediate; for the other, represent the sum over $F$ as the average of two sums of the second type.

- We will use two notations for the action of continuous linear functionals $x^*$ on $X$, either $x^*(x)$ or $\langle x, x^* \rangle$.

- When a Banach space is isometrically embedded in another Banach space, sometimes our notation will make explicit the isometric isomorphism, and sometimes the former space will simply be regarded as a subspace of the latter. We will use whichever notation is most convenient, and we may use both in the same proof ! (eg the proof of Proposition 4.0.12.)

- Given two isomorphic Banach spaces $X$ and $Y$, their Banach-Mazur distance $d(X,Y)$ is defined as

$$d(X,Y) = \inf\{\|T\|\|T^{-1}\|\},$$

where the infimum is taken over all isomorphisms $T$ from $X$ onto $Y$. Thus $d(X,Y) \geq 1$, $d(X,Y) = 1$ if and only if $X$ and $Y$ are isometrically isomorphic, $d(X,Z) \leq d(X,Y)d(Y,Z)$. Then $\log \circ d$ is a metric on the class of Banach spaces isomorphic to a given Banach space $X$, modulo isometric isomorphism.

- A separable Banach space $X$ has the *unconditional metric approximation property (UMAP)* [5] if there is a sequence of finite rank operators such that $\lim_{n\to\infty} T_n x = x$ for $x \in X$ and $\lim_{n\to\infty} \|I - 2T_n\| = 1$; if additionally $\lim_{n\to\infty} T_n^* x^* = x^*$ for $x^* \in X^*$ we say that $X$ has shrinking (UMAP).

- Given two Banach spaces $X$ and $Y$, $\mathcal{K}(X,Y)$ will denote the space of compact linear operators from $X$ into $Y$.

- A theorem of Mazur, which we will use several times, states that if $C$ is a convex set in a normed space $X$, then the weak closure $\overline{C}^w$ of $C$ coincides with the norm closure $\overline{C}$ of $C$.

# Chapter 3

# Asymptotic Unconditionality

Throughout, unless stated otherwise, $X$ will be a separable, real Banach space.

**Definition 3.0.1.** We will say that $X$ is *asymptotically unconditional (au)* if and only if given any $x \in X$ and $\delta > 0$ there is a closed finite co-dimensional subspace $W$ of $X$ such that

$$\|x - w\| \leq (1+\delta)\|x + w\|, \qquad w \in W.$$

**Remark** Note that since $w \in W \Leftrightarrow -w \in W$, this condition is equivalent to

$$\frac{1}{(1+\delta)}\|x - w\| \leq \|x + w\| \leq (1+\delta)\|x - w\|, \ w \in W.$$

**Proposition 3.0.2.** *The following are equivalent.*

*(i) $X$ has property (au).*

*(ii) Whenever $x \in X$ and $(u_d)_{d \in D}$ is a bounded weakly null net,*

$$\lim_{d \in D} (\|x + u_d\| - \|x - u_d\|) = 0.$$

**Proof: (i) $\implies$ (ii):** Suppose (i) holds, and choose an $x \in X$, and a bounded weakly null net $(u_d)_{d \in D}$. Suppose that $\|u_d\| \leq M < \infty$ for all $d \in D$. Let $\delta > 0$, to be chosen later, and let $W$ be as in the definition of (au), so that

$$\|x - w\| \leq (1+\delta)\|x + w\|, \ w \in W.$$

Let $x_1^*, \ldots, x_n^*$ be a basis for $W^\perp \subset X^*$. Then

$$|||x + W||| = \max_{1 \le i \le n} |\langle x, x_i^* \rangle|$$

defines a norm on the $n$-dimensional space $X/W$. Since all norms on a finite dimensional space are equivalent, this norm is equivalent to the usual quotient norm on $X/W$, defined by

$$\|x + W\| = \inf\{\|x - w\| : w \in W\}.$$

Since $(u_d)$ is weakly null, $\lim_{d \in D} |||u_d + W||| = 0$, hence $\lim_{d \in D} \|u_d + W\| = 0$. Let $\gamma > 0$, to be chosen later, and choose $d_0 \in D$ and a net $(y_d)_{d \in D, d \ge d_0} \subset W$ such that

$$\|u_d - y_d\| < \gamma, \; d \ge d_0.$$

Then

$$\begin{aligned} |\|x + u_d\| - \|x - u_d\|| &< |\|x + y_d\| - \|x - y_d\|| + 2\gamma \\ &\le \delta(\|x\| + \|y_d\|) + 2\gamma \\ &< \delta(\|x\| + M + \gamma) + 2\gamma, \end{aligned}$$

whenever $d \ge d_0$.

Given $\varepsilon > 0$, let

$$\gamma = \min(1, \varepsilon/3),$$

and let

$$\delta = \frac{\varepsilon}{3(\|x\| + M + 1)}.$$

Then

$$|\|x + u_d\| - \|x - u_d\|| < \varepsilon, \; d \ge d_0,$$

so $\lim_{d\in D}(\|x+u_d\| - \|x-u_d\||) = 0$, as required.

**(ii) $\Rightarrow$ (i):** Conversely, suppose the first condition fails. Then there exist an $x \in X$ and a $\delta > 0$, such that for every closed finite co-dimensional subspace $W$ of $X$, there exists a $u_W \in W$ such that

$$\|x - u_W\| > (1+\delta)\|x + u_W\|.$$

The closed finite co-dimensional subspaces $W$ of $X$ form a directed set with respect to the partial order of reverse inclusion, thus $(u_W)$ is a net in $X$.

Also

$$\|x\| + \|u_W\| \geq \|x - u_W\| > (1+\delta)\|x + u_W\| \geq (1+\delta)|\|x\| - \|u_W\||,$$

so

$$\|x\| + \|u_W\| > (1+\delta)(\|x\| - \|u_W\|),$$

and

$$\|x\| + \|u_W\| > (1+\delta)(\|u_W\| - \|x\|).$$

So $(2+\delta)\|u_W\| > \delta\|x\|$, or $\|u_W\| > (\delta/(2+\delta))\|x\|$, and $\delta\|u_W\| < (2+\delta)\|x\|$, or $\|u_W\| < ((2+\delta)/\delta))\|x\|$. That is,

$$\frac{\delta}{2+\delta}\|x\| < \|u_W\| < \frac{2+\delta}{\delta}\|x\|.$$

In particular, $(u_W)$ is bounded.

We can assume that $\|x + u_W\|$ does not tend to zero, otherwise $u_W$ would converge in norm to $-x$, and since its weak limit must equal its norm limit, it would follow that $x = 0$. But this would contradict all the inequalities $\|x - u_W\| >$

$(1+\delta)\|x+u_W\|$. Passing to a subnet $(u_d)_{d\in D}$ of $(u_W)$, we can assume that there is some $\varepsilon>0$ such that for all $d\in D$, $\|x+u_d\|\geq\varepsilon$.

It follows that

$$\|x-u_d\|-\|x+u_d\|>\delta\|x+u_d\|\geq\delta\varepsilon>0,\ d\in D.$$

Therefore $\|x+u_d\|-\|x-u_d\|\nrightarrow 0$. Since $(u_d)_{d\in D}$ is bounded and weakly null, (ii) fails, as required. □

**Definition 3.0.3.** We shall say that $X$ is *sequentially asymptotically unconditional ($\omega$-au)* if and only if

$$\lim_{n\to\infty}(\|x+u_n\|-\|x-u_n\|)=0$$

whenever $x\in X$ and $(u_n)_{n=1}^{\infty}$ is a weakly null sequence.

This condition has already been considered by Sims [28] under the acronym WORTH, by Neuwirth [22] under the name "$\tau$-unconditionality", and by Kalton [15] under the name "unconditional type". Note that if $X^*$ is separable then the weak topology is metrizable on bounded sets and in this case $X$ is ($\omega$-au) if and only if $X$ is (au).

**Definition 3.0.4.** We shall say that $X$ is **-asymptotically unconditional (au*)* if and only if

$$\lim_{n\to\infty}(\|x^*+x_n^*\|-\|x^*-x_n^*\|)=0$$

whenever $x^*\in X^*$ and $(x_n^*)_{n=1}^{\infty}$ is a weak*-null sequence in $X^*$.

This condition has been considered under the name $(wM^*)$ by Lima [18]; later Oja [25] considered a family of more general conditions. Also Kalton [15] considered (au$^*$) under the name "shrinking unconditional type". Since $X$ is assumed separable, the weak$^*$-topology on bounded sets is metrizable, and so $X^*$ is $^*$-asymptotically unconditional if and only if either given any $x^* \in X^*$ and $\delta > 0$ there is a weak$^*$-closed finite co-dimensional subspace $W$ of $X^*$ such that

$$\|x^* - w^*\| \leq (1+\delta)\|x^* + w^*\|, \qquad w^* \in W,$$

or, equivalently,

$$\lim_{d \in D}(\|x^* + u_d^*\| - \|x^* - u_d^*\|) = 0$$

whenever $x^* \in X^*$ and $(u_d^*)_{d \in D}$ is a bounded weak$^*$-null net. The equivalence of these last two formulations can be proved exactly in the way that Proposition 3.0.2 was proved.

We state a principle based on compactness that will be used frequently:

**Lemma 3.0.5.** *(i) Let $X$ be a separable Banach space. Then $X$ has property (au) if and only if, given any finite-dimensional subspace $E$ of $X$ and $\delta > 0$ there is a closed finite co-dimensional subspace $F$ of $X$ such that*

$$\|e - f\| \leq (1+\delta)\|e + f\|, \qquad e \in E, f \in F.$$

*(ii) Let $X$ be a separable Banach space. Then $X$ has property (au$^*$) if and only if, given any finite-dimensional subspace $E$ of $X^*$ and $\delta > 0$ there is a weak$^*$-closed finite co-dimensional subspace $F$ of $X^*$ such that*

$$\|e^* - f^*\| \leq (1+\delta)\|e^* + f^*\|, \qquad e^* \in E, f^* \in F.$$

**Proof:** In both statements, one implication is immediate from the definitions. We prove the converse implications.

(i) By homogeneity of the norm, it is enough to prove the inequality in the case $e \in S_E$. Let $x_1, \dots, x_n \in S_E$ be an $\varepsilon$-net, where $\varepsilon > 0$ will be chosen later. For $1 \le i \le n$, let $F_i \subset X$ be a closed finite co-dimensional subspace such that $\|x_i - f_i\| \le (1 + \frac{\delta}{2})\|x_i + f_i\|$ for all $f_i \in F_i$. Let $F = \cap_{i=1}^n F_i$. Then $F$ is a closed finite co-dimensional subspace of $X$.

Let $e \in S_E$ and choose $i$ so that $\|e - x_i\| < \varepsilon$. Let $f \in F$. Then

$$\begin{aligned}\|e - f\| - \|e + f\| &< \|x_i - f\| - \|x_i + f\| + 2\varepsilon \\ &\le \tfrac{\delta}{2}\|x_i + f\| + 2\varepsilon \\ &< \tfrac{\delta}{2}(\|e + f\| + \varepsilon) + 2\varepsilon,\end{aligned}$$

hence

$$\|e - f\| < (1 + \tfrac{\delta}{2})\|e + f\| + \varepsilon(2 + \tfrac{\delta}{2}). \tag{3.1}$$

*Case I:* If $\varepsilon(2 + \frac{\delta}{2}) \le \frac{\delta}{2}\|e + f\|$ then there is nothing left to prove.

*Case II:* Suppose instead that $\varepsilon(2 + \frac{\delta}{2}) > \frac{\delta}{2}\|e + f\|$. Note that

$$1 = \|e\| \le \tfrac{1}{2}(\|e + f\| + \|e - f\|) \;\Rightarrow\; \|e - f\| \ge 2 - \|e + f\|.$$

Therefore

$$\|e - f\| \ge 2 - \frac{\varepsilon(2 + \frac{\delta}{2})}{\frac{\delta}{2}} = \frac{2\delta - \varepsilon(4 + \delta)}{\delta}.$$

Now (3.1) implies that

$$\begin{aligned}\frac{2\delta - \varepsilon(4 + \delta)}{\delta} &< (1 + \tfrac{\delta}{2})\|e + f\| + \varepsilon(2 + \tfrac{\delta}{2}) \\ &< (1 + \tfrac{\delta}{2})\frac{\varepsilon(2 + \frac{\delta}{2})}{\frac{\delta}{2}} + \varepsilon(2 + \tfrac{\delta}{2})\end{aligned}$$

$$= \varepsilon \frac{(4+\delta)(1+\delta)}{\delta},$$

hence

$$\varepsilon > \frac{2\delta}{(4+\delta)(2+\delta)}.$$

So if $\varepsilon$ does not exceed this quantity, then Case II cannot arise. Setting $\varepsilon = \frac{2\delta}{(4+\delta)(2+\delta)}$ is sufficient.

(ii) The same argument works. □

The following result is a consequence of [18] Proposition 4.1, but we give an independent proof.

**Proposition 3.0.6.** *If $X$ is a separable Banach space with (au$^*$) then $X^*$ has no proper closed norming subspace and hence is separable.*

**Proof:** Let $M$ be a closed norming subspace of $X^*$. Note that $B_{X^*} \cap M$ is weak$^*$-dense in $B_{X^*}$. Indeed, if not then there exist an $x^* \in B_{X^*}$ and an $x \in X$ such that

$$\langle x, x^* \rangle > \sup_{y^* \in B_{X^*} \cap M} \langle x, y^* \rangle = \|x\|,$$

which implies that $\|x^*\| > 1$, a contradiction. Moreover, $X$ being separable, the weak$^*$ topology is metrizable on bounded sets in $X^*$, so given any $x^* \in B_{X^*}$ we may find a sequence $(x_n^*) \subset B_{X^*} \cap M$ tending weak$^*$ to $x^*$.

If $M \neq X^*$ then by Riesz's lemma (see [1] or [21],) there exists an $x^* \in X^*$ with $\|x^*\| = 1$ and $d(x^*, M) > 1/2$. Let $(x_n^*)$ be a sequence in $B_{X^*} \cap M$ which weak$^*$-converges to $x^*$. Then by property (au$^*$)

$$\limsup \|2x^* - x_n^*\| = \limsup \|x^* - (x_n^* - x^*)\| = \limsup \|x^* + (x_n^* - x^*)\| = \limsup \|x_n^*\| \leq 1.$$

Hence $\limsup \|x^* - \frac{x_n^*}{2}\| \leq \frac{1}{2}$, a contradiction.

Finally, let $(x_n) \subset X$ be a dense sequence in $X$. Using the Hahn-Banach theorem, choose $x_n^* \in S_{X^*}$ such that $\langle x_n, x_n^* \rangle = \|x_n\|$. Then the closed linear span $[x_n^*]$ of $(x_n^*)$ is a closed separable norming subspace for $X$ in $X^*$. Indeed, let $x \in X$, let $\varepsilon > 0$ and choose $n$ so that $\|x - x_n\| < \frac{\varepsilon}{2}$. Then

$$\begin{aligned}
\langle x, x_n^* \rangle &= \langle x_n, x_n^* \rangle - \langle x_n - x, x_n^* \rangle \\
&= \|x_n\| - \langle x_n - x, x_n^* \rangle \\
&\geq \|x_n\| - \|x_n - x\| \\
&\geq \|x\| - 2\|x_n - x\| \\
&> \|x\| - \varepsilon.
\end{aligned}$$

Then $[x_n^*] = X^*$, hence $X^*$ is itself separable. □

Part (a) of the next proposition appears as Lemma 2.4 in [15].

**Proposition 3.0.7.** *Suppose $X$ is a separable Banach space. Then*

*(a) If $X$ has ($au^*$) then $X$ has ($au$).*

*(b) If $X$ is reflexive then $X$ has ($au^*$) if and only if $X$ has ($au$).*

**Proof:** (a): We want to show that if $x \in X$ and $(u_d)_{d\in D}$ is a bounded weakly null net, then $\lim_{d\in D}(\|x + u_d\| - \|x - u_d\|) = 0$. It is enough to show that if $\lim_{d\in D} \|x + u_d\|$ and $\lim_{d\in D} \|x - u_d\|$ exist, then $\lim_{d\in D} \|x + u_d\| = \lim_{d\in D} \|x - u_d\|$. In this case, by symmetry ($(-u_d)_{d\in D}$ is weakly null if and only if $(u_d)_{d\in D}$ is weakly null,) it is enough to show that $\lim_{d\in D} \|x - u_d\| \geq \lim_{d\in D} \|x + u_d\|$. By homogeneity of the norm, we can assume that $\lim_{d\in D} \|x + u_d\| = 1$.

In summary, it is enough to show that if $x \in X$ and $(u_d)_{d \in D}$ is a bounded weakly null net with

$$\lim_{d \in D} \|x + u_d\| = 1 \quad \text{and} \quad \lim_{d \in D} \|x - u_d\| = \theta$$

then $\theta \geq 1$. To do this we may by the Hahn-Banach theorem pick $(x_d^*)_{d \in D}$ with $x_d^*(x + u_d) = \|x + u_d\|$ and $\|x_d^*\| = 1$. By weak$^*$-compactness of $B_{X^*}$ we may then pass to subnets of $(x_d^*)_{d \in D}$ and $(u_d)_{d \in D}$ and assume that $(x_d^*)_{d \in D}$ is weak$^*$-convergent to some $x^*$. Let $x_d^* = x^* + u_d^*$, so that $(u_d^*)_{d \in D}$ is weak$^*$-null. Then

$$\begin{aligned}
1 &= \lim_{d \in D} (x^* + u_d^*)(x + u_d) \\
&= \lim_{d \in D} (x^*(x) + x^*(u_d) + u_d^*(x) + u_d^*(u_d)) \\
&= \lim_{d \in D} (x^*(x) - x^*(u_d) - u_d^*(x) + u_d^*(u_d)) \\
&= \lim_{d \in D} (x^* - u_d^*)(x - u_d) \\
&\leq \limsup_{d \in D} \|x^* - u_d^*\| \|x - u_d\| \\
&= \limsup_{d \in D} \|x^* + u_d^*\| \|x - u_d\| \\
&= \limsup_{d \in D} \|x - u_d\| \\
&= \theta.
\end{aligned}$$

This proves (a).

(b): One direction is given by (a). It remains to show that if $X$ is reflexive and has (au), then $X$ has (au$^*$). But in this case $X$ has (au) $\Leftrightarrow$ $X^*$ has (au$^*$) $\overset{(a)}{\Rightarrow}$ $X^*$ has (au) $\Leftrightarrow$ $X$ has (au$^*$). $\square$

The next proposition duplicates some parts of Lemmas 2.5 and 2.6 of [15].

**Proposition 3.0.8.** *(i) If $X$ is a Banach space with a 1-unconditional UFDD then $X$ has (au).*

*(ii) If $X$ is a Banach space with a shrinking 1-unconditional UFDD then $X$ has $(au^*)$.*

*(iii) If $Y$ is a Banach space with (au) then any subspace $X$ of $Y$ also has (au).*

*(iv) If $Y$ is a separable Banach space with $(au^*)$ then any subspace or quotient $X$ of $Y$ also has $(au^*)$.*

**Proof: (i):** Denote by $(S_n)_{n=1}^{\infty}$ the UFDD for $X$. Let $x \in X \setminus \{0\}$ and let $\delta > 0$. We want to find a closed, finite-codimensional subspace $W$ of $X$ such that $\|x - w\| \leq (1+\delta)\|x + w\|$ for all $w \in W$. We imitate part of the proof of Lemma 3.0.5.

By homogeneity of the norm, we assume without loss of generality that $\|x\| = 2$. We also assume without loss of generality that $\delta \leq 2$. Choose an $n \in \mathbb{N}$ such that $\|x - S_n x\| \leq \frac{\delta}{2}$. Let $W = (I - S_n)X = \ker S_n$. Then $W$ is a finite-codimensional subspace of $X$, and as the kernel of a bounded linear operator, $W$ is closed. Let $w \in W$. Since the FDD is 1-unconditional, $\|S_n x - w\| = \|S_n x + w\|$. Therefore

$$\|x - w\| \leq \|S_n x - w\| + \tfrac{\delta}{2} = \|S_n x + w\| + \tfrac{\delta}{2} \leq \|x + w\| + \delta. \tag{3.2}$$

But $2 = \|x\| \leq \frac{1}{2}(\|x - w\| + \|x + w\|)$, which with (3.2) implies that

$$\|x + w\| \geq 4 - \|x - w\| \geq 4 - \|x + w\| - \delta \geq 2 - \|x + w\|,$$

hence $\|x + w\| \geq 1$. Therefore $\|x + w\| + \delta \leq (1+\delta)\|x + w\|$, and (3.2) implies that

$$\|x - w\| \leq (1+\delta)\|x + w\|,$$

as required.

The proof of (ii) is similar.

**(iii):** Denote by $i : X \to Y$ the inclusion map. Then $i^* : Y^* \to X^*$ is the restriction map of $Y^*$ onto $X^*$. If $(u_d)_{d\in D} \subset X$ is a bounded weakly null net in $X$, and $y^* \in Y^*$, then $\langle ix_d, y^*\rangle = \langle x_d, i^*y^*\rangle \to 0$. Thus $(ix_d)_{d\in D} \subset Y$ is weakly null in $Y$, and (iii) follows.

**(iv):** First suppose that $Q : Y \to X$ is a quotient map from $Y$ onto $X$. Then $Q^* : X^* \to Y^*$ is an isometric isomorphism from $X^*$ into $Y^*$. Fix $x^* \in X^*$ and let $(u_n^*)_{n\in\mathbb{N}} \subset X^*$ be a weak*-null sequence in $X^*$. Then $(Q^*u_n^*)_{n\in\mathbb{N}}$ is a weak*-null sequence in $Y^*$, so

$$\lim_{n\to\infty} (\|Q^*x^* + Q^*u_n^*\| - \|Q^*x^* - Q^*u_n^*\|) = 0.$$

But $\|Q^*x^* \pm Q^*u_n^*\| = \|x^* \pm u_n^*\|$, so we are done.

Next consider the case when $X$ is a subspace of $Y$.

Suppose $x^* \in X^*$ and $(u_n^*)$ is a weak* null sequence in $X^*$ such that $\lim_{n\to\infty} \|x^* + u_n^*\| = 1$ and $\lim_{n\to\infty} \|x^* - u_n^*\|$ exists. It is enough to show that $\lim_{n\to\infty} \|x^* - u_n^*\| \le 1$. Let $y_n^* \in Y^*$ be Hahn-Banach extensions of $x^* + u_n^*$ to $Y$ with $\|y_n^*\| = \|x^* + u_n^*\|$. Passing to a subsequence we can suppose $(y_n^*)$ converges weak* to some $y^* \in Y^*$. Then for all $x \in X$,

$$\begin{aligned}
\langle x, 2y^* - y_n^*\rangle &= 2 \lim_{m\to\infty} \langle x, y_m^*\rangle - \langle x, y_n^*\rangle \\
&= 2 \lim_{m\to\infty} \langle x, x^* + u_m^*\rangle - \langle x, x^* + u_n^*\rangle \\
&= 2\langle x, x^*\rangle - \langle x, x^* + u_n^*\rangle \\
&= \langle x, x^* - u_n^*\rangle,
\end{aligned}$$

thus $(2y^* - y_n^*)|_X = x^* - u_n^*$, and $\|x^* - u_n^*\| \le \|2y^* - y_n^*\|$. But

$$\begin{aligned}
\limsup_{n\to\infty} \|2y^* - y_n^*\| &= \limsup_{n\to\infty} \|y^* - (y_n^* - y^*)\| \\
&= \limsup_{n\to\infty} \|y^* + (y_n^* - y^*)\| \quad \text{since } Y \text{ has (au}^*\text{)}, \\
&= \limsup_{n\to\infty} \|y_n^*\| \\
&= \limsup_{n\to\infty} \|x^* + u_n^*\| \\
&= 1,
\end{aligned}$$

and similarly, $\liminf_{n\to\infty} \|2y^* - y_n^*\| = 1$, so $\lim_{n\to\infty} \|2y^* - y_n^*\|$ exists and equals 1. Therefore $\lim_{n\to\infty} \|x^* - u_n^*\| \le \lim_{n\to\infty} \|2y^* - y_n^*\| = 1$. □

**Remark** Note that property (au) does not pass to quotients since every separable Banach space is a quotient of $\ell_1$, (see, eg, Corollary 2.3.2 of [1].) In fact, $\ell_1$ has a 1-unconditional basis, namely its standard basis, so by Proposition 3.0.8 (i) above, $\ell_1$ has (au). However, James's space $\mathcal{J}$ is a separable Banach space which fails (au).

For each $\xi = (\xi(n))_{n=1}^\infty$ in $c_0$, let

$$\|\xi\|_{\mathcal{J}} = \frac{1}{\sqrt{2}} \sup \left\{ \left( (\xi(p_0) - \xi(p_n))^2 + \sum_{k=1}^n (\xi(p_k) - \xi(p_{k-1}))^2 \right)^{1/2} \right\},$$

where the supremum is taken over all $n \in \mathbb{N}$ and all sequences $(p_j)_{j=0}^n \subset \mathbb{N}$ with $1 \le p_0 < p_1 < \cdots < p_n$. James's space $\mathcal{J}$ is the subspace of $c_0$ comprising all $\xi$ such that $\|\xi\|_{\mathcal{J}}$ is finite, equipped with the norm $\|\cdot\|_{\mathcal{J}}$.

The standard unit vectors $(e_n)_{n=1}^\infty$ are a shrinking basis for $\mathcal{J}$ (see, eg, [1] or [21].) Therefore $(e_n)_{n=1}^\infty$ is a weakly null sequence in $\mathcal{J}$. However, for $n \ge 3$, $\|e_1 + e_n\|_{\mathcal{J}} = \sqrt{2}$ and $\|e_1 - e_n\|_{\mathcal{J}} = 2$, so $\mathcal{J}$ does not have (au).

We end this section with a Lemma which will be useful later.

**Lemma 3.0.9.** *(i) Let $X$ be a separable Banach space with property (au), and suppose that $(x_n)_{n=1}^{\infty}$ is a weakly null sequence which is not norm convergent to* 0*. Then, given $\delta > 0$, there is a subsequence $(y_n)_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that the sequence $(y_n)_{n=1}^{\infty}$ is $(1+\delta)-$unconditional.*
*(ii) Let $X$ be a separable Banach space with property (au$^*$), and suppose that $(x_n^*)_{n=1}^{\infty}$ is a weak$^*$-null sequence in $X^*$ which is not norm convergent to* 0*. Then, given $\delta > 0$, there is a subsequence $(y_n^*)_{n=1}^{\infty}$ of $(x_n^*)_{n=1}^{\infty}$ such that the sequence $(y_n^*)_{n=1}^{\infty}$ is $(1+\delta)-$unconditional.*

**Proof: (i):** Since $(x_k)_{k=1}^{\infty}$ is weak$^*$-null as a sequence in $X^{**}$, the Uniform Boundedness Principle guarantees a $C < \infty$ such that $\|x_k\| \leq C$. We may suppose by passing to a subsequence, that there exists a $c$ such that $0 < c \leq \|x_k\| \leq C < \infty$. We may therefore pass to a further subsequence such that $(x_n)_{n=1}^{\infty}$ is basic (see e.g. [1] Theorem 1.5.2, and note again that $X \hookrightarrow X^{**}$.) Let $K$ be the basis constant for the sequence $(x_n)_{n=1}^{\infty}$.

Choose $(\delta_n)_{n=1}^{\infty} \subset (0,1)$ to be a decreasing sequence so that $\prod_{j=1}^{\infty}(1+\delta_j) < 1+\delta$. We will construct a subsequence $(y_n)_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ and a sequence $(F_n)_{n=1}^{\infty}$ of closed finite-codimensional subspaces inductively.

Let $y_1 = x_1$ and $F_1 = X$. If $y_1, \ldots, y_{n-1}$ and $F_1, \ldots, F_{n-1}$ have been chosen then by Lemma 3.0.5 we may choose a closed subspace $F_n$ of finite codimension so that if $w \in [y_j]_{j=1}^{n-1}$ and $z \in F_n$ then

$$\|w - z\| \leq (1 + \tfrac{1}{4}\delta_n)\|w + z\|.$$

Let $Q_j : X \to X/F_j$ denote the quotient map for $1 \le j \le n$. Since $(x_n)_{n=1}^{\infty}$ is weakly null and $Q_j$ is finite-rank, $Q_j x_n \xrightarrow{n\to\infty} 0$. So if $y_{n-1} = x_{m_n}$ we may pick $y_n = x_{m_{n+1}}$ with $m_{n+1} > m_n$ so that

$$\|Q_j y_n\| \le \frac{2^{j-n-1} c\delta_j}{10K}, \qquad 1 \le j \le n.$$

Thus we construct $(y_n)_{n=1}^{\infty}$ and $(F_n)_{n=1}^{\infty}$.

Now suppose $2 \le n \le N$, $w = \sum_{j=1}^{n-1} a_j y_j$ and $z = \sum_{j=n}^{N} a_j y_j$ where $\|w+z\| = 1$. Note that for $1 \le j \le N$,

$$|a_j| c \le |a_j| \|y_j\| = \|a_j y_j\| \le 2K \|\sum_{k=1}^{N} a_k y_k\| = 2K,$$

so $|a_j| \le 2Kc^{-1}$. Therefore

$$\|Q_n z\| = \|\sum_{j=n}^{N} a_j Q_n y_j\| \le \sum_{j=n}^{N} |a_j| \|Q_n y_j\| \le \frac{2K}{c} \sum_{j=n}^{\infty} \|Q_n y_j\| \le \frac{\delta_n}{5}.$$

Hence there exists $z' \in F_n$ such that $\|z - z'\| \le \delta_n/4$ and thus

$$\begin{aligned}
\|w - z\| &\le \|w - z'\| + \tfrac{1}{4}\delta_n \\
&\le (1 + \tfrac{1}{4}\delta_n)\|w + z'\| + \tfrac{1}{4}\delta_n \\
&\le (1 + \tfrac{1}{4}\delta_n)(\|w + z\| + \tfrac{1}{4}\delta_n) + \tfrac{1}{4}\delta_n \\
&= (1 + \tfrac{1}{4}\delta_n)(1 + \tfrac{1}{4}\delta_n) + \tfrac{1}{4}\delta_n \\
&< 1 + \tfrac{13}{16}\delta_n \\
&< 1 + \delta_n \\
&= (1 + \delta_n)\|w + z\|.
\end{aligned}$$

Thus we have the inequality

$$\|\sum_{j=1}^{n-1} a_j y_j - \sum_{j=n}^{N} a_j y_j\| \le (1 + \delta_n)\|\sum_{j=1}^{N} a_j y_j\|. \tag{3.3}$$

Then we claim that if $2 \leq k \leq n$ and $\epsilon_j = \pm 1$ with $\epsilon_j = 1$ for $1 \leq j < k$ we have

$$\|\sum_{j=1}^{n} \epsilon_j a_j y_j\| \leq \prod_{j=k}^{\infty}(1+\delta_j)\|\sum_{j=1}^{n} a_j y_j\|. \tag{3.4}$$

This is proved for fixed $n$ by backwards induction on $k$.

Indeed for $k = n$ it follows from (3.3). Next suppose it is proved for $k+1$ and suppose that $\epsilon_k = -1$ but $\epsilon_j = 1$ for $1 \leq j < k$. Then by (3.3),

$$\|\sum_{j=1}^{n} \epsilon_j a_j y_j\| = \|\sum_{j=1}^{k-1} a_j y_j + \sum_{j=k}^{n} \epsilon_j a_j y_j\| \leq (1+\delta_k)\|\sum_{j=1}^{k-1} a_j y_j - \sum_{j=k}^{n} \epsilon_j a_j y_j\|. \tag{3.5}$$

But

$$\sum_{j=1}^{k-1} a_j y_j - \sum_{j=k}^{n} \epsilon_j a_j y_j = \sum_{j=1}^{k} a_j y_j - \sum_{j=k+1}^{n} \epsilon_j a_j y_j = \sum_{j=1}^{k} a_j y_j + \sum_{j=k+1}^{n} (-\epsilon_j) a_j y_j, \tag{3.6}$$

since $\epsilon_k = -1$. Then by the induction hypothesis,

$$\|\sum_{j=1}^{k} a_j y_j + \sum_{j=k+1}^{n} (-\epsilon_j) a_j y_j\| \leq \prod_{j=k+1}^{\infty} (1+\delta_j)\|\sum_{j=1}^{n} a_j y_j\|. \tag{3.7}$$

Combining (3.5), (3.6) and (3.7),

$$\|\sum_{j=1}^{n} \epsilon_j a_j y_j\| \leq (1+\delta_k) \prod_{j=k+1}^{\infty} (1+\delta_j)\|\sum_{j=1}^{n} a_j y_j\| = \prod_{j=k}^{\infty}(1+\delta_j)\|\sum_{j=1}^{n} a_j y_j\|,$$

completing the proof of (3.4).

It follows from (3.4) that if $\epsilon_j = \pm 1$ for $1 \leq j \leq n$, then

$$\|\sum_{j=1}^{n} \epsilon_j a_j y_j\| \leq \prod_{j=1}^{\infty}(1+\delta_j)\|\sum_{j=1}^{n} a_j y_j\| < (1+\delta)\|\sum_{j=1}^{n} a_j y_j\|,$$

as required. Note that since $\|-x\| = \|x\|$ we were able to assume without loss of generality that $\epsilon_1 = 1$.

The proof of (ii) is similar. $\square$

# Chapter 4

# Embedding in a Space with Unconditional Basis

**Lemma 4.0.10.** *Suppose that $X$ and $Y$ are normed spaces and that $T : X \to Y$ is an isomorphism from $X$ into $Y$. Choose $M$ so that for all $x \in X$,*

$$\|x\| \leq M\|Tx\|.$$

*Then the quantity*

$$|||y||| = \inf\{\|x\| + M\|y - Tx\| : x \in X\}$$

*is an equivalent norm on $Y$ with respect to which $T$ is an isometry. In particular, for all $y \in Y$*

$$\frac{1}{\|T\|} \cdot \|y\| \leq |||y||| \leq M \cdot \|y\|.$$

**Proof:** Evidently $||| \cdot ||| \geq 0$. Since $X$ is a vector space, it is straightforward to check that $||| \cdot |||$ is a seminorm.

Note that $M \geq \|T\|^{-1}$. So if $x = 0$, then

$$\|x\| + M\|y - Tx\| = M\|y\| \geq \frac{\|y\|}{\|T\|}.$$

Let $x \in X \setminus \{0\}$. Then

$$\begin{aligned} \|x\| + M\|y - Tx\| &\geq \|x\| + M(\|y\| - \|Tx\|) \\ &\geq \|x\| + \frac{\|x\|}{\|Tx\|}(\|y\| - \|Tx\|) \\ &= \frac{\|x\|}{\|Tx\|}\|y\| \\ &\geq \frac{\|y\|}{\|T\|}. \end{aligned}$$

In either case, $\|x\| + M\|y - Tx\| \geq \|y\|/\|T\|$, and taking the infimum over $x \in X$ we have

$$|||y||| \geq \frac{\|y\|}{\|T\|}.$$

On the other hand,

$$|||y||| \leq \|0\| + M\|y - 0\| = M\|y\|.$$

So for all $y \in Y$,

$$\frac{1}{\|T\|} \cdot \|y\| \leq |||y||| \leq M \cdot \|y\|.$$

In particular, the seminorm $|||\cdot|||$ is a norm on $Y$ which is equivalent to the original norm.

Let $x, x' \in X$. If $x' = x$, then

$$\|x'\| + M\|Tx - Tx'\| = \|x'\| = \|x\|,$$

so $|||Tx||| \leq \|x\|$. If $x' \neq x$, then

$$\begin{aligned} \|x'\| + M\|Tx - Tx'\| &= \|x'\| + M\|T(x - x')\| \\ &\geq \|x'\| + \frac{\|x - x'\|}{\|T(x - x')\|}\|T(x - x')\| \end{aligned}$$

$$
\begin{aligned}
&= \|x'\| + \|x - x'\| \\
&\geq \|x\|,
\end{aligned}
$$

so $|||Tx||| \geq \|x\|$. Therefore

$$|||Tx||| = \|x\|,$$

as required. □

Let $Y$ be a space with an FDD $(Q_j)_{j=1}^{\infty}$ and let $X$ be a subspace of $Y$. Then we will say that $X$ satisfies the *density condition* with respect to $(Q_j)_{j=1}^{\infty}$ if there is a dense subset $D$ of $X$ such that if $x \in D$ we have

$$x = \sum_{j=1}^{n} Q_j x$$

for some $n = n(x) \in \mathbb{N}$. The following Lemma is similar to Lemma 2.1 in [8].

**Lemma 4.0.11.** *Let $Y$ be a space with an FDD $(Q_j)_{j=1}^{\infty}$ and let $X$ be a subspace of $Y$. Then given $\delta > 0$ there exists an automorphism $T : Y \to Y$ so that $\|T - I\| < \delta$ and $X$ satisfies the density condition with respect to the FDD $(TQ_jT^{-1})_{j=1}^{\infty}$.*

We first prove the following claim:

*Claim: Suppose $(Q'_j)_{j=1}^{\infty}$ is any FDD of $Y$ and $x \in Y$. Then given $n \in \mathbb{N}$ and $\nu > 0$ there exists an automorphism $S : Y \to Y$, with $\|S - I\| < \nu$, such that $SQ'_jS^{-1}(Y) = Q'_j(Y)$ for $1 \leq j \leq n$ and for some $m \geq n$ we have $x = \sum_{j=1}^{m} SQ'_jS^{-1}x$.*

**Proof of the claim:** Let $K$ be the FDD-constant of $(Q'_j)_{j=1}^{\infty}$. If $\sum_{j=1}^{n} Q'_j x = x$ we take $m = n$ and $S = I$, and the claim is proved in this case. Otherwise we have that $x - \sum_{j=1}^{n} Q'_j x \neq 0$. Then $\|\sum_{j=n+1}^{m} Q'_j x\| \xrightarrow{m\to\infty} \|x - \sum_{j=1}^{n} Q'_j x\| > 0$, and $\|x - \sum_{j=1}^{m} Q'_j x\| \xrightarrow{m\to\infty} 0$. Therefore we can choose $m > n$ so that

$$\frac{\|x - \sum_{j=1}^{m} Q'_j x\|}{\|\sum_{j=n+1}^{m} Q'_j x\|}$$

is as small as we like. We choose $m > n$ so that

$$\|x - \sum_{j=1}^{m} Q'_j x\| < \frac{\nu}{2K} \| \sum_{j=n+1}^{m} Q'_j x\|.$$

Pick $y^* \in Y^*$ with $\|y^*\| = 1$ and $y^*(\sum_{j=n+1}^{m} Q'_j x) = \|\sum_{j=n+1}^{m} Q'_j x\|$. Then let

$$Sy = y + \|\sum_{j=n+1}^{m} Q'_j x\|^{-1} y^* \left(\sum_{j=n+1}^{m} Q'_j y\right)\left(x - \sum_{j=1}^{m} Q'_j x\right).$$

Then $\|S - I\| < \nu$.

To see that $S$ is invertible, denote by $x^*$ the continuous linear functional

$$y \mapsto \|\sum_{j=n+1}^{m} Q'_j x\|^{-1} y^* \left(\sum_{j=n+1}^{m} Q'_j y\right),$$

by $v$ the vector

$$x - \sum_{j=1}^{m} Q'_j x$$

and by $x^* \otimes v$ the rank-one operator

$$y \mapsto x^*(y)v.$$

Then

$$S = I + x^* \otimes v.$$

Now

$$x^*(v) = \|\sum_{j=n+1}^{m} Q'_j x\|^{-1} y^* \left(\sum_{j=n+1}^{m} Q'_j \left(x - \sum_{j=1}^{m} Q'_j x\right)\right) = 0,$$

so for all $y$ in $Y$,

$$(x^* \otimes v)^2 y = (x^* \otimes v)(x^*(y)v) = x^*(y)x^*(v)v = 0,$$

so $(x^* \otimes v)^2 = 0$. Therefore

$$(I + x^* \otimes v)(I - x^* \otimes v) = (I - x^* \otimes v)(I + x^* \otimes v) = I^2 - (x^* \otimes v)^2 = I,$$

and $S$ is invertible, with

$$S^{-1} = I - x^* \otimes v.$$

Also $SQ'_j = Q'_j$ if $j = 1, 2, \ldots, n$ so that $SQ'_j S^{-1}(Y) = Q'_j(Y)$. We also have $S \sum_{k=1}^m Q'_k x = x$. Hence $S^{-1}x = \sum_{k=1}^m Q'_k x$ and

$$\left(\sum_{j=1}^m Q'_j\right) S^{-1}x = \left(\sum_{j=1}^m Q'_j\right) \sum_{k=1}^m Q'_k x = \sum_{k=1}^m Q'_k x = S^{-1}x,$$

so

$$x = SS^{-1}x = S\left(\sum_{j=1}^m Q'_j\right) S^{-1}x = \sum_{j=1}^m SQ'_j S^{-1}x.$$

This conclude the proof of the claim. □

We now turn to the Lemma.

**Proof of Lemma 4.0.11:** Now suppose $\nu_n > 0$ are such that $\prod_{j=1}^\infty (1+\nu_j) < 1+\delta$. Since $Y$ has an FDD, $Y$ is separable, hence $X$ is separable. Let $(x_n)_{n=1}^\infty$ be a dense sequence in $X$. We inductively define automorphisms $S_n : Y \to Y$ with $\|S_n - I\| < \nu_n$ and a nondecreasing sequence of integers $(m_n)_{n=0}^\infty$ such that if $T_0 = I$ and then $T_n = S_n S_{n-1} \cdots S_1$ we have

$$T_n Q_j T_n^{-1}(Y) = T_{n-1} Q_j T_{n-1}^{-1}(Y) \qquad \text{for } 1 \le j \le m_{n-1}, \tag{4.1}$$

and

$$x_n = \sum_{j=1}^{m_n} T_n Q_j T_n^{-1} x_n. \tag{4.2}$$

To do this pick $m_0 = 1$, say and then proceed inductively using the previous claim. If $m_0, \ldots, m_{n-1}$ and $S_1, \ldots, S_{n-1}$ have been chosen, we pick $S_n$ by the claim so that $\|S_n - I\| < \nu_n$,

$$T_n Q_j T_n^{-1}(Y) = S_n T_{n-1} Q_j T_{n-1}^{-1} S_n^{-1}(Y) = T_{n-1} Q_j T_{n-1}^{-1}(Y) \qquad \text{for } 1 \leq j \leq m_{n-1},$$

and for suitable $m_n \geq m_{n-1}$ we have

$$x_n = \sum_{j=1}^{m_n} S_n T_{n-1} Q_j T_{n-1}^{-1} S_n^{-1} x_n = \sum_{j=1}^{m_n} T_n Q_j T_n^{-1} x_n.$$

Next we show that the sequence $(T_n)_{n=1}^{\infty}$ is Cauchy. First recall that the convergence of $\prod_{j=1}^{\infty}(1+\nu_j)$ is equivalent to the convergence of $\sum_{j=1}^{\infty} \nu_j$, since for all $N \in \mathbb{N}$, $1 + \sum_{j=1}^{N} \nu_j \leq \prod_{j=1}^{N}(1+\nu_j) \leq \exp\left(\sum_{j=1}^{N} \nu_j\right)$. Then for $j \geq 2$,

$$\begin{aligned} \|T_j - T_{j-1}\| &\leq \|S_j - I\| \|\prod_{k=1}^{j-1} S_k\| \\ &< \nu_j \prod_{k=1}^{j-1} \|S_k\| \\ &< \nu_j \prod_{k=1}^{\infty}(1+\nu_k) \\ &< \nu_j(1+\delta). \end{aligned}$$

So for $m > n$

$$\begin{aligned} \|T_m - T_n\| &\leq \sum_{j=n+1}^{m} \|T_j - T_{j-1}\| \\ &< (1+\delta) \sum_{j=n+1}^{m} \nu_j \end{aligned}$$

$$< (1+\delta) \sum_{j=n+1}^{\infty} \nu_j$$
$$\xrightarrow{n\to\infty} 0,$$

as required. Therefore the sequence $(T_n)_{n-1}^{\infty}$ converges in operator norm to an operator $T$.

By induction, for all $n \in \mathbb{N}$, $\|T_n - I\| < \prod_{j=1}^{n}(1+\nu_j) - 1$. Indeed, when $n = 1$, we have

$$\|S_1 - I\| < \nu_1 = \prod_{j=1}^{n}(1+\nu_j) - 1.$$

Now suppose the inequality holds for some $n$. Then

$$\begin{aligned}
\|T_{n+1} - I\| &\leq \|T_{n+1} - T_n\| + \|T_n - I\| \\
&< \|(S_{n+1} - I)\prod_{j=1}^{n} S_j\| + \prod_{j=1}^{n}(1+\nu_j) - 1 \\
&\leq \|S_{n+1} - I\| \prod_{j=1}^{n} \|S_j\| + \prod_{j=1}^{n}(1+\nu_j) - 1 \\
&< \nu_{n+1}\prod_{j=1}^{n}(1+\nu_j) + \prod_{j=1}^{n}(1+\nu_j) - 1 \\
&= (1+\nu_{n+1})\prod_{j=1}^{n}(1+\nu_j) - 1 \\
&= \prod_{j=1}^{n+1}(1+\nu_j) - 1,
\end{aligned}$$

as required.

Therefore

$$\|T - I\| = \lim_n \|T_n - I\| \leq \lim_n \prod_{j=1}^{n}(1+\nu_j) - 1 = \prod_{j=1}^{\infty}(1+\nu_j) - 1 < 1 + \delta - 1 = \delta.$$

By induction on (4.1) and by taking a limit,

$$TQ_jT^{-1}(Y) = T_nQ_jT_n^{-1}(Y), \qquad \text{for } 1 \leq j \leq m_n.$$

Since $(TQ_jT^{-1})(TQ_kT^{-1}) = \delta_{jk}TQ_jT^{-1}$,

$$\left(\sum_{j=1}^{m_n} TQ_jT^{-1}\right)(Y) = \sum_{j=1}^{m_n} TQ_jT^{-1}(Y),$$

and since $(T_nQ_jT_n^{-1})(T_nQ_kT_n^{-1}) = \delta_{jk}T_nQ_jT_n^{-1}$,

$$\left(\sum_{j=1}^{m_n} T_nQ_jT_n^{-1}\right)(Y) = \sum_{j=1}^{m_n} T_nQ_jT_n^{-1}(Y).$$

Therefore

$$\left(\sum_{j=1}^{m_n} T_nQ_jT_n^{-1}\right)(Y) = \left(\sum_{j=1}^{m_n} TQ_jT^{-1}\right)(Y).$$

Equation (4.2) says that $x_n$ is in the range of the projection $\sum_{j=1}^{m_n} T_nQ_jT_n^{-1}$, hence $x_n$ is in the range of the projection $\sum_{j=1}^{m_n} TQ_jT^{-1}$, that is,

$$x_n = \sum_{j=1}^{m_n} TQ_jT^{-1}x_n, \qquad n \in \mathbb{N}.$$

□

**Proposition 4.0.12.** *Let $X$ be a separable Banach space containing no complemented copy of $\ell_1$ (respectively a separable reflexive Banach space) which is isometrically embedded in a Banach space $Y$ with a 1-UFDD $(Q_j)_{j=1}^{\infty}$. Suppose $X$ satisfies the density condition with respect to $(Q_j)_{j=1}^{\infty}$. Then $X$ can be isometrically embedded into a Banach space $Z$ (respectively a reflexive Banach space) with a shrinking 1-UFDD $(Q'_j)_{j=1}^{\infty}$ with* $\operatorname{rank} Q'_j \leq \operatorname{rank} Q_j$.

*If further $X$ is $\lambda-$complemented in $Y$ then $X$ is $\lambda-$complemented in $Z$.*

**Proof:** We assume without loss of generality that $Q_j(Y) = Q_j(X)$ for each $j$, if necessary replacing $Y$ by its closed subspace $[\cup_{j=1}^{\infty} Q_j(X)]$. Note that $[\cup_{j=1}^{\infty} Q_j(X)]$ still contains $X$, since if $x \in X$, then $x = \sum_{j=1}^{\infty} Q_jx$.

Let $J : X \to Y$ be an isometric embedding. Define on $Y^*$ the norm

$$|||y^*||| = \sup_n \sup_{\epsilon_j = \pm 1} \| \sum_{j=1}^n \epsilon_j J^* Q_j^* y^* \|.$$

Note that since $(Q_j)_{j=1}^\infty$ is 1-unconditional, we have $|||y^*||| \leq \|y^*\|$, in particular $|||y^*||| < \infty$.

It is straightforward to check that $||| \cdot |||$ is a seminorm on $Y^*$. Now suppose $|||y^*||| = 0$. Then in particular, for all $j \in \mathbb{N}$, $J^* Q_j^* y^* = 0$. Let $y \in Y$. Since $Q_j(Y) = Q_j(J(X))$ for all $j$, there exist $x_j \in X$ such that $Q_j y = Q_j J x_j$. Then

$$\langle y, y^* \rangle = \langle \sum_{j=1}^\infty Q_j y, y^* \rangle = \sum_{j=1}^\infty \langle Q_j y, y^* \rangle = \sum_{j=1}^\infty \langle Q_j J x_j, y^* \rangle = \sum_{j=1}^\infty \langle x_j, J^* Q_j^* y^* \rangle = 0.$$

Since $y$ is arbitrary, $y^* = 0$, hence $||| \cdot |||$ is a norm. Although we have $||| \cdot ||| \leq \| \cdot \|$, we have no inequality in the other direction, so the normed space $(Y, ||| \cdot |||)$ need not be complete.

Although $\frac{|||y^*|||}{\|y^*\|} \leq 1$ for all $y^*$, there need not be any lower bound on this ratio. In particular, given a $z \in Y$, $\{|y^*(z)| : \ |||y^*||| \leq 1\}$ need not be bounded. Denote by $\tilde{Z}$ the vector space of elements $z$ of $Y$ such that

$$\|z\|_Z = \sup\{|y^*(z)| : \ |||y^*||| \leq 1\} < \infty.$$

Then $\| \cdot \|_Z$ is a norm on $\tilde{Z}$ and the inclusion map from $(\tilde{Z}, \| \cdot \|_Z)$ into $(Y, \| \cdot \|)$ is a bounded linear map, indeed,

$$\|z\|_Y = \sup\{|y^*(z)| : \|y^*\| \leq 1\} \leq \sup\{|y^*(z)| : |||y^*||| \leq 1\} = \|z\|_Z, \qquad z \in \tilde{Z}.$$

In fact, $(\tilde{Z}, \| \cdot \|_Z)$ is complete: Suppose that $(z_n)_{n=1}^\infty \subset \tilde{Z}$ is Cauchy in the norm $\| \cdot \|_Z$. Since $\| \cdot \|_Y \leq \| \cdot \|_Z$, $(z_n)_{n=1}^\infty$ is also Cauchy in $(Y, \| \cdot \|)$, hence convergent to some $y \in Y$.

Next we show that $y \in \tilde{Z}$. Note that since $(z_n)_{n=1}^{\infty}$ is Cauchy in the $\|\cdot\|_Z$ norm, there exists $M$ such that for all $n$, $\|z_n\| \leq M < \infty$. Let $y \in Y^*$, with $|||y^*||| \leq 1$. Then for all $n$, $|y^*(z_n)| \leq \|z_n\|_Z \leq M < \infty$. Therefore

$$|y^*(y)| = \lim_n |y^*(z_n)| \leq M < \infty,$$

and taking the supremum over $y^*$ with $|||y^*||| = 1$, $\|y\|_Z \leq M < \infty$, so $y \in \tilde{Z}$ as required.

Now we show that $\|z_n - y\|_Z \to 0$. Since $\|z_n - y\|_Y \to 0$, certainly $y^*(z_n - y) \to 0$ for all $y^* \in Y^*$, in particular for all $y^* \in Y^*$ such that $|||y^*||| \leq 1$. Let $\varepsilon > 0$. Then there exists an $N \in \mathbb{N}$ such that $m, n > N \implies \|z_m - z_n\|_Z < \varepsilon$, therefore for all $y^* \in B_{(Y^*, |||\cdot|||)}$, $|y^*(z_m - z_n)| < \varepsilon$. Taking the limit as $m \to \infty$, we have that if $n > N$ then for all $y^* \in B_{(Y^*, |||\cdot|||)}$, $|y^*(y - z_n)| \leq \varepsilon$. Thus $\|y - z_n\|_Z \to 0$, as required.

Next we show that $|||\cdot|||$ is weak$^*$-lower semicontinuous: Fix an $n \in \mathbb{N}$ and a sequence $(\epsilon_j)_{j=1}^n \in \{-1, 1\}^n$. The operator $\sum_{j=1}^n \epsilon_j J^* Q_j^* : Y^* \to X^*$ is weak$^*$ to weak$^*$ continuous, and the norm on $X^*$ is weak$^*$ lower semicontinuous (see eg Theorem 2.6.14 of [21],) hence

$$y^* \mapsto \|\sum_{j=1}^n \epsilon_j J^* Q_j^* y^*\|$$

is a weak$^*$ lower semicontinuous function on $Y^*$. The supremum of any family of lower semicontinuous functions is also lower semicontinuous, hence $|||\cdot|||$ is weak$^*$ lower semicontinuous, as claimed.

Let $x \in X$ and $y^* \in Y^*$ with $|||y^*||| \leq 1$. Then

$$\langle Q_j J x, y^* \rangle = \langle x, J^* Q_j^* y^* \rangle \leq \|x\| \|J^* Q_j^* y^*\| \leq \|x\| |||y^*||| \leq \|x\|.$$

Taking the supremum over $y^*$ with $|||y^*||| \le 1$, it follows that $Q_j Jx \in \tilde{Z}$, hence $Q_j(Y) = Q_j(X) \subset \tilde{Z}$.

Let $Z$ be the closed linear span of $\cup_{j=1}^{\infty} Q_j(Y)$ in $\tilde{Z}$ with respect to $\|\cdot\|_Z$. Then since $|||\cdot|||$ is weak* lower semicontinuous, $Z^*$ can be identified with the completion of $(Y^*, |||\cdot|||)$ (see eg the proof of Theorem 2.6.15 in [21], but note that in our case, $|||\cdot|||$ need not be equivalent to $\|\cdot\|$ on $Y^*$, nor need $\|\cdot\|_Z$ be equivalent to $\|\cdot\|$ on $Z$.)

Then $(Q_j)_{j=1}^{\infty}$ is a 1-UFDD for $Z$. We must check that $(Q_j)_{j=1}^{\infty}$ is shrinking for $Z$. Indeed if not we can find a blocked sequence $z_j^* \in \sum_{i=N_{j-1}+1}^{N_j} Q_i^*(Y^*)$ where $N_0 = 0 < N_1 < N_2 < \cdots$ which is equivalent to the canonical basis $(e_j)_{j=1}^{\infty}$ of $c_0$. See, eg, Theorems 3.2.12 and 3.3.2 of [1].

Let $S : c_0 \to (Y^*, |||\cdot|||)$ be an isomorphism into $(Y^*, |||\cdot|||)$ such that $Se_j = z_j^*$.

Choose $\epsilon_i = \pm 1$ so that

$$\|J^* \sum_{i=N_{j-1}+1}^{N_j} \epsilon_i Q_i^* z_j^*\| = |||z_j^*|||.$$

Note that we can do this, since $z_j^*$ is supported on finitely many $Q_i^*(Y^*)$, so the supremum in the definition of $|||z_j^*|||$ is attained. Let

$$x_j^* = J^* \left( \sum_{i=N_{j-1}+1}^{N_j} \epsilon_i Q_i^* z_j^* \right).$$

Denote by $R : [z_j^*] \to X^*$ the bounded linear operator such that $Rz_j^* = x_j^*$. Then the bounded linear operator $R \circ S : c_0 \to X^*$ satisfies $R \circ Se_j = x_j^*$. Since $X^*$ contains no copy of $c_0$, it must be that $\|x_j^*\| \to 0$, hence $|||z_j^*||| \to 0$, contrary to assumption.

Also if $x \in X$ then $\|x\|_Y = \|x\|_Z$ so that $X$ is isometrically embedded in $\tilde{Z}$. By the assumption that $X$ satisfies the density condition with respect to $(Q_j)_{j=1}^{\infty}$, a dense subset of $X$ lies in the linear span of $Q_j(Y)$ and it follows that $X \subset Z$. Further since $\|z\|_Z \geq \|z\|_Y$ in general, if there is a projection $P : Y \to X$ with $\|P\| = \lambda$ then $\|P\|_{Z \to X} \leq \lambda$.

Finally if $X$ is reflexive we show that $Z$ is reflexive. To do this it is necessary and sufficient to show that the UFDD of $Z^*$ given by $(Q_j^*(Z^*))_{j=1}^{\infty}$ is also shrinking. See eg Theorem 3.2.13 of [1].

Suppose not. Then we can find a blocked sequence $z_j^* \in \sum_{i=N_{j-1}+1}^{N_j} Q_i^*(Y^*)$ where $N_0 = 0 < N_1 < N_2 < \cdots$ which is equivalent to the canonical basis $(e_j)_{j=1}^{\infty}$ of $\ell_1$. See eg Theorem 3.3.1 of [1].

Let $\Delta$ denote the Cantor set $\{-1, +1\}^{\mathbb{N}}$ of all sequences $\epsilon = (\epsilon_i)_{i=1}^{\infty}$ and consider the Hausdorff space $\Omega = \Delta \times B_X$ where $B_X$ has the weak topology. Note that $B_X$ is weakly compact because $X$ is reflexive, and $\Delta$ is compact by Tychonoff's theorem, hence $\Omega$ is a compact Hausdorff space.

Let $f_j \in C(\Omega)$ be defined by

$$f_j(\epsilon, x) = \left\langle x, J^* \left( \sum_{i=N_{j-1}+1}^{N_j} \epsilon_i Q_i^* z_j^* \right) \right\rangle .$$

If $n \in \mathbb{N}$ and $a_j \in \mathbb{R}$, then

$$\begin{aligned} \| \sum_{j=1}^{n} a_j f_j \| &= \sup_{\epsilon \in \Delta, x \in B_X} \left| \left\langle x, \sum_{j=1}^{n} a_j J^* \left( \sum_{i=N_{j-1}+1}^{N_j} \epsilon_i Q_i^* z_j^* \right) \right\rangle \right| \\ &= \sup_{\epsilon \in \Delta} \left\| \sum_{j=1}^{n} a_j J^* \left( \sum_{i=N_{j-1}+1}^{N_j} \epsilon_i Q_i^* z_j^* \right) \right\|_{X^*} \end{aligned}$$

$$= \sup_{m\in\mathbb{N}} \sup_{\epsilon_i=\pm 1} \left\| \sum_{j=1}^{n} a_j J^* \sum_{i=1}^{m} \epsilon_i Q_i^* z_j^* \right\|_{X^*}, \qquad \text{since} z_j^* \in \sum_{i=N_{j-1}+1}^{N_j} Q_i^*(Y^*),$$

$$= \sup_{m\in\mathbb{N}} \sup_{\epsilon_i=\pm 1} \left\| \sum_{i=1}^{m} \epsilon_i J^* Q_i^* \sum_{j=1}^{n} a_j z_j^* \right\|_{X^*}$$

$$= ||| \sum_{j=1}^{n} a_j z_j^* |||.$$

Therefore, since $(z_j)_{j=1}^{\infty}$ is equivalent to the canonical basis $(e_j)_{j=1}^{\infty}$ of $\ell_1$, so is $(f_j)_{j=1}^{\infty}$.

Since $[f_j]_{j=1}^{\infty} \subset C(\Omega)$ is isomorphic to $\ell_1$, there is an element of $C(\Omega)^*$ whose restriction to $[f_j]_{j=1}^{\infty}$ corresponds to the element $(1, 1, 1, \ldots) \in \ell_\infty = \ell_1^*$. In particular, its action on the vectors $f_j$ is uniformly bounded below. Let $\nu$ be the standard representation of this functional as a signed regular Borel measure on $\Omega$. Then there exists a probability measure $\mu$ on $\Omega$, that is, a positive regular Borel measure of total variation 1, and a Borel function $\varphi \in L_1(\mu)$ such that $d\nu = \varphi d\mu$. That is, $\varphi$ is the Radon-Nikodym derivative of $\nu$ with respect to $\mu$. By scaling we may assume that

$$\int_\Omega f_j \varphi \, d\mu \geq 1, \qquad j = 1, 2, \ldots.$$

We argue that $\lim_{j\to\infty} f_j(\epsilon, x) = 0$ for every $(\epsilon, x) \in \Omega$ and this contradicts the Dominated Convergence Theorem. Note that since the sequence $(f_j)_{j=1}^{\infty}$ is uniformly bounded above, and since $\varphi \in L_1(\mu)$, the sequence $(f_j\varphi)_{j=1}^{\infty}$ is dominated by an $L_1(\mu)$ function.

Indeed

$$|f_j(\epsilon, x)| = |\langle \sum_{i=N_{j-1}+1}^{N_j} \epsilon_i Q_i J x, z_j^* \rangle|$$

$$\leq \| \sum_{i=N_{j-1}+1}^{N_j} \epsilon_i Q_i J x \|_Z \| z_j^* \|_{Z^*} \xrightarrow{j\to\infty} 0$$

since $\sum_{i=1}^{\infty} \epsilon_i Q_i J x$ converges. □

The proof of the next Proposition is standard.

**Proposition 4.0.13.** *Let $X$ be a separable Banach space. Then the following conditions on $X$ are equivalent:*

*(i) Given $\delta > 0$ there exists a Banach space $Y$ with a 1-UFDD and a subspace $X_\delta$ of $Y$ with $d(X, X_\delta) < 1 + \delta$.*

*(ii) Given $\delta > 0$ there exists a Banach space $Y$ with a 1-unconditional basis and a subspace $X_\delta$ of $Y$ with $d(X, X_\delta) < 1 + \delta$.*

*(iii) Given $\delta > 0$ there exists a Banach space $Y$ containing $X$ (isometrically) and a sequence of finite-rank operators $A_n : X \to Y$ such that*

$$\| \sum_{j=1}^{n} \epsilon_j A_j \| < 1 + \delta, \qquad \epsilon_j = \pm 1, \; n = 1, 2, \ldots$$

*and*

$$x = \sum_{j=1}^{\infty} A_j x, \qquad x \in X.$$

**Proof:** (i) $\implies$ (ii): It is essentially contained in [19] Theorem 1.g.5 (p. 51) that every Banach space with a 1-UFDD is $(1+\delta)$-isomorphic to a subspace of a space with a 1-unconditional basis; in [19] the constants are not tracked, but the same argument proves this more precise statement. In fact, the norm of the isomorphism in Theorem 1.g.5 of [19] is bounded above by 1, and the norm of its inverse is bounded above by $\frac{1+\sum_{n=1}^{\infty} 4^{-n}}{1-\sum_{n=1}^{\infty} 4^{-n}}$. Replacing 4 by $b$, where $b > 2$, we have

$$\frac{1 + \sum_{n=1}^{\infty} b^{-n}}{1 - \sum_{n=1}^{\infty} b^{-n}} = \frac{b}{b-2},$$

and provided that $b > 2\frac{1+\delta}{\delta}$, we have $\frac{b}{b-2} < 1 + \delta$. Now composing the two embeddings, $d(X, X_\delta) < (1 + \delta)^2$, and we may choose $\delta$ as small as we like.

(ii) $\implies$ (iii): By Lemma 4.0.10 $Y$ can be equivalently renormed so that $X_\delta$ is isometrically isomorphic to $X$, and so that the 1-unconditional basis for $Y$ is at most $(1 + \delta)$-unconditional with respect to the new norm. Then let the operators $A_n$ be the coordinate projections for the basis.

(iii) $\implies$ (i): Note that the conditions on $(A_j)_{j=1}^\infty$ imply that for all $x \in X$, $\sum_{j=1}^\infty A_j x$ is a WUC series in $Y$, but they fall short of guaranteeing that $\sum_{j=1}^\infty A_j x$ converges unconditionally. However, by passing to a blocking of $(A_j)_{j=1}^\infty$, we will ensure that even a dense subset of $X$ has an absolutely convergent, in particular, unconditionally convergent expansion with respect to $(A_j)_{j=1}^\infty$.

Let $(x_n)_{n=1}^\infty \subset X$ be a dense sequence in $X$. We will use a diagonal argument to choose a blocking of $(A_j)_{j=1}^\infty$.

Set $A_j^0 = A_j$ for all $j \in \mathbb{N}$. Choose $N_1^1 \in \mathbb{N}$ so that for all $N > N_1^1$, $\|\sum_{i=N_1^1+1}^N A_i^0 x_1\| < \frac{1}{2^{1+2}}$. Now choose $N_2^1 > N_1^1$ so that for all $N > N_2^1$, $\|\sum_{i=N_2^1+1}^N A_i^0 x_1\| < \frac{1}{2^{1+3}}$. Thus define inductively a sequence $N_0^1 = 0 < N_1^1 < N_2^1 < \cdots$ such that

$$\|\sum_{i=N_{j-1}^1+1}^{N_j^1} A_i^0 x_1\| < \frac{1}{2^{1+j}}, \qquad j \geq 2.$$

For all $j \in \mathbb{N}$, let $A_j^1 = \sum_{i=N_{j-1}^1+1}^{N_j^1} A_i^0$. So

$$\|A_j^1 x_1\| < \frac{1}{2^{1+j}}, \qquad j \geq 2.$$

Then $(A_j^1)_{j=1}^\infty$ is a blocking of $(A_j^0)_{j=1}^\infty$. In particular, $\sum_{j=1}^\infty A_j^1 x$ converges to $x$, for all $x \in X$.

Now suppose that for some $n \in \mathbb{N}$, $n$ sequences $((N_j^k)_{j=0}^\infty)_{k=1}^n$ have been chosen, with $N_0^k = 0 < N_1^k < N_2^k < \cdots$ for $1 \le k \le n$, and suppose that if $A_j^k = \sum_{i=N_{j-1}^k+1}^{N_j^k} A_i^{k-1}$ for $1 \le k \le n$ and $j \in \mathbb{N}$, then

$$\|A_j^k x_m\| < \frac{1}{2^{m+j}}, \qquad 1 \le m \le k \le n, \qquad j \ge 2.$$

As above, choose a sequence $N_0^{n+1} = 0 < N_1^{n+1} < N_2^{n+1} < \cdots$ such that if $A_j^{n+1} = \sum_{i=N_{j-1}^{n+1}+1}^{N_j^{n+1}} A_i^n$ for $j \in \mathbb{N}$, then

$$\|A_j^{n+1} x_{n+1}\| < \frac{1}{2^{n+1+j}}, \qquad j \ge 2.$$

Then if $1 \le m \le n$ and $j \ge 2$ we have

$$\|A_j^{n+1} x_m\| \le \sum_{i=N_{j-1}^{n+1}+1}^{N_j^{n+1}} \|A_i^n x_m\| < \sum_{i=N_{j-1}^{n+1}+1}^{N_j^{n+1}} \frac{1}{2^{m+i}} < \frac{2}{2^{m+N_{j-1}^{n+1}+1}} = \frac{1}{2^{m+N_{j-1}^{n+1}}} \le \frac{1}{2^{m+j}},$$

assuming, as we may, that $N_{j-1}^{n+1} \ge j$ for all $j \ge 2$. Hence

$$\|A_j^{n+1} x_m\| < \frac{1}{2^{m+j}}, \qquad 1 \le m \le n+1, \qquad j \ge 2.$$

Therefore

$$\|A_j^k x_m\| < \frac{1}{2^{m+j}}, \qquad 1 \le m \le k \le n+1, \qquad j \ge 2.$$

Thus we construct inductively a sequence $((A_j^k)_{j=1}^\infty)_{k=1}^\infty$ of successive blockings of $(A_j)_{j=1}^\infty$ such that for all $n \in \mathbb{N}$,

$$\|A_j^n x_m\| < \frac{1}{2^{m+j}}, \qquad 1 \le m \le n, \qquad j \ge 2.$$

Next we consider the diagonal sequence $(A_j^j)_{j=1}^\infty$, where $A_j^j = \sum_{i=N_{j-1}^j+1}^{N_j^j} A_i^{j-1}$, and modify it to ensure that it forms a blocking of $(A_j)_{j=1}^\infty$ with no gaps. Let

$$A'_j = \sum_{i=N_{j-1}^j+1}^{N^j_{N_j^{j+1}}} A_i^{j-1}, \qquad j \in \mathbb{N}.$$

Then $(A'_j)_{j=1}^\infty$ is a blocking of $(A_j)_{j=1}^\infty$, and for $1 \le m \le j-1$ we have

$$\|A'_j x_m\| \le \sum_{i=N_{j-1}^j+1}^{N^j_{N_j^{j+1}}} \|A_i^{j-1} x_m\| < \sum_{i=N_{j-1}^j+1}^{N^j_{N_j^{j+1}}} \frac{1}{2^{m+i}} < \frac{2}{2^{m+N_{j-1}^j+1}} = \frac{1}{2^{m+N_{j-1}^j}} \le \frac{1}{2^{m+j}},$$

assuming, as we may, that $N_{j-1}^j \ge j$ for all $j \ge m+1$.

Therefore

$$\begin{aligned}
\sum_{j=1}^\infty \|A'_j x_m\| &= \sum_{j=1}^m \|A'_j x_m\| + \sum_{j=m+1}^\infty \|A'_j x_m\| \\
&< \sum_{j=1}^m \|A'_j x_m\| + \sum_{j=m+1}^\infty \frac{1}{2^{m+j}} \\
&= \sum_{j=1}^m \|A'_j x_m\| + \frac{1}{4^m} < \infty, \qquad m \in \mathbb{N}.
\end{aligned}$$

Since $(A'_j)_{j=1}^\infty$ is a blocking of $(A_j)_{j=1}^\infty$, it shares with $(A_j)_{j=1}^\infty$ all of its hypothetical properties. Therefore by relabeling $A'_j$ as $A_j$, we can assume that

$$\sum_{j=1}^\infty \|A_j x_m\| < \infty \qquad \text{for all } m \in \mathbb{N}.$$

Note that although $(x_n)_{n=1}^\infty$ is dense in $X$ and $\sum_{j=1}^\infty A_j x_n$ converges absolutely for all $n$, we cannot conclude that $\sum_{j=1}^\infty A_j x$ converges absolutely, or even unconditionally, for all $x \in X$.

We define $W$ to be the Banach space of all sequences $(y_j)_{j=1}^\infty$ with $y_j \in A_j(X)$ such that $\sum_{j=1}^\infty y_j$ is a WUC series in $Y$, under the norm

$$\|(y_j)_{j=1}^\infty\| = \sup_n \sup_{\epsilon_j = \pm 1} \|\sum_{j=1}^n \epsilon_j y_j\|.$$

We also define the subspace $Z$ of $W$ to be the space of all sequences $(y_j)_{j=1}^\infty$ with $y_j \in A_j(X)$ such that $\sum_{j=1}^\infty y_j$ converges unconditionally in $Y$. Then $Z$ is closed in $W$, and in particular, $Z$ is complete.

By definition of its norm, the natural coordinate projections on $Z$ are a 1-UFDD for $Z$. Define $\tilde{T}$ as the unique bounded linear operator on $\langle x_n : x \in \mathbb{N}\rangle$ such that

$$\tilde{T}x_n = (A_j x_n)_{j=1}^{\infty}, \qquad n \in \mathbb{N}.$$

Since $\sum_{j=1}^{\infty} A_j x_n$ converges absolutely, in particular unconditionally, in $Y$, we have $\tilde{T}(\langle x_n : n \in \mathbb{N}\rangle) \subset Z$.

Let $T$ be the unique continuous extension of $\tilde{T}$ to $X$. Since for all $x \in X$, $\sum_{j=1}^{\infty} A_j x$ is a WUC series in $Y$, we have that $T(X) \subset W$. Moreover, since $Z$ is closed in $Y$, we have $T(X) \subset Z$. Setting $\epsilon_j = 1$ for $1 \leq j \leq n$ and letting $n$ tend to infinity, we have that $\|Tx\| \geq \|x\|$ for all $x \in X$, hence $T$ is invertible and $\|T^{-1}\| \leq 1$. Finally, by the conditions on $(A_j)_{j=1}^{\infty}$, we have that $\|T\| \leq 1 + \delta$. Therefore $d(X, T(X)) \leq \|T\|\|T^{-1}\| \leq 1 + \delta$. □

Combining Proposition 4.0.12, Proposition 4.0.13 and Lemma 4.0.11 gives the following result. Part (ii) is contained in Corollary IV.4 of [7] (where the proof is inaccurate); by an example on pg 51 of [19] one cannot hope for (iii) to hold with $Z$ having an unconditional basis. The example cited is a Banach space $X$ with a UFDD, such that $X$ cannot be embedded complementedly in any Banach space with an unconditional basis.

**Proposition 4.0.14.** *Suppose $X$ is a separable Banach space containing no complemented copy of $\ell_1$. Suppose, given $\delta > 0$ there exists a Banach space $Y$ containing $X$ (isometrically) and a sequence of finite-rank operators $A_n : X \to Y$ such that*

$$\|\sum_{j=1}^{n} \epsilon_j A_j\| < 1 + \delta, \qquad n \in \mathbb{N},\ \epsilon_j = \pm 1$$

*and*

$$x = \sum_{j=1}^{\infty} A_j x, \qquad x \in X.$$

*Then*

*(i) For any $\delta > 0$, there is a Banach space $Z$ with a shrinking 1-unconditional basis and a subspace $X_\delta$ of $Z$ such that $d(X, X_\delta) < 1 + \delta$.*

*(ii) If $X$ is reflexive then we may take $Z$ reflexive in (i).*

*(iii) If for every $\delta > 0$ we can take $Y = X$ (i.e. $X$ has (UMAP)) then for any $\delta > 0$, there is a Banach space $Z$ with a shrinking 1-UFDD and a $(1+\delta)-$complemented subspace $X_\delta$ of $Z$ such that $d(X, X_\delta) < 1 + \delta$.*

*(iv) If $X$ is reflexive then we may take $Z$ reflexive in (iii).*

# Chapter 5

# The Main Result

**Lemma 5.0.15.** *Let $Y$ be a Banach space and suppose $X$ is a closed subspace of $Y$. Denote by $Q$ the quotient map $Q : Y \to Y/X$. Suppose $(B_n)_{n=1}^{\infty}$ is a uniformly bounded sequence of operators on $Y$ such that*

$$\lim_{n\to\infty} \|QB_n\|_{X\to Y/X} = 0 \tag{5.1}$$

*and*

$$\limsup_{n\to\infty} \|B_n\|_{X\to Y} \leq 1. \tag{5.2}$$

*Then given $\delta > 0$ there is an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that if $n_1 < n_2 < \cdots < n_k$ with $n_j \in \mathbb{M}$ for $1 \leq j \leq k$ then*

$$\|B_{n_1} B_{n_2} \cdots B_{n_k}\|_{X\to Y} < 1 + \delta.$$

**Proof:** We suppose $\|B_n\| \leq M$ for all $n$. We assume $\delta < 1/2$. By passing to a subsequence of $(B_n)_{n=1}^{\infty}$, it suffices to prove this for $\mathbb{M} = \mathbb{N}$ when $\|QB_n\|_{X\to Y/X} < \nu_n/3$ and $\|B_n\|_{X\to Y} \leq 1 + \nu_n/3$ where $(\nu_n)_{n=1}^{\infty}$ is the decreasing positive sequence given by $\nu_n = (3M+6)^{-n+1}\delta$. We will prove by induction on $k$ that

$$\|QB_{n_1} \cdots B_{n_k}\|_{X\to Y/X} < \nu_{n_1} \tag{5.3}$$

and

$$\|B_{n_1}\cdots B_{n_k}\|_{X\to Y} < 1+\nu_{n_1}. \tag{5.4}$$

Under these hypotheses the conclusion is true for $k=1$. We next assume it is true for some $k\in\mathbb{N}$ and prove it for products of length $k+1$. Consider $m<m_1<\cdots<m_k$. Then if $S=B_{m_1}\cdots B_{m_k}$ we have, by the induction hypothesis,

$$\|QS\|_{X\to Y/X} < \nu_{m_1} \le \nu_{m+1} \tag{5.5}$$

and

$$\|S\|_{X\to Y} < 1+\nu_{m_1} \le 1+\nu_{m+1}. \tag{5.6}$$

By (5.5), if $x\in X$ with $\|x\|\le 1$ then there exists $x'\in X$ so that

$$\|x'-Sx\| < \nu_{m+1}$$

and therefore by (5.6),

$$\|x'\| \le \|Sx\| + \|x'-Sx\| < 1+2\nu_{m+1}.$$

Now

$$B_mSx = B_mx' + B_m(Sx-x')$$

and so we have

$$\begin{aligned}\|QB_mSx\| &\le \|QB_m\|_{X\to Y/X}\|x'\| + \|Q\|\|B_m\|_{Y\to Y}\|Sx-x'\| \\ &< \tfrac{1}{3}\nu_m(1+2\nu_{m+1}) + M\nu_{m+1} \\ &< \tfrac{2}{3}\nu_m + M\nu_{m+1} \\ &< \nu_m\end{aligned}$$

and

$$\begin{aligned}\|B_mSx\| &\le \|B_m\|_{X\to Y}\|x'\| + \|B_m\|_{Y\to Y}\|Sx - x'\| \\ &\le (1+\tfrac{1}{3}\nu_m)(1+2\nu_{m+1}) + M\nu_{m+1} \\ &< 1+\tfrac{2}{3}\nu_m + (M+2)\nu_{m+1} \\ &= 1+\nu_m\end{aligned}$$

establishing both inductive hypotheses (5.3) and (5.4). □

Our main result is:

**Theorem 5.0.16.** *Let $X$ be a separable Banach space. Then the following conditions are equivalent:*

*(i) $X$ has (au*).*

*(ii) For any $\delta > 0$ there is a Banach space $Y$ with a shrinking 1-unconditional basis and a subspace $X_\delta$ of $Y$ such that $d(X, X_\delta) < 1+\delta$.*

**Proof:** (ii) $\Longrightarrow$ (i): Let $T : X \to X_\delta$ be an isomorphism onto $X_\delta$, with $\|T\| < 1+\delta$ and $\|T^{-1}\| = 1$. By the second part of Proposition 3.0.8, $Y$ has (au*), and by the fourth part of that proposition, so does $X_\delta$. Now if $x^* \in X^*$ and $(x_n^*)_{n=1}^\infty \subset X^*$ is weak*-null, then

$$\begin{aligned}\limsup_n \|x^* + x_n^*\| &= \limsup_n \|T^*((T^*)^{-1}x^* + (T^*)^{-1}x_n^*)\| \\ &\le (1+\delta)\limsup_n \|(T^*)^{-1}x^* + (T^*)^{-1}x_n^*\| \\ &= (1+\delta)\limsup_n \|(T^*)^{-1}x^* - (T^*)^{-1}x_n^*\| \\ &\le (1+\delta)\limsup_n \|x^* - x_n^*\|.\end{aligned}$$

Since $\delta > 0$ is arbitrary, $\limsup_n \|x^* + x_n^*\| \leq \limsup_n \|x^* - x_n^*\|$.

It follows that $\limsup_n(\|x^* + x_n^*\| - \|x^* - x_n^*\|) \leq 0$. Indeed, if not, then $\limsup_n(\|x^* + x_n^*\| - \|x^* - x_n^*\|) = c > 0$. Note that any subsequence of $(x_n^*)_{n=1}^\infty$ is still weak*-null. We may pass to a subsequence of $(x_n^*)_{n=1}^\infty$ such that $\lim_n(\|x^*+x_n^*\| - \|x^* - x_n^*\|)$ exists and equals $c$. It follows that $\limsup_n \|x^* + x_n^*\| > \limsup_n \|x^* - x_n^*\|$, which contradicts the foregoing argument.

Replacing $(x_n^*)_{n=1}^\infty$ by $(-x_n^*)_{n=1}^\infty$, we also have that $\limsup_n(\|x^* - x_n^*\| - \|x^* + x_n^*\|) \leq 0$. Therefore

$$\begin{aligned}\liminf_n(\|x^* + x_n^*\| - \|x^* - x_n^*\|) &= \liminf_n(-(\|x^* - x_n^*\| - \|x^* + x_n^*\|)) \\ &= -\limsup_n(\|x^* - x_n^*\| - \|x^* + x_n^*\|) \\ &\geq 0,\end{aligned}$$

that is, $\liminf_n(\|x^* + x_n^*\| - \|x^* - x_n^*\|) \geq 0$. Thus $\lim_n(\|x^* + x_n^*\| - \|x^* - x_n^*\|)$ exists and equals 0. Therefore $X$ has (au*).

We turn to the proof of (i) $\implies$ (ii).

By Proposition 3.0.6 $X^*$ is separable. We start by using the result of Zippin [31] that $X$ can be embedded in a space $Y$ with a shrinking basis (by Lemma 4.0.10 we can assume the embedding is isometric). Let $S_n$ denote the partial sum operators with respect to this basis, and let $Q : Y \to Y/X$ be the quotient map. We also denote by $J$ the inclusion $J : X \to Y$.

We will prove the following Lemma:

**Lemma 5.0.17.** *Given a $\nu > 0$ and an $n \in \mathbb{N}$ there exists a $T$ in the convex hull of $\{S_k : k > n\}$ such that $\|QT\|_{X \to Y/X} < \nu$ and $\|I - 2T\|_{X \to Y} < 1 + \nu$.*

**Proof of Lemma 5.0.17:** First we will argue that for every $n \in \mathbb{N}$ there exists an $m > n$ such that

$$\|J^*(S_n^* + S_m^* - I)y^*\| \le \|J^*(S_n^* - S_m^* + I)y^*\| + \tfrac{1}{2}\nu\|y^*\|, \qquad y^* \in Y^*. \tag{5.7}$$

If (5.7) fails we may find a sequence $(y_m^*)_{m>n}$ such that $\|y_m^*\| = 1$ and

$$\|J^*(S_n^* y_m^* + S_m^* y_m^* - y_m^*)\| > \|J^*(S_n^* y_m^* - S_m^* y_m^* + y_m^*)\| + \tfrac{1}{2}\nu, \qquad m > n.$$

By the Banach-Alaoglu theorem we may pass to a subsequence $\mathbb{M}$ of $\{n+1, n+2, \ldots\}$ so that $\lim_{m\in\mathbb{M}} y_m^* = y^*$ weak$^*$ for some $y^* \in Y^*$. Since $S_n$ is finite rank $\lim_{m\in\mathbb{M}} \|S_n^*(y^* - y_m^*)\| = 0$. Hence

$$\liminf_{m\in\mathbb{M}} (\|J^*S_n^*y^* + J^*(S_m^*y_m^* - y_m^*)\| - \|J^*S_n^*y^* - J^*(S_m^*y_m^* - y_m^*)\|) \ge \tfrac{1}{2}\nu. \tag{5.8}$$

Now $(S_m^*y_m^* - y_m^*)_{m=1}^\infty$ is weak$^*$-null in $Y^*$ since for $y \in Y$,

$$|\langle y, S_m^*y_m^* - y_m^*\rangle| = |\langle S_m y - y, y_m^*\rangle| \le \|S_m y - y\|.$$

Hence the sequence $(J^*(S_m^*y_m^* - y_m^*))_{m=1}^\infty$ is weak$^*$-null in $X^*$. Thus (5.8) contradicts (au$^*$) for $X$. This shows that (5.7) holds for some $m = m(n) > n$.

Let us put $R_n = S_{m(n)} - S_n$. Thus we have

$$\begin{aligned}
\|J^*(2S_n - I)^*y^*\| &= \|J^*(S_n^* + S_m^* - I - R_n^*)y^*\| \\
&\le \|J^*(S_n^* + S_m^* - I)y^*\| + \|J^*R_n^*y^*\| \\
&\le \|J^*(S_n^* - S_m^* + I)y^*\| + \|R_n^*y^*\| + \tfrac{1}{2}\nu\|y^*\| \\
&\le \|y^* - R_n^*y^*\| + \|R_n^*y^*\| + \tfrac{1}{2}\nu\|y^*\| \\
&\le (1 + \tfrac{1}{2}\nu)\|y^*\| + 2\|R_n^*y^*\|, \qquad y^* \in Y^*.
\end{aligned}$$

That is,

$$\|J^*(I-2S_n)^*y^*\| \le (1+\tfrac{1}{2}\nu)\|y^*\| + 2\|R_n^*y^*\|, \qquad y^* \in Y^*. \tag{5.9}$$

We next consider two sequences of finite-rank operators. First we consider the sequence $(QS_nJ)_{n=1}^{\infty}$ in $\mathcal{K}(X, Y/X)$. Note that if $z^* \in (Y/X)^*$ then $J^*Q^*z^* = 0$. Since $\lim_{n\to\infty} \|S_n^*Q^*z^* - Q^*z^*\| = 0$ (as the basis is shrinking) we conclude that $\lim_{n\to\infty} \|J^*S_n^*Q^*z^*\| = 0$. This implies [13] that $(QS_nJ)_{n=1}^{\infty}$ is a weakly null sequence in $\mathcal{K}(X, Y/X)$.

Next consider $\tilde{R}_n : c_0(Y) \to Y$ defined by $\tilde{R}_n(y_k)_{k=1}^{\infty} = R_ny_n$. Then $\tilde{R}_n^* : Y^* \to \ell_1(Y^*)$ is given by $\tilde{R}_n^*y^* = (0, \ldots, 0, R_n^*y^*, 0, \ldots)$ with the non-zero entry in the $n$th slot. Indeed, if $(y_k)_{k=1}^{\infty} \in c_0(Y)$ and $y^* \in Y^*$, then

$$\begin{aligned}
\langle (y_1, y_2, \ldots), \tilde{R}_n^*y^* \rangle &= \langle \tilde{R}_n(y_1, y_2, \ldots), y^* \rangle \\
&= \langle R_ny_n, y^* \rangle \\
&= \langle y_n, R_n^*y^* \rangle \\
&= \langle (y_1, y_2, \ldots), (0, \ldots, 0, R_n^*y^*, 0, \ldots) \rangle.
\end{aligned}$$

In particular,

$$\|\tilde{R}_n^*y^*\| = \|(0, \ldots, 0, R_n^*y^*, 0, \ldots)\| = \|R_n^*y^*\|.$$

Since the basis of $Y$ is shrinking we have $\lim_{n\to\infty} \|R_n^*y^*\| = 0$ for $y^* \in Y^*$ and so also $\lim_{n\to\infty} \|\tilde{R}_n^*y^*\| = 0$. This implies that $(\tilde{R}_n)_{n=1}^{\infty}$ is weakly null in $\mathcal{K}(c_0(Y), Y)$ again using [13].

Since $(QS_nJ)_{n=1}^{\infty}$ is weakly null in $\mathcal{K}(X, Y/X)$ and $(\tilde{R}_n)_{n=1}^{\infty}$ is weakly null in $\mathcal{K}(c_0(Y), Y)$, the sequence $((QS_nJ, \tilde{R}_n))_{n=1}^{\infty}$ is weakly null in $\mathcal{K}(X, Y/X) \oplus \mathcal{K}(c_0(Y), Y)$.

Here $\|(S,T)\| = N(\|S\|,\|T\|)$ for all $(S,T) \in \mathcal{K}(X,Y/X) \oplus \mathcal{K}(c_0(Y),Y)$, where $N$ is any norm on $\mathbb{R}^2$. Therefore for all $n \in \mathbb{N}$, $(0,0) \in \overline{\{(QS_jJ,\tilde{R}_j) : j > n\}}^w$ and certainly $(0,0) \in \overline{\mathrm{co}\{(QS_jJ,\tilde{R}_j) : j > n\}}^w$. By Mazur's theorem, the weak and norm closures of $\mathrm{co}\{(QS_jJ,\tilde{R}_j) : j > n\}$ coincide, so for all $n \in \mathbb{N}$, $0 \in \overline{\mathrm{co}\{(QS_jJ,\tilde{R}_j) : j > n\}}$.

Therefore for any $n$ we can find $r > n$ and $(\alpha_j)_{j=n+1}^r$ with $\alpha_j \geq 0$ and $\sum_{j=n+1}^r \alpha_j = 1$ such that

$$\|\sum_{j=n+1}^r \alpha_j QS_jJ\| < \nu \qquad \text{and} \qquad \|\sum_{j=n+1}^r \alpha_j \tilde{R}_j\| < \tfrac{1}{4}\nu.$$

Here we allow $\alpha_j = 0$ in order to have a consecutive sequence of operators, indexed by $n+1 \leq j \leq r$.

Let $T = \sum_{j=n+1}^r \alpha_j S_j$. Then $\|QT\|_{X\to Y/X} = \|QTJ\| < \nu$.

Also if $y^* \in Y^*$, using (5.9),

$$\begin{aligned}\|J^*(I-2T)^*y^*\| &\leq \sum_{j=n+1}^r \alpha_j \|J^*(I-2S_j)^*y^*\| \\ &\leq (1+\tfrac{1}{2}\nu)\|y^*\| + 2\sum_{j=n+1}^r \alpha_j\|R_j^*y^*\| \\ &= (1+\tfrac{1}{2}\nu)\|y^*\| + 2\|\sum_{j=n+1}^r \alpha_j \tilde{R}_j^* y^*\|.\end{aligned}$$

by definition of $\tilde{R}_j^*$.

Therefore

$$\begin{aligned}\|(I-2T)J\| &= \|J^*(I-2T)^*\| \\ &\leq 1 + \tfrac{1}{2}\nu + 2\|\sum_{j=n+1}^r \alpha_j \tilde{R}_j^*\| \\ &= 1 + \tfrac{1}{2}\nu + 2\|\sum_{j=n+1}^r \alpha_j \tilde{R}_j\|\end{aligned}$$

$$< 1 + \tfrac{1}{2}\nu + \tfrac{1}{2}\nu$$

$$= 1 + \nu.$$

This completes the proof of Lemma 5.0.17. $\square$

We now return to the proof of Theorem 5.0.16. Using Lemma 5.0.17, we choose a sequence of convex combinations

$$T_j = \sum_{i=N_{j-1}+1}^{N_j} \alpha_i S_i$$

where $N_0 = 0 < N_1 < N_2 < \cdots$ and $\alpha_i \geq 0$ are such that $\sum_{i=N_{j-1}+1}^{N_j} \alpha_i = 1$, such that for all $j$,

$$\|QT_j\|_{X \to Y/X} < \frac{1}{j}$$

and

$$\|I - 2T_j\|_{X \to Y} < 1 + \frac{1}{j}.$$

Note that $Q(I - 2T_j)|_X = -2QT_j|_X$, so

$$\|Q(I - 2T_j)\|_{X \to Y/X} = 2\|QT_j\|_{X \to Y/X} < \frac{2}{j}.$$

Setting $B_n = (I - 2T_n)$ in Lemma 5.0.15, we pass to a subsequence of $(T_n)_{n=1}^{\infty}$ such that

$$\|(I - 2T_{n_1})(I - 2T_{n_2}) \cdots (I - 2T_{n_k})\|_{X \to Y} < 1 + \delta \tag{5.10}$$

whenever $n_1 < n_2 < \cdots < n_k$.

Let $A_j = T_j - T_{j-1}$ where $T_0 = 0$. Suppose $\epsilon_j = \pm 1$. Since $S_m S_n = S_{\min(m,n)}$ for all $m, n \in \mathbb{N}$, and since distinct operators $T_j$ and $T_k$ are "disjointly supported" in the sequence $(S_n)_{n=1}^{\infty}$, we have $T_j T_k = T_{\min(j,k)}$, whenever $j \neq k$.

We now repeat a calculation in [5] Theorem 3.8 with a correction to a small misprint. (Here the index $n-j+1$ replaces $n-j-1$.) We prove by induction that

$$\epsilon_n T_n \prod_{j=1}^{n-1}(I - T_{n-j} + \epsilon_{n-j+1}\epsilon_{n-j}T_{n-j}) = \sum_{j=1}^{n} \epsilon_j A_j, \qquad n \geq 2. \tag{5.11}$$

(Note that the product notation is unambiguous, since $(I-2T_j)(I-2T_k) = (I-2T_k)(I-2T_j)$ for all $j, k \in \mathbb{N}$.)

Indeed, if $n = 2$, then the left hand side of (5.11) becomes

$$\epsilon_2 T_2(I - T_1 + \epsilon_2\epsilon_1 T_1) = \epsilon_2 T_2 - \epsilon_2 T_1 + \epsilon_1 T_1 = \epsilon_2(T_2 - T_1) + \epsilon_1(T_1 - T_0) = \epsilon_2 A_2 + \epsilon_1 A_1,$$

as required. Now suppose that (5.11) holds for some $n \geq 2$. Then

$$\begin{aligned}
&\epsilon_{n+1}T_{n+1}\prod_{j=1}^{n}(I - T_{n+1-j} + \epsilon_{n+2-j}\epsilon_{n+1-j}T_{n+1-j}) \\
&\quad = \epsilon_{n+1}T_{n+1}(I - T_n + \epsilon_{n+1}\epsilon_n T_n)\prod_{j=2}^{n}(I - T_{n+1-j} + \epsilon_{n+2-j}\epsilon_{n+1-j}T_{n+1-j}) \\
&\quad = \epsilon_{n+1}(T_{n+1} - T_n + \epsilon_{n+1}\epsilon_n T_n)\prod_{j=1}^{n-1}(I - T_{n-j} + \epsilon_{n+1-j}\epsilon_{n-j}T_{n-j}) \\
&\quad = \epsilon_{n+1}\left(T_{n+1}\prod_{j=1}^{n-1}(I - T_{n-j} + \epsilon_{n+1-j}\epsilon_{n-j}T_{n-j}) - T_n\prod_{j=1}^{n-1}(I - T_{n-j} + \epsilon_{n+1-j}\epsilon_{n-j}T_{n-j})\right) \\
&\qquad + \epsilon_n T_n\prod_{j=1}^{n-1}(I - T_{n-j} + \epsilon_{n+1-j}\epsilon_{n-j}T_{n-j}) \\
&\quad = \epsilon_{n+1}(T_{n+1} - T_n) + \sum_{j=1}^{n}\epsilon_j A_j \\
&\quad = \sum_{j=1}^{n+1}\epsilon_j A_j,
\end{aligned}$$

so that (5.11) holds for $n+1$.

Intuitively, (5.11) says that flipping the signs of finitely many "coordinates", and chopping off the infinite tail, is the same as flipping the signs of the first $n_1$

coordinates, then flipping the signs of the first $n_2$ resulting coordinates, etc, finally flipping the signs of the first $n_k$ coordinates as they stand after the previous step, and then chopping off the infinite tail.

Note that for all $x \in X$, $x = \sum_{j=1}^{\infty} A_j x$. Our aim is to prove that $\| \sum_{j=1}^{n} \epsilon_j A_j \|_{X \to Y} < 1 + \delta$, for all $n \in \mathbb{N}$ and all $\epsilon_j = \pm 1$.

In fact, if $n = 1$ then by (5.10),

$$
\begin{aligned}
\|\epsilon_1 A_1\|_{X\to Y} &= \|T_1\|_{X\to Y} \\
&= \|\tfrac{1}{2}(I-(I-2T_1))\|_{X\to Y} \\
&\le \tfrac{1}{2}(\|I\|_{X\to Y} + \|I-2T_1\|_{X\to Y}) \\
&< \tfrac{1}{2}(1+1+\delta) \\
&= 1+\tfrac{\delta}{2} \\
&< 1+\delta.
\end{aligned}
$$

Next, if $n \ge 2$, then (5.10) and (5.11) imply that

$$
\begin{aligned}
\|\sum_{j=1}^{n} \epsilon_j A_j\|_{X\to Y} &= \|T_n \prod_{j=1}^{n-1}(I - T_{n-j} + \epsilon_{n-j+1}\epsilon_{n-j}T_{n-j})\|_{X\to Y} \\
&= \|\tfrac{1}{2}(I-(I-2T_n))\prod_{j=1}^{n-1}(I - T_{n-j} + \epsilon_{n-j+1}\epsilon_{n-j}T_{n-j})\|_{X\to Y} \\
&= \|\tfrac{1}{2}\left(\prod_{j=1}^{n-1}(I - T_{n-j} + \epsilon_{n-j+1}\epsilon_{n-j}T_{n-j})\right. \\
&\left. -(I-2T_n)\prod_{j=1}^{n-1}(I - T_{n-j} + \epsilon_{n-j+1}\epsilon_{n-j}T_{n-j})\right)\|_{X\to Y} \\
&\le \tfrac{1}{2}\left(\|\prod_{j=1}^{n-1}(I - T_{n-j} + \epsilon_{n-j+1}\epsilon_{n-j}T_{n-j})\|_{X\to Y}\right. \\
&\left. +\|(I-2T_n)\prod_{j=1}^{n-1}(I - T_{n-j} + \epsilon_{n-j+1}\epsilon_{n-j}T_{n-j})\|_{X\to Y}\right)
\end{aligned}
$$

$$< 1 + \delta.$$

Note that since $X^*$ is separable, $X$ contains no complemented copy of $\ell_1$. That is, if $Z \subset X$ is a subspace of $X$ isomorphic to $\ell_1$, and if $P : X \to Z$ is a bounded linear projection, then $X/\ker\, P$ is isomorphic to $P(X) = Z$, and the subspace $(\ker\, P)^\perp \subset X^*$ of $X^*$ is isomorphic to $(X/\ker/P)^*$, hence isomorphic to $\ell_1^* = \ell_\infty$. Since $\ell_\infty$ is not separable, there can be no such $Z$. The result now follows by Proposition 4.0.14. □

**Corollary 5.0.18.** *Let $X$ be a separable reflexive Banach space. Then the following conditions are equivalent:*

*(i) $X$ has (au).*

*(ii) For any $\delta > 0$ there is a reflexive Banach space $Y$ with a 1-unconditional basis and a subspace $X_\delta$ of $Y$ such that $d(X, X_\delta) < 1 + \delta$.*

**Proof:** This follows from the proof of Theorem 5.0.16 and from Propositions 3.0.7 and 4.0.14. □

The next Corollary is due to Johnson and Zheng [12] by a quite different proof.

**Corollary 5.0.19.** *Any quotient of a Banach space $X$ with a shrinking unconditional basis is isomorphic to a subspace of a Banach space with a shrinking unconditional basis.*

**Proof:** Let $E$ be a closed subspace of $X$. Renorm $X$ with an equivalent norm $|||\cdot|||$ so that its shrinking basis is 1-unconditional (pg 19 of [19].) Then by part (ii) of Proposition 3.0.8 $(X, |||\cdot|||)$ has (au*). By part (iv) of Proposition 3.0.8

$(X, |||\cdot|||)/E$ has (au$^*$). Now Theorem 5.0.16 implies that $(X, |||\cdot|||)/E$ is isomorphic to a subspace $X'$ of a Banach space $Y$ with a shrinking 1-unconditional basis. Therefore $(X, \|\cdot\|)/E$ is isomorphic to $X'$. $\square$

# Chapter 6

# Skipped Unconditional Bases

Let us say that a basic sequence $(e_k)_{k=1}^N$ (where $1 \leq N \leq \infty$) in a (finite or infinite-dimensional) Banach space $X$ is *skipped $\lambda$-unconditional* if whenever $0 = m_0 < m_1 < \cdots < m_n < \infty$ with $m_j - m_{j-1} \geq 2$ for $1 \leq j \leq n$ and $y_j \in [e_i]_{m_{j-1}+1}^{m_j-1}$ then for any choice of signs $(\epsilon_j)_{j=1}^n$,

$$\|\sum_{j=1}^n \epsilon_j y_j\| \leq \lambda \|\sum_{j=1}^n y_j\|,$$

that is to say, that the basic sequence $(y_j)_{j=1}^n$ is $\lambda$-unconditional.

We shall say that $(e_k)_{k=1}^\infty$ is *asymptotically skipped 1-unconditional* if for every $\lambda > 1$ there exists an $n$ so that if $x \in [e_k]_{k=1}^n \setminus \{0\}$ then the basic sequence $\{x, e_{n+1}, e_{n+2}, \ldots\}$ is skipped $\lambda$-unconditional.

We will define a basis $(f_k)_{k=1}^N$ of a finite-dimensional Banach space to be *dual skipped $\lambda$-unconditional* when the dual basis $(f_k^*)_{k=1}^N$ is skipped $\lambda$-unconditional. We will need the following Lemma:

**Lemma 6.0.20.** *Let $X$ be a Banach space with a basis $(e_k)_{k=1}^N$ where $1 \leq N \leq \infty$. Suppose $1 \leq m_1 < m_2 < \cdots < m_n < N$, and that for every $x \in [e_j]_{j=1}^{m_1}$ the basic sequence $\{x, (e_k)_{k=m_1+1}^N\}$ is skipped $\lambda$-unconditional. Suppose $x^*, y^* \in X^* \setminus \{0\}$ are* *such that* $x^* \in [e_k^*]_{k=1}^{m_1}$, $y^*(e_j) = 0$ *for* $1 \le j \le m_n$. *Then* $\{x^*, e_{m_2}^*, \ldots, e_{m_{n-1}}^*, y^*\}$ *is a dual skipped* $\lambda$*-unconditional basis of its linear span.*

*In particular if* $(e_k)_{k=1}^N$ *is skipped* $\lambda$*-unconditional then the finite sequence* $\{x^*, e_{m_2}^*, \ldots, e_{m_{n-1}}^*, y^*\}$ *is a dual skipped* $\lambda$*-unconditional basis of its linear span.*

**Proof:** Define a map $T : X \to \mathbb{R}^n$ by

$$Tx = (x^*(x), e_{m_2}^*(x), \ldots, e_{m_{n-1}}^*(x), y^*(x))$$

and consider the quotient norm $\|\xi\|_T = \inf\{\|x\| : Tx = \xi\}$ on $\mathbb{R}^n$.

Let $(f_i)_{i=1}^n$ denote the canonical basis of $\mathbb{R}^n$. Then $T^*$ is an isometric isomorphism from $(\mathbb{R}^n, \|\cdot\|_T)^*$ into $X^*$, which maps $(f_1^*, f_2^*, \ldots, f_{n-1}^*, f_n^*)$ onto $(x^*, e_{m_2}^*, \ldots, e_{m_{n-1}}^*, y^*)$. Thus it suffices to show that $(f_1^*, \ldots, f_n^*)$ is a dual skipped $\lambda$-unconditional basis for $(R_n^*, \|\cdot\|_T)^*$, i.e. that $(f_1, \ldots, f_n)$ is a skipped $\lambda$-unconditional basis for $(R_n, \|\cdot\|_T)$.

In fact, suppose that $0 = k_0 < k_1 < \cdots < k_p < \infty$, with $k_j - k_{j-1} \ge 2$ for $1 \le j \le p$. Suppose that for $1 \le j \le p$, $\xi_j \in [f_i]_{i=k_{j-1}+1}^{k_j-1}$ and $\epsilon_i = \pm 1$. Choose $x \in X$ such that $Tx = \sum_{j=1}^p \xi_j$. Then $e_{m_{k_j}}^*(x) = 0$ for all $1 \le j \le p$. Thus

$$x = \sum_{j=1}^{p+1} x_j, \qquad \text{where} \qquad x_j \in [e_i]_{i=m_{k_{j-1}}+1}^{m_{k_j}-1} \qquad \text{for } 1 \le j \le p,$$

and

$$x_{p+1} \in [e_i]_{i=m_{k_p}+1}^N.$$

Then $T(\sum_{j=1}^p \epsilon_j x_j + x_{p+1}) = \sum_{j=1}^p \epsilon_j \xi_j$. By assumption,

$$\|\sum_{j=1}^p \epsilon_j x_j + x_{p+1}\| \le \lambda \|\sum_{j=1}^{p+1} x_j\|. \tag{6.1}$$

Now $\sum_{j=1}^{p+1} x_j \mapsto \sum_{j=1}^p \epsilon_j x_j + x_{p+1}$ is a 1-1 correspondence between the coset $T^{-1}(\sum_{j=1}^p \xi_j)$ and the coset $T^{-1}(\sum_{j=1}^p \epsilon_j \xi_j)$, so taking the infimum over $x \in T^{-1}(\sum_{j=1}^p \xi_j)$

on the left hand side of (6.1), and then on the right hand side, we have $\| \sum_{j=1}^{p} \epsilon_j \xi_j \| \leq \lambda \| \sum_{j=1}^{p} \xi_j \|$, as required. □

Our next result concerns the unconditionality of the biorthogonal sequence $(e_k^*)_{k=1}^{\infty}$ in $X^*$. If $\mathbb{A}$ is a finite subset of $\mathbb{N}$ we denote by $\text{ubc}(e_j^*)_{j\in\mathbb{A}}$ the unconditional basis constant of $(e_j^*)_{j\in\mathbb{A}}$.

**Lemma 6.0.21.** *Let $X$ be a finite-dimensional Banach space with a skipped $1-$unconditional basis $(e_k)_{k=1}^{2N+1}$. Suppose $\text{ubc}(e_{2j-1}^*)_{j=1}^{N+1} = \mu > 1$. Then*

$$\text{ubc}(e_j^*)_{j=1}^{2N+1} \geq 1 + 2(\mu - 1).$$

**Proof:** By assumption there exist real numbers $(\alpha_j)_{j=1}^{N+1}$ and signs $(\epsilon_j)_{j=1}^{N+1}$ so that

$$\| \sum_{j=1}^{N+1} \alpha_j e_{2j-1}^* \| = 1$$

and

$$\| \sum_{j=1}^{N+1} \epsilon_j \alpha_j e_{2j-1}^* \| = \mu.$$

Let $E = [e_{2j-1}]_{j=1}^{N+1}$. Then we have

$$\| \sum_{j=1}^{N+1} \alpha_j e_{2j-1}^* |_E \| \leq 1.$$

By the skipped unconditionality condition $(e_{2j-1})_{j=1}^{N+1}$ is 1-unconditional in $E$, moreover its biorthogonal functionals are isometrically isomorphic to $(e_{2j-1}^*|_E)_{j=1}^{N+1}$, so we have

$$\| \sum_{j=1}^{N+1} \epsilon_j \alpha_j e_{2j-1}^* |_E \| \leq 1.$$

By the Hahn-Banach theorem there exists $(\beta_j)_{j=1}^{N}$ such that

$$\| \sum_{j=1}^{N+1} \epsilon_j \alpha_j e_{2j-1}^* + \sum_{j=1}^{N} \beta_j e_{2j}^* \| = \| \sum_{j=1}^{N+1} \epsilon_j \alpha_j e_{2j-1}^* |_E \| \leq 1.$$

Thus

$$2\mu = \| \left(\sum_{j=1}^{N+1} \epsilon_j \alpha_j e_{2j-1}^* + \sum_{j=1}^{N} \beta_j e_{2j}^*\right) + \left(\sum_{j=1}^{N+1} \epsilon_j \alpha_j e_{2j-1}^* - \sum_{j=1}^{N} \beta_j e_{2j}^*\right) \|$$
$$\leq 1 + \| \sum_{j=1}^{N+1} \epsilon_j \alpha_j e_{2j-1}^* - \sum_{j=1}^{N} \beta_j e_{2j}^* \|$$

so that

$$\operatorname{ubc}(e_j^*)_{j=1}^{2N+1} \geq \frac{\| \sum_{j=1}^{N+1} \epsilon_j \alpha_j e_{2j-1}^* - \sum_{j=1}^{N} \beta_j e_{2j}^* \|}{\| \sum_{j=1}^{N+1} \epsilon_j \alpha_j e_{2j-1}^* + \sum_{j=1}^{N} \beta_j e_{2j}^* \|} \geq 2\mu - 1.$$

□

**Lemma 6.0.22.** *Suppose $N \in \mathbb{N}$. Let $X$ be a Banach space of dimension $2^N + 1$ with a dual skipped $1-$unconditional basis $(f_k)_{k=1}^{2^N+1}$. Suppose $\operatorname{ubc}(f_1, f_{2^N+1}) = \mu > 1$. Then*

$$\operatorname{ubc}(f_j)_{j=1}^{2^N+1} \geq 1 + 2^N(\mu - 1).$$

**Proof:** This is proved by induction on $N$. If $N = 1$ it is immediate from Lemma 6.0.21. Suppose now that the Lemma is proved for $N-1$. Then $\{f_1, f_3, \ldots, f_{2^N+1}\}$ is a dual skipped $1-$unconditional basis of its linear span by Lemma 6.0.20. By the inductive hypothesis

$$\operatorname{ubc}(f_{2j-1})_{j=1}^{2^{N-1}+1} \geq 1 + 2^{N-1}(\mu - 1).$$

Now applying Lemma 6.0.21 we have

$$\operatorname{ubc}(f_j)_{j=1}^{2^N+1} \geq 1+2(\operatorname{ubc}(f_{2j-1})_{j=1}^{2^{N-1}+1}-1) \geq 1+2(1+2^{N-1}(\mu-1)-1) = 1+2^N(\mu-1).$$

□

**Proposition 6.0.23.** *Let $X$ be a Banach space containing no copy of $\ell_1$ and with a skipped unconditional basis $(e_k)_{k=1}^{\infty}$. Then:*

*(i)* $(e_k)_{k=1}^{\infty}$ *is shrinking, and*

*(ii) If* $X$ *contains no copy of* $c_0$ *then either* $X$ *is reflexive or* $X$ *is quasi-reflexive of order one.*

**Proof:** (i) Let $(u_k)_{k=1}^{\infty}$ be any normalized block basic sequence with respect to $(e_k)_{k=1}^{\infty}$. Then $(u_{2k})_{k=1}^{\infty}$ (respectively $(u_{2k-1})_{k=1}^{\infty}$) is an unconditional basic sequence and hence weakly null; thus $(u_k)_{k=1}^{\infty}$ is weakly null. Hence $(e_k)_{k=1}^{\infty}$ is shrinking, by Proposition 3.2.7 of [1], say.

(ii) We may assume $\|e_k\| = 1$ for all $k$. Suppose $x^{**} \in X^{**}$ is such that $\lim_{k\to\infty} x^{**}(e_k^*) = 0$. Select a strictly increasing sequence $(m_k)_{k=0}^{\infty}$ (with $m_0 = 0$) such that $|x^{**}(e_{m_k}^*)| < 2^{-k}$ for $k \geq 1$. Then the series

$$\sum_{k=1}^{\infty} \left( \sum_{i=m_{k-1}+1}^{m_k-1} x^{**}(e_i^*)e_i \right)$$

is a WUC series and hence convergent in $X$. On the other hand the series $\sum_{k=1}^{\infty} x^{**}(e_{m_k}^*)e_k$ is absolutely convergent and so $x^{**} \in X$.

Now suppose $X$ is non-reflexive and $x_0^{**} \in X^{**} \setminus X$. Then $\liminf_k |x_0^{**}(e_k^*)| > 0$. For any $x^{**} \in X^{**}$ we may find $\lambda \in \mathbb{R}$ so that $\liminf_k |(x^{**} - \lambda x_0^{**})(e_k^*)| = 0$ and hence $x^{**} - \lambda x_0^{**} \in X$. Thus $\dim X^{**}/X = 1$. □

**Proposition 6.0.24.** *Let* $X$ *be a Banach space containing no copy of* $\ell_1$ *and with a normalized asymptotically skipped 1-unconditional basis* $(e_k)_{k=1}^{\infty}$*. Suppose* $X$ *fails to have property (au*). Then:*

*(i) No subsequence of* $(e_k^*)_{k=1}^{\infty}$ *is unconditional, and*

*(ii) Every spreading model of* $(e_k)_{k=1}^{\infty}$ *is equivalent to the standard* $\ell_1$*−basis.*

**Proof:** Since $(e_k)_{k=1}^{\infty}$ is shrinking by Proposition 6.0.23 we can assume the existence of $\mu > 1$, $r \in \mathbb{N}$, $\alpha, \beta \in \mathbb{R}$, $x^* \in [e_k^*]_{k=1}^r$ and a sequence $(y_n^*)_{n>r}$ with $y_n^*(e_j) = 0$ for $j < n$ such that $\|x^*\| = \|y_n^*\| = 1$ and $\|\alpha x^* - \beta y_n^*\| = 1$ for all $n$ but $\|\alpha x^* + \beta y_n^*\| \geq \mu$. Let $K$ be the basis constant of $(e_k)_{k=1}^{\infty}$. We first argue that for any $n \in \mathbb{N}$ with $n > 80K/(\mu - 1)$ there exists a $k = k(n)$ so that if $k < m_1 < m_2 < \cdots < m_n$ then

$$\mathrm{ubc}(e_{m_1}^*, \ldots, e_{m_n}^*) \geq \frac{(\mu - 1)n}{10K^2}. \tag{6.2}$$

Assume not. Then for each $k > r$ we may select $k < m_{k,1} < \ldots < m_{k,n}$ so that

$$\mathrm{ubc}(e_{m_{k,1}}^*, \ldots, e_{m_{k,n}}^*) \leq \frac{(\mu - 1)n}{10K^2}.$$

Hence since the basis constant of $(x^*, e_{m_{k,1}}^*, \ldots, e_{m_{k,n}}^*, y_{m_{k,n}+1}^*)$ is at most $K$ we have that if $\epsilon_1, \epsilon_2, \ldots, \epsilon_{n+2} = \pm 1$ and $\xi_1, \ldots, \xi_{n+2}$ are real numbers,

$$\begin{aligned}
&\|\epsilon_1 \xi_1 x^* + \sum_{j=1}^{n} \epsilon_{j+1} \xi_{j+1} e_{m_{k,j}}^* + \epsilon_{n+2} \xi_{n+2} y_{m_{k,n}+1}^*\| \\
&\leq |\xi_1| + |\xi_{n+2}| + \frac{(\mu - 1)n}{10K^2} \|\sum_{j=1}^{n} \xi_{j+1} e_{m_{k,j}}^*\| \\
&\leq \left(4K + \frac{(\mu - 1)n}{5}\right) \|\xi_1 x^* + \sum_{j=1}^{n} \xi_{j+1} e_{m_{k,j}}^* + \xi_{n+2} y_{m_{k,n}+1}^*\| \\
&\leq \frac{(\mu - 1)n}{4} \|\xi_1 x^* + \sum_{j=1}^{n} \xi_{j+1} e_{m_{k,j}}^* + \xi_{n+2} y_{m_{k,n}+1}^*\|.
\end{aligned}$$

Thus

$$\mathrm{ubc}(x^*, e_{m_{k,1}}^*, \ldots, e_{m_{k,n}}^*, y_{m_{k,n}+1}^*) \leq \frac{\mu - 1}{4} n.$$

Note the basis $(x^*, e_{m_{k,1}}^*, \ldots, e_{m_{k,n}}^*, y_{m_{k,n}+1}^*)$ is dual $\lambda_k-$skipped where $\lim_k \lambda_k = 1$.

Let us define a norm on $\mathbb{R}^{n+2}$ by

$$\|(\xi_1, \ldots, \xi_{n+2})\| = \lim_{\mathcal{U}} \|\xi_1 x^* + \sum_{j=1}^{n} \xi_{j+1} e_{m_{k,j}}^* + \xi_{n+2} y_{m_{k,n}+1}^*\|$$

where $\mathcal{U}$ is some non-principal ultrafilter. The canonical basis $(f_1, \ldots, f_{n+2})$ is then dual skipped 1-unconditional and

$$\mathrm{ubc}(f_1, \ldots, f_{n+2}) \leq \frac{1}{4}(\mu - 1)n.$$

Also $\mathrm{ubc}(f_1, f_{n+2}) \geq \mu$. Hence by Lemma 6.0.22 (and utilizing Lemma 6.0.20 since $n+1$ need not be a power of 2)

$$\mathrm{ubc}(f_1, \ldots, f_{n+2}) \geq \frac{1}{2}(\mu - 1)(n + 1).$$

This gives a contradiction and (6.2) is established.

(i) is now immediate.

For (ii) observe that any spreading model of $(e_k)_{k=1}^{\infty}$ is 1-unconditional. For any $n$ there exists $k(n)$ so that (6.2) holds. Suppose $k(n) < m_1 < \cdots < m_n$. Then there exist $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ and $(\epsilon_1, \ldots, \epsilon_n) \in \{-1, 1\}^n$ so that

$$\|\sum_{j=1}^{n} \alpha_j e_{m_j}^*\| = 1, \qquad \|\sum_{j=1}^{n} \epsilon_j \alpha_j e_{m_j}^*\| \geq \frac{\mu - 1}{10K^2} n.$$

Since $\|e_k^*\| \leq 2K$ we thus have

$$\sum_{j=1}^{n} |\alpha_j| \geq \frac{\mu - 1}{20K^3} n$$

and so for a suitable choice of signs $\eta_j$ we have

$$\|\sum_{j=1}^{n} \eta_j e_{m_j}\| \geq \frac{\mu - 1}{20K^3} n.$$

Thus in any spreading model with basis $(f_j)_{j=1}^{\infty}$ we have $\|f_1 + \cdots + f_n\| \geq cn$ for suitable $c > 0$. This implies that $(f_j)_{j=1}^{\infty}$ is equivalent to the canonical basis of $\ell_1$ (since it is a 1-unconditional spreading model). □

# Chapter 7

# The Weak Alternating Banach-Saks Property

We recall that a Banach space $X$ is said to have the *Alternating Banach-Saks (ABS) property* if every bounded sequence $(x_n)_{n=1}^{\infty}$ in $X$ has a subsequence $(y_n)_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \sup_{r_1<r_2<\cdots<r_n} \|\frac{1}{n}\sum_{j=1}^{n}(-1)^j y_{r_j}\| = 0. \tag{7.1}$$

This is equivalent to the requirement that some spreading model of $(x_n)_{n=1}^{\infty}$ is not equivalent to the $\ell_1-$basis (see [2]).

We shall say that $X$ has the *Weak Alternating Banach-Saks (WABS) property* if every bounded sequence $(x_n)_{n=1}^{\infty}$ in $X$ has a convex block sequence $(y_n)_{n=1}^{\infty}$ such that (7.1) holds. Here $(y_n)_{n=1}^{\infty}$ is a convex block sequence if

$$y_n = \sum_{j=p_{n-1}+1}^{p_n} \lambda_j x_j$$

where $p_0 = 0 < p_1 < p_2 < \cdots$, $\lambda_j \geq 0$, and $\sum_{p_{n-1}+1}^{p_n} \lambda_j = 1$ for every $n$. Note that if $(y_n)_{n=1}^{\infty}$ satisfies (7.1) then so does every further sequence of convex blocks.

Let us recall at this point that a Banach space $X$ has Pełczyński's property (u) if for every weakly Cauchy sequence $(x_n)_{n=1}^{\infty}$ there is a weakly null sequence

$(z_n)_{n=1}^\infty$ so that if $u_n = x_n - z_n$ then the series $\sum_{n=1}^\infty (u_n - u_{n-1})$ (where $u_0 = 0$) is weakly unconditionally Cauchy (WUC). Any Banach space with an unconditional basis has property (u) ([26], [20] p.31) Let us note the following, which shows the connection with the (WABS) property:

**Proposition 7.0.25.** *Let $X$ be a separable Banach space. Then $X$ contains no copy of $\ell_1$ and has property (u) if and only if every bounded sequence $(x_n)_{n=1}^\infty$ has a convex block sequence $(y_n)_{n=1}^\infty$ such that*

$$\sup_n \sup_{r_1<r_2<\cdots<r_n} \|\sum_{j=1}^n (-1)^j y_{r_j}\| < \infty. \tag{7.2}$$

**Proof:** If $X$ contains no copy of $\ell_1$, we can assume $(x_n)_{n=1}^\infty$ is weakly Cauchy [27]. If $X$ has property (u) we can write $x_n = u_n + z_n$ where $(z_n)$ is weakly null and $\sum(u_n - u_{n-1})$ is a WUC series. We may then pass to convex blocks $(\hat{x}_n)_{n=1}^\infty$ so that the corresponding convex blocks $(\hat{z}_n)_{n=1}^\infty$ and $(\hat{u}_n)_{n=1}^\infty$ satisfy $\|\hat{z}_n\| < 2^{-n}$. Then $(\hat{x}_n)_{n=1}^\infty$ satisfies our requirements.

Conversely it is clear $X$ cannot contain $\ell_1$. If $(x_n)_{n=1}^\infty$ is weakly Cauchy we may pass to convex blocks $(y_n)_{n=1}^\infty$ verifying (7.2). But then $\sum(y_n - y_{n-1})$ is a WUC series and $x_n - y_n$ is weakly null. □

In [9] Haydon, Odell and Rosenthal introduced the class of Baire-1/2 functions: If $\Omega$ is a compact metric space then a bounded function $f$ on $\Omega$ is Baire-1/2 if for every $\epsilon > 0$ there exist bounded lower-semicontinuous functions $\varphi, \psi$ such that $|f(s) - (\varphi(s) - \psi(s))| < \epsilon$ for $s \in \Omega$.

Suppose $X$ is a separable Banach space and $x^{**} \in X^{**} \setminus X$. We can generate a sequence $\chi_n = \chi_n(x^{**}) \in X^{(2n)}$ by $\chi_1 = x^{**}$ and then $\chi_n = j_{n-1}^{**} x^{**}$ where $j_{n-1}$

is the canonical embedding $X \subset X^{**} \subset \cdots \subset X^{2(n-1)}$. The sequence $(\chi_n)_{n=1}^{\infty}$ is considered in the transfinite dual $X^{\omega}$ defined as the completion of $\cup_{n\geq 1} X^{(2n)}$.

The following theorem follows from [9] and [6]:

**Theorem 7.0.26.** *If $X$ is a separable Banach space then the following are equivalent:*
*(i) $X$ has the (WABS) property.*
*(ii) Every $x^{**} \in X^{**}$ is Baire-1/2 as a function on $B_{X^*}$ with the weak$^*$-topology.*
*(iii) There is no $x^{**} \in X^{**} \setminus X$ so that $(\chi_n(x^{**}))_{n=1}^{\infty}$ is equivalent to the unit vector basis of $\ell_1$.*

**Proof:** (i) $\implies$ (ii). Since $X$ contains no copy of $\ell_1$, every $x^{**} \in X^{**} \setminus X$ is the weak$^*$-limit of a sequence $(x_n)_{n=1}^{\infty}$ [23]. We pass to a sequence of convex blocks $(y_n)_{n=1}^{\infty}$ so that (7.1) holds. Now apply Theorem B of [9] to deduce that $x^{**}$ is Baire-1/2.

(ii) $\iff$ (iii). This is Theorem 11 of Farmaki [6] (since (iii) also implies that $X$ contains no copy of $\ell_1$ by Proposition 6 of [6]).

(ii) $\implies$ (i). Let $(x_n)_{n=1}^{\infty}$ be a bounded sequence in $X$. If $(x_n)_{n=1}^{\infty}$ has a weakly convergent subsequence then Mazur's theorem quickly yields a sequence of convex blocks satisfying (7.1). By Rosenthal's theorem [27] we may therefore pass to the case when $(x_n)_{n=1}^{\infty}$ is weakly Cauchy and converging weak$^*$ to some $x^{**} \in X^{**} \setminus X$. By Theorem 3.7 and Lemma 3.8 of [9] there is a bounded sequence $(f_n)_{n=1}^{\infty}$ in $C(B_{X^*})$ converging pointwise to $x^{**}$ so that $(f_n)_{n=1}^{\infty}$ satisfies (7.1). By Mazur's theorem, we may find a sequence of convex blocks $(y_n)_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ and a sequence of convex blocks $(g_n)_{n=1}^\infty$ of $(f_n)_{n=1}^\infty$ such that $\|y_n - g_n\| < 2^{-n}$ (considering $X$ as a subspace of $C(B_{X^*})$). Then $(y_n)_{n=1}^\infty$ satisfies (7.1). □

We next give a very similar argument to Lemma 3.0.9 for the case when $(x_n)_{n=1}^\infty$ converges weak* to some $x^{**} \in X^{**} \setminus X$.

**Lemma 7.0.27.** *Let $X$ be a separable Banach space with property (au), and suppose that $(x_n)_{n=1}^\infty$ is a weakly Cauchy sequence in $X$ converging weak* to some $x^{**} \in X^{**} \setminus X$. Then there is a subsequence $(y_n)_{n=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that the sequence $(y_n - y_{n-1})_{n=1}^\infty$ (where $y_0 = 0$) is an asymptotically skipped 1-unconditional basic sequence.*

**Proof:** We may suppose, by passing to a subsequence, that $(x_n)_{n=1}^\infty$ is basic (see e.g. [1] Theorem 1.5.6), and that if $x^* \in X^*$ is such that $x^{**}(x^*) = 1$ then $|x^*(x_n) - 1| < 2^{-n}$. This implies the existence of $y^* \in X^*$ with $y^*(x_n) = 1$ for all $n$ and so $(x_n - x_{n-1})_{n=1}^\infty$ (with $x_0 = 0$) is also a basic sequence (see [29] pp. 308-311); note this remark applies to all subsequences of $(x_n)_{n=1}^\infty$. Let $K$ be the basis constant for the sequence $(x_n)_{n=1}^\infty$ and assume that $0 < c \le \|x_k\| \le C < \infty$ for all $k$.

Let $(\delta_n)_{n=1}^\infty$ be a decreasing sequence of positive numbers with the property that $\sum_{n=1}^\infty \delta_n < \infty$. We will construct a subsequence $(y_n)_{n=1}^\infty$ and a sequence $(F_n)_{n=1}^\infty$ of closed finite-codimensional subspaces inductively.

Let $y_1 = x_1$ and $F_1 = X$. If $y_1, \ldots, y_{n-1}$ and $F_1, \ldots, F_{n-1}$ have been chosen then we may choose a closed subspace $F_n$ of finite codimension so that if $w \in [y_j]_{j=1}^{n-1}$ and $z \in F_n$ then

$$\|w - z\| \le (1 + \tfrac{1}{4}\delta_n)\|w + z\|.$$

Let $Q_j : X \to X/F_j$ denote the quotient map for $1 \le j \le n$. If $y_{n-1} = x_{m_n}$ we may pick $y_n = x_{m_{n+1}}$ with $m_{n+1} > m_n$ so that

$$\|Q_j y_n - Q_j^{**} x^{**}\| \le \frac{2^{j-n-1} c\delta_j}{10K}, \qquad 1 \le j \le n.$$

Now suppose $w = \sum_{j=1}^{n-1} a_j y_j$ and $z = \sum_{j=n}^{N} a_j y_j$ where $\|w + z\| = 1$ and $\sum_{j=n}^{N} a_j = 0$. Then we have

$$\|Q_n z\| = \|\sum_{j=n}^{N} a_j (Q_n y_j - Q_n^{**} x^{**})\| \le 2Kc^{-1} \sum_{j=n}^{\infty} \|Q_n y_j - Q_n^{**} x^{**}\| \le \delta_n/5.$$

Hence there exists $z' \in F_n$ such that $\|z - z'\| \le \delta_n/4$ and thus

$$\|w - z\| \le \|w - z'\| + \tfrac{1}{4}\delta_n \le (1 + \tfrac{1}{4}\delta_n)\|w + z'\| + \tfrac{1}{4}\delta_n \le 1 + \delta.$$

Thus we have the inequality

$$\|\sum_{j=1}^{n-1} a_j y_j - \sum_{j=n}^{N} a_j y_j\| \le (1 + \delta_n)\|\sum_{j=1}^{N} a_j y_j\|, \qquad \text{if} \sum_{j=n}^{N} a_j = 0. \tag{7.3}$$

Now let $z_n = y_n - y_{n-1}$ and suppose $v_j = \sum_{m_{j-1}+1}^{m_j - 1} a_j z_j$ for $1 \le j \le n$ where $m_0 = 0 < m_1 < \cdots < m_n$ with $m_j - m_{j-1} \ge 2$ for $j \ge 2$. Then we claim that if $\epsilon_j = \pm 1$ we have

$$\|\sum_{j=1}^{n} \epsilon_j v_j\| \le \prod_{j=1}^{n} (1 + \delta_{m_j})\|\sum_{j=1}^{n} v_j\|. \tag{7.4}$$

This is proved by induction on $n \ge 2$. For $n = 2$ it follows from (7.3). Assume it is proved for $n - 1$. Then

$$\begin{aligned}\|\sum_{j=1}^{n} \epsilon_j v_j\| &\le (1 + \delta_{m_1})\|v_1 + v_2 + \sum_{j=3}^{n} \epsilon_2 \epsilon_j v_j\| \\ &\le \prod_{j=1}^{n} (1 + \delta_{m_j})\|\sum_{j=1}^{n} v_j\|.\end{aligned}$$

Hence $(y_j - y_{j-1})_{j=1}^{\infty}$ is asymptotically skipped 1-unconditional. □

**Theorem 7.0.28.** *Let $X$ be a separable Banach space. Then the following are equivalent:*

*(i) $X$ has properties (au) and (WABS),*

*(ii) For any $\delta > 0$ there is a Banach space $Y$ with a shrinking 1-unconditional basis and a subspace $X_\delta$ of $Y$ such that $d(X, X_\delta) < 1 + \delta$.*

**Proof:** Of course by Theorem 5.0.16 (ii) is equivalent to the fact that $X$ has (au$^*$).

(ii) $\implies$ (i). We observe that (ii) implies that $X$ has property (u) and hence (WABS). Property (au) follows trivially from (ii).

(i) $\implies$ (ii). Clearly $X$ contains no copy of $\ell_1$. Suppose $x^{**} \in X^{**} \setminus X$; by the Odell-Rosenthal theorem [23] and property (WABS) there is a sequence $(x_n)_{n=1}^{\infty}$ converging weak$^*$ to $x^{**}$ with the property that

$$\lim_{n\to\infty} \sup_{r_1<r_2<\cdots<r_n} \left\| \frac{1}{n} \left( \sum_{k=1}^{n} (-1)^k x_{r_k} \right) \right\| = 0.$$

According to Lemma 7.0.27, by passing to a subsequence we can assume that $(x_n - x_{n-1})_{n=1}^{\infty}$ is asymptotically skipped 1-unconditional. But then no spreading model of $(x_n - x_{n-1})_{n=1}^{\infty}$ (with $x_0 = 0$) is equivalent to the $\ell_1-$basis. Thus, by Proposition 6.0.24 we have that the space $E = [x_n - x_{n-1}]_{n=1}^{\infty}$ has property (au$^*$). In particular by Theorem 5.0.16 $E$ has property (u). Since $x^{**}$ is in the weak$^*$-closure of $E$ we conclude that $X$ has property (u).

We next show that $X$ has property (au$^*$). Suppose not. Then there exists $x^* \in X^*$ and a weak$^*$-null sequence $(x_n^*)_{n=1}^{\infty}$ such that $\|x^* + x_n^*\| \leq 1$ and $\|x^* - x_n^*\| > 1 + \delta$ for some $\delta > 0$. Pick $x_n \in X$ so that $\|x_n\| = 1$ but $x^*(x_n) - x_n^*(x_n) > 1 + \delta$. If

$(x_n)_{n=1}^\infty$ is weakly convergent to some $x$ then we obtain a contradiction since

$$\lim_{n\to\infty} x^*(2x - x_n) + x_n^*(2x - x_n) = \lim_{n\to\infty} x^*(x_n) - x_n^*(x_n) > 1 + \delta$$

but

$$\lim_{n\to\infty} \|2x - x_n\| = 1.$$

Thus we can assume, passing to a subsequence, that $(x_n)_{n=1}^\infty$ is a basic sequence which converges weak$^*$ to some $x^{**} \in X^{**} \setminus X$. Since $X$ has property (u) there is sequence $(y_n)_{n=1}^\infty$ in $X$ so that $(y_n)_{n=1}^\infty$ also converges weak$^*$ to $x^{**}$ and is equivalent to the summing basis of $c_0$. Let $G = [y_n]_{n=1}^\infty$. By Sobczyk's theorem (see [30] or e.g. [1] Theorem 2.5.8) there is a projection $P : X \to G$. Then $(P^{**}x_n)_{n=1}^\infty$ converges weak$^*$ to $x^{**}$ and so if $Q = I - P$ the sequence $(Qx_n)_{n=1}^\infty$ is weakly null.

Now, by Lemma 3.0.9, passing to a further subsequence of $(x_n)_{n=1}^\infty$ we can suppose that either (a) $\|Qx_n\| < 2^{-n}$ or (b) $(Qx_n)_{n=1}^\infty$ is an unconditional basis for its closed linear span $Z$. We may also suppose that $z_n = x_n - x_{n-1}$ (where $x_0 = 0$) defines an asymptotically skipped 1-unconditional basis of $Z$. In case (a) the space $E = [x_n]_{n=1}^\infty$ is isomorphic to a subspace of $c_0$. In case (b) $E$ is isomorphic to a subspace of $Z \oplus G$. In either case $E$ embeds (isomorphically, not isometrically) into a space with a shrinking unconditional basis. In particular the biorthogonal sequence $(z_n^*)_{n=1}^\infty$ in $Z^*$ (which is weak$^*$-null) has a subsequence which is an unconditional basic sequence (again by Lemma 3.0.9). By Proposition 6.0.24 this means that $Z$ has property (au$^*$). Now $\|(x^* + x_n^*)|_Z\| \le 1$ and so $\limsup_{n\to\infty} \|(x^* - x_n^*)|_Z\| \le 1$. However $(x^* - x_n^*)(x_n) > 1 + \delta$ and we have a contradiction. □

**Remark** We do not know whether it is possible to replace the (WABS)-

condition in (i) by the assumption that $X$ contains no copy of $\ell_1$ (or even that $X^*$ is separable). This problem reduces to the question of whether one can find a space $Y$ with an asymptotically skipped 1-unconditional basis, which contains no copy of $\ell_1$ but does not have property (au*). If one further imposes the condition that $Y$ contains no copy of $c_0$ then $Y$ would be quasi-reflexive of order one by Proposition 6.0.23. It is certainly possible to find such quasi-reflexive spaces which fail the (WABS) property; this is the requirement that the transfinite dual $Y^\omega \approx Y \oplus \ell_1$. Examples have been given by Bellenot [3] and by Haydon, Odell and Rosenthal [9]. However it seems difficult to impose the extra condition that $Y$ has an asymptotically skipped 1-unconditional basis and therefore leads us to speculate that Theorem 7.0.28 can be improved.

Note that the James space [10] (or see [1] p.62) is quasi-reflexive and does have (WABS). It therefore fails $(au^*)$ (it does not even have property (u)). In fact we showed in chapter 3 that the James space fails $(au)$, and since it is separable, it follows that it fails (au*). By Theorem 7.0.28 the James space cannot have $(au)$ under any equivalent norming. However it does have the (UTP) of Johnson and Zheng [11].

**Remark** The (WABS)-condition also appears implicitly in [16] where Theorem 4.5 could be rephrased as saying that a separable Banach space with the (WABS) property and the $\mathcal{Q}-$property is reflexive; this implies that if $X$ is a space with the (WABS) property such that $X$ coarsely embeds into a reflexive space or $B_X$ uniformly embeds into a reflexive space then $X$ is reflexive. There is a clear link with the problems considered here. For example if $X$ is a separable Banach space with an unconditional basis containing no copy of $c_0$ then $B_X$ uniformly embeds in a reflexive space (Theorem 3.8 of [16]).

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